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Chapter 2

Real numbers and their properties

Real numbers and functions of real numbers form an integral part of mathematics. Certainly all students in the sciences receive basic training in these ideas, normally in the form of courses on calculus and differential equations. In this chapter we establish the basic properties of the set of real numbers and of functions defined on this set. In particular, using the construction of the integers in Section 1.4 as a starting point, we *define* the set of real numbers, thus providing a fairly firm basis on which to develop the main ideas in these volumes. We follow this by discussing various structural properties of the set of real numbers. These cover both algebraic properties (Section 2.2.1) and topological properties (Section 2.5). After this, we discuss important ideas like continuity and differentiability of real-valued functions of a real variable.

Do I need to read this chapter? Yes you do, unless you already know its contents. While the construction of the real numbers in Section 2.1 is perhaps a little bit of an extravagance, it does set the stage for the remainder of the material. Moreover, the material in the remainder of the chapter is, in some ways, the backbone of the mathematical presentation. We say this for two reasons.

1. The technical material concerning the structure of the real numbers is, very simply, assumed knowledge for reading everything else in the series.
2. The *ideas* introduced in this chapter will similarly reappear constantly throughout the volumes in the series. But here, many of these ideas are given their most concrete presentation and, as such, afford the inexperienced reader the opportunity to gain familiarity with useful techniques (e.g., the $\epsilon - \delta$ formalism) in a setting where they presumably possess some degree of comfort. This will be crucial when we discuss more abstract ideas in Chapters 5, II-2, and II-3, to name a few. •

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Section 2.1

Construction of the real numbers

In this section we undertake to define the set of real numbers, using as our starting point the set \mathbb{Z} of integers constructed in Section 1.4. The construction begins by building the rational numbers, which are defined, loosely speaking, as fractions of integers. We know from our school days that every real number can be arbitrarily well approximated by a rational number, e.g., using a decimal expansion. We use this intuitive idea as our basis for defining the set of real numbers from the set of rational numbers.

Do I need to read this section? If you feel comfortable with your understanding of what a real number is, then this section is optional reading. However, it is worth noting that in Section 2.1.2 we first use the $\epsilon - \delta$ formalism that is so important in the analysis featured in this series. Readers unfamiliar/uncomfortable with this idea may find this section a good place to get comfortable with this idea. It is also worth mentioning at this point that the $\epsilon - \delta$ formalism is one with which it is difficult to become fully comfortable. Indeed, PhD theses have been written on the topic of how difficult it is for students to fully assimilate this idea. We shall not adopt any unusual pedagogical strategies to address this matter. However, students are well-advised to spend some time understanding $\epsilon - \delta$ language, and instructors are well-advised to appreciate the difficulty students have in coming to grips with it. •

2.1.1 Construction of the rational numbers

The set of rational numbers is, roughly, the set of fractions of integers. However, we do not know what a fraction is. To define the set of rational numbers, we introduce an equivalence relation \sim in $\mathbb{Z} \times \mathbb{N}$ by

$$(j_1, k_1) \sim (j_2, k_2) \iff j_1 \cdot k_2 = j_2 \cdot k_1.$$

We leave to the reader the straightforward verification that this is an equivalence relation. Using this relation we define the rational numbers as follows.

2.1.1 Definition (Rational numbers) A *rational number* is an element of $(\mathbb{Z} \times \mathbb{N}) / \sim$. The set of rational numbers is denoted by \mathbb{Q} . •

2.1.2 Notation (Notation for rationals) For the rational number $[(j, k)]$ we shall typically write $\frac{j}{k}$, reflecting the usual fraction notation. We shall also often write a typical rational number as “ q ” when we do not care which equivalence class it comes from. We shall denote by 0 and 1 the rational numbers $[(0, 1)]$ and $[(1, 1)]$, respectively. •

The set of rational numbers has many of the properties of integers. For example, one can define addition and multiplication for rational numbers, as well as a total order in the set of rationals. However, there is an important construction that can be made for rational numbers that cannot generally be made for integers, namely that of division. Let us see how this is done.

2.1.3 Definition (Addition, multiplication, and division in \mathbb{Q}) Define the operations of *addition*, *multiplication*, and *division* in \mathbb{Q} by

- (i) $[(j_1, k_1)] + [(j_2, k_2)] = [(j_1 \cdot k_2 + j_2 \cdot k_1, k_1 \cdot k_2)]$,
- (ii) $[(j_1, k_1)] \cdot [(j_2, k_2)] = [(j_1 \cdot j_2, k_1 \cdot k_2)]$, and
- (iii) $[(j_1, k_1)] / [(j_2, k_2)] = [(j_1 \cdot k_2, k_1 \cdot j_2)]$ (we will also write $\frac{[(j_1, k_1)]}{[(j_2, k_2)]}$ for $[(j_1, k_1)] / [(j_2, k_2)]$), respectively, where $[(j_1, k_1)], [(j_2, k_2)] \in \mathbb{Q}$ and where, in the definition of division, we require that $j_2 \neq 0$. We will sometimes omit the “ \cdot ” when in multiplication. •

We leave to the reader as Exercise 2.1.1 the straightforward task of showing that these definitions are independent of choice of representatives in $\mathbb{Z} \times \mathbb{N}$. We also leave to the reader the assertion that, with respect to Notation 2.1.2, the operations of addition, multiplication, and division of rational numbers assume the familiar form:

$$\frac{j_1}{k_1} + \frac{j_2}{k_2} = \frac{j_1 \cdot k_2 + j_2 \cdot k_1}{k_1 \cdot k_2}, \quad \frac{j_1}{k_1} \cdot \frac{j_2}{k_2} = \frac{j_1 \cdot j_2}{k_2 \cdot k_2}, \quad \frac{\frac{j_1}{k_1}}{\frac{j_2}{k_2}} = \frac{j_1 \cdot k_2}{k_1 \cdot j_2}.$$

For the operation of division, it is convenient to introduce a new concept. Given $[(j, k)] \in \mathbb{Q}$ with $j \neq 0$, we define $[(j, k)]^{-1} \in \mathbb{Q}$ by $[(k, j)]$. With this notation, division then can be written as $[(j_1, k_1)] / [(j_2, k_2)] = [(j_1, k_1)] \cdot [(j_2, k_2)]^{-1}$. Thus division is really just multiplication, as we already knew. Also, if $q \in \mathbb{Q}$ and if $k \in \mathbb{N}_0$, then we define $q^k \in \mathbb{Q}$ inductively by $q^0 = 1$ and $q^{k+1} = q^k \cdot q$. The rational number q^k is the k th **power** of q .

Let us verify that the operations above satisfy the expected properties. Note that there are now some new properties, since we have the operation of division, or multiplicative inversion, to account for. As we did for integers, we shall write $-q$ for $-1 \cdot q$.

2.1.4 Proposition (Properties of addition and multiplication in \mathbb{Q}) *Addition and multiplication in \mathbb{Q} satisfy the following rules:*

- (i) $q_1 + q_2 = q_2 + q_1$, $q_1, q_2 \in \mathbb{Q}$ (**commutativity of addition**);
- (ii) $(q_1 + q_2) + q_3 = q_1 + (q_2 + q_3)$, $q_1, q_2, q_3 \in \mathbb{Q}$ (**associativity of addition**);
- (iii) $q + 0 = q$, $q \in \mathbb{Q}$ (**additive identity**);
- (iv) $q + (-q) = 0$, $q \in \mathbb{Q}$ (**additive inverse**);
- (v) $q_1 \cdot q_2 = q_2 \cdot q_1$, $q_1, q_2 \in \mathbb{Q}$ (**commutativity of multiplication**);
- (vi) $(q_1 \cdot q_2) \cdot q_3 = q_1 \cdot (q_2 \cdot q_3)$, $q_1, q_2, q_3 \in \mathbb{Q}$ (**associativity of multiplication**);
- (vii) $q \cdot 1 = q$, $q \in \mathbb{Q}$ (**multiplicative identity**);
- (viii) $q \cdot q^{-1} = 1$, $q \in \mathbb{Q} \setminus \{0\}$ (**multiplicative inverse**);
- (ix) $r \cdot (q_1 + q_2) = r \cdot q_1 + r \cdot q_2$, $r, q_1, q_2 \in \mathbb{Q}$ (**distributivity**);
- (x) $q^{k_1} \cdot q^{k_2} = q^{k_1+k_2}$, $q \in \mathbb{Q}$, $k_1, k_2 \in \mathbb{N}_0$.

Moreover, if we define $i_{\mathbb{Z}}: \mathbb{Z} \rightarrow \mathbb{Q}$ by $i_{\mathbb{Z}}(k) = [(k, 1)]$, then addition and multiplication in \mathbb{Q} agrees with that in \mathbb{Z} :

$$i_{\mathbb{Z}}(k_1) + i_{\mathbb{Z}}(k_2) = i_{\mathbb{Z}}(k_1 + k_2), \quad i_{\mathbb{Z}}(k_1) \cdot i_{\mathbb{Z}}(k_2) = i_{\mathbb{Z}}(k_1 \cdot k_2).$$

Proof All of these properties follow directly from the definitions of addition and multiplication, using Proposition 1.4.19. ■

Just as we can naturally think of \mathbb{N}_0 as being a subset of \mathbb{Z} , so too can we think of \mathbb{Z} as a subset of \mathbb{Q} . Moreover, we shall very often do so without making explicit reference to the map $i_{\mathbb{Z}}$.

Next we consider on \mathbb{Q} the extension of the partial order \leq and the strict partial order $<$.

2.1.5 Proposition (Order on \mathbb{Q})

On \mathbb{Q} define two relations $<$ and \leq by

$$\begin{aligned} [(j_1, k_1)] < [(j_2, k_2)] &\iff j_1 \cdot k_2 < k_1 \cdot j_2, \\ [(j_1, k_1)] \leq [(j_2, k_2)] &\iff j_1 \cdot k_2 \leq k_1 \cdot j_2. \end{aligned}$$

Then \leq is a total order and $<$ is the corresponding strict partial order.

Proof First let us show that the relations defined make sense, in that they are independent of choice of representative. Thus we suppose that $[(j_1, k_1)] = [(\tilde{j}_1, \tilde{k}_1)]$ and that $[(j_2, k_2)] = [(\tilde{j}_2, \tilde{k}_2)]$. Then

$$\begin{aligned} [(j_1, k_1)] \leq [(j_2, k_2)] &\iff j_1 \cdot k_2 \leq k_1 \cdot j_2 \\ &\iff j_1 \cdot k_2 \cdot j_2 \cdot \tilde{k}_2 \cdot \tilde{j}_1 \cdot k_1 \leq k_1 \cdot j_2 \cdot \tilde{j}_2 \cdot k_1 \cdot j_1 \cdot \tilde{k}_1 \\ &\iff (\tilde{j}_1 \cdot \tilde{k}_2) \cdot (j_1 \cdot j_2 \cdot k_1 \cdot k_2) \leq (\tilde{j}_2 \cdot \tilde{k}_1) \cdot (j_1 \cdot j_2 \cdot k_1 \cdot k_2) \\ &\iff \tilde{j}_1 \cdot \tilde{k}_2 \leq \tilde{j}_2 \cdot \tilde{k}_1. \end{aligned}$$

This shows that the definition of \leq is independent of representative. Of course, a similar argument holds for $<$.

That \leq is a partial order, and that $<$ is its corresponding strict partial order, follow from a straightforward checking of the definitions, so we leave this to the reader.

Thus we only need to check that \leq is a total order. Let $[(j_1, k_1)], [(j_2, k_2)] \in \mathbb{Q}$. Then, by the Trichotomy Law for \mathbb{Z} , either $j_1 \cdot k_2 < k_1 \cdot j_2$, $k_1 \cdot j_2 < j_1 \cdot k_2$, or $j_1 \cdot k_2 = k_1 \cdot j_2$. But this directly implies that either $[(j_1, k_1)] < [(j_2, k_2)]$, $[(j_2, k_2)] < [(j_1, k_1)]$, or $[(j_1, k_1)] = [(j_2, k_2)]$, respectively. ■

The total order on \mathbb{Q} allows a classification of rational numbers as follows.

2.1.6 Definition (Positive and negative rational numbers)

A rational number $q \in \mathbb{Q}$ is:

- (i) *positive* if $0 < q$;
- (ii) *negative* if $q < 0$;
- (iii) *nonnegative* if $0 \leq q$;
- (iv) *nonpositive* if $q \leq 0$.

The set of positive rational numbers is denoted by $\mathbb{Q}_{>0}$ and the set of nonnegative rational numbers is denoted by $\mathbb{Q}_{\geq 0}$. •

As we did with natural numbers and integers, we isolate the Trichotomy Law.

2.1.7 Corollary (Trichotomy Law for \mathbb{Q})

For $q, r \in \mathbb{Q}$, exactly one of the following possibilities holds:

- (i) $q < r$;
- (ii) $r < q$;
- (iii) $q = r$.

The following result records the relationship between the order on \mathbb{Q} and the arithmetic operations.

2.1.8 Proposition (Relation between addition and multiplication and $<$)

For $q, r, s \in \mathbb{Q}$, the following statements hold:

- (i) if $q < r$ then $q + s < r + s$;

- (ii) if $q < r$ and if $s > 0$ then $s \cdot q < s \cdot r$;
- (iii) if $q < r$ and if $s < 0$ then $s \cdot r < s \cdot q$;
- (iv) if $0 < q, r$ then $0 < q \cdot r$;
- (v) if $q < r$ and if either

(a) $0 < q, r$ or

(b) $q, r < 0$,

then $r^{-1} < q^{-1}$.

Proof (i) Write $q = [(j_q, k_q)]$, $r = [(j_r, k_r)]$, and $s = [(j_s, k_s)]$. Since $q < r$, $j_q \cdot k_r \leq j_r \cdot k_q$. Therefore,

$$\begin{aligned} j_q \cdot k_r \cdot k_s^2 &< j_r \cdot k_q \cdot k_s^2 \\ \implies j_q \cdot k_r \cdot k_s^2 + j_s \cdot k_q \cdot k_r \cdot k_s &< j_r \cdot k_q \cdot k_s^2 + j_s \cdot k_q \cdot k_r \cdot k_s, \end{aligned}$$

using Proposition 1.4.22. This last inequality is easily seen to be equivalent to $q + s < r + s$.

(ii) Write $q = [(j_q, k_q)]$, $r = [(j_r, k_r)]$, and $s = [(j_s, k_s)]$. Since $s > 0$ it follows that $j_s > 0$. Since $q \leq r$ it follows that $j_q \cdot k_r \leq j_r \cdot k_q$. From Proposition 1.4.22 we then have

$$j_q \cdot j_s \cdot j_s \cdot k_s \leq j_r \cdot k_q \cdot j_s \cdot k_s,$$

which is equivalent to $s \cdot q \leq s \cdot r$ by definition of multiplication.

(iii) The result here follows, as does (ii), from Proposition 1.4.22, but now using the fact that $j_s < 0$.

(iv) This is a straightforward application of the definition of multiplication and $<$.

(v) This follows directly from the definition of $<$. ■

The final piece of structure we discuss for rational numbers is the extension of the absolute value function defined for integers.

2.1.9 Definition (Rational absolute value function) The *absolute value function* on \mathbb{Q} is the map from \mathbb{Q} to $\mathbb{Q}_{\geq 0}$, denoted by $q \mapsto |q|$, defined by

$$|q| = \begin{cases} q, & 0 < q, \\ 0, & q = 0, \\ -q, & q < 0. \end{cases} \quad \bullet$$

The absolute value function on \mathbb{Q} has properties like that on \mathbb{Z} .

2.1.10 Proposition (Properties of absolute value on \mathbb{Q}) *The following statements hold:*

- (i) $|q| \geq 0$ for all $q \in \mathbb{Q}$;
- (ii) $|q| = 0$ if and only if $q = 0$;
- (iii) $|r \cdot q| = |r| \cdot |q|$ for all $r, q \in \mathbb{Q}$;
- (iv) $|r + q| \leq |r| + |q|$ for all $r, q \in \mathbb{Q}$ (*triangle inequality*);
- (v) $|q^{-1}| = |q|^{-1}$ for all $q \in \mathbb{Q} \setminus \{0\}$.

Proof Parts (i), (ii), and (v), follow directly from the definition, and part (iii) follows in the same manner as the analogous statement in Proposition 1.4.24. Thus we have only to prove part (iv). We consider various cases.

1. $|r| \leq |q|$:

(a) $0 \geq r, q$: Since $|r + q| = r + q$, and $|r| = r$ and $|q| = q$, this follows directly.

- (b) $r < 0, 0 \leq q$: Let $r = [(j_r, k_r)]$ and $q = [(j_q, k_q)]$. Then $r < 0$ gives $j_r < 0$ and $0 \leq q$ gives $j_q \geq 0$. We now have

$$|r + q| = \left| \frac{j_r \cdot k_q + j_q \cdot k_r}{k_r \cdot k_q} \right| = \frac{|j_r \cdot k_q + j_q \cdot k_r|}{k_r \cdot k_q}$$

and

$$|r| + |q| = \frac{|j_r| \cdot k_q + |j_q| \cdot k_r}{k_r \cdot k_q}.$$

Therefore,

$$\begin{aligned} |r + q| &= \frac{|j_r \cdot k_q + j_q \cdot k_r|}{k_r \cdot k_q} \\ &\leq \frac{|j_r| \cdot k_q + |j_q| \cdot k_r}{k_r \cdot k_q} \\ &= |r| + |q|, \end{aligned}$$

where we have used Proposition 2.1.8.

- (c) $r, q < 0$: Here $|r + q| = |-r + (-q)| = |-(r + q)| = -(r + q)$, and $|r| = -r$ and $|q| = -q$, so the result follows immediately.

2. $|q| \leq |r|$: This argument is the same as above, swapping r and q . ■

2.1.11 Remark Having been quite fussy about how we arrived at the set of integers and the set of rational numbers, and about characterising their important properties, we shall now use standard facts about these, some of which we may not have proved, but which can easily be proved using the definitions of \mathbb{Z} and \mathbb{Q} . Some of the arithmetic properties of \mathbb{Z} and \mathbb{Q} that we use without comment are in fact proved in Section 4.2 in the more general setting of rings. However, we anticipate that most readers will not balk at the instances where we use unproved properties of integers and rational numbers. ●

2.1.2 Construction of the real numbers from the rational numbers

Now we use the rational numbers as the building block for the real numbers. The idea of this construction, which was originally due to Cauchy¹, is the intuitive idea that the rational numbers may be used to approximate well a real number. For example, we learn in school that any real number is expressible as a decimal expansion. However, any finite length decimal expansion (and even some infinite length decimal expansions) is a rational number. So one could *define* real numbers as a limit of decimal expansions in some way. The problem is that there may be multiple decimal expansions giving rise to the same real number. For example, the decimal expansions 1.0000 and 0.9999... represent the same real number. The way one gets around this potential problem is to use equivalence classes, of course. But equivalence classes of what? This is where we begin the presentation, proper.

2.1.12 Definition (Cauchy sequence, convergent sequence) Let $\{q_j\}_{j \in \mathbb{N}}$ be a sequence in \mathbb{Q} . The sequence:

- (i) is a **Cauchy sequence** if, for each $\epsilon \in \mathbb{Q}_{>0}$, there exists $N \in \mathbb{N}$ such that $|q_j - q_k| < \epsilon$ for $j, k \geq N$;
- (ii) **converges to q_0** if, for each $\epsilon \in \mathbb{Q}_{>0}$, there exists $N \in \mathbb{N}$ such that $|q_j - q_0| < \epsilon$ for $j \geq N$.

¹The French mathematician Augustin Louis Cauchy (1789–1857) worked in the areas of complex function theory, partial differential equations, and analysis. His collected works span twenty-seven volumes.

(iii) is **bounded** if there exists $M \in \mathbb{Q}_{>0}$ such that $|q_j| < M$ for each $j \in \mathbb{N}$. •

The set of Cauchy sequences in \mathbb{Q} is denoted by $\text{CS}(\mathbb{Q})$. A sequence converging to q_0 has q_0 as its **limit**. •

The idea of a Cauchy sequence is that the terms in the sequence can be made arbitrarily close as we get to the tail of the sequence. A convergent sequence, however, gets closer and closer to its limit as we get to the tail of the sequence. Our instinct is probably that there is a relationship between these two ideas. One thing that is true is the following.

2.1.13 Proposition (Convergent sequences are Cauchy) *If a sequence $\{q_j\}_{j \in \mathbb{N}}$ converges to q_0 , then it is a Cauchy sequence.*

Proof Let $\epsilon \in \mathbb{Q}_{>0}$ and choose $N \in \mathbb{N}$ such that $|q_j - q_0| < \epsilon$ for $j \geq N$. Then, for $j, k \geq N$ we have

$$|q_j - q_k| = |q_j - q_0 - q_k + q_0| = |q_j - q_0| + |q_k - q_0| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

using the triangle inequality of Proposition 2.1.10. ■

Cauchy sequences have the property of being bounded.

2.1.14 Proposition (Cauchy sequences are bounded) *If $\{q_j\}_{j \in \mathbb{N}}$ is a Cauchy sequence, then it is bounded.*

Proof Choose $N \in \mathbb{N}$ such that $|q_j - q_k| < 1$ for $j, k \in \mathbb{Q}$. Then take M_N to be the largest of the nonnegative rational numbers $|q_1|, \dots, |q_N|$. Then, for $j \geq N$ we have, using the triangle inequality,

$$|q_j| = |q_j - q_N + q_N| \leq |q_j - q_N| + |q_N| < 1 + M_N,$$

giving the result by taking $M = M_N + 1$. ■

The question as to whether the converse of this is true is now the obvious one.

2.1.15 Example (Nonconvergent Cauchy sequences in \mathbb{Q} exist) If one already knows the real numbers exist, it is somewhat easy to come up with Cauchy sequences in \mathbb{Q} . However, to fabricate one “out of thin air” is not so easy.

For $k \in \mathbb{N}$, since $2k + 5 > k + 4$, it follows that $2^{2k+5} - 2^{k+4} > 0$. Let m_k be the smallest nonnegative integer for which

$$m_k^2 \geq 2^{2k+5} - 2^{k+4}. \quad (2.1)$$

The following contains a useful property of m_k .

1 Lemma $m_k^2 \leq 2^{2k+5}$.

Proof First we show that $m_k \leq 2^{k+3}$. Suppose that $m_k > 2^{k+3}$. Then

$$(m_k - 1)^2 > (2^{k+3} - 1)^2 = 2^{2k+6} - 2^{k+4} + 1 = 2(2^{2k+5} - 2^{k+4}) + 1 > 2^{2k+5} - 2^{k+4},$$

which contradicts the definition of m_k .

Now suppose that $m_k^2 > 2^{2k+5}$. Then

$$(m_k - 1)^2 = m_k^2 - 2m_k + 1 > 2^{2k+5} - 2^{k+4} + 1 > 2^{2k+5} - 2^{k+4},$$

again contradicting the definition of m_k . ■

Now define $q_k = \frac{m_k}{2^{k+2}}$.

2 Lemma $\{q_k\}_{k \in \mathbb{N}}$ is a Cauchy sequence.

Proof By Lemma 1 we have

$$q_k^2 = \frac{m_k^2}{2^{2k+4}} \leq \frac{2^{2k+5}}{2^{2k+4}} = 2, \quad k \in \mathbb{N},$$

and by (2.1) we have

$$q_k^2 = \frac{m_k^2}{2^{2k+4}} \geq \frac{2^{2k+5}}{2^{2k+4}} - \frac{2^{k+4}}{2^{2k+4}} = 2 - \frac{1}{2^k}, \quad k \in \mathbb{N}.$$

Summarising, we have

$$2 - \frac{1}{2^k} \leq q_k^2 \leq 2, \quad k \in \mathbb{N}. \quad (2.2)$$

Then, for $j, k \in \mathbb{N}$ we have

$$2 - \frac{1}{2^k} \leq q_k^2 \leq 2, \quad 2 - \frac{1}{2^j} \leq q_j^2 \leq 2 \quad \implies \quad -\frac{1}{2^j} \leq q_j^2 - q_k^2 \leq \frac{1}{2^k}.$$

Next we have, from (2.1),

$$q_k^2 = \frac{m_k^2}{2^{2k+4}} \geq \frac{2^{2k+5}}{2^{2k+4}} - \frac{2^{k+4}}{2^{2k+4}} = 2 - \frac{1}{2^k}, \quad k \in \mathbb{N},$$

from which we deduce that $q_k^2 \geq 1$, which itself implies that $q_k \geq 1$. Next, using this fact and $(q_j - q_k)^2 = (q_j + q_k)(q_j - q_k)$ we have

$$-\frac{1}{2^j} \frac{1}{q_j + q_k} \leq q_j - q_k \leq \frac{1}{2^j} \frac{1}{q_j + q_k} \quad \implies \quad -\frac{1}{2^{j+1}} \leq q_j - q_k \leq \frac{1}{2^{k+1}}, \quad j, k \in \mathbb{N}. \quad (2.3)$$

Now let $\epsilon \in \mathbb{Q}_{>0}$ and choose $N \in \mathbb{N}$ such that $\frac{1}{2^{N+1}} < \epsilon$. Then we immediately have $|q_j - q_k| < \epsilon$, $j, k \geq N$, using (2.3). ■

The following result gives the character of the limit of the sequence $\{q_k\}_{k \in \mathbb{N}}$, were it to be convergent.

3 Lemma *If q_0 is the limit for the sequence $\{q_k\}_{k \in \mathbb{N}}$, then $q_0^2 = 2$.*

Proof We claim that if $\{q_k\}_{k \in \mathbb{N}}$ converges to q_0 , then $\{q_k^2\}_{k \in \mathbb{N}}$ converges to q_0^2 . Let $M \in \mathbb{Q}_{>0}$ satisfy $|q_k| < M$ for all $k \in \mathbb{N}$, this being possible by Proposition 2.1.14. Now let $\epsilon \in \mathbb{Q}_{>0}$ and take $N \in \mathbb{N}$ such that

$$|q_k - q_0| < \frac{\epsilon}{M + |q_0|}.$$

Then

$$|q_k^2 - q_0^2| = |q_k - q_0| |q_k + q_0| < \epsilon,$$

giving our claim.

Finally, we prove the lemma by proving that $\{q_k^2\}_{k \in \mathbb{N}}$ converges to 2. Indeed, let $\epsilon \in \mathbb{Q}_{>0}$ and note that, if $N \in \mathbb{N}$ is chosen to satisfy $\frac{1}{2^N} < \epsilon$. Then, using (2.2), we have

$$|q_k^2 - 2| \leq \frac{1}{2^k} < \epsilon, \quad k \geq N,$$

as desired. ■

Finally, we have the following result, which is contained in the mathematical works of Euclid.

4 Lemma *There exists no $q_0 \in \mathbb{Q}$ such that $q_0^2 = 2$.*

Proof Suppose that $q_0^2 = [(j_0, k_0)]$ and further suppose that there is no integer m such that $q_0 = [(mj_0, mk_0)]$. We then have

$$q_0^2 = \frac{j_0^2}{k_0^2} = 2 \quad \implies \quad j_0^2 = 2k_0^2.$$

Thus j_0^2 is even, and then so too is j_0 (why?). Therefore, $j_0 = 2\tilde{j}_0$ and so

$$q_0^2 = \frac{4\tilde{j}_0^2}{k_0^2} = 2 \quad \implies \quad k_0^2 = 2\tilde{j}_0^2$$

which implies that k_0^2 , and hence k_0 is also even. This contradicts our assumption that there is no integer m such that $q_0 = [(mj_0, mk_0)]$. ■

With these steps, we have constructed a Cauchy sequence that does not converge. •

Having shown that there are Cauchy sequences that do not converge, the idea is now to define a real number to be, essentially, that which a nonconvergent Cauchy sequence would converge to if only it could. First we need to allow for the possibility, realised in practice, that different Cauchy sequences may converge to the same limit.

2.1.16 Definition (Equivalent Cauchy sequences) Two sequences $\{q_j\}_{j \in \mathbb{N}}, \{r_j\}_{j \in \mathbb{Q}} \in \text{CS}(\mathbb{Q})$ are *equivalent* if the sequence $\{q_j - r_j\}_{j \in \mathbb{N}}$ converges to zero. We write $\{q_j\}_{j \in \mathbb{N}} \sim \{r_j\}_{j \in \mathbb{N}}$ if the two sequences are equivalent. •

We should verify that this notion of equivalence of Cauchy sequences is indeed an equivalence relation.

2.1.17 Proposition *The relation \sim defined in $\text{CS}(\mathbb{Q})$ is an equivalence relation.*

Proof It is clear that the relation \sim is reflexive and symmetric. To prove transitivity, suppose that $\{q_j\}_{j \in \mathbb{N}} \sim \{r_j\}_{j \in \mathbb{N}}$ and that $\{r_j\}_{j \in \mathbb{N}} \sim \{s_j\}_{j \in \mathbb{N}}$. For $\epsilon \in \mathbb{Q}_{>0}$ let $N \in \mathbb{N}$ satisfy

$$|q_j - r_j| < \frac{\epsilon}{2}, \quad |r_j - s_j| < \frac{\epsilon}{2}, \quad j \geq N.$$

Then, using the triangle inequality,

$$|q_j - s_j| = |q_j - r_j + r_j - s_j| \leq |q_j - r_j| + |r_j - s_j| < \epsilon, \quad j \geq N,$$

showing that $\{q_j\}_{j \in \mathbb{N}} \sim \{s_j\}_{j \in \mathbb{N}}$. ■

We are now prepared to define the set of real numbers.

2.1.18 Definition (Real numbers) A *real number* is an element of $\text{CS}(\mathbb{Q}) / \sim$. The set of real numbers is denoted by \mathbb{R} . •

The definition encodes, in a precise way, our intuition about what a real number is. In the next section we shall examine some of the properties of the set \mathbb{R} .

Let us give the notation we will use for real numbers, since clearly we do not wish to write these explicitly as equivalence classes of Cauchy sequences.

2.1.19 Notation (Notation for reals) We shall frequently write a typical element in \mathbb{R} as “ x ”. We shall denote by 0 and 1 the rational numbers associated with the Cauchy sequences $\{0\}_{j \in \mathbb{N}}$ and $\{1\}_{j \in \mathbb{N}}$. •

Exercises

2.1.1 Show that the definitions of addition, multiplication, and division of rational numbers in Definition 2.1.3 are independent of representative.

2.1.2 Prove the **Binomial Theorem** which states that, for $x, y \in \mathbb{R}$ and $k \in \mathbb{N}$,

$$(x + y)^k = \sum_{j=0}^k B_{k,j} x^j y^{k-j},$$

where

$$B_{k,j} = \binom{k}{j} \triangleq \frac{k!}{j!(k-j)!}, \quad j, k \in \mathbb{N}, j \leq k,$$

are the **binomial coefficients**, and $k! = 1 \cdot 2 \cdot \dots \cdot k$ is the **factorial** of k . We take the convention that $0! = 1$.

2.1.3 Show that the order and absolute value on \mathbb{Q} agree with those on \mathbb{Z} . That is to say, show the following:

- (a) for $j, k \in \mathbb{Z}$, $j < k$ if and only if $i_{\mathbb{Z}}(j) < i_{\mathbb{Z}}(k)$;
- (b) for $k \in \mathbb{Z}$, $|k| = |i_{\mathbb{Z}}(k)|$.

