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Automorphic Forms, Representations, and L-functions

A. Borel
W. Casselman

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Foreword

The twenty-fifth AMS Summer Research Institute was devoted to automorphic forms, representations and L-functions. It was held at Oregon State University, Corvallis, from July 11 to August 5, 1977, and was financed by a grant from the National Science Foundation. The Organizing Committee consisted of A. Borel, W. Casselman (cochairmen:), P. Deligne, H. Jacquet, R. P. Langlands, and J. Tate. The papers in this volume consist of the Notes of the Institute, mostly in revised form, and of a few papers written later.

A main goal of the Institute was the discussion of the L-functions attached to automorphic forms on, or automorphic representations of, reductive groups, the local and global problems pertaining to them, and of their relations with the L-functions of algebraic number theory and algebraic geometry, such as Artin L-functions and Hasse-Weil zeta functions. This broad topic, which goes back to E. Hecke, C. L. Siegel and others, has undergone in the last few years and is undergoing even now a considerable development, in part through the systematic use of infinite dimensional representations, in the framework of adelic groups. This development draws on techniques from several areas, some of rather difficult access. Therefore, besides seminars and lectures on recent and current work and open problems, the Institute also featured lectures (and even series of lectures) of a more introductory character, including background material on reductive groups, their representations, number theory, as well as an extensive treatment of some relatively simple cases.

The papers in this volume are divided into four main sections, reflecting to some extent the nature of the prerequisites. I is devoted to the structure of reductive groups and infinite dimensional representations of reductive groups over local fields. Five of the papers supply some basic background material, while the others are concerned with recent developments. II is concerned with automorphic forms and automorphic representations, with emphasis on the analytic theory. The first four papers discuss some basic facts and definitions pertaining to those, and the passage from one to the other. Two papers are devoted to Eisenstein series and the trace formula, first for GL_2 and there in more general cases. In fact, the trace formula and orbital integrals turned out to be recurrent themes for the whole Institute and are featured in several papers in the other sections as well. The main theme of the last four papers is the restriction of the oscillator representation of the metaplectic group to dual reductive pairs of subgroups, first in general and then in more special cases.

III begins with the background material on number theory, chiefly on Weil groups and their L-functions. It then turns to the L-functions attached to automorphic representations, various ways to construct them, their (conjectured or proven) properties and local and global problems pertaining to them. The remaining papers are mostly devoted to the base change problem for GL_2 and its applications to the proof of holomorphy of certain nonabelian Artin series.

Finally, IV relates automorphic representations and arithmetical algebraic geometry. Over function fields, it gives an introduction to the work of Drinfeld for

GL_2, which constructs systems of l-adic representations whose L-series is a given automorphic L-function. Over number fields, it is mainly concerned with problems on Shimura varieties: canonical models, the point of their reductions modulo prime ideals, and Hasse-Weil zeta functions.

This Institute emphasized representations so that, at least formally, the primary object of concern was an automorphic representation rather than an automorphic form. However, there is no substantial difference between the two, and this should not hide the fact that the theory is a direct outgrowth of the classical theory of automorphic forms. In order to give a comprehensive treatment of our subject matter and yet not produce too heavy a schedule, it was decided to omit a number of topics on automorphic forms which do not fit well at present into the chosen framework. For example, the Institute was planned to have little overlap with the Conference on Modular Functions of One Variable held in Bonn (1976). The reader is referred to the Proceedings of the latter (Springer Lecture Notes 601, 627) and to those of its predecessor (Springer Lecture Notes 320, 350, 476) for some of those topics and a more classical point of view. Also, some topics of considerable interest in themselves such as reductive groups, their infinite dimensional representations, or moduli varieties, were discussed chiefly in function of the needs of the main themes of the Institute.

These Proceedings appear in two parts, the first one contains sections I and II, and the second one sections III and IV.

A. Borel
W. Casselman

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REDUCTIVE GROUPS

T. A. SPRINGER

This contribution contains a review of the theory of reductive groups. Some knowledge of the theory of linear algebraic groups is assumed, to the extent covered in §§1–5 of Borel's report **[2]** in the 1965 Boulder conference.

§§1 and 2 contain a discussion of notion of the "root datum" of a reductive group. This is quite important for the theory of L-groups. Since the relevant results are not too easily accessible in the literature (they are dealt with, in a more general context, in the latter part of the Grothendieck-Demazure seminar **[17]**), it is shown how one can deduce these results from the theory of semisimple groups (which is well covered in the literature). In §§3 and 4 we review facts about the relative theory of reductive groups. There is more overlap with **[2**, §6], which deals with the same material.

§5 contains a discussion of a useful class of Lie groups (the "selfadjoint" ones). We indicate how the familiar properties of these groups can be established, assuming the algebraic theory of reductive groups.

I am grateful to A. Borel for valuable suggestions and to J. J. Duistermaat for comments on the material of §5.

1. Root data and root systems. The notion of root datum (introduced in **[17**, Exposé XXI] under the name of "donnée radicielle") is a slight generalization of the notion of root system, which is quite useful for the theory of reductive groups. Below is a brief discussion of root data. For more details see [loc. cit.]. For the theory of root systems we refer to **[7]**.

1.1. *Root data.* A *root datum* is a quadruple $\Psi = (X, \Phi, X^\vee, \Phi^\vee)$ where: X and $X^\vee$ are free abelian groups of finite type, in duality by a pairing $X \times X^\vee \to \mathbf{Z}$ denoted by $\langle\ ,\ \rangle$, Φ and $\Phi^\vee$ are finite subsets of X and $X^\vee$ and there is a bijection $\alpha \mapsto \alpha^\vee$ of Φ onto $\Phi^\vee$. If $\alpha \in \Phi$ define endomorphisms s_α and $s_{\alpha^\vee}$ of X, $X^\vee$, respectively, by

$$s_\alpha(x) = x - \langle x, \alpha^\vee\rangle \alpha, \qquad s_{\alpha^\vee}(u) = u - \langle \alpha, u\rangle \alpha^\vee.$$

Then the following two axioms are imposed:

(RD1) *For all* $\alpha \in \Phi$ *we have* $\langle \alpha, \alpha^\vee\rangle = 2$;

(RD2) *For all* $\alpha \in \Phi$ *we have* $s_\alpha(\Phi) \subset \Phi$, $s_{\alpha^\vee}(\Phi^\vee) \subset \Phi^\vee$.

It follows from (RD1) that $s_\alpha^2 = \mathrm{id}$, $s_\alpha(\alpha) = -\alpha$ (and similarly for $s_{\alpha^\vee}$). It is clear from the definition of a root datum that if $\Psi = (X, \Phi, X^\vee, \Phi^\vee)$ is one, then $\Psi^\vee = (X^\vee, \Phi^\vee, X, \Phi)$ is also one, the *dual* of Ψ.

Let Ψ be as above. Let Q be the subgroup of X generated by Φ and denote by X_0

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the subgroup of X orthogonal to $\Phi^\times$. Put $V = Q \otimes \mathbf{Q}$, $V_0 = X_0 \otimes \mathbf{Q}$. Define similarly subgroups $Q^\vee$, $X_0^\vee$ of $X^\vee$ and vector spaces $V^\vee$, $V_0^\vee$.

We say that Ψ is *semisimple* if $X_0 = \{0\}$ and *toral* if Φ is empty.

1.2. Lemma. *$Q \cap X_0 = \{0\}$ and $Q + X_0$ has finite index in X.*

This is contained in **[17]**. We sketch a proof. Define a homomorphism $p: X \to X^\vee$ by

$$p(x) = \sum_{\alpha \in \Phi} \langle x, \alpha^\vee \rangle \alpha^\vee$$

Since $\langle x, p(x) \rangle = \sum_{\alpha \in \Phi} \langle x, \alpha^\vee \rangle^2$ we have $X_0 = \operatorname{Ker} p$.

Next observe that if $\alpha \in \Phi$ we have $p(\alpha) = \frac{1}{2} \langle \alpha, p(\alpha) \rangle \alpha^\vee$, as follows by summation over $\beta \in \Phi$ from the identity

$$\langle \alpha, \beta^\vee \rangle^2 \alpha^\vee = \langle \alpha, \beta^\vee \rangle \beta^\vee + \langle \alpha, s_{\alpha^\vee}(\beta^\vee) \rangle s_{\alpha^\vee}(\beta^\vee).$$

This shows that $p \otimes \mathrm{id}$ is a surjection $V \to V^\vee$, whence $\dim V^\vee \leqq \dim V$. By symmetry we then have $\dim V = \dim V^\vee$, whence $Q \cap \operatorname{Ker} p = \{0\}$. The assertion now follows readily.

1.3. *Root systems.* It follows from the proof of 1.2 that $V^\vee$ can be identified with the dual of the vector space V. We write again $\langle\ ,\ \rangle$ for the pairing. Also identify Φ with $\Phi \otimes 1 \subset V$ and assume that $\Phi \neq \varnothing$. We then see that Φ is a root system in V in the sense of [7]. Recall that this means that the following conditions are satisfied:

(RS1) *Φ is finite and generates V, moreover $0 \notin \Phi$;*

(RS2) *for all $\alpha \in \Phi$ there is $\alpha^\vee \in V^\vee$ such that $\langle \alpha, \alpha^\vee \rangle = 2$ and that s_α (defined as before) stabilizes Φ;*

(RS3) *for all $\alpha \in \Phi$ we have $\alpha^\vee(\Phi) \subset \mathbf{Z}$.*

The s_α then generate a finite group of linear transformations of V, the *Weyl group* $W(\Phi)$ of Φ.

If $\Psi = (X, \Phi, X^\vee, \Phi^\vee)$ is a root datum which is not toral, we call the root system $\Phi \subset V$ the *root system of* Ψ. The Weyl group $W(\Phi)$ is identified with the group of automorphisms of X generated by the s_α of 1.1 and with the group of automorphisms of $X^\vee$ generated by the $s_{\alpha^\vee}$.

The following observation is sometimes useful.

1.4. Lemma. *Axiom* (RD2) *is equivalent to*:

(RD2′)(a) *For all $\alpha \in \Phi$ we have $s_\alpha(\Phi) \subset \Phi$;*

(b) *the s_α ($\alpha \in \Phi$) generate a finite group.*

It suffices to prove that (RD2′) implies the second assertion of (RD2). Let α, $\beta \in \Phi$. Then $s_\alpha s_\beta s_\alpha$ and $s_{s_\alpha(\beta)}$ are involutions in the group generated by the s_α. We have by an easy computation,

$$s_{s_\alpha(\beta)} s_\alpha s_\beta s_\alpha(x) = x + (\langle x, {}^t s_\alpha(\beta^\vee) \rangle - \langle x, s_\alpha(\beta)^\vee \rangle) s_\alpha(\beta),$$

where ${}^t s_\alpha$ is the transpose of s_α. Since $\langle s_\alpha(\beta), {}^t s_\alpha(\beta^\vee) \rangle - \langle s_\alpha(\beta), s_\alpha(\beta)^\vee \rangle = \langle \beta, \beta^\vee \rangle - \langle s_\alpha(\beta), s_\alpha(\beta)^\vee \rangle = 0$, we see that the above automorphism of X is unipotent. Since it lies in a finite group it must be the identity. Hence $s_\alpha(\beta)^\vee = {}^t s_\alpha(\beta^\vee)$, and the assertion follows by observing that $s_{\alpha^\vee} = {}^t s_\alpha$.

1.5. *Properties of root systems.* Let $\Phi \subset V$ be a root system. Proofs of the properties reviewed below can be found in [7].

(a) If $\alpha \in \Phi$ and $\lambda\alpha \in \Phi$ then $\lambda = \pm 1, \pm\frac{1}{2}, \pm 2$. The root system Φ is called *reduced* if for all $\alpha \in \Phi$ the only multiples of α in Φ are $\pm\alpha$. To every root system, there belong two reduced root systems, obtained by removing for every $\alpha \in \Phi$ the longer (or shorter) multiples of α.

(b) Φ is the direct sum of root systems $\Phi' \subset V'$ and $\Phi'' \subset V''$ if $V = V' \oplus V''$ and $\Phi = \Phi' \cup \Phi''$. A root system is *irreducible* if it is not the direct sum of two subsystems.

Every root system is a direct sum of irreducible ones.

(c) The only reduced irreducible root systems are the usual ones: $\boldsymbol{A}_n$ $(n \geqq 1)$, $\boldsymbol{B}_n$ $(n \geqq 2)$, $\boldsymbol{C}_n$ $(n \geqq 3)$, $\boldsymbol{D}_n$ $(n \geqq 4)$, $\boldsymbol{E}_6, \boldsymbol{E}_7, \boldsymbol{E}_8, \boldsymbol{F}_4, \boldsymbol{G}_2$.

(d) For each dimension n there exists one irreducible nonreduced root system, denoted by $\boldsymbol{BC}_n$ (see below).

EXAMPLES. (i) $\boldsymbol{B}_n$. Take $V = \boldsymbol{Q}^n$ with standard basis $\{e_1, \cdots, e_n\}$. Then $V^\vee = \boldsymbol{Q}^n$, with dual basis $\{e_1^\vee, \ldots, e_n^\vee\}$. We have $\boldsymbol{B}_n = \{\pm e_i \pm e_j \ (i < j)$ and $\pm e_i$ $(1 \leqq i, j \leqq n)\}$. If $\alpha = \pm e_i \pm e_j$ then $\alpha^\vee = \pm e_i^\vee \pm e_j^\vee$; if $\alpha = \pm e_i$ then $\alpha^\vee = \pm 2e_i$. The Weyl group $W(\boldsymbol{B}_n)$ consists of the linear transformations which permute the coordinates and change their signs in all possible ways.

The $\alpha^\vee \in V^\vee$ form an irreducible root system of type $\boldsymbol{C}_n$. We have $W(\boldsymbol{C}_n) \simeq W(\boldsymbol{B}_n)$.

(ii) With the same notations we have $\boldsymbol{BC}_n = \{\pm e_i \pm e_j \ (i < j), \pm e_i$, and $\pm 2e_i$ $(1 \leqq i, j \leqq n)\}$. Then $W(\boldsymbol{BC}_n) = W(\boldsymbol{B}_n)$.

1.6. *Weyl chambers.* Let $\Phi \subset V$ be a root system. We now view it as a subset of $V_{\boldsymbol{R}} = V \otimes_{\boldsymbol{Q}} \boldsymbol{R}$. A hyperplane H of $V_{\boldsymbol{R}}$ is *singular* if it is orthogonal to an $\alpha^\vee$. A *Weyl chamber* C in $V_{\boldsymbol{R}}$ is a connected component of the complement of the union of the singular hyperplanes. To a Weyl chamber one associates an ordering of the roots: $\alpha > 0 \Leftrightarrow \langle x, \alpha^\vee\rangle > 0$ for all $x \in C$.

$\alpha \in \Phi$ is *simple* (for this ordering) if it is not the sum of two positive roots. The set of simple roots Δ is called a *basis* of Φ. We have the following properties.

(a) The Weyl group $W(\Phi)$ acts simply transitively on the set of Weyl chambers.

(b) The s_α $(\alpha \in \Delta)$ generate $W(\Phi)$. More precisely, $(W, (s_\alpha)_{\alpha\in\Delta})$ is a Coxeter system (see [7]).

(c) Every root is an integral linear combination of simple roots, with coefficients all of the same sign.

(d) Say that Δ is connected if it cannot be written as a disjoint union $\Delta = \Delta' \cup \Delta''$, where $(\Delta' + \Delta'') \cap \Delta = \varnothing$.

Then we have: Φ is irreducible $\Leftrightarrow \Delta$ is connected.

A connected Δ leads to a connected Dynkin graph. These are described in [7].

1.7. We collect a few facts about root data to be needed later. First, there is the notion of *direct sum* of root data. This is clear and we skip the definition.

Next we have to say something about morphisms of root data. The following suffices. For more general cases see [7]; see also 2.11(ii) and 2.12.

Let $\Psi = (X, \Phi, X^\vee, \Phi^\vee)$ and $\Psi' = (X', \Phi', (X')^\vee, (\Phi')^\vee)$ be two root data. A homomorphism $f\colon X' \to X$ is called an *isogeny of* Ψ' *into* Ψ *if*:

(a) f is injective and $\operatorname{Im} f$ has finite index in X,

(b) f induces a bijection of Φ' onto Φ and its transpose ${}^tf$ induces a bijection of $\Phi^\vee$ onto $(\Phi')^\vee$.

Notice that then ${}^tf$ is also an injection $X^\vee \to (X')^\vee$ with finite cokernel. Also, coker f and coker$({}^tf)$ are in duality.

EXAMPLE. Given Ψ, we shall construct a Ψ' and an isogeny of X into X', which we shall call the *canonical isogeny* associated to Ψ.

If L is a subgroup of X we denote by $\tilde{L}$ the largest subgroup containing L such that $\tilde{L}/L$ is finite. Then $L = \tilde{L}$ if and only if L is a direct summand.

Let X_0 and Q be as in 1.1. By 1.2 we can view Φ as a subset of X/X_0. It follows that $\Psi_1' = (X/X_0, \Phi, \tilde{Q}^\vee, \Phi^\vee)$ is a semisimple root datum. Likewise, $\Psi_1'' = (X/\tilde{Q}, \varnothing, X_0, \varnothing)$ is a toral root datum. Put $\Psi' = \Psi_1' \oplus \Psi_2''$. Then the *canonical isogeny* f: $X \to (X/X_0) \oplus (X/\tilde{Q})$ is the canonical homomorphism of X into the right-hand side.

1.8. Let Φ be a root system in the $\boldsymbol{Q}$-vector space V. The $\alpha^\vee$ ($\alpha \in \Phi$) in the dual $V^\vee$ of V also form a root system $\Phi^\vee$ (the dual or inverse root system). There are finitely many semisimple root data $(X, \Phi, X^\vee, \Phi^\vee)$ where $X \subset V$, $X^\vee \subset V^\vee$. In fact, let Q and $Q^\vee$ be the lattices in V and $V^\vee$ generated by Φ and $\Phi^\vee$, respectively, and define $P = \{x \in V | \langle x, \alpha^\vee \rangle \in \boldsymbol{Z} \text{ for all } \alpha \in \Phi\}$. $P^\vee \subset V^\vee$ is defined similarly. Then $Q \subset P$ and P/Q is a finite group, in duality with $P^\vee/Q^\vee$. An X as above is then contained between P and Q and for each such X there is a unique $X^\vee$ between $P^\vee$ and $Q^\vee$ such that $(X, \Phi, X^\vee, \Phi^\vee)$ is a root datum.

1.9. Let $\Psi = (X, \Phi, X^\vee, \Phi^\vee)$ be a root datum. Assume that its root system $\Phi \subset V$ is *reduced*. Let Δ be a basis of Φ. Then $\Delta^\vee = \{\alpha^\vee | \alpha \in \Delta\}$ is a basis of the dual root system $\Phi^\vee \subset V^\vee$.

We call *based root datum* a sextuple $\Psi_0 = (X, \Phi, \Delta, X^\vee, \Phi^\vee, \Delta^\vee)$, where $(X, \Phi, X^\vee, \Phi^\vee)$ is a root datum with reduced root system Φ and where Δ is a basis of Φ. However, since Δ and $\Delta^\vee$ determine Φ and $\Phi^\vee$ uniquely, it also makes sense to view a based root system as a quadruple $\Psi_0 = (X, \Delta, X^\vee, \Delta^\vee)$. This we shall do.

2. Reductive groups (absolute theory).

2.1. Let G be a connected reductive linear algebraic group. In this section we consider the absolute case, where fields of definition do not come in. So we can view G as a subgroup of some $\mathbf{GL}(n, \Omega)$, Ω an algebraically closed field (see [**2**]). Let S be a subtorus of G. We define the root system $\Phi(G, S)$ of G with respect to S to be the set of nontrivial characters of S which appear when one diagonalizes the representation of S in the Lie algebra $\mathfrak{g}$ of G, S operating via the adjoint representation.

2.2. *The root datum of* G. Fix a maximal torus T of G. We shall associate to the pair (G, T) a root datum $\psi(G, T) = (X, \Phi, X^\vee, \Phi^\vee)$ (also denoted by $\psi(G)$).

X is the group of rational characters $X^*(T)$ of T. This is a free abelian group of finite rank. $X^\vee$ is the group $X_*(T)$ of 1-parameter multiplicative subgroups of T, i.e., the group of homomorphisms (of algebraic groups) $\mathbf{GL}_1 \to T$. Then $X^\vee$ can be put in duality with X by a pairing $\langle \ , \ \rangle$ defined as follows: if $x \in X^*(T)$, $u \in X_*(T)$, then $x(u(t)) = t^{\langle x, u \rangle}$ ($t \in \Omega^*$).

We take $\Phi = \Phi(G, T)$, the root system of G with respect to T. To complete the definition we have to describe $\Phi^\vee$. If $\alpha \in \Phi$, let T_α be the identity component of the kernel of α. This is a subtorus of codimension 1. The centralizer Z_α of T_α in G is a connected reductive group with maximal torus T, whose derived group G_α is semisimple of rank 1, i.e., is isomorphic to either $\mathbf{SL}(2)$ or $\mathbf{PSL}(2)$. There is a unique homomorphism $\alpha^\vee\colon \mathbf{GL}_1 \to G_\alpha$ such that $T = (\operatorname{Im} \alpha^\vee)T_\alpha$, $\langle \alpha, \alpha^\vee \rangle = 2$. These $\alpha^\vee$ make up $\Phi^\vee$.

The axiom (RD1) is built into the definition of $\alpha^\vee$. We use 1.4 to establish (RD2). Let $n_\alpha \in G_\alpha - T_\alpha$ normalize T_α. Then $n_\alpha^2 \in T_\alpha$ and, s_α being as in 1.1, we have, for $x \in X$, $t \in T$,

$$x(n_\alpha t n_\alpha^{-1}) = t^{s_\alpha(x)}.$$

In fact, working in G_α one shows that there is $u \in X^\vee$ such that the left-hand side equals $t^{x - \langle x, u\rangle \alpha}$. One then shows that $\langle \alpha, u \rangle = 2$ and that $\langle x, u \rangle = 0$ if $\langle x, \alpha^\vee \rangle = 0$. It follows that (RD2′) holds. So $\Phi(G)$ is a root datum. The root system Φ is reduced (for all these facts see [3] or [**14**]).

2.3. To each $\alpha \in \Phi$ there is associated a unique homomorphism of algebraic groups $x_\alpha\colon \mathbf{G}_a \to G_\alpha$ such that

$$t x_\alpha(u) t^{-1} = x_\alpha(t^\alpha u) \qquad (t \in T,\ u \in \Omega).$$

Put $U_\alpha = \operatorname{Im}(x_\alpha)$ and let $X_\alpha \in \mathfrak{g}$ be a nonzero tangent vector to U_α. Then

$$\mathfrak{g} = \operatorname{Lie}(T) \oplus \sum_{\alpha \in \Phi} \Omega X_\alpha.$$

Let B be a Borel subgroup containing T. There is a unique ordering of Φ (as in 1.6) such that B is generated by T and the U_α with $\alpha > 0$, and any $B \supset T$ is so obtained. It follows that we can associate to the triple (G, B, T) a based root system $\psi_0(G, B, T) = (X^*(T), \Delta, X_*(T), \Delta^\vee)$ (or $\psi_0(G)$), where Δ is the basis of Φ determined by the ordering associated to B.

2.4. *Isogenies*. An *isogeny* $\phi\colon G \to G'$ of algebraic groups is a surjective rational homomorphism with finite kernel.

EXAMPLES. (i) The canonical homomorphism $\mathbf{SL}(2) \to \mathbf{PSL}(2)$ ($\mathbf{PSL}(2)$ is to be viewed as the group of linear transformations of the space of 2×2-matrices of the form $x \mapsto gxg^{-1}$, where $g \in \mathbf{SL}(2)$). If char $\Omega = 2$ this is an isomorphism of abstract groups, but not of algebraic groups.

(ii) Let G be defined over the finite field $\boldsymbol{F}_q$. The Frobenius isogeny $G \to G$ raises all coordinates to the qth power. It is again an isomorphism of groups, but not of algebraic groups.

Let G and G' be connected reductive and let T be a maximal torus of G. *A central isogeny* $\phi\colon G \to G'$ is an isogeny which (with the notations of 2.3) induces an isomorphism in the sense of algebraic groups of U_α onto its image, for all $\alpha \in \Phi$. Equivalently, $d\phi(X_\alpha) \neq 0$ for all $\alpha \in \Phi$ (where $d\phi$ is the induced Lie algebra homomorphism). The image $T' = \phi(T)$ is a maximal torus of G'. We shall say that ϕ is a central isogeny of (G, T) onto (G', T').

Let $f(\phi)$ be the homomorphism $X^*(T') \to X^*(T)$ defined by ϕ.

2.5. PROPOSITION. (i) *If ϕ is a central isogeny then $f(\phi)$ is an isogeny of $\psi(G', T')$ into $\psi(G, T)$;*

(ii) *if ϕ and ϕ' are central isogenies of (G, T) onto (G', T') such that $f(\phi) = f(\phi')$ then there is $t \in T$ with $\phi' = \phi \circ \operatorname{Int}(t)$.*

That $f(\phi)$ has property (a) of 1.7 is equivalent to the fact that ϕ induces a surjection $T \to T'$ with finite kernel. There is a bijection $\alpha \mapsto \alpha'$ of root systems such that $\phi(U_\alpha) = U_{\alpha'}$ or that $d\phi(X_\alpha) = X_{\alpha'}$ (choosing $X_{\alpha'}$ properly). We then have $\mathrm{Ad}(\phi(t))X_{\alpha'} = \alpha(t)X_{\alpha'}$, whence $f(\phi)(\alpha') = \alpha$. Then ${}^t f(\phi)(\alpha') = (\alpha')^\vee$, as follows for example from the equality ${}^t s_\alpha = s_{\alpha^\vee}$ established in the proof of 1.8. This proves (i).

Let Δ be a basis of Φ. One knows that the U_α with $\alpha \in \Delta$ together with T generate G. So an isogeny ϕ is completely determined by its restriction to T and to the U_α $(\alpha \in \Delta)$. Since $f(\phi)$ determines $T \to T'$, the only freedom one has when $f(\phi)$ is given, is in the choice of the isomorphisms $U_\alpha \xrightarrow{\sim} U_{\alpha'}$ $(\alpha \in \Delta)$. The assertion of (ii) then readily follows.

2.6. Let ϕ be a central isogeny of (G, T) onto (G', T'). Then $\mathrm{Ker}\,\phi$ lies in T. It is a finite group isomorphic to $\mathrm{Hom}(X/\mathrm{Im}\,f(\phi), \Omega^*)$. Let p be the characteristic exponent of Ω. Then this kernel is isomorphic to the p-regular part of $X/\mathrm{Im}\,f(\phi)$. It follows that there is a factorization of ϕ: $G \xrightarrow{\pi} G/\mathrm{Ker}\ \phi \xrightarrow{\rho} G'$, where π is the canonical homomorphism and where ρ is an isomorphism if $p = 1$ and ρ is a purely inseparable isogeny if $p > 1$ (i.e., such that ρ is an isomorphism of groups). Let $\mathfrak{t}$ be the Lie algebra of T; we have $\mathrm{Ker}(d\phi) \subset \mathfrak{t}$. Now $\mathfrak{t}$ can be identified with $X^\vee \otimes_{\mathbf{Z}} \Omega$. It follows that $\mathrm{Ker}(d\phi)$ is isomorphic to the kernel of $f(\phi)^\vee \otimes \mathrm{id}$: $X^\vee \otimes_{\mathbf{Z}} \Omega \to (X')^\vee \otimes_{\mathbf{Z}} \Omega$, which is isomorphic to $((X')^\vee/p(X')^\vee + \mathrm{Im}\,f(\phi)^\vee) \otimes_{\mathbf{F}_p} \Omega$. Hence $\mathrm{Ker}(d\phi) = 0$ if and only if $\mathrm{Coker}(f(\phi))$ (which is dual to $\mathrm{Coker}({}^t f(\phi))$) has order prime to p.

If $p > 1$ then $\mathrm{Ker}(d\phi)$ is a central restricted subalgebra of $\mathfrak{g}$, which is stable under $\mathrm{Ad}(G)$. Let $G/\mathrm{Ker}(d\phi)$ be the quotient of G by $\mathrm{Ker}\,d\phi$ (see [**3**, p. 376].) It follows that we can factor ϕ, $G \xrightarrow{\sigma} G/\mathrm{Ker}\ (d\phi) \xrightarrow{\tau} G'$, where σ is the canonical (central) isogeny of [loc. cit.]. These remarks imply that we can factorize ϕ as follows:

$$(1) \qquad G = G_0 \xrightarrow{\pi_0} G_1 \xrightarrow{\pi_1} G_2 \xrightarrow{\pi_2} \cdots \longrightarrow G_{s-1} \xrightarrow{\pi_{s-1}} G_s = G',$$

where $\phi = \pi_{s-1} \circ \cdots \circ \pi_0$. Put $\phi_i = \pi_{s-1} \circ \cdots \circ \pi_i$ $(i \geqq 1)$. Then $G_1 = G/\mathrm{Ker}\ \phi$, $G_{i+1} = G_i/\mathrm{Ker}(d\phi_i)$ $(1 \leqq i \leqq s - 1)$ and the π_i are canonical isogenies.

Also, if

$$\begin{array}{ccc} G & \xrightarrow{\phi} & G' \\ \downarrow & & \downarrow \\ \tilde{G} & \xrightarrow{\phi} & \tilde{G}' \end{array}$$

is a commutative diagram of isogenies, we can arrange the factorizations of ϕ and $\tilde{\phi}$ such that there is a diagram with commuting squares

$$(2) \qquad \begin{array}{ccccccccc} G = G_0 & \longrightarrow & G_1 & \longrightarrow & \cdots & \longrightarrow & G_{s-1} & \longrightarrow & G_s = G' \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ \tilde{G} = \tilde{G}_0 & \longrightarrow & \tilde{G}_1 & \longrightarrow & \cdots & \longrightarrow & \tilde{G}_{s-1} & \longrightarrow & \tilde{G}_s = \tilde{G}' \end{array}$$

Notice that the vertical arrows are uniquely determined once the first one is given.

2.7. Lemma. *Let ϕ and ϕ_1 be central isogenies of (G, T) onto (G', T') and (G_1', T_1'), respectively. Assume that* $\mathrm{Im}\,f(\phi) = \mathrm{Im}\,f(\phi_1)$. *Then (G', T') and (G_1', T_1') are isomorphic.*

Let $G'' \subset G' \times G_1'$ be the image of the homomorphism $g \mapsto (\phi(g), \phi_1(g))$. Let $\psi\colon G \to G''$ be the induced homomorphism. Then ψ is a central isogeny of connected reductive groups and so are the projections $\pi_1\colon G'' \to G'$, $\pi_2\colon G'' \to G_1'$. A straightforward check (working in tori) shows that Ker $\pi_i = \{e\}$, Ker $d\pi_i = \{0\}$ ($i = 1, 2$). Hence π_1 and π_2 are isomorphisms, whence the lemma.

2.8. LEMMA. *Let Ψ' be a root datum and let f be an isogeny of Ψ' into $\psi(G, T)$. Then there exist a pair (G', T') and a central isogeny ϕ of (G, T) onto (G', T') such that $\Psi' = \psi(G', T'), f(\phi) = f$.*

From the knowledge of f we can recover successively the groups G_i figuring in (1). This allows one to define G'. We omit the details.

2.9. THEOREM. (i) *For any root datum Ψ with reduced root system there exist a connected reductive group G and a maximal torus T in G such that $\Psi = \psi(G, T)$. The pair (G, T) is unique up to isomorphism*;

(ii) *let $\Psi = \psi(G, T), \Psi' = \psi(G', T')$. If f is an isogeny of Ψ' into Ψ there exists a central isogeny ϕ of (G, T) onto (G', T') with $f(\phi) = f$. Two such ϕ differ by an automorphism* Int(t) ($t \in T$) *of G.*

Let f be the canonical isogeny $\Psi \to \Psi'$ of 1.7. Using 2.8 we see that it suffices to prove the existence statement of (i) for the two cases that Ψ is semisimple or toral. The second case ($\Phi = \phi$) is easily dealt with: take for G the torus $T = \mathrm{Hom}(X, \Omega^*)$. In the semisimple case the statement follows from the existence theorem of the theory of semisimple groups which can be dealt with using the theory of Chevalley groups. (See [**18**] or [**6**, part A]. The uniqueness statement of (i) is part of (ii). To prove (ii) one first reduces to the case that f is an isomorphism (using 2.7 and 2.8).) In the case that G is semisimple the statement of (ii) is Chevalley's fundamental isomorphism theorem, proved, e.g., in [**10**, Exposé 24], or in [**14**, Chapter XI]. The case of a torus G is easy. In the general case, there are central isogenies $G_1 \times S \to G$, $G_1' \times S' \to G'$, where G_1 and G_1' are the derived groups of G and G', and where S and S' are tori, such that the corresponding isogenies of root data are just the canonical ones of 1.7 (see also 2.15).

Now f defines an isogeny f_1 of $\psi(G_1' \times S')$ into $\psi(G \times S)$ and we may assume that there exists a central isogeny $\phi_1\colon G_1 \times S \to G_1' \times S'$ with $f(\phi_1) = f_1$.

We can then complete the diagram like (2), with $G_1 \times S$, $G_1' \times S'$, G, G' instead of $G, \tilde{G}, G', \tilde{G}'$, respectively, and with ϕ_1 as first arrow. The right-hand arrow, which is uniquely determined by ϕ_1, is then the required isomorphism. The last point of (ii) follows from 2.5 (ii).

2.10. REMARKS. (i) In the semisimple case the existence statement of 2.9(i) is due to Chevalley [Séminaire Bourbaki, Exposé 219, 1960–1961]. He constructs a group scheme G_0 over $\mathbf{Z}$ such that $G = G_0 \times_{\mathbf{Z}} k$. This construction is also discussed in [**6**, part A].

A generalization of 2.9, where the field k is replaced by a base scheme, is contained in [**17**, Exposé XXIV].

(ii) The result on the existence of central isogenies of reductive groups contained in 2.9(ii) is a special case of one on arbitrary isogenies, which we shall briefly indicate. Let $\phi\colon G \to G'$ be an isogeny of connected reductive groups with $\phi(T) = T'$. Let f be the induced homomorphism $X^*(T') \in X^*(T)$. Let x_α, $x_{\alpha'}$ ($\alpha \in \Phi, \alpha' \in \Phi'$)

be as in 2.3. One shows that there is a bijection $\alpha \mapsto \alpha'$ of Φ onto Φ' and a function $q: \Phi \to \{p^n | n \in N\}$ (p the characteristic exponent) such that the x_α and $x_{\alpha'}$ can be so normalized that $\phi(x_\alpha(t)) = x_{\alpha'}(t^{q(\alpha)})$. It follows that

$$f(\alpha') = q(\alpha)\alpha, \qquad {}^t f(\alpha^\vee) = q(\alpha)(\alpha')^\vee.$$

If ϕ is a central isogeny then all $q(\alpha)$ are 1. If ϕ is the Frobenius isogeny of 2.4 then f is multiplication by q and all $q(\alpha)$ are q.

Such an f is called a *p-morphism* in [**17**, Exposé XXI]. The analogue in question of 2.9(ii) is obtained by assuming f to be a p-morphism and admitting in the conclusion an arbitrary isogeny ϕ. The proof can be given along similar lines, reducing to the case of an isomorphism. For semisimple G the result is due to Chevalley [**10**, Expose 23].

EXAMPLE OF A p-MORPHISM. $p = 2$, G is semisimple of type $\boldsymbol{B}_2$ and, with the notations of the example in 1.5, we have $f(e_1) = e_1 + e_2, f(e_2) = e_1 - e_2$.

A classification of the possible p-morphisms can be found, e.g., in [**17**, Exposé XXI, p.71].

2.11. Let $\phi: G \to G'$ be a homomorphism of connected reductive algebraic groups. Let T and T' be maximal tori in G, G' with $\phi(T) \subset T'$. Assume that Im ϕ is a *normal* subgroup of G'. We shall briefly describe the relation between the root data $\psi(G, T) = (X, \Phi, X^\vee, \Phi^\vee)$ and $\psi(G', T') = (X', \Phi, (X')^\vee, (\Phi')^\vee)$. Let $f: X' \to X$ be the dual of $\phi: T \to T'$. In general, f is neither injective nor surjective.

Put $\Phi_1 = \Phi \cap \operatorname{Im} f$, $\Phi_2 = \Phi - \Phi_1$. Then $\Phi = \Phi_1 \cup \Phi_2$ is a decomposition into orthogonal subsets (i.e., $\langle \Phi_1, \Phi_2^\vee \rangle = \langle \Phi_2, \Phi_1^\vee \rangle = 0$).

Likewise, if $\Phi'_2 = \Phi' \cap \operatorname{Ker} f$, $\Phi'_1 = \Phi' - \Phi'_2$, then $\Phi' = \Phi'_1 \cup \Phi'_2$ is a decomposition into orthogonal subsets. There is a bijection $\alpha \mapsto \alpha'$ of Φ_1 onto Φ'_1 and a function $q : \Phi_1 \to \{p^n | n \in N\}$, such that for $\alpha \in \Phi_1$ we have

$$f(\alpha') = q(\alpha)\alpha, \qquad {}^t f(\alpha^\vee) = q(\alpha)\,(\alpha')^\vee.$$

Moreover $f(\alpha) = 0$ if $\alpha \in \Phi'_2$ and ${}^t f(\alpha) = 0$ if $\alpha \in \Phi_2$.

2.12. It follows readily from 2.9(i) that if Ψ_0 is a based root datum with reduced root system there exists a triple (G, B, T) as in 2.3 with $\psi_0(G, B, T) = \Psi_0$, which is unique up to isomorphism. There is no canonical isomorphism of one such triple onto another one. In fact, based root systems have nontrivial automorphisms.

In this connection the following results should be mentioned. Let $\psi_0(G, B, T) = \psi_0(G)$. For each $\alpha \in \Delta$ fix an element $u_\alpha \neq e$ in the group U_α. The following is then an easy consequence of 2.5(ii).

2.13. PROPOSITION. Aut $\psi_0(G)$ *is isomorphic to the group* $\operatorname{Aut}(G, B, T, \{u_\alpha\}_{\alpha \in \Delta})$ *of automorphisms of G which stabilize B, T and the set of* u_α.

2.14. COROLLARY. *There is a split exact sequence*

$$\{1\} \longrightarrow \operatorname{Int}(G) \longrightarrow \operatorname{Aut}(G) \longrightarrow \operatorname{Aut} \psi_0(G) \longrightarrow \{1\}.$$

In fact, an isomorphism as in 2.13 defines a splitting. Any two such splittings differ by an automorphism $\operatorname{Int}(t)$ ($t \in T$).

2.15. Let G be a connected reductive group, with a maximal torus T. Put $\psi(G, T) = (X, \Phi, X^\vee, \Phi^\vee)$.

We have a decomposition $G = G' \cdot S$, where G' is the derived group of G (which is semisimple) and where S is a torus, viz. the identity component of the center C of G. We have $T = T' \cdot S$, where T' is a maximal torus of G'.

We use the notations of 1.1 and 1.7.

The following facts can be checked without difficulty.

(a) $\psi(G', T') = (X/X_0, \Phi, \tilde{Q}^\vee, \Phi^\vee)$ (we view Φ as a subset of X/X_0, as we may by 1.2).

(b) The character group $\mathrm{Hom}_{\mathrm{alg.gr.}}(C, \Omega^*)$ is X/Q, and we have $X^*(S) = X/\tilde{Q}$, $X^\vee(S) = X_0^\vee$.

(c) The isogeny $G' \times S \to G$ defines the canonical isogeny of (G, T) (see 1.7). This fact was already used in the proof of 2.5(ii).

It follows that G is semisimple if and only if $\psi(G, T)$ is semisimple. In that case we say that G is *adjoint* if $X = Q$ and *simply connected* if $X = P$ (notation of 1.8). From 2.5(ii), using what was said in 1.8, we see that a semisimple group G is adjoint (resp. simply connected) if and only if a central isogeny $\phi: G \to G'$ (resp. $\psi: G' \to G$) is an isomorphism.

In the case of a general reductive G we have the following facts.

(d) The derived group G' is adjoint $\Leftrightarrow X = Q \oplus X_0 \Leftrightarrow X^\vee = P^\vee \oplus X_0^\vee$.

(e) G' is simply connected $\Leftrightarrow P \subset X + (X_0 \otimes Q) \Leftrightarrow \tilde{Q}^\vee = Q^\vee$.

(f) The center of G is connected $\Leftrightarrow Q = \tilde{Q} \Leftrightarrow P^\vee \subset X^\vee + (X_0^\vee \otimes Q)$.

3. Reductive groups (relative theory). Here we let a ground field $k \subset \Omega$ come into play. We denote by $\bar{k}$ the algebraic closure of k in Ω and by k_s its separable closure.

A linear algebraic group G which is defined over k will be called a *k-group*. We then denote by $G(k)$ the group of its k-rational points (and not by G_k, as in **[2]**). If A is a k-algebra, we denote by $G(A)$ the group $\mathrm{Hom}_k(k[G], A)$ (see **[2]**).

3.1. *Forms of algebraic groups* [**16**, III, §1]. Let G and G' be k-groups. G' is said to be a *k-form* of G if G and G' are isomorphic over Ω.

EXAMPLE. $k = \boldsymbol{R}$. Then $\boldsymbol{U}(n)$ is an $\boldsymbol{R}$-form of $\mathbf{GL}(n)$.

To describe k-forms one proceeds as follows. The k-group G is completely determined by the group $G(k_s)$ of k_s-rational points. This means the following: if $G \to \mathbf{GL}(n)$ is an isomorphism of G onto a closed subgroup of $\mathbf{GL}(n)$, everything being defined over k, then the subgroup $G(k_s)$ of $\mathbf{GL}(n, k_s)$ determines G, up to k-isomorphism. The fact that G is defined over k is reflected in an action of the Galois group $\Gamma_k = \mathrm{Gal}(k_s/k)$ on $G(k_s)$. The k-forms G' of G can be described as follows (up to k-isomorphism). We have $G'(k_s) = G(k_s)$ and there is a continuous function $c: s \mapsto c_s$ of Γ_k to the group of k_s-automorphisms of G (the Galois group being provided with the Krull topology and the second group with the discrete topology), satisfying

$$(*) \qquad c_{st} = c_s \cdot s(c_t) \qquad (s, t \in \Gamma_k),$$

such that the action of Γ_k on $G'(k_s)$ (denoted by $(s, g) \mapsto s * g$) is obtained by "twisting" the original action with c: $(s * g) = c_s(s \cdot g)$. G' is k-isomorphic to G if and only if there exists an automorphism c such that $c_s = c^{-1} \cdot sc$.

We say that G' is an *inner form* of G if all c_s are inner automorphisms.

If C is a group on which Γ_k acts, the continuous functions $s \mapsto c_s$ of Γ_k to C which satisfy $(*)$ are called 1-cocycles of Γ_k with values in C. The equivalence classes of these cocycles for the relation: $(c_s) \sim (c'_s)$ if and only if there is $c \in C$ such that $c'_s = c^{-1} \cdot c_s \cdot (sc)$, form the 1-cohomology set $H^1(k, C)$. It has a privileged element 1, coming from the constant function $c_s = e$.

3.2. *Reductive k-groups.* Now let G be a connected reductive k-group. It is said to be *quasi-split* if it contains a Borel subgroup which is defined over k (this is a very restrictive property). G is *split* (over k) if it has a maximal torus which is defined over k and k-split. In this case G is quasi-split.

EXAMPLE. $G = SO(F)$ (see [**2**, pp. 15–16]). This is quasi-split but not split if and only if the dimension n of the underlying vector space is even and the index equals $\frac{1}{2}n - 1$.

From the splitting of 2.14 one concludes that G is an inner form of a quasi-split group.

Now let B be a Borel subgroup of G and $T \subset B$ a maximal torus, both defined over k_s. Let $\psi_0(G) = (X, \Delta, X^\vee, \Delta^\vee)$ be the based root datum defined by (G, B, T). If $s \in \Gamma_k$ there is $g_s \in G(k_s)$ such that

$$\operatorname{int}(g_s)(sB) = B, \quad \operatorname{int}(g_s)(sT) = T.$$

Then $\operatorname{int}(g_s)\circ s$ defines an automorphism of T depending only on s (since the coset Tg_s is uniquely determined). This automorphism determines an automorphism $\mu_G(s)$ of X, permuting the elements of Δ (since $\operatorname{int}(g_s) \circ s$ fixes B). It is easy to check that μ_G defines a homomorphism $\mu_G \colon \Gamma_k \to \operatorname{Aut} \psi_0(G)$. Let G' be a k-form of G. Then $\mu_G = \mu_{G'}$ if and only if G and G' are inner forms of each other.

3.3. *Restriction of the base field* [**22**, 1.3]. Let $l \subset k_s$ be a finite separable extension of k. Let G be an l-group. Then there exists a k-group $H = R_{l/k}G$ characterized by the following property [**2**, 1.4]: for any k-algebra A we have $H(A) = G(A \otimes_k l)$. In particular, $H(k) = G(l)$. Let Σ be the set of k-isomorphisms $l \to k_s$. We then have $H(k_s) = G(k_s)^\Sigma$. The action of $\Gamma_k = \operatorname{Gal}(k_s/k)$ on $H(k_s)$ is as follows.

If $\phi \in G(k_s)^\Sigma$ is a function on Σ with values in $G(k_s)$, and $s \in \Gamma_k$, then, for $\sigma \in \Sigma$,

$$(s \cdot \phi)(\sigma) = \phi(s \cdot \sigma).$$

$R_{l/k}G$ is obtained from G by *restriction of the ground field* from l to k. If G is connected or reductive then so is $R_{l/k}\, G$.

Now let G be connected and reductive. Fix B and T (defined over k_s) as in 3.2 and let $\psi_0(G)$ be the based root datum defined by (G, B, T). Then $H = R_{l/k}G$ contains the Borel subgroup $B_1 = B^\Sigma$ and the maximal torus $T_1 = T^\Sigma$. The based root datum $\psi_0(R_{l/k}G)$ (relative to B_1 and T_1) is then $\psi_0(G)^\Sigma$. The action of Γ_k on the lattice X^Σ is like before: if $s \in \Gamma_k$, $\phi \in X^\Sigma$ then $(s \cdot \phi)(\sigma) = \phi(s \cdot \sigma)$.

3.4. *Anisotropic reductive groups.* A connected reductive k-group G is called *anisotropic* (over k) if it has no nontrivial k-split k-subtorus.

EXAMPLES. (i) Let F be a nondegenerate quadratic form on a k-vector space (char $k \neq 2$). Let $G = SO(F)$ be the special orthogonal group of F (the identity component of the orthogonal group $O(F)$). It is anisotropic over k if and only if F does not represent 0 over k (the proof is given in [**2**, p. 13]).

(ii) If k is a locally compact (nondiscrete) field then G is anisotropic over k if and only if $G(k)$ is *compact.*

(iii) If k is any field then G is anisotropic if and only if $G(k)$ has no unipotent elements $\neq e$ and the group of its k-rational characters $\operatorname{Hom}_k(G, \mathbf{GL}_1)$ is trivial.

3.5. *Properties of reductive k-groups.* We next review the properties of reductive groups. The reference for these is [**5**].

Let G be a connected reductive k-group. Let S be a maximal k-split torus of G, i.e., a k-subtorus of G which is k-split and maximal for these properties. Any two such tori are conjugate over k, i.e., by an element of $G(k)$. Their dimension is called the *k-rank* of G.

The root system $\Phi(G, S)$ of G with respect to S (see 2.1) is called the *relative root system* of G (notation ${}_k\Phi$ or ${}_k\Phi(G)$). This is indeed a root system in the sense of [**7**], lying in the subspace V of $X^*(S) \otimes \boldsymbol{Q}$ spanned by ${}_k\Phi$. Its Weyl group is the *relative Weyl group* of G (notation ${}_kW$ or ${}_kW(G)$). Let $N(S)$ and $Z(S)$ denote normalizer and centralizer of S in G; these are k-subgroups. Then $N(S)/Z(S)$ operates on ${}_k\Phi$ and in V. In fact it can be identified with ${}_kW$. Any coset of $N(S)/Z(S)$ can be represented by an element in $N(S)\,(k)$.

$Z(S)$ is a connected reductive k-group. Its derived group $Z(S)'$ is a semisimple k-group which is *anisotropic.* To a certain extent G can be recovered from $Z(S)'$ and the relative root system ${}_k\Phi$ (for details see [**19**]). There is a decomposition of the Lie algebra $\mathfrak{g}$ of G:

$$\mathfrak{g} = \mathfrak{g}_0 + \sum_{\alpha \in {}_k\Phi} \mathfrak{g}_\alpha$$

where for $\alpha \in X^*(S)$ we have defined $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \,|\, \mathrm{Ad}(s)X = s^\alpha X,\ s \in S\}$. Then $\mathfrak{g}_0$ is the Lie algebra of $Z(S)$. If $\alpha \in {}_k\Phi$ there is a unique unipotent k-subgroup U_α of G normalized by S, such that its Lie algebra is $\mathfrak{g}_\alpha$.

In the absolute case ($k = \Omega$) S is a maximal torus, Φ is the ordinary root system and the U_α are as in 2.3. If G is split over k then S is a maximal torus of G and ${}_k\Phi$ coincides with the absolute root system Φ.

In the general case ${}_k\Phi$ need not be reduced, nor is $\dim \mathfrak{g}_\alpha = \dim U_\alpha$ always 1.

3.6. *Parabolic subgroups.* Recall that a parabolic subgroup P of an algebraic group G is a closed subgroup such that G/P is a projective variety. Equivalently, P is parabolic if P contains a Borel subgroup of G.

Now let G be as in 3.5. Then the minimal parabolic k-subgroups of G are conjugate over k. If P is one, there is a maximal k-split torus of G such that P is the semidirect product of k-groups $P = Z(S) \cdot R_u(P)$ ($R_u(P)$ denotes the unipotent radical). There is an ordering of ${}_k\Phi$ such that P is generated by $Z(S)$ and the U_α of 3.5 with $\alpha > 0$. The minimal parabolic k-subgroups containing a given S correspond to the Weyl chambers of ${}_k\Phi$. They are permuted simply transitively by the relative Weyl group.

Fix an ordering of ${}_k\Phi$ and let ${}_k\Delta$ be the basis of ${}_k\Phi$ defined by it. For any subset $\theta \subset {}_k\Delta$ denote by P_θ the subgroup of G generated by $Z(S)$ and the U where $\alpha \in {}_k\Phi$ is a linear combination of the roots of ${}_k\Delta$ in which all roots not in θ occur with a coefficient $\geqq 0$. Then $P_{{}_k\Delta} = G$, $P_\phi = P$ and $P_\theta \supset P$.

The P_θ are the *standard parabolic subgroups* of G containing P. Any parabolic k-subgroup Q of G is k-conjugate to a unique P_θ. If S_θ is the identity component of $\bigcap_{\alpha \subset \theta}(\mathrm{Ker}\ \theta)$ then S_θ is a k-split torus of G and we have $P_\theta = Z(S_\theta) \cdot R_u(P_\theta)$, a semidirect product of k-groups. The unipotent radical $R_u(P_\theta)$ is generated by the U_α where α is a positive root which is not a linear combination of elements of θ.

Let Q be any parabolic k-subgroup of G, with unipotent radical V (which is defined over k). A *Levi subgroup* of Q is a k-subgroup L such that Q is the semidirect product of k-groups $Q = L \cdot V$. It follows from the above that such L exist. Two Levi subgroups of Q are k-conjugate. If A is a maximal k-split torus in the centre of L, then $L = Z(A)$. If A is any k-split subtorus of G then there is a parabolic k-subgroup Q of G with Levi subgroup L. Two such Q are not necessarily k-conjugate (as they are when A is a maximal k-split torus). Two parabolic k-subgroups Q_1 and Q_2 are *associated* if they have Levi subgroups which are k-conjugate. This defines an equivalence relation on the set of parabolic k-subgroups.

If Q_1 and Q_2 are two parabolic k-subgroups, then $(Q_1 \cap Q_2) \cdot R_u(Q_1)$ is also a parabolic k-subgroup, contained in Q_1. It is equal to Q_1 if and only if there is a Levi subgroup of Q_1 containing a Levi subgroup of Q_2. Q_1 and Q_2 are called *opposite* if $Q_1 \cap Q_2$ is a Levi subgroup of Q_1 and Q_2.

3.7. *Bruhat decomposition of* $G(k)$. Let P and S be as in 3.5 and put $U = R_u(P)$. If $w \in {}_kW$ denote by n_w a representation in $N(S)(k)$. The *Bruhat decomposition* of $G(k)$ asserts that $G(k)$ is the disjoint union of the double cosets $U(k)n_wP(k)$ $(w \in {}_kW)$.

One can phrase this in a more precise way. If $w \in {}_kW$ there exist two k-subgroups U'_w, U''_w of U such that $U = U'_w \times U''_w$ (product of k-varieties) and that the map $U'_w \times P \to Un_wP$ sending (x, y) onto xn_wy is an isomorphism. We then have

$$(G/P)(k) = G(k)/P(k) = \bigcup_{w \in {}_kW} \pi(U'_w(k)),$$

where π is the projection $G \to G/P$.

If $k = \Omega$ this gives a cellular decomposition of the projective variety G/P.

If $\theta \in {}_k\Delta$ let W_θ be the subgroup of ${}_kW$ generated by the reflections defined by the $\alpha \in {}_k\Delta$. If $\theta, \theta' \in {}_k\Delta$ there is a bijection of double cosets

$$P_\theta(k)\backslash G(k)/P_{\theta'}(k) \simeq W(\theta)\backslash_k W/W(\theta').$$

Let Σ be the set of generators of ${}_kW$ defined by ${}_k\Delta$. The above assertions (except for the algebro-geometric ones) then all follow from the fact that $(G(k), P(k), Z(S)(k), \Sigma)$ is a Tits system in the sense of **[7]**.

3.8. *The Tits building.* Let G be the connected reductive k-group. We define a simplicial complex $\mathscr{B}$, the (simplicial) *Tits building of* (G, k), as follows.

The vertices of $\mathscr{B}$ are the maximal nontrivial parabolic k-subgroups of G. A set $(P_1, \cdots, P_n)$ of distinct vertices determines a simplex of $\mathscr{B}$ if and only if $P = P_1 \cap \cdots \cap P_n$ is parabolic. In that case, the P_i are uniquely determined by P. It follows that the simplices of $\mathscr{B}$ correspond to the nontrivial parabolic k-subgroups of G. Let σ_P be the simplex defined by P. Then σ_P is a face of $\sigma_{P'}$ if and only if $P' \subset P$. The maximal simplices correspond to minimal k-parabolics. These simplices are called *chambers*. A codimension 1 face of a chamber is a *wall*. Two chambers are *adjacent* if they are distinct and have a wall in common. One shows that any two chambers σ, σ' can be joined by a gallery, i.e., a set of chambers $\sigma = \sigma_0, \sigma_1, \cdots, \sigma_s = \sigma'$, such that σ_i and σ_{i+1} are adjacent $(0 \leqq i < s)$.

It is clear that $G(k)$ operates on $\mathscr{B}$.

One can show (using a concrete geometric realization of the abstract simplicial complex $\mathscr{B}$) that $\mathscr{B}$ has the homotopy type of a bouquet of spheres.

For more details about buildings see **[20]**.

3.9. EXAMPLES. (i) The preceding results apply when $k = \Omega$, the absolute case. In particular, we then have the properties of parabolics of 3.6 and the Bruhat decomposition of 3.7.

(ii) $G = \mathbf{GL}(n)$ (k arbitrary). This is indeed a reductive k-group. Its Lie algebra $\mathfrak{g}$ is the Lie algebra of all $n \times n$-matrices.

Let S be the subgroup of diagonal matrices. This is a maximal k-split torus which is also a maximal torus of G (in the absolute sense). Let $e_i \in X = X^*(S)$ map $s \in S$ onto its ith diagonal element. The e_i form a basis of X. The root system $\Phi = \Phi(G, S)$, which coincides with the relative root system ${}_k\Phi$, consists of the $e_i - e_j \in X$ with $i \neq j$. One checks that the root datum of G is given by $X = \mathbf{Z}^n$, $X^\vee = \mathbf{Z}^n$, $\Phi = \{e_i - e_j\}_{i \neq j}$, $\Phi^\vee = \{e_i^\vee - e_j^\vee\}_{i \neq j}$, where $(e_i^\vee)$ is the basis of X dual to (e_i).

The subgroup B of all upper triangular matrices is a minimal parabolic k-subgroup. It is a Borel subgroup. Its unipotent radical U is the group of all upper triangular matrices with ones in the diagonal. The basis Δ of Φ defined by B is $(e_i - e_{i+1})_{1 \leqq i \leqq n-1}$. The Weyl group W (which coincides with the relative Weyl group ${}_kW$) is isomorphic to the symmetric group $\mathfrak{S}_n$, viewed as the group of permutations of the basis (e_i).

The parabolic subgroups $P \supset B$ are the groups of block matrices

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1s} \\ 0 & A_{22} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & A_{ss} \end{pmatrix}$$

where A_{ij} is an $n_i \times n_j$-matrix with $n_1 + \cdots + n_s = n$, the A_{ii} being nonsingular. Its unipotent radical consists of these matrices where $A_{ii} = 1$ $(1 \leqq i \leqq s)$. The subgroup of P of matrices with $A_{ij} = 0$ for $j > i$ is a Levi subgroup of P. The center of L consists of those elements of S at which the elements of Δ different from one of the $e_{n_i} - e_{n_i+1}$ $(1 \leqq i \leqq s)$ are trivial. Hence with the notations of 3.6, we have $P = P_\theta$, where $\theta \in \Delta$ is the complement of the set of these roots.

A more geometric description of parabolic subgroups is as follows.

Let $V = \Omega^n$. A *flag* in V is a sequence $0 = V_0 \subset V_1 \subset \cdots \subset V_s = V$ of distinct subspaces of V. A *k-flag* is one where all V_i are defined over k, i.e., have a basis consisting of vectors in k^n.

G operates on the set of all flags. The parabolic subgroups of G are then the isotropy groups of flags. One sees that there is a bijection of the set of all parabolic subgroups of G onto the set of all flags, under which k-subgroups correspond to k-flags.

If P is a parabolic subgroup, then the points of G/P can be viewed as the flags of the same type as P (i.e., such that the subspaces of the flags have a constant dimension).

The Tits building of (G, k) can then also be described in terms of flags: The simplices correspond to the nontrivial k-flags (i.e., those with $s > 1$). If σ_f is the simplex defined by the flag f, then σ_f is a face of $\sigma_{f'}$ if and only if f' refines f (in the obvious sense). The chambers correspond to the maximal flags ($s = n$, $\dim V_i = i$) and the vertices of $\mathscr{B}$ are described by the nontrivial k-subspaces of V. We see that the combinatorial structure of $\mathscr{B}$ pictures the incidences in the projective space $\mathbf{P}_{n-1}(k)$.

The smallest nontrivial special case is $n = 3$, $k = \boldsymbol{F}_2$. Here $\mathscr{B}$ is a graph with 14 vertices and 21 edges (drawn in [**20**, p. 210]).

(iii) Let char $k \neq 2$. Let V be a vector space over k (in the sense of algebraic geometry). Let F be a nondegenerate quadratic form on V which is defined over k. With respect to a suitable basis of $V(k)$ we have

$$F(x_1, \cdots, x_n) = x_1x_n + x_2x_{n-1} + \cdots + x_qx_{n-q+1} + F_0(x_{q+1}, \cdots, x_{n-q}),$$

where F_0 is anisotropic over k (i.e., does not represent zero nontrivially). The index q of F is the dimension of the maximal isotopic subspaces of $V(k)$.

Let $G = SO(F)$ be the special orthogonal group of F. It is a connected semisimple k-group. A maximal k-split torus S in G is given by the matrices of the form

$$\mathrm{diag}(t_1, \cdots, t_q, 1, \cdots, 1, t_1^{-1}, \cdots, t_q^{-1}).$$

Then $Z(S)$ is the direct product of S and the anisotropic k-group $SO(F_0)$.

For a description of a minimal parabolic k-subgroup and the determination of the relative root system ${}_k\Phi$ we refer to [**2**, p. 16]. The latter is of type B_q if $2q \neq n$ and of type D_q otherwise. If $q < [n/2]$ there are always subgroups U_α of dimension > 1 (notations of 3.5).

A geometric description of parabolic k-subgroups similar to the one for $\mathbf{GL}(n)$ can be given. They are in this case the isotropy groups of *isotropic* k-*flags* in V, i.e., flags all of whose subspaces are isotropic with respect to F.

4. Special fields. Let G be a k-group. In this section we discuss some special features for particular k.

4.1. ***R** and **C**.* If G is a $\boldsymbol{C}$-group then $G(\boldsymbol{C})$ has a canonical structure of complex Lie groups. The latter is connected if and only if G is Zariski-connected (this can be deduced from Bruhat's lemma, compare 4.2).

Now let $k = \boldsymbol{R}$. Then $G(\boldsymbol{R})$ is canonically a Lie group.

4.2. LEMMA. (i) *$G(\boldsymbol{R})$ is compact if and only if the identity component G^0 is a reductive anisotropic $\boldsymbol{R}$-group*;

(ii) *$G(\boldsymbol{R})$ has finitely many connected components.*

(i) is easily established. As to (ii), it suffices to prove this if G is connected reductive. In that case one reduces the statement, via Bruhat's lemma, to the case that G is either compact or a torus. In these cases the assertion is clear.

$G(\boldsymbol{R})$ need not be connected if G is Zariski-connected, as one sees in simple cases (e.g., $G = \mathbf{GL}(n)$).

If G is a $\boldsymbol{C}$-group then the real Lie group $R_{\boldsymbol{C}/\boldsymbol{R}}(G)(\boldsymbol{R})$ (see 3.3) is that defined by the complex Lie group $G(\boldsymbol{C})$.

4.3. *Finite fields.* Let $k = \boldsymbol{F}_q$ and let $\bar{k}$ be an algebraic closure. F denotes the Frobenius automorphism $x \mapsto x^q$ of $\bar{k}/k$. The basic result here is Lang's theorem [**6**, p. 171].

4.4. THEOREM. *If G is a connected k-group then $g \mapsto g^{-1}(Fg)$ is a surjective map of $G(\bar{k})$ onto itself.*

Using that G is an inner form of a quasi-split k-group (see 3.1) one deduces that *a connected reductive k-group is quasi-split.* A complete classification of simple k-groups can then be given.

Before continuing with local and global fields, we must say a little about group schemes over rings.

4.5. *Groups over rings.* If G is a k-group, the product and inversion are described (see [**2**, p. 4]) by morphisms of algebraic varieties $\mu: G \times G \to G$, $\rho: G \to G$, which in turn are given by homomorphisms of k-algebras $\mu^*: k[G] \to k[G] \otimes_k k[G]$ and $\rho^*: k[G] \to k[G]$. These have a number of properties (which we will not write down) expressing the group axioms. We thus obtain a description of the notion of linear algebraic group in terms of the coordinate algebra.

The fact that k is a field does not play any role in this description.

Replacing k by a commutative ring $\mathfrak{o}$, we get a notion of "linear algebraic group G over $\mathfrak{o}$", which is habitually called "affine group scheme G over $\mathfrak{o}$", which we abbreviate to $\mathfrak{o}$-group. It can be viewed as a functor, cf. [**2**, p. 4]. We write $G(\mathfrak{o})$ for the group of $\mathfrak{o}$-points of G (i.e., the value of the functor at $\mathfrak{o}$).

Let $\mathfrak{o}[G]$ be its algebra. If $\mathfrak{o}'$ is an $\mathfrak{o}$-algebra we have, by base extension, an $\mathfrak{o}'$-group $G \times_{\mathfrak{o}} \mathfrak{o}'$, with algebra $\mathfrak{o}[G] \otimes_{\mathfrak{o}} \mathfrak{o}'$.

Let $\mathfrak{m}$ be a maximal ideal of $\mathfrak{o}$ and put $k(\mathfrak{m}) = \mathfrak{o}/\mathfrak{m}$; this is an $\mathfrak{o}$-algebra.

DEFINITION. The $\mathfrak{o}$-group G has *good reduction* at $\mathfrak{m}$ if $G \times_{\mathfrak{o}} k(\mathfrak{m})$ is a $k(\mathfrak{m})$-group. G is *smooth* if it has good reduction at all maximal ideals $\mathfrak{m}$.

EXAMPLE OF BAD REDUCTION. $\mathfrak{o} = \boldsymbol{Z}$, G is the group of matrices

$$\begin{pmatrix} a & b \\ 2b & a \end{pmatrix}$$

with $a^2 - 2b^2 = 1$. Then $\boldsymbol{Z}[G] = \boldsymbol{Z}[X, Y]/(X^2 - 2Y^2 - 1)$ and $\boldsymbol{Z}[G] \otimes \boldsymbol{F}_2 \simeq \boldsymbol{F}_2[X, Y]/(X^2)$, which cannot be the coordinate ring of a linear algebraic group over $\boldsymbol{F}_2$, since it contains nilpotent elements.

Now let G be a k-group and let $\mathfrak{o}$ be a subring of k. We shall say that G is *definable over* $\mathfrak{o}$ if there exists a smooth $\mathfrak{o}$-group G_0 such that $G \simeq G_0 \times_{\mathfrak{o}} k$. By abuse of notation, we sometimes write, if $\mathfrak{o}'$ is an $\mathfrak{o}$-algebra, $G(\mathfrak{o}')$ for $G_0(\mathfrak{o}')$. One can also define when an algebraic variety over k is definable over $\mathfrak{o}$; it is clear how to do this.

EXAMPLE. By a theorem of Chevalley a complex connected semisimple group is definable over $\boldsymbol{Z}$ [**6**, A, §4].

4.6. *Local fields.* Let k be a local field. We denote by $\mathfrak{o}$ its ring of integers and by $\mathfrak{m}$ the maximal ideal of $\mathfrak{o}$. The residue field $\mathfrak{o}/\mathfrak{m}$ is denoted by F.

A profound study of reductive groups over local field has been made by Bruhat and Tits. So far, only part of this has been published [**9**]. For a résumé see [**8**]. In Tits' contribution [**21**] in these PROCEEDINGS more details are given about the Bruhat-Tits theory. In particular, he discusses the building of a reductive k-group and the theory of maximal compact subgroups. Here we mention only a few results.

4.7. LEMMA. *Let G be a connected reductive k-group. Then there is an unramified extension l of k such that G is quasi-split over l.*

This can be deduced from the fact that a maximal unramified extension of k is a field of dimension $\leqq 1$ (see [**16**, p. II-11]).

The group $G(k)$ of k-rational points is a locally compact topological group (even a Lie group over k). It is a compact group if and only if the identity component G^0 is a reductive anisotropic k-group. It is shown in the Bruhat-Tits theory that if G is connected and simply connected simple k-group, there is a finite-dimensional division algebra D with center k such that $G(k) \simeq \mathbf{SL}(1, D)$.

If G is connected and reductive and is definable over $\mathfrak{o}$ (which is always the case if G is k-split, see [**8**, p.31]), then $G(\mathfrak{o})$ is a compact subgroup of $G(k)$. There is a reduction map $G(\mathfrak{o}) \to G(F)$, which is *surjective.*

4.8. *Global fields.* Now let k be a global field. If v is a valuation of k, let k_v be the corresponding completion of k. If v is nonarchimedean, $\mathfrak{o}_v$ denotes the ring of integers of k_v. If S is a nonempty finite set of valuations of k, containing all the archimedean ones, denote by $\mathfrak{o}_S$ the ring of elements of k which are integral outside S. It is a Dedekind ring.

Let G be a connected reductive k-group.

4.9. LEMMA. (i) *There is an S such that G is definable over $\mathfrak{o}_S$;*

(ii) *$G \times_k k_v$ is quasi-split for almost all v.*

(i) is easily established. Let C be the group of inner automorphisms of G; it is a semisimple k-group. We identify it with its group of k_s-rational points. There is $\gamma \in H^1(k, C)$ such that G, twisted by a cocycle c from γ (see 3.1), is quasi-split. This is another way of saying that G is an inner form of a quasi-split group. Now c defines a principal homogeneous space C_c of C over k, i.e., an algebraic variety over k, on which C acts simply transitively, the action being defined over k (see [**16**, p. I-58]). We have $\gamma = 1$ if and only if C_c has a k-rational point. To prove (ii) it now suffices to show that the image of γ in $H^1(k_v, C \times_k k_v)$ is trivial for almost all v, or that C_c has a k_v-rational point for almost all v.

Let v be nonarchimedean such that $C \times_k k_v$ and $C_c \times_k k_v$ are definable over $\mathfrak{o}_v$, say $C \times_k k_v = C_0 \times_{\mathfrak{o}_v} k_v$, $C_c \times_k k_v = C_{c,0} \times_{\mathfrak{o}_v} k_v$. Assume furthermore that the reduced group $C_0 \times_{\mathfrak{o}_v} F_v$, over the residue field F_v, is a connected F_v-group. These conditions are satisfied for almost all v. By 4.4, it follows that $C_{c,0} \times_{\mathfrak{o}_v} F_v$ has an F_v-rational point. A version of Hensel's lemma then gives that $C_{c,0}$ has an $\mathfrak{o}_v$-rational point, which shows that C has a k_v-rational point. This implies (ii), as we have seen.

4.10. *Adelization.* Let A be the adele ring of k. It is a k-algebra, so the group of A-points $G(A)$ of G is defined. Let $G \hookrightarrow \mathbf{GL}_n$ be an embedding over k. Then $g \mapsto (g, g^{-1})$ maps $G(A)$ bijectively onto a closed subset of $A^{n^2+1} \oplus A^{n^2+1}$. Endowed with the induced topology, $G(A)$ is a locally compact group, the *adele group* of G. It has $G(k)$ as a discrete subgroup. The topology on $G(A)$ is independent of the choice of the embedding $G \hookrightarrow \mathbf{GL}(n)$. An alternative way to define $G(A)$ is as follows. Let S_0 have the property of 4.9(i). For each finite set of valuations $S \supset S_0$, the group

$$G(A_S) = \prod_{v \notin S} G(\mathfrak{o}_v) \times \prod_{v \in S} G(k_v)$$

is a locally compact group. If $S \subset S'$ then $G(A_S) \subset G(A_{S'})$. $G(A)$ can also be defined as the limit group $G(A) = \text{inj lim}_{S \supset S_0} G(A_S)$ (this is independent of the choice of S_0).

For each v, we have an injection $G(k_v) \to G(A)$.

EXAMPLES. (a) $G = \mathbf{GL}(1)$. Then $G(A)$ is the group of idèles (the units of A).

(b) $k = \boldsymbol{Q}$, $G = \mathbf{SL}(n)$. One checks that

$$G(A) = \mathbf{SL}(n, \boldsymbol{R}) \cdot \left(\prod_p \mathbf{SL}(n, \boldsymbol{Z}_p)\right) \cdot G_{\boldsymbol{Q}},$$

from which one sees that there is a surjective continuous map $G(A)/G(\boldsymbol{Q}) \to \mathbf{SL}(n, \boldsymbol{R})/\mathbf{SL}(n, \boldsymbol{Z})$. More precisely, if one defines, for a positive integer N, the consequence subgroup $\Gamma(N)$ of $\mathbf{SL}(n, \boldsymbol{Z})$ by $\Gamma(N) = \{\gamma \in \mathbf{SL}(n, \boldsymbol{Z}) \mid \gamma \equiv 1 \,(\mathrm{mod}\, N)\}$, then

$$G(A)/G(\boldsymbol{Q}) = \text{proj lim } \mathbf{SL}(n, \boldsymbol{R})/\Gamma(N).$$

(c) Let G be a $\boldsymbol{Q}$-group and let $G \hookrightarrow \mathbf{GL}(n)$ be an embedding (over $\boldsymbol{Q}$). Fix a lattice L in $\boldsymbol{Q}^n$ and let Γ be the subgroup of $G(\boldsymbol{Q})$ of elements stabilizing L.

There exists a connection, similar to that of the previous examples, between $G(A)/G(\boldsymbol{Q})$ and $G(\boldsymbol{R})/\Gamma$ (see [1]).

The main results about $G(A)/G(k)$ are as follows (G a connected reductive k-group). Let X be the group of k-rational characters of G. For each $\chi \in X$ define a character $|\chi|: G(A) \to \boldsymbol{R}^*$ by $|\chi|((g_v)) = \prod_v |\chi(g_v)|_v$, where $|\ |_v$ is an absolute value, normalized so as to satisfy the product formula. Let $G(A)^0$ be the intersection of kernels of the $|\chi|$, for $\chi \in X$.

The product formula shows that $G(k) \subset G(A)^0$.

4.11. Theorem. (i) *$G(A)^0/G(k)$ has finite invariant volume;*

(ii) (*G semisimple*) *$G(A)^0/G(k)$ is compact if and only if G is anisotropic over k.*

This is a consequence of reduction theory, due to Borel and Harish-Chandra for number fields and to Harder for function fields (see [1] and [11]). Notice, that by restriction of the ground field, it suffices to prove this for $k = \boldsymbol{Q}$ or $k = \boldsymbol{F}_q(T)$.

5. A class of Lie groups. In this section we discuss a class of Lie groups close to the groups of real points of reductive $\boldsymbol{R}$-groups. This is the class of groups occurring in Wallach's paper in these Proceedings (see also [12]). We shall indicate briefly how the properties of these groups can be deduced from the algebraic properties of reductive groups, discussed above.

We shall say that an algebraic group $\boldsymbol{G}$ defined over a field of characteristic zero is reductive if its identity component G^0 (in the Zariski topology) is so.

5.1. Let G be a Lie group, with Lie algebra $\mathfrak{g}$. Its identity component is denoted by G^0. We denote by ${}^0G$ the intersection of the kernels of all continuous homomorphisms $G \to \boldsymbol{R}_+^*$. Then ${}^0G$ is a closed normal subgroup and $G/{}^0G$ is a vector group.

A *split component* of G is a vector subgroup V of G such that $G = {}^0G \cdot V$, ${}^0G \cap V = \{e\}$.

We assume henceforth that G possesses the following properties:

(1) *There is a reductive $\boldsymbol{R}$-group $\boldsymbol{G}$ and a morphism $\nu: G \to \boldsymbol{G}(\boldsymbol{R})$ with finite kernel whose image is an open subgroup of $\boldsymbol{G}(\boldsymbol{R})$.*

It follows that ν induces an isomorphism of $\mathfrak{g}$ onto the set of real points of Lie $\boldsymbol{G}$. We shall often identify $\mathfrak{g}$ and $\nu(\mathfrak{g})$. It also follows that $\nu(G)^0 = \boldsymbol{G}(\boldsymbol{R})^0$ and that G^0 has finite index in G (since this is so for $\boldsymbol{G}(\boldsymbol{R})$, see 4.2(ii), and ker ν is finite).

(2) *The image of G in the automorphism group of $\mathfrak{g}_C = \mathfrak{g} \otimes_{\boldsymbol{R}} \boldsymbol{C}$ lies in the image of the identity component $\boldsymbol{G}^0$ of $\boldsymbol{G}$.*

The main reason to allow for finite coverings of linear groups is to include the metaplectic group and all connected semisimple groups with finite center. The main use of (2) is to insure that G acts trivially on the center of the universal enveloping algebra of $\mathfrak{g}$.

5.2. Let θ_0 be the automorphism of $\mathbf{GL}(n, \boldsymbol{R})$ which sends g to ${}^t g^{-1}$. Let $\mathfrak{s}$ (resp. S_0) be the set of real symmetric (resp. and positive nondegenerate) $n \times n$-matrices. Then $\exp: x \mapsto \exp x = 1 + x + x^2/2! + \cdots$ is an isomorphism of $\mathfrak{s}$ onto S_0. Given $s \in S_0$, there is a unique analytic subgroup of $\mathbf{GL}(n, \boldsymbol{R})$, isomorphic to $\boldsymbol{R}$, contained in S_0, and passing through s. It is contained in any θ_0-stable Lie subgroup of $\mathbf{GL}(n, \boldsymbol{R})$ with finitely many connected components which contains s.

5.3. Lemma. *Let* $\boldsymbol{G} \subset \mathbf{GL}(n, \boldsymbol{C})$ *be an* $\boldsymbol{R}$*-subgroup stable under* θ_0.

(i) *Let* $s \in S_0 \cap \boldsymbol{G}(\boldsymbol{R})$. *Then there exists a* θ_0*-stable* $\boldsymbol{R}$*-split torus* $\boldsymbol{S}$ *of* $\boldsymbol{G}$ *such that* $s \in \boldsymbol{S}(\boldsymbol{R})^0$;

(ii) *let* $X \in \mathfrak{s} \cap \mathrm{Lie}(\boldsymbol{G})$. *There is a* θ_0*-stable* $\boldsymbol{R}$*-split subtorus of* $\boldsymbol{G}$ *whose Lie algebra contains* X;

(iii) $\boldsymbol{G}$ *is reductive.*

The element s generates an infinite subgroup of G, whose Zariski closure is a torus with the required properties. (ii) follows from (i), applied to $\exp x$. Let $\boldsymbol{U}$ be the unipotent radical of $\boldsymbol{G}$, let $s \in \boldsymbol{G}(\boldsymbol{R})$. Then s and $(\theta_0 s)s^{-1}$ are unipotent. By (i) the last element is also semisimple, which implies $s = 1$. Hence $\boldsymbol{U} = \{1\}$. This proves (iii).

5.4. By definition, a Cartan involution of $\mathbf{GL}(n, \boldsymbol{R})$ is an automorphism conjugate to θ_0 by an inner automorphism. Let G and $\boldsymbol{G}$ be as in 5.1. Let $\boldsymbol{G} \subset \mathbf{GL}(n, \boldsymbol{C})$ be an embedding over $\boldsymbol{R}$. Then $\nu(G)$ is stable under some Cartan involution θ of $\mathbf{GL}(n, \boldsymbol{R})$. In other words, we may assume, after conjugation, that $\nu(G)$ is stable under θ_0 (in which case it is said to be selfadjoint) **[1]**, **[15]**. Let $\mathfrak{k}$ (resp. $\mathfrak{s}$) be the fixed point (resp. -1 eigenspace) of θ in $\mathfrak{g}$, and K the inverse image in G of the fixed point set of θ in $\nu(G)$.

5.5. Proposition. *The automorphism* θ *of* $\mathfrak{g}$ *extends uniquely to an automorphism of* G *whose fixed point set is* K. *The map* μ: $(k, x) \mapsto k \cdot \exp x$ *is an isomorphism of analytic manifolds of* $K \times \mathfrak{s}$ *onto* G.

The automorphisms of G thus defined are the Cartan involutions of G. They form one conjugacy class with respect to inner automorphisms by elements of G^0. The decomposition $G = K \cdot S$ ($S = \exp \mathfrak{s}$) is a Cartan decomposition of G.

After conjugation, we may assume that $\theta = \theta_0$. If $G = \mathbf{GL}(n, \boldsymbol{R})$, then $K = \boldsymbol{O}(n)$, $S = S_0$ and our assertion follows from the polar decomposition of real matrices. Assume now that ν is the identity. Write $g \in G$ as a product $g = k \cdot s$ where $k \in \boldsymbol{O}(n)$, $s \in S_0$. Then $s^2 = (\theta_0\, g)^{-1} \cdot g \in G$, and the unique 1-parameter subgroup in S_0 through s^2 (see 5.2) is contained in G. In this group, there is a unique element with square s^2, which must then be equal to s. Thus $s \in G$, hence also $k \in G$. This implies that μ is surjective. Injectivity follows from the uniqueness of the polar decomposition. The decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ implies that the tangent map at any point is bijective; hence μ is an analytic isomorphism. Thus G is the direct product of K and a euclidean space.

This proves the proposition when ν is the identity. Let $\tilde{G}$ be the simply connected group with Lie algebra $\mathfrak{g}$, $\tilde{K}$ the analytic subgroup of $\tilde{G}$ with Lie algebra $\mathfrak{k}$ and $\pi: \tilde{G} \to G^0$ the natural projection. Since the fundamental group of K is that of G^0, the group $\tilde{K}$ is the universal covering of K; hence $\ker \pi \subset \tilde{K}$. The automorphism θ

of $\mathfrak{g}$ extends to one of $\tilde{G}$, which fixes $\tilde{K}$ pointwise, hence acts trivially on ker π, and goes down to an automorphism of G^0. The result for $\nu(G)$ implies that $G = K \cdot S = K \cdot G^0$. Since ker ν acts trivially on $\mathfrak{g}$, hence on G^0, $\theta\colon G^0 \to G^0$ extends obviously to an automorphism of G which fixes K pointwise. The remaining assertions are then obvious.

5.6. COROLLARY. (i) *K is a maximal compact subgroup of G;*
(ii) *K meets every connected component of G.*

G is the topological product of K by a connected space S, whence (ii). The first assertion follows from the fact that every $s \neq 1$ in S generates an infinite discrete subgroup.

Fix a Cartan involution θ of $\boldsymbol{GL}(n, \boldsymbol{R})$ stabilizing $\nu(\boldsymbol{G})$. We also denote by θ the Cartan involution of G defined in 5.5.

5.7. Let C be the center of G. It has again the properties (1), (2) and it is θ-stable. The group corresponding to $\boldsymbol{G}$ is the center of $\boldsymbol{G}$. The subset corresponding to S is now a vector group V. It is, in fact, *the* maximal θ-stable vector subgroup contained in C. Let G_1 be the derived group of G.

5.8. LEMMA. (i) *${}^0G = KG_1$ and V is a split component of G;*
(ii) *${}^0G$ has the properties* (1), (2) *and is θ-stable.*

KG_1 is a θ-stable closed normal subgroup of G, contained in ${}^0G$. The Lie algebra $\mathfrak{g}$ is the direct sum of those of KG_1 and of V, which implies (i). As to (ii), for the algebraic group of (1) we take the Zariski closure of $\nu({}^0G)$ in $\boldsymbol{G}$. Its identity component differs from $\boldsymbol{G}^0$ only in its center. This implies (2), and the final assertion is clear.

5.9. *Parabolic subgroups.* A parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is a subalgebra such that $\mathfrak{p}_C$ is the Lie algebra of a parabolic $\boldsymbol{R}$-subgroup of $\boldsymbol{G}^0$. A *parabolic subgroup P* of G is the normalizer in G of a parabolic subalgebra (which then is the Lie algebra of P). The parabolic subgroups of G correspond to the parabolic $\boldsymbol{R}$-subgroups of $\boldsymbol{G}^0$.

Let $\boldsymbol{P}$ be a parabolic $\boldsymbol{R}$-subgroup of $\boldsymbol{G}^0$. Let $\boldsymbol{N}$ be its unipotent radical. Put $\boldsymbol{L} = \boldsymbol{P} \cap \theta\boldsymbol{P}$.

5.10. LEMMA. (i) *$\boldsymbol{L}$ is a Levi subgroup of $\boldsymbol{P}$;*
(ii) *the Lie algebra of $\boldsymbol{G}$ is the direct sum of those of $\boldsymbol{N}$, $\theta\boldsymbol{N}$ and $\boldsymbol{L}$.*

5.3(iii) shows that $\boldsymbol{L}$ is reductive. $\boldsymbol{LN}$ is a parabolic $\boldsymbol{R}$-subgroup of $\boldsymbol{G}^0$ contained in $\boldsymbol{P}$ (see 3.6) with unipotent radical $\boldsymbol{N}$, hence equal to $\boldsymbol{P}$. This proves (i). Then (ii) follows by using that $\boldsymbol{P}$ and $\theta\boldsymbol{P}$ are opposite parabolics.

5.11. Let $\boldsymbol{S}$ be the maximal $\boldsymbol{R}$-split torus in the center of $\boldsymbol{L}$. Put $A = \nu^{-1}(\boldsymbol{S}(\boldsymbol{R}))^0$, $N = \nu^{-1}(\boldsymbol{N}(\boldsymbol{R}))^0$. These are subgroups of G. Since A is a θ-stable vector group, we have $\theta a = a^{-1}$ for all $a \in A$. Let P be the parabolic subgroup of G defined by $\boldsymbol{P}$ and put $L = P \cap \theta P$, $M = {}^0L$. Then L is the centralizer of A in G. Also, L and M are θ-stable.

5.12. PROPOSITION. (i) *L satisfies* (1), (2) *of* 5.1. *A is a split component of L and of P;*
(ii) *$(m, a, n) \mapsto man$ defines an analytic diffeomorphism of $M \times A \times N$ onto P.*

Let $\boldsymbol{H}$ be the centralizer of $\boldsymbol{S}$ in $\boldsymbol{G}$. Then $\nu(L) \subset \boldsymbol{H}$. Moreover, the identity component $\boldsymbol{H}^0$ is reductive and is equal to $\boldsymbol{H} \cap \boldsymbol{G}^0$ (the last point because centralizers of tori in Zariski-connected algebraic groups are connected, see [3, p. 271]). It follows that (1) and (2) hold for L. The last point of (i) is easy. From $\boldsymbol{P} = \boldsymbol{LN}$ we conclude that $P = LN = MAN$. Now (ii) readily follows.

The decomposition of 5.12(ii) is called the *Langlands decomposition* of P. There is a similar decomposition of the Lie algebra of P: $\mathfrak{p} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n}$.

A *parabolic pair* in G is a pair (G, A) where P is a parabolic subgroup of G and A is as above.

5.13. *Minimal parabolic subgroups.* Now assume that P is a minimal parabolic subgroup. Then $\boldsymbol{P}$ is a minimal parabolic $\boldsymbol{R}$-subgroup of $\boldsymbol{G}^0$. In that case the derived group of $\boldsymbol{L}$ is an anisotropic semisimple $\boldsymbol{R}$-group. It follows that M is compact. We then must have $M \subset K$ (recall that M is θ-stable), so $M = K \cap P$. Let Φ be the root system of $(\boldsymbol{G}, \boldsymbol{S})$ (see 3.5). If $\alpha \in \Phi$ put $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid \mathrm{Ad}(a)X = a^\alpha X, a \in A\}$. Then we also have

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [H, X] = d\alpha(H)X, H \in \mathfrak{a}\}$$

($\mathfrak{a}$ being the Lie algebra of A). Also, there is an ordering of Φ such that $\mathfrak{n} = \sum_{\alpha>0} \mathfrak{g}_\alpha$, $\theta\mathfrak{n} = \sum_{\alpha<0} \mathfrak{g}_\alpha$ ($\mathfrak{n}$ and $\theta\mathfrak{n}$ are the Lie algebras of N and θN).

5.14. Lemma. (i) *We have direct sum decompositions*

$$\mathfrak{g} = \mathfrak{a} + \mathfrak{m} + \mathfrak{n} + \theta\mathfrak{n}, \qquad \mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n};$$

(ii) $\mathfrak{a}$ *is a maximal commutative subalgebra of* $\mathfrak{s}$.

The first decomposition follows by using 5.10(ii). It then follows that $\mathfrak{k}$ is the direct sum of $\mathfrak{m}$ and the space of all $X + \theta X$ ($X \in \mathfrak{n}$). Hence $\mathfrak{k} \cap \mathfrak{n} = \{0\}$, $\mathfrak{k} + \mathfrak{n} = \mathfrak{m} + \mathfrak{n} + \theta\mathfrak{n}$. This gives the second decomposition. We also get that $\mathfrak{s}$ is the direct sum of $\mathfrak{a}$ and the space of all $X - \theta X$ ($X \in \mathfrak{n}$). Since $\mathfrak{a}$ commutes with no nonzero element of $\mathfrak{n} + \theta\mathfrak{n}$, the assertion of (ii) follows.

From the above we see that Φ is also the root system of the symmetric pair (G, K) (see [13, Chapter VII]).

5.15. Proposition (Iwasawa decomposition). $(k, a, n) \mapsto kan$ *is an analytic diffeomorphism of* $K \times A \times N$ *onto* G.

Let ϕ be the map of the statement.

(a) im ϕ is closed. AN is a closed subgroup of G and G/AN is compact (because G/P and M are compact). Let π be the projection $G \to G/AN$. Since K is compact, $\mathrm{im}(\pi \circ \phi)$ is closed. Hence so is $\mathrm{im}\ \phi = \pi^{-1}\ \mathrm{im}(\pi \circ \phi)$.

(b) im ϕ meets all components of G, since k does (see 4.6(ii)).

(c) The tangent map $d\phi$ is bijective at any point (k, a, n). This follows from the direct sum decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$.

(a), (b) and (c) imply that im ϕ is open and closed and meets all components. Hence ϕ is surjective. To finish the proof, it suffices to show that ϕ is injective. It is enough to prove that $kan = a_1$ implies $a = a_1$, $k = n = e$. Now if this is so we have $\theta n = a^2 n a_1^{-2}$. The image under ν of the last element is unipotent. It then follows that $a^2 = a_1^2$, $a = a_1$, whence $\theta n \in N \cap \theta N = \{e\}$.

5.16. Corollary. *For any parabolic subgroup P of G we have $G = KP$.*

Let W be the Weyl group of the root system Φ. This is the relative Weyl group of $(\boldsymbol{G}^0, \boldsymbol{S})$ (see 3.5), i.e.,

$$W = N_{\boldsymbol{G}^0(\boldsymbol{R})}(\boldsymbol{S})/Z_{\boldsymbol{G}^0(\boldsymbol{R})}(\boldsymbol{S}).$$

5.17. LEMMA. *M meets all components of G.*

It suffices to prove this for the case that $\nu(G) = \boldsymbol{G}^0(\boldsymbol{R})$. In that case it follows from Bruhat's lemma that the connected components of $\boldsymbol{G}^0(\boldsymbol{R})$ all meet $N = N_{\boldsymbol{G}^0(\boldsymbol{R})}(\boldsymbol{S})$. Let $\alpha \in \Phi$ be a simple root (for the order defined by $\boldsymbol{P}$), $\boldsymbol{U}_{(\alpha)}$ the unipotent subgroup whose Lie algebra is $(\mathfrak{g}_\alpha)_C + (\mathfrak{g}_{2\alpha})_C$ (where the second term is zero if $2\alpha \notin \Phi$), and $\boldsymbol{U}_{(-\alpha)} = \theta(\boldsymbol{U}_{(\alpha)})$. It is known that $\boldsymbol{U}_{(\alpha)}(\boldsymbol{R}) \cdot \boldsymbol{U}_{(-\alpha)}(\boldsymbol{R}) \cdot \boldsymbol{U}_{(\alpha)}(\boldsymbol{R})$ contains an element of N representing the reflection in W defined by α. It follows that $N \subset Z_{\boldsymbol{G}^0(\boldsymbol{R})}(\boldsymbol{S})G^0$, which implies the assertion.

It follows from the lemma that $W = N_G(A)/Z_G(A)$.

5.18. LEMMA. $W \simeq (K \cap N_G(A))/M$.

Let $g = kan \in N_G(A)$. Then also $(\theta n)^{-1}a^2 n \in N_G(A)$. Let $w_0 \in N_{\boldsymbol{G}^0(\boldsymbol{R})}(\boldsymbol{S})$ represent the element of maximal length of W. Then $\theta n^{-1} = w_0 n_1 w_0^{-1}$, for some $n_1 \in N$. We then have $n_1 w_0^{-1} a^2 n \in w_0^{-1} N_G(A)$. The uniqueness statement of Bruhat's lemma then implies that $n_1 = e$, whence $n = e$. This implies the assertion.

From 5.18 we see that W is the Weyl group of the symmetric spaces G/K ([**13**, p. 244]).

5.19. PROPOSITION (CARTAN DECOMPOSITION). *We have* $G = KAK$.

This follows from 5.5 and the following lemma. Here S is as in 5.3 (observe that K normalizes S).

5.20. LEMMA. $S = \bigcup_{k \in K} kAk^{-1}$.

It suffices to prove this for $\nu(G)$, i.e., when $\nu = \mathrm{id}$. Let $s \in S$. Then s lies in a θ-stable $\boldsymbol{R}$-split torus $\boldsymbol{S}_1 \subset \boldsymbol{G}^0$. Since $\boldsymbol{S}$ is a maximal $\boldsymbol{R}$-split torus in $\boldsymbol{G}^0$ (because $\boldsymbol{P}$ is a minimal parabolic $\boldsymbol{R}$-subgroup of $\boldsymbol{G}^0$, see 3.5) we have that $\boldsymbol{S}_1$ is conjugate to a subtorus of $\boldsymbol{S}$ by an element of $\boldsymbol{G}^0(\boldsymbol{R})$. By 5.17 we may take this element to be in $\boldsymbol{G}^0(\boldsymbol{R})^0$, hence in G. So there is $g \in G$ with $a_1 = g^{-1}sg \in A$. Writing $g = kan$ we obtain

$$n^{-1}a^{-2} \cdot \theta n \cdot a_1 = a_1 n^{-1} \cdot a^{-2} \cdot \theta n.$$

Using again the uniqueness statement of Bruhat's lemma, as in the proof of 5.18, we see that a_1 commutes with n. It follows that s is conjugate to an element of A via K, which is what had to be proved.

5.21. We finally give a brief elementary discussion of the geometric properties of the symmetric space G/K. We identify it with S (cf. 5.5). It is a homogeneous space for G, the action being given by $(x, s) \mapsto x \cdot s = xs(\theta x)^{-1}$. If $x \in \mathfrak{s}$ define $\|X\|^2 = \mathrm{Tr}(X^2)$. This defines a K-invariant Euclidean distance on $\mathfrak{s}$. The exponential map exp defines a diffeomorphism of $\mathfrak{s}$ onto S. Its inverse is denoted by log.

Define a Euclidean metric $d(\ ,\ )$ on A by $d(a, b) = \|\log a - \log b\|$. This determines a structure of Euclidean affine space on A.

We may and shall assume that $\nu = \mathrm{id}$.

5.22. LEMMA. *Let* s, $t \in S$.

(i) *There is* $x \in G$ *such that* $x \cdot s$ *and* $x \cdot t$ *lie in* A;

(ii) *if* x' *is another element with the property of* (i) *then there is* $n \in G$ *normalizing* A *such that* $x^{-1}nx'$ *fixes* s *and* t.

(i) follows from 5.20. To prove (ii) it is sufficient to assume $x' = e$. Then $s, t \in A$. Put $x \cdot s = a$, $x \cdot t = b$. It now follows that $a^{-1/2}xs^{1/2}$ and $b^{-1/2}xt^{1/2}$ lie in K, from which one concludes that st^{-1} and ab^{-1} are conjugate in G. The uniqueness part of Bruhat's lemma then implies that these elements are conjugate by an element of the Weyl group W, from which (ii) follows.

5.23. LEMMA. *If* $x \in G$, $x \cdot A = A$ *then* x *normalizes* A *in* G.

Apply 5.22(ii), taking $x' = e$, s a regular element of A, $t' = e$. It follows that we may assume $x \in K$ and $xs(\theta x)^{-1} = xsx^{-1} = s$. Since s is regular, x centralizes A. The assertion follows.

5.24. The translates $x \cdot A$ of A in S are called *apartments* in S. It follows from 5.23 that for any apartment $\mathscr{A}$ there is a unique structure of Euclidean affine space on $\mathscr{A}$ such that any bijection $A \to \mathscr{A}$ of the form $a \mapsto x \cdot a$ is an isomorphism of such spaces.

5.22(i) shows that for any two elements s, $t \in S$ there is an apartment $\mathscr{A}$ containing them. It follows from 5.22(ii) that, if $s \neq t$, the line in $\mathscr{A}$ containing s and t, together with its structure of 1-dimensional affine space, is *independent* of the choice of $\mathscr{A}$. We call such lines *geodesics* in S.

It now also makes sense to speak of the *geodesic segment* $[st]$, and of the midpoint of $[st]$.

It also follows that there is a unique G-invariant function d on $S \times S$ whose restriction to $A \times A$ is the function of 5.21.

5.25. PROPOSITION. (i) *d is a distance on* S;

(ii) *if* $s, t, u \in S$, $d(s, t) = d(s, u) + d(u, t)$ *then* u *lies on the segment* $[st]$;

(iii) *a closed sphere* $\{x \in X \mid d(x, a) \leqq r\}$ *is compact in* X.

It suffices to prove this for the case $G = \mathbf{GL}(n, \boldsymbol{R})$.

A proof of (i) and (ii) is given in the appendix to this section. The proof of (iii) is easy.

5.26. PROPOSITION. *For each* $s \in S$ *there is a unique involutorial analytic diffeomorphism* σ_s *of* S *with the following properties*:

(a) σ_s *is an isometry* (*for* d),

(b) s *is the only fixed point of* σ_s,

(c) σ_s *stabilizes all geodesics through* s.

We have $\sigma_{x \cdot s} = x \circ \sigma_s \circ x^{-1}$.

We may take $x = e$. The geodesic through e and $\exp X$ consists of the $\exp(\xi X)$ ($\xi \in \boldsymbol{R}$). Observing that $d(e, \exp(\xi X))$ is proportional to $|\xi|$ it follows that the only possibility for σ_e is the map $t \mapsto t^{-1}$. That this satisfies our requirements is clear. The final statement follows from the rest.

5.27. LEMMA. *Let* s, s' *be distinct points of* S, *let* m *be the midpoint* $[ss']$. *Let* t *be* *a point of S not lying on the geodesic through s and s'. Then $d(t, m) < \frac{1}{2}d(t, s) + \frac{1}{2}d(t, s')$.*

Let $\sigma = \sigma_m$, then $\sigma s = s'$. We have

$$2d(t, m) = d(t, \sigma(t)) \leqq d(t, s) + d(\sigma(t), s) = d(t, s) + d(t, s').$$

If the extreme terms are equal we have, by 5.25(ii) that s lies on $[t, \sigma(t)]$. Then so does $s' = \sigma s$, and s, s', t lie on a geodesic, which is contrary to the assumption. The inequality follows.

If C is a subset of X we denote by $I(C)$ the subgroup of the group of isometries of S whose elements stabilize C.

5.28. LEMMA. *If C is compact the group $I(C)$ has a fixed point in S.*

Let $r = \inf_{x \in X} \sup_{y \in C} d(x, y)$. Then $F = \{x \in S \mid \sup_{y \in C} d(x, y) = r\}$ is the intersection of the decreasing family of sets $F_n = \{x \in S \mid \sup_{y \in C} d(x, y) \leqq r + 1/n\}$ $(n = 1, 2, \cdots)$. Since these are nonempty and compact (by 5.25 (iii)) it follows that C is nonempty.

Suppose $a, b \in F$, $a \neq b$ and let m be the midpoint of $[ab]$. We then have by 5.27, for each $y \in C$, $d(m, y) < \frac{1}{2}d(a, y) + \frac{1}{2}d(b, y) = r$, which is impossible. Hence F consists of only one point. It is clearly fixed by $I(C)$.

5.29. THEOREM. *Let M be a compact subgroup of G. Then M fixes a point of S.*

This follows by applying 5.28 to an orbit of M in S.

5.30. COROLLARY. *M is conjugate to a subgroup of K.*

5.31. COROLLARY. *All maximal compact subgroups of G are conjugate.*

5.32. REMARKS. (1) In our discussion of the symmetric space S we wanted to stress, more than is usually done, the analogy with the Bruhat-Tits building $\mathscr{B}$ of a p-adic reductive group. We mention a few features of this analogy.

(a) It is clear from our discussion that S can be obtained, like $\mathscr{B}$, by gluing together apartments (see [**21**, 2.1]).

(b) We have introduced metric and geodesics in S in the same way as is done in the case of $\mathscr{B}$ (see [**9**, 2.5] and [**21**, 2.3]). In the case of $\mathscr{B}$ an important role is played in the discussion of the metric, by the retractions onto an apartment [loc. cit., 2.2]. Such retractions can also be introduced in S (an example is the map ρ used in the appendix).

(c) The fixed point Theorem 5.29 has a counterpart for $\mathscr{B}$ [**8**, 3.2.4].

(d) I owe the proof of 5.29, using the strong convexity property 5.27, to J. J. Duistermaat. 5.29 can also be proved via the argument used in [**8**] (see also [**21**, 2.3]) to prove its counterpart for $\mathscr{B}$. This requires the inequality (m is the midpoint of $[x, y]$)

$$d(x, z)^2 + d(y, z)^2 \geqq 2d(m, z)^2 + \tfrac{1}{2}d(x, y)^2,$$

which can also be established in our situation (e.g. by using that the exponential map $\mathfrak{s} - S$ increases distances).

Appendix. Proof of 5.25 for $G = \mathbf{GL}(n, \boldsymbol{R})$. A is now the group of all diagonal matrices with positive entries. It suffices to show:

if $a, b \in A, s \in S$, then $d(a, b) \leqq d(a, s) + d(s, b)$ equality holding if and only if s lies on the segment $[ab]$.

As is well known, $s \in S$ being given there is a unique $\rho(s) \in A$ and a unique upper triangular unipotent matrix $u(s)$ such that $s = {}^t u(s)\rho(s)u(s)$. If $a \in A$, we have $\rho(asa) = \rho(a)^2\rho(s)$. We shall prove the following.

LEMMA. *$d(\rho(s), e) \leqq d(s, e)$, equality holding if and only if $s \in A$.*

From the lemma it also follows that, for all $a \in A$, $d(\rho(s), a) \leqq d(s, a)$. Hence $d(a, b) \leqq d(a, \rho(s)) + d(\rho(s), b) \leqq d(a, s) + d(s, b)$, proving the triangular inequality. The case of equality is easily dealt with.

It remains to prove the lemma. Let $a_1 \geqq a_2 \geqq \cdots \geqq a_n$ be the eigenvalues of s and $b_1 \geqq b_2 \geqq \cdots \geqq b_n$ those of $\rho(s)$. The lemma asserts that

$$\sum_{i=1}^{n} (\log b_i)^2 \leqq \sum_{i=1}^{n} (\log a_i)^2.$$

Results of this kind are known, they can be found, e.g., in H. Weyl, Ges. Abh. Bd. IV, p. 390. We use Weyl's method. Let $g = \rho(s)^{1/2} u(s)$, so $s = {}^t g \cdot g$. There is a vector $v \in \boldsymbol{R}^n$ with $gv = b^{1/2} v$. Then $(\,,)$ denoting the standard inner product in $\boldsymbol{R}^n$,

$$b_1(v, v) = (v, sv) \leqq a_1(v, v),$$

whence $b_1 \leqq a_1$. Applying a similar argument, working in the exterior powers of $\boldsymbol{R}^n$, we see that $b_1 b_2 \cdots b_p \leqq a_1 a_2 \cdots a_p$ $(1 \leqq p \leqq n - 1)$, and, of course, $b_1 b_2 \cdots b_n = a_1 a_2 \cdots a_n$. We may, and shall, assume that $a_1 a_2 \cdots a_n = 1$.

Let $V \in \boldsymbol{R}^n$ be the subspace of vectors with coordinate sum 0.

Let $\alpha_i = e_i - e_{i+1}$ $(1 \leqq i \leqq n - 1$, (e_i) is the canonical basis), and define $\omega_i \in V$ by $(\omega_i, e_j) = \delta_{ij}$ $(1 \leqq i, j \leqq n - 1)$. Then ω_i is the projection onto V of $e_1 + \cdots + e_i$. Let $x = (\log a_1, \log a_2, \cdots, \log a_n)$, $y = (\log b_1, \log b_2, \cdots, \log b_n)$. We then know that

$$(x, \alpha_i) \geqq 0, \quad (y, \alpha_i) \geqq 0, \quad (x - y, \omega_i) \geqq 0 \qquad (1 \leqq i \leqq n - 1),$$

and we have to prove that $(y, y) \leqq (x, x)$. Now $(x, x) - (y, y) = (x - y, x - y) + 2(y, x - y)$. Since y is a linear combination, with positive coefficients, of the ω_i and $x - y$ is a similar combination of the α_i, we have that $(y, x - y) \geqq 0$. The inequality which we have to prove now becomes obvious.

It is clear that we can only have $(x, x) = (y, y)$ if $x = y$. In that case we have $\mathrm{Tr}(s) = \mathrm{Tr}(\rho(s))$. But it is immediate that if $u(s) \neq e$, we must have

$$\mathrm{Tr}({}^t u(s)\rho(s)u(s)) > \mathrm{Tr}\,\rho(s).$$

So $\mathrm{Tr}(s) = \mathrm{Tr}\,\rho(s)$ implies $\rho(s) = s$. This finishes the proof of the lemma.

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MATHEMATISCH INSTITUUT DER RIJKSUNIVERSITEIT, UTRECHT

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REDUCTIVE GROUPS OVER LOCAL FIELDS

J. TITS

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AMS (MOS) subject classifications (1970). Primary 20G25; Secondary 20G30, 15A63.

0. Preliminaries.

0.1. *Introduction.* This is a survey of some aspects of the structure theory of reductive groups over local fields. Since it is mainly intended for "utilizers", the main emphasis will be on statements and examples. The proofs will mostly be omitted, except for short local arguments which may give a better insight in the way the theory operates. When proofs are available in the literature (which is not always the case!), references will be given; references to **[8]** are often conditional, as explained in §1.5.

We shall not try to give the historical background of the results exposed here. Let us merely recall that the theory was initiated by N. Iwahori and H. Matsumoto **[15]**, who were considering split semisimple groups, that quasi-split and classical groups were later on studied by H. Hijikata **[13]**, and that, in the generality given here, most results are due to F. Bruhat and the author **[6]**, **[7]**, **[8]**, **[9]**. For further information, one may consult the introduction of **[8]**.

0.2. *Notations.* The following notations will be used throughout the paper: K denotes a field endowed with a nontrivial discrete valuation ω, the value group $\omega(K^\times)$ ($\subset \boldsymbol{R}$) is also called Γ, $\mathfrak{o}$ represents the ring of integers, $\mathfrak{p} = \pi\mathfrak{o}$ with $\pi \in \mathfrak{o}$ its prime ideal and $\bar{K} = \mathfrak{o}/\mathfrak{p}$ the residue field. *We always assume K complete and $\bar{K}$ perfect.* We consider an algebraic group G defined over K whose neutral component G° is reductive, and call S a maximal K-split torus of G, N (resp. Z) the normalizer (resp. the centralizer) of S in G, ${}^v\tilde{W}$ the finite group $N(K)/Z(K)$ (as usual, ?(K) stands for the group of rational points of ? over K), $X^* = X^*(S) = \mathrm{Hom}_K(S, \mathrm{Mult})$ (resp. $X_* = X_*(S) = \mathrm{Hom}_K(\mathrm{Mult}, S)$) the group of characters (resp. cocharacters) of S, V the real vector space $X_* \otimes \boldsymbol{R}$, $\Phi = \Phi(G, S) \subset X^*$ the set of roots of G relative to S, ${}^vW$ the Weyl group of the root system Φ which we identify with a normal subgroup of ${}^v\tilde{W}$ (equal to ${}^v\tilde{W}$ if $G = G^\circ$) and U_a, for $a \in \Phi$, the unipotent subgroup of G° normalized by S and corresponding to the root a (i.e., the group called $U_{(a)}$ in [3, 5.2]).

1. The apartment of a maximal split torus and the affine root system.

1.1. *The split case.* As a motivation for what follows, we first consider the case where G° is *split*, that is, where S is a maximal torus of G. Then, the groups U_a are K-isomorphic to the additive group. Indeed, the choice of a "Chevalley basis" in the Lie algebra of G determines a system of K-isomorphisms χ_a: Add $\to U_a$ (an "épinglage") satisfying the commutation relations of Chevalley [10, p. 27]. Since K is a local field, its additive group is filtered and so are the groups $U_a(K)$, "par transport de structure". The terms of those filtrations are conveniently indexed by affine functions on V: for $a \in \Phi$ and $\gamma \in \Gamma$, $a + \gamma$ is such a function and we set

(1) $$X_{a+\gamma} = \chi_a(\omega^{-1}[\gamma, \infty]).$$

If we transform the Chevalley basis by Ad s for an element $s \in S(K)$, the system (χ_a) is replaced by $(\chi_a') = (\chi_a \circ a(s))$ and, setting $X'_{a+\gamma} = \chi_a' (\omega^{-1} [\gamma, \infty])$, we have

(2) $$X'_{a+\gamma} = X_{a+\gamma+\omega(a(s))}.$$

Thus, the terms of the filtrations of the groups $U_a(K)$ are unchanged but their indexation has undergone a translation. The same conclusion holds for an arbitrary change of Chevalley basis (one just has to replace s by a rational element of the image of S in the adjoint group).

We may express that conclusion in a more invariant way as follows. There exist an affine space A under V, a system Φ_{af} of affine functions on A and a mapping $\alpha \mapsto X_\alpha$ of Φ_{af} onto a set of subgroups of $G(K)$ with the following property: to every Chevalley basis, there corresponds a point $0 \in A$ such that Φ_{af} consists of all functions

(3) $$\alpha: x \mapsto a(x - 0) + \gamma \qquad (x \in A; a \in \Phi, \gamma \in \Gamma)$$

and that, if (χ_a) denotes the "épinglage" associated with the given basis, the group X_α corresponding to the function (3) is given by (1). The group $S(K)$ operates by translations on A in such a way that, for $s \in S(K)$, we have

(4) $$s^{-1}X_\alpha s = X_{\alpha\circ s}.$$

From (2) it follows that the translation $\nu(s) \in V$ of A induced by s (i.e., defined by $s(x) = x + \nu(s)$ for $x \in A$) is given by

(5) $$a(\nu(s)) = -\ \omega(a(s)) \quad \text{for every } a \in \Phi.$$

More generally, the normalizer $N(K)$ of $S(K)$ in $G(K)$ operates on A by affine transformations in such a way that (4) holds for any $s \in N(K)$.

1.2. *The apartment* $A(G, S, K)$. Our purpose is to generalize the above results to an arbitrary group G in the following form: *to G, S, K, we want to associate an* *affine space $A = A(G, S, K)$ under V on which $N(K)$ operates, a system $\Phi_{\mathrm{af}} = \Phi_{\mathrm{af}}(G, S, K)$ of affine functions on A and a mapping $\alpha \mapsto X_\alpha$ of Φ_{af} onto a set of subgroups of $G(K)$, such that the relation* 1.1(4) *holds for $s \in N(K)$, that the vector parts $v(\alpha)$ of the functions $\alpha \in \Phi_{\mathrm{af}}$ are the elements of Φ, and that, for $a \in \Phi$, the groups X_α with $v(\alpha) = a$ form a filtration of $U_a(K)$.*

We first proceed with the construction of the space A; the set Φ_{a} and the X_α's will be defined in §§1.6 and 1.4. The relations (5) show us the way. The group $X^*(Z)$ of K-rational characters of Z can be identified with a subgroup of finite index of X^*. Let $\nu: Z(K) \to V$ be the homomorphism defined by

$$\chi(\nu(z)) = -\ \omega(\chi(z)) \quad \text{for } z \in Z(K) \text{ and } \chi \in X^*(Z), \tag{1}$$

and let Z_c denote the kernel of ν. Then, $\Lambda = Z(K)/Z_c$ is a free abelian group of rank $\dim S = \dim V$. The quotient $\tilde{W} = N(K)/Z_c$ is an extension of the finite group ${}^v\tilde{W}$ by Λ. Therefore, there is an affine space A ($= A(G, S, K)$) under V and an extension of ν to a homomorphism, which we shall also denote by ν, of N in the group of affine transformations of A. If G is semisimple, the system (A, ν) is canonical, that is, unique up to unique isomorphism. Otherwise, it is only unique up to isomorphism, but one can, following G. Rousseau [**19**], "canonify" it as follows: calling $\mathscr{D}G^\circ$ the derived group of G° and S_1 the maximal split torus of the center of G°, one takes for A the direct product of $A(\mathscr{D}G^\circ, G^\circ \cap S, K)$ (which is canonical) and $X_*(S_1) \otimes \boldsymbol{R}$ .The affine space A is called the *apartment of S* (relative to G and K). The group $N(K)$ operates on A through $\tilde{W}$.

1.3. *Remark.* Since $V = \mathrm{Hom}(X^*, \boldsymbol{R}) = \mathrm{Hom}(X^*(Z), \boldsymbol{R})$, the groups $\mathrm{Hom}(X^*, \Gamma)$ and $\mathrm{Hom}(X^*(Z), \Gamma)$ are lattices in V and one has

$$\mathrm{Hom}(X^*, \Gamma) \subset \nu(Z(K)) = \Lambda \subset \mathrm{Hom}(X^*(Z), \Gamma). \tag{1}$$

If G is connected and split, both inclusions are equalities, but in general they can be proper. Suppose for instance that $G = R_{L/K}$ Mult, where L is a separable extension of K of degree n, and let Γ_1 be the value group of L. The group $X^*(Z)$ is generated by the norm homomorphism $N_{L/K}$, hence has index n in X^*. On the other end, Λ is readily seen to be equal to $n \cdot \mathrm{Hom}(X^*(Z), \Gamma_1)$. In particular, the first (resp. the second) inclusion (1) is an equality if and only if the extension L/K is unramified (resp. totally ramified). A semisimple example is provided by $G = SU_3$ with splitting field L; exactly the same conclusions as above hold with $n = 2$ (indeed, in that case $Z = R_{L/K}$ Mult). One can prove that *the first inclusion* (1) *is an equality whenever G splits over an unramified extension of K.*

1.4. *Filtration of the groups $U_a(K)$.* Let $a \in \Phi$ and $u \in U_a(K) - \{1\}$. It is known (cf. [**3**, §5]) that the intersection $U_{-a}uU_{-a} \cap N$ consists of a single element $m(u)$ whose image in ${}^v\tilde{W}$ is the reflection r_a associated with a, from which follows that $r(u) = \nu(m(u))$ is an affine reflection whose vector part is r_a. Let $\alpha(a, u)$ denote the affine function on A whose vector part is a and whose vanishing hyperplane is the fixed point set of $r(u)$ and let Φ' be the set of all affine functions whose vector part belongs to Φ. For $\alpha \in \Phi'$, we set $X_\alpha = \{u \in U_a(K) \mid u = 1 \text{ or } \alpha(a, u) \geqq \alpha\}$. The following results are fundamental.

1.4.1. *For every α as above, X_α is a group.*

1.4.2. *If $\alpha, \beta \in \Phi'$, the commutator group (X_α, X_β) is contained in the group generated by all $X_{p\alpha+q\beta}$ for $p, q \in \boldsymbol{N}^*$ and $p\alpha + q\beta \in \Phi'$.*

Clearly, the X_α's with $v(\alpha) = a$ form a filtration of $U_a(K)$. We denote by $X_{\alpha+}$ the union of all $X_{\alpha+\varepsilon}$ for $\varepsilon \in \boldsymbol{R}$, $\varepsilon > 0$ (of course, $X_{\alpha+} = X_{\alpha+\varepsilon}$ for ε sufficiently small). From 1.4.2, it follows that $X_{\alpha+}$ is a normal subgroup of X_α, and we set $\bar{X}_\alpha = X_\alpha/X_{\alpha+}$. Thus, the $\bar{X}_\alpha$'s, for $v(\alpha) = a$, are the quotients of the filtration of $U_a(K)$ in question. It is obvious that for $n \in N(K)$, one has $n^{-1}X_\alpha n = X_{\alpha\circ\nu(n)}$.

1.5. *About proofs, references and generalizations.* Let us identify A with V via the choice of an "origin" 0, and, for every $a \in \Phi$ and $u \in U_a(K)$, set $\varphi_a(u) = \alpha(a, u) - a$ ($\in \boldsymbol{R}$). The assertions 1.4.1 and 1.4.2 essentially mean that the system of functions $(\varphi_a)_{a\in\Phi}$ *is a valuation of the root datum* $(Z(K); (U_a(K))_{a\in\Phi})$, as defined in [**8**, 6.2]. That fact itself is roughly equivalent with (actually somewhat stronger than) the existence of a certain BN-pair in the group generated by all $U_a(K)$ (cf. [**8**, 6.5 and 6.2.3(e)]), and with the existence of the affine building of G over K (cf. §2 below and [**8**, §7]). Those results have been announced in [**6**], [**7**] and [**8**, 6.2.3(c)]), but complete proofs by the same authors have not yet appeared (though the case of classical groups is completely handled in [**8**, §10], and quasi-split groups are essentially taken care of by [**8**, 9.2.3]). In the meantime, proofs of closely related results have been published by H. Hijikata [**14**] and by G. Rousseau [**19**].

In the sequel, quite a few statements will be followed by references to [**8**]; this will usually mean that the quoted section of [**8**] contains a proof of the statement in question *once* 1.4.1 *and* 1.4.2 *are admitted.*

For the sake of simplicity we have assumed that ω is discrete and $\bar{K}$ perfect. In fact, much of what we shall say until §3.3 remains valid (with suitable reformulations) without those assumptions, provided that 1.4.1 and 1.4.2 hold, and this has been shown to be always the case except possibly if char $\bar{K} = 2$ for some groups G whose semisimple part has factors of exceptional type and relative rank $\leqq 2$ (cf. [**8**, §10], [**25**] and [**19**]).

1.6. *The affine root system* Φ_{af}. For every affine function α on A whose vector part $a = v(\alpha)$ belongs to Φ, one has an obvious inclusion $\bar{X}_{2\alpha} \hookrightarrow \bar{X}_\alpha$ (if $2a \notin \Phi$, we set $\bar{X}_{2\alpha} = \{1\}$) and the quotient $\bar{X}_\alpha/\bar{X}_{2\alpha}$ has a natural structure of vector space over $\bar{K}$ (cf. 3.5.1) whose dimension is finite and will be denoted by $d(\alpha)$. In particular, if char $\bar{K} = p$, $\bar{X}_\alpha$ is a p-group. An affine function α such that $a = v(\alpha) \in \Phi$ is called an *affine root* of G (relative to S and K) if $d(\alpha) \neq 0$, that is, if X_α is not contained in $X_{\alpha+\varepsilon} \cdot U_{2a}(K)$ ($= X_{\alpha+\varepsilon}$ if $2a \notin \Phi$) for any strictly positive constant ε. We denote by $\Phi_{\text{af}}(G, S, K) = \Phi_{\text{af}}$ the *affine root system of* G, i.e., the set of all its affine roots. Note that if $2a \notin \Phi$, one has $\alpha(a, u) \in \Phi_{\text{af}}$ for every $u \in U_a(K) - \{1\}$; in particular, if Φ is reduced, $\Phi_{\text{af}} = \{\alpha(a, u) \mid a \in \Phi, u \in U_a(K) - \{1\}\}$.

1.7. *Half-apartments, chambers, affine Weyl group.* For every affine function α such that $a = v(\alpha) \in \Phi$, we denote by A_α the set $\alpha^{-1}([0, \infty))$, by ∂A_α its boundary $\alpha^{-1}(0)$ and by r_α the affine reflection whose vector part is the reflection r_a (cf. 1.4) and whose fixed hyperplane is ∂A_α. The sets A_α (resp. ∂A_α) for $\alpha \in \Phi_{\text{af}}$ are called the *half-apartments* (resp. the *walls*) of A, and the *chambers* are defined as the connected components of the complement in A of the union of all walls. The facets of the chambers are also called the *facets* of A; thus, the chambers are the facets of maximum dimension. If G is quasi-simple the facets (and in particular the chambers) are simplices, if G is semisimple they are polysimplices (i.e., direct products of simplices) and in general they are direct products of a polysimplex and a real affine space.

The group W generated by all r_α with $\alpha \in \Phi_{\mathrm{af}}$ is called the *Weyl group* of the affine root system Φ_{af}. (If G is not semisimple, this is a slight abuse of language since W depends not only on Φ_{af} but also on the subspace of V generated by $X_*(\mathscr{D}G^\circ \cap S)$, where $\mathscr{D}G^\circ$ denotes the derived group of G°.) If G is semisimple, W is the affine Weyl group of a reduced root system (cf. [**5**, VI. 2.1]) whose elements are proportional to those of Φ, *but which is not necessarily proportional to* Φ, *even if* Φ *is reduced* (cf. the examples in §§1.15, 1.16).

Clearly, Φ_{af} is stable by the group $\tilde{W} = \nu(N(K))$ (cf. §1.2). It follows that the half-apartments, the walls and the chambers are permuted by $\tilde{W}$, and that W is a normal subgroup of $\tilde{W}$.

1.8. *Bases, local Dynkin diagram, characteristic dimensions.* The Weyl group W is simply transitive on the set of all chambers (i.e., it permutes the chambers transitively and the stabilizer of a chamber in W is reduced to the identity). Let C be a chamber and let $L_0, \cdots, L_l$ be the walls bounding C. For $i \in \{0, \cdots, l\}$, let α_i be the unique affine root such that $L_i = \partial A_{\alpha_i}$ and $\frac{1}{2}\alpha_i \notin \Phi_{\mathrm{af}}$. The set $\{\alpha_i \mid i = 0, \cdots, l\}$ is called the *basis* of Φ_{af} associated to C.

Let a_i be the vector part of α_i and let us introduce in the dual of V a positive definite scalar product (,) invariant by the (ordinary) Weyl group ${}^vW$. To Φ_{af}, one associates a *(local) Dynkin diagram* $\Delta = \Delta(\Phi_{\mathrm{af}})$ obtained as follows:

The elements α_i of a basis are represented by dots v_i, called the *vertices* of the diagram;

if $2\alpha_i \in \Phi_{\mathrm{af}}$, the vertex v_i is marked with a cross;

two distinct vertices v_i, v_j are joined by an empty, a simple, a double, a triple or a fat segment (*edge* of the diagram) according as the integer $\lambda_{ij} = 4(a_i, a_j)^2/(a_i, a_i)(a_j, a_j)$ equals 0, 1, 2, 3 or 4 (in the latter case, a_j is a positive multiple of $-a_i$);

if $\lambda_{ij} = 2$ or 3 (which implies that $(a_i, a_i) \neq (a_j, a_j)$) or if $\lambda_{ij} = 4$ and $a_j \neq -a_i$, the edge joining v_i and v_j is oriented by an arrow pointing toward the vertex representing the "shortest" of the two roots a_i and a_j.

Since the chambers are permuted simply transitively by W,

1.8.1. *the Dynkin diagram does not depend, up to* canonical *isomorphism, on the choice of the chamber* C.

It is easily seen that the system (A, Φ_{af}) is determined up to isomorphism by the Dynkin diagram Δ and the dimension of A (i.e., the relative rank of G). The Coxeter diagram underlying the Dynkin diagram—i.e., deduced from it by disregarding the crosses and arrows—is the Coxeter diagram of W, hence the Coxeter diagram of an affine reflection group (cf. [**5**, V.3.4, and VI.4.3], where our "diagrams" are called "graphes").

Conversely, consider any Coxeter diagram which is the diagram of an affine reflection group, orient all double and triple edges and possibly some fat ones, and mark some vertices (possibly none) with a cross. Then, *the diagram thus obtained is the local Dynkin diagram* Δ *of some group* G *over some field* K *if and only if, for every vertex* v *marked with a cross, all edges having* v *as an extremity are double or fat and none of them is oriented away from* v.

The necessity of the condition is obvious. As for the sufficiency, the classification of §4 even shows that for any given *locally compact* local field K, every diagram satisfying the above condition is the local Dynkin diagram of some semisimple group G over K: indeed, it is an easy matter to list all irreducible diagrams in question, and one verifies readily that they all appear in the tables of §4. Note that the above statement, or alternatively the tables of §4, provide the classification of all affine root systems, for a suitable "abstract" definition of such systems, which the interested reader will have no difficulty to formulate (cf. also **[8**, 1.4], where the affine root systems are called "échelonnages", and, for the reduced case, **[17]**).

If the vertex v of Δ represents the affine root α, we set $d(v) = d(\alpha) + d(2\alpha)$ $(= d(\alpha)$ if $2\alpha \notin \Phi_{af})$, where the function d is defined as in §1.6. The integer $d(v)$ of course depends not only on Δ and v but on the group G itself. In the tables of §4, the value of $d(v)$ is indicated for every v whenever it is not equal to 1. *If G is split or if the residue field $\bar{K}$ is algebraically closed, all $d(v)$ are equal to* 1.

1.9. *Root system attached to a point of A and special points.* For $x \in A$, we denote by Φ_x the subset of Φ consisting of the vector parts of all affine roots vanishing in x, and by W_x the group generated by all reflections r_α for $\alpha \in \Phi_{af}$ and $\alpha(x) = 0$ (cf. §1.7). To x, we also associate as follows a set I_x of vertices of the local Dynkin diagram Δ: there is an element w of the Weyl group W which carries x in the closure of the "fundamental chamber" C and one sets $I_x = \{v_i \mid wx \notin L_i\}$, with the notations of §1.8; that I_x is independent of the choice of w follows from well-known properties of Coxeter groups: cf., e.g., **[5**, V. 3.3, Proposition 1]. The objects Φ_x, W_x, I_x depend only on the facet F containing x and will also be denoted by Φ_F, W_F, I_F.

The set Φ_x is a (not necessarily closed) subroot system of Φ whose Weyl group is the vector part of W_x and whose (ordinary) Dynkin diagram is obtained by deleting from Δ the vertices belonging to I_x and all edges containing such a vertex. The set I_x has a nonempty intersection with every connected component of Δ and, conversely, every set of vertices with that property is the set I_x for some x.

The point x is called *special* for Φ_{af} if every element of the root system Φ is proportional to some element of Φ_x, that is, if Φ and Φ_x have the same Weyl group. When it is so, W is the semidirect product of W_x by the group of all translations contained in W; similarly, if G is connected, $\tilde{W}$ is the semidirect product of W_x by $\nu(Z(K)) = Z(K)/Z_c$ (cf. 1.2).

The fact for a point x to be special can be recognized from the set of vertices I_x as follows. A vertex of the Coxeter diagram of an irreducible affine reflection group is called *special* if by deleting from the diagram that vertex and all adjoining edges, one obtains the Coxeter diagram of the corresponding finite (spherical) reflection group. (Equivalently: such a diagram being the Coxeter diagram underlying the extended Dynkin diagram—"graphe de Dynkin complété" in the terminology of **[5]**—of a reduced root system, the special vertices are the vertex representing the minimum root and all its transforms by the automorphisms of the diagram.) Clearly, such vertices exist. Now, $x \in A$ is special if and only if I_x consists of one special vertex out of each connected component of Δ. In particular, special points always exist. In the tables of §4, the special vertices are marked with an s or an hs ("hyperspecial points": see below).

1.10. *Behaviour under field extension and hyperspecial points.* Let K_1 be a Galois extension of K with Galois group $\mathrm{Gal}(K_1/K) = \Theta$, and let S_1 be a maximal K_1-split torus of G containing S and defined over K. Such a torus exists for instance in the following cases:

if G is quasi-split over K (obvious!);

if K_1 is the maximal unramified extension of K [**6**(c), 3, Corollaire 1];

if the residue field $\bar{K}$ is finite and K_1/K is unramified.

(The latter condition is necessary as is shown by the following example due to Serre: suppose that Θ has even order and no subgroup of index 2, and that G is the norm one group of a division quaternion algebra; then G splits over K_1 but none of its maximal tori does.) Let $A_1 = A(G, S_1, K_1)$ be the apartment of S_1 and let $\Phi_{1\mathrm{af}} = \Phi_{\mathrm{af}}(G, S_1, K_1)$ be the corresponding affine root system. The Galois group Θ operates on A_1 ("canonified" as in §1.2) "par transport de structure", and A can be identified with the fixed point set A_1^Θ.

That identification is not quite obvious. To characterize it, we have to describe an operation of $N(K)$ on A_1^Θ (cf. 1.2). First observe that A_1^Θ clearly is an affine space under V. Let now $n \in N(K)$, let N_1 be the normalizer of S_1 in G and let ν_1 be the canonical homomorphism of $N_1(K_1)$ into the group of affine transformations of A_1. Since the conjugate ${}^nS_1$ is a maximal K_1-split torus of Z, there exists $z \in Z(K_1)$ with $n' = nz^{-1} \in N_1(K_1)$. Upon multiplying z by a suitable element of $(Z \cap N_1)(K_1)$, one may choose it so that $\nu_1(n')$ stabilizes A_1^Θ. Let now $\nu(z)$ be the element of V defined by the relation 1.2(1) where ω must be replaced by the valuation of K_1. Then $n = n'z$ operates on A_1^Θ through $\nu_1(n') \circ \nu(z)$. That this action is independent of the choices made and indeed defines an operation of $N(K)$ on A_1^Θ is best seen by using the "building" of G over K_1 defined in §2: that building contains A_1^Θ and is operated upon by $G(K_1)$, hence by $N(K)$, and one verifies that $N(K)$ stabilizes A_1^Θ and operates on it as described above. Note that, more generally, the results of §2.6 show that if S_1' is *any* maximal K_1-split torus of G containing S, A can be naturally identified with an affine subspace of $A(G, S_1', K_1)$; much of what we shall say here extends to that situation.

1.10.1. *If K_1/K is unramified, Φ_{af} consists of all nonconstant restrictions $\alpha|_A$, with $\alpha \in \Phi_{1\mathrm{af}}$.*

That is no longer true in general when K_1/K is ramified. An obvious example is provided by the case where G is split over K. Then, $S_1 = S$, $A_1 = A$, and if we identify A with V as in §1.1, we have $\Phi_{\mathrm{af}} = \{a + \gamma | \, a \in \Phi, \ \gamma \in \Gamma\}$ and $\Phi_{1\mathrm{af}} = \{a + \gamma | \, a \in \Phi, \gamma \in \Gamma_1\}$, where Γ_1 denotes the value group of K_1.

From 1.10.1, it follows readily that

1.10.2. *If K_1/K is unramified, every point of A which is special for $\Phi_{1\mathrm{af}}$ is also special for Φ_{af}.*

The above example shows that that assertion becomes false without the assumption on K_1/K. A point $x \in A$ is called *hyperspecial* if there exist K_1, S_1 as above such that K_1/K is unramified, that G splits over K_1 and that x is special for $\Phi_{1\mathrm{af}}$. Then, it is easily seen, using 1.10.2, that the same holds for any Galois unramified splitting field K_1 of G and any choice of S_1 (assuming that such a torus exists). More intrinsic characterizations of the hyperspecial points will be given in 3.8.

If G is quasi-split and splits over an unramified extension of K, hyperspecial points do exist. Indeed, take for K_1 the minimum splitting field of G and (obligatorily) $S_1 = Z$, let $a_1, \cdots, a_l$ be a basis of the root system $\Phi(G, S_1)$ invariant by Θ and choose $\alpha_1, \cdots, \alpha_l \in \Phi_{1\mathrm{af}}$ so that $v(\alpha_i) = a_i$ and that $\{\alpha_1, \cdots, \alpha_l\}$ is stable by Θ (the possibility of such a choice readily follows from the description of A and Φ_{af} given in §1.1). Then, the equations $\alpha_1 = \cdots = \alpha_l = 0$ define an affine subspace of A_1 invariant by Θ (in fact a single point if G is semisimple), and every point invariant by Θ in that subspace belongs to A and is clearly hyperspecial.

Suppose G is quasi-simple. We say that a vertex v of the local Dynkin diagram is *hyperspecial* (with respect to G) if the points $x \in A$ such that $I_x = \{v\}$ are hyperspecial (a property which depends only on v obviously). In the tables of §4, hyperspecial vertices are marked with an *hs*.

Let now K_1 be the maximal unramified extension of K. The group G is said to be *residually quasi-split* over K if there is a chamber of A_1 stable by $\mathrm{Gal}(K_1/K)$, and hence meeting A. We say that G is *residually split* if A_1 is fixed by $\mathrm{Gal}(K_1/K)$, that is, if G has the same rank over K and over K_1, i.e., if $S_1 = S$. For an explanation of the terminology and another definition, cf. 3.5.2.

1.10.3. *If the residue field $\bar{K}$ is finite, G is residually quasi-split. If $\bar{K}$ is algebraically closed, G is residually split.*

By a well-known result of R. Steinberg, if $\bar{K}$ is algebraically closed, G is quasi-split. From that, it follows that:

1.10.4. *Every residually split group is quasi-split.*

If $\bar{K}$ is finite and, more generally, if G is residually quasi-split, G has a "natural splitting field". Indeed, there is a smallest unramified extension K' of K on which G is residually split, namely the smallest splitting field of S_1 (which does not depend on the choice of that torus), and the group G, being quasi-split over K', has a smallest splitting field K'' over K'. The field K'' can also be characterized among all splitting fields of G over K as the unique one for which the pair consisting of the degree $[K'': K]$ and the ramification index $e(K''/K)$ is minimal for the lexicographic ordering.

1.11. *Absolute and relative local Dynkin diagram; the index.* In this section, K_1 denotes the maximal unramified extension of K, and A_1, S_1, $\Phi_{1\mathrm{af}}$ have the same meaning as in §1.10. As in the classical, "global" situation (cf. [**22**] and the references given there), one associates to G, K, S_1 (in fact, to G, K alone: cf. §2.4) a *local index* consisting of

the Dynkin diagram Δ_1 of $\Phi_{1\mathrm{af}}$ (*absolute local Dynkin diagram*),

the action of $\Theta = \mathrm{Gal}(K_1/K)$ on Δ_1 "par transport de structure", and

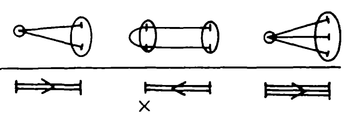
a Θ-invariant set of vertices of Δ_1, called the *distinguished vertices*.

The latter are characterized as follows: to define $\Phi_{1\mathrm{af}}$, one uses a chamber C_1 of A_1 whose closure contains a chamber of A (such a C_1 exists by 1.10.1), and then, the distinguished vertices are those representing the elements of the basis of $\Phi_{1\mathrm{af}}$ associated to C_1 (§1.8) whose restriction to A is not constant.

Residually quasi-split and residually split groups can be characterized as follows in terms of the index. The group G is residually quasi-split if and only if the orbits of Θ in the set of all nondistinguished vertices are unions of full connected components of Δ_1, and G is residually split if and only if Θ operates trivially on Δ_1, all vertices of Δ_1 are distinguished and the smallest splitting field of the connected center of G is totally ramified.

The index of G determines its relative local Dynkin diagram $\Delta = \Delta(G, S, K)$ and the integers $d(v)$ (cf. §1.8) uniquely. We shall indicate an easy algorithm which allows us, in most cases, to deduce the latter from the former. First of all, there is a canonical bijective correspondence $v \mapsto O(v)$ between the vertices of Δ and the orbits of Θ in the set of distinguished vertices of Δ_1. For every vertex v (resp. every pair $\{v, v'\}$ of vertices) of Δ, let $\tilde{\Delta}_{1,v}$ (resp. $\tilde{\Delta}_{1,vv'}$) denote the subdiagram of Δ_1 obtained by removing from it all vertices not belonging to $O(v)$ (resp. $O(v) \cup O(v')$) and all edges containing such vertices, and let $\Delta_{1,v}$ (resp. $\Delta_{1,vv'}$) be the subdiagram of $\bar{\Delta}_{1,v}$ (resp. $\bar{\Delta}_{1,vv'}$) consisting of its connected components which contain at least one distinguished vertex. Then, $\Delta_{1,v}$, together with the action of $\Theta \cong \mathrm{Gal}(\bar{K}_1/\bar{K})$ on it and the set of distinguished orbits it contains, is the index (in the sense of [3] and [22]) of a semisimple group of relative rank one over $\bar{K}$, the integer $d(v)$ is half the total number of absolute roots of that group and v is marked with a cross in Δ if and only if the relative root system of the group in question has type BC_1 (if $\bar{K}$ is finite—or more generally if all vertices of $\Delta_{1,v}$ are distinguished—that means that $\Delta_{1,v}$ is a disjoint union of diagrams of type A_2).

As for the edge of Φ joining v and v', its type is determined by $\Delta_{1,vv'}$, $O(v)$ and $O(v')$. If no connected component of $\Delta_{1,vv'}$ meets both $O(v)$ and $O(v')$, then v and v' are joined by an "empty edge". Otherwise, Θ permutes transitively the connected components of $\Delta_{1,vv'}$ and the result can be described in terms of any one of them, say $\Delta^\circ_{1,vv'}$. If the latter has only two vertices $v_1 \in O(v)$ and $v_1' \in O(v')$, then v and v' are joined in Δ in the same way as v_1 and v_1' in $\Delta^\circ_{1,vv'}$. Thus, we may assume that $\Delta^\circ_{1,vv'}$ has at least three vertices. Suppose first that $\Delta^\circ_{1,vv'}$ is not a full connected component of Δ_1. Then, there is an "admissible index" (i.e., an index appearing in the tables of [22]) of relative rank 2 whose underlying Dynkin diagram is $\Delta^\circ_{1,vv'}$ and whose distinguished orbits are $O(v) \cap \Delta^\circ_{1,vv'}$ and $O(v') \cap \Delta^\circ_{1,vv'}$; indeed, it follows from the assertions 3.5.2 below that to $\{v,v'\}$ is canonically associated a quasi-simple group defined over a certain extension of $\bar{K}$ and having such an index. The relative Dynkin diagram corresponding to that index, which can be computed by simple explicit formulae given in [22, 2.5], provides the nature of the edge joining v and v' in Δ. The following table gives the result in the case where all vertices of $\Delta^\circ_{1,vv'}$ are distinguished (e.g., in the case where the residue field $\bar{K}$ is finite); in the first row, which represents $\Delta^\circ_{1,vv'}$, the sets $O(v) \cap \Delta^\circ_{1,vv'}$ and $O(v') \cap \Delta^\circ_{1,vv'}$ are circled:

$\Delta^\circ_{1,vv'}$	
corresponding edge in Δ	

There remains to consider the case where $\Delta^\circ_{1,vv'}$ is a full connected component of Δ_1, which means that v, v' are the two vertices of the local Dynkin diagram of a quasi-simple factor of relative rank 1 of G (cf. §1.12). Here we shall restrict ourselves to the case where all vertices are distinguished and simply refer the reader to the tables of §4 which give Δ in all the cases that can occur.

1.12. *Reduction to the absolutely quasi-simple case*; *restriction of scalars.* We shall now indicate how the local Dynkin diagram—with the attached integers $d(v)$—and the index of an arbitrary group G can be deduced from those of related absolutely simple groups.

First of all, those data are the same for G and for the adjoint group of G°. Thus, we may assume that G is connected and adjoint, hence is a direct product of K-simple groups. Then, the Dynkin diagram—with the $d(v)$ attached—and the index of G are the disjoint unions of the Dynkin diagrams and the indices of its simple factors.

There remains to consider the case where G is K-simple, which means [3, 6.21] that $G = R_{L/K}\,H$, where L is a separable extension of K, H is an absolutely simple group defined over L and $R_{L/K}$ denotes, as usual, the restriction of scalars. We shall, more generally, assume that $G = R_{L/K}\,H$ for an arbitrary reductive group H; this allows us to decompose the extension L/K into its unramified and its totally ramified parts and to handle the two cases separately.

If L/K is totally ramified, the index, the local Dynkin diagram and the integers $d(v)$ are the same for G, K as for H, L.

If L/K is unramified, the index of G, K consists of $[L\colon K]$ copies of the index of H, L permuted transitively by $\mathrm{Gal}(K_1/K)$ whose operation on the whole diagram is "induced up" from the operation of $\mathrm{Gal}(K_1/L)$ on one copy, the relative local Dynkin diagram of G, K is the same as that of H, L, and the integers $d(v)$ are $[L\colon K]$ times as big.

1.13. *The case of simply connected groups.* In §1.7, we have seen that the Weyl group W of Φ_{af} is a normal subgroup of $\tilde{W} = N(K)/Z(K)$. *When G is semisimple and simply connected, one has $W = \tilde{W}$.* In this and in other instances, *nonsimply connected groups behave with respect to the "local theory" in a way similar to non-connected groups with respect to the classical theory.*

1.14. *Example. General linear groups.* Let D be a finite dimensional central division algebra over K. The unique extension of the valuation ω to D will also be denoted by ω. Suppose that $G = \mathrm{GL}_{n,D}$, the algebraic group defined by $G(L) = \mathrm{GL}_n(D \otimes L)$ for any K-algebra L, and take for S the "group of invertible diagonal matrices with central entries", that is, the split torus whose group of rational points $S(K)$ consists of all diagonal matrices $\mathrm{Diag}(s_1, \cdots, s_n)$ with $s_i \in K^\times$. The homomorphisms $e_i\colon \mathrm{Mult} \to S$ defined by

$$e_i(t) = \mathrm{Diag}(1, \cdots, 1, t^{-1}, 1, \cdots, 1)^1$$

with the coefficient t^{-1} in the ith place ($i = 1, \cdots, n$) form a basis of X_* and hence of $V = X_* \otimes \boldsymbol{R}$. If $(a_i)_{1 \leq i \leq n}$ is the dual basis in the dual of V, the relative roots of G are the characters $a_{ij} = a_j - a_i$ ($i \neq j$), the group $U_{a_{ij}}(K)$ consists of the matrices

$$u_{ij}(d) = 1 + ((g_{rs})) \quad \text{with } g_{rs} = \delta_r^i\, \delta_s^j\, d \quad (d \in D),$$

and $N(K)$ is the group of all invertible monomial matrices

$$n(\sigma; d_1, \cdots, d_n) = ((g_{ij})) \quad \text{with } g_{ij} = \delta_i^{\sigma(j)}\, d_j,$$

where σ is a permutation of $\{1, \cdots, n\}$ and $d_i \in D^\times$ ($= D - \{0\}$). For $d \in D^\times$, one has, with the notations of 1.4,

$$(1) \qquad m(u_{ij}(d)) = u_{ji}(-d^{-1})u_{ij}(d)u_{ji}(-d^{-1}) = n(\sigma; d_1, \cdots, d_n),$$

where σ is the transposition of i and j, $d_j = d$, $d_i = -d^{-1}$ and $d_k = 1$ for $k \neq i, j$. We may identify the apartment A with V in such a way that

$$\nu(n(\sigma; d_1, \cdots, d_n))\left(\sum_{i=1}^n v_i e_i\right) = \sum_{i=1}^n v_i' e_i \quad \text{with } v'_{\sigma(i)} = v_i + \omega(d_i).$$

[1]To avoid confusion, we adhere to the notations of [8, §10] which, unfortunately, impose this somewhat unnatural choice of the basis (e_i) (and, consequently, of (a_i)). This remark also applies to §§1.15 and 1.16.

From (1) and the definition of $\alpha(a, u)$, it now follows that $\alpha(a_{ij}, u_{ij}(d)) = a_{ij} + \omega(d)$. Thus

$$\Phi_{\mathrm{af}} = \{a_{ij} + \gamma \mid i, j \in \{1, \cdots, n\}, i \neq j, \gamma \in \Gamma\}, \tag{2}$$

and the filtration of $U_{a_{ij}}(K)$ by the groups X_α with $v(\alpha) = a_{ij}$ (cf. §1.4) is the image of the natural filtration of D by the isomorphism $d \mapsto u_{ij}(d)$. In particular, for any $\alpha \in \Phi_{\mathrm{af}}$, the integer $d(\alpha)$ of §1.6 is equal to the dimension of the residual algebra of D over $\bar{K}$. The description of the walls and half-apartments is readily deduced from (2). The chambers are prisms with simplicial bases, one of them, call it C, being defined by the inequalities $a_1 < a_2 < \cdots < a_n < a_1 + \omega(\pi_1)$, where π_1 denotes a uniformizing element of D. The corresponding basis consists of the affine roots $a_{i,i+1}$ $(i = 1, \cdots, n-1)$ and $a_{n1} + \omega(\pi_1)$, and we see that the local Dynkin diagram is a cycle of length n (affine diagram of type A_{n-1}). The special points are all the points of the one-dimensional facets of the chambers, that is, all the points $\sum v_i e_i$ where $v_i - v_1$ is an integral multiple of $\omega(\pi_1)$ for all i; they are hyperspecial if and only if $D = K$.

1.15. *Example. Quasi-split special unitary groups in odd dimension.* Let L be a separable quadratic extension of K. The valuation of L extending ω will also be called ω, and we denote by π_1 a uniformizing element of L, by Γ_1 the value group $\omega(L^\times) = Z \cdot \omega(\pi_1)$ and by τ the nontrivial K-automorphism of L. Let n be a strictly positive integer and set $I = \{\pm 1, \cdots, \pm n\}$. In L^{2n+1}, we consider the hermitian form

$$h\colon ((x_{-n}, \cdots, x_n), (y_{-n}, \cdots, y_n)) \mapsto \sum_{i \in I} x_i^\tau y_{-i} + x_0^\tau y_0. \tag{1}$$

Suppose that G is the algebraic group $SU(h)$ and let the torus S be defined by $S(K) = \{\mathrm{Diag}\,(d_{-n}, \cdots, d_n) | d_i \in K \text{ and } d_{-i} d_i = d_0 = 1 \text{ for all } i\}$. The homomorphisms e_i: Mult $\to S$ $(i = 1, \cdots, n)$ defined by $e_i(t) = \mathrm{Diag}(d_{-n}, \cdots, d_n)$ with $d_{-i} = t$, $d_i = t^{-1}$, $d_j = 1$ for $j \neq \pm i$ form a basis of X_*. If we denote by $(a_i)_{1 \leq i \leq n}$ the dual basis and if we set $a_{-i} = -a_i$ and $a_{ij} = a_i + a_j$, we have $\Phi = \{a_{ij} \mid i, j \in I, j \neq \pm i\} \cup \{a_i, 2a_i \mid i \in I\}$. For $c, d \in L$ such that $c^\tau c + d + d^\tau = 0$ and $i, j \in I$ with $j \neq \pm i$, we define the following elements of $G(K)$:

$u_{ij}(c) = 1 + ((g_{rs}))$ with $g_{-j,i} = c^\tau$, $g_{-i,j} = -c$ and all other $g_{rs} = 0$,

$u_i(c, d) = 1 + ((g_{rs}))$ with $g_{-i,0} = -c^\tau$, $g_{-i,i} = d$, $g_{0i} = c$ and all other $g_{rs} = 0$.

Then, $U_{a_{ij}}(K) = \{u_{ij}(c) \mid c \in L\}$, $U_{a_i}(K) = \{u_i(c, d) \mid c, d \in L, c^\tau c + d + d^\tau = 0\}$ and $U_{2a_i}(K) = \{u_i(0, d) \mid d \in L, d + d^\tau = 0\}$. The group $N(K)$ consists of all matrices of determinant one of the form $n(\sigma; d_{-n}, \cdots, d_n) = ((g_{ij}))$ with $g_{ij} = \delta_i^{\sigma(j)} d_j$, where σ is a permutation of $I \cup \{0\} = \{-n, \cdots, n\}$ which fixes 0 and preserves the partition of I in pairs $(-i, i)$, and the d_i's are elements of L such that $d_{-i}^\tau d_i = 1$ for all i.

For $c \in L$, one has, with the notations of 1.14,

$$\begin{aligned} m(u_{ij}(c)) &= u_{-i,-j}(-c^{-1}) u_{ij}(c) u_{-i,-j}(-c^{-1}) \\ &= n(\sigma; d_{-n}, \cdots, d_n) \end{aligned} \tag{2}$$

where σ is the permutation $(i, -j)(j, -i)$, $d_{-i} = c^{-1}$, $d_{-j} = -(c^\tau)^{-1}$, $d_j = -c$, $d_i = c^\tau$ and all other d_r are equal to 1. Similarly, for c, d as above with $c \neq 0$ (and hence $d \neq 0$),

$$\begin{aligned} m(u_i(c, d)) &= u_{-i}(-cd^{-1}, (d^\tau)^{-1}) u_i(c, d) u_{-i}(-c(d^\tau)^{-1}, (d^\tau)^{-1}) \\ &= n(\sigma; d_{-n}, \cdots, d_n) \end{aligned} \tag{3}$$

where σ is the transposition $(i, -i)$, $d_{-i} = (d^{\tau})^{-1}$, $d_0 = -d^{\tau}d^{-1}$, $d_i = d$ and all other d_r are equal to 1.

We may identify the apartment A with V in such a way that, for $v_1, \cdots, v_n \in \boldsymbol{R}$ and setting $v_{-i} = -v_i$, one has

$$(4) \qquad \nu(n(\sigma; d_{-n}, \cdots, d_n))\Big(\sum_{i=1}^{n} v_i e_i\Big) = \sum_{i=1}^{n} v_i' e_i \quad \text{with } v'_{\sigma(i)} = v_i + \omega(d_i).$$

From (2), (3), (4) and the definition of $\alpha(a, u)$, it follows that

for $c \neq 0$, $\alpha(a_{ij}, u_{ij}(c)) = a_{ij} + \omega(c)$,

for c, d as above and $c \neq 0$, $\alpha(a_i, u_i(c, d)) = a_i + \frac{1}{2}\omega(d)$,

for $d \in L^{\times}$ with $d + d^{\tau} = 0$, $\alpha(2a_i, u_i(0, d)) = 2a_i + \omega(d)$.

Setting $\Gamma' = \{\omega(d) | d \in L^{\times}, d + d^{\tau} = 0\}$, we see that for $\gamma \in \Gamma$ (resp. $\gamma \in \Gamma'$) $a_{ij} + \gamma$ (resp. $2a_i + \gamma$) is an affine root for all i, j. Furthermore, the filtration of $U_{a_{ij}}(K)$ (resp. $U_{2a_i}(K)$) by the subgroups X_{α} is the image of the natural filtration of L (resp. its intersection with the subgroup $\{d | d \in L, d + d^{\tau} = 0\}$) by the isomorphism $c \mapsto u_i(c)$ (resp. $d \mapsto u_i(0, d)$). In particular, the corresponding values of the integer $d(\alpha)$ of §1.6 are $d(2a_i + \gamma) = 1$ and $d(a_{ij} + \gamma) = 1$ or 2 according as L/K is ramified or not.

To determine under which condition $a_i + \frac{1}{2}\gamma \in \Phi_{\mathrm{af}}$, we first note that, with the notations of §1.4, $X_{a_i+\gamma/2} = \{u_i(c, d) | c^{\tau}c + d + d^{\tau} = 0, \omega(d) \geqq \gamma\}$. By definition, $a_i + \frac{1}{2}\gamma \in \Phi_{\mathrm{af}}$ if and only if $X_{a_i+\gamma/2} \not\subset X_{a_i+\gamma/2+\varepsilon} \cdot U_{2a_i}$ for every strictly positive ε. That means that there exists $c \in L$ such that

$$(5) \qquad \gamma = \sup\{\omega(d) \mid c^{\tau}c + d + d^{\tau} = 0\}.$$

More precisely, an easy computation shows that, with the notations of §1.6, the group $\bar{X}_{a_i+\gamma/2}/\bar{X}_{2a_i+\gamma}$ is isomorphic to the residue field of L or is trivial according as whether or not γ is given by (5) for some c; thus, we see that, in the first case (i.e., when $a_i + \frac{1}{2}\gamma \in \Phi_{\mathrm{af}}$), $d(a_i + \frac{1}{2}\gamma) = 2$ or 1 according as L/K is unramified or ramified. If we set $\delta = \sup\{\omega(d) \mid d \in L, d + d^{\tau} + 1 = 0\}$, a real number which is strictly negative if L/K is ramified and char $\bar{K} = 2$, and $= 0$ otherwise, the right-hand side of (5) can be written $\omega(c^{\tau}c) + \delta = 2\omega(c) + \delta$, and we conclude that

$$\Phi_{\mathrm{af}} = \{a_{ij} + \gamma \mid i, j \in I, j \neq \pm i, \gamma \in \Gamma_1\} \cup \{2a_i + \gamma \mid i \in I, \gamma \in \Gamma'\}$$
$$\cup \{a_i + \tfrac{1}{2}\gamma \mid i \in I, \gamma \in 2\Gamma_1 + \delta\}.$$

Let us show that

$$(6) \qquad \text{if } L/K \text{ is ramified, } \delta \notin \Gamma'.$$

Indeed, assume the contrary and let $x, y \in L$ be such that $x + x^{\tau} + 1 = y + y^{\tau} = 0$ and $\omega(x) = \omega(y) = \delta$. Upon multiplying y by a suitable unit of K, we may assume that $xy^{-1} + 1 \equiv 0 \pmod{\pi_1}$, but then $(x + y) + (x + y)^{\tau} + 1 = 0$ and $\omega(x + y) > \delta$, which contradicts the maximality of δ.

In view of (6), one of the following holds:

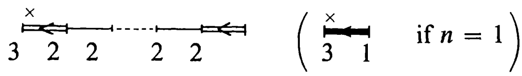
$$(7) \qquad L/K \text{ is unramified and } \Gamma = \Gamma_1 = \Gamma';$$

$$(8) \qquad L/K \text{ is ramified, } \Gamma = 2\Gamma_1 \text{ and } \Gamma' = 2\Gamma_1 + \delta + \omega(\pi_1).$$

In both cases, $\Gamma' \cup (2\Gamma_1 + \delta) = \Gamma_1$; therefore, the walls are the vanishing sets of the affine functions $a_{ij} + \gamma$ and $2a_i + \gamma$, with $\gamma \in \Gamma_1$, and the inequalities $0 < a_1 <$

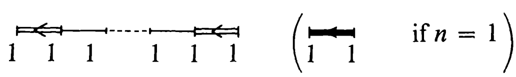
$a_2 < \cdots < a_n < \frac{1}{2}\,\omega(\pi_1)$ define a chamber. The corresponding basis is $\{a_1, a_{-1,2}, \cdots, a_{-n+1,n}, 2a_{-n} + \omega(\pi_1)\}$ if δ is an even multiple of $\omega(\pi_1)$ and $\{a_{-n} + \frac{1}{2}\,\omega(\pi_1), a_{-n+1,n}, \cdots, a_{-1,2}, 2a_1\}$ otherwise. It follows from (7), (8) that in the first case, $2a_1$ is an affine root if and only if L/K is unramified, and in the second case (where L/K is necessarily ramified) $2a_{-n} + \omega(\pi_1)$ is never an affine root. As a result, we see that, whatever the value of δ, the local Dynkin diagram, together with the attached integers $d(v)$ (cf. §1.8) are

(9)



or

(10)



according as L/K is unramified or ramified.

A point $v = \sum_{i=1}^{n} v_i e_i \in A$ is special if and only if either $v_i \in \Gamma_1$ for all i or $v_i - \frac{1}{2}\omega(\pi_1) \in \Gamma_1$ for all i. It is hyperspecial if and only if L/K is unramified and $v_i \in \Gamma_1$ for all i, which means that I_v consists of the vertex at the right end of the diagram (9).

1.16. *Example. Quasi-split but nonsplit orthogonal groups.* Let L be a separable quadratic extension of K and let n be an integer $\geqq 2$. In the space $K^n \oplus L \oplus K^n$, viewed as a $(2n + 2)$-dimensional vector space over K, we consider the quadratic form

$$q\colon (x_{-n}, \cdots, x_n) \mapsto \sum_{i=1}^{n} x_{-i}x_i + N_{L/K}x_0 \qquad (x_0 \in L;\, x_i \in K \text{ for } i \neq 0)$$

(where $N_{L/K}\colon L \to K$ denotes the norm), and we suppose that G is the orthogonal group $O(q)$. The elements of $G(R)$, for any K-algebra R, are conveniently represented by $(2n + 1) \times (2n + 1)$ matrices $((g_{ij}))_{-n\leqq i,\, j\leqq n}$ where $g_{ij} \in R$ if both i and j are not zero, $g_{0j} \in L\otimes_K R$ if $j \neq 0$, $g_{i0} \in \mathrm{Hom}_K(L, R)$ if $i \neq 0$, and $g_{00} \in \mathrm{Hom}_K(L, L) \otimes_K R$. For S, we take the group of diagonal matrices $\mathrm{Diag}(d_{-n}, \cdots, d_n)$ with $d_{-i}d_i = 1$ for $1 \leqq i \leqq n$ and $d_{00} = \mathrm{id}$. The characters a_i: $\mathrm{Diag}(d_{-n}, \cdots, d_n) \mapsto d_{-i}$ for $1 \leqq i \leqq n$ form a basis of $X^*(S)$ and if we set $a_{-i} = -a_i$, $a_{ij} = a_i + a_j$ and $I = \{\pm 1, \cdots, \pm n\}$, we have

$$\Phi = \{a_{ij} \,|\, i, j \in I,\ j \neq \pm i\} \cup \{a_i \,|\, i \in I\},$$

a root system of type B_n.

Here, we shall simply describe the affine root system Φ_{af} and the local Dynkin diagram without giving the details of the calculations, which can be found, in a more general setting (covering also the groups handled in the previous section) in [**8**, 10.1]. Calling again Γ_1 the value group of L, one has, for a suitable identification of A and V,

$$\Phi_{\mathrm{af}} = \{a_{ij} + \gamma \,|\, i, j \in I, j \neq \pm i, \gamma \in \Gamma\} \cup \{a_i + \gamma \,|\, i \in I, \gamma \in \Gamma_1\}.$$

If the extension L/K is unramified, the inequalities $0 < a_1 < \cdots < a_n < \omega(\pi) - a_{n-1}$ define a chamber, the corresponding basis of Φ_{af} is $\{a_1, a_{-1,2}, \cdots,$

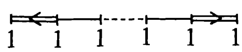
$a_{-n+1,n}, a_{-n+1,-n} + \omega(\pi)\}$ and the local Dynkin diagram, together with the attached integers $d(v)$, is

(1) 

The special vertices are the two endpoints on the ramified side of the diagram (the two endpoints of the diagram if $n = 2$); both correspond to hyperspecial points of A. If L/K is ramified, the inequalities $0 < a_1 < \cdots < a_n < \frac{1}{2}\omega(\pi)$ define a chamber, the corresponding basis is

$$\{a_1, a_{-1,2}, \cdots, a_{-n+1,n}, a_{-n} + \tfrac{1}{2}\omega(\pi)\}$$

and the local Dynkin diagram is



The special vertices are the two endpoints of the diagram and they do not correspond to hyperspecial points. Note that in the unramified case, the Weyl group W is an affine reflection group of type B_n, whereas in the ramified case, it is of type C_n.

2. The building.

2.1. *Definitions.* The *building* $\mathscr{B} = \mathscr{B}(G, K)$ *of* G *over* K can be constructed by "gluing together" the apartments of the various maximal K-split tori of G. More precisely, a definition of $\mathscr{B}$ is provided by the following statement where by "$G(K)$-set", we mean a set with a left action of $G(K)$ on it.

Let $A = A(G, K)$ *be given as in* §1.2. *Then, there exists one and, up to unique isomorphism, only one* $G(K)$*-set* $\mathscr{B}$ *containing* A *and having the following properties*: $\mathscr{B} = \bigcup_{g \in G(K)} gA$, *the group* $N(K)$ *stabilizes* A *and operates on it through* ν (*cf.* §1.2) *and for every affine root* α, *the group* X_α *of* §1.4 *fixes the half-apartment* $A_\alpha = \alpha^{-1}([0, \infty))$ *pointwise.*

(N.B. The "canonicity" of the building $\mathscr{B}$ is the same as that of A: cf. §1.2.)

The proof roughly goes as follows. We assume that G is semisimple (which is no essential restriction). Modulo 1.4.1 and 1.4.2—as explained in §1.5—the existence of $\mathscr{B}$ is proved in [**8**, 7.4]. It is then clear that there is a "universal" $G(K)$-set $\tilde{\mathscr{B}}$ with the given properties, which is obtained by taking the quotient of the direct product $G(K) \times A$ by a certain equivalence relation. The canonical mapping of $\tilde{\mathscr{B}}$ in the building $\mathscr{I}$ defined in [**8**, 7.4.2] is obviously surjective, and it is also injective because, as is readily verified, the stabilizer of a point of $\tilde{\mathscr{B}}$ contains the stabilizer of its image in $\mathscr{I}$. Thus, $\mathscr{I}$ maps onto any $G(K)$-set $\mathscr{B}$ with the required properties and, using [**8**, 7.3.4], one shows that the stabilizers of the points of A cannot be bigger in $\mathscr{B}$ than they are in $\mathscr{I}$ without "eating more of $N(K)$" than they are allowed to by the prescribed action of $N(K)$ on A.

The sets gA with $g \in G(K)$ are called the *apartments* of the building. The apartment gA can be identified with "the" apartment of the maximal split torus ${}^gS$. That gives a one-to-one correspondence between the apartments of $\mathscr{B}$ and the maximal K-split tori of G: indeed, gA is the only apartment stable by ${}^gS(K)$ (the proof of [**8**, 2.8.11] shows that) and ${}^gN(K)$, which determines ${}^gS$, is the stabilizer of

gA in $G(K)$. "In most cases", gA can also be characterized as the fixed-point set in $\mathscr{B}$ of the group of units ${}^gS^\circ = \{s \in {}^gS(K) | \omega(\chi(s)) = 0$ for all characters $\chi \in X^*({}^gS)\}$, but that is not always true (for more precise statements, cf. §3.6).

In this context, it is worthwhile to note also that the half-apartment A_α (for $\alpha \in \Phi_{af}$) is *never* the fixed-point set of the group X_α: indeed, if $\mathscr{B}$ is metrized in the way described below (§2.3), there is a constant c such that for every point $x \in A_\alpha$ at distance d of the wall ∂A_α, the whole ball with center x and radius cd is pointwise fixed by X_α (cf. [**8**, 7.4.33]).

If S_1 is the maximal split torus of the center of G° and if $G_1, \cdots, G_m$ are the almost simple factors of G°, the building $\mathscr{B}$ is canonically isomorphic with the direct product of the buildings $\mathscr{B}(S_1, K)$ (which is an affine space) and $\mathscr{B}(G_i, K)$ $(i = 1, \cdots, m)$. If G is K-anisotropic (i.e., if $S = \{1\}$), $\mathscr{B}$ consists of a single point. If $G = R_{L/K}\, H$, where L is a separable extension of K and H is a reductive group over L, the buildings $\mathscr{B}(G, K)$ and $\mathscr{B}(H, L)$ are canonically isomorphic.

2.2. *Affine structures, facets, retractions, topology and other canonical structures on* $\mathscr{B}$. Since the stabilizer $N(K)$ of A in $G(K)$ preserves its affine structure and its partition in facets, each apartment gA of $\mathscr{B}$ (with $g \in G(K)$) is endowed with a natural structure of real affine space and a partition in facets. Those structures agree on intersections. Indeed,

2.2.1. *If A' and A'' are two apartments, there is an element of $G(K)$ which maps A' onto A'' and fixes the intersection $A' \cap A''$ pointwise; furthermore, $A' \cap A''$ is a closed convex union of facets in A'* (*hence also in A''*) [**8**, 7.4.8].

From that, we deduce a partition of $\mathscr{B}$ in *facets*, among which those which are open in apartments are called *chambers*. In particular, if G° is quasi-simple (resp. semisimple), $\mathscr{B}$ is a simplicial (resp. polysimplicial) complex.

Given two facets of $\mathscr{B}$, there is an apartment containing them both [**8**, 7.4.18]. In particular, given two points $x, y \in \mathscr{B}$, there is an apartment which contains them and it follows from 2.2.1 that, for $t \in [0, 1] \subset \boldsymbol{R}$, the point $(1-t)x + ty$, which is well defined in any such apartment, is independent of it. The set $\{(1-t)x + ty \,|\, t \in [0, 1]\}$ is called the *geodesic segment* joining x and y in $\mathscr{B}$.

Let A' be an apartment and let $C \subset A'$ be a chamber. For every apartment containing C, there is a unique isomorphism of affine spaces of that apartment onto A' which fixes C pointwise. In view of 2.2.1, all those isomorphisms can be glued together in a mapping $\rho_{A';C}\colon \mathscr{B} \to A'$ called the *retraction of* $\mathscr{B}$ *onto* A' with *center* C. Clearly, geodesic segments are mapped by $\rho_{A';C}$ onto broken lines (connected unions of finitely many geodesic segments).

The building $\mathscr{B}$ is commonly endowed with a topology invariant by $G(K)$ which is most naturally defined via the metric considered below (2.3), but which can also be more canonically defined as the weakest topology such that all $\rho_{A';C}$ are continuous. *If the residue field $\bar{K}$ is finite, that topology makes $\mathscr{B}$ into a locally compact space* and coincides with the "CW-topology" (that is, the quotient topology of the natural topology of the disjoint union of all apartments). Otherwise, it is strictly weaker than the latter. In all cases, the topological space $\mathscr{B}$ *is contractible*; indeed, for every point $x \in \mathscr{B}$, the mappings $\varphi_t\colon \mathscr{B} \to \mathscr{B}$ defined by $\varphi_t(y) = tx + (1-t)y$ form a homotopy from the identity to the retraction of $\mathscr{B}$ onto $\{x\}$ [**8**, 7.4.20].

A subset of $\mathscr{B}$ is called *bounded* if its image by some retraction $\rho_{A';C}$ is bounded, in which case its image by every such retraction is bounded, as is easily seen. As usual, a subset H of $G(K)$ is called *bounded* if for every K-regular function f on G, the set $\omega(f(H))$ is bounded from below. The action of $G(K)$ on $\mathscr{B}$ is bounded and "proper" in the following sense: in the mapping $(g, b)\mapsto(gb, b)$ of $G(K) \times \mathscr{B}$ in $\mathscr{B}\times\mathscr{B}$, bounded subsets of $G(K) \times \mathscr{B}$—i.e., subsets of products of bounded sets—are mapped onto bounded sets, and the inverse images of bounded subsets of $\mathscr{B} \times \mathscr{B}$ are bounded. If $\bar{K}$ is finite, "bounded" becomes synonymous with "relatively compact"; in particular, the action of $G(K)$ on $\mathscr{B}$ is proper in the usual sense.

2.3. *Metric and simplicial decomposition.* In various questions, buildings play for p-adic reductive groups the same role as the symmetric spaces in the study of noncompact real simple Lie groups (cf. [**24**, §5] and the references given there). This section shows some aspects of the analogy; cf. also [**18**, 5.32]. Note that, unlike those introduced in §2.2, the structures considered here are *not canonical*, at least when G is not semisimple.

Let us choose in V a scalar product invariant under the Weyl group ${}^{v}W$. If G is quasi-simple, such a scalar product is unique up to a scalar factor, and there are various "natural" ways of normalizing it (Killing form, prescription of the length of short coroots, etc.). Canonical choices are also possible—componentwise—if G is semisimple, but not in general. From the scalar product in question, one deduces a Euclidean distance on A, hence, through the action of $G(K)$, on any apartment. From 2.2.1, it follows that two points x, y of $\mathscr{B}$ have the same distance $d(x, y)$ in all apartments containing them, and the properties of the retractions $\rho_{A';C}$ described in §2.2 readily imply that the building $\mathscr{B}$ endowed with the distance function d: $\mathscr{B} \times \mathscr{B} \to \boldsymbol{R}_{+}$ is a *complete metric space* [**8**, 2.5]. The associated topology coincides with that defined in §2.2. Again using the retractions $\rho_{A';C}$ one shows [**8**, 3.2.1] that d satisfies the following inequality, where $x, y, z, m \in \mathscr{B}$ and $d(x, m) = d(y, m) = \frac{1}{2} d(x, y)$:

$$d(x, z)^2 + d(y, z)^2 \geqq 2d(m, z)^2 + \tfrac{1}{2}d(x, y)^2.$$

In Riemannian geometry, that inequality characterizes the spaces with negative sectional curvatures (hence is valid in noncompact irreducible symmetric spaces!); as in the Riemannian case, it can be used here to prove the following *fixed-point theorem*:

2.3.1. *A bounded group of isometries of $\mathscr{B}$ has a fixed point* [**8**, 3.2.4]. Interesting applications are provided by Galois groups ("Galois descent" of the building) and by bounded subgroups of $G(K)$ (cf. §3.2).

In some applications (cf., e.g., [**2**]), it is useful to dispose of a simplicial decomposition of $\mathscr{B}$ invariant under $G(K)$. To obtain it, it suffices to choose a simplicial decomposition of A invariant under $N(K)$ and finer than the partition in facets—it is easily seen that such a decomposition always exists—and to carry it over to all apartments by means of the $G(K)$-action. If G is semisimple, one can more directly use the canonical barycentric subdivision of the partition of B in polysimplical facets. If G is quasi-simple, that partition itself meets the requirements.

2.4. *Dynkin diagram; special and hyperspecial points.* Let C be a chamber of $\mathscr{B}$. Starting from any apartment containing C, we can, following §1.8, define a local Dynkin diagram $\Delta(G, C)$ which, in view of 2.2.1, does not depend, up to unique isomorphisms, on the choice of the apartment. If C' is another chamber, 1.8.1, applied to any apartment containing C and C', provides an isomorphism $\varphi_{C'C}$:

$\Delta(G,C) \to \Delta(G, C')$ which, again by 2.2.1, is independent of the apartment in question. All those isomorphisms are coherent: if C, C', C'' are three chambers, one has $\varphi_{C''C} = \varphi_{C''C'} \circ \varphi_{C'C}$. Thus, we can talk about *the* local Dynkin diagram $\Delta(G) = \Delta(G, K)$ of G over K, a diagram which is well-defined up to unique isomorphisms. The same is true of the absolute local Dynkin diagram (§1.11), which is nothing else but the diagram $\Delta(G, K_1)$ of G over the unramified closure K_1 of K, and of the local index (§1.11).

The definitions of §§1.9 and 1.10 can be immediately transposed to arbitrary points x and arbitrary facets F of the building $\mathscr{B}$: one chooses an apartment containing x or F, uses the definition under consideration and deduces from 2.2.1 that the result is independent of the apartment chosen. Thus, to every point x (resp. facet F) of $\mathscr{B}$ is canonically associated a set I_x (resp. I_F) of vertices of $\Delta(G)$ and a root system Φ_x (resp. Φ_F), the latter being only defined up to noncanonical isomorphisms. We can also talk about *special* and *hyperspecial* points of $\mathscr{B}$. The criterion in terms of I_x for a point x to be special (last paragraph of §1.9) remains of course valid. A necessary condition for the existence of hyperspecial points is that G split over an unramified extension of K; that condition is also sufficient if G is quasi-split.

To every vertex v of the diagram $\Delta(G)$ is attached an integer $d(v)$: the definition given in §1.8 made reference to an apartment A but the result is independent of its choice, always by 2.2.1. If the residue field $\bar{K}$ is finite, isomorphic with $\boldsymbol{F}_q$, the number $d(v)$ can be interpreted as follows: a facet F of codimension one and "type v", that is, such that I_F is the complement of v in the set of all vertices of $\Delta(G)$, is contained in the closure of exactly $q^{d(v)} + 1$ chambers (cf. §3.5).

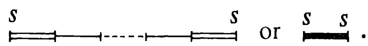
2.5. *Action of* (Aut G)(K) *on* $\mathscr{B}$ *and* Δ; *conjugacy classes of special and hyperspecial points.* The group (Aut G)(K) of all K-automorphisms of G and, in particular, the group $G_{\mathrm{ad}}(K)$ of rational points of the adjoint group G_{ad} of G°, act on $\mathscr{B}$ and on the local Dynkin diagram $\Delta = \Delta(G)$ "par transport de structure". Through the canonical homomorphism int: $G \to \operatorname{Aut} G$, that gives an action of $G(K)$ on $\mathscr{B}$ and on Δ. The action of $G(K)$ on $\mathscr{B}$ provided by the definition of $\mathscr{B}$ as a $G(K)$-set coincides with this one if G is semisimple but not in general; however, the induced actions on Δ are always the same. We call $\Xi = \Xi(G, K)$ the image of $G(K)$ in Aut Δ.

If G is semisimple and simply connected, it operates trivially on Δ, i.e., $\Xi = \{1\}$ (another illustration of the "philosophy" of §1.13).

Suppose G connected. Then, Ξ is also the image of $Z(K)$ in Aut Δ, and it can be computed as follows. We denote by $\tilde{G}$ a simply connected covering of the derived group of G, by $\tilde{S}$ the maximal split torus of $\tilde{G}$ whose image in G is contained in S, by $\tilde{Z}$ the centralizer of $\tilde{S}$ in $\tilde{G}$, by S_1 the maximal subtorus of S which is central in G and by Z_s the image of $\tilde{Z}(K)$ in $Z(K)$. Then, $S_1(K)$, Z_s and $Z_c = \{z \in Z(K) | \omega(\chi(z)) = 0$ for all $\chi \in X^*(Z)\}$ (cf. §1.2) are normal subgroups of $Z(K)$ and their product $S_1(K) \cdot Z_s \cdot Z_c$ is the kernel of the action of $Z(K)$ on Δ; thus $\Xi = Z(K)/(S_1(K) \cdot Z_s \cdot Z_c)$. If G is quasi-split—in particular if $\bar{K}$ is algebraically closed—Z and $\tilde{Z}$ are tori and the computation of Ξ is particularly easy. Note that, in most interesting cases, the subgroup Ξ of Aut Δ is uniquely determined by the underlying "abstract" group.

Two facets F and F' of $\mathscr{B}$ are in the same orbit of $G(K)$—for any one of the two actions of $G(K)$ on $\mathscr{B}$ described above—if and only if I_F and $I_{F'}$ (cf. §2.4) are in the same orbit of Ξ. In particular, *if G is semisimple and simply connected, the orbits of $G(K)$ in the set of special points of $\mathscr{B}$ are in canonical one-to-one correspondence with the sets of vertices of Δ consisting of one special vertex out of each connected component.*

Suppose G semisimple. If G is K-split, the group $G_{\mathrm{ad}}(K)$ permutes transitively the special points of $\mathscr{B}$: that is an immediate consequence of Proposition 2 in [**5**,VI.2.2]. A case analysis shows that, for *any* semisimple G, $G_{\mathrm{ad}}(K)$ permutes transitively the special points except possibly if the Coxeter diagram underlying Δ has a connected component of the form



Suppose now that G is quasi-simple and that the Coxeter diagram in question is one of those above. Then, obvious necessary conditions for $G_{\mathrm{ad}}(K)$ (and even $(\mathrm{Aut}\, G)(K)$) to permute transitively the special points are the existence of an automorphism of Δ permuting its two special vertices, and the equality of the numbers $d(v)$ attached to them. One verifies that if the residue field $\bar{K}$ is finite, those conditions are also sufficient.

For arbitrary G, if $\mathscr{B}$ has hyperspecial points, the facets consisting of such points—hence the points themselves if G is semisimple—are permuted transitively by $G_{\mathrm{ad}}(K)$.

2.6. *Behaviour under field extensions.*

The buildings behave functorially with respect to Galois extensions.

More precisely, for every Galois extension K_1 of K, we can consider the building $\mathscr{B}(G, K_1)$, on which the Galois group $\mathrm{Gal}(K_1/K)$ acts naturally (in the nonsemisimple case, one has to "canonify" the apartments—and hence $\mathscr{B}$—as described in §1.2), and there is a *unique* system of injections

$$\iota_{K_2K_1}\colon \mathscr{B}(G, K_1) \to \mathscr{B}(G, K_2) \qquad (K_1, K_2 \text{ Galois extensions of } K \text{ with } K_1 \subset K_2)$$

with the following properties:

the image of $\iota_{K_2K_1}$ is pointwise fixed by $\mathrm{Gal}(K_2/K_1)$;

the restriction of $\iota_{K_2K_1}$ to any apartment of $\mathscr{B}(G, K_1)$ is an affine mapping into an apartment of $\mathscr{B}(G, K_2)$;

$\iota_{K_2K_1}$ is $G(K_1)$-covariant;

if $K_1 \subset K_2 \subset K_3$, one has has $\iota_{K_3K_1} = \iota_{K_3K_2} \circ \iota_{K_2K_1}$.

The last property allows us to identify coherently every $\mathscr{B}(G, K_1)$ with its image by every $\iota_{K_2K_1}$.

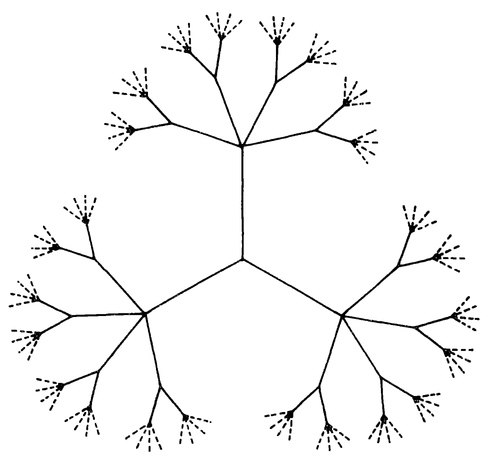
2.6.1. *If K_1/K is unramified* (*or even tamely ramified*: *cf.* [**19**]), *$\mathscr{B}$ is the fixed point set of* $\mathrm{Gal}(K_1/K)$ *in $\mathscr{B}(G, K_1)$ and the apartment $A = A(G, S, K)$ is the intersection of $\mathscr{B}$ with the apartment $A(G, S_1, K_1)$ of any maximal K_1-split K-torus S_1 of G containing S.* Still assuming that K_1/K is unramified, one deduces from 1.10.2 that *a point x of $\mathscr{B}$ which is special in $\mathscr{B}(G, K_1)$ is also special in $\mathscr{B}$*; *if furthermore G is split over K_1, the point x is hyperspecial.*

If K_1/K is wildly ramified, the fixed point set of $\mathrm{Gal}(K_1/K)$ in $\mathscr{B}(G, K_1)$ may be strictly bigger than $\mathscr{B}$: it then looks like the building $\mathscr{B}$ "covered with barbs". Suppose for example that G is split and is not a torus, that $K = \mathbf{Q}_2$ and that K_1 is totally ramified over K and different from K (which implies that K_1/K is wildly ramified). The apartment $A = A(G, S, K)$ of $\mathscr{B}$ is also an apartment of $\mathscr{B}(G, K_1)$.

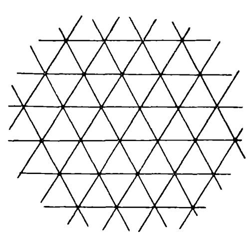
Let F be a facet of codimension one of A with respect to K_1 which is not a facet with respect to K (cf. the example following 1.10.1). By the last assertions of §§1.8 and 2.4, there is exactly one chamber of $\mathscr{B}(G, K_1)$ not contained in A and whose closure contains F; it must of course be fixed by $\mathrm{Gal}(K_1/K)$ and cannot be contained in $\mathscr{B}$ since F is not a facet of $\mathscr{B}$.

For a proof of the above results and a more detailed analysis of the situation, cf. **[19]**.

2.7. *Example. Groups of relative rank* 1. The building of a semisimple group of relative rank 1 is a contractible simplicial complex of dimension 1, i.e., *a tree.* All its vertices are special points. If $\bar{K} \cong F_q$ and if d, d' are the integers $d(v)$ attached to the two vertices of the Dynkin diagram, each edge of the tree has one vertex of order $q^d + 1$ and one vertex of order $q^{d'} + 1$ (cf. §2.4). Consider for instance the special orthogonal group of a nonsplit quadratic form in 5 variables over Q_2: here, $d = 1$, $d' = 2$, and the building looks as suggested by the picture below. In that case, the vertices of order 5 are hyperspecial and the others are not.



2.8. *Example.* SL_3 *and* GL_3. Suppose that $G = \mathrm{SL}_3$. The building $\mathscr{B} = \mathscr{B}(G, K)$ is a 2-dimensional simplicial complex whose maximal simplices are equilateral triangles, for the metric introduced in §2.3. The apartments are Euclidean planes triangulated in the familiar way:



To picture the building itself, one must imagine it "ramifying" along every edge, each edge belonging to $q + 1$ triangles if $q = \operatorname{card} \bar{K}$. The link of each vertex in $\mathscr{B}$

is the "spherical building" of $SL_3(\bar{K})$, that is, the "flag complex" of a projective plane over $\bar{K}$, a picture of which can be found in **[24]** or **[26]** for the special case where $\bar{K} \cong \boldsymbol{F}_2$. All vertices of $\mathscr{B}$ are hyperspecial.

The building $\mathscr{B}(GL_3, K)$ is the direct product of $\mathscr{B}$ and an affine line.

2.9. *Example. General linear groups.* We adopt the hypotheses and the notations of §1.14. In particular, D denotes a finite dimensional central division algebra over K and $G = GL_{n,D}$ (thus $G(K) = GL_n(D)$). Then, the building $\mathscr{B} = \mathscr{B}(G, K)$ can be interpreted as the set $\mathscr{N}$ of all "additive norms" in D^n, that is, of all functions $\varphi\colon D^n \to \boldsymbol{R} \cup \{+\infty\}$ such that

$$\varphi(x + y) \geqq \inf\{\varphi(x), \varphi(y)\} \qquad (x, y \in D^n),$$
$$\varphi(xd) = \varphi(x) + \omega(d) \qquad (x \in D^n,\ d \in D).$$

More precisely, if we identify the apartment A with V as in §1.14, the mapping $A \to \mathscr{N}$ which maps $\sum_{i=1}^n v_i e_i$ onto the norm

$$(x_1, \cdots, x_n) \mapsto \inf\{\omega(x_i) - v_i | i = 1, \cdots, n\}$$

extends—of course uniquely—to an isomorphism of $G(K)$-sets $\mathscr{B} \to \mathscr{N}$, where $G(K)$ operates on $\mathscr{N}$ by $(g\varphi)(x) = \varphi(g^{-1}x)$. A norm φ is special—i.e., corresponds to a special point of B—if and only if there is a basis $(b_i)_{1 \leqq i \leqq n}$ of the vector space D^n and a real number f such that

$$\varphi\Big(\sum_{i=1}^n b_i d_i\Big) = f + \inf\{\omega(d_i) | i = 1, \cdots, n\} \qquad (d_i \in D);$$

φ is hyperspecial if and only if it is special and $D = K$.

A similar interpretation of $\mathscr{B}(SL_{n,D}, K)$ can be found in **[8**, p. 238]. The space $\mathscr{N}$ has been first considered by O. Goldman and N. Iwahori **[12]**.

2.10. *Example. Special unitary groups.* Let L, ω, π_1, Γ_1, τ and δ have the same meaning as in §1.15. In particular, L is a separable quadratic extension of K and $\delta = \sup\{\omega(d) \mid d \in L,\ d + d^\tau = 1\}$. Let E be a finite dimensional vector space over L endowed with a nondegenerate hermitian form h relative to τ, and suppose that $G = SU(h)$. Then, the building $\mathscr{B}$ of G over K, which is also, by the way, the building of $U(h)$, can be interpreted as the set $\mathscr{N}_h$ of all additive norms $\varphi\colon E \to \boldsymbol{R} \cup \{+\infty\}$ satisfying the inequalities

$$\omega(h(x, x)) \geqq 2\varphi(x) - \delta \qquad (x \in E),$$
$$\omega(h(x, y)) \geqq \varphi(x) + \varphi(y) \qquad (x, y \in E),$$

and maximal with that property (cf. **[8**, p. 239] for a more general result).

Suppose further that $E = L^{2n+1}$ and that h is as in 1.15(1). Then, the identification of $\mathscr{B}$ and $\mathscr{N}_h$ can be described more explicitly as follows: with the notations of §1.15, the mapping $A \to \mathscr{N}_h$ which maps $\sum_{i=1}^n v_i e_i$ onto the norm

$$(x_{-n}, \cdots, x_n) \mapsto \inf\{\omega(x_i) - v_i,\ \omega(x_{-i}) + v_i,\ \omega(x_0) - \delta | 1 \leqq i \leqq n\}$$

extends uniquely to an isomorphism of $G(K)$-sets $\mathscr{B} \to \mathscr{N}_h$. A norm $\varphi \in \mathscr{N}_h$ is special—i.e., corresponds to a special point of $\mathscr{B}$—if and only if there is a basis $(b_i)_{1 \leqq i \leqq n}$ of E with respect to which h has the form 1.15(1) and a constant $f \in \frac{1}{2}\Gamma_1$ such that, for $x_i \in L$,

(1) $$\varphi\Big(\sum_{i=1}^{n} b_i x_i\Big) = \inf\{\omega(x_i) - f,\ \omega(x_{-i}) + f,\ \omega(x_0) - \delta \mid 1 \leqq i \leqq n\}$$

(one can then choose the basis (b_i) so that $f = 0$ or $\frac{1}{2}\omega(\pi_1)$). The norm φ is hyperspecial if L/K is unramified and if there is a basis (b_i) of E such that (1) holds for $f = 0$.

3. Stabilizers and centralizers. *From now on, G is assumed to be connected.*

3.1. *Notations; a BN-pair.* For every algebraic extension K_1 of K with finite ramification index and every subset Ω of the building $\mathscr{B}(G, K_1)$, we denote by $G(K_1)^\Omega$ the group of all elements of $G(K_1)$ fixing Ω pointwise. If Ω is reduced to a point x, we also write $G(K_1)^x$ for $G(K_1)^\Omega$. Note that if F is a facet of $\mathscr{B}(G, K_1)$ and if x is a point of F "in general position", one has $G(K_1)^F = G(K_1)^x$. The stabilizers $G(K)^x$ of special (resp. hyperspecial) points $x \in \mathscr{B}$ are called *special* (resp. *hyperspecial*) *subgroups* of $G(K)$.

We recall that if G is semisimple and simply connected, the group $\tilde{W} = N(K)/Z(K)$ coincides with the Weyl group W of the affine root system Φ_{af}. As before, we set $\mathscr{B} = \mathscr{B}(G, K)$.

3.1.1. *Suppose that $\tilde{W} = W$. Then $G(K)^F = G(K)^x$ for every facet F of $\mathscr{B}$ and every $x \in F$. Furthermore, if C is a chamber of $A = A(G, S)$, the pair $(G(K)^C, N(K))$ is a BN-pair (or Tits system: cf.* [**5**], [**23**]) *in $G(K)$ with Weyl group W.* In that case, the groups $G(K)^x$ for $x \in \mathscr{B}$ are called the *parahoric subgroups* of $G(K)$ (cf. [**8**]), but we shall avoid using that terminology here in order not to prejudge of its most suitable extension to the nonsimply connected case. An alternative construction of the building $\mathscr{B}$ starting from the above BN-pair (which can be defined independently of the building, as we shall see) and using the parahoric subgroups defined by means of that BN-pair is given in [**8**, §2].

Let Ω be a nonempty subset of the apartment A whose projection on the building of the semisimple part of G (cf. last paragraph of 2.1) is bounded. For any root $a \in \Phi$, let $\alpha(a, \Omega)$ denote the smallest affine root whose vector part is a and which is positive on Ω. Let Φ' be the set of all nondivisible roots—i.e., all roots $a \in \Phi$ such that $\frac{1}{2}a \notin \Phi$—and let Φ'^+ (resp. Φ'^-) be the set of all nondivisible roots which are positive (resp. negative) with respect to a basis of Φ, arbitrarily chosen. Set $N(K)^\Omega = N(K) \cap G(K)^\Omega$ and let Z_c and X_α be defined as in §§1.2 and 1.4. Then one has the following group-theoretical description of $G(K)^\Omega$ (cf. [**8**,6 .4.9, 6.4.48, 7.4.4]):

If $X^\pm(\Omega)$ denotes the group generated by all $X_{\alpha(a,\Omega)}$ with $a \in \Phi'^\pm$, the product mapping $\prod_{a\in\Phi'^\pm} X_{\alpha(a,\Omega)} \to X^\pm(\Omega)$ is bijective for every ordering of the factors of the product and one has $G(K)^\Omega = X^-(\Omega) \cdot X^+(\Omega) \cdot N(K)^\Omega$. If Ω contains an open subset of A, the product mapping $\prod_{a\in\Phi'} X_{\alpha(a,\Omega)} \times Z_c \to G(K)^\Omega$ is bijective for every ordering of the factors of the product.

3.2. *Maximal bounded subgroup.* For every nonempty subset Ω of $\mathscr{B}$, $G(K)^\Omega$ is a bounded subgroup of $G(K)$ (cf. §2.2). If the residue field $\bar{K}$ is finite, $G(K)^\Omega$ is even compact and, in what follows, "maximal bounded" can be replaced by "maximal compact".

From 2.3.1, one easily deduces that:

every bounded subgroup of $G(K)$ is contained in a maximal one and every maximal bounded subgroup is the stabilizer $G(K)^x$ of a point x of $\mathscr{B}$.

It is now clear that *if x belongs to a facet of minimal dimension of $\mathscr{B}$, $G(K)^x$ is a maximal bounded subgroup of $G(K)$, in particular, special subgroups are maximal bounded subgroups.* From 3.1.1, it follows that the above two statements give a complete description of the maximal bounded subgroups in the simply connected case:

if G is semisimple and simply connected, the maximal bounded subgroups of $G(K)$ are precisely the stabilizers of the vertices of the building $\mathscr{B}$; they form $\prod_{i=1}^{r}(l_i + 1)$ conjugacy classes, where $l_1, \cdots, l_r$ denote the relative ranks of the quasi-simple factors of G.

For an analysis of the nonsimply connected case, cf. [**8**, 3.3.5].

3.3. *Various decompositions.* Let C be a chamber of $A = A(G, S)$. We identify A with the vector space V in such a way that 0 becomes a special point contained in the closure of C; in particular, *$G(K)^0$ is a special subgroup* of $G(K)$. Set $D = \boldsymbol{R}_+^* \cdot C$ (a "vector chamber") and $B = G(K)^C$; if $\bar{K}$ is finite or, more generally, if G is residually quasi-split, and if G is simply connected, B is an *Iwahori subgroup* of $G(K)$ (cf. §3.7). Let U^+ be the group generated by all U_a for which $a|_C$—and hence $a|_D$—is positive and let Y be the "intersection of V and $\tilde{W}$", that is, the group of all translations of A contained in $\tilde{W}$; thus, Y is the image of $Z(K)$ by the homomorphism ν of §1.2. Set $Y_+ = Y \cap \bar{D}$ (closure of D) and $Z(K)_+ = \nu^{-1}(Y_+)$, a subsemigroup of $Z(K)$.

3.3.1. *Bruhat decomposition.* One has $G(K) = BN(K)B$ and the mapping $BnB \mapsto \nu(n)$ ($n \in N(K)$) is a bijection of the set $\{BgB \mid g \in G(K)\}$ onto $\tilde{W}$.

If $n \in N(K)$ and $\nu(n) = w$, we also write $BnB = BwB$, as usual. If $\bar{K} = \boldsymbol{F}_q$, the cardinality q_w of BwB/B (used for instance in [**1**]) is given by the following formula in terms of the integers $d(v)$ of §1.8: set $w = r_1 \cdots r_l w_0$, where $(r_1, \cdots, r_l)$ is a reduced word in the Coxeter group W and $w_0(C) = C$, and let v_i be the vertex of Δ representing r_i; then $q_w = q^d$ with $d = \sum_{i=1}^{l} d(v_i)$. In particular, we have another interpretation of $d(v)$: $q^{d(v)} = q_{r(v)}$ where $r(v)$ denotes the fundamental reflection corresponding to the vertex v of Δ.

More generally, for any $\bar{K}$, the quotient BwB/B has a natural structure of "perfect variety" over $\bar{K}$, in the sense of Serre [Publ. Math. I.H.E.S. **7** (1960), 1.4], and, as such, it is isomorphic to a $\bar{K}$-vector space of dimension $\sum_{i=1}^{l} d(v_i)$, with the above notations.

3.3.2. *Iwasawa decomposition.* One has $G(K) = G(K)^0 Z(K) U^+(K)$ and the mapping $G(K)^0 z U^+(K) \mapsto \nu(z)$ ($z \in Z(K)$) is a bijection of $\{G(K)^0 g U^+(K) \mid g \in G(K)\}$ onto Y.

3.3.3. *Cartan decomposition.* One has $G(K) = G(K)^0 Z(K) G(K)^0$ and the mapping $G(K)^0 z G(K)^0 \mapsto \nu(z)$ ($z \in Z(K)_+$) is a bijection of $\{G(K)^0 g G(K)^0 \mid g \in G(K)\}$ onto Y_+.

In particular, we see that if $\bar{K}$ is finite, the convolution algebra of all functions $G(K) \mapsto \boldsymbol{C}$ with compact support which are bi-invariant under $G(K)^0$ (Hecke algebra) has a canonical basis indexed by Y_+. *That algebra is commutative.*

For the proofs and some generalizations of the above results, cf. [**8**, §4].

3.4. *Some group schemes.* The results of this section and the next are special cases of results which will be established in [**9**].

It is well known that the maximal bounded subgroups of $\mathrm{SL}_n(K)$ are the group $\mathrm{SL}_n(\mathfrak{o})$ and its conjugates under $\mathrm{GL}_n(K)$. It is natural to ask whether, more generally, the maximal bounded subgroups of $G(K)$ can always be interpreted as the groups of units of some naturally defined $\mathfrak{o}$-structures on G. A positive answer is provided by the statement 3.4.1 below. In this section, we denote by G_{ss} the derived group of G and by pr_{ss} the canonical projection $\mathscr{B}(G, K_1) \to \mathscr{B}(G_{ss}, K_1)$ (cf. the last paragraph of 2.1) for any K_1.

3.4.1. *If Ω is a nonempty subset of an apartment of $\mathscr{B}$ whose projection* $\mathrm{pr}_{ss}(\Omega)$ *is bounded, there is a smooth affine group scheme $\mathscr{G}_\Omega$ over $\mathfrak{o}$, unique up to unique isomorphism, with the following properties*:

the generic fiber $\mathscr{G}_{\Omega,K}$ of $\mathscr{G}_\Omega$ is G;

for every unramified Galois extension K_1 of K with ring of integers $\mathfrak{o}_{K_1}$, the group $\mathscr{G}_\Omega(\mathfrak{o}_{K_1})$ is equal to $G(K_1)^\Omega$ (cf. 3.1), *where Ω is identified with its canonical image in the building $\mathscr{B}(G, K_1)$ (cf.* 2.6).

Clearly, $\mathscr{G}_\Omega$ depends only on the closed convex hull of $\mathrm{pr}_{ss}(\Omega)$.

The following two statements are easy consequences of the definitions.

3.4.2. *If G is split, the group schemes $\mathscr{G}_x$ associated to the special points x of $\mathscr{B}$ are the Chevalley group schemes with generic fiber G.*

3.4.3. *Let K_1 be an unramified Galois extension of K with ring of integers $\mathfrak{o}_{K_1}$, let $\Omega \subset \mathscr{B}$ be as above and let Ω_1 be the canonical image of Ω in $\mathscr{B}_1 = \mathscr{B}(G, K_1)$* (2.6). *Then $\mathscr{G}_{\Omega_1}$ is the group scheme over $\mathfrak{o}_{K_1}$ deduced from $\mathscr{G}_\Omega$ by change of base. Conversely, let $\mathscr{G}$ be a smooth group scheme over $\mathfrak{o}$ with generic fiber G and suppose that, by change of base from $\mathfrak{o}$ to $\mathfrak{o}_{K_1}$, $\mathscr{G}$ becomes a group scheme $\mathscr{G}_{\Omega_1}$ with $\Omega_1 \subset \mathscr{B}_1$ as in* 3.4.1; *then* $\mathrm{pr}_{ss}(\Omega_1)$ *is stable by* $\mathrm{Gal}(K_1/K)$, *and if it is pointwise fixed by* $\mathrm{Gal}(K_1/K)$, *hence can be identified with a subset of* $\mathscr{B}(G_{ss}, K)$ (*cf.* 2.6.1) *whose inverse image by* pr_{ss} *in $\mathscr{B}$ we denote by Ω, one has $\mathscr{G} = \mathscr{G}_\Omega$.*

If Ω' is any nonempty subset of the closure of Ω, the inclusion homomorphisms $G(K_1)^\Omega \to G(K_1)^{\Omega'}$, for K_1 as in 3.4.1, define a morphism of group schemes $\mathscr{G}_\Omega \to \mathscr{G}_{\Omega'}$ which we denote by $\rho_{\Omega'\Omega}$. We represent by $\bar{G}_\Omega$ and $\bar{\rho}_{\Omega'\Omega}$ the algebraic group defined over $\bar{K}$ and the $\bar{K}$-homomorphism obtained from $\mathscr{G}_\Omega$ and $\rho_{\Omega'\Omega}$ by reduction mod $\mathfrak{p}$.

3.4.4. *The reduction homomorphism $\mathscr{G}_\Omega(\mathfrak{o}) = G(K)^\Omega \to \bar{G}_\Omega(\bar{K})$ is surjective.*

3.5. *Reduction* mod $\mathfrak{p}$. Let Ω be as in 3.4. Our next purpose is to investigate the group $\bar{G}_\Omega$. We assume, without loss of generality, that $\Omega \subset A(G, S)$. Then, the well-defined split torus scheme whose generic fiber is S is a closed subscheme of $\mathscr{G}_\Omega$, and its reduction mod $\mathfrak{p}$, called $\bar{S}$, is a maximal $\bar{K}$-split torus of $\bar{G}_\Omega$. The character group of $\bar{S}$ is canonically isomorphic with the character group X^* of S and will be identified with it; similarly, we identify the cocharacter group of $\bar{S}$ with X_*. The neutral component $\bar{G}^0$ of $\bar{G}$ possesses a unique Levi subgroup containing $\bar{S}$, which we denote by $\bar{G}_\Omega^{\mathrm{red}}$; it is defined over $\bar{K}$. We suppose Ω convex.

Let F be a facet meeting Ω and of maximal dimension with that property. Since F satisfies all the conditions imposed on Ω, the reductive group $\bar{G}_F^{\mathrm{red}}$ is defined and it also contains $\bar{S}$. One shows that *the identity map of $\bar{S}$ onto itself extends uniquely to an isomorphism $\bar{G}_\Omega^{\mathrm{red}} \to \bar{G}_F^{\mathrm{red}}$*; if $F \subset \Omega$, that is nothing else but the restriction of $\bar{\rho}_{\Omega F}$ to $\bar{G}_\Omega^{\mathrm{red}}$. In the sequel, we shall be mainly concerned with the group $\bar{G}_F^{\mathrm{red}}$.

The notion of *coroot associated with a root a* is usually defined for split groups (cf. [**11**, XX, 2.8], [**20**, §1]). In view of the next statement, we extend it as follows to arbitrary reductive groups: if $2a$ is not a root, we simply take the coroot associated with a in the split subgroup of maximal rank defined in [**3**, §7]; if $2a$ is a root, we define the coroot associated with a as being twice the coroot associated with $2a$ (X_* being written additively).

3.5.1. *The root system of* $\bar{G}_F^{\mathrm{red}}$ *with respect to* $\bar{S}$ *is the system* Φ_F (*cf.* 1.9); *in particular, its Dynkin diagram is obtained from the local Dynkin diagram* $\Delta(G, K)$ *by deleting the vertices belonging to* I_F (*cf.* 1.9) *and all edges containing such vertices. The coroot associated with a root* $a \in \Phi_F$ *is the same for* $\bar{G}_F^{\mathrm{red}}$ *as for* G. *If* $\bar{U}_a$ *denotes the unipotent subgroup of* $\bar{G}_F^{\mathrm{red}}$ *corresponding to* a, *the group* $\bar{U}_a(\bar{K})$ *is nothing else but the group* $\bar{X}_\alpha$ *of* §1.4, *where* α *is the affine root vanishing on* F *and whose vector part is* a.

Applying that to the unramified closure of K, one gets the following immediate consequence.

3.5.2. *The index of* $\bar{G}_F^{\mathrm{red}}$ *over* $\bar{K}$, *in the sense of* [**3**] *and* [**22**], *is obtained from the local index of* G *by deleting from* Δ_1 *all vertices belonging to the orbits* $O(v)$ *with* $v \in I_F$ (*the notations are those of* §1.11) *and all edges containing such vertices. In particular, if* G *is residually quasi-split* (*resp. residually split*), $\bar{G}_F^{\mathrm{red}}$ *is quasi-split* (*resp. split*). *When* F *is a chamber, then* G *is residually quasi-split* (*resp. residually split*) *if and only if* $\bar{G}_F^{\mathrm{red}}$ *is a torus* (*resp. a split torus*).

If G is simply connected, the group $\bar{G}_\Omega$ is connected. In general, the group of components of $\bar{G}_\Omega$ is easily computed when one knows the group $\Xi_1 = \Xi(G, K_1)$ (cf. §2.5), where K_1 is the maximal unramified extension of K. Here, we shall give the result only in the case of a facet.

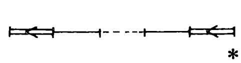
3.5.3. *The group of components of* $\bar{G}_F$ *is canonically isomorphic with the intersection of the stabilizers of the orbits* $O(v)$ *with* $v \in I_F$ *in the group* Ξ_1. *A component is defined over* $\bar{K}$ *if and only if the corresponding element of* Ξ_1 *is centralized by* $\mathrm{Gal}(K_1/K)$. *If* $\bar{K}$ *is finite, every component of* $\bar{G}_F$ *which is defined over* $\bar{K}$ *has a* $\bar{K}$-*rational point* (*by Lang's theorem*).

The groups $\bar{G}_F^{\mathrm{red}}$ give an insight into the geometry of the building through the following statement:

3.5.4. *The link of* F *in* B *is canonically isomorphic with the spherical building of* $\bar{G}_F^{\mathrm{red}}$ *over* $\bar{K}$, *i.e. the "building of* $\bar{K}$-*parabolic subgroups" of* $\bar{G}_F^{\mathrm{red}}$ (*cf.* [**23**, 5.2]).

The groups $\bar{G}_F^{\mathrm{red}}$ also provide an alternative definition of the integers $d(v)$ of §1.8. Suppose F is of codimension one and let v be the complement of I_F in the set of all vertices of Δ. Then, $\bar{G}_F^{\mathrm{red}}$ has semisimple $\bar{K}$-rank 1 and $d(v)$ *is the dimension of its maximal unipotent subgroups, or, equivalently, the dimension of the variety* $\bar{G}_F^\circ/\bar{P}_F$, *where* $\bar{P}_F$ *is a minimal* $\bar{K}$-*parabolic subgroup of* $\bar{G}_F^\circ$, *the neutral component of* $\bar{G}_F$. This, together with 3.5.4, implies the interpretation of $d(v)$ given in 2.4. If G is residually split, $\bar{G}_F^\circ/\bar{P}_F$ is a projective line, hence $d(v) = 1$; in particular, we recover the last statement of §1.8.

While 3.5.2 gives an easy algorithm to determine the *type* of $\bar{G}_F^{\mathrm{red}}$, 3.5.1, applied to the unramified closure of K, actually provides the absolute *isomorphism class* of that group. Here is an immediate application of that. Suppose that G is quasi-simple, simply connected and residually split and that F is a special point. Then, $\bar{G}_F^{\mathrm{red}}$ is a simply connected quasi-simple group except if the local Dynkin diagram is the following one:

(1) 

and if I_F is the vertex marked with a $*$. Indeed, it is readily verified that in all other cases, Φ_F contains all nonmultipliable relative roots of G, and the assertion follows from [**4**, 2.23 and 4.3]. In the exceptional case above, $\bar{G}_F^{\text{red}}$ is a special orthogonal group, hence not simply connected. Using the fact that in a simply connected group the derived group of the centralizer of a torus is also simply connected, one easily deduces from the preceding result the following more general one. Let us say that a special vertex of the absolute local Dynkin diagram Δ_1 is *good* if it is not the vertex $*$ of a connected component of type (1) of that diagram. Then *if G is semisimple and simply connected and if $\bigcup_{v \in I_F} O(v)$ contains a good special vertex out of each connected component of Δ_1, the derived group of $\bar{G}_F^{\text{red}}$ is simply connected.*

3.6. *Fixed points of groups of units of tori.* Let M be a subgroup of the group of units $S_c = \{s \in S(K) \mid \omega(\chi(s)) = 0 \text{ for all } \chi \in X^*\}$ of S. We wish to find under which condition the apartment $A = A(G, S)$ is the full fixed point set $\mathscr{B}^M$ of M in $\mathscr{B}$. From the properties of the building recalled in §2.2, one deduces that $A = \mathscr{B}^M$ if and only if, for every facet F of A of codimension one, the only chambers containing F in their closure and fixed by M are the two chambers of A with those properties. By 3.5.4, that means that the image $\bar{M}$ of M in $\bar{S}(\bar{K})$ has only two fixed points in the spherical building of $\bar{G}_F^{\text{red}}$ over $\bar{K}$. If a is any one of the two nondivisible roots in Φ_F, that condition amounts to $a(\bar{M}) \neq \{1\}$. Thus, we conclude that:

3.6.1. *A necessary and sufficient condition for A to be the full fixed point set of M in $\mathscr{B}$ is that $a(M) \not\subset 1 + \mathfrak{p}$ for every relative root $a \in \Phi$.*

In particular,

if $\bar{K}$ has at least four elements (resp. if $\bar{K} \cong \boldsymbol{F}_2$) A is always (resp. never) the full fixed point set of S_c in B.

The preceding discussion also gives information on the fixed point set of the group of units $S_{1,c}$ of a nonsplit torus S_1 which becomes maximal split over an unramified Galois extension K_1 of K: one applies 3.6.1 to the action of $S_{1,c}$ on $\mathscr{B}(G, K_1)$ and one goes down to $\mathscr{B}$ by Galois descent, using 2.6.1. In that way, one gets the following result for instance:

If S_1 is an anisotropic torus which becomes maximal split over an unramified Galois extension of K, then $S_1(K)$ has a unique fixed point in the building $\mathscr{B}$.

By contrast, it is easily shown that if S_1 is a maximal torus of $G = \mathrm{SL}_2$ whose splitting field is ramified, then $S_1(K)$ necessarily fixes a chamber of $\mathscr{B}$ and possibily more than one[2]; for a similar torus S_1 in PGL_2, $S_1(K)$ may have a single fixed point in $\mathscr{B}$ and may have more than one.

3.7. *Iwahori subgroups; volume of maximal compact subgroups.* In this section, we suppose G residually quasi-split; remember that that is no assumption if the residue field $\bar{K}$ is finite (1.10.3).

To every chamber C of the building $\mathscr{B}$, we associate as follows a subgroup $\mathrm{Iw}(C)$ of $G(K)$, called the *Iwahori subgroup* corresponding to C: if $\bar{G}_C^\circ$ denotes the neutral component of the algebraic group $\bar{G}_C$ (cf. 3.4), $\mathrm{Iw}(C)$ is the inverse image in $\mathscr{G}_C(\mathfrak{o}) = G(K)^C$ of the group $\bar{G}_C^\circ(\bar{K})$ under the reduction homomorphism $\mathscr{G}_C(\mathfrak{o}) \to \bar{G}_C(\bar{K})$. Clearly, *all Iwahori subgroups of $G(K)$ are conjugate.* From 3.5.2, it follows that $\bar{G}_C^\circ$ is a solvable group, hence is the semidirect product of a torus $\bar{T}$ by a uni-

[2] This answers a question of G. Lusztig.

potent group $\bar{U}$. By general results on smooth group schemes, it follows that Iw(C) is the semidirect product of $\bar{T}(\bar{K})$ by a pronilpotent group $\mathrm{Iw}_u(C)$; if $\bar{K}$ has finite characteristic p, $\mathrm{Iw}_u(C)$ is a pro-p-group, and if $\bar{K}$ is finite, $\bar{T}(\bar{K})$ is of course a finite group, of order prime to p.

If x is a point of the closure of C, the image of $\bar{G}_C^{\circ}$ by the homomorphism $\bar{\rho}_{xC}$: $\bar{G}_C \to \bar{G}_x$ is a Borel subgroup $\bar{B}$ of $\bar{G}_x$, and the kernel of $\bar{\rho}_{xC}$ is a connected unipotent group. It follows that $\bar{\rho}_{xC}$ maps $\bar{G}_C^{\circ}(\bar{K})$ onto $\bar{B}(\bar{K})$ and consequently, by 3.4.4, that Iw(C) is the inverse image of $\bar{B}(\bar{K})$ in $G(K)^x = \mathscr{G}_x(\mathfrak{o})$ under the reduction homomorphism. Thus, the Iwahori subgroups of $G(K)$ can also be defined as the inverse images in the stabilizers $G(K)^x$, for $x \in \mathscr{B}$, of the $\bar{K}$-Borel subgroups of the reductions $\bar{G}_x$.

Now, suppose that $\bar{K}$ is finite. Then, $G(K)$ is a unimodular locally compact group of which the Iwahori subgroups are compact open subgroups. Therefore, there is a unique Haar measure μ for which the Iwahori subgroups have volume 1. From the above, it follows that, for any $x \in \mathscr{B}$, the volume of $G(K)^x$ with respect to μ is the index $[\bar{G}_x(\bar{K})\colon \bar{B}(\bar{K})]$ where $\bar{B}$ is any $\bar{K}$-Borel subgroup of $\bar{G}_x$. If x is "in general position" in the facet F of $\mathscr{B}$ containing it, one has $\bar{G}_x = \bar{G}_F$ and the assertions 3.5.2 and 3.5.3 provide an effective way of computing that volume knowing the local index of G (together with the correspondence $v \mapsto O(v)$ of 1.11), the set I_x of vertices of Δ and the group $\Xi_1 = \Xi(G, K_1)$ (cf. 2.5), where K_1 is the unramified closure of K.

3.8. *Hyperspecial points and subgroups.* From 3.5.1 and 3.5.3, one easily deduces the following characterization of the hyperspecial points of $\mathscr{B}$ defined in §1.10:

3.8.1. *A point x of $\mathscr{B}$ is hyperspecial if and only if the neutral component of the group $\bar{G}_x$ is reductive, in which case $\bar{G}_x$ itself is connected and hence reductive.*

One can also show that the schemes $\mathscr{G}_x$ corresponding to the hyperspecial points x are the only smooth group schemes over $\mathfrak{o}$ with generic fiber G and reductive reduction. Thus, the hyperspecial subgroups of $G(K)$ can be characterized as the groups of units of such group schemes.

3.8.2. *Suppose that $\bar{K}$ is finite and that $G(K)$ possesses hyperspecial subgroups* (*a condition satisfied, for instance, if G is quasi-split and has an unramified splitting field*: *cf.* 1.10); *then, the hyperspecial subgroups of $G(K)$ are among all compact subgroups of $G(K)$, those whose volume is maximum.*

The proof, using §§3.5 and 3.7, is not difficult.

3.9. *The global case.* Let L be a global field. For every finite extension L' of L and every place v of L', we denote by $\mathfrak{o}_{L'}$ (resp. $\mathfrak{o}_v$) the ring of integers of L' (resp. of the completion L'_v). Let H be a reductive linear group defined over L. We suppose H embedded in the general linear group GL_n and, for every L' and v as above, we set $H(\mathfrak{o}_v) = H(L'_v) \cap \mathrm{GL}_n(\mathfrak{o}_v)$. Another way of viewing that group consists in considering the $\mathfrak{o}_L$-group scheme structure $\mathscr{H}_{\mathfrak{o}_L}$ "on H" defined by the standard lattice $\mathfrak{o}_L^n$ in L^n—in more precise terms, $\mathscr{H}_{\mathfrak{o}_L}$ is the schematic closure of H in the standard general linear group scheme $\mathscr{GL}_{n,\mathfrak{o}_L}$—; then $H(\mathfrak{o}_v) = \mathscr{H}_{\mathfrak{o}_L}(\mathfrak{o}_v)$. For any ring R containing $\mathfrak{o}_L$, we denote by $\mathscr{H}_R$ the group scheme over R deduced from $\mathscr{H}_{\mathfrak{o}_L}$ by change of base.

3.9.1. *At almost all finite places v of L, $\mathscr{H}_{\mathfrak{o}_v}$ is the group scheme $\mathscr{H}_x$ associated with a hyperspecial point x of the building $\mathscr{B}(H, L_v)$; hence $H(\mathfrak{o}_v)$ is a hyperspecial subgroup of $H(L_v)$.*

Indeed, let L' be a Galois extension of L over which the group H splits, and let $\mathscr{H}'_{\mathfrak{o}_{L'}}$ be a Chevalley group scheme over $\mathfrak{o}_{L'}$ with generic fiber $\mathscr{H}_{L'}$. Since the group schemes $\mathscr{H}_{\mathfrak{o}_{L'}}$ and $\mathscr{H}'_{\mathfrak{o}_{L'}}$ have the same generic fiber, they "coincide" at almost all places of L'. Since almost all places of L' are unramified over their restrictions to L, the assertion now follows from 3.4.2, 3.4.3 and 2.6.1.

3.10. *Example. General linear groups.* Suppose that $G = \mathrm{GL}_n$. The Iwahori subgroups of $\mathrm{GL}_n(K)$ are the subgroups conjugated to

$$B = \{((g_{ij}))|g_{ii} \in \mathfrak{o}^\times,\ g_{ij} \in \mathfrak{o} \text{ for } i < j \text{ and } g_{ij} \in \mathfrak{p} \text{ for } i \geqq j\}.$$

Let $(b_i)_{1\leqq i\leqq n}$ be the canonical basis of K^n. For $1 \leqq r \leqq n$, let Λ_r be the lattice in K^n generated by $\{b_i/\pi | i \leqq r\} \cup \{b_i | i > r\}$ and let P_r be the stabilizer of Λ_r in $\mathrm{GL}_n(K)$. Thus, P_r is the group of all matrices whose determinant is a unit and which have the following form

$$\begin{array}{c c} & \begin{array}{cc} r & n-r \end{array} \\ \begin{array}{c} r \\ n-r \end{array} & \left(\begin{array}{c|c} \mathfrak{o} & \pi^{-1}\mathfrak{o} \\ \hline \pi\mathfrak{o} & \mathfrak{o} \end{array}\right), \end{array}$$

where the notation means that the upper left corner is an $r \times r$ matrix with coefficients in $\mathfrak{o}$, the upper right corner an $r \times (n-r)$ matrix with coefficients in $\pi^{-1}\mathfrak{o}$, etc. The group B is the centralizer in $\mathrm{GL}_n(K)$ of the chamber C described in §1.14. The subgroups P_r are special and every special subgroup is conjugate to any one of them. The P_r's are the stabilizers of the points of $\mathscr{B}$ contained in a one-dimensional facet of C; with the notations of §2.9, the points fixed by P_r correspond to the norms of the form

$$(x_1, \cdots, x_n) \mapsto \inf(\{\omega(x_i) + \omega(\pi) - c | i \leqq r\} \cup \{\omega(x_i) - c | i > r\})$$

for some constant $c \in \boldsymbol{R}$. If v is any such point, the scheme $\mathscr{G}_v$ is the Chevalley scheme "on" GL_n defined by the lattice Λ_r. One can describe the scheme $\mathscr{G}_C$, whose group of units is B, by embedding GL_n in GL_{n^2} by means of the sum of n times the standard representation, and considering in K^{n^2} the lattice $\Lambda_1 \oplus \cdots \oplus \Lambda_n$.

Note that B is the stabilizer of any point of C. The corresponding statement for $G = \mathrm{PGL}_n$ is not true. For instance, the image in $\mathrm{PGL}_n(K)$ of the group generated by B and by the linear transformation

$$(1) \qquad b_1 \mapsto b_2 \mapsto \cdots \mapsto b_n \mapsto b_1\pi^{-1}$$

is the stabilizer of the "center of gravity" y of the chamber of $\mathscr{B}(\mathrm{PGL}_n, K)$ projecting C. That group is also a maximal bounded subgroup of $\mathrm{PGL}_n(K)$. The scheme $\mathscr{G}_y$ can be described by means of a lattice in the Lie algebra of $G = \mathrm{PGL}_n$ on which G acts by the adjoint representation. If F denotes the cyclic group of order n generated by the reduction mod $\mathfrak{p}$ of the image of (1) in PGL_n, the group $\bar{G}_y$ is the semidirect product of F by a connected group; in particular, its group of components is cyclic of order n.

3.11. *Example. Quasi-split special unitary group in odd dimension.* We take over all hypotheses and notations from 1.15 and denote by $\mathfrak{o}_L$ the ring of integers of L. Let $\lambda \in L$ be such that $\lambda + \lambda^\tau + 1 = 0$ and that $\omega(\lambda) = \delta$ (we recall that δ is defined as $\sup\{\omega(d) \mid d \in L,\ d + d^\tau + 1 = 0\}$). We suppose the uniformizing element π_1 chosen in such a way that $(\lambda\pi_1) + (\lambda\pi_1)^\tau = 0$: if L/K is unramified the possibility of such a choice is obvious and if L/K is ramified it follows from 1.15(6). Let $(b_i)_{-n \leqq i \leqq n}$ be the canonical basis of L^{2n+1}. For $0 \leqq r \leqq n$, we consider the basis $(b_i^{(r)})_{-n \leqq r \leqq n}$ defined by $b_i^{(r)} = b_i/\pi_1$ for $i < -r$, $b_i^{(r)} = b_i$ for $-r \leqq i \leqq 0$ and $b_i^{(r)} = \lambda b_i$ for $i > 0$, and we denote by Λ_r the $\mathfrak{o}_L$-lattice generated by that basis. Note that if $\delta = \omega(\lambda) = 0$, which is always the case except if L/K is ramified and char $\bar{K} = 2$, Λ_r is also generated by the basis $\{b_i/\pi_1 \mid i < -r\} \cup \{b_i | i \geqq -r\}$. The stabilizer P_r of Λ_r in $G(K)$ is also the stabilizer of the point $\boldsymbol{v}_r$ of $V = A \subset \mathscr{B}$ (with the conventions of 1.15) determined by

$$a_i(\boldsymbol{v}_r) = \tfrac{1}{2}\delta \qquad \text{if } i \leqq r,$$
$$= \tfrac{1}{2}(\delta + \omega(\pi_1)) \quad \text{if } i > r.$$

The points $\boldsymbol{v}_r$ are the vertices of the chamber defined by the inequalities $\frac{1}{2}\delta < a_1 < \cdots < a_n < \frac{1}{2}\delta + \frac{1}{2}\omega(\pi_1)$; they correspond, by $\boldsymbol{v} \mapsto I_{\mathfrak{v}}$, to the vertices of the diagrams (9) and (10) of 1.15 in the natural order, from left to right. The scheme $\mathscr{G}_{\mathfrak{v}_r}$ is the $\mathfrak{o}_L$-structure on G defined by the lattice Λ_r.

We shall now briefly investigate the algebraic group $\bar{G}_{\mathfrak{v}_r}$ obtained from $\mathscr{G}_{\mathfrak{v}_r}$ by reduction mod $\mathfrak{p}$. We choose r once and for all and use primed letters to designate the coordinates with respect to the basis $(b_i^{(r)})$. With those coordinates, the hermitian form h is given by

$$h((x_i'), (y_i')) = x_0'^{\tau} y_0' + \sum_{i=1}^{r} (\lambda^{\tau} x_i'^{\tau} y_{-i}' + \lambda x_{-i}'^{\tau} y_i')$$
$$+ \frac{\lambda^{\tau}}{\pi_1} \sum_{i=r+1}^{n} (x_i'^{\tau} y_{-i}' - x_{-i}'^{\tau} y_i').$$

We set $\bar{E} = \Lambda_r/\pi\Lambda_r$. That is a $2(2n + 1)$-dimensional vector space over $\bar{K}$ and one shows that the natural morphism $\bar{G}_{\mathfrak{v}_r} \to \mathrm{GL}(\bar{E})$ is a monomorphism; in other words $\bar{G}_{\mathfrak{v}_r}$ can be viewed as a subgroup of $\mathrm{GL}(\bar{E})$. From this point on, we must treat separately the unramified and the ramified case.

First case. L/K is unramified. Then, $\bar{E}$ is also a vector space over the residue field $\bar{L}$ of L. By reduction mod π_1, the antihermitian form $\pi_1 h/\lambda^{\tau}$ becomes the antihermitian form $\bar{h}_1$: $((\bar{x}_i'), (\bar{y}_i')) \mapsto \sum_{i=r+1}^{n} (\bar{x}_i'^{\tau} \bar{y}_{-i}' - \bar{x}_{-i}'^{\tau} \bar{y}_i')$ in $\bar{E}$, with obvious notation. Let $\bar{E}_0$ be the kernel of that form, defined by the equations $\bar{x}_i' = 0$ for $|i| > r$, and let $\Lambda_{r,0}$ be the inverse image of $\bar{E}_0$ in Λ_r. We now consider the restriction of h to $\Lambda_{r,0} \times \Lambda_{r,0}$ which, by reduction mod $\mathfrak{p}$, becomes the hermitian form

$$\bar{h}_2\colon ((\bar{x}_i')_{-r \leqq i \leqq r}, (\bar{y}_i')_{-r \leqq i \leqq r}) \mapsto \bar{x}_0'^{\tau} \bar{y}_0' + \sum_{i=1}^{r} (\lambda^{\tau} \bar{x}_i' \bar{y}_{-i}' + \lambda \bar{x}_{-i}' \bar{y}_i')$$

in $\bar{E}_0$. Finally, $\bar{G}_{\mathfrak{v}_r}$ can be described as the stabilizer of the pair $(\bar{h}_1, \bar{h}_2)$ in the group $R_{\bar{L}/\bar{K}}(\mathrm{SL}_{\bar{L}}(\bar{E}))$, that is, the special linear group of the $\bar{L}$-vector space $\bar{E}$ "considered as an algebraic group over $\bar{K}$" by restriction of scalars. Let $\bar{E}_1$ be the subspace of $\bar{E}$ defined by the equations $\bar{x}_i' = 0$ for $-r \leqq i \leqq r$, and let $\bar{\bar{h}}_1$ denote the restriction of the antihermitian form $\bar{h}_1$ to $\bar{E}_1 \times \bar{E}_1$. Then, $\bar{G}_{\mathfrak{v}_r}$ clearly contains the group $SU(\bar{\bar{h}}_1) \times SU(\bar{h}_2)$, which is nothing else but its Levi subgroup $\bar{G}_{\mathfrak{v}_r}^{\mathrm{red}}$ (cf. 3.5). Observe that, in conformity with 3.5.1, the diagram obtained from the diagram (9) of 1.15 by deleting its $(r + 1)$st vertex and the adjoining edges is a diagram of type $BC_r \times C_{n-r}$, which is indeed the type of the relative root system of $SU(\bar{\bar{h}}_1) \times SU(\bar{h}_2)$.

Second case. L/K is ramified. Then, the scalar multiplication by π_1 in Λ_r, reduced mod π, provides an endomorphism $\nu\colon \bar{E} \to \bar{E}$, obviously centralized by $\bar{G}_{\mathfrak{v}_r}$, and whose kernel is equal to its image $\nu(\bar{E})$. The quotient $\bar{\bar{E}} = \bar{E}/\nu(\bar{E})$ is canonically isomorphic with the quotient $\Lambda_r/\pi_1\Lambda_r$ and will be identified with it. From the fact that ν is centralized by $\bar{G}_{v_r}$, it follows that the projection $\bar{E} \to \bar{\bar{E}}$ induces a homomorphism of $\bar{K}$-algebraic groups $\bar{G}_{v_r} \to \mathrm{GL}(\bar{\bar{E}})$ whose kernel is unipotent. Here, we shall only describe the image $\bar{\bar{G}}_{v_r}$ of that homomorphism, leaving as an exercise the determination of the full structure of $\bar{G}_{v_r}$.

By reduction mod π_1, the antihermitian form $\pi_1 h/\lambda^\tau$ becomes the alternating form $\bar{h}_1$: $((\bar{x}'_i), (\bar{y}'_i)) \mapsto \sum_{i=r+1}^{n} (\bar{x}'_i \bar{y}'_{-i} - \bar{x}'_{-i} \bar{y}'_i)$ in $\bar{\bar{E}}$. Let $\bar{\bar{E}}_0$ be the kernel of that form, defined by the equations $\bar{x}'_i = 0$ for $|i| > r$, and let $\Lambda_{r,0}$ be the inverse image of $\bar{\bar{E}}_0$ in Λ_r. We now consider the function q: $\Lambda_{r,0} \to K$ defined by $q(x) = h(x, x)$. By reduction, it becomes the quadratic form $\bar{q}$: $\bar{\bar{E}}_0 \to \bar{K}$ given by

$$\bar{q}((\bar{x}'_i)_{-r \leqq i \leqq r}) = \bar{x}'^2_0 - \sum_{i=1}^{r} \bar{x}'_{-i}\bar{x}'_i.$$

Finally, the group $\bar{\bar{G}}_{v_r}$ is the group of all elements of $\mathrm{SL}(\bar{\bar{E}})$ stabilizing $\bar{h}_1$ and inducing on $\bar{\bar{E}}_0$ an element of the (reduced) group $SO(\bar{q})$. Let $\bar{\bar{E}}_1$ be the subspace of $\bar{\bar{E}}$ defined by the equations $\bar{x}'_i = 0$ for $-r < i < r$ and let $\bar{\bar{h}}_1$ denote the restriction of $\bar{h}_1$ to $\bar{\bar{E}}_1 \times \bar{\bar{E}}_1$. Then $\mathrm{Sp}(\bar{\bar{h}}_1) \times SO(\bar{q})$ is a Levi subgroup of $\bar{\bar{G}}_{v_r}$, which is the isomorphic image in that group of the Levi subgroup $\bar{G}^{\mathrm{red}}_{v_r}$ of $\bar{G}_{v_r}$. As in the unramified case, we can test the statement 3.5.1, this time by using the diagram (10) of 1.15 which provides, for $\bar{G}^{\mathrm{red}}_{v_r}$, a root system of type $B_r \times C_{n-r}$.

Note that, also in the unramified case, we could have, instead of the restriction of h to $\Lambda_{r,0} \times \Lambda_{r,0}$, considered its "contraction" q: $\Lambda_{r,0} \to K$ defined by $q(x) = h(x, x)$, thus making the treatment of the two cases still more similar. On the other hand, we have introduced λ in order to reduce the case distinction to a minimum; in the unramified case, as well as if char $\bar{K} \neq 2$, we could have replaced λ by 1 everywhere, thus simplifying the equations somewhat.

3.12. *Example. Quasi-split but nonsplit special orthogonal group.* Now, we take over the hypotheses and notations of §1.16 except that we take for G the *special* orthogonal group $SO(q)$. We shall not, as in §3.11, treat that example in any systematic way. Our only aim here is to give an example of a vertex $\boldsymbol{v}$ of the building such that $\bar{G}_v$ is not connected. We suppose that L/K is unramified. Let Λ be the lattice $\mathfrak{o}^n \oplus \pi\mathfrak{o}_L \oplus \mathfrak{o}^n$ in $K^n \oplus L \oplus K^n$ where $\mathfrak{o}_L$ is the ring of integers of L. The stabilizer P of Λ in $G(K)$ is also the stabilizer of the point $\boldsymbol{v} \in V \subset A$ defined by $a_i(\boldsymbol{v}) = \frac{1}{2}\omega(\pi)$ for $1 \leqq i \leqq n$, which is a vertex of the chamber described in §1.16. In the diagram (1) of §1.16, I_v is the vertex at the extreme left. As in §3.11, one can describe $\bar{G}_v$ as a subgroup of $\mathrm{GL}(\bar{E})$, where $\bar{E}$ is the $\bar{K}$-vector space $\Lambda/\pi\Lambda$. By reduction, the form q, restricted to Λ, becomes the quadratic form $\bar{q} : (\bar{x}_i)_{-n \leqq i \leqq n} \mapsto \sum_{i=1}^{n} \bar{x}_{-i}\bar{x}_i$ in $\bar{E}$, with obvious notations ($\bar{x}_i$ belongs to the residue field of L if $i = 0$ and to $\bar{K}$ otherwise). Let $\bar{E}_0$ be the two-dimensional kernel of $\bar{q}$ defined by the equations $\bar{x}_i = 0$ for $i \neq 0$, and let $\bar{\bar{q}}$ be the quadratic form in $\bar{\bar{E}} = \bar{E}/\bar{E}_0$ image of $\bar{q}$. Clearly, $\bar{G}_v$ preserves the form $\bar{q}$. Therefore, the projection $\bar{E} \to \bar{\bar{E}}$ induces a $\bar{K}$-homomorphism $\bar{G}_v \to O(\bar{\bar{q}})$. One verifies that that homomorphism is *surjective* (in particular, $\bar{G}_v$ is not connected) and that it maps $\bar{G}^{\mathrm{red}}_v$ isomorphically onto $SO(\bar{\bar{q}})$. If τ denotes the nontrivial K-automorphism of L, the linear transformation $(x_i)_{-n \leqq i \leqq n} \mapsto (x_n, x_{-n+1}, \cdots, x_{-1}, x_0^\tau, x_1, \cdots, x_{n-1}, x_{-n})$, which belongs to P, provides by reduction an element of $\bar{G}_v(\bar{K})$ which is mapped into $O(\bar{\bar{q}})$ but not into $SO(\bar{\bar{q}})$.

4. Classification.

4.1. *Introduction.* To finish with, we give the classification of simple groups in the case where the residue field $\bar{K}$ is finite, which will be assumed from now on. We recall that, in the characteristic zero case, that classification has been given first by M. Kneser **[16]**. The tables 4.2 and 4.3, together with the comments in §4.5, provide a list of all central isogeny classes of absolutely quasi-simple groups over K. For each type of group, they give the following information, where K_1 denotes the unramified closure of K:

a *name* of the shape ${}^aX$ where the symbol X represents the absolute local Dynkin diagram Δ_1 (1.11) with the notations of **[8**, 1.4.6]—except that our *C-BC* corresponds to the *C-BC*$^{\text{III}}$ of **[8]**—and where a is the order of the automorphism group of Δ_1 induced by $\mathrm{Gal}(K_1/K)$; for residually split groups, $a = 1$ and the superscript a is omitted from the notation; note that the index on the right of X is the relative rank over K_1, hence equal to the number of vertices of Δ_1 minus one; primes, double primes, etc. are used to distinguish types of groups which would otherwise have the same name;

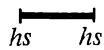
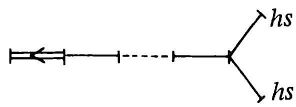
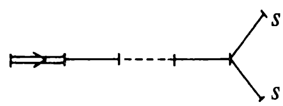
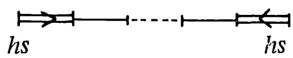
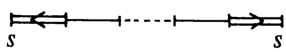
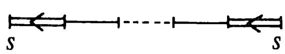
the symbol representing the *affine root system* (or échelonnage) in the notations of **[8]**; in the residually split case, that symbol coincides with the name of the type and is not given separately; note that the right part of the symbol gives the type of the *relative root system* Φ and, in particular, that the index on the right of it is the relative rank over K, hence equal to the number of vertices of the relative local Dynkin diagram Δ minus one;

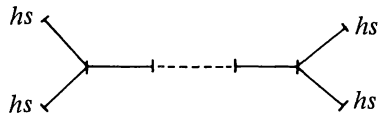
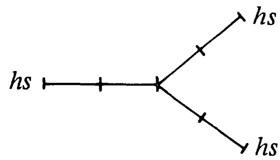
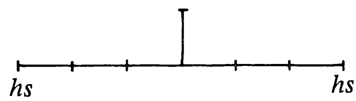
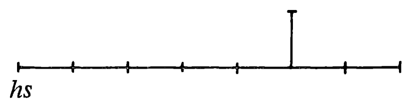
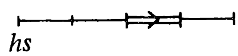
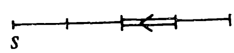
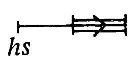
the *local index* (§1.11), the *relative local Dynkin diagram* Δ (§1.8) and *the integers* $d(v)$ attached to its vertices (§1.8); the action of $\mathrm{Gal}(K_1/K)$ on Δ_1—through a cyclic group of order a (see above)—is essentially characterized by its orbits in the set of vertices of Δ_1, orbits which are exhibited as follows: the elements of the orbit $O(v)$ corresponding to a vertex v of Δ (1.11) are placed close together on the same vertical line as v (in the few cases, such as ${}^2D_n$, ${}^2D''_{2m}$, etc., where two vertices of Δ are on the same vertical, the correspondence $v \mapsto O(v)$ should be clear from the way the diagrams are drawn); since $\bar{K}$ is finite G is residually quasi-split (1.10.3), hence all vertices of Δ_1 are distinguished except for the unique anisotropic type ${}^dA_{d-1}$ (§1.11), and there is no need for a special notation like the circling of orbits, as in **[22]**; *hyperspecial vertices* (§1.10) are marked with an *hs* and the other *special vertices* (§1.9) with an *s*;

the *index* of the form, in the "usual" sense of [3] and **[22]**; for simplicity, we do not represent that index by a picture but rather by the corresponding symbol in the notation of **[22]**; we recall that that symbol carries, among other, the following information: the *absolute type* of the group, the *absolute rank*, the *relative rank* (already provided by the symbol representing Φ_{af}) and the order of the automorphism group of the ordinary Dynkin diagram induced by the Galois group of the separable closure of K.

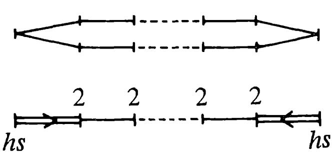
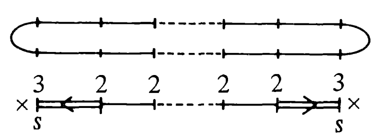
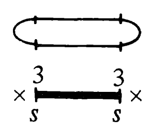
In the case of the inner forms of A_n, the diagrams are, for technical reasons, replaced by explanations in words.

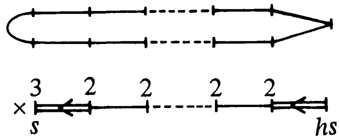
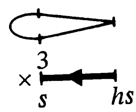
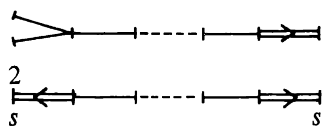
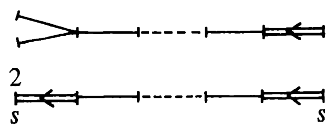
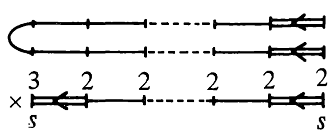
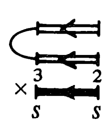
4.2. *Residually split groups.*

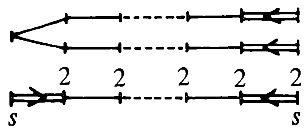
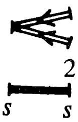
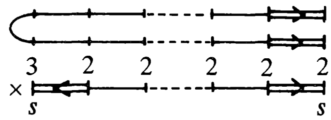
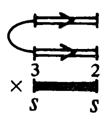
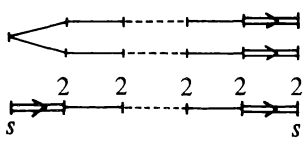
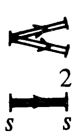
Name	Local Dynkin diagram	Index [22]
A_n ($n \geqq 2$)	A cycle of length $n + 1$ all vertices of which are hyperspecial	$^1A^{(1)}_{n,n}$
A_1		$^1A^{(1)}_{1,1}$
B_n ($n \geqq 3$)		$B_{n,n}$
$B\text{-}C_n$ ($n \geqq 3$)		$^2A^{(1)}_{2n-1,n}$
C_n ($n \geqq 2$)		$C^{(1)}_{n,n}$
$C\text{-}B_n$ ($n \geqq 2$)		$^2D^{(1)}_{n+1,n}$
$C\text{-}BC_n$ ($n \geqq 2$)		$^2A^{(1)}_{2n,n}$
$C\text{-}BC_1$		$^2A^{(1)}_{2,1}$

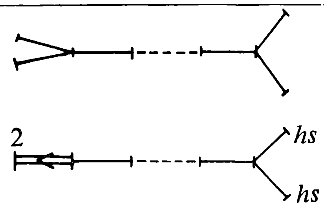
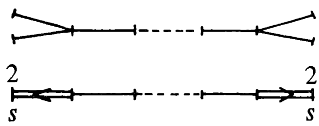
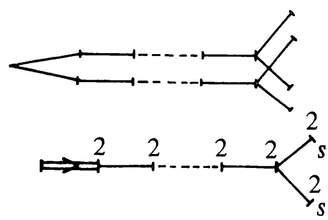
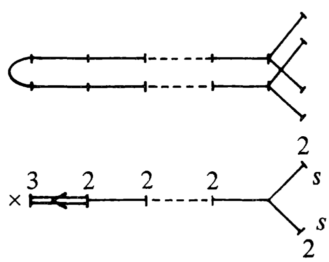
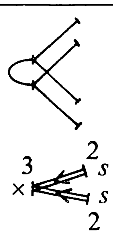
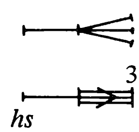
Name	Local Dynkin diagram	Index [22]
D_n $(n \geqq 4)$		${}^1D^{(1)}_{n,n}$
E_6		${}^1E^0_{6,6}$
E_7		$E^0_{7,7}$
E_8		$E^0_{8,8}$
F_4		$F^0_{4,4}$
F^{I}_4		${}^2E^2_{6,4}$
G_2		$G_{2,2}$
G^{I}_2		${}^3D_{4,2}$ or ${}^6D_{4,2}$

4.3. *Nonresidually split groups.*

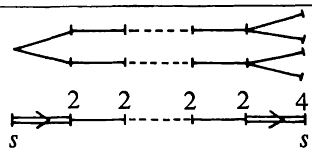
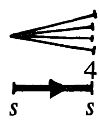
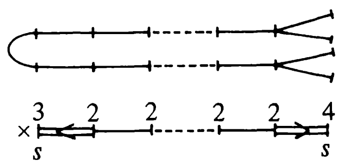
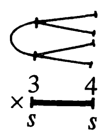
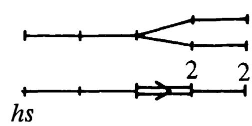
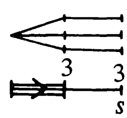
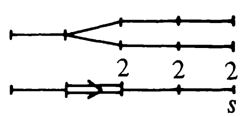
Name	Affine root system (notations of [8, 1.4.6])	Local index and relative local Dynkin diagram.	Index [22]
${}^dA_{md-1}$ $(m \geqq 3,\ d \geqq 2)$	A_{m-1}	The absolute local Dynkin diagram Δ_1 is a cycle of length md on which $\mathrm{Gal}(K_1/K)$ acts through a cyclic group of order d generated by a rotation of the cycle. The relative diagram is a cycle of length m all vertices of which are special but not hyperspecial and carry the number d.	${}^1A^{(d)}_{md-1,m-1}$
${}^dA_{2d-1}$ $(d \geqq 2)$	A_1	Δ_1 is as above, with $m = 2$ Δ is 	${}^1A^{(d)}_{2d-1,1}$
${}^dA_{d-1}$ $(d \geqq 2)$	$\varnothing$	Δ_1 is as above, with $m = 1$, or, if $d = 2$, consists of a fat segment whose vertices are permuted by $\mathrm{Gal}(K_1/K)$ and $\Delta = \varnothing$	${}^1A^{(d)}_{d-1,0}$
${}^2A'_{2m-1}$ $(m \geqq 2)$	C_m		${}^2A^{(1)}_{2m-1,m}$
${}^2A''_{2m-1}$ $(m \geqq 3)$	$C\text{-}BC^{\mathrm{II}}_{m-1}$		${}^2A^{(1)}_{2m-1,m-1}$
${}^2A''_3$	$C\text{-}BC^{\mathrm{II}}_1$		${}^2A^{(1)}_{3,1}$

Name	Affine root system (notations of [8, 1.4.6])	Local index and relative local Dynkin diagram.	Index [22]
${}^2A'_{2m}$ ($m \geqq 2$)	$C\text{-}BC_m^{\mathrm{IV}}$		${}^2A^{(1)}_{2m,m}$
${}^2A'_2$	$C\text{-}BC_1^{\mathrm{IV}}$		${}^2A^{(1)}_{2,1}$
${}^2B_n$ ($n \geqq 3$)	$C\text{-}B_{n-1}$		$B_{n,n-1}$
${}^2B\text{-}C_n$ ($n \geqq 3$)	$C\text{-}BC_{n-1}^{\mathrm{III}}$		${}^2A^{(1)}_{2n-1,n-1}$
${}^2C_{2m-1}$ ($m \geqq 3$)	$C\text{-}BC_{m-1}^{\mathrm{IV}}$		$C^{(2)}_{2m-1,m-1}$
${}^2C_3$	$C\text{-}BC_1^{\mathrm{IV}}$		$C^{(2)}_{3,1}$

Name	Affine root system (notations of [8, 1.4.6])	Local index and relative local Dynkin diagram.	Index [22]
${}^2C_{2m}$ $(m \geqq 2)$	C_m		$C^{(2)}_{2m,\,m}$
${}^2C_2$	A_1		$C^{(2)}_{2,\,1}$
${}^2C\text{-}B_{2m-1}$ $(m \geqq 3)$	$C\text{-}BC^{\mathrm{I}}_{m-1}$		${}^2D^{(2)}_{2m,\,m-1}$
${}^2C\text{-}B_3$	$C\text{-}BC^{\mathrm{I}}_1$		${}^2D^{(2)}_{4,\,1}$
${}^2C\text{-}B_{2m}$ $(m \geqq 2)$	$C\text{-}BC^{\mathrm{III}}_m$		${}^2D^{(2)}_{2m+1,\,m}$
${}^2C\text{-}B_2$	$C\text{-}BC^{\mathrm{III}}_1$		${}^2D^{(2)}_{3,\,1}$

Name	Affine root system (notations of [8, 1.4.6])	Local index and relative local Dynkin diagram.	Index [22]
${}^2D_n$ $(n \geqq 4)$	B_{n-1}		${}^2D^{(1)}_{n,n-1}$
${}^2D'_n$ $(n \geqq 4)$	$C\text{-}B_{n-2}$		${}^1D^{(1)}_{n,n-2}$
${}^2D''_{2m}$ $(m \geqq 3)$	$B\text{-}C_m$		${}^1D^{(2)}_{2m,m}$
${}^2D''_{2m+1}$ $(m \geqq 3)$	$B\text{-}BC_m$		${}^2D^{(2)}_{2m+1,m}$
${}^2D''_5$	$B\text{-}BC_2$[3]		${}^2D^{(2)}_{5,2}$
${}^3D_4$	G_2		${}^3D_{4,2}$

[3]This "échelonnage" is missing in the table of [8, p. 29].

Name	Affine root system (notations of [8, 1.4.6])	Local index and relative local Dynkin diagram.	Index [22]
$^4D_{2m}$ $(m \geqq 3)$	$C\text{-}BC^{\text{III}}_{m-1}$		$^2D^{(2)}_{2m,m-1}$
$^4D_4$	$C\text{-}BC^{\text{III}}_1$		$^2D^{(2)}_{4,1}$
$^4D_{2m+1}$ $(m \geqq 3)$	$C\text{-}BC^{\text{I}}_{m-1}$		$^1D^{(2)}_{2m+1,m-1}$
$^4D_5$	$C\text{-}BC^{\text{I}}_1$		$^1D^{(2)}_{5,1}$
$^2E_6$	F_4		$^2E^2_{6,4}$
$^3E_6$	G^{I}_2		$^1E^{16}_{6,2}$
$^2E_7$	F^{I}_4		$E^5_{7,4}$

4.4. *Interpretation.* We shall now repeat the classification in classical terms. The following enumeration gives a representative of every central isogeny class of absolutely quasi-simple groups over K and, in each case, the name of the corresponding type, with the notations of the first column of the Tables 4.2 and 4.3.

Special linear group SL_m of a d^2-dimensional central division K-algebra ($md \geqq 2$). The type is ${}^dA_{md-1}$.

Special unitary group $SU(h)$ of a hermitian form h in r variables ($r \geqq 3$) with Witt index r' over a quadratic extension L of K. If L/K is ramified, we assume $r \neq 4$ because the case $r = 4$ is more adequately represented by an ordinary special orthogonal group in 6 variables or a quaternionic special orthogonal group in 3 variables according as $r' = 2$ or 1. If the form h is split, the type is ${}^2A'_{r-1}$ if L/K is unramified and $C\text{-}BC_n$ ($r = 2n + 1$) or $B\text{-}C_n$ ($r = 2n$) otherwise. If h is not split, one has $r = 2r' + 2$ and the type is ${}^2A''_{r-1}$ or ${}^2B\text{-}C_{r'+1}$ according as L/K is unramified or ramified.

Special orthogonal group $SO(q)$ of a quadratic form q in r variables ($r \geqq 6$) with Witt index r' over K. If r is even, we denote by L the center of the even Clifford algebra of q which is isomorphic to $K \oplus K$ (form q of discriminant one or Arf invariant zero) or is a quadratic extension of K. If L/K is unramified (in particular if $L = K \oplus K$), we assume $r \neq 6$ because the case $r = 6$ is more adequately represented by a special unitary group in 4 variables. If $r = 2r'$ (resp. $2r' + 1$), the type is $D_{r'}$ (resp. $B_{r'}$). If $r = 2r' + 2$, L is a quadratic extension of K and the type is ${}^2D_{r'+1}$ or $C\text{-}B_{r'}$ according as L/K is unramified or ramified. Finally, if $r = 2r' + 3$ (resp. $2r' + 4$), the type is ${}^2B_{r'+1}$ (resp. ${}^2D'_{r'+2}$).

The *symplectic group* in $2n \geqq 4$ variables is of type C_n.

Special unitary group of a *quaternion* hermitian form in r variables ($r \geqq 2$) relative to the standard involution. The Witt index is always maximal and the type is ${}^2C_r$.

Special orthogonal group $SO(q)$ of a *quaternionic* σ-quadratic form in r variables ($r \geqq 3$) relative to an involution σ of the quaternion algebra whose space of symmetric elements is 3-dimensional (cf. **[21]**, **[23]**; if char $K \neq 2$, σ-quadratic amounts to σ-hermitian and the group is also the *special unitary* group of an *antihermitian* form relative to the standard involution). Let r' be the Witt index of the form and L the center of its "even" Clifford algebra $\mathrm{Cl}(q)$ (cf. **[21]**). If L/K is unramified (in particular if $L \cong K \oplus K$), we assume $r \neq 3$ because the case $r = 3$ is more adequately represented by a special unitary group in 4 variables; if furthermore $r = 2r'$, we also assume $r \neq 4$ because the case $r = 2r' = 4$ is more adequately represented, through the triality principle, by an ordinary special orthogonal group in 8 variables with Witt index 2. One always has $2r' \leqq r \leqq 2r' + 3$. If $r = 2r'$, one has $L \cong K \oplus K$ and the group is of type ${}^2D''_r$. If $r = 2r' + 1$ (resp. $2r' + 2$), L is a quadratic extension of K and the type is ${}^2D''_r$ (resp. ${}^4D_r$) or ${}^2C\text{-}B_{r-1}$ according as L/K is unramified or ramified. If $r = 2r' + 3$, one has $L \cong K \oplus K$ and the type is ${}^4D_r$.

Quasi-split triality D_4. Let L denote the splitting field, which is a cyclic extension of degree 3 or a Galois extension of degree 6 with Galois group $\mathfrak{S}_3$. If L/K is unramified (hence cyclic of degree 3), the type is ${}^3D_4$; otherwise, it is G_2^{I}.

Split exceptional groups. The type has the same name G_2, F_4, E_6, E_7 or E_8 as the absolute type of the group.

Quasi-split groups of type E_6. The type is $^2E_6$ or F_4^1 according as the quadratic splitting field L is unramified or ramified.

Nonquasi-split groups of type E_6 and E_7. They are the forms of E_6 and E_7 constructed by means of a central division algebra of dimension 9 and 4 respectively; their types are $^3E_6$ and $^2E_7$.

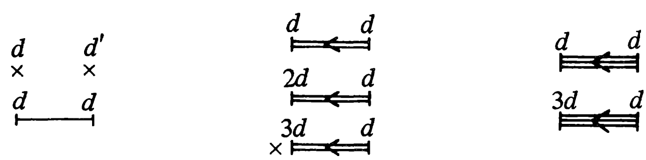
4.5. *Invariants*. All types of groups listed in the Tables 4.2, 4.3 exist over an arbitrary field K with finite residue field. The central isogeny class corresponding to a given name is always unique except in the following cases.

The isogeny classes of type $^dA_{md-1}$ for $d \geqq 5$ are classified by the pairs of opposite central division algebras of dimension d^2 over K; their number is therefore $\frac{1}{2}\varphi(d)$, where φ is the Euler function.

The isogeny classes of the types B-C_n, ^2B-C_n, C-B_n, ^2C-B_n, C-BC_n and F_4^I are classified by the ramified quadratic extensions of K, namely the extension always called L in §4.4.

The groups of type G_2^I are classified by the Galois extensions L of K which are either cyclic of degree 3 or noncyclic of degree 6.

4.6. *The classification kit*. The following experimental facts provide a handy way of reconstructing the classification. First note that, except for $^dA_{d-1}$, each type in the Tables 4.2 and 4.3 is completely characterized by the local Dynkin diagram and the integers $d(v)$ attached. Now, consider a connected Coxeter diagram of affine type and rank (number of vertices) at least three, attach an integer to all vertices, mark some of them (possibly none) with a cross and orient each double or triple link with an arrow. Then a necessary and sufficient condition for the existence of a semisimple group G having the resulting diagram as its relative local Dynkin diagram with the given integers as $d(v)$ is that all subdiagrams formed by the pairs of vertices belong to one of the following types, representing the ordinary Dynkin diagrams of quasi-split groups of relative rank two:



The group G can furthermore be chosen to be absolutely quasi-simple if and only if the integers $d(v)$ are relatively prime or if the underlying Coxeter diagram is a cycle. As for the types of relative rank one, whose underlying Coxeter diagram is $\vdash\!\dashv$, they can be obtained as "limit cases" of types of higher ranks, but we shall not elaborate on that.

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COLLÈGE OF FRANCE, PARIS

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REPRESENTATIONS OF REDUCTIVE LIE GROUPS

NOLAN R. WALLACH

The purpose of this article is to give an introduction to the representation theory of real, reductive Lie groups (see §1 for the precise class of groups to be studied). Our main goal is the Langlands, Harish-Chandra classification of irreducible representations of such groups. §§7 and 8 are devoted to a study of intertwining operators for GL(2, $\boldsymbol{R}$) and GL(2, $\boldsymbol{C}$) and to a detailed exposition of how the classification specializes to GL(2, $\boldsymbol{R}$) and to GL(2, $\boldsymbol{C}$).

The representation theory of reductive Lie groups is a vast subject. A complete survey of the subject would be a very ambitious task. It would also involve a sizeable monograph rather than an article of modest size. It is for this reason that we limit ourselves to a single path through the subject leading to the classification. Many important aspects of the theory are hardly discussed (e.g., the detailed theory of the discrete series **[5]**, **[6]**, **[19]**, intertwining operators **[13]**, **[14]**, the Plancherel theorem **[8]**, **[9]**, **[10]**). However, we feel that the material covered will give the reader an idea of the flavor of the subject. Much of the material that is not covered in this article will appear in other articles in this volume. There are several surveys of the subject of representations of semisimple Lie groups that are available (notably **[20]**). The reader interested in a deeper pursuit of the subject should consult the monographs of G. Warner **[22]**.

1. The class of groups to be studied.

1.1. Let $\mathfrak{G}$ be an algebraic group over $\boldsymbol{C}$ defined over $\boldsymbol{R}$. We denote by $\mathfrak{G}(\boldsymbol{R})$ the real points of $\mathfrak{G}$ and by $\mathfrak{G}^\circ$ the identity component of $\mathfrak{G}$.

1.2. The type of group we will be studying will be a Lie group, G, having the following properties:

(1) There is an algebraic group $\mathfrak{G}$ defined over $\boldsymbol{R}$ and a Lie group homomorphism $\gamma: G \to \mathfrak{G}(\boldsymbol{R})$ so that $\gamma(G)$ is open in $\mathfrak{G}(\boldsymbol{R})$ and $\gamma: G \to \gamma(G)$ is a finite covering.

(2) $\mathfrak{G}^\circ$ is reductive.

(3) $\mathrm{Ad}(\gamma(G)) \subset \mathrm{Ad}(\mathfrak{G}^\circ)$. Here Ad denotes the adjoint representation of $\mathfrak{G}^\circ$ on the Lie algebra of $\mathfrak{G}^\circ$.

1.3. Our basic examples are GL(n, $\boldsymbol{R}$) and GL(n, $\boldsymbol{C}$) (looked upon as the real points of an algebraic group defined over $\boldsymbol{R}$). One of the main reasons for introducing a broader class of groups than linear groups is to include the "metaplectic" group, a two-fold covering of Sp(n, $\boldsymbol{R}$).

AMS (MOS) subject classifications (1970). Primary 22E45.

1.4. Let G be as in 1.2. Let $X(G)$ denote the group of all continuous homomorphisms of G into $R^\times = R - (0)$. Set $°G = \{g \in G \mid |x(g)| = 1, x \in X(G)\}$.

1.5. If H is a Lie group, we will denote by $\mathfrak{h}$ the Lie algebra of H. Ad will denote the adjoint representation of H on $\mathfrak{h}$. $\mathfrak{h}_C$ will denote the complexification of $\mathfrak{h}$.

1.6. Let G satisfy the conditions in 1.2. Then it is not hard to show that there is a nondegenerate, Ad(G)-invariant, symmetric bilinear form, B, on $\mathfrak{g}$ and an involutive automorphism, θ, of G such that:

(1) B is θ-invariant and $(X, Y) = -B(X, \theta Y)$ is positive definite.

(2) There is a maximal compact subgroup, K, of G so that $K^\circ \subset \{g \in G \mid \theta(g) = g\}$ is an open subgroup.

For such G we fix B, θ, K. Let $\mathfrak{c}$ denote the center of $\mathfrak{g}$. Set $°\mathfrak{c} = \{X \in \mathfrak{c} \mid \theta X = X\}$. Then $\exp(°\mathfrak{c}) = K \cap \exp(\mathfrak{c})$. Set $\mathfrak{z} = \{X \in \mathfrak{c} \mid \theta X = -X\}$. Let $Z = \exp(\mathfrak{z})$. Then Z is called the split component of G.

1.7. It is not hard to show that the map $Z \times °G \to G$, $(z, g) \mapsto zg$ is a Lie group isomorphism.

1.8. We now look at our examples. If $G = \mathrm{GL}(n, F)$, $F = \boldsymbol{R}$ or $\boldsymbol{C}$, set $B(X, Y) = \mathrm{Re}(\mathrm{tr}\, XY)$. Set $\theta X = -X^*$ (here X^* denotes the conjugate transpose of the matrix X). $°G = \{g \in G \mid |\det g| = 1\}$, $Z = \{aI \mid a > 0\}$. $K = O(n)$ if $F = \boldsymbol{R}$ and $K = U(n)$ if $F = \boldsymbol{C}$.

2. Admissible representations.

2.1. We fix G as in 1.2, B and K as in 1.6.

2.2. Let H be a separable Hilbert space. Let GL(H) denote the group of all invertible, bounded operators on H. A representation (π, H) of G on H is a homomorphism $\pi: G \to \mathrm{GL}(H)$ so that

(1) The map $G \times H \to H$, $g, v \to \pi(g)v$ is continuous.

(π, H) is said to be unitary if $\pi(g)$ is unitary for $g \in G$.

2.3. Let $\| \cdots \|$ denote the operator norm on End(H). The principle of uniform boundedness implies that if $\omega \subset G$ is a compact subset then $\|\pi(g)\| \leqq C(\omega) < \infty$ for $g \in \omega$. This implies that if (,) is the Hilbert space structure on H and we set $\langle v, w\rangle = \int_K (\pi(k)v, \pi(k)w)\, dk$ (here dk denotes Haar measure with total mass 1 on K) then $\langle\ ,\ \rangle$ gives the same topology on H as (,) and $\pi|_K$ is unitary relative to $\langle\ ,\ \rangle$. We therefore assume that $\pi|_K$ is unitary.

2.4. Fix a Haar measure dg on G (G is unimodular). If $f \in C_c^\infty(G)$ (C^∞ with compact support) we can define $\pi(f): H \to H$ by

(1) $\langle \pi(f)v, w \rangle = \int_G f(g) \langle \pi(g)v, w\rangle\, dg$ and $\|\pi(f)\| \leqq C(\omega)\|f\|_1$, if $\mathrm{supp}\, f \subset \omega$ (see 2.3, $\|f\|_1$ the L^1 norm of f).

If $f \in C^\infty(K)$ set $\pi_K(f)$ equal to $(\pi|_K)(f)$.

2.5. Let $\mathscr{E}(K)$ denote the set of equivalence classes of irreducible, unitary representations of K. If $\gamma \in \mathscr{E}(K)$ fix $(\tau_\gamma, V_\gamma) \in \gamma$. Then $\dim V_\gamma < \infty$. Set $\alpha_\gamma(k) = (\dim V_\gamma)\mathrm{tr}\, \tau_\gamma(k^{-1})$, $k \in K$.

Then $\pi_K(\alpha_\gamma)$ is a projection operator on H. Set $\pi_K(\alpha_\gamma)H = H_\gamma$. Then $H = \bigoplus_\gamma H_\gamma$ Hilbert space direct sum.

2.6. A representation (π, H) of G is said to be *admissible* if $\dim H_\gamma < \infty$ for $\gamma \in \mathscr{E}(K)$.

2.7. A representation (π, H) is said to be irreducible if the only closed $\pi(G)$-invariant subspaces of H are (0) and H.

2.8. THEOREM (HARISH-CHANDRA [2]). (1) *If* (π, H) *is an irreducible unitary representation of* G *then* (π, H) *is admissible.*

(2) *If* (π, H) *is an admissible representation of* G *and if* $F \subset \mathscr{E}(K)$ *is a finite subset then if* $v \in \bigoplus_{\gamma \in F} H_\gamma$ *the map* $g \to \pi(g)v$ *is real analytic.*

2.9. Let (π, H) be an admissible representation of G. Set H_0 equal to the algebraic sum of the H_γ, $\gamma \in \mathscr{E}(K)$. If $v \in H$, set $H(v) = \sum_{k \in K} C\pi(k)v$. Then $H_0 = \{v \in H | \dim H(v) < \infty\}$.

2.10. If $v \in H_0$ then Theorem 2.8 implies:

(1) If $X \in \mathfrak{g}$, $\pi(X)v = (d/dt)\pi(\exp tX)v|_{t=0}$ exists.

(2) If $k \in K$, $X \in \mathfrak{g}$ then $\pi(k)\pi(X)v = \pi(\mathrm{Ad}(k)X)\pi(k)v$.

2.11. 2.10 (2) implies that if $X \in \mathfrak{g}$, $v \in H_0$. Then $\pi(k)\pi(X)v \in \pi(\mathfrak{g}) \cdot H(v)$. Thus $\dim H(\pi(X)v) < \infty$. This implies $\pi(X)v \in H_0$ for $v \in H_0$, $X \in \mathfrak{g}$. Just as in the case $\dim H < \infty$ we find

(1) If $X, Y \in \mathfrak{g}$ then $\pi[X, Y] = \pi(X)\pi(Y) - \pi(Y)\pi(X) = [\pi(X), \pi(Y)]$ on H_0.

2.12. 2.10, 2.11 lead naturally to the notion of a $(\mathfrak{g}, K)$-module. A $(\mathfrak{g}, K)$-module is a complex vector space V such that

(1) V is a $\mathfrak{g}$-module. That is, there is a linear map $\mathfrak{g} \otimes V \to V$, $X \otimes v \to Xv$ such that $[X, Y] \cdot v = X \cdot Yv - Y \cdot Xv$, $X, Y \in \mathfrak{g}$.

(2) V is a K-module. That is there is a map $K \times V \to V$ linear in the V-variable such that $1 \cdot v = v$ and $k_1 \cdot (k_2 v) = (k_1 k_2) \cdot v$, $k_1, k_2 \in K$.

(3) If $v \in V$ then $\dim \sum_{k \in K} Ck \cdot v < \infty$ and if $W \subset V$ is a finite dimensional K-invariant subspace of V then:

(a) K acts completely reducibly on W.

(b) The map $K \times W \to W$, $k, w \to k \cdot w$ is continuous (hence real analytic).

(4) If $X \in \mathfrak{k}$ and $v \in V$ then

$$Xv = (d/dt) \exp tX \cdot v|_{t=0}.$$

(5) If $k \in K$, $X \in \mathfrak{g}$, $v \in V$ then $k \cdot X \cdot v = (\mathrm{Ad}(k)X) \cdot (k \cdot v)$.

2.13. If V, W are K-modules (resp. $\mathfrak{g}$-modules) then we denote by $\mathrm{Hom}_K(W, V)$ (resp. $\mathrm{Hom}_{\mathfrak{g}}(W, V)$) the space of linear maps $A: W \to V$ such that $Ak \cdot w = k \cdot Aw$, $k \in K$, $w \in W$ (resp. $AXw = XAw$, $X \in \mathfrak{g}$, $w \in W$). If V, W are $(\mathfrak{g}, K)$-modules we set $\mathrm{Hom}_{\mathfrak{g},K}(W, V) = \mathrm{Hom}_{\mathfrak{g}}(W, V) \cap \mathrm{Hom}_K(W, V)$.

2.14. A $(\mathfrak{g}, K)$-module V is said to be admissible if for any $\gamma \in \mathscr{E}(K)$, $\dim \mathrm{Hom}_K(V_\gamma, V) < \infty$. We set $V(\gamma) = \sum_{A \in \mathrm{Hom}_K(V_\gamma, V)} AV_\gamma \subset V$. $V(\gamma)$ is called the γ-isotypic subspace of V.

2.15. If (π, H) is an admissible representation of G and if we set $V_\pi = H_0$ with action $X \cdot v = \pi(X)v$, $v \in V$, $X \in \mathfrak{g}$. Then 2.10(1), (2), 2.11(1) imply that V_π is an admissible $(\mathfrak{g}, K)$-module.

2.16. A $(\mathfrak{g}, K)$-module, V, is said to be irreducible if the only $\mathfrak{g}$- and K-invariant subspaces are V and (0).

2.17. THEOREM (HARISH-CHANDRA [2]). *If* (π, H) *is an admissible representation of* G *then* (π, H) *is irreducible if and only if* V_π *is an irreducible* $(\mathfrak{g}, K)$*-moudle.*

2.18. If (π_i, H_i), $i = 1, 2$, are representations of G then an *intertwining operator* from (π_1, H_1) to (π_2, H_2) is a continuous linear map $A: H_1 \to H_2$ so that $A \circ \pi_1(g) =$

$\pi_2(g) \circ A$. π_1, π_2 are said to be equivalent if there is a bijective intertwining operator from π_1 to π_2. If π_1, π_2 are unitary representatives of G then π_1, π_2 are said to be unitarily equivalent if there is a bijective unitary intertwining operator from π_1 to π_2.

2.19. THEOREM (HARISH-CHANDRA **[4]**). *If (π_i, H_i) are unitary, admissible representations of G then π_1, π_2 are unitarily equivalent if and only if V_{π_1} and V_{π_2} are isomorphic (that is, there is a bijective element of* $\mathrm{Hom}_{\mathfrak{g}, K}(V_{\pi_1}, V_{\pi_2})$*).*

2.20. A $(\mathfrak{g}, K)$-module, V, is said to be *unitary* if there is a positive definite inner product $\langle\ ,\ \rangle$ on V so that:
(1) $\langle kv, kw\rangle = \langle v, w\rangle, k \in K$,
(2) $\langle Xv, w\rangle = -\langle v, Xw\rangle, X \in \mathfrak{g}$.

2.21. THEOREM (HARISH-CHANDRA **[2]**, **[3]**). *Let V be an admissible $(\mathfrak{g}, K)$-module. A necessary and sufficient condition for $V = V_\pi$ with π an admissible, unitary representation of G is that V be unitary.*

2.22. The results of this section imply that the determination of the set $\mathscr{E}(G)$ of equivalence classes of irreducible unitary representations of G is the same as finding the set $\mathscr{E}(\mathfrak{g}, K)$ of isomorphism classes of irreducible, unitary, admissible $(\mathfrak{g}, K)$-modules. This can be done in two steps. The first is to determine the set $\Pi(\mathfrak{g}, K)$ of isomorphism classes of irreducible, admissible $(\mathfrak{g}, K)$-modules. The second is to determine which elements of $\Pi(\mathfrak{g}, K)$ have a unitary representative. The first step has been carried out in Langlands' classification (an alternate classification has been given by Vogan which should have important and far-reaching consequences). The second step is far from complete. There should be some discussion of the literature on this problem in the lectures of Knapp and Zuckerman **[14]**.

3. Infinitesimal characters.

3.1. Let $\mathfrak{g}$ be a Lie algebra over a field. Let $T(\mathfrak{g})$ be the tensor algebra over $\mathfrak{g}$. Let $I(\mathfrak{g})$ be the right and left ideal in $T(\mathfrak{g})$ generated by the elements

$$X \otimes Y - Y \otimes X - [X, Y].$$

Set $U(\mathfrak{g}) = T(\mathfrak{g})/I(\mathfrak{g})$. Let $j: \mathfrak{g} \to U(\mathfrak{g})$ be defined by $j(X) = X + I(\mathfrak{g})$.

3.2. $(U(\mathfrak{g}), j)$ or $U(\mathfrak{g})$ is called the universal enveloping algebra of $\mathfrak{g}$. We note that $j[X, Y] = j(X)j(Y) - j(Y)j(X) = [j(X), j(Y)]$. $U(\mathfrak{g})$ is called the universal enveloping algebra of $\mathfrak{g}$ because of the following universal mapping property:

If $\varphi: \mathfrak{g} \to E$, E an associative algebra with unit such that $\varphi[X, Y] = [\varphi(X), \varphi(Y)]$, then there exists a unique algebra homomorphism φ of $U(\mathfrak{g})$ into E so that
(a) $\varphi(1) = 1$,
(b) $\varphi(j(X)) = \varphi(X)$.

3.3. In particular, every $\mathfrak{g}$-module is naturally a $U(\mathfrak{g})$-module.

3.4. THEOREM (CF. **[11]**). *Let $X_1, \cdots, X_n$ be a basis of $\mathfrak{g}$. Then the monomials $X_1^{r_1} \cdots X_n^{r_n}$ form a basis of $U(\mathfrak{g})$.*

3.5. Theorem 3.4 is usually referred to as the Poincaré-Birkhoff-Witt (P-B-W) theorem. In particular it implies that $j: \mathfrak{g} \to U(\mathfrak{g})$ is injective. We therefore suppress the j and look upon $\mathfrak{g}$ as a subalgebra of $U(\mathfrak{g})$ under the commutator of the associative multiplication.

3.6. Let G be as in 1.2. Let $\mathfrak{g}$ be its Lie algebra (as per our conventions). Let $\mathfrak{g}_C$ be the complexification of $\mathfrak{g}$. Let $\mathscr{Z}(\mathfrak{g}_C) = \mathscr{Z}$ be the center of $U(\mathfrak{g}_C)$.

3.7. We note that if $g \in G$ then $\mathrm{Ad}(g)\colon \mathfrak{g}_C \to \mathfrak{g}_C$ extends to an automorphism of $U(\mathfrak{g}_C)$. 1.2 (3) implies that if $z \in \mathscr{Z}$ and $g \in G$ then $\mathrm{Ad}(g)z = z$.

3.8. Recall that a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}_C$ is a subalgebra of $\mathfrak{g}_C$ such that

(1) $\mathfrak{h}$ is abelian.

(2) If $h \in \mathfrak{h}$ then $\mathrm{ad}(h)$ diagonalized on $\mathfrak{g}_C$.

(3) $\mathfrak{h}$ is maximal subject to (1), (2).

3.9. Cartan subalgebras always exist. (E.g., if $G = \mathrm{GL}(n, \boldsymbol{R})$ then $\mathfrak{g} = M_n(\boldsymbol{R})$ with $[X, Y] = XY - YX$ and $\mathfrak{g}_C = M_n(\boldsymbol{C})$. We can take $\mathfrak{h}$ to be the diagonal matrices in $\mathfrak{g}_C$.)

3.10. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}_C$. If $\alpha \in \mathfrak{h}^*$ set $(\mathfrak{g}_C)_\alpha = \{X \in \mathfrak{g}_C | [A, X] = \alpha(H)X, H \in \mathfrak{h}\}$. Set $\Delta = \{\alpha \in \mathfrak{h}^* | \alpha \neq 0, (\mathfrak{g}_C)_\alpha \neq 0\}$. Then Δ is called the root system of $(\mathfrak{g}_C, \mathfrak{h})$. We have $\sum_{\alpha \in \Delta} (\mathfrak{g}_C)_\alpha + \mathfrak{h} = \mathfrak{g}_C$.

3.11. Let B also denote the complex linear extension of B to $\mathfrak{g}_C$ (see 1.6). Then $B|_{\mathfrak{h} \times \mathfrak{h}}$ is nondegenerate. Let $\langle\ ,\ \rangle$ denote the dual bilinear form to $B|_{\mathfrak{h} \times \mathfrak{h}}$ on $\mathfrak{h}^*$. If $\alpha \in \Delta$ set $s_\alpha h = h - 2(\alpha(h)/\langle \alpha, \alpha \rangle) H_\alpha$ where $H_\alpha \in \mathfrak{h}$ is defined by $B(H_\alpha, h) = \alpha(h)$, $h \in \mathfrak{h}$. Let $W(\Delta)$ be the group generated by the s_α, $\alpha \in \Delta$, in $\mathrm{GL}(\mathfrak{h})$. $W(\Delta)$ is called the Weyl group of Δ. We let $W(\Delta)$ act on $\mathfrak{h}^*$ by $s \cdot \lambda = \lambda \circ s^{-1}$.

3.12. If $s \in W(\Delta)$ then it is well known (cf. [**11**]) that there is $g \in \mathfrak{G}^\circ$ so that $s = \mathrm{Ad}(g)|_{\mathfrak{h}}$.

3.13. A system of positive roots for Δ is a subset, Δ^+, of Δ satisfying

(1) $\Delta = \Delta^+ \cup \{-\alpha | \alpha \in \Delta^+\}$ disjoint union.

(2) If $\alpha, \beta \in \Delta^+$ and $\alpha + \beta \in \Delta$ then $\alpha + \beta \in \Delta^+$.

We note that if $\alpha, \beta \in \Delta$ then $\langle \alpha, \beta \rangle \in \boldsymbol{R}$. Let $\mathfrak{h}_{\boldsymbol{R}} = \sum \boldsymbol{R} H_\alpha$. Then $B|_{\mathfrak{h}_{\boldsymbol{R}} \times \mathfrak{h}_{\boldsymbol{R}}}$ is positive definite. Let $\mathfrak{h}'_{\boldsymbol{R}} = \{h \in \mathfrak{h}_{\boldsymbol{R}} | \alpha(h) \neq 0, \alpha \in \Delta\}$. Then $\mathfrak{h}'_{\boldsymbol{R}} \neq \emptyset$. Δ is finite. For $h \in \mathfrak{h}'_{\boldsymbol{R}}$, set $\Delta^+(h) = \{\alpha \in \Delta | \alpha(h) > 0\}$.

3.14. Let Δ^+ be a system of positive roots for Δ. Set $\mathfrak{n}^+ = \sum_{\alpha \in \Delta^+} (\mathfrak{g}_C)_\alpha$. Set $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}^+$. Then $\mathfrak{b}$ is called a Borel subalgebra of $\mathfrak{g}_C$. Set $\mathfrak{n}^- = \sum_{\alpha \in \Delta^+} (\mathfrak{g}_C)_{-\alpha}$.

3.15. By P-B-W, $U(\mathfrak{g}_C) = U(\mathfrak{h}) \oplus (\mathfrak{n}^- U(\mathfrak{g}_C) + U(\mathfrak{g}_C)\mathfrak{n}^+)$. It is easily checked that $\mathscr{Z} \subset U(\mathfrak{h}) \oplus \mathfrak{n}^- U(\mathfrak{g}_C)$ and $\mathscr{Z} \subset U(\mathfrak{h}) \oplus U(\mathfrak{g}_C)\mathfrak{n}^+$. Using these observations we see that if $p\colon \mathscr{Z} \to U(\mathfrak{h})$ is the projection into the $U(\mathfrak{h})$-factor in the above direct sum decomposition then $p\colon \mathscr{Z} \to U(\mathfrak{h})$ is an algebra homomorphism.

3.16. Let $\mu\colon U(\mathfrak{h}) \to U(\mathfrak{h})$ be defined by $\mu(1) = 1$, $\mu(h) = h - \delta(h) \cdot 1$, $\mu(h_1 h_2) = \mu(h_1)\mu(h_2)$. Set $\gamma = \mu \circ p$.

3.17. THEOREM (HARISH-CHANDRA, CF. [**1**]). (1) *The map* $\gamma\colon \mathscr{Z} \to U(\mathfrak{h})$ *is injective with image equal to* $U(\mathfrak{h})^{W(\Delta)} = \{h \in U(\mathfrak{h}) | s \cdot h = h, s \in W(\Delta)\}$.

(2) γ *is independent of the choice of* Δ^+. *Set* $\gamma = \gamma_{\mathfrak{g}_C, \mathfrak{h}}$.

(3) *If* $\chi\colon \mathscr{Z} \to \boldsymbol{C}$ *is a homomorphism such that* $\chi(1) = 1$ *then there is* $\Lambda \in \mathfrak{h}^*$ *so that* $\chi(z) = \Lambda(\gamma(z))$, $z \in \mathscr{Z}$. *Here* $\Lambda\colon U(\mathfrak{h}) \to \boldsymbol{C}$ *is the extension to* $U(\mathfrak{h})$ *of the linear map* $h \to \Lambda(h)$ *of* $\mathfrak{h}$ *into* $\boldsymbol{C}$.

(4) *Set for* $\Lambda \in \mathfrak{h}^*$, $\chi_\Lambda(z) = \Lambda(p(z))$, $z \in \mathscr{Z}$. *Then* $\chi_\Lambda = \chi_{\Lambda'}$ if *and only if* $\Lambda' = s\Lambda$ *for some* $s \in W(\Delta)$.

3.18. The hard part of Theorem 3.17 is (1). (3) follows from the fact that $U(\mathfrak{h})$ is integral over $U(\mathfrak{h})^{W(\Delta)}$. (4) is just a restatement of (1).

3.19. Let V be an admissible $(\mathfrak{g}, K)$-module. Then $V = \bigoplus_{\gamma\in\mathscr{E}(K)} V(\gamma)$. If $z \in \mathscr{Z}$ then $z \cdot V(\gamma) \subset V(\gamma)$, $\gamma \in \mathscr{E}(K)$. Since $\dim V(\gamma) < \infty$ we see that $\mathscr{Z}|_{V(\gamma)}$ can be put in upper triangular form.

3.20. Set for a homomorphism $\chi: \mathscr{Z} \to \boldsymbol{C}$,

$$V_\chi = \{v \in V | (z-\chi(z))^d v = 0 \text{ for } z \in \mathscr{Z},\ d \text{ depending only on } v\}.$$

The discussion in 3.19 implies that V is the $(\mathfrak{g}, K)$-module direct sum of the V_χ's.

3.21. If V is irreducible and $V_\chi \neq 0$, then $V_\chi = V$. Let $W = \{v \in V | z \cdot v = \chi(z)v,\ z \in \mathscr{Z}\}$. Since $V_\chi = V$ it is clear that $W \neq 0$. But W is clearly a $(\mathfrak{g}, K)$-submodule of V. Hence $W = V$. χ is called the infinitesimal character of the $(\mathfrak{g}, K)$-module V.

3.22. THEOREM (CHEVALLEY, HARISH-CHANDRA, CF. **[22]**). *Let w be the order of $W(\Delta)$. Then there exist elements $u_1, \cdots, u_w$ of $U(\mathfrak{h})$ so that $U(\mathfrak{h}) = \sum u_i U(\mathfrak{h})^{W(\Delta)}$.*

3.23. THEOREM (HARISH-CHANDRA, CF. **[22]**). *If V is a $(\mathfrak{g}, K)$-module, finitely generated as a $U(\mathfrak{g})$-module, and if $V = V_\chi$ for some χ, then V is admissible. Furthermore, if V is nonzero, then V has a nonzero irreducible quotient.*

4. The principal series.

4.1. Let G be as in 1.2. Let $P \subset G$ be a subgroup of G. Then P is said to be a parabolic subgroup of G if

(1) P is its own normalizer in G.

(2) $\mathfrak{p}_C$ contains a Borel subalgebra of $\mathfrak{g}_C$ (see 3.14).

4.2. Let $\mathfrak{a}_0$ be a maximal abelian subalgebra of $\mathfrak{g}$ contained in $\mathfrak{k}^\perp = \{X \in \mathfrak{g} | B(X, \mathfrak{k}) = 0\}$. If $h \in \mathfrak{a}_0$ then $\operatorname{ad} h$ diagonalizes on $\mathfrak{g}$ (indeed if $h \in \mathfrak{a}_0$ then ${}^t h = h$). Let, for $\lambda \in \mathfrak{a}_0^*$, $\mathfrak{g}^\lambda = \{X \in \mathfrak{g} | \operatorname{ad} h \cdot X = \lambda(h)X,\ h \in \mathfrak{a}_0\}$. Set $\Lambda = \{\lambda \in \mathfrak{a}_0^* | \mathfrak{g}^\lambda \neq (0), \lambda \neq 0\}$. Fix $h_0 \in \mathfrak{a}_0$ so that $\lambda(h_0) \neq 0$, $\lambda \in \Lambda$. Set $\Lambda^+ = \{\lambda \in \Lambda | \lambda(h_0) > 0\}$. Set $\mathfrak{n}_0 = \sum_{\lambda\in\Lambda^+} \mathfrak{g}^\lambda$. Set M_0 equal to the centralizer of $A_0 = \exp \mathfrak{a}_0$ in G. Then $P_0 = M_0 N_0$ is a parabolic subgroup of G. If P is a parabolic subgroup of G then there is $g \in G$ so that $gP_0g^{-1} \subset P$.

4.3. The Iwasawa decomposition of G says that $G = KP_0$ if P_0 is as above. Thus if P is a parabolic subgroup of G then $P \supset kP_0k^{-1}$ for some $k \in K$. Hence $G = KP$.

4.4. Fix (P_0, A_0) as in 4.2. Then (P_0, A_0) is known as a minimal parabolic pair. ${}^\circ M_0 = M_0 \cap K$ (see 1.5). A_0 is a split component of M_0 chosen as in 1.6.

4.5. Let P be a parabolic subgroup of G. Let N be the unipotent radical of P. Let $M = P \cap \theta P$. Then $P = MN$. $M = {}^\circ M \cdot A$ with A a split component of M chosen as in 1.6. (P, A) is said to be a parabolic pair or p-pair. (P, A) is said to be standard if $A \subset A_0$ and $P \supset P_0$. Each p-pair is conjugate in G to a *unique* standard p-pair.

4.6. If (P, A) is a p-pair, $P = MN$, then M satisfies the conditions of 1.2. Suppose that (P, A) is standard. Then $M \cap K$ plays the role of K for G. The notion of admissible will be relative to $M \cap K$.

4.7. Let (P, A) be a standard p-pair. $P = MN = {}^\circ MAN$. Let (σ, H_σ) be a finitely generated admissible representation of M. Let δ_p denote the modular function of MN. Then $\delta_P({}^\circ man) = a^{2\rho_P} = e^{2\rho_P(H)}$ if $a = \exp H$ and $2\rho_P(H) = \operatorname{tr}(\operatorname{ad} H|_{\mathfrak{n}})$, $H \in \mathfrak{a}$.

4.8. We form a representation $(\pi(P, \sigma), H^{P,\sigma})$ of G as follows:

(1) Set $H^{P,\sigma}$ equal to the space of all measurable functions f: $K \to H_\sigma$ such that $f(km) = \sigma(m)^{-1}f(k)$, $m \in K \cap M$, $k \in K$ and $\|f\|^2 = \int_K \|f(k)\|^2\, dk < \infty$.

(2) If $f \in H^{P,\sigma}$ extend f to G by $f(kp) = \delta_p(p)^{-1/2}\, \sigma(p)^{-1}f(k)$, for $p \in P$, $k \in K$. Set $(\pi(g)f)(x) = f(g^{-1}x)$ $(\pi = \pi(P, \sigma))$.

4.9. THEOREM. $(\pi(P, \sigma), H^{P,\sigma}) = I(P, \sigma)$ *is an admissible representation of* G.

4.10. Theorem 4.9 can be derived from Theorem 3.23.

4.11. Set $V(P, \sigma)$ equal to the corresponding admissible $(\mathfrak{g}, K)$-module (i.e., $V(P, \sigma) = V_\pi$, $\pi = \pi(P, \sigma)$). Let V_σ denote the $(\mathfrak{m}, K \cap M)$-module corresponding to σ.

4.12. If $f \in V(P, \sigma)$, f is real analytic by Theorem 2.8(2). We define $j(f) = f(1)$, $f \in V(P, \sigma)$. Then it is easy to see

(1) j: $V(P, \sigma) \to V_\sigma$ is a surjective $(\mathfrak{m}, K \cap M)$-module homomorphism.

(2) $\operatorname{Ker} j \supset \mathfrak{n} \cdot V(P, \sigma)$.

This implies that V_σ is a quotient of the $(\mathfrak{m}, K \cap M)$-module $V(P, \sigma)/\mathfrak{n}V(P, \sigma)$.

4.13. The following result can be proved using Theorems 3.22, 3.23, the subquotient theorem (Harish-Chandra [3], Lepowsky [16], Rader [18]) combined with a technique of Casselman using the asymptotic expansion of matrix entries of admissible representations (cf. Miličić [17]). For a proof that does not use asymptotic expansions, see [21].

4.14. THEOREM. *Let* V *be an admissible* $(\mathfrak{g}, K)$*-module. Let* (P, A) *be a standard p-pair,* $P = MN$. *Then*

(1) $V \neq \mathfrak{n}V$.

(2) *If* V *is finitely generated then* $V/\mathfrak{n}V$ *is a finitely generated* $(\mathfrak{m}, M \cap K)$*-module.*

4.15. The particular case of 4.14, $(P, A) = (P_0, A_0)$ has an interesting consequence, ${}^\circ M_0 = M_0 \cap K$. Hence a finitely generated, admissible $(\mathfrak{m}, K \cap M)$-module is finite dimensional.

4.16. Let V be a nonzero admissible, finitely generated $(\mathfrak{g}, K)$-module. Then $1 \leqq \dim V/\mathfrak{n}_0 V < \infty$ by the observation of 4.15. Let W be a nonzero irreducible quotient of $V/\mathfrak{n}_0 V$ as an $(\mathfrak{m}, K \cap M_0)$-module. Then $\mathfrak{a}_0$ acts by $\nu \in (\mathfrak{a}_0^*)_C$. We can therefore look upon W as a representation (σ, H_σ) of M_0. Set $\xi = \delta_{P_0}^{-1/2}\sigma$. Let μ: $V \to W$ be the composition of natural maps. Define, for $v \in V$, $C(v)(k) = \mu(k^{-1} \cdot v)$. Then clearly, C: $V \to H^{P_0,\xi} = H$ and $C(k \cdot v) = k \cdot C(v)$.

4.17. LEMMA. *Set* $\pi = \pi_{P_0,\xi}$. *Then* $C((Xv)) = \pi(X)C(v)$ *for* $v \in V$, $X \in \mathfrak{g}$.

PROOF. We note $C(k \cdot X \cdot v) = C(\operatorname{Ad}(k)X \cdot k \cdot v)$. Hence $\pi(k)C(X \cdot v) = C(\operatorname{Ad}(k)X \cdot kv)$. Thus it is enough to show that

(1) $C(X \cdot v)(1) = (\pi(X)C(v))(1)$, for $X \in \mathfrak{g}$, $v \in V$. (1) is not hard in light of

(2) $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$, and the discussion in 4.12.

4.18. THEOREM (THE SUBREPRESENTATION THEOREM OF CASSELMAN). *If* V *is an* *irreducible admissible* ($\mathfrak{g}$, K)*-module then* V *is isomorphic with a* ($\mathfrak{g}$, K)*-submodule of* $V(P_0, \sigma)$ *for an appropriate irreducible representation of* M_0.

PROOF. Let C be as in 4.16. Then $C \neq 0$ and hence C is injective.

4.19. COROLLARY. *Let* V *be an irreducible, admissible* ($\mathfrak{g}$, K)*-module. Then* V *is isomorphic with* V_π *where* (π, H) *is an irreducible, admissible representation of* G.

4.20. Let (P, A) be a standard p-pair, $P = MN$. Let V be an admissible, irreducible ($\mathfrak{g}$, K)-module. Then $V/\mathfrak{n}V$ is a finitely generated, nonzero, admissible ($\mathfrak{m}$, $K \cap M$)-module. Hence $V/\mathfrak{n}V$ has an irreducible, nonzero quotient. Corollary 4.19 implies that this quotient is of the form V_σ for (σ, H_σ) an irreducible admissible representation of M. Arguing as in 4.16 and Lemma 4.17 we have:

4.21. THEOREM. *Let* V *be an irreducible, admissible* ($\mathfrak{g}$, K)*-module. Let* (P, A) *be a standard p-pair* $P = MN$. *Then*

(1) $V/\mathfrak{n}V$ *has a nonzero irreducible* ($\mathfrak{m}$, $K \cap M$)*-module quotient.*

(2) *If* W *is an irreducible* ($\mathfrak{m}$, $K \cap M$)*-module quotient of* $V/\mathfrak{n}V$ *then* $W = V_\sigma$ *and* V *is isomorphic with a* ($\mathfrak{g}$, K)*-submodule of* $V(P, \delta_P^{-1/2}\sigma)$.

5. Exponents.

5.1. We retain the notation of §4.

5.2. Let (P, A), $P = MN$ be a standard p-pair. Then $M \cap P_0 = {}^*P_0$ is a minimal parabolic subgroup of M. Then ${}^*P_0 = {}^\circ M_0 A_0 {}^*N_0$, ${}^*M_0 = {}^\circ M_0 A_0$. $({}^*P_0, A_0)$ will be used for M in the same way as (P_0, A_0) is used for G. We note that $N_0 = {}^*N_0 N$.

5.3. If V is an admissible, finitely generated ($\mathfrak{g}$, K)-module then it is easy to see that

(1) $V/\mathfrak{n}_0 V = (V/\mathfrak{n}V)/{}^*\mathfrak{n}_0(V/\mathfrak{n}V)$.

5.4. Let V be as in 5.3. Let (P, A) be a standard p-pair. Set $E(P, V_j) = \{\nu - \rho_P | \nu$ a weight of $\mathfrak{a}$ on $V/\mathfrak{n}V\}$. 5.3(1) implies that $E(P, V) = \{\mu|_\mathfrak{a} \,|\, \mu \in E(P_0, V)\}$.

5.5. If (P, A) is a p-pair let $\Sigma(P, A)$ be the set of weights of $\mathfrak{a}$ on $\mathfrak{n}$. Then $\Sigma(P, A)$ is called the set of roots of (P, A). An element $\lambda \in \Sigma(P, A)$ is said to be simple if λ cannot be written in the form $\lambda_1 + \cdots + \lambda_r$, $r \geqq 2$, with $\lambda_i \in \Sigma(P, A)$, $i = 1, \cdots, r$. Set ${}^\circ\Sigma(P, A)$ equal to the set of simple roots of $\Sigma(P, A)$.

5.6. Set ${}^\circ\Sigma_0(P_0, A_0) = \{\alpha_1, \cdots, \alpha_l\}$. Let $H_1, \cdots, H_l \in \mathfrak{a}_0$ be defined by $\alpha_i(H_j) = \delta_{ij}$ and $H_i \perp \mathfrak{z}$ ($\mathfrak{z}$ the Lie algebra of Z, the split component of G chosen as in 1.6). Set $D_0 = \sum \boldsymbol{R} H_i$. Then $\mathfrak{a}_0 = \mathfrak{z} \oplus \mathfrak{D}_0$, orthogonal direct sum.

5.7. Let $\beta_i \in D_0^*$ be defined by $\langle \beta_i, \alpha_j \rangle = \delta_{ij}$. If $F \subset \{1, \cdots, l\}$ set $D_0^*(F) = \{\nu \in D_0^* | \nu = -\sum_{i \notin F} x_i \beta_i + \sum_{i \in F} y_i \alpha_i,\ x_i > 0,\ y_i \geqq 0\}$.

5.8. LEMMA (LANGLANDS **[15]**). D_0^* *is the disjoint union of the* $D_0^*(F)$.

5.9. If $\nu \in D_0^*(F)$ set $\nu^\circ = -\sum_{i \notin F} x_i \beta_i$ if $\nu = -\sum_{i \notin F} x_i \beta_i + \sum_{i \in F} y_i \alpha_i$. If $\alpha, \beta \in D_0^*$ we say $\alpha \succeq \beta$ if $\alpha - \beta = \sum z_i \alpha_i$, $z_i \in \boldsymbol{R}$, $z_i \geqq 0$, $i = 1, \cdots, l$.

5.10. LEMMA (LANGLANDS **[15]**). *If* $\lambda, \nu \in D_0^*$ *and* $\lambda \succeq \nu$ *then* $\lambda^\circ \succeq \nu^\circ$.

5.11. Let $\mathscr{E}_t(G)$ denote the set of equivalence classes of irreducible, admissible ($\mathfrak{g}$, K)-modules V, such that if $\nu \in E(V, P_0)$ then $\mathrm{Re}(\nu|_{D_0}) \in D_0^*(\varnothing)$.

5.12. If $F \subset \{1, \cdots, l\}$ then we can construct a standard p-pair (P_F, A_F) as follows:

Set $\mathfrak{a}_F = \{H \in \mathfrak{a}_0 | \alpha_i(H) = 0, i \in F\}$. Set $\Sigma_F = \{\lambda \in \Sigma(P_0, A_0) | \lambda|_{\mathfrak{a}_F} \neq 0\}$. Set $\mathfrak{n}_F = \sum_{\lambda \in \Sigma_F} (\mathfrak{n}_0)_\lambda$. Set M_F equal to the centralizer in G of $A_F = \exp \mathfrak{a}_F$. Then $M_F N_F = P_F$ and (P_F, A_F) is a standard p-pair. It is well known (cf. **[22]**) that every standard p-pair is of the form (P_F, A_F).

5.13. LEMMA. *Let V be an irreducible, admissible* $(\mathfrak{g}, K)$*-module. Then there exist* $F \subset \{1, \cdots, l\}$, *and an irreducible, admissible representation* (σ, H_σ) of M_F, *so that if* $\nu \in E(\sigma, {}^*P_0)$ *then* $\operatorname{Re}(\nu|_{D_0}) \in D_0^*(F)$ *and V is isomorphic with a* $(\mathfrak{g}, K)$*-submodule of* $V(P_F, \sigma)$.

Note. In particular, $V_\sigma \in \mathscr{E}_t(M_F)$.

PROOF. Let $L(V, P_0) = \{\operatorname{Re} \nu|_{D_0} \mid \nu \in E(V, P_0)\}$. Let $\mu \in L(V, P_0)$ be a minimal element. Then $\mu \in D_0^*(F)$ for a unique $F \subset \{1, \cdots, l\}$. Let (P, A) be the corresponding standard p-pair. Since $V/\mathfrak{n}_0 V = (V/nV)/{}^*\mathfrak{n}_0(V/\mathfrak{n}V)$ we see that there is $\eta \in E(V/\mathfrak{n}V, {}^*P_0)$ so that $\operatorname{Re} \eta|_{D_0} = \mu$. This implies that there is a quotient W of $V/\mathfrak{n}V$ so that $\eta \in E(W, {}^*P_0)$. If $\eta' \in E(W, {}^*P_0)$ then $\eta'|_\mathfrak{a} = \eta|_\mathfrak{a}$. Thus if $\mu = -\sum_{i \notin F} x_i \beta_i + \sum_{i \in F} y_i \alpha_i$, $x_i > 0$, $i \in F$, $y_i \geqq 0$ then $\operatorname{Re} \eta'|_{D_0} = -\sum_{i \notin F} x_i \beta_i + \sum_{i \in F} z_i \alpha_i$ with $z_i \in \boldsymbol{R}$. If we show that $z_i \geqq 0$ we will have completed the proof of the lemma. Suppose only $z_{i_1}, \cdots, z_{i_r} < 0$. Then $\operatorname{Re} \eta'|_{D_0} \leqq -\sum_{i \notin F} x_i \beta_i + \sum_{i \in F - \{i_1, \cdots, i_r\}} z_i \alpha_i$. Set $\mu' = \operatorname{Re} \eta'|_{D_0}$, $\mu'' = -\sum_{i \notin F} x_i \beta_i + \sum_{i \in F - \{i_1, \cdots, i_r\}} z_i \alpha_i$. Then $\mu' \preceq \mu''$ and hence ${}^\circ\mu' \preceq {}^\circ\mu''$. But ${}^\circ\mu'' = {}^\circ\mu$. Hence ${}^\circ\mu' \preceq {}^\circ\mu$. But μ was assumed to be minimal. Hence ${}^\circ\mu = {}^\circ\mu'$. This implies the result.