(Note that we see clearly here the abuse of notation that follows from using $<$ for both the order on \mathbb{Z} and \mathbb{Q} and from using $|\cdot|$ as the absolute value both on \mathbb{Z} and \mathbb{Q} . It is expected that the reader can understand where the notational abuse occurs.)

2.1.4 Show that the set of rational numbers is countable using an argument along the following lines.

1. Construct a doubly infinite grid in the plane with a point at each integer coordinate. Note that every rational number $q = \frac{n}{m}$ is represented by the grid point (n, m) .
2. Start at the “centre” of the grid with the rational number 0 being assigned to the grid point $(0, 0)$, and construct a spiral which passes through each grid point. Note that this spiral should hit every grid point exactly once.
3. Use this spiral to infer the existence of a bijection from \mathbb{Q} to \mathbb{N} .

The following exercise leads you through Cantor’s famous “diagonal argument” for showing that the set of real numbers is uncountable.

2.1.5 Fill in the gaps in the following construction, justifying all steps.

1. Let $\{x_j\}_{j \in \mathbb{N}}$ be a countable subset of $(0, 1)$.
2. Construct a doubly infinite table for which the k th column of the j th row contains the k th term in the decimal expansion for x_j .
3. Construct $\bar{x} \in (0, 1)$ by declaring the k th term in the decimal expansion for \bar{x} to be different from the k th term in the decimal expansion for x_k .
4. Show that \bar{x} is not an element of the sequence $\{x_j\}_{j \in \mathbb{N}}$.

Hint: Be careful to understand that a real number might have different decimal expansions.

Section 2.2

Properties of the set of real numbers

In this section we present some of the well known properties as the real numbers, both algebraic and (referring ahead to the language of Chapter II-2) topological.

Do I need to read this section? Many of the properties given in Sections 2.2.1, 2.2.2 and 2.2.3 will be well known to any student with a high school education. However, these may be of value as a starting point in understanding some of the abstract material in Chapters 4 and 5. Similarly, the material in Section 2.2.4 is “obvious.” However, since this material will be assumed knowledge, it might be best for the reader to at least skim the section, to make sure there is nothing new in it for them. •

2.2.1 Algebraic properties of \mathbb{R}

In this section we define addition, multiplication, order, and absolute value for \mathbb{R} , mirroring the presentation for \mathbb{Q} in Section 2.1.1. Here, however, the definitions and verifications are not just trivialities, as they are for \mathbb{Q} .

First we define addition and multiplication. We do this by defining these operations first on elements of $\text{CS}(\mathbb{Q})$, and then showing that the operations depend only on equivalence class. The following is the key step in doing this.

2.2.1 Proposition (Addition, multiplication, and division of Cauchy sequences) *Let $\{q_j\}_{j \in \mathbb{N}}, \{r_j\}_{j \in \mathbb{N}} \in \text{CS}(\mathbb{Q})$. Then the following statements hold.*

- (i) *The sequence $\{q_j + r_j\}_{j \in \mathbb{N}}$ is a Cauchy sequence which we denote by $\{q_j\}_{j \in \mathbb{N}} + \{r_j\}_{j \in \mathbb{N}}$.*
- (ii) *The sequence $\{q_j \cdot r_j\}_{j \in \mathbb{N}}$ is a Cauchy sequence which we denote by $\{q_j\}_{j \in \mathbb{N}} \cdot \{r_j\}_{j \in \mathbb{N}}$.*
- (iii) *If, for all $j \in \mathbb{N}$, $q_j \neq 0$ and if the sequence $\{q_j\}_{j \in \mathbb{N}}$ does not converge to 0, then $\{q_j^{-1}\}_{j \in \mathbb{N}}$ is a Cauchy sequence.*

Furthermore, if $\{\tilde{q}_j\}_{j \in \mathbb{N}}, \{\tilde{r}_j\}_{j \in \mathbb{N}} \in \text{CS}(\mathbb{Q})$ satisfy

$$\{\tilde{q}_j\}_{j \in \mathbb{N}} \sim \{q_j\}_{j \in \mathbb{N}}, \quad \{\tilde{r}_j\}_{j \in \mathbb{N}} \sim \{r_j\}_{j \in \mathbb{N}},$$

then

- (iv) $\{\tilde{q}_j\}_{j \in \mathbb{N}} + \{\tilde{r}_j\}_{j \in \mathbb{N}} = \{q_j\}_{j \in \mathbb{N}} + \{r_j\}_{j \in \mathbb{N}}$,
- (v) $\{\tilde{q}_j\}_{j \in \mathbb{N}} \cdot \{\tilde{r}_j\}_{j \in \mathbb{N}} = \{q_j\}_{j \in \mathbb{N}} \cdot \{r_j\}_{j \in \mathbb{N}}$, and
- (vi) *if, for all $j \in \mathbb{N}$, $q_j, \tilde{q}_j \neq 0$ and if the sequences $\{q_j\}_{j \in \mathbb{N}}, \{\tilde{q}_j\}_{j \in \mathbb{N}}$ do not converge to 0, then $\{\tilde{q}_j\}_{j \in \mathbb{N}} \sim \{q_j\}_{j \in \mathbb{N}}$.*

Proof (i) Let $\epsilon \in \mathbb{Q}_{>0}$ and let $N \in \mathbb{N}$ have the property that $|q_j - q_k|, |r_j - r_k| < \frac{\epsilon}{2}$ for all $j, k \geq N$. Then, using the triangle inequality,

$$|(q_j + r_j) - (q_k + r_k)| \leq |q_j - q_k| + |r_j - r_k| = \epsilon, \quad j, k \geq N.$$

(ii) Let $M \in \mathbb{Q}_{>0}$ have the property that $|q_j|, |r_j| < M$ for all $j \in \mathbb{N}$. For $\epsilon \in \mathbb{Q}_{>0}$ let $N \in \mathbb{N}$ have the property that $|q_j - q_k|, |r_j - r_k| < \frac{\epsilon}{2M}$ for all $j, k \geq N$. Then, using the triangle inequality,

$$|(q_j \cdot r_j) - (q_k \cdot r_k)| = |q_j(r_j - r_k) - r_k(q_k - q_j)| \leq |q_j||r_j - r_k| + |r_k||q_k - q_j| < \epsilon, \quad j, k \geq N.$$

(iii) We claim that if $\{q_j\}_{j \in \mathbb{N}}$ satisfies the conditions stated, then there exists $\delta \in \mathbb{Q}_{>0}$ such that $|q_k| \geq \delta$. Indeed, since $\{q_j\}_{j \in \mathbb{N}}$ does not converge to zero, choose $\epsilon \in \mathbb{Q}_{>0}$ such that, for all $N \in \mathbb{N}$, there exists $j \geq N$ for which $|q_j| \geq \epsilon$. Next take $N \in \mathbb{N}$ such that $|q_j - q_k| < \frac{\epsilon}{2}$ for $j, k \geq N$. Then there exists $\tilde{N} \geq N$ such that $|q_{\tilde{N}}| \geq \epsilon$. For any $j \geq N$ we then have

$$|q_j| = |q_{\tilde{N}} - (q_{\tilde{N}} - q_j)| \geq ||q_{\tilde{N}}| - |q_{\tilde{N}} - q_j|| \geq \epsilon - \frac{\epsilon}{2} = \frac{\epsilon}{2},$$

where we have used Exercise 2.2.6. The claim follows by taking δ to be the smallest of the numbers $\frac{\epsilon}{2}, |q_1|, \dots, |q_N|$.

Now let $\epsilon \in \mathbb{Q}_{>0}$ and choose $N \in \mathbb{N}$ such that $|q_j - q_k| < \delta^2 \epsilon$ for $j, k \geq N$. Then

$$|q_j^{-1} - q_k^{-1}| = \left| \frac{q_k - q_j}{q_j q_k} \right| < \frac{\delta^2 \epsilon}{\delta^2} = \epsilon, \quad j, k \geq N.$$

(iv) For $\epsilon \in \mathbb{Q}_{>0}$ let $N \in \mathbb{N}$ have the property that $|\tilde{q}_j - q_j|, |\tilde{r}_j - r_j| < \frac{\epsilon}{2}$. Then, using the triangle inequality,

$$|(\tilde{q}_j + \tilde{r}_j) - (q_k + r_k)| \leq |\tilde{q}_j - q_k| + |\tilde{r}_k - r_k| < \epsilon, \quad j, k \geq N.$$

(v) Let $M \in \mathbb{Q}_{>0}$ have the property that $|\tilde{q}_j|, |r_j| < M$ for all $j \in \mathbb{N}$. Then, for $\epsilon \in \mathbb{Q}_{>0}$, take $N \in \mathbb{N}$ such that $|\tilde{r}_j - r_k|, |\tilde{q}_j - q_k| < \frac{\epsilon}{2M}$ for $j, k \geq N$. We then use the triangle inequality to give

$$|(\tilde{q}_j \cdot \tilde{r}_j) - (q_k \cdot r_k)| = |\tilde{q}_j(\tilde{r}_j - r_k) - r_k(q_k - \tilde{q}_j)| < \epsilon, \quad j, k \geq N.$$

(vi) Let $\delta \in \mathbb{Q}_{>0}$ satisfy $|q_j|, |\tilde{q}_j| \geq \delta$ for all $j \in \mathbb{N}$. Then, for $\epsilon \in \mathbb{Q}_{>0}$, choose $N \in \mathbb{N}$ such that $|\tilde{q}_j - q_j| < \delta^2 \epsilon$ for $j \geq N$. Then we have

$$|\tilde{q}_j^{-1} - q_j^{-1}| = \left| \frac{q_j - \tilde{q}_j}{q_j \tilde{q}_j} \right| < \frac{\delta^2 \epsilon}{\delta^2}, \quad j \geq N,$$

so completing the proof. ■

The requirement, in parts (iii) and (vi), that the sequence $\{q_j\}_{j \in \mathbb{N}}$ have no zero elements is not really a restriction, as is the requirement that the sequence not converge to zero. The reason for this is that, as we showed in the proof, if the sequence does not converge to zero, then there exists $\epsilon \in \mathbb{Q}_{>0}$ and $N \in \mathbb{N}$ such that $|q_j| > \epsilon$ for $j \geq N$. Thus the tail of the sequence is guaranteed to have no zero elements, and the tail of the sequence is all that matters for the equivalence class.

Now that we have shown how to add and multiply Cauchy sequences in \mathbb{Q} , and that this addition and multiplication depends only on equivalence classes under the notion of equivalence given in Definition 2.1.16, we can easily define addition and multiplication in \mathbb{R} .

2.2.2 Definition (Addition, multiplication, and division in \mathbb{R}) Define the operations of *addition*, *multiplication*, and *division* in \mathbb{R} by

- (i) $[\{q_j\}_{j \in \mathbb{N}}] + [\{r_j\}_{j \in \mathbb{N}}] = [\{q_j + r_j\}_{j \in \mathbb{N}}]$,
- (ii) $[\{q_j\}_{j \in \mathbb{N}}] \cdot [\{r_j\}_{j \in \mathbb{N}}] = [\{q_j \cdot r_j\}_{j \in \mathbb{N}}]$,
- (iii) $[\{q_j\}_{j \in \mathbb{N}}] / [\{r_j\}_{j \in \mathbb{N}}] = [\{q_j / r_j\}_{j \in \mathbb{N}}]$,

respectively, where, in the definition of division, we require that the sequence $\{r_j\}_{j \in \mathbb{N}}$ have no zero elements, and that it not converge to 0. We will sometimes omit the “.” when in multiplication. ●

Similarly to what we have done previously with \mathbb{Z} and \mathbb{Q} , we let $-x = [\{-1\}_{j \in \mathbb{N}}] \cdot x$. For $x \in \mathbb{R} \setminus \{0\}$, we also denote by x^{-1} the real number corresponding to a Cauchy sequence $\{\frac{1}{q_j}\}_{j \in \mathbb{N}}$, where $x = [\{q_j\}_{j \in \mathbb{N}}]$.

As with integers and rational numbers, we can define powers of real numbers. For $x \in \mathbb{R} \setminus \{0\}$ and $k \in \mathbb{N}_0$ we define $x^k \in \mathbb{R}$ inductively by $x^0 = 1$ and $x^{k+1} = x^k \cdot x$. As usual, we call x^k the k th **power** of x . For $k \in \mathbb{Z} \setminus \mathbb{N}_0$, we take $x^k = (x^{-k})^{-1}$. For real numbers, the notion of the power of a number can be extended. Let us show how this is done. In the statement of the result, we use the notion of positive real numbers which are not defined until Definition 2.2.8. Also, in our proof, we refer ahead to properties of \mathbb{R} that are not considered until Section 2.3. However, it is convenient to state the construction here.

2.2.3 Proposition ($x^{1/k}$) For $x \in \mathbb{R}_{>0}$ and $k \in \mathbb{N}$, there exists a unique $y \in \mathbb{R}_{>0}$ such that $y^k = x$. We denote the number y by $x^{1/k}$.

Proof Let $S_x = \{y \in \mathbb{R} \mid y^k < x\}$. Since $x \geq 0$, $0 \in S$ so $S \neq \emptyset$. We next claim that $\max\{1, x\}$ is an upper bound for S_x . First suppose that $x < 1$. Then, for $y \in S_x$, $y^k < x < 1$, and so 1 is an upper bound for S_x . If $x \geq 1$ and $y \in S_x$, then we claim that $y \leq x$. Indeed, if $y > x$ then $y^k > x^k > x$, and so $y \notin S_x$. This shows that S_x is upper bounded by x in this case. Now we know that S_x has a least upper bound by Theorem 2.3.6. Let y denote this least upper bound.

We shall now show that $y^k = x$. Suppose that $y^k \neq x$. From Corollary 2.2.9 we have $y^k < x$ or $y^k > x$.

Suppose first that $y^k < x$. Then, for $\epsilon > 0$ we have

$$(y + \epsilon)^k = \epsilon^k + a_{k-1}y\epsilon^{k-1} + \dots + a_1y^{k-1}\epsilon + y^k$$

for some numbers a_1, \dots, a_{k-1} (these are the binomial coefficients of Exercise 2.1.2). If $\epsilon \leq 1$ then $\epsilon^k \leq \epsilon$ for $k \in \mathbb{N}$. Therefore, if $\epsilon \leq 1$ we have

$$(y + \epsilon)^k \leq \epsilon(1 + a_{k-1}y + \dots + a_1y^{k-1}) + y^k.$$

Now, if $\epsilon < \min\{1, \frac{x-y^k}{1+a_{k-1}y+\dots+a_1y^{k-1}}\}$, then $(y + \epsilon)^k < x$, contradicting the fact that y is an upper bound for S_x .

Now suppose that $y^k > x$. Then, for $\epsilon > 0$, we have

$$(y - \epsilon)^k = (-1)^k\epsilon^k + (-1)^{k-1}a_{k-1}y\epsilon^{k-1} + \dots - a_1y^{k-1}\epsilon + y^k.$$

The sum on the right involves terms that are positive and negative. This sum will be greater than the corresponding sum with the positive terms involving powers of ϵ removed. That is to say,

$$(y - \epsilon)^k > y^k - a_1y^{k-1}\epsilon - a_3y^{k-3}\epsilon^3 + \dots.$$

For $\epsilon \leq 1$ we again have $\epsilon^k \leq \epsilon$ for $k \in \mathbb{N}$. Therefore

$$(y - \epsilon)^k > y^k - (a_1y^{k-1} + a_3y^{k-3} + \dots)\epsilon.$$

Thus, if $\epsilon < \min\{1, \frac{y^k-x}{a_1y^{k-1}+a_3y^{k-3}+\dots}\}$ we have $(y - \epsilon)^k > x$, contradicting the fact that y is the least upper bound for S_x .

We are forced to conclude that $y^k = x$, so giving the result. ■

If $x \in \mathbb{R}_{>0}$ and $q = \frac{j}{k} \in \mathbb{Q}$ with $j \in \mathbb{Z}$ and $k \in \mathbb{N}$, we define $x^q = (x^{1/k})^j$.

Let us record the basic properties of addition and multiplication, mirroring analogous results for \mathbb{Q} . The properties all follow easily from the similar properties for \mathbb{Q} , along with Proposition 2.2.1 and the definition of addition and multiplication in \mathbb{R} .

2.2.4 Proposition (Properties of addition and multiplication in \mathbb{R}) *Addition and multiplication in \mathbb{R} satisfy the following rules:*

- (i) $x_1 + x_2 = x_2 + x_1$, $x_1, x_2 \in \mathbb{R}$ (**commutativity of addition**);
- (ii) $(x_1 + x_2) + x_3 = x_1 + (x_2 + x_3)$, $x_1, x_2, x_3 \in \mathbb{R}$ (**associativity of addition**);
- (iii) $x + 0 = x$, $x \in \mathbb{R}$ (**additive identity**);
- (iv) $x + (-x) = 0$, $x \in \mathbb{R}$ (**additive inverse**);
- (v) $x_1 \cdot x_2 = x_2 \cdot x_1$, $x_1, x_2 \in \mathbb{R}$ (**commutativity of multiplication**);
- (vi) $(x_1 \cdot x_2) \cdot x_3 = x_1 \cdot (x_2 \cdot x_3)$, $x_1, x_2, x_3 \in \mathbb{R}$ (**associativity of multiplication**);
- (vii) $x \cdot 1 = x$, $x \in \mathbb{R}$ (**multiplicative identity**);
- (viii) $x \cdot x^{-1} = 1$, $x \in \mathbb{R} \setminus \{0\}$ (**multiplicative inverse**);
- (ix) $y \cdot (x_1 + x_2) = y \cdot x_1 + y \cdot x_2$, $y, x_1, x_2 \in \mathbb{R}$ (**distributivity**);
- (x) $x^{k_1} \cdot x^{k_2} = x^{k_1+k_2}$, $x \in \mathbb{R}$, $k_1, k_2 \in \mathbb{N}_0$.

Moreover, if we define $i_{\mathbb{Q}}: \mathbb{Q} \rightarrow \mathbb{R}$ by $i_{\mathbb{Q}}(q) = [\{q\}_{j \in \mathbb{N}}]$, then addition and multiplication in \mathbb{R} agrees with that in \mathbb{Q} :

$$i_{\mathbb{Q}}(q_1) + i_{\mathbb{Q}}(q_2) = i_{\mathbb{Q}}(q_1 + q_2), \quad i_{\mathbb{Q}}(q_1) \cdot i_{\mathbb{Q}}(q_2) = i_{\mathbb{Q}}(q_1 \cdot q_2).$$

As we have done in the past with $\mathbb{Z} \subset \mathbb{Q}$, we will often regard \mathbb{Q} as a subset of \mathbb{R} without making explicit mention of the inclusion $i_{\mathbb{Q}}$. Note that this also allows us to think of both \mathbb{N}_0 and \mathbb{Z} as subsets of \mathbb{R} , since \mathbb{N}_0 is regarded as a subset of \mathbb{Z} , and since $\mathbb{Z} \subset \mathbb{Q}$. Of course, this is nothing surprising. Indeed, perhaps the more surprising thing is that it is not actually the case that the definitions do not precisely give $\mathbb{N}_0 \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$!

Now is probably a good time to mention that an element of \mathbb{R} that is not in the image of $i_{\mathbb{Q}}$ is called **irrational**. Also, one can show that the set \mathbb{Q} of rational numbers is countable (Exercise 2.1.4), but that the set \mathbb{R} of real numbers is uncountable (Exercise 2.1.5). Note that it follows that the set of irrational numbers is uncountable, since an uncountable set cannot be a union of two countable sets.

2.2.2 The total order on \mathbb{R}

Next we define in \mathbb{R} a natural total order. To do so requires a little work. The approach we take is this. On the set $\text{CS}(\mathbb{Q})$ of Cauchy sequences in \mathbb{Q} we define a partial order that is *not* a total order. We then show that, for any two Cauchy sequences, in each equivalence class in $\text{CS}(\mathbb{Q})$ with respect to the equivalence relation of Definition 2.1.16, there exists representatives that can be compared using the order. In this way, while the order on the set of Cauchy sequences is not a total order, there is induced a total order on the set of equivalence classes.

First we define the partial order on the set of Cauchy sequences.

2.2.5 Definition (Partial order on $\text{CS}(\mathbb{Q})$) The partial order \preceq on $\text{CS}(\mathbb{Q})$ is defined by

$$\{q_j\}_{j \in \mathbb{N}} \preceq \{r_j\}_{j \in \mathbb{N}} \iff q_j \leq r_j, \quad j \in \mathbb{N}. \quad \bullet$$

This partial order is clearly not a total order. For example, the Cauchy sequences $\{\frac{1}{j}\}_{j \in \mathbb{N}}$ and $\{\frac{(-1)^j}{j}\}_{j \in \mathbb{N}}$ are not comparable with respect to this order. However, what is true is that equivalence classes of Cauchy sequences *are* comparable. We refer the reader to Definition 2.1.16 for the definition of the equivalence relation we denote by \sim in the following result.

2.2.6 Proposition Let $\{q_j\}_{j \in \mathbb{N}}, \{r_j\}_{j \in \mathbb{N}} \in \text{CS}(\mathbb{Q})$ and suppose that $\{q_j\}_{j \in \mathbb{N}} \not\sim \{r_j\}_{j \in \mathbb{N}}$. The following two statements hold:

- (i) There exists $\{\tilde{q}_j\}_{j \in \mathbb{N}}, \{\tilde{r}_j\}_{j \in \mathbb{N}} \in \text{CS}(\mathbb{Q})$ such that
- (a) $\{\tilde{q}_j\}_{j \in \mathbb{N}} \sim \{q_j\}_{j \in \mathbb{N}}$ and $\{\tilde{r}_j\}_{j \in \mathbb{N}} \sim \{r_j\}_{j \in \mathbb{N}}$, and
 - (b) either $\{\tilde{q}_j\}_{j \in \mathbb{N}} \prec \{\tilde{r}_j\}_{j \in \mathbb{N}}$ or $\{\tilde{r}_j\}_{j \in \mathbb{N}} \prec \{\tilde{q}_j\}_{j \in \mathbb{N}}$.
- (ii) There does not exist $\{\tilde{q}_j\}_{j \in \mathbb{N}}, \{\bar{q}_j\}_{j \in \mathbb{N}}, \{\tilde{r}_j\}_{j \in \mathbb{N}}, \{\bar{r}_j\}_{j \in \mathbb{N}} \in \text{CS}(\mathbb{Q})$ such that
- (a) $\{\tilde{q}_j\}_{j \in \mathbb{N}} \sim \{\bar{q}_j\}_{j \in \mathbb{N}} \sim \{q_j\}_{j \in \mathbb{N}}$ and $\{\tilde{r}_j\}_{j \in \mathbb{N}} \sim \{\bar{r}_j\}_{j \in \mathbb{N}} \sim \{r_j\}_{j \in \mathbb{N}}$, and
 - (b) one of the following two statements holds:
 - I. $\{\tilde{q}_j\}_{j \in \mathbb{N}} \prec \{\tilde{r}_j\}_{j \in \mathbb{N}}$ and $\{\bar{r}_j\}_{j \in \mathbb{N}} \prec \{\bar{q}_j\}_{j \in \mathbb{N}}$;
 - II. $\{\tilde{r}_j\}_{j \in \mathbb{N}} \prec \{\tilde{q}_j\}_{j \in \mathbb{N}}$ and $\{\bar{q}_j\}_{j \in \mathbb{N}} \prec \{\bar{r}_j\}_{j \in \mathbb{N}}$.

Proof (i) We begin with a useful lemma.

1 Lemma With the given hypotheses, there exists $\delta \in \mathbb{Q}_{>0}$ and $N \in \mathbb{N}$ such that $|q_j - r_j| \geq \delta$ for all $j \geq N$.

Proof Since $\{q_j - r_j\}_{j \in \mathbb{N}}$ does not converge to zero, choose $\epsilon \in \mathbb{Q}_{>0}$ such that, for all $N \in \mathbb{N}$, there exists $j \geq N$ such that $|q_j - r_j| \geq \epsilon$. Now take $N \in \mathbb{N}$ such that $|q_j - q_k|, |r_k - r_k| \leq \frac{\epsilon}{4}$ for $j, k \geq N$. Then, by our assumption about ϵ , there exists $\tilde{N} \geq N$ such that $|q_{\tilde{N}} - r_{\tilde{N}}| \geq \epsilon$. Then, for any $j \geq N$, we have

$$|q_j - r_j| = |(q_{\tilde{N}} - r_{\tilde{N}}) - (q_{\tilde{N}} - r_{\tilde{N}}) - (q_j - r_j)| \geq ||q_{\tilde{N}} - r_{\tilde{N}}| - |(q_{\tilde{N}} - r_{\tilde{N}}) - (q_j - r_j)|| \geq \epsilon - \frac{\epsilon}{2}.$$

The lemma follows by taking $\delta = \frac{\epsilon}{2}$. ▼

Now take N and δ as in the lemma. Then take $\tilde{N} \in \mathbb{N}$ such that $|q_j - q_k|, |r_j - r_k| < \frac{\delta}{2}$ for $j, k \geq \tilde{N}$. Then, using the triangle inequality,

$$|(q_j - r_j) - (q_k - r_k)| \leq \delta, \quad j, k \geq \tilde{N}.$$

Now take K to be the larger of N and \tilde{N} . We then have either $q_K - r_K \geq \delta$ or $r_K - q_K \geq \delta$. First suppose that $q_K - r_K \geq \delta$ and let $j \geq K$. Either $q_j - r_j \geq \delta$ or $r_j - q_j \geq \delta$. If the latter, then

$$q_j - r_j \leq -\delta \implies (q_j - r_k) - (q_K - r_K) \leq 2\delta,$$

contradicting the definition of K . Therefore, we must have $q_j - r_j \geq \delta$ for all $j \geq K$. A similar argument when $r_K - q_K \geq \delta$ shows that $r_j - q_j \geq \delta$ for all $j \geq K$. For $j \in \mathbb{N}$ we then define

$$\tilde{q}_j = \begin{cases} q_K, & j < K, \\ q_j, & j \geq K, \end{cases} \quad \tilde{r}_j = \begin{cases} r_K, & j < K, \\ r_j, & j \geq K, \end{cases}$$

and we note that the sequences $\{\tilde{q}_j\}_{j \in \mathbb{N}}$ and $\{\tilde{r}_j\}_{j \in \mathbb{N}}$ satisfy the required conditions.

(ii) Suppose that

1. $\{q_j\}_{j \in \mathbb{N}} \not\sim \{r_j\}_{j \in \mathbb{N}}$,
2. $\{\tilde{q}_j\}_{j \in \mathbb{N}} \sim \{\bar{q}_j\}_{j \in \mathbb{N}} \sim \{q_j\}_{j \in \mathbb{N}}$,
3. $\{\tilde{r}_j\}_{j \in \mathbb{N}} \sim \{\bar{r}_j\}_{j \in \mathbb{N}} \sim \{r_j\}_{j \in \mathbb{N}}$, and
4. $\{\tilde{q}_j\}_{j \in \mathbb{N}} \prec \{\tilde{r}_j\}_{j \in \mathbb{N}}$.

From the previous part of the proof we know that there exists $\delta \in \mathbb{Q}_{>0}$ and $N \in \mathbb{N}$ such that $\tilde{q}_j - \tilde{r}_j \geq \delta$ for $j \geq N$. Then take $\tilde{N} \in \mathbb{N}$ such that $|\tilde{q}_j - \bar{q}_j|, |\tilde{r}_j - \bar{r}_j| < \frac{\delta}{4}$ for $j \geq \tilde{N}$. This implies that for $j \geq \tilde{N}$ we have

$$|(\tilde{q}_j - \tilde{r}_j) - (\bar{q}_j - \bar{r}_j)| < \frac{\delta}{2}.$$

Therefore,

$$(\bar{q}_j - \bar{r}_j) > (\tilde{q}_j - \tilde{r}_j) - \frac{\delta}{2}, \quad j \geq \tilde{N}.$$

If additionally $j \geq N$, then we have

$$(\bar{q}_j - \bar{r}_j) > \delta - \frac{\delta}{2} = \frac{\delta}{2}.$$

This shows the impossibility of $\{\bar{r}_j\}_{j \in \mathbb{N}} \prec \{\bar{q}_j\}_{j \in \mathbb{N}}$. A similar argument shows that $\{\tilde{r}_j\}_{j \in \mathbb{N}} \prec \{\tilde{q}_j\}_{j \in \mathbb{N}}$ bars the possibility that $\{\bar{q}_j\}_{j \in \mathbb{N}} \prec \{\tilde{r}_j\}_{j \in \mathbb{N}}$. ■

Using the preceding result, the following definition then makes sense.

2.2.7 Definition (Order on \mathbb{R}) The total order on \mathbb{R} is defined by $x \leq y$ if and only if there exists $\{q_j\}_{j \in \mathbb{N}}, \{r_j\}_{j \in \mathbb{N}} \in \text{CS}(\mathbb{Q})$ such that

- (i) $x = [\{q_j\}_{j \in \mathbb{N}}]$ and $y = [\{r_j\}_{j \in \mathbb{N}}]$ and
- (ii) $\{q_j\}_{j \in \mathbb{N}} \preceq \{r_j\}_{j \in \mathbb{N}}$. •

Note that we have used the symbol “ \leq ” for the total order on \mathbb{Z} , \mathbb{Q} , and \mathbb{R} . This is justified since, if we think of $\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$, then the various total orders agree (Exercises 2.1.3 and 2.2.4).

We have the usual language and notation we associate with various kinds of numbers.

2.2.8 Definition (Positive and negative real numbers) A real number x is:

- (i) **positive** if $0 < x$;
- (ii) **negative** if $x < 0$;
- (iii) **nonnegative** if $0 \leq x$;
- (iv) **nonpositive** if $x \leq 0$.

The set of positive real numbers is denoted by $\mathbb{R}_{>0}$, the set of nonnegative real numbers is denoted by $\mathbb{R}_{\geq 0}$, the set of negative real numbers is denoted by $\mathbb{R}_{<0}$, and the set of nonpositive real numbers is denoted by $\mathbb{R}_{\leq 0}$. •

Now is a convenient moment to introduce some simple notation and concepts that are associated with the natural total order on \mathbb{R} . The **signum function** is the map $\text{sign}: \mathbb{R} \rightarrow \{-1, 0, 1\}$ defined by

$$\text{sign}(x) = \begin{cases} -1, & x < 0, \\ 0, & x = 0, \\ 1, & x > 0. \end{cases}$$

For $x \in \mathbb{R}$, $\lceil x \rceil$ is the **ceiling** of x which is the smallest integer greater than or equal to x . Similarly, $\lfloor x \rfloor$ is the **floor** of x which is the largest integer less than or equal to x . If $S \subset \mathbb{R}$ is a *finite* set, then both $\sup S$ and $\inf S$ are elements of S . In this case we denote $\max S = \sup S$ and $\min S = \inf S$.

A consequence of our definition of order is the following extension of the Trichotomy Law to \mathbb{R} .

2.2.9 Corollary (Trichotomy Law for \mathbb{R}) For $x, y \in \mathbb{R}$, exactly one of the following possibilities holds:

- (i) $x < y$;
- (ii) $y < x$;
- (iii) $x = y$.

2.2.10 Remark (Trichotomy Law and Axiom of Choice)

finish

As with integers and rational numbers, addition and multiplication of real numbers satisfy the expected properties with respect to the total order.

2.2.11 Proposition (Relation between addition and multiplication and $<$)

For $x, y, z \in \mathbb{R}$, the following statements hold:

- (i) if $x < y$ then $x + z < y + z$;
- (ii) if $x < y$ and if $z > 0$ then $z \cdot x < z \cdot y$;
- (iii) if $x < y$ and if $z < 0$ then $z \cdot y < z \cdot x$;
- (iv) if $0 < x, y$ then $0 < x \cdot y$;
- (v) if $x < y$ and if either
 - (a) $0 < x, y$ or
 - (b) $x, y < 0$,
 then $y^{-1} < x^{-1}$.

Proof These statements all follow from the similar statements for \mathbb{Q} , along with Proposition 2.2.6. We leave the straightforward verifications to the reader as Exercise 2.2.3. ■

2.2.3 The absolute value function on \mathbb{R}

In this section we generalise the absolute value function on \mathbb{Q} . As we shall see in subsequent sections, this absolute value function is essential for providing much of the useful structure of the set of real numbers.

The definition of the absolute value is given as usual.

2.2.12 Definition (Real absolute value function)

The *absolute value function* on \mathbb{R} is the map from \mathbb{R} to $\mathbb{R}_{\geq 0}$, denoted by $x \mapsto |x|$, defined by

$$|x| = \begin{cases} x, & 0 < x, \\ 0, & x = 0, \\ -x, & x < 0. \bullet \end{cases}$$

Note that we have used the symbol “ $|\cdot|$ ” for the absolute values on \mathbb{Z} , \mathbb{Q} , and \mathbb{R} . This is justified since, if we think of $\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$, then the various absolute value functions agree (Exercises 2.1.3 and 2.2.4).

The real absolute value function has the expected properties. The proof of the following result is straightforward, and so omitted.

2.2.13 Proposition (Properties of absolute value on \mathbb{R})

The following statements hold:

- (i) $|x| \geq 0$ for all $x \in \mathbb{R}$;
- (ii) $|x| = 0$ if and only if $x = 0$;
- (iii) $|x \cdot y| = |x| \cdot |y|$ for all $x, y \in \mathbb{R}$;
- (iv) $|x + y| \leq |x| + |y|$ for all $x, y \in \mathbb{R}$ (*triangle inequality*);
- (v) $|x^{-1}| = |x|^{-1}$ for all $x \in \mathbb{R} \setminus \{0\}$.

2.2.4 Properties of \mathbb{Q} as a subset of \mathbb{R}

In this section we give some seemingly obvious, and indeed not difficult to prove, properties of the rational numbers as a subset of the real numbers.

The first property bears the name of Archimedes,² but Archimedes actually attributes this to Eudoxus.³ In any case, it is an Ancient Greek property.

2.2.14 Proposition (Archimedean property of \mathbb{R}) *Let $\epsilon \in \mathbb{R}_{>0}$. Then, for any $x \in \mathbb{R}$ there exists $k \in \mathbb{N}$ such that $k \cdot \epsilon > x$.*

Proof Let $\{q_j\}_{j \in \mathbb{N}}$ and $\{e_j\}_{j \in \mathbb{N}}$ be Cauchy sequences in \mathbb{Q} such that $x = [\{q_j\}_{j \in \mathbb{N}}]$ and $\epsilon = [\{e_j\}_{j \in \mathbb{N}}]$. By Proposition 2.1.14 there exists $M \in \mathbb{R}_{>0}$ such that $|q_j| < M$ for all $j \in \mathbb{N}$, and by Proposition 2.2.6 we may suppose that $e_j > \delta$ for $j \in \mathbb{N}$, for some $\delta \in \mathbb{Q}_{>0}$. Let $k \in \mathbb{N}$ satisfy $k > \frac{M+1}{\delta}$ (why is this possible?). Then we have

$$k \cdot e_j > \frac{M+1}{\delta} \cdot \delta = M+1 \geq q_j + 1, \quad j \in \mathbb{N}.$$

Now consider the sequence $\{k \cdot e_j - q_j\}_{j \in \mathbb{N}}$. This is a Cauchy sequence by Proposition 2.2.1 since it is a sum of products of Cauchy sequences. Moreover, our computations show that each term in the sequence is larger than 1. Also, this Cauchy sequence has the property that $[\{k \cdot e_j - q_j\}_{j \in \mathbb{N}}] = k \cdot \epsilon - x$. This shows that $k \cdot \epsilon - x \in \mathbb{R}_{>0}$, so giving the result. ■

The Archimedean property roughly says that there are no real numbers which are greater all rational numbers. The next result says that no real numbers that are smaller than all rational numbers.

2.2.15 Proposition *If $\epsilon \in \mathbb{R}_{>0}$ then there exists $q \in \mathbb{Q}_{>0}$ such that $q < \epsilon$.*

Proof Since $\epsilon^{-1} > 0$ let $k \in \mathbb{N}$ satisfy $k \cdot 1 > \epsilon^{-1}$ by Proposition 2.2.14. Then taking $q = k^{-1} \in \mathbb{Q}_{>0}$ gives $q < \epsilon$. ■

Using the preceding two results, it is then easy to see that arbitrarily near any real number lies a rational number.

2.2.16 Proposition (Real numbers are well approximated by rational numbers I) *If $x \in \mathbb{R}$ and if $\epsilon \in \mathbb{R}_{>0}$, then there exists $q \in \mathbb{Q}$ such that $|x - q| < \epsilon$.*

Proof If $x = 0$ then the result follows by taking $q = 0$. Let us next suppose that $x > 0$. If $x < \epsilon$ then the result follows by taking $q = 0$, so we assume that $x \geq \epsilon$. Let $\delta \in \mathbb{Q}_{>0}$ satisfy $\delta < \epsilon$ by Proposition 2.2.15. Then use Proposition 2.2.14 to choose $k \in \mathbb{N}$ to satisfy $k \cdot \delta > x$. Moreover, since $x > 0$, we will assume that k is the smallest such number. Since $x \geq \epsilon$, $k \geq 2$. Thus $(k-1) \cdot \delta \leq x$ since k is the smallest natural number for which $k \cdot \delta > x$. Now we compute

$$0 \leq x - (k-1) \cdot \delta < k \cdot \delta - (k-1) \cdot \delta = \delta < \epsilon.$$

It is now easy to check that the result holds by taking $q = (k-1) \cdot \delta$. The situation when $x < 0$ is easily shown to follow from the situation when $x > 0$. ■

²Archimedes of Syracuse (287 BC–212 BC) was a Greek mathematician and physicist (although in that era such classifications of scientific aptitude were less rigid than they are today). Much of his mathematical work was in the area of geometry, but many of Archimedes' best known achievements were in physics (e.g., the Archimedean Principle in fluid mechanics). The story goes that when the Romans captured Syracuse in 212 BC, Archimedes was discovered working on some mathematical problem, and struck down in the act by a Roman soldier.

³Eudoxus of Cnidus (408 BC–355 BC) was a Greek mathematician and astronomer. His mathematical work was concerned with geometry and numbers.

The following stronger result is also useful, and can be proved along the same lines as Proposition 2.2.16, using the Archimedean property of \mathbb{R} . The reader is asked to do this as Exercise 2.2.2.

2.2.17 Corollary (Real numbers are well approximated by rational numbers II) *If $x, y \in \mathbb{R}$ with $x < y$, then there exists $q \in \mathbb{Q}$ such that $x < q < y$.*

One can also show that irrational numbers have the same property.

2.2.18 Proposition (Real numbers are well approximated by rational numbers) *If $x \in \mathbb{R}$ and if $\epsilon \in \mathbb{R}_{>0}$, then there exists $y \in \mathbb{R} \setminus \mathbb{Q}$ such that $|x - y| < \epsilon$.*

Proof By Corollary 2.2.17 choose $q_1, q_2 \in \mathbb{Q}$ such that $x - \epsilon < q_1 < q_2 < x + \epsilon$. Then the number

$$y = q_1 + \frac{q_2 - q_1}{\sqrt{2}}$$

is irrational and satisfies $q_1 < y < q_2$. Therefore, $x - \epsilon < y < x + \epsilon$, or $|x - y| < \epsilon$. ■

2.2.5 The extended real line

It is sometimes convenient to be able to talk about the concept of “infinity” in a somewhat precise way. We do so by using the following idea.

2.2.19 Definition (Extended real line) The *extended real line* is the set $\mathbb{R} \cup \{-\infty\} \cup \{\infty\}$, and we denote this set by $\overline{\mathbb{R}}$. •

Note that in this definition the symbols “ $-\infty$ ” and “ ∞ ” are to simply be thought of as labels given to the elements of the singletons $\{-\infty\}$ and $\{\infty\}$. That they somehow correspond to our ideas of what “infinity” means is a consequence of placing some additional structure on $\overline{\mathbb{R}}$, as we now describe.

First we define “arithmetic” in $\overline{\mathbb{R}}$. We can also define some rules for arithmetic in $\overline{\mathbb{R}}$.

2.2.20 Definition (Addition and multiplication in $\overline{\mathbb{R}}$) For $x, y \in \overline{\mathbb{R}}$, define

$$x + y = \begin{cases} x + y, & x, y \in \mathbb{R}, \\ \infty, & x \in \mathbb{R}, y = \infty, \text{ or } x = \infty, y \in \mathbb{R}, \\ \infty, & x = y = \infty, \\ -\infty, & x = -\infty, y \in \mathbb{R} \text{ or } x \in \mathbb{R}, y = -\infty, \\ -\infty, & x = y = -\infty. \end{cases}$$

The operations $\infty + (-\infty)$ and $(-\infty) + \infty$ are undefined. Also define

$$x \cdot y = \begin{cases} x \cdot y, & x, y \in \mathbb{R}, \\ \infty, & x \in \mathbb{R}_{>0}, y = \infty, \text{ or } x = \infty, y \in \mathbb{R}_{>0}, \\ \infty, & x \in \mathbb{R}_{<0}, y = -\infty, \text{ or } x = -\infty, y \in \mathbb{R}_{<0}, \\ \infty, & x = y = \infty, \text{ or } x = y = -\infty, \\ -\infty, & x \in \mathbb{R}_{>0}, y = -\infty, \text{ or } x = -\infty, y \in \mathbb{R}_{>0}, \\ -\infty, & x \in \mathbb{R}_{<0}, y = \infty, \text{ or } x = \infty, y \in \mathbb{R}_{<0}, \\ -\infty, & x = \infty, y = -\infty \text{ or } x = -\infty, y = \infty, \\ 0, & x = 0, y \in \{-\infty, \infty\} \text{ or } x \in \{-\infty, \infty\}, y = 0. \end{cases} \bullet$$

2.2.21 Remarks

1. The above definitions of addition and multiplication on $\overline{\mathbb{R}}$ *do not* make this a field. Thus, in some sense, the operations are simply notation, since they do not have the usual properties we associate with addition and multiplication.
2. Note we *do* allow multiplication between 0 and $-\infty$ and ∞ . This convention is not universally agreed upon, but it will be useful for us to do adopt this convention in Chapter II-1. •

2.2.22 Definition (Order on $\overline{\mathbb{R}}$) For $x, y \in \overline{\mathbb{R}}$, write

$$x \leq y \iff \begin{cases} x = y, & \text{or} \\ x, y \in \mathbb{R}, x \leq y, & \text{or} \\ x \in \mathbb{R}, y = \infty, & \text{or} \\ x = -\infty, y \in \mathbb{R}, & \text{or} \\ x = -\infty, y = \infty. & \bullet \end{cases}$$

This is readily verified to be a total order on $\overline{\mathbb{R}}$, with $-\infty$ being the least element and ∞ being the greatest element of $\overline{\mathbb{R}}$. As with \mathbb{R} , we have the notation

$$\overline{\mathbb{R}}_{>0} = \{x \in \overline{\mathbb{R}} \mid x > 0\}, \quad \overline{\mathbb{R}}_{\geq 0} = \{x \in \overline{\mathbb{R}} \mid x \geq 0\}.$$

Finally, we can extend the absolute value on \mathbb{R} to $\overline{\mathbb{R}}$.

2.2.23 Definition (Extended real absolute value function) The *extended real absolute function* is the map from $\overline{\mathbb{R}}$ to $\overline{\mathbb{R}}_{\geq 0}$, denoted by $x \mapsto |x|$, and defined by

$$|x| = \begin{cases} |x|, & x \in \mathbb{R}, \\ \infty, & x = \infty, \\ \infty, & x = -\infty. \bullet \end{cases}$$

Exercises

- 2.2.1 Let $q \in \mathbb{Q} \setminus \{0\}$ and $x \in \mathbb{R} \setminus \mathbb{Q}$. Show the following:
- (a) $q + x$ is irrational;
 - (b) qx is irrational;
 - (c) $\frac{x}{q}$ is irrational;
 - (d) $\frac{q}{x}$ is irrational.
- 2.2.2 Prove Corollary 2.2.17.
- 2.2.3 Prove Proposition 2.2.11.
- 2.2.4 Show that the order and absolute value on \mathbb{R} agree with those on \mathbb{Q} . That is to say, show the following:
- (a) for $q, r \in \mathbb{Q}$, $q < r$ if and only if $i_{\mathbb{Q}}(q) < i_{\mathbb{Q}}(r)$;
 - (b) for $q \in \mathbb{Q}$, $|q| = |i_{\mathbb{Q}}(q)|$.
- (Note that we see clearly here the abuse of notation that follows from using $<$ for both the order on \mathbb{Z} and \mathbb{Q} and from using $|\cdot|$ as the absolute value both on \mathbb{Z} and \mathbb{Q} . It is expected that the reader can understand where the notational abuse occurs.)
- 2.2.5 Do the following:

- (a) show that if $x \in \mathbb{R}_{>0}$ satisfies $x < 1$, then $x^k < x$ for each $k \in \mathbb{N}$ satisfying $k \geq 2$;
- (b) show that if $x \in \mathbb{R}_{>0}$ satisfies $x > 1$, then $x^k > x$ for each $k \in \mathbb{N}$ satisfying $k \geq 2$.
- 2.2.6 Show that, for $t, s \in \mathbb{R}$, $||t| - |s|| \leq |t - s|$.
- 2.2.7 Show that if $s, t \in \mathbb{R}$ satisfy $s < t$, then there exists $q \in \mathbb{Q}$ such that $s < q < t$.

Section 2.3

Sequences in \mathbb{R}

In our construction of the real numbers, sequences played a key rôle, inasmuch as Cauchy sequences of rational numbers were integral to our definition of real numbers. In this section we study sequences of real numbers. In particular, in Theorem 2.3.4 we prove the result, absolutely fundamental in analysis, that \mathbb{R} is “complete,” meaning that Cauchy sequences of real numbers converge.

Do I need to read this section? If you do not already know the material in this section, then it ought to be read. It is also worth the reader spending some time over the idea that Cauchy sequences of real numbers converge, as compared to rational numbers where this is not the case. The same idea will arise in more abstract settings in Chapter II-3, and so it will pay to understand it well in the simplest case. •

2.3.1 Definitions and properties of sequences

In this section we consider the extension to \mathbb{R} of some of the ideas considered in Section 2.1.2 concerning sequences in \mathbb{Q} . As we shall see, it is via sequences, and other equivalent properties, that the nature of the difference between \mathbb{Q} and \mathbb{R} is spelled out quite clearly.

We begin with definitions, generalising in a trivial way the similar definitions for \mathbb{Q} .

2.3.1 Definition (Cauchy sequence, convergent sequence, bounded sequence, monotone sequence) Let $\{x_j\}_{j \in \mathbb{N}}$ be a sequence in \mathbb{R} . The sequence:

- (i) is a **Cauchy sequence** if, for each $\epsilon \in \mathbb{R}_{>0}$, there exists $N \in \mathbb{N}$ such that $|x_j - x_k| < \epsilon$ for $j, k \geq N$;
- (ii) **converges to s_0** if, for each $\epsilon \in \mathbb{R}_{>0}$, there exists $N \in \mathbb{N}$ such that $|x_j - s_0| < \epsilon$ for $j \geq N$;
- (iii) **diverges** if it does not converge to any element in \mathbb{R} ;
- (iv) is **bounded above** if there exists $M \in \mathbb{R}$ such that $x_j < M$ for each $j \in \mathbb{N}$;
- (v) is **bounded below** if there exists $M \in \mathbb{R}$ such that $x_j > M$ for each $j \in \mathbb{N}$;
- (vi) is **bounded** if there exists $M \in \mathbb{R}_{>0}$ such that $|x_j| < M$ for each $j \in \mathbb{N}$;
- (vii) is **monotonically increasing** if $x_{j+1} \geq x_j$ for $j \in \mathbb{N}$;
- (viii) is **strictly monotonically increasing** if $x_{j+1} > x_j$ for $j \in \mathbb{N}$;
- (ix) is **monotonically decreasing** if $x_{j+1} \leq x_j$ for $j \in \mathbb{N}$;
- (x) is **strictly monotonically decreasing** if $x_{j+1} < x_j$ for $j \in \mathbb{N}$. •

Associated with the notion of convergence is the notion of a limit. We also, for convenience, wish to allow sequences with infinite limits. This makes for some rather subtle use of language, so the reader should pay attention to this.

2.3.2 Definition (Limit of a sequence) Let $\{x_j\}_{j \in \mathbb{N}}$ be a sequence.

- (i) If $\{x_j\}_{j \in \mathbb{N}}$ converges to s_0 , then the sequence has s_0 as a **limit**, and we write $\lim_{j \rightarrow \infty} x_j = s_0$.

- (ii) If, for every $M > 0$, there exists $N \in \mathbb{N}$ such that $x_j > M$ (resp. $x_k < -M$) for $j \geq N$, then the sequence **diverges to ∞** (resp. **diverges to $-\infty$**), and we write $\lim_{j \rightarrow \infty} x_j = \infty$ (resp. $\lim_{j \rightarrow \infty} x_j = -\infty$);
- (iii) If $\lim_{j \rightarrow \infty} x_j \in \mathbb{R}$, then the limit of the sequence $\{x_j\}_{j \in \mathbb{N}}$ **exists**.
- (iv) If the limit of the sequence $\{x_j\}_{j \in \mathbb{N}}$ does not exist, does not diverge to ∞ , or does not diverge to $-\infty$, then the sequence is **oscillatory**. •

The reader can prove in Exercise 2.3.1 that limits, if they exist, are unique.

That convergent sequences are Cauchy, and that Cauchy sequences are bounded follows in exactly the same manner as the analogous results, stated as Propositions 2.1.13 and 2.1.14, for \mathbb{Q} . Let us state the result here for reference.

2.3.3 Proposition (Cauchy sequences are bounded) *If $\{x_j\}_{j \in \mathbb{N}}$ is a Cauchy sequence in \mathbb{R} , then it is bounded.*

Moreover, what is true for \mathbb{R} , and that is not true for \mathbb{Q} , is that every Cauchy sequence converges.

2.3.4 Theorem (Cauchy sequences in \mathbb{R} converge) *If $\{x_j\}_{j \in \mathbb{N}}$ is a Cauchy sequence in \mathbb{R} then there exists $s_0 \in \mathbb{R}$ such that $\{x_j\}_{j \in \mathbb{N}}$ converges to s_0 .*

Proof For $j \in \mathbb{N}$ choose $q_j \in \mathbb{Q}_{>0}$ such that $|x_j - q_j| < \frac{1}{j}$, this being possible by Proposition 2.2.16. For $\epsilon > 0$ let $N_1 \in \mathbb{N}$ satisfy $|x_j - x_k| < \frac{\epsilon}{2}$ for $j, k \geq N_1$. By Proposition 2.2.14 let $N_2 \in \mathbb{N}$ satisfy $N_2 \cdot 1 > 4\epsilon^{-1}$, and let N be the larger of N_1 and N_2 . Then, for $j, k \geq N$, we have

$$|q_j - q_k| = |q_j - x_j + x_j - x_k + x_k - q_k| \leq |x_j - q_j| + |x_j - x_k| + |x_k - q_k| < \frac{1}{j} + \frac{\epsilon}{2} + \frac{1}{k} < \epsilon.$$

Thus $\{q_j\}_{j \in \mathbb{N}}$ is a Cauchy sequence, and so we define $s_0 = [\{q_j\}_{j \in \mathbb{N}}]$.

Now we show that $\{q_j\}_{j \in \mathbb{N}}$ converges to s_0 . Let $\epsilon > 0$ and take $N \in \mathbb{N}$ such that $|q_j - q_k| < \frac{\epsilon}{2}$, $j, k \geq N$, and rewrite this as

$$\frac{\epsilon}{2} < q_j - q_k + \epsilon, \quad \frac{\epsilon}{2} < -q_k + q_k + \epsilon, \quad j, k \geq N. \quad (2.4)$$

For $j_0 \geq N$ consider the sequence $\{q_j - q_{j_0} + \epsilon\}_{j \in \mathbb{N}}$. This is a Cauchy sequence by Proposition 2.2.1. Moreover, by Proposition 2.2.6, $[\{q_j - q_{j_0} + \epsilon\}_{j \in \mathbb{N}}] > 0$, using the first of the inequalities in (2.4). Thus we have $x - q_{j_0} + \epsilon > 0$, or

$$-\epsilon < x - q_{j_0}, \quad j_0 \geq N.$$

Arguing similarly, but using the second of the inequalities (2.4), we determine that

$$x - q_{j_0} < \epsilon, \quad j_0 \geq N.$$

This gives $|s_0 - q_j| < \epsilon$ for $j \geq N$, so showing that $\{q_j\}_{j \in \mathbb{N}}$ converges to s_0 .

Finally, we show that $\{x_j\}_{j \in \mathbb{N}}$ converges to s_0 . Let $\epsilon > 0$ and take $N_1 \in \mathbb{N}$ such that $|s_0 - q_j| < \frac{\epsilon}{2}$ for $j \geq N_1$. Also choose $N_2 \in \mathbb{N}$ such that $N_2 \cdot 1 > 2\epsilon^{-1}$ by Proposition 2.2.14. If N is the larger of N_1 and N_2 , then we have

$$|s_0 - x_j| = |s_0 - q_j + q_j - x_j| \leq |s_0 - q_j| + |q_j - x_j| < \frac{\epsilon}{2} + \frac{1}{j} < \epsilon,$$

for $j \geq N$, so giving the result. ■

2.3.5 Remark (Completeness of \mathbb{R}) The property of \mathbb{R} that Cauchy sequences are convergent gives, in the more general setting of Section II-2.1.6, \mathbb{R} the property of being **complete**. Completeness is an extremely important concept in analysis. •

2.3.2 Some properties equivalent to the completeness of \mathbb{R}

Using the fact that Cauchy sequences converge, it is easy to prove two other important features of \mathbb{R} , both of which seem obvious intuitively.

2.3.6 Theorem (Bounded subsets of \mathbb{R} have a least upper bound) *If $S \subset \mathbb{R}$ is nonempty and possesses an upper bound with respect to the standard total order \leq , then S possesses a least upper bound with respect to the same total order.*

Proof Since S has an upper bound, there exists $y \in \mathbb{R}$ such that $x \leq y$ for all $x \in S$. Now choose some $x \in S$. We then define two sequences $\{x_j\}_{j \in \mathbb{N}}$ and $\{y_j\}_{j \in \mathbb{N}}$ recursively as follows:

1. define $x_1 = x$ and $y_1 = y$;
2. suppose that x_j and y_j have been defined;
3. if there exists $z \in S$ with $\frac{1}{2}(x_j + y_j) < z \leq y_j$, take $x_{j+1} = z$ and $y_{j+1} = y_j$;
4. if there is no $z \in S$ with $\frac{1}{2}(x_j + y_j) < z \leq y_j$, take $x_{j+1} = x_j$ and $y_{j+1} = \frac{1}{2}(x_j + y_j)$.

A lemma characterises these sequences.

1 Lemma *The sequences $\{x_j\}_{j \in \mathbb{N}}$ and $\{y_j\}_{j \in \mathbb{N}}$ have the following properties:*

- (i) $x_j \in S$ for $j \in \mathbb{N}$;
- (ii) $x_{j+1} \geq x_j$ for $j \in \mathbb{N}$;
- (iii) y_j is an upper bound for S for $j \in \mathbb{N}$;
- (iv) $y_{j+1} \leq y_j$ for $j \in \mathbb{N}$;
- (v) $0 \leq y_j - x_j \leq \frac{1}{2^j}(y - x)$ for $j \in \mathbb{N}$.

Proof We prove the result by induction on j . The result is obviously true for $j = 0$. Now suppose the result true for $j \in \{1, \dots, k\}$.

First take the case where there exists $z \in S$ with $\frac{1}{2}(x_k + y_k) < z \leq y_k$, so that $x_{k+1} = z$ and $y_{k+1} = y_k$. Clearly $x_{k+1} \in S$ and $y_{k+1} \geq y_k$. Since $y_k \geq x_k$ by the induction hypotheses, $\frac{1}{2}(x_k + y_k) \geq x_k$ giving $x_{k+1} = z \geq x_k$. By the induction hypotheses, y_{k+1} is an upper bound for S . By definition of x_{k+1} and y_{k+1} ,

$$y_{k+1} - x_{k+1} = y_k - z \geq 0$$

and

$$y_{k+1} - x_{k+1} = y_k - z = y_k - \frac{1}{2}(y_k + x_k) = \frac{1}{2}(y_k - x_k),$$

giving $y_{k+1} - x_{k+1} \leq \frac{1}{2^{k+1}}(y - x)$ by the induction hypotheses.

Now we take the case where there is no $z \in S$ with $\frac{1}{2}(x_j + y_j) < z \leq y_j$, so that $x_{k+1} = x_k$ and $y_{k+1} = \frac{1}{2}(x_k + y_k)$. Clearly $x_{k+1} \geq x_k$ and $x_{k+1} \in S$. If y_{k+1} were not an upper bound for S , then there exists $a \in S$ such that $a > y_{k+1}$. By the induction hypotheses, y_k is an upper bound for S so $a \leq y_k$. But this means that $\frac{1}{2}(y_k + x_k) < a \leq y_k$, contradicting our assumption concerning the nonexistence of $z \in S$ with $\frac{1}{2}(x_j + y_j) < z \leq y_j$. Thus y_{k+1} is an upper bound for S . Since $x_k \leq y_k$ by the induction hypotheses,

$$y_{k+1} = \frac{1}{2}(y_k + x_k) \leq y_k.$$

Also

$$y_{k+1} - x_{k+1} = \frac{1}{2}(y_k - x_k)$$

by the induction hypotheses. This completes the proof. ▼

The following lemma records a useful fact about the sequences $\{x_j\}_{j \in \mathbb{N}}$ and $\{y_j\}_{j \in \mathbb{N}}$.

2 Lemma *Let $\{x_j\}_{j \in \mathbb{N}}$ and $\{y_j\}_{j \in \mathbb{N}}$ be sequences in \mathbb{R} satisfying:*

- (i) $x_{j+1} \geq x_j$, $j \in \mathbb{N}$;
- (ii) $y_{j+1} \leq y_j$, $j \in \mathbb{N}$;
- (iii) the sequence $\{y_j - x_j\}_{j \in \mathbb{N}}$ converges to 0.

Then $\{x_j\}_{j \in \mathbb{N}}$ and $\{y_j\}_{j \in \mathbb{N}}$ converge, and converge to the same limit.

Proof First we claim that $x_j \leq y_k$ for all $j, k \in \mathbb{N}$. Indeed, suppose not. Then there exists $j, k \in \mathbb{N}$ such that $x_j > y_k$. If N is the larger of j and k , then we have $y_N \leq y_k < x_j \leq x_N$. This implies that

$$x_m - y_m \geq x_j - y_m \geq x_j - y_k > 0, \quad m \geq N,$$

which contradicts the fact that $\{y_j - x_j\}_{j \in \mathbb{N}}$ converges to zero.

Now, for $\epsilon > 0$ let $N \in \mathbb{N}$ satisfy $|y_j - x_j| < \epsilon$ for $j \geq N$, or, simply, $y_j - x_j < \epsilon$ for $j \geq N$. Now let $j, k \geq N$, and suppose that $j \geq k$. Then

$$0 \leq x_j - x_k \leq x_j - y_k < \epsilon.$$

Similarly, if $j \leq k$ we have $0 \leq x_k - x_j < \epsilon$. In other words, $|x_j - x_k| < \epsilon$ for $j, k \geq N$. Thus $\{x_j\}_{j \in \mathbb{N}}$ is a Cauchy sequence. In like manner one shows that $\{y_j\}_{j \in \mathbb{N}}$ is also a Cauchy sequence. Therefore, by Theorem 2.3.4, these sequences converge, and let us denote their limits by s_0 and t_0 , respectively. However, since $\{x_j\}_{j \in \mathbb{N}}$ and $\{y_j\}_{j \in \mathbb{N}}$ are equivalent Cauchy sequences in the sense of Definition 2.1.16, it follows that $s_0 = t_0$. \blacktriangledown

Using Lemma 1 we easily verify that the sequences $\{x_j\}_{j \in \mathbb{N}}$ and $\{y_j\}_{j \in \mathbb{N}}$ satisfy the hypotheses of Lemma 2. Therefore these sequences converge to a common limit, which we denote by s . We claim that s is a least upper bound for S . First we show that it is an upper bound. Suppose that there is $x \in S$ such that $x > s$ and define $\epsilon = x - s$. Since $\{y_j\}_{j \in \mathbb{N}}$ converges to s , there exists $N \in \mathbb{N}$ such that $|s - y_j| < \epsilon$ for $j \geq N$. Then, for $j \geq N$,

$$y_j - s < \epsilon = x - s,$$

implying that $y_j < x$, and so contradicting Lemma 1.

Finally, we need to show that s is a least upper bound. To see this, let b be an upper bound for S and suppose that $b < s$. Define $\epsilon = s - b$, and choose $N \in \mathbb{N}$ such that $|s - x_j| < \epsilon$ for $j \geq N$. Then

$$s - x_j < \epsilon = s - b,$$

implying that $b < x_j$ for $j \geq N$. This contradicts the fact, from Lemma 1, that $x_j \in S$ and that b is an upper bound for S . \blacksquare

As we shall explain more fully in Aside 2.3.8, the least upper bound property of the real numbers as stated in the preceding theorem is actually *equivalent* to the completeness of \mathbb{R} . In fact, the least upper bound property forms the basis for an alternative definition of the real numbers using *Dedekind cuts*.⁴ Here the idea is that one defines a real number as being a splitting of the rational numbers into two halves, one corresponding to the rational numbers less than the real number one is defining, and the other corresponding to the rational numbers greater than the real number one is defining. Historically, Dedekind cuts provided the first rigorous construction of the real numbers. We refer to for the details of this construction. We also comment, as we discuss in Aside 2.3.8, that any construction of the real numbers with the property of completeness, or an equivalent, will produce something that is “essentially” the real numbers as we have defined them.

Another consequence of Theorem 2.3.4 is the following.

⁴After Julius Wilhelm Richard Dedekind (1831–1916), the German mathematician, did work in the areas of analysis, ring theory, and set theory. His rigorous mathematical style has had a strong influence on modern mathematical presentation.