5.14. Let $\mathscr{E}_d(G)$ be the set of equivalence classes of irreducible, admissible $(\mathfrak{g}, K)$-modules V such that if $\nu \in E(V, P_0)$ then $\operatorname{Re} \nu(H_i) > 0$ for $i = 1, \cdots, l$.

5.15. LEMMA. *Let V be an irreducible* $(\mathfrak{g}, K)$*-module such that the equivalence class of V is in $\mathscr{E}_t(G)$ but is not in $\mathscr{E}_d(G)$. Then there are a standard p-pair, (P, A), and an irreducible admissible representation (σ, H_σ) of M so that*

(1) *if $a \in A$ then $\sigma(a) = \chi(a)I$ and if $H \in \mathfrak{a} \cap D_0$ then $|\chi(\exp H)| = 1$,*

(2) *$V_\sigma \in \mathscr{E}_d(M)$, and V is isomorphic with a subrepresentation of $V(P, \sigma)$.*

5.16. The proof of this lemma is essentially the same as the proof of Lemma 5.13.

6. The discrete series and the Langlands classification.

6.1. We maintain the notation of §5. Let (π, H) be an irreducible unitary representation of G. If $v, w \in H$ then $|\langle \pi(g)v, w \rangle| = \varphi_{v,w}(g)$ satisfies

(1) $\varphi_{v,w}(gz) = \varphi_{v,w}(g)$ for $g \in G$, $z \in Z$.

This is because $\pi(z) = \chi(z)I$, χ a unitary character of Z.

(π, H) is said to be square-integrable if $\varphi_{v,w} \in L^2(G/Z)$ for all $v, w \in H$.

6.2. Let $\mathscr{E}_2(G)$ denote the set of isomorphism classes of the V_π, (π, H) an irreducible square-integrable representation of G.

6.3. THEOREM (HARISH-CHANDRA **[6]**, CASSELMAN, CF. MILIČIĆ **[17]**). (1) $\mathscr{E}_2(G) \subset \mathscr{E}_d(G)$.

(2) *If V is a representative of an element of $\mathscr{E}_d(G)$ then there is a 1-dimensional* $(\mathfrak{g}, K)$*-module W so that the class of $V \otimes W \in \mathscr{E}_2(G)$.*

6.4. Theorem 6.3 is proved using the relationship between the elements of $E(V, P_0)$ and the leading exponents of V in the sense of Harish-Chandra (cf. Warner [**22**, Chapter 9]). This relationship is due to Harish-Chandra and Casselman (see also Miličić [**17**]).

6.5. COROLLARY. *If V is a representative of an element of $\mathscr{E}_t(G)$ then there is a 1-dimensional* $(\mathfrak{g}, K)$*-module W so that $V \otimes W$ is unitary.*

6.6 This result follows from the fact that $V(P, \sigma)$ is unitary if σ is unitary and 5.15.

6.7. THEOREM (LANGLANDS [**15**]). *Let $F \subset \{1, \cdots, l\}$ (see 5.6, 5.7). Let (σ, H_σ) be an irreducible admissible representation of M_F so that if $\nu \in E(V_\sigma, {}^*(P_F)_0)$ then* $\mathrm{Re}\, \nu|_{D_0} \in D_0^*(F)$. *Then $V(P_F, \sigma)$ has a unique nonzero, irreducible, submodule denoted $J(P_F, \sigma)$. Furthermore if $F' \subset \{1,\cdots, l\}$ and σ' satisfies the above properties for $M_{F'}$ then $J(P_F, \sigma)$ is isomorphic with $J(P_{F'}, \sigma')$ if and only if $F' = F$ and σ is equivalent with σ'.*

6.8. The first part of Theorem 6.7 is proved by using the relationship between intertwining operators and certain limit formulae. The second statement is proved using the limit formulae alluded to above and an argument similar to the proof of Lemma 5.13.

6.9. Theorem 6.7 combined with Harish-Chandra's classification of the elements of $\mathscr{E}_2(G)$ (see [**6**]), Lemma 5.13 and Lemma 5.15 give the complete classification of irreducible $(\mathfrak{g}, K)$-modules.

7. The principal series for GL(2, ***R***) **and** GL(2, ***C***).

7.1. Let $G = \mathrm{GL}(2, F)$, $F = \boldsymbol{R}$ or $\boldsymbol{C}$. We take θ, B, K, etc. as in 1.8. We take P_0 to be the group of upper triangular matrices in G and $\bar{P}_0$ to be the group of lower triangular matrices in G. M_0 is then the group of all diagonal matrices in G.

7.2. We denote by $[h_1, h_2]$ the diagonal matrix with diagonal entries h_1, h_2. We also use the notation

$$n(x) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \bar{n}(x) = \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}.$$

7.3. We note that $\sum(P_0, A_0) = \{\lambda\}$ and that $\sum(\bar{P}_0, A_0) = \{-\lambda\}$ with $\lambda([x_1, x_2]) = x_1 - x_2$.

7.4. Up to normalization of Haar measure on N_0 (resp. $\bar{N}_0$), $dn(x) = dx$ (resp. $d\bar{n}(x) = dx$) where dx denotes Lebesgue measure on F.

7.5. If $F = \boldsymbol{C}$ (resp. $\boldsymbol{R}$) we define for $k_1, k_2 \in \boldsymbol{Z}$ (resp. $k_1, k_2 \in \{0, 1\}$), $z_1, z_2 \in \boldsymbol{C}$, $\xi_{k_1, k_2, z_1, z_2}([h_1, h_2]) = h_1^{k_1} h_2^{k_2} |h_1|^{z_1 - k_1} |h_2|^{z_2 - k_2}$. Then ξ_{k_1, k_2, z_1, z_2} is the most general irreducible admissible representation of M_0. We extend ξ_{k_1, k_2, z_1, z_2} to P_0 (resp. $\bar{P}_0$) by making it take the value 1 on $n(x)$ (resp. $\bar{n}(x)$).

7.6. LEMMA. *If k_1, k_2, z_1, z_2 are as in 7.5 and if* $\mathrm{Re}(z_1 - z_2) < 0$, *if $f \in V(\bar{P}_0, \xi_{k_1, k_2, z_1, z_2})$ then*

$$(A(k_1, k_2, z_1, z_2)f)(k) = \int_{N_0} f(kn)\, dn = A(f)$$

converges absolutely and uniformly for $k \in K$ and defines a $(\mathfrak{g}, K)$*-module homomorphism of $V(\bar{P}_0, \sigma)$ into $V(P_0, \sigma)$ $(\sigma = \xi_{k_1, k_2, z_1, z_2})$.*

PROOF. $(A(f))(k) = \int_{N_0} \delta_{\bar{P}_0}(n)^{-1/2} \sigma(a(n))^{-1} f(k(kn))\, dn$ where $\delta_{\bar{P}_0}(k\bar{p}) = \delta_{\bar{P}_0}(\bar{p})$, $k \in K$, $\bar{p} \in \bar{P}_0$ and if $g \in G$ then $g = k(g)a(g)n$ with $k(g) \in K$, $a(g) \in A_0$ and $n \in N_0$.

Now, $|f(k(kn))| \leqq \|f\|_\infty$ ($\| \dots \|_\infty$ is the L_∞-norm). Thus to prove the absolute and uniform convergence we must only show that

$$\text{(I)} \qquad \int_{N_0} \delta_{\bar{P}_0}(n)^{-1/2} \left|\sigma(a(n))\right|^{-1} dn < \infty.$$

Writing n as $n(x)$ a direct computation gives

$$\text{(II)} \qquad a(n(x)) = [(1 + x^2)^{-1/2}, (1 + x^2)^{1/2}].$$

Set $d = \dim_{\boldsymbol{R}} F$. Then (II) implies that the integrand in (I) is given by

$$\text{(III)} \qquad (1 + |x|^2)^{-d/2 + \mathrm{Re}(z_1 - z_2)/2}$$

which is integrable on F if $\mathrm{Re}(z_1 - z_2) < 0$.

The fact that $Af \in V(P_0, \sigma)$ is proved by a simple change of variables. The uniform convergence of the integral implies that A is a $(\mathfrak{g}, K)$-module homomorphism.

7.7. We now focus our attention on the case $F = \boldsymbol{R}$. In this case $K = O(2)$. We use the notation ε_i for elements of $\{0, 1\}$.

7.8. Set $\eta = \mathrm{diag}(1, -1) \in K$ and set

$$k(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \qquad \theta \in \boldsymbol{R}.$$

$SO(2) = \{k(\theta) | \theta \in \boldsymbol{R}\}$, $K = SO(2) \cup \eta SO(2)$. $\eta k(\theta)\eta^{-1} = k(-\theta)$.

7.9. We now describe $\mathscr{E}(K)$. Let W be an irreducible finite dimensional K-module. Then there is $w_0 \in W$ so that $k(\theta) \cdot w_0 = e^{il\theta} w_0$, $\theta \in \boldsymbol{R}$ and $l \in Z$, fixed. $k(\theta)\eta w_0 = e^{-ilw_0}\eta w_0$. Thus if $l \neq 0$, w is completely determined by l. Since we can replace w_0 by ηw_0 we can take $l > 0$. Let W_l, $l = 1, 2, \cdots$, denote the K-module we just described. If $l = 0$, set $W_0 = \{w \in W | k(\theta)w = w, \theta \in \boldsymbol{R}\}$. Then $\eta \cdot W_0 = W_0$. Hence since $\eta^2 = 1$ we see $W = W_0$, $\dim W = 1$ and $\eta w = w$, $w \in W$ or $\eta w = -w$, $w \in W$. We use the notation W_0^0 for the trivial representation of K and W_0^1 for the case $\eta \to -1$.

7.10. We leave it to the reader to check that $\mathrm{Ad}|_K$ on $\mathfrak{g}_C$ is a direct sum of two copies of W_0^0 and one copy of W_2.

7.11. We note that

(1) $W_l \otimes W_k = W_{l+k} \oplus W_{|l-k|}$ for $l, k > 0$.

(2) $W_l \otimes W_0^\varepsilon = W_l$, $\varepsilon = 0$ or 1.

(3) $W_0^\varepsilon \otimes W_0^{\varepsilon'} = W_0^{\varepsilon''}$ with $\varepsilon'' = \varepsilon + \varepsilon' \bmod 2$.

7.12. Let γ_l denote the equivalence class of W_l, $l > 0$. Let γ_0^ε denote the equivalence class of W_0^ε, $\varepsilon = 0, 1$.

7.13. Using the formulas of 7.5 we see that if V is an admissible $(\mathfrak{g}, K)$-module then

(1) $\mathfrak{g} \cdot V(\gamma_l) \subset V(\gamma_{l+2}) + V(\gamma_{|l-2|}) + V(\gamma_l)$ if $l > 0$.

(2) $\mathfrak{g} \cdot V(\gamma_0^\varepsilon) \subset V(\gamma_0^\varepsilon) + V(\gamma_2)$, $\varepsilon = 0, 1$.

7.14. Since (P_0, A_0) is a minimal p-pair we can take ${}^\circ M_0 = M_0 \cap K = \{[\eta_1, \eta_2] | \eta_i = \pm 1, i = 1, 2\}$. Let $V = V(\xi_{\varepsilon_1, \varepsilon_2, z_1, z_2}, P_0)$. $\xi_{\varepsilon_1, \varepsilon_2, z_1, z_2}|_{{}^\circ M_0} = \xi_{\varepsilon_1, \varepsilon_2}$ with

$\xi_{\varepsilon_1,\varepsilon_2}([\eta_1, \eta_2]) = \eta_1^{\varepsilon_1}\eta_2^{\varepsilon_2}$. Frobenius reciprocity implies that $\dim V(\gamma)$ is the number of times $\xi_{\varepsilon_1,\varepsilon_2}$ occurs in $\gamma|_{{}^\circ M_0}$.

7.15. We leave it to the reader to check that if $\tau_l \in \gamma_l$, $l = 1, 2, \cdots$, $\tau_0^\varepsilon \in \gamma_0^\varepsilon$, $\varepsilon = 0, 1$, then

(1) $\tau_0^0|_{{}^\circ M_0} = \xi_{0,0}$.
(2) $\tau_0^1|_{{}^\circ M_0} = \xi_{1,1}$.
(3) $\tau_l|_{{}^\circ M_0} = \xi_{0,0} \oplus \xi_{1,1}$ if l is even, $l > 0$.
(4) $\tau_l|_{{}^\circ M_0} = \xi_{1,0} \oplus \xi_{0,1}$ if l is odd.

7.16. Let $V = V(\xi_{\varepsilon_1,\varepsilon_2,z_1,z_2}, P_0)$. Then 7.15 implies $\dim V(\gamma) \leqq 1$ for $\gamma \in \mathscr{E}(K)$ and

(I) If $\varepsilon_1 + \varepsilon_2$ is even then $V = V(\gamma_0^{\varepsilon_1}) \oplus \bigoplus_{l>0} V(\gamma_{2l})$.
(II) If $\varepsilon_1 + \varepsilon_2$ is odd then $V = \bigoplus_{l\geqq 0} V(\gamma_{2l+1})$.

7.17. LEMMA. *Set, for z in $\boldsymbol{C}$, k in $\boldsymbol{Z}$*

$$a_k(z) = \pi^{1/2}\Gamma(z/2)\Gamma((z+1)/2)/\Gamma((1+z+k)/2)\Gamma((1+z-k)/2)$$

where Γ is the gamma function (cf. [23]).

Set $\sigma = \xi_{\varepsilon_1,\varepsilon_2,z_1,z_2}$. Then if $V = V(\bar{P}_0, \sigma)$ then as a K-module V is isomorphic with $V(P_0, \sigma)$. If $\operatorname{Re}(z_1 - z_2) < 0$ *set $A = A(\varepsilon_1, \varepsilon_2, z_1, z_2)$. Let us denote by γ_0 either γ_0^1 or γ_0^0. If $f \in V(\gamma_k)$ then $Af = a_k(z_2 - z_1)f$.*

PROOF. 7.16 implies that A is a scalar on each $V(\gamma)$. Using the formula at the beginning of the proof of Lemma 7.6 and an easy computation, it follows that, on $V(\gamma_k)$, A is given by the following scalar:

$$\int_{-\infty}^{\infty} ((1+ix)/(1+x^2)^{1/2})^{-k}(1+x^2)^{(-1+z_1-z_2)/2}\,dx. \tag{1}$$

In (1) use the change of variables $e^{it} = (1+ix)/(1+x^2)^{1/2}$, $-\pi/2 \leqq t \leqq \pi/2$. Then the integral becomes

$$2\int_0^{\pi/2} \cos(kt)(\cos(t))^{-(1+z_1-z_2)}\,dt. \tag{2}$$

The lemma now follows from Whittaker and Watson [23, p. 263, Exercise 39] and the duplication formula for the gamma function [23, p. 240].

7.18. We now concentrate our attention on the case $F = \boldsymbol{C}$. For the remainder of this section G will denote $GL(2, \boldsymbol{C})$ and K will denote $U(2)$.

7.19. As is well known, the map $S^1 \times SU(2) \to U(2)$ (S^1 the circle group) given by $z, k \mapsto zk$ is a two-fold covering of $U(2)$.

If (τ, V) is an irreducible unitary representation of K and if z is in S^1 then $\tau(zI) = z^kI$ for some k depending only on τ. Hence the restriction of τ to $SU(2)$ is irreducible.

The irreducible, unitary representations of $SU(2)$ are determined up to equivalence by their dimension. If (τ, V) is the $d + 1$ dimensional representation of $SU(2)$ then it is well known that $\tau(\varepsilon I) = \varepsilon^d I$, $\varepsilon = \pm 1$. Hence $k + d$ is even.

7.20. We have completely described $\mathscr{E}(K)$. $\mathscr{E}(K)$ is the set of all $\gamma_{m,n}$ with $m \in \boldsymbol{Z}$, $n \in \boldsymbol{Z}$ and $n \geqq 0$ and $m + n$ even. Here $\gamma_{m,n}$ denotes the class of all irreducible (τ, W) with $\dim W = n + 1$ and $\tau(zI) = z^mI$ for $z \in S^1$.

7.21. Let $\xi_{k_1, k_2} = \xi_{k_1, k_2, z_1, z_2}|_{{}^\circ M_0}$. It is an easy consequence of the classification of representations of $SU(2)$ that

$$\gamma_{m,n}|_{{}^\circ M_0} = \bigoplus_{j=0}^{n} \xi_{(n+m-2j)/2,\,(m-n+2j)/2}.$$

7.22. Using 7.21 and Frobenius reciprocity we find that if $V = V(\bar{P}_0, \xi_{k_1, k_2, z_1, z_2})$ then

$$V = \bigoplus_{j=0} V(\gamma_{k_1+k_2,\, |k_1-k_2|+2j});$$

furthermore each $V(\gamma)$ is irreducible as a representation of K.

7.23. LEMMA. *Set, for* $0 \leqq r \leqq n$, *r*, *n integers and for z in* $\boldsymbol{C}$

$$b_{n,r}(z) = 2\pi(z+n)^{-1} \prod_{j=1}^{r} (z - n + 2(j-1))/(z + n - 2j).$$

If $V = V(\bar{P}_0, \xi_{k_1, k_2, z_1, z_2})$ *and if* $\operatorname{Re}(z_1 - z_2) < 0$ *then*

$$A(k_1, k_2, z_1, z_2)|_{V(\gamma_{k_1+k_2,\, |k_1+k_2|+2j})} = b_{|k_1-k_2|+2j,\, j}(z_2 - z_1)I.$$

7.24. This lemma is proved using an argument similar to the proof of Lemma 7.17. The proof is more complicated and even less enlightening. It is therefore omitted.

7.25. Lemmas 7.17 and 7.23 imply that the $A(k_1, k_2, z_1, z_2)$ originally defined for $\operatorname{Re}(z_1 - z_2) < 0$ have a meromorphic continuation to $\boldsymbol{C} \times \boldsymbol{C}$. The reader is advised to consult the lectures of Knapp and Zuckerman for more information on these intertwining operators.

8. The representations of $\mathrm{GL}(2, \boldsymbol{R})$ **and** $\mathrm{GL}(2, \boldsymbol{C})$**.**

8.1. We retain the notation of §7. We first look at the case of $G = \mathrm{GL}(2, \boldsymbol{R})$.

8.2. LEMMA. $V = V(P_0, \xi_{\varepsilon_1, \varepsilon_2, z_1, z_2})$ *is reducible if and only if*

(1) $k = z_1 - z_2 \in \boldsymbol{Z} - \{0\}$.

(2) $\varepsilon_1 + \varepsilon_2 + 1 = 0 \bmod 2$.

If $V(P_0, \xi_{\varepsilon_1, \varepsilon_2, z_1, z_2})$ *is reducible then*

(a) *If* $z_1 - z_2 = -k - 1$, $k \geqq 0$, $\varepsilon_1 + \varepsilon_2 = k \bmod 2$, *then* $J(P_0, \xi_{\varepsilon_1, \varepsilon_2, z_1, z_2}) = J$ *is finite dimensional and indeed* $\dim J = k + 1$ *and* V/J *is irreducible.*

(b) *If* $z_1 - z_2 = k + 1$, $k \geqq 0$, $\varepsilon_1 + \varepsilon_2 = k \bmod 2$, *then V contains* $V(\xi_{\varepsilon_2, \varepsilon_1, z_2, z_1})/J(\xi_{\varepsilon_2, \varepsilon_1, z_2, z_1})$ *as its unique nontrivial submodule and* $J(\xi_{\varepsilon_2, \varepsilon_1, z_2, z_1})$ *as its unique nontrivial quotient module.*

PROOF. Let $W \subset V$ be a nonzero submodule. Then $W = \bigoplus W(\gamma)$

(1) If $W(\gamma_{j_r}) \neq 0$ with $\lim_{r\to\infty} j_r = \infty$ then W contains $V(\gamma_{j+2k})$ for all $k \geqq 0$. Hence $\dim V/W < \infty$.

Indeed, suppose $j_i \geqq 2$ and $j_{i+1} > j_i + 2$. Then $j_{i+1} - j_i = 2m$, $m \geqq 2$. Hence $(V/W)(\gamma_{j_i+2}), \cdots, (V/W)(\gamma_{j_i+2m-2})$ are nonzero. But $(V/W)(\gamma_{j_i}) = 0$, $(V/W)(\gamma_{j_i+2m}) = 0$. Now 7.13 implies $\mathfrak{g} \cdot \bigoplus_{u=1}^{m-1}(V/W)(\gamma_{j_i+2u}) \subset \bigoplus_{u=1}^{m-1}(V/W)(\gamma_{j_i+2u})$. Hence $\bigoplus_{u=1}^{m-1}(V/W)(\gamma_{j_i+2u})$ is a finite dimensional $(\mathfrak{g}, K)$-submodule of V/W. The finite dimensional representations of G restricted to K are of one of the following forms:

(a) $W_0^{\varepsilon} \oplus \bigoplus_{l \leqq j} W_{2l}$ or

(b) $\bigoplus_{0\leqq l\leqq j} W_{2l+1}$.

This gives a contradiction proving (1).

(1) easily implies that if V is reducible then V has a nonzero finite dimensional submodule or a nonzero finite dimensional quotient module.

(2) If W is a finite dimensional $(\mathfrak{g}, K)$-module, then $W/\mathfrak{n}_0 W$ is isomorphic as an M_0-module with $\delta_{P_0}^{1/2}\xi_{\varepsilon_1,\varepsilon_2,z_1,z_2}$, with $z_1 - z_2 = -k-1$, $k \geqq 0$, $k \in \boldsymbol{Z}$ and $\varepsilon_1 + \varepsilon_2 = k \bmod 2$.

This is just the theorem of the lowest weight for G.

Combining (1), (2) and the fact that the dual $(\mathfrak{g}, K)$-module to $V(P_0, \xi_{\varepsilon_1,\varepsilon_2,z_1,z_2})$ is $V(P_0, \xi_{\varepsilon_2,\varepsilon_1,z_2,z_1})$ gives the lemma.

8.3. LEMMA. *Set* $V = V(P_0, \xi_{\varepsilon_1,\varepsilon_2,z_1,z_2})$. *Suppose that* $z_1 - z_2 = -k - 1$, $k \in \boldsymbol{Z}$, $k \geqq 0$ *and* $\varepsilon_1 + \varepsilon_2 = k \bmod 2$. *Set* $J = J(\xi_{\varepsilon_1,\varepsilon_2,z_1,z_2}, P_0)$. *Then the class of* V/J *is in* $\mathscr{E}_d(G)$.

PROOF. $E(V/J, P_0) = \langle\lambda, \lambda\rangle^{-1}(z_2 - z_1)\lambda$ by 8.2 and 4.21.

8.4. Using the results of §§4, 5, 6 we have a complete classification of irreducible, admissible, $(\mathfrak{g}, K)$-modules (see also Jacquet-Langlands [**12**]).

(I) The finite dimensional $(\mathfrak{g}, K)$-modules $J(\xi_{\varepsilon_1,\varepsilon_2,z_1,z_2}, P_0)$ with $z_1 - z_2 = -k-1$, $k \geqq 0$, $k \in \boldsymbol{Z}$ and $\varepsilon_1 + \varepsilon_2 = k \bmod 2$.

(II) The $V(\xi_{\varepsilon_1,\varepsilon_2,z_1,z_2}, P_0)$ with $z_1 - z_2 \notin \boldsymbol{Z} - \{0\}$ or if $z_1 - z_2 = k \in Z$ then $\varepsilon_1 + \varepsilon_2 \neq k + 1 \bmod 2$. Furthermore $V(\xi_{\varepsilon_1,\varepsilon_2,z_1,z_2}, P_0) = V(\xi_{\varepsilon_2,\varepsilon_1,z_2,z_1}, P_0)$.

(III) $\mathscr{E}_d(G)$: For $z_1 - z_2 = -k-1$ and $k \geqq 0$, $k \in \boldsymbol{Z}$ and $\varepsilon_1 + \varepsilon_2 = k \bmod 2$, $D_{\varepsilon_1,\varepsilon_2,k,z_1+z_2} = V(\xi_{\varepsilon_1,\varepsilon_2,z_1,z_2}, P_0)/J(\xi_{\varepsilon_1,\varepsilon_2,z_1,z_2}, P_0)$.

8.5. We now give a classification of the irreducible, unitary $(\mathfrak{g}, K)$-modules. We first need a lemma.

8.6. LEMMA. *If* $\operatorname{Re}(z_1 - z_2) < 0$ *then* $J(P_0, \xi_{\varepsilon_1,\varepsilon_2,z_1,z_2})$ *is unitary if and only if the following conditions are satisfied*:

(1) $z_1 + z_2$ *is pure imaginary*,

(2) $z_1 - z_2 \in \boldsymbol{R}$,

(3) $A(\varepsilon_1, \varepsilon_2, z_1, z_2)$ *is positive semidefinite*.

PROOF. We observe that if $\operatorname{Re}(z_1 - z_2) < 0$ then $A(\varepsilon_1, \varepsilon_2, z_1, z_2)V(\bar{P}_0, \xi_{\varepsilon_1,\varepsilon_2,z_1,z_2}) = J(P_0, \xi_{\varepsilon_1,\varepsilon_2,z_1,z_2})$ (use Lemma 8.2 and Lemma 7.17). The conjugate dual admissible $(\mathfrak{g}, K)$-module to $V(P_0, \xi_{\varepsilon_1,\varepsilon_2,z_1,z_2})$ is $V(\bar{P}_0, \xi_{\varepsilon_1,\varepsilon_2,-\bar{z}_2,-\bar{z}_1})$ (this is a computation). Thus $J(P_0, \xi_{\varepsilon_1,\varepsilon_2,z_1,z_2})$ is equivalent with its conjugate dual if and only if $z_1 = -\bar{z}_2$ and $z_2 = -\bar{z}_1$. This implies the necessity of conditions (1) and (2). To complete the proof we observe that $A(\varepsilon_1, \varepsilon_2, z_1, z_2)$ composed with the sesquilinear pairing of the V's gives a sesquilinear pairing of the corresponding J with itself.

8.7. Lemmas 8.6 and 8.4 combined with Lemma 7.17 immediately give a complete classification of the elements of $\mathscr{E}(G)$. We give the list:

(1) *The unitary principal series.* $V(P_0, \xi_{\varepsilon_1,\varepsilon_2,z_1,z_2})$ with $z_1, z_2 \in i\boldsymbol{R}$.

(2) *The discrete series.* $D_{\varepsilon_1,\varepsilon_2,k,z}$ with $k \in Z$, $k \geqq 0$, $\varepsilon_1 + \varepsilon_2 = k \bmod 2$ and $z \in i\boldsymbol{R}$.

(3) *The complementary series.* $V(\xi_{\varepsilon_1,\varepsilon_2,z_1,z_2})$ with $z_1 + z_2 \in i\boldsymbol{R}$, $\varepsilon_1 + \varepsilon_2 = 0 \bmod 2$ and $z_1 - z_2 \in \boldsymbol{R}$, $-1 < z_1 - z_2 < 0$.

(4) *The unitary characters of* G.

8.8. We now look at the case when $G = \mathrm{GL}(2, \boldsymbol{C})$. We use the notation of 7.20. We note that:

$$\gamma_{n,m} \otimes \gamma_{n',m'} = \bigoplus_{0 \leq j \leq \min(m,m')} \gamma_{n+n',m+m'-2j} \quad \text{and } \mathrm{Ad}|_K = \gamma_{0,2} + \gamma_{0,2}.$$

8.9. Using the formulae in 8.8 the following result is provided in precisely the same way as Lemma 8.2.

8.10. LEMMA. *Let* $\mathrm{Re}(z_1 - z_2) \leqq 0$. *Then* $V(P_0, \xi_{k_1,k_2,z_1,z_2})$ *is reducible if and only if*

(1) $z_1 - z_2 = -k - 2$, $k \geqq 0$, $k \in \boldsymbol{Z}$,

(2) $|k_1 - k_2| \leqq k$.

If V is reducible then the corresponding J is finite dimensional.

8.11. Lemma 8.10 combined with the results of §6 give the classification of the irreducible, admissible $(\mathfrak{g}, K)$-modules:

(I) $\mathscr{E}_d(G) = \emptyset$.

(II) The finite dimensional representations: $J(P_0, \xi_{k_1,k_2,z_1,z_2})$ with $z_1 - z_2 = -k - 2$, $k \geqq 0$, $k \in \boldsymbol{Z}$ and $|k_1 - k_2| \leqq k$.

(III) The $V(\xi_{k_1,k_2,z_1,z_2}, P_0)$ with $\mathrm{Re}(z_1 - z_2) < 0$ and $z_1 - z_2 \neq -k - 2$, $k \in \boldsymbol{Z}$, $k \geqq 0$ or $z_1 - z_2 = -k - 2$ and $|k_1 - k_2| > k$.

(IV) The $V(P_0, \xi_{k_1,k_2,z_1,z_2})$ with $\mathrm{Re}(z_1 - z_2) = 0$ and $V(P_0, \xi_{k_1,k_2,z_1,z_2}) = V(P_0, \xi_{k_2,k_1,z_2,z_1})$ if $\mathrm{Re}\, z_1 - z_2 = 0$.

8.12. Using the analogue of Lemma 4.6 for $\mathrm{GL}(2, \boldsymbol{C})$ (the proof is exactly the same) and Lemma 7.23 it is not hard to give the following classification of the irreducible, admissible, unitary $(\mathfrak{g}, K)$-modules:

(I) *The unitary characters of G.*

(II) *The unitary principal series*: $V(P_0, \xi_{k_1,k_2,z_1,z_2})$, $z_j \in i\boldsymbol{R}$, $j = 1, 2$.

(III) *The complementary series*: $V(P_0, \xi_{k_1,k_2,z_1,z_2})$, $z_1 + z_2 \in i\boldsymbol{R}$, $z_1 - z_2 \in \boldsymbol{R}$, $0 > z_1 - z_2 > -2$ and $k_1 - k_2 = 0$.

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BRANDEIS UNIVERSITY

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REPRESENTATIONS OF $GL_2(\boldsymbol{R})$ AND $GL_2(\boldsymbol{C})$

A. W. KNAPP

1. $SL_2(\boldsymbol{R})$**.** We shall give lists of the irreducible finite-dimensional representations, the irreducible unitary representations, and the nonunitary principal series. Then we discuss reducibility questions, asymptotic expansions, and the Langlands classification. Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be a typical element of $G = SL_2(\boldsymbol{R})$.

Irreducible finite-dimensional representations. $\mathscr{F}_n$, $n \geqq 0$, an integer.

$$\text{Space} = \{f \text{ polynomial on } \boldsymbol{R} \text{ of degree } n\},$$
$$\mathscr{F}_n(g)f(x) = (bx + d)^n f((ax + c)/(bx + d)).$$

Finite-dimensional representations of G are fully reducible.

Unitary representations. The irreducible unitary representations were classified by Bargmann [**1**]. We give realizations in function spaces on the line or upper half-plane. Realizations on the circle or disc are possible also.

(1) Discrete series $\mathscr{D}_n^+$ and $\mathscr{D}_n^-$, $n \geqq 2$.

$$\text{Space for } \mathscr{D}_n^+ = \left\{ f \text{ analytic for Im } z > 0 \,\middle|\, \|f\|^2 = \iint_{\operatorname{Im} z > 0} |f(z)|^2 y^{n-2}\, dx\, dy < \infty \right\},$$
$$\mathscr{D}_n^+(g)f(z) = (bz + d)^{-n} f\left(\frac{az + c}{bz + d}\right).$$

The space for $\mathscr{D}_n^+$ is not 0 because $(z + i)^{-n}$ is in it. The representation $\mathscr{D}_n^-$ is obtained by using complex conjugates. All these representations are irreducible, unitary, and square-integrable. The square-integrability (of a matrix coefficient) will be shown below.

(2) Principal series $\mathscr{P}^{+,iv}$ and $\mathscr{P}^{-,iv}$, $v \in \boldsymbol{R}$.

Space for $\mathscr{P}^{\pm,iv} = L^2(\boldsymbol{R})$,

$$\begin{aligned} \mathscr{P}^{\pm,iv}(g)f(x) &= |bx + d|^{-1-iv} f((ax + c)/(bx + d)) && \text{if } +, \\ &= \operatorname{sgn}(bx + d)|bx + d|^{-1-iv} f((ax + c)/(bx + d)) && \text{if } -. \end{aligned}$$

These representations are all unitary, and all but $\mathscr{P}^{-,0}$ are irreducible. Equivalences $\mathscr{P}^{+,iv} \cong \mathscr{P}^{+,-iv}$ and $\mathscr{P}^{-,iv} \cong \mathscr{P}^{-,-iv}$ are implemented by analytic continuations of intertwining operators that we give below. $\mathscr{P}^{\pm,iv}$ is really the induced representation $\operatorname{Ind}_{MAN}^G(\sigma \otimes e^{iv\cdot} \otimes 1)$ with G acting by right translation and with the functions restricted to $\bar{N} = \left(\begin{smallmatrix} 1 & 0 \\ * & 1 \end{smallmatrix}\right)$. Here MAN is the upper triangular group, σ is trivial or signum on $M = \{\pm I\}$, and the character of A is

$$\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \to e^{ivt}.$$

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(3) Complementary series $\mathscr{C}^s$, $0 < s < 1$.

$$\text{Space for } \mathscr{C}^s = \left\{ f\colon \mathbf{R} \to \mathbf{C} \,\middle|\, \|f\|^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{f(x)\overline{f(y)}\,dx\,dy}{|x-y|^{1-s}} < \infty \right\},$$

$$\mathscr{C}^s(g)f(x) = |bx + d|^{-1-s} f\left(\frac{ax + c}{bx + d}\right).$$

These are irreducible unitary. They arise from certain nonunitary principal series (see below) by redefining the inner product.

(4) Others. There is the trivial representation, and there are two "limits of discrete series," $\mathscr{D}_1^+$ and $\mathscr{D}_1^-$. The group action with $\mathscr{D}_1^+$ and $\mathscr{D}_1^-$ is like that in discrete series, but the norm is different. We have the relation $\mathscr{P}^{-,0} \cong \mathscr{D}_1^+ \oplus \mathscr{D}_1^-$.

Nonunitary principal series. $\mathscr{P}^{\pm,\zeta}$, $\zeta \in \mathbf{C}$.

Space $= L^2(\mathbf{R}, (1 + x^2)^{\operatorname{Re}\zeta} dx)$

$$\begin{aligned}\mathscr{P}^{\pm,\zeta}(g)f(x) &= |bx + d|^{-1-\zeta} f((ax + c)/(bx + d)) && \text{if } +,\\ &= \operatorname{sgn}(bx + d)|bx + d|^{-1-\zeta} f((ax + c)/(bx + d)) && \text{if } -.\end{aligned}$$

Reducibility. We can see some reducibility in $\mathscr{P}^{\pm,\zeta}$ on a formal level by specializing the parameter ζ and by passing from z in the upper half-plane to x on the real axis. We obtain the following continuous inclusions:

$$\begin{aligned}\mathscr{F}_n &\subseteq \mathscr{P}^{+,-(n+1)} && \text{if } n \text{ even},\\ &\subseteq \mathscr{P}^{-,-(n+1)} && \text{if } n \text{ odd}, n \geqq 0;\\ \mathscr{D}_n^+ \oplus \mathscr{D}_n^- &\subseteq \mathscr{P}^{+,n-1} && \text{if } n \text{ even},\\ &\subseteq \mathscr{P}^{-,n-1} && \text{if } n \text{ odd}, n \geqq 1.\end{aligned}$$

There is no other reducibility. The quotient by an $\mathscr{F}$ is the sum of two $\mathscr{D}$'s, and vice versa.

Asymptotics. Let k_θ be the rotation

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

The maximal compact subgroup $K = \{k_\theta\}$ is abelian, and its irreducible representations are one-dimensional, $k_\theta \to e^{im\theta}$ with m an integer. We have

$$G = KA^+K \quad \text{with} \quad A^+ = \left\{ a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, t \geqq 0 \right\}$$

and Haar measure is of the form $dg = c \sinh 2t\, dk_\theta\, dk_{\theta'}\, dt$ if $g = k_\theta a_t k_{\theta'}$. Let $U(g)$ be an admissible representation of G, and let φ_1 and φ_2 transform under K according to $k_\theta \to e^{im_1\theta}$ and $k_\theta \to e^{im_2\theta}$. Then

$$(U(g)\varphi_1, \varphi_2) = (U(k_\theta a_t k_{\theta'})\varphi_1, \varphi_2) = \exp(i(m_1\theta' + m_2\theta))\,(U(a_t)\varphi_1, \varphi_2).$$

Thus to test whether a matrix coefficient is in some L^p class on G, it is enough to test $(U(a_t)\varphi_1, \varphi_2)$ and use the measure $\sinh 2t\,dt$, $t \geqq 0$.

EXAMPLE. $\mathscr{D}_n^+(k_\theta)(z + i)^{-n} = e^{in\theta}(z + i)^{-n}$. Then

$$(\mathcal{D}_n^+(a_t)(z+i)^{-n}, (z+i)^{-n})$$
$$= \iint_{\mathrm{Im}\, z>0} e^{nt}[x + i(y+1)]^{-n}[e^{2t}x - i(e^{2t}y+1)]^{-n} y^{n-2}\, dx\, dy.$$

By residues the right side is

$$= c_n \int_0^{\infty} e^{-nt}(y + 1 + e^{-2t})^{1-2n} y^{n-2}\, dy,$$

and this in turn, after the change of variables $y = y'(1 + e^{-2t})$, is $= c_n'(\cosh t)^{-n}$. Then

$$\int_G |\ldots|^2\, dg = cc_n' \int_0^{\infty} (\cosh t)^{-2n} \sinh 2t\, dt,$$

which is finite for $n > 1$. Thus this matrix coefficient is square-integrable on G. A theorem in functional analysis due to Godement [3] implies that all matrix coefficients are square-integrable on G.

In the example, we could see the matrix coefficient was square-integrable by computation. There is a general technique, due to Harish-Chandra, for getting at the behavior of matrix coefficients by means of differential equations. Let

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

be a basis for the Lie algebra of G. The *Casimir operator* $\Omega = \frac{1}{2}h^2 + ef + fe$ is a member of the universal enveloping algebra. For $SL_2(\boldsymbol{R})$, Ω generates the center of the universal enveloping algebra. (For larger groups, it must be replaced in this discussion by the whole center of the universal enveloping algebra.) It acts as a scalar on each representation in our lists, hence on each matrix coefficient. Take a matrix coefficient whose two K-dependences are according to known characters of K, and regard the matrix coefficient as an unknown function. Then the equation Ω(coefficient) $= c$(coefficient) leads to a second order ordinary differential equation on A^+, with t as independent variable. The classical substitution is $s = \cosh t$, and the resulting differential equation has three singularities, all regular; we are interested in the behavior at $s = \infty$. (If the "known characters" of K are trivial, this is Legendre's equation.) This substitution does not generalize well, and Harish-Chandra's treatment of this equation amounts to making the substitution $z = e^{-2t}$ instead. The resulting differential equation has four singularities, all regular, and we expand about $z = 0$, using standard regular-singular-point theory. The result is that

$$\text{coefficient}(a_t) = e^{-(1+\zeta)t}\sum_{n=0}^{\infty} c_n e^{-2nt} + e^{-(1-\zeta)t}\sum_{n=0}^{\infty} d_n e^{-2nt}$$

except when ζ is an integer, in which case there may be factors of t that arise from factors $\log z$ in the solution. If one of the leading terms vanishes, the whole corresponding infinite sum vanishes.

The eigenvalue of Ω determines ζ, and in particular the matrix coefficients of $\mathcal{P}^{\pm,\zeta}$ lead to the expansion with ζ present. From this expansion, we can read off L^p-integrability conditions, since we are to integrate for $t \geqq 0$ the pth power against $\sinh 2t\, dt$, which is comparable with $e^{2t}dt$. We see that $\mathcal{P}^{\pm,iv}$ has coefficients in

$L^{2+\varepsilon}(G)$ for every $\varepsilon > 0$, but not in $L^2(G)$. *Discrete series* $\mathscr{D}_n^{\pm}$ with $n \geqq 2$ have one sum absent, in order to have coefficients in $L^2(G)$. Representations with coefficients in $L^{2+\varepsilon}$ for every $\varepsilon > 0$ are said to be *tempered.* The tempered representations are $\mathscr{P}^{\pm,iv}$ and $\mathscr{D}_n^{\pm}$ with $n \geqq 1$. Notice how in general the two leading terms give some information about where imbeddings occur as subrepresentations in the nounitary principal series; Wallach dealt with this point in his lectures.

Langlands classification. For general G, Langlands parametrizes the irreducible admissible representations by triples (P, π, ν), where $P = MAN$ is a standard parabolic, π is (the class of) an irreducible tempered representation of M, and ν is a complex-valued linear functional on the Lie algebra of A with real part in the open positive Weyl chamber. The Langlands representation $J_P(\pi, \nu)$ is the unique irreducible quotient of $\operatorname{Ind}_P^G(\pi \otimes e^\nu \otimes 1)$. In our case, $P = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$ is minimal parabolic, or $P = G$.

Case P minimal. There are two (one-dimensional) representations of $M = \{\pm I\}$, and the functional ν enters as the complex number ζ with $\operatorname{Re} \zeta > 0$; the character of A is $a_t = \exp(\nu \log a_t) = \exp(\zeta t)$. The Langlands list then includes the unique irreducible quotient of $\mathscr{P}^{\pm,\zeta}$ for each ζ with $\operatorname{Re} \zeta > 0$.

Case $P = G$. Here ν is irrelevant, and $M = G$. We simply get the irreducible tempered representations of G. The Langlands classification itself does not address the question of what these are, though one of the theorems implies for our G that they are subrepresentations of discrete series or unitary principal series.

Intertwining operators. The Langlands classification theorem describes the unique irreducible quotient more precisely than we have done. Kunze and Stein [4] showed in 1960 that the operator

$$f \to \int_{-\infty}^{\infty} \frac{f(y)\, dy}{|x - y|^{1-\zeta}} \qquad \text{for } \mathscr{P}^{+,\zeta},$$

$$\to \int_{-\infty}^{\infty} \frac{\operatorname{sgn}(x - y) f(y)\, dy}{|x - y|^{1-\zeta}} \qquad \text{for } \mathscr{P}^{-,\zeta},$$

intertwines $\mathscr{P}^{\zeta}$ with $\mathscr{P}^{-\zeta}$. Note that the integral is convergent only if $\operatorname{Re} \zeta > 0$. Later [5] they found a formula in the induced picture, namely $f \to \int_{\bar N} f(\bar n w^{-1} g)\, d\bar n$, where $\bar N = \left(\begin{smallmatrix} 1 & 0 \\ * & 1 \end{smallmatrix}\right)$ and $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. This is the composition of two operators, $f \to \int_{\bar N} f(\bar n g)\, d\bar n$ and a relatively trivial translation operator by w^{-1}. Define $A(\bar P : P : \pi : \nu) f(x) = \int_{\bar N} f(\bar n x)\, d\bar n$. Under the Langlands conditions on ν, this integral is convergent if f is K-finite. The theorem is that

$$J_P(\pi, \nu) = \operatorname{Ind}_P^G(\pi \otimes e^\nu \otimes 1)/\ker A(\bar P : P : \pi : \nu) \cong \operatorname{Image} A(\bar P : P : \pi : \nu).$$

2. Other groups.

$\mathrm{GL}_2(\boldsymbol{R})$. To pass from $\mathrm{SL}_2(\boldsymbol{R})$ to the group $\mathrm{SL}_2^{\pm}(\boldsymbol{R})$ of matrices of determinant ± 1, we first induce the representations of $\mathrm{SL}_2(\boldsymbol{R})$. The $\mathscr{P}$'s and $\mathscr{F}$'s split into two equivalent pieces, and the $\mathscr{D}$'s yield irreducibles on $\mathrm{SL}_2^{\pm}(\boldsymbol{R})$ that restrict back to $\mathscr{D}^+ \oplus \mathscr{D}^-$ on $\mathrm{SL}_2(\boldsymbol{R})$. This construction gives us the representations of $\mathrm{SL}_2^{\pm}(\boldsymbol{R})$. Then to pass to $\mathrm{GL}_2(\boldsymbol{R})$, we paste on a character of the group $\boldsymbol{R}^+\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

$\mathrm{SL}_2(\boldsymbol{C})$. This group has finite-dimensional representations given by two integer parameters; the representations can be realized in spaces of polynomials in z and $\bar z$ on $\boldsymbol{C}$. The group has no discrete series. We have

$$M = \left\{\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{pmatrix}\right\} \quad \text{and} \quad A = \left\{\begin{pmatrix} e^{t} & 0 \\ 0 & e^{-t}\end{pmatrix}\right\},$$

so that the nonunitary principal series is parametrized by an integer n (for M) and a complex number ζ (for A); by restriction of functions to $\bar{N} = \left(\begin{smallmatrix} 1 & 0 \\ * & 1\end{smallmatrix}\right)$, we can realize these representations in spaces of functions on $\boldsymbol{C}$. See [2] for more detail. The unitary principal series is all irreducible and provides the only tempered irreducibles, and parameters (n, iv) and $(-n, -iv)$ lead to equivalent representations. The Langlands classification points to the Langlands quotients of the nonunitary principal series with Re $\zeta > 0$ and to the irreducible tempered representations.

$GL_2(\boldsymbol{C})$. To an irreducible representation of $SL_2(\boldsymbol{C})$, we paste on a character of $\boldsymbol{C}^{\times}\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1\end{smallmatrix}\right)$ that agrees with the representation on $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1\end{smallmatrix}\right)$. In this way we obtain all irreducible representations of $GL_2(\boldsymbol{C})$.

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CORNELL UNIVERSITY

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NORMALIZING FACTORS, TEMPERED REPRESENTATIONS, AND L-GROUPS

A. W. KNAPP* AND GREGG ZUCKERMAN*

Several lecturers have alluded to the intertwining operators associated with principal series representations, particularly for SL(2, $\boldsymbol{R}$). These operators and their normalizations play a role in the trace formula, in reducibility questions for principal series, and in the construction of an inner product that exhibits certain representations as unitary. We shall review the development of these operators and their normalizations in the context of a linear real reductive group G (as defined in Wallach's lectures) whose identity component has compact center.

We shall be especially interested in the application to reducibility questions for unitary principal series and other continuous series of representations coming from parabolic subgroups, since the answers to these questions lead to a classification of irreducible tempered representations and thereby complement the Langlands classification **[12]**. The answers concerning reducibility are reviewed in §2 in terms of three easy-to-calculate finite groups, denoted W, W', and R.

In lectures during 1975–76, the authors mentioned how, in some special cases, the groups W, W', R, initially defined in terms of roots, could be defined in terms of co-roots. Building on this presentation, Langlands **[14]** was able to redefine these groups in general in terms of the L-group. We present his definitions, along with an example, in §3. In §4 we summarize earlier work **[8]**, **[9]** that leads from the R-group to the classification of irreducible tempered representations.

1. Intertwining operators and normalizing factors. In the group G, fix a maximal compact subgroup K and Cartan involution θ. To each parabolic subgroup P, we associate the Langlands decomposition $P = MAN$ with MA θ-stable and with M a linear real reductive group whose identity component has compact center. To the pair (ξ, ν), where ξ is an irreducible unitary representation of M and ν is a complex-valued linear functional on the Lie algebra $\mathfrak{a}$ of A, we associate the representation $U_P(\xi, \nu, x)$, with x in G, given by

$$U_P(\xi, \nu) = \operatorname{ind}_{MAN}^G(\xi \otimes e^\nu \otimes 1). \tag{1.1}$$

We adopt the convention that G acts on the left in the induced representation. A member f of the representation space satisfies

$$f(xman) = \exp(-(\nu + \rho_P)\log a)\,\xi(m)^{-1}f(x),$$

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where ρ_P is the usual half-sum of restricted roots associated with P. If $P = P_0$ is minimal parabolic, these representations comprise the *nonunitary principal series*. In formula (1.1), we have assumed that ξ is an irreducible unitary representation of M, but we shall allow also that ξ is a nonunitary principal series representation of M, provided we are not working with formulas involving adjoints of operators.

For SL(2, $\boldsymbol{R}$), one can restrict the functions in the representation space for the nonunitary principal series to the lower triangular group $\bar{N} = \theta N = \{\left(\begin{smallmatrix}1 & 0\\ x & 1\end{smallmatrix}\right)\}$ and realize the representations in spaces of functions on $\boldsymbol{R}$. The representations become

$$\mathscr{P}^{\pm,\zeta}(g)f(x) = |-bx+d|^{-1-\zeta} f\left(\frac{ax-c}{-bx+d}\right) \qquad \text{if } +,$$

$$= \operatorname{sgn}(-bx+d)|-bx+d|^{-1-\zeta} f\left(\frac{ax-c}{-bx+d}\right) \quad \text{if } -,$$

if $g = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$. Kunze and Stein **[10]** showed that the operator

$$f \to \int_{-\infty}^{\infty} \frac{f(x-y)\,dy}{|y|^{1-\zeta}} \qquad \text{for } \mathscr{P}^{+,\zeta}, \tag{1.2a}$$

$$\to \int_{-\infty}^{\infty} \frac{(\operatorname{sgn} y) f(x-y)\,dy}{|y|^{1-\zeta}} \qquad \text{for } \mathscr{P}^{-,\zeta}, \tag{1.2b}$$

intertwines $\mathscr{P}^{\zeta}$ with $\mathscr{P}^{-\zeta}$ when it is convergent, namely for Re $\zeta > 0$. Later **[11]** they found a formula in the induced picture, namely

$$f \to \int_{\bar{N}} f(gw\bar{n})\,d\bar{n}, \quad \text{where } w = \begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}.$$

This is the composition of two operators, $f \to \int_{\bar{N}} f(g\bar{n})\,d\bar{n}$ and a relatively trivial translation operator by w. The first operator intertwines the representation induced from $P = MAN$ with the one induced from $\bar{P} = MA\bar{N}$ and the same data on MA.

In the general case, let $P_1 = MAN_1$ and $P_2 = MAN_2$ be two parabolics with the same MA, and define

$$A(P_2 : P_1 : \xi : \nu) f(x) = \int_{\bar{N}_1 \cap N_2} f(x\bar{n})\,d\bar{n}. \tag{1.3}$$

Formally

$$U_{P_2}(\xi, \nu, g) A(P_2 : P_1 : \xi : \nu) = A(P_2 : P_1 : \xi : \nu) U_{P_1}(\xi, \nu, g). \tag{1.4}$$

In general, the integral (1.3) will not converge but will be defined by analytic continuation. To accomplish this analytic continuation, we need the representations to occur in a single space, as ν varies. This space is obtained by restricting the functions to K. Say ξ operates in the space H^{ξ}. We consider functions $f: K \to H^{\xi}$ satisfying $f(km) = \xi(m)^{-1} f(k)$ for $k \in K$, $m \in M \cap K$. Under the action by g, we replace $f(k)$ by $f(g^{-1}k)$ and restrict back to K. Thus we are led to define

$$U_P(\xi, \nu, g) f(k) = \exp(-(\rho_P + \nu)\, H(g^{-1}k))\, \xi(\mu(g^{-1}k))^{-1} f(\kappa(g^{-1}k)), \tag{1.5}$$

where x decomposes within $G = KMAN$ as $\kappa(x)\mu(x)e^{H(x)}n$.

The analytic continuation of (1.3) is accomplished in three stages. In the first stage the essence of the argument can be seen by continuing (1.2a) if f is smooth and compactly supported. Since f has compact support, we can integrate over a finite interval. Expand $f(x - y)$ about $y = 0$ in a finite Taylor series with remainder. Each of the main terms can then be integrated explicitly and continues meromorphically to the whole plane. The error term is integrable for $\operatorname{Re} \zeta > -n - 1$ if n is the order of the Taylor series. Hence the integral continues meromorphically to the whole plane.

In Schiffmann [**15**] and in [**6**], it is shown that this style of argument yields a meromorphic continuation of (1.3) if P_1 and P_2 are minimal parabolic and A is one-dimensional. In the second stage, for minimal parabolics with $\dim A > 1$, Schiffmann shows how to decompose $\bar{N}_1 \cap N_2$ into a product of subgroups and to write (1.3) as an iterated integral; the analytic continuation is essentially then reduced to the case $\dim A = 1$.

The third and final stage of the analytic continuation was obtained independently by Wallach [**17**] and in [**7**]. We can use Casselman's subrepresentation theorem to imbed ξ as an $(\mathfrak{m}, K \cap M)$ module in a nonunitary principal series representation of M. It follows from the double induction theorem that (1.1) is a subrepresentation of a nonunitary principal series and (1.3) is a restriction of an intertwining operator for the nonunitary principal series. Then (1.3) has an analytic continuation for K-finite f, and (1.4) holds if g is replaced by a member of K or of the Lie algebra $\mathfrak{g}$ of G. Moreover, with Haar measures normalized suitably, we obtain

$$A(P_2 : P_1 : \xi : \nu)^* = A(P_1 : P_2 : \xi : -\bar{\nu}) \tag{1.6}$$

for ξ unitary, if the adjoint is defined K-space by K-space, and

$$A(P_2 : P_1 : \xi : \nu) = R(w^{-1})\, A(wP_2w^{-1} : wP_1w^{-1} : w\xi : w\nu)\, R(w),$$

where $R(w)$ denotes right translation by an element w in K representing a member of the Weyl group $W(\mathfrak{a})$.

These operators tend to have poles at many interesting values of ν. We introduce scalar normalizing factors—in part to eliminate some of these poles, in part to make the operators unitary for ν imaginary, and in part to make the operators behave nicely under composition. The normalizing factors are not unique, and different choices are useful for different purposes.

Again the construction is in several stages. We impose the condition that ξ have a real infinitesimal character. Matters are based on the following lemma [**6**, p. 544], proved using Weierstrass canonical products.

LEMMA 1.1. *If $\eta(z)$ is a meromorphic function in the plane such that*
(i) *$\eta(z)$ is real on the real axis,*
(ii) *$\eta(z) \geqq 0$ on the imaginary axis,*
(iii) *$\eta(z) = \eta(-z)$ for all z,*
then there exists a meromorphic function $\gamma(z)$ in the plane such that $\eta(z) = \gamma(-z)\gamma(z)$ and $\gamma(z)$ is real for real z.

The first stage of the construction deals with $P = MAN$ a minimal parabolic and G of real rank one ($\dim A = 1$). In this case $A(P:\bar{P}:\xi:\nu)\, A(\bar{P}:P:\xi:\nu)$ is a self-intertwining operator for $U_P(\xi, \nu)$. It is a result due to Bruhat [**3**] that, for ν nonzero imaginary, the unitary representation $U_P(\xi, \nu)$ is irreducible. It follows readily that

$$A(P : \bar{P} : \xi : \nu) A(\bar{P} : P : \xi : \nu) = \eta(\bar{P} : P : \xi : \nu) I \tag{1.7}$$

with $\nu = z\rho_P$ meromorphic and scalar valued for z in $\mathbf{C}$. One checks that

(i) $\eta(\bar{P}:P:\xi:\nu)$ is real for ν real.

(ii) $\eta(\bar{P}:P:\xi:\nu) \geqq 0$ for ν imaginary.

(iii) $\eta(\bar{P}:P:\xi:\nu) = \eta(\bar{P}:P:\xi:-\nu)$ for all ν.

(iv) $\eta(\bar{P}:P:\xi:\nu)$ depends only on the class of ξ, and if φ is an automorphism of G leaving K and P stable, $\eta(\bar{P}:P:\xi^\varphi:\nu) = \eta(\bar{P}:P:\xi:\nu)$. Then we can apply the lemma to obtain $\gamma(\bar{P}:P:\xi:\nu)$ and to define normalized operators by

$$\mathscr{A}(\bar{P}:P:\xi:\nu) = \gamma(\bar{P}:P:\xi:\nu)^{-1} A(\bar{P}:P:\xi:\nu).$$

The normalized operators satisfy

$$\mathscr{A}(P:\bar{P}:\xi:\nu)\,\mathscr{A}(\bar{P}:P:\xi:\nu) = I$$

and

$$\mathscr{A}(\bar{P}:P:\xi:\nu)^* = \mathscr{A}(P:\bar{P}:\xi:-\bar{\nu}).$$

These two relations together imply $\mathscr{A}$ is unitary for ν imaginary.

The second stage is to handle a minimal parabolic for general G. Use Schiffmann's decomposition of a general intertwining operator into operators that are essentially rank one operators, and use the product of the normalizing factors as normalizing factor for the given operator. Then one proves the relation

$$\mathscr{A}(P_3:P_1:\xi:\nu) = \mathscr{A}(P_3:P_2:\xi:\nu)\,\mathscr{A}(P_2:P_1:\xi:\nu). \tag{1.8}$$

The third stage is to handle a general parabolic $P = MAN$ with $\dim A = 1$. We again use the trick of imbedding ξ as a subrepresentation of a nonunitary principal series representation of M. If we combine this trick with formula (1.8) for minimal parabolics, we are led to the conclusion that (1.7) holds for our P with $\dim A = 1$ and that η has the same properties as before. Again we apply the lemma to obtain a normalizing factor γ, and we set $\mathscr{A} = \gamma^{-1}A$.

The final stage is for a general parabolic with G general and is handled in the same way as with a minimal parabolic and G general. If the γ's are chosen compatibly, the result is as follows [7, p. 2460].

THEOREM 1.2. *The normalized intertwining operators satisfy*

(i) $\mathscr{A}(P_3:P_1:\xi:\nu) = \mathscr{A}(P_3:P_2:\xi:\nu)\,\mathscr{A}(P_2:P_1:\xi:\nu)$.

(ii) $\mathscr{A}(P_2:P_1:\xi:\nu)^* = \mathscr{A}(P_1:P_2:\xi:-\bar{\nu})$, *K-space by K-space.*

(iii) $\mathscr{A}(P_2:P_1:\xi:\nu)$ *is unitary for* ν *imaginary.*

(iv) *If* w *in* K *represents a member of the Weyl group* $W(\mathfrak{a})$, *then*

$$\mathscr{A}(P_2:P_1:\xi:\nu) = R(w^{-1})\,\mathscr{A}(wP_2w^{-1}:wP_1w^{-1}:w\xi:w\nu)R(w).$$

For w in K representing a member of $W(\mathfrak{a})$, let

$$\mathscr{A}_P(w,\xi,\nu) = R(w)\mathscr{A}(w^{-1}Pw:P:\xi:\nu).$$

From (i) and (iv) we obtain the cocycle relation

$$\mathscr{A}_P(w_1w_2,\xi,\nu) = \mathscr{A}_P(w_1, w_2\xi, w_2\nu)\mathscr{A}_P(w_2,\xi,\nu). \tag{1.9}$$

From (ii) we find that

$$\mathscr{A}_P(w,\xi,\nu)^* = \mathscr{A}_P(w^{-1}, w\xi, -w\bar{\nu}) \tag{1.10}$$

and hence that $\mathscr{A}_P(w,\xi,\nu)$ is unitary for ν imaginary. The intertwining relation is

(1.11) $$U_P(w\xi, w\nu, g)\mathscr{A}_P(w, \xi, \nu) = \mathscr{A}_P(w, \xi, \nu)U_P(\xi, \nu, g)$$

for g in K or in $\mathfrak{g}$.

For application to adelic situations, a more specific normalization is needed. At almost all places the representation has a K-fixed vector, and the infinite tensor product of local normalized intertwining operators will be defined only if the local operator fixes the K-fixed vector. This condition determines the normalizing factor. For example, with P_1 and P_2 minimal parabolic and $\xi = 1$, we are led to normalize (1.3) by taking

$$\gamma(P_2 : P_1 : \xi : \nu) = \int_{\bar{N}_1 \cap N_2} \exp(-(\rho_P + \nu) H(\bar{n}))\, d\bar{n},$$

which can be computed in terms of Harish-Chandra's **c**-functions. Schiffmann pursued this idea further at an early stage in his work leading to [**15**]. Langlands was led to conjecture [**13**, p. 282] in general that a valid normalization is obtained by using the quotient of two L-functions:

(1.12) $$\gamma(P_2 : P_1 : \xi : \nu) = L(0, \tilde{\rho}_{P_2|P_1} \circ \varphi)/L(1, \tilde{\rho}_{P_2|P_1} \circ \varphi).$$

Here φ is the homomorphism of the Weil group of $\boldsymbol{R}$ into the L-group ${}^L(MA)$ of MA corresponding to $\xi \otimes e^{\nu}$, and $\tilde{\rho}_{P_2|P_1}$ is the contragredient of the representation of ${}^L(MA)$ on the Lie algebra of ${}^L\bar{N}_1 \cap {}^LN_2$. Arthur [**1**] proved that (1.12) is a valid normalization if the Haar measures are normalized suitably.

2. The R-group, a first formulation. Fix a discrete series representation ξ of M, and consider the corresponding continuous series representation $U_P(\xi, \nu)$ with ν imaginary. In [**7**], it is stated how the problem of determining the algebra $\mathscr{C}(\xi, \nu)$ of operators that commute with $U_P(\xi, \nu)$ can be reduced to algebraic problems involving certain finite groups.

Fix ν. Let W be the subgroup of elements s in $W(\mathfrak{a})$ that fix ν and the class of ξ. If w is a representative in K of an s in W, then one can define $\xi(w)$ in such a way that ξ extends to a representation of the smallest group containing M and w; the definition of $\xi(w)$ is unique up to a scalar factor equal to a root of unity. Then $\xi(w)\, \mathscr{A}_P(w, \xi, \nu)$ is independent of the representative w, and we can write

(2.1) $$\xi(s)\, \mathscr{A}_P(s, \xi, \nu)$$

for it. One sees from (1.11) that the unitary operator (2.1) intertwines $U_P(\xi, \nu)$ with itself. The essence of the next theorem is due to Harish-Chandra [**4**].

THEOREM 2.1. *The operators* (2.1), *for s in W, span the commuting algebra* $\mathscr{C}(\xi, \nu)$.

Despite formula (1.9), it does not follow that the map

(2.2) $$s \in W \to \xi(s)\mathscr{A}_P(s, \xi, \nu)$$

is a homomorphism into unitary operators; the $\xi(s)$ factors need to be chosen compatibly, and there may a priori be an obstruction to making such a choice. However, the map (2.1) is at least a homomorphism into the projective unitary group. Let W' be its kernel.

THEOREM 2.2. *The group W' is the Weyl group of a root system Δ' contained in the* *set of roots of* $(\mathfrak{g}, \mathfrak{a})$; *consequently* $W = W'R$ *is a semidirect product if* R *is defined as the subgroup of* W *leaving stable the positive roots in* Δ'. *The image of the map* (2.2) *consists of linearly independent operators*; *consequently the dimension of* $\mathscr{C}(\xi, \nu)$ *equals the order of* R.

It turns out that the elements of R are characterized in W as those elements for which the normalizing factor of (2.1) is holomorphic at ν. Thus W' and R are intimately connected with the functions $\eta(P_2: P_1: \xi: \nu)$ defined by

$$A(P_1:P_2:\xi:\nu)A(P_2:P_1:\xi:\nu) = \eta(P_2:P_1:\xi:\nu)I.$$

In turn, these functions can be connected by analytic methods with the Plancherel measures of subgroups of G; these measures have been determined by Harish-Chandra. Consequently we can describe Δ' very concretely, as follows.

Adjoin to $\mathfrak{a}$ a compact Cartan subalgebra of the Lie algebra $\mathfrak{m}$ of $\mathscr{M}$, and denote a typical root by β. Let λ be the Harish-Chandra parameter for the discrete series ξ. Let α be an $\mathfrak{a}$-root and suppose 2α is not an $\mathfrak{a}$-root. If α has even multiplicity and 2α is not an $\mathfrak{a}$-root, define

$$\mu_{\xi,\alpha}(\nu) = \prod_{\beta|\mathfrak{a}=\alpha} \langle \lambda + \nu, \beta \rangle.$$

If α has odd multiplicity, define

$$\mu_{\xi,\alpha/2}(\nu) = \mu_{\xi,\alpha}(\nu) = \Big(\prod_{\beta|\mathfrak{a}=c\alpha,\, c>0} \langle \lambda + \nu, \beta \rangle \Big) f_{\xi,\alpha}(\nu),$$

where $f_{\xi,\alpha}(\nu) = \tan(\pi\langle \nu, \alpha\rangle / |\alpha|^2)$ or $\cot(\pi\langle \nu, \alpha\rangle / |\alpha|^2)$ according as

$$\xi(\gamma_\alpha) = -(-1)^{2\langle \rho_\alpha, \alpha\rangle/|\alpha|^2} I \quad \text{or} \quad +(-1)^{2\langle \rho_\alpha, \alpha\rangle/|\alpha|^2} I.$$

Here α is a real root and there is a corresponding homomorphism of SL(2, $\boldsymbol{R}$) into G, and γ_α denotes the image of $-I$.

Now we can characterize Δ' as

$$\Delta' = \{\text{roots } \alpha \text{ of } \mathfrak{a} \,|\, s_\alpha \text{ is in } W \text{ and } \mu_{\xi,\alpha}(\nu) = 0\},$$

where ξ_α denotes the reflection corresponding to α. As before,

$$R = \{p \in W \,|\, p\alpha > 0 \text{ for every } \alpha > 0 \text{ in } \Delta'\}.$$

Then we have the following result [5], [8].

THEOREM 2.3. (a) *R is a finite direct sum of 2-element groups* $\boldsymbol{Z}_2$.

(b) *For s in R the operators $\xi(s)$ can be selected so that the mapping* (2.2) *is a homomorphism into unitary operators*; *consequently the algebra* $\mathscr{C}(\xi, \nu)$ *is commutative.*

(c) *There exists a set of positive orthogonal real roots* $\mathscr{H} = \{\alpha_1, \ldots, \alpha_q\}$ *such that* (i) *the only roots in the span of* $\mathscr{H}$ *are the* $\pm\alpha_j$; (ii) *each r in R is of the form* $s_{\alpha_{j_1}} \cdots s_{\alpha_{j_n}}$; *and* (iii) *each α_j occurs in the decomposition of some r in R.*

3. The R-group, a second formulation. An admissible representation of G is *tempered* if its K-finite matrix coefficients are in $L^{2+\varepsilon}(G)$ for every $\varepsilon > 0$. It is known that every irreducible tempered representation is a summand in some representation of the type considered in §2, induced from discrete series on M and a unitary character on A. The R group can be used to decompose these induced representations and give a classification of the irreducible tempered representations. We return to this point in §4.

For now, we want to consider W, W', and R from a different point of view. We begin with some motivation in the case that G is split and our parabolic is minimal. Then M is finite abelian and is generated by the elements γ_α that are the images of $\left(\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right)$ in the SL(2, $\boldsymbol{R}$)'s that correspond to each root. Here ξ is a character of M, and we assume $\nu = 0$. Let the root system be Δ. It is a simple matter to see that $\Delta' = \{\alpha \in \Delta \mid \xi(\gamma_\alpha) = 1\}$. The elements γ_α transform under Weyl group elements differently from what one might expect. The correct rule is

$$s_\beta \gamma_\alpha s_\beta^{-1} = \gamma_{s_\beta \alpha} = \gamma_\alpha \gamma_\beta^{2\langle\beta,\alpha\rangle/|\alpha|^2}.$$

As a result, Δ' need not be closed under addition within the set of all roots. It is in the co-root system that there is closure under addition. The co-root of α is $\alpha^\vee = 2\alpha/\langle\alpha,\alpha\rangle$, and $\Delta'^\vee$ is closed under addition within $\Delta^\vee$. We introduce a kind of dual group $G^\sim$, not the L-group just yet. $G^\sim$ is a connected real group with a compact Cartan subgroup and root system $\Delta^\vee$, arranged so that the roots of $\Delta'^\vee$ are compact and the others are noncompact; it is to have a centerless complexification $G^\sim_C$, a Lie algebra $\mathfrak{g}^\sim$, and a Cartan decomposition $\mathfrak{g}^\sim = \mathfrak{k}^\sim \oplus i\mathfrak{p}^\sim$. Elements of W' lead to elements in the Weyl group of $\mathfrak{k}^\sim$, with representatives in $G^\sim$. Elements of W lead to elements in the Weyl group of $\mathfrak{g}^\sim_C$ that leave the compact roots stable, hence normalize $\mathfrak{k}^\sim_C$ and $\mathfrak{k}^\sim \oplus i\mathfrak{p}^\sim$, hence normalize $\mathfrak{k}^\sim$ and $\mathfrak{p}^\sim$ and therefore $\mathfrak{g}^\sim$. It follows that R injects onto the quotient $\mathrm{Norm}_{G^\sim_C}(\mathfrak{g})/G^\sim$. This style of argument can be pushed to yield a proof that $R = \sum \boldsymbol{Z}_2$ in this case.

Langlands **[14]** built on these ideas and gave a formulation in general of W, W', and R in terms of the L-group. We use the notation of Borel **[2]** approximately and will follow the L-group constructions given by Langlands **[12]** when we need them.

First let us see what the above example has to do with the L-group. Of course, ${}^LG^0$ is just $G^\sim_C$, up to coverings. The character ξ is specified by giving $\xi(\gamma_\alpha)$, and γ_α behaves like the (character associated to the) co-root $\alpha^\vee$. To know $\alpha^\vee$ on ξ for each $\alpha^\vee$ is to know ξ as an element of the dual torus ${}^LT^0$. There is a corresponding homomorphism φ of the Weil group into ${}^LG$ given by $\varphi(z) = 1$ for $z \in \boldsymbol{C}^\times$ and $\varphi(\tau) = \xi \times \sigma \in {}^LT$; here σ is the trivial outer automorphism of ${}^LG^0$, trivial since G is split. The elements of the Weyl group that centralize the image $\{1, \xi \times \sigma\}$ of φ are those of W; this statement motivates Theorem 3.1 below. The roots whose root vectors are centralized by the image of φ are those in Δ'; this statement motivates Theorem 3.3 below. The observation above that R is isomorphic to $\mathrm{Norm}_{G^\sim_C}(\mathfrak{g})/G^\sim$ could have been stated in terms of $K^\sim$ and then motivates Theorem 3.4 below.

We pass to the general case. Fix a minimal parabolic $P_0 = M_0A_0N_0$ in G, and let T be a Cartan subgroup of G containing A_0. Let ${}^LG^0 \rtimes \{1,\sigma\}$ be the L-group of G, with its maximal torus ${}^LT^0$ and Borel subgroup ${}^LB^0$. The "standard relevant parabolics" ${}^LP$, in the sense of **[2]**, correspond to the standard parabolics $P \supseteq P_0$ of G. Let ${}^L(MA)$ be the Levi component of ${}^LP$. According to **[12]**, the classes of L-indistinguishable irreducible admissible representations of G correspond to conjugacy classes (suitably defined) of certain kinds of homomorphisms φ of the Weil group into ${}^LG$. A representation is tempered if and only if the image of φ is bounded. It comes from the series of §2 for P if, after conjugation, φ has image in the Levi component ${}^L(MA)$ of ${}^LP$ but not in any smaller such Levi component. We may and shall assume that $\varphi(C^\times) \subseteq {}^LT^0$ and $\varphi(\tau)$ is in the normalizer of ${}^LT^0$. We shall give an example of these notions at the end of this section, illustrating the three theorems below.

Fix a summand in a representation $U_P(\xi, \nu)$. The corresponding φ goes with a whole L-class of representations in G. Since the image of φ is in ${}^L(MA)$, the L-class consists of those representations obtained by inducing from a single L-class for MA. The latter class corresponds with the map φ^{MA} obtained by regarding φ as having image in ${}^L(MA)$, and it is characterized by the data ν, the central character of ξ, and the infinitesimal character of ξ. But the definitions of W, W', and R depend only on ν, the central character of ξ, and the infinitesimal character of ξ. Thus the question arises how to define W, W', and R in terms of φ directly.

[*Digression.* A simple consequence of the discussion above and a lemma due to D. Shelstad is that the cardinalities of R and the L-classes corresponding to φ and φ^{MA} are related by the formula

$$|L\text{-class for } \varphi| = |R|\, |L\text{-class for } \varphi^{MA}|.$$

In fact, the representations in the L-class for φ are all the irreducible constituents of all representations $U_P(\xi', \nu)$ with $\xi' \otimes \nu$ in the L-class for φ^{MA}. The R-group for all these $U_P(\xi', \nu)$ is the same as ξ' varies. Moreover, Shelstad's lemma says that $U_P(\xi_1, \nu)$ and $U_P(\xi_2, \nu)$ are disjoint if $\xi_1 \otimes \nu$ and $\xi_2 \otimes \nu$ are L-equivalent but not equivalent. The formula follows.]

Let $S = \mathrm{Cent}(\mathrm{Image}\ \varphi)$, the centralizer being taken in ${}^LG^0$, and let S^0 be the identity component of S and $\mathfrak{s}$ its Lie algebra.

The Weyl group $W(\mathfrak{a})$ imbeds in the Weyl group of $({}^LG^0, {}^LT^0)$. Namely any element of $W(\mathfrak{a})$ has a representative that normalizes the Cartan subalgebra of $\mathfrak{m}$ and preserves positive roots of $\mathfrak{m}$; in this way an element of $W(\mathfrak{a})$ leads to a unique member of the Weyl group of $({}^LG^0, {}^LT^0)$. With this identification in mind, we have the following result.

THEOREM 3.1 (LANGLANDS). $W = (\mathrm{Norm}({}^LT^0) \cap S)/({}^LT^0 \cap S)$, *the normalizer being taken in* ${}^LG^0$.

We denote the lattices that are given as part of the L-group data by L and $L^\vee$:

$$L = \mathrm{Hom}\,(T_C, C^\times) \cong \mathrm{Hom}\,(C^\times, {}^LT^0),$$
$$L^\vee = \mathrm{Hom}\,({}^LT^0, C^\times) \cong \mathrm{Hom}\,(C^\times, T_C).$$

These are in duality as follows: If λ is in $L = \mathrm{Hom}(C^\times, {}^LT^0)$ and $\lambda^\vee$ is in $L^\vee = \mathrm{Hom}({}^LT^0, C^\times)$, then $\lambda^\vee \circ \lambda$ is a power of z in $C^\times$, and $\langle \lambda, \lambda^\vee \rangle$ denotes this power. The homomorphism φ on $C^\times$, with values in ${}^LT^0$, can be written symbolically as

$$\varphi(z) = z^\mu \bar{z}^\nu = z^{\mu-\nu}(z\bar{z})^\nu, \tag{3.1}$$

where μ and ν are in $L \otimes C$ and $\mu - \nu$ is in L. Formula (3.1) means that

$$\lambda^\vee(\varphi(z)) = z^{\langle \mu-\nu, \lambda^\vee \rangle}\, (z\bar{z})^{\langle \nu, \lambda^\vee \rangle}$$

for all $\lambda^\vee$ in $L^\vee$.

The condition that the image of φ lies in no proper parabolic within P means that $\varphi(\tau)\, \alpha^\vee = -\alpha^\vee$ exactly for the roots of ${}^L(MA)$ and that $\langle \mu, \alpha^\vee \rangle \neq 0$ for all roots of ${}^L(MA)$. (See Lemma 3.3 and the paragraph before Lemma 3.1 in **[12]**.)

Let ${}^L\mathfrak{t}$ be the Lie algebra of ${}^LT$ and decompose ${}^L\mathfrak{t}$ into the $+1$ and -1 eigenspaces for $\varphi(\tau)$ as ${}^L\mathfrak{t} = {}^L\mathfrak{t}^+ \oplus {}^L\mathfrak{t}^-$.

LEMMA 3.2 (LANGLANDS). *The Lie algebra ${}^L\mathfrak{t}^+$, which lies in $\mathfrak{S}$, is a Cartan subalgebra of $\mathfrak{S}$.*

PROOF. If X is in $\mathfrak{S}$, then the condition that $X^{\varphi(\tau)} = X$ means that X is the sum of a member of ${}^L\mathfrak{t}^+$ and a sum Y of root vectors. Moreover, Y must centralize $\varphi(\boldsymbol{C}^\times)$. If $Y = \sum X_{\alpha^\vee}$, then

$$\sum Y_{\alpha^\vee} = Y = Y^{\varphi(z)} = \sum z^{\langle\mu-\nu,\alpha^\vee\rangle} (z\bar{z})^{\langle\nu,\alpha^\vee\rangle} Y_{\alpha^\vee}.$$

Hence each $\alpha^\vee$ with $Y_{\alpha^\vee} \neq 0$ satisfies $\langle\mu, \alpha^\vee\rangle = \langle\nu, \alpha^\vee\rangle = 0$. Thus

$$\mathfrak{S} \subseteq {}^L\mathfrak{t}^+ + \sum_{\langle\mu,\alpha^\vee\rangle=\langle\nu,\alpha^\vee\rangle=0} \boldsymbol{C}X_{\alpha^\vee}.$$

We are to prove that ${}^L\mathfrak{t}^+$ is maximal abelian in $\mathfrak{S}$. It is enough to see that no $X_{\alpha^\vee}$ with $\langle\mu, \alpha^\vee\rangle = \langle\nu, \alpha^\vee\rangle = 0$ centralizes ${}^L\mathfrak{t}^+$, i.e., that no such $\alpha^\vee$ vanishes on ${}^L\mathfrak{t}^+$. However, $\langle\mu, \alpha^\vee\rangle = 0$ implies $\alpha^\vee$ is not a root of ${}^L(MA)$, as we noted above, and then $\varphi(\tau)\,\alpha^\vee \neq -\alpha^\vee$. Thus $\alpha^\vee$ does not vanish on ${}^L\mathfrak{t}^+$.

THEOREM 3.3 (LANGLANDS). *The group W' is canonically isomorphic with the Weyl group of $(\mathfrak{S}, {}^L\mathfrak{t}^+)$.*

THEOREM 3.4 (LANGLANDS). *Suppose $G_{\boldsymbol{C}}$ is semisimple and simply-connected. Then the group $F = ({}^LT^0 \cap S)/({}^LT^0 \cap S^0)$ injects into S/S^0, and the quotient is isomorphic to the R group. Moreover, $F = \{1\}$ if G is a split group and P is a minimal parabolic.*

EXAMPLE. Let G be $SU(2, 1)$, which we conjugate for convenience by

$$g = \begin{pmatrix} 2^{-1/2} & 0 & 2^{-1/2} \\ 0 & 1 & 0 \\ -2^{-1/2} & 0 & 2^{-1/2} \end{pmatrix},$$

so as to be able to take

$$T = \begin{pmatrix} e^{t+i\theta} & & \\ & e^{-2i\theta} & \\ & & e^{t-i\theta} \end{pmatrix}.$$

The Borel subgroup B we use is the upper triangular group. The group G is quasi-split but not split, and the L-group ${}^LG$ is

$${}^LG = {}^LG^0 \rtimes \boldsymbol{Z}_2 = \mathrm{PGL}(3, \boldsymbol{C}) \rtimes \{1, \sigma\},$$

where σ is a particular realization of the nontrivial outer automorphism of $\mathrm{PGL}(3, \boldsymbol{C})$. Specifically ${}^LT^0$ is the diagonal group, ${}^LT$ is ${}^LT^0 \rtimes \{1, \sigma\}$, and ${}^LB^0$ is the upper triangular group. We are led to the standard simple co-roots and use standard root vectors in $\mathrm{PGL}(3, \boldsymbol{C})$; the root vectors for the simple co-roots are

$$X_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

and we require $X_1^\sigma = X_2$, $X_2^\sigma = X_1$. Explicitly the action of σ on ${}^LG^0$ is $x^\sigma = w_0(x^{tr})^{-1}w_0^{-1}$, where

$$w_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

and $\sigma^2 = 1$. On the Lie algebra ${}^L\mathfrak{t}$, we have $\operatorname{diag}(c_1, c_2, c_3)^\sigma = (-c_3, -c_2, -c_1)$.

We assume that the homomorphism φ maps $\boldsymbol{C}^\times$ into ${}^LT^0$ and maps τ into $w \times \sigma$, where w is in $\operatorname{Norm}({}^LT^0)$. Up to conjugation of φ, there are two possibilities for w as a member of $\operatorname{Norm}({}^LT^0)/{}^LT^0$, namely the cosets of 1 and w_0. In either case, $\varphi(\tau)$ yields an involution of ${}^L\mathfrak{t}$, with $+1$ and -1 eigenspaces ${}^L\mathfrak{t} = {}^L\mathfrak{t}^+ \oplus {}^L\mathfrak{t}^-$. If $[w] = [w_0]$, then ${}^L\mathfrak{t}^+ = 0$ and we are led to discrete series. We shall be interested in the contrary case $[w] = [1]$. Say $w = x^\vee$ with $x^\vee$ in ${}^LT^0$. The action of $\varphi(\tau)$ on diagonal matrices ${}^LT^0$ is in this case the same as the action of σ.

We write members of $L = \operatorname{Hom}(T_{\boldsymbol{C}}, \boldsymbol{C}^\times)$ by abbreviating

$$\lambda(\operatorname{diag}(z, z^{-1}w^{-1}, w)) = z^a w^b$$

as

$$\lambda = a[z] + b[w].$$

Here $\{[z], [w]\}$ is a basis of L. Let $\{\delta_z, \delta_w\}$ denote the dual basis of $L^\vee = \operatorname{Hom}({}^LT^0, \boldsymbol{C}^\times)$. If

$$\lambda^\vee(\operatorname{diag}(a_1, a_2, a_3) \text{ mod center}) = (a_1/a_2)^c\,(a_3/a_2)^d, \tag{3.2}$$

then

$$\lambda^\vee = c\delta_z + d\delta_w.$$

The action of $\varphi(\tau)$ on ${}^LT^0$ yields an action on $L^\vee$ and then one on L, by duality, namely

$$(a[z] + b[w])^{\varphi(\tau)} = -b[z] - a[w].$$

Let $\varphi(z)$ be as in (3.1), for z in $\boldsymbol{C}^\times$. Since $\bar z = \tau z \tau^{-1}$ is in the Weil group, we obtain $\varphi(\bar z) = \varphi(z)^{\varphi(\tau)}$, and then (3.1) implies $\nu = \mu^{\varphi(\tau)}$. (Here the action of $\varphi(\tau)$ on $L \otimes \boldsymbol{C}$ occurs only in the L part.)

Write

$$\mu = c_1[z] + c_2[w] \quad \text{with } c_1, c_2 \text{ in } \boldsymbol{C};$$
$$\nu = \mu^{\varphi(\tau)} = -c_2[z] - c_1[w].$$

The condition that $\mu - \nu$ be in L says that $c_1 + c_2$ is in $\boldsymbol{Z}$. From p. 27 of [**12**], we see that the character of T that is to provide data for a principal series representation is in this case

$$e^H \to e^{\langle \mu, H + \sigma\bar H\rangle/2} = e^{\langle \mu, H\rangle}.$$

If $H = \operatorname{diag}(t + i\theta, -2i\theta, t - i\theta)$, we are led to the character

$$\operatorname{diag}(e^t e^{i\theta}, e^{-2i\theta}, e^{t-i\theta}) \to e^{t(c_1-c_2)}\, e^{\,i\theta(c_1+c_2)}.$$

If $\operatorname{Re}(c_1 - c_2) > 0$, we are led to nontempered representations. We shall specialize to the principal series with $c_1 - c_2 = 0$, which corresponds to $e^{in\theta}$ on the M part and trivial on the A part, by taking $\mu = n[z]/2 + n[w]/2$. These principal series representations are reducible (splitting into two parts) if and only if n is even and $\neq 0$. To see what $\varphi(z)$ is, we realize ${}^LT^0$ as

$$ {}^LT^0 = \operatorname{diag}(a_1, a_2, a_3) \text{ mod center.} $$

Apply (3.2) to the element $\varphi(z)$ with $c = 1$, $d = 0$, and then with $c = 0$, $d = 1$. Then

$$ \begin{aligned} (a_1/a_2)(z) &= \delta_z\varphi(z) = z^{\langle \mu-\nu, \delta_z\rangle}(z\bar{z})^{\langle \nu, \delta_z\rangle} \\ &= z^{\langle 2\mu, \delta_z\rangle}(z\bar{z})^{\langle -\mu, \delta_z\rangle} = z^n/|z|^n \end{aligned} $$

and

$$ (a_3/a_2)(z) = z^n/|z|^n. $$

Hence

$$ \varphi(z) = \begin{pmatrix} z^n/|z|^n & 0 & 0 \\ 0 & (z^n/|z|^n)^{-2} & 0 \\ 0 & 0 & z^n/|z|^n \end{pmatrix}^{1/3} \text{ mod center,} $$

in the sense that if the cube roots are extracted compatibly, their ambiguity disappears when we pass to $PGL(3, \boldsymbol{C})$.

To compute the image of φ completely, there is one more step—to determine $\varphi(\tau) = x^\vee \times \sigma$. We must have $\varphi(\tau)^2 = \varphi(-1)$, and we conclude $(x^\vee)(x^\vee)^\sigma = \varphi(-1)$. We can take

$$ \begin{aligned} x^\vee &= 1 && \text{if } n \text{ is even,} \\ &= \operatorname{diag}(i, 1, -i) && \text{if } n \text{ is odd.} \end{aligned} $$

Now we can compute the groups S and S^0 that are the subjects of the theorems of this section. There are three distinct cases for the image of φ, corresponding to $n = 0$, n odd, and n nonzero even. The pattern of the computation is to compute $S_1 = \operatorname{Cent} \varphi(\boldsymbol{C}^\times)$, which is connected, being the centralizer of a torus, and then to compute the centralizer of $\varphi(\tau)$ in S_1. The idea is to work as much as possible on the Lie algebra level. For $n = 0$ and n odd, we use a trick to obtain S; S is contained in the normalizer in ${}^LG^0$ of $\mathfrak{s}$, and $\mathfrak{s}$ is found to be $\mathfrak{sl}(2, \boldsymbol{C})$. Since $\mathfrak{sl}(2, \boldsymbol{C})$ has no outer automorphisms, S must conjugate S^0 by inner automorphisms. Then it is easy to see that $S = S^0$. The results are as follows.

$$ n = 0: \qquad \mathfrak{s} = \begin{pmatrix} a & b & 0 \\ c & 0 & b \\ 0 & c & -a \end{pmatrix}, \quad S = S^0, $$

$$ n \text{ odd}: \qquad \mathfrak{s} = \begin{pmatrix} a & 0 & b \\ 0 & 0 & 0 \\ c & 0 & -a \end{pmatrix}, \quad S = S^0, $$

$$ n \text{ even} \neq 0: \quad \mathfrak{s} = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -a \end{pmatrix}, \quad S = S^0 \rtimes \{1, w_0\}. $$

These results are compatible with the theorems of this section and the facts that

$$n = 0 \text{ or odd:} \quad W' = \boldsymbol{Z}_2, \quad R = \{1\},$$

$$n \text{ even} \neq 0: \quad W' = \{1\}, \quad R = \boldsymbol{Z}_2.$$

4. Irreducible tempered representations. We continue to investigate the representations $U_P(\xi, \nu)$ discussed in §2. Here ξ is a discrete series representation of M and ν is imaginary on $\mathfrak{A}$. We need a parametrization for the discrete series, and we use Harish-Chandra's. Let T^- be a compact Cartan subgroup of M, let $\mathfrak{t}^-$ be its Lie algebra, and let Z_M be the center of M. Each discrete series representation of M is determined by a nonsingular linear form λ on $i\mathfrak{t}^-$ and a character χ on Z_M. The form λ satisfies the integrality condition that $\lambda - \rho$ (with ρ equal to half the sum of the positive roots of M in some order) lifts to a character $e^{\lambda-\rho}$ on T^-, and λ and χ satisfy the compatibility condition that $e^{\lambda-\rho}$ and χ agree on their common domain. We write $\xi = \xi(\lambda, C, \chi)$, where C is the unique Weyl chamber of $i\mathfrak{t}^-$ with respect to which λ is dominant. Two such ξ's are equivalent if and only if $\chi = \chi'$ and there is some w in the Weyl group $W(T^-, M) = \text{Norm}_M(T^-)/\text{Cent}_M(T^-)$ with $w\lambda = \lambda'$ and $wC = C'$.

Representations that are "limits of discrete series" are discussed in **[18]**. We can parametrize them in the same fashion, writing $\xi = \xi(\lambda, C, \chi)$, except that λ is allowed to be singular and there is more than one choice of C that makes λ dominant. These representations are irreducible or zero, and the criterion for equivalence of nonzero ones is the same as for discrete series.

A *basic representation* is an induced representation $U_P(\xi(\lambda, C, \chi), \nu)$, with $\xi(\lambda, C, \chi)$ a limit of discrete series on M. If λ is nonsingular, so that ξ is in the discrete series, then we say that the basic representation is *induced from discrete series*. A basic representation $U_P(\xi(\lambda, C, \chi), \nu)$, has *nondegenerate data* if for each root α of $(\mathfrak{m}, \mathfrak{t}^-)$, $\langle\lambda, \alpha\rangle = 0$ implies that s_α is not in $W(T^-: M)$. A representation induced from discrete series automatically has nondegenerate data.

Nondegeneracy accomplishes several things. For one thing, it eliminates the 0 representation from consideration. For a second thing, it assigns a definite parabolic to the data for a basic representation. For example, in $\text{SL}^{\pm}(2, \boldsymbol{R})$, there is a single limit of discrete series representation, and it imbeds as a full principal series. The nondegeneracy assumption requires that this representation be viewed in the principal series. In general, it requires that a basic representation be attached to as small a parabolic as possible. A third thing that nondegeneracy accomplishes is to allow the whole discussion of W, W', Δ', and R in §2 to extend to basic representations with nondegenerate data. The theorem from **[8]**, **[9]** is as follows.

THEOREM 4.1. (a) *A basic representation with nondegenerate data is necessarily tempered, and it is irreducible if and only if its R-group is trivial.*

(b) *Two irreducible basic representations $U_P(\xi(\lambda, C, \chi), \nu)$ and $U_{P'}(\xi(\lambda', C', \chi'), \nu')$ with nondegenerate data are equivalent if and only if there is an element w in G carrying M to M', A to A', $\mathfrak{t}$ to $\mathfrak{t}'$, and (λ, C, χ, ν) to $(\lambda', C', \chi', \nu')$.*

(c) *Every irreducible tempered representation is basic and can be written with nondegenerate data.*

The R-group can be used to point to those basic representations with nondegenerate data that are needed to exhibit the reducibility of a representation induced from discrete series. The constituents of a given representation induced from discrete series, or even of all induced from an L-class of discrete series, are L-indistinguishable. A precise description of how this reducibility may be read off from the data, from the R-group, and from the set of orthogonal roots in Theorem 2.3(c) is given in [**8**].

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CORNELL UNIVERSITY

YALE UNIVERSITY

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ORBITAL INTEGRALS FOR $\mathrm{GL}_2(\boldsymbol{R})$*

D. SHELSTAD

We report briefly on the characterization of orbital integrals of smooth (C^∞) functions of compact support on $\mathrm{GL}_2(\boldsymbol{R})$, following [3]. A similar argument applies to $\mathrm{GL}_2(\boldsymbol{C})$ [3]. We begin by recalling some well-known properties of these integrals in a form convenient for the characterization, indicating the proof afterwards; a more elegant formulation is given in [3].

We fix an invariant 4-form ω_G on $G = \mathrm{GL}_2(\boldsymbol{R})$. If T is a Cartan subgroup of G we take ω_T to be the form $C_T d\gamma_1 d\gamma_2/\gamma_1\gamma_2$ where γ_1, γ_2 are the eigenvalues of γ under some order (prescribed by a diagonalization of T) and C_T is a constant as follows:

$C_T = 1$ if T is split, and $C_T = i$ otherwise. If $f \in C_c^\infty(G)$ and $\gamma \in T_{\mathrm{reg}}$, the set of regular elements of G lying in T, we set

$$\Phi_f^T(\gamma) = \int_{G/T} f(g\gamma g^{-1}) \frac{dg}{dt}$$

where dg, dt are the Haar measures defined by ω_G, ω_T respectively. Then Φ_f^T is a well-defined C^∞ function on T_{reg}, invariant under the Weyl group and vanishing off some set relatively compact in T. Let Z be the group of scalar matrices in G; thus $Z = T - T_{\mathrm{reg}}$. The behavior of Φ_f^T near $z \in Z$ is described as follows: there exist a neighborhood N_z of z in T (invariant under the Weyl group) and C^∞ functions $\Lambda_f^0(z, \ )$ and $\Lambda_f^1(z, \ )$, each defined on N_z and invariant under the Weyl group, such that

$$\Phi_f^T(\gamma) = \Lambda_f^0(z, \gamma) + \Lambda_f^1(z, \gamma)\, |D(\gamma)|^{-1/2} \tag{1}$$

for $\gamma \in N_z \cap T_{\mathrm{reg}}$. Here $D(\gamma) = (\gamma_1 - \gamma_2)^2/\gamma_1\gamma_2$ where, as before, γ_1 and γ_2 are the eigenvalues of γ. The functions $\Lambda_f^0(z, \ )$ and $\Lambda_f^1(z, \ )$ depend on T although we omit this in notation. Note that the equation (1) determines uniquely the restriction to $Z \cap N_z$ of $\Lambda_f^i(z, \ )$, and of all its derivatives. Thus we may set $\Lambda_f^i(z) = \Lambda_f^i(z', z)$ for any z' such that $z \in N_{z'}$, with a similar definition for derivatives. Further

(a) if T is split then we may take $\Lambda_f^0(\ , \ ) \equiv 0$

and if X_T denotes the image under the Harish-Chandra isomorphism of the operator X in the center of the universal enveloping algebra of $\mathfrak{gl}_2(\boldsymbol{C})$ then

(b) for each $z \in Z$, $X_T\Lambda_f^1(z)$ is independent of T.

To determine the restriction to $Z \cap N_z$ of the derivatives of $\Lambda_f^i(z, \ )$ it is suffi-

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cient to compute $X_T \Lambda_f^i(z)$ for each X; $X_T \Lambda_f^1(z)$, which we will not need explicitly, is the appropriately defined integral of Xf over the conjugacy class of $\left(\begin{smallmatrix} z & 1 \\ 0 & z \end{smallmatrix}\right)$ and if T is not split then $X_T \Lambda_f^0(z) = c_G Xf(z)$, where c_G is a constant depending only on our choice of Haar measure on G.

We recall the proof. It is sufficient to consider the Cartan subgroups A, the diagonal group, and

$$B = \left\{\begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a^2 + b^2 \neq 0\right\}.$$

We find it more convenient to write an element of B_{reg} as

$$\gamma(\lambda, \theta) = \begin{pmatrix} \lambda \cos\theta & \lambda \sin\theta \\ -\lambda \sin\theta & \lambda\cos\theta \end{pmatrix}, \qquad \lambda > 0, \theta \not\equiv 0(\pi).$$

Proceeding formally, we may choose ω_G so that

$$|D(\gamma)|^{1/2}\Phi_f^A(\gamma) = \frac{1}{2}\left|\frac{\gamma_2}{\gamma_1}\right|^{1/2}\int_N f_0(n\gamma)\,dn \tag{2}$$

where

$$N = \left\{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right\}, \qquad dn = dx, \qquad f_0(x) = \int_{K_0} f(kxk^{-1})\,dk,$$
$$K_0 = \left\{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right\}, \quad dk = d\theta$$

and

$$\Phi_f^B(\gamma(\lambda,\theta)) = \frac{1}{4}\int_0^\infty (H_\lambda(e^t\theta, e^{-t}\theta) + H_\lambda(-e^t\theta, -e^{-t}\theta))\,(e^t - e^{-t})\,dt \tag{3}$$

where

$$H_\lambda(u, v) = \int_{K_0} f\left(\lambda k \exp\begin{pmatrix} 0 & u \\ -v & 0 \end{pmatrix} k^{-1}\right) dk.$$

We obtain (2) from the Iwasawa decomposition of $\mathrm{SL}_2(\boldsymbol{R})$ and (3) from the Cartan decomposition. The function on the right-hand side of (3) can be analyzed as in **[1]**. It is easy then to see that this function can be expanded as on the right-hand side of (1). The proof is now straightforward. To compute $X_T \Lambda_f^i(z)$ note that $|D|^{1/2}\Phi_{Xf}^T = X_T(|D|^{1/2}\Phi_f^T)$. This is essentially Harish-Chandra's formula $F_{Xf}^T = X_T F_f^T$ **[2]**; $|D|^{1/2}\Phi_f^A$ is the function F_f^A and $(e^{i\theta} - e^{-i\theta})\,\Phi_f^B(\gamma(\lambda,\theta)) = F_f^B(\gamma(\lambda,\theta))$.

We come then to the characterization. Suppose that for each Cartan subgroup T we are given a function Φ^T, defined and C^∞ on T_{reg}, invariant under the Weyl group and vanishing off some set relatively compact in T. Suppose that Φ^T and $\Phi^{T'}$ satisfy the obvious consistency requirements when T and T' are conjugate. Finally, suppose that for each T and $z \in Z$ there exist a neighborhood N_z of z in T invariant under the Weyl group and C^∞ functions $\Lambda^0(z, \ )$ and $\Lambda^1(z, \ )$ on N_z, also invariant under the Weyl group, such that

$$\Phi^T(\gamma) = \Lambda^0(z, \gamma) + \frac{\Lambda^1(z,\gamma)}{|D(\gamma)|^{1/2}} \tag{4}$$

for $\gamma \in N_z \cap T_{\text{reg}}$; the functions $\Lambda^i(z, \ )$ are assumed to have the following two properties:

(a) $\Lambda^0(\ ,\ ) \equiv 0$ if T is split and

(b) for each X in the center of the universal enveloping algebra of $\mathfrak{gl}_2(C)$ the restriction to $Z \cap N_z$ of $X_T\Lambda^1(z,\ )$ is independent of T.

Then Lemma 4.1 of [3] asserts that there exists $f \in C_c^\infty(G)$ such that $\Phi_f^T = \Phi^T$ for each T. We sketch the argument.

Let $G_r = G - Z$ and $Y = \{(x_1, x_2) \in \boldsymbol{R}^2;\ x_2 \neq 0\}$. Define $\pi: G_r \to Y$ by $\pi(g) =$ (trace g, det g); π is submersive and each fiber is a conjugacy class in G. Let $S = \{(2x_1, x_1^2);\ x_1 \neq 0\}$. Then we define a function ψ on $Y - S$ by $\psi(\pi(\gamma)) = |D(\gamma)|^{1/2}\Phi^T(\gamma)$, $\gamma \in T_{\text{reg}}$, allowing T to vary among the Cartan subgroups of G. If ψ vanishes near S, that is, if each Φ^T vanishes in a neighborhood of Z, then it is easy to find $f \in C_c^\infty(G)$ such that $\Phi_f^T = \Phi^T$ for all T (via the coverings $T_{\text{reg}} \times G/T \to T_{\text{reg}}^G$). Suppose now that ψ extends to a smooth function on Y. Since ψ has compact support we may apply a partition of unity argument on Y and assume that ψ has support in some neighborhood (to be specified) of a point in S.

Fix $a \in S$ and $g \in \pi^{-1}(a)$. We can find a neighborhood N_1 of g in G_r with a coordinate system $y_1, \cdots, y_4$ such that $y_1 = x_1 \circ \pi$, $y_2 = x_2 \circ \pi$; we may as well assume that (y_i) maps N_1 to a cube in $\boldsymbol{R}^4$. Set $N_2 = \pi(N_1)$ and assume that $\psi \in C_c^\infty(N_2)$. We lift the form $|x_2|^{-3/2}dx_1dx_2$ to N_1. Using this and the invariant form ω_G we construct a G-invariant measure on each fiber of π. It is easy to find $f \in C_c^\infty(N_1)$ such that $\int_{\pi^{-1}(x)} f = \psi(x)$, $x \in N_2$. On the other hand, suppose that $\gamma \in N_1 \cap T$ and that $x = \pi(\gamma)$. Then we find that $\int_{\pi^{-1}(x)} h = |D(\gamma)|^{1/2}\,\Phi_h^T(\gamma)$, $h \in C_c^\infty(N_1)$. This is a straightforward computation with coordinates. Hence $\psi(x) = |D(\gamma)|^{1/2}\Phi_f^T(\gamma)$ and our argument is complete in the case that ψ is smooth.

We observe next that ψ extends smoothly to Y when the functions $\Lambda^0(z,\ )$ attached to the Cartan subgroup B satisfy

$$\lim_{\theta\to 0;(\theta\to\pi)} \frac{d^n}{d\theta^n}\left(|D(\gamma(\lambda, \theta))|^{1/2}\,\Lambda^0(\pm\lambda, \gamma(\lambda, \theta))\right) = 0$$

for each n or, more simply, $X_B\Lambda^0(z', z) \equiv 0$ for each X, z' and z. As before we suppress z' and write just $X_B\Lambda^0(z)$. To compute $X_B\Lambda^0(z)$ in general we resort to rapidly decreasing functions and their orbital integrals.

On G, or any real reductive group, we may introduce the space of rapidly decreasing (Schwartz) functions, as defined by Harish-Chandra [2]; [2], together with earlier papers listed there, contains an extensive analysis of the "F_f" (normalized orbital integral) transform on this space. If now f is rapidly decreasing on G then Φ_f^T has the properties listed earlier, except that in place of the statement about the support of Φ_f^T we have that $|D|^{1/2}\Phi_f^T$ is "rapidly decreasing on T_{reg}" [2]. For the characterization we take $\{\Phi^T\}$ as before, but allow $|D|^{1/2}\Phi^T$ to be just rapidly decreasing on T_{reg}. The argument is straightforward (since there are many rapidly decreasing functions with computable orbital integrals [2]), but lengthy. Here is our procedure. Consider the rapidly decreasing function $|D|^{1/2}\Phi^A$ on A. Using wave-packets [2] we find f_1 such that $F_{f_1}^A = |D|^{1/2}\Phi^A$; then $\Phi_{f_1}^A = \Phi^A$. Consider $\sin\theta\,\Phi^B(\gamma(\lambda, \theta)) - F_{f_1}^B(\gamma(\lambda, \theta))$. From (1) we see that this function is C^∞ on B (and rapidly decreasing as a function of λ). Then using essentially matrix coefficients of the discrete series representations of $\{x \in G:\ |\det x| = 1\}$ we find f_2 such that $F_{f_2}^B = \sin\theta\,\Phi^B - F_{f_1}^B$ and $F_{f_2}^A \equiv 0$. If $f = f_1 + f_2$ then $\Phi_f^T = \Phi^T$ for all T (... for a general group this argument characterizes only stable orbital integrals). We refer to [4] for details.

Returning to our original family $\{\Phi^T\}$, where Φ^T vanishes off some set relatively compact in T, we can find a rapidly decreasing function f such that $\Phi^T_f = \Phi^T$. Then $X_B\Lambda^0(z) = c_G Xf(z)$, $z \in Z$. Multiplying f, if necessary, by a suitable function of det, we may assume that $\{x \in \boldsymbol{R}^\times;\ x = \det g,\ f(g) \neq 0\}$ is relatively compact in $\boldsymbol{R}^\times$. This allows us to find in $C_c^\infty(G)$ a function f_1 which coincides with f on a neighborhood of Z. Then $Xf(z) = Xf_1(z)$ for all z and X, and the function ψ attached to $\{\Phi^T - \Phi^T_{f_1}\}$ is smooth on Y. We now argue as earlier and the proof of the characterization is complete.

Finally, fix a quasi-character χ on Z. Suppose that $f \in C^\infty(G)$ satisfies $f(zg) = \chi(z)f(g)$ for $z \in Z$, $g \in G$ and has support compact modulo Z. Then Φ^T_f is well defined and has the properties listed earlier, with the necessary modifications concerning support and transformation under Z. To characterize $\{\Phi^T_f\}$ we can argue as follows. Let $\{\Phi^T\}$ have those properties. We can easily find Φ^T_0, C^∞ on T_{reg}, invariant under the Weyl group, vanishing off a set relatively compact in T, satisfying (4) and such that $\Phi^T(\gamma) = \int_Z \chi(z^{-1})\Phi^T_0(z\gamma)\,dz$, $\gamma \in T_{\mathrm{reg}}$. For example, we may take Φ^T_0 as Φ^T multiplied by a suitable function of $|\det|$. We pick $f_0 \in C_c^\infty(G)$ such that $\Phi^T_{f_0} = \Phi^T_0$ for each T. Then f defined by $f(g) = \int_Z \chi(z^{-1})f_0(zg)\,dz$, $g \in G$, satisfies $\Phi^T_f = \Phi^T$ for each T and is of the desired form.

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Columbia University

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REPRESENTATIONS OF $\mathfrak{p}$-ADIC GROUPS: A SURVEY

P. CARTIER

Introduction. The aim of this article is to (partially) survey the present state of knowledge about the representations (mostly infinite-dimensional) of reductive algebraic groups over a local field. This includes the familar p-adic groups like $GL_n(\boldsymbol{Q}_p)$, $Sp_{2n}(\boldsymbol{Q}_p)$, ...

This theory evolved slowly and lately. The first steps were taken around 1960 by Mautner and his students who concerned themselves with a detailed study of the particular group $GL_2(\boldsymbol{Q}_p)$. The first general results were obtained by Bruhat **[8]** who imitated the 'real' methods of his thesis [7] and by Satake who determined the spherical functions **[38]**. But the next developments had to await the deep results of Bruhat and Tits **[10]**, **[11]**, **[12]** and **[13]** about the structure of $\mathfrak{p}$-adic reductive groups.

In their reference work in which they were basically concerned with the group GL_2, Jacquet and Langlands **[34]** introduced the important notion of an admissible representation. They thus opened the way towards a purely algebraic theory of these representations. The basic results about induced representations were soon after obtained by Jacquet **[32]**, who considered the case of GL_n only, but used perfectly general methods. These results have been generalized by Casselman and Harish-Chandra.

The main goal of this article will be the description and study of the principal series and the spherical functions. There shall be almost no mention of two important lines of research which are still actively pursued today:

(a) *Plancherel theorem* and detailed harmonic analysis on $\mathfrak{p}$-adic Lie groups. Here Harish-Chandra is the uncontested leader. We refer the reader to Harish-Chandra's own description of his results **[26]**, **[27]** and also to my Bourbaki lecture **[14]** for more recent results due to Harish-Chandra and Roger Howe.

(b) *Explicit construction of absolutely cuspidal representations* (the so-called 'discrete series'). Here important progress has been made by Shintani **[40]**, Gérardin **[21]** and Howe (forthcoming papers in the Pacific J. Math.). One can expect to meet here difficult and deep arithmetical questions which are barely uncovered.

Let us give a brief description of the contents of these notes. In §I, we describe the various classes of representations in a very general framework. Following Harish-Chandra **[25]**, we give the definitions for totally disconnected locally com-

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pact groups. Number theory and automorphic functions provide us with a host of such groups and their representations. Special attention is paid to various forms of *Frobenius reciprocity* and various notions of *induced representations*. Our exposition is based on an apparently novel method using tensor products (over rings without unit element, alas!).

In §II, we build up the machinery which enables us, after Jacquet, to reduce the classification of the irreducible admissible representations for a $\mathfrak{p}$-adic group G to the two following problems:

(a) *Construct the absolutely cuspidal irreducible representations for G and the Levi components of its parabolic subgroups as well.*

(b) *Study the representations induced from parabolic subgroups to the whole group.*

Our presentation of Jacquet's fundamental construction (the two 'Jacquet's functors') is based on our previous description of the induced representations and is slightly more symmetrical than usual. After specializing the previous results to the classical case of GL_n, we turn to the relation between unitary representations and admissible representations. Here the basic results are due to Harish-Chandra **[28]** (generalizing earlier results of R. Howe **[29]** for GL_n) and Bernshtein **[1]**. They show that any reductive algebraic group over a local field is of type I in the sense of von Neumann-Murray classification. The foundations are thus laid down for Plancherel theorem.

§III is devoted to the *unramified principal series*. These representations are parametrized by the so-called *unramified characters* of the Levi component of a minimal parabolic subgroup. Let us mention that those characters provide the crux of the applications to Langlands theory of L-groups (see Borel lectures in these PROCEEDINGS). In general, the representations $I(\chi)$ in this series are irreducible. They have always a nonzero vector invariant under a special maximal compact subgroup. The main result concerns the explicit construction of an equivalence between $I(\chi)$ and $I(w\chi)$ where w is an element of the (relative) Weyl group. In the case of real Lie groups, this question led to the introduction of singular integral operators (Stein, Knapp). The deep analytical problems involved in the construction of these operators are bypassed by a very ingenious trick of Casselman, making full use of Jacquet's construction $V \Rightarrow V_N$. An important role is played by the so-called Iwahori subgroups and their geometrical interpretation via buildings.

In §IV, we culminate with the theory of spherical functions. Using the results expounded in §III, one recovers Macdonald's formula **[36]**, **[37]** for these spherical functions. This theory has been highly developed in the case of real Lie groups. From the point of view of representation theory, the spherical principal series is quite special, but plays a prominent role in the applications to L-functions à la Langlands.

The whole approach leading to spherical functions through §§III and IV has been developed by Casselman in a still unpublished paper **[18]**. I borrowed extensively from this paper as well as from a preliminary paper **[17]** by the same author developing the foundations of representation theory in the $\mathfrak{p}$-adic case. The reader will have to consult these papers for the details of proofs and for numerous generalizations. It is hoped that they shall appear soon.

I have to thank several friends. Anna Helversen-Pasotto tape-recorded my lectures and made out of her tapes a transcript of the spoken words. This ungra-

tifying task proved very helpful to me when transforming the sketchy notes distributed during the conference into the present report.

Serge Lang and John Tate allowed me to use freely their notes about spherical transforms. My treatment in §IV has been influenced by them. Also John Tate corrected my misinterpretations about Frobenius reciprocity (see §1.7).

Notations and conventions. 1. Let A be a ring with unit element 1. Let $A^\times$ be the set of elements a in A for which there exists b in A with $ab = 1$. Endowed with ring multiplication, $A^\times$ is a group, the "*multiplicative group*" of A.

2. Let M be a module over a ring A without unit element. One says that M is *nondegenerate* if every element in M can be written in the form $a_1 \cdot m_1 + \cdots + a_k \cdot m_k$ with $a_1, \cdots, a_k$ in A and $m_1, \cdots, m_k$ in M.[1]

One says a sequence of elements $m_1, \cdots, m_k$ in M generates M if every element in M is of the form $a_1 \cdot m_1 + \cdots + a_k \cdot m_k + n_1 \cdot m_1 + \cdots + n_k \cdot m_k$ with $a_1, \cdots, a_k$ in A and integers $n_1, \cdots, n_k$ (of either sign). If M is nondegenerate, we can omit the terms $n_1 \cdot m_1, \cdots, n_k \cdot m_k$. The module M is *finitely generated* if there exists a finite sequence generating M.

3. By a *local field* we mean a nonarchimedean local field with finite residue field. Such a field F comes equipped with a subring $\mathfrak{O}_F$, whose elements are called the *integers* in F. There exists in $\mathfrak{O}_F$ a unique nonzero prime ideal $\mathfrak{p}_F$. Moreover there is in $\mathfrak{O}_F$ a *prime element* $\tilde{\omega}_F$ such that $\mathfrak{p}_F = \mathfrak{O}_F \cdot \tilde{\omega}_F$. Every element x in $F^\times$ can be uniquely written as $\tilde{\omega}_F^n \cdot u$ with some integer n and some u in $\mathfrak{O}_F^\times = \mathfrak{O}_F \backslash \mathfrak{p}_F$. We set $\mathrm{ord}_F(x) = n$ in this case. By convention, $\mathrm{ord}_F(0)$ is put equal to $+\infty$. The index $q = (\mathfrak{O}_F : \mathfrak{p}_F)$ is finite and a power of a prime number p. We set $|x|_F = q^{-\mathrm{ord}_F(x)}$ for x in $F^\times$ and $|0|_F = 0$. The index 'F' may be omitted in $\mathfrak{O}_F$, $\mathfrak{p}_F$, $\mathfrak{O}_F^\times$, $\tilde{\omega}_F$, $\mathrm{ord}_F(x)$ and $|x|_F$ when no confusion can arise.

There are two possibilities:

(a) If F is of characteristic 0, then F is a finite algebraic extension of the p-adic field $\boldsymbol{Q}_p$ and $\mathfrak{O}_F$ consists of the elements integrally dependent on the ring $\boldsymbol{Z}_p$ of p-adic integers.

(b) If F is of characteristic p, then F is the quotient field of the ring $\mathfrak{O}_F = \boldsymbol{F}_q[[t]]$ of formal power series in one indeterminant t with coefficients in the Galois field $\boldsymbol{F}_q$ (also denoted as $\mathrm{GF}(q)$ by various authors).

4. We use the standard notations:

$\boldsymbol{Z}$ for the ring of integers,

$\boldsymbol{Q}$ for the field of rational numbers,

$\boldsymbol{R}$ for the field of real numbers,

$\boldsymbol{C}$ for the field of complex numbers.

5. Let G be a topological group, whose unit element shall be denoted by 1. By a *character* of G we mean any continuous homomorphism $\chi: G \to \boldsymbol{C}^\times$. We say χ is *unitary* in case $\chi(g)$ is a complex number of modulus 1 for every g in G, that is [2]

$$\overline{\chi(g)} = \chi(g)^{-1} = \chi(g^{-1}).$$

[1]We make the convention that any unitary module over a ring with unit element is nondegenerate.

[2]Some authors call 'quasi-character' what we call 'character' and 'character' what we call 'unitary character'.

Assume that G is locally compact. We use the symbol $C_c(G)$ to denote the space of continuous compactly supported complex-valued functions on G. If μ is a (left invariant) Haar measure on G, we denote usually by $\int_G f(g)dg$ the integral of a function f in $C_c(G)$ w.r.t. μ. The *modular function* Δ_G is characterized by the integration rule

$$\int_G f(g)\,dg = \Delta_G(g_0)\int_G f(gg_0)\,dg \quad \text{for } f \text{ in } C_c\,(G).$$

The group G is called *unimodular* in case $\Delta_G = 1$.

One has sometimes to integrate over homogeneous spaces G/H. Suppose that f is a continuous function on G and there exists a function f_1 in $C_c(G/H)$ such that $f(g) = f_1(gH)$ for any g in G. Choose an invariant measure μ on G/H. The integral of f_1 w.r.t. μ shall often be denoted by $\int_{G/H} f(g)\,d\bar{g}$.

If G is a Lie group (real or $\mathfrak{p}$-adic) we use the corresponding German letter $\mathfrak{g}$ to denote its Lie algebra.

6. Let X be any set. The *identity map* in X shall be denoted by 1_X. If A is any subset of X, the *characteristic function* $I_A\colon X \to \{0, 1\}$ is defined by

$$\begin{aligned} I_A(x) &= 1 \quad \text{for } x \text{ in } A,\\ &= 0 \quad \text{for } x \text{ in } X\backslash A, \end{aligned}$$

where we denote by $X\backslash A$ the set-theoretic difference.

If X is finite, its cardinality shall be denoted by $|X|$.

7. We use the abbreviations 'iff' for 'if and only if' and 'w.r.t.' for 'with respect to'.

I. Totally disconnected groups and their representations.

1.1. *Groups of td-type*. Let G be a topological group. We say G is of *td-type* if every neighborhood of its unit element 1 contains a compact open subgroup. Such a group is a locally compact Hausdorff space. Moreover it is totally disconnected (hence *td*), that is there is no connected subset of G with more than one element.

Let G be a group of td-type. If X_1 and X_2 are nonempty compact open sets in G, there exists a compact open subgroup K of G such that X_1 and X_2 are unions of finitely many cosets $x_{1,1}K, \cdots, x_{1,n_1}K$ for X_1 and $x_{2,1}K, \cdots, x_{2,n_2}K$ for X_2. We set

$$(X_1\colon X_2) = n_1/n_2. \tag{1}$$

For instance, if X_1 and X_2 are compact open subgroups and X_2 is contained in X_1, then $(X_1\colon X_2)$ is the index of X_2 in X_1. The chain rule holds, namely

$$(X_1\colon X_3) = (X_1\colon X_2)\cdot(X_2\colon X_3). \tag{2}$$

Let μ be a left invariant Haar measure on G. The following formula is obvious:

$$(X_1\colon X_2) = \mu(X_1)/\mu(X_2). \tag{3}$$

Hence if $\mu(X)$ is rational for some compact open set $X \neq \varnothing$, the same is true for every such set. In this case one calls the Haar measure μ *rational*.

1.2. *Examples of groups of td-type.* (a) Let G be of td-type. Then every open subgroup, every closed subgroup of G is of td-type. A factor group of G by a closed invariant subgroup is of td-type.

(b) If $G_1, \cdots, G_n$ are groups of td-type, so is their direct product $G_1 \times \cdots \times G_n$ endowed with the product topology.

(c) Let $(G_i)_{i \in I}$ be any infinite family of groups of td-type, and let $K_i \subset G_i$ be a compact open subgroup for each i in I. In the direct product $\prod_{i \in I} G_i$, let G be the subgroup consisting of the families $(g_i)_{i \in I}$ such that the set $\{i \in I | g_i \notin K_i\}$ is finite. Then $K = \prod_{i \in I} K_i$ is a subgroup of G. We endow K with the product topology. A set U in G is open iff $gU \cap K$ is open in K for every g in G. Then G with this topology is a group of td-type and K is a compact open subgroup of G. The group G is known as the *restricted product of the groups G_i w.r.t. the groups K_i*.

(d) Let F be a local field. Then $|x - y|_F$ defines a distance in F, hence a topology. Then F as an additive group is of td-type, with $\mathfrak{O}_F$ as a compact open subgroup. Moreover $F^\times$ is open in F, hence, as a multiplicative group, it is of td-type with $\mathfrak{O}_F^\times$ as a compact open subgroup.

(e) Let $n \geqq 1$ be an integer. The linear group $\mathrm{GL}_n(F)$ is the open subspace of the n^2-dimensional space over F with coordinates $x_{11}, x_{12}, \cdots, x_{nn}$, defined by $\det(x_{ij}) \neq 0$. It is a group of td-type. A compact open subgroup is $\mathrm{GL}_n(\mathfrak{O}_F)$, the set of n-by-n matrices with entries in $\mathfrak{O}_F$, and determinant in $\mathfrak{O}_F^\times$.

(f) Let G be a subgroup of $\mathrm{GL}_n(F)$ defined as the set of common zeroes of a set of polynomials in the coordinates x_{ij} with coefficients in F. For short, G is an *algebraic subgroup of* $\mathrm{GL}_n(F)$ (more precisely, the set of F-rational points of an algebraic group defined over F). It is a closed subgroup of $\mathrm{GL}_n(F)$, hence a group of td-type on its own merits. A neighborhood basis of the unit matrix I_n in G is given by the subgroups

$$K_m = \{g = (g_{ij}) \text{ in } G \mid |g_{ij} - \delta_{ij}|_F \leqq q^{-m} \text{ for } 1 \leqq i, j \leqq n\}.$$

Let G' be an algebraic subgroup of $\mathrm{GL}_{n'}(F)$ for some integer $n' \geqq 1$. Assume that the group homomorphism $\rho: G \to G'$ is rational, that is, there exist polynomials ρ_{kl} in $F[X_{11}, \cdots, X_{nn}]$ and an integer $m \geqq 1$ such that

$$\rho(g) = (\rho_{kl}(g_{11}, \cdots, g_{nn})/(\det g)^m)_{1 \leq k, l \leq n'}$$

for $g = (g_{ij})$ in G. Then ρ is continuous w.r.t. the topologies defined on G and G' by their embedding in the linear groups $\mathrm{GL}_n(F)$ and $\mathrm{GL}_{n'}(F)$ respectively. In particular if ρ is a biregular isomorphism, i.e., ρ is a group isomorphism and ρ, ρ^{-1} are both rational, then ρ is a homeomorphism.

In more intrinsic terms, the algebraic structure on G is defined by the ring $F[G]$ of polynomial functions[3] and the topology defined above is the coarsest for which the elements of $F[G]$ are continuous mappings from G to F (F is given the topology defined in (d)).

(g) By (b) and (f), the product of finitely many algebraic groups defined over the

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[3]This ring consists of the functions $g \mapsto u(g_{11}, \cdots, g_{nn})/(\det g)^m$ for a polynomial u in $F[X_{11}, \cdots, X_{nn}]$ and an integer $m \geqq 0$.

same or distinct local fields is of td-type. Similarly, by (c) and (f), adelic groups without archimedean components are of td-type.

(h) Let F be a local field and F_{sep} a separably algebraic closure of F. Let F_{unr} be the maximal unramified extension of F contained in F_{sep} and σ the Frobenius automorphism of F_{unr} over F. Let $G_F = \text{Gal}(F_{\text{sep}}/F)$ be the Galois group of F_{sep} over F endowed with the Krull topology. It is a compact group of td-type. Let $W_F \subset G_F$ be the subgroup of automorphisms φ of F_{sep} which induce some power σ^n of σ in F_{unr}. In a unique way we can consider W_F, the *Weil group* of F, as a group of td-type in which $\text{Gal}(F_{\text{sep}}/F_{\text{unr}})$ (with Krull topology) is a compact open subgroup of W_F.

1.3. *Hecke algebra*. Let G be a group of td-type. If K is any compact open subgroup of G, we denote by $\mathscr{H}(G, K)$ the complex vector space consisting of the complex-valued functions f on G which satisfy the following two conditions:

(a) *f is bi-invariant under K, that is $f(kgk') = f(g)$ for g in G and k, k' in K.*

(b) *f vanishes off a finite union of double cosets KgK.*

Moreover, let us choose a (left invariant) Haar measure μ on G. One defines a bilinear multiplication in the complex vector space $\mathscr{H}(G, K)$ by the customary convolution formula

$$(f_1 * f_2)(g) = \int_G f_1(x) f_2(x^{-1}g)\, dx. \tag{4}$$

This integral makes sense since, as a function of x, the integrand is locally constant and compactly supported. For this multiplication $\mathscr{H}(G, K)$ becomes an associative algebra over the complex field $\boldsymbol{C}$.

Let us choose a set of representatives $\{g_\alpha\}$ for the double cosets of G modulo K, i.e., G is the disjoint union of the sets $Kg_\alpha K$. For any index α, let u_α be defined by

$$\begin{aligned} u_\alpha(g) &= \mu(K)^{-1} && \text{if } g \in Kg_\alpha K, \\ &= 0 && \text{otherwise.} \end{aligned} \tag{5}$$

In particular we may assume that $g_0 = 1$ for some index 0 and the corresponding function u_0 shall be denoted by e_K. Hence

$$\begin{aligned} e_K(g) &= \mu(K)^{-1} && \text{if } g \in K, \\ &= 0 && \text{otherwise.} \end{aligned} \tag{6}$$

The family $\{u_\alpha\}$ is a basis of the vector space $\mathscr{H}(G, K)$. Moreover, e_K is the unit element of this algebra and the multiplication table is given by $u_\alpha * u_\beta = \sum_\gamma c_{\alpha\beta\gamma} u_\gamma$ where the coefficients $c_{\alpha\beta\gamma}$ are computed as follows. The group $K_\alpha = K \cap g_\alpha K g_\alpha^{-1}$ is compact and open, hence of finite index in K. There exist therefore elements $x_1, \cdots, x_m$ of K such that K is the disjoint union of the sets $x_1 K_\alpha, \cdots, x_m K_\alpha$. Then $Kg_\alpha K$ is the disjoint union of the sets $x_1 g_\alpha K, \cdots, x_m g_\alpha K$. Define similarly K_β and the elements $y_1, \cdots, y_n$. Then $c_{\alpha\beta\gamma}$ is the number of pairs (i, j) such that $g_\gamma^{-1} x_i g_\alpha y_j g_\beta$ belong to K (see Shimura [**39**]).

When K' is a compact open subgroup of K, then $\mathscr{H}(G, K)$ is a subring of $\mathscr{H}(G, K')$ but with a different unit element if $K \neq K'$. Define $\mathscr{H}(G) = \bigcup_K \mathscr{H}(G, K)$ where K runs through a neighborhood basis of 1 consisting of compact open subgroups. Then *$\mathscr{H}(G)$ is the space of locally constant and compactly supported functions on G.* For the convolution product defined by (4), $\mathscr{H}(G)$ is an associative algebra. It has no unit element unless G is discrete, and it is commutative iff G is commutative.

The algebra $\mathscr{H}(G)$ is called the *Hecke algebra* of G and $\mathscr{H}(G, K)$ is called the Hecke algebra of G w.r.t. K.[4]

For later purposes, we need a generalization of $\mathscr{H}(G)$. Namely, let Z be a closed subgroup of the center of G and χ a character of Z. We denote by $\mathscr{H}_\chi(G)$ the set of complex-valued functions which satisfy the following conditions:

(a) *f is locally constant;*

(b) *one has $f(zg) = \chi(z)^{-1}f(g)$ for z in Z and g in G;*

(c) *f is compactly supported modulo Z.*

More explicitly, assertion (c) means that f vanishes off a set of the form $\Omega \cdot Z$ where Ω is compact in G.

Choose a Haar measure ν on G/Z. The convolution product is defined in $\mathscr{H}_\chi(G)$ by

$$(7) \qquad (f_1 *_\chi f_2)(g) = \int_{G/Z} f_1(x) f_2(x^{-1}g)\, d\bar{x}$$

(notice that the integrand takes the same value for x and xz if z belongs to Z). This product is bilinear and associative. There is no unit element in the algebra $\mathscr{H}_\chi(G)$ unless Z is open in G. When $Z = \{1\}$, this construction brings us back to $\mathscr{H}(G)$.

1.4. *Smooth representations.* By a *representation* of G, we mean as customary a pair (π, V) where V is a complex vector space and π a homomorphism from G into the group of invertible linear maps in V. If H is a subgroup of G, we denote by V^H the space of vectors v in V such that $\pi(h) \cdot v = v$ for any h in H, that is, vectors whose stabilizer in G contains H.

DEFINITION 1.1. *A representation (π, V) of G is smooth iff the stabilizer of every vector in V is open, equivalently if $V = \bigcup_K V^K$ where K runs over the compact open subgroups of G.*

Let (π, V) be a smooth representation of G and V^* the space of all linear forms on V. The *coefficient* π_{v,v^*} of π (for v in V and v^* in V^*) is defined by

$$(8) \qquad \pi_{v,v^*}(g) = \langle v^*, \pi(g) \cdot v \rangle.$$

It is a locally constant function on G.

For f in $\mathscr{H}(G)$ there exists a linear operator $\pi(f)$ acting on V and such that

$$(9) \qquad \langle v^*, \pi(f) \cdot v \rangle = \int_G f(g) \pi_{v,v^*}(g)\, dg.$$

It is computed as follows: given v in V there exist a compact open subgroup K of G, constants $c_1, \cdots, c_m$, elements $g_1, \cdots, g_m$ of G such that $v \in V^K$ and $f = \sum_{i=1}^m c_i I_{g_i K}$. Then $\pi(f) \cdot v$ is equal to $\mu(K) \cdot \sum_{i=1}^m c_i \pi(g_i) \cdot v$.

Using standard calculations one checks that $f \mapsto \pi(f)$ is an algebra homomorphism from $\mathscr{H}(G)$ into $\mathrm{End}_C(V)$; hence we may consider V as an $\mathscr{H}(G)$-module. For every compact open subgroup K of G, the operator $\pi(e_K)$ is a projection of V onto V^K; hence every vector v in V satisfies $v = \pi(e_K) \cdot v$ for a suitable K. As a corollary, V is a nondegenerate $\mathscr{H}(G)$-module.

The following facts are easily proved:

[4] For $G = \mathrm{GL}_2(\mathbf{Q}_p)$ and $K = \mathrm{GL}_2(\mathbf{Z}_p)$, this algebra is just the classical algebra of Hecke operators attached to p.

(a) *A subspace V_1 of V is stable under the operators $\pi(g)$ for g in G iff it is invariant under the operators $\pi(f)$ for f in $\mathscr{H}(G)$.*

Otherwise stated, the subrepresentations (π_1, V_1) of (π, V) correspond to the submodules of the $\mathscr{H}(G)$-module V. In particular, the representation (π, V) is (algebraically) *irreducible* iff V is a simple $\mathscr{H}(G)$-module.

The representation (π, V) is said to be *finitely generated* if there exist finitely many vectors $v_1, \cdots, v_m$ such that the transforms $\pi(g) \cdot v_i$ for g in G and $1 \leqq i \leqq m$ generate the complex vector space V. By (a), this amounts to the assertion that $(v_1, \cdots, v_m)$ generates the $\mathscr{H}(G)$-module V.

(b) *Let (π, V) and (π', V') be smooth representations of G, and let $u: V \to V'$ be a linear map. Then u satisfies $\pi'(g)u = u\pi(g)$ for every g in G iff it satisfies $\pi'(f)u = u\pi(f)$ for every f in $\mathscr{H}(G)$.*

A map $u: V \to V'$ such that $\pi'(g)u = u\pi(g)$ for every g in G is called an *intertwining map* or a *G-homomorphism*. By (b), it is nothing else but a homomorphism of $\mathscr{H}(G)$-modules.

(c) *Let (π, V) be any irreducible smooth representation of G. Assume that the topology of G has a countable*[5] *basis.*[6] *Then every intertwining map $u : V \to V$ is a scalar.*

This version of *Schur's lemma* is proved as follows. Since G has a countable basis, the index $(G: K)$ is countable for every compact open subgroup K of G; hence $\mathscr{H}(G, K)$ has a countable dimension over $\boldsymbol{C}$. Moreover, there exists a countable basis of neighborhoods of 1; hence $\mathscr{H}(G) = \bigcup_K \mathscr{H}(G, K)$ has a countable dimension. For any $v \neq 0$ in V, the map $f \mapsto \pi(f) \cdot v$ from $\mathscr{H}(G)$ to V is surjective; hence the dimension of V is countable. Let A be the algebra of intertwining maps from V into V. For any $v \neq 0$ in V, the map $u \mapsto u(v)$ of A into V is injective because (π, V) is irreducible; hence the dimension of A is countable. But A is a division algebra over the algebraically closed field $\boldsymbol{C}$. If $A \neq \boldsymbol{C}$, there exists a subfield of A isomorphic over $\boldsymbol{C}$ to the field of rational fractions $\boldsymbol{C}(x)$. In this field, the uncountably many elements $(x - \lambda)^{-1}$, for λ running over $\boldsymbol{C}$, are linearly independent. Contradiction! [This proof is due to Jacquet **[33]**.]

For instance, let z belong to the center $Z(G)$ of G. Then $\pi(z)$ commutes to $\pi(g)$ for every g in G. Hence there exists a character ω_π of $Z(G)$ such that $\pi(z) = \omega_\pi(z) \cdot 1_V$ for every z in $Z(G)$. One refers to ω_π as the *central character of π.*

(d) *Any nondegenerate $\mathscr{H}(G)$-module is associated to a unique smooth representation of G.*

To summarize, *the category of nondegenerate $\mathscr{H}(G)$-modules is identical to the category of smooth representations of G and intertwining maps.*

To conclude this section, we define the contragredient representation to a smooth representation (π, V) of G. Let K be a compact open subgroup of G. Denote by $V^*(K)$ the space of linear forms v^* on V such that $\langle v^*, \pi(e_K) \cdot v \rangle = \langle v^*, v \rangle$ for every v in V. The space $\tilde{V} = \bigcup_K V^*(K)$ is called the *smooth dual* to V. In $\tilde{V}$ there exists a smooth representation $\tilde{\pi}$ of G characterized by the relation

$$\langle \tilde{\pi}(g) \cdot \tilde{v}, v \rangle = \langle \tilde{v}, \pi(g^{-1}) \cdot v \rangle \tag{10}$$

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[5] A finite set is countable!

[6] This condition is satisfied for the algebraic groups over a local field.

for g in G, v in V and $\tilde{v}$ in $\tilde{V}$. The representation $(\tilde{\pi}, \tilde{V})$ is called the representation *contragredient* to (π, V).

1.5. *Admissible representations and their characters.* We come now to a more restricted class of representations.

DEFINITION 1.2. *A representation (π, V) of G is called admissible if it is smooth and the space V^K of vectors invariant under K is finite-dimensional for every compact open subgroup K of G.*

Fix a smooth representation (π, V) of G and a compact open subgroup K of G. Let $\mathscr{E}(K)$ be the set of (equivalence classes of) continuous irreducible finite-dimensional representations of K; every neighborhood of 1 in K contains a subgroup L of K which is compact, open and invariant in K. It follows that every representation in $\mathscr{E}(K)$ factors through the finite group K/L for such a subgroup L.

Let v be a vector in V. Since π is smooth, there exists a subgroup L as above fixing v. Let $k_1, \cdots, k_m$ be coset representatives for K modulo L. The subspace spanned by $\pi(k_1) \cdot v, \cdots, \pi(k_m) \cdot v$ is stable under K and affords a representation of the finite group K/L. It is therefore the direct sum of subspaces affording irreducible representations of K/L, hence of K. In other words, the restriction of π to K is a semisimple representation. For any $\mathfrak{d}$ in $\mathscr{E}(K)$, let $V_\mathfrak{d}$ be the subspace of V generated by the minimal K-invariant subspaces of V affording a representation of K of class $\mathfrak{d}$. The space $V_\mathfrak{d}$ is called the *isotypic component of class* $\mathfrak{d}$ of π; if ε is the one-dimensional representation of K given by $\varepsilon(k) = 1$ for k in K, then $V_\varepsilon = V^K$. More generally for any character χ of K, the space V_χ consists of the vectors v in V such that $\pi(k) \cdot v = \chi(k) \cdot v$ for every k in K.

We state now the elementary properties of admissible representations.

(a) *If (π, V) is a smooth representation of G and K is a compact open subgroup of G, then V is the direct sum $\bigoplus_{\mathfrak{d} \in \mathscr{E}(K)} V_\mathfrak{d}$. Moreover (π, V) is admissible iff $V_\mathfrak{d}$ is finite-dimensional for every $\mathfrak{d}$ in $\mathscr{E}(K)$.*

Let now $\mathfrak{d}$ in $\mathscr{E}(K)$ and let $\tilde{\mathfrak{d}}$ be the representation of K contragredient to $\mathfrak{d}$. Since the space E of $\mathfrak{d}$ is finite-dimensional, the space of $\tilde{\mathfrak{d}}$ is the dual E^* of E.

(b) *Assume (π, V) is admissible. The restriction of the pairing between $\tilde{V}$ and V to $\tilde{V}_{\tilde{\mathfrak{d}}} \times V_\mathfrak{d}$ defines a nondegenerate K-invariant bilinear form on $\tilde{V}_{\tilde{\mathfrak{d}}} \times V_\mathfrak{d}$.*

Since $V_\mathfrak{d}$ is finite-dimensional, so is $\tilde{V}_{\tilde{\mathfrak{d}}}$ and each of the spaces $V_\mathfrak{d}$ and $\tilde{V}_{\tilde{\mathfrak{d}}}$ can be identified to the dual of the other. So we get from (b) the following result:

(c) *If (π, V) is admissible, then $(\tilde{\pi}, \tilde{V})$ is admissible and the pairing between $\tilde{V}$ and V enables one to identify (π, V) with the representation contragredient to $(\tilde{\pi}, \tilde{V})$.*

We come now to the characters.

(d) *Let (π, V) be a smooth representation of G. Then π is admissible iff the operator $\pi(f)$ is of finite rank for every f in $\mathscr{H}(G)$.*

By definition a *distribution* (in the sense of Bruhat [9]) on G is a linear form on the space $\mathscr{H}(G)$ of "test functions". According to property (d) above, we can associate to any admissible representation (π, V) of G a distribution Θ_π on G by

$$\Theta_\pi(f) = \mathrm{Tr}(\pi(f)) \quad \text{for } f \text{ in } \mathscr{H}(G). \tag{11}$$

One refers to Θ_π as the *character* of π.

(e) *Let $(\pi_\alpha, V_\alpha)_{\alpha \in I}$ be a family of admissible and irreducible representations of G. Assume that π_α is inequivalent to π_β for $\alpha \neq \beta$. Then the characters Θ_{π_α} are linearly independent, hence mutually distinct.*

For the *calculation* of Θ_π we may proceed as follows. Assume to simplify matters that G has a countable basis of open sets, hence a countable basis $(K_m)_{m\geqq 1}$ for the neighborhoods of 1 consisting of a decreasing sequence of compact open subgroups. Let V_m be the space of vectors in V invariant under $\pi(K_m)$. Since π is admissible, each V_m is finite-dimensional, and $V_1 \subset V_2 \subset \cdots \subset V_m \subset \cdots$ with $V = \bigcup_m V_m$. Let g be in G. For every m, the operator $\pi(e_{K_m})\cdot\pi(g)\cdot\pi(e_{K_m})$ maps V_m into itself. Since V_m is finite-dimensional, this operator has a trace, to be denoted by $\Theta_{\pi,m}(g)$. Notice that as a function of g, $\Theta_{\pi,m}(g)$ is bi-invariant under K, hence locally constant. Let f be in $\mathscr{H}(G)$. There exists an integer $m_0 \geqq 1$ such that f belongs to $\mathscr{H}(G, K_{m_0})$ and then one gets

$$\Theta_\pi(f) = \int_G \Theta_{\pi,m}(g)f(g)\,dg \tag{12}$$

for every integer $m \geqq m_0$. This fact can also be stated as

$$\Theta_\pi = \lim_{m\to\infty} \Theta_{\pi,m} \quad \text{(weak limit in the space dual to } \mathscr{H}(G)). \tag{13}$$

For each m, let B_m be a basis of V_m over $\boldsymbol{C}$; assume that $B_1 \subset B_2 \subset \cdots \subset B_m \subset B_{m+1} \subset \cdots$. Then $B = \bigcup_m B_m$ is a basis of V over $\boldsymbol{C}$. Put $B = \{v_\alpha\}_{\alpha\in I}$; hence $B_m = \{v_\alpha\}_{\alpha\in I_m}$ for some finite subset I_m of I. Define the matrix $(\pi_{\alpha\beta}(g))$ by

$$\pi(g)\cdot v_\beta = \sum_{\alpha\in I} \pi_{\alpha\beta}(g)v_\alpha. \tag{14}$$

Then $\Theta_{\pi,m}$ can be calculated as

$$\Theta_{\pi,m}(g) = \sum_{\alpha\in I_m} \pi_{\alpha,\alpha}(g). \tag{15}$$

Hence we get the series expansion

$$\Theta_\pi = \sum_{\alpha\in I} \pi_{\alpha,\alpha} \tag{16}$$

which converges in the weak sense.

REMARK. The definitions of the convolution product by (4) and of $\pi(f)$ by (9) depend on a Haar measure μ on G. To free them from this dependence, we can proceed as follows. Let $C^\infty(G)$ be the space of locally constant complex-valued functions on G. By $C_c^\infty(G)$ we denote the space of linear forms T on $C^\infty(G)$ which satisfy the following two properties:

(1) *There exists a compact open subgroup K of G such that T is bi-invariant under K.* Namely, for f in $C^\infty(G)$ and k_1, k_2 in K, then $T(f) = T(f')$ where $f'(g) = f(k_1gk_2)$.

(2) *T is compactly supported*, namely there exists a compact open subset Ω of G such that $T(f) = 0$ for every f in $C^\infty(G)$ which vanishes identically on Ω.

By a generalized function on G we mean a linear form on $C_c^\infty(G)$; they make up a vector space $C^{-\infty}(G)$ over $\boldsymbol{C}$. We can embed in a natural way $C^\infty(G)$ as a subspace of $C^{-\infty}(G)$.

The *convolution product* on $C_c^\infty(G)$ is defined as follows: for T_i $(i = 1, 2)$ in $C_c^\infty(G)$ choose Ω_i as in (2) above and let $\Omega = \Omega_1 \cup \Omega_2$. For f in $C^\infty(G)$ there exist functions $f'_1, \cdots, f'_m, f''_1, \cdots, f''_m$ in $C^\infty(G)$ such that

$$(17)\qquad f(g_1g_2) = \sum_{i=1}^{m} f_i'(g_1)\cdot f_i''(g_2) \quad \text{for } g_1, g_2 \text{ in } \Omega.$$

Then $(T_1 * T_2)(f)$ is defined by

$$(18)\qquad (T_1 * T_2)(f) = \sum_{i=1}^{m} T_1(f_i')\cdot T_2(f_i'').$$

For T in $C_c^\infty(G)$, the operator $\pi(T)$ in V is defined in such a way that

$$(19)\qquad \langle v^*, \pi(T)\cdot v\rangle = T(\pi_{v,v^*}) \quad \text{for } v \text{ in } V,\ v^* \text{ in } V^*$$

(notice that π_{v,v^*} belongs to $C^\infty(G)$). If π is admissible, its character is the generalized function defined by $\Theta_\pi(T) = \mathrm{Tr}(\pi(T))$ for T in $C_c^\infty(G)$.

If we choose a left invariant Haar measure μ on G, we get an isomorphism of $\mathscr{H}(G)$ onto $C_c^\infty(G)$ which associates to $u \in \mathscr{H}(G)$ the linear form $f \mapsto \int_G f(g)u(g)\,dg$ on $C^\infty(G)$. By duality, one gets an isomorphism of $C^{-\infty}(G)$ with the space of distributions. This brings us back to our previous constructions.

1.6. *Absolutely cuspidal representations.* In this section, we denote by Z a closed subgroup of the center of G. We fix a character χ of Z and a Haar measure on G/Z and assume that G is unimodular. A *χ-representation* of G is a representation (π, V) of G such that $\pi(z) = \chi(z)\cdot 1_V$ for every z in Z. If π is smooth and irreducible, this means that the restriction to Z of the central character ω_π is equal to χ (at least, when G has a countable basis of open sets).

Let π be a smooth χ-representation of G. For f in $\mathscr{H}_\chi(G)$, the linear operator $\pi(f)$ in V is defined in such a way that

$$(20)\qquad \langle v^*, \pi(f)\cdot v\rangle = \int_{G/Z} \langle v^*, \pi(g)\cdot v\rangle\, f(g)\, d\bar{g}$$

holds for v in V and v^* in V^*.

It is then easily proved that the category of smooth χ-representations of G is isomorphic to the category of nondegenerate $\mathscr{H}_\chi(G)$-modules. Moreover the irreducible smooth representations correspond to the simple $\mathscr{H}_\chi(G)$-modules.

DEFINITION 1.3. *Let χ be a character of Z. A χ-representation (π, V) of G is called absolutely cuspidal* (or supercuspidal, or parabolic according to some authors) *if it is admissible and each coefficient $\pi_{v,\tilde{v}}$* (v in V, $\tilde{v}$ in $\tilde{V}$) *is compactly supported modulo* Z (hence belongs to $\mathscr{H}_{\chi^{-1}}(G)$).

A representation is called absolutely cuspidal (w.r.t. Z) if it is an absolutely cuspidal χ-representation for some character χ of Z.

One of the main properties of absolutely cuspidal representations is embodied in the so-called *Schur orthogonality relations.*

THEOREM 1.1. *Let (π, V) be any irreducible absolutely cuspidal χ-representation of G. Assume that the character $|\chi|$ of Z can be extended to a character of G. Then there exists a constant $d(\pi) > 0$ such that the following identity*

$$(21)\qquad \int_{G/Z} \langle \tilde{v}_1, \pi(g)\cdot v_1\rangle \langle \tilde{v}_2, \pi(g^{-1})\cdot v_2\rangle\, d\bar{g} = d(\pi)^{-1} \langle \tilde{v}_1, v_2\rangle \langle \tilde{v}_2, v_1\rangle$$

holds for v_1, v_2 in V and $\tilde{v}_1$, $\tilde{v}_2$ in $\tilde{V}$.

The assumption about χ is satisfied if χ is unitary or else if G is a connected reductive algebraic group over a local field.

The number $d(\pi)$ is called the *formal degree of* π. It depends on the choice of the Haar measure on G/Z. Indeed, multiplying the Haar measure by a constant $c > 0$ amounts to replacing $d(\pi)$ by $d(\pi)/c$. More invariantly, to π is associated a Haar measure ν_π on G/Z such that

$$\int_{G/Z} \langle \tilde{v}_1, \pi(g)\cdot v_1\rangle \langle \tilde{v}_2, \pi(g^{-1})\cdot v_2\rangle \, d_\pi \bar{g} = \langle \tilde{v}_1, v_2\rangle\langle \tilde{v}_2, v_1\rangle \tag{22}$$

holds for v_1, v_2 in V and $\tilde{v}_1$, $\tilde{v}_2$ in $\tilde{V}$. We denote by $d_\pi g$ the integration w.r.t. ν_π. If K is a compact open subgroup of G/Z, the number $\nu_\pi(K)$ is well defined and may be called the formal degree of π w.r.t. K. For instance assume that π is induced from a finite-dimensional representation λ of K. Then $\nu_\pi(K)$ is the degree of λ.

Theorem 1.1 has a number of interesting corollaries. Generally speaking, let (π, V) be any admissible irreducible χ-representation of G. For any linear map $u: V \to V$, the following conditions are equivalent:

(a) There exists a function f in $\mathscr{H}_\chi(G)$ such that $u = \pi(f)$ holds.

(b) There exists a compact open subgroup K of G such that $u = \pi(k)\cdot u\cdot \pi(k')$ holds for k, k' in K.

(c) There exist vectors $v_1, \cdots, v_m$ in V and linear forms $\tilde{v}_1, \cdots, \tilde{v}_m$ in $\tilde{V}$ such that

$$u(v) = \sum_{i=1}^{m} \langle \tilde{v}_i, v\rangle \cdot v_i \tag{23}$$

holds for any v in V. The set $\mathscr{L}(\pi)$ of all such operators is a subalgebra of the algebra of all linear operators in V.

Assume now that π is absolutely cuspidal. Denote by $\mathscr{K}(\pi)$ the two-sided ideal in $\mathscr{H}_\chi(G)$ consisting of the functions f such that $\pi(f) = 0$. The vector subspace $\mathscr{A}(\pi)$ of $\mathscr{H}_\chi(G)$ generated by the functions of the form $g \mapsto \langle \tilde{v}, \pi(g^{-1})\cdot v\rangle$ is then a two-sided ideal in $\mathscr{H}_\chi(G)$ and $\mathscr{H}_\chi(G)$ *is the direct sum of* $\mathscr{A}(\pi)$ and $\mathscr{K}(\pi)$. Hence we get an isomorphism $f \mapsto \pi(f)$ of the algebra $\mathscr{A}(\pi)$ with the algebra $\mathscr{L}(\pi)$. *The inverse isomorphism* is given by $u \mapsto \varphi_{\pi,u}$ where

$$\varphi_{\pi,u}(g) = d(\pi)\cdot \mathrm{Tr}\,(u\cdot \pi(g^{-1})) \tag{24}$$

for g in G and u in $\mathscr{L}(\pi)$. Notice also that we have an isomorphism $\theta: V \otimes \tilde{V} \to \mathscr{L}(\pi)$ given by

$$\theta(v \otimes \tilde{v})\cdot v' = \langle \tilde{v}, v'\rangle \cdot v \tag{25}$$

for v, v' in V and $\tilde{v}$ in $\tilde{V}$. *The three spaces* $V \otimes \tilde{V}$, $\mathscr{L}(\pi)$ *and* $\mathscr{A}(\pi)$ *carry natural representations of* $G \times G$ *and the previous isomorphisms are equivariant w.r.t. these actions of* $G \times G$.

Let Θ_π be the character of π. The projection Λ_π of $\mathscr{H}_\chi(G)$ onto $\mathscr{A}(\pi)$ with kernel $\mathscr{K}(\pi)$ is given by $\Lambda_\pi(f) = \varphi_{\pi,\pi(f)}$; hence more explicitly[7]

[7] We abuse notation by treating Θ_π as a function!

(26) $$\Lambda_\pi(f)(g) = d(\pi) \int_{G/Z} \Theta_\pi(xg^{-1})f(x)\, d\bar{x}$$

for g in G and f in $\mathscr{H}_\chi(G)$. To simplify assume that $Z = \{1\}$. Then we can rewrite formula (26) as a convolution (of a distribution and a compactly supported function!)

(27) $$\Lambda_\pi(f) = d(\pi)\,(\Theta_\pi^\vee * f),$$

where $\Theta_\pi^\vee$ is the distribution on G deduced from Θ_π by the symmetry $g \mapsto g^{-1}$ of G.

From the decomposition $\mathscr{H}_\chi(G) = \mathscr{K}(\pi) \oplus \mathscr{A}(\pi)$ one deduces easily the following theorem, due to Casselman **[16]**:

THEOREM 1.2. *Let (π, V) be an irreducible absolutely cuspidal χ-representation of G. Then V is projective in the category of nondegenerate $\mathscr{H}_\chi(G)$-modules.*

There are two important corollaries.

COROLLARY 1.1. *Any absolutely cuspidal χ-representation of G is a direct sum of irreducible absolutely cuspidal χ-representations, each counted with finite multiplicity.*

The following is a converse of Schur's lemma and follows immediately from Corollary 1.1.

COROLLARY 1.2. *Let (π, V) be any absolutely cuspidal χ-representation of G. Assume every intertwining map $u\colon V \to V$ is a scalar. Then π is irreducible.*

So far we considered one absolutely cuspidal representation at a time. We give now the second half of Schur's orthogonality relations.

THEOREM 1.3. *Assume (π, V) and (π', V') are inequivalent absolutely cuspidal χ-representations of G. Then the following relation holds*

(28) $$\int_{G/Z} \langle \tilde{v}, \pi(g)\cdot v\rangle \langle \tilde{v}', \pi(g^{-1})\cdot v'\rangle\, d\bar{g} = 0$$

whatever be v in V, v' in V', $\tilde{v}$ in $\tilde{V}$ and $\tilde{v}'$ in $\tilde{V}'$.

Let Λ be a complete set of mutually inequivalent irreducible absolutely cuspidal χ-representations of G. From Theorem 1.3, it follows immediately that the sum of the two-sided ideals $\mathscr{A}(\pi)$ (for π running over Λ) is direct in $\mathscr{H}_\chi(G)$. This sum is called the *cuspidal part of* $\mathscr{H}_\chi(G)$ to be denoted by $\mathscr{H}_\chi(G)^\circ$.

1.7. *Change of groups and Frobenius reciprocity.* Let G and G' be two groups of td-type and φ a continuous homomorphism from G into G'. Denote by $\mathscr{S}_G$ ($\mathscr{S}_{G'}$) the category of smooth representations of G (G').

The *restriction functor* φ^* from $\mathscr{S}_{G'}$ to $\mathscr{S}_G$ takes the smooth representation (π', V') of G' to the smooth representation (π, V) of G, where $V = V'$ and $\pi(g) = \pi'(\varphi(g))$ for g in G.[8]

To define the *extension functor* φ_* from $\mathscr{S}_G$ to $\mathscr{S}_{G'}$, we take the view that $\mathscr{S}_G$ consists of the nondegenerate $\mathscr{H}(G)$-modules and similarly for $\mathscr{S}_{G'}$. We consider $\mathscr{H}(G')$ both as a left $\mathscr{H}(G')$-module in the obvious way and as a right $\mathscr{H}(G)$-module

[8] If φ is the injection of G into a larger group G', we write $\mathrm{Res}_G^{G'}$ instead of φ^*.

as follows: for f' in $\mathscr{H}(G')$ and f in $\mathscr{H}(G)$, we define the convolution $f' *_\varphi f$ as the function in $\mathscr{H}(G')$ whose values are given by

$$(29) \qquad (f' *_\varphi f)(g') = \int_G f'(g' \cdot \varphi(g)^{-1})\, f(g)\, \Delta_{G'}(\varphi(g)^{-1})\, dg.$$

We define now φ_* via tensor products. Indeed for any V in $\mathscr{S}_G$, we put $\varphi_* V = \mathscr{H}(G') \otimes_{\mathscr{H}(G)} V$ and define the action of $\mathscr{H}(G')$ on $\varphi_* V$ by the rule

$$(30) \qquad f_1' \cdot (f_2' \otimes v) = (f_1' * f_2') \otimes v$$

for f_1', f_2' in $\mathscr{H}(G')$ and v in V.

Let V be in $\mathscr{S}_G$ and V' in $\mathscr{S}_{G'}$. Using the universal property of tensor products, we produce a canonical map

$$(31) \qquad \Theta\colon \mathrm{Hom}_G(V, \varphi^* V') \to \mathrm{Hom}_{G'}(\varphi_* V, V')$$

as follows: for $u\colon V \to \varphi^* V'$, the linear map $\Theta(u)$ takes $f' \otimes v$ into $f' \cdot u(v)$ (f' in $\mathscr{H}(G')$, v in V).

Frobenius reciprocity is the assertion that Θ is an isomorphism. Unfortunately, as John Tate pointed out to me, this is not true in general due to the lack of unit elements in our rings. We have to introduce another functor $\varphi^!$. First of all let us define the notion of a *generalized vector* in a representation space (π, V) for G. One defines in $\mathscr{H}(G)$ the translation operators by

$$(32) \qquad L_g f(g_1) = f(g^{-1} g_1), \qquad R_g f(g_1) = \Delta_G(g) \cdot f(g_1 g)$$

for g, g_1 in G and f in $\mathscr{H}(G)$. The space $V^{-\infty}$ of generalized vectors consists of the linear maps $u : \mathscr{H}(G) \to V$ such that $uL_g = \pi(g)u$ for every g in G. We identify any vector v in V to the generalized vector $f \mapsto \pi(f) \cdot v$. The representation π of G into V is extended to $V^{-\infty}$ by $\pi^{-\infty}(g)(u) = uR_{g^{-1}}$. It is easy to show that V consists of the 'smooth' vectors in $V^{-\infty}$, that is the generalized vectors u for which there exists a compact open subgroup K of G such that $\pi^{-\infty}(k)(u) = u$ for every k in K.

We apply this construction to a smooth representation (π', V') of G'. We get a representation $(\pi'^{-\infty}, V'^{-\infty})$ of G', hence a representation of G on the same space by the operators $\pi'^{-\infty}(\varphi(g))$. One defines $\varphi^! V'$ as the set of smooth vectors for G in the space $V'^{-\infty}$. It is clear that $\varphi^* V'$ is a subspace of $\varphi^! V'$ and the carrier of a subrepresentation for G.

Let (π, V) be a smooth representation of G. One establishes easily the following facts:

(a) *The map Θ extends to an isomorphism*

$$(33) \qquad \Theta'\colon \mathrm{Hom}_G(V, \varphi^! V') \to \mathrm{Hom}_{G'}(\varphi_* V, V').$$

Hence the functor $\varphi^!$ from $\mathscr{S}_{G'}$ to $\mathscr{S}_G$ is a right adjoint to φ_*.

By general functorial results it follows that φ_* is a right exact functor and $\varphi^!$ a left exact functor.

(b) *In order that Θ be an isomorphism for every V in $\mathscr{S}_G$ and every V' in $\mathscr{S}_{G'}$, it is necessary and sufficient that $\varphi^* V' = \varphi^! V'$ for every V' in $\mathscr{S}_{G'}$.*

In order to get a counterexample to Frobenius reciprocity, it is enough to find a smooth representation (π', V') of G' such that $V'^{-\infty} \neq V'$ and to consider the group $G = \{1\}$ since $\varphi^! V' = V'^{-\infty}$ in this case. Consider the left translations acting on

$\mathscr{H}(G')$. The identity map is a generalized vector in $\mathscr{H}(G')^{-\infty}$ and is not of the form $f \mapsto f * f_0$ for a fixed f_0 in $\mathscr{H}(G')$ unless G' is discrete. Hence $\mathscr{H}(G') \neq \mathscr{H}(G')^{-\infty}$ in this case.

(c) *Let φ' be a continuous homomorphism from G' into another group G'' of td-type. Then the functors $\varphi'_* \circ \varphi_*$ and $(\varphi' \circ \varphi)_*$ from $\mathscr{S}_G$ to $\mathscr{S}_{G''}$ are naturally isomorphic.*

This follows from the associativity of tensor products.

Assume now that φ is open, that is $\varphi(U)$ is open in G' for every open set U in G. Then, for any module V' in $\mathscr{S}_{G'}$, a generalized vector in $V'^{-\infty}$ is smooth for G iff it is smooth for G', that is belongs to V'. Hence one has $\varphi^! V' = \varphi^* V'$ and by property (b) above, *Frobenius reciprocity holds.*

Let (π, V) be any smooth representation of G. We can describe more explicitly $\varphi_* V$ as follows. Let H be the kernel of φ and let $V(H)$ be the subspace of V generated by the vectors $\pi(h) \cdot v - v$ for h in H and v in V. Moreover, by Frobenius reciprocity, there is a G-homomorphism $\iota : V \to \varphi^* \varphi_* V$ such that $\Theta(\iota)$ is the identity map in $\varphi_* V$. Finally, let $\{g'_\alpha\}_{\alpha \in A}$ be a set of representatives for the cosets $g' \cdot \varphi(G)$ in G' (notice that $\varphi(G)$ is open in G').

(d) *With the previous notations, the kernel of the linear map ι from V into $\varphi_* V$ is equal to $V(H)$ and $\varphi_* V$ is the direct sum $\bigoplus_{\alpha \in A} \pi(g'_\alpha) \cdot \iota(V)$ where π' is the representation of G' in $\varphi_* V'$ deduced from its $\mathscr{H}(G')$-module structure.*

(e) *Assume that the kernel H of φ is the union of its open compact subgroups* (for instance H is a unipotent algebraic group over a local field, or φ is injective). *Then the functor φ_* is exact.*

For every compact open subgroup K of G, the operator $\pi(e_K)$ (see §1.3, formula (6)) is a projection of V onto the set V^K of vectors invariant under K, with kernel $V(K)$. Hence $V = V^K \oplus V(K)$. Under the assumptions made in (e) one has

$$V(H) = \bigcup_K V(K) = \bigcup_K \ker \pi(e_K)$$

where K runs over the compact open subgroups of H. Hence, for every G-invariant subspace W of V, one gets $W(H) = W \cap V(H)$, hence the exactness of φ_*.

There are two special instances of the previous results. Assume first that G is an open subgroup of G' and φ is the injection of G into G'. Then we can identify V to its image by ι in $\varphi_* V$ and then $\varphi_* V = \bigoplus_{\alpha \in A} \pi'(g'_\alpha) \cdot V$.

Assume now that $\varphi(G) = G'$. Then φ defines an isomorphism of topological groups from G/H onto G'. Moreover, ι defines an isomorphism of the linear space $V_H = V/V(H)$ onto $\varphi_* V$.

1.8. *Induced representations.* In this section, we denote by G a group of td-type and by H a closed subgroup of G.

Let (π, V) be a smooth representation of H. We denote by $\mathscr{V}_\pi$ the space of functions $f : G \to V$ satisfying the following assumptions:

(a) *One has $f(hg) = \pi(h) \cdot f(g)$ for h in H and g in G.*

(b) *There exists a compact open subgroup K of G such that $f(gk) = f(g)$ for g in G and k in K.*

The group G acts on $\mathscr{V}_\pi$ by right translations, namely

$$(\theta_\pi(g) \cdot f)(g_1) = f(g_1 g) \quad \text{for } g, g_1 \text{ in } G, f \text{ in } \mathscr{V}_\pi. \tag{34}$$

The representation $(\theta_\pi, \mathscr{V}_\pi)$ of G is smooth. It is called the representation induced from π and usually denoted by $\mathrm{Ind}_H^G\,\pi$. One has a kind of *dual Frobenius reciprocity*, namely an isomorphism

$$\Theta^* : \mathrm{Hom}_H(\mathrm{Res}_H^G\,\lambda, \pi) \xrightarrow{\sim} \mathrm{Hom}_G(\lambda, \mathrm{Ind}_H^G\,\pi) \tag{35}$$

for every smooth representation (λ, W) of G. The proof is trivial.

The functions in $\mathscr{V}_\pi$ are locally constant, but assumption (b) is usually stronger than just local constancy. Denote by $\mathscr{V}_\pi^c$ the subspace of $\mathscr{V}_\pi$ consisting of the functions which vanish off a subset of the form $H\Omega$ where $\Omega \subset G$ is compact. For g in G, the translation operator $\theta_\pi(g)$ induces an operator $\theta_\pi^c(g)$ in $\mathscr{V}_\pi^c$. The representation $(\theta_\pi^c, \mathscr{V}_\pi^c)$ is called the c-induced representation from π and is usually denoted by $c\text{-}\mathrm{Ind}_H^G\,\pi$. If G/H is compact, there is no need to distinguish between $\mathscr{V}_\pi$ and $\mathscr{V}_\pi^c$, and $c\text{-}\mathrm{Ind}_H^G\,\pi = \mathrm{Ind}_H^G\,\pi$.

The adjoint of a composite functor being the composite of the adjoints in reverse order, one deduces from Frobenius reciprocity the possibility of *inducing by stage*. Namely, if L is a closed subgroup of H, there is a canonical isomorphism

$$\mathrm{Ind}_H^G\,\mathrm{Ind}_L^H\,\lambda \xrightarrow{\sim} \mathrm{Ind}_L^G\,\lambda \tag{36}$$

for any smooth representation (λ, W) of L. A similar property holds for the c-induced representations.

Let φ be the injection of H into G. We want to compare our functor φ_* to the induced representations. Define a character δ of H by

$$\delta(h) = \Delta_H(h)/\Delta_G(h) \quad \text{for } h \text{ in } H. \tag{35}$$

If χ is any character of H and (π, V) a smooth representation of H, the *twisted representation* $(\pi \otimes \chi, V)$ acts on the same space as π via the operators

$$(\pi \otimes \chi)(h) = \chi(h)\,\pi(h) \quad \text{for } h \text{ in } H. \tag{36}$$

THEOREM 1.4. *Let H be a closed subgroup of G and φ the injection of H into G. For every smooth representation (π, V) of H, $\varphi_* V$ is isomorphic to the c-induced representation $c\text{-}\mathrm{Ind}_H^G(\pi \otimes \delta^{-1})$.*

COROLLARY 1.3. *The functor $c\text{-}\mathrm{Ind}_H^G$ from $\mathscr{S}_H$ to $\mathscr{S}_G$ is exact.*

We know that φ_* is right exact (see above, p. 124). It is clear that $c\text{-}\mathrm{Ind}_H^G$ is a left exact functor, hence the corollary.

An explicit isomorphism $P_\pi \colon \varphi_* V \to \mathscr{V}_{\pi\otimes\delta^{-1}}^c$ is given as follows:

$$P_\pi(f \otimes v)(g) = \int_H f(g^{-1}h)\,\delta(h)^{-1}\,\pi(h)\cdot v\,dh \tag{37}$$

for f in $\mathscr{H}(G)$, v in V and g in G.

II. The structure of representations of $\mathfrak{p}$-adic reductive groups.

2.1. *Properties of algebraic groups*. We summarize here a few properties of algebraic groups. For a more complete exposition we refer the reader to the lectures by Springer **[41]** in these PROCEEDINGS. As usual, F is a local field.

Let $n \geqq 1$ be an integer and let G be an algebraic subgroup of $\mathrm{GL}_n(F)$. We say:

G is *unipotent* if it consists of unipotent matrices (all eigenvalues in some algebraic closure of F equal to 1);

G is a *torus* if it is connected, commutative and any element of G can be put in diagonal form in some extension of F;

G is a *split torus* if it is a torus and the eigenvalues of every element of G belong to F;

G is *reductive* if there exists no invariant connected unipotent algebraic subgroup of G with more than one element;

G is *semisimple* if it is reductive and its center is finite.

A connected reductive algebraic group G is called *split* if there exists in G a maximal torus which is split. Then every maximal split torus in G is a maximal torus.

From now on, we assume G is reductive and connected. Any split torus in G is contained in some maximal split torus of G. Any two maximal split tori in G are conjugate by an element of G, their common dimension is called the *split rank* of G. There exists in the center of G a largest split torus Z.

We do not repeat the definitions of a parabolic subgroup of G, a Borel subgroup and a quasi-split group (see **[41]**). A *parabolic pair* (P, A) consists of a parabolic subgroup P of G and a split torus A subjected to the following assumption:

If N is the unipotent radical of P (its largest unipotent invariant algebraic subgroup), there exists a connected reductive algebraic subgroup M of G such that $P = M\cdot N$ (semidirect product)[9] and A is the largest split torus contained in the center of M.

Any parabolic subgroup P can be embedded into a parabolic pair. Given P, the split torus A is unique up to conjugation by an element of N. Given (P, A), the group M is the centralizer of A in G.

One says the parabolic pair (P, A) *dominates* the parabolic pair (P', A') in case $P \supset P'$ and $A \subset A'$ hold. There exists then a parabolic subgroup P_1 of M such that $P' = P_1\cdot N$ and (P_1, A') is a parabolic pair in M. This result is used very often in proofs by induction on the dimension of G.

2.2. *Jacquet's functors.* Let be given G, P, A, M and N as above. Define two homomorphisms

$$G \xleftarrow{\ \iota\ } P \xrightarrow{\ \rho\ } M$$

were ι is the injection of P into G and $\rho(mn) = m$ for m in M and n in N. The group P is thus a kind of link between the groups G and M, and will be used to define functors relating the categories of G-modules and M-modules.

The *first Jacquet's functor* is $J_{G,M} = \rho_*\iota^*\colon \mathscr{S}_G \to \mathscr{S}_M$. Let (π, V) be a smooth representation of G. Since ι^* is simply the restriction Res^G_P, the space of the representation $J_{G,M}\,\pi$ is $V_N = V/V(N)$ where $V(N)$ is generated as a vector space by the elements $\pi(n)\cdot v - v$ for n in N and v in V. The representation of M on V_N is obtained from the restriction of π to M, which leaves $V(N)$ invariant since M normalizes N.

[9] The equation $P = M\cdot N$ is called the *Levi decomposition of P*.

THEOREM 2.1. *Suppose* (π, V) *is an admissible finitely generated representation of* G. *Then* $J_{G,M}(\pi, V) = (\pi_N, V_N)$ *is an admissible finitely generated representation of* M.

This theorem is a deep result essentially due to Jacquet **[32]** (see also **[17]**).

COROLLARY 2.1. *Assume* G *is quasi-split and* P *is a Borel subgroup of* G. *Then* V_N *is finite-dimensional.*

If G is quasi-split and P a Borel subgroup, M is a maximal torus in G, hence is commutative. It is then easy to check that any admissible finitely generated representation of M is finite-dimensional, hence the corollary.

The *second Jacquet's functor* is $J_{M,G} = \mathrm{Ind}_P^G \rho^* : \mathscr{S}_M \to \mathscr{S}_G$. More explicitly, this functor takes a smooth representation (λ, W) of M into $(\pi_\lambda, \mathscr{V}_\lambda)$ where $\mathscr{V}_\lambda$ is the space of functions $f : G \to W$ such that

(1) $$f(mng) = \lambda(m)\cdot f(g) \quad \text{for } g \text{ in } G, m \text{ in } M \text{ and } n \text{ in } N.$$

The group G acts via right translations, namely

(2) $$(\pi_\lambda(g)\cdot f)(g_1) = f(g_1 g)$$

for g, g_1 in G and f in $\mathscr{V}_\lambda$.

The following result follows easily from the compactness of G/P.

THEOREM 2.2. *Assume that* (λ, W) *is an admissible finitely generated representation of* M. *Then* $(\pi_\lambda, \mathscr{V}_\lambda)$ *is an admissible finitely generated representation of* G.

This construction is especially interesting when G is quasi-split and P is a Borel subgroup of G. We may take for (λ, W) a one-dimensional representation corresponding to a character of the maximal torus M. The corresponding representations of G comprise the *principal series* (see also §III).

The Jacquet's functors are *exact* by the results quoted in §§1.7 and 1.8. They are adjoint to each other, giving rise to a canonical isomorphism

$$\mathrm{Hom}_M(J_{G,M}\, V, W) \xrightarrow{\sim} \mathrm{Hom}_G(V, J_{M,G}\, W)$$

for any smooth representation (λ, W) of M and any smooth representation (π, V) of G.

2.3. *The main theorems.* Let G, P, A, M and N be as before. The parabolic pair (P, A) dominates a parabolic pair (P', A') where P' is a minimal parabolic subgroup of G; hence A' is a maximal split torus in G. Let Φ be the root system of G w.r.t. A' and Δ be the basis of Φ associated to P'. There exists a subset Θ of Δ such that A is the largest torus contained in the intersection of the kernels of the elements of Θ (we view the roots α in Φ as homomorphisms from A' to $F^\times$). For any real number $\varepsilon > 0$, let $A^-(\varepsilon)$ consist of the elements a in A such that $|\alpha(a)|_F \leqq \varepsilon$ for every root α in $\Delta \backslash \Theta$. Moreover, let $\bar{P}$ be the parabolic subgroup of G opposite to P. The roots associated to $\bar{P}$ are the roots $-\alpha$ where α runs over the roots associated to P. Let $\bar{N}$ be the unipotent radical of $\bar{P}$.

Let now (π, V) be any admissible finitely generated representation of G. Let $(\tilde{\pi}, \tilde{V})$ be the representation contragredient to (π, V). Using first Jacquet's functor we get representations (π_N, V_N) and $(\pi_{\bar{N}}, V_{\bar{N}})$ of M.

The following theorem is due to Casselman [**17**]. It plays a crucial role in the representation theory of reductive $\mathfrak{p}$-adic groups.

THEOREM 2.3. *There exists a unique M-invariant nondegenerate pairing $\langle \cdot, \cdot \rangle_N$ between V_N and $\tilde{V}_{\bar{N}}$ with the following property:*

Given v in V and $\tilde{v}$ in $\tilde{V}$, with canonical images u in V_N and $\tilde{u}$ in $\tilde{V}_{\bar{N}}$ respectively, there exists a real number $\varepsilon > 0$ such that

$$\langle \tilde{v}, \pi(a) \cdot v \rangle = \langle \tilde{u}, \pi_N(a) \cdot u \rangle_N \tag{4}$$

holds for every a in $A^-(\varepsilon)$.

As a matter of fact, the previous pairing identifies $(\tilde{\pi}_N, \tilde{V}_{\bar{N}})$ to the representation of M contragredient to (π_N, V_N).

The following criterion is due to Jacquet [**32**] and results easily from Theorem 2.3.

THEOREM 2.4. *Let (π, V) be any irreducible admissible representation of G. The following assertions are equivalent:*

(1) *(π, V) is absolutely cuspidal.*

(2) *For every parabolic subgroup $P \neq G$ of G, with unipotent radical N, we have $V_N = 0$.*

Note that 'absolutely cuspidal' is with reference to the maximal split torus Z in the center $Z(G)$ of G.[10]

An alternate formulation of Theorem 2.4 is as follows. Choose a character χ of Z and recall that $\mathscr{H}_\chi(G)^\circ$ is the subspace of $\mathscr{H}_\chi(G)$ generated by the coefficients of the irreducible absolutely cuspidal χ-representations of G. Then the following conditions are equivalent:

(a) *f belongs to $\mathscr{H}_\chi(G)^\circ$;*

(b) *f belongs to $\mathscr{H}_\chi(G)$ and*

$$\int_N f(gn)\, dn = 0 \tag{5}$$

holds for g in G and every subgroup N as in Theorem 2.4.

The next result is again due to Jacquet [**32**]. It is easily proved by induction on the split rank of G/Z.

THEOREM 2.5. *Let (π, V) be any irreducible admissible representation of G. There exist a parabolic pair (P, A) with associated Levi decomposition $P = M \cdot N$ and an irreducible absolutely cuspidal representation (λ, W) of M such that π is isomorphic to a subrepresentation of $\mathrm{Ind}_P^G \lambda_1$ (where λ_1 is the extension of λ to $P = M \cdot N$ given by $\lambda_1(mn) = \lambda(m)$).*

In principle, the classification problem for irreducible admissible representations of G is split into two problems:

(a) Find all irreducible absolutely cuspidal representations for G and for the groups M which occur as centralizer of A for some parabolic pair (P, A) in G.

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[10] It is well known that $Z(G)/Z$ is a compact group. It makes no essential difference to take Z or $Z(G)$ as central subgroup. The choice of Z is quite convenient however.

(b) Study the decomposition of the induced representations $\mathrm{Ind}_P^G \lambda_1$ as above, in particular look for irreducibility criteria.

Needless to say, a general answer to these problems is not yet in sight. The construction of some absolutely cuspidal representations has been given by Shintani **[40]** for $\mathrm{GL}_n(F)$ and in more general cases by Gérardin in his thesis **[21]**. The idea is to induce from some compact open subgroups.

As to problem (b), let us mention the notion of *associated parabolic subgroups.* Two parabolic pairs (P, A) and (P', A') are associated iff A and A' are conjugate by some element in G. Let $\mathscr{T}(P, A)$ be the set of irreducible admissible representations of G which occur in a composition series of some induced representation $\mathrm{Ind}_P^G \lambda_1$ where λ is an irreducible absolutely cuspidal representation of M and M is the centralizer of A in G. If (P, A) and (P', A') are associated, then $\mathscr{T}(P, A) = \mathscr{T}(P', A')$.

2.4. *An example.* Take for instance the group $G = \mathrm{GL}_n(F)$. For any (ordered) partition $n = n_1 + \cdots + n_r$ of n, let $P_{n_1, \cdots, n_r}$ consist of the matrices in block form $g = (G_{kl})_{1 \leq k, l \leq r}$ where G_{kl} is an $n_k \times n_l$ -matrix and $G_{kl} = 0$ if $k > l$. Let $A_{n_1, \cdots, n_r}$ be the set of matrices which in block form are such that $G_{kl} = 0$ for $k \neq l$ and G_{kk} is a scalar matrix $a \cdot I_{n_k}$. For $M_{n_1, \cdots, n_r}$ take the matrices in diagonal block form, i.e., $G_{kl} = 0$ for $k \neq l$ and let finally $N_{n_1, \cdots, n_r}$ be the subgroup of $P_{n_1, \cdots, n_r}$ defined by the conditions $G_{11} = I_{n_1}, \cdots, G_{rr} = I_{n_r}$. Then $(P_{n_1, \cdots, n_r}, A_{n_1, \cdots, n_r})$ is a parabolic pair, with associated Levi decomposition $P_{n_1, \cdots, n_r} = M_{n_1, \cdots, n_r} \cdot N_{n_1, \cdots, n_r}$. Up to conjugation, the pairs $(P_{n_1, \cdots, n_r}, A_{n_1, \cdots, n_r})$ comprise all parabolic pairs in $\mathrm{GL}_n(F)$. The pairs corresponding to partitions $n = n_1 + \cdots + n_r$ and $n = m_1 + \cdots + m_s$ are associated iff $r = s$ and $n_1, \cdots, n_r$ is a permutation of $m_1, \cdots, m_s$. Notice that the group $M_{n_1, \cdots, n_r}$ is isomorphic to $\mathrm{GL}_{n_1}(F) \times \cdots \times \mathrm{GL}_{n_r}(F)$.

The well-known operation $\alpha \circ \beta$ on characters of the various finite groups $\mathrm{GL}_n(q)$, introduced by Green **[24]**, may now be generalized to the case of a local field. Namely, let $n = n' + n''$, let (π', V') be an admissible representation of $\mathrm{GL}_{n'}(F)$ and (π'', V'') an admissible representation of $\mathrm{GL}_{n''}(F)$. Define a representation λ_1 of $P_{n', n''}$ acting on the space $V' \otimes V''$ by

$$\lambda_1 \begin{pmatrix} G_{11} & G_{12} \\ 0 & G_{22} \end{pmatrix} = \pi'(G_{11}) \otimes \pi''(G_{22}).$$

By induction from $P_{n', n''}$ to $\mathrm{GL}_n(F)$ one gets a representation $\pi' \circ \pi''$ of $\mathrm{GL}_n(F)$. This product is commutative and associative. In particular, given characters $\alpha_1, \cdots, \alpha_n$ of $F^\times$ one gets a representation $\alpha_1 \circ \cdots \circ \alpha_n$ of $\mathrm{GL}_n(F)$; these representations comprise the *principal series.* Given a partition $n = n_1 + \cdots + n_r$, the *intermediate series* associated to the parabolic group $P_{n_1, \cdots, n_r}$ consists of the representations of the form $\pi_1 \circ \cdots \circ \pi_r$ where π_j is an irreducible absolutely cuspidal representation of $\mathrm{GL}_{n_j}(F)$ for $j = 1, \cdots, r$. Finally the *discrete series* is the family of irreducible absolutely cuspidal representations of $\mathrm{GL}_n(F)$.

From the general results summarized in §2.3, one infers that any irreducible admissible representation of $\mathrm{GL}_n(F)$ is contained in some representation of the form $\pi_1 \circ \cdots \circ \pi_r$ where $n = n_1 + \cdots + n_r$ and π_j belongs to the discrete series of $\mathrm{GL}_{n_j}(F)$. The question of the irreducibility of the representations $\pi_1 \circ \cdots \circ \pi_r$ has now been completely settled by Bernshtein and Zhelevinski **[3]**.

2.5. *Square-integrable representations.* For the results expounded in this section, see Harish-Chandra and van Dijk [**28**, especially part I].

Let again Z be the maximal split torus contained in the center of G. Fix a unitary character χ of Z. We let $\mathscr{E}_2(G, \chi)$ denote the (equivalence classes of) irreducible unitary representations (π, V) of G which satisfy the following two conditions:

$$\pi(gz) = \chi(z)\,\pi(g) \quad \text{for } z \in Z,\ g \in G, \tag{6}$$

$$\int_{G/Z} |(u|\pi(g)\cdot v)|^2\, d\bar{g} < +\infty. \tag{7}$$

Here $(u|v)$ denotes the scalar product in the Hilbert space V.[11] Due to assumption (6), one gets

$$|(u|\pi(gz)\cdot v)|^2 = |(u|\pi(g)\cdot v)|^2;$$

hence we can integrate over G/Z in formula (7). The integral (7) is equal to $d(\pi)\|u\|^2\|v\|^2$ where the constant $d(\pi) > 0$, the *formal degree of* π, is independent from u and v. These representations are called as usual *square-integrable.*

Fix now a compact open subgroup K of G and a class $\mathfrak{d}$ of irreducible continuous representations of K. Every representation of class $\mathfrak{d}$ acts on a space of finite dimension, to be denoted by $\deg \mathfrak{d}$. Moreover, for any unitary representation (π, V) of G, denote by $(\pi : \mathfrak{d})$ the multiplicity of $\mathfrak{d}$ in the restriction of π to K.

The following theorem is easy to prove (see [**28**, p. 6]).

THEOREM 2.6. *Given* K, $\mathfrak{d}$ *and* χ *as above, one gets*

$$\sum_{\pi \in \mathscr{E}_2(G,\chi)} d(\pi)\,(\pi : \mathfrak{d}) \leqq \deg \mathfrak{d}/\mathrm{meas}(K/(Z \cap K)) \tag{8}$$

and in particular

$$(\pi : \mathfrak{d}) \leqq \frac{\deg \mathfrak{d}}{d(\pi)\cdot \mathrm{meas}(K/(Z \cap K))} \tag{9}$$

for every square-integrable irreducible representation π *of* G.

Let (π, V) be in $\mathscr{E}_2(G, \chi)$. If $\mathfrak{d}$ is the unit representation of K, the integer $(\pi : \mathfrak{d})$ is the dimension of the space V^K of vectors in V invariant under $\pi(K)$. Let $V_\infty = \bigcup_K V^K$ where K runs over the compact open subgroups of G. It is easy to check that V_∞ is dense in V and stable under $\pi(G)$. By the previous theorem we get an admissible χ-representation (π_∞, V_∞) of G. Moreover for any function f in $\mathscr{H}(G)$, the operator $\pi(f) = \int_G f(g)\pi(g)\,dg$ in the Hilbert space V has a finite-dimensional range (contained in V^K if f belongs to $\mathscr{H}(G, K)$), hence a trace $\Theta_\pi(f) = \mathrm{Tr}(\pi(f))$. To sum up, *every square-integrable irreducible representation has a character* Θ_π, *a distribution on* G.

2.6. *The fundamental estimate.* Fix a parabolic pair (P, A) where P is a minimal parabolic subgroup of G; hence A is a maximal split torus in G. Let N be the unipotent radical of P and $\tilde{N}$ the unipotent radical of the parabolic subgroup $\tilde{P}$ op-

[11] The scalar product $(u\,|\,v)$ is assumed to be linear in the *second* argument v.

posite to P. Hence there exists a basis Δ of the root system of G w.r.t. A such that N (resp. $\tilde{N}$) is associated with the roots α with positive (resp. negative) coefficients when expressed in terms of Δ.

Let A^- be the set of elements a in A such that $|\alpha(a)|_F \leqq 1$ for every root α in Δ. According to Bruhat and Tits **[13]**, there exist a compact subgroup L of A and a finitely generated semigroup S in A such that $A^- = ZLS$. Moreover there exist finitely many elements $g_1, \cdots, g_m$ in G and a compact open subgroup K_0 of G such that $G = \bigcup_{1 \leqq i \leqq m} K_0\, SZg_iK_0$ ('Cartan decomposition').

Let K be a compact open subgroup of G. We make the following assumptions:

(a) *K is invariant in K_0;*

(b) *one has $K = (K \cap P)\cdot(K \cap \tilde{N})$ and*

$$a^{-1}(K \cap \tilde{N})a \subset K \cap \tilde{N}, \qquad a(K \cap P)a^{-1} \subset K \cap P$$

for every a in S (hence the inner automorphism $g \mapsto aga^{-1}$ of G expands $K \cap \tilde{N}$ and contracts $K \cap P$).

It is known that every neighborhood of the unit element in G contains such a subgroup K.

The following estimate is due to Bernshtein **[1]**.

THEOREM 2.7. *The compact open subgroup K of G is as above. Then there exists a constant $N = N(G, K) > 0$ such that every simple module over $\mathscr{H}(G, K)$ either is infinite-dimensional (over $\mathbf{C}$) or else has a dimension bounded by N.*

Let χ be a unitary character of Z and $\mathscr{H}_\chi(G, K)^\circ$ be the subalgebra of $\mathscr{H}_\chi(G)^\circ$ consisting of the functions invariant under right and left translation by an element of K. If (π, V) is an irreducible absolutely cuspidal χ-representation of G, then V^K is a simple module over $\mathscr{H}(G, K)$; hence $\dim V^K \leqq N$ by Theorem 2.7. By a well-known argument due to Godement **[23]**, one infers from this bound the following corollary:

COROLLARY 2.2. *There is an integer $p \geqq 2$ such that the higher commutator*

$$[f_1, \cdots, f_p] = \sum_{\sigma \in S_p} \operatorname{sgn}(\sigma) f_{\sigma(1)} \cdots f_{\sigma(p)} \tag{10}$$

vanishes for arbitrary elements $f_1, \cdots, f_p$ in $\mathscr{H}_\chi(G, K)^\circ$.

2.7. *Properties of unitary representations.* In his lectures **[28]**, Harish-Chandra was unable to prove Corollary 2.2, and had to assume it[12] in order to establish the following result:

THEOREM 2.8. *Let K be a compact open subgroup of G and $\mathfrak{d}$ be a class of irreducible continuous representations of K. There exists a constant $N = N(G, K, \mathfrak{d}) > 0$ such that $(\pi: \mathfrak{d}) \leqq N$ for every irreducible unitary representation (π, V) of G.*

In the proof, one may assume that K is as in Bernshtein's Theorem 2.7 and that $\mathfrak{d}$ is the unit class of representations of K.[12]

As before (see end of §2.5), one deduces from Theorem 2.8 the following corollaries:

[12] Harish-Chandra assumes apparently the stronger 'Conjecture I' (p. 16 of **[28]**). But the proof (p. 18, end of first paragraph) uses only Corollary 2.2 above.

COROLLARY 2.3. *Let (π, V) be any irreducible unitary representation of G. Let V_∞ be the space of vectors in V stable by $\pi(K)$ for some compact open subgroup K of G. Then V_∞ is dense in V and stable under G, hence affords a smooth representation π_∞ of G. The representation (π_∞, V_∞) is admissible.*

COROLLARY 2.4. *Let (π, V) be any irreducible unitary representation of G. For any function f in $\mathscr{H}(G)$ the operator $\pi(f) = \int_G f(g)\pi(g)\, dg$ in V has a finite-dimensional range, hence a trace $\Theta_\pi(f)$.*

The distribution Θ_π is called the *character* of π.

By an easy limiting process one deduces from Corollary 2.4 that $\pi(f)$ is a compact (= completely continuous) operator in the Hilbert space V for every integrable function f on G. Hence the group G belongs to the category *CCR* of Kaplansky. In particular (see Dixmier **[19]**) every factor unitary representation of G is a multiple of an irreducible representation, there exists a Plancherel formula, ...

2.8. *Some other results*. Much more is now known about the characters of the irreducible unitary representations of G. The character Θ_π is for instance represented by a locally integrable function on G, which is locally constant on the set of regular elements (see Harish-Chandra **[26]** and my Bourbaki report **[14]**).

Moreover it is now known that Conjecture II in Harish-Chandra's lectures holds true. More precisely, if the Haar measure on G/Z is suitably normalized, the formal degree of any irreducible absolutely cuspidal representation of G is an integer. As a corollary (see **[28]**, part III]), the algebra $\mathscr{H}_\chi(G, K)^\circ$ is finite-dimensional and its dimension is bounded by a constant depending on G and K, but not on χ. Also for any character χ of Z, any compact open subgroup K of G and any class $\mathfrak{d}$ of irreducible continuous representations of K, there exist only finitely many irreducible absolutely cuspidal representations π of G such that $\omega_\pi = \chi$ and $(\pi : \mathfrak{d}) \neq 0$.

Let (π, V) be a smooth representation of G. We say (π, V) is *preunitary* if there exists a hermitian form Φ on V such that $\Phi(v, v) > 0$ for $v \neq 0$ in V and

$$\Phi(\pi(g)\cdot v, \pi(g)\cdot v') = \Phi(v, v') \tag{11}$$

for v, v' in V and g in G. We can then complete V to a Hilbert space $\hat{V}$ and extend by continuity $\pi(g)$ to a unitary operator $\hat{\pi}(g)$ in $\hat{V}$. Then $(\hat{\pi}, \hat{V})$ is a unitary representation of G. If (π, V) is admissible, V is exactly the set of smooth vectors in $\hat{V}$, that is $V = \bigcup_K \hat{V}^K$ where K runs over the compact open subgroups in G.

It follows from Theorem 2.8 and these remarks that *the classification of the irreducible unitary representations of G amounts to the search of the preunitary representations among the irreducible admissible representations of G.*

III. Unramified principal series of representations.

3.1. *Preliminaries about tori*. For this section only, we denote by k a (commutative) infinite field. Let $\boldsymbol{G}_m$ be the *multiplicative group in one variable*, considered as an algebraic group defined over k. To a connected algebraic group H defined over k, we associate two finitely generated free $\boldsymbol{Z}$-modules, namely

$$X^*(H) = \mathrm{Hom}_{k\text{-gr}}(H, \boldsymbol{G}_m), \qquad X_*(H) = \mathrm{Hom}_{\boldsymbol{Z}}(X^*(H), \boldsymbol{Z}).$$

In these formulas, $\mathrm{Hom}_{k\text{-gr}}$ (resp. $\mathrm{Hom}_{\boldsymbol{Z}}$) means the group of homomorphisms of algebraic groups defined over k (resp. of $\boldsymbol{Z}$-modules). We denote by $\langle \varphi, \lambda \rangle$ (for

φ in $X_*(H)$ and λ in $X^*(H)$) the pairing between $X_*(H)$ and $X^*(H)$. We can as well use this pairing to identify $X^*(H)$ to $\mathrm{Hom}_{\boldsymbol{Z}}(X_*(H), \boldsymbol{Z})$.

Let now S be a split torus defined over k. A sequence $(\lambda_1, \cdots, \lambda_n)$ is a basis of the free $\boldsymbol{Z}$-module $X^*(S)$ iff the mapping $s \mapsto (\lambda_1(s), \cdots, \lambda_n(s))$ is an isomorphism from S onto the product $(\boldsymbol{G}_m)^n = \boldsymbol{G}_m \times \cdots \times \boldsymbol{G}_m$ (n factors). Moreover we may identify $X_*(S)$ to $\mathrm{Hom}_{k\text{-gr}}(\boldsymbol{G}_m, S)$ in such a way that the following relation holds

$$\lambda(\varphi(t)) = t^{\langle\varphi, \lambda\rangle} \tag{1}$$

for φ in $X_*(S)$, λ in $X^*(S)$ and t in k.

By construction, the elements of $X^*(S)$ are polynomial functions on S and it is easily shown that they form a basis of the k-algebra A of such functions. Otherwise stated, S is the spectrum of the group algebra $A = k[X^*(S)]$ of the group $X^*(S)$ with coefficients in k. For any commutative k-algebra L, the L-points of S correspond therefore to the k-algebra homomorphisms from A into L, hence an isomorphism $S(L) \simeq \mathrm{Hom}(X^*(S), L^\times)$. From the duality between $X^*(S)$ and $X_*(S)$ we get another isomorphism $S(L) \simeq X_*(S) \otimes_{\boldsymbol{Z}} L^\times$.

3.2. *Unramified characters.* Let H be a connected algebraic group defined over our local field F. There exists a homomorphism $\mathrm{ord}_H : H \to X_*(H)$ characterized by

$$\langle \mathrm{ord}_H(h), \lambda\rangle = \mathrm{ord}_F(\lambda(h)) \tag{2}$$

for h in H and λ in $X^*(H)$. In the right-hand side of this formula, $\mathrm{ord}_F(\lambda(h))$ is the valuation of the element $\lambda(h)$ of $F^\times$. We denote by ${}^\circ H$ the kernel and by $\Lambda(H)$ the image of the homomorphism ord_H. By construction, one gets an exact sequence

$$1 \longrightarrow {}^\circ H \longrightarrow H \xrightarrow{\mathrm{ord}_H} \Lambda(H) \longrightarrow 1. \tag{S}$$

We can also describe ${}^\circ H$ as the set of elements h in H such that $\lambda(h) \in \mathfrak{O}_F^\times$ for any rational homomorphism λ from H into $F^\times$. Therefore ${}^\circ H$ is an open subgroup of H.

A character χ of H is called *unramified* if it is trivial on ${}^\circ H$. Otherwise stated, an unramified character is of the form $u \circ \mathrm{ord}_H$ where u is a homomorphism from $\Lambda(H)$ into $\boldsymbol{C}^\times$. Introduce the complex algebraic torus $T = \mathrm{Spec}\, \boldsymbol{C}[\Lambda(H)]$. By definition, one has $\Lambda(H) = X^*(T)$ and $T(\boldsymbol{C}) = \mathrm{Hom}(\Lambda(H), \boldsymbol{C}^\times)$. Thus, there exists a well-defined isomorphism $t \mapsto \chi_t$ between the group $T(\boldsymbol{C})$ of complex points of the torus T and the group of unramified characters of H. If H is a torus, one has

$$\chi_t(\varphi(\tilde{\omega}_F)) = \varphi(t) \tag{3}$$

for t in $T(\boldsymbol{C})$, φ in $X^*(H) = X_*(T)$ and any prime element $\tilde{\omega}_F$ of the field F.

Let again G be a connected reductive algebraic group defined over the local field F. We fix a *maximal split torus* A in G and denote by M its centralizer in G. We let $N(A)$ be the normalizer of A in G and $W = N(A)/M$ be the corresponding Weyl group. We choose also a parabolic group P such that (P, A) is a parabolic pair. Hence P is a *minimal parabolic subgroup* of G and $P = M \cdot N$ where N is the unipotent radical of P.

We denote by Φ the set of roots of G w.r.t. A and by Λ the group $\Lambda(M)$. Hence Φ is a subset of $X^*(A)$ and P defines a basis Δ of Φ. Let Δ^- be the set of roots opposite to the roots in Δ. The basis Δ^- of Φ corresponds to a parabolic subgroup

$P^- = M \cdot N^-$ of G. Let $\mathfrak{n}$ be the Lie algebra of N. For any m in M, the adjoint representation defines an automorphism $\mathrm{Ad}_{\mathfrak{n}}(m)$ of $\mathfrak{n}$. We set

(4) $$\delta(m) = |\det \mathrm{Ad}_{\mathfrak{n}}(m)|_F \quad \text{for } m \text{ in } M.$$

The group ${}^\circ A$ (resp. ${}^\circ M$) is the largest compact subgroup of A (resp. M) and ${}^\circ A$ is equal to ${}^\circ M \cap A$. Thus the inclusion of A into M gives rise to an injective homomorphism of $A/{}^\circ A$ into $M/{}^\circ M$. We do not know in general if this map is surjective (see however Borel [5, 9.5]). More precisely, the inclusion of A into M gives rise to a commutative diagram with exact lines

(D) $$\begin{array}{ccccccccc} 1 & \longrightarrow & {}^\circ A & \longrightarrow & A & \xrightarrow{\mathrm{ord}_A} & X_*(A) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & {}^\circ M & \longrightarrow & M & \xrightarrow{\mathrm{ord}_M} & \Lambda & \longrightarrow & 1 \end{array}$$

Indeed, since A is a split torus, ord_A is surjective. The inclusion of A into M enables us to identify $X_*(A)$ to a subgroup of finite index in $X_*(M)$, hence the relation $X_*(A) \subset \Lambda \subset X_*(M)$.

We denote by X the group of unramified characters of M. We may (and shall) introduce as before a complex torus T such that $X^*(T) = \Lambda$ and an isomorphism $t \mapsto \chi_t$ of $T(\mathbf{C})$ onto X. This isomorphism enables us to consider X as a complex Lie group.

The subgroup $N(A)$ of G acts on M, ${}^\circ M$, A, ${}^\circ A$ via inner automorphisms. Using diagram (D) above, we may let the Weyl group $W = N(A)/M$ operate on $X_*(M)$ so as to leave invariant the subgroups $X_*(A)$ and Λ of $X_*(M)$. The group W acts therefore on X and T by automorphisms of complex Lie groups. For instance, if χ is any unramified character of M and w any element of the Weyl group W, the transformed character $w\chi$ is given by

(5) $$(w\chi)(m) = \chi(x_w^{-1} m x_w) \quad \text{for } m \text{ in } M,$$

where x_w is any representative of w in $N(A)$. The unramified character χ of M is called *regular* if $w\chi \neq \chi$ for every element $w \neq 1$ of W.

For the applications to automorphic functions, one has to examine the case where G is *unramified over* F, that is the following hypotheses are fulfilled:

(a) *G is quasi-split over F.*

(b) *There exists an unramified extension F' of F, of finite degree d, such that G splits over F'.*

Let σ denote the Frobenius transformation of F' over F. In this situation, the L-group associated to G is defined. It is a complex connected reductive algebraic group ${}^LG^\circ$ endowed (at least) with a complex torus T', an automorphism $g' \mapsto g'^\sigma$ such that $T'^\sigma = T'$ and a homomorphism $t' \mapsto \chi'_{t'}$ of T' onto X. We say two elements g_1' and g_2' of ${}^LG^\circ$ are *σ-conjugate* if there exists h in ${}^LG^\circ$ such that $g_2' = h^{-1} g_1' h^\sigma$.

The following theorem has been proved by Gantmacher [20] and Langlands [35].

THEOREM 3.1. (a) *Any semisimple element in ${}^LG^\circ$ is σ-conjugate to an element of T'.*

(b) *Two elements t_1' and t_2' of T' are σ-conjugate iff the unramified characters $\chi'_{t'_1}$ and $\chi'_{t'_2}$ of M are conjugate under the action of the Weyl group W.*

Otherwise stated, the orbits of W in the group X of unramified characters of M are in a bijective correspondence to the σ-conjugacy classes of semisimple elements in ${}^LG^\circ$.

For more details, we refer the reader to Borel's lectures [**5**, §6, 9.5] in these PROCEEDINGS.

3.3. *The unramified principal series.* This series shall presently be defined via induction from P with the slight adjustment of $\delta^{1/2}$.

DEFINITION 3.1. *Let χ in X be any unramified character of M. We define the representation $(\nu_\chi, I(\chi))$ of G as follows:*

(a) *The space $I(\chi)$ consists of the locally constant functions f: $G \to \mathbf{C}$ such that*

$$f(mng) = \delta(m)^{1/2}\,\chi(m)f(g) \quad \text{for } m \text{ in } M,\ n \text{ in } N,\ g \text{ in } G. \tag{6}$$

(b) *The group G acts by right translations on $I(\chi)$, namely*

$$(\nu_\chi(g)\cdot f)(g') = f(g'g) \quad \text{for } f \text{ in } I(\chi),\ g,\ g' \text{ in } G. \tag{7}$$

It is important to give an alternate description of $I(\chi)$ as a factor space of $\mathscr{H}(G)$. Indeed one defines a surjective linear map $P_\chi : \mathscr{H}(G) \to I(\chi)$ by

$$P_\chi f(g) = \int_M \int_N \delta^{1/2}(m)\chi^{-1}(m) f(mng)\, dm\, dn. \tag{8}$$

(See formula (37) in §1.8.) The groups M and N are unimodular, hence the Haar measures dm on M and dn on N are left *and* right invariant. The map P_χ intertwines the right translations on $\mathscr{H}(G)$ with the representation ν_χ acting on the space $I(\chi)$.

From the general results described in §II, one gets immediately the following theorem.

THEOREM 3.2. (a) *For every χ in X, the representation $(\nu_\chi, I(\chi))$ of G is admissible.*

(b) *The representation $I(\chi^{-1})$ is isomorphic to the contragredient $(I(\chi))^\sim$ of $I(\chi)$.*

(c) *If χ is a unitary unramified character of M, the representation $(\nu_\chi, I(\chi))$ of G is preunitary.*

One of the reasons for inserting the factor $\delta^{1/2}$ in the definition of $I(\chi)$ is to get assertions (b) and (c) above. We state them more precisely: there exists a linear form J on $I(\delta^{1/2})$ invariant under the right translations by the elements of G and characterized by

$$J(P_{\delta^{1/2}} f) = \int_G f(g)\, dg \quad \text{for } f \text{ in } \mathscr{H}(G) \tag{9}$$

(see Bourbaki [**6**, p. 41] for similar calculations). For f in $I(\chi)$ and f' in $I(\chi^{-1})$, the function ff' belongs to $I(\delta^{1/2})$ and the pairing is given by

$$\langle f, f' \rangle = J(ff'). \tag{10}$$

Similarly, for χ unitary and f_1, f_2 in $I(\chi)$ the function $\bar{f}_1 f_2$ belongs to $I(\delta^{1/2})$ and the unitary scalar product in $I(\chi)$ is given by

$$(f_1 | f_2) = J(\bar{f}_1 f_2). \tag{11}$$

We now state one of the main results about irreducibility and equivalence (see also Theorem 3.10 below).

THEOREM 3.3. *Let χ be any unramified character of M.*

(a) *If χ is unitary and regular, the representation $(\nu_\chi, I(\chi))$ is irreducible.*

(b) *Let w be in W. The representations $(\nu_\chi, I(\chi))$ and $(\nu_{w\chi}, I(w\chi))$ have the same character, hence are equivalent if they are irreducible.*

(c) *The $\mathscr{H}(G)$-module $I(\chi)$ is of finite length.*

In general, if V is a module of finite length over any ring, with a Jordan-Hölder series $0 = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = V$, the semisimple module $V_{(s)} = \bigoplus_{i=1}^n V_i/V_{i-1}$ is called the *semisimplified* form of V. According to Jordan-Hölder theorem, it is uniquely defined by V up to isomorphism.

Let $I(\chi)_{(s)}$ be the semisimplified form of $I(\chi)$. It exists by Theorem 3.3(c) above. It is clear that $I(\chi)$ and $I(\chi)_{(s)}$ have the same character. Hence for any w in W, the representations $I(\chi)_{(s)}$ and $I(w\chi)_{(s)}$ are semisimple and have the same character. By the linear independence of characters, they are therefore isomorphic.

3.4. *Structure of Jacquet's module $I(\chi)_N$.* For any unramified character χ of M, let $\boldsymbol{C}_\chi$ denote the one-dimensional complex space $\boldsymbol{C}^1$ on which M acts via χ, viz. by $(m, z) \mapsto \chi(m)\cdot z$. Frobenius reciprocity takes here a simple form, namely (see §2.2):

THEOREM 3.4. *Let (π, V) be any smooth representation of G. For any unramified character χ of M, one gets an isomorphism* $\mathrm{Hom}_G(V, I(\chi)) \xrightarrow{\sim} \mathrm{Hom}_M(V_N, \boldsymbol{C}_{\chi\delta^{1/2}})$.

The proof is obvious. Indeed the relation

$$\Phi(v)(g) = \langle \varphi, \pi(g)\cdot v\rangle \quad (\text{for } g \text{ in } G,\ v \text{ in } V) \tag{12}$$

expresses an isomorphism $\Phi \leftrightarrow \varphi$ of $\mathrm{Hom}_G(V, I(\chi))$ with the space of linear forms φ on V such that

$$\langle \varphi, \pi(mn)\cdot v\rangle = \delta^{1/2}(m)\chi(m)\langle \varphi, v\rangle \tag{13}$$

for v in V, m in M and n in N. Recall that $V_N = V/V(N)$ where $V(N)$ is generated by the vectors $\pi(n)\cdot v - v$ for n in N and v in V. Any solution φ of (13) vanishes on $V(N)$, hence factors through V_N.

The previous theorem exemplifies the relevance of Jacquet's module $I(\chi)_N$ in the study of the intertwining operators between representations of the unramified principal series. We know by Theorems 2.1, 3.2 and 3.3(c) that $I(\chi)_N$ is a finite-dimensional complex vector space.

The following basic result is due to Casselman **[17]**.

THEOREM 3.5. *For any unramified character χ of M, the semisimplified form of the M-module $I(\chi)_N$ is $\bigoplus_{w\in W} \boldsymbol{C}_{(w\chi)\cdot\delta^{1/2}}$. Moreover, the group ${}^\circ M$ acts trivially on $I(\chi)_N$.*

COROLLARY 3.1. *The dimension of $I(\chi)_N$ over $\boldsymbol{C}$ is equal to the order $|W|$ of the Weyl group W.*

COROLLARY 3.2. *Assume χ regular. Then $I(\chi)_N$ as an M-module is isomorphic to $\bigoplus_{w\in W} \boldsymbol{C}_{(w\chi)\cdot\delta^{1/2}}$.*

For the proof of Corollary 3.2, notice that M acts on $I(\chi)_N$ through the commutative group $M/{}^\circ M$ isomorphic to Λ. By Schur's lemma, the semisimplified form of $I(\chi)_N$ is therefore of the form $\bigoplus_{i=1}^{s} C_{\chi_i}$ for some sequence of unramified characters $\chi_1, \cdots, \chi_s$ of M; by a well-known lemma, each representation C_{χ_i} occurs as a subrepresentation of $I(\chi)_N$. Hence for w in W, $C_{(w\chi)\cdot\delta^{1/2}}$ occurs as a subrepresentation of $I(\chi)_N$ by Theorem 3.5. Corollary 3.2 follows at once from this remark as well as the following corollary (use Frobenius reciprocity in the form of Theorem 3.4):

COROLLARY 3.3. *Let χ be any unramified character of M and w be any element of the Weyl group W. There exists a nonzero intertwining operator T_w: $I(\chi) \to I(w\chi)$. If χ is regular this operator T_w is unique up to a scalar.*

A similar argument shows that if (π, V) is an irreducible subquotient of $I(\chi)$, then there exists $w \in W$ such that (π, V) is isomorphic to a subrepresentation of $I(w\cdot\chi)$ (see 6.3.9 in [**17**]).

We shall describe more explicitly the operators T_w in §3.7.

We sketch now a proof of Theorem 3.5, a streamlined version of Casselman's proof of more general results in [**17**]. Basically, it is Mackey's double coset technique extended from finite groups to $\mathfrak{p}$-adic groups. The case of real Lie groups has been considered by Bruhat in his thesis [**7**], it is much more elaborate.

(A) We remind the reader of Bruhat's decomposition $G = \bigcup_w PwP$, where PwP means PgP for any g in $N(A)$ representing the element w in $W = N(A)/M$. For w in W, the set PwP is irreducible and locally closed in the Zariski topology, hence has a well-defined dimension. Set $d(w) = \dim(PwP) - \dim(P)$. For any integer $r \geqq 0$, let F_r be the union of the double cosets PwP such that $d(w) < r$. The Zariski closure of any double coset PwP is a union of double cosets $Pw'P$ of smaller dimension, hence F_r is Zariski closed, hence closed in the $\mathfrak{p}$-adic topology. We let I_r be the subspace of $I(\chi)$ consisting of the functions vanishing identically on F_r. We have then a decreasing filtration

$$I(\chi) = I_0 \supset I_1 \supset \cdots \supset I_r \supset I_{r+1} \supset \cdots \tag{14}$$

of $I(\chi)$ by *P-stable subspaces.*

(B) *The next step is to prove that any function on F_{r+1} which satisfies the relation*

$$f(mng) = \delta^{1/2}(m)\chi(m)f(g) \quad \text{for } m \text{ in } M,\ n \text{ in } N,\ g \text{ in } F_{r+1} \tag{15}$$

is the restriction of some function belonging to $I(\chi)$. Indeed, one proves easily (using local cross-sections of G fibered over $P\backslash G$) that such a function is of the form

$$f(g) = \int_M \int_N \delta^{1/2}(m)\chi^{-1}(m)\varphi(mng)\, dm\, dn$$

for a suitable locally constant and compactly supported function φ on F_{r+1}. Extend φ to a function φ' in $\mathscr{H}(G)$. Then $P_\chi\varphi'$ belongs to $I(\chi)$ and restricts to f in F_{r+1}.

(C) From this it follows that I_r/I_{r+1} is the space of functions f on F_{r+1} which satisfy the following conditions:

(a) *f is locally constant*;

(b) *f vanishes on F_r*;

(c) *relation* (15).

Moreover, F_{r+1} is the union of F_r and the various double cosets PwP such that $d(w) = r$, which are *open* in F_{r+1}; hence one gets an isomorphism

(16) $$I_r/I_{r+1} \simeq \bigoplus_{d(w)=r} J_w.$$

Here J_w is the space of functions f on PwP such that

$$f(mng) = \delta^{1/2}(m)\chi(m)f(g) \quad \text{for } m \text{ in } M,\ n \text{ in } N, g \text{ in } PwP,$$

and which vanish outside a set of the form $P\Omega$ where Ω is compact. Since Jacquet's functor $V \Rightarrow V_N$ is exact, one infers from (16) that the *$\mathscr{H}(M)$-modules $I(\chi)_N$ and $\bigoplus_{w\in W}(J_w)_N$ have isomorphic semisimplified forms.*

(D) It remains to identify the representation of M on the space $(J_w)_N$. Here we are paid off the dividends of our approach to induced representations via tensor products.

Choose a representative x_w of w in $N(A)$ and put $P(w) = P \cap x_w^{-1}Px_w$; hence $P(w) = M \cdot N(w)$ with a suitable subgroup $N(w)$ of N. It is then easy to show that, as a P-module, J_w carries the representation $c\text{-Ind}_{P(w)}^{P}\,\sigma_w$ where the character σ_w of $P(w)$ is defined by

(17) $$\sigma_w(mn) = w^{-1}(\delta^{1/2}\chi)\,(m) \quad \text{for } m \text{ in } M,\ n \text{ in } N(w).$$

Consider the group homomorphisms $P(w) \xrightarrow{\alpha} P \xrightarrow{\beta} M$ where α is the injection and $\beta\,(mn) = m$ for m in M and n in N. By Theorem 1.4, one gets $c\text{-Ind}_{P(w)}^{P}\,\sigma_w = \alpha_*(\sigma_w \cdot \delta_w)$ where the character δ_w of $P(w)$ is defined by

(18) $$\delta_w(p) = \Delta_{P(w)}(p)/\Delta_P(p) \quad \text{for } p \text{ in } P(w).$$

Since β^* is Jacquet's functor $V \Rightarrow V_N$ and $\beta_* \circ \alpha_* = (\beta \circ \alpha)_*$ one gets that $(J_w)_N$ is the carrier of the representation $(\beta \circ \alpha)_*(\sigma_w \cdot \delta_w)$. Since $\beta \circ \alpha$ is the projection of $P(w)$ onto M with kernel $N(w)$ and the characters σ_w and δ_w are trivial on $N(w)$ one gets an isomorphism $(J_w)_N \simeq \boldsymbol{C}_\lambda$ where $\lambda = \sigma_w\delta_w|M$. It remains to prove the formula (see formula (24_c) below) $\delta_w^{-1}|M = \delta^{1/2}(w^{-1}\delta)^{-1/2}$ to be able to conclude

(19) $$(J_w)_N \simeq \boldsymbol{C}_{(w^{-1}\chi)\cdot\delta^{1/2}}.$$

(E) From (C) and (D) above, we know that $I(\chi)_N$ and $\bigoplus_{w\in W} \boldsymbol{C}_{(w\chi)\cdot\delta^{1/2}}$ have isomorphic semisimplified forms. It remains to show that *$^\circ M$ acts trivially on $I(\chi)_N$*. But $^\circ M$ is a *compact* subgroup of M. Hence its action on $I(\chi)_N$ and its semisimplified form are equivalent. Since any unramified character of M (including δ) is trivial on $^\circ M$, this group acts trivially on the semisimplified form of $I(\chi)_N$, hence on $I(\chi)_N$.

This concludes our proof of Theorem 3.5.

3.5. *Buildings and Iwahori subgroups.* Our aim in this section is mainly to fix notations. For more details, we refer the reader to the lectures by Tits in these Proceedings **[42]** or to the book by Bruhat and Tits **[13]**.

Let $\mathscr{B}$ be the building associated to G and let $\mathscr{A}$ be the apartment in $\mathscr{B}$ associated to the split torus A. We choose once for all a special vertex x_0 in $\mathscr{A}$. Among the conical chambers in $\mathscr{A}$ with apex at x_0 there is a unique one, $\mathscr{C}$ say, enjoying the following property: for every n in N, the intersection $\mathscr{C} \cap n\mathscr{C}$ contains a translate of $\mathscr{C}$. There is then a unique chamber C contained in $\mathscr{C}$ having x_0 for one of its vertices (see Figure 1). The stabilizer of x_0 in G shall be denoted by K; it is called by Bruhat and Tits a *special, good, maximal compact subgroup* of G. The interior points of C all have the same stabilizer B in G, called the *Iwahori subgroup of G* attached to C.

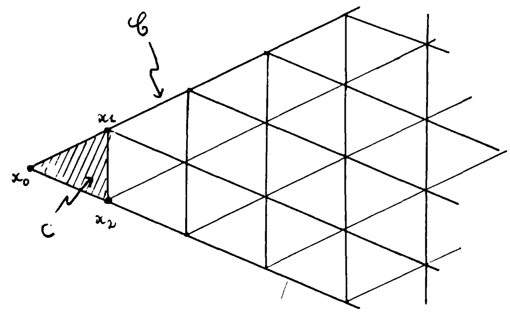

FIGURE 1

Any element g in $N(A)$ takes the apartment $\mathscr{A}$ to itself, it fixes every point of $\mathscr{A}$ iff it belongs to $^\circ M$. We may therefore identify the group $W_1 = N(A)/{}^\circ M$, called the *modified Weyl group*, to a group of affine linear transformations in $\mathscr{A}$. The group W_1 can be represented in two different ways as a semidirect product:

(a) Let Ω be the subgroup of W_1 consisting of the w's taking the chamber C to itself. The walls in $\mathscr{A}$ are certain hyperplanes and to each of them is associated a certain reflection. These reflections generate the invariant subgroup W_{aff} of W_1. Since W_{aff} acts simply transitively on the set of chambers contained in $\mathscr{A}$, the group W_1 is the semidirect product $\Omega \cdot W_{\text{aff}}$.

(b) Any element w of W has a representative $\omega(w)$ in $K \cap N(A)$ and $^\circ M = K \cap M$. We may therefore identify the Weyl group $W = N(A)/M$ to the stabilizer $(K \cap N(A))/(K \cap M)$ of x_0 in W_1. The intersection of W_1 with the group of translations in $\mathscr{A}$ is $M/{}^\circ M$ which we identify to Λ by means of the exact sequence (D), p. 135. Then W_1 is the semidirect product $W \cdot \Lambda$ where Λ is an invariant subgroup.

The fundamental structure theorems may now be formulated as follows.

IWASAWA DECOMPOSITION. *$G = PK$ and, more precisely, G is the disjoint union of the sets $P\omega(w)B$ for w running over the Weyl group W.*

BRUHAT-TITS DECOMPOSITION. *G is the disjoint union of the sets Bw_1B for w_1 running over the modified Weyl group W_1.*

CARTAN DECOMPOSITION. *Let Λ^- be the subset of Λ consisting of the elements of Λ taking the conical chamber $\mathscr{C}^-$ of $\mathscr{A}$ opposite to $\mathscr{C}$ into itself. Then G is the disjoint union of the sets $K \cdot \operatorname{ord}_M^{-1}(\lambda) \cdot K$ for λ running over Λ.*

IWAHORI DECOMPOSITION. *$B = (B \cap N^-) \cdot (B \cap M) \cdot (B \cap N)$ (unique factorization). Moreover one has $B \cap M = {}^\circ M$ and*

$$m(B \cap N)m^{-1} \subset B \cap N, \qquad m^{-1}(B \cap N^-)m \subset B \cap N^-$$

for any m in M such that $\operatorname{ord}_M(m) \in \Lambda^-$.

As before, we denote by Φ the system of roots of G w.r.t. A. We identify the elements of Φ to affine linear functions on $\mathscr{A}$ in such a way that $\alpha(x_0 + \lambda) = \langle \lambda, \alpha \rangle$

for α in Φ and λ in Λ. The conical chamber $\mathscr{C}$ is then defined as the set of points x in $\mathscr{A}$ such that $\alpha(x) > 0$ for every root α in the basis Δ of Φ associated to the parabolic group P.

We denote by Φ_0 the set of affine linear functions α on $\mathscr{A}$, vanishing at x_0 and such that the hyperplane $\alpha^{-1}(r)$ is a wall in $\mathscr{A}$ iff the real number r is an integer. An *affine root* is a function on $\mathscr{A}$ of the form $\alpha + k$ where α belongs to Φ_0 and k is an integer; their set is denoted by Φ_{aff}. For any affine root α, the reflection in the wall $\alpha^{-1}(0)$ is denoted by S_α. The reflections S_α for α running over Φ_0 (resp. Φ_{aff}) generate the group W (resp. W_{aff}). For α in Φ_0, there exists a unique vector t_α in Λ such that

$$S_\alpha(x) = x - \alpha(x)\cdot t_\alpha \quad \text{for any } x \text{ in } \mathscr{A}. \tag{20}$$

We denote by a_α any element in M such that $t_\alpha = \text{ord}_M(a_\alpha)$. The set Φ_1 is obtained by adjoining to Φ_0 the set of functions $\alpha/2$ for α in Φ_0 such that $(B\omega(S_\alpha)B\colon B) \neq \delta(a_\alpha)^{-1/2}$.

The sets Φ, Φ_0 and Φ_1 are root systems in the customary sense (see for instance [**37**, p. 14 sqq.]). When the group G is split, the sets Φ, Φ_0 and Φ_1 are identical. In the nonsplit case, all we can assert is that, for any α in Φ, there exists a unique root $\lambda(\alpha)$ in Φ_0 proportional to α, and that any element of the reduced root system Φ_0 is of the form $\lambda(\alpha)$ for a suitable α in Φ.

Let w_1 be any element of W_1. Since w_1 is a coset modulo the subgroup ${}^\circ M$ of B the set Bw_1B is a double coset modulo B. We put

$$q(w_1) = (Bw_1B\colon B). \tag{21}$$

Since $K = BWB$, one gets

$$(K\colon B) = \sum_{w\in W} q(w). \tag{22}$$

Let Δ_1 be the set of affine roots in $\mathscr{A}$ which are positive on the chamber C and whose null set is a wall of C. The group W_1 is generated by Ω and the reflections S_α for α in Δ_1. The value of $q(w_1)$ is given by

$$q(w_1) = q(S_{\alpha_1}) \cdots q(S_{\alpha_m}) \tag{23}$$

where $w_1 = \omega S_{\alpha_1} \cdots S_{\alpha_m}$ is a decomposition of minimal length m (ω in Ω, $\alpha_1, \cdots, \alpha_m$ in Δ_1).

To each root β in Φ_1 is associated a real number $q_\beta > 0$. This association is characterized by the following set of properties

$$q_{w\cdot\beta} = q_\beta \quad \text{for } \beta \text{ in } \Phi_1 \text{ and } w \text{ in } W, \tag{24$_\text{a}$}$$

$$q(w) = \prod_{\beta>0;\, w^{-1}\cdot\beta<0} q_\beta, \tag{24$_\text{b}$}$$

$$\delta(m) = \prod_{\beta>0} q_\beta^{-\beta(m\cdot x_0)} \quad \text{for } m \text{ in } M. \tag{24$_\text{c}$}$$

In the previous formulas β is a variable element in Φ_1 and the notation $\beta > 0$ means that β takes only positive values on $\mathscr{C}$. We make the convention that $q_{\alpha/2} = 1$ for α in Φ_0 if $\alpha/2$ does not belong to Φ_1. Two corollaries of the previous relations are worth mentioning

$$q(S_\alpha) = q_\alpha q_{\alpha/2}, \tag{24$_\text{d}$}$$

$$\delta(a_\alpha) = q_\alpha^{-2}\, q_{\alpha/2}^{-1}, \tag{24$_e$}$$

for any $\alpha > 0$ in Φ_0. When G is split, q_β is equal to the order q of the residue field $\mathfrak{O}_F/\mathfrak{p}_F$ for any root β in $\Phi = \Phi_0 = \Phi_1$.

The structure of the Hecke algebra $\mathscr{H}(G, B)$ has been described by Iwahori and Matsumoto **[30]**, **[31]**.

THEOREM 3.6. *For w_1 in W_1, let $C(w_1)$ be the characteristic function of the double coset Bw_1B.*

(a) *The family $\{C(w_1)\}_{w_1 \in W_1}$ is a basis of the complex vector space $\mathscr{H}(G, B)$ (Bruhat-Tits decomposition!).*

(b) *Let w_1 be any element of W_1 and let $\omega S_{\alpha_1} \cdots S_{\alpha_m}$ (ω in Ω, $\alpha_1, \cdots, \alpha_m$ in Δ_1) be a decomposition of w_1 of minimal length m. Then*

$$C(w_1) = C(\omega)C(S_{\alpha_1})\cdots C(S_{\alpha_m}). \tag{25}$$

(c) *For each α in Δ_1, one has*

$$(C(S_\alpha) - 1)\cdot(C(S_\alpha) + q(S_\alpha)) = 0, \tag{26}$$

where $q(S_\alpha)$ has been defined by formula (21) *above.*

(d) *If α and β are distinct elements in Δ_1, there exists an integer $m_{\alpha\beta} \geqq 2$ such that*

$$\underbrace{C(S_\alpha)C(S_\beta)C(S_\alpha)\cdots}_{m_{\alpha\beta}\ factors} = \underbrace{C(S_\beta)C(S_\alpha)C(S_\beta)\cdots}_{m_{\alpha\beta}\ factors}. \tag{27}$$

Moreover the relations (26) *and* (27) *are a complete set of relations among the $C(S_\alpha)$'s.*

3.6. *Action of the Iwahori subgroups on the representations.* Here is the main result, due to Casselman **[18]** and Borel **[4]**.

THEOREM 3.7. *Let (π, V) be any admissible representation of G. The natural projection of V onto V_N defines an isomorphism of V^B onto $(V_N)^{\circ M}$.*

One proves first that V^B maps onto $(V_N)^{\circ M}$ using Iwahori decomposition of B and the methods used in the proof of Theorem 2.3. There are some simplifications due to the fact that P is a minimal parabolic subgroup of G.

To prove that V^B maps injectively in $(V_N)^{\circ M}$, one first proves that, for any given vector v in $V(N) \cap V^B$, there exists a real number $\varepsilon > 0$ such that $\pi(C(a))\cdot v = 0$ for every a in A satisfying $|\alpha(a)|_F \leqq \varepsilon$ whenever the root $\alpha \in \Phi$ is positive on P. But this relation implies $v = 0$ by Theorem 3.6, since one has $q(S_\alpha) > 0$ there.

COROLLARY 3.4. *Given any unramified character χ of M, one has a direct sum decomposition $I(\chi) = I(\chi)^B \oplus I(\chi)(N)$.*

This follows from Theorem 3.7 since $^\circ M$ acts trivially on $I(\chi)_N$. A direct proof can also be obtained using the methods used in the proof of Theorem 3.5.

Using the decomposition of G into the pairwise disjoint *open* subsets $P\omega(w)B$ (w in W), one gets easily a basis for $I(\chi)^B$. Indeed, $\omega(w)$ normalizes M and $B \cap M = {}^\circ M$ lies in the kernel of $\chi\delta^{1/2}$. Hence the following function is well defined on G

$$\begin{aligned} \Phi_{w,\chi}(g) &= \delta^{1/2}(m)\chi(m) && \text{if } g = mn\omega(w)b, \\ &= 0 && \text{if } g \notin P\omega(w)B. \end{aligned} \tag{28}$$

The family $\{\Phi_{\omega,\chi}\}_{w\in W}$ is the sought-for basis.

The next result is due to Casselman **[18]** (see also Borel **[4]**).

THEOREM 3.8. *Let (π, V) be any admissible irreducible representation of G. The following assertions are equivalent*:

(a) *There are in V nonzero vectors invariant under B (that is $V^B \neq 0$).*

(b) *There exists some unramified character χ of M such that (π, V) is isomorphic to a subrepresentation of $(\nu_\chi, I(\chi))$.*

By Theorem 3.7, assertion (a) means that $(V_N)^{\circ M} \neq 0$. By Frobenius reciprocity (Theorem 3.4), assertion (b) means that there exists in the space dual to V_N a nonzero vector invariant under $^\circ M$ which is an eigenvector for the group M. Since $^\circ M$ is compact, acts continuously on V_N and $M/^\circ M$ is commutative, the equivalence follows immediately.

COROLLARY 3.5. *Let χ be any unramified character of M. The space $I(\chi)^B$ generates $I(\chi)$ as a G-module.*

We prove Corollary 3.5 by reductio ad absurdum. Assume that $I(\chi)^B$ does not generate $I(\chi)$. Since $I(\chi)$ is finitely generated, there exist an irreducible admissible representation (π, V) of G and a G-homomorphism $u\colon I(\chi) \to V$ which is nonzero and contains $I(\chi)^B$ in its kernel. Since B is compact, one gets $V^B = u(I(\chi)^B) = 0$. By duality, one gets an injective G-homomorphism $\tilde{u}\colon \tilde{V} \to (I(\chi))^\sim$. Since $(I(\chi))^\sim$ is isomorphic to $I(\chi^{-1})$ by Theorem 3.2, it follows from Theorem 3.8 that $\tilde{V}^B \neq 0$. But the finite-dimensional spaces V^B and $\tilde{V}^B$ are dual to each other and this is clearly impossible.

3.7. *Intertwining operators*. In this section, *we assume the unramified character χ to be regular*, that is the characters $w\chi$, for w running over W, are all distinct.

Corollary 3.2 may be reformulated as follows: given w in W, there exists a linear form $L_w \neq 0$ on $I(\chi)$, such that

$$L_w(\nu_\chi(mn)\cdot f) = \delta^{1/2}(m)w\chi(m)L_w(f) \tag{29}$$

for m in M, n in N and f in $I(\chi)$, unique up to multiplication by a constant. We normalize L_w by

$$L_w(f) = \int_{N(w)\backslash N} f(\omega(w)^{-1}n)\, d\bar{n} \tag{30}$$

for a function f whose support does not meet F_r ($r = d(w^{-1})$). The Haar measure is chosen in such a way that $N(w)\backslash N(w)\cdot(N \cap B)$ be of measure 1.

To L_w we associate an intertwining operator $T_w\colon I(\chi) \to I(w\chi)$ defined by

$$T_w f(g) = L_w(\nu_\chi(g)\cdot f) \quad \text{for } f \text{ in } I(\chi) \text{ and } g \text{ in } G. \tag{31}$$

It is easily checked that L_w, hence T_w, depends only on w, not on the representative $\omega(w)$ for w.

Since $G = PK$ (Iwasawa decomposition) and the character $\delta^{1/2}\chi$ of M is trivial on $M \cap K = {}^\circ M$, there exists in $I(\chi)$ a unique function $\Phi_{K,\chi}$ invariant under K and normalized by $\Phi_{K,\chi}(1) = 1$. Explicitly, one has

$$\Phi_{K,\chi}(mnk) = \delta^{1/2}(m)\chi(m) \quad \text{for } m \text{ in } M,\ n \text{ in } N \text{ and } k \text{ in } K. \tag{32}$$

If the Haar measures on M and N are normalized by $\int_{M\cap K} dm = \int_{N\cap K} dn = 1$, one may also define $\Phi_{K,\chi}$ as $P_\chi(I_K)$ where P_χ is defined as on p. 136 and I_K is the characteristic function of K. The space $I(\chi)^K$ consists of the constant multiples of $\Phi_{K,\chi}$.

The next theorem again is due to Casselman **[18]**.

THEOREM 3.9. *The operator T_w takes $I(\chi)^K$ into $I(w\chi)^K$. More precisely, one has*

$$T_w(\Phi_{K,\chi}) = c_w(\chi)\cdot\Phi_{K,w\chi}, \tag{33}$$

where the constant $c_w(\chi)$ is defined by

$$c_w(\chi) = \prod_\alpha c_\alpha(\chi), \tag{34}$$

$$c_\alpha(\chi) = \frac{(1 - q_{\alpha/2}^{-1/2} q_\alpha^{-1}\chi(a_\alpha))(1 + q_{\alpha/2}^{-1/2}\chi(a_\alpha))}{1 - \chi(a_\alpha)^2}. \tag{35}$$

The product in (34) *is extended over the affine roots α in Φ_0 which are positive over $\mathscr{C}$, but such that $w\alpha$ is negative over $\mathscr{C}$.*

When G is split, formula (35) takes the simpler form

$$c_\alpha(\chi) = \frac{1 - q^{-1}\chi(a_\alpha)}{1 - \chi(a_\alpha)}. \tag{36}$$

The bulk of the proof of Theorem 3.9 rests with the case where w is the reflection associated to a simple root β in Φ_0. In this case, there exists exactly one positive root α in Φ_0 taken by w into a negative root, namely $\alpha = \beta$. The general case follows then since $T_{w_1w_2}$ is equal to $T_{w_1}T_{w_2}$ when the lengths of w_1 and w_2 add to the length of w_1w_2.

COROLLARY 3.6. *The intertwining map T_w is an isomorphism from $I(\chi)$ onto $I(w\chi)$ iff $c_w(\chi)$ and $c_{w^{-1}}(w\chi)$ are nonzero.*

Indeed $T_{w^{-1}}T_w$ is multiplication by $c_w(\chi)c_{w^{-1}}(w\chi)$ by Theorem 3.9.

One may strengthen the irreducibility criterion (Theorem 3.3).

THEOREM 3.10. *Assume χ is any unramified regular character of M. Then the representation $(\nu_\chi, I(\chi))$ of G is irreducible iff $c_\alpha(\chi) \neq 0$, $c_\alpha(w_0\chi) \neq 0$ for every positive root α in Φ_0 positive over $\mathscr{C}$, where w_0 is the unique element in W which takes $\mathscr{C}$ into $\mathscr{C}^-$.*

IV. Spherical functions.

4.1. *Some integration formulas.* We keep the notation of the previous part.

If Γ is any of the groups G, M, N, K, then Γ is unimodular. We normalize the (left and right invariant) Haar measure on Γ by $\int_{\Gamma\cap K} d\gamma = 1$. The group $P = M\cdot N$ is not unimodular. One checks immediately that the formulas

$$\int_P f(p)\, d_lp = \int_M \int_N f(mn)\, dm\, dn, \tag{1}$$

$$\int_P f(p)\, d_rp = \int_M \int_N f(nm)\, dm\, dn \tag{2}$$

define a left invariant Haar measure d_lp and a right invariant Haar measure

$d_r p$ on P. Since ${}^\circ M = M \cap K$ is the largest compact subgroup of M, one gets $(P \cap K) = (M \cap K)\cdot(N \cap K)$, hence the normalization

$$\int_{P\cap K} d_l p = \int_{P\cap K} d_r p = 1. \tag{3}$$

The Haar measures are related to the Iwasawa decomposition by the formulas

$$\int_G f(g)\, dg = \int_K \int_P f(pk)\, dk\, d_l p, \tag{4}$$

$$\int_G f(g)\, dg = \int_K \int_P f(kp)\, dk\, d_r p \tag{5}$$

for f in $C_c(G)$. Let us prove for instance formula (4). One defines a linear map $h \mapsto u_h$ from $C_c(K \times P)$ into $C_c(G)$ by the rule

$$u_h(pk^{-1}) = \int_{P\cap K} h(kp_1, pp_1)\, d_l p_1. \tag{6}$$

The linear form $h \mapsto \int_G u_h(g)\, dg$ on $C_c(K \times P)$ is then a left invariant Haar measure on $K \times P$; hence by our normalizations, one gets

$$\int_G u_h(g)\, dg = \int_K \int_P h(k, p)\, dk\, d_l p. \tag{7}$$

It suffices to substitute $f(pk^{-1})$ for $h(k, p)$ in formulas (6) and (7) to get formula (4)!

From the definition of δ (see p. 135), one gets

$$\int_N f(mnm^{-1})\, dn = \delta(m)^{-1} \int_N f(n)\, dn \tag{8}$$

for any function f in $C_c(N)$. As a corollary, we get the alternate expressions for the Haar measures on P

$$\int_P f(p)\, d_l p = \int_M \int_N \delta(m)^{-1} f(nm)\, dm\, dn, \tag{9}$$

$$\int_P f(p)\, d_r p = \int_M \int_N \delta(m) f(mn)\, dm\, dn. \tag{10}$$

Otherwise stated, the modular function of P is given by

$$\Delta_P(mn) = \delta(m)^{-1} \quad \text{for } m \text{ in } M,\ n \text{ in } N. \tag{11}$$

The group N is unipotent and M acts on N via inner automorphisms. It is then easy to construct a sequence of subgroups of N, say $N = N_0 \supset N_1 \supset \cdots \supset N_{r-1} \supset N_r = 1$, which are invariant under M and such that N_{j-1}/N_j is isomorphic (for $j = 1, \cdots, r$) to a vector space over F on which M acts linearly. Putting

$$\Delta(m) = |\det(\mathrm{Ad}_{\mathfrak{n}}(m) - 1_{\mathfrak{n}})|_F \tag{12}$$

for m in M, one gets by induction on r the integration formula

$$\int_N f(n)\, dn = \Delta(m) \int_N f(nmn^{-1} m^{-1})\, dn \tag{13}$$

for f in $C_c(N)$ and any m in M such that $\Delta(m) \neq 0$ (see [**28**, Lemma 22]).

Let us define now the so-called *orbital integrals*. Let m be any element of M

such that $\Delta(m) \neq 0$, let $Z(m)$ be the centralizer of m in G and G_m the conjugacy class of m in G. Then $A \subset M$ and $Z(m) \cap M$ has finite index in $Z(m)$; since M/A is a compact group, the same is true of $Z(m)/A$. Moreover, the conjugacy class G_m is closed in G; hence the mapping $gZ(m) \mapsto gmg^{-1}$ is a homeomorphism from $G/Z(m)$ onto G_m. Therefore, the mapping $gA \mapsto gmg^{-1}$ from G/A into G is proper and we may set after Harish-Chandra

$$F_f(m) = \Delta(m)\int_{G/A} f(gmg^{-1})\, d\bar{g} \tag{14}$$

for any function f in $C_c(G)$. The Haar measure on A is normalized in such a way that $\int_{M/A} d\bar{m} = 1$.

LEMMA 4.1. *Let f be any function in $\mathscr{H}(G, K)$ and m an element of M such that $\Delta(m) \neq 0$. Then $F_f(m)$ is equal to $\int_N f(nm)\, dn$.*

From formulas (2) and (5), one gets

$$\int_G u(g)\, dg = \int_M \int_N u(nm_1)\, dm_1\, dn \tag{15}$$

for any function u in $C_c(G)$ which is invariant under left translation by the elements in K. Putting $u(g) = f(gmg^{-1})$, one gets the following representation for $F_f(m)$

$$F_f(m) = \int_{M/A} h(m_1)\, d\bar{m}_1, \tag{16}$$

$$h(m_1) = \Delta(m) \int_N f(nm_1mm_1^{-1}\, n^{-1})\, dn. \tag{17}$$

Fix m_1 and set $m_2 = m_1mm_1^{-1}$. From the definition (12) of Δ, one gets $\Delta(m) = \Delta(m_2)$. We get therefore

$$\begin{aligned} h(m_1) &= \Delta(m_2) \int_N f((nm_2\, n^{-1}\, m_2^{-1})m_2)\, dn &&\text{by (17)}\\ &= \int_N f(nm_2)\, dn &&\text{by (13).}\end{aligned}$$

Notice that the group $M/{}^\circ M = M/(M \cap K)$ is commutative. Hence we get $m_2 \in mK$ and since the function f is invariant under right translation by the elements of K, one gets $f(nm_2) = f(nm)$ for any n in N. It follows that $h(m_1)$ is equal to $\int_N f(nm)\, dn$ for any m_1 in M. The contention of Lemma 4.1 follows from formula (16) since M/A is of measure 1.

4.2. *Satake isomorphism.* The construction we are going to expound now is due to Satake [**38**]. It is the $\mathfrak{p}$-adic counterpart of a well-known construction of Harish-Chandra in the set-up of real Lie groups.

For any λ in Λ, let $\mathrm{ch}(\lambda)$ be the characteristic function of the subset $\mathrm{ord}_M^{-1}(\lambda)$ of M. Since $\int_{M\cap K} dm = 1$, one gets

$$\mathrm{ch}(\lambda) * \mathrm{ch}(\lambda') = \mathrm{ch}(\lambda + \lambda') \tag{18}$$

for λ, λ' in Λ. Moreover the elements $\mathrm{ch}(\lambda)$ (for λ in Λ) form a basis of the complex algebra $\mathscr{H}(M, {}^\circ M)$, which may be therefore identified to the group algebra $\mathbf{C}[\Lambda]$.

We define now a linear map $S\colon \mathscr{H}(G, K) \to \mathscr{H}(M, {}^\circ M)$ by the formula

$$Sf(m) = \delta(m)^{1/2} \int_N f(mn)\, dn = \delta(m)^{-1/2} \int_N f(nm)\, dn \tag{19}$$

for f in $\mathscr{H}(G, K)$ and m in M. The two integrals are equal by formula (8). It is immediate that the function Sf on M belongs to $C_c(M)$. That it is bi-invariant under $^\circ M$ follows from the fact that f is bi-invariant under K and that $^\circ M = M \cap K$.

The following fundamental theorem is due to Satake **[38]**.

THEOREM 4.1. *The Satake transformation S is an algebra isomorphism from $\mathscr{H}(G, K)$ onto the subalgebra $\mathbf{C}[\Lambda]^W$ of $\mathbf{C}[\Lambda]$ consisting of the invariants of the Weyl group W.*

Here are the main steps in the proof.

(A) *S is a homomorphism of algebras.* By construction, S is the composition of three linear maps

$$\mathscr{H}(G, K) \xrightarrow{\alpha} \mathscr{H}(P) \xrightarrow{\beta} \mathscr{H}(M) \xrightarrow{\gamma} \mathscr{H}(M).$$

Here α is simply the restriction of functions from G to P. It is compatible with convolution by an easy corollary of (4). The map β is given by $(\beta u)(m) = \int_N u(mn)\, dn$ and one checks easily that it is compatible with convolution. The map γ is given by $(\gamma f)(m) = f(m)\delta(m)^{1/2}$; since δ is a character of M, it is compatible with convolution.

(B) *The image of S is contained in $\mathbf{C}[\Lambda]^W$.* Since $W = (N(A) \cap K)/^\circ M$, this property is equivalent to

$$Sf(xmx^{-1}) = Sf(m) \tag{20}$$

for m in M and x in $N(A) \cap K$.

The function $m \mapsto \det(\mathrm{Ad}_{\mathfrak{n}}(m) - 1_{\mathfrak{n}})$ from M to F is polynomial and nonzero. The elements of M which do not annihilate this function are therefore dense in M; they are called the regular elements. Hence by continuity it suffices to prove (20) for m regular.

From Lemma 4.1, one gets

$$Sf(m) = D(m) \int_{G/A} f(gmg^{-1})\, d\bar{g} \tag{21}$$

for m regular in M. Here $D(m)$ is equal to $\Delta(m)\delta(m)^{-1/2}$; hence

$$\begin{aligned} D(m)^2 &= |\det(\mathrm{Ad}_{\mathfrak{n}}(m) - 1_{\mathfrak{n}})|_F^2 \cdot |\det \mathrm{Ad}_{\mathfrak{n}}(m)|_F^{-1} \\ &= |\det(\mathrm{Ad}_{\mathfrak{n}}(m) - 1_{\mathfrak{n}})|_F \cdot |\det(\mathrm{Ad}_{\mathfrak{n}}(m^{-1}) - 1_{\mathfrak{n}})|_F \\ &= |\det(\mathrm{Ad}_{\mathfrak{n}}(m) - 1_{\mathfrak{n}})|_F \cdot |\det(\mathrm{Ad}_{\mathfrak{n}^-}(m) - 1_{\mathfrak{n}^-})|_F. \end{aligned}$$

The last equality follows for instance from the fact that the weights in $\mathfrak{n} \otimes_F \bar{F}$ ($\bar{F}$ an algebraic closure of F) are the inverses of the weights in $\mathfrak{n}^- \otimes_F \bar{F}$ of any maximal torus of M. Since $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{m} \oplus \mathfrak{n}^-$, one gets

$$D(m) = |\det(\mathrm{Ad}_{\mathfrak{g}/\mathfrak{m}}(m) - 1_{\mathfrak{g}/\mathfrak{m}})|_F^{1/2} \tag{22}$$

and therefore

$$D(xmx^{-1}) = D(m) \quad \text{for } m \text{ in } M,\ x \text{ in } N(A). \tag{23}$$

On the other hand, the compact group $N(A) \cap K$ acts by inner automorphisms on G and A. It leaves therefore invariant the Haar measures on G and A, hence the G-invariant measure on G/A. Let m in M be regular, x in $N(A) \cap K$ and f in $\mathscr{H}(G, K)$. One has $f(xgx^{-1}) = f(g)$ for any g in G; hence

$$\int_{G/A} f(g(xmx^{-1})g^{-1})\, d\bar{g} = \int_{G/A} f((x^{-1}gx)m(x^{-1}gx)^{-1})\, d\bar{g}$$
$$= \int_{G/A} f(gmg^{-1})\, d\bar{g}.$$

The invariance property (20) follows from the representation (21), the invariance property (23) and the just-established invariance.

(C) *The linear map* S: $\mathscr{H}(G, K) \to \boldsymbol{C}[\Lambda]^W$ *is bijective.* We let as before Λ^- denote the subset of Λ consisting of the translations in $\mathscr{A}$ which take $\mathscr{C}^-$ into itself. For λ in Λ^-, let φ_λ be the characteristic function of the double coset $K \cdot \mathrm{ord}_M^{-1}(\lambda) \cdot K$; by Cartan decomposition, the family $\{\varphi_\lambda\}_{\lambda \in \Lambda^-}$ is a basis of $\mathscr{H}(G, K)$. Moreover, any element of Λ is conjugate under W to a unique element in Λ^-. We get therefore a basis $\{\mathrm{ch}'(\lambda)\}_{\lambda \in \Lambda^-}$ of $\boldsymbol{C}[\Lambda]^W$ by putting

$$\mathrm{ch}'(\lambda) = \frac{1}{|W(\lambda)|} \sum_{w \in W} \mathrm{ch}(w \cdot \lambda), \tag{24}$$

where $W(\lambda)$ is the stabilizer of λ in W. Define the matrix $\{c(\lambda, \lambda')\}$ by

$$S\varphi_{\lambda'} = \sum_{\lambda} c(\lambda, \lambda') \cdot \mathrm{ch}'(\lambda) \tag{25}$$

where λ, λ' run over Λ^-.

To calculate $c(\lambda, \lambda')$, choose representatives m and m' respectively of λ, λ' in M. Then we get (μ is the Haar measure on G)

$$c(\lambda, \lambda') = S\varphi_{\lambda'}(m) = \delta(m)^{-1/2}\, \mu(Km'K \cap NmK). \tag{26}$$

It is clear that $KmK \cap NmK \supset mK$; hence

$$c(\lambda, \lambda') \geqq \delta(m)^{-1/2}. \tag{27}$$

Moreover $Km'K \cap NmK$ *is empty unless* $\lambda' - \lambda$ *is a linear combination with non-negative real coefficients of the positive roots.* Using a suitable lexicographic ordering $\geqq$, we conclude that $c(\lambda, \lambda') = 0$, unless $0 \geqq \lambda' \geqq \lambda$. Since $c(\lambda, \lambda) \neq 0$ by (27), this remark shows that the elements $S\varphi_{\lambda'}$, for λ' in Λ, form a basis of $\boldsymbol{C}[\Lambda]^W$, hence our contention (C).

COROLLARY 4.1. *The algebra* $\mathscr{H}(G, K)$ *is commutative and finitely generated over* $\boldsymbol{C}$.

The algebra $\boldsymbol{C}[\Lambda]$ is commutative and generated by $\mathrm{ch}(\lambda_1), \cdots, \mathrm{ch}(\lambda_m), \mathrm{ch}(\lambda_1^{-1}), \cdots, \mathrm{ch}(\lambda_m^{-1})$ if $\{\lambda_1, \cdots, \lambda_m\}$ is a basis of Λ over $\boldsymbol{Z}$. Since the group W is finite, it follows from well-known results in commutative algebra[13] that $\boldsymbol{C}[\Lambda]^W$ is finitely generated as an algebra over $\boldsymbol{C}$, and that $\boldsymbol{C}[\Lambda]$ is finitely generated as a module over $\boldsymbol{C}[\Lambda]^W$.

We determine now the algebra homomorphisms from $\mathscr{H}(G, K)$ to $\boldsymbol{C}$. Let χ be any unramified character of M. Since $\int_{{}^\circ M} dm = 1$, the map $f \mapsto \int_M f(m)\chi(m)\, dm$ is a unitary homomorphism from $\mathscr{H}(M, {}^\circ M)$ to $\boldsymbol{C}$, and we get in this way all such homomorphisms. Define a linear map ω_χ: $\mathscr{H}(G, K) \to \boldsymbol{C}$ by

[13] See for instance N. Bourbaki, *Commutative algebra*, Chapter V, p. 323, Addison-Wesley, 1972.

$$\omega_\chi(f) = \int_M Sf(m)\cdot\chi(m)\, dm. \tag{28}$$

COROLLARY 4.2. *Any unitary homomorphism from $\mathscr{H}(G, K)$ into $\mathbf{C}$ is of the form ω_χ for some unramified character χ of M. Moreover, one has $\omega_\chi = \omega_{\chi'}$ iff there exists an element w in W such that $\chi' = w\cdot\chi$.*

This corollary follows from Theorem 4.1 and the classical properties of invariants of finite groups acting on polynomial algebras (see previous footnote).

Otherwise stated, the set of unitary homomorphisms from $\mathscr{H}(G, K)$ to $\mathbf{C}$ is in a bijective correspondence with the set X/W of orbits of W in the set X of unramified characters of M. We refer the reader to §3.2 for a discussion of this set X/W. Notice that since X is isomorphic to a complex torus T such that $X^*(T) = \Lambda$, then X/W is a complex algebraic affine variety and Satake isomorphism defines an isomorphism of $\mathscr{H}(G, K)$ with the algebra of polynomial functions on X/W.

4.3. *Determination of the spherical functions.* Since the characteristic functions of the double cosets KgK form a basis of the complex vector space $\mathscr{H}(G, K)$, one defines as follows an isomorphism of the dual to the space $\mathscr{H}(G, K)$ onto the space of functions on G, bi-invariant under K:

$$\omega(f) = \int_G f(g)\Gamma(g)\, dg \qquad \text{for } f \text{ in } \mathscr{H}(G, K), \tag{29}$$

$$\Gamma(g) = \omega(I_{KgK})\Big/\int_{KgK} dg_1 \qquad \text{for } g \text{ in } G. \tag{30}$$

I claim that the following conditions are equivalent:

(a) *ω is a homomorphism of algebras from $\mathscr{H}(G, K)$ to $\mathbf{C}$.*

(b) *One has*

$$\Gamma(g_1)\Gamma(g_2) = \int_K \Gamma(g_1kg_2)\, dk \tag{31}$$

for g_1, g_2 in G.

(c) *For any function f in $\mathscr{H}(G, K)$, there exists a constant $\lambda(f)$ such that*

$$f * \Gamma = \Gamma * f = \lambda(f)\cdot\Gamma. \tag{32}$$

The equivalence of (a) and (b) follows from the following calculation

$$\begin{aligned}\omega(f_1 * f_2) &= \iint \Gamma(g_1g_2)f_1(g_1)f_2(g_2)\, dg_1\, dg_2 \\ &= \int_G f_1(g_1)\, dg_1 \int_G f_2(g_2)\, dg_2 \int_K \Gamma(g_1kg_2)\, dk\end{aligned}$$

and the fact that the function $(g_1, g_2) \mapsto \int_K\Gamma(g_1kg_2)\, dk$ on $G \times G$ is invariant under left and right translations by elements of $K \times K$.

The equivalence of (a) and (c) follows from the following formula:

$$\omega(f_1 * f_2) = \int f_2(g)(\check{f}_1*\Gamma)(g)\, dg = \int f_1(g)(\Gamma*\check{f}_2)(g)\, dg$$

for f_1, f_2 in $\mathscr{H}(G, K)$ where $\check{f}(g) = f(g^{-1})$. Hence $\lambda(f) = \omega(\check{f})$.

DEFINITION 4.1. *A* (zonal) *spherical function on G w.r.t. K is a function Γ on G, bi-invariant under K, such that $\Gamma(1) = 1$ and enjoying the equivalent properties* (a), (b) *and* (c) *above.*

We may now translate our previous results in terms of spherical functions. Let χ be any unramified character of M. Recall that the function $\Phi_{K,\chi}$ is defined by

$$\Phi_{K,\chi}(mnk) = \chi(m)\delta^{1/2}(m) \quad \text{for } m \text{ in } M, n \text{ in } N \text{ and } k \text{ in } K. \tag{33}$$

We put

$$\Gamma_\chi(g) = \int_K \Phi_{K,\chi}(kg)\, dk \quad \text{for } g \text{ in } G. \tag{34}$$

THEOREM 4.2. (a) *The spherical functions on G w.r.t. K are the functions Γ_χ.*

(b) *Let χ and χ' be unramified characters of M. The spherical functions Γ_χ and $\Gamma_{\chi'}$ are equal iff there exists an element w in the Weyl group W such that $\chi' = w\cdot\chi$.*

It is clear that Γ_χ is bi-invariant under K, and $\Gamma_\chi(1) = 1$. Theorem 4.2 follows from Corollary 4.2 and the formula

$$\omega_\chi(f) = \int_G \Gamma_\chi(g)\cdot f(g)\, dg \quad \text{for } f \text{ in } \mathscr{H}(G, K). \tag{35}$$

This in turn is proved as follows:

$$\begin{aligned}
\int_G \Gamma_\chi(g) f(g)\, dg &= \int_G \Phi_{K,\chi}(g) f(g)\, dg \\
&= \int_K \int_M \int_N \Phi_{K,\chi}(mnk) f(mnk)\, dk\, dm\, dn \\
&= \int_M \chi(m)\delta(m)^{1/2}\, dm \int_N f(mn)\, dn \\
&= \int_M \chi(m) Sf(m)\, dm = \omega_\chi(f).
\end{aligned}$$

We used the integration relations (1) and (4).

4.4. *The spherical principal series of representations.*

DEFINITION 4.2. *A representation of G is called spherical* (w.r.t. K) *if it is smooth, irreducible and contains a nonzero vector invariant under K.*

Let Γ be a spherical function on G (w.r.t. K). We denote by V_Γ the space of functions f on G of the form $f(g) = \sum_{i=1}^n c_i\Gamma(gg_i)$ for $c_1, \cdots, c_n$ in $\mathbf{C}$ and $g_1, \cdots, g_n$ in G. From the functional equation of the spherical functions (formula (31)), one deduces

$$\int_K f(gkg')\, dk = \Gamma(g)\cdot f(g') \quad \text{for } f \text{ in } V_\Gamma \text{ and } g, g' \text{ in } G. \tag{36}$$

We let G operate on V_Γ by right translations, namely

$$(\pi_\Gamma(g)\cdot f)(g_1) = f(g_1 g) \quad \text{for } f \text{ in } V_\Gamma \text{ and } g, g_1 \text{ in } G. \tag{37}$$

I claim that *the representation (π_Γ, V_Γ) is spherical and that the elements of V_Γ invariant under $\pi_\Gamma(K)$ are the constant multiples of Γ.* Indeed, it is clear that for any function f in V_Γ there exists a compact open subgroup K_f of G such that f is invariant under right translation by the elements of K_f; hence the representation (π_Γ, V_Γ) is smooth. Let $f \neq 0$ in V_Γ and choose an element g' in G such that $f(g') \neq 0$. The functional equation (36) may be rewritten as

$$(38) \qquad \Gamma = f(g')^{-1} \int_K \pi_\Gamma(kg')\cdot f\, dk = f(g')^{-1}\pi_\Gamma(I_{Kg'})f.$$

Any vector subspace of V_Γ containing f and invariant under $\pi_\Gamma(G)$ contains therefore Γ, hence is identical to V_Γ. Finally, if a function f in V_Γ is invariant under $\pi_\Gamma(K)$, one gets $f = f(1)\cdot\Gamma$ by substituting $g' = 1$ in the functional equation (36).

THEOREM 4.3. *Let (π, V) be any spherical representation of G. There exists a unique spherical function Γ such that (π, V) is isomorphic to (π_Γ, V_Γ).*

As usual, let V^K denote the subspace of V consisting of the vectors invariant under $\pi(K)$. If f is any function in $\mathscr{H}(G, K)$, the operator $\pi(f)$ takes V^K into itself; hence V^K is a module over $\mathscr{H}(G, K)$. I claim that this module is *simple*: indeed, let $v \neq 0$ and v' be two elements of V^K. Since the $\mathscr{H}(G)$-module V is simple, there exists a function f in $\mathscr{H}(G)$ such that $v' = \pi(f)\cdot v$. The function $f_K = I_K * f * I_K$ belongs to $\mathscr{H}(G, K)$ and $v' = \pi(f_K)\cdot v$, substantiating our claim.

The algebra $\mathscr{H}(G, K)$ over the field $\boldsymbol{C}$ is commutative and of countable dimension. By the reasoning used to prove Schur's lemma (see p. 118) (or by Hilbert's Zero Theorem), one concludes that any simple module over $\mathscr{H}(G, K)$ is of dimension 1 over $\boldsymbol{C}$. Hence V^K is of dimension 1 over $\boldsymbol{C}$ and there exists a unitary homomorphism $\omega\colon \mathscr{H}(G, K) \to \boldsymbol{C}$ such that

$$(39) \qquad \pi(f)\cdot v = \omega(f)v \quad \text{for any } f \text{ in } \mathscr{H}(G, K) \text{ and any } v \text{ in } V^K.$$

Let $(\tilde{\pi}, \tilde{V})$ be the representation of G contragredient to (π, V). The space $\tilde{V}^K$ of vectors in $\tilde{V}$ invariant under $\tilde{\pi}(K)$ is dual to V^K, hence of dimension 1. Choose a vector v in V^K and a vector $\tilde{v}$ in $\tilde{V}^K$ such that $\langle \tilde{v}, v\rangle = 1$ and define the function Γ on G by

$$(40) \qquad \Gamma(g) = \langle \tilde{v}, \pi(g)\cdot v\rangle \quad \text{for } g \text{ in } G.$$

From (39) and (40) one deduces $\omega(f) = \int_G \Gamma(g)\cdot f(g)\, dg$ for any f in $\mathscr{H}(G, K)$. It is obvious that $\Gamma(1) = 1$ and that Γ is bi-invariant under K. Hence Γ is a spherical function.

The map which associates to any vector v' in V the coefficient $\pi_{v',\tilde{v}}$ defines an isomorphism of (π, V) with (π_Γ, V_Γ). Moreover, for any spherical function Γ' on G, the representation (π, V) is isomorphic to $(\pi_{\Gamma'}, V_{\Gamma'})$ iff the following relation holds

$$(41) \qquad \pi(I_K)\pi(g)\pi(I_K) = \Gamma'(g)\cdot\pi(I_K) \quad \text{for } g \text{ in } G.$$

This holds for $\Gamma' = \Gamma$ only. Q.E.D.

The definition of spherical functions as well as the results obtained so far in this section depend only on the fact that K is a compact open subgroup of G and that the Hecke algebra $\mathscr{H}(G, K)$ is commutative. We use now the classification of spherical functions on G afforded by Theorem 4.2. Let χ be any unramified character of M; when the spherical function Γ is set equal to Γ_χ, we write (π_χ, V_χ) instead of

(π_Γ, V_Γ). The family of representations $\{(\pi_\chi, V_\chi)\}_{\chi\in X}$ is called the *spherical principal series of representations of G.*

We summarize now the main properties of the spherical principal series; they are immediate corollaries of the results obtained so far.

(a) *Any representation* (π_χ, V_χ) *is irreducible, admissible, and the only functions in* V_χ *invariant under* $\pi_\chi(K)$ *are the constant multiples of* Γ_χ.

(b) *Let* χ *and* χ' *be unramified characters of M. The representations* (π_χ, V_χ) *and* $(\pi_{\chi'}, V_{\chi'})$ *are isomorphic iff there exists an element w in the Weyl group W such that* $\chi' = w\cdot\chi$.

(c) *Assume that the representation* $(\nu_\chi, I(\chi))$ *in the unramified principal series is irreducible. For any function f in* $I(\chi)$ *define the function* $f^\natural$ *by* $f^\natural(g) = \int_K f(kg)\,dk$. *Then the map* $f \mapsto f^\natural$ *is an isomorphism of the representation* $(\nu_\chi, I(\chi))$ *with the representation* (π_χ, V_χ).

(d) *In general, let* $0 = V_0 \subset V_1 \subset \cdots \subset V_{r-1} \subset V_r = I(\chi)$ *be a Jordan-Hölder series of the* $\mathscr{H}(G)$*-module* $I(\chi)$. *There exists a unique index j such that* $1 \leqq j \leqq r$ *and that the representation of G in* V_j/V_{j-1} *is spherical. Then this representation is isomorphic to* (π_χ, V_χ).

The last two statements come from the fact that $\Phi_{K,\chi}$ is, up to constant multiples, the unique function in $I(\chi)$ invariant under $\nu_\chi(K)$ and from the relation (34) which can be expressed as $\Gamma_\chi = \Phi^\natural_{K,\chi}$.

REMARKS. (1) It is easy to show without recourse to Satake's Theorem 4.1 that the representation $(\nu_\chi, I(\chi))$ is spherical *provided it is irreducible.* Since then the representations $(\nu_\chi, I(\chi))$ and $(\nu_{w\cdot\chi}, I(w\cdot\chi))$ are equivalent for any w in W, this provides another proof of step (B) in Satake's theorem (see criterion 3.10, p. 144).

(2) We refer the reader to Macdonald [**37**, p. 63] for a characterization of the *bounded* spherical functions, that is the spectrum of the Banach algebra of integrable functions on G which are bi-invariant under K. It does not seem to be known which spherical functions are positive-definite, or stated in other terms, which spherical representations are preunitary.

4.5. *The explicit formula for the spherical functions.* The following result is due to Macdonald [**36**], [**37**].

THEOREM 4.4. *Suppose that the unramified character* χ *of M is regular. For any m in* M^-, *the value of the spherical function* Γ_χ *is given as follows:*

$$\Gamma_\chi(m) = Q^{-1}\,\delta(m)^{1/2} \sum_{w\in W} \mathbf{c}(w\cdot\chi)\, w\cdot\chi(m) \tag{42}$$

where

$$Q = \sum_{w\in W} q(w)^{-1}, \tag{43}$$

$$\mathbf{c}(\lambda) = \prod_{\alpha>0} (1 - q_{\alpha/2}^{-1/2}\, q_\alpha^{-1}\lambda(a_\alpha)^{-1})(1 + q_{\alpha/2}^{-1/2}\,\lambda(a_\alpha)^{-1})(1 - \lambda(a_\alpha)^{-2})^{-1} \tag{44}$$

for any regular unramified character λ *of M (product extended over the roots* α *in* Φ_0 *which are positive on the conical chamber* $\mathscr{C}$).

We sketch the proof given by Casselman [**18**], which rests on the properties of the intertwining operators. We use the notations of §3.7.

For each w in W, the linear form L_w on $I(\chi)$ transforms by M according to the character $\delta^{1/2}(w\cdot\chi)$ and is trivial on $I(\chi)(N)$. Since χ is regular, these characters of M are distinct; hence the linear forms L_w are linearly independent. The vector space $I(\chi)^B$ is supplementary to $I(\chi)(N)$ in $I(\chi)$ and its dimension is equal to $|W|$. Hence there exists in $I(\chi)^B$ a basis $\{f_{w,\chi}\}_{w\in W}$ characterized by

$$(45)\qquad L_w(f_{w',\chi}) = 1 \quad \text{if } w' = w, \qquad = 0 \quad \text{if } w' \neq w.$$

As a corollary of Theorem 3.9 one gets $\Phi_{K,\chi} = \sum_{w\in W} c_w(\chi)\cdot f_{w,\chi}$. On the other hand, one has $\Gamma_\chi(m) = \int_K \Phi_{K,\chi}(km)\,dk$. Since $f_{w,\chi}$ is invariant under $\nu_\chi(B)$ and B is a subgroup of K, one gets from these remarks the relation

$$(46)\qquad \Gamma_\chi(m) = \sum_{w\in W} c_w(\chi)\int_K g_w(k)\,dk$$

where[14] $g_w = \mu(BmB)^{-1}\pi(I_{BmB})\cdot f_{w,\chi}$. By the methods used in Theorem 2.3, one proves, in general, that, for any admissible representation (π, V) of G and any m in M, the operator $\pi(I_{BmB}) - \mu(BmB)\pi(m)$ maps V into $V(N)$. From (45), one infers

$$(47)\qquad g_w = \delta(m)^{1/2}\, w\cdot\chi(m)\, f_{w,\chi}.$$

From (46) and (47), one deduces that, on M^-, the spherical function Γ_χ agrees with a linear combination of the characters $\delta^{1/2}(w\cdot\chi)$ of M. Taking into account the invariance property $\Gamma_{w\cdot\chi} = \Gamma_\chi$, it suffices to calculate one of these coefficients, for instance the coefficient of $\delta^{1/2}(w_0\cdot\chi)$ where w_0 is the (unique) element of W which takes the conical chamber $\mathscr{C}$ into its opposite $\mathscr{C}^-$ (or any positive root in Φ_0 to a negative root). In this case, one proves without difficulty that $f_{w_0,\chi}$ is equal to the function $\Phi_{w_0,\chi}$ defined by formula (28) in §3.6. The sought-for coefficient is obtained by multiplying $c_{w_0}(w_0\cdot\chi)$ by

$$\int_K f_{w_0,\chi}(k)\,dk = \int_K \Phi_{w_0,\chi}(k)\,dk = \mu(Bw_0B).$$

It remains to show that the measure of Bw_0B, that is the index $(Bw_0B\colon K)$, is equal to Q^{-1}. This follows easily from the formulas (21) to (24) in §3.5. Q.E.D.

Note added in proof. As I was told by the editors, my conventions about algebraic groups differ slightly from those of other authors in these PROCEEDINGS. The local field F being infinite, the set $G(F)$ of F-points of any of the algebraic groups G used in the previous paper is Zariski-dense in G and I allowed myself to identify G to $G(F)$.

[14] The Haar measure μ on G is normalized by $\mu(K) = 1$.

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INSTITUT DES HAUTES ETUDES SCIENTIFIQUES, 91440 BURES SUR YVETTE, FRANCE

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CUSPIDAL UNRAMIFIED SERIES FOR CENTRAL SIMPLE ALGEBRAS OVER LOCAL FIELDS

PAUL GERARDIN

1. Statement of the theorem.

1.1. Let A be a central simple algebra over a nonarchimedean local field F. Call E an unramified splitting field for A, and Γ the Galois group of E over F. A quasi-character of $E^\times$ is called regular if all its conjugates by the action of Γ are distinct.

For any quasi-character χ of $F^\times$, let $\chi_{A/F}$ and $\chi_{E/F}$ be respectively the quasi-characters of $A^\times$ and $E^\times$ defined by: $\chi_{A/F} = \chi \circ N_{A/F}$ and $\chi_{E/F} = \chi \circ N_{E/F}$, with $N_{A/F}$ the reduced norm of A over F, and $N_{E/F}$ the norm of E over F.

For any nontrivial character ψ of F, the characters $\psi_{A/F} = \psi \circ T_{A/F}$ of A and $\psi_{E/F} = \psi \circ T_{E/F}$ of E are nontrivial: here, $T_{A/F}$ is the reduced trace of A over F, and $T_{E/F}$ the trace of E over F.

Let n be the rank of A, that is the degree of E over F.

1.2. We have the following theorem:

THEOREM. *With the above notations, for any regular quasi-character θ of $E^\times$, there is a well-defined irreducible admissible representation $\theta_{A/F}$ of $A^\times$ such that*:

(a) *the representations $\dot\theta_{A/F}$ and $\ddot\theta_{A/F}$ associated to two regular quasi-characters $\dot\theta$ and $\ddot\theta$ are equivalent if and only if $\dot\theta$ and $\ddot\theta$ are conjugate under Γ*;

(b) *the restriction of $\theta_{A/F}$ to the center $F^\times$ of $A^\times$ is given by the quasi-character $x \in F^\times \mapsto (-1)^{(n-1)\operatorname{val} x}\,\theta(x)$*;

(c) *for any quasi-character χ of $F^\times$, the twisted representation $\theta_{A/F} \otimes \chi_{A/F}$ is equivalent to $(\theta\chi_{E/F})_{A/F}$*;

(d) *the contragredient representation of $\theta_{A/F}$ is equivalent to $(\theta^{-1})_{A/F}$*;

(e) *the L-function of $\theta_{A/F}$ is* 1;

(f) *the ε-factor of $\theta_{A/F}$ is $\varepsilon(\theta_{A/F}, \psi) = (-1)^{(n-1)\operatorname{ord}\psi}\,\varepsilon(\theta, \psi_{E/F})$.*

Moreover, when A is not a division algebra, the representation $\theta_{A/F}$ is cuspidal.

Here, cuspidal means that the coefficients of the representation are compactly supported modulo the center.

1.3. Recall how the L-function and ε-factor are defined [**6**]. The reduced trace $T_{A/F}$ defines a nondegenerate bilinear form on A by its value on the product of two elements in A; hence, for any nontrivial additive character ψ on F, the mapping

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$$a, b \in A \mapsto \psi_{A/F}(ab) = \psi(T_{A/F}(ab)),$$

identifies A with its Pontryagin dual group; the corresponding self-dual measure on A will be written $d_{A,\psi}$, and the Fourier transform on the Schwartz-Bruhat space $\mathscr{S}(A)$ by $f \mapsto f_\psi$:

$$f_\psi(x) = \int_A f(y)\psi_{A/F}(xy)\, d_{A,\psi}y.$$

Let $\mathscr{A}(A)$ be the set of classes of irreducible admissible representations of the multiplicative group $A^\times$ of A.

Godement and Jacquet have proved in **[6]** that the integrals $\int_{A^\times} f(x)\pi(x)\,dx$, $f \in \mathscr{S}(A)$, dx any Haar measure on A, where π is a representation of $A^\times$ with class in $\mathscr{A}(A)$, can be defined as meromorphic functions of $s \in \boldsymbol{C}$ from the integrals $\int_{A^\times} f(x)c(x)\,|x|^s_{A/F}\,dx$ where c is a coefficient of π, and the convergence holds for $\mathrm{Re}(s)$ large. They proved too that there is an eulerian factor $L(\pi)$ and a factor $\varepsilon(\pi, \psi)$ which is an exponential in s, such that

$$\frac{\int_{A^\times} f_\psi(a)\pi(a)^{-1}\,|a|_{A/F}^{-(n-1)/2}\, d_{A,\psi}\, a}{L(\pi^\vee \otimes |\ \ |_{A/F})} = (-1)^{n-m}\, \varepsilon(\pi, \psi)\, \frac{\int_{A^\times} f(a)\pi(a)|a|_{A/F}^{-(n+1)/2} d_{A,\psi}\, a}{L(\pi)},$$

where $\pi^\vee$ is the contragredient representation of π, m is the rank over F of the group $A^\times$, and $|\ \ |_{A/F}$ comes from the normalized absolute value on F by composition with the reduced norm $N_{A/F}$. For a cuspidal π in $\mathscr{A}(A)$, $L(\pi) = 1$.

1.4. We give here the proof of the theorem when A is the split algebra of all endomorphisms on an n-dimensional vector space over F; for the division algebras, the proof is similar but simpler; the general case is a mixture of these two.

For the construction of the representation, we use basically the same method as in **[4]**; it may be generalized to any reductive group over a p-adic field [3]. For $\mathrm{GL}(n, F)$, constructions of cuspidal representations associated to tamely ramified extensions of degree n of F have been made by T. Shintani in some special cases **[10]**, and by R. Howe in general [7], both assuming that the residual characteristic of F is odd. What is new here, is that there is no assumption on the residual characteristic, that the construction of the representation is canonical, and that we compute the ε-factors. For normalizers of Iwahori subgroups in multiplicative groups of central simple algebras, R. Howe gave in **[8]** a general construction. It would be interesting to know the relations of our representations with those obtained by G. Lusztig using l-adic cohomology **[9]**.

2. Unramified induced representations of the Weil group. The one-dimensional admissible representations of the Weil group W_E of $E^\times$ are exactly the quasi-characters of $E^\times$. From the exact sequence

$$1 \longrightarrow E^\times \longrightarrow W_{E/F} \longrightarrow \Gamma \longrightarrow 1$$

a quasi-character θ of $E^\times$ defines an n-dimensional representation of $W_{E/F}$ by induction; but $W_{E/F}$ is a quotient of W_F, so this gives an n-dimensional representation of W_F, denoted $\mathrm{Ind}_E^F\,\theta$. Moreover, this representation is irreducible if and only if all the transforms of θ by Γ are distinct, that is for θ being regular. In general, let Γ_0 be the stabilizer of θ in Γ, and E_0 be the corresponding subfield of E; there are $n_0 = [E:E_0]$ quasi-characters θ' of $E_0^\times$ such that $\theta = \theta'_{E/E_0}$, and each of them is regular with respect to F; so the representation $\mathrm{Ind}_E^F\ \theta$ is the direct sum of the n_0 representations $\mathrm{Ind}_{E_0}^F\ \theta'$, each one being irreducible. This gives part (a) of the following proposition. For part (e), we use the property of inductivity of the L-function; for part (f), there is only a degree zero inductivity for the ε-factor, and the factor $\lambda(E/F, \psi)$ is computed from the identity $\varepsilon(\mathrm{Ind}_E^F\theta, \psi) = \lambda(E/F, \psi)\varepsilon(\theta, \psi_{E/F})$ by taking for θ the trivial character, so that $\mathrm{Ind}_E^F 1$ is the regular representation of Γ, that is the sum of the n unramified characters $x \mapsto \zeta^{\mathrm{val}\, x}$ where ζ are the n different nth roots of 1; so:

$$\prod_\zeta \varepsilon(\zeta^{\mathrm{val}}, \psi) = \lambda(E/F, \psi)\, \varepsilon(1, \psi_{E/F});$$

but $\varepsilon(\zeta^{\mathrm{val}}, \psi) = (\zeta q^{1/2})^{\mathrm{ord}\,\psi}$ and $\varepsilon(1, \psi_{E/F}) = q^{n\ \mathrm{ord}\,\psi/2}$, if q is the order of the residue field $\bar{F}$ of F. This gives the formula $\lambda(E/F, \psi) = (-1)^{(n-1)\,\mathrm{ord}\,\psi}$.

PROPOSITION 1. *The representations of W_F induced by the quasi-characters of $E^\times$ satisfy the following properties*:

(a) *the representations* $\mathrm{Ind}_E^F\dot{\theta}$ *and* $\mathrm{Ind}_E^F\ddot{\theta}$ *induced by two quasi-characters $\dot{\theta}$ and $\ddot{\theta}$ are equivalent if and only if $\dot{\theta}$ and $\ddot{\theta}$ are conjugate under Γ*;

(b) *the one-dimensional representation* $\det \mathrm{Ind}_E^F\theta$ *is given by the quasi-character*: $x \in F^\times \mapsto (-1)^{(n-1)\,\mathrm{val}\, x}\,\theta(x)$;

(c) *for any quasi-character χ of $F^\times$, the representation* $\mathrm{Ind}_E^F\theta$ *twisted by χ is equivalent to* $\mathrm{Ind}_E^F(\theta\chi_{E/F})$;

(d) *the contragredient representation of* $\mathrm{Ind}_E^F\theta$ *is equivalent to* $\mathrm{Ind}_F^E(\theta^{-1})$;

(e) *the L-function* $L(\mathrm{Ind}_E^F\theta)$ *is the L-function* $L(\theta)$;

(f) *the ε-factor* $\varepsilon(\mathrm{Ind}_E^F\theta, \psi)$ *is* $(-1)^{(n-1)\,\mathrm{ord}\,\psi}\ \varepsilon(\theta, \psi_{E/F})$.

3. Construction of the representation.

3.1. For any quasi-character θ of $E^\times$, we have the n characters $\theta/\theta^\gamma, \gamma \in \Gamma$, with Γ the Galois group $\mathrm{Gal}(E/F)$. They are trivial on $F^\times$. Let $a(\theta, \gamma) = a(\gamma)$ be the conductor of θ/θ^γ; it is a nonnegative integer, and the following relations hold:

$a(\gamma) = a(\gamma^{-1})$ for any $\gamma \in \Gamma$,

$a(\gamma\gamma') \leqq \mathrm{Max}(a(\gamma), a(\gamma'))$, for any $\gamma, \gamma' \in \Gamma$, equality for $a(\gamma) \neq a(\gamma')$,

$a(\gamma) \geqq 1$ for any $\gamma \neq 1$, if and only if θ is regular.

For each integer $r \geqq 0$, let Γ_r be the subgroup of Γ formed by the elements γ with $a(\gamma) \leqq r$, and E_r be its fixed points in E; call A_r the algebra $\mathrm{End}_{E_r} E$; we have the inclusions:

$$E \subset A_0 \subset A_1 \subset A_2 \subset \cdots A_r \subset \cdots A.$$

Observe that $A_r = A_s$ for $r \geqq s \geqq \mathrm{Max}_{\Gamma_r}\, a(\gamma)$.

The algebra A_r can be seen too as the algebra over E of the group Γ_r; this leads to A_s-linear projections for $r \geqq s$: $T_{A_r/A_s}: A_r \to A_s$ and decompositions in direct sum of A_s-submodules: $A_r = A_s + A_r^s = \sum_{\Gamma_r/\Gamma_s} \gamma A_s$ where $A_r^s = \sum_{\Gamma_r^s} A_s = \mathrm{Ker}\ T_{A_r/A_s}$, due to the partition of Γ_r in Γ_s and its complement, written Γ_r^s, moreover,

$$T_{A_r/A_s}(xax^{-1}) = x(T_{A_r/A_s}\, a)x^{-1}, \quad \text{for } x \in A_s^\times, a \in A_r, r \geqq s,$$

the group $A_s^\times$ acts by conjugation on A_r and leaves invariant each term of this decomposition.

The natural filtration on E by its ideals $\mathfrak{p}_E^m$ induces filtrations on each of the algebras A_r: the lattice $A_r(\mathfrak{p}^m)$ is the set of $a \in A_r$ such that $a\mathfrak{p}_E^m \in \mathcal{O}_E$, the ring of integers in E; it is too the set of functions from Γ_r to $\mathfrak{p}_E^m$. As the extension E is unramified over F, these filtrations are compatible with the maps T_{A_r/A_s}.

The residue field of E_r is written $\bar{E}_r$, and $\bar{A}_r = A_r(\mathcal{O}_E)/A_r(\mathfrak{p}_E)$ is the algebra $\mathrm{End}_{\bar{E}_r} \bar{E}$. The finite group $\bar{A}_s^\times$ acts by conjugacy on each $\bar{E}_r$-algebra $A_r(\mathfrak{p}^m)/A_r(\mathfrak{p}^{m+1}) = A_r(\mathfrak{p}^m/\mathfrak{p}^{m+1})$, and leaves invariant each of the subspaces $\gamma A_s(\mathfrak{p}^m/\mathfrak{p}^{m+1})$, $\gamma \in \Gamma_r/\Gamma_s$, for $r \geqq s$.

3.2. Assume now that θ is a regular quasi-character of $E^\times$, i.e., that A_0 is E. For each $r \geqq 0$, define a subgroup K_r of $A_r^\times$ as follows; we denote by r'(resp. r'') the largest (resp. the smallest) integer such that $2r' \leqq r \leqq 2r''$; then $K_0 = F^\times$, K_1 is the isotropy subgroup in $A_1^\times$ of the lattice $\mathcal{O}_E$ in E, and for $r > 1$, $K_r = 1 + A_{r-1}(\mathfrak{p}^{r''}) + A_r^{r-1}(\mathfrak{p}^{r'})$. For each $r \geqq 1$, the set $K_r^+ = 1 + \mathfrak{p}_E^r + A_r^0(\mathfrak{p}^{r''})$ is an invariant subgroup of K_r, normalized by each of the groups K_s, $s \leqq r$, moreover, by conjugation K_s acts on K_r, and sends K_r in K_r^+ for $r > s \geqq 2$. The factor groups K_r/K_r^+ are the following:

$K_1/K_1^+ = \bar{A}_1^\times$,

K_r/K_r^+ for $r > 1$ is the group $(1 + \mathfrak{p}_E^{r''})/(1 + \mathfrak{p}_E^r) \simeq \mathfrak{p}_E^{r''}/\mathfrak{p}_E^r$, unless r is odd and $A_r \neq A_{r-1}$ where it is the central extension:

$$0 \longrightarrow \mathfrak{p}_E^{r''}/\mathfrak{p}_E^r \longrightarrow K_r/K_r^+ \longrightarrow A_r^{r-1}(\mathfrak{p}^{r''}/\mathfrak{p}^{r'}) \longrightarrow 0,$$

defined by the 2-cocycle

$$a, b \in A_r^{r-1}(\mathfrak{p}^{r'}/\mathfrak{p}^{r''}) \mapsto T_{A_r/E}(ab) \in \mathfrak{p}_E^{r-1}/\mathfrak{p}_E^r.$$

3.3. For each $r \geqq 1$, choose a character θ_r of $E^\times$ with conductor equal to $\mathrm{Max}_{\gamma \in \Gamma_r} a(\gamma)$, such that θ/θ_r is fixed by the group Γ_r: this means that there is a quasi-character $\chi^{(r)}$ of $E_r^\times$ with $\theta = \theta_r \chi^{(r)}_{E/E_r}$. For $r = 1$, the character θ_1 defines a character of $\bar{E}^\times$ regular with respect to $\bar{E}_1$. For $r > 1$, the restriction of θ_r to $1 + \mathfrak{p}_E^{r-1}$ is fixed by Γ_{r-1}, so there is a character $\psi^{(r)}$ of E_{r-1} such that

$$\theta_r(1 + x) = \psi^{(r)}_{E/E_{r-1}}(x) \quad \text{for } x \in \mathfrak{p}_E^{r-1}/\mathfrak{p}_E^r;$$

moreover, the mapping defined by

$$a, b \in A_r^{r-1}(\mathfrak{p}^{r'}/\mathfrak{p}^{r''}) \mapsto \psi^{(r)}_{E/E_{r-1}}(T_{A_r/E}ab) \quad \text{for } r', r'' \text{ as in 3.2},$$

is a self-duality; this gives the following result:

LEMMA 1. *For each $r > 1$, there is a unique class of irreducible representations κ_r' of K_r trivial on K_r^+ and such that the restriction to $E^\times \cap K_r$ is isotypic, given by θ_r; define then a representation κ_r of K_r by $\kappa_r(x) = \chi^{(r)}_{A_r/E_r}(x)\kappa_r'(x)$, $x \in K_r$; then, the class of κ_r depends only on θ, and is fixed by the action of any K_s, $s < r$.*

For r even or with $A_r = A_{r-1}$, this is clear. In the remaining cases, we use r' and r'' from r as above; we introduce the subgroup $H(r)$ of K_r/K_r^+ defined by the central extension by $\mathfrak{p}_E^{r-1}/\mathfrak{p}_E^r$ of $A_r^{r-1}(\mathfrak{p}^{r'}/\mathfrak{p}^{r''})$ given by the same cocycle; it is a Heisenberg group, and if $K(r)$ is its pull-back in K_r, then $1 + \mathfrak{p}_E^{r''}$ normalizes $K(r)$ and we have $K_r = (1 + \mathfrak{p}_E^{r''})K(r)$. The first part of the lemma comes from a well-known property of Heisenberg groups; for the second, we observe that the restriction of

κ'_r to $K(r)$ factors through any group $H_s(r)$, $1 \leqq s < r$, central extension by $\mathfrak{p}_{E_s}^{r-1}/\mathfrak{p}_{E_s}^r$ of $A_r^{r-1}(\mathfrak{p}^{r'}/\mathfrak{p}^{r''})$ obtained from $H(r)$ by the map T_{E/E_s} on its center:

$$0 \longrightarrow \mathfrak{p}_{E_s}^{r''}/\mathfrak{p}_{E_s}^r \longrightarrow H_s(r) \longrightarrow A_r^{r-1}(\mathfrak{p}^{r'}/\mathfrak{p}^{r''}) \longrightarrow 0$$

defined by the 2-cocycle

$$a, b \in A_r^{r-1}(\mathfrak{p}^{r'}/\mathfrak{p}^{r''}) \mapsto T_{A_s/E_s}T_{A_r/A_s}(ab) \in \mathfrak{p}_{E_s}^{r-1}/\mathfrak{p}_{E_s}^r \subset \mathfrak{p}_{E_s}^{r''}/\mathfrak{p}_{E_s}^r;$$

it suffices now to observe that the action of $A_s^\times$ by conjugation on A_r fixes the form $u \in A_r \mapsto T_{A_s/E_s}T_{A_r/A_s}a$.

3.4. More precisely, for $r > s > 1$, the action of K_s on K_r gives a trivial action on $H_s(r)$. For $s = 1$, the group $H_1(r)$ is a central product of the group's $H_1(\gamma)$ inverse images of the subspaces $\gamma A_1(\mathfrak{p}^{r'}/\mathfrak{p}^{r''}) + \gamma^{-1} A_1(\mathfrak{p}^{r'}/\mathfrak{p}^{r''})$, the group $H_1(\gamma)$ is a central extension of this subspace by $\mathfrak{p}_{E_1}^{r''}/\mathfrak{p}_{E_1}^r$, the 2-cocycle being given by

$$a, b \in \gamma A_1(\mathfrak{p}^{r'}/\mathfrak{p}^{r''}) \mapsto T_{A_1/E_1}(ab) \quad \text{for } \gamma\Gamma_1 = \gamma^{-1}\Gamma_1,$$

$$a, b \in \gamma A_1(\mathfrak{p}^{r'}/\mathfrak{p}^{r''}) + \gamma^{-1}A_1(\mathfrak{p}^{r'}/\mathfrak{p}^{r''}) \mapsto T_{A_1/E_1}(a'b''+a''b')$$

with $a = a' + a''$, $b = b' + b''$, a', $b' \in \gamma A_1(\mathfrak{p}^{r'}/\mathfrak{p}^{r''})$, a'', $b'' \in \gamma^{-1}A_1(\mathfrak{p}^{r'}/\mathfrak{p}^{r''})$, in general; the action of K_1 on K_r gives an action of $K_1/K_1^+ = \bar{A}_1^\times$ on each of these subgroups $H_1(\gamma)$; for $\gamma\Gamma_1 = \gamma^{-1}\Gamma_1$, the above 2-cocycle is a unitary form on the $\bar{E}_1$-vector space $\gamma A_1(\mathfrak{p}^{r'}/\mathfrak{p}^{r''})$ with respect to the field $\bar{E}'_1$ of fixed points of γ in $\bar{E}_1$, and $\bar{A}_1$ preserves this form; from [5, Corollary 4.8.2], there is a canonical extension to $\bar{A}_1^\times \rtimes H_1(\gamma)$ of any irreducible representation κ'_γ of $H_1(\gamma)$ which is given on the center by $\psi_{E_1/E_{r-1}}^{(r)}$; for a $\gamma \in \Gamma_r^{r-1}$ with $\gamma\Gamma_1 \neq \gamma^{-1}\Gamma_1$, the group $\bar{A}_1^\times$ leaves stable each of the $\bar{A}_1$-submodules of the quotient of $H_1(\gamma)$; hence any irreducible representation κ_γ of this group which is given on $\mathfrak{p}_{E_1}^{r-1}/\mathfrak{p}_{E_1}^r$ by $\psi_{E_1/E_{r-1}}^{(r)}$ has a canonical extension to the group $\bar{A}_1 \ltimes H_1(\gamma)$ ([5, Proposition 1.4], in fact, the character of the restriction to $\bar{A}_1^\times$ is positive). The product of the representations κ'_γ defines a representation κ'_r of K_r as in the lemma, and the product of the extensions defines an extension $\tilde{\kappa}'_r$ of κ'_r to the group K_1K_r. Now, the representation $\tilde{\kappa}_r$ of this group obtained from $\tilde{\kappa}'_r$ by twisting with $\chi^{(r)}$ as in the lemma is, up to equivalence, independent of the choice of $\psi^{(r)}$.

For $r = 1$, there is a unique class of irreducible representations κ'_1 of K_1 which is trivial on K_1^+ and such that its tensor product with the pull-back of the Steinberg representation of $\bar{A}_1^\times$ is the representation induced by the one-dimensional represantation $tx \mapsto \theta_1(t)$, $t \in \mathcal{O}_E^\times$, $x \in K_1^+$, of the subgroup $\mathcal{O}_E^\times K_1^+$: it is the representation $R_{E_1}(\theta_1)$ of [2, Theorem 8.3], up to a sign $(-1)^{n_1-1}$, $n_1 = [E : E_1]$, and it is a cuspidal representation. Define then κ_1 as the twist of κ'_1 by $\chi^{(1)}$:

$$\kappa_1(x) = \chi_{A_1/E_1}^{(1)}(x)\kappa'_1(x), \qquad x \in K_1;$$

its class does not depend on the choice of $\chi^{(1)}$.

Finally, the representation κ_0 of $K_0 = F^\times$ is the restriction of θ to $F^\times$.

PROPOSITION 2. *A regular quasi-character θ of $E^\times$ defines canonically a class of irreducible representations of K_θ by the formula*

$$\kappa_\theta(x_0x_1 \cdots x_r \cdots) = \kappa_0(x_0) \otimes \kappa_1(x_1) \otimes \cdots \otimes \tilde{\kappa}_r(x_1)\kappa_r(x_r) \otimes \cdots$$

for x_r in K_r, where κ_r is defined as above, $\tilde{\kappa}_r$ is its canonical extension to K_1 for r odd > 1 with $A_r \neq A_{r-1}$, and $\tilde{\kappa}_r$ is the trivial representation for the other $r > 1$.

4. Commuting algebra.

4.1. Let π_θ be the representation of $A^\times$ induced by the representation κ_θ of K_θ in functions on $A^\times$ with compact support modulo the center $F^\times$. The commuting algebra is identified with the set of applications from $A^\times$ to the space of κ_θ which transform on both sides under the action of K_θ according to the representation κ_θ.

4.2. For a quasi-character θ of $E^\times$ such that any character θ/θ^γ, for $\gamma \in \Gamma$, $\gamma \neq 1$, has conductor 1, the representation κ_θ is obtained by twisting from a cuspidal irreducible representation κ' of $\bar{A}^\times$; if T is a maximal diagonalizable subgroup of $A^\times$ such that $A = K_\theta T K_\theta$, then a function ϕ in the commuting algebra is determined by its values on T; for a $t \in T$ not in K_θ, there exists a unipotent radical U of a proper parabolic subgroup of $A^\times$ such that $t(U \cap K)t^{-1}$ is contained in the kernel of $K_\theta \to \bar{A}^\times$; we have then, for $u \in U \cap K$: $\phi(tu) = \kappa_\theta(t)\phi(u) = \phi(tut^{-1}t) = \phi(t)$; hence, from the cusp property of κ', $\phi(t) = 0$. This shows that the commuting algebra is reduced to the scalars, and π_θ is irreducible; its coefficients being compactly supported modulo the center, the representation π_θ is cuspidal too.

4.3. Let now r be such that $r = \mathrm{Max}_{\Gamma_r} a(\gamma) > 1$. Let ϕ be a nonzero function in the commuting algebra of the representation π_θ; choose a in $A^\times$ with $\phi(a) \neq 0$. Suppose we know that $a \in K_\theta a_r K_\theta$ for some a_r in $A_r^\times$; as the values of ϕ are determined by its values on a set of representatives modulo K_θ on both sides, we replace a by a_r. From the definition of κ_θ, its restriction to the subgroup $1 + A_r(\mathfrak{p}^{r''})$ of K_r (r'' is the smallest integer with $2r'' \geqq r$), is a scalar operator, defined by the restriction of κ_r, that is

$$1 + x \in 1 + A_r(\mathfrak{p}^{r''}) \mapsto \chi^{(r)}_{A_r/E_r}(1 + x)\theta_r(1 + T_{A_r/E}x).$$

As $\phi(a)$ is nonzero, we have, for $x \in A_r(\mathfrak{p}^{r''}) \cap a^{-1}A_r(\mathfrak{p}^{r''})\, a$, the identity:

$$\theta_r(1 + T_{A_r/E}x) = \theta_r(1 + T_{A_r/E}(axa^{-1})).$$

Choose a character α of E such that $\alpha(t) = \theta_r(1 + t)$ for $t \in \mathfrak{p}_E^{r''}$; then the character of A_r defined by $x \mapsto \alpha_{A_r/E}(axa^{-1})\alpha_{A_r/E}(x)^{-1}$ is trivial on the intersection $a^{-1}A_r(\mathfrak{p}^{r''})a \cap A_r(\mathfrak{p}^{r''})$ so it can be written as $\eta(axa^{-1})^{-1}\,\xi(x)$ with two characters ξ and η of A_r trivial on $A_r(\mathfrak{p}^{r''})$, so that

$$\alpha_{A_r/E}(x)\,\xi(x) = \alpha_{A_r/E}(axa^{-1})\eta(axa^{-1}) \quad \text{for any } x \in A_r.$$

4.4. Fix a character ψ of E_r with order $-r$; the restriction to $\mathfrak{p}_E^{r-1}$ of the character α is trivial on $\mathfrak{p}_E^r$, and its stabilizer in Γ_r is Γ_{r-1}; so, there is an element τ in the multiplicative pull-back of $\bar{E}^\times$ in $E^\times$ such that $E_r(\tau) = E_{r-1}$ and $\alpha(t) = \psi_{E/E_r}(\tau t)$ for t in $\mathfrak{p}_E^{r-1}$.

LEMMA 2. *With these notations, then*

(a) *any character of A_r equal to α_{A_r/E_r} on $A_r(\mathfrak{p}^{r''})$ has a conjugate by an element of K_r which is equal to $x \mapsto \psi_{A_r/E_r}(\tau x)$ on $A_{r-1}(\mathfrak{p}^{r''}) + A_r^{r-1}$;*

(b) *if an element of $A_r^\times$ conjugates two characters of A_r equal to $x \mapsto \alpha_{A_r/E_r}(\tau x)$ on $A_{r-1}(\mathfrak{p}^{r''}) + A_r^{r-1}$, then this element belongs to $A_{r-1}^\times$.*

The proof uses the identification of A_r with its group of characters obtained from the application $a, b \in A_r \mapsto \alpha_{A_r/E_r}(ab)$, where the action o A_r is the conjugation. Part (a) is easy, and for part (b), we have to show that an element $b \in A_r^\times$ such that $b(\tau + X)b^{-1} = \tau + X'$ with X, Y in $A_{r-1}(\mathfrak{p}^{r'})$, $r' + r'' = r$, is in fact in $A_{r-1}^\times$: as X and Y centralize τ, taking the successive powers q^{nm} of both sides, we get at the limit m infinite, the equality $b\tau b^{-1} = \tau$, so b centralizes the center E_{r-1} of A_{r-1}, hence is itself in $A_{r-1}^\times$.

We apply this lemma to the above situation: the characters of A_r, $\alpha_{A_r/E_r}\,\xi$ and $\alpha_{A_r/E_r}\,\eta$, are conjugate by suitable elements from K_r to characters equal to $x \mapsto \psi_{A_r/E_r}(\tau x)$ on $A_{r-1}(\mathfrak{p}^{r''}) + A_r^{r-1}$; as they were conjugate by $a \in A_r^\times$, part (b) of Lemma 2 shows that a belongs to $K_r A_{r-1}^\times K_r$.

4.5. As for r large, $A_r = A$, we have shown that the relation $\phi(a) \neq 0$ implies $a \in KA_1^\times K$. If $A_1 = E$, then a is in K_θ, so $\phi(a) = \kappa_\theta(a)\phi(1)$, and $\phi(1)$ is an intertwining operator on the representation κ_θ; the irreducibitity of κ_θ implies that this operator is scalar, and the representation is irreducible, and cuspidal for its coefficients are compactly supported modulo the center.

Suppose now that A_1 is not E. Let a be an element of $A_1^\times$ which is not in K_θ, i.e. not in $K_1F^\times$.Then, there is a unipotent radical U of a proper parabolic subgroup of $A_1^\times$ such that $a(U \cap K_1)a^{-1}$ is in K_1^+, and even in the kernel of any κ_r for any $r \geqq 1$, for the determinant is trivial on any unipotent element. For each $\gamma \in \Gamma$ with $a(\gamma) = r$, an odd number > 1, the group $U \cap K_1$ acts on the$\bar{E}_1$-vector space $\gamma A_1(\mathfrak{p}^{r'}/\mathfrak{p}^{r''}) + \gamma^{-1}A_1(\mathfrak{p}^{r'}/\mathfrak{p}^{r''})$; its fixed points form a subspace which can be lifted as a commutative subgroup $\tilde{W}_\gamma$ of $H_1(\gamma)$ (notations as in 3.4), and there is a subgroup W_γ of unipotent elements in K_r with projection $\tilde{W}_\gamma$ on $H(r)$.

For r even, or without γ such that $a(\gamma) = r$, define $W_r = 1$; for $r = 1$, put $W_1 = U \cap K_1$; for the other r's, W_r is the product of W_γ for γ with $a(\gamma) = r$, taken in a chosen order, and without repetition. Then, for any r, aW_ra^{-1} is in K_r^+ and made of unipotent elements so is in the kernel of κ_θ. For $x = w_1 \cdots w_r \cdots$, $w_r \in W_r$, we have:

$$\phi(a) = \phi(ax) = \phi(a)\kappa_1(w_1) \otimes \cdots \otimes \tilde{\kappa}_r(w_1)\kappa_r(w_r) \otimes \cdots.$$

From the next lemma, it results that the operators $\int_{W_r} \kappa_r(w_r)\,dw_r$ are fixed by $\kappa_r(w_1)$, so that the average

$$\int_{W_1 \cdots W_r \cdots} \kappa_1(w_1) \otimes \cdots \otimes \tilde{\kappa}_r(w_1)\kappa_r(w_r) \otimes \cdots dw_1 \cdots dw_r \cdots$$

is equal to

$$\int_{W_1} \kappa_1(w_1)\,dw_1 \otimes \cdots \otimes \int_{W_r} \kappa_r(w_r)\,dw_r \otimes \cdots$$

which is 0 from the cusp property of κ_1.