2.3.7 Theorem (Bounded, monotonically increasing sequences in \mathbb{R} converge) *If $\{x_j\}_{j \in \mathbb{N}}$ is a bounded, monotonically increasing sequence in \mathbb{R} , then it converges.*

Proof The subset $\{x_j\}_{j \in \mathbb{N}}$ of \mathbb{R} has an upper bound, since it is bounded. By Theorem 2.3.6 let b be the least upper bound for this set. We claim that $\{x_j\}_{j \in \mathbb{N}}$ converges to b . Indeed, let $\epsilon > 0$. We claim that there exists some $N \in \mathbb{N}$ such that $b - x_N < \epsilon$ since b is a least upper bound. Indeed, if there is no such N , then $b \geq x_j + \epsilon$ for all $j \in \mathbb{N}$ and so $b - \frac{\epsilon}{2}$ is an upper bound for $\{x_j\}_{j \in \mathbb{N}}$ that is smaller than b . Now, with N chosen so that $b - x_N < \epsilon$, the fact that $\{x_j\}_{j \in \mathbb{N}}$ is monotonically increasing implies that $|b - x_j| < \epsilon$ for $j \geq N$, as desired. ■

It turns out that Theorems 2.3.4, 2.3.6 and 2.3.7 are equivalent. But to make sense of this requires one to step outside the concrete representation we have given for the real numbers to a more axiomatic one. This can be skipped, so we present it as an aside.

2.3.8 Aside (Complete ordered fields) An *ordered field* is a field \mathbb{F} (see Definition 4.3.1 for the definition of a field) equipped with a total order satisfying the conditions

1. if $x < y$ then $x + z < y + z$ for $x, y, z \in \mathbb{F}$ and
2. if $0 < x, y$ then $0 < x \cdot y$.

Note that in an ordered field, or in any field, one can define the absolute value exactly as we have done for \mathbb{Z} , \mathbb{Q} , and \mathbb{R} (although it will not generally have all the properties that the absolute value has in these special cases). There are many examples of ordered fields, of which \mathbb{Q} and \mathbb{R} are two that we have seen. However, if one adds to the conditions for an ordered field an additional condition, then this turns out to essentially uniquely specify the set of real numbers. (We say “essentially” since the uniqueness is up to a bijection that preserves the field structure as well as the order.) This additional structure comes in various forms, of which three are as stated in Theorems 2.3.4, 2.3.6 and 2.3.7. To be precise, we have the following theorem.

Theorem *If \mathbb{F} is an ordered field, then the following statements are equivalent:*

- (i) *every Cauchy sequence converges;*
- (ii) *each set possessing an upper bound possesses a least upper bound;*
- (iii) *each bounded, monotonically increasing sequence converges.*

We have almost proved this theorem with our arguments above. To see this, note that in the proof of Theorem 2.3.6 we use the fact that Cauchy sequences converge. Moreover, the argument can easily be adapted from the special case of \mathbb{R} to a general ordered field. This gives the implication (i) \implies (ii) in the theorem above. In like manner, the proof of Theorem 2.3.7 gives the implication (ii) \implies (iii), since the proof is again easily seen to be valid for a general ordered field. The argument for the implication (iii) \implies (i) is outlined in Exercise 2.3.6. An ordered field satisfying any one of the three equivalent conditions (i), (ii), and (iii) is called a **complete ordered field**. Thus there is essentially only one complete ordered field, and it is \mathbb{R} . ♠

2.3.3 Tests for convergence of sequences

There is generally no algorithmic way, other than checking the definition, to ascertain when a sequence converges. However, there are a few simple results that are often useful, and here we state some of these.

2.3.9 Proposition (Squeezing Principle) *Let $\{x_j\}_{j \in \mathbb{N}}$, $\{y_j\}_{j \in \mathbb{N}}$, and $\{z_j\}_{j \in \mathbb{N}}$ be sequences in \mathbb{R} satisfying*

(i) $x_j \leq z_j \leq y_j$ for all $j \in \mathbb{N}$ and

(ii) $\lim_{j \rightarrow \infty} x_j = \lim_{j \rightarrow \infty} y_j = \alpha$.

Then $\lim_{j \rightarrow \infty} z_j = \alpha$.

Proof Let $\epsilon > 0$ and let $N_1, N_2 \in \mathbb{N}$ have the property that $|x_j - \alpha| < \frac{\epsilon}{3}$ for $j \geq N_1$ and $|y_j - \alpha| < \frac{\epsilon}{3}$. Then, for $j \geq \max\{N_1, N_2\}$,

$$|x_j - y_j| = |x_j - \alpha + \alpha - y_j| \leq |x_j - \alpha| + |y_j - \alpha| < \frac{2\epsilon}{3},$$

using the triangle inequality. Then, for $j \geq \max\{N_1, N_2\}$, we have

$$|z_j - \alpha| = |z_j - x_j + x_j - \alpha| \leq |z_j - x_j| + |x_j - \alpha| \leq |y_j - x_j| + |x_j - \alpha| = \epsilon,$$

again using the triangle inequality. ■

The next test for convergence of a series is sometimes useful.

2.3.10 Proposition (Ratio Test for sequences) Let $\{x_j\}_{j \in \mathbb{N}}$ be a sequence in \mathbb{R} for which $\lim_{j \rightarrow \infty} \left| \frac{x_{j+1}}{x_j} \right| = \alpha$. If $\alpha < 1$ then the sequence $\{x_j\}_{j \in \mathbb{N}}$ converges to 0, and if $\alpha > 1$ then the sequence $\{x_j\}_{j \in \mathbb{N}}$ diverges.

Proof For $\alpha < 1$, define $\beta = \frac{1}{2}(\alpha + 1)$. Then $\alpha < \beta < 1$. Now take $N \in \mathbb{N}$ such that

$$\left| \left| \frac{x_{j+1}}{x_j} \right| - \alpha \right| < \frac{1}{2}(1 - \alpha), \quad j > N.$$

This implies that

$$\left| \frac{x_{j+1}}{x_j} \right| < \beta.$$

Now, for $j > N$,

$$|x_j| < \beta |x_{j-1}| < \beta^2 |x_{j-2}| < \cdots < \beta^{j-N} |x_N|.$$

Clearly the sequence $\{x_j\}_{j \in \mathbb{N}}$ converges to 0 if and only if the sequence obtained by replacing the first N terms by 0 also converges to 0. If this latter sequence is denoted by $\{y_j\}_{j \in \mathbb{N}}$, then we have

$$0 \leq y_j \leq \frac{|x_N|}{\beta^N} \beta^j.$$

The sequence $\left\{ \frac{|x_N|}{\beta^N} \beta^j \right\}_{j \in \mathbb{N}}$ converges to 0 since $\beta < 1$, and so this part of the result follows from the Squeezing Principle.

For $\alpha > 1$, there exists $N \in \mathbb{N}$ such that, for all $j \geq N$, $x_j \neq 0$. Consider the sequence $\{y_j\}_{j \in \mathbb{N}}$ which is 0 for the first N terms, and satisfies $y_j = x_j^{-1}$ for the remaining terms. We then have $\left| \frac{y_{j+1}}{y_j} \right| < \alpha^{-1} < 1$, and so, from the first part of the proof, the sequence $\{y_j\}_{j \in \mathbb{N}}$ converges to 0. Thus the sequence $\{|y_j|\}_{j \in \mathbb{N}}$ converges to ∞ , which prohibits the sequence $\{y_j\}_{j \in \mathbb{N}}$ from converging. ■

In Exercise 2.3.4 the reader can explore the various possibilities for the ratio test when $\lim_{j \rightarrow \infty} \left| \frac{x_{j+1}}{x_j} \right| = 1$.

2.3.4 lim sup and lim inf

Associated with the least upper bound property of \mathbb{R} is a useful notion that weakens the usual idea of convergence. We recall from Definition 1.5.11 the notation $\sup S$ and $\inf S$ for the least upper bound and greatest lower bound, respectively, associated to a partial order. In order for us to make a sensible definition, we first prove a simple result.

2.3.11 Proposition For any sequence $\{x_j\}_{j \in \mathbb{N}}$ in \mathbb{R} , the limits

$$\lim_{N \rightarrow \infty} \left(\sup \{x_j \mid j \geq N\} \right), \quad \lim_{N \rightarrow \infty} \left(\inf \{x_j \mid j \geq N\} \right)$$

exist, diverge to ∞ , or diverge to $-\infty$.

Proof Note that the sequences $\{\sup \{x_j \mid j \geq N\}\}_{N \in \mathbb{N}}$ and $\{\inf \{x_j \mid j \geq N\}\}_{N \in \mathbb{N}}$ in $\overline{\mathbb{R}}$ are monotonically decreasing and monotonically increasing, respectively, with respect to the natural order on $\overline{\mathbb{R}}$. Moreover, note that a monotonically increasing sequence in $\overline{\mathbb{R}}$ is either bounded by some element of \mathbb{R} , or it is not. If the sequence is upper bounded by some element of \mathbb{R} , then by Theorem 2.3.7 it either converges or is the sequence $\{-\infty\}_{j \in \mathbb{N}}$. If it is not bounded by some element in \mathbb{R} , then either it diverges to ∞ , or it is the sequence $\{\infty\}_{j \in \mathbb{N}}$ (this second case cannot arise in the specific case of the monotonically increasing sequence $\{\sup \{x_j \mid j \geq N\}\}_{N \in \mathbb{N}}$. In all cases, the limit $\lim_{N \rightarrow \infty} (\sup \{x_j \mid j \geq N\})$ exists or diverges to ∞ . A similar argument holds for $\lim_{N \rightarrow \infty} (\inf \{x_j \mid j \geq N\})$. ■

2.3.12 Definition (lim sup and lim inf) For a sequence $\{x_j\}_{j \in \mathbb{N}}$ in \mathbb{R} denote

$$\begin{aligned} \limsup_{j \rightarrow \infty} x_j &= \lim_{N \rightarrow \infty} \left(\sup \{x_j \mid j \geq N\} \right), \\ \liminf_{j \rightarrow \infty} x_j &= \lim_{N \rightarrow \infty} \left(\inf \{x_j \mid j \geq N\} \right). \end{aligned} \quad \bullet$$

Before we get to characterising lim sup and lim inf, we give some examples to illustrate all the cases that can arise.

2.3.13 Examples (lim sup and lim inf)

1. Consider the sequence $\{x_j = (-1)^j\}_{j \in \mathbb{N}}$. Here we have $\limsup_{j \rightarrow \infty} x_j = 1$ and $\liminf_{j \rightarrow \infty} x_j = -1$.
2. Consider the sequence $\{x_j = j\}_{j \in \mathbb{N}}$. Here $\limsup_{j \rightarrow \infty} x_j = \liminf_{j \rightarrow \infty} x_j = \infty$.
3. Consider the sequence $\{x_j = -j\}_{j \in \mathbb{N}}$. Here $\limsup_{j \rightarrow \infty} x_j = \liminf_{j \rightarrow \infty} x_j = -\infty$.
4. Define

$$x_j = \begin{cases} j, & j \text{ even,} \\ 0, & j \text{ odd.} \end{cases}$$

We then have $\limsup_{j \rightarrow \infty} x_j = \infty$ and $\liminf_{j \rightarrow \infty} x_j = 0$.

5. Define

$$x_j = \begin{cases} -j, & j \text{ even,} \\ 0, & j \text{ odd.} \end{cases}$$

We then have $\limsup_{j \rightarrow \infty} x_j = 0$ and $\liminf_{j \rightarrow \infty} x_j = -\infty$.

6. Define

$$x_j = \begin{cases} j, & j \text{ even,} \\ -j, & j \text{ odd.} \end{cases}$$

We then have $\limsup_{j \rightarrow \infty} x_j = \infty$ and $\liminf_{j \rightarrow \infty} x_j = -\infty$. ■

There are many ways to characterise lim sup and lim inf, and we shall indicate but a few of these.

2.3.14 Proposition (Characterisation of lim sup) For a sequence $\{x_j\}_{j \in \mathbb{N}}$ in \mathbb{R} and $\alpha \in \mathbb{R}$, the following statements are equivalent:

- (i) $\alpha = \limsup_{j \rightarrow \infty} x_j$;
- (ii) $\alpha = \inf \{ \sup \{ x_j \mid j \geq k \} \mid k \in \mathbb{N} \}$;
- (iii) for each $\epsilon > 0$ the following statements hold:
 - (a) there exists $N \in \mathbb{N}$ such that $x_j < \alpha + \epsilon$ for all $j \geq N$;
 - (b) for an infinite number of $j \in \mathbb{N}$ it holds that $x_j > \alpha - \epsilon$.

Proof (i) \iff (ii) Let $y_k = \sup \{ x_j \mid j \geq k \}$ and note that the sequence $\{y_k\}_{k \in \mathbb{N}}$ is monotonically decreasing. Therefore, the sequence $\{y_k\}_{k \in \mathbb{N}}$ converges if and only if it is lower bounded. Moreover, if it converges, it converges to $\inf \{y_k\}_{k \in \mathbb{N}}$. Putting this all together gives the desired implications.

(i) \implies (iii) Let y_k be as in the preceding part of the proof. Since $\lim_{k \rightarrow \infty} y_k = \alpha$, for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|y_k - \alpha| < \epsilon$ for $k \geq N$. In particular, $y_N < \alpha + \epsilon$. Therefore, $x_j < \alpha + \epsilon$ for all $j \geq N$, so (iii a) holds. We also claim that, for every $\epsilon > 0$ and for every $N \in \mathbb{N}$, there exists $j \geq N$ such that $x_j > y_N - \epsilon$. Indeed, if $x_j \leq y_N - \epsilon$ for every $j \geq N$, then this contradicts the definition of y_N . Since $y_N \geq \alpha$ we have $x_j > y_N - \epsilon \geq \alpha - \epsilon$ for some j . Since N is arbitrary, (iii b) holds.

(iii) \implies (i) Condition (iii a) means that there exists $N \in \mathbb{N}$ such that $y_k < \alpha + \epsilon$ for all $k \geq N$. Condition (iii b) implies that $y_k > \alpha - \epsilon$ for all $k \in \mathbb{N}$. Combining these conclusions shows that $\lim_{k \rightarrow \infty} y_k = \alpha$, as desired. \blacksquare

The corresponding result for \liminf is the following. The proof follows in the same manner as the result for \limsup .

2.3.15 Proposition (Characterisation of \liminf) For a sequence $\{x_j\}_{j \in \mathbb{N}}$ in \mathbb{R} and $\alpha \in \mathbb{R}$, the following statements are equivalent:

- (i) $\alpha = \liminf_{j \rightarrow \infty} x_j$;
- (ii) $\alpha = \sup \{ \inf \{ x_j \mid j \geq k \} \mid k \in \mathbb{N} \}$;
- (iii) for each $\epsilon > 0$ the following statements hold:
 - (a) there exists $N \in \mathbb{N}$ such that $x_j > \alpha - \epsilon$ for all $j \geq N$;
 - (b) for an infinite number of $j \in \mathbb{N}$ it holds that $x_j < \alpha + \epsilon$.

Finally, we characterise the relationship between \limsup , \liminf , and \lim .

2.3.16 Proposition (Relationship between \limsup , \liminf , and \lim) For a sequence $\{x_j\}_{j \in \mathbb{N}}$ and $s_0 \in \mathbb{R}$, the following statements are equivalent:

- (i) $\lim_{j \rightarrow \infty} x_j = s_0$;
- (ii) $\limsup_{j \rightarrow \infty} x_j = \liminf_{j \rightarrow \infty} x_j = s_0$.

Proof (i) \implies (ii) Let $\epsilon > 0$ and take $N \in \mathbb{N}$ such that $|x_j - s_0| < \epsilon$ for all $j \geq N$. Then $x_j < s_0 + \epsilon$ and $x_j > s_0 - \epsilon$ for all $j \geq N$. The current implication now follows from Propositions 2.3.14 and 2.3.15.

(ii) \implies (i) Let $\epsilon > 0$. By Propositions 2.3.14 and 2.3.15 there exists $N_1, N_2 \in \mathbb{N}$ such that $x_j - s_0 < \epsilon$ for $j \geq N_1$ and $s_0 - x_j < \epsilon$ for $j \geq N_2$. Thus $|x_j - s_0| < \epsilon$ for $j \geq \max\{N_1, N_2\}$, giving this implication. \blacksquare

2.3.5 Multiple sequences

It will be sometimes useful for us to be able to consider sequences indexed, not by a single index, but by multiple indices. We consider the case here of two indices, and extensions to more indices are done by induction.

2.3.17 Definition (Double sequence) A *double sequence* in \mathbb{R} is a family of elements of \mathbb{R} indexed by $\mathbb{N} \times \mathbb{N}$. We denote a double sequence by $\{x_{jk}\}_{j,k \in \mathbb{N}}$, where x_{jk} is the image of $(j, k) \in \mathbb{N} \times \mathbb{N}$ in \mathbb{R} . •

It is not *a priori* obvious what it might mean for a double sequence to converge, so we should carefully say what this means.

2.3.18 Definition (Convergence of double sequences) Let $s_0 \in \mathbb{R}$. A double sequence $\{x_{jk}\}_{j,k \in \mathbb{N}}$:

- (i) *converges to* s_0 , and we write $\lim_{j,k \rightarrow \infty} x_{jk} = s_0$, if, for each $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|s_0 - x_{jk}| < \epsilon$ for $j, k \geq N$;
- (ii) has s_0 as a *limit* if it converges to s_0 .
- (iii) is *convergent* if it converges to some member of \mathbb{R} ;
- (iv) *diverges* if it does not converge;
- (v) *diverges to* ∞ (resp. *diverges to* $-\infty$), and we write $\lim_{j,k \rightarrow \infty} x_{jk} = \infty$ (resp. $\lim_{j,k \rightarrow \infty} x_{jk} = -\infty$) if, for each $M > 0$, there exists $N \in \mathbb{N}$ such that $x_{jk} > M$ (resp. $x_{jk} < -M$) for $j, k \geq N$;
- (vi) has a limit that *exists* if $\lim_{j,k \rightarrow \infty} x_{jk} \in \mathbb{R}$;
- (vii) is *oscillatory* if the limit of the sequence does not exist, does not diverge to ∞ , or does not diverge to $-\infty$. •

Note that the definition of convergence requires that one check both indices at the same time. Indeed, if one thinks, as it is useful to do, of a double sequence as assigning a real number to each point in an infinite grid defined by the set $\mathbb{N} \times \mathbb{N}$, convergence means that the values on the grid can be made arbitrarily small outside a sufficiently large square (see Figure 2.1). It is useful, however, to have means of computing limits of double sequences by

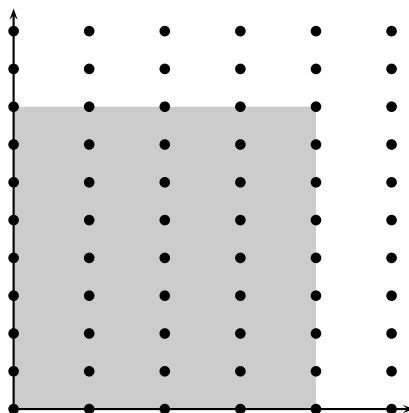


Figure 2.1 Convergence of a double sequence: by choosing the square large enough, the values at the unshaded grid points can be arbitrarily close to the limit

computing limits of sequences in the usual sense. Our next results are devoted to this.

2.3.19 Proposition (Computation of limits of double sequences I) Suppose that for the double sequence $\{x_{jk}\}_{j,k \in \mathbb{N}}$ it holds that

- (i) the double sequence is convergent and
- (ii) for each $j \in \mathbb{N}$, the limit $\lim_{k \rightarrow \infty} x_{jk}$ exists.

Then the limit $\lim_{j \rightarrow \infty} (\lim_{k \rightarrow \infty} x_{jk})$ exists and is equal to $\lim_{j,k \rightarrow \infty} x_{jk}$.

Proof Let $s_0 = \lim_{j,k \rightarrow \infty} x_{jk}$ and denote $s_j = \lim_{k \rightarrow \infty} x_{jk}$, $j \in \mathbb{N}$. For $\epsilon > 0$ take $N \in \mathbb{N}$ such that $|x_{jk} - s_0| < \frac{\epsilon}{2}$ for $j, k \geq N$. Also take $N_j \in \mathbb{N}$ such that $|x_{jk} - s_j| < \frac{\epsilon}{2}$ for $k \geq N_j$. Next take $j \geq N$ and let $k \geq \max\{N, N_j\}$. We then have

$$|s_j - s_0| = |s_j - x_{jk} + x_{jk} - s_0| \leq |s_j - x_{jk}| + |x_{jk} - s_0| < \epsilon,$$

using the triangle inequality. ■

2.3.20 Proposition (Computation of limits of double sequences II) Suppose that for the double sequence $\{x_{jk}\}_{j,k \in \mathbb{N}}$ it holds that

- (i) the double sequence is convergent,
- (ii) for each $j \in \mathbb{N}$, the limit $\lim_{k \rightarrow \infty} x_{jk}$ exists, and
- (iii) for each $k \in \mathbb{N}$, the limit $\lim_{j \rightarrow \infty} x_{jk}$ exists.

Then the limits $\lim_{j \rightarrow \infty} (\lim_{k \rightarrow \infty} x_{jk})$ and $\lim_{k \rightarrow \infty} (\lim_{j \rightarrow \infty} x_{jk})$ exist and are equal to $\lim_{j,k \rightarrow \infty} x_{jk}$.

Proof This follows from two applications of Proposition 2.3.19. ■

Let us give some examples that illustrate the idea of convergence of a double sequence.

2.3.21 Examples (Double sequences)

1. It is easy to check that the double sequence $\{\frac{1}{j+k}\}_{j,k \in \mathbb{N}}$ converges to 0. Indeed, for $\epsilon > 0$, if we take $N \in \mathbb{N}$ such that $\frac{1}{2N} < \epsilon$, it follows that $\frac{1}{j+k} < \epsilon$ for $j, k \geq N$.
2. The double sequence $\{\frac{j}{j+k}\}_{j,k \in \mathbb{N}}$ does not converge. To see this we should find $\epsilon > 0$ such that, for any $N \in \mathbb{N}$, there exists $j, k \geq N$ for which $\frac{j}{j+k} \geq \epsilon$. Take $\epsilon = \frac{1}{2}$ and let $N \in \mathbb{N}$. Then, if $j, k \geq N$ satisfy $j \geq 2k$, we have $\frac{j}{j+k} \geq \epsilon$.

Note that for this sequence, the limits $\lim_{j \rightarrow \infty} \frac{j}{j+k}$ and $\lim_{k \rightarrow \infty} \frac{j}{j+k}$ exist for each fixed k and j , respectively. This cautions about trying to use these limits to infer convergence of the double sequence.

3. The double sequence $\{\frac{(-1)^j}{k}\}_{j,k \in \mathbb{N}}$ is easily seen to converge to 0. However, the limit $\lim_{j \rightarrow \infty} \frac{(-1)^j}{k}$ does not exist for any fixed k . Therefore, one needs condition (ii) in Proposition 2.3.19 and conditions (ii) and (iii) in Proposition 2.3.20 in order for the results to be valid. ●

2.3.6 Algebraic operations on sequences

It is of frequent interest to add, multiply, or divide sequences and series. In such cases, one would like to ensure that convergence of the sequences or series is sufficient to ensure convergence of the sum, product, or quotient. In this section we address this matter.

2.3.22 Proposition (Algebraic operations on sequences) Let $\{x_j\}_{j \in \mathbb{N}}$ and $\{y_j\}_{j \in \mathbb{N}}$ be sequences converging to s_0 and t_0 , respectively, and let $\alpha \in \mathbb{R}$. Then the following statements hold:

- (i) the sequence $\{\alpha x_j\}_{j \in \mathbb{N}}$ converges to αs_0 ;
- (ii) the sequence $\{x_j + y_j\}_{j \in \mathbb{N}}$ converges to $s_0 + t_0$;
- (iii) the sequence $\{x_j y_j\}_{j \in \mathbb{N}}$ converges to $s_0 t_0$;
- (iv) if, for all $j \in \mathbb{N}$, $y_j \neq 0$ and if $s_0 \neq 0$, then the sequence $\{\frac{x_j}{y_j}\}_{j \in \mathbb{N}}$ converges to $\frac{s_0}{t_0}$.

Proof (i) Let $\epsilon > 0$ and choose $N \in \mathbb{N}$ such that $|x_j - s_0| < \frac{\epsilon}{|\alpha|}$. Then, for $j \geq N$,

$$|\alpha x_j - \alpha s_0| = |\alpha| |x_j - s_0| < \epsilon.$$

(ii) Let $\epsilon > 0$ and take $N_1, N_2 \in \mathbb{N}$ such that

$$|x_j - s_0| < \frac{\epsilon}{2}, \quad j \geq N_1, \quad |y_j - t_0| < \frac{\epsilon}{2}, \quad j \geq N_2.$$

Then, for $j \geq \max\{N_1, N_2\}$,

$$|x_j + y_j - (s_0 + t_0)| \leq |x_j - s_0| + |y_j - t_0| = \epsilon,$$

using the triangle inequality.

(iii) Let $\epsilon > 0$ and define $N_1, N_2, N_3 \in \mathbb{N}$ such that

$$\begin{aligned} |x_j - s_0| < 1, \quad j \geq N_1, & \implies |x_j| < |s_0| + 1, \quad j \geq N_1, \\ |x_j - s_0| < \frac{\epsilon}{2(|t_0| + 1)}, \quad j \geq N_2, & \\ |y_j - t_0| < \frac{\epsilon}{2(|s_0| + 1)}, \quad j \geq N_3. & \end{aligned}$$

Then, for $j \geq \max\{N_1, N_2, N_3\}$,

$$\begin{aligned} |x_j y_j - s_0 t_0| &= |x_j y_j - x_j t_0 + x_j t_0 - s_0 t_0| \\ &= |x_j(y_j - t_0) + t_0(x_j - s_0)| \\ &\leq |x_j| |y_j - t_0| + |t_0| |x_j - s_0| \\ &\leq (|s_0| + 1) \frac{\epsilon}{2(|s_0| + 1)} + (|t_0| + 1) \frac{\epsilon}{2(|t_0| + 1)} = \epsilon. \end{aligned}$$

(iv) It suffices using part (iii) to consider the case where $x_j = 1$, $j \in \mathbb{N}$. For $\epsilon > 0$ take $N_1, N_2 \in \mathbb{N}$ such that

$$\begin{aligned} |y_j - t_0| < \frac{|t_0|}{2}, \quad j \geq N_1, & \implies |y_j| > \frac{|t_0|}{2}, \quad j \geq N_1, \\ |y_j - t_0| < \frac{|t_0|^2 \epsilon}{2}, \quad j \geq N_2. & \end{aligned}$$

Then, for $j \geq \max\{N_1, N_2\}$,

$$\left| \frac{1}{y_j} - \frac{1}{t_0} \right| = \left| \frac{y_j - t_0}{y_j t_0} \right| \leq \frac{|t_0|^2 \epsilon}{2} \frac{2}{|t_0|} \frac{1}{|t_0|} = \epsilon,$$

as desired. ■

As we saw in the statement of Proposition 2.2.1, the restriction in part (iv) that $y_j \neq 0$ for all $j \in \mathbb{N}$ is not a real restriction. The salient restriction is that the sequence $\{y_j\}_{j \in \mathbb{N}}$ not converge to 0.

2.3.7 Convergence using more general index sets

Up to this point in this section we have talked about convergence of sequences. However, in practice it is often useful to take limits of more general objects where the index set is not \mathbb{N} , but a subset of \mathbb{R} . This will be particularly useful when considering the relationships between limits and functions. Therefore, in this section we indicate how one can do this. As we shall see, this more general notion of convergence can be reduced to standard convergence

of sequences. We comment that the setup in this section can be generalised by the notion of a net, and we refer the reader to for details.

what?

We shall introduce a slight logical inconsistency in our presentation at this point by supposing the reader to be familiar with the notion of an interval, and the notation used for intervals. For the (presumably few) readers not familiar with this, we refer to Section 2.5.1, and particularly Example 2.5.3.

Our objective is to understand what is meant by an expression like $\lim_{x \rightarrow a} \phi(x)$, where $\phi: A \rightarrow \mathbb{R}$ is a map from a subset A of \mathbb{R} to \mathbb{R} . We will be interested in subsets A of a rather specific form. Specifically, we will be interested in sets A having one of the following forms:

1. A is an interval that is unbounded on the left and equipped with the partial order $\preceq = \geq$;
2. A is an interval unbounded on the right and equipped with the partial order $\preceq = \leq$;
3. A is an interval not containing a , but for which $\inf A = a$, and equipped with the partial order $\preceq = \geq$;
4. A is an interval not containing a , but for which $\sup A = a$, and equipped with the partial order $\preceq = \leq$;
5. A is the union of two intervals I_l and I_r , neither of which contains a , but which satisfy $\sup I_l = a$ and $\inf I_r = a$, and equipped with the partial order \preceq defined by $x \preceq y$ if $|x - a| \leq |y - a|$.

In Figure 2.2 we depict the various sets A that we consider.

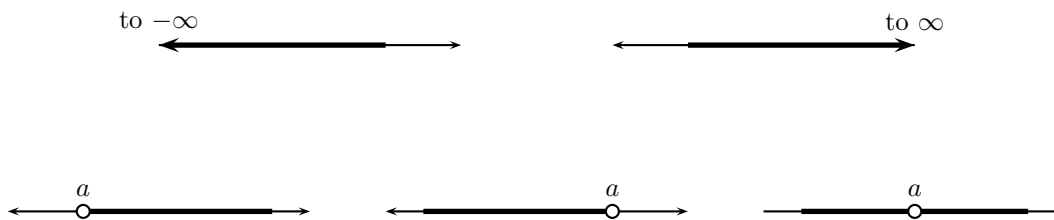


Figure 2.2 The five kinds of \mathbb{R} -directed sets

Let us give a definition that encodes the preceding constructions.

2.3.23 Definition (\mathbb{R} -directed set) A pair (A, \preceq) of the form of one of the preceding five cases is a \mathbb{R} -*directed set*. •

The language here mirrors the more general notion of a directed set that will be introduced in . Note that, like (\mathbb{N}, \leq) , a \mathbb{R} -directed set has the property that it is a partially ordered set with no upper bound. We next wish to generalise to \mathbb{R} -directed sets the idea of a sequence.

2.3.24 Definition (\mathbb{R} -net) Let (A, \preceq) be a \mathbb{R} -directed set and let S be a set. A \mathbb{R} -*net* in S on (A, \preceq) is a map $\phi: A \rightarrow \mathbb{R}$. •

The idea is a clear generalisation of a sequence, which is just a map whose domain is \mathbb{N} . The final part of the generalisation concerns convergence of \mathbb{R} -nets. This is a straightforward generalisation of convergence of sequences.

2.3.25 Definition (Convergence in \mathbb{R} -nets) Let (A, \preceq) be a \mathbb{R} -directed set and let $s_0 \in \mathbb{R}$. A net $\phi: A \rightarrow \mathbb{R}$ in \mathbb{R} defined on (A, \preceq) :

- (i) **converges** to s_0 , and we write $\lim_{x \in A} \phi(x) = s_0$, if, for any $\epsilon > 0$ there exists $y \in A$ such that $|\phi(x) - s_0| < \epsilon$ for $y \preceq x$;
- (ii) has s_0 as a **limit** if it converges to s_0 ;
- (iii) is **convergent** if it converges to some member of \mathbb{R} ;
- (iv) **diverges** if it does not converge;
- (v) **diverges to ∞** (resp. **diverges to $-\infty$**), and we write $\lim_{x \in A} \phi(x) = \infty$ (resp. $\lim_{x \in A} \phi(x) = -\infty$) if, for each $M > 0$, there exists $y \in A$ such that $\phi(x) > M$ (resp. $\phi(x) < -M$) for $y \preceq x$;
- (vi) has a limit that **exists** if $\lim_{x \in A} \phi(x) \in \mathbb{R}$;
- (vii) is **oscillatory** if the limit of the sequence does not exist, does not diverge to ∞ , or does not diverge to $-\infty$. •

Now we come to the real point of this discussion which is to make precise some familiar notation. For the five sorts of \mathbb{R} -directed sets listed above, we introduce the following notation, where the order here follows the order above.

1. If $\phi: A \rightarrow \mathbb{R}$ is a \mathbb{R} -net in \mathbb{R} of the first sort, then we write $\lim_{x \rightarrow -\infty} \phi(x)$ for $\lim_{x \in A} \phi(x)$.
2. If $\phi: A \rightarrow \mathbb{R}$ is a \mathbb{R} -net in \mathbb{R} of the second sort, then we write $\lim_{x \rightarrow \infty} \phi(x)$ for $\lim_{x \in A} \phi(x)$.
3. If $\phi: A \rightarrow \mathbb{R}$ is a \mathbb{R} -net in \mathbb{R} of the third sort, then we write $\lim_{x \downarrow a} \phi(x)$ for $\lim_{x \in A} \phi(x)$.
4. If $\phi: A \rightarrow \mathbb{R}$ is a \mathbb{R} -net in \mathbb{R} of the first sort, then we write $\lim_{x \uparrow a} \phi(x)$ for $\lim_{x \in A} \phi(x)$.
5. If $\phi: A \rightarrow \mathbb{R}$ is a \mathbb{R} -net in \mathbb{R} of the first sort, then we write $\lim_{x \rightarrow a} \phi(x)$ for $\lim_{x \in A} \phi(x)$.

It will also be convenient to adopt language that will allow for either of the possibilities $\lim_{x \uparrow a} \phi(x)$, $\lim_{x \downarrow a} \phi(x)$, or $\lim_{x \rightarrow a} \phi(x)$, depending on the character of the set A in question. For example, if $a = 0$, then we might have $A = [-b, 0)$ (in which case we would want $\lim_{x \uparrow 0} \phi(x)$), we might have $A = (0, b]$ (in which case we would want $\lim_{x \downarrow 0} \phi(x)$), or we might have $A = [-b, 0) \cup (0, b]$ (in which case we would want $\lim_{x \rightarrow 0} \phi(x)$). To allow for any of these possibilities, we shall adopt the following notation.

2.3.26 Notation (Limit in an interval) Let $I \subset \mathbb{R}$ be an interval, let $\phi: I \rightarrow \mathbb{R}$ be a map, and let $a \in I$. We define $\lim_{x \rightarrow a} \phi(x)$ by

- (i) $\lim_{x \rightarrow a} \phi(x) = \lim_{x \uparrow a} \phi(x)$ if $a = \sup I$,
- (ii) $\lim_{x \rightarrow a} \phi(x) = \lim_{x \downarrow a} \phi(x)$ if $a = \inf I$, and
- (iii) $\lim_{x \rightarrow a} \phi(x) = \lim_{x \rightarrow a} \phi(x)$ otherwise. •

We expect that most readers will be familiar with this notation, but here we merely give it a precise meaning. Let us also give the notation a precise characterisation in terms of limits of sequences.

2.3.27 Proposition (Convergence in \mathbb{R} -nets in terms of sequences) Let (A, \preceq) be a \mathbb{R} -directed set and let $\phi: A \rightarrow \mathbb{R}$ be a \mathbb{R} -net in \mathbb{R} on (A, \preceq) . Then, depending on which of the five forms is taken by A , one of the following five statements holds.

- (i) For (A, \preceq) of the first sort, the following statements are equivalent:
 - (a) $\lim_{x \rightarrow -\infty} \phi(x) = s_0$;
 - (b) $\lim_{j \rightarrow \infty} \phi(x_j) = s_0$ for every sequence $\{x_j\}_{j \in \mathbb{N}}$ in A satisfying $\lim_{j \rightarrow \infty} x_j = -\infty$.
- (ii) For (A, \preceq) of the second sort, the following statements are equivalent:

- (a) $\lim_{x \rightarrow \infty} \phi(x) = s_0$;
 (b) $\lim_{j \rightarrow \infty} \phi(x_j) = s_0$ for every sequence $\{x_j\}_{j \in \mathbb{N}}$ in A satisfying $\lim_{j \rightarrow \infty} x_j = \infty$.
- (iii) For (A, \preceq) of the third sort, the following statements are equivalent:
 (a) $\lim_{x \downarrow a} \phi(x) = s_0$;
 (b) $\lim_{j \rightarrow \infty} \phi(x_j) = s_0$ for every sequence $\{x_j\}_{j \in \mathbb{N}}$ in A satisfying $\lim_{j \rightarrow \infty} x_j = a$.
- (iv) For (A, \preceq) of the first sort, the following statements are equivalent:
 (a) $\lim_{x \uparrow a} \phi(x) = s_0$;
 (b) $\lim_{j \rightarrow \infty} \phi(x_j) = s_0$ for every sequence $\{x_j\}_{j \in \mathbb{N}}$ in A satisfying $\lim_{j \rightarrow \infty} x_j = a$.
- (v) For (A, \preceq) of the first sort, the following statements are equivalent:
 (a) $\lim_{x \rightarrow a} \phi(x) = s_0$;
 (b) $\lim_{j \rightarrow \infty} \phi(x_j) = s_0$ for every sequence $\{x_j\}_{j \in \mathbb{N}}$ in A satisfying $\lim_{j \rightarrow \infty} x_j = a$.

Proof These are all proved in essentially the same way, so let us prove just, say, the fourth part.

First suppose that $\lim_{x \downarrow a} \phi(x) = s_0$, and let $\{x_j\}_{j \in \mathbb{N}}$ be a sequence in A converging to a . Let $\epsilon > 0$ and choose $y \in A$ such that $|\phi(x) - s_0| < \epsilon$ whenever $y \leq x$. Then there exists $N \in \mathbb{N}$ such that $y \leq x_j$ for all $j \in \mathbb{N}$. Clearly, $|\phi(x_j) - s_0| < \epsilon$, so giving convergence of $\{\phi(x_j)\}_{j \in \mathbb{N}}$ to s_0 for every sequence $\{x_j\}_{j \in \mathbb{N}}$ in A converging to a .

Now let $\{x_j\}_{j \in \mathbb{N}}$ be a sequence in A converging to a , and suppose that $\{\phi(x_j)\}_{j \in \mathbb{N}}$ does not converge to s_0 . Then there exists $\epsilon > 0$ such that, for any $N \in \mathbb{N}$, there exists $j \geq N$ for which $|\phi(x_j) - s_0| \geq \epsilon$. Therefore, for any $\delta > 0$ there exists $x \in B(\delta, a) \cap A$ such that $|\phi(x) - s_0| \geq \epsilon$. This clearly prohibits having $\lim_{x \uparrow a} \phi(x) = s_0$. ■

Of course, similar conclusions hold when “convergence to s_0 ” is replaced with “divergence,” “convergence to ∞ ,” “convergence to $-\infty$,” or “oscillatory.” We leave the precise statements to the reader.

Let us give some examples to illustrate that this is all really nothing new.

2.3.28 Examples (Convergence in nets)

1. Consider the \mathbb{R} -directed set $([0, \infty), \leq)$ and the corresponding \mathbb{R} -net ϕ defined by $\phi(x) = \frac{1}{1+x^2}$. This \mathbb{R} -net then converges to 0. Let us verify this using the formal definition of convergence of a \mathbb{R} -net. For $\epsilon > 0$ choose $y > 0$ such that $y^2 = \frac{1}{\epsilon} > \frac{1}{\epsilon} - 1$. Then, if $y \leq x$, we have

$$\left| \frac{1}{1+x^2} - 0 \right| < \frac{1}{1+y^2} < \epsilon,$$

giving convergence to $\lim_{x \rightarrow \infty} \phi(x) = 0$ as stated.

2. Next consider the \mathbb{R} -directed set $((0, 1], \geq)$ and the corresponding \mathbb{R} -net ϕ defined by $\phi(x) = \sin \frac{1}{x}$. We claim that this \mathbb{R} -net converges to 0. To see this, let $\epsilon > 0$ and let $y < \epsilon$ with $y > 0$. Then we have, for $y \geq x$,

$$\left| x \sin \frac{1}{x} - 0 \right| = x \leq y < \epsilon,$$

giving $\lim_{x \downarrow 0} \phi(x) = 0$ as desired.

3. Consider the \mathbb{R} -directed set $([0, \infty), \leq)$ and the associated \mathbb{R} -net ϕ defined by $\phi(x) = x$. In this case we have $\lim_{x \rightarrow \infty} \phi(x) = \infty$.
4. Consider the \mathbb{R} -directed set $([0, \infty), \leq)$ and the associated \mathbb{R} -net ϕ defined by $\phi(x) = x \sin x$. In this case, due to the oscillatory nature of \sin , $\lim_{x \rightarrow \infty} \phi(x)$ does not exist, nor does it diverge to either ∞ or $-\infty$.

5. Take the \mathbb{R} -directed set $(A = \mathbb{R} \setminus \{0\}, \preceq)$ where \preceq is defined by $x \preceq y$ if $|x| \leq |y|$. In A define the \mathbb{R} -net ϕ by $\phi(x) = x$. Clearly, $\lim_{x \rightarrow 0} \phi(x) = 0$. •

There are also generalisations of \limsup and \liminf to nets. We let (A, \preceq) be a \mathbb{R} -directed set and let $\phi: A \rightarrow \mathbb{R}$ be a \mathbb{R} -net in \mathbb{R} defined on this \mathbb{R} -directed set. We write

$$\limsup_{x \in A} \phi(x) = \lim_{x \in A} (\sup \{ \phi(y) \mid x \preceq y \})$$

and

$$\liminf_{x \in A} \phi(x) = \lim_{x \in A} (\inf \{ \phi(y) \mid x \preceq y \}).$$

These allow us to talk of limits in cases where limits in the usual sense do not exist. Let us consider this via an example.

2.3.29 Example We consider the \mathbb{R} -directed set $([0, \infty), \leq)$ and let ϕ be the \mathbb{R} -net defined by $\phi(x) = e^{-x} + \sin x$.⁵ We claim that $\limsup_{x \in [0, \infty)} \phi(x) = 1$ and that $\liminf_{x \in [0, \infty)} \phi(x) = -1$. Let us prove the first claim, and leave the second as an exercise. Let $\epsilon > 0$ and take $y > \ln \epsilon$. We then have

$$\sup \{ e^{-x} + \sin x \mid y \leq x \} = \sup \{ e^{-x} + 1 \mid y \leq x \} \leq 1 + \epsilon,$$

as desired. •

Exercises

- 2.3.1 Show that if $\{x_j\}_{j \in \mathbb{N}}$ is a sequence in \mathbb{R} and if $\lim_{j \rightarrow \infty} x_j = x_0$ and $\lim_{j \rightarrow \infty} x_j = x'_0$, then $x_0 = x'_0$.
- 2.3.2 Answer the following questions:
- find a subset $S \subset \mathbb{Q}$ that possesses an upper bound in \mathbb{Q} , but which has no least element;
 - find a bounded monotonic sequence in \mathbb{Q} that does not converge in \mathbb{Q} .
- 2.3.3 For $A, B \subset \mathbb{R}$, show that

$$\sup \{ x + y \mid x \in A, y \in B \} \leq \sup A + \sup B$$

and

$$\inf \{ x + y \mid x \in A, y \in B \} \geq \inf A + \inf B$$

2.3.4 Do the following.

- Find a sequence $\{x_j\}_{j \in \mathbb{N}}$ for which $\lim_{j \rightarrow \infty} \left| \frac{x_{j+1}}{x_j} \right| = 1$ and which converges in \mathbb{R} .
- Find a sequence $\{x_j\}_{j \in \mathbb{N}}$ for which $\lim_{j \rightarrow \infty} \left| \frac{x_{j+1}}{x_j} \right| = 1$ and which diverges to ∞ .
- Find a sequence $\{x_j\}_{j \in \mathbb{N}}$ for which $\lim_{j \rightarrow \infty} \left| \frac{x_{j+1}}{x_j} \right| = 1$ and which diverges to $-\infty$.
- Find a sequence $\{x_j\}_{j \in \mathbb{N}}$ for which $\lim_{j \rightarrow \infty} \left| \frac{x_{j+1}}{x_j} \right| = 1$ and which is oscillatory.

2.3.5

simple exercise on limits

In the next exercise you will show that the property that a bounded, monotonically increasing sequence converges implies that Cauchy sequences converge. This completes the argument needed to prove the theorem stated in Aside 2.3.8 concerning characterisations of complete ordered fields.

⁵We have not yet defined e^{-x} or $\sin x$. The reader who is unable to go on without knowing what these functions really are can skip ahead to Section 3.10.

2.3.6 Assume that every bounded, monotonically increasing sequence in \mathbb{R} converges, and using this show that every Cauchy sequence in \mathbb{R} converges using an argument as follows.

1. Let $\{x_j\}_{j \in \mathbb{N}}$ be a Cauchy sequence.
2. Let $I_0 = [a, b]$ be an interval that contains all elements of $\{x_j\}_{j \in \mathbb{N}}$ (why is this possible?)
3. Split $[a, b]$ into two equal length closed intervals, and argue that in at least one of these there is an infinite number of points from the sequence. Call this interval I_1 and let $x_{k_1} \in \{x_j\}_{j \in \mathbb{N}} \cap I_1$.
4. Repeat the process for I_1 to find an interval I_2 which contains an infinite number of points from the sequence. Let $x_{k_2} \in \{x_j\}_{j \in \mathbb{N}} \cap I_2$.
5. Carry on doing this to arrive at a sequence $\{x_{k_j}\}_{j \in \mathbb{N}}$ of points in \mathbb{R} and a sequence $\{I_j\}_{j \in \mathbb{N}}$.
6. Argue that the sequence of left endpoints of the intervals $\{I_j\}_{j \in \mathbb{N}}$ is a bounded monotonically increasing sequence, and that the sequence of right endpoints is a bounded monotonically decreasing sequence. and so both converge.
7. Show that they converge to the same number, and that the sequence $\{x_{k_j}\}_{j \in \mathbb{N}}$ also converges to this limit.
8. Show that the sequence $\{x_j\}_{j \in \mathbb{N}}$ converges to this limit.

Section 2.4

Series in \mathbb{R}

From a sequence $\{x_j\}_{j \in \mathbb{N}}$ in \mathbb{R} , one can consider, in principle, the infinite sum $\sum_{j=1}^{\infty} x_j$. Of course, such a sum *a priori* makes no sense. However, as we shall see in Chapter III-2, such infinite sums are important for characterising certain discrete-time signal spaces. In this section we outline some of the principle properties of these sums.

Do I need to read this section? Most readers will probably have seen much of the material in this section in their introductory calculus course. What might be new for some readers is the fairly careful discussion in Theorem 2.4.5 of the difference between convergence and absolute convergence of series. Since absolute convergence will be of importance to us, it might be worth understanding in what ways it is different from convergence. The material in Section 2.4.7 can be regarded as optional until it is needed during the course of reading other material in the text. •

2.4.1 Definitions and properties of series

A *series* in \mathbb{R} is an expression of the form

$$S = \sum_{j=1}^{\infty} x_j,$$

where $x_j \in \mathbb{R}$. Of course, a series is meaningless as an element of \mathbb{R} unless it possesses additional features. For example, if $x_j = 1$, $j \in \mathbb{N}$, then the sum is infinite. Also, if $x_j = (-1)^j$, $j \in \mathbb{N}$, then it is not clear what the sum is: perhaps it is 0 or perhaps it is 1. Therefore, we need some sort of condition placed on a series if it is to represent a real number.

2.4.1 Definition (Convergence and absolute convergence of series) Let $\{x_j\}_{j \in \mathbb{N}}$ be a sequence in \mathbb{R} and consider the series

$$S = \sum_{j=1}^{\infty} x_j.$$

The corresponding sequence of *partial sums* is the sequence $\{S_k\}_{k \in \mathbb{N}}$ defined by

$$S_k = \sum_{j=1}^k x_j.$$

Let $s_0 \in \mathbb{R}$. The series:

- (i) **converges to s_0** , and we write $\sum_{j=1}^{\infty} x_j = s_0$, if the sequence of partial sums converges to s_0 ;
- (ii) has s_0 as a **limit** if it converges to s_0 ;
- (iii) is **convergent** if it converges to some member of \mathbb{R} ;
- (iv) **converges absolutely**, or is **absolutely convergent**, if the series

$$\sum_{j=1}^{\infty} |x_j|$$

converges;

- (v) *converges conditionally*, or is *conditionally convergent*, if it is convergent, but not absolutely convergent;
- (vi) *diverges* if it does not converge;
- (vii) *diverges to ∞* (resp. *diverges to $-\infty$*), and we write $\sum_{j=1}^{\infty} x_j = \infty$ (resp. $\sum_{j=1}^{\infty} x_j = -\infty$), if the sequence of partial sums diverges to ∞ (resp. diverges to $-\infty$);
- (viii) has a limit that *exists* if $\lim_{j \rightarrow \infty} x_j \in \mathbb{R}$;
- (ix) is *oscillatory* if the sequence of partial sums is oscillatory. •

Let us consider some examples of series in \mathbb{R} .

2.4.2 Examples (Series in \mathbb{R})

1. First we consider the *geometric series* $\sum_{j=1}^{\infty} x^{j-1}$ for $x \in \mathbb{R}$. We claim that this series converges if and only if $|x| < 1$. To prove this we claim that the sequence $\{S_k\}_{k \in \mathbb{N}}$ of partial sums is defined by

$$S_k = \begin{cases} \frac{1-x^{k+1}}{1-x}, & x \neq 1, \\ k, & x = 1. \end{cases}$$

The conclusion is obvious for $x = 1$, so we can suppose that $x \neq 1$. The conclusion is obvious for $k = 1$, so suppose it true for $j \in \{1, \dots, k\}$. Then

$$S_{k+1} = \sum_{j=1}^{k+1} x^j = x^{k+1} + \frac{1-x^{k+1}}{1-x} = \frac{x^{k+1} - x^{k+2} + 1 - x^{k+1}}{1-x} = \frac{1-x^{k+2}}{1-x},$$

as desired. It is clear, then, that if $x = 1$ then the series diverges to ∞ . If $x = -1$ then the series is directly checked to be oscillatory; the sequence of partial sums is $\{1, 0, 1, \dots\}$. For $x > 1$ we have

$$\lim_{k \rightarrow \infty} S_k = \lim_{k \rightarrow \infty} \frac{1-x^{k+1}}{1-x} = \infty,$$

showing that the series diverges to ∞ in this case. For $x < -1$ it is easy to see that the sequence of partial sums is oscillatory, but increasing in magnitude. This leaves the case when $|x| < 1$. Here, since the sequence $\{x^{k+1}\}_{k \in \mathbb{N}}$ converges to zero, the sequence of partial sums also converges, and converges to $\frac{1}{1-x}$. (We have used the results concerning the swapping of limits with algebraic operations as described in Section 2.3.6.)

2. We claim that the series $\sum_{j=1}^{\infty} \frac{1}{j}$ diverges to ∞ . To show that, we show that the sequence $\{S_k\}_{k \in \mathbb{N}}$ is not upper bounded. To show this, we shall show that $S_{2^k} \geq 1 + \frac{1}{2}k$ for all $k \in \mathbb{N}$. This is true directly when $k = 1$. Next suppose that $S_{2^j} \geq 1 + \frac{1}{2}j$ for $j \in \{1, \dots, k\}$. Then

$$\begin{aligned} S_{2^{k+1}} &= S_{2^k} + \frac{1}{2^k+1} + \frac{1}{2^k+2} + \dots + \frac{1}{2^{k+1}} \\ &\geq 1 + \frac{1}{2}k + \underbrace{\frac{1}{2^{k+1}} + \dots + \frac{1}{2^{k+1}}}_{2^k \text{ terms}} \\ &= 1 + \frac{1}{2}k + \frac{2^k}{2^{k+1}} = 1 + \frac{1}{2}(k+1). \end{aligned}$$

Thus the sequence of partial sums is indeed unbounded, and since it is monotonically increasing, it diverges to ∞ , as we first claimed.

3. We claim that the series $S = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j}$ converges. To see this, we claim that, for any $m \in \mathbb{N}$, we have

$$S_2 \leq S_4 \leq \cdots \leq S_{2m} \leq S_{2m-1} \leq \cdots \leq S_3 \leq S_1.$$

That $S_2 \leq S_4 \leq \cdots \leq S_{2m}$ follows since $S_{2k} - S_{2k-2} = \frac{1}{2k-1} - \frac{1}{2k} > 0$ for $k \in \mathbb{N}$. That $S_{2m} \leq S_{2m-1}$ follows since $S_{2m-1} - S_{2m} = \frac{1}{2m}$. Finally, $S_{2m-1} \leq \cdots \leq S_3 \leq S_1$ since $S_{2k-1} - S_{2k+1} = \frac{1}{2k} - \frac{1}{2k+1} > 0$ for $k \in \mathbb{N}$. Thus the sequences $\{S_{2k}\}_{k \in \mathbb{N}}$ and $\{S_{2k-1}\}_{k \in \mathbb{N}}$ are monotonically increasing and monotonically decreasing, respectively, and their terms are getting closer and closer together since $\lim_{m \rightarrow \infty} S_{2m-1} - S_{2m} = \frac{1}{2m} = 0$. By Lemma 2 from the proof of Theorem 2.3.6, it follows that the sequences $\{S_{2k}\}_{k \in \mathbb{N}}$ and $\{S_{2k-1}\}_{k \in \mathbb{N}}$ converge and converge to the same limit. Therefore, the sequence $\{S_k\}_{k \in \mathbb{N}}$ converges as well to the same limit. One can moreover show that the limit of the series is $\ln 2$, where \ln denotes the natural logarithm.

Note that we have now shown that the series $\sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j}$ converges, but does not converge absolutely; therefore, it is conditionally convergent.

4. We next consider the **harmonic series** $\sum_{j=1}^{\infty} j^{-k}$ for $k \in \mathbb{N}_0$. For $k = 1$ this agrees with our example of part 2. We claim that this series converges if and only if $k > 1$. We have already considered the case of $k = 1$. For $k < 1$ we have $j^{-k} \geq j^{-1}$ for $j \in \mathbb{N}$. Therefore,

$$\sum_{j=1}^{\infty} j^{-k} \geq \sum_{j=1}^{\infty} \frac{1}{j} = \infty,$$

showing that the series diverges to ∞ .

For $k > 1$ we note that the sequence of partial sums is monotonically increasing. Thus, to show convergence of the series it suffices by Theorem 2.3.7 to show that the sequence of partial sums is bounded above. Let $N \in \mathbb{N}$ and take $j \in \mathbb{N}$ such that $N < 2^j - 1$. Then the N th partial sum satisfies

$$\begin{aligned} S_N &\leq S_{2^j-1} = 1 + \frac{1}{2^k} + \frac{1}{3^k} + \cdots + \frac{1}{(2^j-1)^k} \\ &= 1 + \underbrace{\left(\frac{1}{2^k} + \frac{1}{3^k}\right)}_{2 \text{ terms}} + \underbrace{\left(\frac{1}{4^k} + \cdots + \frac{1}{7^k}\right)}_{4 \text{ terms}} + \underbrace{\left(\frac{1}{(2^{j-1})^k} + \cdots + \frac{1}{(2^j-1)^k}\right)}_{2^{j-1} \text{ terms}} \\ &< 1 + \frac{2}{2^k} + \frac{4}{4^k} + \cdots + \frac{2^{j-1}}{(2^{j-1})^k} \\ &= 1 + \frac{1}{2^{k-1}} + \left(\frac{1}{2^{k-1}}\right)^2 + \cdots + \left(\frac{1}{2^{k-1}}\right)^{j-1}. \end{aligned}$$

Now we note that the last expression on the right-hand side is bounded above by the sum $\sum_{j=1}^{\infty} (2^{k-1})^{j-1}$, which is a convergent geometric series as we saw in part 1. This shows that S_N is bounded above by this sum for all N , so showing that the harmonic series converges for $k > 1$.

5. The series $\sum_{j=1}^{\infty} (-1)^{j+1}$ does not converge, and also does not diverge to ∞ or $-\infty$. Therefore, it is oscillatory. •

Let us next explore relationships between the various notions of convergence. First we relate the notions of convergence and absolute convergence in the only possible way, given that the series $\sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j}$ has been shown to be convergent, but not absolutely convergent.

2.4.3 Proposition (Absolutely convergent series are convergent) *If a series $\sum_{j=1}^{\infty} x_j$ is absolutely convergent, then it is convergent.*

Proof Denote

$$s_k = \sum_{j=1}^k x_j, \quad \sigma_k = \sum_{j=1}^k |x_j|,$$

and note that $\{\sigma_k\}_{k \in \mathbb{N}}$ is a Cauchy sequence since the series $\sum_{j=1}^{\infty} x_j$ is absolutely convergent. Thus let $\epsilon > 0$ and choose $N \in \mathbb{N}$ such that $|\sigma_k - \sigma_l| < \epsilon$ for $k, l \geq N$. For $m > k$ we then have

$$|s_m - s_k| = \left| \sum_{j=k+1}^m x_j \right| \leq \sum_{j=k+1}^m |x_j| = |\sigma_m - \sigma_k| < \epsilon,$$

where we have used Exercise 2.4.3. Thus, for $m > k \geq N$ we have $|s_m - s_k| < \epsilon$, showing that $\{s_k\}_{k \in \mathbb{N}}$ is a Cauchy sequence, and so convergent by Theorem 2.3.4. ■

The following result is often useful.

2.4.4 Proposition (Swapping summation and absolute value) *For a sequence $\{x_j\}_{j \in \mathbb{N}}$, if the series $S = \sum_{j=1}^{\infty} x_j$ is absolutely convergent, then*

$$\left| \sum_{j=1}^{\infty} x_j \right| \leq \sum_{j=1}^{\infty} |x_j|.$$

Proof Define

$$S_m^1 = \left| \sum_{j=1}^m x_j \right|, \quad S_m^2 = \sum_{j=1}^m |x_j|, \quad m \in \mathbb{N}.$$

By Exercise 2.4.3 we have $S_m^1 \leq S_m^2$ for each $m \in \mathbb{N}$. Moreover, by Proposition 2.4.3 the sequences $\{S_m^1\}_{m \in \mathbb{N}}$ and $\{S_m^2\}_{m \in \mathbb{N}}$ converge. It is then clear (why?) that

$$\lim_{m \rightarrow \infty} S_m^1 \leq \lim_{m \rightarrow \infty} S_m^2,$$

which is the result. ■

It is not immediately clear on a first encounter why the notion of absolute convergence is useful. However, as we shall see in Chapter III-2, it is the notion of absolute convergence that will be of most use to us in our characterisation of discrete signal spaces. The following result indicates why mere convergence of a series is perhaps not as nice a notion as one would like, and that absolute convergence is in some sense better behaved.

2.4.5 Theorem (Convergence and rearrangement of series) *For a series $S = \sum_{j=1}^{\infty} x_j$, the following statements hold:*

- (i) *if S is conditionally convergent then, for any $s_0 \in \mathbb{R}$, there exists a bijection $\phi: \mathbb{N} \rightarrow \mathbb{N}$ such that the series $S_{\phi} = \sum_{j=1}^{\infty} x_{\phi(j)}$ converges to s_0 ;*
- (ii) *if S is conditionally convergent then there exists a bijection $\phi: \mathbb{N} \rightarrow \mathbb{N}$ such that the series $S_{\phi} = \sum_{j=1}^{\infty} x_{\phi(j)}$ diverges to ∞ ;*
- (iii) *if S is conditionally convergent then there exists a bijection $\phi: \mathbb{N} \rightarrow \mathbb{N}$ such that the series $S_{\phi} = \sum_{j=1}^{\infty} x_{\phi(j)}$ diverges to $-\infty$;*
- (iv) *if S is conditionally convergent then there exists a bijection $\phi: \mathbb{N} \rightarrow \mathbb{N}$ such that the limit of the partial sums for the series $S_{\phi} = \sum_{j=1}^{\infty} x_{\phi(j)}$ is oscillating;*
- (v) *if S is absolutely convergent then, for any bijection $\phi: \mathbb{N} \rightarrow \mathbb{N}$, the series $S_{\phi} = \sum_{j=1}^{\infty} x_{\phi(j)}$ converges to the same limit as the series S .*

Proof We shall be fairly “descriptive” concerning the first four parts of the proof. More precise arguments can be tediously fabricated from the ideas given. We shall use the fact, given as Exercise 2.4.1, that if a series is conditionally convergent, then the two series formed by the positive terms and the negative terms diverge.

(i) First of all, rearrange the terms in the series so that the positive terms are arranged in decreasing order, and the negative terms are arranged in increasing order. We suppose that $s_0 \geq 0$, as a similar argument can be fabricated when $s_0 < 0$. Take as the first elements of the rearranged sequence the enough of the first few positive terms in the sequence so that their sum exceeds s_0 . As the next terms, take enough of the first few negative terms in the series such that their sum, combined with the already chosen positive terms, is less than s_0 . Now repeat this process. Because the series was initially rearranged so that the positive and negative terms are in descending and ascending order, respectively, one can show that the construction we have given yields a sequence of partial sums that starts greater than s_0 , then monotonically decreases to a value less than s_0 , then monotonically increases to a value greater than s_0 , and so on. Moreover, at the end of each step, the values get closer to s_0 since the sequence of positive and negative terms both converge to zero. An argument like that used in the proof of Proposition 2.3.9 can then be used to show that the resulting sequence of partial sums converges to s_0 .

(ii) To get the suitable rearrangement, proceed as follows. Partition the negative terms in the sequence into disjoint finite sets S_j^- , $j \in \mathbb{N}$. Now partition the positive terms in the sequence as follows. Define S_1^+ to be the first N_1 positive terms in the sequence, where N_1 is sufficiently large that the sum of the elements of S_1^+ exceeds by at least 1 in absolute value the sum of the elements from S_1^- . This is possible since the series of positive terms in the sequence diverges to ∞ . Now define S_2^+ by taking the next N_2 positive terms in the sequence so that the sum of the elements of S_2^+ exceeds by at least 1 in absolute value the sum of the elements from S_2^- . Continue in this way, defining S_3^+, S_4^+, \dots . The rearrangement of the terms in the series is then made by taking the first collection of terms to be the elements of S_1^+ , the second collection to be the elements of S_1^- , the third collection to be the elements of S_2^+ , and so on. One can verify that the resulting sequence of partial sums diverges to ∞ .

(iii) The argument here is entirely similar to the previous case.

(iv) This result follows from part (i) in the following way. Choose an oscillating sequence $\{y_j\}_{j \in \mathbb{N}}$. For y_1 , by part (i) one can find a finite number of terms from the original series whose sum is as close as desired to y_1 . These will form the first terms in the rearranged series. Next, the same argument can be applied to the remaining elements of the series to yield a finite number of terms in the series that are as close as desired to y_2 . One carries on in this way, noting that since the sequence $\{y_j\}_{j \in \mathbb{N}}$ is oscillating, so too will be the sequence of partial sums for the rearranged series.