The lemma is the following, with the same notations as in **[5]**:

LEMMA 3. (a) *Let V be a finite dimensional vector space over finite field, and η be an irreducible representation of the Heisenberg group $H(V)$ nontrivial on the center; let U be the unipotent radical of a proper parabolic subgroup of $G(V)$, and W the space of its fixed points in $V \times V^*$; then W has a lift in $H(V)$ as a subgroup, and the operator $\sum_W \eta(w)$ is fixed by the action of U through the Weil representation W^V;*

(b) *let F be a K/k-unitary vector space over a finite filed, and η be an irreducible* *representation of the associated Heisenberg group $H(F, i)$ which is nontrivial on the center; let U be the unipotent radical of a proper parabolic subgroup of U (the unitary group $U(F, i)$), and W the corresponding totally isotropic subspace in F; then W has a lift in $H(V)$ as a subgroup, and the operator $\sum_W \eta(w)$ is fixed under the action of U through the Weil representation $W^{(F,i)}$.*

The proof is easy for (a), and for (b) it suffices to use a realisation of η as functions on W with values in the space of the representation associated to the Heisenberg group $H(W^\perp/W, i)$.

Finally, we have proved the following result:

PROPOSITION 3. *Let θ be a regular quasi-character of $E^\times$. Then the representation π_θ induced by κ_θ is a cuspidal irreducible admissible representation of $A^\times$.*

5. Intertwining operators.

5.1. Let now $\dot\theta$ and $\ddot\theta$ be two regular quasi-characters of $E^\times$; we denote with one (resp. two) dot(s) every object attached to $\dot\theta$ (resp. $\ddot\theta$): $\dot\pi = \pi_{\dot\theta}$, $\dot K = K_{\dot\theta}$, $\dot\kappa = \kappa_{\dot\theta}$, $\cdots$; when the object is the same for $\dot\theta$ and $\ddot\theta$, we take off the dots. The intertwining operators between the two representations $\dot\pi$ and $\ddot\pi$ are identified with the applications ϕ from $A^\times$ to the vector space of linear maps between the space of $\dot\kappa$ and the space of $\ddot\kappa$ satisfying the relation: $\phi(xay) = \dot\kappa(x)\phi(a)\ddot\kappa(y)$, $x \in \dot K$, $y \in \ddot K$.

5.2. When there is a γ such that $\ddot\theta = \dot\theta^\gamma$, then for any r, the groups $\dot\Gamma_r$ and $\ddot\Gamma_r$ are the same; we can take $\ddot\chi^{(r)} = \dot\chi^{(r)}$ and $\ddot\theta_r = \dot\theta_r^\gamma$; moreover, for $r > 1$, the representation $\dot\kappa_r^\gamma$ and its extension $\dot{\bar\kappa}_r^\gamma$ have the required properties for the corresponding representations for $\ddot\kappa$, so are equivalent to $\ddot\kappa_r$ and $\ddot{\bar\kappa}_r$ respectively. The representations $\dot\kappa_1$ and $\ddot\kappa_1$ of K_1 are trivial on K_1^+ and their pull-back with a Steinberg representation of $\bar A_1^\times$ is equivalent to the representation induced from $\mathcal{O}_E^\times K_1^+$ by the one-dimensional representation $tx \mapsto \ddot\theta_1(t) = \dot\theta_1^\gamma(t)$ for $t \in \mathcal{O}_E$, $x \in K_1^+$, so they too are equivalent; finally, on $F^\times$, $\dot\theta$ is $\ddot\theta$ so $\ddot\kappa_0$ is $\dot\kappa_0^\gamma$; this shows that the representations $\ddot\kappa$ and $\dot\kappa^\gamma$ of $\dot K = \ddot K = K$ are equivalent, so too are the representations $\ddot\pi$ and $\dot\pi^\gamma$; but the operator $\dot\pi(\gamma)$ sends the representation $\dot\pi$ on $\dot\pi^\gamma$, so $\ddot\pi$ and $\dot\pi$ are equivalent. We have proved that two regular quasi-characters of $E^\times$ which are conjugate under Γ give equivalent representations of $A^\times$.

5.3. Suppose now that there exists a nonzero intertwining map ϕ between $\dot\pi$ and $\ddot\pi$; choose $a \in A^\times$ such that $\phi(a) \neq 0$. Consider the following property for a non-negative integer r:

$P(r)$: for any $t \geqq r$, $\dot\Gamma_t = \ddot\Gamma_t$ and there is $\gamma \in \Gamma$ such that $a \in \dot K A_r^\times \gamma \ddot K$. We observe that $P(0)$ implies $\dot K = \ddot K$, and that $\phi(\gamma)$ intertwines $\ddot\kappa$ and $\dot\kappa^\gamma$ so that $\ddot\theta = \dot\theta$ on $F^\times$; on $1 + \mathfrak{p}_E$, the restriction of $\dot\kappa_\theta$ is the scalar operator defined by the restriction of θ to $1 + \mathfrak{p}_E$, so $\ddot\theta$ and $\dot\theta^\gamma$ coincide on $1 + \mathfrak{p}_E$; this gives the equivalence between each of the representations $\ddot\kappa_r$ and $\dot\kappa_r^\gamma$ of K_r for $r > 1$, and of the extensions $\ddot{\bar\kappa}_r$ and $\dot{\bar\kappa}_r^\gamma$ to K_1; the isomorphism $\phi(\gamma)$ intertwines the restrictions of $\ddot\kappa_\gamma$ and $\dot\kappa_\gamma$ to K_1; as the character of $\ddot{\bar\kappa}_r$ restricted to K_1 does not vanish, this implies that $\ddot\kappa_1$ and $\dot\kappa_1^\gamma$ are equivalent, having the same character as $\ddot\chi^{(1)} = \dot\chi^{(1)}$ on $1 + \mathfrak{p}_E$, this implies that the corresponding representations of $\bar A_1^\times$ are equivalent, so $\ddot\theta_1$ and $\dot\theta_1^\gamma$ are conjugate by Γ_1: $\ddot\theta_1 = \dot\theta_1^{\gamma\gamma_1}$, $\gamma_1 \in \Gamma_1$; as Γ_1 acts trivially on $F^\times$ and on the restriction of $\ddot\theta$ to $1 + \mathfrak{p}_E$, we obtain that $\ddot\theta = \dot\theta^{\gamma\gamma_1}$: the quasi-characters $\dot\theta$ and $\ddot\theta$ are conjugate under Γ. We note that $P(r)$ is satisfied for r large; moreover $P(r)$ is $P(s)$ if $s \leqq r$ is any integer larger than all the conductors $\dot{a}(\gamma)$ and $\ddot{a}(\gamma)$ for $\gamma \in \Gamma_r$.

5.4. LEMMA 4. *With these notations, we have the implication, for* $r \geqq 1$: $P(r) \Rightarrow P(r-1)$.

We may assume that $r = \operatorname{Max}_{\Gamma_r} \dot{a}(\gamma) \geqq s = \operatorname{Max}_{\Gamma_r} \ddot{a}(\gamma)$. Suppose first that r is not 1. As $\phi(a)$ is determined by the double class $\dot{K}a\ddot{K}$, we may take a in $A_r^\times \Gamma$, from the property $P(r)$. Let r'', resp. s'', be the smallest integer such that $2r'' \geqq r$, resp. $2s'' \geqq s$. We keep the notations as in 3.3. These integers r'', s'' are larger than 1; the nonvanishing of $\phi(a)$ implies that, on the lattice $L = A_r(\mathfrak{p}^{s''}) \cap a^{-1}A_r(\mathfrak{p}^{r''})a$, the two scalar operators $\ddot{\kappa}(x)$ and $\dot{\kappa}(axa^{-1})$ are equal:

$$\ddot{\theta}_r(1 + T_{A_r/E}\, x)\, \chi^{(r)}_{A_r/E_r}(1 + x) = \dot{\theta}_r(1 + T_{A_r/E}\, axa^{-1})\chi^{(r)}_{A_r/E_r}(1 + x);$$

the map $x \in L \mapsto \dot{\chi}^{(r)}_{A_r/E_r}(1 + x)\, \ddot{\chi}^{(r)}_{A_r/E_r}(1 + x)^{-1}$ is a character, trivial on $A_r(\mathfrak{p}^s) \cap a^{-1}A_r(\mathfrak{p}^r)a \supset L \cdot L$, so this character factors through T_{A_r/E_r} as a character ζ of E_r, trivial on $\mathfrak{p}^r_{E_r}$. Let now $\dot{\alpha}$ be a character of E such that $\ddot{\alpha}(t) = \zeta_{E/E_r}(t)\dot{\theta}_r(1 + t)$ for $t \in \mathfrak{p}^{r''}_E$, and let $\ddot{\alpha}$ be a character of E such that $\ddot{\alpha}(t) = \ddot{\theta}_r(1 + t)$ for $t \in \mathfrak{p}^{s''}_E$. The two characters $\ddot{\alpha}_{A_r/E}$ and $x \mapsto \ddot{\alpha}_{A_r/E}(axa^{-1})$ of A_r coincide on L, so there are two characters $\dot{\xi}$ and $\ddot{\xi}$ of A_r such that $\dot{\xi}$ is trivial on $A_r(\mathfrak{p}^{r''})$ and $\ddot{\xi}$ trivial on $A_r(\mathfrak{p}^{s''})$ satisfying the identity:

$$\ddot{\alpha}_{A_r/E}(x)\, \ddot{\xi}\,(x) = \dot{\alpha}_{A_r/E}(axa^{-1})\dot{\xi}(axa^{-1}) \quad \text{for all } x \in A_r.$$

Applying now part (b) of Lemma 2, the right-hand side can be written $\psi_{A_r/E_r}((\dot{\tau} + \dot{X})baxa^{-1}b^{-1})$, for some $b \in K_r$, with $\dot{X} \in \dot{A}_{r-1}(\mathfrak{p}^{r'})$, and $\dot{\tau}$ in the multiplicative pull-back of $\dot{E}^\times_{r-1}$ in E, primitive over E_r. The left-hand side is $\psi_{A_r/E_r}((T + X)x)$ for some T in $\mathfrak{p}^{r-s}_E$ and X in $A_r(\mathfrak{p}^{r-s''})$. We have the relation: $a^{-1}b^{-1}(\dot{\tau} + \dot{X})ba = T + X$ as the sequence of q^{nm} powers of the left-hand side has a limit, $a^{-1}b^{-1}\, xba$, we must have $r = s$; so, by part (b) of Lemma 2 again, there is a $c \in K_r$ such that

$$\psi_{A_r/E_r}((T + X)x) = \psi_{A_r/E_r}((\ddot{\tau} + \ddot{X})cxc^{-1}),$$

with $\ddot{X}$ in $A_{r-1}(\mathfrak{p}^{r'})$ and $\ddot{\tau}$ in the multiplicative pull-back of $\ddot{E}^\times_{r-1}$ in E, primitive over E_r. The equality $a^{-1}b^{-1}(\dot{\tau} + \dot{X})ba = c^{-1}(\ddot{\tau} + \ddot{X})c$ gives by taking the successive powers q^{nm}, the relation $a^{-1}b^{-1}\, \dot{\tau}\, ba = c^{-1}\, \ddot{\tau}\, c$; this proves that $\dot{\tau}$ and $\ddot{\tau}$ generate the same subfield of E, hence that $\dot{E}_{r-1} = \ddot{E}_{r-1}$, and that the conjugation by bac^{-1} is an automorphism of E_{r-1}; as it is F-linear, it is given by a $\gamma \in \Gamma$; finally $bac^{-1}\gamma^{-1}$ fixing a primitive element of E_{r-1} over E_r belongs to A_{r-1}. This proves the lemma for $r > 1$.

Suppose now $r = 1$; we know that $\dot{\Gamma}_r = \ddot{\Gamma}_r$ for any r; we may suppose that $\Gamma_1 \neq 1$. Let $a_1 \in A_1^\times$ be such that $a \in Ka_1\gamma K$. As in §4.5 if $a_1 \notin K_1F^\times$, there is a subgroup $W \subset K$ such that $a_1\gamma W\gamma^{-1}a_1^{-1}$ is contained in the kernel of $\dot{\kappa}$, and with $\int_W \ddot{\kappa}(w)\, dw = 0$. But this implies that $\phi(a_1\gamma) = 0$, so a_1 is in K and a in $K\Gamma K$: this is $P(0)$, for we know already that the groups Γ_r are the same.

We have proved the following result.

PROPOSITION 4. *Two regular quasi-characters of* $E^\times$ *define the same class of irreducible representations of* $A^\times$ *if and only if they are conjugate under the group* Γ.

6. Computation of the ε-factors.

6.1. The L-functions of the representations π_θ defined by regular quasi-characters θ of $E^\times$ are identically 1 [**6**, Proposition 5.11]. If we apply then the formula of 1.3 giving the ε-factor of π_θ to the subspace of type κ_θ, we get the identity:

$$\int_{K_\theta} f_\psi(x)\kappa_\theta(x)^{-1}|x|_{A/F}^{-(n-1)/2}\, d_{A,\psi}\, x = \varepsilon(\pi_\theta, \psi)\int_{\kappa_\theta} f(x)\kappa_\theta(x)\, |x|_{A/F}^{-(n+1)/2}\, d_{A,\psi}x;$$

for the irreducibility of π_θ; representation induced by κ_θ means too that κ_θ occurs with multiplicity one in the restriction of π_θ to K_θ.

Taking in this formula for f the characteristic function of the subgroup $1 + A(\mathfrak{p}^a)$ with a any integer larger than the conductor of θ, the representation κ_θ is trivial on this group, and the formula gives

$$\varepsilon(\pi_\theta, \psi) = \int_{K_\theta} \psi_{A/F}(x)\kappa_\theta(x)^{-1}|x|_{A/F}^{-(n-1)/2}\, d_{A,\psi}\, x, \quad \text{scalar operator,}$$

the meaning of the integral being by principal value for $|x|_{A/F}$ large, and by analytic continuation for $|x|_{A/F}$ small.

6.2. Let θ be a regular character of $E^\times$ with conductor 1. Then the representation κ_θ of K_θ comes from a representation κ of $K_\theta/K^+ = \bar{A}^\times$, so we have:

$$\varepsilon(\pi_\theta, \psi) = \sum_{K_\theta/K^+}\left(\int_{K^+} \psi_{A/F}(xu)\, d_{A,\psi}\, u\right)\kappa_\theta(x)^{-1}\, |x|_{A/F}^{(n+1)/2}.$$

The measure $d_{A,\psi}$ gives to $A(\mathcal{O})$ the volume $q^{-n^2\operatorname{ord}\psi/2}$, and $K^+ = 1 + A(\mathfrak{p})$, so the inner integral is 0 unless x belongs to $\mathfrak{p}^{-\operatorname{ord}\psi-1}A(\mathcal{O})$ where it is the measure $q^{-n^2(1+\operatorname{ord}\psi/2)}$ of K^+. Our expression is now:

$$\varepsilon(\pi_\theta, \psi) = q^{n\operatorname{ord}\psi/2}q^{-N}\sum_{\bar{A}^\times} \psi_{A/F}(\pi^{-1-\operatorname{ord}\psi}a)\theta(\pi^{-1-\operatorname{ord}\psi}a)^{-1},$$

with $N = n(n-1)/2$ and a a prime element in E. The cusp property of π_θ implies that this sum is in fact on the semisimple elements of $\bar{A}^\times$ where the character of κ is known (3.5); the computations have been made by Springer [**11**], and give

$$\begin{aligned}\varepsilon(\pi_\theta, \psi) &= q^{n\operatorname{ord}\psi/2}(-1)^{n-1}\sum_{\bar{E}^\times} \psi_{E/F}(\pi^{-1-\operatorname{ord}\psi}t)\theta(\pi^{-1-\operatorname{ord}\psi}t)^{-1}\\ &= (-1)^{n-1}\int_{E^\times} \psi_{E/F}(t)\theta(t)^{-1}\, d_{E,\psi}\, t = (-1)^{n-1}\,\varepsilon(\theta, \psi_{E/F}).\end{aligned}$$

6.3. Suppose now that θ is a regular quasi-character of $E^\times$ with even conductor $2m$: the integral over K_θ giving $\varepsilon(\pi_\theta, \psi)$ can be split with respect to the subgroup $K' = 1 + A(\mathfrak{p}^m)$:

$$\varepsilon(\pi_\theta, \psi) = \sum_{K_\theta/K'}\left(\int_{K'} \psi_{A/F}(xy)\kappa_\theta(y)^{-1}\, d_{A,\psi}\, y\right)\kappa_\theta(x)^{-1}|x|_{A/F}^{(n+1)/2};$$

choose an element θ_ψ in E such that

$$\theta(1 + t) = \psi_{E/F}(\theta_\psi t), \quad t \in \mathfrak{p}_E^m;$$

then, from the definition of κ_θ, we see that the restriction of κ_θ to K' is given by the scalar

$$y = 1 + u \in 1 + A(\mathfrak{p}^m) \mapsto \theta(1 + T_{A/E}u) = \psi_{A/F}(\theta_\psi u).$$

The inner integral above is the product by $\psi(x)$ of the integral over $A(\mathfrak{p}^m)$ of the character $\psi_{A/F}((x - \theta_\psi)u)$ for the measure $d_{A,\psi}$, so is 0 unless $x - \theta_\psi$ belongs to $\mathfrak{p}^{-\operatorname{ord}\psi - m} A(\mathcal{O})$ where it is the measure $q^{-n^2(m+\operatorname{ord}\psi)}$ of K'. This condition means $x \in \theta_\psi K'$, hence:

$$\begin{aligned}\varepsilon(\pi_\theta, \psi) &= q^{-n^2(m+\operatorname{ord}\psi/2)} |\theta_\psi|_{E/F}^{(n+1)/2} \psi_{E/F}(\theta_\psi)\theta(\theta_\psi)^{-1} \\ &= |\theta_\psi|_{E/F}^{1/2} \psi_{E/F}(\theta_\psi)\theta(\theta_\psi)^{-1} = \int_{E^\times} \psi_{E/F}(t)\theta(t)^{-1} \, d_{E,\psi}\, t \\ &= \varepsilon(\pi_\theta, \psi_{E/F}).\end{aligned}$$

6.4. Take now for θ a regular quasi-character of $E^\times$ with conductor m, an odd integer greater than 1; call m' and m'' the integers such that $m'' = m' + 1$ and $2m' < m < 2m''$. Choose an element θ_ψ in E such that $\theta(1 + t) = \psi_{E/F}(\theta_\psi t)$ for $t \in \mathfrak{p}_E^{m''}$; the restriction of θ to the subgroup $1 + \mathfrak{p}_E^{m'}$ defines a second degree character of $\mathfrak{p}_E^{m'}$ such that $\theta(1 + t)\theta(1 + t') = \theta(1 + t + t')\psi_{E/F}(\theta_\psi tt')$. The application $t \in \mathfrak{p}_E^{m'} \mapsto \psi(\theta_\psi t)\theta(1 + t)^{-1}$ is trivial on $\mathfrak{p}_E^{m''}$, and the ε-factor of the quasi-character θ on $E^\times$ is

$$\begin{aligned}\varepsilon(\theta, \psi_{E/F}) &= \int_{E^\times} \psi_{E/F}(t)\theta(t)^{-1} \, d_{E,\psi}\, t \\ &= |\theta_\psi|_{E/F}^{1/2} \psi_{E/F}(\theta_\psi)\theta(\theta_\psi)^{-1} \hat{\int}_{\mathfrak{p}_E^{m'}/\mathfrak{p}_E^{m''}} \psi_{E/F}(\theta_\psi t)\theta(1 + t)^{-1},\end{aligned}$$

where $\hat{\int}_A$ for a finite set A means $(\operatorname{Card} A)^{1/2} \sum_A$.

The integral giving $\varepsilon(\pi_\theta, \psi)$ is split now with respect to the subgroup $K'' = 1 + A(\mathfrak{p}^{m''})$:

$$\varepsilon(\pi_\theta, \psi) = \sum_{K_\theta/K''} \left(\int_{K''} \psi_{A/F}(xy)\kappa_\theta(y)^{-1} \, d_{A,\psi}\, y \right) \eta_\theta(x)^{-1} |x|_{A/F}^{(n+1)/2}$$

call K' the subgroup $1 + A(\mathfrak{p}^{m'})$; the inner integral is 0 unless x belongs to $\theta_\psi K'$, so the formula can be written:

$$\varepsilon(\pi_\theta, \psi) = |\theta_\psi|_{E/F}^{1/2} \hat{\int}_{\theta_\psi K'/K''} \psi_{A/F}(x)\kappa_\theta(x)^{-1}.$$

But K'/K'' is isomorphic to $A(\mathfrak{p}^{m'}/\mathfrak{p}^{m''}) = \mathfrak{p}_E^{m'}/\mathfrak{p}_E^{m''} + A^\circ(\mathfrak{p}^{m'}/\mathfrak{p}^{m''})$, with the kernel A° of $T_{A/E}$, so the expression splits in two terms and gives, using the formula for $\varepsilon(\theta, \psi_{E/F})$:

$$\varepsilon(\pi_\theta, \psi) = \varepsilon(\theta, \psi_{E/F}) \hat{\int}_{A^\circ(\mathfrak{p}^{m'}/\mathfrak{p}^{m''})} \kappa_\theta(1 + x)^{-1} \theta(\theta_\psi)\kappa_\theta(\theta_\psi)^{-1}.$$

Call r the maximum of all the conductors $a(\gamma)$ of θ/θ^γ for $\gamma \neq 1$. We examine the situation in two cases, according to the respective places of m and r, the conductor of θ.

(a) Suppose first $m > r$. But $\theta = \theta^{(r)} \chi_{E/F}^{(r)}$ as in 3.3. The restriction of κ_θ to K' is a scalar, given by

$$1 + x \in K' = 1 + A(\mathfrak{p}^{m'}) \mapsto \theta_r(1 + T_{A/E}x)\chi^{(r)}_{A/F}(1 + x).$$

For x in $A^\circ(\mathfrak{p}^{m'})$ it is only $\chi^{(r)}_{A/F}(1 + x) = \chi^{(r)}(1 + \sigma_2(x))$, where $\sigma_2(x)$ is the trace of the action of x on $\Lambda^2 E$, as an F-vector space. But now the application $x \mapsto \chi^{(r)}(1 + \sigma_2(x))$ is a nondegenerate second degree character on the $(n-1)$-dimensional vector space $A^\circ(\mathfrak{p}^{m'}/\mathfrak{p}^{m''})$ over $\bar{E}$, so its sum $\hat{\int}_{A^\circ(\mathfrak{p}^{m'}/\mathfrak{p}^{m''})} \chi^{(r)}(1 + \sigma_2(x))$ is $(-1)^{n-1}$; this gives

$$\varepsilon(\pi_\theta, \psi) = (-1)^{n-1}\, \varepsilon(\theta, \psi_{E/F})\theta(\theta_\psi)\kappa_\theta(\theta_\psi)^{-1};$$

from the definition of the representation κ_θ, we know that the operator $\theta(\theta_\psi)\kappa_\theta(\theta_\psi)^{-1}$ depends only on the class of θ_ψ modulo $1 + \mathfrak{p}_E$; but the condition $m > r$ means that this class is fixed by Γ, so there is a representative in $F^\times$; but on $F^\times$, κ is θ, so $\kappa_\theta(\theta_\psi) = \theta(\theta_\psi)$. We have proved the formula: $\varepsilon(\pi_\theta, \psi) = (-1)^{n-1}\, \varepsilon(\theta, \psi_{E/F})$.

(b) Suppose now that $m = r$; put $r' = m'$, $r'' = m''$. The group K' is contained in $K^{\pm}_{r-1}K_r$ (notations as in 3.2). From the decomposition $A^\circ = A^\circ_{r-1} + A^{r-1}$ defined by $T_{A/A_{r-1}}$, we get, for $x \in A^\circ(\mathfrak{p}^{r'})$, $y \in A^\circ_{r-1}(\mathfrak{p}^{r'})$, $z \in A^{r-1}(\mathfrak{p}^{r'})$, $x = y + z$:

$$\kappa_\theta(1 + x) = I \otimes \chi^{(r-1)}_{A_{r-1}/E_{r-1}}(1 + y)\kappa_r(1 + z).$$

As in (a) the map $y \mapsto \chi^{(r-1)}_{A_{r-1}/E_{r-1}}(1 + y) = \chi^{(r-1)}(1 + \sigma_2^{r-1}(y))$ is a nondegenerate second degree character on the vector space $A^\circ_{r-1}(\mathfrak{p}^{r'}/\mathfrak{p}^{r''})$ over $\bar{E}_{r-1}$ and we have:

$$\hat{\int}_{\bar{A}^\circ_{r-1}(\mathfrak{p}^{r'}/\mathfrak{p}^{r''})} \chi^{(r-1)}_{A_{r-1}/E_{r-1}}(1+y)^{-1} = -(-1)^{n_{r-1}}$$

where n_{r-1} is the dimension of A_{r-1} over E_{r-1}, that is the order of Γ_{r-1}. The element θ_ψ of E is fixed modulo $1 + \mathfrak{p}_E$ by Γ_{r-1}, so the operator $\theta(\theta_\psi)\kappa_\theta(\theta_\psi)^{-1}$ is simply $I \otimes \tilde{\kappa}_r(\theta_\psi)^{-1}$, by definition of the representation κ_θ (Proposition 2). This means that $\varepsilon(\pi_\theta, \psi)$ differs from $\varepsilon(\theta, \psi_{E/F})$ only by the scalar defined by the operator:

$$\hat{\int}_{A^{r-1}\mathfrak{p}^{r'}/\mathfrak{p}^{r''}} \kappa_r(1 + z)^{-1}\, \tilde{\kappa}_r(\theta_\psi)^{-1}.$$

In the semidirect product $H(r) \rtimes \bar{E}^\times$, the image of $\theta_\psi(1 + z)$ is conjugate to the image of θ_ψ: this comes from the fact that if $\gamma \in \Gamma$ fixes θ_ψ modulo $1 + \mathfrak{p}_E$ then $\gamma \in \Gamma_{r-1}$, so $z \mapsto \theta_\psi z \theta_\psi^{-1} - z$ is an automorphism of the $\bar{E}$-vector space $A^{r-1}(\mathfrak{p}^{r'}/\mathfrak{p}^{r''})$. To compute the above scalar, we compute its trace, which is the trace of $\tilde{\kappa}_r(\theta_\psi)^{-1}$, equal to $(-1)^{n-n_{r-1}}$ ([4, Theorem 1, p.16], or [5]). This shows that in case (b), we have the formula: $\varepsilon(\pi_\theta, \psi) = (-1)^{n-1}\, \varepsilon(\theta, \psi_{E/F})$.

6.5. The discussion in the preceding section gave the following result:

PROPOSITION 4. *Let θ be a regular character of $E^\times$ and π_θ be the representation constructed in* §3. *Then, if $a(\theta)$ is the conductor of θ, we have*: $\varepsilon(\pi_\theta, \psi) = (-1)^{(n-1)a(\theta)}\varepsilon(\theta, \psi_{E/F})$.

This formula can be written

$$\varepsilon(\pi_\theta, \psi) = (-1)^{(n-1)\operatorname{ord}\psi}(-1)^{(n-1)(\operatorname{ord}\psi + a(\theta))}\, \varepsilon(\theta, \psi_{E/F}).$$

Define a quasi-character $\tilde{\theta}$ from θ by the formula $\tilde{\theta} = (-1)^{(n-1)\mathrm{val}}\,\theta$ (we observe that the character $(-1)^{(n-1)\mathrm{val}}$ on $E^\times$ is the unique unramified extension to $E^\times$ of the character of $F^\times$ defined by the determinant of the regular representation of Γ, $\det \operatorname{Ind}^E_F 1$). Now, we can define the representation $\theta_{A/F}$:

DEFINITION. The representation $\theta_{A/F}$ associated to the regular quasi-character θ of $E^\times$ is $\pi_{\bar{\theta}}$, where $\pi_{\bar{\theta}}$ has been defined in §3.

Then, from Propositions 1, 2, 3, we get the theorem for the split algebra $A = \mathrm{End}_F E$.

7. Remark. Keep the notations as above. For a nonregular quasi-character θ of $E^\times$, the field E_0 is strictly contained in E, and there are $n_0 = [E: E_0]$ quasi-characters θ' of $E_0^\times$ such that $\theta = \theta'_{E/E_0}$, and they are regular with respect to F. From the theorem, they define irreducible admissible representations $\theta'_{A'/F}$ of the multiplicative group of the algebra $A' = \mathrm{End}_F E_0$. The direct product of n_0 copies of $A'^\times$ is F-isomorphic to an Levi subgroup of an F-parabolic subgroup P of $A^\times$, and by Theorem 3 of **[1]**, the representation $\theta_{A/F}$ of $A^\times$ induced by the representation of P defined by the product of the representations $\theta'_{A'/F}$ for all choices of θ', is irreducible; by Theorem 3.4 of **[6]**, this representation $\theta_{A/F}$ has its L-function and ε-factor given as products of the L-functions and ε-factors of the representations $\theta'_{A'/F}$. From the properties of inductivity and the theorem, we have the formulas, with n' the rank n/n_0 of A':

$$\begin{aligned}
L(\theta_{A/F}) &= \prod L(\theta'_{A'/F}) = \prod L(\theta'_{E_0/F}) = L(\theta'_{E/E_0}) = L(\theta), \\
\varepsilon(\theta_{A/F}, \psi) &= \prod \varepsilon(\theta'_{A'/F}, \psi) = \prod (-1)^{(n'-1)\operatorname{ord}\psi}\, \varepsilon(\theta', \psi_{E_0/F}) \\
&= (-1)^{n_0(n'-1)\operatorname{ord}\psi}\, \varepsilon(\theta'_{E/E_0}, \psi_{E/F})\, \lambda(E/E_0, \psi_{E_0/F}) \\
&= (-1)^{n_0(n'-1)\operatorname{ord}\psi + (n_0-1)\operatorname{ord}\psi}\, \varepsilon(\theta, \psi_{E/F}) \\
&= (-1)^{(n-1)\operatorname{ord}\psi}\, \varepsilon(\theta, \psi_{E/F}).
\end{aligned}$$

The other properties of functoriality given in the theorem for a regular quasi-character θ are still true here.

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UNIVERSITÉ PARIS VII

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SOME REMARKS ON THE SUPERCUSPIDAL REPRESENTATIONS OF p-ADIC SEMISIMPLE GROUPS

G. LUSZTIG

1. Let $\boldsymbol{G}$ be a semisimple group over a local nonarchimedean field K with ring of integers $\mathcal{O}$ and finite residue field k. We define $G = \boldsymbol{G}(K)$, $\tilde{G} = \boldsymbol{G}(\tilde{K})$ where $\tilde{K}$ is a maximal unramified extension of K (with ring of integers $\tilde{\mathcal{O}}$ and residue field $\tilde{k}$, an algebraic closure of k).

Let $\boldsymbol{T} \subset \boldsymbol{G}$ be a maximal torus, defined and anisotropic over K. We shall assume that $\boldsymbol{T}$ satisfies the following condition. There exists a Borel subgroup $\boldsymbol{B}$ of $\boldsymbol{G}$ containing $\boldsymbol{T}$ and defined over $\tilde{K}$. (This condition is certainly satisfied if $\boldsymbol{T}$ is split over $\tilde{K}$.) Let $\check{T}$ be the set of homomorphisms of $T = \boldsymbol{T}(K)$ into $\bar{Q}_l^*$ (l a prime $\neq$ char k) which factor through a finite quotient of T. One expects that to any $\theta \in \check{T}$ satisfying some regularity condition, there corresponds an irreducible admissible supercuspidal representation of G (over $\bar{Q}_l$). Such a correspondence has been established by Gérardin (for $\boldsymbol{T}$ split over $\tilde{K}$, with certain restrictions, see [**2**]) using methods of Shintani, Howe and Corwin. (For SL_2 the correspondence was established by Gelfand, Graev, Shalika and others.)

I would like to suggest another possible approach to the question of constructing this correspondence. This would use l-adic cohomology (or homology) of a certain infinite dimensional variety X over $\tilde{k}$. (The fact that l-adic cohomology might be used to construct representations of G is made plausible by the work of Drinfeld and by that of Deligne and myself, in the case of finite fields [**1**]. Note that, even in Gérardin's approach, one has to appeal, in the case where the conductor is very small, to the representation theory of a reductive group over a finite field, which is, itself, based on l-adic cohomology.)

Let $\boldsymbol{U}$ be the unipotent radical of $\boldsymbol{B}$ and let $\tilde{U} = \boldsymbol{U}(\tilde{K})$. Consider the Frobenius map $F: \tilde{G} \to \tilde{G}$ defined by the Frobenius element $\phi \in \mathrm{Gal}(\tilde{K}/K)$, so that $G = \tilde{G}^F$.

Let $X = \{g \in \tilde{G} \mid g^{-1} F(g) \in \tilde{U}\}/\tilde{U} \cap F^{-1}\tilde{U}$. (The action of $\tilde{U} \cap F^{-1}\tilde{U}$ on $\tilde{G}$ is by right multiplication.) Now $G \times T$ acts on X by $(g_0, t)\colon g \to g_0 g t^{-1}$ ($g_0 \in G$, $t \in T$, $g \in \tilde{G}$).

I believe that, by regarding X as an infinite dimensional variety over $\tilde{k}$, one can define l-adic homology groups $H_i(X)$ on which $G \times T$ acts in such a way that $H_i(X) = \bigoplus_{\theta \in T} H_i(X)_\theta$ (where $H_i(X)_\theta$ is the subspace of $H_i(X)$ on which T acts according to θ). Moreover, for θ fixed, $H_i(X)_\theta$ should be zero for large i, while if θ is not fixed

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by any nontrivial element in the small Weyl group of T, $H_i(X)_\theta$ should be nonzero for exactly one index $i = i_0$ (depending on θ) and the resulting representation of G on $H_{i_0}(X)_\theta$ should be an irreducible admissible supercuspidal representation. This should establish the required correspondence between characters of T and representations of G.

2. Consider, for example, the case where $\tilde{G} = \mathrm{SL}_n(\tilde{K})$ and let $F\colon \tilde{G} \to \tilde{G}$ be the homomorphism defined by

$$F(A) = \begin{pmatrix} 0 & 1 & & 0 \\ & 0 & 1 & \\ & 0 & \ddots & 1 \\ \pi & & & 0 \end{pmatrix} A^{\phi} \begin{pmatrix} 0 & 1 & & 0 \\ & 0 & 1 & \\ & 0 & \ddots & 1 \\ \pi & & & 0 \end{pmatrix}^{-1} \tag{2.1}$$

where π is a uniformizing element of $\mathcal{O}$ and, for any matrix A over $\tilde{K}$, A^ϕ is obtained by applying ϕ to each entry of A. The fixed point set $\tilde{G}^F$ is the group of elements of reduced norm one in a central division algebra of dimension n^2 over K. Let $\tilde{T}$ be the group of diagonal matrices in $\mathrm{SL}_n(\tilde{K})$; it is invariant under F. Let $\tilde{U} \subset \mathrm{SL}_n(\tilde{K})$ (resp. $\tilde{U}^-$) be the subgroup consisting of all upper (lower) triangular matrices with 1's on the diagonal. If A is a matrix in $\mathrm{SL}_n(\tilde{K})$ satisfying $A^{-1}F(A) \in \tilde{U}$, we can find a unique $B \in \tilde{U} \cap F^{-1}\tilde{U}$ such that $(AB)^{-1}F(AB) \in \tilde{U} \cap F\tilde{U}^-$. Thus,

$$\{A \in \mathrm{SL}_n(\tilde{K}) \mid A^{-1}F(A) \in \tilde{U}\}/\tilde{U} \cap F^{-1}(\tilde{U})$$

can be identified with

$$X = \{A \in \mathrm{SL}_n(\tilde{K}) \mid A^{-1}F(A) \in \tilde{U} \cap F\tilde{U}^-\}$$

on which $\tilde{G}^F$ acts by left multiplication and $\tilde{T}^F$ acts by right multiplication. X is just the set of all $n \times n$ matrices of the form

$$\begin{pmatrix} a_1 & a_n^{\phi^{-1}} & \cdots & a_2^{\phi^{-(n-1)}} \\ \pi a_2 & a_1^{\phi^{-1}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_n^{\phi^{-(n-1)}} \\ \pi a_n & \cdots & \pi a_2^{\phi^{-(n-2)}} & a_1^{\phi^{-(n-1)}} \end{pmatrix} \tag{2.2}$$

with $a_i \in \tilde{K}$ and determinant equal to 1. For such a matrix, we have automatically $a_i \in \tilde{\mathcal{O}}$ $(1 \leqq i \leqq n)$ and $a_1 \notin \pi\tilde{\mathcal{O}}$.

It follows that X may be regarded as the projective limit $\operatorname{proj\,lim}_h X_h$ of the algebraic varieties X_h over $\bar{k}$, where X_h $(h \geqq 1)$ is the set of all $n \times n$ matrices of the form (2.2) with $a_1 \in (\tilde{\mathcal{O}}/\pi^h\tilde{\mathcal{O}})^*$, $a_i \in \tilde{\mathcal{O}}/\pi^{h-1}\tilde{\mathcal{O}}$ $(2 \leqq i \leqq n)$, with determinant equal to 1. (We regard πa_i $(2 \leqq i \leqq n)$ as elements of $\pi\tilde{\mathcal{O}}/\pi^h\tilde{\mathcal{O}}$; the determinant of such a matrix has an obvious meaning as an element of $\tilde{\mathcal{O}}/\pi^h\tilde{\mathcal{O}}$.)

Let G_h be the set of all $n \times n$ matrices (a_{ij}) with $a_{ij} \in \pi\,\tilde{\mathcal{O}}/\pi^h\tilde{\mathcal{O}}$ $(\forall i > j)$, $a_{ij} \in \tilde{\mathcal{O}}/\pi^{h-1}\tilde{\mathcal{O}}$ $(\forall i < j)$, $a_{ii} \in (\tilde{\mathcal{O}}/\pi^h\tilde{\mathcal{O}})^*$ $(\forall i)$, with determinant equal to 1, as an element of $\tilde{\mathcal{O}}/\pi^h\tilde{\mathcal{O}}$. Define $F\colon G_h \to G_h$ by the formula (2.1). G_h is an algebraic group over $\bar{k}$ and $F\colon G_h \to G_h$ is the Frobenius map for a k-rational structure on G_h. Let T_h be the group of diagonal matrices in G_h and let U_h (resp. U_h^-) be the group of upper (lower) triangular matrices in G_h with 1's on the diagonal. We may identify X_h with the variety

$$\{A \in G_h \mid A^{-1}F(A) \in U_h \cap FU_h^-\}.$$

G_h^F acts on X_h by left multiplication and T_h^F acts on X_h by right multiplication. (Note that $\tilde{G}^F = \text{proj lim}_h G_h^F$, $\tilde{T}^F = \text{proj lim}_h T_h^F$). The following lemma is crucial.

LEMMA. *The action of* $K_h = \ker(T_h \to T_{h-1})$ $(h \geqq 2)$, *on* X_h *preserves each fibre of the natural map* $X_h \to X_{h-1}$, *and each fibre of* $X_h/K_h \to X_{h-1}$ *is isomorphic to the affine space of dimension* $(n - 1)$ *over* k.

Let us define $H_i(Y)$ for any smooth algebraic variety Y over k of pure dimension d, to be $H_c^{2d-i}(Y, \bar{Q}_l(d))$, where l is a fixed prime $\neq$ char $\bar{k}$. The previous lemma shows that, for $h \geqq 2$, $H_i(X_{h-1})$ is canonically isomorphic to $H_i(X_h)^{K_h}$ (fixed points of K_h on $H_i(X_h)$). In particular, we have a well-defined embedding $H_i(X_{h-1}) \to H_i(X_h)$. Using these embeddings, we form the direct limit inj $\lim_h H_i(X_h)$. We define $H_i(X)$ to be this direct limit. $H_i(X)$ decomposes naturally in a direct sum $\bigoplus H_i(X)_\theta$, where θ runs through the set $(\tilde{T}^F)^\vee$. It is clear that, for fixed θ, $H_i(X)_\theta$ is of finite dimension and is zero for large i. On the other hand, it is in a natural way a $\tilde{G}^F$-module ($\tilde{G}^F$ acting via a finite quotient). Let $R_\theta = \sum_i (-1)^i H_i(X)_\theta$.

THEOREM. *For each* $\theta \in (\tilde{T}^F)^\vee$, $\pm R_\theta$ *is an irreducible* $\tilde{G}^F$-*module. If* $\theta \neq \theta'$, *then* $\pm R_\theta$, $\pm R'_\theta$ *are distinct.*

It suffices to prove that, if $\Sigma = G_h^F \backslash (X_h \times X_h)$ (with G_h^F acting diagonally on $X_h \times X_h$) we have

$$\sum_i (-1)^i \dim H_i(\Sigma)_{\theta, \theta'^{-1}} = 1 \quad \text{if } \theta' = \theta,$$
$$= 0 \quad \text{if } \theta' \neq \theta.$$

(Here $T_h^F \times T_h^F$ acts on Σ by right multiplication and $H_i(\Sigma)_{\theta, \theta'^{-1}}$ denotes the (θ, θ'^{-1})-eigenspace of $T_h^F \times T_h^F$.)

To compute the (equivariant) Euler characteristic of Σ we use the principle that the Euler characteristic of a space can be computed from the zeros of a nice vector field.

The map $(g, g') \to (x, x', y)$, $x = g^{-1}F(g)$, $x' = g'^{-1}F(g')$, $y = g^{-1}g'$ defines an isomorphism

$$\Sigma \simeq \{(x, x', y) \in (U_h \cap FU_h^-) \times (U_h \cap FU_h^-) \times G_h \mid xF(y) = yx'\}$$

(compare [**1**, 6.6]). Now, any $y \in G_h$ can be written uniquely in the form

$$y = y_1' y_2' y_1'' y_2'', \qquad y_1' \in U_h^- \cap F^{-1}U_h,\ y_2' \in T_h(U_h^- \cap F^{-1}U_h^-),$$
$$y_1'' \in U_h \cap F^{-1}U_h^-,\ y_2'' \in U_h \cap F^{-1}U_h.$$

We now make the substitution $xF(y_1') = \tilde{x} \in U_h$, so the equation of Σ becomes

$$\tilde{x}F(y_2')F(y_1'')F(y_2'') = y_1' y_2' y_1'' y_2'' x'.$$

Any element $z \in U_h$ can be written uniquely in the form $z = y_2'' x' F(y_2'')^{-1}$ with $y_2'' \in U_h \cap F^{-1}U_h$, $x' \in U_h \cap FU_h^-$, so the equation of Σ becomes

$$\tilde{x}F(y_2')F(y_1'') \in y_1' y_2' y_1'' U_h = y_1' y_2' U_h.$$

Thus, we may identify

$$\Sigma = \{(\tilde{x}, y_1', y_2' y_i'') \in (U_h \cap FU_h^-) \times (U_h^- \cap F^{-1}U_h) \times T_h(U_h^- \cap F^{-1}U_h^-) \times (U_h \cap F^{-1}U_h^-) \,|\, y_2'^{-1} y_1'^{-1} \tilde{x}F(y_2') F(y_1'') \in U_h\}.$$

The action of $T_h^F \times T_h^F$ is given by

$$(t, t'): (\tilde{x}, y_1', y_2', y_1'') \to (t\tilde{x}t^{-1}, ty_1' t^{-1}, ty_2' t'^{-1}, t'y_1'' t'^{-1}).$$

This action extends to an action of a larger group H, consisting of all pairs (t, t') of $n \times n$-diagonal matrices with diagonal entries in $(\tilde{\mathcal{O}}/\pi^h\tilde{\mathcal{O}})^*$ such that $\det t = \det t'$ and $t^{-1}F(t) = t'^{-1}F(t') \in$ centralizer of $T_h(U_h^- \cap F^{-1}U_h^-)$; the action of H is given by

$$(t, t'): (\tilde{x}, y_1', y_2', y_1'') \to (F(t)\tilde{x}F(t)^{-1}, F(t)y_1'F(t)^{-1}, ty_2't'^{-1}, t'y_1''t'^{-1}).$$

A diagonal matrix of the form

$$\begin{pmatrix} 1 & & & & \\ & 1 & & 0 & \\ & & \ddots & & \\ 0 & & & 1 & \\ & & & & a \end{pmatrix}, \qquad a \in (\tilde{\mathcal{O}}/\pi^h\tilde{\mathcal{O}})^*,$$

certainly centralizes $T_h(U_h^- \cap F^{-1}U_h^-)$. Thus, any pair (t, t) with t of the form

$$\begin{pmatrix} \xi^{\phi^{n-1}} & & 0 \\ & \ddots \quad \xi^{\phi} & \\ 0 & & \xi \end{pmatrix}$$

(with $\xi \in (\tilde{\mathcal{O}}/\pi^h\tilde{\mathcal{O}})^*$ a root of 1 of order prime to char k) is in H. The set of all such pairs (t, t) is a one dimensional subtorus $\mathcal{T}$ of H (over $\bar{k}$). This torus acts on Σ commuting with $T_h^F \times T_h^F$. Its fixed point set on Σ is the finite set given by $\tilde{x} = y_1' = y_1'' = e$, $y_2' \in T_h^F$, hence it is isomorphic to T_h^F with $T_h^F \times T_h^F$ acting by left and right multiplication. It follows that

$$\begin{aligned}\textstyle\sum(-1)^i \dim H_i(\Sigma)_{\theta,\theta'^{-1}} &= \textstyle\sum(-1)^i \dim H_i(\Sigma^{\tau})_{\theta,\theta'^{-1}} \\ &= \dim H_0(T_h^F)_{\theta,\theta'^{-1}} = 1 \quad \text{if } \theta = \theta', \\ &= 0 \quad \text{if } \theta \neq \theta',\end{aligned}$$

as required.

3. Let V be a 2-dimensional vector space over K and let $\tilde{V} = V \otimes_k \tilde{K}$. Let F: $\tilde{V} \to \tilde{V}$ be defined by $F = 1 \otimes \phi$. Assume that we are given a nonzero element $\omega \in \Lambda^2 V$. The set $X = \{x \in \tilde{V} \,|\, x \wedge Fx = \omega\}$ is invariant under the obvious action of $\mathrm{SL}_2(V)$ and under the action of the group $T = \{\lambda \in \tilde{K}^* \,|\, \lambda \cdot \lambda^{\phi} = 1\}$ which acts by scalar multiplication. (This set X can be identified with the set X defined in §1 for $\boldsymbol{G} = \mathrm{SL}_2$ and $\boldsymbol{T}$ a maximal torus associated to the unramified quadratic extension of K.) Objects similar to this X appear in the work of Drinfeld. We now

show what is the meaning of the homology groups $H_i(X)$ in the present case. It can be easily checked that if $x \in X$, the $\tilde{\mathcal{O}}$-module spanned by x and Fx is invariant under F, hence it comes from a lattice L in V.

Thus X is a disjoint union $X = \coprod_L X_L$ over all lattices $L \subset V$ with determinant $\mathcal{O}^*\omega$, where

$$X_L = \{x \in L \otimes_{\mathcal{O}} \tilde{\mathcal{O}} \mid x \wedge Fx = \omega\}.$$

Each X_L can be regarded in a natural way as proj $\lim_h X_{L,h}$ where $X_{L,h}$ is the h-dimensional variety over $\bar{k}$ defined by

$$X_{L,h} = \{x \in L \otimes_{\mathcal{O}} (\tilde{\mathcal{O}}/\pi^L\tilde{\mathcal{O}}) \mid x \wedge Fx = \omega\}.$$

The fibres of $X_{L,h} \to X_{L,h-1}$ have a property similar to that in the Lemma in §2. This allows us to define $H_i(X_L) = \text{inj} \lim_h H_i(X_{L,h})$ as in §2. We also define $H_i(X) = \bigoplus_L H_i(X_L)$.

Similar arguments should apply in the general case.

4. Let $\boldsymbol{G}$, $\boldsymbol{T}$, $\boldsymbol{B}$, $\boldsymbol{U}$ be as in §1. Assume that $\boldsymbol{G}$ comes from a Chevalley group over $\boldsymbol{Z}$ by extension of scalars so that $\boldsymbol{G}(\mathcal{O})$, $\boldsymbol{G}(\tilde{\mathcal{O}})$ are well defined. Let $\tilde{G}_h = \boldsymbol{G}(\tilde{\mathcal{O}}/\pi^h\tilde{\mathcal{O}})$. Assume that $\boldsymbol{T}(\tilde{K}) \subset \boldsymbol{G}(\tilde{\mathcal{O}})$. Let $\tilde{T}_h$, $\tilde{U}_h$ be the images of $\boldsymbol{T}(\tilde{K})$, $\boldsymbol{U}(\tilde{K}) \cap \boldsymbol{G}(\tilde{\mathcal{O}})$ under $\boldsymbol{G}(\tilde{\mathcal{O}}) \to \tilde{G}_h$. $F: \boldsymbol{G}(\tilde{K}) \to \boldsymbol{G}(\tilde{K})$ induces $F: \tilde{G}_h \to \tilde{G}_h$. This gives a $\bar{k}$-rational structure on the $\bar{k}$-algebraic group $\tilde{G}_h$. Define

$$X_h = \{g \in \tilde{G}_h \mid g^{-1} F(g) \in \tilde{U}_h\}/\tilde{U}_h \cap F^{-1}\tilde{U}_h.$$

The finite group $\tilde{G}_h^F \times \tilde{T}_h^F$ acts on X_h, as before, by left and right multiplication. For each character θ: $\tilde{T}_h^F \to \boldsymbol{Q}_l^{-1}$ we form

$$R_\theta = \sum_i (-1)^i H_i(X_h)_\theta.$$

THEOREM. *If θ is sufficiently regular, the virtual $\tilde{G}_h^F$-module $\pm R_\theta$ is irreducible. It is independent of the choice of* $\boldsymbol{B}$.

In the case $h = 1$, this follows from [**1**].

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UNIVERSITY OF WARWICK, COVENTRY, GREAT BRITAIN

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DECOMPOSITION OF REPRESENTATIONS INTO TENSOR PRODUCTS

D. FLATH

1. In this paper some generalizations of the classical theorem which classifies the irreducible representations of the direct product of two finite groups in terms of those of the factors will be discussed. The first generalization consists in expanding the class of groups considered.

THEOREM 1. *Let G_1, G_2 be locally compact totally disconnected groups and let $G = G_1 \times G_2$.*

(1.1) *If π_i is an admissible irreducible representation of G_i, $i = 1, 2$, then $\pi_1 \otimes \pi_2$ is an admissible irreducible representation of G.*

(1.2) *If π is an admissible irreducible representation of G, then there exist admissible irreducible representations π_i of G_i such that $\pi \simeq \pi_1 \otimes \pi_2$. The classes of the π_i are determined by that of π.*

We recall some notation. For a locally compact totally disconnected group G, the *Hecke algebra* $H(G)$ of G is the convolution algebra of locally constant complex valued functions on G with compact support. For a compact open subgroup K of G, let e_K be the function $(\text{meas } K)^{-1} \cdot \text{ch}_K$, where ch_K is the characteristic function of K and meas is the Haar measure on G which has been used to define convolution in $H(G)$. Then e_K is an idempotent of $H(G)$. The subalgebra $e_K H(G) e_K$ of $H(G)$ will be denoted $H(G, K)$. A smooth G-module W is in a natural way an $H(G)$-module, and for every compact open subgroup $K \subset G$ the space $W^K = e_K W$ is an $H(G, K)$-module.

Before proving Theorem 1 we state an *Irreducibility Criterion. A smooth G-module W is irreducible if and only if W^K is an irreducible $H(G, K)$-module for all compact open subgroups K of G.*

PROOF. This follows from the fact that if U is an $H(G, K)$-submodule of W^K, then $(H(G) \cdot U)^K = U$. □

Remark that in applying the irreducibility criterion it is sufficient to check that W^K is an irreducible $H(G, K)$-module for a set of K which forms a neighborhood base of the identity in G.

COROLLARY. *Let K be a compact open subgroup of G such that $H(G, K)$ is commutative, and let W be an admissible irreducible G-module. Then* $\dim W^K \leqq 1$.

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PROOF OF THEOREM 1. It is straightforward that
(i) $H(G_1 \times G_2) \simeq H(G_1) \otimes H(G_2)$,
(ii) $H(G_1 \times G_2, K_1 \times K_2) \simeq H(G_1, K_1) \otimes H(G_2, K_2)$ and
(iii) $(W_1 \otimes W_2)^{K_1 \times K_2} \simeq W_1^{K_1} \otimes W_2^{K_2}$
for every pair of compact open subgroups K_i of G_i and every pair of smooth G_i-modules W_i.

Assertion (1.1) follows from (iii) and the irreducibility criterion.

Conversely, let W be an admissible irreducible G-module. Let $K = K_1 \times K_2$, where K_i is a compact open subgroup of G_i, $i = 1, 2$, be such that $W^K \neq 0$. The space W^K is finite dimensional, so by the corollary on p. 94 of [2] there exist irreducible $H(G_i, K_i)$-modules $W_i^{K_i}$ and an $H(G, K)$ isomorphism a_K from W^K to $W_1^{K_1} \otimes W_2^{K_2}$. Similar remarks apply to every open subgroup $K' = K_1' \times K_2'$ of K. There exist $H(G_i, K_i)$-maps $b_i = b_i(K, K')$: $W_i^{K_i} \to W_i^{K_i'}$ such that the following diagram is commutative.

$$\begin{array}{ccc} W^K & \xrightarrow{a_K} & W_1^{K_1} \otimes W_2^{K_2} \\ \Big\downarrow{\scriptstyle \text{incl.}} & & \Big\downarrow{\scriptstyle b_1 \otimes b_2} \\ W^{K'} & \xrightarrow{a_{K'}} & W_1^{K_1'} \otimes W_2^{K_2'} \end{array}$$

Moreover, the maps $b_i(K, K')$ can be chosen for every pair of compact open subgroups K, K' of this type in such a way as to form an inductive system. Then $W \simeq W_1 \otimes W_2$, where $W_i = \text{ind}\lim_{K_i} W_i^{K_i}$, and W_i is an admissible irreducible representation of G_i, $i = 1, 2$.

The class of W_i is determined by that of W, for the restriction of W to G_i is W_i-isotypic. □

An analysis of the proof of Theorem 1 reveals that the groups G and G_i enter only through their Hecke algebras. This leads one to define an *idempotented algebra* (A, E) to be an algebra A with a directed family of idempotents E such that $A = \bigcup_{e \in E} eAe$. An *admissible module* W for (A, E) is an A-module W which is *nondegenerate* in the sense that $AW = W$ and is such that dim eW is finite for all $e \in E$. The tensor product of two idempotented algebras is naturally idempotented. The proof of Theorem 1 is readily adapted to establish a similar theorem about the admissible irreducible modules of the tensor product of two idempotented algebras.

2. The study of the representations of adelic groups, which are infinite restricted products of groups, requires the notion of restricted tensor product of vector spaces which was introduced in [4].

Let $\{W_v \mid v \in V\}$ be a family of vector spaces. Let V_0 be a finite subset of V. For each $v \in V \backslash V_0$, let x_v be a nonzero vector in W_v. For each finite subset S of V containing V_0, let $V_S = \bigotimes_{v \in S} W_v$; and if $S \subset S'$, let f_S: $W_S \to W_{S'}$ be defined by $\bigotimes_{v \in S} w_v \mapsto \bigotimes_{v \in S} w_v \bigotimes_{v \in S' \backslash S} x_v$. Then $W = \bigotimes_{x_v} W_v$, the *restricted tensor product* of the W_v with respect to the x_v, is defined by $W = \text{ind}\lim_S W_S$. The space W is spanned by elements written in the form $w = \bigotimes w_v$, where $w_v = x_v$ for almost all $v \in V$.

The ordinary constructions with finite tensor products extend easily to restricted tensor products.

(1) Given linear maps $B_v\colon W_v \to W_v$ such that $B_v x_v = x_v$ for almost all $v \in V$, then one can define $B = \bigotimes B_v\colon W \to W$ by $B(\bigotimes w_v) = \bigotimes B_v w_v$.

(2) Given a family of algebras $\{A_v \mid v \in V\}$ and given nonzero idempotents $e_v \in A_v$ for almost all v, then $A = \bigotimes_{e_v} A_v$ is an algebra in the obvious way.

(3) If W_v is an A_v-module for each $v \in V$ such that $e_v \cdot x_v = x_v$ for almost all v, then $\bigotimes_{x_v} W_v$ is an A-module. The isomorphism class of W depends on $\{x_v\}$. However, if $\{x'_v\}$ is another collection of nonzero vectors such that x_v and x'_v lie on the same line in W_v for almost all v, then the A-modules $\bigotimes_{x_v} W_v$ and $\bigotimes_{x'_v} W_v$ are isomorphic.

EXAMPLE 1. The polynomial ring in an infinite number of variables $C[X_1, X_2, \ldots]$ is isomorphic to $\bigotimes_{e_i} C[X_i]$, where e_i is the identity element of $C[X_i]$.

EXAMPLE 2. Let $G = \prod'_{K_v} G_v$ be the restricted product of locally compact totally disconnected groups G_v, restricted with respect to the compact open subgroups K_v. Then G itself is locally compact and totally disconnected, and $H(G)$ is isomorphic to $\bigotimes_{e_{K_v}} H(G_v)$.

For each $v \in V$ let W_v be an admissible G_v-module. Assume that $\dim W_v^{K_v} = 1$ for almost all v. Choosing for almost all v a nonzero vector $x_v \in W_v^{K_v}$, we may form the G-module $W = \bigotimes_{x_v} W_v$. The isomorphism class of W is in fact independent of the choice of $x_v \in W_v^{K_v}$ and will be called the tensor product of the representations W_v. One sees that W is admissible, and that it is irreducible if and only if each W_v is. The admissible irreducible representations of G isomorphic to ones constructed in this way are said to be *factorizable.*

THEOREM 2. *Suppose that $H(G_v, K_v)$ is commutative for almost all v. Then every admissible irreducible representation W of G is factorizable, $W \simeq \bigotimes W_v$. The isomorphism classes of the factors W_v are determined by that of W. For almost all v, $\dim W_v^{K_v} = 1$.*

PROOF. One first factorizes the finite dimensional spaces $W^{K'}$ for compact open subgroups $K' = \prod K'_v$ of G, then continues as in the proof of Theorem 1. □

3. Let G be a connected reductive algebraic group over a global field F. Let A be the adele ring of F, and let V be the set of places of F. The adelic group $G(A)$ is isomorphic to a restricted product $\prod'_{K_v} G(F_v)$, where the subgroups K_v are defined for all finite v and are certain maximal compact subgroups of $G(F_v)$. For almost all finite $v \in V$, $G(F_v)$ is a quasi-split group over F_v, and K_v is a special maximal compact subgroup. For these places v, $H(G(F_v), K_v)$ is commutative. See [**5**]. So the function field case of the following theorem, whose meaning has yet to be explained in the number field case, is a special case of Theorem 2.

THEOREM 3. *Every admissible irreducible representation of $G(A)$ is factorizable. The factors are unique up to equivalence.*

Let F be a number field. Then the class of admissible representations of $G(A)$ has yet to be defined. For each archimedean place $v \in V$, let K_v be a maximal compact subgroup of $G(F_v)$, and let $\mathfrak{g}_v$ be the real Lie algebra of $G(F_v)$. Let $K_\infty = \prod_{\text{arch } v} K_v$, $K = \prod_{\text{all } v} K_v$, and $G_\infty = \prod_{\text{arch } v} G(F_v)$. Let $\mathfrak{g}_\infty$ be the real Lie algebra of G_∞. Let A_f be the ring of finite adeles of F.

DEFINITION. An *admissible* $G(A)$-module W is a vector space W which is both a $(\mathfrak{g}_\infty, K_\infty)$-module and a smooth $G(A_f)$-module such that

(1) the action of $G(A_f)$ commutes with the action of $\mathfrak{g}_\infty$ and K_∞, and

(2) for each isomorphism class γ of continuous irreducible representations of K, the γ-isotypic component of W has finite dimension.

In Theorem 3, the factors at the archimedean places v are to be admissible $(\mathfrak{g}_v, K_v)$-modules. The proof when F is a number field is the same as that when it is a function field once an idempotented algebra is found for each archimedean place v whose admissible modules are the same as admissible $(\mathfrak{g}_v, K_v)$-modules.

Let G be a Lie group, and let K be a compact subgroup. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the real Lie algebras of G and K. Let $U(\mathfrak{g}_C)$ and $U(\mathfrak{k}_C)$ be the universal enveloping algebras of the complexified Lie algebras. Define the *Hecke algebra* $H(\mathfrak{g}, K)$ of $(\mathfrak{g}, K)$ to be the algebra of left and right K-finite distributions on G with support in K. It contains the algebra A_K of K-finite measures on K viewed as distributions on G. The map $(X, \mu) \to X * \mu$ from $U(\mathfrak{g}_C) \times A_K$ to the space of distributions on G induces a vector space isomorphism of $U(\mathfrak{g}_C) \otimes_{U(\mathfrak{k}_C)} A_K$ with $H(\mathfrak{g}, K)$. With the set E of central idempotents of A_K, $H(\mathfrak{g}, K)$ is an idempotented algebra.

Let (π, W) be a $(\mathfrak{g}, K)$-module. By means of the formula $X \otimes \mu \cdot w = \pi(X)\pi(\mu)w$ for $X \in U(\mathfrak{g}_C)$, $\mu \in A_K$, and $w \in W$, the space W becomes a nondegenerate $H(\mathfrak{g}, K)$-module. Moreover, it is not difficult to verify that this construction establishes an isomorphism between the categories of admissible $(\mathfrak{g}, K)$-modules and of admissible $(H(\mathfrak{g}, K), E)$-modules.

4. In practice, a more analytic theory than that described above is needed as well. Let $\{H_v \mid v \in V\}$ be a family of Hilbert spaces. For almost all $v \in V$, let x_v be a unit vector in H_v. The *Hilbert restricted product* $H = \widehat{\bigotimes}_{x_v} H_v$ is a Hilbert space which can be conveniently described by giving an orthonormal basis. Let P_v be an orthonormal basis for H_v for each $v \in V$ which extends $\{x_v\}$ for almost all v. The set of symbols $\widehat{\otimes} h_v$ such that $h_v \in P_v$ for all v and $h_v = x_v$ for almost all v is an orthonormal basis for H. Constructions analogous to those described above with reference to the ordinary restricted tensor product are available in the Hilbert space context.

For the following theorem see [**3**] and [**1**].

THEOREM 4. *Let $G(A)$ be as in Theorem* 3. *Let π be a continuous irreducible unitary Hilbert space representation of $G(A)$. Then*

(1) *There exist continuous irreducible unitary Hilbert space representations π_v of $G(F_v)$, almost all of which are unramified, such that $\pi \simeq \widehat{\otimes}\pi_v$. The factors π_v are unique up to isomorphism.*

(2) *For each isomorphism class γ of continuous irreducible representations of K, the γ-isotypic component of π has finite dimension.*

(3) *The space of K-finite vectors of π is in a natural way an admissible irreducible $G(A)$-module π^K. Let $\pi^K \simeq \otimes\pi_v^K$ be the factorization of π^K given by Theorem* 3. *Then π_v^K is isomorphic as an admissible $G(F_v)$-module to the space of K_v-finite vectors of π_v.*

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Duke University

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CLASSICAL AND ADELIC AUTOMORPHIC FORMS. AN INTRODUCTION

I. PIATETSKI-SHAPIRO*

1. Classical Hecke theory. Let H be the upper half-plane $\{z \in C | \operatorname{Im} z > 0\}$, Γ a congruence subgroup of $\mathbf{SL}(2, Z)$, i.e., $\Gamma \supset \Gamma_N$ for some integer $N \geqq 0$ (where

$$\Gamma_N = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}(2, Z) \text{ such that } \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\operatorname{mod} N)\right\}\Big).$$

The group $\mathbf{SL}(2, \boldsymbol{R})$ acts on H by $z \to (az + b)/(cz + d)$. We say that a function f: $H \to C$ is a *modular form of weight* k (with respect to Γ) iff

(a) f is holomorphic on H.

(b) $f((az + b)/(cz + d))(cz + d)^{-k} = f(z)$, for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, and k some strictly positive integer k.

(c) f is holomorphic also at the cusps of H with respect to Γ.

For example, at ∞, this means that f has the Fourier expansion

$$f(z) = \sum_{n \geqq 0} a_n \, e^{2\pi i n \lambda z}.$$

We say that f is a *cusp form* if in the Fourier expansion at each cusp, $a_0 = 0$.

Let $\Omega(f)$ be the C-linear span of the set

$$\left\{ f\left(\frac{az + b}{cz + d}\right)(cz + d)^{-k} \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{GL}^+(2, \boldsymbol{Q})\right\}.$$

Here $G = \mathbf{GL}^+(2, \boldsymbol{Q}) = \{g \in \mathbf{GL}(2, \boldsymbol{Q}) \mid \det(g) > 0\}$. Note that $\mathbf{GL}^+(2, \boldsymbol{R})$ acts on H. There is an obvious representation of G on $\Omega(f)$.

Hecke defined the L-function attached to the modular form f by the formula:

$$L(f, s) = \frac{(2\pi\lambda)^s}{\Gamma(s)} \int_0^\infty f(iy) y^{(s-1)/2} \, dy = \sum_{n=1}^{\infty} a_n n^{-s}$$

(the Dirichlet series corresponding to f). Hecke **[1]** proved the following theorem:

THEOREM. (1) *$L(f, s)$ is a "nice" entire function when f is a cusp form ("nice" means that $L(f, s)$ has a functional equation).*

(2) *$L(f, s)$ has an Euler product if $\Omega(f)$ is algebraically irreducible, i.e., has no invariant linear subspaces under the action of G.*

The second statement is the more interesting; it is equivalent to Hecke's actual

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statement. He proved that $L(f, s)$ has an Euler product if f is an eigenfunction of the "Hecke operators". This is equivalent to $\Omega(f)$ being irreducible. Already (2) suggests that it might be better to study modular forms from the point of view of representation theory.

This leads us to the main purpose of this paper, which is to motivate the transition to the adelic setting and the systematic use of representation theory in the study of modular forms.

First let us explain the notion of Euler product. $L(s) = L(f, s)$ has an Euler product means that

$$L(s) = \prod_{p \text{ a prime}} L_p(s),$$

where

$$L_p(s) = (1 - \alpha_p p^{-s})^{-1} (1 - \beta_p p^{-s})^{-1}, \qquad \alpha_p, \beta_p \in \boldsymbol{C}.$$

In Hecke's theorem we have a functional equation of the sort: $L(s) = \varepsilon(s)\tilde{L}(1 - s)$, where $L(s)$ (resp. $\varepsilon(s)$) can be written as an infinite product $\prod L_p(s)$ (resp. $\prod \varepsilon_p(s)$). Now Tate's theory of L-functions associated to Grossencharakters and Artin L-functions suggests the problem of finding objects such that the local factors $L_p(s)$ and $\varepsilon_p(s)$ are ε- and L-functions for these objects. It was the beautiful idea of Jacquet-Langlands [2] to take as such objects irreducible representations of $\mathbf{GL}(2,\boldsymbol{Q}_p)$. More precisely, let us look at $\Omega(f)$. Any element $h \in \Omega(f)$ is fixed by a congruence subgroup $\Gamma \subseteq G$. Now we define a topology on G by taking a basis of neighborhoods of the identity to be the set of congruence subgroups of G, and let $\bar{G}$ be the completion of G in this topology. Then $\bar{G} = \{g \in \prod_{\text{finite primes}} \mathbf{GL}(2, \boldsymbol{Q}_p) | \det g_p = r > 0,\ r \in \boldsymbol{Q}\}$, where $\prod$ means restricted direct product, acts on $\Omega(f)$, via the action of G, and it can be shown that this representation π is an infinite tensor product (interpreted in a suitable sense) $\pi = \bigotimes_{\text{finite primes}} \pi_p$, where π_p is a representation of $\mathbf{GL}(2, \boldsymbol{Q}_p)$.

In the author's interpretation, it is the idea of Jacquet-Langlands to attach an L-function to the irreducible representation $\pi = \Omega(f)$, rather than to f, and then to interpret $L_p(s)$ and $\varepsilon_p(s)$ as the L- and ε-factors for the representations π_p of $\mathbf{GL}(2, \boldsymbol{Q}_p)$.

Finally, let us give the following additional motivation for introducing the adelic framework into the study of modular forms. We wish to study modular forms with respect to different congruence subgroups simultaneously, and every such form is stabilized by some congruence subgroup. To each inclusion relation $\Gamma \subset \Gamma_1$ corresponds a projection $H/\Gamma \to H/\Gamma_1$. A small computation shows that the projective limit of this system,

$$\operatorname*{proj\,lim}_{\Gamma \text{ a congruence subgroup}} H/\Gamma = K_\infty \backslash \mathbf{SL}(2, A)/\mathbf{SL}(2, \boldsymbol{Q}).$$

Hence the study of modular forms has been transformed into the study of a certain space of functions on this double coset space.

2. Automorphic forms for adelic groups. Assume G is a connected reductive algebraic group over a global field k. For such a G we can define the group G_A (cf. Springer's paper for the construction), and $G_k \subset G_A$ as a discrete subgroup.

We need some basic facts from reduction theory. First let $G_A^0 = \bigcap\{\ker|\chi|, \chi$ a rational character of $G\}$. Then $G_k\backslash G_A^0$ is compact iff there are no additive unipotent elements. (This is due to Borel-Harish-Chandra-Mostow-Tamagawa-Harder.)

Let $G_A = NAK$ be the global Iwasawa decomposition (which follows directly from the local Iwasawa decompositions), where K is a standard maximal compact subgroup of G_A, A the set of adelic points of a maximal split torus, and N is a maximal unipotent subgroup.

EXAMPLE. $G = \mathbf{GL}(2, k)$. Then

$$\begin{aligned} K_p &= O(2) && \text{if } p \text{ is real,} \\ &= U(2) && \text{if } p \text{ is complex,} \\ &= \mathbf{GL}(2, \mathbf{Q}_p) && \text{for a finite place;} \end{aligned}$$

$$N = \left\{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\right\}, \quad A = \left\{\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}\right\}.$$

Note that $A \supset C = Z(G_A)$, and that we can write $A = CA^0$ for some subgroup A^0 s.t. $C \cap A^0$ is finite.

We say that a set $S \subset A^0$ is semibounded iff for any relatively compact subset $N_0 \subset N$, the set $\bigcup_{s \in S} sN_0s^{-1}$ is relatively compact.

EXAMPLE. $G = \mathrm{GL}(2)$. Then $A^0 = \{(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix})\}$, $S = \{(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}) \in A^0 \mid |a_p|_p \geqq c_p$ for all places p, and $c_p = 1$ for almost all $p\}$. Then S is semibounded.

We now give the important definition of a Siegel set $\mathfrak{S}$.[1]

$\mathfrak{S} = N_0SK$, when N_0 is an open compact subset of N, S is semibounded, and K as above. Then the main result of reduction theory says:

THEOREM. *There exists a Siegel set $\mathfrak{S}$ s.t. $G_A = G_k\mathfrak{S}C$. Hence there exists a fundamental domain for G_k contained in $\mathfrak{S}C$.*

We now come to the definition of a cusp form (the definition of an automorphic form is given elsewhere in these PROCEEDINGS): A function $f\colon G_k\backslash G_A \to \mathbf{C}$ is said to be a cusp form iff

(1) f is an eigenfunction with respect to C: $f(cg) = \omega(c)f(g)$.

(2) f is smooth (i.e., f is C^∞ at archimedean places and locally constant at the finite places).

(3) $\int_{CG_k\backslash G_A} |f(g)|^2\, dg < \infty$ (here we suppose that ω is unitary).

(4) $\int_{Z_k\backslash Z_A} f(zg)\, dz = 0$, where Z is the unipotent radical of any parabolic subgroup P.

EXAMPLE. For $\mathbf{GL}_2$ it is sufficient to consider the standard unipotent group $(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix})$.

For $\mathbf{GL}$ one has the following conjugacy classes of unipotent radicals:

$$\begin{pmatrix} 1 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}.$$

In general, for $\mathbf{GL}_n$, one associates to each partition $n_1 + \dots + n_r, = n$, a conjugacy class of unipotent radicals

[1]For simplicity we give the definition only for split groups. There is a similar definition in general.

$$\begin{pmatrix} I_{n_1} & & * \\ & I_{n_2} & \\ 0 & & \\ & & I_{n_r} \end{pmatrix}.$$

We shall write $L_0^2(\omega)$ for the space of cusp forms with respect to ω.

The first main result is the following.

THEOREM (GELFAND, PIATETSKI-SHAPIRO, HARDER). $L_0^2(\omega)$ *is a countable sum of irreducible admissible representations of* G_A, *each occurring with finite multiplicity.*

We now describe a naive form of Langlands' philosophy. Let π be a representation of G_A occurring in $L_0^2(\omega)$. Then one can attach to π a function $L(\pi, s)$ which is a product $\prod L(\pi_p, s)$, and furthermore $L(\pi, s)$ is "nice" (i.e., it is meromorphic, with a finite set of poles, and satisfies a functional equation), and the ε-factor $\varepsilon(\pi, s) = \prod \varepsilon(\pi_p, s)$.

A precise exposition of Langlands' philosophy will be given by Borel in his article [5].

3. The case of $\mathbf{GL}_2$. Now we shall explain what follows from Langlands' conjecture for $\mathbf{GL}_2$.

Let π be an irreducible cuspidal automorphic representation of $\mathbf{GL}_2(A)$. Then $\pi = \bigotimes_{\text{all primes}} \pi_p$, and for almost all primes $\pi_p \cong \mathrm{Ind}_B(\mu_1 \otimes \mu_2)$, where μ_1, μ_2 are unramified characters of $k_p^\times$. Such μ_1, μ_2 are described by $\mu_1(\tilde{p}), \mu_2(\tilde{p})$, where $\tilde{p}$ is the generator of the prime ideal p. Let n be an integer $n \geqq 1$. We can define local factors of the form

$$L^n(\pi_p, s) = (1 - \mu_1^n|\tilde{p}|^s)^{-1}(1 - \mu_1^{n-1}\mu_2|\tilde{p}|^s)^{-1} \ldots (1 - \mu_2^n|\tilde{p}|^s)^{-1}.$$

Here we are writing μ_1, μ_2 for $\mu_1(\tilde{p}), \mu_2(\tilde{p})$.

Conjecture. For π, n as above we can attach to *every* prime p a local factor $L^n(\pi_p, s)$, agreeing with the definition above at the unramified primes, such that the function $L^n(\pi, s) = \prod L^n(\pi_p, s)$ exists, and is "nice".

For $n = 1$ this conjecture is proved in the book by Jacquet-Langlands. For $n = 2$, it has been proved by Gelbart-Jacquet and Shimura. For $n = 3, 4$ it can be proved that $L^n(\pi, s)$ exists and has a meromorphic continuation. Beyond these cases, the situation is unresolved.

Notes of this talk were made by Lawrence Morris (I. H. E. S.) and Ben Seifert (U. C. Berkeley).

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UNIVERSITY OF TEL AVIV

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AUTOMORPHIC FORMS AND AUTOMORPHIC REPRESENTATIONS

A. BOREL AND H. JACQUET

Originally, the theory of automorphic forms was concerned only with holomorphic automorphic forms on the upper half-plane or certain bounded symmetric domains. In the fifties, it was noticed (first by Gelfand and Fomin) that these automorphic forms could be viewed as smooth vectors in certain representations of the ambient group G, on spaces of functions on G invariant under the given discrete group Γ. This led to the more general notion of automorphic forms on real semisimple groups, with respect to arithmetic subgroups, on adelic groups, and finally to the direct consideration of the underlying representations. The main purpose of this paper is to discuss the notions of automorphic forms on real or adelic reductive groups, of automorphic representations of adelic groups, and the relations between the two. We leave out completely the passage from automorphic forms on bounded symmetric domains to automorphic forms on groups, which has been discussed in several places (see, e.g., **[2]**, or also **[5]**, **[6]**, **[15]** for modular forms).

1. Automorphic forms on a real reductive group.

1.1. Let G be a connected reductive group over $\boldsymbol{Q}$, Z the greatest $\boldsymbol{Q}$-split torus of the center of G and K a maximal compact subgroup of $G(\boldsymbol{R})$. Let $\mathfrak{g}$ be the Lie algebra of $G(\boldsymbol{R})$, $U(\mathfrak{g})$ its universal enveloping algebra over $\boldsymbol{C}$ and $Z(\mathfrak{g})$ the center of $U(\mathfrak{g})$. We let $\mathscr{H}$ or $\mathscr{H}(G(\boldsymbol{R}), K)$ be the convolution algebra of distributions on $G(\boldsymbol{R})$ with support in K **[4]** and A_K the algebra of finite measures on K. We recall that $\mathscr{H}$ is isomorphic to $U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} A_K$. An idempotent in $\mathscr{H}$ is, by definition, one of A_K, i.e., a finite sum of measures of the form $d(\sigma)^{-1} \cdot \chi_\sigma \cdot dk$, where σ is an irreducible finite dimensional representation of K, $d(\sigma)$ its degree, χ_σ its character, and dk the normalized Haar measure on K. The algebra $\mathscr{H}$ is called the Hecke algebra of $G(\boldsymbol{R})$ and K.

1.2. A norm $\| \ \|$ on $G(\boldsymbol{R})$ is a function of the form $\|g\| = (\operatorname{tr} \sigma(g)^* \cdot \sigma(g))^{1/2}$, where $\sigma \colon G(\boldsymbol{R}) \to \mathrm{GL}(E)$ is a finite dimensional complex representation with finite kernel and image closed in the space $\mathrm{End}(E)$ of endomorphisms of E and $*$ denotes the adjoint with respect to a Hilbert space structure on E invariant under K. It is easily seen that if τ is another such representation, then there exist a constant $C > 0$ and a positive integer n such that

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(1) $$\|x\|_\sigma \leqq C\cdot\|x\|_\tau^n, \quad \text{for all } x\in G(\boldsymbol{R}).$$

A function f on $G(\boldsymbol{R})$ is said to be *slowly increasing* if there exist a norm $\|\ \|$ on $G(\boldsymbol{R})$, a constant C and a positive integer n such that

(2) $$|f(x)| \leqq C\cdot\|x\|^n, \quad \text{for all } x\in G(\boldsymbol{R}).$$

In view of (1) this condition does not depend on the norm (but n does).

REMARK. If $\sigma: G\to \mathrm{GL}(E)$ has finite kernel, but does not have a closed image in End E, then we can either add one coordinate and det $\sigma(g)^{-1}$ as a new entry, or consider the sum of σ and $\sigma^*: g\mapsto \sigma(g^{-1})^*$. The associated square norms are then $|\det\sigma(g)^{-1}|^2 + \mathrm{tr}(\sigma(g)^*\sigma(g))$ and $\mathrm{tr}(\sigma(g)^*\sigma(g)) + \mathrm{tr}(\sigma(g^{-1})\sigma(g^{-1})^*)$.

1.3. Let Γ be an arithmetic subgroup of $G(\boldsymbol{Q})$. A smooth complex valued function f on $G(\boldsymbol{R})$ is an *automorphic form* for (Γ, K), or simply for Γ, if it satisfies the following conditions:

(a) $f(\gamma\cdot x) = f(x)$ $(x\in G(\boldsymbol{R}); \gamma\in\Gamma)$.

(b) There exists an idempotent $\xi\in\mathscr{H}$(cf. 1.1) such that $f*\xi = f$.

(c) There exists an ideal J of finite codimension of $Z(\mathfrak{g})$ which annihilates f: $f*X = 0$ $(X\in J)$.

(d) f is slowly increasing (1.2).

An automorphic form satisfying those conditions is said to be of type (ξ, J). We let $\mathscr{A}(\Gamma, \xi, J, K)$ be the space of all automorphic forms for (Γ, K) of type (ξ, J).

1.4. EXAMPLE. Let $G = \mathbf{GL}_2$, $\Gamma = \mathbf{GL}_2(\boldsymbol{Z})$, $K = \boldsymbol{O}(2, \boldsymbol{R})$, $\xi = 1$, J the ideal of $Z(\mathfrak{g})$ generated by $(C-\lambda)$, where C is the Casimir operator and $\lambda\in\boldsymbol{C}$. Then f is an eigenfunction of the Casimir operator which is left invariant under Γ, right invariant under K, and invariant under the center Z of $G(\boldsymbol{R})$. The quotient $G(\boldsymbol{R})/Z\cdot K$ may be identified with the Poincaré upper half-plane H. The function f may then be viewed as a Γ-invariant eigenfunction of the Laplace-Beltrami operator on X. It is a "wave-form," in the sense of Maass. See **[2]**, **[5]**, **[6]** for a similar interpretation of modular forms.

1.5. REMARKS. (1) Condition 1.3(b) is equivalent to: f is K-finite on the right, i.e., the right translates of f by elements of K span a finite dimensional space of functions. These functions clearly satisfy 1.3(a), (c), (d) if f does.

(2) Let $r: K\to \mathrm{GL}(V)$ be a finite dimensional unitary representation of K. One can similarly define the notion of a V-valued automorphic form: a smooth function φ: $G\to V$ satisfying (a), (c), (d), where $|\ |$ now refers to the norm in V, and

(b′) $$f(x\cdot k) = r(k^{-1})\cdot f(x) \qquad (x\in G(\boldsymbol{R}); k\in K).$$

For semisimple groups, this is Harish-Chandra's definition (cf. **[2]**, **[11]**). For $v\in V$, the functions $x\mapsto(\varphi(x), v)$ are then scalar automorphic forms. Conversely, a finite dimensional space E of scalar automorphic forms stable under K yields an E-valued automorphic form.

(3) A similar definition can be given if $G(\boldsymbol{R})$ is replaced by a finite covering H of an open subgroup of $G(\boldsymbol{R})$ and Γ by a discrete subgroup of H whose image in $G(\boldsymbol{R})$ is arithmetic. For instance, modular forms of rational weight can be viewed as automorphic forms on finite coverings of $\mathbf{SL}_2(\boldsymbol{R})$.

1.6. *Growth condition.* (1) Let A be the identity component of a maximal $\boldsymbol{Q}$-split torus S of G, and ${}_{\boldsymbol{Q}}\Phi$ the system of $\boldsymbol{Q}$-roots of G with respect to S. Fix an ordering on ${}_{\boldsymbol{Q}}\Phi$ and let Δ be the set of simple roots. Given $t > 0$ let

$$A_t = \{a \in A : |\alpha(a)| \geq t, (a \in \Delta)\}. \tag{1}$$

Let f be a function satisfying 1.3(a), (b), (c). Then the growth condition (d) is equivalent to:

(d′) Given a compact set $R \subset G(\boldsymbol{R})$, and $t > 0$, there exist a constant $C > 0$ and a positive integer m such that

$$|f(x \cdot a)| \leq C \cdot |\alpha(a)|^m, \quad \text{for all } a \in A_t, \alpha \in \Delta, x \in R. \tag{2}$$

This follows from reduction theory [**11**, §2]. More precisely, let G' be the derived group of G. Then A is the direct product of $Z(\boldsymbol{R})^\circ$ and $A' = A \cap G'(\boldsymbol{R})$. For a function satisfying 1.3(a), (b), the growth condition (d′) is equivalent to (d) for $a \in A'_t$; but says nothing for $a \in Z(\boldsymbol{R})^\circ$. However condition (c) implies that f depends polynomially on $z \in Z(\boldsymbol{R})$, and this takes care of the growth condition on $Z(\boldsymbol{R})$.

(2) Assume f satisfies 1.3(a), (b), (c) and

$$f(z \cdot x) = \chi(z) f(x) \qquad (z \in Z(\boldsymbol{R}),\ x \in G(\boldsymbol{R})) \tag{3}$$

where χ is a character of $Z(\boldsymbol{R})/(Z(\boldsymbol{R}) \cap \Gamma)$. Then $|f|$ is a function on $Z(\boldsymbol{R}) \cdot \Gamma\backslash G(\boldsymbol{R})$. If $|f| \in L^p(Z(\boldsymbol{R})\Gamma\backslash G(\boldsymbol{R}))$ for some $p \geqq 1$, then f is slowly increasing, hence is an automorphic form. In view of the fact that $Z(\boldsymbol{R})\Gamma\backslash G(\boldsymbol{R})$ has finite invariant volume, it suffices to prove this for $p = 1$. In that case, it follows from the corollary to Lemma 9 in [**11**], and from the existence of a K-invariant function $\alpha \in C_c^\infty(G(\boldsymbol{R}))$ such that $f = f * \alpha$ (a well-known property of K-finite and $Z(\mathfrak{g})$-finite elements in a differentiable representation of $G(\boldsymbol{R})$, which follows from 2.1 below).

1.7. THEOREM [**11**, THEOREM 1]. *The space $\mathscr{A}(\Gamma, \xi, J, K)$ is finite dimensional.*

This theorem is due to Harish-Chandra. Actually the proof given in [**11**] is for semisimple groups, but the extension to reductive groups si easy. In fact, it is implicitly done in the induction argument of [**11**] to prove the theorem. For another proof, see [**13**, Lemma 3.5]. At any rate, it is customary to fix a quasi-character χ of $Z(\boldsymbol{R})/(\Gamma \cap Z(\boldsymbol{R}))$ and consider the space $\mathscr{A}(\Gamma, \xi, J, K)_\chi$ of elements in $\mathscr{A}(\Gamma, \xi, J, K)$ which satisfy 1.6(3). For those, the reduction to the semisimple case is immediate. Note that since the identity component $Z(\boldsymbol{R})^\circ$ of $Z(\boldsymbol{R})$ (sometimes called the split component of $G(\boldsymbol{R})$) has finite index in $Z(\boldsymbol{R})$ and $Z(\boldsymbol{R})^\circ \cap \Gamma = \{1\}$, it is substantially equivalent to require 1.6(3) for an arbitrary quasi-character of $Z(\boldsymbol{R})^\circ$.

The space $\mathscr{A}(\Gamma, \xi, J, K)$ is acted upon by the center $C(G(\boldsymbol{R}))$ of $G(\boldsymbol{R})$, by left or right translations. Since it is finite dimensional, we see that *any automorphic form is $C(G(\boldsymbol{R}))$-finite.*

1.8. *Cusp forms.* A continuous (resp. measurable) function on $G(\boldsymbol{R})$ is cuspidal if

$$\int_{(\Gamma \cap N(\boldsymbol{R}))\backslash N(\boldsymbol{R})} f(n \cdot x)\, dn = 0, \tag{1}$$

for all (resp. almost all) x in $G(\boldsymbol{R})$, where N is the unipotent radical of any proper parabolic $\boldsymbol{Q}$-subgroup of G. It suffices in fact to require this for any proper maximal parabolic $\boldsymbol{Q}$-subgroup [**11**, Lemma 3].

A *cusp form* is a cuspidal automorphic form. We let ${}^\circ\mathscr{A}(\Gamma, \xi, J, K)$ be the space of cusp forms in $\mathscr{A}(\Gamma, \xi, J, K)$.

Let f be a smooth function on $G(\boldsymbol{R})$ satisfying the conditions (a), (b), (c) of 1.3. Assume that f is cuspidal and that there exists a character χ of $Z(\boldsymbol{R})$ such that 1.6(3) is satisfied. Then the following conditions are equivalent:

(i) f is slowly increasing, i.e., f is a cusp form;

(ii) f is bounded;

(iii) $|f|$ is square-integrable modulo $Z(\boldsymbol{R})\cdot\Gamma$

(cf. [**11**, §4]). In fact, one has much more: $|f|$ decreases very fast to zero at infinity on $Z(\boldsymbol{R})\Gamma\backslash G(\boldsymbol{R})$, so that if g is any automorphic form satisfying 1.6(3), then $|f\cdot g|$ is integrable on $Z(\boldsymbol{R})\cdot\Gamma\backslash G(\boldsymbol{R})$ (loc. cit.).

The space ${}^\circ\mathscr{A}(\Gamma, \xi, J, K)_\chi$ of the functions in ${}^\circ\mathscr{A}(\Gamma, \xi, J, K)$ satisfying 1.6(3) may then be viewed as a closed subspace of bounded functions in the space $L^2(\Gamma\backslash G(\boldsymbol{R}))_\chi$ of functions on $\Gamma\backslash G(\boldsymbol{R})$ satisfying 1.6(3), whose absolute value is square-integrable on $Z(\boldsymbol{R})\Gamma\backslash G(\boldsymbol{R})$. Since $Z(\boldsymbol{R})\Gamma\backslash G(\boldsymbol{R})$ has finite measure, this space is finite dimensional by a well-known lemma of Godement [**11**, Lemma 17]. This proves 1.7 for $\mathscr{A}(\Gamma, \xi, J, K)_\chi$ when $Z(\boldsymbol{R})\Gamma\backslash G(\boldsymbol{R})$ is compact, and is the first step of the proof of 1.7 in general.

1.9. Let $a \in G(\boldsymbol{Q})$. Then ${}^a\Gamma = a\cdot\Gamma\cdot a^{-1}$ is an arithmetic subgroup of $G(\boldsymbol{Q})$, and the left translation l_a by a induces an isomorphism of $\mathscr{A}(\Gamma, \xi, J, K)$ onto $\mathscr{A}({}^a\Gamma, \xi, J, K)$. Let Σ be a family of arithmetic subgroups of $G(\boldsymbol{Q})$, closed under finite intersection, whose intersection is reduced to $\{1\}$. The union $\mathscr{A}(\Sigma, \xi, J, K)$ of the spaces $\mathscr{A}(\Gamma, \xi, J, K)$ ($\Gamma \in \Sigma$) may be identified to the inductive limit of those spaces:

$$(1) \qquad \mathscr{A}(\Sigma, \xi, J, K) = \operatorname{ind\,lim}_{\Gamma\in\Sigma} \mathscr{A}(\Gamma, \xi, J, K),$$

where the inductive limit is taken with respect to the inclusions

$$(2) \qquad j_{\Gamma''\Gamma'}\colon \mathscr{A}(\Gamma', \xi, J, K) \to \mathscr{A}(\Gamma'', \xi, J, K) \qquad (\Gamma'' \subset \Gamma')$$

associated to the projections $\Gamma''\backslash G(\boldsymbol{R}) \to \Gamma'\backslash G(\boldsymbol{R})$.

Assume Σ to be stable under conjugation by $G(\boldsymbol{Q})$. Then $G(\boldsymbol{Q})$ operates on $\mathscr{A}(\Sigma, \xi, J, K)$ by left translations. Let us topologize $G(\boldsymbol{Q})$ by taking the elements of Σ as a basis of open neighborhoods of 1. Then this representation is admissible (every element is fixed under an open subgroup, and the fixed point set of every open subgroup is finite dimensional). By continuity, it extends to a continuous admissible representation of the completion $G(\boldsymbol{Q})_\Sigma$ of $G(\boldsymbol{Q})$ for the topology just defined. For suitable Σ, the passage to $\mathscr{A}(\Sigma, \xi, J, K)$ amounts essentially to considering all adelic automorphic forms whose type at infinity is prescribed by ξ, J, K; the group $G(\boldsymbol{Q})_\Sigma$ may be identified to the closure of $G(\boldsymbol{Q})$ in $G(\boldsymbol{A}_f)$ and its action comes from one of $G(\boldsymbol{A}_f)$. See 4.7.

1.10. Finally, we may let ξ and J vary and consider the space $\mathscr{A}(\Sigma, J, K)$ spanned by the $\mathscr{A}(\Sigma, \xi, J, K)$ and the space $\mathscr{A}(\Sigma, K)$ spanned by the $\mathscr{A}(\Sigma, J, K)$. They are $G(\boldsymbol{Q})_\Sigma$-modules and $(\mathfrak{g}, K)$-modules, and these actions commute. Again, this has a natural adelic interpretation (4.8).

1.11. *Hecke operators.* Let $\mathscr{H}(G(\boldsymbol{Q}), \Gamma)$ be the Hecke algebra, over $\boldsymbol{C}$, of $G(\boldsymbol{Q})$

mod Γ. It is the space of complex valued functions on $G(\boldsymbol{Q})$ which are bi-invariant under Γ and have support in a finite union of double cosets mod Γ. The product may be defined directly in terms of double cosets (see, e.g., **[17]**) or of convolution (see below). This algebra operates on $\mathscr{A}(\Gamma, \xi, J, K)$. The effect of $\Gamma a \Gamma$ $(a \in G(\boldsymbol{Q}))$ is given by $f \mapsto \sum_{b \in (\Gamma \cap {}^a\Gamma)\backslash \Gamma} l_b f$. More generally, let $\mathscr{H}(G(\boldsymbol{Q}), \Sigma)$ be the Hecke algebra spanned by the characteristic functions of the double cosets $\Gamma' a \Gamma''$ $(\Gamma', \Gamma'' \in \Sigma$, $a \in G(\boldsymbol{Q}))$ **[17**, Chapter 3]. It may be identified with the Hecke algebra $\mathscr{H}(G(\boldsymbol{Q})_\Sigma)$ of locally constant compactly supported functions on $G(\boldsymbol{Q})_\Sigma$. This identification carries $\mathscr{H}(G(\boldsymbol{Q}), \Gamma)$ onto $\mathscr{H}(G(\boldsymbol{Q})_\Sigma, \bar{\Gamma})$, where $\bar{\Gamma}$ is the closure of Γ in $G(\boldsymbol{Q})_\Sigma$ **[12]**. The product here is ordinary convolution (which amounts to finite sums in this case). Since $\mathscr{A}(\Sigma, \xi, J, K)$ is an admissible module for $G(\boldsymbol{Q})_\Sigma$, the action of $G(\boldsymbol{Q})_\Sigma$ extends in the standard way to one of $\mathscr{H}(G(\boldsymbol{Q})_\Sigma)$. The space $\mathscr{A}(\Gamma, \xi, J, K)$ is the fixed point set of $\bar{\Gamma}$, and the previous operation of $\mathscr{H}(G(\boldsymbol{Q}), \Gamma)$ on this space may be viewed as that of $\mathscr{H}(G(\boldsymbol{Q})_\Sigma, \bar{\Gamma})$. For an adelic interpretation, see 4.8.

2. Automorphic forms and representations of $G(\boldsymbol{R})$. The notion of automorphic form has a simple interpretation in terms of representations (which in fact suggested its present form). To give it, we need the following known lemma (cf. **[18]** for the terminology).

2.1. LEMMA. *Let (π, V) be a differentiable representation of $G(\boldsymbol{R})$. Let $v \in V$ be K-finite and $Z(\mathfrak{g})$-finite. Then the smallest $(\mathfrak{g}, K)$-submodule of V containing v is admissible.*

Indeed, $\mathscr{H} \cdot v$ is a finite sum of spaces $\mathscr{H}^\circ \cdot w$, where $\mathscr{H}^\circ$ is the Hecke algebra of the identity component $G(\boldsymbol{R})^\circ$ of $G(\boldsymbol{R})$ and $K^\circ = K \cap G(\boldsymbol{R})^\circ$, and w is K°-finite and $Z(\mathfrak{g})$-finite. It suffices therefore to show that $\mathscr{H}^\circ \cdot v$ is an admissible $(\mathfrak{g}, K^\circ)$-module. By assumption, there exist an ideal R of finite codimension of the enveloping algebra $U(\mathfrak{k})$ of the Lie algebra $\mathfrak{k}$ of K and an ideal J of finite codimension of $Z(\mathfrak{g})$ which annihilate v and moreover $U(\mathfrak{k})/R$ is a semisimple $\mathfrak{k}$-module. Then $\mathscr{H}^\circ \cdot v$ may be identified with $U(\mathfrak{g})/U(\mathfrak{g}) \cdot R \cdot J$. By a theorem of Harish-Chandra (see **[19**, 2.2.1.1]), $U(\mathfrak{g})/U(\mathfrak{g}) \cdot R$ is $\mathfrak{k}$-semisimple and its $\mathfrak{k}$-isotypic submodules are finitely generated $Z(\mathfrak{g})$-modules. Hence their quotients by J are finite dimensional.

2.2. We apply this to $C^\infty(\Gamma \backslash G(\boldsymbol{R}))$, acted upon by $G(\boldsymbol{R})$ via right translations. Therefore, if f is automorphic form, then $f * \mathscr{H}$ is an admissible $\mathscr{H}$- or $(\mathfrak{g}, K)$-module. This module consists of automorphic forms. In fact, 1.3(a) is clear, and 1.3(b) follows from 2.1; its elements are annihilated by the same ideal of $Z(\mathfrak{g})$ as v, whence (d). Finally, there exists $\alpha \in C_c^\infty(G)$ such that $f * \alpha = f$ so that $f * X$ satisfies 1.2(c) (with the same exponent as f) for all $X \in U(\mathfrak{g})$ **[11**, Lemma 14]. Thus the spaces

$$\mathscr{A}(\Gamma, J, K) = \Sigma_\xi \mathscr{A}(\Gamma, \xi, J, K), \qquad \mathscr{A}(\Gamma, K) = \Sigma_J \mathscr{A}(\Gamma, J, K),$$

are $(\mathfrak{g}, K)$-modules and unions of admissible $(\mathfrak{g}, K)$-modules.

If f is a cusp form, then $f * \mathscr{H}$ consists of cusp forms. Thus the subspace ${}^\circ\mathscr{A}(\Gamma, K)$ of cusp forms is also an $\mathscr{H}$-module. If χ is a quasi-character of Z, then the space ${}^\circ\mathscr{A}(\Gamma, K)_\chi$ of eigenfunctions for Z with character χ is a direct sum of irreducible admissible $(\mathfrak{g}, K)$-modules, with finite multiplicities. In fact, after a twist by $|\chi|^{-1}$, we may assume χ to be unitary, and we are reduced to the Gelfand-Piatetski-Shapiro theorem (**[7]**, see also **[11**, Theorem 2], **[13**, pp. 41–42]) once we identify ${}^\circ\mathscr{A}(\Gamma, K)_\chi$ to the space of K-finite and $Z(\mathfrak{g})$-finite elements in the space ${}^\circ L^2(\Gamma\backslash G(\boldsymbol{R}))_\chi$ of cuspidal functions in $L^2(\Gamma\backslash G(\boldsymbol{R}))_\chi$ (see 1.8 for the latter).

3. Some notation. We fix some notation and conventions for the rest of this paper.

3.1. F is a global field, O_F the ring of integers of F, V or V_F (resp. V_∞, resp. V_f) the set of places (resp. archimedean places, resp. nonarchimedean places) of F, F_v the completion of F at $v \in V$, O_v the ring of integers of F_v if $v \in V_f$. As usual, $\boldsymbol{A}$ or $\boldsymbol{A}_F$ (resp. $\boldsymbol{A}_f$) is the ring of adèles (resp. finite adèles) of F.

3.2. G is a connected reductive group over F, Z the greatest F-split torus of the center of G, $\mathscr{H}_v$ the Hecke algebra of $G_v = G(F_v)$ $(v \in V)$ [4]. Thus $\mathscr{H}_v$ is of the type considered in §1 if $v \in V_\infty$ and is the convolution algebra of locally constant compactly supported functions on $G(F_v)$ if $v \in V_f$. We set

$$\mathscr{H}_\infty = \bigotimes_{v\in V_\infty} \mathscr{H}_v, \qquad \mathscr{H}_f = \bigotimes_{v\in V_f} \mathscr{H}_v, \qquad \mathscr{H} = \mathscr{H}_\infty \otimes \mathscr{H}_f, \tag{1}$$

where the second tensor product is the restricted tensor product with respect to a suitable family of idempotents [4]. Thus $\mathscr{H}$ is the global Hecke algebra of $G(\boldsymbol{A})$ [4]. If F is a function field, then V_∞ is empty and $\mathscr{H} = \mathscr{H}_f$.

If L is a compact open subgroup of $G(\boldsymbol{A}_f)$, we denote by ξ_L the associated idempotent, i.e., the characteristic function of L divided by the volume of L (relative to the Haar measure underlying the definition of $\mathscr{H}_f$). Thus $f * \xi_L = f$ if and only if f is right invariant under L.

The right translation by $x \in G(\boldsymbol{A})$ on $G(\boldsymbol{A})$, or on functions on $G(\boldsymbol{A})$, is denoted r_x or $r(x)$.

3.3. A continuous (resp. measurable) function on $G(\boldsymbol{A})$ is cuspidal if

$$\int_{N(F)\backslash N(\boldsymbol{A})} f(nx)\, dn = 0,$$

for all (resp. almost all) $x \in G(\boldsymbol{A})$, where N is the unipotent radical of any proper parabolic F-subgroup P of G. It suffices to chcek this condition when P runs through a set of representatives of the conjugacy classes of proper maximal parabolic F-subgroups.

4. Groups over number fields.

4.1. In this section, F is a number field. An element $\xi \in \mathscr{H}$ is said to be *simple* if it is of the form

$$\xi = \xi_\infty \otimes \xi_f, \qquad \xi_f \in \mathscr{H}_f,\ \xi_\infty \text{ idempotent in } \mathscr{H}_\infty. \tag{1}$$

We let $G_\infty = \prod_{v\in V_\infty} G_v$ and $\mathfrak{g}_\infty$ be the Lie algebra of G_∞, viewed as a real Lie group. We recall that G_∞ may be viewed canonically as the group of real points $H(\boldsymbol{R})$ of a connected reductive group H, namely the group $H = R_{F/\boldsymbol{Q}}\, G$ obtained from G by restriction of scalars from F to $\boldsymbol{Q}$. This identification is understood when we apply the results and definitions of §§1, 2 to G_∞.

The group $G(\boldsymbol{A})$ is the direct product of G_∞ by $G(\boldsymbol{A}_f)$. A complex valued function on $G(\boldsymbol{A})$ is *smooth* if it is continuous and, if viewed as a function of two arguments $x \in G_\infty$, $y \in G(\boldsymbol{A}_f)$, it is C^∞ in x (resp. locally constant in y) for fixed y (resp. x).

4.2. *Automorphic forms.* Fix a maximal compact subgroup K_∞ of G_∞. A smooth function f on $G(\boldsymbol{A})$ is a K_∞-automorphic form on $G(\boldsymbol{A})$ if it satisfies the following conditions:

(a) $f(\gamma \cdot x) = f(x)$ ($\gamma \in G(F)$, $x \in G(A)$).

(b) There is a simple element $\xi \in \mathscr{H}$ such that $f * \xi = f$.

(c) There is an ideal J of finite codimension of $Z(\mathfrak{g}_\infty)$ which annihilates f.

(d) For each $y \in G(A_f)$, the function $x \mapsto f(x \cdot y)$ on G_∞ is slowly increasing.

We shall sometimes say that f is then of type (ξ, J, K_∞). We let $\mathscr{A}(\xi, J, K_\infty)$ be the space of automorphic forms of type (ξ, J, K_∞).

There exists a compact open subgroup L of $G(A_f)$ such that $\xi_f * \xi_L = \xi_f$. Then $\mathscr{A}(\xi, J, K_\infty) \subset \mathscr{A}(\xi_\infty * \xi_L, J, K_\infty)$. We could therefore assume $\xi_f = \xi_L$ for some L without any real loss of generality.

4.3. We want now to relate these automorphic forms to automorphic forms on G_∞. For this we may (and do) assume $\xi = \xi_\infty \otimes \xi_L$ for some compact open subgroup L of $G(A_f)$. There exists a finite set $C \subset G(A)$ such that $G(A) = G(F) \cdot C \cdot G_\infty \cdot L$ [**1**]. We assume that C is a set of representatives of such cosets and is contained in $G(A_f)$. Then the sets $G(F) \cdot c \cdot G_\infty \cdot L$ form a partition of $G(A)$ into open sets. For $c \in C$, let

$$\Gamma_c = G(F) \cap (G_\infty \times c \cdot L \cdot c^{-1}). \tag{1}$$

It is an arithmetic subgroup of G_∞.

Given a function f on $G(A)$ and $c \in C$, let f_c be the function $x \mapsto f(c \cdot x)$ on G_∞. Suppose f is right invariant under L. Then it is immediately checked that f is left invariant under $G(F)$ if and only if f_c is left invariant under Γ_c for every $c \in C$. More precisely, the map $f \mapsto (f_c)_{c \in C}$ yields a bijection between the spaces of functions on $G(F)\backslash G(A)/L$ and on $\coprod_c(\Gamma_c\backslash G_\infty)$. It then follows from the definitions that it also induces an isomorphism

$$\mathscr{A}(\xi_\infty \otimes \xi_L, J, K_\infty) \xrightarrow{\sim} \bigoplus_{c \in C} \mathscr{A}(\Gamma_c, \xi_\infty, J, K_\infty), \tag{2}$$

so that the results mentioned in §1 transcribe immediately to properties of adelic automorphic forms. In particular, 1.6, 1.7 imply:

(i) The space $\mathscr{A}(\xi, J, K_\infty)$ is finite dimensional.

(ii) A smooth function f on $G(A)$ satisfying 4.2(a), (b), (c), and

$$f(z \cdot x) = \chi(z) \cdot f(x) \qquad (z \in Z(A),\ x \in G(A)), \tag{3}$$

for some character χ of $Z(A)/Z(F)$, such that $|f| \in L^p(Z(A)G(F)\backslash G(A))$ for some $p \geqq 1$, is slowly increasing.

(iii) For a smooth function satisfying 4.2(a), (b), (c), the growth condition 4.2(d) is equivalent to 1.6(d′), where R is now a compact subset of $G(A)$.

(iv) Any automorphic form is $C(G(A))$-finite, where $C(G(A))$ is the center of $G(A)$.

We note also that one can also define directly slowly increasing functions on $G(A)$, as in 1.2, using adelic norms: given an F-morphism $G \to \mathbf{GL}_n$ with finite kernel define, for $x \in G(A)$,

$$\|x\| = \sup_{v \in V} \max_{ij} \left(|\sigma(g)_{ij}|_v, |\sigma(g^{-1})_{ij}|_v\right) \tag{4}$$

(or simply $\max |\sigma(g_{ij})|_v$ if $\sigma(G)$ is closed as a subset of the space of $n \times n$ matrices). For continuous functions satisfying 4.2(a), (b), this is equivalent to 4.2(d).

4.4. A *cusp form* is a cuspidal automorphic form. We let $^\circ\mathscr{A}(\xi, J, K_\infty)$ be the space of cusp forms of type (ξ, J, K_∞).

The group N is unipotent, hence satisfies strong approximation, i.e., we have for *any* compact open subgroup Q of $N(A_f)$

(2) $N(A) = N(F)\cdot N_\infty \cdot Q$; hence $N(F)\backslash N(A) \cong (N(F) \cap (N_\infty \times Q))\backslash N_\infty$

[1]. Let now f be a continuous function on $G(A)$ which is left invariant under $G(F)$ and right invariant under L. From (2) it follows by elementary computations that f is cuspidal if and only if the f_c's (notation of 4.3) are cuspidal on G_∞. Hence, the isomorphism of 4.3(2) induces an isomorphism

$$(3) \qquad {}^\circ\mathscr{A}(\xi_\infty \otimes \xi_L, J, K_\infty) \xrightarrow{\sim} \bigoplus_{c\in C} {}^\circ\mathscr{A}(\Gamma_c, \xi_\infty, J, K_\infty),$$

so that the results of 1.8 extend to adelic cusp forms. In particular, assume that f satisfies 4.2(a), (b), (c), and also 4.3(3) for a character χ of $Z(A)/Z(F)$. Then the following conditions are equivalent:

(i) f is slowly increasing, i.e., f is a cusp form;

(ii) f is bounded;

(iii) $|f|$ is square-integrable modulo $Z(A)\cdot G(F)$.

REMARK. In 4.3, 4.4, we have reduced statements on adelic automorphic forms to the corresponding ones for automorphic forms on $G(\mathbf{R})$, chiefly for the convenience of references. However, it is also possible to prove them directly in the adelic framework, and then deduce the results at infinity as corollaries via 4.3(2), 4.4(3). In particular, as in 1.8, one proves using (ii) above and Godement's lemma that $^\circ\mathscr{A}(\xi, J, K_\infty)$ is finite dimensional.

4.5. PROPOSITION. *Let f be a smooth function on $G(A)$ satisfying* 4.2(a), (b), (d). *Then the following conditions are equivalent*:

(1) *f is an automorphic form.*

(2) *For each infinite place v, the space $f * \mathscr{H}_v$ is an admissible $\mathscr{H}_v$-module.*

(3) *For each place $v \in V$, the space $f * \mathscr{H}_v$ is an admissible $\mathscr{H}_v$-module.*

(4) *The space $f * \mathscr{H}$ is an admissible $\mathscr{H}$-module.*

PROOF. The implications (4) ⇒ (3) ⇒ (2) ⇒ (1) are obvious; (1) ⇒ (4) follows from 4.3(i).

4.6. *Automorphic representations.* An irreducible representation of $\mathscr{H}$ is *automorphic* (resp. *cuspidal*) if it is isomorphic to a subquotient of a representation of $\mathscr{H}$ in the space of automorphic (resp. cusp) forms on $G(A)$. It follows from 4.5 that such a representation is always admissible. It will also be called an automorphic representation of G or $G(A)$, although, strictly speaking, it is not a $G(A)$-module. However, it is always a $G(A_f)$-module. More generally, a topological $G(A)$-module E will be said to be *automorphic* if the subspace of admissible vectors in E is an automorphic representation of $\mathscr{H}$. In particular, if χ is a character of $Z(A)/Z(F)$, any G-invariant irreducible closed subspace of

$$L^2(G(F)\backslash G(A))_\chi = \{f \in L^2(G(F)\cdot Z(A)\backslash G(A))|, f(z\cdot x) = \chi(z)\cdot f(x),\ z \in Z(A),\ x \in G(A))\}$$

is automorphic in this sense, in view of [4, Theorem 4]. By a theorem of Gelfand and

Piatetski-Shapiro [7] (see also [8]), the subspace ${}^{\circ}L^2(G(F)\backslash G(A))_\chi$ of cuspidal functions of $L^2(G(F)\backslash G(A))_\chi$ is a discrete sum with finite multiplicities of closed irreducible invariant subspaces. Those give then, up to isomorphisms, all cuspidal automorphic representations in which $Z(A)$ has character χ. The admissible vectors in those subspaces are all the cusp forms satisfying 4.3(3).

4.7. Let $L \supset L'$ be compact open subgroups of $G(A_f)$. Then $\xi_L * \xi_{L'} = \xi_L$; hence $\mathscr{A}(\xi_\infty \otimes \xi_L; J, K_\infty) \subset \mathscr{A}(\xi_\infty \otimes \xi_{L'}, J, K_\infty)$. The space $\mathscr{A}(\xi_\infty, J, K_\infty)$ spanned by all automorphic forms of types $(\xi_\infty \otimes \xi_f, J, K_\infty)$, with ξ_f arbitrary in $\mathscr{H}_f$, may then be identified to the inductive limit

$$(1) \qquad \mathscr{A}(\xi_\infty, J, K_\infty) = \operatorname*{ind\,lim}_L \mathscr{A}(\xi_\infty \otimes \xi_L, J, K_\infty),$$

where L runs through the compact open subgroups of $G(A_f)$, the inductive limit being taken with respect to the above inclusions. The group $G(A_f)$ operates on $\mathscr{H}_f$ by inner automorphisms. Let us denote by ${}^x\xi$ the transform of $\xi \in \mathscr{H}_f$ by Int x. We have in particular

$$(2) \qquad {}^x(\xi_L) = \xi_{({}^xL)} \qquad (x \in G(A_f),\ \xi_L \text{ as in } 4.1);$$

if f is a continuous (or measurable) function on $G(A)$, then

$$(3) \qquad (r_x f) * \xi = r_x(f * {}^{x^{-1}}\xi) \qquad (x \in G(A_f),\ \xi \in \mathscr{H}_f).$$

Therefore, $G(A_f)$ operates on $\mathscr{A}(\xi_\infty, J, K_\infty)$ by right translations. It follows from 4.3(i) that this representation is admissible.

In view of 4.3(3), $\mathscr{A}(\xi_\infty, J, K_\infty)$ is the adelic analogue of the space $\mathscr{A}(\Sigma, \xi_\infty, J, K_\infty)$ of 1.9, where Σ is the family of arithmetic subgroups of $G(F)$ of the form $\Gamma_L = G(F) \cap (G_\infty \times L)$, where L is a compact open subgroup of $G(A_f)$. These are the *congruence arithmetic subgroups* of $G(F)$, i.e., those subgroups which, for an embedding $G \hookrightarrow \mathbf{GL}_n$ over F, contain a congruence subgroup of $G \cap \mathbf{GL}_n(O_F)$. This analogy can be made more precise when G satisfies strong approximation, which is the case in particular when G is semisimple, simply connected, almost simple over F, and G_∞ is noncompact [16]. In that case, as recalled in 4.4(2), we have $G(A) = G(F) \cdot G_\infty \cdot L$ for any compact open subgroup of $G(A_f)$, so that we may take $C = \{1\}$ in 4.3. Then 4.3(2) provides an isomorphism

$$(4) \qquad \mathscr{A}(\xi_\infty \otimes \xi_L, J, K_\infty) \xrightarrow{\sim} \mathscr{A}(\Gamma_L, \xi_\infty, J, K_\infty),$$

for any L, whence

$$(5) \qquad \mathscr{A}(\xi_\infty, J, K_\infty) \cong \mathscr{A}(\Sigma, \xi_\infty, J, K_\infty),$$

where Σ is the set of congruence arithmetic subgroups of $G(F)$. Moreover, the projection of $G(F)$ in $G(A_f)$ is dense in $G(A_f)$ and $G(A_f)$ may be identified to the completion $G(F)_\Sigma$ of $G(F)$ with respect to the topology defined by the subgroups Γ_L. It is easily seen that the isomorphism (5) commutes with $G(\boldsymbol{Q})$, where, on the left-hand side $x \in G(\boldsymbol{Q})$ acts as in 1.9, via left translations, and, on the right-hand side, x acts as an element of $G(A_f)$ by right translations. It follows that the isomorphism (5) commutes with the actions of $G(F)_\Sigma = G(A_f)$ defined here and in 1.9. Also, the isomorphism $G(F)_\Sigma \xrightarrow{\sim} G(A_f)$ induces one of the Hecke algebra $\mathscr{H}(G(F)_\Sigma)$ (see 1.10) onto $\mathscr{H}_f$ and, again, (5) is compatible with the actions defined here and in 1.10. Note also that Γ_L is dense in L, by strong approximation, so that this isomorphism of Hecke algebras induces one of $\mathscr{H}(G(F)_\Sigma, \bar{\Gamma}_L)$, which is equal to $\mathscr{H}(G(F), \Gamma_L)$, onto $\mathscr{H}(G(A_f), L)$. [Strictly speaking, this applies at first for $F = \boldsymbol{Q}$, but we can reduce the general case to that one, if we replace G by $R_{F/\boldsymbol{Q}}G$ (4.1).]

In the general case, the isomorphisms 4.3(2) for various ξ_L are compatible with the action of $G(F)$ defined here and in 1.9 respectively, and this extends by continuity to the closure in $G(A_f)$ of the projection of $G(F)$.

4.8. Let $\mathscr{A}(J, K_\infty)$ be the span of the spaces $\mathscr{A}(\xi_\infty, J, K_\infty)$ and $\mathscr{A}(K_\infty)$ the span of the $\mathscr{A}(J, K_\infty)$. These spaces are $\mathscr{H}$-modules, and union of admissible $\mathscr{H}$-submodules. When 4.6(5) holds, they are isomorphic to the spaces $\mathscr{A}(\Sigma, J, K_\infty)$ and $\mathscr{A}(\Sigma, K_\infty)$ of automorphic forms on $G(\boldsymbol{R})$ defined in 1.10. Otherwise, the relationship is more complex, and would have to be expressed by means of the isomorphisms 4.3(2).

5. Groups over function fields. In this section, F is a function field of one variable over a finite field. A function on $G(A)$ is said to be smooth if it is locally constant.

5.1. Let χ be a quasi-character of $Z(A)/Z(F)$ and K an open subgroup of $G(A)$. We let ${}^\circ\mathscr{V}(\chi, K)$ be the space of complex valued functions on $G(A)$ which are right invariant under K, left invariant under $G(F)$, satisfy

$$(1) \qquad f(z\cdot x) = \chi(z)\cdot f(x) \qquad (z \in Z(A),\ x \in G(A)),$$

and are cuspidal (3.3). [These functions are cusp forms, in a sense to be defined below (5.7), but the latter notion is slightly more general.] We need the following:

5.2. PROPOSITION (G. HARDER). *Let K and χ be given. Then there exists a compact subset C of $G(A)$ such that every element of ${}^\circ\mathscr{V}(\chi, K)$ has support in $Z(A)\cdot G(F)\cdot C$. In particular, ${}^\circ\mathscr{V}(\chi, K)$ is finite dimensional.*

This follows from Corollary 1.2.3 in **[10]** when G is split over F and semisimple (the latter restriction because (1) is not the condition imposed in **[10]** with respect to the center). However, since $G(A)$ can be covered by finitely many Siegel sets, the argument is general (see **[9**, p. 142] for $\mathbf{GL}_n$).

5.3. COROLLARY. *Let X be a finite set of quasi-characters of $Z(A)/Z(F)$ and m a positive integer. Then the space ${}^\circ\mathscr{V}(X, m, K)$ of cuspidal functions which are right invariant under K and satisfy the condition*

$$(1) \qquad \prod_{\chi\in X} (r(z) - \chi(z))^m \cdot f = 0$$

is finite dimensional.

PROOF. The space ${}^\circ\mathscr{V}(X, m, K)$ is the direct sum of the spaces ${}^\circ\mathscr{V}(\{\chi\}, m, K)$; hence we may assume that X consists of one quasi-character χ. By 5.3, ${}^\circ\mathscr{V}(X, 1, K)$ is finite dimensional. We then proceed by induction and assume that ${}^\circ\mathscr{V}(X, s, K)$ is finite dimensional for some $s \geqq 1$.

The group $Z(A)/Z(F)\cdot K'$, where $K' = Z(A) \cap K$, is finitely generated. Let $(z_j)_{1\leqq j\leqq q}$ be a generating set. Set $A_{s+1,0} = {}^\circ\mathscr{V}(X, s+1, K)$ and, for $t = 1, \ldots, q$:

$$A_{s+1,t} = \{f \in A_{s+1,0} | (r(z_j) - \chi(z_j))^s \cdot f = 0 \ (j = 1, \ldots, t)\}.$$

Then

$$A_{s+1,0} \supset A_{s+1,1} \supset \dots \supset A_{s+1,t} \supset \dots \supset A_{s+1,q} = {}^\circ\mathscr{V}(X, s, K).$$

Fix t ($0 \leqq t < q$). Then $f \mapsto (r(a_{t+1}) - \chi(a_{t+1}))\cdot f$ maps $A_{s+1,t}$ into $A_{s+1,t+1}$ and its kernel is contained in $A_{s+1,t+1}$. It follows, by descending induction on t, that $A_{s+1,0}$ is finite dimensional.

5.4. REMARK. If H is a commutative group, let $C[H]$ be its group algebra over C. Every complex representation (π, W) of H extends canonically to one of $C[H]$. An element $w \in W$ is H-finite if and only if it is annihilated by some ideal I of finite codimension of $C[H]$. If I is such an ideal, there exist a finite set X of quasi-characters of H and a positive integer m such that

$$(1)\qquad \prod_{\chi \in X} (\pi(h) - \chi(h))^m \cdot w = 0, \quad \text{for all } h \in H \text{ and all } w \text{ annihilated by } I.$$

If H is finitely generated, then, conversely, all elements $w \in W$ satisfying (1) for all $h \in H$ are H-finite, and annihilated by some ideal of finite codimension of $C[H]$. Therefore, 5.3 implies that the space ${}^\circ\mathscr{V}(I, K)$ of cuspidal functions on $G(F)\backslash G(A)/K$ which are annihilated by some ideal I of finite codimension of $C[Z(A)/Z(F)]$ is finite dimensional. In fact, since $Z(A)/Z(F)\cdot(K \cap Z(A))$ is finitely generated, these two statements are equivalent.

5.5. Let E be a local field, R a connected reductive group over E, $\mathscr{H}_R$ the Hecke algebra of $R(E)$. A left ideal J of $\mathscr{H}_R$ is said to be *admissible* if the natural representation of $\mathscr{H}_R$ on $\mathscr{H}_R/J$ is admissible.

LEMMA. *Let J be an admissible ideal of $\mathscr{H}_R$ and K a compact open subgroup of $R(E)$. Let P be a parabolic E-subgroup of H, N the unipotent radical of P and M a Levi E-subgroup of P. There is an admissible ideal J_M in the Hecke algebra $\mathscr{H}_M$ of $M(E)$ with the following property: if π is a smooth representation of $R(E)$ on a space W, and $w \in W$ is K-fixed, annihilated by J, then the image $\bar{w}$ of w in the space W_N (cf.* [3]) *is annihilated by J_M.*

PROOF. Let $\varphi_0 \in \mathscr{H}_R$ be the characteristic function of K, v_0 its image in $\mathscr{H}_R/J$ and $\bar{v}_0$ the image of v_0 in $(\mathscr{H}_R/J)_N$. The representation of $R(E)$ on $\mathscr{H}_R/J$ is admissible by assumption; therefore the representation of $M(E)$ on $(\mathscr{H}_R/J)_N$ is admissible [3], and the annihilator J_M of $\bar{v}_0$ in $\mathscr{H}_M$ is admissible. We claim that it has the required properties. In fact, if π and w are as in the lemma, then there is a unique $R(E)$-morphism $\mathscr{H}_R \to W$ taking φ_0 to w. It maps $\xi \in \mathscr{H}_R$ onto a scalar multiple of $\pi(\xi)\cdot w$. Therefore, it factors through an $R(E)$-morphism $\mathscr{H}_R/J \to W$, mapping v_0 onto w, whence an $M(E)$-morphism $(\mathscr{H}_R/J)_N \to W_N$ mapping $\bar{v}_0$ onto $\bar{w}$. It follows that $\bar{w}$ is annihilated by J_M.

5.6. THEOREM. *Let K be a compact open subgroup of $G(A)$. Let $v \in V$ and J an admissible ideal of $\mathscr{H}_v$. Then the space $\mathscr{V}(G, v, J, K)$ of complex valued functions f on $G(A)$ which are left invariant under $G(F)$, right invariant under K and annihilated by J is finite dimensional.*

PROOF. Since the representation of G_v on $\mathscr{H}_v/J$ is admissible (and finitely generated), there exists an ideal I_v of finite codimension of $C[Z(F_v)]$ which annihilates $\mathscr{H}_v/J$.

Let $f \in \mathscr{V}(G, v, J, K)$. Then $h \mapsto f * \check{h}$ ($h \mapsto \check{h}$ being the canonical involution of $\mathscr{H}_v$) gives a G_v-morphism $\mathscr{H}_v/J \to f * \mathscr{H}_v$. Therefore $\mathscr{V}(G, v, J, K)$ is annihilated by I_v. Let $Z' = Z(A) \cap K$ and regard $Z(F_v)$ as a subgroup of $Z(A)$. Then $Z(F) \cdot Z(F_v) \cdot Z'$ has *finite index* in $Z(A)$. As a consequence, there exists an ideal I of finite codimension of $Z(A)/Z(F)$ which annihilates $\mathscr{V}(G, v, J, K)$. The space ${}^\circ\mathscr{V}$ of cuspidal elements in $\mathscr{V}(G, v, J, K)$ is then contained in ${}^\circ\mathscr{V}(I, K)$ (notation of 5.4), hence is finite dimensional.

We now prove the theorem by induction on the F-rank $\mathrm{rk}_F G'$ of the derived group G' of G. If $\mathrm{rk}_F G' = 0$, then $\mathscr{V}(G, v, J, K) = {}^\circ\mathscr{V}$, and our assertion is already proved. So assume $\mathrm{rk}_F G' \geqq 1$ and the theorem proved for groups of strictly smaller semisimple F-rank. Let now $P = M \cdot N$ vary through a set $\mathscr{P}$ of representatives of the conjugacy classes of proper parabolic F-subgroups of G, where M is a Levi F-subgroup and N the unipotent radical of P. For each such P, let C_P be a set of representatives of $P(A)\backslash G(A)/K$. It is finite. The intersection of the kernels of the maps $f \mapsto f_{P,c}$, where $f_{P,c}(m) = \int_{N(F)\backslash N(A)} f(n \cdot m \cdot c)\, dn$ ($c \in C_P$, $P \in \mathscr{P}$) ($f \in \mathscr{V}(G, v, J, K)$) is then ${}^\circ\mathscr{V}$, hence is finite dimensional. It suffices therefore to show that, for given $P \in \mathscr{P}$, $c \in C_P$, the functions $f_{P,c}$ vary in a fixed finite dimensional space. After having replaced J and K by conjugates, we may assume that $c = 1$. We write f_P for $f_{P,1}$. Let now U_G (resp. U_M) be the space of functions on $G(F)\backslash G(A)$ (resp. $M(F)\backslash M(A)$) which are right invariant under some compact open subgroup (depending on the function). The representation r of $G(A)$ by right translations on U_G is smooth. If $x \in N(A)$ then $f_P = (r_x f)_P$; hence $\mu_P : f \mapsto f_P$ factors through $(U_G)_{N(F_v)}$. It follows then from 5.5 that the elements f_P ($f \in \mathscr{V}(G, v, J, K)$) are all annihilated by some admissible ideal J' of the Hecke algebra of $M(F_v)$. Since these elements are right invariant under $K' = K \cap M(A)$, it follows that μ_P maps $\mathscr{V}(G, v, J, K)$ into $\mathscr{V}(M, v, J', K')$. Since this last space is finite dimensional by our induction assumption, the proof is now complete.

5.7. COROLLARY. *Let f be a function on $G(A)$ which is left invariant under $G(F)$ and right invariant under some compact open subgroup of $G(A)$. Then the following conditions are equivalent*:

(1) *There is a place v of F such that the representation of G_v on the $G(F_v)$-invariant subspace generated by $r(G_v) \cdot f$ is admissible.*

(2) *For each place v of F, the representation of G_v on the space generated by $r(G_v) \cdot f$ is admissible.*

(3) *The representation of $G(A)$ on the space spanned by $r(G(A)) \cdot f$ is admissible.*

PROOF. Clearly (3) $\Rightarrow$ (2) $\Rightarrow$ (1). Assume (1). Let v be as in (1). Then f is annihilated by an admissible ideal J of $\mathscr{H}_v$. Let $U = f * \mathscr{H}$. We have to prove that U^K is finite dimensional for any compact open subgroup K of $G(A)$. There is no harm in replacing K by a smaller group, so we may assume that K fixes f. We may also assume that $K = K_v \times K^v$, where K_v is compact open in G_v and K^v is compact open in the subgroup G^v of elements in $G(A)$ with v-component equal to 1. We also have $\mathscr{H} = \mathscr{H}_v \otimes \mathscr{H}^v$ where $\mathscr{H}^v$ is the Hecke algebra of G^v. Let ξ_v (resp. ξ^v) be the idempotents associated to K_v (resp. K^v) (3.3). Then $\xi_v \otimes \xi^v = \xi_K$ is the idempotent associated to K. Any element g in U is a finite linear combination of elements of the form $f * \alpha * \beta$ ($\alpha \in \mathscr{H}^v$, $\beta \in \mathscr{H}_v$). If such an element is fixed under K, then $g * \xi_K =$

g; hence we may assume that each summand is fixed under K, and that $\alpha * \xi^v = \alpha$, $\beta * \xi_v = \beta$. Since f is fixed under K, it follows that $f * \alpha$ is fixed under K. The elements $f * \alpha$ then belong to the space $\mathscr{V}(G, v, J, K)$, which is finite dimensional by the theorem. For each such element $f * \alpha * \beta$ is contained in the space of K_v-fixed vectors in the admissible $\mathscr{H}_v$-module $f * \alpha * \mathscr{H}_v = f * \mathscr{H}_v * \alpha$, whence our assertion.

5.8. Definitions. An *automorphic form* on $G(A)$ is a function which is left invariant under $G(F)$, right invariant under some compact open subgroup, and satisfies the equivalent conditions of 5.7. A *cusp form* is a cuspidal automorphic form. Any automorphic form is $Z(A)$-finite (as follows from 5.7(3)).

An *irreducible representation of* $G(A)$ *is automorphic* if it is isomorphic to a subquotient of the $G(A)$-module $\mathscr{A}$ of all automorphic forms on $G(F)\backslash G(A)$. It follows from 5.7 that it is always admissible.

More generally, a topologically irreducible continuous representation of $G(A)$ in a topological vector space is automorphic if the submodule of smooth vectors is automorphic.

As in 4.6, it follows from [4, Theorem 4] that if χ is a character of $Z(A)/Z(F)$, then any G-invariant closed irreducible subspace of $L^2(G(F)\backslash G(A))_\chi$ is automorphic.

5.9. Proposition. *Let f be a function on $G(F)\backslash G(A)$. Then the following conditions are equivalent*:

(1) *f is a cusp form.*

(2) *f is $Z(A)$-finite, cuspidal* (3.3), *and right invariant under some compact open subgroup of $G(A)$.*

Proof. That (1) $\Rightarrow$ (2) is clear. Assume (2). Then f is annihilated by an ideal I of finite codimension of $C[Z(A)/Z(F)]$. Let U be the space of functions spanned by $r(G(A)) \cdot f$. Every element of U is cuspidal, annihilated by I and right invariant under some compact open subgroup. If L is any compact open subgroup, then U^L is contained in the space ${}^\circ\mathscr{V}(I, L)$ (notation of 5.4), hence is finite dimensional. Therefore U is an admissible $G(A)$-module and (1) holds.

5.10. It also follows in the same way that the space ${}^\circ\mathscr{A}(I)$ (resp. ${}^\circ\mathscr{A}(X, m)$) of all cusp forms which are annihilated by an ideal I of finite codimension of $C[Z(A)/Z(F)]$ (resp. which satisfy 5.3(1)) is an admissible $G(A)$-module. Moreover, if X consists of one element χ, and if $m = 1$, in which case we put $\mathscr{A}(X, m) = {}^\circ\mathscr{A}_\chi$, then this space is a direct sum of irreducible admissible $G(A)$-modules, with finite multiplicities. To see this we may, after twisting with $|\chi|^{-1}$, assume that χ is unitary. Then, since ${}^\circ\mathscr{A}_\chi$ consists of elements with compact support modulo $Z(A)$,

$$(f, g) = \int_{Z(A)G(F)\backslash G(A)} f(x) \cdot \overline{g(x)}\, dx$$

defines a nondegenerate positive invariant hermitian form on ${}^\circ\mathscr{A}_\chi$. Our assertion follows from this and admissibility. This is the counterpart over function fields of the Gelfand-Piatetski-Shapiro theorem (4.6).

We note that, by **[14]**, every automorphic representation transforming according to χ is a constituent of a representation induced from a cuspidal automorphic representation of a Levi subgroup of some parabolic F-subgroup, for any global field F.

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INSTITUTE FOR ADVANCED STUDY

COLUMBIA UNIVERSITY

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ON THE NOTION OF AN AUTOMORPHIC REPRESENTATION. A SUPPLEMENT TO THE PRECEDING PAPER

R. P. LANGLANDS

The irreducible representations of a reductive group over a local field can be obtained from the square-integrable representations of Levi factors of parabolic subgroups by induction and formation of subquotients **[2]**, **[4]**. Over a global field F the same process leads from the cuspidal representations, which are analogues of square-integrable representations, to all automorphic representations.

Suppose P is a parabolic subgroup of G with Levi factor M and $\sigma = \bigotimes \sigma_v$ a cuspidal representation of $M(\mathbf{A})$. Then $\operatorname{Ind} \sigma = \bigotimes_v \operatorname{Ind} \sigma_v$ is a representation of $G(\mathbf{A})$ which may not be irreducible, and may not even have a finite composition series. As usual an irreducible subquotient of this representation is said to be a constituent of it.

For almost all v, $\operatorname{Ind} \sigma_v$ has exactly one constituent π_v° containing the trivial representation of $G(\mathbf{O}_v)$. If $\operatorname{Ind} \sigma_v$ acts on X_v then π_v° can be obtained by taking the smallest $G(F_v)$-invariant subspace V_v of X_v containing nonzero vectors fixed by $G(\mathbf{O}_v)$ together with the largest $G(F_v)$-invariant subspace U_v of V_v containing no such vectors and then letting $G(F_v)$ act on V_v/U_v.

LEMMA 1. *The constituents of* $\operatorname{Ind} \sigma$ *are the representations* $\pi = \bigotimes \pi_v$ *where* π_v *is a constituent of* $\operatorname{Ind} \sigma_v$ *and* $\pi_v = \pi_v^\circ$ *for almost all* v.

That any such representation is a constituent is clear. Conversely let the constituent π act on V/U with $0 \subseteq U \subseteq V \subseteq X = \bigotimes X_v$. Recall that to construct the tensor product one chooses a finite set of places S_0 and for each v not in S_0 a vector x_v° which is not zero and is fixed by $G(\mathbf{O}_v)$. We can find a finite set of places S, containing S_0, and a vector x_S in $X_S = \bigotimes_{v \in S} X_v$ which are such that $x = x_S \otimes (\bigotimes_{v \notin S} x_v^\circ)$ lies in V but not in U.

Let V_S be the smallest subspace of X_S containing x_S and invariant under $G_S = \prod_{v \in S} G(F_v)$. There is clearly a surjective map

$$V_S \otimes (\bigotimes_{v \notin S} V_v) \longrightarrow V/U,$$

and if $v_0 \notin S$ the kernel contains $V_S \otimes U_{v_0} \otimes (\bigotimes_{v \notin S \cup \{v_0\}} V_v)$. We obtain a surjection $V_S \otimes (\bigotimes_{v \in S} V_v/U_v) \to V/U$ with a kernel of the form $U_S \otimes (\bigotimes_{v \in S} V_v/U_v)$, U_S lying in V_S. The representation of G_S on V_S/U_S is irreducible and, since $\operatorname{Ind} \sigma_v$ has a finite composition series, of the form $\bigotimes_{v \in S} \pi_v$, π_v being a constituent of $\operatorname{Ind} \sigma_v$. Thus $\pi = \bigotimes \pi_v$ with $\pi_v = \pi_v^\circ$ when $v \notin S$.

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The purpose of this supplement is to establish the following proposition.

PROPOSITION 2. *A representation π of $G(A)$ is an automorphic representation if and only if π is a constituent of* Ind σ *for some P and some σ.*

The proof that every constituent of Ind σ is an automorphic representation will invoke the theory of Eisenstein series, which has been fully developed only when the global field F has characteristic zero [3]. One can expect however that the analogous theory for global fields of positive characteristic will appear shortly, and so there is little point in making the restriction to characteristic zero explicit in the proposition. Besides, the proof that every automorphic representation is a constituent of some Ind σ does not involve the theory of Eisenstein series in any serious way.

We begin by remarking some simple lemmas.

LEMMA 3. *Let Z be the centre of G. Then an automorphic form is $Z(A)$-finite.*

This is verified in [1].

LEMMA 4. *Suppose K is a maximal compact subgroup of $G(A)$ and φ an automorphic form with respect to K. Let P be a parabolic subgroup of G. Choose $g \in G(A)$ and let K' be a maximal compact subgroup of $M(A)$ containing the projection of $gKg^{-1} \cap P(A)$ on $M(A)$. Then*

$$\varphi_P(m; g) = \int_{N(F)\backslash N(A)} \varphi(nmg)\, dn$$

is an automorphic form on $M(A)$ with respect to K'.

It is clear that the growth conditions are hereditary and that $\varphi_P(\cdot\,; g)$ is smooth and K'-finite. That it transforms under admissible representations of the local Hecke algebras of M is a consequence of theorems in [2] and [4].

We say that φ is orthogonal to cusp forms if $\int_{\Omega G_{\mathrm{der}}(A)} \varphi(g)\psi(g)\, dg = 0$ whenever ψ is a cusp form and Ω is a compact set in $G(A)$. If P is a parabolic subgroup we write $\psi \perp P$ if $\varphi_P(\cdot\,; g)$ is orthogonal to cusp forms on $M(A)$ for all g. We recall a simple lemma [3].

LEMMA 5. *If $\varphi \perp P$ for all P then φ is zero.*

We now set about proving that any automorphic representation π is a constituent of some Ind σ. We may realize π on V/U, where U, V are subspaces of the space $\mathscr{A}$ of automorphic forms and V is generated by a single automorphic form φ. Totally order the conjugacy classes of parabolic subgroups in such a way that $\{P\} < \{P'\}$ implies rank $P \leqq$ rank P' and rank $P <$ rank P' implies $\{P\} < \{P'\}$. Given φ let $\{P_\varphi\}$ be the minimum $\{P\}$ for which $\{P\} < \{P'\}$ implies $\varphi \perp P'$. Amongst all the φ serving to generate π choose one for which $\{P\} = \{P_\varphi\}$ is minimal. If $\psi \in \mathscr{A}$ let $\psi_P(g) = \psi_P(1, g)$ and consider the map $\psi \to \psi_P$ on V. If U and V had the same image we could realize π as a constituent of the kernel of the map. But this is incompatible with our choice of φ, and hence if U_P and V_P are the images of U and V we can realize π in the quotient V_P/U_P.

Let $\mathscr{A}_P^\circ$ be the space of smooth functions ψ on $N(A)P(F)\backslash G(A)$ satisfying

(a) ψ is K-finite.

(b) For each g the function $m \to \psi(m, g) = \psi(mg)$ is automorphic and cuspidal.

Then $V_P \subseteq \mathscr{A}_P^\circ$. Since there is no point in dragging the subscript P about, we change notation, letting π be realized on V/U with $U \subseteq V \subseteq \mathscr{A}_P^\circ$. We suppose that V is generated by a single function φ.

LEMMA 6. *Let A be the centre of M. We may so choose φ and V that there is a character χ of $A(\boldsymbol{A})$ satisfying $\varphi(ag) = \chi(a)\varphi(g)$ for all $g \in G(\boldsymbol{A})$ and all $a \in A(\boldsymbol{A})$.*

Since $P(\boldsymbol{A})\backslash G(\boldsymbol{A})/K$ is finite, Lemma 3 implies that any φ in $\mathscr{A}_P^\circ$ is $A(\boldsymbol{A})$-finite. Choose V and the φ generating it to be such that the dimension of the span Y of $\{l(a)\varphi | a \in A(\boldsymbol{A})\}$ is minimal. Here $l(\text{a})$ is left translation by a. If this dimension is one the lemma is valid. Otherwise there is an $a \in A(\boldsymbol{A})$ and $\alpha \in \boldsymbol{C}$ such that $0 < \dim(l(a) - \alpha)Y < \dim Y$.

There are two possibilities. Either $(l(a) - \alpha)U = (l(a) - \alpha)V$ or $(l(a) - \alpha)U \neq (l(a) - \alpha)V$. In the second case we may replace φ by $(l(a) - \alpha)\varphi$, contradicting our choice. In the first we can realize π as a subquotient of the kernel of $l(a) - \alpha$ in V.

What we do then is choose a lattice B in $A(\boldsymbol{A})$ such that $BA(F)$ is closed and $BA(F)\backslash A(\boldsymbol{A})$ is compact. Amongst all those φ and V for which Y has the minimal possible dimension we choose one φ for which the subgroup of B, defined as $\{b \in B | l(b)\varphi = \beta\varphi, \beta \in \boldsymbol{C}\}$, has maximal rank. What we conclude from the previous paragraph is that this rank must be that of B. Since φ is invariant under $A(F)$ and $BA(F)\backslash A(\boldsymbol{A})$ is compact, we conclude that Y must now have dimension one. The lemma follows.

Choosing such a φ and V we let ν be that positive character of $M(\boldsymbol{A})$ which satisfies

$$\nu(a) = |\chi(a)|, \qquad a \in A(\boldsymbol{A}),$$

and introduce the Hilbert space $L_2^\circ = L_2^\circ(M(F)\backslash M(\boldsymbol{A}), \chi)$ of all measurable functions ψ on $M(\boldsymbol{Q})\backslash M(\boldsymbol{A})$ satisfying:

(i) For all $m \in M(\boldsymbol{A})$ and all $a \in A(\boldsymbol{A})$, $\psi(ag) = \chi(a)\psi(g)$.

(ii) $\int_{A(\boldsymbol{A})M(\boldsymbol{Q})\backslash M(\boldsymbol{A})} \nu^{-2}(m)|\psi(m)|^2 \, dm < \infty$.

L_2° is a direct sum of irreducible invariant subspaces, and if $\psi \in V$ then $m \to \psi(m, g)$ lies in L_2° for all $g \in G(\boldsymbol{A})$. Choose some irreducible component H of L_2° on which the projection of $\varphi(\cdot, g)$ is not zero for some $g \in G(\boldsymbol{A})$.

For each ψ in V define $\psi'(\cdot, g)$ to be the projection of $\psi(\cdot, g)$ on H. It is easily seen that, for all $m_1 \in M(\boldsymbol{A})$, $\psi'(mm_1, g) = \psi'(m, m_1 g)$. Thus we may define $\psi'(g)$ by $\psi'(g) = \psi'(1, g)$. If $V' = \{\psi' | \psi \in V\}$, then we realize π as a quotient of V'. However if δ^2 is the modular function for $M(\boldsymbol{A})$ on $N(\boldsymbol{A})$ and σ the representation of $M(\boldsymbol{A})$ on H then V' is contained in the space of $\text{Ind}\, \delta^{-1}\, \sigma$.

To prove the converse, and thereby complete the proof of the proposition, we exploit the analytic continuation of the Eisenstein series associated to cusp forms. Suppose π is a representation of the global Hecke algebra $\mathscr{H}$, defined with respect to some maximal compact subgroup K of $G(\boldsymbol{A})$. Choose an irreducible representation κ of K which is contained in π. If E_κ is the idempotent defined by K let $\mathscr{H}_k = E_\kappa \mathscr{H} E_\kappa$ and let π_κ be the irreducible representation of $\mathscr{H}_\kappa$ on the κ-isotypical subspace of π. To show that π is an automorphic representation, it is sufficient to show that π_κ is a constituent of the representation of $\mathscr{H}_\kappa$ on the space of automorphic forms of type κ. To lighten the burden on the notation, we henceforth denote π_κ by π and $\mathscr{H}_\kappa$ by $\mathscr{H}$.

Suppose P and the cuspidal representation σ of $M(A)$ are given. Let L be the lattice of rational characters of M defined over F and let $L_C = L \otimes C$. Each element μ of L_C defines a character ξ_μ of $M(A)$. Let I_μ be the κ-isotypical subspace of $\mathrm{Ind}\, \xi_\mu\sigma$ and let $I = I_0$. We want to show that if π is a constituent of the representation on I then π is a constituent of the representation of $\mathscr{H}$ on the space of automorphic forms of type κ.

If $\{g_i\}$ is a set of coset representations for $P(A)\backslash G(A)/K$ then we may identify I_μ with I by means of the map $\varphi \to \varphi_\mu$ with

$$\varphi_\mu(nmg_i\, k) = \xi_\mu(m)\, \varphi(nmg_i k).$$

In other words we have a trivialization of the bundle $\{I_\mu\}$ over L_C, and we may speak of a holomorphic section or of a section vanishing at $\mu = 0$ to a certain order. These notions do not depend on the choice of the g_i, although the trivialisation does.

There is a neighbourhood U of $\mu = 0$ and a finite set of hyperplanes passing through U so that for μ in the complement of these hyperplanes in U the Eisenstein series $E(\varphi)$ is defined for φ in I_μ. To make things simpler we may even multiply E by a product of linear functions and assume that it is defined on all of U. Since it is only the modified function that we shall use, we may denote it by E, although it is no longer the true Eisenstein series. It takes values on the space of automorphic forms and thus $E(\varphi)$ is a function $g \to E(g, \varphi)$ on $G(A)$. It satisfies

$$E(\rho_\mu(h)\varphi) = r(h)E(\varphi)$$

if $h \in \mathscr{H}$ and ρ_μ is $\mathrm{Ind}\, \xi_\mu\sigma$. In addition, if φ_μ is a holomorphic section of $\{I_\mu\}$ in a neighbourhood of 0 then $E(g, \varphi_\mu)$ is holomorphic in μ for each g, and the derivatives of $E(\varphi_\mu)$, taken pointwise, continue to be in $\mathscr{A}$.

Let I_r be the space of germs of degree r at $\mu = 0$ of holomorphic sections of I. Then $\varphi_\mu \to \rho_\mu(h)\varphi_\mu$ defines an action of $\mathscr{H}$ on I_r. If $s \leqq r$ the natural map $I_r \to I_s$ is an $\mathscr{H}$-homomorphism. Denote its kernel by I_r^s. Certainly $I_0 = I$. Choosing a basis for the linear forms on L_C we may consider power series with values in the κ-isotypical subspace of $\mathscr{A}$, $\sum_{|\alpha| \leqq r} \mu^\alpha \psi_\alpha$. $\mathscr{H}$ acts by right translation in this space and the representation so obtained is, of course, a direct sum of that on the κ-isotypical automorphic forms. Moreover $\varphi_\mu \to E(\varphi_\mu)$ defines an $\mathscr{H}$-homomorphism λ from I_r to this space. To complete the proof of the proposition all one needs is the Jordan-Hölder theorem and the following lemma.

LEMMA 7. *For r sufficiently large the kernel of* λ *is contained in* I_r^0.