(v) Let $y_j = x_{\phi(j)}$ for $j \in \mathbb{N}$. Then define sequences $\{x_j^+\}_{j \in \mathbb{N}}$, $\{x_j^-\}_{j \in \mathbb{N}}$, $\{y_j^+\}_{j \in \mathbb{N}}$, and $\{y_j^-\}_{j \in \mathbb{N}}$ by

$$x_j^+ = \max\{x_j, 0\}, \quad x_j^- = \max\{-x_j, 0\}, \quad y_j^+ = \max\{y_j, 0\}, \quad y_j^- = \max\{-y_j, 0\}, \quad j \in \mathbb{N},$$

noting that $|x_j| = \max\{x_j^-, x_j^+\}$ and $|y_j| = \max\{y_j^-, y_j^+\}$ for $j \in \mathbb{N}$. By Proposition 2.4.8 it follows that the series

$$S^+ = \sum_{j=1}^{\infty} x_j^+, \quad S^- = \sum_{j=1}^{\infty} x_j^-, \quad S_{\phi}^+ = \sum_{j=1}^{\infty} y_j^+, \quad S_{\phi}^- = \sum_{j=1}^{\infty} y_j^-$$

converge. We claim that for each $k \in \mathbb{N}$ we have

$$\sum_{j=1}^k x_j^+ \leq \sum_{j=1}^{\infty} y_j^+.$$

To see this, we need only note that there exists $N \in \mathbb{N}$ such that

$$\{x_1^+, \dots, x_k^+\} \subset \{y_1^+, \dots, y_N^+\}.$$

With N having this property,

$$\sum_{j=1}^k x_j^+ \leq \sum_{j=1}^N y_j^+ \leq \sum_{j=1}^{\infty} y_j^+,$$

as desired. Therefore,

$$\sum_{j=1}^{\infty} x_j^+ \leq \sum_{j=1}^{\infty} y_j^+.$$

Reversing the argument gives

$$\sum_{j=1}^{\infty} y_j^+ \leq \sum_{j=1}^{\infty} x_j^+ \implies \sum_{j=1}^{\infty} x_j^+ = \sum_{j=1}^{\infty} y_j^+.$$

A similar argument also gives

$$\sum_{j=1}^{\infty} x_j^- = \sum_{j=1}^{\infty} y_j^-.$$

This then gives

$$\sum_{j=1}^{\infty} y_j = \sum_{j=1}^{\infty} y_j^+ - \sum_{j=1}^{\infty} y_j^- = \sum_{j=1}^{\infty} x_j^+ - \sum_{j=1}^{\infty} x_j^- = \sum_{j=1}^{\infty} x_j,$$

as desired. ■

The theorem says, roughly, that absolute convergence is necessary and sufficient to ensure that the limit of a series be independent of rearrangement of the terms in the series. Note that the necessity portion of this statement, which is parts (i)–(iv) of the theorem, comes in a rather dramatic form which suggests that conditional convergence behaves maximally poorly with respect to rearrangement.

2.4.2 Tests for convergence of series

In this section we give some of the more popular tests for convergence of a series. It is infeasible to expect an easily checkable general condition for convergence. However, in some cases the tests we give here are sufficient.

First we make a simple general observation that is very often useful; it is merely a reflection that the convergence of a series depends only on the tail of the series. We shall often make use of this result without mention.

2.4.6 Proposition (Convergence is unaffected by changing a finite number of terms) *Let $\sum_{j=1}^{\infty} x_j$ and $\sum_{j=1}^{\infty} y_j$ be series in \mathbb{R} and suppose that there exists $K \in \mathbb{Z}$ and $N \in \mathbb{N}$ such that $x_j = y_{j+K}$ for $j \geq N$. Then the following statements hold:*

- (i) *the series $\sum_{j=1}^{\infty} x_j$ converges if and only if the series $\sum_{j=1}^{\infty} y_j$ converges;*
- (ii) *the series $\sum_{j=1}^{\infty} x_j$ diverges if and only if the series $\sum_{j=1}^{\infty} y_j$ diverges;*
- (iii) *the series $\sum_{j=1}^{\infty} x_j$ diverges to ∞ if and only if the series $\sum_{j=1}^{\infty} y_j$ diverges to ∞ ;*
- (iv) *the series $\sum_{j=1}^{\infty} x_j$ diverges to $-\infty$ if and only if the series $\sum_{j=1}^{\infty} y_j$ diverges to $-\infty$.*

The next convergence result is also a more or less obvious one.

2.4.7 Proposition (Sufficient condition for a series to diverge) *If the sequence $\{x_j\}_{j \in \mathbb{N}}$ does not converge to zero, then the series $\sum_{j=1}^{\infty} x_j$ diverges.*

Proof Suppose that the series $\sum_{j=1}^{\infty} x_j$ converges to s_0 . Then, for $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\left| s_0 - \sum_{j=N+1}^{\infty} x_j \right| < \epsilon \quad \implies \quad \left| |s_0| - \left| \sum_{j=N+1}^{\infty} x_j \right| \right| < \epsilon,$$

where we have used Exercise 2.2.6. ■

Note that Example 2.4.2–2 shows that the converse of this result is false. That is to say, for a series to converge, it is not sufficient that the terms in the series go to zero. For this reason, checking the convergence of a series numerically becomes something that must be done carefully, since the blind use of the computer with a prescribed numerical accuracy will suggest the false conclusion that a series converges if and only if the terms in the series go to zero as the index goes to infinity.

Another more or less obvious result is the following.

2.4.8 Proposition (Comparison Test) *Let $\{x_j\}_{j \in \mathbb{N}}$ and $\{y_j\}_{j \in \mathbb{N}}$ be sequences of nonnegative numbers for which there exists $\alpha \in \mathbb{R}_{>0}$ satisfying $y_j \leq \alpha x_j$, $j \in \mathbb{N}$. Then the following statements hold:*

- (i) *the series $\sum_{j=1}^{\infty} y_j$ converges if the series $\sum_{j=1}^{\infty} x_j$ converges;*
- (ii) *the series $\sum_{j=1}^{\infty} x_j$ diverges if the series $\sum_{j=1}^{\infty} y_j$ diverges.*

Proof We shall show that, if the series $\sum_{j=1}^{\infty} x_j$ converges, then the sequence $\{T_k\}_{k \in \mathbb{N}}$ of partial sums for the series $\sum_{j=1}^{\infty} y_j$ is a Cauchy sequence. Since the sequence $\{S_k\}_{k \in \mathbb{N}}$ for $\sum_{j=1}^{\infty} x_j$ is convergent, it is Cauchy. Therefore, for $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that whenever $k, m \geq N$, with $k > m$ without loss of generality,

$$S_k - S_m = \sum_{j=m+1}^k x_j < \epsilon \alpha^{-1}.$$

Then, for $k, m \geq N$ with $k > m$ we have

$$T_k - T_m = \sum_{j=m+1}^k y_j \leq \alpha \sum_{j=m+1}^k x_j < \epsilon,$$

showing that $\{T_k\}_{k \in \mathbb{N}}$ is a Cauchy sequence, as desired.

The second statement is the contrapositive of the first. ■

Now we can get to some less obvious results for convergence of series. The first result concerns series where the terms alternate sign.

2.4.9 Proposition (Alternating Test) *Let $\{x_j\}_{j \in \mathbb{N}}$ be a sequence in \mathbb{R} satisfying*

- (i) $x_j > 0$ for $j \in \mathbb{N}$,
- (ii) $x_{j+1} \leq x_j$ for $j \in \mathbb{N}$, and
- (iii) $\lim_{j \rightarrow \infty} x_j = 0$.

Then the series $\sum_{j=1}^{\infty} (-1)^{j+1} x_j$ converges.

Proof The proof is a straightforward generalisation of that given for Example 2.4.2–3, and we leave for the reader the simple exercise of verifying that this is so. ■

Our next result is one that is often useful.

2.4.10 Proposition (Ratio Test for sequences) *Let $\{x_j\}_{j \in \mathbb{N}}$ be a sequence in \mathbb{R} with $\sum_{j=1}^{\infty} x_j$ the corresponding series. Then the following statements hold:*

- (i) if $\limsup_{j \rightarrow \infty} \left| \frac{x_{j+1}}{x_j} \right| < 1$, then the series converges absolutely;
- (ii) if there exists $N \in \mathbb{N}$ such that $\left| \frac{x_{j+1}}{x_j} \right| > 1$ for all $j \geq N$, then the series diverges.

Proof (i) By Proposition 2.3.14 there exists $\beta \in (0, 1)$ and $N \in \mathbb{N}$ such that $\left| \frac{a_{j+1}}{a_j} \right| < \beta$ for $j \geq N$. Then

$$\left| \frac{x_j}{x_N} \right| = \left| \frac{x_{N+1}}{x_N} \right| \left| \frac{x_{N+2}}{x_{N+1}} \right| \cdots \left| \frac{x_j}{x_{j-1}} \right| < \beta^{j-N}, \quad j > N,$$

implying that

$$|x_j| < \frac{|x_N|}{\beta^N} \beta^j.$$

Since $\beta < 1$, the geometric series $\sum_{j=1}^{\infty} \beta^j$ converges. The result for $\alpha < 1$ now follows by the Comparison Test.

(ii) The sequence $\{x_j\}_{j \in \mathbb{N}}$ cannot converge to 0 in this case, and so this part of the result follows from Proposition 2.4.7. ■

The following simpler test is often stated as the Ratio Test.

2.4.11 Corollary (Weaker version of the Ratio Test) *If $\{x_j\}_{j \in \mathbb{N}}$ is a sequence in \mathbb{R} for which $\lim_{j \rightarrow \infty} \left| \frac{x_{j+1}}{x_j} \right| = \alpha$, then the series $\sum_{j=1}^{\infty} x_j$ converges absolutely if $\alpha < 1$ and diverges if $\alpha > 1$.*

Our next result has a similar character to the previous one.

2.4.12 Proposition (Root Test) *Let $\{x_j\}_{j \in \mathbb{N}}$ be a sequence for which $\limsup_{j \rightarrow \infty} |x_j|^{1/j} = \alpha$. Then the series $\sum_{j=1}^{\infty} x_j$ converges absolutely if $\alpha < 1$ and diverges if $\alpha > 1$.*

Proof First take $\alpha < 1$ and define $\beta = \frac{1}{2}(\alpha + 1)$. Then, just as in the proof of Proposition 2.4.10, $\alpha < \beta < 1$. By Proposition 2.3.14 there exists $N \in \mathbb{N}$ such that $|x_j|^{1/j} < \beta$ for $j \geq N$. Thus $|x_j| < \beta^j$ for $j \geq N$. Note that $\sum_{j=N+1}^{\infty} \beta^j$ converges by Example 2.4.2-1. Now $\sum_{j=0}^{\infty} |x_j|$ converges by the Comparison Test.

Next take $\alpha > 1$. In this case we have $\lim_{j \rightarrow \infty} |x_j| \neq 0$, and so we conclude divergence from Proposition 2.4.7. ■

The following obvious corollary is often stated as the Root Test.

2.4.13 Corollary (Weaker version of Root Test) *Let $\{x_j\}_{j \in \mathbb{N}}$ be a sequence for which $\lim_{j \rightarrow \infty} |x_j|^{1/j} = \alpha$. Then the series $\sum_{j=1}^{\infty} x_j$ converges absolutely if $\alpha < 1$ and diverges if $\alpha > 1$.*

The Ratio Test and the Root Test are related, as the following result indicates.

2.4.14 Proposition (Root Test implies Ratio Test) *If $\{p_j\}_{j \in \mathbb{N}_0}$ is a sequence in $\mathbb{R}_{>0}$ then*

$$\begin{aligned} \liminf_{j \rightarrow \infty} \frac{p_{j+1}}{p_j} &\leq \liminf_{j \rightarrow \infty} p_j^{1/j} \\ \limsup_{j \rightarrow \infty} p_j^{1/j} &\leq \limsup_{j \rightarrow \infty} \frac{p_{j+1}}{p_j}. \end{aligned}$$

In particular, for a sequence $\{a_j\}_{j \in \mathbb{N}}$, if $\lim_{j \rightarrow \infty} \left| \frac{a_{j+1}}{a_j} \right|$ exists, then $\lim_{j \rightarrow \infty} |a_j|^{1/j} = \lim_{j \rightarrow \infty} \left| \frac{a_{j+1}}{a_j} \right|$.

Proof For the first inequality, let $\alpha = \liminf_{j \rightarrow \infty} \frac{p_{j+1}}{p_j}$. First consider the case where $\alpha = \infty$. Then, given $M > 0$, there exists $N \in \mathbb{N}$ such that $\frac{p_{j+1}}{p_j} > M$ for $j \geq N$. Then we have

$$\left| \frac{p_j}{p_N} \right| = \left| \frac{p_{N+1}}{p_N} \right| \left| \frac{p_{N+1}}{p_{N+1}} \right| \cdots \left| \frac{p_j}{p_{j-1}} \right| > M^{j-N}, \quad j > N.$$

This gives

$$p_j > \frac{p_N}{M^N} M^j, \quad j > N.$$

Thus $p_j^{1/j} > (\frac{p_N}{M^N})^{1/j} M$. Since $\lim_{j \rightarrow \infty} (p_N \beta^{-N})^{1/j} = 1$ (cf. the definition of P_a in Section 3.10.3), we have $\liminf_{j \rightarrow \infty} p_j^{1/j} > M$, giving the desired conclusion in this case, since M is arbitrary. Next consider the case when $\alpha \in \mathbb{R}_{>0}$ and let $\beta < \alpha$. By Proposition 2.3.15 there exists $N \in \mathbb{N}$ such that $\frac{p_{j+1}}{p_j} \geq \beta$ for $j \geq N$. Performing just the same computation as above gives $p_j \geq \beta^{j-N} p_N$ for $j \geq N$. Therefore, $p_j^{1/j} \geq (p_N \beta^{-N})^{1/j} \beta$. Since $\lim_{j \rightarrow \infty} (p_N \beta^{-N})^{1/j} = 1$ we have $\liminf_{j \rightarrow \infty} p_j^{1/j} \geq \beta$. The first inequality follows since $\beta < \alpha$ is arbitrary.

Now we prove the second inequality. Let $\alpha = \limsup_{j \rightarrow \infty} \frac{p_{j+1}}{p_j}$. If $\alpha = \infty$ then the second inequality in the statement of the result is trivial. If $\alpha \in \mathbb{R}_{>0}$ then let $\beta > \alpha$ and note that there exists $N \in \mathbb{N}$ such that $\frac{p_{j+1}}{p_j} \leq \beta$ for $j \geq N$ by Proposition 2.3.14. In particular, just as in the proof of Proposition 2.4.10, $p_j \leq \beta^{j-N} p_N$ for $j \geq N$. Therefore, $p_j^{1/j} \leq (p_N \beta^{-N})^{1/j} \beta$. Since $\lim_{j \rightarrow \infty} (p_N \beta^{-N})^{1/j} = 1$ we then have $\limsup_{j \rightarrow \infty} p_j^{1/j} \leq \beta$. The second inequality follows since $\beta > \alpha$ is arbitrary.

The final assertion follows immediately from the two inequalities using Proposition 2.3.16. ■

In Exercises 2.4.6 and 2.4.7 the reader can explore the various possibilities for the ratio test and root test when $\lim_{j \rightarrow \infty} \left| \frac{x_{j+1}}{x_j} \right| = 1$ and $\lim_{j \rightarrow \infty} |x_j|^{1/j} = 1$, respectively.

The final result we state in this section can be thought of as the summation version of integration by parts.

2.4.15 Proposition (Abel's⁶ partial summation formula) For sequences $\{x_j\}_{j \in \mathbb{N}}$ and $\{y_j\}_{j \in \mathbb{N}}$ of real numbers, denote

$$S_k = \sum_{j=1}^k a_j.$$

Then

$$\begin{aligned} \sum_{j=1}^k a_j b_j &= S_k b_{k+1} - \sum_{j=1}^k S_j (b_{j+1} - b_j) \\ &= S_k b_1 + \sum_{j=1}^k (S_k - S_j) (b_{j+1} - b_j). \end{aligned}$$

Proof Let $S_0 = 0$ by convention. Since $a_j = S_j - S_{j-1}$ we have

$$\sum_{j=1}^n a_j b_j = \sum_{j=1}^n (S_j - S_{j-1}) b_j = \sum_{j=1}^n S_j b_j - \sum_{j=1}^n S_{j-1} b_j.$$

⁶Niels Henrik Abel (1802–1829) was a Norwegian mathematician who worked in the area of analysis. An important theorem of Abel, one that is worth knowing for people working in application areas, is a theorem stating that there is no expression for the roots of a quintic polynomial in terms of the coefficients that involves only the operations of addition, subtraction, multiplication, division and taking roots.

Trivially,

$$\sum_{j=1}^n S_{j-1} b_j = \sum_{j=1}^n S_j b_{j+1} - S_n b_{n+1}.$$

This gives the first equality of the lemma. The second follows from a substitution of

$$b_{n+1} = \sum_{j=1}^n (b_{j+1} - b_j) + b_1$$

into the first equality. ■

2.4.3 e and π

In this section we consider two particular convergent series whose limits are among the most important of “physical constants.”

2.4.16 Definition (e) $e = \sum_{j=0}^{\infty} \frac{1}{j!}$. •

Note that the series defining e indeed converges, for example, by the Ratio Test. Another common representation of e as a limit is the following.

2.4.17 Proposition (Alternative representations of e) $e = \lim_{j \rightarrow \infty} \left(1 + \frac{1}{j}\right)^j = \lim_{j \rightarrow \infty} \left(1 + \frac{1}{j}\right)^{j+1}$.

Proof First note that if the limit $\lim_{j \rightarrow \infty} \left(1 + \frac{1}{j}\right)^j$ exists, then, by Proposition 2.3.22,

$$\lim_{j \rightarrow \infty} \left(1 + \frac{1}{j}\right)^{j+1} = \lim_{j \rightarrow \infty} \left(1 + \frac{1}{j}\right) \left(1 + \frac{1}{j}\right)^j = \lim_{j \rightarrow \infty} \left(1 + \frac{1}{j}\right)^j.$$

Thus we will only prove that $e = \lim_{j \rightarrow \infty} \left(1 + \frac{1}{j}\right)^j$.

Let

$$S_k = \sum_{j=0}^k \frac{1}{j!}, \quad A_k = \left(1 + \frac{1}{k}\right)^k, \quad B_k = \left(1 + \frac{1}{k}\right)^{k+1},$$

be the k th partial sum of the series for e and the k th term in the proposed sequence for e. By the Binomial Theorem (Exercise 2.1.2) we have

$$A_k = \left(1 + \frac{1}{k}\right)^k = \sum_{j=0}^k \binom{k}{j} \frac{1}{k^j}.$$

Moreover, the exact form for the binomial coefficients can directly be seen to give

$$A_k = \sum_{j=0}^k \frac{1}{j!} \left(1 - \frac{1}{k}\right) \left(1 - \frac{2}{k}\right) \cdots \left(1 - \frac{j-1}{k}\right).$$

Each coefficient of $\frac{1}{j!}$, $j \in \{0, 1, \dots, k\}$ is then less than 1. Thus $A_k \leq S_k$ for each $k \in \mathbb{N}_0$. Therefore, $\limsup_{k \rightarrow \infty} A_k \leq \limsup_{k \rightarrow \infty} S_k$. For $m \leq k$ the same computation gives

$$A_k \geq \sum_{j=0}^m \frac{1}{j!} \left(1 - \frac{1}{k}\right) \left(1 - \frac{2}{k}\right) \cdots \left(1 - \frac{j-1}{k}\right).$$

Fixing m and letting $k \rightarrow \infty$ gives

$$\liminf_{k \rightarrow \infty} A_k \geq \sum_{j=0}^m \frac{1}{j!} = S_m.$$

Thus $\liminf_{k \rightarrow \infty} A_k \geq \liminf_{m \rightarrow \infty} S_m$, which gives the result when combined with our previous estimate $\limsup_{k \rightarrow \infty} A_k \leq \limsup_{k \rightarrow \infty} S_k$. ■

It is interesting to note that the series representation of e allows us to conclude that e is irrational.

2.4.18 Proposition (Irrationality of e) $e \in \mathbb{R} \setminus \mathbb{Q}$.

Proof Suppose that $e = \frac{l}{m}$ for $l, m \in \mathbb{N}$. We compute

$$(m-1)!l = m!e = m! \sum_{j=0}^{\infty} \frac{1}{j!} = \sum_{j=0}^m \frac{m!}{j!} + \sum_{j=m+1}^{\infty} \frac{m!}{j!},$$

which then gives

$$\sum_{j=m+1}^{\infty} \frac{m!}{j!} = (m-1)!l - \sum_{j=0}^m \frac{m!}{j!},$$

which implies that $\sum_{j=m+1}^{\infty} \frac{m!}{j!} \in \mathbb{N}$. We then compute, using Example 2.4.2-1,

$$0 < \sum_{j=m+1}^{\infty} \frac{m!}{j!} < \sum_{j=m+1}^{\infty} \frac{1}{(m+1)^{j-m}} = \sum_{j=1}^{\infty} \frac{1}{(m+1)^j} = \frac{\frac{1}{m+1}}{1 - \frac{1}{m+1}} = \frac{1}{m} \leq 1.$$

Thus $\sum_{j=m+1}^{\infty} \frac{m!}{j!} \in \mathbb{N}$, being an integer, must equal 1, and, moreover, $m = 1$. Thus we have

$$\sum_{j=2}^{\infty} \frac{1}{j!} = e - 2 = 1 \quad \implies \quad e = 3.$$

Next let

$$\alpha = \sum_{j=1}^{\infty} \left(\frac{1}{2^{j-1}} - \frac{1}{j!} \right),$$

noting that this series for α converges, and converges to a positive number since each term in the series is positive. Then, using Example 2.4.2-1,

$$\alpha = (2 - (e - 1)) \quad \implies \quad e = 3 - \alpha.$$

Thus $e < 3$, and we have arrived at a contradiction. ■

Next we turn to the number π . Perhaps the best description of π is that it is the ratio of the circumference of a circle with the diameter of the circle. Indeed, the use of the Greek letter “ π ” (i.e., π) has its origins in the word “perimeter.” However, to make sense of this definition, one must be able to talk effectively about circles, what the circumference means, etc. This is more trouble than it is worth for us at this point. Therefore, we give a more analytic description of π , albeit one that, at this point, is not very revealing of what the reader probably already knows about it.

2.4.19 Definition (π) $\pi = 4 \sum_{j=0}^{\infty} \frac{(-1)^j}{2j+1}$. •

By the Alternating Test, this series representation for π converges. We can also fairly easily show that π is irrational.

2.4.20 Proposition (Irrationality of π) $\pi \in \mathbb{R} \setminus \mathbb{Q}$.

Proof In Section 3.10.4 we will give a definition of the trigonometric functions, \sin and \cos , and prove that, on $(0, \pi)$, \sin is positive, and that $\sin 0 = \sin \pi = 0$. We will also prove the rules of differentiation for trigonometric functions necessary for the proof we now present.

Note that if π is rational, then π^2 is also rational. Therefore, it suffices to show that π^2 is irrational.

Let us suppose that $\pi^2 = \frac{l}{m}$ for $l, m \in \mathbb{N}$. For $k \in \mathbb{N}$ define $f_k: [0, 1] \rightarrow \mathbb{R}$ by

$$f_k(x) = \frac{x^k(1-x)^k}{k!},$$

noting that $\text{image}(f) \subset [0, \frac{1}{k!}]$. It is also useful to write

$$f_k(x) = \frac{1}{k!} \sum_{j=k}^{2k} c_j x^j,$$

where we observe that $c_j, j \in \{k, k+1, \dots, 2k\}$ are integers. Define $g_j: [0, 1] \rightarrow \mathbb{R}$ by

$$g_k(x) = k^j \sum_{j=0}^k (-1)^j \pi^{2(k-j)} f^{(2j)}(x).$$

A direct computation shows that

$$f_k^{(j)}(0) = 0, \quad j < k, \quad j > 2k,$$

and that

$$f_k^{(j)}(0) = \frac{j!}{k!} c_j, \quad j \in \{k, k+1, \dots, 2k\},$$

is an integer. Thus f and all of its derivatives take integer values at $x = 0$, and therefore also at $x = 1$ since $f_k(x) = f_k(1-x)$. One also verifies directly that $g_k(0)$ and $g_k(1)$ are integers.

Now we compute

$$\begin{aligned} \frac{d}{dx}(g_k'(x) \sin \pi x - \pi g_k(x) \cos \pi x) &= (g_k''(x) + \pi^2 g_k(x)) \sin \pi x \\ &= m^k \pi^{2k+2} f(x) \sin \pi x = \pi^2 l^k f(x) \sin \pi x, \end{aligned}$$

using the definition of g_k and the fact that $\pi^2 = \frac{l}{m}$. By the Fundamental Theorem of Calculus we then have, after a calculation,

$$\pi l^k \int_0^1 f(x) \sin \pi x \, dx = g_k(0) + g_k(1) \in \mathbb{N}.$$

But we then have, since the integrand in the above integral is nonnegative,

$$0 < \pi l^k \int_0^1 f(x) \sin \pi x \, dx < \frac{\pi l^k}{k!}$$

given the bounds on f_k . Note that $\lim_{k \rightarrow \infty} \frac{l^k}{k!} = 0$. Since the above computations hold for any k , if we take k sufficiently large that $\frac{\pi l^k}{k!} < 1$, we arrive at a contradiction. ■

2.4.4 Doubly infinite series

We shall frequently encounter series whose summation index runs not from 1 to ∞ , but from $-\infty$ to ∞ . Thus we call a collection $\{x_j\}_{j \in \mathbb{Z}}$ of elements of \mathbb{R} a **doubly infinite sequence** in \mathbb{R} , and a sum of the form $\sum_{j=-\infty}^{\infty} x_j$ a **doubly infinite series**. A little care need to be shown when defining convergence for such series, and here we give the appropriate definitions.

2.4.21 Definition (Convergence and absolute convergence of doubly infinite series) Let $\{x_j\}_{j \in \mathbb{Z}}$ be a doubly infinite sequence and let $S = \sum_{j=-\infty}^{\infty} x_j$ be the corresponding doubly infinite series. The sequence of *single partial sums* is the sequence $\{S_k\}_{k \in \mathbb{N}}$ where

$$S_k = \sum_{j=-k}^k x_j,$$

and the sequence of *double partial sums* is the double sequence $\{S_{k,l}\}_{k,l \in \mathbb{N}}$ defined by

$$S_{k,l} = \sum_{j=-k}^l x_j.$$

Let $s_0 \in \mathbb{R}$. The doubly infinite series:

- (i) *converges to* s_0 if the double sequence of partial sums converges to s_0 ;
- (ii) has s_0 as a *limit* if it converges to s_0 ;
- (iii) is *convergent* if it converges to some element of \mathbb{R} ;
- (iv) *converges absolutely*, or is *absolutely convergent*, if the double series

$$\sum_{j=-\infty}^{\infty} |x_j|$$

converges;

- (v) *converges conditionally*, or is *conditionally convergent*, if it is convergent, but not absolutely convergent;
- (vi) *diverges* if it does not converge;
- (vii) *diverges to* ∞ (resp. *diverges to* $-\infty$), and we write $\sum_{j=-\infty}^{\infty} x_j = \infty$ (resp. $\sum_{j=-\infty}^{\infty} x_j = -\infty$), if the sequence of double partial sums diverges to ∞ (resp. diverges to $-\infty$);
- (viii) has a limit that *exists* if $\sum_{j=-\infty}^{\infty} x_j \in \mathbb{R}$;
- (ix) is *oscillatory* if the limit of the double sequence of partial sums is oscillatory. •

2.4.22 Remark (Partial sums versus double partial sums) Note that the convergence of the sequence of partial sums is not a very helpful notion, in general. For example, the series $\sum_{j=-\infty}^{\infty} j$ possesses a sequence of partial sums that is identically zero, and so obviously converges to zero. However, it is not likely that one would wish this doubly infinite series to qualify as convergent. Thus partial sums are not a particularly good measure of convergence. However, there are situations—for example, the convergence of Fourier series (see Chapter III-5)—where the standard notion of convergence of a doubly infinite series is made using the partial sums. However, in these cases, there is additional structure on the setup that makes this a reasonable thing to do. •

The convergence of a doubly infinite series has the following useful, intuitive characterisation.

2.4.23 Proposition (Characterisation of convergence of doubly infinite series) For a doubly infinite series $S = \sum_{j=-\infty}^{\infty} x_j$, the following statements are equivalent:

- (i) S converges;
- (ii) the two series $\sum_{j=0}^{\infty} x_j$ and $\sum_{j=1}^{\infty} x_{-j}$ converge.

Proof For $k, l \in \mathbb{N}$, denote

$$S_{k,l} = \sum_{-k}^l x_j, \quad S_k^+ = \sum_{j=0}^k x_j, \quad S_k^- = \sum_{-k}^{-1} x_j,$$

so that $S_{k,l} = S_k^- + S_l^+$.

(i) \implies (ii) Let $\epsilon > 0$ and choose $N \in \mathbb{N}$ such that $|S_{j,k} - s_0| < \frac{\epsilon}{2}$ for $j, k \geq N$. Now let $j, k \geq N$, choose some $l \geq N$, and compute

$$|S_j^+ - S_k^+| \leq |S_j^+ + S_l^- - s_0| + |S_k^+ + S_l^- - s_0| < \epsilon.$$

Thus $\{S_j^+\}_{j \in \mathbb{N}}$ is a Cauchy sequence, and so is convergent. A similar argument shows that $\{S_j^-\}_{j \in \mathbb{N}}$ is also a Cauchy sequence.

(ii) \implies (i) Let s^+ be the limit of $\sum_{j=0}^{\infty} x_j$ and let s^- be the limit of $\sum_{j=1}^{\infty} x_{-j}$. For $\epsilon > 0$ define $N^+, N^- \in \mathbb{N}$ such that $|S_j^+ - s^+| < \frac{\epsilon}{2}$, $j \geq N^+$, and $|S_j^- - s^-| < \frac{\epsilon}{2}$, $j \leq -N^-$. Then, for $j, k \geq \max\{N^-, N^+\}$,

$$|S_{j,k} - (s^+ + s^-)| = |S_k^+ - s^+ + S_j^- - s^-| \leq |S_k^+ - s^+| + |S_j^- - s^-| < \epsilon,$$

thus showing that S converges. ■

Thus convergent doubly infinite series are really just combinations of convergent series in the sense that we have studied in the preceding sections. Thus, for example, one can use the tests of Section 2.4.2 to check for convergence of a doubly infinite series by applying them to both “halves” of the series. Also, the relationships between convergence and absolute convergence for series also hold for doubly infinite series. And a suitable version of Theorem 2.4.5 also holds for doubly infinite series. These facts are so straightforward that we will assume them in the sequel without explicit mention; they all follow directly from Proposition 2.4.23.

2.4.5 Multiple series

Just as we considered multiple sequences in Section 2.3.5, we can consider multiple series. As we did with sequences, we content ourselves with double series.

2.4.24 Definition (Double series) A *double series* in \mathbb{R} is a sum of the form $\sum_{j,k=1}^{\infty} x_{jk}$ where $\{x_{jk}\}_{j,k \in \mathbb{N}}$ is a double sequence in \mathbb{R} . ●

While our definition of a series was not entirely sensible since it was not really identifiable as anything unless it had certain convergence properties, for double series, things are even worse. In particular, it is not clear what $\sum_{j,k=1}^{\infty} x_{jk}$ means. Does it mean $\sum_{j=1}^{\infty} \left(\sum_{k=1}^{\infty} x_{jk} \right)$? Does it mean $\sum_{k=1}^{\infty} \left(\sum_{j=1}^{\infty} x_{jk} \right)$? Or does it mean something different from both of these? The only way to rectify our poor mathematical manners is to define convergence for double series as quickly as possible.

2.4.25 Definition (Convergence and absolute convergence of double series) Let $\{x_{jk}\}_{j,k \in \mathbb{N}}$ be a double sequence in \mathbb{R} and consider the double series

$$S = \sum_{j,k=1}^{\infty} x_{jk}.$$

The corresponding sequence of **partial sums** is the double sequence $\{S_{jk}\}_{j,k \in \mathbb{N}}$ defined by

$$S_{jk} = \sum_{l=1}^j \sum_{m=1}^k x_{lm}.$$

Let $s_0 \in \mathbb{R}$. The double series:

- (i) **converges to s_0** , and we write $\sum_{j,k=1}^{\infty} x_{jk} = s_0$, if the double sequence of partial sums converges to s_0 ;
- (ii) has s_0 as a **limit** if it converges to s_0 ;
- (iii) is **convergent** if it converges to some member of \mathbb{R} ;
- (iv) **converges absolutely**, or is **absolutely convergent**, if the series

$$\sum_{j,k=1}^{\infty} |x_{jk}|$$
 converges;
- (v) **converges conditionally**, or is **conditionally convergent**, if it is convergent, but not absolutely convergent;
- (vi) **diverges** if it does not converge;
- (vii) **diverges to ∞** (resp. **diverges to $-\infty$**), and we write $\sum_{j,k=1}^{\infty} x_{jk} = \infty$ (resp. $\sum_{j,k=1}^{\infty} x_{jk} = -\infty$), if the double sequence of partial sums diverges to ∞ (resp. diverges to $-\infty$);
- (viii) has a limit that **exists** if $\sum_{j,k=1}^{\infty} x_{jk} \in \mathbb{R}$;
- (ix) is **oscillatory** if the sequence of partial sums is oscillatory. •

Note that the definition of the partial sums, S_{jk} , $j, k \in \mathbb{N}$, for a double series is unambiguous since

$$\sum_{l=1}^j \sum_{m=1}^k x_{lm} = \sum_{m=1}^k \sum_{l=1}^j x_{lm},$$

this being valid for finite sums. The idea behind convergence of double series, then, has an interpretation that can be gleaned from that in Figure 2.1 for double sequences.

Let us state a result, derived from similar results for double sequences, that allows the computation of limits of double series by computing one limit at a time.

2.4.26 Proposition (Computation of limits of double series I) Suppose that for the double series

$\sum_{j,k=1}^{\infty} x_{jk}$ it holds that

- (i) the double series is convergent and
- (ii) for each $j \in \mathbb{N}$, the series $\sum_{k=1}^{\infty} x_{jk}$ converges.

Then the series $\sum_{j=1}^{\infty} (\sum_{k=1}^{\infty} x_{jk})$ converges and its limit is equal to $\sum_{j,k=1}^{\infty} x_{jk}$.

Proof This follows directly from Proposition 2.3.19. ■

2.4.27 Proposition (Computation of limits of double series II) Suppose that for the double series

$\sum_{j,k=1}^{\infty} x_{jk}$ it holds that

- (i) the double series is convergent,
- (ii) for each $j \in \mathbb{N}$, the series $\sum_{k=1}^{\infty} x_{jk}$ converges, and
- (iii) for each $k \in \mathbb{N}$, the limit $\sum_{j=1}^{\infty} x_{jk}$ converges.

Then the series $\sum_{j=1}^{\infty} (\sum_{k=1}^{\infty} x_{jk})$ and $\sum_{k=1}^{\infty} (\sum_{j=1}^{\infty} x_{jk})$ converge and their limits are both equal to $\sum_{j,k=1}^{\infty} x_{jk}$.

Proof This follows directly from Proposition 2.3.20. ■

2.4.6 Algebraic operations on series

In this section we consider the manner in which series interact with algebraic operations. The results here mirror, to some extent, the results for sequences in Section 2.3.6. However, the series structure allows for different ways of thinking about the product of sequences. Let us first give these definitions. For notational convenience, we use sums that begin at 0 rather than 1. This clearly has no affect on the definition of a series, or on any of its properties.

2.4.28 Definition (Products of series) Let $S = \sum_{j=0}^{\infty} x_j$ and $T = \sum_{j=0}^{\infty} y_j$ be series in \mathbb{R} .

(i) The **product** of S and T is the double series $\sum_{j,k=0}^{\infty} x_j y_k$.

(ii) The **Cauchy product** of S and T is the series $\sum_{k=0}^{\infty} \left(\sum_{j=0}^k x_j y_{k-j} \right)$. •

Now we can state the basic results on algebraic manipulation of series.

2.4.29 Proposition (Algebraic operations on series) Let $S = \sum_{j=0}^{\infty} x_j$ and $T = \sum_{j=0}^{\infty} y_j$ be series in \mathbb{R} that converges to s_0 and t_0 , respectively, and let $\alpha \in \mathbb{R}$. Then the following statements hold:

- (i) the series $\sum_{j=0}^{\infty} \alpha x_j$ converges to αs_0 ;
- (ii) the series $\sum_{j=0}^{\infty} (x_j + y_j)$ converges to $s_0 + t_0$;
- (iii) if S and T are absolutely convergent, then the product of S and T is absolutely convergent and converges to $s_0 t_0$;
- (iv) if S and T are absolutely convergent, then the Cauchy product of S and T is absolutely convergent and converges to $s_0 t_0$;
- (v) if S or T are absolutely convergent, then the Cauchy product of S and T is convergent and converges to $s_0 t_0$;
- (vi) if S and T are convergent, and if the Cauchy product of S and T is convergent, then the Cauchy product of S and T converges to $s_0 t_0$.

Proof (i) Since $\sum_{j=0}^k \alpha x_j = \alpha \sum_{j=0}^k x_j$, this follows from part (i) of Proposition 2.3.22.

(ii) Since $\sum_{j=0}^{\infty} (x_j + y_j) = \sum_{j=0}^k x_j + \sum_{j=0}^k y_j$, this follows from part (ii) of Proposition 2.3.22.

(iii) and (iv) To prove these parts of the result, we first make a general argument. We note that $\mathbb{N}_0 \times \mathbb{N}_0$ is a countable set (e.g., by Proposition 1.7.16), and so there exists a bijection, in fact many bijections, $\phi: \mathbb{N} \rightarrow \mathbb{N}_0 \times \mathbb{N}_0$. For such a bijection ϕ , suppose that we are given a double sequence $\{x_{jk}\}_{j,k \in \mathbb{N}_0}$ and define a sequence $\{x_j^{\phi}\}_{j \in \mathbb{N}}$ by $x_j^{\phi} = x_{kl}$ where $(k, l) = \phi(j)$. We then claim that, for any bijection $\phi: \mathbb{N} \rightarrow \mathbb{N}_0 \times \mathbb{N}_0$, the double series $A = \sum_{k,l=1}^{\infty} x_{kl}$ converges absolutely if and only if the series $A^{\phi} = \sum_{j=1}^{\infty} x_j^{\phi}$ converges absolutely.

Indeed, suppose that the double series $|A| = \sum_{k,l=1}^{\infty} |x_{kl}|$ converges to $\beta \in \mathbb{R}$. For $\epsilon > 0$ the set

$$\{(k, l) \in \mathbb{N}_0 \times \mathbb{N}_0 \mid ||A|_{kl} - \beta| \geq \epsilon\}$$

is then finite. Therefore, there exists $N \in \mathbb{N}$ such that, if $(k, l) = \phi(j)$ for $j \geq N$, then $||A|_{kl} - \beta| < \epsilon$. It therefore follows that $||A^{\phi}|_j - \beta| < \epsilon$ for $j \geq N$, where $|A^{\phi}|$ denotes the series $\sum_{j=1}^{\infty} |x_j^{\phi}|$. This shows that the series $|A^{\phi}|$ converges to β .

For the converse, suppose that the series $|A^{\phi}|$ converges to β . Then, for $\epsilon > 0$ the set

$$\{j \in \mathbb{N} \mid ||A^{\phi}|_j - \beta| \geq \epsilon\}$$

is finite. Therefore, there exists $N \in \mathbb{N}$ such that

$$\{(k, l) \in \mathbb{N}_0 \mid k, l \geq N\} \cap \{(k, l) \in \mathbb{N}_0 \mid ||A^{\phi}|_{\phi^{-1}(k,l)} - \beta| \geq \epsilon\} = \emptyset.$$

It then follows that for $k, l \geq N$ we have $||A|_{kl} - \beta| < \epsilon$, showing that $|A|$ converges to β .

Thus we have shown that A is absolutely convergent if and only if A^ϕ is absolutely convergent for any bijection $\phi: \mathbb{N} \rightarrow \mathbb{N}_0 \times \mathbb{N}_0$. From part (v) of Theorem 2.4.5, and its generalisation to double series, we know that the limit of an absolutely convergent series or double series is independent of the manner in which the terms in the series are arranged.

Consider now a term in the product of S and T . It is easy to see that this term appears exactly once in the Cauchy product of S and T . Conversely, each term in the Cauchy product appears exactly one in the product. Thus the product and Cauchy product are simply rearrangements of one another. Moreover, each term in the product and the Cauchy product appears exactly once in the expression

$$\left(\sum_{j=0}^N x_j\right)\left(\sum_{k=0}^N y_k\right)$$

as we allow N to go to ∞ . That is to say,

$$\sum_{j,k=0}^{\infty} x_j y_k = \sum_{k=0}^{\infty} \left(\sum_{j=k}^{\infty} x_j y_{k-j}\right) = \lim_{N \rightarrow \infty} \left(\sum_{j=0}^N x_j\right)\left(\sum_{k=0}^N y_k\right).$$

However, this last limit is exactly $s_0 t_0$, using part (iii) of Proposition 2.3.22.

(v) Without loss of generality, suppose that S converges absolutely. Let $\{S_k\}_{k \in \mathbb{N}}$, $\{T_k\}_{k \in \mathbb{N}}$, and $\{(ST)_k\}_{k \in \mathbb{N}}$ be the sequences of partial sums for S , T , and the Cauchy product, respectively. Also define $\tau_k = T_k - t_0$, $k \in \mathbb{N}_0$. Then

$$\begin{aligned} (ST)_k &= x_0 y_0 + (x_0 y_1 + x_1 y_0) + \cdots + (x_0 y_k + \cdots + x_k y_0) \\ &= x_0 T_k + x_1 T_{k-1} + \cdots + x_k T_0 \\ &= x_0(t_0 + \tau_k) + x_1(t_0 + \tau_{k-1}) + \cdots + x_k(t_0 + \tau_0) \\ &= S_k t_0 + x_0 \tau_k + x_1 \tau_{k-1} + \cdots + x_k \tau_0. \end{aligned}$$

Since $\lim_{k \rightarrow \infty} S_k t_0 = s_0 t_0$ by part (i), this part of the result will follow if we can show that

$$\lim_{k \rightarrow \infty} (x_0 \tau_k + x_1 \tau_{k-1} + \cdots + x_k \tau_0) = 0. \quad (2.5)$$

Denote

$$\sigma = \sum_{j=0}^{\infty} |x_j|,$$

and for $\epsilon > 0$ choose $N_1 \in \mathbb{N}$ such that $|\tau_j| \leq \frac{\epsilon}{2\sigma}$ for $j \geq N_1$, this being possible since $\{\tau_j\}_{j \in \mathbb{N}}$ clearly converges to zero. Then, for $k \geq N_1$,

$$\begin{aligned} |x_0 \tau_k + x_1 \tau_{k-1} + \cdots + x_k \tau_0| &\leq |x_0 \tau_k + \cdots + x_{k-N_1+1} \tau_{N_1+1}| + |x_{k-N_1} \tau_{N_1} + \cdots + x_k \tau_0| \\ &\leq \frac{\epsilon}{2} + |x_{k-N_1} \tau_{N_1} + \cdots + x_k \tau_0|. \end{aligned}$$

Since $\lim_{k \rightarrow \infty} x_k = 0$, choose $N_2 \in \mathbb{N}$ such that

$$|x_{k-N_1} \tau_{N_1} + \cdots + x_k \tau_0| < \frac{\epsilon}{2}$$

for $k \geq N_2$. Then

$$\begin{aligned} \limsup_{k \rightarrow \infty} |x_0 \tau_k + x_1 \tau_{k-1} + \cdots + x_k \tau_0| &= \lim_{k \rightarrow \infty} \sup \{ |x_0 \tau_j + x_1 \tau_{j-1} + \cdots + x_j \tau_0| \mid j \geq k \} \\ &\leq \lim_{k \rightarrow \infty} \sup \left\{ \frac{\epsilon}{2} + |x_{k-N_1} \tau_{N_1} + \cdots + x_k \tau_0| \mid j \geq k \right\} \\ &\leq \sup \left\{ \frac{\epsilon}{2} + |x_{k-N_1} \tau_{N_1} + \cdots + x_k \tau_0| \mid j \geq N_2 \right\} \leq \epsilon. \end{aligned}$$

Thus

$$\limsup_{k \rightarrow \infty} |x_0\tau_k + x_1\tau_{k-1} + \cdots + x_k\tau_0| \leq 0,$$

and since clearly

$$\liminf_{k \rightarrow \infty} |x_0\tau_k + x_1\tau_{k-1} + \cdots + x_k\tau_0| \geq 0,$$

we infer that (2.5) holds by Proposition 2.3.16.

(vi) The reader can prove this as Exercise 3.9.3. ■

examples?

2.4.7 Series with arbitrary index sets

It will be helpful on a few occasions to be able to sum series whose index set is not necessarily countable, and here we indicate how this can be done. This material should be considered optional until one comes to that point in the text where it is needed.

2.4.30 Definition (Sum of series for arbitrary index sets) Let A be a set and let $\{x_a\}_{a \in A}$ be a family of elements of \mathbb{R} . Let $A_+ = \{a \in A \mid x_a \in [0, \infty]\}$ and $A_- = \{a \in A \mid x_a \in [-\infty, 0]\}$.

- (i) If $x_a \in [0, \infty]$ for $a \in A$, then $\sum_{a \in A} x_a = \sup \{ \sum_{a \in A'} x_a \mid A' \subset A \text{ is finite} \}$.
- (ii) For a general family, $\sum_{a \in A} x_a = \sum_{a_+ \in A_+} x_{a_+} - \sum_{a_- \in A_-} (-x_{a_-})$, provided that at least one of $\sum_{a_+ \in A_+} x_{a_+}$ or $\sum_{a_- \in A_-} (-x_{a_-})$ is finite.
- (iii) If both $\sum_{a_+ \in A_+} x_{a_+}$ and $\sum_{a_- \in A_-} (-x_{a_-})$ are finite, then $\{x_a\}_{a \in A}$ is **summable**. •

We should understand the relationship between this sort of summation and our existing notion of the sum of a series in the case where the index set is \mathbb{N} .

2.4.31 Proposition (A summable series with index set \mathbb{N} is absolutely convergent) A sequence $\{x_j\}_{j \in \mathbb{N}}$ in \mathbb{R} is summable if and only if the series $S = \sum_{j=1}^{\infty} x_j$ is absolutely convergent.

Proof Consider the sequences $\{x_j^+\}_{j \in \mathbb{N}}$ and $\{x_j^-\}_{j \in \mathbb{N}}$ defined by

$$x_j^+ = \max\{x_j, 0\}, \quad x_j^- = \max\{-x_j, 0\}, \quad j \in \mathbb{N}.$$

Then $\{x_j\}_{j \in \mathbb{N}}$ is summable if and only if both of the expressions

$$\sup \left\{ \sum_{j \in A'} x_j^+ \mid A' \subset \mathbb{N} \text{ is finite} \right\}, \quad \sup \left\{ \sum_{j \in A'} x_j^- \mid A' \subset \mathbb{N} \text{ is finite} \right\} \quad (2.6)$$

are finite.

First suppose that $\{x_j\}_{j \in \mathbb{N}}$ is summable. Therefore, if $\{S_k^+\}_{k \in \mathbb{N}}$ and $\{S_k^-\}_{k \in \mathbb{N}}$ are the sequences of partial sums

$$S_k^+ = \sum_{j=1}^k x_j^+, \quad S_k^- = \sum_{j=1}^k x_j^-,$$

then these sequences are increasing and so convergent by (2.6). Then, by Proposition 2.3.22,

$$\sum_{j=1}^{\infty} |x_j| = \sum_{j=1}^{\infty} x_j^+ + \sum_{j=1}^{\infty} x_j^-$$

giving absolute convergence of S .

Now suppose that S is absolutely convergent. Then the subsets $\{S_k^+\}_{k \in \mathbb{N}}$ and $\{S_k^-\}_{k \in \mathbb{N}}$ are bounded above (as well as being bounded below by zero) so that both expressions

$$\sup \{S_k^+\}_{k \in \mathbb{N}}, \quad \sup \{S_k^-\}_{k \in \mathbb{N}}$$

are finite. Then for any finite set $A' \subset \mathbb{N}$ we have

$$\sum_{j \in A'} x_j^+ \leq S_{\sup A'}^+, \quad \sum_{j \in A'} x_j^- \leq S_{\sup A'}^-.$$

From this we deduce that

$$\sup \left\{ \sum_{j \in A'} x_j^+ \mid A' \subset \mathbb{N} \text{ is finite} \right\} \leq \sup \{S_k^+\}_{k \in \mathbb{N}},$$

$$\sup \left\{ \sum_{j \in A'} x_j^- \mid A' \subset \mathbb{N} \text{ is finite} \right\} \leq \sup \{S_k^-\}_{k \in \mathbb{N}},$$

which implies that $\{x_j\}_{j \in \mathbb{N}}$ is summable. ■

Now we can actually show that, for a summable family of real numbers, only countably many of them can be nonzero.

2.4.32 Proposition (A summable family has at most countably many nonzero members) *If $\{x_a\}_{a \in A}$ is summable, then the set $\{a \in A \mid x_a \neq 0\}$ is countable.*

Proof Note that for any $k \in \mathbb{N}$, the set $\{a \in A \mid |x_a| \geq \frac{1}{k}\}$ must be finite if $\{x_a\}_{a \in A}$ is summable (why?). Thus, since

$$\{a \in A \mid |x_a| \neq 0\} = \cup_{k \in \mathbb{N}} \left\{ a \in A \mid |x_a| \geq \frac{1}{k} \right\},$$

the set $\{a \in A \mid x_a \neq 0\}$ is a countable union of finite sets, and so is countable by Proposition 1.7.16. ■

A legitimate question is, since a summable family reduces to essentially being countable, why should we bother with the idea at all? The reason is simply that it will be notationally convenient in Section 3.3.5.

Exercises

2.4.1 Let $S = \sum_{j=1}^{\infty} x_j$ be a series in \mathbb{R} , and, for $j \in \mathbb{N}$, define

$$x_j^+ = \max\{x_j, 0\}, \quad x_j^- = \max\{0, -x_j\}.$$

Show that, if S is conditionally convergent, then the series $S^+ = \sum_{j=1}^{\infty} x_j^+$ and $S^- = \sum_{j=1}^{\infty} x_j^-$ diverge to ∞ .

2.4.2 In this exercise we consider more carefully the paradox of Zeno given in Exercise 1.9.2. Let us attach some symbols to the relevant data, so that we can say useful things. Suppose that the tortoise travels with constant velocity v_t and that Achilles travels with constant velocity v_a . Suppose that the tortoise gets a head start of t_0 seconds.

(a) Compute directly using elementary physics (i.e., time/distance/velocity relations) the time at which Achilles will overtake the tortoise, and the distance both will have travelled during that time.

(b) Consider the sequences $\{d_j\}_{j \in \mathbb{N}}$ and $\{t_j\}_{j \in \mathbb{N}}$ defined so that

1. d_1 is the distance travelled by the tortoise during the head start time t_0 ,
2. t_j , $j \in \mathbb{N}$, is the time it takes Achilles to cover the distance d_j ,
3. d_j , $j \geq 2$, is the distance travelled by the tortoise in time t_{j-1} .

Find explicit expressions for these sequences in terms of t_0 , v_t , and v_a .

(c) Show that the series $\sum_{j=1}^{\infty} d_j$ and $\sum_{j=1}^{\infty} t_j$ converge, and compute their limits.

- (d) What is the relationship between the limits of the series in part (c) and the answers to part (a).
- (e) Does this shed some light on how to resolve Zeno's paradox?

2.4.3 Show that

$$\left| \sum_{j=1}^m x_j \right| \leq \sum_{j=1}^m |x_j|$$

for any set $\{x_1, \dots, x_m\} \subset \mathbb{R}$.

2.4.4 State the correct version of Proposition 2.4.4 in the case that $S = \sum_{j=1}^{\infty} x_j$ is not absolutely convergent, and indicate why it is not a very interesting result.

2.4.5 For a sum

$$S = \sum_{j=1}^{\infty} s_j,$$

answer the following questions.

- (a) Show that if S converges then the sequence $\{s_j\}_{j \in \mathbb{N}}$ converges to 0.
- (b) Is the converse of part (a) true? That is to say, if the sequence $\{s_j\}_{j \in \mathbb{N}}$ converges to zero, does S converge? If this is true, prove it. If it is not true, give a counterexample.

2.4.6 Do the following.

- (a) Find a series $\sum_{j=1}^{\infty} x_j$ for which $\lim_{j \rightarrow \infty} \left| \frac{x_{j+1}}{x_j} \right| = 1$ and which converges in \mathbb{R} .
- (b) Find a series $\sum_{j=1}^{\infty} x_j$ for which $\lim_{j \rightarrow \infty} \left| \frac{x_{j+1}}{x_j} \right| = 1$ and which diverges to ∞ .
- (c) Find a series $\sum_{j=1}^{\infty} x_j$ for which $\lim_{j \rightarrow \infty} \left| \frac{x_{j+1}}{x_j} \right| = 1$ and which diverges to $-\infty$.
- (d) Find a series $\sum_{j=1}^{\infty} x_j$ for which $\lim_{j \rightarrow \infty} \left| \frac{x_{j+1}}{x_j} \right| = 1$ and which is oscillatory.

2.4.7 Do the following.

- (a) Find a series $\sum_{j=1}^{\infty} x_j$ for which $\lim_{j \rightarrow \infty} |x_j|^{1/j} = 1$ and which converges in \mathbb{R} .
- (b) Find a series $\sum_{j=1}^{\infty} x_j$ for which $\lim_{j \rightarrow \infty} |x_j|^{1/j} = 1$ and which diverges to ∞ .
- (c) Find a series $\sum_{j=1}^{\infty} x_j$ for which $\lim_{j \rightarrow \infty} |x_j|^{1/j} = 1$ and which diverges to $-\infty$.
- (d) Find a series $\sum_{j=1}^{\infty} x_j$ for which $\lim_{j \rightarrow \infty} |x_j|^{1/j} = 1$ and which is oscillatory.

The next exercise introduces the notion of the decimal expansion of a real number. An *infinite decimal expansion* is a series in \mathbb{Q} of the form

$$\sum_{j=0}^{\infty} \frac{a_j}{10^j}$$

where $a_0 \in \mathbb{Z}$ and where $a_j \in \{0, 1, \dots, 9\}$, $j \in \mathbb{N}$. An infinite decimal expansion is *eventually periodic* if there exists $k, m \in \mathbb{N}$ such that $a_{j+k} = a_j$ for all $j \geq m$.

- 2.4.8 (a) Show that the sequence of partial sums for an infinite decimal expansion is a Cauchy sequence.
- (b) Show that, for every Cauchy sequence $\{q_j\}_{j \in \mathbb{N}}$, there exists a sequence $\{d_j\}_{j \in \mathbb{N}}$ of partial sums for a decimal expansion having the property that $[\{q_j\}_{j \in \mathbb{N}}] = [d_j]_{j \in \mathbb{N}}$ (the equivalence relation is that in the Cauchy sequences in \mathbb{Q} as defined in Definition 2.1.16).
- (c) Give an example that shows that two distinct infinite decimal expansions can be equivalent.

- (d) Show that if two distinct infinite decimal expansions are equivalent, and if one of them is eventually periodic, then the other is also eventually periodic.

The previous exercise shows that every real number is the limit of a (not necessarily unique) infinite decimal expansion. The next exercises characterise the infinite decimal expansions that correspond to rational numbers. First you will show that an eventually periodic decimal expansion corresponds to a rational number. Let $\sum_{j=0}^{\infty} \frac{a_j}{10^j}$ be an eventually periodic infinite decimal expansion and let $k, m \in \mathbb{N}$ have the property that $a_{j+k} = a_j$ for $j \geq m$. Denote by $x \in \mathbb{R}$ the number to which the infinite decimal expansion converges.

- (e) Show that

$$10^{m+k}x = \sum_{j=0}^{\infty} \frac{b_j}{10^j}, \quad 10^m x = \sum_{j=0}^{\infty} \frac{c_j}{10^j}$$

are decimal expansions, and give expressions for b_j and c_j , $j \in \mathbb{N}$, in terms of a_j , $j \in \mathbb{N}$. In particular, show that $b_j = c_j$ for $j \geq 1$.

- (f) Conclude that $(10^{m+k} - 10^m)x$ is an integer, and so x is therefore rational.

Next you will show that the infinite decimal expansion of a rational number is eventually periodic. Thus let $q \in \mathbb{Q}$.

- (g) Let $q = \frac{a}{b}$ for $a, b \in \mathbb{Z}$ and with $b > 0$. For $j \in \{0, 1, \dots, b-1\}$, let $r_j \in \{0, 1, \dots, b-1\}$ satisfy $\frac{10^j}{b} = s_j + \frac{r_j}{b}$ for $s_j \in \mathbb{Z}$, i.e., r_j is the remainder after dividing 10^j by b . Show that at least two of the numbers $\{r_0, r_1, \dots, r_{b-1}\}$ must agree, i.e., conclude that $r_m = r_{m+k}$ for $k, m \in \mathbb{N}_0$ satisfying $0 \leq m < m+k \leq b$.

Hint: There are only b possible values for these $b+1$ numbers.

- (h) Show that b exactly divides $10^{m+k} - 10^k$ with k and m as above. Thus $bc = 10^{m+k} - 10^k$ for some $c \in \mathbb{Z}$.

- (i) Show that

$$\frac{a}{b} = 10^{-m} \frac{ac}{10^k - 1},$$

and so write

$$q = 10^{-m} \left(s + \frac{r}{10^k - 1} \right)$$

for $s \in \mathbb{Z}$ and $r \in \{0, 1, \dots, 10^k - 1\}$, i.e., r is the remainder after dividing ac by $10^k - 1$.

- (j) Argue that we can write

$$b = \sum_{j=1}^k b_j 10^j,$$

for $b_j \in \{0, 1, \dots, 9\}$, $j \in \{1, \dots, k\}$.

- (k) With b_j , $j \in \{1, \dots, k\}$ as above, define an infinite decimal expansion $\sum_{j=0}^{\infty} \frac{a_j}{10^j}$ by asking that $a_0 = 0$, that $a_j = b_j$, $j \in \{1, \dots, k\}$, and that $a_{j+km} = a_j$ for $j, m \in \mathbb{N}$. Let $d \in \mathbb{R}$ be the number to which this decimal expansion converges. Show that $(10^k - 1)d = b$, so $d \in \mathbb{Q}$.

- (l) Show that $10^m q = s + d$, and so conclude that $10^m q$ has the eventually periodic infinite decimal expansion $s + \sum_{j=1}^{\infty} \frac{a_j}{10^j}$.

- (m) Conclude that q has an eventually periodic infinite decimal expansion, and then conclude from (d) that any infinite decimal expansion for q is eventually periodic.

Section 2.5

Subsets of \mathbb{R}

In this section we study in some detail the nature of various sorts of subsets of \mathbb{R} . The character of these subsets will be of some importance when we consider the properties of functions defined on \mathbb{R} , and/or taking values in \mathbb{R} . Our presentation also gives us an opportunity to introduce, in a fairly simple setting, some concepts that will appear later in more abstract settings, e.g., open sets, closed sets, compactness.

Do I need to read this section? Unless you know the material here, it is indeed a good idea to read this section. Many of the ideas are basic, but some are not (e.g., the Heine–Borel Theorem). Moreover, many of the not-so-basic ideas will appear again later, particularly in Chapter II-2, and if a reader does not understand the ideas in the simple case of \mathbb{R} , things will only get more difficult. Also, the ideas expressed here will be essential in understanding even basic things about signals as presented in Chapter III-2. •

2.5.1 Open sets, closed sets, and intervals

One of the basic building blocks in the understanding of the real numbers is the idea of an open set. In this section we define open sets and some related notions, and provide some simple properties associated to these ideas.

First, it is convenient to introduce the following ideas. For $r > 0$ and $x \in \mathbb{R}$, the *open ball* in \mathbb{R} of radius r about x is the set

$$B(r, x) = \{y \in \mathbb{R} \mid |x - y| < r\},$$

and the *closed ball* of radius r about x is the set

$$\bar{B}(r, x) = \{y \in \mathbb{R} \mid |x - y| \leq r\}.$$

These sets are simple to understand, and we depict them in Figure 2.3. With the notion of



Figure 2.3 An open ball (left) and a closed ball (right) in \mathbb{R}

an open ball, it is easy to give some preliminary definitions.

2.5.1 Definition (Open and closed sets in \mathbb{R}) A set $A \subset \mathbb{R}$ is:

- (i) *open* if, for every $x \in A$, there exists $\epsilon > 0$ such that $B(\epsilon, x) \subset A$ (the empty set is also open, by declaration);
- (ii) *closed* if $\mathbb{R} \setminus A$ is open. •

A trivial piece of language associated with an open set is the notion of a neighbourhood.

2.5.2 Definition (Neighbourhood in \mathbb{R}) A *neighbourhood* of an element $x \in \mathbb{R}$ is an open set U for which $x \in U$. •

Some authors allow a “neighbourhood” to be a set A which contains a neighbourhood in our sense. Such authors will then frequently call what we call a neighbourhood an “open neighbourhood.”

Let us give some examples of sets that are open, closed, or neither. The examples we consider here are important ones, since they are all examples of *intervals*, which will be of interest at various times, and for various reasons, throughout these volumes. In particular, the notation we introduce here for intervals will be used a great deal.

2.5.3 Examples (Intervals)

1. For $a, b \in \mathbb{R}$ with $a < b$ the set

$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$$

is open. Indeed, let $x \in (a, b)$ and let $\epsilon = \frac{1}{2} \min\{b - x, x - a\}$. It is then easy to see that $B(\epsilon, x) \subset (a, b)$. If $a \geq b$ we take the convention that $(a, b) = \emptyset$.

2. For $a \in \mathbb{R}$ the set

$$(a, \infty) = \{x \in \mathbb{R} \mid a < x\}$$

is open. For example, if $x \in (a, \infty)$ then, if we define $\epsilon = \frac{1}{2}(x - a)$, we have $B(\epsilon, x) \subset (a, \infty)$.

3. For $b \in \mathbb{R}$ the set

$$(-\infty, b) = \{x \in \mathbb{R} \mid x < b\}$$

is open.

4. For $a, b \in \mathbb{R}$ with $a \leq b$ the set

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$$

is closed. Indeed, $\mathbb{R} \setminus [a, b] = (-\infty, a) \cup (b, \infty)$. The sets $(-\infty, a)$ and (b, ∞) are both open, as we have already seen. Moreover, it is easy to see, directly from the definition, that the union of open sets is also an open set. Therefore, $\mathbb{R} \setminus [a, b]$ is open, and so $[a, b]$ is closed.

5. For $a \in \mathbb{R}$ the set

$$[a, \infty) = \{x \in \mathbb{R} \mid a \leq x\}$$

is closed since its complement in \mathbb{R} is $(-\infty, a)$ which is open.