Since we are dealing with Eisenstein series associated to a fixed P and σ we may replace E by the sum of its constant terms for the parabolic associated to P, modifying λ accordingly. All of these constant terms vanish identically if and only if E itself does. If $Q_1, \ldots, Q_m$ is a set of representatives for the classes of parabolics associated to P let $E_i(\varphi)$ be the constant term along Q_i. We may suppose that M is a Levi factor of each Q_i. Define $\nu(m)$ for $m \in M(A)$ by $\xi_\mu(m) = e^{\langle \mu, \nu(m) \rangle}$. Thus $\nu(m)$ lies in the dual of L_R. If $\varphi \in I_\mu$, the function $E_i(\varphi)$ has the form

$$E_i(nmg_jk, \varphi) = \sum_{\alpha=1}^{a} \sum_{\beta=1}^{b} p_\alpha(\nu(m))\xi_{\nu_\beta(\mu)}(m)\psi_{\alpha\beta}(m, k).$$

Here $\psi_{\alpha\beta}$ lies in a finite-dimensional space independent of μ and g_j; ν_β, $1 \leqq \beta \leqq b$ is a holomorphic function of μ; and $\{p_\alpha\}$ is a basis for the polynomials of some degree. This representation may not be unique. The next lemma implies that we may shrink the open set U and then find a finite set $h_1, \ldots, h_n$ in $G(A)$ such that $E(\varphi_\mu)$ is 0 for $\mu \in U$, $\varphi_\mu \in I_\mu$ if and only if the numbers $E_i(h_j, \varphi_\mu)$, $1 \leqq i \leqq m$, $1 \leqq j \leqq n$ are all 0.

LEMMA 8. *Let U be a neighbourhood of* 0 *in $\mathbf{C}^l$, $\nu_1, \ldots, \nu_k$, holomorphic functions on U, and $p_1, \ldots, p_a$ a basis for the polynomials of some given degree. Then there is a neighbourhood V of* 0 *contained in U and a finite set $y_1, \ldots, y_b$ in $\mathbf{C}^l$ such that if $\mu \in V$ then*

$$(*) \qquad \sum p_i(y)\, e^{\nu_j(\mu)\cdot y} = 0$$

for all y if and only if it is 0 *for $y = y_1, \ldots, y_b$.*

To prove this lemma one has only to observe that the analytic subset of U defined by the equations $(*)$, $y \in \mathbf{C}^l$, is defined in a neighbourhood of 0 by a finite number of them.

We may therefore regard E as a function on U with values in the space of linear transformations from the space I, which is finite-dimensional, to the space $\mathbf{C}^{mn}$. One knows from the theory of Eisenstein series that E_μ is injective for μ in an open subset of U. Then to complete the proof of the proposition, we need only verify the following lemma.

LEMMA 9. *Suppose E is a holomorphic function in U, a neighbourhood of* 0 *on $\mathbf{C}^l$, with values in* Hom(I, J), *where I and J are finite-dimensional spaces, and suppose that E_μ is injective on an open subset of U. Then there is an integer r such that if φ_μ is analytic near $\mu = 0$ and the Taylor series of $E_\mu\varphi_\mu$ vanishes to order r then $\varphi_0 = 0$.*

Projecting to a quotient of J, we may assume that $\dim I = \dim J$ and even that $I = J$. Let the first nonzero term of the power-series expansion of $\det E_\mu$ have degree s. Then we will show that r can be taken to be $s + 1$. It is enough to verify this for $l = 1$, for we can restrict to a line on which the leading term of E_μ still has degree s. But then multiplying E fore and aft by nonsingular matrices we may suppose it is diagonal with entries z^α, $0 \leqq \alpha \leqq s$. Then the assertion is obvious.

In conclusion I would like to thank P. Deligne, who drew my attention to a blunder in the first version of the paper.

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MULTIPLICITY ONE THEOREMS

I. PIATETSKI-SHAPIRO*

1. Let $G = \mathbf{GL}_n$ over a global field k. We shall discuss the following two results.

(1) MULTIPLICITY ONE THEOREM. *Let π be an irreducible smooth admissible representation of $G(\boldsymbol{A})$. Then the multiplicity of π in the space of cusp forms is equal to one or zero.*

(2) Recall that any irreducible admissible smooth representation π can be written $\pi = \bigotimes_p \pi_p$, where each π_p is an irreducible admissible smooth representation of the local group G_p.

STRONG MULTIPLICITY ONE THEOREM. *Let $\pi_1 = \bigotimes_p \pi_{1,p}$ and $\pi_2 = \bigotimes_p \pi_{2,p}$ be two irreducible representations; suppose $\pi_{1,p} \cong \pi_{2,p}$ for every $p \notin S$, where S is a finite set, which in case $n > 2$ is assumed to contain only finite places. Then $\pi_{1,p} \cong \pi_{2,p}$ for all p. (Hence $\pi_1 = \pi_2$.)*

We begin by sketching the proof of the first Theorem (1). The basic tool is the Whittaker model. We introduce this first in the case k a local field, and (π, V) an irreducible smooth representation. In the case of k archimedean, we mean by "smooth representation" the representation of G on the space V of C^∞-vectors in some Hilbert space H on which G acts unitarily; for k nonarchimedean, this notion was introduced in Cartier's lectures. Let ψ be an additive character of k. Let

$$X = \begin{pmatrix} 1 & & * \\ & 1 & \\ 0 & & 1 \end{pmatrix}$$

be the standard maximal unipotent subgroup of G. Then a Whittaker model $W(\pi, \psi)$ for (π, V) is the image of V under an element of $\mathrm{Hom}_G(V, \mathrm{Ind}_X^G(\psi))$ where $\psi(x) = \psi(x_1 + \cdots + x_{n-1})$ if

$$x = \begin{pmatrix} 1 & x_1 & & * \\ & & x_2 & \\ & & & x_{n-1} \\ 0 & & & 1 \end{pmatrix}.$$

More explicitly, it is given by a set of smooth functions $\{W_v : G \to \boldsymbol{C}, v \in V\}$ for which

(i) $W_v(xg) = \psi(x) W_v(g)$, for all $x \in X$, $g \in G$.

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(ii) $W_{\pi(h)v}(g) = W_v(gh)$, for all $g, h \in G$.

We have the following important result due to Gelfand-Kazhdan for the case k nonarchimedean, and Shalika for general local fields ([**1**], [**2**]).

UNIQUENESS THEOREM. *For each irreducible admissible smooth representation* (π, V), *there exists at most one* $W(\pi, \psi)$ (*for fixed* ψ).

For k archimedean we assume that (π, V) is a unitarizable representation and $V = \{x \in H \,|\, (\mathscr{D}x, \mathscr{D}x) < \infty \;\forall\, \mathscr{D} \in \text{enveloping algebra}\}$. Here H means the completion of V with respect to the inner product (x, x). We assume also that $W_v(1)$ is a continuous linear functional on V with respect to the topology defined by seminorms $(\mathscr{D}x, \mathscr{D}x)$, $\mathscr{D} \in$ enveloping algebra.

Returning to the global case, we point out that the preceding discussion easily implies uniqueness of global Whittaker models (defined in the obvious way).

2. Global Fourier analysis. Let (π, V) be admissible irreducible cuspidal as before, $\varphi \in V$. Then we can define

$$W_\varphi(g) = \int_{X_k \backslash X_A} \varphi\,(xg)\psi^{-1}(x)\,dx.$$

Global Fourier analysis says that this "Fourier transform" defines a cusp-form uniquely. In the classical setting this is due to Hecke; for $n = 2$ it is proved in Jacquet-Langlands [**3**]; for $n > 2$ it is due independently to Piatetski-Shapiro [**4**] and Shalika [**2**]. The proof is motivated by a corresponding result over a finite field due to S. I. Gelfand [**5**]

It is now easy to see that these results imply Theorem (1), since

$$\dim \operatorname{Hom}_G(V, W(\pi, \psi)) = 1 \geqq \dim \operatorname{Hom}_G(V, L_0^2).$$

We now turn to the proof of the strong multiplicity one theorem. First we discuss the case $n = 2$; we need the following

SMALL LEMMA. (1) *Assume k local,* (π_1, V_1), (π_2, V_2) *two irreducible admissible representations with Whittaker models. Then there exist* $v_1 \in V_1$, $v_2 \in V_2$ *such that*

$$W_{v_1}\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} = W_{v_2}\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \quad (W_{v_i} \in W(\pi_i, \lambda)).$$

(2) *If* $k = \boldsymbol{R}$ *or* $\boldsymbol{C}$ *we assume that* (π_1, H_1) *and* (π_2, H_2) *are irreducible infinite-dimensional unitary representations. Denote by* V_1 (V_2) *the set of all smooth vectors in* H_1 (H_2). *Then there exist* $v_1 \in V_1$, $v_2 \in V_2$ *such that*

$$W_{v_1}\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} = W_{v_2}\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}, \qquad W_{v_i} \in W(\pi_i, \psi).$$

PROOF. For k a local nonarchimedean field it is known that V contains all Schwartz-Bruhat functions with compact support in k^*. Hence we have what we want.

Now let $k = \boldsymbol{R}$ or $\boldsymbol{C}$. The Kirillov theorem (see [**8**, p. 221]) says that each irreducible infinite-dimensional unitary representation of GL(2, k) remains irreducible after restriction on the subgroup $\{(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix})\} = P$ and hence as a representation of P is isomorphic to the standard representation of P. Hence, if $\varphi(x)$ is a C^∞-function with compact support then there exist $v_1 \in V_1$, $v_2 \in V_2$ such that

$$W_{v_i}\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} = \varphi(x).$$

REMARK. Assume that for a unitary representation with a Whittaker model the inner product can be written as an integral similar to the case for $n = 2$. Using this result we can prove the "small lemma" for any n as we did for $n = 2$. This implies the strong multiplicity one theorem for any n.

Next we give the formula for recovering φ from its Whittaker model due to Jacquet-Langlands, for $\mathbf{GL}(2, A)$:

$$(*) \qquad \varphi(g) = \sum_{\lambda \in k^*} W_\varphi\left(\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} g\right).$$

Now suppose π_1, π_2 satisfy the hypotheses of the theorem. To prove the assertion, it is enough to produce a nonzero $\varphi \in V_1 \cap V_2$, since then the irreducibility of (π_i, V_i) implies equality. Further, since $B_k \backslash B_A$ is dense in $G_k \backslash G_A$, it is enough to produce two functions (nonzero) $\varphi_i \in V_i$ which are equal on B_A (as usual B is the group of upper triangular matrices).

From the properties of Whittaker models and $(*)$, it is enough to produce Whittaker functions W_1, W_2 such that $W_1(\left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right)) = W_2(\left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right))$, $x \in A^*$. One can suppose such W_i are of the form $\prod_p W_i^p$ and then it suffices to construct the appropriate W_i at a finite number of places (by assumption). But then one can use the small lemma. This type of argument was found independently here by Shalika and in Moscow.

For $n \geqq 3$, we need a similar small lemma (Gelfand-Kazhdan): Suppose k local, nonarchimedean, (π_i, V_i), $i = 1, 2$, irreducible admissible representations with Whittaker models. There exist $v_i \in V_i$ such that

$$W_{v_1}\begin{pmatrix} h & 0 \\ 0 & I \end{pmatrix} = W_{v_2}\begin{pmatrix} h & 0 \\ 0 & I \end{pmatrix}, \quad \text{all } h \in \mathbf{GL}(n-1).$$

One can then employ induction using arguments similar to the case $n = 2$, in order to prove the general case. It should be possible to prove this lemma also for k archimedean; then the restriction we made that S contains no infinite places could be removed.

Now suppose G is quasi-split and satisfies the *transitivity condition*:

$T(A)$ acts transitively on $\prod_{\alpha \text{ a simple root}} X_\alpha^*(A)$. Here T is a maximal k-torus in a Borel group, $X_\alpha^* = X_\alpha - \{I\}$ where X_α is the root group associated to the simple root α.

Define an automorphic cuspidal irreducible representation (π, V) to be *hypercuspidal* (degenerate cuspidal) if

$$W_\varphi(g) = \int_{X_k \backslash X_A} \varphi(xg)\psi^{-1}(x)\, dx = 0$$

for all $\varphi \in V$. Holomorphic cusp forms lifted from symmetric spaces which contain no copies of $H = \{\operatorname{Im} z > 0\}$ are of this type.

A cuspidal automorphic form will be called *generic* if it is orthogonal to all hypercuspidal automorphic forms (under the usual scalar product $\int_{CG_k \backslash G_A} \varphi\psi\, dg$).

Counterexamples to the Ramanujan conjecture given during this conference by Howe and the author are hypercuspidal forms [6]. The author does not wish to kill all belief in the Ramanujan conjecture; he conjectures it to be true for the generic cuspidal automorphic irreducible representation.

Now I shall sketch the proof of the multiplicity one theorem for generic cuspidal automorphic forms. First, the uniqueness theorem for local Whittaker models is true **[2]**. But of course now the Whittaker function does not define an arbitrary cusp form uniquely. Now it follows immediately from the definition that a generic cusp form is uniquely defined by its Whittaker function. This implies multiplicity one just as before. It can be proved that for each quasi-split reductive group there exist generic cusp forms. It also can be proved that for such groups there exists a unipotent subgroup U such that $\int_{U_k \backslash U_A} \varphi(ug)\, du$ can be expressed in terms of the Whittaker function. Since for any group except $\mathrm{GL}(n)$ there exist hypercuspidal forms, we cannot of course expect to be able to recover φ itself from its Whittaker function. Details concerning this will be given in a forthcoming publication of Novodvorskii and the author. (Notes prepared by B. Seifert and L. Morris.)

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UNIVERSITY OF TEL AVIV

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FORMS OF GL(2) FROM THE ANALYTIC POINT OF VIEW

STEPHEN GELBART* AND HERVÉ JACQUET

Introduction. Suppose G is a reductive group defined over an A-field F, ρ_{cusp} is the representation of $G(A)$ in the space of cusp forms, and φ is a well-behaved function on $G(A)$. Then φ defines a trace-class operator $\rho_{\text{cusp}}(\varphi)$, and the goal of the trace formula is to give an explicit formula for this trace. In most cases, one takes $\varphi = \prod_v \varphi_v$ and wants to compute tr $\rho_{\text{cusp}}(\varphi)$ in terms of *local invariant distributions.* Although it is desirable to have *only* invariant distributions, this is not necessary for all applications. In any case, contrary to common belief, the goal is *not* to express the trace in terms of characters of irreducible representations. Indeed the distribution $\varphi(e)$ always appears, and one is quite content to leave it as is.

In the anisotropic case, ρ_{cusp} is actually an induced representation. In this case a general technique—valid for all induced representations and all cocompact groups—gives the trace of $\rho_{\text{cusp}}(\varphi)$ in terms of local orbital integrals. In general, ρ_{cusp} is a subrepresentation of an induced one ρ.

Our purpose here is to describe the analytic theory of the trace formula for forms of GL(2). Since the trace of $\rho_{\text{cusp}}(\varphi)$ is always computed as the integral of the kernel which defines it, we need an explicit formula for this kernel. For GL(2) this turns out to be the kernel of $\rho(\varphi)$ minus the sum and integral of the product of two Eisenstein series. But since only the difference of these kernels is integrable—not one or the other—we need to use a truncation process. For this we closely follow J. Arthur's exposition ([**Ar 1**] and [**Ar 2**]). Any improvement in clarity over earlier references (such as [**DL**], [**JL**], or [**Ge**]) should be credited to him.

A large part of these notes—§§3 through 5—is devoted to the analytic continuation of Eisenstein series. In particular, nearly complete proofs are given for the analytic continuation, functional equation, and location of poles of the *constant term* of the Eisenstein series. This analysis enters precisely because we have to subtract off the continuous spectrum of $\rho(\varphi)$ before we can compute its trace.

§1 is included to show how easy things are when there is no continuous spectrum. §8 is included to give some idea of the power and range of applicability of the trace formula; here discussions with D. Flath on his thesis [**Fl**] were invaluable.

1. Division algebras. First we derive a statement of the trace formula for a division

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*Alfred P. Sloan Fellow.

algebra whose rank is the square of a prime. Then we give a simple, but in a sense typical, application.

A. *Some generalities.* Let F denote a number field and H a division algebra of degree p^2 over H. The multiplicative group G of H may be regarded as an F-group whose center Z is isomorphic to F^x. Set $\tilde{G} = G/Z$. If ω is a character of $F^x \backslash A^x$ denote by $L^2(\omega, G)$ the space of (classes of) functions f on $G(A)$ such that

$$f(\gamma z g) = \omega(z) f(g), \qquad \gamma \in G(F),\ z \in Z(A) \tag{1.1}$$

and

$$\int_{\tilde{G}(F)\backslash\tilde{G}(A)} |f(g)|^2\, dg < +\infty. \tag{1.2}$$

Let ρ_ω denote the natural representation of $G(A)$ in $L^2(\omega, G)$ given by right translation.

Suppose now that φ is a function on $G(A)$ satisfying

$$\varphi(zg) = \omega^{-1}(z)\varphi(g) \tag{1.3}$$

for z in $Z(A)$. Suppose also that φ is C^∞ and of compact support mod $Z(A)$. More precisely, let c_i, $1 \leqq i \leqq p^2$, be a basis of H over F. Then, for almost all v, the module generated over the ring of integers R_v of F_v is a maximal order O_v. In particular, $K_v = O_v^*$ is a maximal compact subgroup of H_v^x for almost all v. We assume that $\varphi(g) = \prod \varphi_v(g_v)$ where, for each v, φ_v is a C^∞-function of compact support satisfying the analogue of (1.3); moreover, for almost all v,

$$\begin{aligned} \varphi_v(g_v) &= \omega_v^{-1}(z_v) \quad \text{if } g_v = k_v z_v,\ k_v \in K_v,\ z_v \in Z_v, \\ &= 0 \qquad\quad \text{otherwise.} \end{aligned}$$

Then the operator

$$\rho_\omega(\varphi) = \int_{G(A)} \varphi(g)\rho_\omega(g)\, dg \tag{1.4}$$

is an integral operator in $L^2(\tilde{G}(F)\backslash\tilde{G}(A))$ with kernel

$$K(x, y) = \sum_{\gamma \in \tilde{G}(F)} \varphi(x^{-1}\gamma y). \tag{1.5}$$

When x and y lie in fixed compact sets the sum in (1.5) is actually finite. Therefore $K(x, y)$ is continuous.

THEOREM (1.6). *The operator $\rho_\omega(\varphi)$ is of trace class.*

PROOF. Since $\tilde{G}(F)\backslash\tilde{G}(A)$ is compact, and $K(x, y)$ is continuous, $\rho_\omega(\varphi)$ is at least of Hilbert-Schmidt class. But φ can be written as $\sum \varphi_i^* f_i$ with f_i, φ_i satisfying the same conditions as φ (except at infinity where f_i may only be of class C^m with m large; cf. [**DL**, p. 199]). Thus $\rho_\omega(\varphi)$ is a sum of *products* of Hilbert-Schmidt operators, and hence is of trace class.

COROLLARY (1.7). *The representation ρ_ω decomposes as a (discrete) direct sum of irreducible representations each occurring with finite multiplicity.*

PROOF. According to (1.6) the operator $\rho_\omega(\varphi)$ is compact for well-behaved φ.

Thus the corollary follows from a basic result of functional analysis (cf. **[GGPS]** or **[La 1]**.

If we denote by $m(\pi)$ the multiplicity (possibly zero) with which the irreducible representation π of $G(A)$ occurs in ρ_ω then we have

$$\text{(1.8)} \qquad \operatorname{tr} \rho_\omega(\varphi) = \sum m(\pi) \operatorname{tr} \pi(\varphi).$$

Note that each component of ρ_ω perforce admits ω as central character, i.e., $\pi(z) = \omega(z)I$, $z \in Z(A)$. Thus the sum in (1.8) need only be extended over representations of $G(A)$ with central character ω. On the other hand, we can also compute the trace of $\rho_\omega(\varphi)$ in terms of the kernel K, viz.,

$$\text{(1.9)} \qquad \operatorname{tr} \rho_\omega(\varphi) = \int_{\tilde{G}(F)\backslash\tilde{G}(A)} K(x, x)\, dx.$$

B. *The trace formula.* From now on assume that p is a prime. Then any element of $G(F) - Z(F)$ is regular. On the other hand, if $\gamma = e$ (in $\tilde{G}(F)$) then $\varphi(x^{-1}\gamma x) = \varphi(e)$. Thus we may rewrite the right-hand side of (1.9) as

$$\text{(1.10)} \qquad \int K(x, x)\, dx = \operatorname{vol}(\tilde{G}(F)\backslash\tilde{G}(A))\varphi(e) + \int \sum_{\gamma \neq e} \varphi(x^{-1}\gamma x)\, dx.$$

But every element ξ of $G(F) - Z(F)$ generates an extension L of F of degree p in H. Moreover, if ξ' is conjugate to ξ in $G(F)$, then the extension L' generated by ξ' is F-isomorphic to L, and there is an F-isomorphism of L' onto L taking ξ' to ξ. Thus, if we let X be a set of representatives for the isomorphism classes of extensions of degree p of F which imbed in H, any element $\gamma \neq e$ of $\tilde{G}(F)$ can be expressed in the form

$$(*) \qquad \gamma = \eta^{-1}\xi\eta$$

where ξ is in $L^\times/F^\times - \{e\}$ for some L in X and η belongs to a set of representatives for

$$(F^\times\backslash L^\times)\backslash\tilde{G}(F) \simeq L^\times\backslash G(F).$$

Note that the extension $L \in X$ is uniquely determined by γ and, if g_L is the number of F-automorphisms of L, γ admits g_L decompositions like $(*)$. Thus

$$K(x, x) = \varphi(e) + \sum_{L \in X} (g_L)^{-1} \sum_{\xi} \sum_{\eta} \varphi(x^{-1}\eta^{-1}\xi\eta x), \qquad \xi \in L^\times/F^\times - \{e\},\ \eta \in L^\times\backslash G(F)$$

and

$$\int K(x, x)\, dx = \operatorname{vol}(\tilde{G}(F)\backslash\tilde{G}(A))\varphi(e) + \sum_{L \in X} (g_L)^{-1} \sum_{\xi} \int_{L^\times Z(A)\backslash G(A)} \varphi(x^{-1}\xi x)\, dx.$$

In the last integral, the integrand depends only on the class of x mod $L^\times(A)$. Thus we also have

$$\text{(1.11)} \qquad \begin{aligned} \int K(x, x)\, dx &= \operatorname{vol}(\tilde{G}(F)\backslash\tilde{G}(A))\varphi(e) + \sum_{L \in X} (g_L)^{-1} \operatorname{vol}(F^\times(A)L^\times\backslash L^\times(A)) \\ &\qquad \sum_{\xi \in (L^\times - F^\times)/F^\times} \int_{L^\times(A)\backslash\tilde{G}(A)} \varphi(x^{-1}\xi x)\, dx = \operatorname{tr} \rho_\omega(\varphi). \end{aligned}$$

In this formula the measures on $\tilde{G}(A)$ and $F^x(A)\backslash L^x(A)$ are chosen arbitrarily and $L^x(A)\backslash\tilde{G}(A)$ is given the quotient measure.

Suppose now that the measure on $L^x(A)\backslash\tilde{G}(A)$ is a product measure; given ξ, for almost all v, ξ is in K_v and $\varphi_v(x_v^{-1}\xi x_v) = 0$ unless $x_v \in K_v L_v^x$. Thus we have

$$\int \varphi(x^{-1}\xi x)\, dx = \prod_v \int_{L_v^x\backslash G_v} \varphi_v(x_v^{-1}\xi x_v)\, dx_v$$

almost all factors being equal to $\mathrm{vol}(L_v^x\backslash L_v^x K_v)$.

Thus we have expressed $\mathrm{tr}\,\rho_\omega(\varphi)$ in terms of the local invariant distributions $\varphi_v \mapsto \varphi_v(e)$ and $\varphi_v \mapsto \int_{L_v^x\backslash G_v} \varphi_v(x_v^{-1}\xi x_v)\, dx_v$.

C. *An application.* Let H' denote another division algebra of degree p^2 and suppose H' and H both fail to split at the same set of places S. For $v \notin S$ there is an isomorphism $H_v \to H_v'$ which is uniquely determined up to inner automorphism. For almost all v, we may suppose that this isomorphism takes O_v to a similarly defined maximal order O_v' (and hence $K_v = O_v^x$ to $K_v' = (O_v')^x$). The resulting isomorphisms $G_v \to G_v'$ then give rise to an isomorphism of the restricted product groups $G^S = \prod_{v\notin S} G_v$ and $G'^S = \prod_{v\notin S} G_v'$ which is again determined up to inner automorphism.

Let V denote the space of functions on $L^2(\omega, G)$ which are invariant under $G_S = \prod_{v\in S} G_v$. Since G^S and G_S commute, V is invariant for the action of G^S. Thus we have a representation σ of G^S on V and (similarly) a representation σ' of G'^S on V'. Via the isomorphism $G^S \to G'^S$ we may transport σ' to a representation of G^S which we again denote by σ'. (Its class does not depend on the choice of isomorphism.)

THEOREM (1.12). *Suppose $\omega_v = 1$ for all $v \in S$. Then the representations σ and σ' are equivalent.*

PROOF. Let $\theta = \prod_{v\notin S} \theta_v$ be any function on G^S satisfying conditions analogous to those satisfied by φ in §1.A. To prove our theorem it will suffice to show that

$$\mathrm{tr}\,\sigma(\theta) = \mathrm{tr}\,\sigma'(\theta). \tag{1.13}$$

Indeed a basic result in harmonic analysis asserts that σ is a subrepresentation of σ' if and only if

$$\mathrm{tr}\,\sigma(f)\sigma(f)^* \leqq \mathrm{tr}\,\sigma'(f)\sigma'(f)^* \tag{1.14}$$

for sufficiently many "nice" f. Cf. Lemma 16.1.1 of [JL]; it will suffice to apply (1.13) with $\theta = \theta_1 * \theta_1^*$ and $\theta_1^*(g) = \bar{\theta}_1(g^{-1})$.

To prove (1.13), extend θ to a function φ on $G(A)$ by setting $\varphi(g) = \prod_{v\notin S} \theta(g_v)$ and extend θ similarly to φ' by identifying G'^S with G^S. Now take the volume of $\tilde{G}_v$ (resp. $\tilde{G}_v'$) to be one for each $v \in S$ and the measure on $\tilde{G}(A)$ (resp. $\tilde{G}'(A)$) to be a product measure. Furthermore, assume that the isomorphism $G_v \to G_v'$ takes the Haar measure of G_v to the Haar measure of G_v'. Then we find that

$$\mathrm{tr}\,\rho_\omega(\varphi) = \mathrm{tr}\,\sigma(\theta) \quad (\text{resp. } \mathrm{tr}\,\rho_\omega'(\varphi') = \mathrm{tr}\,\sigma'(\theta)). \tag{1.15}$$

Thus (1.13) holds if and only if

$$\mathrm{tr}\,\rho_\omega(\varphi) = \mathrm{tr}\,\rho_\omega'(\varphi'). \tag{1.16}$$

To prove (1.16) we apply the trace formula (1.11) as follows.

Assume that the measure on $F^x(A)\backslash L^x(A)$ is a product measure so that

$$\int_{L^x(A)\backslash \tilde{G}(A)} \varphi(x^{-1}\xi x)\,dx = \Big(\prod_{v\in S}\int_{L_v^x\backslash \tilde{G}_v} \theta_v(x^{-1}\xi x)\,dx_v\Big)\prod_{v\in S}\operatorname{vol}(L_v^x\backslash \tilde{G}_v).$$

A similar identity holds for $\tilde{G}'$, and the extensions of F which imbed in H or H' are the same. Thus (1.11) implies that

$$\text{(1.17)} \qquad \operatorname{tr}\rho_\omega(\varphi) - \operatorname{tr}\rho_\omega(\varphi') = (c - c')\theta(e).$$

Here $c = \operatorname{vol}(\tilde{G}(F)\backslash\tilde{G}(A))$ and c' is defined similarly. To prove the theorem it remains to show $c = c'$.

Suppose $c > c'$. If $\theta = \theta_1 * \theta_1^*$ with $\theta_1^*(g) = \bar{\theta}_1(g^{-1})$, then the right side of (1.17) is strictly positive. So by (1.15)—and the result described by (1.14)—σ' must be a subrepresentation of σ, and the quotient representation $\pi = \sigma/\sigma'$ must satisfy the identity

$$\operatorname{tr}(\pi(\theta_1)\pi(\theta_1)^*) = (c - c')\,\|\theta_1\|_2$$

for all nice θ_1.

This implies that the regular representation of G^S is quasi-equivalent to π and hence that the regular representation of G^S decomposes discretely. Since the same is then also true of the regular representation of each G_v, $v \notin S$, we obtain an obvious contradiction. The assumption $c < c'$ yields a similar contradiction and the theorem is proved.

REMARK (1.18). Suitably modified, the proof above shows that $\tilde{G}$ and $\tilde{G}'$ have the same Tamagawa number. The restriction on ω is not really necessary.

2. Cusp forms on GL(2). Henceforth G will denote the group GL(2). The center Z of G is

$$Z = \left\{\begin{pmatrix} a & 0 \\ 0 & a\end{pmatrix}\right\}.$$

Again $Z \cong F^x$ and we set $\tilde{G} = G/Z$. We also introduce the subgroups

$$P = \left\{\begin{pmatrix} * & * \\ 0 & *\end{pmatrix}\right\}, \quad A = \left\{\begin{pmatrix} * & 0 \\ 0 & *\end{pmatrix}\right\}, \quad N = \left\{\begin{pmatrix} 1 & * \\ 0 & 1\end{pmatrix}\right\}.$$

If φ is a function on $G(F)\backslash G(A)$ we say that φ is cuspidal if

$$\int_{N(F)\backslash N(A)} \varphi(ng)\,dn \equiv 0, \qquad g \in G(A).$$

As in §1, we introduce a space $L^2(\omega, G)$ and a representation ρ_ω. We denote by $L_0^2(\omega, G)$ the subspace of cuspidal elements in $L^2(\omega, G)$.

THEOREM (2.1). *Let $\rho_{\omega,0}$ denote the restriction of ρ_ω to the invariant subspace $L_0^2(\omega, G)$. Let φ denote a C^∞-function which satisfies* (1.3) *and is of compact support* mod $Z(A)$. *Then the operator $\rho_{\omega,0}(\varphi)$ is Hilbert-Schmidt.*

SKETCH OF PROOF. For each $c > 0$, recall that a Siegel domain $\mathcal{S}$ in $G(A)$ is a set of points of the form

$$g = \begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix}\begin{pmatrix} tab & 0 \\ 0 & a\end{pmatrix}k$$

where x is in a compact subset of A, t is an idele whose finite components are trivial and whose infinite components equal some fixed number $u > c > 0$, a is arbitrary in A^x, b lies in a compact subset of A^x, and k is in the standard maximal compact subgroup K of $G(A)$. If $\mathscr{S}$ is sufficiently large then $G(A) = G(F)\mathscr{S}$. For a proof of this fact see **[Go 3]**.

By abuse of language, if $\mathscr{S}$ is a Siegel domain in $G(A)$, we call its image in $N(F)\backslash G(A)$ a Siegel domain in $N(F)\backslash G(A)$. If $\mathscr{S}$ is such a domain, we denote by $L^2(\omega, \mathscr{S})$ the space of functions on $\mathscr{S}$ such that $f(zg) = \omega(z)f(g)$ and $\int_{N(F)Z(A)\backslash\mathscr{S}} |f(g)|^2\, dg < +\infty$. Clearly $L^2(\omega, G)$ can be identified with a closed subspace V of $L^2(\omega, \mathscr{S})$ *if* $\mathscr{S}$ is large enough. In this case there is a bounded operator with bounded inverse A from $L^2(\omega, G)$ onto V.

If f is in $L^2(\omega, G)$ and φ is as in Theorem (2.1), then

$$\rho_\omega(\varphi)f(x) = \int_{N(F)\backslash\bar{G}(A)} H(x, y)f(y)\, dy$$

with

$$H(x, y) = \sum_{\xi\in N(F)} \varphi(x^{-1}\xi y).$$

However, if f is actually cuspidal, we also have

$$\rho_\omega(\varphi)f(x) = \int \tilde{H}(x, y)f(y)\, dy \tag{2.2}$$

with

$$\tilde{H}(x, y) = \sum_{\xi\in N(F)} \varphi(x^{-1}\xi y) - \int_{N(A)} \varphi(x^{-1}ny)\, dn. \tag{2.3}$$

Indeed f cuspidal implies

$$\int_{N(F)\backslash\bar{G}(A)} f(y) \int_{N(A)} \varphi(x^{-1}ny)\, dn\, dy = 0.$$

To prove the theorem we need to estimate $\tilde{H}(x, y)$.

If x is in a fixed Siegel domain $\mathscr{S}$ (of $N(F)\backslash G(A)$) it is easy to see that $\tilde{H}(x, y) = 0$ unless y is in another such domain $\mathscr{S}'$. Then using the fact that the term subtracted in (2.3) is precisely the Fourier transform of $\varphi(x^{-1}\left(\begin{smallmatrix}1 & *\\ 0 & 1\end{smallmatrix}\right)y)$ at 0 we see that

$$\iint_{(N(F)Z(A)\backslash\mathscr{S})\,(N(F)Z(A)\backslash\mathscr{S}')} |\tilde{H}(x, y)|^2\, dx\, dy < +\infty,$$

i.e., $\tilde{H}$ defines a Hilbert-Schmidt operator B from $L^2(\omega, \mathscr{S}')$ to $L^2(\omega, \mathscr{S})$. For a detailed discussion of this type of argument see **[Go 1]** or **[La 1]**.

Now enlarge $\mathscr{S}'$ (if necessary) so that $G(A) = G(F)\mathscr{S}'$. Then $\rho_{\omega,0}(\varphi)$ can be written as the composition of the injection $L_0^2(\omega, G) \to L^2(\omega, \mathscr{S}')$, the operator B, the projection of $L^2(\omega, \mathscr{S})$ onto V, and the operator A^{-1}. This proves $\rho_{\omega,0}(\varphi)$ itself is Hilbert-Schmidt.

COROLLARY (2.4). *$\rho_{\omega,0}(\varphi)$ is of trace class.*

PROOF. The conclusion of (2.1) is valid if φ is highly (as opposed to infinitely) differentiable at infinity. Thus one argues as in §1.A (cf. [**DL**, p.199]).

COROLLARY (2.5). *The representation $\rho_{\omega,0}$ decomposes discretely with finite multiplicities.*

3. *P*-series. Suppose f is a function on $G(A)$ satisfying

$$f(n\gamma zg) = \omega(z)f(g) \tag{3.1}$$

with $n \in N(A)$, $\gamma \in P(F)$, and $z \in Z(A)$. Then we call the series

$$F(G) = \sum_{P(F)\backslash G(F)} f(\gamma g) \tag{3.2}$$

a *P-series*. To study the convergence of such a series we need a well-known lemma. Let $H(g)$ be the function on $G(A)$ defined by

$$H(g) = \left|\frac{a}{b}\right| \quad \text{if } g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}k.$$

LEMMA (3.3) (CF. [**JL**, P.197]). *Fix a Siegel domain $\mathfrak{S}$ and a positive number c'. Then the set of γ in $G(F)$ such that $H(\gamma g) \geqq c'$ for some $g \in \mathfrak{S}$ is finite modulo $P(F)$.*

From this lemma we see that if the support of f is contained in a set $\{(g: H(g) \geqq c'\}$—in particular if it is compact mod $N(A)Z(A)P(F)$—then the series (3.2) is finite. Moreover, in the latter case, the function F has compact support mod $G(F)Z(A)$.

Our goal in this section is to analyze the orthocomplement of $L_0^2(\omega, G)$ in terms of these P-series.

A. *The scalar product of two P-series.* First we compute the scalar product of a P-series with a function ϕ satisfying (1.1):

$$\begin{aligned}
(\phi, F) &= \int_{\bar{G}(F)\backslash\bar{G}(A)} \phi(g)\bar{F}(g)\,dg \\
&= \int_{\bar{G}(F)\backslash\bar{G}(A)} \phi(g) \sum_{P(F)\backslash G(F)} \bar{f}(\gamma g)\,dg \\
&= \int \sum \phi(\gamma g)\bar{f}(\gamma g)\,dg = \int_{P(F)Z(A)\backslash G(A)} \phi(g)\bar{f}(g)\,dg \\
&= \int_{P(F)N(A)Z(A)\backslash G(A)} dg \int_{N(F)\backslash N(A)} \phi(ng)\bar{f}(g)\,dg,
\end{aligned}$$

i.e.,

$$(\phi, F) = \int_{P(F)N(A)Z(A)\backslash G(A)} \phi_N(g)\bar{f}(g)\,dg \tag{3.4}$$

where ϕ_N is the *constant term* of ϕ:

$$\phi_N(g) = \int_{N(F)\backslash N(A)} \phi(ng)\,dn. \tag{3.5}$$

From (3.4) it follows that if ϕ is cuspidal then $(\phi, F) = 0$ for any P-series F.

Conversely, if $\phi \in L^2(\omega, G)$ is orthogonal to all P-series (with f of compact support mod $N(A)P(F)Z(A)$), then $\phi_N = 0$ and ϕ is cuspidal. Thus the P-series (with f of compact support mod ...) span a dense subspace of the orthocomplement of $L_0^2(\omega, G)$.

Now we compute the scalar product of two P-series. According to (3.4) we need first to compute the constant term of F. But by Bruhat's decomposition for GL(2), $F(g) = f(g) + \sum_{\gamma \in N(F)} f(w\gamma g)$ with $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Thus

$$F_N(g) = \int_{N(F)\backslash N(A)} f(ng)\, dn + \int_{N(F)\backslash N(A)} dn \sum f(w\gamma ng).$$

Normalizing the Haar measure on $N(A)$ by the condition $\mathrm{vol}(N(F)\backslash N(A)) = 1$ we see that

$$F_N(g) = f(g) + f'(g) \tag{3.6}$$

where

$$f'(g) = \int_{N(A)} f(wng)\, dn. \tag{3.7}$$

Thus

$$(F_1, F_2) = \int f_1 \cdot \bar{f}_2(g)\, dg + \int f_1' \cdot \bar{f}_2(g)\, dg \tag{3.8}$$

where both integrals extend over the space $P(F)N(A)Z(A)\backslash G(A)$.

To further analyze formula (3.8) we need to carefully investigate the map $f \mapsto f'$. Note that f' still satisfies (3.1). Note also that all the computations above are merely formal unless certain assumptions on f_1 and f_2 are made to insure convergence. For instance, if both f_1 and f_2 have compact support (mod ...) then these computations *are* valid. Under these same conditions, however, the support of f_1' need no longer (in general) be compact.

Another formal relation which is easy to prove is this:

$$\int f_1'(g)\bar{f}_2(g)\, dg = \int f_1(g)\bar{f}_2'(g)\, dg. \tag{3.9}$$

Indeed the left- (resp. right-) hand side of (3.9) may be written as

$$\int_{P(F)Z(A)\backslash G(A)} f_1(wg)\bar{f}_2(g)\, dg$$

(resp. $\int_{P(F)Z(A)\backslash G(A)} f_1(g)\bar{f}_2(wg)\, dg$) and these integrals are equal since w normalizes $P(F)Z(A)$.

B. *Induced representations and intertwining operators.* To investigate the map $f \mapsto f'$ (and the scalar product (3.8)) we need to perform a Mellin transform on the group $Z(A)A(F)\backslash A(A)$. Right now it will suffice to deal with a subgroup of this group. Thus we let F_∞^+ denote the group of ideles whose finite components all equal 1 and whose infinite components all equal some positive number u (independent of the infinite place). By $F^0(A)$ we denote the ideles of norm 1 and by A^0 (resp. A_∞^+) the group of diagonal matrices with entries in $F^0(A)$ (resp. F_∞^+). For convenience *we assume that ω is trivial on F_∞^+*. We also normalize Haar measures as follows.

The Haar measure on F_∞^+ is obtained by transporting the usual measure on $\boldsymbol{R}_+^\times$ through the map $t \mapsto |t|$. Thus

$$\frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} ds \int_{F_\infty^+} f(t)|t|^{-s}\, d^\times t = f(e).$$

Recall that the measure on $N(A)$ is chosen so that $\mathrm{vol}(N(F)\backslash N(A)) = 1$. On $A^\times$ we select any Haar measure and give $F^\times\backslash A^\times$ the quotient measure. Then we use the isomorphism $F^\times\backslash A^\times \cong F_\infty^+ \times F^\times\backslash F^0(A)$ to get a Haar measure on $F^\times\backslash F^0(A)$ (which in general will not have volume 1). We also use the isomorphism $\left(\begin{smallmatrix} a & 0 \\ 0 & 1\end{smallmatrix}\right) \mapsto a$ to get a measure on $Z(A)\backslash A(A)$. Finally we select measures on $\tilde{G}(A)$ and K such that

$$\int_{\tilde{G}(A)} f(g)\, dg = \int_K \int_{A^\times} \int_{N(A)} f\left[\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix} k\right] dx |a|^{-1}\, d^\times a\, dk. \tag{3.10}$$

Note that vol (K) is not necessarily 1.

Now for each complex number s we introduce a Hilbert space $\mathbf{H}(s)$ of (classes of) functions φ on $G(A)$ such that

$$\varphi\left[\begin{pmatrix} \alpha a u & x \\ 0 & \beta a v\end{pmatrix} g\right] = \omega(a)\left|\frac{u}{v}\right|^{s+1/2}\varphi(g) \tag{3.11}$$

for $\alpha, \beta \in F^\times$, $a \in A^\times$, $u, v \in F_\infty^+$, and $x \in A$. Such functions are completely determined by their restriction to the set of matrices of the form

$$g = \begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix} k, \qquad a \in F^0(A),\ k \in K. \tag{3.12}$$

For φ to belong to $\mathbf{H}(s)$ we require that

$$\int_K \int_{F^\times\backslash F^0(A)} |\varphi|^2 \left[\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix} k\right] da\ dk < +\infty. \tag{3.13}$$

The restriction of φ to matrices of the form (3.12) satisfies

$$\varphi\left[\begin{pmatrix} \alpha a b & 0 \\ 0 & 1\end{pmatrix} k\right] = \varphi\left[\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} b & 0 \\ 0 & 1\end{pmatrix} k\right] \tag{3.14}$$

each time $\alpha \in F^\times$ and $\left(\begin{smallmatrix} b & 0 \\ 0 & 1\end{smallmatrix}\right) \in A(A) \cap K$. Moreover, the group $G(A)$ operates on $\mathbf{H}(s)$ by right translation and the resulting representation is denoted π_s. This representation is continuous but not always unitary. It is, however, unitary when s is purely imaginary.

We may think of the collection $\mathbf{H}(s)$ as a holomorphic fibre bundle of base $\boldsymbol{C}$. The sections over an open set U of $\boldsymbol{C}$ are the functions $\varphi(g, s)$ on $G(A) \times U$ satisfying

$$\varphi\left[\begin{pmatrix} \alpha a u & x \\ 0 & \beta a v\end{pmatrix} g, s\right] = \omega(a)\left|\frac{u}{v}\right|^{s+1/2}\varphi[g, s].$$

Of course this bundle is trivial since every φ in $\mathbf{H}(s)$ is uniquely determined by its restriction to the set of matrices of the form (3.12). Thus we may think of $\mathbf{H}(s)$ as the Hilbert space of functions (*independent of s*) satisfying (3.13) and (3.14).

Put another way, we may set $\mathbf{H} = \mathbf{H}(0)$. Then every element φ of $\mathbf{H}$ defines a section of our bundle over C, namely $(g, s) \mapsto \varphi(g)H(g)^s$.

With this understanding let f be a function satisfying (3.1), say of compact support mod $N(A)Z(A)P(F)$. Then define the *Mellin transform* of f by

$$\hat{f}(g, s) = \int_{F_\infty^+} f\left[\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} g\right] |t|^{-s-1/2} d^\times t.$$

Since the integrand has compact support, $\hat{f}$ clearly defines a section of our fibre bundle. The Haar measure is chosen so that

$$f(g) = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} \hat{f}(g, s)\, ds$$

with x any real number.

Now it is easy to see that the first term of the inner product formula (3.8) is

$$\int_K \int_{(F^\times \backslash F^0(A))} \int_{F_\infty^+} f_1 \cdot \bar{f}_2\left[\begin{pmatrix} at & 0 \\ 0 & 1 \end{pmatrix} k\right] |t|^{-1}\, d^\times t\, da\, dk$$

$$= \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} \int_{F^\times \backslash F^0} \int_K \hat{f}_1\left[\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k, s\right] \bar{\hat{f}}_2\left[\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k, -\bar{s}\right] da\, dk\, ds$$

$$= \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} (\hat{f}_1(s), \hat{f}_2(-\bar{s}))\, ds.$$

Here $\hat{f}_1(s)$ denotes the value of the function $\hat{f}_1(g, s)$ at s and the scalar product $(\hat{f}_1(s), \hat{f}_2(-\bar{s}))$ is taken by identifying all fibres with $\mathbf{H}$. Alternatively, observe that if φ_1 is in $\mathbf{H}(s)$ and φ_2 is in $\mathbf{H}(-\bar{s})$ then $\varphi_1\bar{\varphi}_2$ transforms on the left according to the modular function of the group $N(A)Z(A)A_\infty^+ A(F)$. Thus if we set

$$(\varphi_1, \varphi_2) = \int_{F^\times \backslash F^0(A)} \int_K \varphi_1 \cdot \bar{\varphi}_2\left[\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k\right] da\, dk \tag{3.15}$$

we obtain a nondegenerate sesquilinear pairing between $\mathbf{H}(s)$ and $\mathbf{H}(-\bar{s})$ such that

$$(\pi_s(g)\varphi_1, \pi_{-\bar{s}}(g)\varphi_2) = (\varphi_1, \varphi_2). \tag{3.16}$$

This is the pairing which appears in our formula. (For more details see **[Go 2**, pp. 1.24–1.26].)

Similarly we find that the second term of (3.8) is

$$(2\pi i)^{-1} \int_{x-i\infty}^{x+i\infty} (\hat{f}'_1(-s), \hat{f}_2(\bar{s}))\, ds.$$

Here we must integrate on a line where $\hat{f}'_1(-s)$ is given by a convergent integral, i.e., we must have $x > \frac{1}{2}$. It remains then to compute $\hat{f}'(-s)$ in terms of $\hat{f}(s)$.

For $\mathrm{Re}(s) > \frac{1}{2}$ we find that

$$\hat{f}'(-s) = \int f'\left[\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} g\right] |t|^{s-1/2}\, d^\times t$$

$$= \int |t|^{s-1/2}\, d^\times t \int f\left[w \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & t^{-1}x \\ 0 & 1 \end{pmatrix} g\right] dx$$

$$= \int |t|^{s+1/2}\, d^\times t \int f\left[\begin{pmatrix}1 & 0\\ 0 & t\end{pmatrix} w \begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right] dx$$

$$= \int |t|^{-s-1/2}\, d^\times t \int f\left[\begin{pmatrix}t & 0\\ 0 & 1\end{pmatrix} w \begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right] dx$$

$$= \int \hat{f}\left[w \begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g, s\right] dx.$$

(Recall that ω was assumed to be trivial on F_∞^+; therefore

$$f\left[\begin{pmatrix} & 0\\ 0 & t^{-1}\end{pmatrix} g\right] = f\left[\begin{pmatrix}t & 0\\ 0 & 1\end{pmatrix} g\right]$$

for $t \in F_\infty^+$.) If we define an operator $M(s)$ from $\mathbf{H}(s)$ to $\mathbf{H}(-s)$ by

$$M(s)\varphi(g) = \int_{N(A)} \varphi[wng]\, dn \tag{3.17}$$

when $\operatorname{Re}(s) > \frac{1}{2}$, we conclude that

$$\hat{f}'(-s) = M(s)\hat{f}(s). \tag{3.18}$$

Summing up, we find that the scalar product of two P-series F_1, F_2 (with f_1, f_2 of compact support) is given by the formula

$$\begin{aligned}(F_1, F_2) &= \frac{1}{2\pi i}\int (\hat{f}_1(s), \hat{f}_2(-\bar{s}))\, ds \\ &\quad + \frac{1}{2\pi i}\int (M(s)\hat{f}_1(s), \hat{f}_2(\bar{s}))\, ds.\end{aligned} \tag{3.19}$$

The integrals are taken over any vertical line with $\operatorname{Re}(s) > \frac{1}{2}$ and the pairings we have written are on $\mathbf{H}(s) \times \mathbf{H}(-\bar{s})$ and $\mathbf{H}(-s) \times \mathbf{H}(\bar{s})$ (or simply $\mathbf{H} \times \mathbf{H}$). It is worth noting that

$$(M(s)\varphi_1, \varphi_2) = (\varphi_1, M(\bar{s})\varphi_2) \tag{3.20}$$

for φ_1 in $\mathbf{H}(s)$ and φ_2 in $\mathbf{H}(\bar{s})$. (This is analogous to the identity (3.9) and proved just the same way.) If we think of $M(s)$ as an operator from $\mathbf{H}$ to $\mathbf{H}$ then (3.20) simply asserts that

$$M(s)^* = M(\bar{s}). \tag{3.21}$$

Note finally that if f is replaced by $h \mapsto f(hg)$ then F (resp. $\hat{f}(s)$) is replaced by its right translate, i.e., by $\rho_\omega(g)F$ (resp. $\pi_s(g)\hat{f}(s)$). On the other hand, it is also clear that

$$M(s)\pi_s(g) = \pi_{-s}(g)M(s). \tag{3.22}$$

These facts play a key role in the next section.

4. Analytic continuation of $M(s)$. Our goal is to use the inner product formula (3.19) to construct an intertwining operator between a subrepresentation of ρ_ω and a continuous sum of the representations π_s *with s purely imaginary.* To this end we need to analytically continue the operator $M(s)$. Indeed the integration in (3.19) is over a line $\operatorname{Re}(s) = x$ with $x > \frac{1}{2}$. So if we want to rewrite this formula using purely imaginary s we need to know something about the poles of $M(s)$.

A. *Prelimminary rearks.* Now it will be more convenient to replace π_s and $M(s)$ with the operators we get by performing a Mellin transform on the whole group $F^\times\backslash A^\times$ rather than just F_∞^+. Let $\eta = (\mu, \nu)$ denote a pair of quasi-characters of $F^\times\backslash A^\times$ with $\mu\nu = \omega$. Let $\mathbf{H}(\eta)$ be the space of functions on $G(A)$ such that

$$\varphi\left[\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} g\right] = \mu(a)\nu(b)\left|\frac{a}{b}\right|^{1/2}\varphi(g) \tag{4.1}$$

and

$$\int_K |\varphi|^2(k)\, dk < +\infty. \tag{4.2}$$

Once more we can think of the collection $\mathbf{H}(\eta)$ as a fibre-bundle over the space of pairs η. These pairs make up a complex manifold of dimension 1 with infinitely many connected components and our fibre-bundle is trivial over any such component because the character $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) \mapsto \mu(a)\nu(b)$ of $A(A) \cap K$ is fixed there. In particular, $\mathbf{H}(\eta)$ may be regarded as the subspace of functions in $L^2(K)$ such that

$$\varphi\left[\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} k\right] = \mu(a)\nu(b)\varphi(k)$$

when $\left(\begin{smallmatrix} a & x \\ 0 & b \end{smallmatrix}\right) \in K$. In any case, we denote by π_η the natural representation of $G(A)$ on $\mathbf{H}(\eta)$. The fiber $\mathbf{H}(s)$ decomposes as the sum of the fibres $\mathbf{H}(\eta)$, with $\eta = (\mu, \nu)$, and $\mu \circ \nu^{-1}(a) = |a|^s$ for $a \in F_\infty^+$; the representation π_s decomposes correspondingly as the sum of the "principal series representations π_η"; see [Ge, p. 67].

As far as pairings are concerned, we have a natural one between $\mathbf{H}(\eta)$ and $\mathbf{H}(\bar{\eta}^{-1})$ defined by

$$(\varphi_1, \varphi_2) = \int_K \varphi_1(k)\bar{\varphi}_2(k)\, dk.$$

As before, $(\pi_\eta(g)\varphi_1, \varphi_2) = (\varphi_1, \pi_{\bar{\eta}^{-1}}(g^{-1})\varphi_2)$. Moreover, if we set $\tilde{\eta} = (\nu, \mu)$, then formula (3.17) defines an operator $M(\eta)$ from $\mathbf{H}(\eta)$ to $\mathbf{H}(\tilde{\eta})$ satisfying

$$M(\eta)\pi_\eta(g) = \pi_{\tilde{\eta}}(g)M(\eta) \tag{4.3}$$

and

$$(M(\eta)\varphi_1, \varphi_2) = (\varphi_1, M(\bar{\tilde{\eta}}^{-1})\varphi_2). \tag{4.4}$$

Here φ_1 is in $\mathbf{H}(\eta)$ and φ_2 is in $\mathbf{H}(\bar{\tilde{\eta}}^{-1})$; cf. formulas (3.20) and (3.22). In particular, if we identitfy $\mathbf{H}(\eta)$ with $\mathbf{H}(\bar{\eta}^{-1})$ and $\mathbf{H}(\tilde{\eta})$ with $\mathbf{H}(\bar{\tilde{\eta}}^{-1})$ (keeping in mind that η and $\bar{\eta}^{-1}$ belong to the same connected component) then (4.4) reads

$$M(\eta)^* = M(\bar{\tilde{\eta}}^{-1}). \tag{4.5}$$

Note that (3.21) is essentially a direct sum of identities like (4.5).

To continue analytically $M(\eta)$ we are going to decompose it as a tensor product of local intertwining operators and thereby (essentially) reduce the problem to a local one.

B. *Local intertwining operators.* Let v be a place of F and F_v the corresponding local field. If $\eta_v = (\mu_v, \nu_v)$ is a pair of quasi-characters of $F_v^\times$ such that $\mu_v \nu_v = \omega_v$ we can form the space $\mathbf{H}(\eta_v)$ analogous to $\mathbf{H}(\eta)$, the representation π_{η_v}, and the operator $M(\eta_v)\colon \mathbf{H}(\eta_v) \to \mathbf{H}(\tilde{\eta}_v)$ defined by

$$(M(\eta_v)\varphi)(g) = \int_{F_v} \varphi\left[w\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}g\right]dx. \tag{4.6}$$

Since the group G_v operates by right matrix multiplication on $F_v \times F_v$, and the stabilizer of the line $F_v(0, 1)$ is the group P_v, we may define global sections of the bundle $\mathbf{H}(\eta_v)$ by the formula

$$\varphi(g, \eta_v) = \frac{\mu_v \alpha_v^{1/2}(\det g)}{L(1, \mu_v\nu_v^{-1})} \int_{F_v^\times} \Phi[(0, t)g]\mu_v\nu_v^{-1}(t)\,|t|\, d^\times t. \tag{4.7}$$

Here $L(s, \mu_v\nu_v^{-1})$ is the usual Euler factor attached to the character $\mu_v\nu_v^{-1}$, $\alpha_v(x) = |x|_v$, and Φ is a Schwartz-Bruhat function on $F_v \times F_v$. Recall that if χ is a quasi-character of $F_v^\times$, and f is Schwartz-Bruhat *on* F_v, the integral $\int_{F_v^\times} f(t)\,|t|^s\chi(t)\,d^\times t = Z(f, \chi, s)$ converges for $\mathrm{Re}(s) \gg 0$ (> 0 if χ is unitary) and the ratio $Z(f, \chi, s)/L(s, \chi)$ extends to an entire function of s. (This is Tate's theory of the local zeta-function.) Thus the formula (4.7) actually makes sense for all η and indeed defines a section of our fibre-bundle.

Now we want to apply the operator $M(\eta_v)$ defined by (4.6) to an element of $H(\eta_v)$ given by the section (4.7). After a change of variables we get

$$\omega_v(-1)\frac{\mu_v\alpha_v^{1/2}(\det g)}{L(1, \mu_v\nu_v^{-1})} \int_{F_v}\int_{F_v^\times} \Phi[(t, x)g]\mu_v\nu_v^{-1}(t)\, d^\times t\, dx. \tag{4.8}$$

Next recall the functional equation for the quotient $Z(f, \chi, s)/L(s, \chi)$:

$$\frac{Z(\hat{f}, \chi^{-1}, 1-s)}{L(1-s, \chi^{-1})} = \varepsilon(s, \chi, \psi_v)\,\frac{Z(f, \chi, s)}{L(s, \chi)}.$$

Here $\hat{f}$ denotes the usual Fourier transform taken with respect to the fixed additive character ψ_v, and $\varepsilon(s, \chi, \psi_v)$ is an exponential function which also depends on ψ_v. If we define the Fourier transform of Φ to be

$$\hat{\Phi}(x, y) = \iint \Phi(u, v)\, \psi_v\,(yu - xv)\; du\; dv$$

we see that (4.8) can be written as the product of

$$\frac{L(0, \mu_v\nu_v^{-1})}{L(1, \mu_v\nu_v^{-1})\,\varepsilon(0, \mu_v\nu_v^{-1}, \psi_v)} \tag{4.9}$$

and

$$\frac{\omega_v(-1)}{L(1, \mu_v^{-1}\nu_v)} \int \hat{\Phi}[(0, t)g]\mu_v^{-1}\nu_v(t)\,|t|\, d^\times t(\nu_v\alpha_v^{1/2}(\det g)). \tag{4.10}$$

Note that we pass from (4.7) to (4.10) by the substitutions $\eta \to \tilde{\eta}$, $\Phi \to \hat{\Phi}$, and multiplication by $\omega_v(-1)$.

Next we write $\eta_v = (\mu_v, \nu_v)$, $\mu_v = \chi_1\alpha_v^{s/2}$, $\nu_v = \chi_2\alpha_v^{-s/2}$, where χ_1 and χ_2 are characters. For $\mathrm{Re}\, s > 0$, we can write $M(\eta_v)$ as the product of the scalar (4.9) with an operator $R(\eta_v)$ which (for each η_v) takes the element (4.7) of $\mathbf{H}(\eta_v)$ to the element (4.10) of $\mathbf{H}(\tilde{\eta}_v)$. But for $\mathrm{Re}(s) > -\frac{1}{2}$, every element of $\mathbf{H}(\eta_v)$ can be represented by an integral of the form (4.7) ([**JL**, pp. 97-98]). An easy argument then shows that $R(\eta_v)$ extends to a holomorphic operator valued function of η_v in the domain $\mathrm{Re}(s) > -\frac{1}{2}$ which again takes (4.7) to (4.10). Moreover, from the obvious relations $(\tilde{\eta})^{\sim} = \eta$, $(\Phi^{\wedge})^{\wedge} = \Phi$, and $\omega_v^2(-1) = 1$, we find that

$$R(\tilde{\eta}_v)R(\eta_v) = \mathrm{Id}. \tag{4.11}$$

Since (4.9) is clearly meromorphic, this also gives the analytic continuation of the operator $M(\eta_v)$ in the domain $\mathrm{Re}(s) > -\frac{1}{2}$. Although we do not need to, we note that $R(\eta_v)$ (and hence $M(\eta_v)$) extends to a meromorphic function of all η_v (or $s \in \boldsymbol{C}$).

It is important to note that the operator $R(\eta_v)$ is "normalized" in the following sense. If η_v is unramified then $\mathbf{H}(\eta_v)$ (resp. $\mathbf{H}(\tilde{\eta}_v)$) contains a unique function φ_v (resp. $\tilde{\varphi}_v$) which is invariant under K_v and equal to one on K_v. Then, provided ψ_v has order zero,

$$R(\eta_v)\varphi_v = \tilde{\varphi}_v. \tag{4.12}$$

Moreover, $R(\eta_v)$ is unitary whenever η_v is unitary. Indeed if $\tilde{\eta}_v = \bar{\eta}_v^{-1}$, the operators $M(\eta_v)\colon \mathbf{H}(\eta_v) \to \mathbf{H}(\tilde{\eta}_v)$, $\omega_v(-1)M(\tilde{\eta}_v)\colon \mathbf{H}(\tilde{\eta}_v) \to \mathbf{H}(\eta_v)$ are adjoint to one another. Therefore, since the scalar (4.9) changes to its imaginary conjugate times $\omega_v(-1)$ when η_v is replaced by $\tilde{\eta}_v$, we conclude from (4.11) that $R(\eta_v)^* = R(\eta_v)^{-1}$.

REMARK (4.13) (ON THE RANGE OF $R(\eta_v)$). If $\mu_v\nu_v^{-1} = \alpha_v$ then $\mu_v = \chi_v\alpha_v^{1/2}$ and $\nu_v = \chi_v\alpha_v^{-1/2}$ with $\chi_v^2 = \omega_v$. In this case the space $\mathbf{H}(\eta_v)$ consists of functions φ satisfying

$$\varphi\left[\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} g\right] = \chi_v(ab)\left|\frac{a}{b}\right|\varphi(g),$$

and the kernel of $M(\eta_v)$ (or $R(\eta_v)$) has codimension one. The space $\mathbf{H}(\tilde{\eta}_v)$ consists of functions φ satisfying

$$\varphi\left[\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} g\right] = \chi_v(ab)\varphi(g);$$

so the range of $M(\eta_v)$ (or $R(\eta_v)$) must be the one-dimensional space spanned by the function $g \mapsto \chi_v(\det g)$. Moreover, for the sesquilinear pairing between $\mathbf{H}(\eta)$ and $\mathbf{H}(\tilde{\eta})$,

$$(R(\eta)\varphi_1, \varphi_2) = c'(\varphi_1, \chi \circ \det g)\,\overline{(\varphi_2, \chi \circ \det g)}, \tag{4.14}$$

where c' is a known constant.

C. *Global theory*. Our task is to piece together the local intertwining operators $M(\eta_v)$ in order to analytically continue $M(s)$. First we define an operator $R(\eta)$ as the "infinite tensor product" of the local operators $R(\eta_v)$. If φ in $\mathbf{H}(\eta)$ has the form $\prod \varphi_v$, with φ_v invariant under K_v and equal to one on K_v for almost all v, then $R(\eta)\varphi = \prod_v R(\eta_v)\varphi_v$. Because each $R(\eta_v)$ is normalized (cf. (4.12)), $R(\eta)\varphi$ is indeed well-defined in $\mathbf{H}(\tilde{\eta})$. Moreover, we can write $M(\eta)$ as

$$M(\eta) = \frac{L(0, \mu\nu^{-1})}{L(1, \mu\nu^{-1})\,\varepsilon(0, \mu\nu^{-1})}\, R(\eta). \tag{4.15}$$

This gives the analytic continuation of $M(\eta)$ since the scalar factor in (4.15) has a known meromorphic behavior and the operator $R(\eta)$ is a meromorphic function of η.

To get the functional equation of $M(\eta)$ we use the functional equation of the L-function $L(s, \mu\nu^{-1})$. This allows us to write the scalar in (4.15) as

$$\frac{L(1, \mu^{-1}\nu)}{L(1, \mu\nu^{-1})}. \tag{4.16}$$

Therefore $M(\eta)$ satisfies the functional equation

$$M(\bar{\eta})M(\eta) = \text{Id}. \tag{4.17}$$

(Cf. (4.11).) By (4.5) we also have

$$M(\eta)^* = M(\eta^{-1}) \quad \text{if } \bar{\eta} = \eta^{-1}. \tag{4.18}$$

Thus $M(\eta)$ is *unitary* when η is unitary. Recall that $R(\eta)$ also satisfies (4.17) and (4.18).

We sum up the analytic behavior of $M(\eta)$ as follows. Write $|\mu\nu^{-1}| = \alpha^t$ with t real. Then $M(\eta)$ is meromorphic in the half-plane $t \geqq 0$ and its only poles there are simple ones which occur for $\mu = \chi\alpha^{1/2}$, $\nu = \chi\alpha^{-1/2}$, $\chi^2 = \omega$. The residues are scalar multiples of the operator $R(\eta)$ and the ranges are the one-dimensional spaces spanned by the functions $g \mapsto \chi(\det g)$. Going back to $M(s)$ we get (from (4.14)):

THEOREM (4.19). *As a function of s the operator $M(s)$ is meromorphic in the whole complex plane and satisfies the functional equation*

$$M(-s)M(s) = \text{Id}. \tag{4.20}$$

Its only pole in the half-plane $\text{Re}(s) \geqq 0$ *is at* $s = \frac{1}{2}$ *and the residue there is such that*

$$(\text{Re}\, s_{1/2}\, M(\tfrac{1}{2})\hat{f}_1(\tfrac{1}{2}), \hat{f}_2(\tfrac{1}{2})) = c \sum_{\chi^2=\omega} (\hat{f}_1(\tfrac{1}{2}), \chi \circ \det)\, \overline{(\hat{f}_2(\tfrac{1}{2}), \chi \circ \det)}$$

where c is a known constant.

The last assertion follows from (4.14).

D. *Analysis of the continuous spectrum.* Suppose F_1, F_2 are two P-series belonging to f_1, f_2. We know that F_1 and F_2 lie in the orthocomplement of $L^2_0(\omega, G)$ and their scalar product is given by the formula (3.19). If we use the residue theorem, Theorem (4.19), and some simple estimates, we can shift the integration in (3.19) to the imaginary axis and write

$$\begin{aligned}(F_1, F_2) = {} & \frac{1}{2\pi}\int_{-\infty}^{\infty} \{(\hat{f}_1(iy), \hat{f}_2(iy)) + (M(iy)\hat{f}_1(iy), \hat{f}_2(-iy))\}\, dy \\ & + c \sum_{\chi^2=\omega} (\hat{f}_1(\tfrac{1}{2}), \chi \circ \det)\, \overline{(\hat{f}_2(\tfrac{1}{2}), \chi \circ \det)}.\end{aligned} \tag{4.21}$$

Here the sum is extended over all characters χ of $F^\times\backslash A^\times$ such that $\chi^2 = \omega$ and the scalar product is the pairing between $\mathbf{H}(\frac{1}{2})$ and $\mathbf{H}(-\frac{1}{2})$. (Note that since ω is assumed to be trivial on F^+_∞, so is χ if $\chi^2 = \omega$; thus $\chi \circ \det$ indeed belongs to $\mathbf{H}(-\frac{1}{2})$.) But (cf. (3.15))

$$(\hat{f}_1(\tfrac{1}{2}), \chi \circ \det) = \int_{P(F)N(A)Z(A)\backslash G(A)} f_1(g)\, \bar{\chi}(\det g)\, dg$$
$$= \int_{G(F)Z(A)\backslash G(A)} F_1(g)\, \bar{\chi}(\det g)\, dg$$

since F_1 is the P-series attached to f_1. Therefore the *second* term in (4.21) can also be written as

$$c \sum_{\chi^2=\omega} (F_1, \chi \circ \det)\, \overline{(F_2, \chi \circ \det)}, \tag{4.22}$$

the scalar product now being taken in $L^2(\omega, G)$.

Now note that (4.20) and (3.21) together imply that, for $y \in \boldsymbol{R}$, $M(iy)^*M(iy) =$ Id. In particular, $M(s)$ is unitary on the imaginary axis and, if we set

$$a(iy) = \tfrac{1}{2}\{\hat{f}(iy) + M(-iy)\hat{f}(-iy)\}, \tag{4.23}$$

then

$$M(-iy)a(-iy) = a(iy), \tag{4.24}$$

and the first term in (4.21) can be written as

$$\frac{1}{\pi}\int_{-\infty}^{\infty} (a_1(iy), a_2(iy))\, dy. \tag{4.25}$$

This is significant for the decomposition of ρ_ω because if we replace f_1 by $h \mapsto f_1(hg)$ then F_1 is replaced by $\rho_\omega(g)F_1$, $y \mapsto \hat{f}(iy)$ by $y \mapsto \pi_s(g)\hat{f}(iy)$, and $y \mapsto a(iy)$ by $y \mapsto \pi_s(h)a(iy)$.

To sum up, let $\mathscr{L}$ denote the Hilbert space of square-integrable functions a on $i\boldsymbol{R}$ with values in H (i.e., square-integrable sections of our bundle over $i\boldsymbol{R}$) *satisfying* (4.24). Equip $\mathscr{L}$ with the inner product (4.25) and let π denote the representation of $G(\boldsymbol{A})$ on $\mathscr{L}$ given by $\pi(g)a(iy) = \pi_s(g)a(iy)$. Because of (4.24), we may also regard $\mathscr{L}$ as the space of square-integrable functions $a(y)$ from $\boldsymbol{R}_+$ to $\mathbf{H}$, the scalar product being given by

$$(a_1, a_2) = \frac{2}{\pi}\int_0^{+\infty} (a_1(y), a_2(y))\, dy.$$

Thus π is a continuous sum of the representations π_{iy}.

For each χ such that $\chi^2 = \omega$ we let $\mathscr{L}_\chi$ denote the space of the one-dimensional representation $g \mapsto \chi(\det g)$. Then, since (F_1, F_2) is the sum of (4.22) and (4.25), it follows that there is an isometric map (with dense domain) from $(L_0^2)^\perp$ to $\mathscr{L} \oplus (\bigoplus_\chi \mathscr{L}_\chi)$. This map (given by $F \mapsto (a(iy), \hat{f}(\tfrac{1}{2}))$) extends by continuity to a map T from $(L_0^2)^\perp$ to a dense subspace of $\mathscr{L} \oplus (\bigoplus_\chi \mathscr{L}_\chi)$ and, by the remarks above, it is also an intertwining operator.

To conclude, we see that $L^2(\omega, G)$ decomposes as a direct sum

$$L_0^2(\omega, G) \oplus L_{\text{cont}}^2(\omega, G) \oplus L_{\text{sp}}^2(\omega, G)$$

where L_{sp}^2 is the space spanned by the functions $\chi(\det g)$, i.e., L_{sp}^2 is the space $\bigoplus \mathscr{L}_\chi$, and L_{cont}^2 is isomorphic to $\mathscr{L}$ via the intertwining operator S: $L_{\text{cont}}^2(\omega, G) \to \mathscr{L}$.

Moreover, (4.22) is the scalar product of the orthogonal projections of F_1 and F_2 on L_{sp}^2. Therefore $c = [\text{vol}(\tilde{G}(F)\backslash\tilde{G}(\boldsymbol{A}))]^{-1}$ and we have computed this volume.

5. Eisenstein series and the truncation process. Let P_{cusp} denote the orthogonal projection onto $L_0^2(\omega, G)$. Since we need an explicit formula for the kernel of the Hilbert-Schmidt operator $P_{\text{cusp}}\rho_\omega(\varphi)P_{\text{cusp}}$ we need an explicit description of the operator S of §4. This description is given in terms of Eisenstein series.

A. *Eisenstein series.* If φ is a section of the fibre-bundle $\mathbf{H}(s)$ we set

$$E(\varphi(s), g) = \sum_{\gamma \in P(F)\backslash G(F)} \varphi(\gamma g, s). \tag{5.1}$$

This series converges only for $\operatorname{Re}(s) > \frac{1}{2}$ and in general has to be defined by analytic continuation.

Recall that if f satisfies (3.1) and has compact support mod $N(A)Z(A)P(F)$ then $f(g) = (2\pi i)^{-1} \int_{x-i\infty}^{x+i\infty} \hat{f}(g, s)\, ds$. Interchanging summation and integration we see that the P-series F defined by f is given by

$$F(g) = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} E(\hat{f}(s), g)\, ds \tag{5.2}$$

provided $x > \frac{1}{2}$. Thus the functions of a dense subspace of $(L_0^2)^\perp$ are "continuous sums of Eisenstein series". However, as before, to obtain a useful formula we must analytically continue E and shift the integration in (5.2) to the imaginary axis.

Since E is a P-series, its constant term is easily computed (cf. (3.6) and (3.7)):

$$E_N(\varphi(s), g) = \varphi(g, s) + \int \varphi[wng, s]\, dn$$

or

$$E_N(\varphi(s), g) = \varphi(s)(g) + [M(s)\varphi(s)](g). \tag{5.3}$$

In both equations, $\operatorname{Re}(s) > \frac{1}{2}$.

To analytically continue E we shall primarily deal with sections obtained from identifying $\mathbf{H}(s)$ with $\mathbf{H}$. In other words, if $h \in \mathbf{H} = \mathbf{H}(0)$, we define a section $h(s)$ by the formula

$$h(g, s) = h(g)H(g)^s. \tag{5.4}$$

The corresponding Eisenstein series will be denoted $E(h(s), g)$.

B. *Properties of the truncation operator.* We shall use the truncation operator to obtain the analytic continuation of E. If $c > 1$ we let χ_c denote the characteristic function of $[c, +\infty)$. For any function φ on $G(F)\backslash G(A)$ set

$$\Lambda^c\varphi(g) = \varphi(g) - \sum_{P(F)\backslash G(F)} \varphi_N(\gamma g)\chi_c(H(\gamma g)). \tag{5.5}$$

The second term here is a P-series attached to a function with support in the set $\{g: H(g) > c\}$. Thus if g is in a Siegel domain *the series has only finitely many terms* (cf. Lemma (3.3)).

If φ is a cuspidal function, i.e., $\varphi_N \equiv 0$, then clearly $\Lambda^c\varphi = \varphi$. In general, we need to appeal to the following lemma:

LEMMA (5.6) (CF. [**DL**, P. 197]). *If there is a g in $G(A)$ such that $H(g) > 1$ and $H(\gamma g) > 1$ then γ belongs to $P(F)$.*

This lemma shows that, for a given g, the series in (5.5) has at most one term for $c > 1$. In particular,

$$\begin{aligned}(5.7)\qquad \Lambda^c\varphi(g) &= \varphi(g) - \varphi_N(g)\chi_c(H(g)) \quad \text{if } H(g) > 1,\ c > 1,\\ &= \varphi(g) - \varphi_N(g) \quad\quad\quad\quad \text{if } H(g) > c > 1.\end{aligned}$$

Now let $\mathscr{G}$ be a Siegel domain and Ω the set of g in $\mathscr{G}$ such that $H(g) \leqq c$. Then Ω is compact (mod $Z(A)$) and, if $c > 1$, $\Lambda^c\varphi$ is given on $\mathscr{G} - \Omega$ by the second formula in (5.7). Thus $\Lambda^c\varphi(g)$ is bounded on $\mathscr{G} - \Omega$ *under very mild assumptions* on φ. For instance, this is the case if φ is the convolution of a "slowly increasing" or square-integrable function on $G(F)\backslash G(A)$ with a C^∞-function of compact support. Actually (5.7) will then be "rapidly decreasing".

On the other hand, if Ω is any compact set, then

$$(5.8)\qquad \Lambda^c\varphi(g) = \varphi(g) \quad \text{for } g \in \Omega \textit{ and } c \textit{ large}.$$

Indeed suppose $g \in \Omega$ and $\Lambda^c\varphi(g) \neq \varphi(g)$ with $c > 1$. Then there is a finite nonempty set of elements γ of $P(F)\backslash G(F)$ such that $H(\gamma g) > c > 1$ (cf. Lemma (3.3) again; this finite set depends on Ω but not on c). Since the resulting element γg must belong to a compact set mod $P(F)N(A)Z(A)$, and since H is continuous, we must have $H(\gamma g) < c_0$. But if $c > c_0$ we get a contradiction. Therefore (5.8) must hold. In other words, $\Lambda^c\varphi \to \varphi$ uniformly on compact sets as $c \to +\infty$.

We also want to point out that Λ^c is a continuous *hermitian* operator on $L^2(\omega, G)$:

$$(5.9)\qquad (\Lambda^c\varphi_1, \varphi_2) = (\varphi_1, \Lambda^c\varphi_2).$$

Indeed the left-hand side is (φ_1, φ_2) minus the scalar product of a P-series with φ_2. In particular, by (3.4) the left-hand side equals

$$(\varphi_1, \varphi_2) - \int_{P(F)N(A)Z(A)\backslash G(A)} \varphi_{1,N}(g)\chi_c(H(g))\, \overline{\varphi_{2,N}(g)\, dg}.$$

Similarly the right-hand side is $(\varphi_1, \varphi_2) - \int \varphi_{1,N}(g)\bar{\varphi}_{2,N}(g)\bar{\chi}_c(H(g))\, dg$; since χ_c is real the desired equality follows.

We also have, for $c > 1$,

$$(5.10)\qquad ((1 - \Lambda^c)\varphi_1, \Lambda^2\varphi_2) = 0,$$

i.e., *Λ^c is an orthogonal projection in $L^2(\omega, G)$*. Indeed $(1 - \Lambda^c)\varphi_1$ is also a P-series. So the left-hand side of (5.10) is

$$\int_{N(A)P(F)Z(A)\backslash G(A)} \varphi_{1,N}(g)\chi_c(H(g)) \left(\int_{N(F)\backslash N(A)} \Lambda^c\varphi_2(ng)\, dn\right) dn.$$

But $\chi_c(H(g)) = 0$ unless $H(g) > c > 1$, in which case the inner integral is (by (5.7))

$$\int_{N(F)\backslash N(A)} (\varphi_2(ng) - \varphi_{2,N}(ng))\, dn = 0$$

and (5.10) follows. (In fact (5.10) is sometimes true even when φ_i does not belong to $L^2(\omega, G)$.)

C. *Analytic continuation of Eisenstein series.* Using the truncation operator (5.5) we can write our Eisenstein series as

$$(5.11) \qquad E(h(s), g) = \Lambda^c E(h(s), g) + \sum_{P(F)\backslash G(F)} E_N(h(s), \gamma g)\chi_c(H(\gamma g)).$$

The second term on the right side is the P-series attached to a function with support in the set $\{g: H(g) \geqq c\}$. Thus, as noted before, for g in a Siegel domain the second term has only finitely many terms. In particular, it represents a meromorphic function of s whose singularities are at most those of $M(s)$ (cf. (5.3)).

On the other hand, the first term on the right side of (5.11) is initially defined only for $\mathrm{Re}(s) > \frac{1}{2}$. However, since it is square-integrable (cf. (5.7) and the remarks immediately following it), it will suffice to continue it analytically as a $L^2(\omega, G)$-valued function. Thus we need to examine the inner product

$$(5.12) \qquad (\Lambda^c E(h_1(s_1)), \Lambda^c E(h_2(\bar{s}_2))).$$

By (5.10) (which is true in this case) the inner product (5.12) is just $(E(h_1(s_1)), \Lambda^c E(h_2(\bar{s}_2)))$, which, since $\Lambda^c E$ is a difference of P-series, we compute (using (3.4)) to be

$$\int_{N(A)P(F)Z(A)\backslash G(A)} E_N(h_1(s_1), g)\{\bar{h}_2(g, \bar{s}_2) - \bar{E}_N(h_2(\bar{s}_2), g)\chi_c(H(g))\}\, dg.$$

Now recall that E_N is given by (5.3) with $\varphi(g, s) = h(g, s)$ and

$$h\left[\begin{pmatrix} ta & x \\ 0 & 1 \end{pmatrix} k, s\right] = |t|^{s+1/2} h\left[\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k\right]$$

for $t \in F_\infty^+$, $a \in F^0(A)$, and $k \in K$. So by Iwasawa's decomposition, we compute

$$\begin{aligned} &(\Lambda^c E(h_1, (s_1)), \Lambda^c E(h_2(\bar{s}_2))) \\ (5.13) \qquad &= \{(h_1, h_2)c^{s_1+s_2} - (M(s_1)h_1, M(\bar{s}_2)h_2)c^{-(s_1+s_2)}\} \frac{1}{s_1 + s_2} \\ &+ \{(h_1, M(\bar{s}_2)h_2)c^{s_1-s_2} - (M(s_1)h_1, h_2)^{-(s_1-s_2)}\} \frac{1}{s_1 - s_2}. \end{aligned}$$

Here $\mathrm{Re}(s_1) > \mathrm{Re}(s_2) > \frac{1}{2}$.

Note that the right side of (5.13) is a meromorphic function of (s_1, s_2) which seems to have a singularity on the line $s_1 + s_2 = 0$. However, by (3.21) and (4.20), the expression in the first bracket vanishes precisely along this line. Similarly the expression in the second bracket vanishes when $s_1 - s_2 = 0$. Thus we conclude that (5.13) is meromorphic in s_1, s_2 with singularities at most those of $M(s_1)$ and $M(s_2)$. In particular, for $s = s_1 = -s_2 \neq 0$,

$$\begin{aligned} &(\Lambda^c E(h_1(s)), \Lambda^c E(h_2(-\bar{s}))) = 2(h_1, h_2)\log c + (M(-s)M'(s)h_1, h_2) \\ (5.14) \qquad &+ \{(h_1, M(-\bar{s})h_2)c^{2c} - (M(s)h_1, h_2)c^{-2s}\} \frac{1}{2s}. \end{aligned}$$

Now suppose that for $h_1 = h_2 = h$, the right-hand side of (5.13) is analytic in the polydisc $|s_1 - s_0| < R$, $|s_2 - s_0| < R$, while $E(h(s))$ is holomorphic in some small disc centered at s_0. Then the double Taylor series of the left-hand side converges in this polydisc. Since

$$\left\| \frac{\partial^n}{\partial s^n} \Lambda^c E(h(s)) \right\|^2 = \left(\frac{\partial^n}{\partial s^n} \Lambda^c E(h(s)), \frac{\partial^n}{\partial s^n} \Lambda^c E(h(s)) \right)$$
$$= \frac{\partial^{2n}}{\partial s_1^n \partial \bar{s}_2^n} (\Lambda^c E(h(s_1)), \Lambda^c E(h(\bar{s}_2))) \Big\|_{s_1=s_2=s}$$

it is easy to conclude that the Taylor series of $\Lambda^c E(h(s))$ converges in the disc $|s-s_0| < R$, i.e., $\Lambda^c E(h(s))$ is analytic in this disc. Thus we conclude $\Lambda^c E(h(s))$ (and hence $E(h(s), g)$) is meromorphic in $\mathrm{Re}(s) \geqq 0$ with singularities at most those of $M(s)$. Note that in (5.13) and (5.14) we needed to identify $\mathbf{H}(s)$ with $\mathbf{H}$ to define the scalar products and the derivative $M'(s)$ of $M(s)$.

Summing up, we know that if $\varphi(s)$ is any meromorphic section of our bundle $\mathbf{H}(s)$ then $E(\varphi(s), g)$ is defined as a meromorphic function of s (at least for $\mathrm{Re}(s) \geqq 0$). Moreover,

$$E(M(s)\varphi(s), g) = E(\varphi(s), g). \tag{5.15}$$

Indeed since $\varphi(s)$ belongs to $\mathbf{H}(s)$ and $M(s)\varphi(s)$ belongs to $\mathbf{H}(-s)$, the function $g \to E_N(M(s)\varphi(s), g)$ is equal to

$$M(s)\varphi(s) + M(-s)M(s)\varphi(s) = \varphi(s) + M(s)\varphi(s) = E_N(\varphi(s), g).$$

Thus $E_N(M(s)\varphi(s), g) = E_N(\varphi(s), g)$, i.e., the difference between the two sides of (5.15) is a cuspidal function. Therefore, since any P-series (or its analytic continuation) is orthogonal to all cuspidal functions, this difference vanishes as claimed.

In general, say for a group whose derived group has F-rank one, the same facts can be proved. The only difference is that the operator $M(s)$ may have a finite number of poles and these poles are not necessarily known. Nevertheless, the operator $M(s)$ still similarly controls the analytic behavior of the Eisenstein series (cf. **[Ar 1]** and **[La 1]**).

D. *The kernel of* $\rho_\omega(\varphi)$ *in* L^2_{cont}. Suppose $F_1(g)$ is a P-series attached to f_1 (of compact support mod $Z(A)N(A)P(F)$). Our immediate goal is to prove that, for all $h \in \mathbf{H}$,

$$(h, SF_1(iy)) = \frac{1}{2} \int_{G(F)Z(A)\backslash G(A)} E(h(iy), g)\bar{F}_1(g)\, dg \tag{5.16}$$

for almost every y. Here $E(h(iy), g)$ is defined by analytic continuation and $SF_1(iy) = a_1(iy)$ as in (4.23).

Note first that if F_2 is the P-series attached to f_2 we get from (5.2) that

$$(F_1, F_2) = \int_{\bar{G}(F)\backslash\bar{G}(A)} \left\{ \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} E(\hat{f}_1(s), g)\, ds \right\} \bar{F}_2(g)\, dg$$
$$= \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} ds \int_{\bar{G}(F)\backslash\bar{G}(A)} E(\hat{f}_1(s), g)\bar{F}_2(g)\, dg$$

for $\mathrm{Re}(s) = x > \frac{1}{2}$. Shifting the integration to the imaginary axis we then get

$$(F_1, F_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dy \int E(\hat{f}_1(iy), g)\bar{F}_2(g)\, dg + c \sum_{\chi^2=\omega} (F_1, \chi)\,(\chi, F_2). \tag{5.17}$$

But (5.15) says that the integral in (5.17) is unchanged if we replace $\hat{f}_1(iy)$ by $M(iy)\hat{f}_1(iy)$ and then change y to $-y$. Thus, with $a_1(iy)$ as in (4.23), formula (5.17) reads

$$(F_1, F_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dy \int E(a_1(iy), g)\bar{F}_2(g)\, dg + c \sum_{\chi^2=\omega} (F_1, \chi)(\chi, F_2).$$

On the other hand, computing the inner product as in (4.21), we also have

$$(F_1, F_2) = \frac{1}{\pi}\int_{-\infty}^{\infty} (a_1(iy), a_2(iy))\, dy + c \sum_{\chi^2=\omega} (F_1, \chi)(\chi, F_2).$$

Thus we conclude that

$$\int_{-\infty}^{\infty} (a(iy), SF(iy))\, dy = \frac{1}{2}\int_{-\infty}^{\infty} \left\{\int_{\tilde{G}(F)\backslash\tilde{G}(A)} E(a(iy), g)\, \bar{F}(g)\, dg\right\} dy \tag{5.18}$$

for all a in $\mathscr{L}$. This formula is then also true for any square-integrable function $a(y)$ with values in $\mathbf{H}$ since both sides remain unchanged when $a(iy)$ is replaced by $\frac{1}{2}(a(iy) + M(-iy)a(-iy))$. Thus we may take $a(iy) = c(y)h(iy)$ with c a scalar function and h a "constant section" to conclude from (5.18) that (5.16) indeed holds.

Now extend the map $S\colon L^2_{\mathrm{cont}}(\omega, G) \to \mathscr{L}$ to $S\colon L^2(\omega, G) \to \mathscr{L}$ by setting it equal to 0 on L^2_0 and $L^2_{\mathrm{sp}}(\omega, G)$. Then S^*S is the orthogonal projection of $L^2(\omega, G)$ onto L^2_{cont}. Since S is an intertwining operator, $S^*S\rho_\omega(\varphi)S^*S = S^*\pi(\varphi)S$ for any C^∞ function φ of compact support.

Therefore, if F_1 and F_2 are in $L^2(\omega, G)$ then

$$\begin{aligned}(S^*S\rho_\omega(\varphi)S^*SF_1, F_2) &= (S^*\pi(\varphi)SF_1, F_2) = (\pi(\varphi)SF_1, SF_2)\\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} (\pi_{iy}(\varphi)SF_1(iy), SF_2(iy))\, dy.\end{aligned} \tag{5.19}$$

Before applying the identity (5.16) we make the following observation. Although (5.16) was proved only for F_1 in the orthocomplement of $L^2_0(\omega, G)$ *it is actually true for all* F_1 *in* $L^2(\omega, G)$. Indeed if F is in $L^2_0(\omega, G)$ then $SF \equiv 0$ by definition. On the other hand, since every Eisenstein series is orthogonal to any cuspidal function (in the domain of convergence at first but for all s by analytic continuation), the right side of (5.16) is also identically zero.

If $\{\Phi_\alpha\}$ is an orthonormal basis for $\mathbf{H}$ we can finally apply (5.16) to (5.19) to get

$$\begin{aligned}(S^*S\rho_\omega(\varphi)S^*SF_1, F_2) &= \frac{1}{\pi}\int_{-\infty}^{\infty} \sum_\alpha (\pi_{iy}(\varphi)SF_1(iy), \Phi_\alpha)(\Phi_\alpha, SF_2(iy))\, dy\\ &= \frac{1}{4\pi}\int_{-\infty}^{\infty} dy \sum_\alpha \int F_1(g)\bar{E}(\pi^*_{iy}(\varphi)\Phi_\alpha(iy), g)\, dg\\ &\quad\cdot \int \bar{F}_2(h) E(\Phi_\alpha(iy), h)\, dh\end{aligned}$$

with g, h in $\tilde{G}(F)\backslash\tilde{G}(A)$. Interchanging the integrations and summations then yields

$$\begin{aligned}&(S^*S\rho_\omega(\varphi)S^*SF_1, F_2)\\ &= \iint F_1(g)\bar{F}_2(h)\, dg\, dh \left\{\sum_\alpha \frac{1}{4\pi}\int_{-\infty}^{\infty} E(\Phi_\alpha(iy), h)\,\bar{E}(\pi^*_{iy}(\varphi)\Phi_2(iy), g)\, dy\right\}.\end{aligned} \tag{5.20}$$

So if we let P_{cont} denote the projection S^*S onto L^2_{cont} we find that the kernel of

$P_{\text{cont}}\,\rho_\omega(\varphi)P_{\text{cont}}$ is precisely the expression in brackets in (5.20). Alternately, since $\pi^*_{iy}(\varphi)\Phi_\alpha = \sum(\pi^*_{iy}(\varphi)\Phi_\alpha, \Phi_\beta)\,\Phi_\beta$, we also have

$$(5.21)\qquad K_{\text{cont}}(h, g) = \frac{1}{4\pi}\sum_{\alpha,\beta}\int_{-\infty}^{\infty}(\pi_{iy}(\varphi)\Phi_\beta, \Phi_\alpha)\,E(\Phi_\alpha(iy), h)\,\bar{E}(\Phi_\beta(iy), g)\,dg.$$

On the other hand, if P_{sp} is the projection onto L^2_{sp}, then the operator $P_{\text{sp}}\rho_\omega(\varphi)P_{\text{sp}}$ is defined by the kernel

$$K_{\text{sp}}(h, g) = [\text{vol}(\tilde{G}(F)\backslash\tilde{G}(A))]^{-1}\sum_{\chi^2=\omega}\chi(h)\bar{\chi}(g)\int_{\tilde{G}(A)}\varphi(g)\bar{\chi}(\det g)\,dg.$$

The operator $\rho_\omega(\varphi)$ of course is still defined by the kernel $K(h, g) = \sum_{\tilde{G}(F)}\varphi(h^{-1}\gamma g)$. Thus we conclude that

$$(P_{\text{cusp}}\,\rho_\omega(\varphi)P_{\text{cusp}}F)(h) = \int K_{\text{cusp}}(h, g)\bar{F}(g)\,dg,$$

where $K_{\text{cusp}}(h, g) = K(h, g) - K_{\text{cont}}(h, g) - K_{\text{sp}}(h, g)$.

6. The trace formula. Suppose φ is a C^∞-function on $G(A)$ which has compact support mod $Z(A)$ and satisfies (1.3) and the conditions immediately following it. Then we have observed that $P_{\text{cusp}}\,\rho_\omega(\varphi)P_{\text{cusp}}$ is of Hilbert-Schmidt class and also of trace class. In fact the technique we used can also be used to show that the kernel K_{cusp} is square-integrable, continuous, and integrable over the diagonal (cf. **[DL]**). Moreover,

$$(6.1)\qquad \text{tr}(P_{\text{cusp}}\rho_\omega(\varphi)P_{\text{cusp}}) = \int_{\tilde{G}(F)\backslash\tilde{G}(A)} K_{\text{cusp}}(x, x)\,dx.$$

What we are going to do now is give an explicit formula for the right-hand side of (6.1). Note that for f in $L^2_0(\omega, G)^\perp$, $\int K_{\text{cusp}}(x, y)f(y)\,dy = 0$. Thus for each x, the function $y \mapsto \bar{K}(x, y)$ is orthogonal to $L^2_0(\omega, G)^\perp$ and hence in $L^2_0(\bar{\omega}, G)$. In other words, it is a cuspidal function of y.

If we denote by Λ^c_2 the truncation operator *with respect to the second variable* then

$$(6.2)\qquad K_{\text{cusp}}(x, y) = \Lambda^c_2 K_{\text{cusp}}(x, y) = \Lambda^c_2 K(x, y) - \Lambda^c_2 K_{\text{cont}}(x, y) - \Lambda^c_2 K_{\text{sp}}(x, y).$$

But one can show that each term in (6.2) is integrable over the diagonal. Thus

$$(6.3)\quad \text{tr}(P_{\text{cusp}}\rho_\omega(\varphi)P_{\text{cusp}}) = \int \Lambda^c_2 K(x, x)\,dx - \int \Lambda^c_2 K_{\text{cont}}(x, x)\,dx - \int \Lambda^c_2 K_{\text{sp}}(x, x)\,dx.$$

We shall content ourselves with computing each of these integrals.

Note that since the left-hand side of (6.3) does not depend on c, we can let c tend to $+\infty$ and—when computing—*ignore all terms which tend to zero.*

A. *Contribution from the kernel $\Lambda^c_2 K$.* When $x = y$, $\Lambda^c_2 K(x, y)$ is by definition

$$(6.4)\qquad \begin{aligned}\Lambda^c_2 K(x, x) = &\sum_{\gamma\in\tilde{G}(F)}\varphi(x^{-1}\gamma x)\\ &-\sum_{\xi\in P(F)\backslash G(F)}\int_{N(F)\backslash N(A)} dn\sum_{\gamma\in\tilde{G}(F)}\varphi(x^{-1}\gamma n\xi x)\chi_c(H(\xi x)).\end{aligned}$$

Let us recall the following lemma:

LEMMA (6.5) (CF. **[Ge**, P. 201**]**). *If Ω is a compact set in $Z(A)\backslash G(A)$ then there exists* *a number d_Ω with the property that if γ in $G(F)$ and n in $N(A)$ are such that $x^{-1}\gamma n x \in \Omega$ for some x in $G(A)$ with $H(x) > d_\Omega$ then $\gamma \in P(F)$.*

We shall apply this to the support Ω of φ. After changing γ to $\xi^{-1}\gamma$ we may write the second term in (6.4) as

$$\sum_{\xi \in P(F)\backslash G(F)} \int_{N(F)\backslash N(A)} dn \sum_{\gamma \in \tilde{G}(F)} \varphi(x^{-1}\xi^{-1}\gamma n \xi x)\chi_c(H(\xi x)). \tag{6.6}$$

Here we need only sum over those γ for which there is x, ξ, n such that $g = \xi x$ and $g^{-1}\gamma n g$ is in Ω. Thus if c is sufficiently large (φ being given) it follows from Lemma (6.5) that γ must belong to $P(F)$, i.e., if c is large enough, we need only sum over those γ which lie in the image of $P(F)$ in $\tilde{G}(F)$. But for such γ we can write

$$\gamma = \mu\nu \quad \text{with } \mu = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \text{and } \nu \in N(F). \tag{6.7}$$

Thus in (6.6) we can combine the integral over $N(F)\backslash N(A)$ with a sum over $N(F)$ and rewrite (6.4) as

$$\sum_{\gamma \in \tilde{G}(F)} \varphi(x^{-1}\gamma x) - \sum_{\xi \in P(F)\backslash G(F)} \int_{N(A)} \sum_{\mu} \varphi(x^{-1}\xi^{-1}\mu n \xi x)\chi_c(H(\xi x))\, dn \tag{6.8}$$

with μ as in (6.7).

Now we break up the sums in (6.8) and write

$$\Lambda_2^c K(x, x) =$$

$$\varphi(e) \tag{6.9}$$

$$+ \sum \varphi(x^{-1}\gamma x) \qquad (\gamma\ F\text{-elliptic}) \tag{6.10}$$

$$+ \sum_{\gamma} \varphi(x^{-1}\gamma x) - \sum_{\xi \in P(F)\backslash G(F)} \int_{N(A)} \varphi(x^{-1}\xi^{-1}n\xi x)\chi_c(H(\xi x))\, dn \tag{6.11}$$

(γ nilpotent regular)

$$+ \sum_{\gamma} \varphi(x^{-1}\gamma x) - \sum_{\xi \in P(F)\backslash G(F)} \int_{N(A)} \sum_{\alpha \neq 1} \varphi\left(x^{-1}\xi^{-1}\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} n\xi x\right)\chi_c(H(\xi x))\, dn \tag{6.12}$$

(γ F-hyperbolic regular).

Here γ *F-elliptic* means γ is not $G(F)$-conjugate to anything in $P(F)$; γ *nilpotent* regular means γ is conjugate to a nontrivial element of $N(F)$ and γ *F-hyperbolic* regular means γ is conjugate to some $\left(\begin{smallmatrix} \alpha & 0 \\ 0 & 1 \end{smallmatrix}\right)$ with $\alpha \neq 1$ in $F^\times$. One can see that each of the terms (6.9)—(6.12) is integrable over $\tilde{G}(F)\backslash\tilde{G}(A)$; our task is to evaluate the resulting integrals.

The integral of (6.9) is clearly

$$\operatorname{vol}(\tilde{G}(F)\backslash\tilde{G}(A))\varphi(e). \tag{6.13}$$

As for (6.10), it has compact support mod $\tilde{G}(F)$. Indeed if $\varphi(x^{-1}\gamma x) \neq 0$ then $x^{-1}\gamma x$ belongs to $\Omega = \text{support}(\varphi)$. So since γ is elliptic, Lemma (6.5) implies $H(x) < d_\Omega$. The integral of (6.10) over $\tilde{G}(F)\backslash\tilde{G}(A)$ is

$$\int_{\bar{G}(F)\backslash\bar{G}(A)} \sum_{\gamma} \varphi(x^{-1}\gamma x)\,dx \qquad (\gamma \text{ elliptic}) \tag{6.14}$$

and this can be transformed further just as in the division algebra case.

In (6.11) we can write γ in the form $\gamma = \xi^{-1}\left(\begin{smallmatrix}1 & \eta\\ 0 & 1\end{smallmatrix}\right)\xi$ with $\xi \in P(F)\backslash G(F)$, $\eta \neq 0$. Thus (6.11) takes the form

$$\sum_{\xi \in P(F)\backslash G(F)} \left\{ \sum_{\eta \neq 0} \varphi\left[x^{-1}\xi^{-1}\begin{pmatrix}1 & \eta\\ 0 & 1\end{pmatrix}\xi x\right] - \int_{N(A)} \varphi(x^{-1}\xi^{-1}n\xi x)\,\chi_c(H(\xi x))\,dn \right\}$$

which we now have to integrate over $\bar{G}(F)\backslash\bar{G}(A)$. This is the same as the integral over $P(F)Z(A)\backslash G(A)$ of

$$\sum_{\eta\neq 0} \varphi\left[x^{-1}\begin{pmatrix}1 & \eta\\ 0 & 1\end{pmatrix}x\right] - \int_A \varphi\left[x^{-1}\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}x\right]\chi_c(H(x))\,du. \tag{6.15}$$

To evaluate this integral we shall use Poisson's summation formula.

Set

$$F(x) = \int_K \varphi\left[k^{-1}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}k\right]dk \tag{6.16}$$

so that $F(x)$ is a Schwartz-Bruhat function on A. Using Iwasawa's decomposition we get that the integral of (6.15) over $P(F)Z(A)\backslash G(A)$ is

$$\int_{F^\times\backslash A^\times} \left\{ \sum_{\eta\neq 0} F(a\eta) - \hat{F}(0)\chi_c(|a|^{-1})|a|^{-1} \right\} |a|\,d^\times a. \tag{6.17}$$

Using Poisson's summation formula we find this is

$$\int_{|a|\geq 1}\left(\sum_{\eta\neq 0} F(a\eta)\right)|a|\,d^\times a + \int_{|a|\leq 1}\sum_{\eta\neq 0}\hat{F}(a^{-1}\eta)\,d^\times a + \hat{F}(0)$$
$$\cdot \int_{|a|\leq 1}(1-\chi_c(|a|^{-1}))\,d^\times a - F(0)\int_{|a|\leq 1}|a|\,d^\times a.$$

This shows that the integral converges. The term involving $\hat{F}(0)$ depends on c and equals $(\log c)\hat{F}(0)\mathrm{vol}(F^\times\backslash F^0(A))$.

For $\mathrm{Re}(s) > 1$ we also have

$$\int_{A^\times} F(a)|a|^s\,d^\times a = \int_{F^\times\backslash A^\times}\sum_{\eta\neq 0} F(a\eta)|a|^s\,d^\times a$$
$$= \int_{|a|\geq 1}\left(\sum_{\eta\neq 0} F(a\eta)\right)|a|^s\,d^\times a + \int_{|a|\leq 1}\left(\sum_{\eta\neq 0}\hat{F}(a^{-1}\eta)\right)|a|^{s-1}\,d^\times a$$
$$+ \hat{F}(0)\int_{|a|\leq 1}|a|^{s-1}d^\times a - F(0)\int_{|a|\leq 1}|a|^s\,d^\times a.$$

Here each term is analytic at $s = 1$ except the term involving $\hat{F}(0)$ which equals $(\hat{F}(0)\,\mathrm{vol}(F^\times\backslash F^0A))/s - 1$. Thus we conclude that (6.17) equals

$$\text{f.p.}\left(\int_{A^\times} F(a)|a|\,d^\times a\right) + (\log c)\hat{F}(0)\mathrm{vol}(F^\times\backslash F^0(A))$$

where the symbol f.p. means f.p.$(\ldots)$ = value at 1 of $\{\int_{A^\times} F(a)|a|^s\,d^\times a$ − principal part at 1}. Changing a to a^{-1} in the "f.p. integral" we get (finally) that the integral of (6.11) is

$$\text{(6.18)}\quad \begin{aligned} &\text{f.p.} \int_{A^\times}\int_K \varphi\left[k^{-1}\begin{pmatrix} a^{-1} & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1\end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}k\right]|a|^{-1}\, d^\times a\, dk \\ &+ (\log c)\mathrm{vol}(F^\times\backslash F^0(A)) \int_K\int_A \varphi\left(k^{-1}\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix}k\right] dx\, dk. \end{aligned}$$

Now we deal with the term (6.12). In this term we can write γ in the form $\gamma = \xi^{-1}\left(\begin{smallmatrix} \alpha & 0 \\ 0 & 1\end{smallmatrix}\right)\xi$ with ξ in $A(F)\backslash G(F)$, α in $F^\times$, $\alpha \neq 1$. But when ξ varies through a system of representatives for $A(F)\backslash G(F)$ we obtain each γ *twice* since

$$w\begin{pmatrix} \alpha & 0 \\ 0 & 1\end{pmatrix}w^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \alpha\end{pmatrix} = \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & 1\end{pmatrix} \quad \text{in } \tilde{G}.$$

Thus the first term in (6.12) is

$$\begin{aligned} &\frac{1}{2}\sum_{\xi\in A(F)\backslash G(F)}\ \sum_{\alpha\neq 1}\varphi\left[x^{-1}\xi^{-1}\begin{pmatrix} \alpha & 0 \\ 0 & 1\end{pmatrix}\xi x\right] \\ &\quad = \frac{1}{2}\sum_{\xi\in P(F)\backslash G(F)}\ \sum_{\nu\in N(F)}\ \sum_{\alpha\neq 1}\varphi\left[x^{-1}\xi^{-1}\nu^{-1}\begin{pmatrix} \alpha & 0 \\ 0 & 1\end{pmatrix}\nu\xi x\right]. \end{aligned}$$

The integral of all of (6.12) is therefore the integral over $\tilde{G}(F)\backslash\tilde{G}(A)$ of

$$\begin{aligned} &\frac{1}{2}\sum_{\xi\in P(F)\backslash G(F)}\ \sum_{\nu\in N(F)}\ \sum_{\alpha\neq 1}\varphi\left[x^{-1}\xi^{-1}\nu^{-1}\begin{pmatrix} \alpha & 0 \\ 0 & 1\end{pmatrix}\nu\xi x\right] \\ &\quad - \sum_{\xi\in P(F)\backslash G(F)}\int_{N(A)}\sum_{\alpha\neq 1}\varphi\left[x^{-1}\xi^{-1}\begin{pmatrix} \alpha & 0 \\ 0 & 1\end{pmatrix}n\xi x\right]\chi_c(H(\xi x)). \end{aligned}$$

Making the change of variables on $N(A)$ given by

$$n \to \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & 1\end{pmatrix}n^{-1}\begin{pmatrix} \alpha & 0 \\ 0 & 1\end{pmatrix}n \qquad (\alpha \neq 1)$$

we see that the integral of (6.12) over $\tilde{G}(F)\backslash\tilde{G}(A)$ equals the integral over $P(F)Z(A)\backslash G(A)$ of

$$\begin{aligned} &\frac{1}{2}\sum_{\alpha\neq 1}\ \sum_{\nu\in N(F)}\varphi\left[x^{-1}\nu^{-1}\begin{pmatrix} \alpha & 0 \\ 0 & 1\end{pmatrix}\nu x\right] \\ &\quad - \int_{N(A)}\sum_{\alpha\neq 1}\varphi\left[x^{-1}n^{-1}\begin{pmatrix} \alpha & 0 \\ 0 & 1\end{pmatrix}nx\right]\chi_c(H(x))\, dn. \end{aligned}$$

To compute this last integral we have to first integrate over $N(F)\backslash N(A)$ and then over $N(A)A(F)Z(A)\backslash G(A)$. The integration over $N(F)\backslash N(A)$ gives

$$\begin{aligned} &\frac{1}{2}\sum_{\alpha\neq 1}\int_{N(A)}\sum\varphi\left[x^{-1}n^{-1}\begin{pmatrix} \alpha & 0 \\ 0 & 1\end{pmatrix}nx\right]dn \\ &\quad - \int_{N(A)}\sum_{\alpha\neq 1}\varphi\left[x^{-1}n^{-1}\begin{pmatrix} \alpha & 0 \\ 0 & 1\end{pmatrix}nx\right]\chi_c(H(x))\, dn \end{aligned}$$

and the subsequent integral over $N(A)A(F)Z(A)\backslash G(A)$ is the same as the integral of

$$(6.19)\qquad \sum_{\alpha\neq 1}\varphi\left[x^{-1}\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}x\right]\left(\frac{1}{2}-\chi_c(H(x))\right)$$

over $A(F)Z(A)\backslash G(A)$.

Note now that the first factor in (6.19) does not change when x is replaced by wx but the second factor does. In any case, w normalizes $A(F)$. Thus we see that the integral of (6.12) is also the integral over $A(F)Z(A)\backslash G(A)$ of

$$\frac{1}{2}\sum_{\alpha\neq 1}\varphi\left[x^{-1}\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}x\right](1-\chi_c(H(x))-\chi_c(H(wx))).$$

Using Iwasawa's decomposition we get that this integral is

$$(6.20)\qquad \begin{aligned}&\frac{1}{2}\int_K\int_{N(A)}\sum_{\alpha\neq 1}\varphi\left[k^{-1}n^{-1}\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}nk\right]\\ &\qquad\cdot\left(\int_{F^\times\backslash A^\times}(1-\chi_c(|a|)-\chi_c(|a|^{-1}H(wn)))\,d^\times a\right)dk\,du.\end{aligned}$$

But $c > 1$ and $H(wn) \leqq 1$. Therefore in (6.20) the integrand in the inner integral vanishes unless $c^{-1}H(wn) < |a| < c$ in which case it equals 1. Thus the integral of (6.12) over $\tilde{G}(F)\backslash\tilde{G}(A)$ is

$$(6.21)\qquad \begin{aligned}&(\log c)\mathrm{vol}(F^\times\backslash F^0(A))\int_K\int_{N(A)}\sum_{\alpha\neq 1}\varphi\left[k^{-1}\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}nk\right]dk\,dn\\ &-\frac{1}{2}\mathrm{vol}(F^\times\backslash F^0(A))\int_K\int_{N(A)}\sum_{\alpha\neq 1}\varphi\left[k^{-1}n^{-1}\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}nk\right]\log H(wnk)\,dn\,dk.\end{aligned}$$

Summing up (6.13), (6.14), (8.16), and (6.21), we get

PROPOSITION (6.22).

$$\int \Lambda_2^c K(x,x)\,dx$$

$$= \mathrm{vol}(\tilde{G}(F)\backslash\tilde{G}(A))\varphi(e) + \int_{\tilde{G}(F)\backslash\tilde{G}(A)}\sum_{\gamma\text{ elliptic}}\varphi(x^{-1}\gamma x)\,dx$$

$$(6.23)\qquad + \text{f.p.}\int_{Z(A)N(A)\backslash G(A)}\varphi\left[g^{-1}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}g\right]dg$$

$$(6.24)\qquad + (\log c)\mathrm{vol}(F^\times\backslash F^0(A))\int_K\int_{N(A)}\sum_{\alpha\in F^\times}\varphi\left[k^{-1}\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}nk\right]dn\,dk$$

$$(6.25)\qquad -\frac{1}{2}\mathrm{vol}(F^\times\backslash F^0(A))\int_K\int_{N(A)}\sum_{\alpha\neq 1}\varphi\left[h^{-1}n^{-1}\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}nh\right]\log H(wnk)\,dn\,dk.$$

REMARK. The term (6.24) depends on c but $\mathrm{tr}\,\rho_\omega(\varphi)$ does not. Thus we can expect (6.24) to cancel with another term later on.

B. *Contribution from the continuous spectrum.* Recall that

$$\Lambda_2^c K_{\mathrm{cont}}(g,h) = \Lambda_2^c\left\{\frac{1}{4\pi}\sum_{\alpha,\beta}\int_{-\infty}^{\infty}(\pi_{iy}(\varphi)\Phi_\beta,\Phi_\alpha)E(\Phi_\alpha(iy),g)\bar{E}(\Phi_\beta(iy),h)\,dy\right\}.$$

It is not hard to see that this also equals

$$\frac{1}{4\pi}\sum_{\alpha,\beta}\int_{-\infty}^{\infty}(\pi_{iy}(\varphi)\Phi_\beta,\Phi_\alpha)E(\Phi_\alpha(iy),g)\Lambda^c\bar{E}(\Phi_\beta(iy),h)\,dy.$$

So taking it for granted that we can interchange orders of integration, we find

$$\int \Lambda_2^c K_{\text{cont}}(x,x)\,dx$$

$$= \frac{1}{4\pi}\sum_{\alpha,\beta}\int_{-\infty}^{\infty}(\pi_{iy}(\varphi)\Phi_\beta,\Phi_\alpha)\,dy\int_{\bar{G}(F)\backslash\bar{G}(A)}E(\Phi_\alpha(iy),x)\Lambda^c\bar{E}(\Phi_\beta(iy),x)\,dx.$$

Note that the inner integral here has already been computed in §5 (cf. (5.14) with $h_1(s)=\Phi_\alpha(s)$, $h_2(s)=\Phi_\beta(s)$). Plugging in (5.14) then gives

$$\int \Lambda_2^c K_{\text{cont}}(x,x)\,dx$$

$$= \frac{(\log c)}{2\pi}\sum_{\alpha,\beta}\int_{-\infty}^{\infty}(\pi_{iy}(\varphi)\Phi_\beta,\Phi_\alpha)\,(\Phi_\alpha,\Phi_\beta)\,dy \tag{6.26}$$

$$-\frac{1}{4\pi}\sum_{\alpha,\beta}\int_{-\infty}^{\infty}(M(-iy)M'(iy)\Phi_\alpha,\Phi_\beta)(\pi_{iy}(\varphi)\Phi_\beta,\Phi_\alpha)\,dy \tag{6.27}$$

$$+\frac{1}{4\pi}\sum_{\alpha,\beta}\int_{-\infty}^{\infty}(\pi_{iy}(\varphi)\Phi_\beta,\Phi_\alpha)\left\{(\Phi_\alpha,M(iy)\Phi_\beta)\frac{c^{2iy}}{2iy}-(M(iy)\Phi_\alpha,\Phi_\beta)\frac{c^{-2iy}}{2iy}\right\}dy. \tag{6.28}$$

After exchanging $\sum$ and $\int$ the term (6.26) can also be written $(\log c/2\pi)\cdot\int_{-\infty}^{\infty}\operatorname{tr}\pi_{iy}(\varphi)\,dy$. But π_{iy} is an induced representation whose trace is easily computed to be

$$\operatorname{vol}(F^\times\backslash F^0(A))\int_K\int_{N(A)}\sum_{\alpha\in F^\times}\left(\int_{F_\infty^+}\varphi\left(k^{-1}\begin{pmatrix}t\alpha & 0\\ 0 & 1\end{pmatrix}nk\right)|t|^{iy+1/2}d^\times t\right)dn\,dk.$$

So after using the Fourier inversion formula we find that (6.26) precisely equals (6.24), i.e., it cancels (6.24) just as expected.

Now rewrite (6.27) as

$$-\frac{1}{4\pi}\int_{-\infty}^{\infty}\operatorname{tr}(M(-iy)M'(iy)\pi_{iy}(\varphi))\,dy. \tag{6.29}$$

As for (6.28), it can be written as

$$\frac{1}{4\pi}\sum_{\beta}\int_{-\infty}^{\infty}\left\{(\pi_{iy}(\varphi)\Phi_\beta,M(iy)\Phi_\beta)\frac{c^{2iy}}{2iy}-(\pi_{iy}(\varphi)\Phi_\beta,M(-iy)\Phi_\beta)\frac{c^{-2iy}}{2iy}\right\}dy$$

or

$$\begin{aligned}&\frac{1}{4\pi}\sum_{\beta}\int_{-\infty}^{\infty}(M(-iy)\pi_{iy}(\varphi)\Phi_\beta,\Phi_\beta)\frac{c^{2iy}-c^{-2iy}}{2iy}\,dy\\&+\frac{1}{4\pi}\sum_{\beta}\int_{-\infty}^{\infty}\{(\pi_{iy}(\varphi)\Phi_\beta,M(iy)\Phi_\beta)-(\pi_{iy}(\varphi)\Phi_\beta,M(-iy)\Phi_\beta)\}\frac{c^{-2iy}}{2iy}\,dy.\end{aligned} \tag{6.30}$$

But the last term here is the Fourier transform of an integrable function (namely $\sum(\pi_{iy}(\varphi)\Phi_\beta,M(iy)\Phi_\beta)\ldots$) at $\log c/\pi$ (cf. Lemma 9.14 of [Ge]). Thus it tends to zero as $c\to+\infty$. To evaluate the first term we need the following lemma:

LEMMA (6.31). *If F is continuous on $\mathbf{R}$, and F, $\hat{F}$ are integrable, then*

$$\lim_{x\to+\infty}\int \frac{e^{2\pi ixy}-e^{-2\pi ixy}}{y}F(y)\,dy = 2\pi iF(0).$$

PROOF. Set $G(x) = \int_{-\infty}^{\infty}((e^{2\pi ixy}-e^{-2\pi ixy}/y))F(y)\,dy$. Then $G(0)=0$, and

$$G'(x) = 2\pi i\int (e^{2\pi ixy}+e^{-2\pi ixy})F(y)\,dy$$
$$= 2\pi i(\hat{F}(x)+\hat{F}(-x)).$$

Therefore

$$G(x) = 2\pi i\int_0^x (\hat{F}(t)+\hat{F}(-t))\,dt$$
$$= 2\pi i\int_{-x}^{x}\hat{F}(t)\,dt \qquad \text{for } x>0.$$

So as $x\to+\infty$, $G(x)$ tends to $2\pi iF(0)$.

Applying Lemma (6.31) to (6.30) we conclude that (6.28) tends to

$$(6.32)\qquad \tfrac{1}{4}\sum_{\beta}(M(0)\pi_0(\varphi)\Phi_\beta,\Phi_\beta) = \tfrac{1}{4}\operatorname{tr} M(0)\pi_0(\varphi)$$

as $c\to+\infty$.

C. *Trace formula.* We leave it to the reader to check that

$$\int \Lambda_2^c K_{\mathrm{sp}}(x,x)\,dx \to \int K_{\mathrm{sp}}(x,x)\,dx$$
$$= [\operatorname{vol}(\tilde{G}(F)/\tilde{G}(A))]^{-1}\sum_{\chi^2=\omega}\int \varphi(x)\bar{\chi}(x)\,dx$$

as $c\to+\infty$. Therefore, by combining (6.3), Proposition (6.22), (6.26), (6.27), and (6.32) we obtain

THEOREM (6.33).

$$\operatorname{tr}\rho_{\mathrm{cusp}}(\varphi)+\operatorname{tr}\rho_{\mathrm{sp}}(\varphi) = \operatorname{vol}(\tilde{G}(F)\backslash\tilde{G}(A))\varphi(e)$$
$$+\int_{G(F)\backslash\bar{G}(A)}\Big(\sum_{\gamma\ elliptic}\varphi(x^{-1}\gamma x)\Big)\,dx$$
$$(6.34)\qquad +\ \mathrm{f.p.}\int_{Z(A)N(A)\backslash G(A)}\varphi\left[g^{-1}\begin{pmatrix}1&1\\0&1\end{pmatrix}g\right]dg$$
$$(6.35)\qquad -\frac{1}{2}\operatorname{vol}(F^\times\backslash F^0(A))\int_K\int_{N(A)}\sum_{\alpha\neq 1}\varphi\left[k^{-1}n^{-1}\begin{pmatrix}\alpha&0\\0&1\end{pmatrix}nk\right]\log H[wnk]\,dn\,dk$$
$$(6.36)\qquad +\frac{1}{4\pi}\int_{-\infty}^{\infty}\operatorname{tr}(M(-iy)M'(iy)\pi_{iy}(\varphi))\,dy$$
$$(6.37)\qquad -\tfrac{1}{4}\operatorname{tr}(M(0)\pi_0(\varphi)).$$

Here the f.p. term is computed as the value at $s=1$ of

$$\left\{\int \varphi\left[k^{-1}\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}k\right]|a|^s\, d^\times a\, dk - \text{principal part at } s = 1\right\}.$$

Recall also that we have identified all fibres $\mathbf{H}(s)$ with $\mathbf{H}$ in order to define $M'(s) = dM/ds$.

7. A second form of the trace formula. Our next goal is to express the right-hand side of (6.33) in terms of *local* distributions, most of them invariant. This involves replacing the Mellin transform on F_∞^+ by a Mellin transform on $F^\times\backslash F^\times(\mathbf{A})$, the fiber bundle $\mathbf{H}(s)$ by the bundle $\mathbf{H}(\eta)$, and the operator $M(s)$ by the operator $M(\eta)$.

To express the right-hand side of (6.33) as a sum *only* of invariant distributions entails applying a form of Poisson summation to some nonsmooth functions. This analysis is carried out in §7 of **[La 2]** but is not needed for the application we have in mind.

A. *Normalization of Haar measures*. We normalize the Haar measure on $\mathbf{A}^\times$ as follows. Select (in any way) a nontrivial additive character $\psi = \pi\psi_v$ of $F\backslash\mathbf{A}$. For each place v let dx_v be the self-dual Haar measure on F_v with respect to ψ_v. Then the Haar measure we take on $F_v^\times$ is $d^\times x_v = L(1, 1_v)dx_v/|x_v|$. Note that on $\mathbf{A}$ the Haar measure $dx = \bigotimes dx_v$ is self-dual. On $\mathbf{A}^\times$ the (normalized) Tamagawa measure is

$$d^\times x = \frac{1}{\lambda_{-1}} \otimes d^\times x_v \tag{7.1}$$

where

$$\lambda_{-1} = \lim_{s\to 1}(s-1)L(s, 1_F). \tag{7.2}$$

The map $x \mapsto |x|$ allows us to identify $F^0(\mathbf{A})\backslash\mathbf{A}^\times$ with $\mathbf{R}_+^\times$. The Tamagawa measure μ on $F^0(\mathbf{A})$ is such that the quotient measure $d^\times x/\mu$ (on $F^0(\mathbf{A})\backslash\mathbf{A}^\times$ or $\mathbf{R}_+^\times$) is dt/t and—as shown in Tate's thesis—

$$\operatorname{vol}(F^\times\backslash F^0(\mathbf{A})) = 1. \tag{7.3}$$

To define the Tamagawa measure on $\tilde{G}(\mathbf{A})$ we select in any way an invariant differential form ω of degree 3 defined over F. In particular, we may take ω to be the form whose pull-back through the map

$$(a, x, y) \mapsto \begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ y & 1\end{pmatrix}$$

is the form $(da/a)\, dx\, dy$. Then for $f = \prod f_v$ on $\tilde{G}(\mathbf{A})$,

$$\begin{aligned}\int f(g)\, dg &= \int f(g)|\omega(g)| = \prod_v \int f_v(g_v)|\omega_v(g)|\\ &= \prod_v \int f_v\left[\begin{pmatrix}a_v & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & x_v\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ y_v & 1\end{pmatrix}\right]\frac{da_v}{|a_v|}\, dx_v\, dy_v.\end{aligned} \tag{7.4}$$

On the other hand, the Haar measure on K is normalized by the condition

$$\int f(g)\, dg = \int f\left[\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}k\right]d^\times a\, dx\, dk. \tag{7.5}$$

Similarly the Haar measure on K_v is defined by

$$\int f(g_v)\,dg_v = \int f\left[\begin{pmatrix} a_v & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x_v \\ 0 & 1 \end{pmatrix} k_v\right] d^\times a_v\, dx_v\, dk_v. \tag{7.6}$$

Then a simple computation shows that

$$\operatorname{vol}(K_v) = \frac{1}{L(2, 1_v)\varepsilon(0, 1_v, \psi_v)}. \tag{7.7}$$

Since $dk = \lambda_{-1} \otimes_v dk_v$ we also have

$$\operatorname{vol}(K) = \frac{\lambda_{-1}}{L(2, 1_F)\varepsilon(0, 1_F, \psi_F)}. \tag{7.8}$$

Now we replace the Mellin transform on F_∞^+ (introduced in §3) by a Mellin transform on $A^\times$, namely

$$\hat{f}(\chi)(g) = \int_{A^\times} f\left[\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right] \chi^{-1}(a)|a|^{-1/2}\, d^\times a.$$

Then formula (3.19) should be replaced by

$$\begin{aligned}(F_1, F_2) &= \frac{1}{2\pi i}\sum_\chi \int_{x-i\infty}^{x+i\infty} (\hat{f}_1(\chi\alpha^s), \hat{f}_2(\chi\alpha^{-\bar{s}}))\, ds \\ &+ \frac{1}{2\pi i}\sum_\chi \int_{x-i\infty}^{x+i\infty} (M(\chi\alpha^s, \chi^{-1}\omega\alpha^{-s})\hat{f}_1(\chi\alpha^s), \hat{f}_2(\chi^{-1}\omega\alpha^{\bar{s}}))\, ds.\end{aligned} \tag{7.9}$$

Here χ runs through the set of characters of $F^\times(A)$ trivial on $F^\times F_\infty^+$; cf. [**Ge**, p. 167].

B. *The* "f.p." *integral.* In order to compute (6.34) we have to remove from $\int \varphi[k^{-1}(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix})k]\,|a|^s\, d^\times a\, dk$ the principal part at $s = 1$ and then set $s = 1$. So first we write this integral as the product $L(s, 1_F)\theta(s)$ where

$$\theta(s) = \frac{1}{L(s, 1_F)} \int \varphi\left[k^{-1}\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} k\right] |a|^s\, d^\times a\, dk. \tag{7.10}$$

At $s = 1$ the function $\theta(s)$ is holomorphic. On the other hand, $L(s, 1_F) = \lambda_{-1}/(s - 1) + \lambda_0 + \cdots$. Therefore the f.p. integral is $\lambda_{-1}\theta'(1) + \lambda_0\theta(1)$. But $d^\times a = (\lambda_{-1})^{-1} \otimes d^\times a_v$ and $dk = \lambda_{-1} \otimes dk_v$. So for $\operatorname{Re}(s) > 1$,

$$\theta(s) = \prod_v \frac{1}{L(s, 1_v)} \int \varphi_v\left[k_v^{-1}\begin{pmatrix} 1 & a_v \\ 0 & 1 \end{pmatrix} k_v\right] |a_v|^s\, d^\times a_v\, dk_v$$

and since almost all factors here are equal to one,

$$\begin{aligned}\theta(1) &= \prod_v \frac{1}{L(1, 1_v)} \int \varphi_v\left[k_v^{-1}\begin{pmatrix} a_v^{-1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a_v & 0 \\ 0 & 1 \end{pmatrix} k_v\right] |a_v|\, d^\times a_v\, dk_v \\ &= \prod_v \frac{1}{L(1, 1_v)} \int_{Z_v N_v \backslash \bar{G}_v} \varphi_v\left[g_v^{-1}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} g_v\right] dg_v.\end{aligned}$$

I.e., $\theta(1)$ is the product of the orbital integral for $(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix})$ with a convergence factor (which is "missing" in the original f.p. integral).

Similarly, taking the derivative of $\theta(s)$ at $s = 1$, we get

(7.11)
$$\theta'(1) = \sum_u \prod_{v\neq u} \int_{Z_vN_v\backslash G_v} \varphi_v\left[g_v^{-1}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}g_v\right]dg_v \times \frac{d}{ds}\bigg|_{s=1} \int \varphi_u\left[k_u^{-1}\begin{pmatrix}1 & a_u\\ 0 & 1\end{pmatrix}k_u\right]|a_u|^s\, d^\times a_u / L(s, 1_u).$$

Here the sum is extended over all places u, but only finitely many have a nonzero contribution.

C. *Computation of* (6.35). Recall that $\mathrm{vol}(F^\times\backslash F^0(A)) = 1$, $dk = \lambda_{-1} \otimes dk_v$, $dn = \bigotimes dn_v$, and $H(g) = \prod_v H_v(g_v)$ where

$$H\left[\begin{pmatrix}a_v & x_v\\ 0 & b_v\end{pmatrix}k_v\right] = \left|\frac{a_v}{b_v}\right|.$$

Thus (6.35) is equal to

$$\frac{-\lambda_{-1}}{2}\sum_u \sum_{\alpha\neq 1}\prod_{v\neq u}\int \varphi_v\left[k_v^{-1}n_v^{-1}\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}n_vk_v\right]dn_v\, dk_v \cdot \int \varphi_u\left[k_u^{-1}n_u^{-1}\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}n_uk_u\right]\log(H_u[wn_u])\, dn_u$$

or

(7.12)
$$\frac{-\lambda_{-1}}{2}\sum_{u;\alpha\neq 1}\prod_{v\neq u}\int \varphi_v\left[g_v^{-1}\begin{pmatrix}\alpha & 0\\ 1 & 0\end{pmatrix}g_v\right]dg_v \int \varphi_u\left[g_u^{-1}\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}g_u\right]\mu_u(g_u)\, dg_u$$

where

$$\mu_u\left[\begin{pmatrix}a_v & 0\\ 0 & b_v\end{pmatrix}\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}k\right] = \log H_u[wn_u], \qquad g_v \in A_v\backslash G_v.$$

Since μ_u vanishes on K_u, the sum above has only finitely many nonzero terms.

D. *Computation of* (6.36) *and* (6.37). To begin with, (6.36) can be written as

$$\frac{1}{4\pi}\sum_\chi \int_{-i\infty}^{i\infty} \mathrm{tr}(M(\eta)^{-1}M'(\eta)\pi_\eta(\varphi))\, dy$$

where $\eta = (\chi\alpha^{iy}, \chi^{-1}\omega\alpha^{-iy})$, π_η is the representation of $G(A)$ on $H(\eta)$, and the sum is over all characters χ of $F^\times\backslash A^\times$ whose restriction to F_∞^+ is trivial The derivative is defined by $M'(\eta)\varphi = (d/ds)M(\eta)\varphi$ if $\eta = (\chi\alpha^s, \chi^{-1}\omega\alpha^{-s})$ and $\varphi|_K$ is independent of s. (To define the derivative we have once again trivialized the fibre-bundle.) Since $M(\eta) = m(\eta) \otimes_v R_v(\eta_v)$ (where $m(\eta)$ is the scalar described by (4.15)) we get (by taking the logarithmic derivative) that

$$M'(\eta)M^{-1}(\eta) = m'(\eta)I + \sum_u R_u^{-1}(\eta_u)R_u'(\eta_u) \bigotimes_{v\neq u} I_v.$$

But if $A = \bigotimes_v A_v$ where A_v is an operator on $H(\eta_v)$ then $\mathrm{tr}(A) = \prod_v \mathrm{tr}(A_v)$. Thus (6.36) is equal to

(7.13)
$$\frac{1}{4\pi}\sum_\chi \int_{-i\infty}^{i\infty} m'(\eta)\mathrm{tr}(\pi_\eta(\varphi))\, dy + \sum_u \frac{1}{4\pi}\sum_\chi \int_{-i\infty}^{i\infty}\prod_{v\neq u}\mathrm{tr}(\pi_{\eta_v}(\varphi_v))\mathrm{tr}(R_u(\eta_u)^{-1}R'(\eta_u)\pi_{\eta_u}(\varphi_u))\, dy.$$

Recall that if $\varphi = 1$ on K_u then $R(\eta_u)\varphi = \varphi$ for all η_u. Thus $R'(\eta_u)\varphi = 0$, and the sum above once again has only finitely many nonzero terms.

As for (6.37), we write it as

$$-\frac{1}{4}\sum_{\chi^2=\omega} \operatorname{tr}(M(\chi, \chi)\pi_{(\chi,\chi)}(\varphi)).$$

E. *Summing up.* We have that tr $\rho_{\mathrm{cusp}}(\varphi)$ + tr $\rho_{\mathrm{sp}}(\varphi)$ is equal to the sum of $\mathrm{vol}(\bar{G}(F)\backslash\bar{G}(A))\varphi(e)$,

$$(*)\qquad \sum_{\gamma \text{ elliptic}} \int_{\bar{G}(F)\backslash\bar{G}(A)} \varphi(g^{-1}\gamma g)\, dg$$

and a complementary term. As in the case of division algebras, (∗) can be expressed in terms of local elliptic *orbital integrals* of the form

$$\int_{G_\gamma(F_v)\backslash G_v} \varphi_v(g_v^{-1}\gamma g)\, dg_v, \qquad \gamma \text{ elliptic}.$$

The complementary term can be expressed explicitly in terms of the following local distributions:

$$(7.14)\qquad \int_{Z_vN_v\backslash G_v} \varphi_v\left[g_v^{-1}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}g_v\right] dg_v,$$

$$(7.15)\qquad \int_{A_v\backslash G_v} \varphi_v\left[g_v^{-1}\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}g_v\right] dg_v,$$

$$(7.16)\qquad \operatorname{tr} \pi_{\eta_v}(\varphi_v),$$

$$(7.17)\qquad \frac{d}{ds}\bigg|_{s=1} \frac{1}{L(s, 1_v)} \int \varphi_v\left[k_v^{-1}\begin{pmatrix}1 & a_v\\ 0 & 1\end{pmatrix}k_v\right] |a_v|^s\, d^\times a_v\, dk_v,$$

$$(7.18)\qquad \int \varphi_v\left[g_v^{-1}\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}g_v\right] \mu_v(g_v)\, dg_v \qquad (\alpha \neq 1)$$

and

$$(7.19)\qquad \operatorname{tr}(R_v(\eta_v)^{-1}R_v'(\eta_v)\pi_{\eta_v}(\varphi_v)).$$

The distributions (7.14) and (7.15) are orbital integrals. The distributions (7.16) are invariant and can also be computed in terms of orbital integrals (cf. **[DL]** for example). The distributions (7.17)—(7.19), however, are *not* invariant.

F. *A special case.* The distributions (7.14)—(7.16) enjoy the following property. Suppose

$$(7.20)\qquad \int_{A_v\backslash G_v} \varphi_v\left(g^{-1}\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}g\right) d^\times a = 0$$

for all $a \neq 1$, i.e., the orbital integrals of φ_v vanish for all regular hyperbolic elements. Then each of the distributions (7.14)—(7.16) vanishes. Indeed there is nothing to prove for (7.15), and (7.16) is an integral of orbital integrals of this type. As for (7.14) we note that

$$\int \varphi\left[g^{-1}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}g\right] dg = \lim_{a\to 1} |1 - a^{-1}| \int \varphi\left[g^{-1}\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}g\right] dg,$$

i.e., the nilpotent orbital integral is a limit of hyperbolic orbital integrals.

It follows that if (7.20) is satisfied for a certain place u_1 then the sums in (7.11), (7.12), and (7.13) reduce to the term involving $u = u_1$. On the other hand, tr $\pi_\eta(\varphi)$ vanishes. Thus we have:

THEOREM (7.21). *Suppose condition* (7.20) *is satisfied for two places* $u_1 \neq u_2$. *Then* (7.11)—(7.13) *vanish and we have*

$$\operatorname{tr} \rho_{\text{cusp}}(\varphi) + \operatorname{tr} \rho_{\text{sp}}(\varphi) = \operatorname{vol}(\tilde{G}(F)\backslash\tilde{G}(A))\varphi(e) + \sum_{\gamma \text{ elliptic}} \int_{G(F)\backslash G(A)} \varphi(g^{-1}\gamma g)\, dg.$$

The reader will note that this formula closely resembles formula (1.11).

8. Applications to quaternion algebras. As before, G is the group GL(2) regarded as an F-group and $Z \simeq$ GL(1) is its center. Let D be a quaternion algebra of center F and S the (finite) set of places of F where D does not split. Regard the multiplicative group of D as an algebraic F-group G' and let Z' denote its center. Then for all $v \notin S$ the local groups G_v and G'_v are isomorphic. More precisely, as in §1, the isomorphism $D_v \approx M(2, F_v)$ induces an isomorphism $G_v \approx G'_v$ defined up to inner automorphism. If ε_i, $1 \leqq i \leqq 4$, is an F basis of D then for almost all v we can assume that $\sum R_v\varepsilon_i$ maps to $M(2, R_v)$. Also $K_v = \mathrm{GL}(2, R_v)$ maps to the compact subgroup K'_v of G'_v and the isomorphisms $G_v \simeq G'_v$ give rise to an isomorphism of the restricted products $G^S = \prod_{v \notin S} G_v$, $G'^S = \prod_{v \notin S} G'_v$.

A. *Statement of results.* For the moment, let F be a local field and D a division algebra of center F so that $G'(F) = D^x$; let ν denote the reduced norm on D. Denote by $\mathscr{E}(G(F))$ the set of classes of irreducible admissible representations of $G(F)$ and by $\mathscr{E}_2(G(F))$ the subset of those which are square-integrable (modulo the center). Define $\mathscr{E}(G'(F)) = \mathscr{E}_2(G'(F))$ similarly.

THEOREM (8.1). *There is a unique bijection* $\pi' \leftrightarrow \pi$ *from* $\mathscr{E}(G'(F))$ *to* $\mathscr{E}_2(G(F))$ *such that the characters* $\theta_{\pi'}$ *and* θ_π *of* π' *and* π *satisfy the relation*

$$\theta_{\pi'}(t') = -\theta_\pi(t) \tag{8.2}$$

each time t *and* t' *are regular semisimple elements of* $G'(F)$ *and* $G(F)$ *related by the identities* $\operatorname{tr}(t') = \operatorname{tr}(t)$, $\nu(t') = \det(t)$.

This condition implies that the central quasi-characters of π and π' are the same. Note that G' is an inner twisting of G and $\Phi(G) \subset \Phi(G')$ (cf. [**Bo**]). Thus if $F = \boldsymbol{R}$, the correspondence $\pi' \leftrightarrow \pi$ is the one specified by Langlands; in the *non*-archimedean case one can at least construct the map using "Weil's representation" (as in [**JL**]).

Now let F be a number field and D a division algebra with center F. Let $\mathscr{A}(G')$ be the set of (classes of) automorphic representations of $G'(A)$ and $\mathscr{A}_*(G')$ the subset of those which are not one dimensional. Similarly let $\mathscr{A}_0(G)$ be the set of cuspidal representations of $G(A)$.

THEOREM (8.3) (GLOBAL). *If* π' *is in* $\mathscr{A}_*(G')$ *let* π *be the representation of* $G(A)$ *such that, for* $v \in S$, $\pi'_v \leftrightarrow \pi_v$ *as in* (8.1), *and for* $v \notin S$, $\pi_v \approx \pi'_v$ *via the isomorphism* $G_v \approx G'_v$. *Then* π *is in* $\mathscr{A}_0(G)$. *Moreover, the map* $\pi' \mapsto \pi$ *is a bijection between* $\mathscr{A}_*(G')$ *and the set of* π *in* $\mathscr{A}_0(G)$ *such that* $\pi_v \in \mathscr{E}_2(G_v)$ *for all* $v \in S$.

We shall give two proofs of this result, both somewhat different from the one in **[JL]**. First we shall take Theorem (8.1) for granted and indicate how it together with the trace formula implies Theorem (8.3). Then we shall sketch an alternate proof that essentially implies (rather than depends on) Theorem (8.1). Both proofs use orbital integral techniques (cf. **[Sa]**, **[Sh]**, and **[La 2]**).

B. *Matching orbital integrals* (*and consequences*). Let ω and ω' be differential forms invariant and of maximal degree on $\tilde{G}$ and $\tilde{G}'$ respectively. We assume that ω and ω' are related to one another as in **[JL**, pp. 475 and 503]. Then for each $v \notin S$, $\omega_v = \pm\omega_v'$ and $|\omega_v| = |\omega_v'|$.

For $v \in S$ let us say that $f \in C_c^\infty(G_v, \omega_v^{-1})$ and $\varphi \in C_v^\infty(G_v, \omega_v^{-1})$ have *matching orbital integrals* if

$$\int_{Z_v\backslash G_v} g(g)h(\operatorname{tr} g, \det g)\, dg = \int_{Z_v'\backslash G_v'} \varphi(g)h(\operatorname{tr}(g), \nu(g))\, dg' \tag{8.4}$$

for "any function" h on $F \times F^\times$. In particular, if π_v and π_v' are related as in Theorem (8.1), then

$$\operatorname{tr} \pi_v(f) = -\operatorname{tr} \pi_v'(\varphi). \tag{8.5}$$

On the other hand,

$$\operatorname{tr} \pi_v(f) = \operatorname{tr} \pi_v'(\varphi) \tag{8.6}$$

if $\pi_v = \chi \circ \det$ and $\pi_v' = \chi \circ \nu$.

Condition (8.4) can also be expressed in terms of orbital integrals as follows:

(i) The hyperbolic (regular) orbital integrals of f vanish, i.e.,

$$\int_{A_v\backslash G_v} f\left(g^{-1}\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}g\right) dg = 0 \quad \text{for } a \neq 1. \tag{8.7}$$

(ii) Let $L \subset M(2, F_v)$ and $L' \subset D$ be isomorphic quadratic extensions; then

$$\int_{L_v^\times\backslash G_v} f(g^{-1}tg)\, dg = \int_{L_v'^\times\backslash G_v'} \varphi(g^{-1}t'g)\, dg$$

if there is an isomorphsim $L_v \to L_v'$ taking t in $L^\times - F^\times$ to $t' \in L'^\times - F^\times$. (Here we select in any way a Haar measure on $F_v^\times\backslash L_v^\times$ and transport it to $F_v^\times\backslash L_v'^\times$ via the isomorphism $L_v \to L_v'$; the quotient spaces are then given the quotient measures.) A consequence of the last few identities is

$$f(e) = \varphi(e). \tag{8.8}$$

(This follows, for instance, from the Plancherel formula.) Note too that if π_v is a representation of G_v which is neither square-integrable nor finite dimensional then

$$\operatorname{tr} \pi_v(f) = 0. \tag{8.9}$$

The following lemma results from the characterization of orbital integrals in §4 of **[La 2]**.

LEMMA (8.10). *Given φ in $C_c^\infty(G_v', \omega^{-1})$ there are (many) f in $C_c^\infty(G_v, \omega^{-1})$ with matching orbital integrals.*

REMARK (8.11). Fix a representation π_v' of G_v' and let π_v be the corresponding representation of G_v given by (8.1). There is easily seen to be a φ in $C_c^\infty(G'_v, \omega_v^{-1})$ such that tr $\pi'_v(\varphi) \neq 0$, but tr $\sigma'(\varphi) = 0$ if σ' is any irreducible representation of G'_v (with central quasi-character ω_v) not equivalent to π'_v. Thus if f has matching orbital integrals, tr $\pi_v(f) \neq 0$, but tr $\sigma(f) = 0$ whenever σ is not equivalent to π_v (or finite dimensional).

Now suppose f is a function on $G(A)$ such that $f(zg) = \omega^{-1}(z)f(g)$. Moreover, suppose as before that $f(g) = \prod_v f_v(g_v)$ where $f_v \in C_c^\infty(G_v, \omega_v^{-1})$ and f_v for almost all v is the function defined by

$$\begin{aligned} f_v(g_v) &= \omega_v^{-1}(z) \quad \text{if } g_v = z_v k_v,\ k_v \in K_v,\ z_v \in Z_v, \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

Suppose φ is a function on $G'(A)$ satisfying similar conditions, $\varphi_v \sim f_v$ via the isomorphism $G_v \approx G'_v$ for $v \notin S$, and φ_v and f_v have matching orbital integrals for $v \in S$.

PROPOSITION (8.12). *With φ and f as above,*

$$\text{tr } \rho'_\omega(\varphi) = \text{tr } \rho_{\omega,0}(f) + \text{tr } \rho_{\omega,\mathrm{sp}}(f). \tag{8.13}$$

PROOF. By (1.11) the left-hand side is

$$\begin{aligned} &\mathrm{vol}(Z(A)G'(F)\backslash G'(A))\varphi(e) \\ &+ \frac{1}{2} \sum_{L' \in X'} \mathrm{vol}(A^\times L^\times \backslash L^\times(A)) \sum_{\xi \in F^\times \backslash L'^\times;\xi \neq 1} \int_{L'^\times(A)\backslash G'(A)} \varphi[g^{-1}\xi g]\, dg \end{aligned} \tag{8.14}$$

the sum extending over a set X' of representatives for the classes of quadratic extension L' of F embedded in D. Moreover

$$\int_{L'^\times(A)\backslash G'(A)} \varphi[g^{-1}\xi g]\, dg = \prod_v \int_{L'^\times_v \backslash G'_v} \varphi_v[g_v^{-1}\xi g_v]\, dg_v.$$

Here we have selected for each v a Haar measure on $F_v^\times\backslash L_v^\times$ such that (units $F_v^\times\backslash L_v^\times$) $= 1$ for almost all v; to $A^\times\backslash L^\times(A)$ we have given the product measure.

On the GL(2) side, note that S has at least two places v_1, v_2 and for these places f_{v_1}, f_{v_2} have vanishing hyperbolic orbital integrals. Thus by (7.21), the right-hand side of (8.13) is

$$\begin{aligned} &\mathrm{vol}(Z(A)G(F)\backslash G(A))f(e) \\ &+ \frac{1}{2} \sum_{L} \mathrm{vol}(A^\times L^\times \backslash L^\times(A)) \sum_{\xi \in F^\times \backslash L^\times;\xi \neq 1} \int_{L^\times(A)\backslash G(A)} f(g^{-1}\xi g)\, dg \end{aligned} \tag{8.15}$$

where L runs through a set of representatives for the classes of quadratic extensions of F in $M(2, F)$. The measures are selected as above, and

$$\int f(g^{-1}\xi g^{-1})\, dg = \prod_v \int_{L_v^\times \backslash G_v} f(g_v^{-1}\xi g_v)\, dg_v.$$

Note that if $v \in S$ and L_v is split then the corresponding local integral is 0 by (8.7) (i). Thus we need only sum over the set X of L which do *not* split at any $v \in S$, that is, which embed in D. But for every $L \in X$ there is an L' in X' and an isomorphism

$L \to L'$. So if we assume that the Haar measures of $F_v^\times \backslash L_v^\times$ and $F_v^\times \backslash L_v'^\times$ correspond to one another we have

$$\int_{L_v^\times \backslash G_v} f_v(g^{-1}\xi g)\, dg_v = \int_{L_v'^\times \backslash G_v'} \varphi_v(g^{-1}\xi g)\, dg_v$$

for all v. Indeed for $v \in S$ this is clear, and for $v \notin S$ it follows from the fact that we may modify the isomorphism $G_v \to G_v'$ so as to make it compatible with $L_v^\times \to L_v'^\times$.

We conclude from (8.7) and our choice of φ_v, f_v for $v \notin S$ that in (8.14) and (8.15) the series on L and L' are equal. Since we have used the Tamagawa measures on G and G' we also know that

$$\mathrm{vol}(Z(A)G'(F)\backslash G'(A)) = \mathrm{vol}(Z(A)G(F)\backslash G(A)).$$

Therefore, taking (8.8) into account, (8.12) follows.

C. *Proofs of the main result.* Let $\rho'_{\omega,0}$ be the representation of $G'(A)$ in the orthocomplement of the space spanned by the functions $\chi \circ \nu$ with $\chi^2 = \omega$. From (8.6) and our choice of f_v, φ_v for $v \notin S$ we get

$$\int_{Z(A)\backslash G(A)} f(g)\chi(\det g)\, dg = \int_{Z(A)\backslash G'(A)} \varphi(g)\chi(\nu(g))\, dg.$$

Thus (8.12) implies that

$$\mathrm{tr}\, \rho_{\omega,0}(f) = \mathrm{tr}\, \rho'_{\omega,0}(\varphi). \tag{8.16}$$

FIRST PROOF OF THEOREM (8.2). For each place $v \in S$ fix an irreducible representation σ_v' of G_v' and let σ_v be the corresponding representation of G_v. Choose $\varphi_v \in C_c^\infty(G_v', \omega_v^{-1})$ such that $\mathrm{tr}\, \sigma_v'(\varphi_v) = 1$, and $\mathrm{tr}\, \pi_v'(\varphi_v) = 0$ for π_v' inequivalent to σ_v'. Thus identity (8.16) reads

$$\sum n(\pi')\, \mathrm{tr}\, \pi'(\varphi) = \sum n(\pi)\, \mathrm{tr}\, \pi(f)$$

where the sum on the right-hand (resp. left-hand) side is over all irreducible representations π of $G(A)$ (resp. π' of $G'(A)$) such that $\pi_v \simeq \sigma_v$ (resp. $\pi_v' \simeq \pi_v$) for all v in S, and $n(\pi)$ (resp. $n(\pi')$) is the multiplicity of π in $\rho_{\omega,0}$ (resp. $\rho'_{\omega,0}$). Moreover, if π and π' satisfy these conditions, then, because S has even cardinality, we get $1 = \prod_{v\in S} \pi_v'(\varphi_v) = \prod_{v\in S} \pi_v^v(f_v)$. Thus

$$\sum n(\pi') \prod_{v\notin S} \mathrm{tr}\, \pi_v'(\varphi_v) = \sum n(\pi) \prod_{v\notin S} \mathrm{tr}\, \pi_v(f_v)$$

$$(\pi_v' \simeq \sigma_v \text{ for } v \in S,\ \pi_v \simeq \sigma_v \text{ for } v \in S),$$

and Theorem (8.2) follows from fundamental principles of functional analysis (applied to the isomorphic groups G^S and G'^S). Moreover, if π' corresponds to π, then $n(\pi') = n(\pi)$. So since we already know from [JL] that $n(\pi)$ is at most 1, we have also proved multiplicity one for π'.

SECOND PROOF. We rewrite (8.16) as

$$\sum \prod_v \mathrm{tr}\, \pi_v(f_v) = \sum \prod_v \mathrm{tr}\, \pi_v'(\varphi_v), \tag{8.17}$$

where $\pi = \otimes \pi_v$ (resp. $\pi' = \otimes \pi_v'$) runs through all irreducible subrepresentations of $\rho_{\omega,0}$ (resp. $\rho'_{\omega,0}$). What we shall do now is manipulate (8.17) until it equates the right side to the trace of just *one* representation. (Since we are summing over subrepresentations rather than classes, two representations on the right side of (8.17) may, a priori, be equivalent.)

LEMMA (8.18). *Fix an arbitrary place w outside S and an irreducible unitary representation τ_w of G_w. If $f = \prod f_v$ and $\varphi = \prod \varphi_v$ are as in* (8.12) *then*

$$\sum_{\pi=\otimes\pi_v} \prod_{v\neq w} \operatorname{tr} \pi_v(f) = \sum_{\pi'=\otimes\pi'_v} \prod_{v\neq w} \operatorname{tr} \pi'_v(\varphi_v)$$

$$(\pi \subset \rho_{\omega,0};\ \pi_w \simeq \tau_w)\ (\pi' \subset \rho'_{\omega,0};\ \pi'_w \simeq \tau_w).$$

PROOF. Set

$$a_{\tau_w} = \sum \prod_{v\neq w} \operatorname{tr} \pi_v(f_v) - \sum \prod_{v\neq w} \operatorname{tr} \pi'_v(\varphi_v)$$

$$(\pi \subset \rho_{\omega,0},\ \pi_w \simeq \tau_w)\ (\pi' \subset \rho'_{\omega,0},\ \pi'_w \simeq \tau_w).$$

It suffices to prove $a_{\tau_w} = 0$. But (8.17) can be rewritten as

$$\sum \operatorname{tr} \pi_w(f_w) \prod_{v\neq w} \operatorname{tr} \pi_v(f_v) = \sum_{\pi'\subset\rho'_{\omega,0}} \operatorname{tr}\ \pi'_\omega(\varphi_w) \prod_{v\neq w} \operatorname{tr} \pi'_v(\varphi_v)$$

or, since $G_w \simeq G'_w$ and $f_w \simeq \varphi_w$, as

$$0 = \sum_{\tau_w \text{ in } \mathscr{E}(G_w)} a_{\tau_w} \operatorname{tr} \tau_w(f_w).$$

Moreover, f_w is completely arbitrary in $C_c^\infty(G_w, \omega_w^{-1})$. Therefore a_{τ_w} must be zero by the generalized "linear independence of characters for GL(2)" (cf. [**LL**]).

LEMMA (8.19). *For every $w \notin S$ fix τ_w (which is unramified for almost all w). Then if f_v and φ_v have matching orbital integrals for $v \in S$,*

$$\sum \prod_{v\in S} \operatorname{tr} \pi_v(f_v) = \sum \prod_{v\in S} \operatorname{tr} \pi'_v(\varphi_v) \tag{8.20}$$

$$(\pi \subset \rho_{\omega,0},\ \pi_w \simeq \tau_w,\ w \notin S)\ (\pi' \subset \rho'_{\omega,0},\ \pi_w \simeq \tau_w,\ w \notin S).$$

PROOF. Apply the argument above to the restricted direct product $G^S = \prod_{w\notin S} G_w$.

PROPOSITION (8.21). *Given $\pi' = \otimes\pi'_v$ in $\rho'_{\omega,0}$ there exists a unique $\pi = \otimes\pi_v$ in $\rho_{\omega,0}$ such that $\pi_v \approx \pi'_v$ for all $v \notin S$. Moreover $\pi_v \in \mathscr{E}_2(G_v)$ for all v in S.*

PROOF. Uniqueness follows from the strong multiplicity one theorem for GL(2). What remains to be shown is that there is at least one such π, and that π_v belong to $\mathscr{E}_2(G_v)$ for all v in S.

Suppose no such π exists. Then applying (8.19) with $\tau_w = \pi'_w$ for all $w \notin S$ we conclude

$$\sum \prod_{v\in S} \operatorname{tr} \sigma'_v(\varphi_v) = 0 \qquad (\sigma' \subset \rho'_{\omega,0},\ \sigma'_w \simeq \pi'_w \text{ for } w \in S). \tag{8.22}$$

But (8.22) is a sum of characters of the group $G'_S = \prod_{v\in S} G'_v$. Thus by the linear independence of characters for G'_S we get a contradiction.

Now suppose $\pi_{v_0} \notin \mathscr{E}_2(G_v)$ for some v_0 in S. The fact that π is cuspidal excludes the possibility that π_{v_0} is finite dimensional. So by (8.9) we have $\pi_{v_0}(f_{v_0}) = 0$, and again by (8.22) we get a contradiction.

COROLLARY (8.23). *The left-hand side of* (8.20) *reduces to exactly one term when* $\tau_w = \pi'_w$ *and* $\bigotimes \pi'_v$ *is in* $\rho'_{\omega,0}$.

PROPOSITION (8.24). *Given* $\pi = \bigotimes \pi_v$ *in* $\rho_{\omega,0}$ *such that* $\pi_v \in \mathscr{E}_2(G_v)$ *for all* $v \in S$ *there exists a unique* $\pi' = \bigotimes \pi'_v$ *in* $\rho'_{\omega,0}$ *such that* $\pi'_v = \pi_v$ *for all* $v \notin S$. *Moreover, for each* $v \in S$, *there is* $\varepsilon_v = \pm 1$ *such that* $\theta'_{\pi_v}(t') = \varepsilon_v \theta_{\pi_v}(t)$ *each time* $t' \in G_v$ *and* $t \in G_v$ *satisfy the conditions of* (8.1).

Later we shall see that ε_v in fact equals 1.

PROOF. For $v \in S$ let X_v denote a set of representatives for the equivalence classes of quadratic extensions L_v of F_v contained in $M(2, F_v)$. As in [**JL**, §15], we can introduce a measure μ_v on $\bigcup F_v^\times \backslash L_v^\times$ $(L_v \in X_v)$ and we consider the Hilbert space $\mathscr{H}_v = \mathscr{H}_v(\omega_v, \mu_v)$ of all functions h on that union which satisfy $h(zl) = \omega_v(z)h(l)$, for $z \in F_v^\times$, and $\int |h(x)|^2 \, d\mu_v(x) < +\infty$; the functions θ_λ, $\lambda \in \mathscr{E}_2(G_v, \omega_v)$, then determine an orthonormal basis of $\mathscr{H}_v$.

Similarly we can introduce a Hilbert space $\mathscr{H}'_v$ for the group G'_v. But since any quadratic extension of F_v embeds into D_v, we may identify $\mathscr{H}_v$ and $\mathscr{H}'_v$.

Recall $\pi = \bigotimes_v \pi_v \subset \rho_{\omega,0}$ is such that $\pi_v \in \mathscr{E}_2(G_v)$ for all $v \in S$. So for each $v \in S$ there is $\sigma_v \in \mathscr{E}(G'_v, \omega_v)$ such that (when we regard θ_{π_v} and θ_{σ_v} as elements of $\mathscr{H}_v$) $\langle \theta_{\pi_v}, \theta_{\sigma_v} \rangle = a_v \neq 0$. Since both θ_{π_v} and θ_{σ_v} are unit vectors, it follows that $|a_v| \leqq 1$.

Now, fix φ_v to be θ_{σ_v} when $v \in S$. Then for any $\pi'_v \in \mathscr{E}(G'_v, \omega_v)$,

$$\begin{aligned} \operatorname{tr} \pi'_v(\varphi_v) &= 1 \quad \text{if } \pi'_v \simeq \sigma_v, \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

In particular, if $f_v \in C_c^\infty(G_v, \omega_v^{-1})$ and φ_v have matching orbital integrals, then (using the notations of [**JL**, §15])

$$\begin{aligned} \operatorname{tr} \pi_v(f_v) &= \int_{Z_v \backslash G_v} f_v(g) \theta_{\pi_v}(g) \, dg \\ &= \sum_{L_v} \frac{1}{2} \int_{Z_v \backslash L_v^\times} \delta(b) \theta_{\pi_v}(b) \int_{L_v'^\times \backslash G'_v} \theta_{\sigma_v}(g^{-1}bg) \, dg \\ &= \langle \theta_{\pi_v}, \theta_{\sigma_v} \rangle = a_v. \end{aligned}$$

This means (8.20) with $\tau_v = \pi_v$ for $v \notin S$ reduces to the identity

$$\prod_{v \in S} a_v = \sum 1 \qquad (\pi' \subset \rho'_{\omega,0}, \pi'_v \simeq \sigma_v \text{ for } v \in S, \pi'_v \simeq \pi_v \text{ for } v \notin S).$$

From this we conclude that the right-hand side has only 1 term (once again proving multiplicity one for G'). We also conclude that $|a_v| = 1$. So since a simple argument involving π_v and $\tilde{\pi}_v$ implies a_v must be real, we conclude finally that $a_v = \pm 1$, i.e., $\langle \theta_{\pi_v}, \theta_{\sigma_v} \rangle = \pm 1$. But this implies that $\pi'_v = \sigma_v$ is completely determined by π_v and that $\theta_{\pi_v} = \pm \theta_{\pi'_v} = \theta_{\sigma_v}$ (regarded as elements of $\mathscr{H}$). Thus the proposition follows.

It remains to complete the local correspondence asserted by Theorem (8.1) and to show that θ_{π_v} actually equals $-\theta_{\pi'_v}$. For this we need to embed an arbitrary local representation π'_v of G'_v in an automorphic representation π' and exploit the fact that the cardinality of S is even. For details, see the original arguments in [**Fl**].

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Cornell University

Columbia University

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EISENSTEIN SERIES AND THE TRACE FORMULA

JAMES ARTHUR

Part I. Eisenstein series

The spectral theory of Eisenstein series was begun by Selberg. It was completed by Langlands in a manuscript which was for a long time unpublished but which recently has appeared [1]. The main references are

1. Langlands, *On the functional equations satisfied by Eisenstein series*, Springer-Verlag, Berlin, 1976.

2. ------, *Eisenstein series*, Algebraic Groups and Discontinuous Subgroups, Summer Research Institute (Univ. Colorado, 1965), Proc. Sympos. Pure Math., vol. 9, Amer. Math. Soc., Providence, R. I. 1966.

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In the first part of these notes we shall try to describe the main ideas in the theory.

Let G be a reductive algebraic matrix group over $\boldsymbol{Q}$. Then $G(\boldsymbol{A})$ is the restricted direct product over all valuations v of the groups $G(\boldsymbol{Q}_v)$. If v is finite, define K_v to be $G(\mathfrak{o}_v)$ if this latter group is a special maximal compact subgroup of $G(\boldsymbol{Q}_v)$. This takes care of almost all v. For the remaining finite v, we let K_v be *any* fixed special maximal compact subgroup of $G(\boldsymbol{Q}_v)$. We also fix a minimal parabolic subgroup P_0, defined over $\boldsymbol{Q}$, and a Levi component M_0 of P_0. Let A_0 be the maximal split torus in the center of M_0. Let $K_{\boldsymbol{R}}$ be a fixed maximal compact subgroup of $G(\boldsymbol{R})$ whose Lie algebra is orthogonal to the Lie algebra of $A_0(\boldsymbol{R})$ under the Killing form. Then $K = \prod_v K_v$ is a maximal compact subgroup of $G(\boldsymbol{A})$.

For most of these notes we shall deal only with standard parabolic subgroups; that is, parabolic subgroups P, defined over $\boldsymbol{Q}$, which contain P_0. Fix such a P. Let N_P be the unipotent radical of P, and let M_P be the unique Levi component of P which contains M_0. Then the split component, A_P, of the center of M_P is contained in A_0. If $X(M_P)_{\boldsymbol{Q}}$ is the group of characters of M_P defined over $\boldsymbol{Q}$, define $\mathfrak{a}_P = \mathrm{Hom}((X_P)_{\boldsymbol{Q}}, \boldsymbol{R})$. Then if $m = \prod_v m_v$ lies in $M(\boldsymbol{A})$, we define a vector $H_M(m)$ in $\mathfrak{a}_P$ by

$$\exp(\langle H_M(m), \chi\rangle) = |\chi(m)| = \prod_v |\chi(m_v)|_v.$$

Let $M_P(\boldsymbol{A})^1$ be the kernel of the homomorphism H_M. Then $M_P(\boldsymbol{A})$ is the direct

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product of $M_P(\boldsymbol{A})^1$ and $A(\boldsymbol{R})^0$, the connected component of 1 in $A(\boldsymbol{R})$. Since $G(\boldsymbol{A})$ equals $N_P(\boldsymbol{A})M_P(\boldsymbol{A})K$, we can write any $x \in G(\boldsymbol{A})$ as nmk, $n \in N_P(\boldsymbol{A})$, $m \in M_P(\boldsymbol{A})$, $k \in K$. We define $H_P(x)$ to be the vector $H_M(m)$ in $\mathfrak{a}_P$.

Let Ω be the restricted Weyl group of (G, A_0). Ω acts on the dual space of $\mathfrak{a}_0$. We identify $\mathfrak{a}_0$ with its dual by fixing a positive definite Ω-invariant bilinear form $\langle\ ,\ \rangle$ on $\mathfrak{a}_0$. This allows us to embed each $\mathfrak{a}_P$ in $\mathfrak{a}_0$. Let Σ_P be the set of roots of (P, A). These are the elements in $X(A)_{\boldsymbol{Q}}$ obtained by decomposing the Lie algebra of N_P under the adjoint action of A_P. They can be regarded as vectors in $\mathfrak{a}_P$. Let Φ_P be the set of simple roots of (P, A). G itself is a parabolic subgroup. We write Z and $\mathfrak{z}$ for A_G and $\mathfrak{a}_G$ respectively. Then Φ_P is a basis of the orthogonal complement of $\mathfrak{z}$ in $\mathfrak{a}_P$.

Suppose that Q is another (standard) parabolic subgroup, with $Q \subset P$. Then $Q^P = Q \cap M_P$ is a parabolic subgroup of M_P. Its unipotent radical is $N_Q^P = N_Q \cap M_P$. We write $\mathfrak{a}_Q^P$ for the orthogonal complement of $\mathfrak{a}_P$ in $\mathfrak{a}_Q$. In general, we shall index the various objects associated with Q^P by the subscript Q and the superscript P. For example, Φ_Q^P stands for the set of simple roots of (Q^P, A_Q). It is the projection onto $\mathfrak{a}_Q^P$ of $\Phi_{P_0}^P\backslash\Phi_{P_0}^Q$. We shall write $\hat{\Phi}_Q^P$ for the basis of the Euclidean space $\mathfrak{a}_Q^P$ which is dual to Φ_Q^P. If P_i is a parabolic subgroup, we shall often use i instead of P_i for a subscript or superscript. If the letter P alone is used, we shall often omit it altogether as a subscript. Finally, we shall always denote the Lie algebras of groups over $\boldsymbol{Q}$ by lower case Gothic letters.

If P and P_1 are parabolic subgroups, let $\Omega(\mathfrak{a}, \mathfrak{a}_1)$ be the set of distinct isomorphisms from $\mathfrak{a}$ onto $\mathfrak{a}_1$ obtained by restricting elements in Ω to $\mathfrak{a}$. P and P_1 are said to be *associated* if $\Omega(\mathfrak{a}, \mathfrak{a}_1)$ is not empty. Suppose that $\mathscr{P}$ is an associated class, and that $P \in \mathscr{P}$.

$$\mathfrak{a}^+ = \mathfrak{a}_P^+ = \{H \in \mathfrak{a} : \langle \alpha, H\rangle > 0, \alpha \in \Phi_P\}$$

is called the *chamber* of P in $\mathfrak{a}$.

LEMMA 1. $\bigcup_{P_1 \subset \mathscr{P}} \bigcup_{s \subset \Omega(\mathfrak{a}, \mathfrak{a}_1)} s^{-1}(\mathfrak{a}_1^+)$ *is a disjoint union which is dense in* $\mathfrak{a}$. □

Before discussing Eisenstein series, we shall define a certain induced representation. Fix P. Let $\mathscr{H}_P^0$ be the space of functions

$$\Phi : N(\boldsymbol{A}) \cdot M(\boldsymbol{Q}) \cdot A(\boldsymbol{R})^0 \backslash G(\boldsymbol{A}) \to \boldsymbol{C}$$

such that

(i) for any $x \in G(\boldsymbol{A})$ the function $m \to \Phi(mx)$, $m \in M(\boldsymbol{A})$, is $\mathscr{Z}_{M(\boldsymbol{R})}$-finite, where $\mathscr{Z}_{M(\boldsymbol{R})}$ is the center of the universal enveloping algebra of $\mathfrak{m}(\boldsymbol{C})$,

(ii) the span of the set of functions $\Phi_k : x \to \Phi(xk)$, $x \in G(\boldsymbol{A})$, indexed by $k \in K$, is finite dimensional.

(iii) $$\|\Phi\|^2 = \int_K \int_{A(\boldsymbol{R})^0 M(\boldsymbol{Q})\backslash M(\boldsymbol{A})} |\Phi(mk)|^2 dm\, dk < \infty.$$

Let $\mathscr{H}_P$ be the Hilbert space obtained by completing $\mathscr{H}_P^0$. If Λ is in $\mathfrak{a}_{\boldsymbol{C}}$, the complexification of $\mathfrak{a}$, $\Phi \in \mathscr{H}_P$, and $x, y \in G(\boldsymbol{A})$, put

$$(I_P(\Lambda, y)\Phi)(x) = \Phi(xy) \exp(\langle \Lambda + \rho_P, H_P(xy)\rangle) \exp(-\langle \Lambda + \rho_P, H_P(x)\rangle).$$

Here ρ_P is the vector in $\mathfrak{a}$ such that

$$|\det(\operatorname{Ad} m)_{\mathfrak{n}(\boldsymbol{A})}| = \exp(\langle 2\rho_P, H_M(m)\rangle), \qquad m \in M(\boldsymbol{A}).$$

$I_P(\Lambda) = I_P^G(\Lambda)$ is a representation of $G(\boldsymbol{A})$ induced from a representation of

$P(\boldsymbol{A})$, which in turn is the pull-back of a certain representation, $I_M^M(\Lambda)$ in our notation, of $M(\boldsymbol{A})$. We have

$$I_P(\Lambda, y)^* = I_P(-\bar{\Lambda}, y^{-1}), \qquad y \in G(\boldsymbol{A}),$$

and

$$I_P(\Lambda, f)^* = I_P(-\bar{\Lambda}, f^*), \qquad f \in C_c^\infty(G(\boldsymbol{A})),$$

where $f^*(y) = f(y^{-1})$. In particular, $I_P(\Lambda)$ is unitary if Λ is purely imaginary.

REMARK. It is not difficult to show that $I_M^M(0)$ is the subrepresentation of the regular representation of $M(\boldsymbol{A})$ on $L^2(A(\boldsymbol{R})^0 \cdot M(\boldsymbol{Q})\backslash M(\boldsymbol{A}))$ which decomposes discretely. We can write $I_M^M(0) = \bigoplus_\iota \sigma^\iota$, where $\sigma^\iota = \bigotimes_v \sigma_v^\iota$ is an irreducible representation of $M(\boldsymbol{A})$. If v is any prime and σ_v is an irreducible unitary representation of $M(\boldsymbol{Q}_v)$, define

$$\sigma_{v,\Lambda}(m) = \sigma_v(m)\exp(\langle \Lambda, H_M(m)\rangle), \qquad \Lambda \in \mathfrak{a}_{\boldsymbol{C}},\ m \in M(\boldsymbol{Q}_v).$$

If $\sigma_{v,\Lambda}$ is lifted to $P(\boldsymbol{Q}_v)$ and then induced to $G(\boldsymbol{Q}_v)$, the result is a representation $I_P(\sigma_{v,\Lambda})$ of $G(\boldsymbol{Q}_v)$, acting on a Hilbert space $\mathscr{H}_P(\sigma_v)$. In this notation,

$$I_P(\Lambda) = \bigoplus_\iota \bigotimes_v I_P(\sigma_{v,\Lambda}^\iota).$$

Thus $I_P(\Lambda)$ can be completely understood in terms of induced representations and the discrete spectrum of M.

If P and P_1 are fixed, and $s \in \Omega(\mathfrak{a}, \mathfrak{a}_1)$, let w_s be a fixed representative of s in the intersection of $K \cap G(\boldsymbol{Q})$ with the normalizer of A_0. For $\Phi \in \mathscr{H}_P^0$, $\Lambda \in \mathfrak{a}_{\boldsymbol{C}}$, and $x \in G(\boldsymbol{A})$, consider

$$\int_{N_1(\boldsymbol{A}) \cap w_s N(\boldsymbol{A}) w_s^{-1} \backslash N_1(\boldsymbol{A})} \Phi(w_s^{-1}nx)\exp(\langle \Lambda + \rho_P, H_P(w_s^{-1}nx)\rangle)\, dn \exp(-\langle s\Lambda + \rho_1, H_1(x)\rangle).$$

(We adopt the convention that if H is any closed connected subgroup of N_0, dh is the Haar measure on $H(\boldsymbol{A})$ which makes the volume of $H(\boldsymbol{Q})\backslash H(\boldsymbol{A})$ one. This defines a unique quotient measure dn on $N_1(\boldsymbol{A}) \cap w_s N(\boldsymbol{A}) w_s^{-1} \backslash N_1(\boldsymbol{A})$.)

LEMMA 2. *Suppose that* $\langle \alpha, \operatorname{Re} \Lambda - \rho_P \rangle > 0$ *for each* α *in* Σ_P *such that* $s\alpha$ *belongs to* $-\Sigma_{P_1}$. *Then the above integral converges absolutely.* □

The integral, for Λ as in the lemma, defines a linear operator from $\mathscr{H}_P^0$ to $\mathscr{H}_{P_1}^0$, which we denote by $M(s, \Lambda)$. Intertwining integrals play an important role in the harmonic analysis of groups over local fields, so it is not surprising that $M(s, \Lambda)$ arises naturally in the global theory.

LEMMA 3. $M(s, \Lambda)^* = M(s^{-1}, -s\bar{\Lambda})$. *Moreover, if* $f \in C_c^\infty(G(\boldsymbol{A}))^K$, *the K-conjugate invariant functions in* $C_c^\infty(G(\boldsymbol{A}))$,

$$M(s, \Lambda) I_P(\Lambda, f) = I_P(s\Lambda, f)\, M(s, \Lambda). \quad \square$$

We now define Eisenstein series:

LEMMA 4. *If* $\Phi \in \mathscr{H}_P^0$, $x \in G(\boldsymbol{A})$ *and* $\Lambda \in \mathfrak{a}_{\boldsymbol{C}}$, *with* $\operatorname{Re} \Lambda \in \rho_P + \mathfrak{a}^+$, *the series*

$$E(x, \Phi, \Lambda) = \sum_{\delta \in P(\boldsymbol{Q}) \backslash G(\boldsymbol{Q})} \Phi(\delta x) \exp(\langle \Lambda + \rho_P, H_P(\delta x)\rangle)$$

converges absolutely. □

The principal results on Eisenstein series are contained in the following:

MAIN THEOREM. (a) *Suppose that* $\Phi \in \mathscr{H}_P^0$. $E(x, \Phi, \Lambda)$ *and* $M(s, \Lambda)\,\Phi$ *can be analytically continued as meromorphic functions to* $\mathfrak{a}_C$. *On* $i\mathfrak{a}$, $E(x, \Phi, \Lambda)$ *is regular, and* $M(s, \Lambda)$ *is unitary. For* $f \in C_c^\infty(G(A))^K$ *and* $t \in \Omega(\mathfrak{a}_1, \mathfrak{a}_2)$, *the following functional equations hold*:

(i) $E(x, I_P(\Lambda, f)\,\Phi, \Lambda) = \int_{G(A)} f(y)E(xy, \Phi, \Lambda)\,dy$,

(ii) $E(x, M(s, \Lambda)\,\Phi, s\Lambda) = E(x, \Phi, \Lambda)$,

(iii) $M(ts, \Lambda)\,\Phi = M(t, s\Lambda)M(s, \Lambda)\Phi$.

(b) *Let* $\mathscr{P}$ *be an associated class of parabolic subgroups. Let* $\hat{L}_{\mathscr{P}}$ *be the set of collections* $F = \{F_P : P \in \mathscr{P}\}$ *of measurable functions* F_P: $i\mathfrak{a} \to \mathscr{H}_P$ *such that*

(i) *If* $s \in \Omega(\mathfrak{a}, \mathfrak{a}_1)$,

$$F_{P_1}(s\Lambda) = M(s, \Lambda)F_P(\Lambda),$$

(ii)
$$\|F\|^2 = \sum_{P \in \mathscr{P}} n(A)^{-1}\left(\frac{1}{2\pi i}\right)^{\dim A} \int_{i\mathfrak{a}_P} \|F_P(\Lambda)\|^2 d\Lambda < \infty,$$

where $n(A)$ *is the number of chambers in* $\mathfrak{a}$. *Then the map which sends* F *to the function*

$$\sum_{P \in \mathscr{P}} n(A)^{-1}\left(\frac{1}{2\pi i}\right)^{\dim A} \int_{i\mathfrak{a}} E(x, F_P(\Lambda), \Lambda)\, d\Lambda,$$

defined for F *in a dense subspace of* $\hat{L}_{\mathscr{P}}$, *extends to a unitary map from* $\hat{L}_{\mathscr{P}}$ *onto a closed* $G(A)$*-invariant subspace* $L^2_{\mathscr{P}}(G(Q)\backslash G(A))$ *of* $L^2(G(Q)\backslash G(A))$. *Moreover, we have an orthogonal decomposition*

$$L^2(G(Q)\backslash G(A)) = \bigoplus_{\mathscr{P}} L^2_{\mathscr{P}}(G(Q)\backslash G(A)). \quad \square$$

The theorem states that the regular representation of $G(A)$ on $L^2(G(Q)\backslash G(A))$ is the direct sum over a set of representatives $\{P\}$ of associated classes of parabolic subgroups, of the direct integrals $\int_{i\mathfrak{a}_P^+}^{\oplus} I_P(\Lambda)\, d\Lambda$.

The theorem looks relatively straightforward, but the proof is decidedly round about. The natural inclination might be to start with a general Φ in $\mathscr{H}_P^0$ and try to prove directly the analytic continuation of $M(s, \Lambda)\Phi$ and $E(x, \Phi, \Lambda)$. This does not seem possible. One does not get any idea how the proof will go, for general Φ, until p. 231 of **[1]**, the second last page of Langlands' original manuscript. Rather Langlands' strategy was to prove all the relevant statements of the theorem for Φ in a certain subspace $\mathscr{H}^0_{P,\mathrm{cusp}}$ of $\mathscr{H}_P^0$. He was then able to finesse the theorem from this special case.

Let $\mathscr{H}_{P,\mathrm{cusp}}$ be the space of measurable functions Φ on $N(A)M(Q)A(R)^0\backslash G(A)$ such that

(i) $\|\Phi\|^2 = \int_K \int_{M(Q)A(R)^0\backslash M(A)} |\Phi(mk)|^2\, dm\, dk < \infty$,

(ii) for any $Q \supsetneqq P$, and $x \in G(A)$, $\int_{N_Q(Q)\backslash N_Q(A)} \Phi(nx)\, dn = 0$.

It is a right $G(A)$-invariant Hilbert space.

LEMMA 5. *If* $f \in C_c^N(G(A))$ *for some large* N, *the map* $\Phi \to \Phi * f$, $\Phi \in \mathscr{H}_{G,\mathrm{cusp}}$, *is a Hilbert-Schmidt operator on* $\mathscr{H}_{G,\mathrm{cusp}}$. $\square$

This lemma, combined with the spectral theorem for compact operators, leads to

COROLLARY. *$\mathcal{H}_{G,\mathrm{cusp}}$ decomposes into a direct sum of irreducible representations of $G(A)$, each occurring with finite multiplicity.* □

$\mathcal{H}_{G,\mathrm{cusp}}$ is called the space of *cusp forms* on $G(A)$. It follows from the corollary, applied to M, that any function in $\mathcal{H}_{P,\mathrm{cusp}}$ is a limit of functions in $\mathcal{H}_P^0$. Therefore $\mathcal{H}_{P,\mathrm{cusp}}$ is a subspace of $\mathcal{H}_P$. It is closed and invariant under $I_P(\Lambda)$.

Let $\mathcal{S}(G)$ be the collection of triplets $\chi = (\mathcal{P}, \mathcal{V}, W)$, where W is an irreducible representation of K, $\mathcal{P}$ is an associated class of parabolic subgroups, and $\mathcal{V}$ is a collection of subspaces

$$\{V_P \subset \mathcal{H}^M_{M,\mathrm{cusp}}, \text{ the space of cusp forms on } M(A)\}_{P\in\mathcal{P}},$$

such that

(i) if $P \in \mathcal{P}$, V_P is the eigenspace of $\mathcal{H}^M_{M,\mathrm{cusp}}$ associated to a complex valued homomorphism of $\mathcal{Z}_{M(\boldsymbol{R})}$, and

(ii) if $P, P' \in \mathcal{P}$, and $s \in \Omega(\mathfrak{a}, \mathfrak{a}_1)$, V_{P_1} is the space of functions obtained by conjugating functions in V_P by w_s.

If $P \in \mathcal{P}$, define $\mathcal{H}_{P,\chi}$ to be the space of functions Φ in $\mathcal{H}^0_{P,\mathrm{cusp}}$ such that for each $x \in G(A)$,

(i) the function $k \to \Phi(xk)$, $k \in K$, is a matrix coefficient of W, and

(ii) the function $m \to \Phi(mx)$, $m \in M(A)$, belongs to V_P. $\mathcal{H}_{P,\chi}$ is a finite dimensional space which is invariant under $I_P(\Lambda, f)$ for any $f \in C_c^\infty(G(A))^K$. $\mathcal{H}_{P,\mathrm{cusp}}$ is the orthogonal direct sum over all $\chi = (\mathcal{P}, \mathcal{V}, W)$, for which $P \in \mathcal{P}$, of the spaces $\mathcal{H}_{P,\chi}$.

Fix $\chi = (\mathcal{P}, \mathcal{V}, W)$ and fix $P \in \mathcal{P}$. Suppose that we have an analytic function

$$\Lambda \to \Phi(\Lambda) = \Phi(\Lambda, x), \qquad \Lambda \in \mathfrak{a}_C,\ x \in N(A)M(\boldsymbol{Q})A(\boldsymbol{R})^0\backslash G(A),$$

of Paley-Wiener type, from $\mathfrak{a}_C$ to the finite dimensional space $\mathcal{H}_{P,\chi}$. Then

$$\phi(x) = \left(\frac{1}{2\pi i}\right)^{\dim A} \int_{\mathrm{Re}\Lambda=\Lambda_0} \exp(\langle \Lambda + \rho_P, H_P(x)\rangle)\Phi(\Lambda, x)\, d\Lambda$$

is a function on $N(A)M(\boldsymbol{Q})\backslash G(A)$ which is independent of the point $\Lambda_0 \in \mathfrak{a}$. It is compactly supported in the $A(\boldsymbol{R})^0$-component of x.

LEMMA 6. *For Φ as above, the function*

$$\hat{\phi}(x) = \sum_{\delta\in P(\boldsymbol{Q})\backslash G(\boldsymbol{A})} \phi(\delta x)$$

converges absolutely and belongs to $L^2(G(\boldsymbol{Q})\backslash G(A))$. Let $L^2_\chi(G(\boldsymbol{Q})\backslash G(A))$ be the closed subspace generated by all such $\hat{\phi}$. Then there is an orthogonal decomposition

$$L^2(G(\boldsymbol{Q})\backslash GA)) = \bigoplus_{\chi\in\mathcal{S}(G)} L^2_\chi(G(\boldsymbol{Q})\backslash G(A)). \quad \square$$

Suppose that $\Lambda_0 \in \rho_P + \mathfrak{a}^+$. Then

$$\begin{aligned}\hat{\phi}(x) &= \left(\frac{1}{2\pi i}\right)^{\dim A} \int_{\mathrm{Re}\,\Lambda=\Lambda_0} \sum_{\delta\in P(\boldsymbol{Q})\backslash G(\boldsymbol{Q})} \exp(\langle \Lambda + \rho_P, H_P(\delta x)\rangle)\, \Phi(\Lambda, \delta x)\, d\Lambda \\ &= \left(\frac{1}{2\pi i}\right)^{\dim A} \int_{\mathrm{Re}\,\Lambda=\Lambda_0} E(x, \Phi(\Lambda), \Lambda)\, d\Lambda.\end{aligned}$$

Suppose that $\Phi_1(\Lambda_1, x)$ is another function, associated to P_1. We want an inner product formula for

$$\int_{G(\mathbf{Q})\backslash G(\mathbf{A})} \hat{\phi}(x)\overline{\hat{\phi}_1(x)}\,dx$$

in terms of Φ and Φ_1. The inner product is

$$\begin{aligned}\int_{G(\mathbf{Q})\backslash G(\mathbf{A})} &\sum_{\delta\in P_1(\mathbf{Q})\backslash G(\mathbf{Q})} \hat{\phi}(x)\overline{\phi_1(\delta x)}\,dx \\ &= \int_{P_1(\mathbf{Q})\backslash G(\mathbf{A})} \hat{\phi}(x)\overline{\phi_1(x)}\,dx \\ &= \left(\frac{1}{2\pi i}\right)^{\dim A_1} \int_{\mathrm{Re}\Lambda=\Lambda_0}\int_K\int_{A_1(\mathbf{R})^0 M_1(\mathbf{Q})\backslash M_1(\mathbf{A})}\int_{A_1(\mathbf{R})^0} E(namk, \Phi(\Lambda), \Lambda) \\ &\quad\cdot\exp(-2\langle\rho_1, H_1(x)\rangle)\,\overline{\phi_1(amk)}\,da\,dm\,dk.\end{aligned}$$

LEMMA 7. *Suppose that P and P_1 are of the same rank. If $\Phi \in \mathscr{H}_{P,\chi}$ and* $\mathrm{Re}\,\Lambda \in \rho_P + \mathfrak{a}^+$, *then*

$$\int_{N_1(\mathbf{Q})\backslash N_1(\mathbf{A})} E(nx, \Phi, \Lambda)\,dn = \sum_{s\in\Omega(\mathfrak{a},\mathfrak{a}_1)} (M(s,\Lambda)\Phi)(x)\exp(\langle s\Lambda + \rho_1, H_1(x)\rangle).$$

(*Of course, the right-hand side is* 0 *if* $\Omega(\mathfrak{a}, \mathfrak{a}_1)$ *is empty; that is, if P and P_1 are not associated.*)

PROOF. Let $\{\Omega\}$ be the set of $s \in \Omega$ such that $s^{-1}\alpha > 0$ for every $\alpha \in \Phi_0^P$. Then $\{\Omega\}$ is a set of representatives in Ω of the left cosets of Ω modulo the Weyl group of M. By the Bruhat decomposition,

$$\begin{aligned}\int_{N_1(\mathbf{Q})\backslash N_1(\mathbf{A})} &E(nx, \Phi, \Lambda)\,dn \\ &= \sum_{t\in\{\Omega\}} \int_{N_1(\mathbf{Q})\backslash N_1(\mathbf{A})} \sum_{\{\nu\in w_t^{-1}P(\mathbf{Q})w_t\cap N_0(\mathbf{Q})\backslash N_0(\mathbf{Q})\}} \Phi(w_t\,\nu nx) \\ &\quad\cdot\exp(\langle\Lambda + \rho_P, H_P(w_t\nu nx)\rangle)\,dn,\end{aligned}$$

$w_t N_1 w_t^{-1} \cap M$ is the unipotent radical of a standard parabolic subgroup of M. If the group is not M itself the term corresponding to s above is 0, since Φ is cuspidal. The group is M itself if and only if $s = t^{-1}$ maps $\mathfrak{a}$ onto $\mathfrak{a}_1$. In this case $w_t^{-1}Pw_t \cap N_0\backslash N_0$ is isomorphic to $w_t^{-1}Nw_t \cap N_1\backslash N_1$. Therefore the above formula equals

$$\begin{aligned}\sum_{s\in\Omega(\mathfrak{a},\mathfrak{a}_1)} &\mathrm{vol}(w_s N(\mathbf{Q}) w_s^{-1} \cap N(\mathbf{Q})\backslash w_s N(\mathbf{A}) w_s^{-1}\backslash N(\mathbf{A})) \\ &\cdot\int_{w_s N(\mathbf{A}) w_s^{-1}\cap N(\mathbf{A})\backslash N(\mathbf{A})} \Phi(w_s^{-1}\,nx)\exp(\langle\Lambda + \rho_P, H_P(w_s^{-1}nx)\rangle)\,dn.\end{aligned}$$

The volume is one by our choice of measure. The lemma therefore follows. □

Combining the lemma with the Fourier inversion formula, one obtains

COROLLARY. *Suppose that P and P_1 are associated, and that ϕ and ϕ_1 are as in the discussion preceding Lemma* 7. *Then*

$$\int_{G(\mathbf{Q})\backslash G(\mathbf{A})} \hat{\phi}(x)\overline{\hat{\phi}_1(x)}\, dx$$
$$= \left(\frac{1}{2\pi i}\right)^{\dim A} \int_{\Lambda_0+i\mathfrak{a}} \sum_{s\in\Omega(\mathfrak{a},\mathfrak{a}_1)} (M(s,\Lambda)\Phi(\Lambda), \Phi_1(-s\bar{\Lambda}))\, d\Lambda,$$

where Λ_0 is any point in $\rho_P + \mathfrak{a}^+$. □

The proofs of Lemmas 1–6 are based on rather routine and familiar estimates. This is the point at which the serious portion of the proof of the Main Theorem should begin. There are two stages. The first stage is to complete the analytic continuation and functional equations for Φ a vector in $\mathscr{H}^0_{P,\text{cusp}}$. This is nicely described in **[2]**, so we shall skip it altogether. The second stage, done in Chapter 7 of **[1]**, is the spectral decomposition of $L^2_\chi(G(\mathbf{Q})\backslash G(\mathbf{A}))$. From this all the remaining assertions of the Main Theorem follow. We shall try to give an intuitive description of the argument.

Fix $\chi = (\mathscr{P}_\chi, \mathscr{V}_\chi, W)$. The argument in Chapter 7 of **[1]** is based on an intricate induction on the dimension of (A/Z), for $P \in \mathscr{P}_\chi$. For $\Phi \in \mathscr{H}_{P,\chi}$, $P \in \mathscr{P}_\chi$, we are assuming that $E(x,\Phi,\Lambda)\Phi$ and $M(s,\Lambda)\Phi$ are meromorphic on $\mathfrak{a}_C$. In the process of proving this, one shows that the singularities of these two functions are hyperplanes of the form $\mathfrak{r} = \{\Lambda \in \mathfrak{a}_C : \langle\alpha,\Lambda\rangle = \mu,\ \mu \in \mathbf{C},\ \alpha \in \Sigma_P\}$, and that only finitely many $\mathfrak{r}$ meet $\mathfrak{a}^+ + i\mathfrak{a} = \{\Lambda \ni \mathfrak{a}_C : \langle\alpha, \operatorname{Re}\Lambda\rangle > 0,\ \alpha \in \Phi_P\}$.

Let $L^2_{\mathscr{P}_\chi,\chi}(G(\mathbf{Q})\backslash G(\mathbf{A}))$ be the closed subspace of $L^2_\chi(G(\mathbf{Q})\backslash G(\mathbf{A}))$ generated by functions $\hat{\phi}(x)$, where $\hat{\phi}$ comes from a function $\Phi(\Lambda)$, as above, which vanishes on the finite set of singular hyperplanes which meet $\mathfrak{a}^+ + i\mathfrak{a}$. If $\phi_1(x)$ comes from $\Phi_1(\Lambda_1)$, for $P_1 \in \mathscr{P}_\chi$, we can apply Cauchy's theorem to the formula for

$$\int_{G(\mathbf{Q})\backslash G(\mathbf{A})} \hat{\phi}(x)\overline{\hat{\phi}_1(x)}\, dx.$$

The result is

$$\left(\frac{1}{2\pi i}\right)^{\dim A} \int_{i\mathfrak{a}} \sum_{s\in\Omega(\mathfrak{a},\mathfrak{a}_1)} (M(s,\Lambda)\Phi(\Lambda), \Phi_1(s\Lambda))\, d\Lambda,$$

since $-s\bar{\Lambda} = s\Lambda$ on $i\mathfrak{a}$. Changing variables in the integral and sum, we obtain

$$\left(\frac{1}{2\pi i}\right)^{\dim A} n(A)^{-1} \sum_{P_2\in\mathscr{P}_\chi} \sum_{t\in\Omega(\mathfrak{a}_2,\mathfrak{a})} \sum_{s\in\Omega(\mathfrak{a},\mathfrak{a}_1)} \cdot \int_{i\mathfrak{a}_2} (M(s,t\Lambda)\Phi(t\Lambda), \Phi_1(t\Lambda))\, d\Lambda$$
$$= \left(\frac{1}{2\pi i}\right)^{\dim A} n(A)^{-1} \sum_{P_2} \sum_{r\in\Omega(\mathfrak{a}_2,\mathfrak{a}_1)} \sum_{t\in\Omega(\mathfrak{a}_2,\mathfrak{a})} \cdot \int_{i\mathfrak{a}_2} (M(rt^{-1},t\Lambda)\Phi(t\Lambda), \Phi_1(r\Lambda))\, d\Lambda$$
$$= \left(\frac{1}{2\pi i}\right)^{\dim A} n(A)^{-1} \sum_{P_2}\sum_r\sum_t \int_{i\mathfrak{a}_2} (M(t,\Lambda)^{-1}\Phi(t\Lambda), M(r,\Lambda)^{-1}\Phi_1(r\Lambda))\, d\Lambda$$
$$= \sum_{P_2\in\mathscr{P}_\chi} n(A)^{-1}\left(\frac{1}{2\pi i}\right)^{\dim A} \int_{i\mathfrak{a}_2} (F_{P_2}(\Lambda), F_{1,P_2}(\Lambda))\, d\Lambda,$$

where

(1) $$F_{1,P_2}(\Lambda) = \sum_{r\in\Omega(\mathfrak{a}_2,\mathfrak{a}_1)} M(r,\Lambda)^{-1}\Phi_1(r\Lambda),$$

and F_{P_2} is defined similarly. Define $\hat{L}_{\mathscr{P}_\chi,\chi}$ to be the subspace of the space $\hat{L}_{\mathscr{P}_\chi}$ (defined in the statement of the Main Theorem) consisting of those collections $\{F_{P_2}: P_2 \in \mathscr{P}_\chi\}$ such that F_{P_2} takes values in $\hat{\mathscr{H}}_{P_2,\chi}$. We have just exhibited an isometric isomorphism from a dense subspace of $\hat{L}_{\mathscr{P}_\chi,\chi}$ to a dense subspace of $L^2_{\mathscr{P}_\chi,\chi}(G(Q)\backslash G(A))$. Suppose that $\{F_{P_2}\}$ is a collection of functions in $\hat{L}_{\mathscr{P}_\chi,\chi}$ each of which is smooth and compactly supported. Let $h(x)$ be the corresponding function in $L^2_{\mathscr{P}_\chi,\chi}(G(Q)\backslash G(A))$ defined by the above isomorphism. We would like to prove that $h(x)$ equals

$$h'(x) = \sum_{P_2\in\mathscr{P}_\chi} n(A_2)^{-1}\left(\frac{1}{2\pi i}\right)^{\dim A}\int_{i\mathfrak{a}_2} E(x, F_{P_2}(\Lambda), \Lambda)\, d\Lambda.$$

If $\hat{\phi}_1(x)$ is as above, the same argument as that of the corollary to Lemma 7 shows that

$$\int_{G(Q)\backslash G(A)} h'(x)\overline{\hat{\phi}_1(x)}\, dx = \sum_{P_2} n(A_2)^{-1}\left(\frac{1}{2\pi i}\right)^{\dim A_2} \cdot\int_{i\mathfrak{a}_2}\sum_{r\in\Omega(\mathfrak{a}_2,\mathfrak{a}_1)}(M(r,\Lambda)F_{P_2}(\Lambda), \Phi_1(r\Lambda))\, d\Lambda.$$

Since the projection of $\hat{\phi}_1(x)$ onto $L^2_{\mathscr{P}_\chi,\chi}(G(Q)\backslash G(A))$ corresponds to the collection $\{F_{1,P_2}\}$ defined by (1), this equals

$$\sum_{P_2} n(A_2)^{-1}\left(\frac{1}{2\pi i}\right)^{\dim A_2}\int_{i\mathfrak{a}_2}(F_{P_2}(\Lambda), F_{1,P_2}(\Lambda))\, d\Lambda = \int_{G(Q)\backslash G(A)} h(x)\overline{\hat{\phi}_1(x)}\, dx.$$

We have shown that $h(x) = h'(x)$. This completes the first stage of Langlands' induction.

To begin the second stage, Langlands lets Q be the projection of $L^2_\chi(G(Q)\backslash G(A))$ onto the orthogonal complement of $L^2_{\mathscr{P}_\chi,\chi}(G(Q)\backslash G(A))$. Then for any $\hat{\phi}(x)$ and $\hat{\phi}_1(x)$ corresponding to $\Phi(\Lambda)$ and $\Phi_1(\Lambda_1)$, $(Q\hat{\phi}, \hat{\phi}_1)$ equals

$$\left(\frac{1}{2\pi i}\right)^{\dim A}\left(\int_{\Lambda_0+i\mathfrak{a}}\sum_{s\in\Omega(\mathfrak{a},\mathfrak{a}_1)}(M(s,\Lambda)\Phi(\Lambda), \Phi_1(-s\bar{\Lambda}))\, d\Lambda - \int_{i\mathfrak{a}}\sum_{s}(M(s,\Lambda)\Phi(\Lambda), \Phi_1(-s\bar{\Lambda}))\, d\Lambda\right).$$

Choose a path in $\mathfrak{a}^+$ from Λ_0 to 0 which meets any singular hyperplane $\mathfrak{r}$ of $\{M(s,\Lambda): s\in\Omega(\mathfrak{a},\mathfrak{a}_1)\}$ in at most one point $Z(\mathfrak{r})$. Any such $\mathfrak{r}$ is of the form $X(\mathfrak{r}) + \mathfrak{r}^\vee_C$, where $\mathfrak{r}^\vee$ is a real vector subspace of $\mathfrak{a}$ of codimension one, and $X(\mathfrak{r})$ is a vector in $\mathfrak{a}$ orthogonal to $\mathfrak{r}^\vee$. The point $Z(\mathfrak{r})$ belongs to $X(\mathfrak{r}) + \mathfrak{r}^\vee$. By the residue theorem $(Q\hat{\phi}, \hat{\phi}_1)$ equals

$$\left(\frac{1}{2\pi i}\right)^{\dim A-1}\sum_{\mathfrak{r}}\int_{Z(\mathfrak{r})+i\mathfrak{r}^\vee}\sum_{s\in\Omega(\mathfrak{a},\mathfrak{a}_1)}\operatorname{Res}_{\mathfrak{r}}(M(s,\Lambda)\Phi(\Lambda), \Phi_1(-s\bar{\Lambda}))\, d\Lambda.$$

The obvious tactic at this point is to repeat the first stage of the induction with Λ_0 replaced by $Z(\mathfrak{r})$, 0 replaced by $X(\mathfrak{r})$, and $E(x, \Phi, \Lambda)$ by $\operatorname{Res}_{\mathfrak{r}}E(x, \Phi, \Lambda)$.

Suppose that $\mathfrak{r}^\vee = \{H \in \mathfrak{a} : \langle \alpha, H \rangle = 0\}$ for a simple root $\alpha \in \Phi_P$. Then $\mathfrak{r}^\vee = \mathfrak{a}_R$, for R a parabolic subgroup of G containing P. If $\Phi \in \mathscr{H}_{P,\chi}$, define

$$E^R(x, \Phi, \Lambda) = \sum_{P(\mathbf{Q})\backslash R(\mathbf{Q})} \Phi(\delta x) \exp(\langle \Lambda + \rho_P, H(\delta x) \rangle).$$

This is essentially a cuspidal Eisenstein series on the group $M_R(\mathbf{A})$. It converges for suitable $\Lambda \in \mathfrak{a}_C$ and can be meromorphically continued. It is clear that

$$E(x, \Phi, \Lambda) = \sum_{R(\mathbf{Q})\backslash G(\mathbf{Q})} E^R(\delta x, \Phi, \Lambda)$$

whenever the right-hand side converges. Suppose that $\Lambda \in \mathfrak{r}$, and $\Lambda = X(\mathfrak{r}) + \Lambda^\vee$, $\Lambda^\vee \in \mathfrak{r}_C^\vee$, Re $\Lambda^\vee \in \rho_R + \mathfrak{a}_R^+$. Then for any small positive ε,

$$\begin{aligned}
\operatorname{Res}_{\mathfrak{r}} E(x, \Phi, \Lambda) &= \frac{1}{2\pi i} \int_0^{2\pi} E(x, \Phi, \Lambda + \varepsilon e^{2\pi i\theta} X(\mathfrak{r}))\, d\theta \\
&= \sum_{R(\mathbf{Q})\backslash G(\mathbf{Q})} \left(\frac{1}{2\pi i} \int_0^{2\pi} E^R(\delta x, \Phi, \Lambda + \varepsilon e^{2\pi i\theta} X(\mathfrak{r}))\, d\theta \right) \\
&= \sum_{R(\mathbf{Q})\backslash G(\mathbf{Q})} \Phi^\vee(\delta x) \exp(\langle \Lambda^\vee + \rho_R, H_R(\delta x) \rangle),
\end{aligned}$$

where

$$\Phi^\vee(y) = \frac{1}{2\pi i} \int_0^{2\pi} E^R(y, \Phi, (1 + \varepsilon) X(\mathfrak{r}) e^{2\pi i\theta})\, d\theta,$$

the residue at $X(\mathfrak{r})$ of an Eisenstein series in one variable. One shows that the function $m \to \Phi^\vee(my)$, $m \in M_R(\mathbf{Q})\backslash M_R(\mathbf{A})$, is in the discrete spectrum. Thus

$$\operatorname{Res}_{\mathfrak{r}} E(x, \Phi, \Lambda) = E(x, \Phi^\vee, \Lambda^\vee),$$

the Eisenstein series over $R(\mathbf{Q})\backslash G(\mathbf{Q})$ associated to a vector $\Phi^\vee$ in $\mathscr{H}_R \backslash \mathscr{H}_{R,\mathrm{cusp}}$. Its analytic continuation is immediate. Let $\mathscr{H}_{R,\chi}$ be the finite dimensional subspace of $\mathscr{H}_R$ consisting of all those vectors $\Phi^\vee$. By examining $\operatorname{Res}_{\mathfrak{r}} M(s, \Lambda)$, one obtains the operators

$$M(s, \Lambda^\vee) : \mathscr{H}_{R,\chi} \to \mathscr{H}_{Q,\chi} \qquad \Lambda^\vee \in \mathfrak{a}_{R,C},$$

for $s \in \Omega(\mathfrak{a}_R, \mathfrak{a}_Q)$. Their analytic continuation then comes without much difficulty.

Let $\mathscr{P}'$ be the class of parabolic subgroups associated to R. In carrying out the second stage of the induction one defines subspaces $L^2_{\mathscr{P}',\chi}(G(\mathbf{Q})\backslash G(\mathbf{A})) \subset L^2_\chi(G(\mathbf{Q})\backslash G(\mathbf{A}))$ and $\hat{L}^2_{\mathscr{P}',\chi} \subset \hat{L}^2_{\mathscr{P}'}$, and as above, obtains an isomorphism between them. In the process, one proves the functional equations in (a) of the Main Theorem for vectors $\Phi^\vee \in \mathscr{H}_{R,\chi}$.

The pattern is clear. For R now *any* standard parabolic subgroup and $\mathscr{P}$ *any* associated class, one eventually obtains a definition of spaces $\mathscr{H}_{R,\chi}$, $\hat{L}_{\mathscr{P},\chi}$ and $L^2_{\mathscr{P},\chi}(G(\mathbf{Q})\backslash G(\mathbf{A}))$. By definition, $\mathscr{H}_{R,\chi}$ is $\{0\}$ unless R contains an element of $\mathscr{P}_\chi$, and the other two spaces are $\{0\}$ unless an element of $\mathscr{P}$ contains an element of $\mathscr{P}_\chi$. If $\mathscr{P}$ is the associated class of R, there corresponds a stage of the induction in which one proves part (a) of the Main Theorem for vectors $\Phi^\vee$ in $\mathscr{H}_{R,\chi}$ and part (b) for the spaces $\hat{L}_{\mathscr{P},\chi}$ and $L^2_{\mathscr{P},\chi}(G(\mathbf{Q})\backslash G(\mathbf{A}))$. Finally, one shows that

$$L^2_\chi(G(\mathbf{Q})\backslash G(\mathbf{A})) = \bigoplus_{\mathscr{P}} L^2_{\mathscr{P},\chi}(G(\mathbf{Q})\backslash G(\mathbf{A})).$$

Notice that if R and $\mathscr{P}$ are fixed,

$$\mathscr{H}_R = \bigoplus_\chi \mathscr{H}_{R,\chi}, \quad \hat{L}_{\mathscr{P}} = \bigoplus_\chi \hat{L}_{\mathscr{P},\chi},$$

and

$$L^2_{\mathscr{P}}(G(\boldsymbol{Q})\backslash G(\boldsymbol{A})) = \bigoplus_\chi L^2_{\mathscr{P},\chi}(G(\boldsymbol{Q}\backslash G(\boldsymbol{A})).$$

The last decomposition together with the Main Theorem yields

$$L^2(G(\boldsymbol{Q})\backslash G(\boldsymbol{A})) = \bigoplus_{\mathscr{P},\chi} L^2_{\mathscr{P},\chi}(G(\boldsymbol{Q})\backslash G(\boldsymbol{A})). \tag{2}$$

This completes our description of the proof of the Main Theorem. It is perhaps a little too glib. For one thing, we have not explained why it suffices to consider only those $\mathfrak{r}$ above such that $\mathfrak{r}^\vee = \{\Lambda \in \mathfrak{a} : \langle \alpha, \Lambda\rangle = 0\}$. Moreover, we neglected to mention a number of serious complications that arise in higher stages of the induction. Some of them are described in Appendix III of **[1]**. We shall only remark that most of the complications exist because eventually one has to study points $X(\mathfrak{r})$ and $Z(\mathfrak{r})$ which lie *outside* the chamber $\mathfrak{a}^+$, where the behavior of the functions $M(s, \Lambda)$ is a total mystery.

Part II. The trace formula

In this section we shall describe a trace formula for G. We have not yet been able to prove as explicit a formula as we would like for general G. We shall give a more explicit formula for GL_3 in the next section. In the past most results have been for groups of rank one. The main references are

4. H. Jacquet and R. Langlands, *Automorphic forms on* GL(2), Springer-Verlag, Berlin, 1970.

5. M. Duflo and J. Labesse, *Sur la formule des traces de Selberg*, Ann. Sci. École Norm. Sup. (4) **4** (1971), 193–284.

6. J. Arthur, *The Selberg trace formula for groups of F-rank one*, Ann. of Math. (2) **100** (1974), 326–385.

We shall also quote from

7. J. Arthur, *The characters of discrete series as orbital integrals*, Invent. Math. **32** (1976), 205–261.

Let R be the regular representation of $G(\boldsymbol{A})$ on $L^2(Z(\boldsymbol{R})^0 \cdot G(\boldsymbol{Q})\backslash G(\boldsymbol{A}))$. If $\xi \in i\mathfrak{z}$, recall that R_ξ is the twisted representation on $L^2(Z(\boldsymbol{R})^0 \cdot G(\boldsymbol{Q})\backslash G(\boldsymbol{A}))$ given by $R_\xi(x) = R(x)\exp(\langle \xi, H_G(x)\rangle)$. We are really interested in the regular representation of $G(\boldsymbol{A})$ on $L^2(G(\boldsymbol{Q})\backslash G(\boldsymbol{A}))$; but this representation is a direct integral over $\xi \in i\mathfrak{z}$ of the representations R_ξ, so it is good enough to study these latter ones. The decomposition (2), quoted in Part I, is equivalent to

$$L^2(Z(\boldsymbol{R})^0 \cdot G(\boldsymbol{Q})\backslash G(\boldsymbol{A})) = \bigoplus_{\mathscr{P},\chi} L^2_{\mathscr{P},\chi}(Z(\boldsymbol{R})^0 \cdot G(\boldsymbol{Q})\backslash G(\boldsymbol{A})).$$

Suppose that $f \in C_c^\infty(G(\boldsymbol{A}))^K$. Then this last decomposition is invariant under the operator $R_\xi(f)$. $R_\xi(f)$ is an integral operator with kernel

$$K(x, y) = \sum_{\gamma \in G(\boldsymbol{Q})} f_\xi(x^{-1}\gamma y),$$

where

$$f_\xi(u) = \int_{Z(\mathbf{R})^0} f(zu)\exp(\langle \xi, H_G(zu)\rangle)\,dz, \qquad u \in Z(\mathbf{R})^0\backslash G(\mathbf{A}).$$

The following result is essentially due to Duflo and Labesse.

LEMMA 1. *For every $N \geqq 0$ we can express f as a finite sum of functions of the form*

$$f^1 * f^2, \qquad f^i \in C_c^N(G(\mathbf{A}))^K. \quad \square$$

Let $\mathscr{S}_E(G)$ be the set of $\chi = (\mathscr{P}, \mathscr{V}, W)$ in $\mathscr{S}(G)$ such that $\mathscr{P} \neq \{G\}$. Let $R_{E,\xi}(f)$ be the restriction of $R_\xi(f)$ to $\bigoplus_{\mathscr{P}}\bigoplus_{\chi\in\mathscr{S}_E(G)} L^2_{\mathscr{P},\chi}\,(Z(\mathbf{R})^0G(\mathbf{Q})\backslash G(\mathbf{A}))$. Then

$$R_{\mathrm{cusp},\xi}(f) = R_\xi(f) - R_{E,\xi}(f)$$

is the restriction of $R_\xi(f)$ to the space of cusp forms. It is a finite sum of compositions $R_{\mathrm{cusp},\xi}(f^1)R_{\mathrm{cusp},\xi}(f^2)$ of Hilbert-Schmidt operators and so is of trace class. For each P and χ let $\mathscr{B}_{P,\chi}$ be a fixed orthonormal basis of the finite dimensional space $\mathscr{H}_{P,\chi}$. Finally, recall that $\mathfrak{a}^G = \mathfrak{a}_P^G$ is the orthogonal complement of $\mathfrak{z} = \mathfrak{a}_G$ in $\mathfrak{a}$. If Λ is in $i\mathfrak{a}^G$, we shall write Λ_ξ for the vector $\Lambda + \xi$ in $i\mathfrak{a}$.

LEMMA 2. *$R_E(f)$ is an integral operator with kernel*

$$K_E(x, y) = \sum_{\chi\in\mathscr{S}_E(G)}\sum_P n(A)^{-1}\left(\frac{1}{2\pi i}\right)^{\dim(A/Z)} \cdot \int_{i\mathfrak{a}^G}\left\{\sum_{\Phi\in\mathscr{B}_{P,\chi}} E(x, I_P(\Lambda_\xi, f)\Phi, \Lambda)\overline{E(y, \Phi, \Lambda)}\right\}d\Lambda.$$

The lemma would follow from the spectral decomposition described in the last section if we could show that the integral over Λ and sum over χ converged and was locally bounded. We can assume that $f = f^1 * f^2$. If

$$K_{P,\chi}(f, x, y) = \sum_{\Phi\in\mathscr{B}_{P,\chi}} E(x, I_P(\Lambda, f)\,\Phi, \Lambda)\overline{E(y, \Phi, \Lambda)},$$

a finite sum, then one easily verifies that

$$|K_{P,\chi}(f, x, y)| \leq K_{P,\chi}(f^1 * (f^1)^*, x, x)^{1/2}\,K_{P,\chi}((f^2)^* * f^2, y, y)^{1/2}.$$

By applying Schwartz' inequality to the sum over χ, P and the integral over Λ, we reduce to the case that $f = f^1 * (f^1)^*$ and $x = y$. But then $R_{E,\xi}(f)$ is the restriction of the positive semidefinite operator $R_\xi(f)$ to an invariant subspace. The integrand in the expression for $K_E(x, x)$ is nonnegative, and the integral itself is bounded by $K(x, x)$. This proves the lemma. $\square$

The proof of the lemma can be modified to show that $K_E(x, y)$ is continuous in each variable. The same is therefore true of

$$K_{\mathrm{cusp}}(x, y) = K(x, y) - K_E(x, y).$$

One proves without difficulty

LEMMA 3. *The trace of $R_{\mathrm{cusp},\xi}(f)$ equals*

$$\int_{Z(\mathbf{R})^0G(\mathbf{Q})\backslash G(\mathbf{A})} K_{\mathrm{cusp}}(x, x)\,dx. \quad \square$$

Of course neither K nor K_E is integrable over the diagonal. It turns out, however, that there is a natural way to modify the kernels so that they are integrable. Given P, let τ_P (resp. $\hat{\tau}_P$) be the characteristic function of $\{H \in \mathfrak{a}_0 \colon \alpha(H) > 0,\ \alpha \in \Phi_P$ (resp. $\mu(H) > 0,\ \mu \in \hat{\Phi}_P)\}$. (Recall that $\hat{\Phi}_P$ is the basis of $\mathfrak{a}^G$ which is dual to Φ_P.) Then $\tau_P \leqq \hat{\tau}_P$. Suppose that $T \in \mathfrak{a}_0$.

LEMMA 4. *For any P, $\sum_{\delta \in P(Q)\backslash G(Q)} \hat{\tau}_P(H(\delta x) - T)$ is a locally bounded function of $x \in G(A)$. In particular the sum is finite.* □

We will now take $T \in \mathfrak{a}_0^+$. We shall assume that the distance from T to each of the walls of $\mathfrak{a}_0^+$ is arbitrarily large. We shall modify $K(x, x)$ by regarding it as the term corresponding to $P = G$ in a sum of functions indexed by P. If x remains within a large compact subset of $Z(\mathbf{R})^0G(\mathbf{Q})\backslash G(\mathbf{A})$, depending on T, the functions corresponding to $P \neq G$ will vanish. They are defined in terms of

$$K^P(x, y) = \int_{N(A)} \sum_{\mu \subset M(Q)} f_\xi(x^{-1} n \mu y)\, dn,$$

the kernel of $R_\xi^P(f)$, where R^P is the regular representation of $G(A)$ on $L^2\,(Z(\mathbf{R})^0 N(A) M(\mathbf{Q})\backslash G(A))$. Define the modified function to be

$$k^T(x) = \sum_P (-1)^{\dim(A/Z)} \sum_{\delta \subset P(Q)\backslash G(Q)} K^P(\delta x, \delta x)\hat{\tau}_P(H(\delta x) - T).$$

For each x this is a finite sum. The function obtained turns out to be integrable over $Z(\mathbf{R})^0 G(\mathbf{Q})\backslash G(A)$. We shall give a fairly detailed sketch of the proof of this fact because it is typical of the proofs of later results, which we shall only state.

We begin by partitioning $Z(\mathbf{R})^0 \cdot G(\mathbf{Q})\backslash G(A)$ into disjoint sets, indexed by P, which depend on T. Fix a Siegel set $\mathfrak{r}$ in $Z(\mathbf{R})^0\backslash G(A)$ such that $Z(\mathbf{R})^0\backslash G(A) = G(\mathbf{Q})\mathfrak{r}$. Consider the set of $x \in \mathfrak{r}$ such that $\mu(H_0(x) - T) < 0$ for every $\mu \in \hat{\Phi}_0$. It is a compact subset of $\mathfrak{r}$. The projection, $G(T)$, of this set onto $Z(\mathbf{R})^0 G(\mathbf{Q})\backslash G(A)$ remains compact. For any P we can repeat this process on M, to obtain a compact subset of $A(\mathbf{R})^0 M(\mathbf{Q})\backslash M(A)$, which of course depends on T. Let $F^P(m, T)$ be its characteristic function. Extend it to a function on $G(A)$ by

$$F^P(nmk, T) = F^P(m, T), \qquad n \in N(A),\ m \in M(A),\ k \in K.$$

This gives a function on $N(A)M(\mathbf{Q})A(\mathbf{R})^0\backslash G(A)$. If $Q \subset P$, define τ_Q^P (resp. $\hat{\tau}_Q^P$) to be the characteristic function of

$$\{H \in \mathfrak{a}_0 \colon \alpha(H) > 0,\ \alpha \in \Phi_Q^P\ (\text{resp. } \mu(H) > 0,\ \mu \in \hat{\Phi}_Q^P)\}.$$

The following lemma gives our partition of $N(A)M(\mathbf{Q})A(\mathbf{R})^0\backslash G(A)$. It is essentially a restatement of standard results from reduction theory.

LEMMA 5. *Given P,*

$$\sum_{\{Q : P_0 \subset Q \subset P\}} \sum_{\delta \in Q(Q)\backslash P(Q)} F^Q(\delta x,\ T)\tau_Q^P(H(\delta x) - T)$$

equals 1 for almost all $x \in G(A)$. □

We can now study the function $k^T(x)$. It equals

$$\sum_{\{Q, P: P_0 \subset Q \subset P\}} (-1)^{\dim(A/Z)} \sum_{\delta \in Q(\boldsymbol{Q})\backslash G(\boldsymbol{Q})} F^Q(\delta x, T)\tau_Q^P(H(\delta x) - T)$$
$$\cdot \hat{\tau}_P(H(\delta x) - T)K^P(\delta x, \delta x).$$

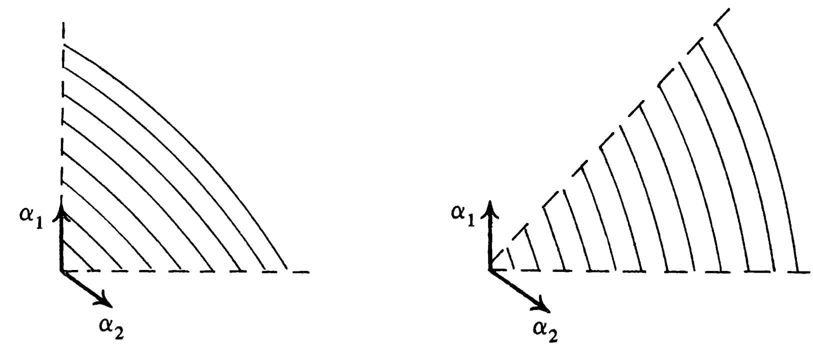
Now for any $H \in \mathfrak{a}_0$ we can surely write

$$\tau_Q^P(H)\hat{\tau}_P(H) = \sum_{\{P_1, P_2: P \subset P_1 \subset P_2\}} (-1)^{\dim(A_1/A_2)} \tau_Q^2(H)\hat{\tau}_2(H)$$
$$= \sum_{\{P_1: P_1 \supset P\}} \sigma_Q^1(H),$$

where

$$\sigma_Q^1(H) = \sigma_Q^{P_1}(H) = \sum_{\{P_2: P_2 \supset P_1\}} (-1)^{\dim(A_1/A_2)} \tau_Q^2(H)\hat{\tau}_2(H).$$

To study σ_Q^1, consider the case that $G = \mathrm{GL}_3$. Then $\mathfrak{a}_0/\mathfrak{z}$ is two dimensional, spanned by simple roots α_1 and α_2. Let $Q = P_0$, and let P_1 be the maximal parabolic subgroup such that $\mathfrak{a}_1 = \{H \in \mathfrak{a}_0 \colon \alpha_1(H) = 0\}$. Then σ_Q^1 is the difference of the characteristic functions of the following sets:



$P_2 = P_1$ $\qquad\qquad$ $P_2 = G$

The next lemma generalizes what is clear from the diagrams.

LEMMA 6. *Suppose that H is a vector in the orthogonal complement of $\mathfrak{z}$ in $\mathfrak{a}_Q$. If $\sigma_Q^1(H) \neq 0$, and $H = H_* + H^*$, $H_* \in \mathfrak{a}_Q^1$, $H^* \in \mathfrak{a}_1$, then $\alpha(H_*) > 0$ for each $\alpha \in \Phi_Q^1$ and $\|H^*\| < c\|H_*\|$ for a constant c depending only on G. In other words H^* belongs to a compact set, while H_* belongs to the positive chamber in $\mathfrak{a}_Q^1$.* □

We write $k^T(x)$ as

$$\sum_{Q \subset P_1} \sum_{\delta \in Q(\boldsymbol{Q})\backslash G(\boldsymbol{Q})} F^Q(\delta x, T)\,\sigma_Q^1(H(\delta x) - T) \sum_{\{P: Q \subset P \subset P_1\}} (-1)^{\dim(A/Z)} K^P(\delta x, \delta x).$$

The integral over $Z(\boldsymbol{R})^0 G(\boldsymbol{Q})\backslash G(\boldsymbol{A})$ of the absolute value of $k^T(x)$ is bounded by the sum over $Q \subset P_1$ of the integral over $Q(\boldsymbol{Q})Z(\boldsymbol{R})^0\backslash G(\boldsymbol{A})$ of the product of $F^Q(x, T)\sigma_Q^1(H(x) - T)$ and

$$(1) \qquad \left| \sum_{\{P: Q \subset P \subset P_1\}} (-1)^{\dim(A/Z)} \int_{N(A)} \sum_{\mu \in M(\boldsymbol{Q})} f_\xi(x^{-1} n\mu x)\, dn \right|.$$

We can assume that for a given x the first function does not vanish. Then by the last lemma, the projection of $H_Q(x)$ onto $\mathfrak{a}_Q^1$ is large. Conjugation by x^{-1} tends to stretch any element $\mu \in M(Q)$ which does not lie in $Q(Q)$. Since f is compactly supported, we can choose T so large that the only μ which contribute nonzero summands in (1) belong to $Q^P(Q) = Q(Q) \cap M(Q) = M_Q(Q)N_Q^P(Q)$. Thus, (1) equals

$$\left| \sum_{\mu \in M_Q(Q)} \sum_{\{P: Q \subset P \subset P_1\}} (-1)^{\dim(A/Z)} \int_{N(A)} \sum_{\nu \in N_Q^P(Q)} f_\xi(x^{-1}\mu\nu n x)\, dn \right|,$$

which is bounded by

$$\sum_{\mu \in M_Q(Q)} \left| \sum_{\{P: Q \subset P \subset P_1\}} (-1)^{\dim(A/Z)} \int_{N(A)} \sum_{\nu \in N_Q^P(Q)} f_\xi(x^{-1}\mu\nu n x)\, dn \right|.$$

Recall that $\mathfrak{n}_0$ is the Lie algebra of N_0. Let $\langle\ ,\ \rangle$ denote the canonical bilinear form on $\mathfrak{n}_0$. If $X \in \mathfrak{n}_0$, let $e(X)$ be the matrix in N_0 obtained by adding X to the identity. Then e is an isomorphism (of varieties over Q) from $\mathfrak{n}_0$ onto N_0. Let ψ be a nontrivial character on A/Q. Applying the Poisson summation formula to $\mathfrak{n}_Q^P$, we see that (1) is bounded by

$$\sum_\mu \left| \sum_P (-1)^{\dim(A/Z)} \sum_{\zeta \in \mathfrak{n}_Q^P(Q)} \int_{\mathfrak{n}_Q(A)} f_\xi(x^{-1}\mu e(X) x)\psi(\langle X, \zeta\rangle)\, dX \right|.$$

If $\mathfrak{n}_Q^1(Q)'$ is the set of elements in $\mathfrak{n}_Q^1(Q)$, which do not belong to any $\mathfrak{n}_Q^P(Q)$ with $Q \subset P \subsetneqq P_1$, this expression equals

$$\sum_\mu \left| \sum_{\zeta \in \mathfrak{n}_Q^1(Q)'} \int_{\mathfrak{n}_Q(A)} f_\xi(x^{-1}\mu e(X) x)\psi(\langle X, \zeta\rangle)\, dX \right|.$$

Therefore the integral of $|k^T(x)|$ is bounded by the sum over $Q \subset P_1$ and $\mu \in M_Q(Q)$ of the integral over (n^*, n_*, a, m, k) in

$$N_1(Q)\backslash N_1(A) \times N_Q^1(Q)\backslash N_Q^1(A) \times Z(R)^0\backslash A_Q(R)^0 \times A_Q(R)^0 M_Q(Q)\backslash M_Q(A) \times K$$

of

$$F^Q(m, T)\, \sigma_Q^1(H(a) - T) \exp(-2\langle \rho_Q, H(a)\rangle)$$
$$\cdot \sum_{\zeta \in \mathfrak{n}_Q^1(Q)'} \left| \int_{\mathfrak{n}_Q(A)} f_\xi(k^{-1}m^{-1}a^{-1}n_*^{-1}n^{*-1} \cdot \mu e(x) \cdot n^* n_* a m k)\psi(\langle X, \zeta\rangle)\, dX \right|.$$

The integral over n^* goes out. The integrals over k and m can be taken over compact sets. It follows from the last lemma that the set of points $\{a^{-1} n_* a\}$, indexed by those n_* and a for which the integrand is not zero, is relatively compact. Therefore there is a compact set C in $Z(R)^0\backslash G(A)$ such that the integral of $|k^T(x)|$ is bounded by the sum over $Q \subset P_1$, $\mu \in M_Q(Q)$ and the integral over $x \in C$ of

$$\int_{Z(R)^0\backslash A_Q(R)^0} \exp(-2\langle \rho_Q, H(a)\rangle)\, \sigma_Q^1(H(a) - T)$$
$$\cdot \sum_{\zeta \in \mathfrak{n}_Q^1(Q)'} \left| \int_{\mathfrak{n}_Q(A)} f_\xi(x^{-1}\mu a^{-1} e(X) a x)\psi(\langle X, \zeta\rangle)\, dX \right| da$$
$$= \int_{Z(R)^0\backslash A_Q(R)^0} \sigma_Q^1(H(a) - T) \sum_\zeta \left| \int_{\mathfrak{n}_Q(A)} f_\xi(x^{-1}\, \mu e(X) x) \cdot \psi(\langle X, \mathrm{Ad}(a)\zeta\rangle)\, dX \right| da.$$

Since f is compactly supported, the sum over μ is finite. If $Q = P_1$, σ_Q^1 equals 0, unless of course $Q = P_1 = G$, when it equals 1. If $Q \subsetneqq P_1$,

$$Y \to \int_{\mathfrak{n}_Q(A)} f_\xi(x^{-1}\mu e(X)x)\psi(\langle X, Y\rangle)\, dX, \qquad Y \in \mathfrak{n}_Q^1(A),$$

is the Fourier transform of a Schwartz-Bruhat function on $\mathfrak{n}_Q^1(A)$, and is continuous in x. If $H(a) = H_* + H^*$, $H_* \in \mathfrak{a}_Q^1$, $H^* \in \mathfrak{a}_1/\mathfrak{z}$, H^* must remain in a compact set. Since H_* lies in the positive chamber of $\mathfrak{a}_Q^1$, far from the walls, $\operatorname{Ad}(a)$ stretches any element ζ in $\mathfrak{n}_Q^1(Q)'$. In fact as H_* goes to infinity in any direction, $\operatorname{Ad}(a)\zeta$ goes to infinity. Here it is crucial that ζ not belong to any $\mathfrak{n}_Q^P(Q)$, $Q \subset P \subsetneqq P_1$. It follows that if $Q \subsetneqq P_1$, the corresponding term is finite and goes to 0 exponentially in T. Thus the dominant term is the only one left, that corresponding to $Q = P_1 = G$. It is an integral over the compact set $G(T)$. We have sketched the proof of

THEOREM 1. *We can choose $\varepsilon > 0$ such that for any $T \in \mathfrak{a}_0^+$, sufficiently far from the walls,*

$$\int_{Z(\mathbf{R})^0 G(Q)\backslash G(A)} k^T(x)\, dx = \int_{G(T)} K(x, x)\, dx + O(e^{-\varepsilon\|T\|}). \quad \square$$

We would expect the integral of $k^T(x)$ to break up into a sum of terms corresponding to conjugacy classes in $G(Q)$. It seems, however, that a certain equivalence relation in $G(Q)$, weaker than conjugacy, is more appropriate. If $\mu \in G(Q)$, let μ_s be its semisimple component relative to the Jordan decomposition. Call two elements μ and μ' in $G(Q)$ *equivalent* if μ_s and μ_s' are $G(Q)$-conjugate. Let $\mathscr{C}$ be the set of equivalence classes in $G(Q)$. If $\mathfrak{o} \in \mathscr{C}$, define

$$K_{\mathfrak{o}}^P(x, y) = \sum_{\mu \in M(Q) \cap \mathfrak{o}} \int_{N(A)} f_\xi(x^{-1}\mu n y)\, dn,$$

and

$$k_{\mathfrak{o}}^T(x) = \sum_P (-1)^{\dim(A/Z)} \sum_{\delta \in P(Q)\backslash G(Q)} K_{\mathfrak{o}}^P(\delta x, \delta x)\hat{\tau}_P(H(\delta x) - T).$$

Then

$$k^T(x) = \sum_{\mathfrak{o} \in \mathscr{C}} k_{\mathfrak{o}}^T(x).$$

If $\mu \in G$ and H is a closed subgroup of G, let H_μ denote the centralizer of μ in H.

LEMMA 7. *Fix P. Then for $\mu \in M(Q)$ and $\phi \in C_c^\infty(N(A))$,*

$$\sum_{\zeta \in N_{\mu_s}(Q)\backslash N(Q)} \sum_{\nu \in N_{\mu_s}(Q)} \phi(\mu^{-1}\zeta^{-1}\mu\nu\zeta) = \sum_{\nu \in N(Q)} \phi(\nu). \quad \square$$

This lemma is easily proved. It implies that for $\mathfrak{o} \in \mathscr{C}$, $P(Q) \cap \mathfrak{o} = (M(Q) \cap \mathfrak{o})N(Q)$. If this fact is combined with the proof of Theorem 1 we obtain a stronger version.

THEOREM 1*. *There are positive constants C and ε such that*

$$\sum_{\mathfrak{o} \in \mathscr{C}} \int_{Z(\mathbf{R})^0 G(Q)\backslash G(A)} \left| k_{\mathfrak{o}}^T(x) - F^G(x, T) K_{\mathfrak{o}}^G(x, x) \right| dx \leq C e^{-\varepsilon\|T\|}. \quad \square$$

The integral of $k_{\mathfrak{o}}^T(x)$ cannot be computed yet. What we must do is replace $k_{\mathfrak{o}}^T(x)$ by a different function. Define

$$J_{\mathfrak{o}}^P(x, y) = \sum_{\mu \in M(\boldsymbol{Q}) \cap \mathfrak{o}} \sum_{\zeta \in N_{\mu_s}(\boldsymbol{Q}) \backslash N(\boldsymbol{Q})} \int_{N_{\mu_s}(A)} f_\xi(x^{-1}\zeta^{-1}\mu n\zeta y)\, dn.$$

It is obtained from $K_{\mathfrak{o}}^P(x, y)$ by replacing a part of the integral over $N(A)$ by the corresponding sum over $\boldsymbol{Q}$-rational points. Define

$$j_{\mathfrak{o}}^T(x) = \sum_P (-1)^{\dim(A/Z)} \sum_{\delta \in P(\boldsymbol{Q}) \backslash G(\boldsymbol{Q})} J_{\mathfrak{o}}^P(\delta x, \delta x)\hat{\tau}_P(H(\delta x) - T).$$

The proof of the following is similar to that of Theorem 1*.

THEOREM 2. *There are positive constants C and* ε *such that*

$$\sum_{\mathfrak{o} \in \mathscr{C}} \int_{Z(\boldsymbol{R})^0 G(\boldsymbol{Q}) \backslash G(A)} |j_{\mathfrak{o}}^T(x) - F^G(x, T) K_{\mathfrak{o}}^G(x, x)|\, dx \leqq C e^{-\varepsilon \|T\|}. \quad \square$$

Suppose that T_1 is a point in $T + \mathfrak{a}_0^+$. By integrating the difference of $k_{\mathfrak{o}}^T(x)$ and $k_{\mathfrak{o}}^{T_1}(x)$ one proves inductively that the integral of $k_{\mathfrak{o}}^T(x)$ is a polynomial in T. The same goes for the integral of $j_{\mathfrak{o}}^T(x)$. Since the integrals of $k_{\mathfrak{o}}^T(x)$ and $j_{\mathfrak{o}}^T(x)$ differ by an expression which approaches 0 as T approaches ∞, they must be equal. Summarizing what we have said so far, we have

THEOREM 3.

$$\int_{Z(\boldsymbol{R})^0 G(\boldsymbol{Q}) \backslash G(A)} k^T(x)\, dx = \sum J_{\mathfrak{o}}^T(f_\xi),$$

where $J_{\mathfrak{o}}^T(f_\xi) = \int_{Z(\boldsymbol{R})^0 G(\boldsymbol{Q}) \backslash G(A)} j_{\mathfrak{o}}^T(x)\, dx$. $\square$

Suppose that the class $\mathfrak{o}$ consists entirely of semisimple elements, so that $\mathfrak{o}$ is an actual conjugacy class. The centralizer of any element in $\mathfrak{o}$ is anisotropic modulo its center. The split component of the center is $G(\boldsymbol{Q})$-conjugate to the split component of a standard parabolic subgroup. Thus, we can find $\mu_0 \in \mathfrak{o}$ and a standard parabolic P_0 such that the identity component of G_{μ_0} is contained in M_0 and is anisotropic modulo A_0. We shall say that the class $\mathfrak{o}$ is *unramified* if G_{μ_0} itself is contained in M_0. Assume that this is the case. We shall show how to express $J_{\mathfrak{o}}^T(f_\xi)$ as a weighted orbital integral of f_ξ.

Given any P, suppose that $\mu \in \mathfrak{o} \cap M(\boldsymbol{Q})$. Then by the same argument, we can choose an element s in $\bigcup_{P_1} \Omega(\mathfrak{a}_0, \mathfrak{a}_1)$ and an element $\eta \in M(\boldsymbol{Q})$ such that $s\mathfrak{a}_0$ contains $\mathfrak{a}$, and $\mu = \eta w_s \mu_0 w_s^{-1} \eta^{-1}$. Let $\Omega(\mathfrak{a}_0; P)$ be the set of elements s in $\bigcup_{P_1} \Omega(\mathfrak{a}_0, \mathfrak{a}_1)$ such that if $\mathfrak{a}_1 = s\mathfrak{a}_0$, $\mathfrak{a}_1$ contains $\mathfrak{a}$, and $s^{-1}\alpha$ is positive for every root α in $\Phi_{P_1}^P$. Then if we demand that the element s above lie in $\Omega(\mathfrak{a}_0; P)$, it is uniquely determined. Thus $J_{\mathfrak{o}}^P(\delta x, \delta x)$ equals

$$\sum_{s \in \Omega(\mathfrak{a}_0; P)} \sum_{\eta \in M_{w_s \mu_0 w_s^{-1}}(\boldsymbol{Q}) \backslash M(\boldsymbol{Q})} \sum_{\zeta \in N(\boldsymbol{Q})} f_\xi(x^{-1}\delta^{-1}\zeta^{-1}\eta^{-1} w_s \mu_0 w_s^{-1} \eta \zeta \delta x)$$

$$= \sum_{s \in \Omega(\mathfrak{a}_0; P)} \sum_{\zeta \in M_{w_s \mu_0 w_s}(\boldsymbol{Q}) \backslash P(\boldsymbol{Q})} f_\xi(x^{-1}\delta^{-1}\zeta^{-1} w_s \mu_0 w_s^{-1} \zeta \delta x).$$

Therefore $j_{\mathfrak{o}}^T(x)$ equals

$$\sum_P (-1)^{\dim(A/Z)} \sum_{s \in \Omega(\mathfrak{a}_0;P)} \sum_{\delta \in M_{w_s\mu_0 w_s^{-1}}(\mathbf{Q})\backslash G(\mathbf{Q})} f_\xi(x^{-1}\delta^{-1}w_s\mu_0 w_s^{-1}\delta x)\hat{\tau}_P(H(\delta x) - T).$$

Since the centralizer of $w_s\mu_0 w_s^{-1}$ in G is contained in M, this equals

$$\sum_{\delta \in G_{\mu_0}(\mathbf{Q})\backslash G(\mathbf{Q})} f_\xi(x^{-1}\delta^{-1}\mu\delta x) \sum_P (-1)^{\dim(A/Z)} \sum_{s \in \Omega(\mathfrak{a}_0;P)} \hat{\tau}_P(H(w_s\delta x) - T).$$

Then $J_{\mathfrak{o}}^T(f_\xi)$ equals

$$\mathrm{vol}(A_0(\mathbf{R})^0 G_{\mu_0}(\mathbf{Q})\backslash G_{\mu_0}(\mathbf{A})) \int_{G_{\mu_0}(\mathbf{A})\backslash G(\mathbf{A})} f_\xi(x^{-1}\mu x)v(x, T)\, dx, \tag{2}$$

where

$$v(x, T) = \int_{Z(\mathbf{R})^0\backslash A_0(\mathbf{R})^0} \left\{ \sum_P (-1)^{\dim(A/Z)} \sum_{s \in \Omega(\mathfrak{a}_0;P)} \hat{\tau}_P(H(w_s ax) - T)\, da \right\}.$$

The expression in the brackets is compactly supported in a. In fact it follows from the results of [7, §§2, 3] that $v(x, T)$ equals the volume in $\mathfrak{a}_0/\mathfrak{z}$ of the convex hull of the projection onto $\mathfrak{a}_0/\mathfrak{z}$ of $\{s^{-1}T - s^{-1}H(w_s x);\ s \in \bigcup_{P_1}\Omega(\mathfrak{a}_0, \mathfrak{a}_1)\}$. It was Langlands who surmised that the volume of a convex hull would play a role in the trace formula.

By studying how far $J_{\mathfrak{o}}^T(f_\xi)$ differs from an invariant distribution, I hope to express $J_{\mathfrak{o}}^T(f_\xi)$, for general $\mathfrak{o}$, as a limit of the distributions for which $\mathfrak{o}$ is as above, at least modulo an invariant distribution that lives on the unipotent set of $G(\mathbf{A})$. However, this has not yet been done.

The study of $K_E(x, x)$ parallels what we have just done. The place of $\{\mathfrak{o} \in \mathscr{C}\}$ is now taken by $\{\chi \in \mathscr{S}_E(G)\}$. Given P, and $\chi \in \mathscr{S}_E(G)$, define

$$K_\chi^P(x, y) = \sum_{Q \subset P} n^P(A_Q)^{-1}\left(\frac{1}{2\pi i}\right)^{\dim(A_Q/Z)}$$
$$\cdot \int_{i\mathfrak{a}_Q^G} \left\{ \sum_{\Phi \in \mathscr{B}_{Q,\chi}} E^P(x, I_Q(\Lambda_\xi, f)\Phi, \Lambda)\, \overline{E^P(y, \Phi, \Lambda)} \right\} d\Lambda,$$

where $n^P(A_Q)$ is the number of chambers in $\mathfrak{a}_Q/\mathfrak{a}$. Then if $P \neq G$,

$$K^P(x, y) = \sum_{\chi \in \mathscr{S}_E(G)} K_\chi^P(x, y).$$

The convergence of the sum over χ and the above integral over Λ is established by the argument of Lemma 2. Define

$$k_\chi^T(x) = \sum_P (-1)^{\dim(A/Z)} \sum_{\delta \in P(\mathbf{Q})\backslash G(\mathbf{Q})} K_\chi^P(\delta x, \delta x)\hat{\tau}_P(H(\delta x) - T).$$

Then

$$K_{\mathrm{cusp}}(x, x) = \sum_{\mathfrak{o} \in \mathscr{C}} k_{\mathfrak{o}}^T(x) - \sum_{\chi \in \mathscr{S}_E(G)} k_\chi^T(x).$$

We would like to be able to integrate the function $k_\chi^T(x)$. But as before, we will have to replace it with a new function $j_\chi^T(x)$ before this can be done.

To define the new function, we need to introduce a truncation operator. In form it resembles the way we modified the kernel $K(x, x)$, but it applies to any continuous function ϕ on $Z(\mathbf{R})^0 \cdot G(\mathbf{Q})\backslash G(\mathbf{A})$. Define a new function on $Z(\mathbf{R})^0 \cdot G(\mathbf{Q})\backslash G(\mathbf{A})$ by

$$(\Lambda^T\phi)(x) = \sum_{P \supset P_0} (-1)^{\dim(A/Z)} \sum_{\delta \in P(\mathbf{Q})\backslash G(\mathbf{Q})} \hat{\tau}_P(H(\delta x) - T) \cdot \int_{N(\mathbf{Q})\backslash N(\mathbf{A})} \phi(n\delta x)\, dn.$$

Λ^T has some agreeable properties. It leaves any cusp form invariant. It is a self-adjoint operator. These facts are clear. It is less clear, but true, that $\Lambda^T \circ \Lambda^T = \Lambda^T$. If $\chi \in \mathcal{S}_E(G)$, define

$$J_\chi^T(x, y) = \sum_P n(A)^{-1}\left(\frac{1}{2\pi i}\right)^{\dim(A/Z)} \cdot \int_{i\mathfrak{a}^G} \left\{ \sum_{\phi \in \mathcal{B}_{P,\chi}} E(x, I_P(\Lambda_\xi, f)\Phi, \Lambda)\overline{\Lambda^T E(y, \Phi, \Lambda)} \right\} d\Lambda.$$

If we apply Λ^T to the second variable in $K_E(x, y)$ we obtain $\sum_{\chi \in \mathcal{S}_E(G)} J_\chi^T(x, y)$. Define $j_\chi^T(x) = J_\chi^T(x, x)$.

THEOREM 4. *The function* $\sum_{\chi \in \mathcal{S}_E(G)} |j_\chi^T(x)|$ *is integrable over* $Z(\mathbf{R})^0 G(\mathbf{Q})\backslash G(\mathbf{A})$. *For any* χ, *the integral of* $j_\chi^T(x)$ *equals*

$$\sum_P n(A)^{-1}\left(\frac{1}{2\pi i}\right)^{\dim(A/Z)} \cdot \int_{i\mathfrak{a}^G} \sum_{\Phi \in \mathcal{B}_{P,\chi}} \int_{Z(\mathbf{R})^0 G(\mathbf{Q})\backslash G(\mathbf{A})} \Lambda^T E(x, I_P(\Lambda_\xi, f)\Phi, \Lambda)\overline{\Lambda^T E(x, \Phi, \Lambda)}\, dx\, d\Lambda.$$

The first statement of the theorem comes from a property of Λ^T. Namely, if ϕ is a smooth function on $Z(\mathbf{R})^0 \cdot G(\mathbf{Q})\backslash G(\mathbf{A})$ any of whose derivatives (with respect to the universal enveloping algebra of $\mathfrak{g}(\mathbf{C})$) are slowly increasing in a certain sense, then $\Lambda^T\phi$ is rapidly decreasing. The proof of this property is similar to the proof of Theorem 2. The Poisson formula can no longer be used, but one uses [3, Lemma 10] instead. Given the fact that $\Lambda^T \circ \Lambda^T = \Lambda^T$, the other half of the theorem is a statement about the interchange of the integrals over x and Λ. By the proof of Lemma 2 we can essentially assume the integrand is nonnegative. The result follows. □

THEOREM 5. *There are positive constants C and ε such that*

$$\sum_{\chi \in \mathcal{S}_E(G)} \int_{Z(\mathbf{R})^0 G(\mathbf{Q})\backslash G(\mathbf{A})} \left|k_\chi^T(x) - j_\chi^T(x)\right| dx \leqq Ce^{-\varepsilon\|T\|}.$$

It turns out that this theorem can be proved by studying the function $k^T(x) - \Lambda_y^T K(x, y)$, at $x = y$. This is essentially the sum over all χ of the above integrand (without the absolute value bars). The point is that the new function is easier to study because it has a manageable expression in terms of f. □

Combining Theorems 4 and 5, we see that $\int_{Z(\mathbf{R})^0 G(\mathbf{Q})\backslash G(\mathbf{A})} \sum_{\chi \in \mathcal{S}_E(G)} |k_\chi^T(x)|\, dx$ is finite. In particular, each $k_\chi^T(x)$ is integrable. With a little more effort it can be shown that the integrals of $k_\chi^T(x)$ and $j_\chi^T(x)$ are actually equal. We shall denote the common value by $J_\chi^T(f_\xi)$. It is a polynomial in T. We have shown that

$$\operatorname{tr} R_{\mathrm{cusp},\xi}(f) = \sum_{\mathfrak{o}\in\mathscr{C}} J_{\mathfrak{o}}^T(f_\xi) - \sum_{\chi\in\mathscr{S}_E(G)} J_\chi^T(f_\xi)$$

for any suitably large $T \in \mathfrak{a}_0^+$. The right-hand side is a polynomial in T while the left-hand side is independent of T. Letting $J_{\mathfrak{o}}(f_\xi)$ and $J_\chi(f_\xi)$ be the constant terms of the polynomials $J_{\mathfrak{o}}^T(f_\xi)$ and $J_\chi^T(f_\xi)$ we have

THEOREM 6. *For any* $f \in C_c^\infty(G(A))^K$,

$$\operatorname{tr} R_{\mathrm{cusp},\xi}(f) = \sum_{\mathfrak{o}\in\mathscr{C}} J_{\mathfrak{o}}(f_\xi) - \sum_{\chi\in\mathscr{S}_E(G)} J_\chi(f_\xi). \quad \square$$

Suppose that $\chi = (\mathscr{P}, \mathscr{V}, W)$ and that $P \in \mathscr{P}$. Then if $\Phi \in \mathscr{H}_{P,\chi}$, $\Lambda^T E(x, \Phi, \Lambda)$ equals the function denoted $E''(x, \Phi, \Lambda)$ by Langlands in [**2**, §9]. This is, in fact, what led us to the definition of Λ^T in the first place. If Φ' is another vector in $\mathscr{H}_{P,\chi}$, Langlands has proved the elegant formula

$$\int_{Z(\mathbf{R})^0 \cdot G(\mathbf{Q})\backslash G(\mathbf{A})} E''(x, \Phi', \Lambda')\overline{E''(x, \Phi, \Lambda)}\, dx$$
$$= \sum_{P_2} \sum_{t\in\Omega(\mathfrak{a},\mathfrak{a}_2)} \sum_{s\in\Omega(\mathfrak{a},\mathfrak{a}_2)} \frac{e^{\langle t\Lambda' + s\bar{\Lambda}, T\rangle}}{\prod_{\alpha\in\Phi_2}\langle t\Lambda' + s\bar{\Lambda}, \alpha\rangle} (M(t, \Lambda')\Phi', M(s, \Lambda)\Phi)$$

(see [**2**, §9]. Actually the formula quoted by Langlands is slightly more complicated, but it can be reduced to what we have stated.) In this formula, we can set $\Lambda' = \Lambda$ and $\Phi' = I_P(\Lambda, f)\Phi$. We can then sum over all Φ in $\mathscr{B}_{P,\chi}$ and integrate over $\Lambda \in i\mathfrak{a}^G$. The result is *not* a polynomial in T. To obtain $J_\chi^T(f_\xi)$ we would have to consider all P, not just those in the associated class $\mathscr{P}$. The best hope seems to be to calculate residues in Λ and Λ' separately in the above formula. For GL_3 the result turns out to be relatively simple.

PART III. THE CASE OF GL_3

In this last section we shall give the results of further calculations. They can be stated for general G but at this point they can be proved only for $G = GL_3$. Our aim is to express the trace of $R_{\mathrm{cusp},\xi}(f)$ in terms of the invariant distributions defined in

8. J. Arthur, *On the invariant distributions associated to weighted orbital integrals*, preprint.

A trace formula for K-bi-invariant functions has also been proved in

9. A. B. Venkov, *On Selberg's trace formula for* $SL_3(\mathbf{Z})$, Soviet Math. Dokl. **17** (1976), 683–687.

First we remark that the distributions $J_{\mathfrak{o}}^T(f_\xi)$ and $J_\chi^T(f_\xi)$ are independent of our minimal parabolic subgroup so there is no further need to fix P_0. If A is any $\mathbf{Q}$-split torus in G, let $\mathscr{P}(A)$ denote the set of parabolic subgroups with split component A. They are in bijective correspondence with the chambers in A. In fact, if P_0 is a minimal parabolic subgroup contained in an element P of $\mathscr{P}(A)$, then

$$\mathscr{P}(A) = \bigcup_{A_1} \bigcup_{s\in\Omega(\mathfrak{a},\mathfrak{a}_1)} w_s^{-1} P_1 w_s.$$

If $P' \in \mathscr{P}(A)$, and $P' = w_s^{-1} P_1 w_s$, define

$$(M_{P'|P}(\Lambda)\Phi)(x) = (M(s, \Lambda)\Phi)(w_s x), \qquad \Phi \in \mathscr{H}(P).$$

Then $M_{P'|P}(\Lambda)$ is a map from $\mathscr{H}(P)$ to $\mathscr{H}(P')$, which is independent of P_0. In fact, if Re $\Lambda \in \rho_P + \mathfrak{a}_P^+$,

$$(1) \qquad (M_{P'|P}(\Lambda)\Phi)(x) = \int_{N(A)\cap N'(A)\backslash N'(A)} \Phi(nx)\exp(\langle \Lambda + \rho_P, H_P(nx)\rangle) \cdot \exp(-\langle \Lambda + \rho_{P'}, H_{P'}(x)\rangle)\, dn.$$

We have changed our notation to agree with that of **[8]**.

A is said to be a *special subgroup* of G if $\mathscr{P}(A)$ is not empty. Suppose that A and A_1 are special subgroups, with $A \supset A_1$. We write $\Omega^{M_1}(\mathfrak{a}, \mathfrak{a})_{\text{reg}}$ for the set of elements $s \in \Omega(\mathfrak{a}, \mathfrak{a})$ whose space of fixed vectors in $\mathfrak{a}$ is $\mathfrak{a}_1$. Suppose that $P_1 \in \mathscr{P}(A_1)$ and $Q \in \mathscr{P}^{M_1}(A)$, the set of parabolic subgroups of M_1 with split component A. Then there is a unique group in $\mathscr{P}(A)$, which we denote by $P_1(Q)$, such that $P_1(Q) \subset P_1$ and $P_1(Q) \cap M_1 = Q$.

LEMMA 1. *Suppose that A and A_1 are as above and that $P = P_1(Q)$ for some $P_1 \in \mathscr{P}(A_1)$ and $Q \in \mathscr{P}^{M_1}(A)$. Then if $\Lambda \in i\mathfrak{a}_1$, the limit as λ approaches 0 of*

$$\sum_{P_1 \in \mathscr{P}(A_1)} M_{P_1(Q)|P}(\Lambda)^{-1} M_{P_1(Q)|P}(\Lambda + \lambda)\Big(\prod_{\alpha \in \Phi_{P_1}} \langle \lambda, \alpha\rangle\Big)^{-1}$$

exists as an operator on $\mathscr{H}_P$. We denote it by $M(P, A_1, \Lambda)$.

This lemma follows from **[8]**. □

Fix a maximal special subgroup A_0 of G. From Langlands' inner product formula, quoted at the end of Part II, one can prove

LEMMA 2. *For any $\chi \in \mathscr{S}_E(G)$, $J_\chi(f_\xi)$ equals the sum over all special subgroups A_1 and A of G, with $A_1 \subset A \subset A_0$, and over $s \in \Omega^{M_1}(\mathfrak{a}, \mathfrak{a})_{\text{reg}}$ of*

$$c_s \int_{i\mathfrak{a}_1^G} \sum_{\Phi \in \mathscr{B}_{P,\chi}} (M(P, A_1, \Lambda_\xi) M(s, 0)\, I_P(\Lambda_\xi, f)\Phi, \Phi)\, d\Lambda.$$

Here c_s is the product of

$$\left(\frac{1}{2\pi i}\right)^{\dim(A_1/Z)} n^M(A_0)\cdot n(A_0)^{-1}\cdot |\det(1 - \mathrm{Ad}(s))_{\mathfrak{a}/\mathfrak{a}_1}|^{-1}$$

with the volume of $\mathfrak{a}_P^1$ modulo the lattice generated by Φ_P^1, and P is any element in $\mathscr{P}(A_1)$ which contains some group in $\mathscr{P}(A)$. □

Recall the decomposition $I_P(\Lambda) = \bigoplus_\iota \bigotimes_v I_P(\sigma_{v,\Lambda}^\iota)$. If Φ_v is a smooth vector in $\mathscr{H}_P(\sigma_v^\iota)$ and Re $\Lambda \in \rho_P + \mathfrak{a}_P^+$, define

$$(M_{P'|P}(\sigma_{v,\Lambda}^\iota)\Phi_v)(x) = \int_{N(\mathbf{Q}_v)\cap N'(\mathbf{Q}_v)\backslash N'(\mathbf{Q}_v)} \Phi_v(nx)\exp(\langle \Lambda + \rho_P, H_P(nx)\rangle) \cdot \exp(-\langle \Lambda + \rho_{P'}, H_{P'}(x)\rangle)\, dn,$$

for $x \in G(\mathbf{Q}_v)$. This is the usual unnormalized intertwining operator for a group over a local field. Then

$$M_{P'|P}(\Lambda) = \bigoplus_\iota \bigotimes_v M_{P'|P}(\sigma_{v,\Lambda}^\iota).$$

LEMMA 3. *If σ_v is an irreducible unitary representation of $M(\mathbf{Q}_v)$, we can define meromorphic functions $r_{P'\backslash P}(\sigma_{v,\Lambda})$, $P, P' \in \mathscr{P}(A)$, $\Lambda \in \mathfrak{a}_{\mathbf{C}}$, so that the operators*

$R_{P'|P}(\sigma_{v,\Lambda}) = M_{P'|P}(\sigma_{v,\Lambda}) \cdot r_{P'|P}(\sigma_{v,\Lambda})^{-1}$ *can be analytically continued in* Λ, *with the following functional equations holding*:

$$R_{P''|P}(\sigma_{v,\Lambda}) = R_{P''|P'}(\sigma_{v,\Lambda})R_{P'|P}(\sigma_{v,\bar{\Lambda}}), \qquad P, P', P'' \in \mathscr{P}(A),$$

and

$$R_{P'|P}(\sigma_{v,\Lambda})^* = R_{P|P'}(\sigma_{v,-\bar{\Lambda}}).$$

Moreover if σ_v *is of class* 1 *and* Φ_v *is the* K_v*-invariant function*,

$$R_{P'|P}(\sigma_{v,\Lambda})\Phi_v = \Phi_v. \quad \square$$

Suppose that $\sigma = \bigotimes_v \sigma_v$ is an irreducible unitary representation of $M(A)$. If $\Phi = \bigotimes_v \Phi_v$ is a smooth vector in $\mathscr{H}_P(\sigma) = \bigotimes_v \mathscr{H}_P(\sigma_v)$, define

$$R_{P'|P}(\sigma)\Phi = \bigotimes_v R_{P'|P}(\sigma_v)\Phi_v.$$

For almost all v, the right-hand vector is the characteristic function of K_v. Define M^1 to be the kernel of the set of rational characters of M defined over $\boldsymbol{Q}$. Then M^1 is defined over $\boldsymbol{Q}$, and $M(A) = M^1(A)A(A)$. Let f be a function in $C_c^\infty(G(A))^K$, as in Part II.

LEMMA 4. *There is a function* $\phi_A(f)$ *in* $C^\infty(M(A))^{K\cap M(A)}$ *such that*

$$(m, a) \to \phi_A(f, ma), \qquad m \in M^1(A),\ a \in A(A),$$

is compactly supported in m and a Schwartz function in a, so that the following property holds. If $\sigma = \bigotimes_v \sigma_v$ *is any irreducible unitary representation of* $M(A)$,

$$\operatorname{tr} \sigma(\phi_A(f)) = \lim_{\lambda \to 0} \sum_{P' \in \mathscr{P}(A)} \Big(\prod_{\alpha \in \Phi_{P'}} \langle \lambda, \alpha \rangle \Big)^{-1} \cdot \operatorname{tr}(R_{P'|P}(\sigma)^{-1} R_{P'|P}(\sigma_\lambda) I_P(\sigma, f)).$$

(*The limit on the right exists and is independent of the fixed group* $P \in \mathscr{P}(A)$.) $\square$

Suppose that $\mathfrak{o}$ is an equivalence class in $G(\boldsymbol{Q})$. Then $M(\boldsymbol{Q}) \cap \mathfrak{o}$ is a finite union (possibly empty), $\mathfrak{o}_1^M \cup \cdots \cup \mathfrak{o}_n^M$, of equivalence classes relative to the group $M(\boldsymbol{Q})$. If $F_{\mathfrak{o}^M}$ is a function defined on the equivalence classes relative to the group $M(\boldsymbol{Q})$, let us write

$$F_{\mathfrak{o}} = n^M(A_0)n(A_0)^{-1} \sum_{i=1}^n F_{\mathfrak{o}_i^M}.$$

Now we shall define an invariant distribution $I_{\mathfrak{o}}$ for each $\mathfrak{o} \in \mathscr{C}$. The definition is inductive; we assume that the invariant distributions $I_{\mathfrak{o}^M}^M$ have been defined for each special subgroup A, with $Z \subsetneq A \subset A_0$. We then define

$$I_{\mathfrak{o}}(f_\xi) = J_{\mathfrak{o}}(f_\xi) - \sum_{\{A: Z \subsetneq A \subset A_0\}} I_{\mathfrak{o}}^M(\phi_A(f)_\xi).$$

LEMMA 5. $I_{\mathfrak{o}}$ *is invariant.*

This is essentially Theorem 5.3 of [**8**]. Note that this lemma is necessary for our inductive definition, since $\phi_A(f)$ is only defined up to conjugation by an element in $M(A)$. $\square$

Suppose that $\sigma = \bigotimes_v \sigma_v$ is a unitary *automorphic* representation of $M(A)$. Then

$$r_{P'|P}(\sigma) = \prod_v r_{P'|P}(\sigma_v), \qquad P, P' \in \mathscr{P}(A),$$

is essentially a quotient of two Euler products and is defined by analytic continuation. If $\Lambda \in i\mathfrak{a}$, let $r_{P'|P}(\Lambda)$ be the operator on $\mathscr{H}_P$ which acts on the subspace determined by $I_P(\sigma^\iota)$ by the scalar $r_{P'|P}(\sigma_v^\iota)$. If $A \supset A_1$ define the operator $r(P, A_1, \Lambda)$ on $\mathscr{H}_P$ by the limit in Lemma 1, with $M_{P_1(Q)|P}(\Lambda)^{-1}M_{P_1(Q)|P}(\Lambda + \lambda)$ replaced by $r_{P_1(Q)|P}(\Lambda)^{-1}r_{P_1(Q)|P}(\Lambda + \lambda)$. It commutes with the action of $G(\boldsymbol{A})$. Finally, for $\chi \in \mathscr{S}_E(G)$, define $i_\chi(f_\xi)$ by the formula for $J_\chi(f_\xi)$ in Lemma 2, with $M(P, A_1, \Lambda_\xi)$ replaced by $r(P, A_1, \Lambda_\xi)$. Then i_χ is an invariant distribution.

THEOREM 1. *For any* $f \in C_c^\infty(G(\boldsymbol{A}))^K$,

$$\operatorname{tr} R_{\mathrm{cusp},\xi}(f) = \sum_{\mathfrak{o}\in\mathscr{C}} I_\mathfrak{o}(f_\xi) - \sum_{\chi\in\mathscr{S}_E(G)} i_\chi(f_\xi). \qquad \square$$

REMARK. It follows from formula (2) of Part II that if $\mathfrak{o}$ consists entirely of semisimple elements, $I_\mathfrak{o}$ is one of the invariant distributions studied in **[8]**. Moreover since $G = \mathrm{GL}_3$, it is possible to show that for *any* $\mathfrak{o}$, $I_\mathfrak{o}$ is a sum of limits of the invariant distributions in **[8]**. Suppose that $f = \prod_v f_v$ and that for *two* places v,

$$\int_{T(\boldsymbol{Q}_v)\backslash G(\boldsymbol{Q}_v)} f_v(x^{-1}tx)\, dx = 0, \qquad t \in T(\boldsymbol{Q}_v)_{\mathrm{reg}},$$

for all maximal tori T in G such that $Z(\boldsymbol{Q}_v)\backslash T(\boldsymbol{Q}_v)$ is not compact. It follows from the last theorem of **[8]** that if there exists an element γ in a given $\mathfrak{o}$ which is $\boldsymbol{Q}$-elliptic mod Z,

$$I_\mathfrak{o}(f_\xi) = \operatorname{vol}(Z(\boldsymbol{R})^0\cdot G_\gamma(\boldsymbol{Q})\backslash G_\gamma(\boldsymbol{A}))\int_{G_\gamma(\boldsymbol{A})\backslash G(\boldsymbol{A})} f_\xi(x^{-1}\gamma x)\, dx,$$

and that $I_\mathfrak{o}(f_\xi) = 0$ if no such γ exists. Moreover, it is easy to see that each $i_\chi(f_\xi) = 0$. From this it follows that if $\{\gamma\}$ is a set of representatives of $G(\boldsymbol{Q})$-conjugacy classes of elements in $G(\boldsymbol{Q})$ which are elliptic mod Z,

$$\operatorname{tr} R_{\mathrm{cusp},\xi}(f) = \sum_{\{\gamma\}} \operatorname{vol}(Z(\boldsymbol{R})^0\cdot G_\gamma(\boldsymbol{Q})\backslash G_\gamma(\boldsymbol{A}))\int_{G_\gamma(\boldsymbol{A})\backslash G(\boldsymbol{A})} f_\xi(x^{-1}\gamma x)\, dx,$$

for f as above.

DUKE UNIVERSITY

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θ-SERIES AND INVARIANT THEORY

R. HOWE

0. Introduction. Historically θ-series have been one of the major methods of constructing automorphic forms. In **[Wi 1]** Weil reported the discovery that a certain unitary representation, also important in quantum mechanics, underlay the theory of θ-series, especially as developed by Siegel **[Si 1]**, **[Si 2]**. This note is a summary of work which extends Weil's representation theoretic formulation of the theory of θ-series. So far, this formulation provides a systematic framework in which to place recent work by several authors as well as several important older computations (see **[G]** for examples). A key feature of this approach is its strong ties with Classical Invariant Theory, as exposed in **[Wy]**. Indeed, it would seem that the theory of θ-series is in a sense a transcendental version of Classical Invariant Theory.

Convention. Throughout F will denote a field, never of characteristic 2. More precise specifications on F will vary from paragraph to paragraph, and will be stated at the beginning of each.

1. The Heisenberg group and the Stone-von Neumann Theorem; local case. Let F be a local field. Let W be a symplectic vector space over F, with form $\langle\ ,\ \rangle$. Define $H(W)$, the *Heisenberg group* attached to F by: $H(W) = W \oplus F$ as set, and has group law $(w, s)(w', s') = (w + w', s + s' + \frac{1}{2}\langle w, w'\rangle)$.

Let χ be a nontrivial character of F.

STONE-VON NEUMANN THEOREM. *There is only one equivalence class of irreducible unitary representations of H with central character χ.*

We denote this representation by ρ_χ. The corresponding representation on the smooth vectors **[C 2]**, **[Wa]** will be written ρ_χ^∞.

Although this representation is unique up to equivalence, it has, as already noted by Cartier **[C 1]**, many different concrete realizations which can be organized into smooth families. This circumstance seems to be at the heart of the theory of the oscillator representation.

2. The oscillator representation, local case. F continues to be a local field.

Let Sp be the isometry group of $\langle\ ,\ \rangle$. The action $g((w,s)) = (g(w),s)$ for $g \in \mathrm{Sp}$, and $(w, s) \in H(W)$ embeds Sp into the automorphism group of $H(V)$.

According to **[M]** there is a unique nontrivial two-fold cover $\widetilde{\mathrm{Sp}}$ of Sp (except

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if $F = C$, when we will understand $\tilde{\mathrm{Sp}} = \mathrm{Sp}$). Let $\tilde{g} \to g$ denote the covering map.

THEOREM (SHALE-WEIL). *There is a representation ω_χ (unique up to unitary equivalence) of $\tilde{\mathrm{Sp}}$ on the space of ρ_χ such that*

$$\omega_\chi(\tilde{g})\rho_\chi(h)\omega_\chi(\tilde{g}^{-1}) = \rho_\chi(g(h)) \tag{2.1}$$

for $\tilde{g} \in \tilde{\mathrm{Sp}}$ and $h \in H$.

To ω_χ, there is a corresponding smooth representation ω_χ^∞. The smooth vectors of ρ_χ and of ω_χ are equal (in any common realization) so ρ_χ^∞ and ω_χ^∞ act on the same space.

We call ω_χ, or the corresponding multiplier representation of Sp, the *oscillator representation.*

3. Adèlization. Let F now denote a global or A-field. Let v be a typical place of F and F_v the completion of F at v. Let $\boldsymbol{A}$ be the adèles of F. If $W = W_F$ is a symplectic vector space over F then we can form $H(W_F)$ and also its adèlization $H(W_{\boldsymbol{A}})$. As usual $H(W_F)$ is embedded as a discrete subgroup of $H(W_{\boldsymbol{A}})$, with compact quotient. We will suppress W, and just write $H(W_F) = H_F$, and $H(W_{F_v}) = H_v$, etc.

Let $\{e_i, f_i\}$ be a symplectic basis for W. Let Γ_v be the closure in H_v of the subgroup generated by the elements $\{(e_i, 0), (f_i, 0)\}$ of H_F. Then $H_{\boldsymbol{A}}$ is the restricted direct product of the H_v with respect to the Γ_v.

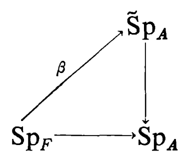
Let χ be a nontrivial character of $\boldsymbol{A}$ with $F \subseteq \ker \chi$. Let $\{\chi_v\}$ be the local factors of χ. At almost all places, there will be a unique vector ϕ_v invariant under $\rho_{\chi_v}(\Gamma_v)$. Thus we may form a representation ρ_χ of $H(W_{\boldsymbol{A}})$ as the restricted tensor product of the ρ_{χ_v}. Since each representation $\rho_{\chi_v}^\infty$ is admissible [C 2] and each ϕ_v is a smooth vector, the smooth subrepresentation ρ_χ^∞ of ρ_χ is admissible and is the restricted tensor product of the $\rho_{\chi_v}^\infty$ [F].

Let $J_v \subseteq \mathrm{Sp}_v$ be the normalizer of Γ_v. Then $\mathrm{Sp}_{\boldsymbol{A}}$ is the restricted direct product of the Sp_v with respect to the J_v. We may fit local actions together to obtain an action of $\mathrm{Sp}_{\boldsymbol{A}}$ on $H_{\boldsymbol{A}}$. As locally, there is a two-fold cover $\tilde{\mathrm{Sp}}_{\boldsymbol{A}}$ and a representation ω_χ of $\mathrm{Sp}_{\boldsymbol{A}}$ such that the global analogue of (2.1) holds. There is a corresponding smooth representation ω_χ^∞. Although $\tilde{\mathrm{Sp}}_{\boldsymbol{A}}$ is not the restricted product of the $\tilde{\mathrm{Sp}}_v$, there is a commutative diagram

$$\begin{array}{ccc} \prod_v \tilde{\mathrm{Sp}}_v & \longrightarrow & \tilde{\mathrm{Sp}}_{\boldsymbol{A}} \\ \downarrow & & \downarrow \\ \prod_v \mathrm{Sp}_v & \longrightarrow & \mathrm{Sp}_{\boldsymbol{A}} \end{array} \tag{3.1}$$

The products in (3.1) are restricted products with respect to the J_v. (For almost all v (in fact for v not dividing 2), there is a unique lifting of J_v to $\tilde{\mathrm{Sp}}_v$. We use this lifting to consider J_v a subgroup of $\tilde{\mathrm{Sp}}_v$ when convenient.) For almost all v the vector ϕ_v will be the unique vector fixed by $\omega_{\chi_v}^\infty(J_v)$, and the pull-back of ω_χ^∞ to $\prod_v \tilde{\mathrm{Sp}}_v$ is the restricted tensor product of the ω_χ^∞ with respect to the ϕ_v.

THEOREM (WEIL). *There is a unique homomorphism β: $\mathrm{Sp}_F \to \tilde{\mathrm{Sp}}_{\boldsymbol{A}}$ such that the triangle*



commutes, where the horizontal and vertical arrows are the canonical injection and projection respectively.

Thus we may speak unambiguously of $\omega_\chi(g)$, for g belonging to Sp_F.

4. The Θ-distribution. F continues as in §3.

THEOREM 4.1. *There is a linear functional Θ on (the space Y of) ρ_χ^∞ which is invariant by H_F. That is, $\Theta(\rho_\chi^\infty(h)(y)) = \Theta(y)$ for $h \in H_F$ and $y \in Y$. The functional Θ is unique up to multiples.*

This result is essentially a strong form of the Poisson Summation Formula. Results of the same type are in **[C 1]**. Existence of Θ is implicit in Weil **[Wi 1]**, and exploited classically. Existence of Θ is more or less responsible for the lifting β of §3.

Since Sp_F normalizes H_F, we see by formula (2.1) and the uniqueness of Θ that $\omega_\chi(\mathrm{Sp}_F)$ preserves Θ up to multiples. Since Sp_F is its own commutator subgroup, θ will in fact be invariant by Sp_F. Therefore, if u is a vector in the space of ρ_χ^∞, the function on $\tilde{\mathrm{Sp}}_A$ defined by $\theta(u)(g) = \theta(\omega_\chi(g)u)$, $g \in \tilde{\mathrm{Sp}}_A$, actually factors to the coset space $X = \mathrm{Sp}_F \backslash \tilde{\mathrm{Sp}}_A$. It may also be computed that $\theta(u)(g)$ is of "moderate growth" at ∞ on X, so that actually $\theta(u)$ is an automorphic form on $\tilde{\mathrm{Sp}}_A$. We call $\theta(u)$ the *theta-series* attached to u.

If $\dim W = 2$ and u is chosen appropriately, then $\theta(u)$ will be the classical θ-series of Jacobi. With somewhat more general u, but still a restricted class, one will obtain "θ-constants with characteristic" **[I]**. Thus the θ-distribution gives rise to automorphic forms which generalize well-known θ-series. However, the functions $\theta(u)$ are still very special automorphic forms. But if $G \subseteq \mathrm{Sp}_F$ is a reductive algebraic subgroup, then we may restrict $\theta(u)$ to $\tilde{G}_A$ (where this is the inverse image in $\tilde{\mathrm{Sp}}_A$ of $G_A \subseteq \mathrm{Sp}_A$) and thereby obtain automorphic forms on $\tilde{G}_A$. If G is a relatively small subgroup, this will produce fairly general automorphic forms on $\tilde{G}_A$. For general G it will be difficult to be very precise about the nature of $\theta(u) \mid \tilde{G}_A$, but if G belongs to a reductive dual pair as defined below then the structure of the $\theta(u) \mid \tilde{G}_A$ should be specifiable in considerable detail. It is in this circumstance that the relevance of θ-series to the theory of automorphic forms lies.

5. Reductive dual pairs. Here F is arbitrary (except still not of characteristic 2).

Let (G, G') be a pair of subgroups of Sp. We say (G, G') form a *reductive dual pair* if

(i) G and G' act absolutely reductively on W; and

(ii) G is the centralizer of G' in Sp and vice versa.

Reductive dual pairs may be classified as follows. If (G, G') is a reductive dual pair in Sp, and if $W = W_1 \oplus W_2$ is an orthogonal direct sum decomposition where W_1 and W_2 are invariant by $G \cdot G'$, then we say (G, G') is *reducible*. The restrictions (G_i, G_i') of (G, G') to the W_i define reductive dual pairs in the $\mathrm{Sp}(W_i)$. We say that

(G, G') is the *direct sum* of the (G_i, G'_i). If (G, G') is not reducible, it is *irreducible.* Any pair is a direct sum in an essentially unique way of irreducible pairs. The irreducible pairs may be described as follows.

Type II. There are maximal isotropic subspaces $X, Y \subseteq W$ with $X \oplus Y = W$, and X and Y invariant by $G \cdot G'$. In this case there exist:

(a) a division algebra D over F (not necessarily central);

(b) a right D-module X_1 and a left D-module X_2; such that

(c) $X \simeq X_1 \otimes_D X_2$ in such fashion that (G, G') are identified to $(\mathrm{GL}_D(X_1), \mathrm{GL}_D(X_2))$.

Type I. The joint action of $G \cdot G'$ is irreducible on V. Then there exist

(a) a division algebra D,

(b) with involution $\natural$, and

(c) D-modules V_1 and V_2

(d) with forms $(\ ,\ )_1$ and $(\ ,\ )_2$, one $\natural$-Hermitian and the other $\natural$-skew-Hermitian; such that

(e) $W \simeq V_1 \otimes V_2$ in such a way that (G, G') are identified to the isometry groups of $(\ ,\ )_1$ and $(\ ,\ )_2$, and

(f) $\langle\ ,\ \rangle = \mathrm{tr}_{D/F}((\ ,\ )_1 \otimes (\ ,\ )_2^{\natural})$. Here $\mathrm{tr}_{D/F}$ is reduced trace. The algebra D is not necessarily central over F. Although the tensor product of forms over D does not make sense, when you take traces you get a good F-bilinear form.

REMARKS. (a) Essentially the same classification can be found in Weil **[Wi 2]** and elsewhere. It essentially goes back to Albert. It may also be deduced from the results in **[Sa]**.

(b) Over a local field F a division algebra with involution is either a quadratic extension or a quaternion algebra over some extension field of F.

(c) Consider a pair (G, G') over a number field F. Suppose (G, G') is irreducible of type I, that $\natural$ is a positive involution on D, that $V_2 \simeq D$ with $\natural$-Hermitian form $(x, y)_2 = xy^{\natural}$. Then $V_1 \simeq W$ has a $\natural$-skew-Hermitian form $(\ ,\ )_1$. The data D, $\natural$, V_1 and $(\ ,\ )_2$ are 4 of the 7 things Shimura **[Sh]** uses to specify his PEL-types. The rest of Shimura's data arises further on in the development of the theory of reductive dual pairs. Thus Shimura's PEL-types arise naturally in the context of reductive dual pairs. It would be an interesting study to see how much of the work of Shimura and others on these objects can be understood in terms of the oscillator representation.

6. Local duality. F is now again a local field.

Let $H \subseteq \widetilde{\mathrm{Sp}}$ be a closed subgroup. Define

$$\mathscr{R}(H) = \{\sigma : \sigma \text{ is an irreducible smooth representation of } H, \text{ and there exists a nontrivial } H\text{-intertwining map } \alpha : \omega_\chi^\infty \to \sigma\}.$$

As before, if $G \subseteq \mathrm{Sp}$, then $\tilde{G}$ is the inverse image of G in $\widetilde{\mathrm{Sp}}$. If $(G, G') \subseteq \mathrm{Sp}$ is a reductive dual pair, then $\tilde{G}$ and $\tilde{G}'$ commute in $\widetilde{\mathrm{Sp}}$, and so an irreducible admissible representation of $\tilde{G} \cdot \tilde{G}'$ will pull back to an irreducible admissible representation of $\tilde{G} \times \tilde{G}'$. Such a representation has the form $\sigma \otimes \sigma'$ where σ and σ' are irreducible admissible representations of $\tilde{G}$ and $\tilde{G}'$ respectively. Thus $\mathscr{R}(\tilde{G} \cdot \tilde{G}')$ defines (is the graph of) a correspondence between representations of $\tilde{G}$ and representations of $\tilde{G}'$.

Conjecture (*local duality*). If (G, G') is a reductive dual pair in Sp, then $\mathscr{R}(\tilde{G} \cdot \tilde{G}')$ is the graph of a bijection between $\mathscr{R}(\tilde{G})$ and $\mathscr{R}(\tilde{G}')$.

REMARKS. (a) If duality holds for irreducible pairs, it holds for all pairs.

(b) By invariant theoretic methods, this conjecture can be shown to "almost" be true. See §11.

(c) For various pairs (G, G') such that G or G' is anisotropic, the duality conjecture has been considered by many authors (see **[G]** for some examples).

It can be established for such pairs by the method of (b). For $F = \boldsymbol{R}$, the infinitesimal methods of Classical Invariant Theory apply directly to prove duality when G or G' is anisotropic. See **[H 2]**.

(d) Duality holds also if G or G' is $\mathrm{GL}_1(D)$ or O_2, or if $G = G' = T$ is a torus. The theory in these cases amounts more or less to the local theory for L-functions attached to grössencharaktere.

(e) For the pair $(\mathrm{GL}_n(F), \mathrm{GL}_n(F))$ the duality conjecture amounts essentially to the conjecture made by Weil **[Wi 3]** about the left and right actions of GL_n on M_n. Weil formulated his conjecture in order to provide a distribution-theoretic rationale for the theory of zeta-functions of local algebras.

7. The spherical case; correspondences between Hecke algebras. F will be a non-Archimedean local field of odd residual characteristic.

In general the local duality conjecture may be very difficult; however for applications to the theory of automorphic forms weaker results may suffice. We will discuss one such partial result that is particularly relevant for the global theory.

Let $\mathcal{O}$ be the integers of F. Let U be a vector space over F, with a form $(\ ,\ )$, either skew, or symmetric, or Hermitian (with respect to some automorphism of F of order 2). If $S \subseteq U$ is a set, then we put

$$S^* = \{u \in U \colon (u, s) \in \mathcal{O} \text{ for all } s \text{ in } S\}.$$

A *lattice* in U is a compact open $\mathcal{O}$-module. If L is a lattice, so is L^*. We say L is self-dual if $L = L^*$. We say $(\ ,\ )$ is *unramified* if self-dual lattices for $(\ ,\ )$ exist. If G is the isometry group of an unramified form, then a *standard maximal compact subgroup* is the isotropy group of a self-dual lattice for $(\ ,\ )$. In $\mathrm{GL}_n(D)$, with D a division algebra not necessarily central over F, a standard maximal compact is the isotropy group of any lattice which is invariant under the integers of D.

By an *unramified* classical group over F we mean

(i) $\mathrm{GL}_n(D)$, with D unramified over F, or

(ii) the isometry group of an unramified form over F' where F' is an unramified extension of F.

Returning to our symplectic vector space W, we say an irreducible pair $(G, G') \subseteq$ Sp is unramified if both G and G' are unramified. A general pair is unramified if each of its irreducible summands is.

The group Sp is always unramified, and the group J defined in §3 (there subscripted J_v) is a standard maximal compact subgroup of Sp. As in §3, we can write $\tilde{J} = J \times \{1, -1\}$ unambiguously. If (G, G') is an unramified reductive dual pair in Sp, and (K, K') are standard maximal compacts in (G, G'), then up to conjugacy we can assume K and K' are contained in J. Thus we may write $\tilde{K} =$

$K \times \{1, -1\}$ unambiguously, and likewise for K'. So we may consider K and K' to be well-defined subgroups of $\tilde{\mathrm{S}}\mathrm{p}$.

For an unramified pair (G, G') with standard maximal compacts K, K', put

$$\mathscr{R}(G, K) = \{\sigma \in \mathscr{R}(\tilde{G}) : \sigma \text{ admits a } K\text{-fixed vector}\}$$

and similarly for G', and for $G \cdot G'$.

We assume the character χ defining ω_χ is unramified, in the sense that $O \subseteq \ker \chi \not\supseteq \pi^{-1}O$, where π is a prime of F.

Let $C_c^\infty(\tilde{G}//K)$ be the (Hecke) algebra of K-bi-invariant functions on $\tilde{G}$. Define $C_c^\infty(\tilde{G}'//K')$ similarly. Let $I(K, K')$ be the vectors fixed by $\omega_\chi^\infty(K)$ and by $\omega_\chi^\infty(K')$. Then obviously $\omega_\chi^\infty(C_c^\infty(G//K))$ leaves $I(K, K')$ invariant so that we may consider the restrictions $\omega_\chi^\infty(C_c^\infty(G//K)) \mid I(K, K')$.

THEOREM 7.1. (a) *$\mathscr{R}(\tilde{G} \cdot \tilde{G}', K \cdot \tilde{K}')$ is the graph of a bijection between $\mathscr{R}(\tilde{G}, K)$ and $\mathscr{R}(\tilde{G}', K')$.*

(b) *If $\sigma \otimes \sigma' \in \mathscr{R}(\tilde{G} \cdot \tilde{G}')$ and σ is in $\mathscr{R}(\tilde{G}, K)$, then σ' is in $\mathscr{R}(\tilde{G}', K')$.*

(c) *The restrictions $\omega_\chi^\infty(C_c^\infty(\tilde{G}//K)) \mid I(K, K')$ and $\omega_\chi^\infty(C_c^\infty(\tilde{G}'//K')) \mid I(K, K')$ are the same algebra of operators.*

8. Global duality. F is now again a global field. Associated paraphernalia are as defined in §3.

If $G \subseteq \mathrm{Sp}$ is an algebraic group over F then we can consider $\tilde{G}_A \subseteq \tilde{\mathrm{S}}\mathrm{p}_A$. In analogy with the local case, we can define

$$\mathscr{R}(\tilde{G}_A) = \{\sigma : \sigma \text{ is an irreducible admissible representation of } \tilde{G}_A \text{ and there exists a nontrivial } \tilde{G}_A \text{ intertwining operator } \alpha : \omega_\chi^\alpha \to \sigma\}.$$

Each σ in $\mathscr{R}(\tilde{G}_A)$ will be a tensor product of local representations in the same sense that ω_χ^∞ is, and it is easy to see that $\mathscr{R}(\tilde{G}_A)$ itself will be the restricted product of the $\mathscr{R}(\tilde{G}_v)$ with respect to the compact subsets $\mathscr{R}(\tilde{G}_v, K_v)$ where $K_v = G \cap J_v$.

Conjecture (*global duality, part* I). If $(G, G') \subseteq \mathrm{Sp}$ is a reductive dual pair, then $\mathscr{R}(\tilde{G}_A \cdot \tilde{G}'_A)$ is the graph of a bijection between $\mathscr{R}(\tilde{G}_A)$ and $\mathscr{R}(\tilde{G}'_A)$.

We note that this conjecture would follow directly from the local duality conjecture together with the theorem of §7.

Let $G \subseteq \mathrm{Sp}$ as above. By the theorem of §3, we can consider G to be a subgroup of $\tilde{G}_A$. Let $\mathscr{A}(\tilde{G}_A)$ denote the slowly growing smooth functions on $G\backslash\tilde{G}_A$. Let σ be an irreducible admissible representation of $\tilde{G}_A$. We will call σ *automorphic* if there exists a $\tilde{G}_A$ embedding $\beta : \sigma \to \mathscr{A}(\tilde{G}_A)$. (This notion of automorphic representation is slightly more restrictive than the usual one **[P]**.)

A natural complement to the conjecture just above is

Conjecture (*global duality, part* II). In the above bijection between $\mathscr{R}(\tilde{G}_A)$ and $\mathscr{R}(\tilde{G}'_A)$, an automorphic representation of $\tilde{G}_A$ is matched with an automorphic representation of $\tilde{G}'_A$ and vice versa.

This conjecture is in fact more than just a convenient addition to the first conjecture. Like the local duality conjecture, the theorem of §7, and indeed, the whole of the theory, it has a basis in invariant theory.

9. Transcendental classical invariant theory (non-Archimedean case). Here F will be a non-Archimedean local field of odd residual characteristic and integers O.

We will sketch here how the conjectures and results of §§6, 7, 8 are related to invariant-theoretic considerations. For economy of space, we will only consider a non-Archimedean field. Unfortunately, this case is more than one short step removed from the algebraic classical invariant theory of **[Wy]**, and the strong relations may not be immediately clear. For a recasting of classical invariant theory in terms of reductive dual pairs, including a proof of the local duality for pairs (G, G') over $\boldsymbol{R}$ where G or G' is compact, see **[H 2]**.

The first necessity for invariant theory is a supply of "natural" invariants. To provide these we return to the symplectic vector space W and the Heisenberg group $H = H(W)$. Let χ be a nontrivial character of the center of H. For any subgroup $A \subseteq H$, we define A^* to be the centralizer of A in H modulo the kernel of χ. If $A = A^*$ we say A *polarizes* χ. If A polarizes χ, then F (here considered as the center of H) is contained in A, and $A = (A \cap W) \oplus F$. The set $A \cap W$ will be a subgroup of W, hence a $\boldsymbol{Z}_p$-module. If $A \cap W$ is an $\mathcal{O}$-module, $\mathcal{O}$ being the integers in F, we will say A is an *F-polarization* or a polarization over F. Let Ω denote the set of all polarizations and Ω_F the subset of F-polarizations. The action of Sp on H permutes the polarizations. If dim $W = n$, then Ω_F consists of $n + 1$ orbits for Sp.

If $A \subseteq H$ polarizes χ, then there is a unique extension χ_A of χ to A such that $A \cap W \subseteq \ker \chi_A$.

THEOREM 9.1. *Let the representation ρ_χ^∞ be realized on a vector space Y. For each polarization A of χ there is a linear form λ_A on Y such that*

$$\lambda_A(\rho_\chi^\infty(a)y) = \chi_A(a)\lambda_A(y) \quad \textit{for } a \in A, y \in Y. \tag{9.1}$$

The functional λ_A is specified up to multiples by equation (9.1). The resulting map $\mathcal{Q}\colon \Omega \to \boldsymbol{P}(Y^*)$ embeds Ω as a compact subset in $\boldsymbol{P}(Y^*)$. The map $\mathcal{Q}$ is equivariant for the natural actions of Sp on Ω and on $\boldsymbol{P}(Y^*)$.

Since $\boldsymbol{P}(Y^*)$ has a natural topology, the map $\mathcal{Q}$ implicitly topologizes Ω. In this topology Ω_F is a closed subset of Ω and each Sp orbit in Ω_F has its natural topology. We can label the Sp-orbits in Ω_F by $\{\Omega_F^i\}_{i=0}^n$ in such a way that Ω_F^i is open and dense in $\bigcup_{j=i}^n \Omega_F^j$. Thus Ω_F^n is the unique closed orbit. It consists of polarizing A such that $A \cap W$ is a maximal isotropic subspace of W. Also Ω_F^0 is the unique open orbit. It consists of A such that $A \cap W$ is a lattice (an open compact O-module) in W.

The way Ω_F is used to produce invariants is as follows. If $G \subseteq$ Sp, and $A \in \Omega_F$, and A is invariant by G, then λ_A must be invariant up to multiples by $\omega_\chi^{\infty *}(\tilde{G})$. Thus there is a quasi-character ψ_A on $\tilde{G}$ such that

$$\omega_\chi^{\infty *}(\tilde{g})\, \lambda_A = \psi_A(\tilde{g})\, \lambda_A \quad \text{for } \tilde{g} \in \tilde{G}. \tag{9.2}$$

Thus the existence of G-fixed points in Ω_F leads to the existence of at least quasi-invariant linear forms on Y, and these forms will often be invariant. The analogy with construction of automorphic forms (θ-series) by means of the Θ-distribution as formulated in §4 is clear. This analogy will be developed further in §12. This is also essentially the basis of the First Fundamental Theorems of classical invariant theory, though the parallel is perhaps not so immediately seen.

Let $(G, G') \subseteq$ Sp be a reductive dual pair. If $A \in \Omega_F^n$ is G-invariant, let ψ_A be the quasi-character of $\tilde{G}$ attached to A as in (9.2). For any g' in G', we see that $g'(A)$ will again be G-invariant, and that ψ_A and $\psi_{g'(A)}$ will be equal. Thus if we let $\Omega_F^n(G, \psi)$ be the set of points A in Ω_F^n which are G-invariant and such that $\psi = \psi_A$, then $\Omega_F^n(G, \psi)$ is a union of G' orbits.

Since $\mathcal{Q}(\Omega_F^n(G, \psi))$ is a compact "subvariety" of $\boldsymbol{P}(Y^*)$, the restriction of the hyperplane section bundle defines a line bundle

$$\begin{array}{c} L(G, \psi) = L \\ \downarrow \\ \mathcal{Q}(\Omega_F^n(G, \psi)) \end{array}$$

Let $\mathrm{Sec}^\infty(L)$ denote the space of smooth sections of L. The evaluation of points of the inverse image in Y^* of $\mathcal{Q}(\Omega_F^n(G, \psi))$ defines a linear map $e\colon Y \to \mathrm{Sec}^\infty(L(G, \psi))$. The evaluation map e is clearly $\tilde{G}'$ equivariant. We have the dual to e, e^*: $\mathrm{Sec}^{\infty *}(L) \to Y^*$. Clearly the image of e^* will consist of eigenfunctionals for $\omega_\chi^{\infty *}(\tilde{G})$, with eigencharacter ψ.

THEOREM 9.2. *If $\Omega_F^n(G, \psi)$ is nonempty, then it consists of a single G' orbit. Further $L(G, \psi)$ is then a $\tilde{G}'$ homogeneous line bundle over $\Omega^0(G, \psi)$. Finally, if μ is any linear functional on Y such that $\omega_\chi^{\infty *}(\tilde{g})\mu = \psi(\tilde{g})\mu$ for $\tilde{g} \in \tilde{G}$, then $\mu = e^*(\nu)$ for some distribution ν in* $\mathrm{Sec}^{\infty *}(L)$.

10. The spherical case. F and other objects are as in §9.

The subgroups of H comprising Ω_F^0 are compact modulo the center; because of this, Theorems 9.1 and 9.2 have analogues for Ω_F^0 which are somewhat more direct. We formulate them.

THEOREM 10.1. *For every $A \in \Omega_F^0$, there is a vector y_A in Y, unique up to multiples, such that $\rho_\chi^\infty(a)y_A = \chi_A(a)y_A$ for $a \in A$. The map $A \to y_A$ is equivariant for the relevant actions of* Sp *on Ω_F^0 and $\boldsymbol{P}(Y)$.*

Let (G, G') be an unramified reductive dual pair in Sp and let (K, K') be a pair of standard maximal compacts of K and K', both contained in the standard maximal compact J of Sp. As explained in §7, we can consider J, hence K and K', to be embedded in $\tilde{\mathrm{S}}\mathrm{p}$. Let $L \subseteq W$ be the self-dual lattice fixed by J. Assume that χ is unramified in the sense of §7. Then $A = L \oplus F \subseteq H$ polarizes χ. Since J normalizes A, the vector y_A will be $\omega_\chi^\infty(J)$-invariant up to multiples. Since (except when the residue class field of F is 3 and dim $W = 2$, a case which can be dealt with separately, but which we will exclude) J is perfect, we see y_A will actually be invariant under ω_χ^∞ by J, and a fortiori by K and K'.

THEOREM 10.2. *For $f \in C_c^\infty(\tilde{G}'/K')$, define $\gamma(f) = \omega_\chi^\infty(f)(y_A)$. Then γ is a surjection from $C_c^\infty(G'/K')$ to $I(K)$, the space of K-fixed vectors in Y.*

Since every irreducible admissible representation of $\tilde{G}'$ with a K' fixed vector is a quotient of $C_c^\infty(\tilde{G}'/K')$ in a unique way, and since $C_c^\infty(\tilde{G}'/K')^{K'} = C_c^\infty(\tilde{G}'//K')$ is a commutative algebra with 1 (hence a cyclic module for itself) Theorem 7.1 follows from Theorem 9.4.

11. Doubling variables and local duality. F and other objects continue as in §§9 and 10.

We will now describe how Theorem 9.2 can be used to attack the local duality conjecture. For an irreducible admissible representation σ of $\tilde{G}$, let Y_σ be the quotient of Y by the kernel of all $\tilde{G}$-intertwining maps from Y to (the space of) σ. Then $\omega_\chi^\infty(\tilde{G})$ evidently factors to a multiple of σ on Y_σ, and it is not hard to see that the kernel of the quotient map from Y to Y_σ is $\tilde{G}'$-invariant, so Y_σ is actually a $\tilde{G} \times \tilde{G}'$-module. As such, it must have the form $\sigma \otimes \tau$ where τ is some smooth module for $\tilde{G}'$. The condition that $\mathscr{R}(\tilde{G} \cdot \tilde{G}')$ is an injection from $\mathscr{R}(\tilde{G})$ to $\mathscr{R}(\tilde{G}')$ (terminology as in §6) amounts to saying that whenever $Y_\sigma \neq \{0\}$, the representation τ should have a unique nontrivial irreducible admissible $\tilde{G}'$-module σ' as quotient.

We will consider the question of uniqueness of σ, which is more crucial than existence. We will suppose that σ^*, the (smooth) contragredient of σ is a quotient of $\omega_{\bar{\chi}}^\infty$, where $\bar{\chi} = \chi^{-1}$ is the character of F inverse to χ. This will be automatically so if σ is unitary or even quasi-unitary. Let $Y^\vee$ be the space of $\omega_{\bar{\chi}}^\infty$ and let $Y_{\sigma^*}^\vee$ be the maximal quotient of $Y^\vee$ on which $\tilde{G}$ acts by a multiple of σ^*. Then $Y_{\sigma^*}^\vee$ is a $\tilde{G} \times \tilde{G}'$-module of the form $\sigma^* \otimes \tau^\vee$ where $\tau^\vee$ is another $\tilde{G}'$-module. Therefore $Y_\sigma \otimes Y_{\sigma^*}^\vee$ is a $(\tilde{G} \otimes \tilde{G}) \times (\tilde{G}' \otimes \tilde{G}')$-module of the form $(\sigma \otimes \sigma^*) \otimes (\tau \otimes \tau^\vee)$. Let $\tilde{G}_\Delta$ be the diagonal subgroup in $\tilde{G} \times \tilde{G}$. Then the restriction of $\sigma \otimes \sigma^*$ to $\tilde{G}_\Delta$ contains the trivial representation uniquely as a quotient. Therefore if $(Y \otimes Y^\vee)_1$ is the maximal quotient of $Y \otimes Y^\vee$ on which $\tilde{G}_\Delta$ acts trivially, we see that $\tau \otimes \tau^\vee$ will be a $\tilde{G}' \times \tilde{G}'$ quotient of $(Y \otimes Y^\vee)_1$. Therefore if we can analyze $(Y \otimes Y^\vee)_1$ as $\tilde{G}' \times \tilde{G}'$-module, and in particular if we can show that the only irreducible quotients it admits are of the form $\sigma' \otimes \sigma'^*$, where σ' is an irreducible admissible representation of $\tilde{G}'$, then we shall have established a reasonable approximation to the local duality conjecture. Indeed, passing back through the above analysis, one can easily extract 3 or so technical conjectures which together would be equivalent to the local duality conjecture.

The point of the above reduction is that $(Y \otimes Y^\vee)_1$ can be described by means of Theorem 9.2. Specifically, let W^- be the symplectic vector space whose underlying space is W, but whose symplectic form is $-\langle\ ,\ \rangle$, the negative of the form on W. By way of contrast, we will then also write W^+ for W. Put $2W = W^+ \oplus W^-$ as symplectic vector space. We call $2W$ the *double* of W. We can form $H(2W)$ according to the recipe of §1. We may define injections i_1 and i_2 of $H(W)$ into $H(2W)$ by the formulas

$$\begin{aligned} i_1(w, t) &= ((w, 0)t), \\ i_2(w, t) &= ((0, -w), -t) \end{aligned} \qquad \text{for } w \in W,\ t \in F. \tag{11.1}$$

Thus $i_1 \times i_2 \colon H(W) \times H(W) \to H(2W)$ is a surjective homomorphism with the diagonal of the centers as kernel. Let χ be a character of F and let $2\rho_\chi$ be the corresponding representation of $H(2W)$. Then it is easy to see that

$$2\rho_\chi \circ (i_1 \times i_2) \simeq \rho_\chi \otimes \rho_{\bar{\chi}} \tag{11.2}$$

(outer tensor product).

Let $\mathrm{Sp}(2W)$ be the symplectic group of $2W$, and $\mathrm{Sp}(W)$ the group of W. We have two embeddings of $\mathrm{Sp}(W)$ into $\mathrm{Sp}(2W)$, compatible with the maps of (11.1):

$$(11.3)\qquad \begin{aligned} i_1(g)(w_1, w_2) &= (g(w_1), w_2) \\ i_2(g)(w_1, w_2) &= (w_1, g(w_2)) \end{aligned} \quad \text{for } g \in \mathrm{Sp}(W), \text{ and } (w_1, w_2) \in 2W.$$

These maps lift uniquely to maps between $\tilde{\mathrm{S}}$p's. Let $2\,\omega_\chi$ be the representation of $\tilde{\mathrm{S}}\mathrm{p}(2W)$ defined by (2.1). Just as for (11.2), it is easy to see that

$$(11.4)\qquad 2\omega_\chi \circ (i_1 \times i_2) = \omega_\chi \otimes \omega_{\bar{\chi}}$$

(outer tensor product).

Let (G, G') be a reductive dual pair in $\mathrm{Sp}(W)$. Then it can be shown there is a group $2G'$ such that $(i_1 \times i_2(G_\Delta), 2G')$ form a dual reductive pair in $\mathrm{Sp}(2W)$. The group $2G'$ is to G' as $\mathrm{Sp}(2W)$ is to $\mathrm{Sp}(W)$. In particular $i_1(G') \times i_2(G') \subseteq 2G'$.

From this doubling construction, we see that the space $(Y \otimes Y^\vee)_1$ defined above is described as a $2\tilde{G}'$-module by Theorem 9.2. By restriction we can investigate its structure as a $\tilde{G}' \times \tilde{G}'$-module. Doing so we find that the duality conjecture is certainly true if G or G' is compact, and in general is "almost" true. It remains in the general case to remove the "almost". For more details in the case of finite fields, see [**H 1**].

12. Global invariants. In this section, F is a global field, and has associated structures as in §3.

Given an F-module $M \subseteq W_A$, put

$$M^* = \{x \in W_A : \langle x, m\rangle \in F, \text{ all } m \in M\}.$$

If $M = M^*$, we will call M a *global polarization* of W. To each global polarization M we can associate the closed subgroup $M \oplus F \subseteq H_A(W)$. The following analogue of Theorems 9.1 and 10.1 extends Theorem 4.1. Let the space of global polarizations be denoted Ω_A.

THEOREM 12.1. *Let the representation ρ_χ of H_A be realized on a space Y. For every $M \in \Omega_A$, there is a linear functional θ_M on Y, unique up to multiples, such that*

$$\rho_\chi^{\infty *}(x)\,\theta_M = \theta_M \quad \textit{for } x \in M \oplus F \subseteq H_A.$$

The resulting mapping $\mathcal{Q}\colon \Omega_A \to \boldsymbol{P}Y^*$ is an Sp_A *equivariant embedding of Ω_A with compact image.*

Evidently $W_F \subseteq W_A$ is a global polarization and $\mathcal{Q}(W_F) = \Theta$ is the θ-distribution of §4. The isotropy group of W_F in Sp_A is Sp_F, so the Sp_A orbit of Θ is $\sim \mathrm{Sp}_A/\mathrm{Sp}_F$, a global version of the Siegel upper half-plane. Also the Sp_A orbit of Θ is the unique open orbit in $\mathcal{Q}(\Omega_A)$ and is dense, so that Ω_A is a compactification of $\mathrm{Sp}_A/\mathrm{Sp}_F$. The embedding $\mathcal{Q}$ is a global analogue of the projective embeddings of the Siegel modular varieties studied by Igusa [**I**], and they can be derived from it. F. Herman is studying this problem. It seems it may also be fruitful to study the canonical models of Shimura [**Sh**] from this viewpoint.

As in §9, the hyperplane section bundle over $\boldsymbol{P}Y^*$ induces a line bundle L over $\mathcal{Q}(\Omega_A)$, and there is an evaluation map from Y to sections of L. These sections are just the Θ-series of §4 and antiquity. (Since θ is actually invariant by Sp_F, the bundle L is $\tilde{\mathrm{S}}\mathrm{p}_A$-equivariantly trivial over the orbit of Θ, and an $\tilde{\mathrm{S}}\mathrm{p}_A$-invariant cross-section of L results by taking the $\tilde{\mathrm{S}}\mathrm{p}_A$ orbit of Θ. The restriction of sections of L to this cross-section is the θ-series.) The fact that the θ-series extend to sections over all of $\mathcal{Q}(\Omega_A)$ guarantees their good behavior "at ∞" on $\mathrm{Sp}_A/\mathrm{Sp}_F$.

Let (G, G') be a reductive dual pair in $\widetilde{\mathrm{Sp}}_F$. The θ-distribution is Sp_F-invariant, hence G-invariant. Thus the whole $\tilde{G}'_A$ orbit of Θ in $\mathscr{Q}(\Omega_A)$ consists of G-invariant functionals. In analogy with Theorem 9.1, which itself is in analogy with classical invariant theory, we might expect that we would get all G-invariant functionals this way. We formulate a weak version of this precisely.

Conjecture (*weak global invariants*). Let $Y_1 \subset Y$ be the kernel of all G_F-invariant functionals on Y_1 and let Y_2 be the intersection of the kernels of $\omega_\chi^{\infty *}(g')\Theta$ for all $\tilde{g}'$ in $\tilde{G}'_A$. Then $Y_2 = Y_1$.

If $G'_F \backslash \tilde{G}'_A$ is compact, then this conjecture implies that every automorphic representation in $\mathscr{R}(\tilde{G}_A)$ is obtainable by taking θ-series on $\tilde{G}_A \cdot \tilde{G}'_A$ and integrating against some function on $G'_F \backslash \tilde{G}'_A$. In particular, the weak global invariants conjecture plus the first global duality conjecture would imply that if $\sigma \in \mathscr{R}(\tilde{G}_A)$ is automorphic then so is $\sigma' \in \mathscr{R}(\tilde{G}'_A)$, proving one direction of the second global duality conjecture. Still when G'_F is anisotropic, another point of view towards the global invariants conjecture is that it provides a local-global principle: if σ is an automorphic representation whose local components are everywhere quotients of the local oscillator representation, then σ is obtained by θ-series. In conclusion, it seems fair to say that the major facts one would like to know about correspondences between θ-series are intimately related to invariant-theoretic considerations.

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YALE UNIVERSITY

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EXAMPLES OF DUAL REDUCTIVE PAIRS

STEPHEN GELBART*

This note is meant to complement Howe's paper **[Ho 1]** by giving concrete down-to-earth examples of dual reductive pairs and the automorphic forms and group representations which arise from them; I am grateful to Howe for explaining his theory of the oscillator representation to me.

CONTENTS

1. Historical remarks. In 1964 Weil singled out a natural representation of the symplectic group (the Segal-Shale-Weil-metaplectic oscillator representation) in order to give a group theoretical foundation for the classical theta function

$$\theta(z) = \sum_{n=-\infty}^{\infty} \exp(2\pi i n^2 z).$$

For $\mathrm{SL}(2, F)$ this representation acts in $L^2(F)$ according to the formulas

$$T_\chi\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \Phi(x) = \chi(\tfrac{1}{2} bx^2)\Phi(x)$$

and

$$T_\chi\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\Phi(x) = \gamma\hat{\Phi}(x).$$

Here χ is a fixed character of F and γ is an eighth root of unity which depends on χ. For $\mathrm{Sp}(2m, \boldsymbol{R})$ the oscillator representation acts in $L^2(\boldsymbol{R}^m)$ through similar operators.

In 1966 Shalika and Tanaka tensored T_χ with $T_{\chi'}$ to produce a representation of $\mathrm{SL}(2, F)$ in $L^2(F^2)$ which commutes with the natural action of $O(2)$ and decomposes

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*Alfred P. Sloan Fellow.

according to the latter group's representation theory. In this way Shalika and Tanaka constructed discrete series representations of SL(2, F) and reformulated earlier results of Hecke and Maass on the construction of theta series with grossencharacter. For Sp($2m$, $\boldsymbol{R}$) an analogous $2m$-fold tensor product of oscillator representations acts in $L^2(\boldsymbol{R}^{2m\times m})$ through the operators

$$\omega_\chi\begin{pmatrix} I_m & N \\ 0 & I_m \end{pmatrix}\Phi(X) = \exp(\pi i \operatorname{tr} XNX^t)\Phi(X), \tag{1.1}$$
$$\omega_\chi\begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix}\Phi(X) = |\det A|^m\,\Phi(XA).$$

and

$$\omega_\chi\begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}\Phi(X) = i^{-m^2}\hat{\Phi}(X). \tag{1.2}$$

This representation decomposes into discrete series representations of Sp($2m$, $\boldsymbol{R}$) indexed by certain irreducible representations of $O(2m, \boldsymbol{R})$ (namely those which occur in the natural representation of $O(2m, \boldsymbol{R})$ in $L^2(\boldsymbol{R}^{2m\times m})$ given by left matrix multiplication); cf. **[Ge 1]** and **[Sa]**.

Howe's recently developed theory of the oscillator representation reformulates and refines Weil's theory so as to "explain" the above results and point the way towards new ones. A key notion of the theory—the concept of a dual reductive pair—formalizes that natural duality (already experimentally observed) between the symplectic and orthogonal groups. To describe this phenomenon more concretely the notion of a Schrödigner model is crucial.

2. Schrödinger models. We adopt the notation and definitions of **[Ho 1]**. Thus F is a local field not of characteristic 2, V is a symplectic space defined over F, $H(V)$ is the Heisenberg group attached to V, χ is a fixed nontrivial character of F, ρ_χ is the irreducible unitary representation of H attached to χ, and Sp(V) is the isometry group of the symplectic form $\langle\ ,\ \rangle$ on F. Recall that $H(V)$ is a central extension of V by F.

Let $\bar{\mathrm{S}}\mathrm{p}$ denote the nontrivial 2-fold cover of Sp(V) (whose existence and uniqueness is assured by **[Mo]**). The *oscillator representation* ω_χ is the unitary representation of $\bar{\mathrm{S}}\mathrm{p}$ in the space of ρ_χ such that

$$\omega_\chi(g)\rho_\chi(h)\omega_\chi(g^{-1}) = \rho_\chi(g(h)) \tag{2.1}$$

for all $g \in \bar{\mathrm{S}}\mathrm{p}$ and $h \in H$. This representation is unique up to unitary equivalence, and it is "genuine" in the sense that it does not factor through Sp(V). The corresponding multiplier representation of Sp(V) is again denoted by ω_χ.

To describe a *Schrödinger model* for ω_χ (and ρ_χ) we need the notion of a *complete polarization* of V. This is a pair of subspaces (X, Y) of V satisfying the following properties:

(i) X and Y are isotropic for $\langle\ ,\ \rangle$, i.e., $\langle\ ,\ \rangle$ is trivial on X and Y; and

(ii) $X \oplus Y = V$.

With respect to this polarization, the space of ρ_χ is $L^2(X)$ and the action of $H(V)$ is given by

$$\begin{aligned}(2.2)\qquad &\rho_\chi((x,0))f(x') = f(x'-x), && x, x' \in X,\\ &\rho_\chi((y,0))f(x') = \chi(\langle y, x'\rangle)f(x'), && y \in V, x' \in X,\\ &\rho_\chi((0,t))f(x') = \chi(t)f(x'), && t \in F, x' \in X.\end{aligned}$$

The corresponding Schrödinger model for ω_χ is more difficult to describe. Thus we first restrict our attention to some distinguished subgroups of $\mathrm{Sp}(V)$.

Let $P(Y)$ (resp. $P(X)$) denote the subgroup of $\mathrm{Sp}(V)$ which preserves Y (resp. X), and let $N(Y)$ denote the subgroup which acts trivially on Y. If $M = M(X, Y) = P(X) \cap P(Y)$, then $P(Y)$ is a semidirect product of M and $N(Y)$. Using (2.1) and (2.2) we find that

$$(2.3)\qquad \omega_\chi(n)f(x) = \chi(-\tfrac{1}{2}\langle x, nx\rangle)f(x), \qquad x \in X, n \in N(Y),$$

and

$$(2.4)\qquad \omega_\chi(m)f(x) = \mu(m)f(m^{-1}x,) \qquad x \in X, m \in M.$$

In the Schrödinger model for ω_χ then, N acts by multiplication by a quadratic character, and M simply acts linearly (the factor $\mu(m)$ being roughly the square root of the modulus of m acting on X; cf. (1.1)).

Rather than attempt a general description of ω_χ *outside* $P(Y)$, we refer the reader to the many examples of this paper; recall that matrices of the form $\left[\begin{smallmatrix}1 & b\\ 0 & 1\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right]$ generate $\mathrm{SL}(2, F)$.

3. The pair $(\mathrm{Sp}(V_0), O(U_0))$. The general notion of a dual reductive pair is described in **[Ho 1]**. The basic example we treat in this paper simply formalizes the special features of the $\mathrm{Sp}(2m, \boldsymbol{R})$ example just discussed.

Let $(V_0, \langle\ ,\ \rangle_0)$ denote any symplectic vector space, and $(U_0, (\ ,\ ))$ any orthogonal one. Put $V = V_0 \otimes U_0$, and define a form $\langle\ ,\ \rangle$ on V by

$$\langle v \otimes u, v' \otimes u'\rangle = \langle v, v'\rangle_0 \langle u, u'\rangle_0$$

for $v, v' \in V_0$, and $u, u' \in U_0$. Then $(V, \langle\ ,\ \rangle)$ is a symplectic space, and $(\mathrm{Sp}(V_0). O(U_0))$ is a natural dual reductive pair in $\mathrm{Sp}(V)$. In particular, the groups $\mathrm{Sp}(V_0)$ and $O(U_0)$ are each other's centralizers in $\mathrm{Sp}(V)$. ($(\mathrm{Sp}(2m, \boldsymbol{R}), O(2m, \boldsymbol{R}))$ is such a pair in $\mathrm{Sp}(V)$, with $V = \boldsymbol{R}^{2m} \otimes \boldsymbol{R}^{2m} = V_0 \otimes U_0$.

If (X_0, Y_0) is a complete polarization of V_0, let $X = X_0 \otimes U_0$ and $Y = Y_0 \otimes U_0$. Then (X, Y) is a complete polarization of V. Moreover—according to §2—the Schrödinger model for ω_χ in $L^2(X)$ is such that $O(U_0)$ acts linearly. On the other hand, if $(X_0. Y_0)$ is a complete polarization of $O(U_0)$—which is possible only when U_0 is split—then the spaces $X = V_0 \otimes X_0$ and $Y = V_0 \otimes Y_0$ provide a polarization of V *with respect to which* $\omega_\chi(V_0)$ (*not* $O(U_0)$) *acts linearly*. In either case, the problem is to describe the decomposition of ω_χ restricted to $\mathrm{Sp}(V_0) \cdot O(U_0)$.

Over $\boldsymbol{R}$ one has the following generalization of the results alluded to in §1; cf. **[Ho 2]**. Let $\bar{\mathrm{S}}\mathrm{p}_0$ and $\bar{O}_0$ denote the inverse images of $\mathrm{Sp}(V_0)$ and $O(U_0)$ in $\bar{\mathrm{S}}\mathrm{p}(V)$. Then $\omega_\chi(\bar{\mathrm{S}}\mathrm{p}_0)$ and $\omega_\chi(\bar{O}_0)$ generate each other's commutants (in the sense of von Neumann algebras) in the algebra of bounded operators in the space of ω_χ. Furthermore, there is a discrete direct sum decomposition

$$\omega|_{\bar{\mathrm{S}}\mathrm{p}_0\cdot\bar{O}_0} \approx \sum_i \tau_i \otimes \sigma_i$$

where τ_i is an irreducible representation of $\bar{\mathrm{Sp}}_0$ of holomorphic type, and σ_i is an irreducible finite-dimensional representation of $\bar{O}_0$; the representations σ_i and τ_i determine each other, and the resulting bijection

$$\sigma_i \leftrightarrow \tau_i \tag{3.1}$$

may be computed using classical invariant theory.

These results, already valid for the pairs

(i) $(\mathrm{Sp}(2m, \boldsymbol{R}), O(n, \boldsymbol{R}))$ in $\mathrm{Sp}(2nm, \boldsymbol{R})$, with n, m positive, also extend to cover the pairs

(ii) $(U_n, U_{p,q})$ in $\mathrm{Sp}(2n(p+q), \boldsymbol{R})$, and the pairs

(iii) $(\mathrm{Sp}(p, O), O^*(2n))$ in $\mathrm{Sp}(4np, \boldsymbol{R})$.

Thus—in addition to generalizing **[Ge 1]**—these results also contain earlier works of **[GK]** (which treats (i) with $n \geqq 2m$, (ii) with $n \geqq 2p$ and $p = q$, and (iii) with $p \geqq 2n$) and **[KV]** (which treats (i) and (ii) with no restrictions).

We shall call the correspondence which results from the "duality" of the pair (Sp, O) the *duality correspondence.*

Now consider an arbitrary local field F and an arbitrary subgroup H of $\bar{\mathrm{Sp}}$. Let $R(H)$ denote the set of irreducible smooth representations of H for which there is a nontrivial H-intertwining map from the smooth vectors of ω_χ. Then if (G, G') is a dual reductive pair in Sp, $R(G \cdot G)'$ should be the graph of a bijection (or duality correspondence) between $R(G)$ and $R(G')$; cf. (3.1) and **[Ho 1]**. Though unproved in general, this conjecture has been established when G or G' is compact. The examples later on give more motivtiaon for the theory.

4. Adelization. Now let F denote an A-field not of characteristic 2, v an arbitrary place of F, O_v the ring of integers of F_v, A the adele ring of F, and $\chi = \prod \chi_v$ a nontrivial character of $F\backslash A$. If V is a symplectic space defined over F, then for each v the groups and representations $H_v = H(V/F_v)$, $\mathrm{Sp}_v = \mathrm{Sp}(V/F_v)$, $\bar{\mathrm{Sp}}_v$, ρ_{χ_v}, and ω_{χ_v} are defined as in §2.

As explained in **[Ho 1]**, one can make sense out of the restricted direct product $\bigotimes \omega_{\chi_v}$ and use it to define a multiplier representation ω_χ of $\mathrm{Sp}_A = \mathrm{Sp}(V_A)$. If $\bar{\mathrm{Sp}}_A$ denotes the 2-fold cover of Sp_A determined by the product of the cocycles defining $\bar{\mathrm{Sp}}_v$ then ω_χ also defines an ordinary representation of $\bar{\mathrm{Sp}}_A$.

A fundamental property of the global representation ω_χ is that it splits over the rational points $\mathrm{Sp}(F) = \mathrm{Sp}(V/F)$; cf. **[We]**. If (X, Y) is a complete polarization of V and θ is the distribution on the Schwartz-Bruhat space $\mathscr{S}(X(A))$ defined by $\theta(\Phi) = \sum_{\xi \in X(F)} \Phi(\xi)$, then θ is $\mathrm{Sp}(F)$-invariant. In particular, the functions

$$\begin{aligned} \theta_\Phi(g) &= \theta(\omega_\chi(g)\Phi) \\ &= \sum_{\xi \in X(F)} (\omega_\chi(g)\Phi)(\xi) \end{aligned} \tag{4.1}$$

are automorphic functions on $\bar{\mathrm{Sp}}_A$ (slowly increasing continuous functions which are left-$\mathrm{Sp}(F)$-invariant).

The basic goal of the global theory of the oscillator representation is to describe the automorphic forms and representations which arise from the θ-distribution through functions of the form (4.1). As in the local theory, a great deal of structure is introduced by considering dual reductive pairs in Sp_A. If (G, G') is such a pair, it follows from the local theory that there should be a *duality correspondence* between representations of G and G' which intertwine with ω_χ. Moreover, this bijection should pair *automorphic* representations of G with *automorphic* representations of G'. For a precise description of what to expect in general, see **[Ho 1]**. Rather than pursue the general theory, we refer the reader to the examples of §6.

5. Local examples. Throughout this section, F is a local field not of characteristic 2, χ is a fixed character of F, V_0 is the space F^2 equipped with the skew form $x_1y_2 - x_2y_1$, U_0 is the orthogonal space F^n with quadratic form q, and $V = V_0 \otimes U_0$. Then $\mathrm{Sp}(V_0) = \mathrm{SL}(2, F)$, $O(U_0) = O(q)$, and the problem is to describe ω_χ restricted to $\mathrm{SL}(2, F) \cdot O(U_0)$. Recall that we may choose a polarization in V so that ω_χ acts in $L^2(F^n)$ according to the formulas

$$\omega_\chi\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}f(X) = \chi(bq(X))f(X), \tag{5.1}$$

and

$$\omega_\chi\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}f(X) = \gamma(q, \chi)\hat{f}(X). \tag{5.2}$$

The orthogonal group acts linearly in $L^2(F^n)$ through its natural left action in F^n.

A. *The anisotropic case.* Assume that F is nonarchimedean and q is anisotropic. Then $n \leqq 4$, $O(q)$ is compact, and

$$\omega_\chi|_{\mathrm{SL}(2, F)\cdot O(q)} = \sum \pi(\sigma) \otimes \sigma. \tag{5.3}$$

Here the sum is over σ in $R(O(q))$ and the multiplicity of each $\pi(\sigma) \otimes \sigma$ is one (cf. **[RS 1]**). It remains to describe $R(O(q))$, $R(\mathrm{SL}(2, F))$, and the resulting duality correspondence.

Case (i): $n = 1$. In this case, $q(x) = ax^2$ with $a \in F^\times$, $O(q) = \{\pm 1\}$, and the corresponding representations of $\mathrm{SL}(2, F)$ act on the space of even or odd functions in $L^2(F)$. More precisely, if σ is the trivial representation of $O(q)$, then $\pi(\sigma)$ (defined by formulas (5.1), (5.2) restricted to the space of even functions) is the unique subrepresentation of an appropriate nonunity principal series representation of $\overline{\mathrm{SL}}(2, F)$ at $s = -1/2$; if σ is the nontrivial representation of $O(q)$, then $\pi(\sigma)$ (defined by (5.1), (5.2) acting in the space of odd functions) is a supercuspidal representation of $\overline{\mathrm{SL}}(2, F)$ which is "exceptional" in the sense explained later in §6; for more details, see **[Ge 2]**, **[G-PS]**, and **[Ho 3]**. Note that ax^2 and bx^2 lead to equivalent oscillator representations if and only if $ab^{-1} \in (F^\times)^2$. In case $F = \boldsymbol{R}$, the pieces of ω_χ are square-integrable with extreme vectors of weight 1/2 and 3/2.

Case (ii): $n = 2$. Let U_0 denote a quadratic extension K of F equipped with the inner product derived from its norm form q. Then $O(q)$ is the semidirect product of the norm 1 group K^1 in K with the Galois group of K over F. In this case each σ in $R(O(q))$ can be described in terms of a character of K^1 and most of the representations of $R(\mathrm{SL}(2, F))$ are supercuspidal. For further details, see **[ST]**, **[Cas]**, **[G]**, or **[RS 1]**; in **[ST]** the correspondence is 2-to-1 because K^1 (the special orthogonal group) is used in place of $O(U_0)$. This example is historically important because Shalika and Tanaka were the first to use the oscillator representation to construct an interesting class of irreducible representations.

Case (iii): $n = 3$ **[RS 1]**. Let H denote the unique division quaternion algebra over F, H^0 the subspace of pure quaternions, and q_3 the restriction of the reduced norm form on H to H^0. Then every anisotropic ternary form q over F is equivalent to one of the forms aq_3 with a in $F^\times/(F^\times)^2$, and every irreducible representation σ of $O(q)$ belongs to $R(O(q))$ ($SO(q)$ is isomorphic to $(H^0)^\times$). In **[RS 1]** it is also shown that the corresponding representations $\pi(\sigma)$ of $\overline{\mathrm{SL}}(2, F)$ are square-integrable, and in fact exhaust the class of all such "genuine" representations (at least when the residual characteristic of F is odd).

Case (iv): $n = 4$. Let U_0 denote the division quaternion algebra H defined over F and equipped with the inner product derived from the reduced norm form q. Then though $O(q)$ is not the same as $H^\times$, each σ in $R(O(q))$ corresponds naturally to an irreducible representation of $H^\times$, all such representations of $H^\times$ thus arise, and each $\pi(\sigma_v)$ in $R(\mathrm{SL}(2, F))$ is square-integrable. Actually, $\pi(\sigma_v)$ extends in a unique way to a representation $\pi'(\sigma_v)$ of $\mathrm{GL}(2, F_v)$ such that

$$\pi'(\sigma_v)\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \sigma_v(a)I$$

and the resulting duality correspondence $\sigma_v \leftrightarrow \pi'(\sigma_v)$ is *onto* the set of classes of square-integrable irreducible representations of $\mathrm{GL}(2, F_v)$; for further details, see **[JL]**.

Note that Case (iv) is analogous to Case (iii) in that the $\pi(\sigma)$'s of the local duality correspondence are characterized by their square-integrability. Only in Case (ii) is a neat characterization of $R(O(q))$ lacking.

B. *The noncompact* (*real*) *case.* Assume that $F = \boldsymbol{R}$ and $\chi(u) = e^{2\pi i u}$. Let $O(q) = O(k, l)$ denote the isometry group of the standard quadratic form of signature (k, l) on $\boldsymbol{R}^n$, with $n = k + l$. In [ST] Strichartz has essentially given a decomposition of the natural action of $O(q)$ in $L^2(\boldsymbol{R}^n)$, i.e., of ω_χ restricted to $O(q)$. Using results of **[Re]** on the tensor product of representations of $\mathrm{SL}(2, \boldsymbol{R})$, Howe has proved the following version of his local duality conjecture:

$$\omega_\chi \Big|_{\overline{\mathrm{SL}(2, \boldsymbol{R})} \cdot O(q)} \approx \int \sigma_s' \otimes \sigma_s \, ds$$

where ds is a Borel measure on the unitary dual of $\mathrm{SL}(2, \boldsymbol{R})$, σ_s and σ_s' are irreducible unitary representations of $O(k, l)$ and $\overline{\mathrm{SL}}(2, \boldsymbol{R})$ respectively, and σ_s and σ_s' determine each other almost everywhere with respect to s. The resulting correspondence $\sigma_s \to \pi(\sigma_s) = \sigma_s'$ is particularly interesting for the discrete series representations in $R(\overline{\mathrm{SL}}(2, \boldsymbol{R}))$ and $R(O(q))$. For complete details, see **[Ho 4]** and **[RS 2]**; the case $(k, l) = (2.1)$ is described in **[Ge 2]**.

C. *Noncompact p-adic case.* An interesting example arises when we replace the quadratic extension of example 5.A(ii) by the hyperbolic plane F^2 equipped with inner product $((x, y), (x', y')) = xx' - yy'$. In this case $O(U_0)$ splits. Thus we can find a polarization of V left fixed by $S_p(V_0) = \mathrm{SL}(2, F)$ (take $X_0 = \{(x, x) : x \in F\}$ and $Y_0 = \{(x, -x) : x \in F\}$). The corresponding Schrödinger model for ω_χ then makes $\mathrm{SL}(2, F)$ act linearly in $L^2(F^2)$. More precisely, $\omega_\chi(g)f\binom{x}{y} = f(g^{-1}\binom{x}{y})$. So since functions on $F^2 - \{0\}$ can be identified with functions on G invariant by $N - \{\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)\}$, the oscillator representation in this case leads to a direct integral of principal series representations of $\mathrm{SL}(2, F)$. The orthogonal group, however, no longer acts linearly. Indeed the operator corresponding to reflection (the element taking (x, y) to (y, x)) is the Fourier transform (in $L^2(F^2)$) taken with respect to the skew form in F^2.

The significance of these facts for the representation theory of SL(2, F) has already been discussed in Cartier's lectures (see also **[G]**).

6. Global examples.

A. *The anisotropic case. Case* (i): $O(1)$. By tensoring the pieces of the oscillator representation in $L^2(F_v)$ we obtain some very interesting automorphic representations of SL(2) and (by inducing) GL(2). The idea that nontrivial pieces of the oscillator representation could define interesting cusp forms was first communicated to me by Howe (cf. **[Ho 3]**). We shall be content to merely sketch the subsequent development and refer the reader to **[G-PS]** for details.

If $\sigma = \bigotimes \sigma_v$ is an automorphic representation of $O(1)$ then σ_v is the trivial representation for all v except in a finite set S whose cardinality is even. If $\chi = \prod \chi_v$ is a character of A/F, let ω_{χ_v} denote the oscillator representation of SL(2, F_v) in $L^2(F_v)$ corresponding to χ_v. Fix a character $\lambda = \prod \lambda_v$ of A^x/F^x such that $\lambda_v(-1) = 1$ (resp. -1) if $v \notin S$ (resp. $v \in S$) and let ω'_{χ_v} denote the even (resp. odd) piece of ω_{χ_v}. Then extend ω'_{χ_v} to

$$G_v^* = \{g \in \mathrm{GL}(2, F_v) : \det(g) \in (F_v^x)^2\}$$

by defining $\omega'_{\chi_v}\left(\begin{smallmatrix} 1 & 0 \\ 0 & a^2 \end{smallmatrix}\right)$ to be the operator

$$\Phi(x) \mapsto \lambda(a)\,|\,a\,|^{-1/2}\,\Phi(a^{-1}x),$$

and induce ω'_{χ_v} up to GL(2, F_v) to obtain the representation ω'_v of the double cover $\bar{G}_v$ of GL(2, F_v). This representation ω'_v is an irreducible unitary "genuine" representation of $\bar{G}_v$ *which is independent of* χ_v *for all* v and class 1 for almost all v.

The representation $\pi(\lambda) = \bigotimes \omega'_v$ is an automorphic genuine representation of $\overline{\mathrm{GL}}(2, A)$ which is cuspidal precisely when $S \neq \varnothing$. In general, when S is empty, these $\pi(\lambda)$ generate the discrete *non*cuspidal spectrum of $\overline{\mathrm{GL}}(2, A)$ (cf. **[GS]**); in particular, when $F = \boldsymbol{Q}$, $\pi(\lambda)$ generalizes the classical theta-function $\theta(z) = \sum_{n=-\infty}^{\infty} \exp(2\pi i n^2 z)$. On the other hand, when $S \neq \varnothing$, $\pi(\lambda)$ generalizes such classical cusp forms as Dedekind's η-function. In this case, $\pi(\lambda)$ is *exceptional* in the sense that the Fourier expansion of any function in its representation space contains "only one orbit of characters", generalizing the fact that the expansion of the η-function has "only square terms": $\eta(24z) = \sum_{n=1}^{\infty}(3/n)\exp(2\pi i n^2 z)$.

Since the Fourier expansion of a cusp form $\bar{\pi}$ on $\overline{\mathrm{GL}}(2, A)$ is carried out with respect to a group which is isomorphic to the product of F with the double cover $\bar{F}^x$ of F^x, we can say $\bar{\pi}$ is exceptional if—given χ on F—the Whittaker model for $\bar{\pi}$ with respect to $(\chi, \bar{\mu})$ exists for *only one* character $\bar{\mu}$ of $\bar{F}^x$. Thus the notion of exceptionality makes sense locally too; see **[G-PS]** for details. Our conjecture is that the oscillator representation produces all possible exceptional representations, both locally and globally. In other words, for $O(1)$ the image of the duality correspondence $\sigma \mapsto \pi(\sigma)$ should be all the exceptional genuine representations of $\bar{G}$.

The classical analogue of the global part of the conjecture just described is that every cusp form of weight $k/2$ whose Fourier expansion involves only square terms must be of weight 1/2 or 3/2 and (a linear combination of translates of functions) of the form

$$\sum_{n=1}^{\infty}\psi(n)\, n^{\nu} \exp(2\pi i n^2 z), \qquad \nu = 0 \text{ or } 1.$$

At this conference we learned that M.-F. Vigneras has proved this assertion in **[V]**.

Case (ii): $O(2)$. Let K denote a quadratic extension of the field F, χ a character of $F\backslash \boldsymbol{A}$, and $\sigma = \bigotimes \sigma_v$ a character of $K^1(F)\backslash K^1(\boldsymbol{A})$. Piecing together the resulting local representations $\pi(\sigma_v)$ of SL(2, F_v) (described in §5.A(ii)) produces automorphic cuspidal representations of SL(2, $\boldsymbol{A}$) (provided σ is nontrivial). These automorphic forms are related to the theta series with grossencharacter constructed earlier by E. Hecke **[H]** and H. Maass **[M]**; for more details, see **[ST]** and **[Ge 3**, Chapter 7**]**.

Case (iii): $O(4)$. Let D denote a division quaternion algebra over F and S_D the set of places where D ramifies. Suppose $\sigma = \bigotimes \sigma_v$ belongs to $R(D^{\times}(A))$. If $v \in S_D$ then $D_v^{\times}$ is isomorphic to the unique division quaternion algebra defined over F_v and $\pi'(\sigma_v)$ is defined as in §5. On the other hand, when $v \notin S_D$, $D_v^{\times}$ is isomorphic to GL(2, F_v), and the natural duality correspondence is the identity map.

Now suppose σ is actually automorphic, i.e., there is an embedding of σ into the space of automorphic forms on $G'(\boldsymbol{A}) = D^{\times}(\boldsymbol{A})$. Then according to **[JL]** the representation $\pi'(\sigma) = \bigotimes_v \pi'(\sigma_v)$ of GL(2, $\boldsymbol{A}$) will also be automorphic. Conversely, if $\pi = \bigotimes \pi_v$ belongs to R(GL(2, $\boldsymbol{A}$)), i.e., $\pi_v = \pi'(\sigma_v)$ for some σ_v in $R(D_v^{\times})$ (equivalently π_v is square-integrable for each $v \in S_D$), then π will be automorphic *only if* $\sigma = \bigotimes \sigma_v$ is automorphic; cf. Theorems 14.4 and 16.1 of **[JL]**. These results confirm the main global conjecture of Howe's theory in the special context of division quaternion algebras. Whereas the proof in **[JL]** uses the trace formula, Howe apparently has an independent proof by different, more elementary, methods.

B. $O(2, 1)$. Let U_0 denote the three-dimensional space of trace zero 2×2 matrices over F equipped with the inner product derived from the determinant function. Set $V_0 = (F^2, \langle\ ,\ \rangle)$ and $V = V_0 \otimes U_0$. Then for each place v of F, the pair (SL(2, F_v), $O(U_0)$) is a dual reductive pair in Sp(V).

For $F_v = \boldsymbol{R}$, the duality correspondence $\sigma_v \leftrightarrow \pi(\sigma_v)$ pairs discrete series representations of PGL(2, $\boldsymbol{R}$) = $SO(2, \boldsymbol{R})$ of weight $k - 1$ with discrete series representations of $\overline{\text{SL}}(2, \boldsymbol{R})$ of weight $k/2$. For $F = \boldsymbol{Q}$ it is conjectured in **[Ge 2]** that the resulting global correspondence should generalize Shimura's correspondence **[Shm]** between classical forms of weight $k/2$ and $k - 1$. Evidence for this is provided by Howe's recent description of the local duality correspondence *for class* 1 *representations* (in **[Ho 1]**), and by Shintani and Niwa's work on the global correspondence using integrals of theta-functions (**[Sht]** and **[N]**). For further work on this problem, see also **[G-PS]**.

If F_v is nonarchimedean, and σ_v is a discrete series representation of PGL(2, F_v) of the form $\pi'(\sigma_v')$, with σ_v' an appropriate finite-dimensional representation of the orthogonal group of an anisotropic ternary form, then $\pi(\sigma_v)$ should coincide with the representation $\pi(\sigma_v')$ of $\overline{\text{SL}}(2, F_v)$ described in §5A, Case (iii).

C. $O(2, 2)$ ($F = \boldsymbol{Q}$). Let $K = \boldsymbol{Q}(\sqrt{\Delta})$, $\Delta > 0$ the discriminant, and let τ denote the Galois automorphism of $K/\boldsymbol{Q}$. Let

$$U_0 = \left\{ X = \begin{pmatrix} x_1 & x_4 \\ x_3 & -x_1^{\tau} \end{pmatrix} : x_1 \in K,\ x_3, x_4 \in \boldsymbol{Q} \right\} \approx \boldsymbol{Q}^4$$

and define $q : U_0 \to \boldsymbol{Q}$ by $q(X) = -2\det(X)$. Then q has signature (2, 2), and the corresponding global duality correspondence should pair together modular forms of integral weight with Hilbert modular forms (since $SL(2, \boldsymbol{R}) \times SL(2, \boldsymbol{R})$ has a natural representation in $SO(2, 2)$). Work on this correspondence (in the classical language of forms in the upper half-plane) has been carried out by Oda **[O]**, Asai [A], Kudla [K], Rallis and Schiffmann [RS 3], Zagier [Z], and, in a somewhat different light (and earlier) by Doi-Naganuma [DN].

Note added in proof. We have now proved the global "exceptionality conjecture" described at the bottom of page 293. The local conjecture has been proved by James Meister and will appear in his forthcoming Cornell Ph.D. thesis.

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CORNELL UNIVERSITY

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ON A RELATION BETWEEN $\widetilde{\mathrm{SL}}_2$ CUSP FORMS AND AUTOMORPHIC FORMS ON ORTHOGONAL GROUPS

S. RALLIS

This lecture is devoted to showing how, starting from the ideas of A. Weil in **[22]**, one can construct a correspondence or lifting between $\widetilde{\mathrm{SL}}_2$ cusp forms and automorphic forms on orthogonal groups.

1. Siegel formula and the lifting of modular forms. Let $\boldsymbol{R}^k$ be k dimensional Euclidean space with standard basis $e_1, \ldots, e_k$. Let $[\ ,\ ]$ be the bilinear form on $\boldsymbol{R}^k$ given by $[X, Y] = \sum_{i=1}^k x_i y_i$ where $X = \sum_{i=1}^k x_i e_i$, $Y = \sum_{i=1}^k y_i e_i$. Let Q be the quadratic form on $\boldsymbol{R}^k$ given by

$$Q(X, Y) = \sum_{i=1}^{a} x_i y_i - \sum_{i=a+1}^{k} x_i y_i \quad \text{with } a < k.$$

Then for every $X \in \boldsymbol{R}^k$ we have $Q(X, X) = \|X_+\|^2 - \|X_-\|^2$, with $\|X_+\|^2 = \sum_{i=1}^a x_i^2$ and $\|X_-\|^2 = \sum_{i=a+1}^k x_i^2$. Thus Q has signature (a, b) with $a + b = k$ and $b \geqq 1$.

A lattice $L \subseteq \boldsymbol{R}^k$ is Q *integral* (Q-even, resp.) if $Q(L, L) \subseteq \boldsymbol{Z}$, the integers ($Q(L, L) \subseteq 2\cdot\boldsymbol{Z}$, resp.). The Q dual lattice to L is $L_*(Q) = \{\xi \in \boldsymbol{R}^k \mid Q(\xi, L) \subseteq \boldsymbol{Z}\}$. Then $L_*(Q)/L$ is a finite Abelian group. We let n_L be the *exponent* of this group, i.e. the smallest positive integer n_L so that $n_L \cdot \xi \in L$ for all $\xi \in L_*(Q)$. If $\xi_1, \ldots, \xi_k$ is any $\boldsymbol{Z}$ basis of L, let $D_{Q(L)} = \det\{Q(\xi_i, \xi_j)\}$. This number, the *discriminant* of Q relative to L, is independent of the choice of basis of L.

Then define the function:

$$\psi(X, z, g) = e^{\pi\sqrt{-1}\{xQ(X,X) + \sqrt{-1}y[gX, gX]\}} (\operatorname{Im} z)^{b/2}, \tag{1–1}$$

with $z = x + \sqrt{-1}\ y \in H$ the upper half-plane (i.e. $y > 0$) and $g \in O(Q)$, the orthogonal group of Q. Let $\eta \in L_*(Q)$ and consider the θ series

$$\tilde{\Theta}^L_{\psi,\eta}(z, g) = \sum_{\xi \in L} \psi(\xi + \eta, z, g). \tag{1–2}$$

Then let η run over a set of representatives of $L_*(Q)/L$. We define the column matrix

$$\Psi^L_\psi(z, g) = \left(\tilde{\Theta}^L_{\psi,\eta}(z, g)\right)_{\eta \in L_*(Q)/L}. \tag{1–3}$$

Then Siegel proves the following functional equation: (L, Q-even)

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(1–4) $$\Psi_\psi^L(\Gamma\cdot z, g\gamma) = c(\Gamma,\gamma)(c_\Gamma z + d_\Gamma)^{(a-b)/2}\Psi_\psi^L(z,g)$$

(for $\Gamma\in \mathrm{SL}_2(\boldsymbol{Z}), \gamma\in O(Q)_L = \{g\in O(Q)\mid g(L) = L\}$), where $c(\Gamma,\gamma)$ is a $[L_*(Q)/L]^2$ unitary matrix depending only on Γ and γ. In general we know that the map $(\Gamma,\gamma) \rightsquigarrow c(\Gamma,\gamma)$ defines a *projective unitary representation* of the group $\widetilde{\mathrm{SL}}_2(\boldsymbol{Z}) \times O(Q)_L$.

The main difficulty in studying $\Psi_\psi^L(z,g)$ is that this function is not automorphic as it is not an eigenfunction for the center of the enveloping algebra of $\mathrm{SL}_2 \times O(Q)$. However this can be remedied by integrating $\tilde{\Theta}_{\psi,\eta}^L(z,g)$ against the constant function on $O(Q)/O(Q)_L$. More precisely, the main point of the analytic part of *Siegel's formula* is to introduce a suitably normalized $O(Q)$ invariant measure $d\mu_\eta$ on the homogeneous space

$$O(Q)/\{\gamma\in O(Q)_L\mid \gamma\eta\equiv\eta \bmod L\} \cong O(Q)/O(Q)_{L,\eta}$$

so that we have the identity (valid for $k\geqq 5$ and L, Q-even **[20]**)

(1–5) $$\int_{O(Q)/O(Q)_{L,\eta}} \tilde{\Theta}_{\psi,\eta}^L(z,g)\; d_{\mu_\eta}(g) = E_\eta(z),$$

where

(1–6) $$E_\eta(z) = (\mathrm{Im}\, z)^{b/2}\sum_{M\in \mathrm{SL}_2(\boldsymbol{Z})_\infty\backslash \mathrm{SL}_2(\boldsymbol{Z})} c_{\eta,0^-}((M,1))\,(c_M z + d_M)^{(b-a)/2}|c_M z + d_M|^{-b}$$

with $\mathrm{SL}_2(\boldsymbol{Z})_\infty = \{\Gamma\in \mathrm{SL}_2(\boldsymbol{Z})\mid c_\Gamma = 0\}$, and $c_{\eta,0^-}((M,1))$ represents an element of the first column of the matrix $c((M,1))$ (1, identity element of $O(Q)$). Then we know that $E_\eta(z)$ is an eigenfunction of the center of the enveloping algebra of $\widetilde{\mathrm{SL}}_2$; in concrete terms this means that E_η satisfies the equation:

(1–7) $$\Delta_{k/2}(E_\eta) \equiv 0,$$

with

$$\Delta_{k/2} = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \tfrac{1}{2}ky\frac{\partial}{\partial y} + \sqrt{-1}\,\tfrac{1}{2}(b-a)y\frac{\partial}{\partial x}.$$

In terms of representation theory, Siegel's formula expresses the fact that the identity representation of $O(Q)$ corresponds (in a sense to be made precise below) to the representation of $\widetilde{\mathrm{SL}}_2$ determined by the nonanalytic Eisenstein series $(E_\eta)_\eta$. We then note that it is easy to extend this correspondence to other automorphic representations of $O(Q)$. Namely we can do this in two ways. First we can vary the kernel function. We let P be a homogeneous polynomial on $\boldsymbol{R}^a$ ($= \boldsymbol{R}$ span of $\{e_1,\dots,e_a\}$) of degree t satisfying $\partial(Q)(P)\equiv 0$, where $\partial(Q)$ is the differential operator $Q(\partial/\partial x_1,\dots,\partial/\partial x_n)$. Then we form

$$\psi_*(X,z,g) = (\mathrm{Im}\, z)^{t/2}P(g\cdot(\mathrm{Im}\, z)^{1/2}X)\,\psi(X,z,g)$$

and as above define $\Psi_{\psi_*}^L(z,g)$. Thus $\Psi_{\psi_*}^L$ satisfies a functional equation similar to (1–4). The second step is then to integrate $\tilde{\Theta}_{\psi_*,\eta}^L$ against a *cusp form* β on $O(Q)/O(Q)_{L,\eta}$. This is possible because $\tilde{\Theta}_{\psi_*,\eta}^L$ is a "slowly increasing" function on

$H \times O(Q)/O(Q)_{L,\eta}$. Let $\langle \tilde{\Theta}^L_{\phi_*,\eta}(z,g) \mid \beta(g)\rangle_{O(Q)/O(Q)_{L,\eta}}$ denote the Hilbert space inner product of $\tilde{\Theta}^L_{\phi_*,\eta}$ and β (relative to $d\mu_\eta$). We deduce that if $\omega_{O(Q)} * \beta = \lambda\beta$, where $\omega_{O(Q)}$ is the Casimir element of $O(Q)$, then $z \rightsquigarrow \langle \tilde{\Theta}^L_{\phi_*,\eta}(z,g) \mid \beta(g)\rangle$ is an eigenfunction of an operator similar to $\Delta_{k/2}$. However the cuspidal behavior of the function $z \rightsquigarrow \langle \Theta^L_{\phi_*,\eta}(\ \ ,g) \mid \beta(g)\rangle$ is yet another question. In **[19]** (for the case $O(2, 1)$) it is shown that if β is a "holomorphic" cusp form on $\mathrm{PGL}_2(\boldsymbol{R})$ ($\cong O(2, 1)$ at least locally), then $z \rightsquigarrow \langle \tilde{\Theta}^L_{\phi_*,\eta}(\ \ ,g) \mid \beta(g)\rangle$ is a holomorphic cusp form of semi-integral weight in the sense of **[18]** for suitable Hecke congruence of $\mathrm{SL}_2(\boldsymbol{Z})$ (depending on the lattice L). We note that in **[9]** (for the case $O(1, 1)$, an anisotropic lattice L and β a *unitary* character on $O(Q)/O(Q)_{L,\eta}$) $z \rightsquigarrow \langle \tilde{\Theta}_{\phi_*,\eta}(\ \ ,g) \mid \beta(g)\rangle$ corresponds to a (nonanalytic) cusp form which is determined by a principal series representation of $\mathrm{SL}_2(\boldsymbol{R})$ (see also **[8]**).

On the other hand one can try to invert the above correspondence in the following formal manner. We let $h(z)$ be a cusp form on $\widetilde{\mathrm{SL}}_2$ (again in the sense of **[17]** for a suitable Hecke congruence subgroup Γ of $\mathrm{SL}_2(\boldsymbol{Z})$), and we form the Petersson inner product of $h(\ \ )$ with $\tilde{\Theta}^L_{\phi_*,\eta}(\ \ ,\ \ )$,

$$\langle \tilde{\Theta}^L_{\phi_*,\eta}(\ \ ,g) \mid h(\ \ )\rangle = \int_{\Gamma\backslash H} \tilde{\Theta}^L_{\phi_*,\eta}(z,g)\,\overline{h(z)}\,(\mathrm{Im}\, z)^{\nu-2}\,dx\,dy \tag{1–8}$$

(with ν, the *weight* of h). Then it is simple to deduce that $g \rightsquigarrow \langle \tilde{\Theta}^L_{\phi_*,\eta}(\ \ ,g) \mid h(\ \ )\rangle$ is an eigenfunction for the center of the enveloping algebra of $O(Q)$. Again the cuspidal behavior of this function is another question. Niwa in **[10]** shows (for the case $O(2, 1)$) how such a correspondence explains the construction of Shimura in **[18]** of cusp forms of integral weight from cusp forms of semi-integral weight.

In any case the main object of study is the kernel function $\tilde{\Theta}^L_{\phi_*,\eta}$. We adopt the following point of view. One can analyze the "cuspidal components" of $\tilde{\Theta}^L_{\phi_*,\eta}$ (that is, the "projections" of $\tilde{\Theta}^L_{\phi_*,\eta}$ on the various spaces of cusp forms discussed above) by studying *in detail* the spectral decomposition of the Weil representation. Then knowing precisely the "cuspidal components" gives a correspondence or lifting between modular forms on the two groups in question.

2. Discrete spectrum of the Weil representation. Let $\wedge$, Fourier transform, be defined as follows: if $f \in L^1 \cap L^2$, then

$$\hat{f}(Y) = \int_{\boldsymbol{R}^k} f(X)\, e^{2\pi\sqrt{-1}[X,Y]}\, dX.$$

Then let π_Q be the Weil representation of $\widetilde{\mathrm{SL}}_2 \times O(Q)$ in $L^2(\boldsymbol{R}^k)$, where $\widetilde{\mathrm{SL}}_2$ is the two-fold cover of $\mathrm{SL}_2(\boldsymbol{R})$. In particular, if

$$\left(\left(\begin{bmatrix} a & b \\ 0 & a^{-1}\end{bmatrix}, 1\right), g\right) \in \widetilde{\mathrm{SL}}_2 \times O(Q) \quad \text{with } a > 0,$$

then

$$\pi_Q\left(\left(\left(\begin{bmatrix} a & b \\ 0 & a^{-1}\end{bmatrix}, 1\right), g\right)\right)\varphi(Z) = |a|^{k/2}\, e^{\pi\sqrt{-1}baQ(Z,Z)}\,\varphi(ag^{-1}\cdot Z).$$

Also if

$$w_0 = \left(\left(\begin{bmatrix} 0 & -1 \\ 1 & 0\end{bmatrix}, -1\right), 1\right),$$

then $\pi_Q(w_0)\,\varphi(Z) = \delta_Q\hat{\varphi}(-M_Q(Z))$ with $M_Q \in \mathrm{Aut}(\boldsymbol{R}^k)$, so that $Q(X, Y) = [X, M_Q(Y)]$ for all $X, Y \in \boldsymbol{R}^k$ and δ_Q a certain eighth root of unity. Then π_Q is determined by the above formulae.

We let $\tilde{K}$ be the maximal compact subgroup of $\widetilde{\mathrm{SL}}_2$ given by

$$\left\{k(\theta, \varepsilon) = \left(\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \varepsilon\right) \Big| -\pi \leqq \theta < \pi,\ \varepsilon = \pm 1\right\}.$$

Also let

$$A = \left\{a(r) = \left(\begin{bmatrix} r & 0 \\ 0 & r^{-1} \end{bmatrix}, 1\right), r > 0\right\}$$

and

$$N = \left\{n(x) = \left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}, 1\right) \Big| x \in \boldsymbol{R}\right\}.$$

Then if $\mathfrak{a}$, $\mathfrak{n}$, and $\mathfrak{k}$ are the infinitesimal generators of A, N, and $\tilde{K}$, then $\omega_{\mathrm{SL}_2} = -\mathfrak{k}^2 + \mathfrak{a}^2 + (\mathfrak{n} + \mathrm{Ad}(w_0)\mathfrak{n})^2$.

We let K ($\cong O(a) \times O(b)$) be the maximal compact subgroup of $O(Q)$. If π is a differentiable representation of $\widetilde{\mathrm{SL}}_2 \times O(Q)$ on a Banach space $\mathfrak{B}$, then let $\mathfrak{B}_{\tilde{K}\times K}$ be the space of $\tilde{K} \times K$ finite vectors in $\mathfrak{B}$.

Let $\boldsymbol{F}_Q$ be the space of C^∞ vectors of π_Q in $L^2(\boldsymbol{R}^k)$. An easy use of the Sobolev regularity theorem shows that if $\varphi \in \boldsymbol{F}_Q$, then for any y belonging to $u(\widetilde{\mathrm{SL}}_2 \times O(Q))$, the enveloping algebra of $\widetilde{\mathrm{SL}}_2 \times O(Q)$, $\pi_Q(y)\varphi$ is a C^∞ function on Ω_+ and Ω_-, where $\Omega_+ = \{X \in \boldsymbol{R}^k \mid Q(X) > 0\}$ and $\Omega_- = \{X \in \boldsymbol{R}^k \mid Q(X) < 0\}$.

Then by a simple exercise in invariant theory, one can verify that the centers of the enveloping algebras of $\widetilde{\mathrm{SL}}_2$ and $O(Q)$ collapse on each other in the representation π_Q. That is, $\pi_Q(\mathfrak{z}(\widetilde{\mathrm{SL}}_2)) = \pi_Q(\mathfrak{z}(O(Q)))$, where $\mathfrak{z}(\ ) =$ the center of the enveloping algebra of ( ). In particular we have that $\pi_Q(\omega_{\mathrm{SL}_2}) + \pi_Q(\omega_{O(Q)}) = (\frac{1}{4}k^2 - k)\cdot I$, where ω_{SL_2} and $\omega_{O(Q)}$ are the Casimir elements of $\widetilde{\mathrm{SL}}_2$ and $O(Q)$, respectively. Then it follows that the spectral analysis of the algebra $\pi_Q(\mathfrak{z}(\widetilde{\mathrm{SL}}_2 \times O(Q)))$ in $L^2(\boldsymbol{R}^k)$ reduces to the spectral analysis of a single operator $\pi_Q(\omega_{\mathrm{SL}_2})$.

For the rest of this paper we assume that $k \geqq 3$ and $a \geqq b \geqq 1$.

We let $\boldsymbol{F}_Q(\lambda) = \{\varphi \in \boldsymbol{F}_Q \mid \pi_Q(\omega_{\mathrm{SL}_2})\,\varphi = \lambda\varphi\}$ and let $\mathrm{Spec}(\pi_Q) = \{\lambda \in \boldsymbol{C} \mid \boldsymbol{F}_Q(\lambda) \neq \{0\}\}$. Then we know that if π is any unitary irreducible representation of $\widetilde{\mathrm{SL}}_2 \times O(Q)$ which occurs "discretely" in $L^2(\boldsymbol{R}^k)$, then π is a subrepresentation of $\boldsymbol{F}_Q(\lambda)$ for some λ. More generally we have

THEOREM 2.1 ([**12**], [**14**]). (a) $\mathrm{Spec}(\pi_Q) = \{s^2 - 2s \mid s \equiv \frac{1}{2}k \bmod 1$ *and* $s > 1\}$.

(b) *Let* $\boldsymbol{F}_Q^+(\lambda) = \{\varphi \in \boldsymbol{F}_Q(\lambda) \mid \varphi_{|\Omega_-} \equiv 0\}$ *and* $\boldsymbol{F}_Q^-(\lambda) = \{\varphi \in \boldsymbol{F}_Q(\lambda) \mid \varphi_{|\Omega_+} \equiv 0\}$. *Then* $\boldsymbol{F}_Q^+(\lambda)$, $\boldsymbol{F}_Q^-(\lambda)$ *are* $\widetilde{\mathrm{SL}}_2 \times O(Q)$ *topologically irreducible modules; there is a direct sum* $\boldsymbol{F}_Q(\lambda) = \boldsymbol{F}_Q^+(\lambda) \oplus \boldsymbol{F}_Q^-(\lambda)$. *Moreover* $\boldsymbol{F}_Q^+(\lambda)$ *and* $\boldsymbol{F}_Q^-(\lambda)$ *are inequivalent* $\widetilde{\mathrm{SL}}_2 \times O(Q)$ *irreducible representations.*

(c) $\boldsymbol{F}_Q^+(s^2 - 2s)$ *is* $\widetilde{\mathrm{SL}}_2 \times O(Q)$ *equivalent to the tensor product* $L^2(\mathrm{Whit})_{s^2-2s} \hat{\otimes} \mathscr{A}_s^+$, *where* $L^2(\mathrm{Whit})_{s^2-2s}$ *is the eigenspace of* ω_{SL_2} *(with eigenvalue* $s^2 - 2s$*) in the space of* C^∞ *vectors of the unitarily induced representation of* SL_2 *(from the unitary character*

$$\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}, 1\right) \rightsquigarrow e^{\pi\sqrt{-1}x}\right)$$

and $\mathscr{A}_s^+$ *is the eigenspace* (*with eigenvalue* $s^2 - 2s - \frac{1}{4}k^2 + k$) *of* W_ξ^+, *the Laplace-Beltrami operator on the space of* C^∞ *vectors of the standard representation of* $O(Q)$ *in* $L^2(\Gamma_{+1}, d\mu_+)$, *where* $\Gamma_{+1} = \{X \in \boldsymbol{R}^k \mid Q(X, X) = 1\}$ *and* $d\mu_+$ *an* $O(Q)$ *invariant measure on* Γ_{+1}. *Here* W_ξ^+ *is determined by the separation of variables of*

$$\partial(Q) = \frac{\partial^2}{\partial t^2} + \frac{k-1}{t}\frac{\partial}{\partial t} - \frac{1}{t^2} W_\xi^+,$$

with $X = t \cdot \xi$, $\xi \in \Gamma_{+1}$.

REMARK 2.1. $\boldsymbol{F}_Q^-(s^2 - 2s)$ is $\widetilde{\mathrm{SL}}_2 \times O(Q)$ equivalent to the tensor product $L^2(\mathrm{Whit})_{s^2-2s}^A \hat{\otimes} \mathscr{A}_s^-$, where $L^2(\mathrm{Whit})_{s^2-2s}^A$ is the representation of $\widetilde{\mathrm{SL}}_2$ on $L^2(\mathrm{Whit})_{s^2-2s}$ twisted by the unique outer automorphism A of $\widetilde{\mathrm{SL}}_2$, and $\mathscr{A}_s^-$ is the eigenspace (with eigenvalue $s^2 - 2s - \frac{1}{4}k^2 + k$) of W_ξ^-, the Laplace-Beltrami operator on the space of C^∞ vectors on the standard representation of $O(Q)$ in $L^2(\Gamma_{-1}, d\mu_{-1})$.

The first point in analyzing the discrete spectrum of π_Q is simply to observe that a $\tilde{K}$ eigenfunction φ (also K finite) in $\boldsymbol{F}_Q(\lambda)$ has a separation of variables property on $\Omega_\pm$, i.e. $\varphi(X) = \rho_1^\pm(t)\rho_2^\pm(\xi)$ with $\rho_1^\pm$, $\rho_2^\pm$ C^∞ functions on Ω_+ (Ω_- respectively) satisfying certain differential equations (with $X = t \cdot \xi$, $\xi \in \Gamma_{+1}$, $t \in \boldsymbol{R}_+$). The second point is to realize that the map $\varphi \rightsquigarrow [G \to \pi_Q(G^{-1})(\varphi)(\xi)$ with $G \in \widetilde{\mathrm{SL}}_2]$ defines an $\widetilde{\mathrm{SL}}_2$ infinitesimal intertwining map of the $\tilde{K}$ finite functions in $\boldsymbol{F}_Q(\lambda)$ to the "square-integrable" functions ($L^2(\mathrm{Whit})^\pm$ space, the positive and negative Whittaker models) of the unitarily induced representation of $\widetilde{\mathrm{SL}}_2$ from the unitary character $n(x) \rightsquigarrow e^{\pi\sqrt{-1}\,x\,\mathrm{sgn}\,Q(\xi,\xi)}$. Then it is possible to analyze the spectrum of ω_{SL_2} in $L^2(\mathrm{Whit})^\pm$ and determine for which values λ one has $L^2(\mathrm{Whit})_\lambda^\pm \neq 0$. We see that the discrete spectrum of $\widetilde{\mathrm{SL}}_2$ in $L_2(\mathrm{Whit})^+$ is disjoint from the discrete spectrum of $\widetilde{\mathrm{SL}}_2$ in $L_2(\mathrm{Whit})^-$. Hence we conclude that $\boldsymbol{F}_Q^+(\lambda)$ and $\boldsymbol{F}_Q^-(\lambda)$ are $\widetilde{\mathrm{SL}}_2 \times O(Q)$ stable and inequivalent submodules of $\boldsymbol{F}_Q(\lambda)$ (see (b) above). In concrete terms this means that the functions f in the eigenspaces $\boldsymbol{F}_Q^+(\lambda)$ and $\boldsymbol{F}_Q^-(\lambda)$ have the remarkable feature that both f and its Fourier transform $\hat{f}$ are supported exclusively on *one* side of the light cone.

The next step is to analyze the $\rho_2^\pm$ part of the function φ. We note that $\rho_2^\pm$ is simply an L^2 eigenfunction of the Laplace-Beltrami operator $W_\xi^\pm$ on L^2 (hyperboloid). Using the relation between $\pi_Q(\omega_{O(Q)})$ and $\pi_Q(\omega_{\mathrm{SL}_2})$ above and any of the various methods of "hyperboloid" analysis **[21]**, [3], and **[12]**, the remaining questions of irreducibility in Theorem 2.1 reduce to similar questions about the discrete spectrum of $O(Q)$ in L^2 (hyperboloid).

One of the more interesting consequences of the discussion above is the determination of the regularity and growth properties of $\tilde{K} \times K$ finite functions φ of $\boldsymbol{F}_Q(s^2 - 2s)$. In particular any such $\tilde{K} \times K$ finite function extends uniquely to a continuous function on $\boldsymbol{R}^k - \{0\}$. The problem is the behavior of such functions at the origin $\vec{0}$. However if $s > \frac{1}{2}k$, then, in fact, such functions are continuous at $\vec{0}$. These results are deduced by studying the explicit form of the function φ. In a similar fashion we deduce the following growth property of φ, a $\tilde{K} \times K$ finite function in $\boldsymbol{F}_Q^+(s^2 - 2s)$:

$$\text{(2–1)} \qquad |\pi_Q((G,g))^{-1}(\varphi)(Z)| \leq r_G^{s-2}\|g^{-1}\|_k^{s+k/2-2}\exp(-\tfrac{1}{2}\pi r_G^2 Q(Z,Z))\left(\frac{1}{\|Z_+\|}\right)^{s+k/2-2}$$

(with $Z \in \Omega_+$), where $Q(Z) = \|Z_+\|^2 - \|Z_-\|^2$, $\| \ \|_k$ the Frobenius norm of the linear operator $g \in O(Q)$, and r_G the A-part of the Iwasawa decomposition of G, i.e. $G = K(G)A(r_G)N(x_G)$.

3. $\widetilde{\mathrm{SL}}_2$ cusp forms. The key property of the Weil representation π_Q as set forth in **[22]** is the following one: given a lattice $L \subseteq \boldsymbol{R}^k$ so that $Q(L, L) \subseteq \boldsymbol{Z}$, there exists an "arithmetical discrete" subgroup $\Gamma(Q, L)$ of $\widetilde{\mathrm{SL}}_2 \times O(Q)$ such that for any Schwartz function φ on $\boldsymbol{R}^k$

$$\Theta^L_{\varphi,\eta}(G, g) = \sum_{\xi \in L} \pi_Q((G, g))^{-1}(\varphi)(\xi + \eta) \qquad (\eta \in L_*/L) \tag{3–1}$$

is a C^∞ function on $\widetilde{\mathrm{SL}}_2 \times O(Q)$, transforming on the right according to a finite dimensional unitary representation of $\Gamma(Q, L)$. (Note (1–4) is a special case of this.) The main analytic content of this statement is that the family of Poisson distributions $G_\eta = \sum_{\xi \in L} \delta_{\xi+\eta}$ defines a continuous intertwining map (relative to $\widetilde{\mathrm{SL}}_2 \times O(Q)$) of $S(\boldsymbol{R}^k)$, the Schwartz space, to the C^∞ sections of a certain vector bundle over $\widetilde{\mathrm{SL}}_2 \times O(Q)/\Gamma(Q, L)$.

However the above methods do not directly apply to the $\tilde{K} \times K$ finite functions on $\boldsymbol{F}^\pm_Q(s^2 - 2s)$, since such functions are not Schwartz functions. But the growth and continuity properties discussed above for such φ coupled with the classical Poisson summation formula give the following theorem.

THEOREM 3.1 **[13]**. *Let $s > \frac{1}{2}k$. Then for any φ, a $\tilde{K} \times K$ finite function in $\boldsymbol{F}^\pm_Q(s^2 - 2s)$*

$$\Theta^L_\varphi(G, g) = \sum_{\xi \in L} \pi_Q((G, g))^{-1}(\varphi)(\xi) \tag{3–2}$$

is an absolutely convergent series. Then for any $(\Gamma, \gamma) \in \Gamma^L(Q) \times O(Q)_L$ with

$$\Gamma^L(Q) = \left\{ \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \varepsilon \right) \middle| \, a, b, c, d \in \boldsymbol{Z}, ad - bc = 1, \right.$$
$$\left. c \equiv 0 \bmod 2N_L, \, b \equiv 0 \bmod 2 \right\},$$

we have the functional equation:

$$\Theta^L_\varphi(G\Gamma, g\gamma) = c(\Gamma, \gamma)\Theta^L_\varphi(G, g), \tag{3–3}$$

with $c(\Gamma, \gamma)$, a unitary character on $\Gamma^L(Q) \times O(Q)_L$, taking values in $S_4 = \{z \in \boldsymbol{C} \mid z^4 = 1\}$. Moreover Θ^L_φ is a C^∞ function on $\widetilde{\mathrm{SL}}_2 \times O(Q)$ satisfying $D * \Theta^L_\varphi(G, g) = \Theta^L_{\pi_Q(D)\varphi}(G, g)$ for any D in the universal enveloping algebra of $\widetilde{\mathrm{SL}}_2 \times O(Q)$ ($*$ denotes differentiation on the left).

In particular it follows from Theorem 3.1 that Θ^L_φ is an eigenfunction of the center $\mathfrak{z}(\widetilde{\mathrm{SL}}_2 \times O(Q))$ of the universal enveloping algebra of $\widetilde{\mathrm{SL}}_2 \times O(Q)$. That is, $\omega_{\mathrm{SL}_2} * \Theta^L_\varphi(G, g) = (s^2 - 2s)\,\Theta^L_\varphi(G, g)$. On the other hand, from the growth estimates (2–1) we have

$$|\Theta^L_\varphi(G, g)| \leqq M \,\|g^{-1}\|_k^{s+k/2-2} \begin{cases} r_G^{s-1/2} & \text{if } r_G \geqq 1, \\ r_G^{s}\, e^{-c(r_G)^{-2}} & \text{if } r_G < 1 \end{cases} \tag{3–4}$$

(with M and c positive constants independent of (G, g)). This allows us to show

COROLLARY TO THEOREM 3.1 (WITH THE SAME HYPOTHESES AS IN THEOREM 3.1). *$\Theta^L_\varphi(G, g)$ is a cusp form in the $\widetilde{\mathrm{SL}}_2$ variable.*

EXAMPLE 3–1. An example of a $\tilde{K} \times K$ finite function in $F_Q^+(s^2 - 2s)$:

$$\begin{aligned}\varphi(X) &= 0 \quad \text{on } \Omega_-,\\ &= Q(X)^{s-1}e^{-\pi Q(X)} \left\| X_+ \right\|^{-(s+s_1+s_2+k/2-2)} Q(X, \xi_+)^{s_1}\\ &\quad Q(X, \xi_-)^{s_2} \quad \text{with } X \in \Omega_+,\end{aligned} \tag{3–5}$$

where ξ_+ (ξ_- resp.) is a *nonzero* complex isotropic vector in $\boldsymbol{C}^a$ ($\boldsymbol{C}^b$ resp.). Here isotropic is relative to $\| \ _+\|^2$ on $\boldsymbol{C}^a$($\| \ _-\|^2$ on $\boldsymbol{C}^b$ resp.), and s_1 and s_2 are positive integers satisfying $s_1 - s_2 = s - \frac{1}{2}(a - b)$. Then we put Θ_φ^L into classical coordinates relative to the upper half-plane H by setting

$$\tilde{\Theta}_\varphi^L(z,g) = \Theta_\varphi^L\left(\left(\begin{bmatrix} x & y \\ 0 & x^{-1}\end{bmatrix}, 1\right), g\right)(\operatorname{Im} z)^{s/2}, \tag{3–6}$$

with $z = \sqrt{-1}/x^2 - y/x \in H$. Then

$$\tilde{\Theta}_\varphi^L(z, g) = \sum_{n \in \boldsymbol{Z},\, n \geqq 1} n^{s-1}\, e^{\pi\sqrt{-1}zn} \cdot \varphi_n^{s_1}(g), \tag{3–7}$$

where

$$\varphi_n^{s_1}(g) = \sum_{\{M \in L \mid Q(M,M)=n\}} Q(M, g^{-1}\xi_+)^{s_1} Q(M, g^{-1}\,\xi_-)^{s_2} \left\| (gM)_+ \right\|^{-(s+s_1+s_2+k/2-2)}.$$

Then $\tilde{\Theta}_\varphi^L(z, g)$ is a *holomorphic cusp form* in z of weight s with multiplier v_Q for the arithmetic group $\Delta_{N_L} = \{G \in \mathrm{SL}_2(\boldsymbol{Z}) \mid c_G \equiv 0 \bmod 2N_L,\, b_G \equiv 0 \bmod 2\}$, that is

$$\tilde{\Theta}_\varphi^L\left(\frac{az+b}{cz+d}, g\right) = v_Q(G)\,(cz+d)^s\, \tilde{\Theta}_\varphi^L(z, g), \tag{3–8}$$

where

$$G = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Delta_{N_L}$$

and

$$v_Q(G) = (\bar{\varepsilon}_{d_G})^k \left(\frac{2c_G}{d_G}\right)^k \left(\frac{D_{Q(L)}}{d_G}\right)$$

with $c_G \neq 0$ (where

$$\begin{aligned}\varepsilon_m &= 1 && \text{if } m \equiv 1 \bmod 4,\\ &= \sqrt{-1} && \text{if } m \equiv 3 \bmod 4),\end{aligned}$$

and where $(-)$ is the Legendre symbol.

The expression of $\tilde{\Theta}_\varphi^L$ above is precisely the Fourier expansion of $\tilde{\Theta}_\varphi^L$ in the z variable at $\{\infty\}$; thus $n^{s_1-1}\varphi_n^{s_1}(g)$, the nth Fourier coefficient, is an automorphic form on $O\,(Q)$ relative to the group $O\,(Q)_L$.

4. Constant term of $\tilde{\Theta}_\varphi^L$ in $O(Q)$ variable. The main problem is to determine the Fourier coefficients of $\Theta_\varphi^L(G, g)$ in the $O(Q)$ variable. In general a formal answer can be given, which resembles the Fourier coefficients of Poincaré series, i.e. an infinite sum involving Bessel functions and certain trigonometric sums (similar to Kloosterman sums). Such an expression is unsatisfactory from the arithmetic point of view.

However, the zeroth Fourier coefficients of $\Theta_\varphi^L(\ ,\ )$, the constant term along the different unipotent radicals of rational $O(Q)$ parabolics, can be determined in a rather elementary fashion.

If $\boldsymbol{Q}$ is the set of rational numbers, then let $O(Q)_{\boldsymbol{Q}\cdot L} = \{\gamma \in O(Q) \mid \gamma(\boldsymbol{Q}\cdot L) = \boldsymbol{Q}\cdot L\}$, where $\boldsymbol{Q}\cdot L =$ the Q span of $\{r\cdot\xi \mid r \in \boldsymbol{Q}, \xi \in L\}$; $\boldsymbol{Q}\cdot L$ is a $\boldsymbol{Q}$ vector space of $\boldsymbol{Q}$-dimension k.

We know that every *maximal* parabolic subgroup of $O(Q)$ is $O(Q)$ conjugate to a group of the form p_F, where $p_F = \{g \in O(Q) \mid g(F) = F\}$ with F a nonzero Q isotropic subspace of $\boldsymbol{R}^k$, i.e. $Q(X, X') = 0$ for all $X, X' \in F$. Then there exists (by Witt's Theorem) another Q isotropic subspace F' so that $\dim F = \dim F'$, $F \cap F' = (0)$, and Q is nondegenerate on the direct sum $F + F'$. The Levi decomposition of p_F is given as follows: $p_F = \mathrm{Gl}(F) \times O(Q, (F + F')^\perp)\cdot N_F$, where $\mathrm{Gl}(F)$ is the group of all $\boldsymbol{R}$-linear transformations of F, $O(Q, (F+F')^\perp)$ is the orthogonal group of Q restricted to $(F + F')^\perp$, and N_F, the unipotent radical of p_F. Here $\mathrm{Gl}(F)$ embeds into $O(Q)$ via the map $g(u_1 + u_2 + u_3) = g(u_1) + (g^t)^{-1}(u_2) + u_3$ with $u_1 \in F$, $u_2 \in F'$, and $u_3 \in (F + F')^\perp$.

We say that p_F is *compatible* with the lattice L if there exist F' and $\boldsymbol{Z}$ lattices L_F, $L_{F'}$, and $L_{(F+F')^\perp}$ in F, F', and $(F + F')^\perp$, respectively, so that L is commensurate to $L_F \oplus L_{F'} \oplus L_{(F+F')^\perp}$ in $\boldsymbol{R}^k$. In particular this means that $p_F \cap O(Q)_L$ is an arithmetic subgroup of p_F (relative to the $\boldsymbol{Q}$ structure $\boldsymbol{Q}\cdot L$ on $\boldsymbol{R}^k$).

In order to study the cuspidal behavior of $\Theta_\varphi^L(G, g)$, we must examine integrals of the following form:

$$\text{(4–1)}\qquad \int_{N_F/N_F \cap \gamma O(Q)_L \gamma^{-1}} \Theta_\varphi^L(G, gv\gamma)\, d\sigma(v)$$

with $g \in O(Q)$, $\gamma \in O(Q)_{\boldsymbol{Q}\cdot L}$ and $d\sigma$, some N_F invariant measure on $N_F/N_F \cap \gamma O(Q)_L \gamma^{-1}$. Here we assume that p_F is *compatible with* L. In such a case $N_F/N_F \cap \gamma O(Q)_L\gamma^{-1}$ is compact. We know that $\Theta_\varphi^L(G, g)$ is a *cusp form* on $O(Q)$ relative to the group $O(Q)_L$ if the family of above integrals vanish for all $g \in O(Q)$, all $\gamma \in O(Q)_{\boldsymbol{Q}\cdot L}$, and all p_F *compatible* with L (see [5]).

On the other hand let $\varphi \in \boldsymbol{F}_Q^{\pm}(s^2 - 2s)_{\tilde{K}\times K}$ and define (for $s > \frac{1}{2}k$)

$$\text{(4–2)}\qquad \Phi_\varphi^F(W) = \int_F \varphi(W + U)\, dU,$$

with $W \in (F + F')^\perp$, dU some Euclidean measure on F. Wc know that the affine plane $W + F$ (with $W \in (F + F')^\perp$) is the N_F orbit of the nonzero vector W in $\boldsymbol{R}^k$. So, in particular, Φ_φ^F can be interpreted as the integral of φ over the N_F orbit of W (the orbit here carries an N_F invariant measure which can be identified to dU above). Then we have

LEMMA 4.1 [13]. *Let $s > \frac{1}{2}k$. Then the map $\varphi \rightsquigarrow \Phi_\varphi^F$ is an infinitesimal $\widetilde{\mathrm{SL}}_2 \times O(Q, (F+F')^\perp)$ intertwining map of $\boldsymbol{F}_Q^{\pm}(s^2 - 2s)_{\tilde{K}\times K}$ onto $\boldsymbol{F}_{Q_F}^{\pm}(s^2 - 2s)_{\tilde{K}\times K_F}$, where $K_F = K \cap O(Q, (F + F')^\perp)$ and Q_F is Q restricted to $(F + F')^\perp$. (Infinitesimal means relative to the associated representations of the universal enveloping algebra.)*

REMARK 4.1. If Q_F is positive definite, then $\boldsymbol{F}_{Q_F}^{+}(s^2 - 2s)$ is defined in a similar way as $\boldsymbol{F}_{Q_F}^{\pm}(\ldots)$ (when Q is indefinite).

The most important case of Lemma 4.1 occurs when $\boldsymbol{F}_{Q_F}^{\pm}(s^2 - 2s) = \{0\}$. Such is the case when Q_F has signature $(m, 1)$. Then $\boldsymbol{F}_{Q_F}^{-}(s^2 - 2s) = \{0\}$. Thus

COROLLARY TO LEMMA 4.1. *Let* $(G, g) \in \widetilde{\mathrm{SL}}_2 \times O(Q)$. *Then* $\Phi^F_{\pi_Q((G,g)^{-1})(\varphi)} \equiv 0$ *in the following cases*:

(a) $\varphi \in \boldsymbol{F}_Q^-(s^2 - 2s)_{\tilde{K}\times K}$ *and* $\dim F = b - 1$,

(b) $\varphi \in \boldsymbol{F}_Q^-(s^2 - 2s)_{\tilde{K}\times K}$ *and* $\dim F = b$,

(c) $\varphi \in \boldsymbol{F}_Q^+(s^2 - 2s)_{\tilde{K}\times K}$ *and* $\dim F = b$ (*with* $b = a - 1$ *or* $b = a$),

(d) $\varphi \in \boldsymbol{F}_Q^+(s^2 - 2s)_{\tilde{K}\times K}$ *and* $\dim F = b - 1$ (*with* $b = a$ *or* $a = 2$).

We emphasize here that in the statement of the Corollary to Lemma 4.1, $g \in O(Q)$, although in Lemma 4.1 the map $\varphi \to \Phi_\varphi^F$ is intertwining only for $O(Q, (F + F')^\perp)$. The reason is that $\Phi_\varphi^F(W) \equiv 0$ for all $W \in (F + F')^\perp$ and all $\varphi \in \boldsymbol{F}_Q^+(s^2 - 2s)_{\tilde{K}\times K}$ implies that

$$\Phi^F_{\pi_Q(g)^{-1}\varphi}(W) = \Phi^F_{\pi_Q(k)^{-1}(\varphi)}\big((pW)_{(F+F')^\perp}\big) \equiv 0.$$

(We use the *polar decomposition* of $g = k \cdot p$ with $k \in K, p \in p_F$, and $(pW)_{(F+F')^\perp}$, the projection of pW on $(F + F')^\perp$.)

Thus we can determine the cuspidal behavior of Θ_φ^L completely.

THEOREM 4.2 [13]. *Let* p_F *be a maximal parabolic of* $O(Q)$ *compatible with* L. *Let* $\varphi \in \boldsymbol{F}_Q^+(s^2 - 2s)_{\tilde{K}\times K}$ *with* $s > \frac{1}{2}k$. *Let* $\gamma \in O(Q)_{Q\cdot L}$. *Let* $S_{\gamma,F}(L)$ *be the lattice in* $(F + F')^\perp$ *obtained by projecting* $\gamma(L) \cap (F + (F + F')^\perp)$ *onto* $(F + F')^\perp$ (*relative to the* Q *splitting of* $\boldsymbol{R}^k = F + F' + (F + F')^\perp$). *Then for any* $(G, g) \in \widetilde{\mathrm{SL}}_2 \times O(Q)$

$$\text{(4–3)} \qquad \int_{N_F/N_F \cap \gamma O(Q)_L \gamma^{-1}} \Theta_\varphi^L(G, gv\gamma)\, d\sigma(v) = c_1 \sum_{\xi \in S_{\gamma,F}(L)} \Phi^F_{\pi_Q((G,g)^{-1})(\varphi)}(\xi)$$

where c_1 *is a positive constant depending only on* p_F.

A schematic way to convey the inductive nature of (4–3) is the following commutative diagram:

$$\text{(4–4)} \qquad \begin{array}{ccc}
\pi_Q((G,g)^{-1})(\varphi) & \rightsquigarrow & \Phi^F_{\pi_Q((G,g)^{-1})(\varphi)} = \pi_{Q_F}((G, g_1)^{-1})\big(\Phi^F_{\pi_Q(k_1^{-1})\varphi}\big) \\
\Big\downarrow {\scriptstyle \theta \text{ series relative to } L} & & \Big\downarrow {\scriptstyle \theta \text{ series relative to } S_{\gamma,F}(L)} \\
\Theta_\varphi^L(G, g) & \underset{\text{constant term along } N_F/N_F \cap \gamma O(Q)_L \gamma^{-1}}{\rightsquigarrow} & \Theta^{S_{\gamma,F}(L)}_{\Phi^F_{\pi_Q(k_1^{-1})\varphi}}(G, g_1)
\end{array}$$

Here $g = k_1 \cdot g_1 \cdot n$ with $k_1 \in K$, $g_1 \in O(Q, (F + F')^\perp)$, and $n \in N_F$.

From (4–4) we see that the constant term of $\Theta_\varphi^L(G, g)$ along $N_F/N_F \cap \gamma O(Q)_L \gamma^{-1}$ is a θ series of a function on a space of smaller dimension.

The main idea in the proof of Theorem 4.2 is to substitute (3–2) for Θ_φ^L in (4–1) and reorder the summation and integration to obtain a sum of "N_F-horocyclic" projection maps of φ over equivalence classes of $N_F \cap \gamma O(Q)_L \gamma^{-1}$ in L. Here a horocyclic projection map is the integral of φ over an N_F orbit (carrying an N_F invariant measure) in $\boldsymbol{R}^k$. An example of a horocyclic projection map is given by

(4–2). Essentially there are two types of N_F orbits in $\boldsymbol{R}^k$. The first type of N_F orbit corresponds to a "curved conic section", called a regular horocycle. The projection map of φ corresponding to a *regular* horocycle is zero. (This statement is called the First Cusp Vanishing Theorem in **[13]**.) This follows essentially from the fact that $\hat{\varphi}$ is continuous on $\boldsymbol{R}^k$ and that $\hat{\varphi}$ vanishes on the light cone. The second type of N_F orbit is the affine type discussed above. The contribution of the horocyclic projection maps of φ corresponding to the affine type N_F orbit is then determined by Lemma 4.1.

Then using Corollary to Lemma 4.1 and Theorem 4.2, we have

COROLLARY TO THEOREM 4.2. *Let $\varphi \in \boldsymbol{F}_Q^-(s^2 - 2s)_{\tilde{K}\times K}$ with $s > \frac{1}{2}k$. Then if p_F is compatible with L and* dim $F = b - 1$ *or b, the constant term of $\Theta_\varphi^L(G, g)$ in the direction of $N_F/N_F \cap \gamma O(Q)_L\gamma^{-1}$ ($\gamma \in O(Q)_{Q\cdot L}$) vanishes. In particular if $b = 2$ and $\varphi \in \boldsymbol{F}_Q^-(s^2 - 2s)_{\tilde{K}\times K}$, then $\Theta_\varphi^L(G, g)$ is a cusp form on $O(Q)$ relative to $O(Q)_L$.*

REMARK 4.2. Let $\varphi \in \boldsymbol{F}_Q^+(s^2 - 2s)_{\tilde{K}\times K}$ with $s > \frac{1}{2}k$. Let p_F be compatible with L. If dim $F = b - 1$ (when either $a = 2$ or $a = b$) or dim $F = b$ (when either $b = a - 1$ or $b = a$), then the constant term of $\Theta_\varphi^L(G, g)$ in the direction of $N_F/N_F \cap \gamma O(Q)\gamma^{-1}$ ($\gamma \in O(Q)_{Q\cdot L}$) vanishes. In particular if $a = 2$, $b = 1$ or 2, and $\varphi \in \boldsymbol{F}_Q^+(s^2 - 2s)_{\tilde{K}\times K}$, then $\Theta_\varphi^L(G, g)$ is a cusp form on $O(Q)$ relative to $O(Q)_L$.

The case $b = 2$ is thus the main case of interest. In particular we know that $O(Q)/K_1$, where $K_1 = O(a) \times SO(2)$, is then a Hermitian symmetric space. Let $\mathscr{F} = \mathfrak{k} + p$ be the Cartan decomposition of the Lie algebra of $O(Q)$. Then complexifying $\mathscr{F}$ to $\mathscr{F}_C = \mathscr{F} \otimes_{\boldsymbol{R}} \boldsymbol{C}$, we have the direct sum $\mathscr{F}_C = \mathfrak{k}_C \oplus p^+ \oplus p^-$, where p^+ and p^- span the holomorphic and antiholomorphic tangent vectors at the "origin" in $O(Q)/K_1$. We recall the construction of a family of holomorphic discrete series representations of $O(Q)$. We let $\chi_n : K_1 \to S'$ be the unitary character on K_1 which is trivial on $O(a)$ and maps

$$SO(2) = \left\{\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{bmatrix} \middle| -\pi < \theta \leqq \pi\right\}$$

to $e^{-\sqrt{-1}n\theta}$. Then we form the holomorphically induced representation space

$$\mathscr{H}(O(Q)/K_1, \chi_n) = \Big\{ \varphi : O(Q) \to \boldsymbol{C} \mid \varphi \in C^\infty(O(Q)),\ \varphi(gk) = \varphi(g)\chi_n(k)$$
$$\text{for all } g \in O(Q), k \in K_1,\ \varphi * W \equiv 0 \text{ for all } W \in p^+ \text{ and}$$
$$\int_{O(Q)/K_1} |\varphi(g)|^2\, d\tau(g) < \infty \Big\}$$

with $* W$ convolution on the right and $d\tau$ some $O(Q)$ invariant measure on $O(Q)/K_1$. Then we have

THEOREM 4.3 (**[13]**, **[15]**). *The representation of $O(Q)$ in $\mathscr{A}_s^-$ (see Remark 2.1) is equivalent to the "holomorphic" induced representation of $O(Q)$ in $\mathscr{H}(O(Q)/K_1, \chi_{s_2})_\infty$. where $s_2 = |s| + \frac{1}{2}k - 2$ (($\ \ )_\infty$ denotes C^∞ vectors).*

5. The Shimura-Niwa correspondence. Having determined the cuspidal properties of Θ_φ^L in both variables ($\varphi \in \boldsymbol{F}_Q^\pm(s^2 - 2s)_{\tilde{K}\times K}$ with $s > \frac{1}{2}k$), we now go back to the question of using Θ_φ^L as a kernel function to set up a correspondence between modular forms on the groups involved. In particular we let φ be as in Example 3–1, and consider $\tilde{\Theta}_\varphi^L(z,g)$ (see (3–7)). Then we let f be a *holomorphic cusp form* of weight s $(s > \frac{1}{2}k)$ on H satisfying $f(\gamma \cdot z) = v_Q(\gamma)(c_\gamma z + d_\gamma)^s f(z)$ with $\gamma \in \Delta_{N_L}$. As in §1 we consider the Petersson inner product of f with $\tilde{\Theta}_\varphi^L$:

$$(5\text{–}1) \qquad \langle \tilde{\Theta}_\varphi^L(\ ,g) | f(\ ) \rangle = \int_{\Delta_{N_L} \backslash H} \tilde{\Theta}_\varphi^L(z,g)\overline{f(z)} \,|\operatorname{Im} z|^{s-2}\, dx\, dy.$$

The definition given in (5–1) does not, at first sight, coincide with (1–8), the Petersson inner product $\langle \tilde{\Theta}_{\phi_*}^L(\ ,g) \,|\, f(\ )\rangle$, where $\Theta_{\phi_*}^L$ is the θ series constructed from the Schwartz function $P(X)e^{-\pi[X,X]}$. (In keeping with the notation of §1, we note that if $\eta \in L$, then we drop the subscript η from $\tilde{\Theta}_{\phi,\eta}^L$.) In particular we must determine the relationship of $\tilde{\Theta}_{\phi_*}^L(\ ,\ )$ to $\tilde{\Theta}_\phi^L(\ ,\ )$ defined above. But we know that the map $\phi \to \Theta_\phi^L(\ ,\ )$ is an $\widetilde{\mathrm{SL}}_2 \times O(Q)$ infinitesimal intertwining map from certain $\widetilde{\mathrm{SL}}_2 \times O(Q)$ stable subspaces of $L^2(\boldsymbol{R}^k)$ to the space of C^∞ functions on $\widetilde{\mathrm{SL}}_2 \times O(Q)$. Thus perhaps the appropriate problem to analyze is the following: for a given $\tilde{K} \times K$ finite function ϕ and the projection ϕ_D of ϕ onto the discrete spectrum of $\widetilde{\mathrm{SL}}_2 \times O(Q)$ in $L^2(\boldsymbol{R}^k)$, what is the relation between the θ series Θ_ϕ and Θ_{ϕ_D}. This we now do partially in

LEMMA 5.1 **[16]**. *Let F be a K finite Schwartz function in $L^2(\boldsymbol{R}^k)$ and suppose that $\pi_{\mathrm{m}}(k(\theta,\varepsilon))(F) = e^{\sqrt{-1}s'\theta}\varepsilon^k F$ for all $-\pi < \theta \leqq \pi$ and $\varepsilon = \pm 1$. Let $s' > \frac{1}{2}k+1$. Let $P_{s'}^+$ be the projection of $\boldsymbol{F}_Q$ onto the subspace $\boldsymbol{F}_Q^+(s'^2 - 2s')$. Then*

$$(5\text{–}2) \qquad \langle \tilde{\Theta}_F^L(\ ,g) | f(\ )\rangle = \langle \tilde{\Theta}_{P_{s'}^+(F)}^L(\ ,g) | f(\ )\rangle.$$

REMARK 5.1. If $s' < -(\frac{1}{2}k + 1)$, then a similar statement is valid where P_s^+ is replaced by $P_{s'}^-$, the projection of $\boldsymbol{F}_Q$ onto $\boldsymbol{F}_Q^-(s'^2 + 2s')$.

Thus the two correspondences (1–8) and (5–1) are essentially the same.

The next main problem is to characterize the image of the map $f \rightsquigarrow \langle \tilde{\Theta}_\varphi^L(\ ,g) | f(\ )\rangle$ as f varies in the space $[\Delta_{N_L}, s, v_Q]_0 = \{f : H \to \boldsymbol{C} \,|\, f \text{ holomorphic}, f(\gamma\cdot z) = v_Q(\gamma)\,(c_\gamma z + d_\gamma)^s f(z)$ for all $\gamma \in \Delta_{N_L}$, $z \in H$, and f vanishes at the cusp points of Δ_{N_L} on $\boldsymbol{Q} \cup \{\infty\}\}$, the holomorphic cusp forms of weight s and multiplier v_Q. The first trivial observation is that if we let

$$f(z) = G_n(z) = \sum_{\gamma \in (\Delta_{N_L})_\infty \backslash \Delta_{N_L}} \left(\frac{1}{c_\gamma z + d_\gamma}\right)^s \overline{v_Q(\gamma)}\, e^{\pi\sqrt{-1}n\gamma(z)},$$

the Eisenstein-Poincaré series, then

$$\langle \Theta_\varphi^L(\ ,g) \,|\, G_n(\ )\rangle = c_1 \cdot \overline{\varphi_n^{s_1}(g)},$$

with c_1 a nonzero constant independent of g and n. However since $s > \frac{1}{2}k$, we know that the functions G_n span $[\Delta_{N_L}, s, v_Q]_0$, and hence the space $\{\langle \tilde{\Theta}_\varphi^L(\ ,g) \,|\, f(\ )\rangle \,|\, f \in [\Delta_{N_L}, s, v_Q]_0\}$ is exactly the complex linear span of the $\overline{\varphi_n^{s_1}}$ as $n \geqq 1$.

From Corollary to Theorem 4.2 we know the cases when all $\varphi_n^{s_1}$ $(n \geqq 1)$ are cusp forms on $O(Q)$ relative to $O(Q)_L$. And the most general Fourier coefficient of $\varphi_n^{s_1}$ is difficult to describe arithmetically, since $\varphi_n^{s_1}$ is essentially very much like a Poincaré series. However using the results of §4, it is possible to determine for which $f \in [\Delta_{N_L}, s, v_Q]_0$, $\langle \tilde{\Theta}_\varphi^L(\ ,g) \,|\, f(\ )\rangle$ will be a cusp form on $O(Q)$ relative to $O(Q)_L$.

We let F_j be a *maximal* Q isotropic subspace so that p_{F_j} is compatible with L. Then we consider a flag of Q isotropic subspaces $F_1 \subseteq F_2 \subseteq \cdots \subseteq F_j$ so that each p_{F_i} is compatible with L and F_{i+1}/F_i has dimension 1. Then we know that $O(Q)_{\mathbf{Q}\cdot L}$, the set of rational points in $O(Q)$, has the following decomposition: $O(Q)_{\mathbf{Q}\cdot L} = O(Q)_L \cdot \Xi_{F_i} \cdot (p_{F_i})_{\mathbf{Q}\cdot L}$, where Ξ_{F_i} is some *finite* subset of $O(Q)_{\mathbf{Q}\cdot L}$ (see [2, p. 104]). Then we have the following criterion for $\langle \tilde{\Theta}^L_\varphi(\ ,g) \mid f(\ )\rangle$ to be a cusp form.

THEOREM 5.2. *Let $s > \frac{1}{2}k$. Let φ be given by* (3–5). *Let $f \in [\Delta_{N_L}, s, \nu_Q]_0$. Let Y_{F_i} be the unique $K_{F_i} = K \cap O(Q, (F_i + F_i')^\perp)$ invariant subspace of $F^+_{Q_{F_i}}(s^2 - 2s)$ given by the complex linear span of $\{\Phi^{F_i}_{\pi_Q(k)\varphi} \mid k \in K\}$. Then $\langle \tilde{\Theta}^L_\varphi(\ ,g) \mid f(\ )\rangle$ is a cusp form on $O(Q)$ relative to the group $O(Q)_L$ if and only if*

$$\langle \tilde{\Theta}^{S_{\lambda_{ji}},F_i(L)}_{\psi}(\ ,g_1) \mid f(\ )\rangle \equiv 0 \tag{5–3}$$

for all $\lambda_{ji} \in \Xi^{-1}_{F_i}$, all $\psi \in Y_{F_i}$, and all $g_1 \in O(Q, (F_i + F_i')^\perp)$ as $i = 1, \cdots, j$.

We note that Y_{F_i} is a finite dimensional space, Ξ_{F_i} is a finite set, and that as g varies in $O(Q, (F_i + F_i')^\perp)$, the functions $\tilde{\Theta}^{S_{\lambda_{ji}},F_i(L)}_{\psi}(z, g)$ span a finite dimensional subspace of $[\Delta_{N_L}, s, \nu_Q]_0$. In particular, if Q has signature $(k - 1, 1)$, then we need only the validity of (5–3) for g_1, the identity element of $O(Q, (F + F')^\perp)$ (in this case $O(Q, (F + F')^\perp) \subseteq K_F$). This fact probably explains the orthogonality condition of [1] required to get a lifting from $\widetilde{\mathrm{SL}}_2$ cusp forms to cusp forms on $O(3, 1)$. We note that the condition for $\langle \tilde{\Theta}^L_\varphi(\ ,g) \mid f(\ )\rangle$ to be a cusp form on $O(Q)$ is determined essentially by the fact that f belongs to the kernel of a *finite* family of correspondences (i.e. $f \rightsquigarrow \langle \tilde{\Theta}^{S_{\lambda_{ij}},F_i(L)}_{\psi}(\ ,g_i) \mid f(\ )\rangle$) coming from lower dimensional quadratic forms and lattices.

6. "Zagier polarization identity". We recall that if $b = 2$, then $O(Q)/K_1$ is a Hermitian symmetric space. In fact we can realize $O(Q)/K_1$ as a tube domain. Let $\mathbf{R}^{k-2}$ be the span of $e_2, e_3, \cdots, e_{k-2}, e_k$ (see §1). Thus *Q restricted to $\mathbf{R}^{k-2}$* is of Lorentz type, i.e. Q has *signature* $(k - 3, 1)$ on $\mathbf{R}^{k-2}$. Then let $\mathscr{E}_+ = \{W \in \mathbf{R}^{k-2} \mid Q(W, W) < 0$ and $Q(W, e_k) < 0\}$ (the *future* or *forward* light cone). The tube domain of $\mathscr{E}_+$ is $T(\mathscr{E}_+) = \{X + \sqrt{-1}\, Y \mid X, Y \in \mathbf{R}^{k-2}$ with $Y \in \mathscr{E}_+\} = \mathbf{R}^{k-2} + \sqrt{-1}\mathscr{E}_+$. Then let $O(Q)^\#$ be the subgroup of $O(Q)$ which is the connected component of the full group of *analytic automorphisms* of $T(\mathscr{E}_+)$. If F_1 denotes the Q isotropic line $\{t\cdot(e_1 - e_{k-1})/\sqrt{2}) \mid t \in \mathbf{R}\}$, then $p^\#_{F_1} = p_{F_1} \cap O(Q)^\#$ acts on $T(\mathscr{E}_+)$ as follows: if $q = m_1 \cdot A(r) \cdot n_1 \in p^\#_{F_1}$ with $m_1 \in O(Q, (F_1 + F_1')^\perp)$, $A(r) \in S_{F_1}$, and $n_1 \in N_F$, then $q(Z) = r\{m_1(Z + Y)\}$ where n_1 is determined uniquely by an element $Y \in \mathbf{R}^{k-2} = (F_1 + F_1')^\perp$.

We recall that there exists a *unique holomorphic automorphy factor* $\mathscr{D}$ on $T(\mathscr{E}_+)$ so that $\mathscr{D}(m_1 A(r)n_1, Z) = 1/r$, where $m_1A(r)n_1 \in p^\#_{F_1}$ as given above.

Then, in light of Theorem 4.3, we let $\varphi \in \mathscr{A}^-_s$ (with $b = 2$ and $s > \frac{1}{2}k$) be the unique (up to scalar multiple) "lowest weight" vector of $O(Q)$ relative to K_1. Then φ can be given explicitly as follows:

$$\begin{aligned} \varphi(X) &= 0 && \text{on } \Omega_+, \\ &= Q(X, X)^{s-1} e^{\pi Q(X)} Q(X, \xi_+)^{-s_2} && \text{on } \Omega_-, \end{aligned} \tag{6–1}$$

with $\xi_+ = e_{k-1} + \sqrt{-1}\, e_k$ and $s_2 = s + \frac{1}{2}k - 2$.

We are going to consider a kernel function (constructed from φ) which is slightly twisted from (5–1). We assume that the Q integral lattice L has a Q orthogonal direct sum decomposition in the form $\mathscr{L} \oplus (\boldsymbol{Z}tv \oplus \boldsymbol{Z}\tilde{v})$ with $\mathscr{L} \subseteq \boldsymbol{R}^{k-2}$ (= the $\boldsymbol{R}$ span of $\{e_2, e_3, \cdots, e_{k-2}, e_k\}$), satisfying the condition that $Q(\mathscr{L}) \subseteq 2\cdot\boldsymbol{Z}$ and $n_{\mathscr{L}}\cdot Q(\mu, \mu) \in 2\cdot\boldsymbol{Z}$ for all $\mu \in \mathscr{L}_*(Q)$, the Q dual to $\mathscr{L}$. Moreover v and $\tilde{v}$ span a hyperbolic plane with $Q(v, v) = Q(\tilde{v}, \tilde{v}) = 0$ and $Q(v, \tilde{v}) = 1$. Also assume t is an integer so that $n_{\mathscr{L}} \mid t$ and $4 \mid t$. Thus the exponent n_L of L equals t.

Then we let χ be a Dirichlet character mod t and consider the formal sum

$$\Theta^L_{\varphi,\chi}(G, g) = \sum_{r \bmod t} \chi(r)\Theta^L_{\varphi, rv}(G, g).$$

Then following Example (3–1) we define

$$\begin{aligned}\tilde{\Theta}^L_{\varphi,\chi}(z, g) &= \Theta^L_{\varphi,\chi}\left(\left(\begin{bmatrix} x & y \\ 0 & x^{-1}\end{bmatrix}, 1\right), g\right)(\operatorname{Im} z)^{-|s|/2} \\ &= \sum_{r \leqq -1} \beta_{r,\chi}(g)|r|^{s-1} e^{\pi\sqrt{-1}rz},\end{aligned} \tag{6–2}$$

with $z = -y/x - \sqrt{-1}\, x^2 \in \bar{H}$, the lower half-plane, and

$$\beta_{r,\chi}(g) = \sum_{u \bmod t} \sum_{\{M \in L \mid Q(M+uv, M+uv) = r\}} \chi(u)\, Q(M + u\cdot v, g^{-1}\cdot\xi_+)^{-s_2}.$$

Then using the holomorphic automorphy factor $\mathscr{D}$, we let

$$\tilde{\beta}_{r,\chi}(Z) = [\mathscr{D}(g, \sqrt{-1}\, e_k)]^{s_2}\, \beta_{r,\chi}(g^{-1}) \quad \text{with } Z = g\, \sqrt{-1}\, (\sqrt{2}e_k) \in T(\mathscr{E}_+).$$

Then we immediately have

$$\begin{aligned}\tilde{\beta}_{r,\chi}(Z) = \sum_{u \bmod t} \Bigg\{ &\sum_{\{M \in L \mid Q(M+uv, M+uv) = r\}} \chi(u) \\ &\cdot\left\{-\tfrac{1}{2}Q\left(M, \frac{v}{\sqrt{2}}\right)Q(Z, Z) + Q(M, Z) + Q(M, \sqrt{2}\,\tilde{v}) + \sqrt{2}u\right\}^{-s_2}\Bigg\}.\end{aligned} \tag{6–3}$$

Moreover $\tilde{\beta}_{r,\chi}$ is a holomorphic function on $T(\mathscr{E}_+)$ satisfying $\tilde{\beta}_{r,\chi}(\gamma\cdot Z) = [\mathscr{D}(\gamma, Z)]^{s_2} \cdot \tilde{\beta}_{r,\chi}(Z)$ for all $Z \in T(\mathscr{E}_+)$ and all $\gamma \in O(Q)^\sharp_{L,v} = \{\gamma \in O(Q)^\sharp_L \mid \gamma(v) \equiv v \bmod L\}$.

On the other hand we note that $\tilde{\Theta}^L_{\varphi,\chi}$ is a holomorphic modular *cusp form* in the z variable ($z \in \bar{H}$), i.e.

$$\Theta^L_{\varphi,\chi}(\gamma\cdot z, g) = j(\gamma, z)^{2|s|}\, \lambda^L_{Q,|s|}(d_\gamma)\, \chi(d_\gamma)\, \Theta^L_{\varphi,\chi}(z, g)$$

$$\text{for all } \gamma \in \Gamma_0(t) = \left\{\begin{bmatrix} a & b \\ c & d\end{bmatrix} \in \mathrm{SL}_2(\boldsymbol{Z}) \,\middle|\, c \equiv 0 \bmod t\right\},$$

where

$$j(g, z) = \left(\frac{c_g}{d_g}\right)\bar{\mathscr{E}}_{d_g}(c_g z + d_g)^{1/2}$$

and

$$\lambda^L_{Q,|s|}(m) = \left(\frac{2}{m}\right)^k \left(\frac{D_{Q(L)}}{m}\right)\left(-\frac{1}{m}\right)^{s-k/2},$$

a Dirichlet character mod t.

Then we define (as in (5–1)) the Petersson inner product $\langle \Theta^L_{\varphi,\chi}(\ \ , g), h_f(\ \ )\rangle$, where $h_f(z) = \overline{(f(\bar{z}))}$ with $f \in S_{2|s|}(\Gamma_0(t), \beta) = \{\phi\colon H \to \boldsymbol{C} \mid \text{(i)}\ \phi$ holomorphic on H; (ii) $\phi(\gamma\cdot z) = j(\gamma, z)^{2|s|}\, \beta(d_\gamma)\phi(z)$ for all $\gamma \in \Gamma_0(t)$ and $z \in H$; (iii) ϕ vanishes at cusps of $\Gamma_0(t)$ on $\boldsymbol{Q} \cup \{\infty\}\}$ and $\beta(m) = (\lambda^L_{Q,|s|} \otimes \bar{\chi})(m)(-1/m)^{2|s|}$.

We then define

$$(6\text{–}4) \qquad F_f(Z \mid \varphi, L, v, \chi, \Gamma_0(t)) = \langle \tilde{\Theta}^L_{\varphi,\chi}(z, g^{-1}) \mid h_f(z) \rangle \, [\mathscr{D}(g, \sqrt{-1}\, e_k)]^{s_2}$$

(with $g(\sqrt{-1}\ \sqrt{2}\ e_k) = Z \in T(\mathscr{E}_+)$ and $s_2 = s + \frac{1}{2}k - 2 > k - 2$). Then $F_f(\gamma \cdot Z \mid \varphi, L, v, \chi, \Gamma_0(t)) = \{\mathscr{D}(\gamma, Z)\}^{s_2} F_f(Z \mid \varphi, L, v, \chi, \Gamma_0(t))$ for all $Z \in T(\mathscr{E}_+)$ and all $\gamma \in O(Q)^{\#}_{L,v}$.

The main problem is to give an adequate characterization of the image of this map. The difficult problem is to determine the Fourier coefficients of $\tilde{\beta}_{r,\chi}$. As shown in **[23]**, **[11]**, and **[13]** such Fourier coefficients are highly transcendental, involving an infinite sum of Bessel functions and certain trigonometric sums, which behave like Kloosterman sums.

We know that N_{F_1} ($\cong \boldsymbol{R}^{k-2}$) acts by translation on $T(\mathscr{E}_+)$. Thus there exists a lattice $\mathscr{L}_{F_1}$ in $\boldsymbol{R}^{k-2} = (F_1 + F_1')^{\perp}$ so that if $\xi \in \mathscr{L}_{F_1}$, then

$$F_f(Z + \xi \mid \varphi, L, v, \chi, \Gamma_0(t)) = F_f(Z \mid \varphi, L, v, \chi, \Gamma_0(t)) \quad \text{for all } Z \in T(\mathscr{E}_+).$$

A simple computation shows that $\mathscr{L}_{F_1} = \sqrt{2} \cdot \{\xi \in \mathscr{L} \mid Q(\xi, \xi) \equiv 0 \bmod 2t,\ Q(\mathscr{L}, \xi) \equiv 0 \bmod t\}$. Hence we have an expansion of $F_f(Z \mid \varphi, L, v, \chi, \Gamma_0(t))$ of the form:

$$(6\text{–}5) \qquad F_f(Z \mid \varphi, L, v, \chi, \Gamma_0(t)) = \sum_{\mu \in (\mathscr{L}_{F_1})_*(Q) \cap \mathscr{E}_+} a(\mu, f)\, e^{-2\pi\sqrt{-1} Q(\mu, Z)},$$

where $(\mathscr{L}_{F_1})_*(Q) = \{\mu \in (F_1 + F_1')^{\perp} \mid Q(\mu, \mathscr{L}_{F_1}) \subseteq \boldsymbol{Z}\}$, the Q dual to $\mathscr{L}_{F_1}$ and

$$(6\text{–}6) \quad a(\mu, f) = \int_{\boldsymbol{R}^{k-2}/\mathscr{L}_{F_1}} F_f(X + \sqrt{-1}\ Y \mid \varphi, L, v, \chi, \Gamma_0(t))\, e^{2\pi\sqrt{-1} Q(X + \sqrt{-1} Y, \mu)}\, dX.$$

The problem is then to determine $a(\mu, f)$ in terms of the Fourier coefficients of f at $\{\infty\}$. We note immediately from the Corollary to Theorem 4.2 that $a(\mu, f) \equiv 0$ if $\mu \notin (\mathscr{L}_{F_1})_*(Q) \cap \mathscr{E}_+$.

We begin with a heuristic discussion. First we know that the space $S_{2s}(\Gamma_0(t), \beta)$ admits a reproducing kernel function $K : H \times H \to \boldsymbol{C}$, a holomorphic (antiholomorphic) function in the first (second) variable which satisfies

$$K(\gamma_1(z_1), \gamma_2(z_2)) = j(\gamma_1, z_1)^{2s}\, \overline{j(\gamma_2, z_2)}^{2s}\, \beta(d_{\gamma_1})\, \overline{\beta(d_{\gamma_2})}\, K(z_1, z_2)$$

$$\text{for all } z_1, z_2 \in H \text{ and } \gamma_1, \gamma_2 \in \Gamma_0(t).$$

Moreover K satisfies the formula $\langle f(z), K(z, w) \rangle = f(w)$ ($\langle\ ,\ \rangle$ the Petersson inner product) for all $f \in S_{2s}(\Gamma_0(t), \beta)$. Then we know for s sufficiently large, there is a convergent expansion of $K(z, w) = \sum_{n \geq 1} n^{s-1} e^{-2\pi\sqrt{-1} n \bar{w}} G_n(z)$ where G_n is an Eisenstein-Poincaré series belonging to $S_{2s}(\Gamma_0(t), \beta)$. On the other hand, we know that $\overline{K(z, w)} = K(w, z)$. Thus we have another expansion of

$$K(z, w) = \sum_{n \geq 1} n^{s-1}\, \overline{G_n(w)}\, e^{2\pi\sqrt{-1} n z}.$$

The second expansion can be interpreted as the Fourier expansion of $K(z, w)$ in z at $\{\infty\}$ (i.e. $z \rightsquigarrow K(z, w)$ is an element of $S_{2s}(\Gamma_0(t), \beta)$, where the nth Fourier coefficient $n^{s-1}\overline{G_n(w)}$ is a modular form of antiholomorphic type.

We, however, note the analogy of this second expansion with the Fourier expansion of $\tilde{\Theta}^L_{\varphi}(z, g)$ in z given in (3–6), where each Fourier coefficient is a modular form on $O(Q)$. Thus it is reasonable to ask if there is a "polarization identity" of $\tilde{\Theta}^L_\varphi$ of the form: $\tilde{\Theta}^L_\varphi(z, g) = \sum_{n\geqq 1} n^{s-1}(\varphi^{s_1}_n)^*(g)G_n(z)$, where $(\varphi^{s_1}_n)^* : O(Q) \to \boldsymbol{C}$ is a function which satisfies $(\varphi^{s_1}_n)^*(g\gamma) = (\varphi^{s_1}_n)(g)$ for all $g \in O(Q)$ *and all* $\gamma \in O(Q)_L \cap p_{F_1}$. Also we should require that $\varphi^{s_1}_n$ is related to $(\varphi^{s_1}_n)^*$ by an averaging over $O(Q)_L/O(Q)_L \cap p_{F_1}$, i.e. $\varphi^{s_1}_n(g) = \sum_{\gamma\in O(Q)_L/O(Q)_L\cap p_{F_1}} (\varphi^{s_1}_n)^*(g\gamma)$.

Zagier in **[23]** proved such a "polarization identity" in the case when $a = b = 2$. However such a formula can be proved generally (by Oda in **[11]** for the cases $k \geqq 6$, and in **[16]** a very general version of the formula is shown to be valid without restriction to the case $b = 2$).

THEOREM 6.1 (ZAGIER IDENTITY). *Let* $s > 2k$. *Let* $\Xi^L_{\varphi,\chi}(z, Z) = \tilde{\Theta}^L_{\varphi,\chi}(z, g^{-1})\,[\mathscr{D}(g, \sqrt{-1}e_k)]^{s_2}$ *(see* (6–2)) *with* $Z = g\sqrt{-1}(\sqrt{2}e_k) \in T(\mathscr{E}_+)$. *Then we have the formula*:

$$(6\text{–}7)\qquad \Xi^L_{\varphi,\chi}(z, Z) = \sum_{\{n\in\boldsymbol{Z}\,|\,n\leqq -1\}} |n|^{s-1}\beta^*_{n,\chi}(Z)G_n(z, \chi),$$

where G_n *is the Eisenstein-Poincaré series on* $\bar{H}$, *the lower half-plane, given by*

$$(6\text{–}8)\qquad G_n(z, \chi) = \sum_{\gamma\in\Gamma_0(t)_\infty\backslash\Gamma_0(t)} \left(\frac{1}{c_\gamma z + d_\gamma}\right)^s \overline{s^L_{Q,\chi}(\gamma)}\, e^{2\pi\sqrt{-1}n\gamma(z)},$$

with

$$s^L_{Q,\chi}(\gamma) = \left(\bar{\varepsilon}_{d_\gamma}\right)^k\left(\frac{2c_\gamma}{d_\gamma}\right)^k\left(\frac{D_{Q(L)}}{d_\gamma}\right)\chi(d_\gamma)$$

(*note here that*

$$\tilde{\Theta}^L_{\varphi,\chi}(\gamma\cdot z, g) = s^L_{Q,\chi}(\gamma)(c_\gamma z + d_\gamma)^s\,\tilde{\Theta}^L_{\varphi,\chi}(z, g) \quad \textit{for } \gamma \in \Gamma_0(t))$$

and

$$(6\text{–}9)\qquad \beta^*_{n,\chi}(Z) = \sum_{\nu \bmod t} \chi(\nu)\left\{\sum_{\{\xi\in\mathscr{L}\,|\,Q(\xi,\,\xi)=n\};\,j\in\boldsymbol{Z}} \{Q(\xi, Z) + \sqrt{2}[jt - \nu]\}^{-s_2}\right\}.$$

REMARK 6.1. The function $\beta^*_{n,\chi}$ is obtained from $\beta_{n,\chi}$ by restriction of the summation in (6–3) to precisely those lattice points of the form $\{\xi + tj\nu\,|\,\xi \in \mathscr{L}$ with $Q(\xi, \xi) = n$, and $j \in \boldsymbol{Z}\}$. This latter set (*singular horocyclic* lattice points) is stable under $O(Q)_L \cap p_{F_1}$.

Then using Theorem 6.1, it is possible to determine $a(\mu, f)$ (see (6–6)). The main idea is to apply the Lipschitz identity to the inner sum of the right-hand side of (6–9), i.e.

$$\sum_{m\in\boldsymbol{Z}}\left(\frac{1}{w + m}\right)^{s_2} = \frac{(2\pi\sqrt{-1})^{s_2}}{(s_2 - 1)!}\sum_{r=1}^{r=+\infty} r^{s_2-1}\, e^{2\pi\sqrt{-1}wr} \quad \text{with } w \in H.$$

Then we deduce

COROLLARY TO THEOREM 6.1. *Let* $s > 2k$. *Let* $f \in S_{2s}(\Gamma_0(t), \sigma)$ *with* σ, *a Dirichlet character* mod t. *Then let* $f(z) = \sum_{n\geqq 1} a_f(n)\, e^{2\pi\sqrt{-1}nz}$ *be the Fourier expansion of* f *at* $\{\infty\}$.

Assume $\mu \in (\mathscr{L}_{F_1})_*(Q) \cap \mathscr{E}_+$ *and* $\mu \notin \mathscr{L}/(\sqrt{2}\cdot t)$. *Then* $a(\mu, f) = 0$.

On the other hand, let $\mu \in (\mathscr{L}_{F_1})_*(Q) \cap \mathscr{E}_+$ *with* $\mu = (m/(\sqrt{2}\cdot t))\cdot\xi$ *with* ξ, *a primitive element of the lattice* $\mathscr{L}$ (*m, a positive integer*). *Then*

$$a(\mu, f) = c_1' G(\chi_\sigma, t) t^{-s_2} \cdot \sum_{\{\nu \mid \nu \mid m\}} \overline{\chi_\sigma(\nu)} \nu^{s_2-1} a_f\left(\frac{m^2}{\nu^2} \mid Q(\xi, \xi) \mid\right), \tag{6–10}$$

with χ_σ *the Dirichlet character given by*

$$\chi_\sigma(x) = \overline{\sigma(x)}(-1/x)^{2s} \lambda^L_{Q,s}(x)$$

and $G(\chi_\sigma, t)$, *the Gauss sum given by*

$$G(\chi_\sigma, t) = \sum_{\nu \bmod t} \chi_\sigma(\nu) e^{2\pi\sqrt{-1}\nu/t}$$

(*here* c_1' *is a nonzero constant depending only on* s).

Then following the well-known methods in automorphic form theory, it is possible to associate a Dirichlet series to the automorphic function $F_f(\quad | \varphi, L, \nu, \chi_\sigma, \Gamma_0(t))$. The cases $k = 3, 4$ have been studied extensively in **[10]** and **[6]**, so in the ensuing discussion we assume that $k \geqq 5$. In particular we let

$$R(\mathfrak{s}, f) = \sum_{\Omega \in \{(\mathscr{L}_{F_1})_*(Q) \cap \mathscr{E}_+\}} a(\Omega, f) \frac{1}{\varepsilon(\Omega)} \mid Q(\Omega, \Omega) \mid^{-\mathfrak{s}}, \tag{6–11}$$

where $\{(\mathscr{L}_{F_1})_*(Q) \cap \mathscr{E}_+\}$ denotes the *set of equivalence classes* of $O(Q, F_1 + F_1')^\# \cap O(Q)_L$ in $(\mathscr{L}_{F_1})_*(Q) \cap \mathscr{E}_+$ and $\mathscr{E}(\Omega)$ = the order of the finite subgroup of $O(Q, F_1 + F_1')^\# \cap O(Q)_L$ which fixes Ω.

Then from (6–10) we deduce immediately

PROPOSITION 6.2 **[16]** (WITH THE SAME HYPOTHESES AS IN THEOREM 6.1 AND $k \geqq 5$). *Let* $\mathfrak{s} \in \boldsymbol{C}$ *so that* $\mathrm{Re}(\mathfrak{s})$ *is sufficiently large. Then we have the identity*:

$$R(\mathfrak{s}, f) = d_1 G(\chi_\sigma, t) t^{2\mathfrak{s}-s_2} 2^{\mathfrak{s}} L(\bar{\chi}_\sigma, 2\mathfrak{s} + 1 - s_2) \sum_{\{n \in \boldsymbol{Z} \mid n \leqq -1\}} a_f(-n) M(Q_1, \mathscr{L}, n) |n|^{-\mathfrak{s}}, \tag{6–12}$$

where $M(Q_1, \mathscr{L}, n)$ *is the Siegel mass number of the form* Q_1 (= Q *restricted to* $F_1 + F_1'$) *relative to the lattice* $\mathscr{L}$ *on the quadric of level* n (*i.e.* $M(Q_1, \mathscr{L}, n) = \sum_{i=1}^{h(n)} \varepsilon(\xi_i)^{-1}$, *where* $\xi_1, \cdots, \xi_{h(n)}$ *run through a set of representatives of* $O(Q, F_1 + F_1')^\# \cap O(Q)_L$ *orbits in* $\mathscr{L} \cap \{X \in \boldsymbol{R}^{k-2} | Q(X, X) = n\}$) *and* d_1 *a nonzero constant dependent only on* s_2 (*recall here that* $s_2 = s + \frac{1}{2}k - 1$). *Also* $L(\delta, \mathfrak{s})$ *is the classical* L *function associated to the Dirichlet character* δ.

Thus we have expressed $R(\mathfrak{s}, f)$ as the product of elementary functions (i.e. $a^{\mathfrak{s}}$), an L function, and the Rankin convolution of 2 Dirichlet series (i.e. the Dirichlet series $D(\mathfrak{s}, f) = \sum_{n \geqq 1} a_f(n) n^{-\mathfrak{s}}$ and Siegel's zeta function $\zeta_-(Q_1, \mathscr{L}, \mathfrak{s}) = \sum_{\{n \in \boldsymbol{Z} \mid n \leqq -1\}} M(Q_1, \mathscr{L}, n) |n|^{-\mathfrak{s}}$).

The analytic nature of the function $R(\mathfrak{s}, f)$ can then be determined easily from Proposition 6.1. If we let $R^*(\mathfrak{s}, f) = \{\pi^{-2\mathfrak{s}} \Gamma(\mathfrak{s} - \frac{1}{2}k + 2) \Gamma(\mathfrak{s}) R(\mathfrak{s}, f)\}$, then $R^*(\mathfrak{s}, f)$ can be analytically continued to the whole $\mathfrak{s}$ plane (Γ, the gamma function).

REMARK 6–2. Using (6–12), it is possible to deduce a type of Euler product expansion of $R(\mathfrak{s}, f)$. It suffices to study the possible Euler product properties of the Rankin convolution of $D(\mathfrak{s}, f)$ and $\zeta_-(Q_1, \mathscr{L}, \mathfrak{s})$. However if k is even and both $D(\mathfrak{s}, f)$ and $\zeta_-(Q_1, \mathscr{L}, \mathfrak{s})$ admit the usual Euler product of the GL_2 theory, then the Rankin convolution of these series can be expressed as an Euler product with numerator of degree 2 and denominator of degree 4 for almost all primes p. (For suitable choice of Q_1 and $\mathscr{L}$, $\zeta_-(Q_1, \mathscr{L}, \mathfrak{s})$ is a finite sum of Euler products of the GL_2 theory, see [7].) On the other hand, if k is odd then $D(\mathfrak{s}, f)$ and $\zeta_-(Q_1, \mathscr{L}, \mathfrak{s})$ do not have the usual GL_2 type Euler product. However in **[18]** a modified theory of Euler products is set forth for $\widetilde{SL}_2$ automorphic forms of semi-integral weight. In particular if $f \in S_{2s}(\Gamma_0(t), \beta)$ is a Hecke eigenfunction in the sense of **[18]**, then the partial Dirichlet series $\sum_{n\geqq 1} a_f(|d|n^2)n^{-\mathfrak{s}}$ (d, the discriminant of an imaginary quadratic extension of $\boldsymbol{Q}$) can be expressed as an Euler product with numerator of degree 1 and denominator of degree 2 for almost all primes p. And it is possible for suitable $\mathscr{L}$ to find a similar Euler product for $\sum_{n\geqq 1} M(Q_1, \mathscr{L}, dn^2)n^{-\mathfrak{s}}$. Then by purely algebraic methods, one can show that the Rankin convolution $\sum_{n\geqq 1} a_f(|d|n^2) \cdot M(Q_1, \mathscr{L}, dn^2)n^{-\mathfrak{s}}$ is an Euler product with numerator of degree 3 and denominator of degree 4 for almost all primes p (see **[16]** for the case $k = 5$ where $O(3, 2)$ is locally isomorphic to $Sp_2(\boldsymbol{R})$).

Notes. For a complete bibliography on "hyperboloid analysis", we refer the reader to **[21]**. Also for an adelic treatment of the Weil representation and automorphic forms, see **[4]**.

We use the terminology that a function ϕ vanishes at a cusp $\gamma(\infty) = a$ if $(\phi\,|\,\gamma)(z) = (cz + d)^{-s}\,\phi(\gamma(z))$ has an expansion of the form

$$\sum_{n\geqq 0} c_n e^{2\pi\sqrt{-1}((n+\kappa)/N)z} \quad \text{with } c_0 = 0 \text{ when } \kappa = 0$$

(here κ is the ramification of the multiplier at a and N is the smallest positive integer so that

$$\gamma\begin{bmatrix}1 & N\\ 0 & 1\end{bmatrix}\gamma^{-1} \in \Gamma_1,$$

Γ_1 the arithmetic group in question) (see **[18]**).

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PRINCETON UNIVERSITY

Current address: Ohio State University

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A COUNTEREXAMPLE TO THE "GENERALIZED RAMANUJAN CONJECTURE" FOR (QUASI-) SPLIT GROUPS

R. HOWE AND I. I. PIATETSKI-SHAPIRO

1. Introduction. In **[Sat]**, Satake explains how the Ramanujan and Ramanujan-Peterson conjectures concerning the coefficients of cuspidal modular forms can be formulated group theoretically. Briefly, the interpretation is that the local constituents (see **[F]**) of the automorphic representation associated to a classical cusp form should be tempered in the sense of Harish-Chandra **[KZ]**. (See also **[GGP]**.) Temperedness is a technical condition on the asymptotic behavior of matrix coefficients that has proved crucial in Harish-Chandra's work on the Plancherel formula. Since it makes sense for any reductive group, the question as to whether the cuspidal automorphic representations of any global algebraic group might not be tempered came to be known as the generalized Ramanujan conjecture. Examples of automorphic cusp forms whose component at ∞ was nontempered have been constructed by several authors (see **[Ga]**, **[HW]**, **[M]**, and the remarks in **[B]**). However these examples were for anisotropic groups, and the question remained open for quasi-split groups. Here we will construct examples of cusp forms for Sp_4 which are nontempered almost everywhere.[1]

Our construction uses the oscillator representation. Regarding Sp_4 as one member of the dual reductive pair (Sp_4, O_2) (see **[H]**) yields an injection of the automorphic forms on O_2 into the automorphic forms on Sp_4. This construction is parallel to those given by Hecke, Maass, and Shalika-Tanaka for cusp forms on $\mathrm{Sp}_2 = \mathrm{Sl}_2$ corresponding to grössencharaktere of quadratic fields. Indeed these are automorphic forms on Sl_2 corresponding to the dual pair (Sp_2, O_2). We remark that although every automorphic representation of O_2 gives rise to a representation of Sp_4, not all such representations of O_2 are involved in the correspondence with Sp_2. It turns out that it is precisely the O_2 representations which were missing in the correspondence with Sp_2 that give rise to cuspidal representations of Sp_4. Locally, at a place where the binary quadratic form is anisotropic, there is precisely one representation of O_2 which is missing from the pairing with Sp_2, namely the nontrivial character which is trivial on SO_2. Thus the nonconnectedness of O_2 plays an im-

AMS (*MOS*) *subject classifications* (1970). Primary 22E55.

[1]Our first examples were for $U_{2,1}$; however the Sp_4 example is easier to present, and in some sense is better, since Sp_4 is not merely quasi-split but split.

portant role in this construction. The local (Sp_4, O_2) correspondence was analyzed by C. Asmuth in his thesis **[A]**.

The cusp forms that we construct are peculiar in some respects. In particular, they fail to have Whittaker models **[P 1]**. Thus the possibility remains open that the generalized Ramanujan conjecture is true for cuspidal automorphic representations with Whittaker models. Probably the best known example of a cuspidal representation without Whittaker model is the representation θ_{10} of Sp_4 over a finite field, discovered by Srinivasan **[Sr]**. The cusp forms constructed here are closely related to θ_{10}. Indeed, let k be a non-Archimedean local field with integers $\mathcal{O}$ and residue class field F. Then θ_{10} is a representation of $Sp_4(F)$ and may be pulled back to a representation of $Sp_4(\mathcal{O})$ via the canonical surjection

$$Sp_4(\mathcal{O}) \longrightarrow Sp_4(F).$$

It is not hard to see by means of the Cartan decomposition that the induced representation

$$\operatorname{ind} \theta_{10} = \operatorname{ind}_{Sp_4(\mathcal{O})}^{Sp_4(k)} \theta_{10}$$

will be irreducible and supercuspidal. It is essentially this ind θ_{10} which will be the local constituent at certain finite places of the automorphic representations we construct. We note that θ_{10} is one of the simplest cuspidal representations of the type Lusztig **[L]** calls unipotent. The examples for $U_{2,1}$ also involve a cuspidal unipotent representation of that group. This suggests that the failure of the generalized Ramanujan conjecture is related to the existence of these cuspidal unipotent representations. Lusztig has classified these representations for the classical groups over finite fields and has shown that although they are rare, they exist for groups of arbitrarily high rank.

The question of computing L-functions attached to the automorphic forms we construct is of some interest. Using the definition of L-functions for GSp_4 in **[P 2]**, **[NPS]** we made some preliminary computations. The feeling is that the L-function for the local cuspidal representation which comes from the oscillator representation has a pole. This is consistent with the expectation that after an appropriate lifting our cuspidal nontempered automorphic representation corresponds to a noncuspidal automorphic form. In this way perhaps our example can be shown to fit consistently in the pattern of Langlands' philosophy. Detailed computations will be given elsewhere.

2. The construction. Let k be a global field (an A-field in the terminology of Weil **[W 2]**). We take char $k \neq 2$. Let $Sp_4(k)$ be the set of matrices, with entries in k, of the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A, B, C, and D are 2×2 matrices satisfying

$$A^tD - C^tD = I, \quad A^tC = C^tA, \quad B^tD = D^tB, \tag{2.1}$$

where I is the 2×2 identity matrix and A^t is the usual transpose of A.

Let K be a quadratic extension of k. The norm form from K to k is a quadratic form, labeled β, on K regarded as a 2-dimensional vector space over k. Denote the isometry group of this form by $O_2(k)$. Let τ denote the nontrivial element of the Galois group Gal of K over k. Then τ is in $O_2(k)$, and $\beta(x, x) = x\tau(x)$ for x in K. Set

$$U = \{x \in K: \beta(x, x) = 1\}. \tag{2.2}$$

Evidently U is a subgroup of $K^\times$, the multiplicative group of K, and so acts on K by multiplication. This action identifies U with SO_2, and we have an exact sequence

$$1 \longrightarrow U \longrightarrow O_2 \longrightarrow \text{Gal} \longrightarrow 1. \tag{2.3}$$

Let $\boldsymbol{A}$ be the adèle ring of k. Let v be a typical place of k, and k_v the completion of k at v. Then $\text{Sp}_4(\boldsymbol{A})$ is the group of matrices with entries in $\boldsymbol{A}$ satisfying (2.1), $K_v = K \otimes_k k_v$ is the completion of K at v, and β_v is the form induced by β on K_v, and so forth.

Following Weil **[W 1]** and Saito **[So]** we know there is a unitary representation ω of $\text{Sp}_4(\boldsymbol{A})$ on $L^2(\boldsymbol{A}^2 \otimes_k K) \simeq L^2(\boldsymbol{A}^2 \otimes_{\boldsymbol{A}} K_{\boldsymbol{A}}) \simeq L^2(K_{\boldsymbol{A}}^2)$. From these authors' work we can glean that this representation will have the following properties.

(i) ω preserves the Schwartz-Bruhat space $\mathscr{S}(K_{\boldsymbol{A}}^2)$. Denote by ω^∞ the restriction of ω to $\mathscr{S}$.

(ii) Just as $\mathscr{S}(K_{\boldsymbol{A}}^2)$ is the (restricted) tensor product of the local Schwartz-Bruhat spaces $\mathscr{S}(K_v^2)$, so also ω^∞ is the restricted tensor product of representations ω_v^∞ of $\text{Sp}_4(k_v)$ acting on $\mathscr{S}(K_v^2)$, and the ω_v^∞ are restrictions of unitary representations ω_v on $L^2(K_v^2)$.

(iii) The action of the subgroup of Sp_4 of elements of the form

$$\begin{pmatrix} A & 0 \\ 0 & A^{t-1} \end{pmatrix}$$

is essentially induced by the linear action on $\boldsymbol{A}^2$, modified to provide unitarity. Precisely, choose a basis e_1, e_2 for K over k. Then for $f \in \mathscr{S}(\boldsymbol{A}^2 \otimes_{\boldsymbol{A}} K_{\boldsymbol{A}})$, we have

$$\begin{aligned} \omega\begin{pmatrix} A & 0 \\ 0 & A^{t-1} \end{pmatrix}(f)(v_1 \otimes e_1 + v_2 \otimes e_2) \\ = s(A)|\det A|^{-1} f(A^{-1}(v_1) \otimes e_1 + A^{-1}(v_2) \otimes e_2). \end{aligned} \tag{2.4}$$

In (2.4), $|\ \ |$ is the standard "module" (cf. **[W 2]**) or absolute value on $\boldsymbol{A}$, v_1, v_2 are in $\boldsymbol{A}^2$, and $s(A)$ is a character on $\text{GL}_2(\boldsymbol{A})$ of order 2. We note that (2.4) breaks up in an obvious way into local actions.

(iv) The action ρ of $O_2(\boldsymbol{A})$ on $\mathscr{S}(K_{\boldsymbol{A}}^2)$ induced by the linear action of $O_2(\boldsymbol{A})$ on $K_{\boldsymbol{A}}$ commutes with $\omega(\text{Sp}_4(\boldsymbol{A}))$. Note this action of $O_2(\boldsymbol{A})$ also breaks up into a product of actions of local $O_2(K_v)$'s.

(v) For $f \in \mathscr{S}(K_{\boldsymbol{A}}^2)$ define

$$\Theta(f) = \sum_{x \in K^2} f(x). \tag{2.5}$$

Then Θ is a distribution on $\mathscr{S}(K^2)$, and Θ is invariant by $\omega^\infty(\text{Sp}_4(k))$. That is, for g in $\text{Sp}_4(k)$ and $f \in \mathscr{S}(K_{\boldsymbol{A}}^2)$, we have

$$\Theta(\omega(g)(f)) = \Theta(f). \tag{2.6}$$

It is obvious that Θ is also invariant by $\rho(O_2(k))$. Therefore, if we define, for f in $\mathscr{S}(K_A^2)$, a function θ_f on $\mathrm{Sp}_4(A) \times O_2(A)$ by the formula

$$\theta_f(g, h) = \Theta(\omega(g)\rho(h)f) \tag{2.7}$$

then θ_f factors to a function on the quotient space

$$\mathrm{Sp}_4(k) \times O_2(k)\backslash \mathrm{Sp}_4(A) \times O_2(A).$$

Furthermore, θ_f will have moderate growth at ∞, so that θ_f is an automorphic form on $\mathrm{Sp}_4(A) \times O_2(A)$.

Our strategy now is the usual one in the theory of θ-series. The coset space $O_2(k)\backslash O_2(A)$ is compact. We will see shortly that for an appropriate choice of f, the restriction of θ_f to $O_2(A)$ is an arbitrary Schwartz-Bruhat function on $O_2(k)\backslash O_2(A)$. Thus in particular, we can arrange that θ_f will transform under some given irreducible representation σ of $O_2(A)$. From a study of the local representations ω_v, we can then conclude that if σ_v is chosen properly at certain places v, then θ_f will transform under $\mathrm{Sp}_4(k_v)$ according to a certain supercuspidal representation.[2] It follows that $\theta_f|\mathrm{Sp}_4(A)$ will be a cusp form. Further examination will show that at almost all places v, the cusp form θ_f will transform according to a certain irreducible nontempered representation σ_v' of $\mathrm{Sp}_4(k_v)$, determined by σ_v. Therefore, any irreducible component in the space of cusp forms on $\mathrm{Sp}_4(A)$ generated by θ_f will transform at almost all places by these same σ_v' and our goal will be achieved. We proceed to details.

PROPOSITION 1. *The map $f \to \theta_f|O_2(A)$ is surjective from $\mathscr{S}(K_A^2)$ to $\mathscr{S}(O_2(k)\backslash O_2(A))$.*

PROOF. Let (x_1, x_2) be a point in K^2 such that x_1 and x_2 span K over k. The map

$$e\colon g \longrightarrow (gx_1, gx_2)$$

of $O_2(A)$ into K_A^2 is a smooth injection. It also has closed image. This can be seen in various ways, but perhaps most easily by noting that the image of $O_2(k)$ consists of rational points and is therefore discrete, and $O_2(k)\backslash O_2(A)$ is compact. Since the image of e is closed, the pull-back

$$e^*\colon C_c^\infty(K_A^2) \longrightarrow C^\infty(O_2(A))$$

is surjective onto $C_c^\infty(O_2(A))$. Let $f \in C_c^\infty(K_A^2)$ vanish at all points of K^2 not in the image of e. Then by Witt's Theorem [J] it is easy to compute that

$$\theta_f(g) = \sum_{h \in O_2(k)} e^*(f)(hg). \tag{2.8}$$

Since it is well known that all f in $C^\infty(O_2(k)\backslash O_2(A))$ are of the form given on the right-hand side of (2.8), the proposition follows.

REMARK. The same result with the same proof holds for an arbitrary irreducible type I dual reductive pair (G, G') (see [H]) when the space with form attached to G has an isotropic subspace of dimension at least equal to the dimension of the formed space attached to G'.

[2]Actually it can be shown that θ_f will transform under $\mathrm{Sp}_4(A)$ according to an irreducible representation σ' determined by σ, but this finer result is unnecessary for the present purpose and would unduly lengthen the paper.

If K_v is a field, i.e., if K does not split at v, then β_v will be anisotropic and $O_2(k_v)$ will be compact. Therefore $\mathscr{S}(K_v^2)$ will break up into a discrete direct sum of spaces transforming under $O_2(k_v)$ according to its (one or two dimensional) irreducible representations. Asmuth has established the following facts **[A]**. (See also **[H]**.)

PROPOSITION 2. *If K does not split over k at v, then there is an orthogonal direct sum decomposition*

$$\mathscr{S}(K_v^2) \simeq \sum_{\sigma} \mathscr{S}(K_v^2, \sigma)$$

over all irreducible representations of $O_2(k_v)$. Each space $\mathscr{S}(K_v^2, \sigma)$ is irreducible under the joint action of $O_2(k_v)$ and $\mathrm{Sp}_4(k_v)$ (which acts by the restriction of ω_v to $\mathscr{S}(K_v^2, \sigma)$) and thus has the form $\sigma \otimes \sigma'$ as $O_2 \times \mathrm{Sp}_4$ module. The mapping $\sigma \to \sigma'$ defines an injection of the representations of $O_2(k_v)$ into the (irreducible admissible unitarizable) representations of $\mathrm{Sp}_4(k_v)$. If σ is the nontrivial 1-dimensional representation of O_2 which is trivial on SO_2, (the signum representation) then σ' is supercuspidal.

If σ is any representation of O_2 other than the signum, the corresponding σ' is not even tempered **[KZ]**. We will show this is true when σ' is the trivial representation, which will suffice for the present purpose. The point is that there are functions $f, f' \in \mathscr{S}(K_v^2)$ which are invariant under $O_2(k_v)$ and which are positive and do not vanish at zero. We compute the matrix coefficient

$$\left(f, \omega_v\begin{pmatrix} r & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & r^{-1} & 0 \\ 0 & 0 & 0 & t^{-1}\end{pmatrix}(f')\right) \tag{2.9}$$
$$= \pm\, |rt|_v^{-1} \int_{K_v} f(x_1, x_2)\overline{f'(r^{-1}x_1, t^{-1}x_2)}\, dx_1\, dx_2.$$

Since f has compact support, we see that for r and t sufficiently large, (2.9) becomes

$$\pm\, |rt|^{-1}\overline{f'(0,0)} \int_{K_v} f(x_1, x_2)\, dx_1\, dx_2.$$

Since the δ-function of Sp_4 is $|r^4t^2|_v$, we see the matrix coefficients (2.9) decay too slowly at ∞ for σ' to be tempered.

If K_v is not a field, then β_v is a hyperbolic plane whose isotropic lines are the two places of K lying above v. The joint action of O_2 and Sp_4 on $\mathscr{S}(K_v^2)$ can be analyzed in a manner directly analogous to Proposition 2, but the work is slightly longer since the decomposition is continuous rather than discrete. Since we cannot refer to the literature, we prove only what we need. We consider only non-Archimedean places. Let C and C' be the subgroups of integer matrices in $O_2(k_v)$ and $\mathrm{Sp}_4(k_v)$. Let $\mathscr{S}(K_v^2)^{C'}$ be the C'-fixed vectors in $\mathscr{S}(K_v^2)$.

PROPOSITION 3. *There is a linear isomorphism*

$$\alpha\colon C_c^\infty(O_2(k_v)/C) \longrightarrow \mathscr{S}(K_2^v)^{C'}.$$

The map α is an $O_2(k_v)$ intertwining map.

PROOF. By performing a partial Fourier transform (see **[RS]** for the analogy in the (O_2, Sl_2) pair) we find ω_v^∞ is equivalent to the representation of $\mathrm{Sp}_4(k_v)$ on $\mathscr{S}(k_v^4)$ induced by the natural linear action of $\mathrm{Sp}_4(k_v)$ on k_v^4. In this realization of ω_v^∞, the action ρ_v^∞ of $\mathrm{SO}_2(k_v)$ becomes the (scalar) dilations of k_v^4 normalized for unitarity. To get O_2, you add Fourier transform on k_v^4 with respect to the symplectic form

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

of which $\mathrm{Sp}_4(k_v)$ is the isometry group.

It is easy to check that the orbits of C' in k_v^4 are the sets $\pi^k(\theta_v^4 - \pi\theta_v^4)$, together with 0. From this one sees that $\mathscr{S}(k_v^4)^{C'}$ has a basis composed of the functions ϕ_k, where ϕ_k is the characteristic function of $\pi^k\theta_v^4$. All the ϕ_k are invariant by dilations of K_v^4 by elements of $\mathcal{O}_v^\times$. Assuming the Fourier transform $\wedge$ on k_v^4 to be normalized properly, as we may, it will be seen that

$$\hat{\phi}_k = |\pi|_v^{4k}\,\phi_{-k}. \tag{2.10}$$

In particular ϕ_0 is invariant under $\mathcal{O}_v^\times$ and under $\wedge$, that is, under C. Define a map

$$\alpha\colon C_c^\infty(O_2(k_v)/C) \longrightarrow \mathscr{S}(k_v^4)^{C'}$$

by

$$\alpha(f) = \rho_v(f)(\phi_0) \quad \text{for } f \in C^\infty(O_2(k_v)/C).$$

Since the dilations by powers of π are a set of coset representatives for $\rho_v(O_2(k_v))/\rho_v(C)$, it is easy to see that α is a linear isomorphism. It is also transparently an $O_2(k_v)$ intertwining map, so the proposition follows.

We will say an irreducible admissible representation of $\mathrm{Sp}_4(k_v)$ is *C'-spherical* if it contains a C'-fixed vector. Similar terminology applies to $O_2(k_v)$.

PROPOSITION 4. *Suppose σ' is an (irreducible admissible) C'-spherical representation of $\mathrm{Sp}_4(k_v)$. Let $Y_{\sigma'}$ be the kernel of all $\mathrm{Sp}_4(k_v)$ intertwining operators from $\mathscr{S}(K_v^2)$ to the space of σ'. Then the joint action of $O_2(k_v) \times \mathrm{Sp}_4(k_v)$ on $Q_{\sigma'} = \mathscr{S}(K_v^2)/Y_{\sigma'}$ is irreducible, and of the form $\sigma \otimes \sigma'$, where σ is an irreducible C-spherical representation of $O_2(k_v)$. The correspondence $\sigma \to \sigma'$ is an injection from the set of all C-spherical representations of $O_2(k_v)$ into the set of C'-spherical representations of $\mathrm{Sp}_4(k_v)$.*

PROOF. Since C' is compact, the natural projection map $\mathscr{S}(K_v^2)^{C'} \to Q_{\sigma'}^{C'}$ is surjective. The space $Q_{\sigma'}^{C'}$ will be invariant by $O_2(k_v)$, and since there is precisely one C' fixed vector in σ' [C], $Q_{\sigma'}$ will be irreducible under $O_2 \times \mathrm{Sp}_4$ if and only if $Q_{\sigma'}^{C'}$ is irreducible under O_2. According to Proposition 3, there is a surjective O_2 intertwining map

$$\alpha'\colon C_c^\infty(O_2/C) \longrightarrow Q_{\sigma'}^{C'}.$$

Therefore the irreducible O_2 representations in $Q_{\sigma'}^{C'}$ will correspond to the C-fixed vectors in $Q_{\sigma'}^{C'}$. So to show irreducibility of Q_σ under $O_2 \times \mathrm{Sp}_4$, it will suffice to show that $Q_\sigma^{C'\times C}$ is one-dimensional. To this end, consider the action of the Hecke algebras [C] $C_c^\infty(O_2//C)$ and $C_c^\infty(\mathrm{Sp}_4//C')$ on $\mathscr{S}(k_v^4)^{C\times C'}$, as described in Proposition 3. (We will pass freely between $\mathscr{S}(k_v^4)$ and $\mathscr{S}(K_v^2)$.) If the ϕ_k are as in (2.10), then it is easy to see that the set of functions $q^{2k}\phi_k + q^{-2k}\phi_{-k}$ where $q = |\pi|_v^{-1}$ form a basis for $\mathscr{S}(k_v^4)^{C \times C'}$. Let T_1 be the Hecke operator on O_2 defined by

$$T_1(f)(x) = q^{-2} f(\pi x) + q^2 f(\pi^{-1}x) \quad \text{for } f \in \mathscr{S}(k_v^4)^{C \times C'}.$$

Then T_1 generates the Hecke algebra $C_c^\infty(O_2//C)$. Let T_1' and T_2' be the Hecke operators on Sp_4, each of total mass 1 and supported on the C' double cosets with respective coset representatives

$$\begin{bmatrix} \pi & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \pi^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \pi & 0 & 0 & 0 \\ 0 & \pi & 0 & 0 \\ 0 & 0 & \pi^{-1} & 0 \\ 0 & 0 & 0 & \pi^{-1} \end{bmatrix}.$$

These operators generate $C_c^\infty(\mathrm{Sp}_4//C')$.

Direct computation yields the equations

$$T_1(\phi_0) = q^{-2}\phi_{-1} + q^2\phi_1,$$

$$(2.11) \quad T_1'(\phi_0) = \frac{1}{q(q^4-1)}((q-1)\phi_{-1} + (q^4 - 2q + 1)\phi_0 + q^4(q-1)\phi_1),$$

$$T_2'(\phi_0) = \frac{1}{q^2(q^4-1)}((q^2-1)\phi_{-1} + (q^4 - 2q^2 + 1)\phi_0 + q^4(q^2-1)\phi_1).$$

From Proposition 3, we know that $\mathscr{S}(k_v^4)^{C \times C'}$ is isomorphic as $C_c^\infty(O_2//C)$ module to the regular action of $C_c^\infty(O_2//C)$ on itself. In particular ϕ_0 is a cyclic vector for $\mathscr{S}(k_v^4)^{C \times C'}$ under the action of T_1. Since T_1 and the T_j' commute with each other we have, for any f in $\mathscr{S}(k_v^4)^{C \times C'}$,

$$(2.12) \quad \begin{aligned} q(q^4-1)T_1'(f) &= (q^4 - 2q + 1)f + q^2(q-1)T_1(f), \\ q^2(q^4-1)T_2'(f) &= (q^4 - 2q^2 + 1)f + q^2(q^2-1)T_1(f). \end{aligned}$$

Now return to $Q_{\sigma'}^{C \times C'}$. It will be the quotient of $\mathscr{S}(K_v^2)^{C \times C'}$. Therefore if $\bar{\phi}_0$ is the image of ϕ_0 in $Q_{\sigma'}^{C \times C'}$, then $\bar{\phi}_0$ is a cyclic vector for T_1 acting on $Q_{\sigma'}^{C \times C'}$. On the other hand, since $Q_{\sigma'}$ is a sum of copies of σ' as Sp_4 module, we see $\bar{\phi}_0$ will be an eigenvector for T_1' and T_2'. Hence by (2.11), we see $\bar{\phi}_0$ will be an eigenvector for T_1, so $Q_{\sigma'}^{C \times C'}$ is one-dimensional as desired. Hence $Q_{\sigma'} \simeq \sigma \otimes \sigma'$. Furthermore (2.12) shows that the eigenvalue of T_1 acting on $\bar{\phi}_0$ determines those of T_1' and T_2', and vice versa. Since these eigenvalues determine in turn σ and σ' [C], we see also that $\sigma \to \sigma'$ is injective, so the proposition is proved.

We note that, from the realization of ω_v^∞ on $\mathscr{S}(k_v^4)$, it is obvious that no C'-spherical quotient of ω_v^∞ can be tempered, because all such quotients will admit a distribution which is invariant by the isotropy group in Sp_4 of a point in k_v^4, and existence of such a distribution precludes temperedness.

We may now proceed as indicated above. Select f in $\mathscr{S}(K_A^2)$ such that θ_f transforms under $O_2(A)$ according to an irreducible representation whose local component is the signum representation at at least one place where K does not split over k. Such representations obviously exist. Then under $\mathrm{Sp}_4(A)$, we know from Propositions 2 and 4 that θ_f will transform according to an irreducible representation σ_v', determined by σ_v, of $\mathrm{Sp}_4(k_v)$ for all places v where K does not split over k, or where K does split and θ_f is fixed by C_v'. Together, these account for almost all places. Since θ_f transforms by a supercuspidal representation at at least one place, it will be a cusp form. Therefore the (L^2-closure of) the $\mathrm{Sp}_4(A)$ translates of θ_f will decompose into a direct sum of cuspidal automorphic representations. At all the places mentioned above, the local components of these automorphic representations will have to be the σ_v' determined by the σ_v. Therefore these automorphic representations will be nontempered at almost all places. This completes the construction.

To conclude, we note that if $k = Q$ and K is an imaginary quadratic extension, then some of the forms we construct will be ordinary Siegel modular forms of weight 1.

Also, since automorphic forms on O_2 obviously fail to satisfy strong multiplicity one, the corresponding forms on Sp_4 will also contradict strong multiplicity one.

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Yale University

University of Tel Aviv