6. For $b \in \mathbb{R}$ the set

$$(-\infty, b] = \{x \in \mathbb{R} \mid x \leq b\}$$

is closed.

7. For $a, b \in \mathbb{R}$ with $a < b$ the set

$$(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$$

is neither open nor closed. To see that it is not open, note that $b \in (a, b]$, but that any open ball about b will contain points not in $(a, b]$. To see that $(a, b]$ is not closed, note that $a \in \mathbb{R} \setminus (a, b]$, and that any open ball about a will contain points not in $\mathbb{R} \setminus (a, b]$.

8. For $a, b \in \mathbb{R}$ with $a < b$ the set

$$[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$$

is neither open nor closed.

9. The set \mathbb{R} is both open and closed. That it is open is clear. That it is closed follows since $\mathbb{R} \setminus \mathbb{R} = \emptyset$, and \emptyset is, by convention, open. We will sometimes, although not often, write $\mathbb{R} = (-\infty, \infty)$. •

We shall frequently denote typical interval by I , and the set of intervals we denote by \mathcal{I} . If I and J are intervals with $J \subset I$, we will say that J is a **subinterval** of I . The expressions “open interval” and “closed interval” have their natural meanings as intervals that are, as subsets of \mathbb{R} , open and closed, respectively. An interval that is neither open nor closed will be called **half-open** or **half-closed**. A **left endpoint** (resp. **right endpoint**) for an interval I is a number $x \in \mathbb{R}$ such that $\inf I = x$ (resp. $\sup I = x$). An endpoint x , be it left or right, is **open** if $x \notin I$ and is **closed** if $x \in I$. If $\inf I = -\infty$ (resp. $\sup I = \infty$), then we saw that I is **unbounded on the left** (resp. **unbounded on the right**). We will also use the interval notation to denote subsets of the extended real numbers $\overline{\mathbb{R}}$. Thus, we may write

1. $(a, \infty] = (a, \infty) \cup \{\infty\}$,
2. $[a, \infty] = [a, \infty) \cup \{\infty\}$,
3. $[-\infty, b) = (-\infty, b) \cup \{-\infty\}$,
4. $[-\infty, b] = (-\infty, b] \cup \{-\infty\}$, and
5. $[-\infty, \infty] = (-\infty, \infty) \cup \{-\infty, \infty\} = \overline{\mathbb{R}}$.

The following characterisation of intervals is useful.

2.5.4 Proposition (Characterisation of intervals) *A subset $I \subset \mathbb{R}$ is an interval if and only if, for each $a, b \in I$ with $a < b$, $[a, b] \subset I$.*

Proof It is clear from the definition that, if I is an interval, then, for each $a, b \in I$ with $a < b$, $[a, b] \subset I$. So suppose that, for each $a, b \in I$ with $a < b$, $[a, b] \subset I$. Let $A = \inf I$ and let $B = \sup I$. We have the following cases to consider.

1. $A = B$: Trivially I is an interval.
2. $A, B \in \mathbb{R}$ and $A \neq B$: Choose $a_1, b_1 \in I$ such that $a_1 < b_1$. Define $a_{j+1}, b_{j+1} \in I$, $j \in \mathbb{N}$, inductively as follows. Let a_{j+1} be a point in I to the left of $\frac{1}{2}(A + a_j)$ and let b_{j+1} be a point in I to the right of $\frac{1}{2}(b_j + B)$. These constructions make sense by definition of A and B . Note that $\{a_j\}_{j \in \mathbb{N}}$ is a monotonically decreasing sequence converging to A and that $\{b_j\}_{j \in \mathbb{N}}$ is a monotonically increasing sequence converging to B . Also,

$$\bigcup_{j \in \mathbb{N}} [a_j, b_j] \subset I.$$

We also have either $\bigcup_{j \in \mathbb{N}} [a_j, b_j] = (A, B)$, $\bigcup_{j \in \mathbb{N}} [a_j, b_j] = [A, B)$, $\bigcup_{j \in \mathbb{N}} [a_j, b_j] = (A, B]$, or $\bigcup_{j \in \mathbb{N}} [a_j, b_j] = [A, B]$. Therefore we conclude that I is an interval with endpoints A and B .

3. $A = -\infty$ and $B \in \mathbb{R}$. Choose $a_1, b_1 \in I$ with $a_1 < b_1 < B$. Define $a_{j+1}, b_{j+1} \in I$, $j \in \mathbb{N}$, inductively by asking that a_{j+1} be a point in I to the left of $a_j - 1$ and that b_{j+1} be a point in I to the right of $\frac{1}{2}(b_j + B)$. These constructions make sense by definition of A and B . Thus $\{a_j\}_{j \in \mathbb{N}}$ is a monotonically decreasing sequence in I diverging to $-\infty$ and $\{b_j\}_{j \in \mathbb{N}}$ is a monotonically increasing sequence in I converging to B . Thus

$$\bigcup_{j \in \mathbb{N}} [a_j, b_j] = I.$$

Note that either $\bigcup_{j \in \mathbb{N}} [a_j, b_j] = (-\infty, B)$ or $\bigcup_{j \in \mathbb{N}} [a_j, b_j] = (-\infty, B]$. This means that either $I = (-\infty, B)$ or $I = (-\infty, B]$.

4. $A \in \mathbb{R}$ and $B = \infty$: A construction entirely like the preceding one shows that either $I = (A, \infty)$ or $I = [A, \infty)$.
5. $A = -\infty$ and $B = \infty$: Choose $a_1, b_1 \in I$ with $a_1 < b_1$. Inductively define $a_{j+1}, b_{j+1} \in I$, $j \in \mathbb{N}$, by asking that a_{j+1} be a point in I to the left of a_j and that b_{j+1} be a point in I to the right of b_j . We then conclude that

$$\bigcup_{j \in \mathbb{N}} [a_j, b_j] = \mathbb{R} = I,$$

and so $I = \mathbb{R}$.

In all cases we have concluded that I is an interval. ■

The following properties of open sets will be useful for us.

2.5.5 Proposition (Open sets in \mathbb{R} are unions of open intervals) *If $U \subset \mathbb{R}$ is a nonempty open set, then U is a countable union of disjoint open intervals.*

Proof Let $x \in U$ and let I_x be the largest open interval containing x and contained in U . This definition of I_x makes sense since the union of open intervals containing x is also an open interval containing x . Now to each interval can be associated a rational number within the interval. Therefore, the number of intervals to cover U can be associated with a subset of \mathbb{Q} , and is therefore countable or finite. This shows that U is indeed a finite or countable union of open intervals. ■

2.5.6 Proposition (Open sets in \mathbb{R} are unions of half-open intervals) *If $U \subset \mathbb{R}$ is a nonempty open set, then there exists a countable number of half-open intervals $\{I_j\}_{j \in \mathbb{N}}$ such that $U = \bigcup_{j \in \mathbb{N}} I_j$.*

Proof For $k \in \mathbb{N}_0$ define

$$\mathcal{C}_k = \{ [j2^{-k}, (j+1)2^{-k}) \mid j \in \mathbb{Z} \}.$$

Note that, for each $k \in \mathbb{N}_0$, the sets from \mathcal{C}_k form a countable partition of \mathbb{R} . Also note that for $k < \ell$, every interval in \mathcal{C}_ℓ is also an interval in \mathcal{C}_k . Now let $U \subset \mathbb{R}$ be open. Let $\mathcal{D}_0 = \emptyset$. Let

$$\begin{aligned} \mathcal{D}_1 &= \{ I \in \mathcal{C}_1 \mid I \subset U \} \\ \mathcal{D}_2 &= \{ I \in \mathcal{C}_2 \mid I \subset U, I \notin \mathcal{D}_1 \} \\ &\vdots \\ \mathcal{D}_k &= \{ I \in \mathcal{C}_k \mid I \subset U, I \notin \mathcal{D}_1 \cap \cdots \cap \mathcal{D}_{k-1} \} \\ &\vdots \end{aligned}$$

The result will follow if we can show that each point $x \in U$ is contained in some \mathcal{D}_k , $k \in \mathbb{N}$. However, this follows since U is open, and so, for each $x \in U$, one can find a smallest $k \in \mathbb{N}_0$ with the property that there exists $I \in \mathcal{C}_k$ with $x \in I$ and $I \subset U$. ■

2.5.2 Partitions of intervals

In this section we consider the idea of partitioning an interval of the form $[a, b]$. This is a construction that will be useful in a variety of places, but since we dealt with intervals in the previous section, this is an appropriate time to make the definition and the associated constructions.

2.5.7 Definition (Partition of an interval) A *partition* of an interval $[a, b]$ is a collection $\{I_1, \dots, I_k\}$ of intervals such that

- (i) $[a, b] = \cup_{j=1}^k I_j$ and
- (ii) $I_j \cap I_l = \emptyset$ for $j \neq l$.

We denote by $\text{Part}([a, b])$ the set of partitions of $[a, b]$. •

We shall always suppose that a partition $\{I_1, \dots, I_k\}$ is totally ordered so that the left endpoint of I_{j+1} agrees with the right endpoint of I_j for each $j \in \{1, \dots, k-1\}$. That is to say, when we write a partition, we shall list the elements of the set according to this total order. Note that associated to a partition $\{I_1, \dots, I_k\}$ are the endpoints of the intervals. Thus there exists a subset $\{x_0, x_1, \dots, x_k\}$ of $[a, b]$, ordered with respect to the natural total order on \mathbb{R} , such that, for each $j \in \{1, \dots, k\}$, t_{j-1} is the left endpoint of I_j and t_j is the right endpoint of I_j . Note that necessarily we have $t_0 = a$ and $t_k = b$. The set of endpoints of the intervals in a partition $P = \{I_1, \dots, I_k\}$ we denote by $\text{EP}(P)$. In Figure 2.4 we show

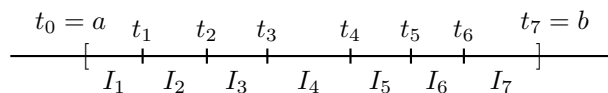


Figure 2.4 A partition

a partition with all ingredients labelled. For a partition P with $\text{EP}(P) = \{x_0, x_1, \dots, x_k\}$, denote

$$|P| = \max \{ |x_j - x_l| \mid j, l \in \{1, \dots, k\} \},$$

which is the *mesh* of P . Thus $|P|$ is the length of the largest interval of the partition.

It is often useful to be able to say one partition is finer than another, and the following definition makes this precise.

2.5.8 Definition (Refinement of a partition) If P_1 and P_2 are partitions of an interval $[a, b]$, then P_2 is a *refinement* of P_1 if $\text{EP}(P_1) \subset \text{EP}(P_2)$. •

Next we turn to a sometimes useful construction involving the addition of certain structure onto a partition. This construction is rarely used in the text, so may be skipped until it is encountered.

2.5.9 Definition (Tagged partition, δ -fine tagged partition) Let $[a, b]$ be an interval and let $\delta: [a, b] \rightarrow \mathbb{R}_{>0}$.

- (i) A *tagged partition* of $[a, b]$ is a finite collection of pairs $\{(c_1, I_1), \dots, (c_k, I_k)\}$ where $\{I_1, \dots, I_k\}$ is a partition and where c_j is contained in the union of I_j with its endpoints.
- (ii) A tagged partition $\{(c_1, I_1), \dots, (c_k, I_k)\}$ is *δ -fine* if the interval I_j , along with its endpoints, is a subset of $\mathbf{B}(\delta(c_j), c_j)$. •

The following result asserts that δ -fine tagged partitions always exist.

2.5.10 Proposition (δ -fine tagged partitions exist) For any positive function $\delta: [a, b] \rightarrow \mathbb{R}_{>0}$, there exists a δ -fine tagged partition.

Proof Let Δ be the set of all points $x \in (a, b]$ such that there exists a δ -fine tagged partition of $[a, x]$. Note that $(a, a + \delta(a)) \subset \Delta$ since, for each $x \in (a, a + \delta(a))$, $\{(a, [a, x])\}$ is a δ -fine tagged partition of $[a, x]$. Let $b' = \sup \Delta$. We will show that $b' = b$ and that $b' \in \Delta$.

Since $b' = \sup \Delta$ there exists $b'' \in \Delta$ such that $b' - \delta(b') < b'' < b'$. Then there exists a δ -fine partition P' of $[a, b']$. Now $P' \cup \{(b', (b'', b'))\}$ is δ -fine tagged partition of $[a, b']$. Thus $b' \in \Delta$.

Now suppose that $b' < b$ and choose $b'' < b$ such that $b' < b'' < b' + \delta(b')$. If P is a tagged partition of $[a, b']$ (this exists since $b' \in \Delta$), then $P \cup \{(b', (b', b''))\}$ is a δ -fine tagged partition of $[a, b'']$. This contradicts the fact that $b' = \sup \Delta$. Thus we conclude that $b' = b$. ■

Some uses of δ -fine tagged partitions in real analysis can be found in the paper of Gordon [1998].

2.5.3 Interior, closure, boundary, and related notions

Associated with the concepts of open and closed are a collection of useful concepts.

2.5.11 Definition (Accumulation point, cluster point, limit point in \mathbb{R}) Let $A \subset \mathbb{R}$. A point $x \in A$ is:

- (i) an **accumulation point** for A if, for every neighbourhood U of x , the set $A \cap (U \setminus \{x\})$ is nonempty;
- (ii) a **cluster point** for A if, for every neighbourhood U of x , the set $A \cap U$ is infinite;
- (iii) a **limit point** of A if there exists a sequence $\{x_j\}_{j \in \mathbb{N}}$ in A converging to x .

The set of accumulation points of A is called the **derived set** of A , and is denoted by $\text{der}(A)$. •

2.5.12 Remark (Conventions concerning “accumulation point,” “cluster point,” and “limit point”) There seems to be no agreed upon convention about what is meant by the three concepts of accumulation point, cluster point, and limit point. Some authors make no distinction between the three concepts at all. Some authors lump two together, but give the third a different meaning. As we shall see in Proposition 2.5.13 below, sometimes there is no need to distinguish between two of the concepts. However, in order to keep as clear as possible the transition to the more abstract presentation of Chapter II-2, we have gone with the most pedantic interpretation possible for the concepts of “accumulation point,” “cluster point,” and “limit point.” •

The three concepts of accumulation point, cluster point, and limit point are actually excessive for \mathbb{R} since, as the next result shall indicate, two of them are exactly the same. However, in the more general setup of Chapter II-2, the concepts are no longer equivalent.

2.5.13 Proposition (“Accumulation point” equals “cluster point” in \mathbb{R}) For a set $A \subset \mathbb{R}$, $x \in \mathbb{R}$ is an accumulation point for A if and only if it is a cluster point for A .

Proof It is clear that a cluster point for A is an accumulation point for A . Suppose that x is not a cluster point. Then there exists a neighbourhood U of x for which the set $A \cap U$ is finite. If $A \cap U = \{x\}$, then clearly x is not an accumulation point. If $A \cap U \neq \{x\}$, then $A \cap (U \setminus \{x\}) \supset \{x_1, \dots, x_k\}$ where the points x_1, \dots, x_k are distinct from x . Now let

$$\epsilon = \frac{1}{2} \min\{|x_1 - x|, \dots, |x_k - x|\}.$$

Clearly $A \cap (B(\epsilon, x) \setminus \{x\})$ is then empty, and so x is not an accumulation point for A . ■

Now let us give some examples that illustrate the differences between accumulation points (or equivalently cluster points) and limit points.

2.5.14 Examples (Accumulation points and limit points)

1. For any subset $A \subset \mathbb{R}$ and for every $x \in A$, x is a limit point for A . Indeed, the constant sequence $\{x_j = x\}_{j \in \mathbb{N}}$ is a sequence in A converging to x . However, as we shall see in the examples to follow, it is not the case that all points in A are accumulation points.
2. Let $A = (0, 1)$. The set of accumulation points of A is then easily seen to be $[0, 1]$. The set of limit points is also $[0, 1]$.
3. Let $A = [0, 1)$. Then, as in the preceding example, both the set of accumulation points and the set of limit points are the set $[0, 1]$.
4. Let $A = [0, 1] \cup \{2\}$. Then the set of accumulation points is $[0, 1]$ whereas the set of limit points is A .
5. Let $A = \mathbb{Q}$. One can readily check that the set of accumulation points of A is \mathbb{R} and the set of limit points of A is also \mathbb{R} . •

The following result gives some properties of the derived set.

2.5.15 Proposition (Properties of the derived set in \mathbb{R}) For $A, B \subset \mathbb{R}$ and for a family of subsets $\{A_i\}_{i \in I}$ of \mathbb{R} , the following statements hold:

- (i) $\text{der}(\emptyset) = \emptyset$;
- (ii) $\text{der}(\mathbb{R}) = \mathbb{R}$;
- (iii) $\text{der}(\text{der}(A)) = \text{der}(A)$;
- (iv) if $A \subset B$ then $\text{der}(A) \subset \text{der}(B)$;
- (v) $\text{der}(A \cup B) = \text{der}(A) \cup \text{der}(B)$;
- (vi) $\text{der}(A \cap B) \subset \text{der}(A) \cap \text{der}(B)$.

Proof Parts (i) and (ii) follow directly from the definition of the derived set.

(iii)

(iv) Let $x \in \text{der}(A)$ and let U be a neighbourhood of x . Then the set $A \cap (U \setminus \{x\})$ is nonempty, implying that the set $B \cap (U \setminus \{x\})$ is also nonempty. Thus $x \in \text{der}(B)$.

(v) Let $x \in \text{der}(A \cup B)$ and let U be a neighbourhood of x . Then the set $U \cap ((A \cup B) \setminus \{x\})$ is nonempty. But

$$\begin{aligned} U \cap ((A \cup B) \setminus \{x\}) &= U \cap ((A \setminus \{x\}) \cup (B \setminus \{x\})) \\ &= (U \cap (A \setminus \{x\})) \cup (U \cap (B \setminus \{x\})). \end{aligned} \quad (2.7)$$

Thus it cannot be that both $U \cap (A \setminus \{x\})$ and $U \cap (B \setminus \{x\})$ are empty. Thus x is an element of either $\text{der}(A)$ or $\text{der}(B)$.

Now let $x \in \text{der}(A) \cup \text{der}(A)$. Then, using (2.7), $U \cap ((A \cup B) \setminus \{x\})$ is nonempty, and so $x \in \text{der}(A \cup B)$.

(vi) Let $x \in \text{der}(A \cap B)$ and let U be a neighbourhood of x . Then $U \cap ((A \cap B) \setminus \{x\}) \neq \emptyset$. We have

$$U \cap ((A \cap B) \setminus \{x\}) = U \cap ((A \setminus \{x\}) \cap (B \setminus \{x\}))$$

Thus the sets $U \cap (A \setminus \{x\})$ and $U \cap (B \setminus \{x\})$ are both nonempty, showing that $x \in \text{der}(A) \cap \text{der}(B)$. ■

Next we turn to characterising distinguished subsets of subsets of \mathbb{R} .

2.5.16 Definition (Interior, closure, and boundary in \mathbb{R}) Let $A \subset \mathbb{R}$.

- (i) The *interior* of A is the set

$$\text{int}(A) = \cup \{U \mid U \subset A, U \text{ open}\}.$$

(ii) The **closure** of A is the set

$$\text{cl}(A) = \bigcap \{C \mid A \subset C, C \text{ closed}\}.$$

(iii) The **boundary** of A is the set $\text{bd}(A) = \text{cl}(A) \cap \text{cl}(\mathbb{R} \setminus A)$. •

In other words, the interior of A is the largest open set contained in A . Note that this definition makes sense since a union of open sets is open (Exercise 2.5.1). In like manner, the closure of A is the smallest closed set containing A , and this definition makes sense since an intersection of closed sets is closed (Exercise 2.5.1 again). Note that $\text{int}(A)$ is open and $\text{cl}(A)$ is closed. Moreover, since $\text{bd}(A)$ is the intersection of two closed sets, it too is closed (Exercise 2.5.1 yet again).

Let us give some examples of interiors, closures, and boundaries.

2.5.17 Examples (Interior, closure, and boundary)

1. Let $A = \text{int}(0, 1)$. Then $\text{int}(A) = (0, 1)$ since A is open. We claim that $\text{cl}(A) = [0, 1]$. Clearly $[0, 1] \subset \text{cl}(A)$ since $[0, 1]$ is closed and contains A . Moreover, the only smaller subsets contained in $[0, 1]$ and containing A are $[0, 1)$, $(0, 1]$, and $(0, 1)$, none of which are closed. We may then conclude that $\text{cl}(A) = [0, 1]$. Finally we claim that $\text{bd}(A) = \{0, 1\}$. To see this, note that we have $\text{cl}(A) = [0, 1]$ and $\text{cl}(\mathbb{R} \setminus A) = (-\infty, 0] \cup [1, \infty)$ (by an argument like that used to show that $\text{cl}(A) = [0, 1]$). Therefore, $\text{bd}(A) = \text{cl}(A) \cap \text{cl}(\mathbb{R} \setminus A) = \{0, 1\}$, as desired.
2. Let $A = [0, 1]$. Then $\text{int}(A) = (0, 1)$. To see this, we note that $(0, 1) \subset \text{int}(A)$ since $(0, 1)$ is open and contained in A . Moreover, the only larger sets contained in A are $[0, 1)$, $(0, 1]$, and $[0, 1]$, none of which are open. Thus $\text{int}(A) = (0, 1)$, just as claimed. Since A is closed, $\text{cl}(A) = A$. Finally we claim that $\text{bd}(A) = \{0, 1\}$. Indeed, $\text{cl}(A) = [0, 1]$ and $\text{cl}(\mathbb{R} \setminus A) = (-\infty, 0] \cup [1, \infty)$. Therefore, $\text{bd}(A) = \text{cl}(A) \cap \text{cl}(\mathbb{R} \setminus A) = \{0, 1\}$, as claimed.
3. Let $A = (0, 1) \cup \{2\}$. We have $\text{int}(A) = (0, 1)$, $\text{cl}(A) = [0, 1] \cup \{2\}$, and $\text{bd}(A) = \{0, 1, 2\}$. We leave the simple details of these assertions to the reader.
4. Let $A = \mathbb{Q}$. One readily ascertains that $\text{int}(A) = \emptyset$, $\text{cl}(A) = \mathbb{R}$, and $\text{bd}(A) = \mathbb{R}$. •

Now let us give a characterisation of interior, closure, and boundary that are often useful in practice. Indeed, we shall often use these characterisations without explicitly mentioning that we are doing so.

2.5.18 Proposition (Characterisation of interior, closure, and boundary in \mathbb{R}) For $A \subset \mathbb{R}$, the following statements hold:

- (i) $x \in \text{int}(A)$ if and only if there exists a neighbourhood U of x such that $U \subset A$;
- (ii) $x \in \text{cl}(A)$ if and only if, for each neighbourhood U of x , the set $U \cap A$ is nonempty;
- (iii) $x \in \text{bd}(A)$ if and only if, for each neighbourhood U of x , the sets $U \cap A$ and $U \cap (\mathbb{R} \setminus A)$ are nonempty.

Proof (i) Suppose that $x \in \text{int}(A)$. Since $\text{int}(A)$ is open, there exists a neighbourhood U of x contained in $\text{int}(A)$. Since $\text{int}(A) \subset A$, $U \subset A$.

Next suppose that $x \notin \text{int}(A)$. Then, by definition of interior, for any open set U for which $U \subset A$, $x \notin U$.

(ii) Suppose that there exists a neighbourhood U of x such that $U \cap A = \emptyset$. Then $\mathbb{R} \setminus U$ is a closed set containing A . Thus $\text{cl}(A) \subset \mathbb{R} \setminus U$. Since $x \notin \mathbb{R} \setminus U$, it follows that $x \notin \text{cl}(A)$.

Suppose that $x \notin \text{cl}(A)$. Then x is an element of the open set $\mathbb{R} \setminus \text{cl}(A)$. Thus there exists a neighbourhood U of x such that $U \subset \mathbb{R} \setminus \text{cl}(A)$. In particular, $U \cap A = \emptyset$.

(iii) This follows directly from part (ii) and the definition of boundary. ■

Now let us state some useful properties of the interior of a set.

2.5.19 Proposition (Properties of interior in \mathbb{R}) For $A, B \subset \mathbb{R}$ and for a family of subsets $\{A_i\}_{i \in I}$ of \mathbb{R} , the following statements hold:

- (i) $\text{int}(\emptyset) = \emptyset$;
- (ii) $\text{int}(\mathbb{R}) = \mathbb{R}$;
- (iii) $\text{int}(\text{int}(A)) = \text{int}(A)$;
- (iv) if $A \subset B$ then $\text{int}(A) \subset \text{int}(B)$;
- (v) $\text{int}(A \cup B) \supset \text{int}(A) \cup \text{int}(B)$;
- (vi) $\text{int}(A \cap B) = \text{int}(A) \cap \text{int}(B)$;
- (vii) $\text{int}(\cup_{i \in I} A_i) \supset \cup_{i \in I} \text{int}(A_i)$;
- (viii) $\text{int}(\cap_{i \in I} A_i) \subset \cap_{i \in I} \text{int}(A_i)$.

Moreover, a set $A \subset \mathbb{R}$ is open if and only if $\text{int}(A) = A$.

Proof Parts (i) and (ii) are clear by definition of interior. Part (v) follows from part (vii), so we will only prove the latter.

(iii) This follows since the interior of an open set is the set itself.

(iv) Let $x \in \text{int}(A)$. Then there exists a neighbourhood U of x such that $U \subset A$. Thus $U \subset B$, and the result follows from Proposition 2.5.18.

(vi) Let $x \in \text{int}(A) \cap \text{int}(B)$. Since $\text{int}(A) \cap \text{int}(B)$ is open by Exercise 2.5.1, there exists a neighbourhood U of x such that $U \subset \text{int}(A) \cap \text{int}(B)$. Thus $U \subset A \cap B$. This shows that $x \in \text{int}(A \cap B)$. This part of the result follows from part (viii).

(vii) Let $x \in \cup_{i \in I} \text{int}(A_i)$. By Exercise 2.5.1 the set $\cup_{i \in I} \text{int}(A_i)$ is open. Thus there exists a neighbourhood U of x such that $U \subset \cup_{i \in I} \text{int}(A_i)$. Thus $U \subset \cup_{i \in I} A_i$, from which we conclude that $x \in \text{int}(\cup_{i \in I} A_i)$.

(viii) Let $x \in \text{int}(\cap_{i \in I} A_i)$. Then there exists a neighbourhood U of x such that $U \subset \cap_{i \in I} A_i$. It therefore follows that $U \subset A_i$ for each $i \in I$, and so that $x \in \text{int}(A_i)$ for each $i \in I$.

The final assertion follows directly from Proposition 2.5.18. ■

Next we give analogous results for the closure of a set.

2.5.20 Proposition (Properties of closure in \mathbb{R}) For $A, B \subset \mathbb{R}$ and for a family of subsets $\{A_i\}_{i \in I}$ of \mathbb{R} , the following statements hold:

- (i) $\text{cl}(\emptyset) = \emptyset$;
- (ii) $\text{cl}(\mathbb{R}) = \mathbb{R}$;
- (iii) $\text{cl}(\text{cl}(A)) = \text{cl}(A)$;
- (iv) if $A \subset B$ then $\text{cl}(A) \subset \text{cl}(B)$;
- (v) $\text{cl}(A \cup B) = \text{cl}(A) \cup \text{cl}(B)$;
- (vi) $\text{cl}(A \cap B) \subset \text{cl}(A) \cap \text{cl}(B)$;
- (vii) $\text{cl}(\cup_{i \in I} A_i) \supset \cup_{i \in I} \text{cl}(A_i)$;
- (viii) $\text{cl}(\cap_{i \in I} A_i) \subset \cap_{i \in I} \text{cl}(A_i)$.

Moreover, a set $A \subset \mathbb{R}$ is closed if and only if $\text{cl}(A) = A$.

Proof Parts (i) and (ii) follow immediately from the definition of closure. Part (vi) follows from part (viii), so we will only prove the latter.

(iii) This follows since the closure of a closed set is the set itself.

(iv) Suppose that $x \in \text{cl}(A)$. Then, for any neighbourhood U of x , the set $U \cap A$ is nonempty, by Proposition 2.5.18. Since $A \subset B$, it follows that $U \cap B$ is also nonempty, and so $x \in \text{cl}(B)$.

(v) Let $x \in \text{cl}(A \cup B)$. Then, for any neighbourhood U of x , the set $U \cap (A \cup B)$ is nonempty by Proposition 2.5.18. By Proposition 1.1.3, $U \cap (A \cup B) = (U \cap A) \cup (U \cap B)$. Thus the sets $U \cap A$ and $U \cap B$ are not both nonempty, and so $x \in \text{cl}(A) \cup \text{cl}(B)$. That $\text{cl}(A) \cup \text{cl}(B) \subset \text{cl}(A \cup B)$ follows from part (vii).

(vi) Let $x \in \text{cl}(A \cap B)$. Then, for any neighbourhood U of x , the set $U \cap (A \cap B)$ is nonempty. Thus the sets $U \cap A$ and $U \cap B$ are nonempty, and so $x \in \text{cl}(A) \cap \text{cl}(B)$.

(vii) Let $x \in \cup_{i \in I} \text{cl}(A_i)$ and let U be a neighbourhood of x . Then, for each $i \in I$, $U \cap A_i \neq \emptyset$. Therefore, $\cup_{i \in I} (U \cap A_i) \neq \emptyset$. By Proposition 1.1.6, $\cup_{i \in I} (U \cap A_i) = U \cap (\cup_{i \in I} A_i)$, showing that $U \cap (\cup_{i \in I} A_i) \neq \emptyset$. Thus $x \in \text{cl}(\cup_{i \in I} A_i)$.

(viii) Let $x \in \text{cl}(\cap_{i \in I} A_i)$ and let U be a neighbourhood of x . Then the set $U \cap (\cap_{i \in I} A_i)$ is nonempty. This means that, for each $i \in I$, the set $U \cap A_i$ is nonempty. Thus $x \in \text{cl}(A_i)$ for each $i \in I$, giving the result. ■

Note that there is a sort of “duality” between int and cl as concerns their interactions with union and intersection. This is reflective of the fact that open and closed sets themselves have such a “duality,” as can be seen from Exercise 2.5.1. We refer the reader to Exercise 2.5.4 to construct counterexamples to any missing opposite inclusions in Propositions 2.5.19 and 2.5.20.

Let us state some relationships between certain of the concepts we have thus far introduced.

2.5.21 Proposition (Joint properties of interior, closure, boundary, and derived set in \mathbb{R}) For $A \subset \mathbb{R}$, the following statements hold:

- (i) $\mathbb{R} \setminus \text{int}(A) = \text{cl}(\mathbb{R} \setminus A)$;
- (ii) $\mathbb{R} \setminus \text{cl}(A) = \text{int}(\mathbb{R} \setminus A)$.
- (iii) $\text{cl}(A) = A \cup \text{bd}(A)$;
- (iv) $\text{int}(A) = A - \text{bd}(A)$;
- (v) $\text{cl}(A) = \text{int}(A) \cup \text{bd}(A)$;
- (vi) $\text{cl}(A) = A \cup \text{der}(A)$;
- (vii) $\mathbb{R} = \text{int}(A) \cup \text{bd}(A) \cup \text{int}(\mathbb{R} \setminus A)$.

Proof (i) Let $x \in \mathbb{R} \setminus \text{int}(A)$. Since $x \notin \text{int}(A)$, for every neighbourhood U of x it holds that $U \not\subset A$. Thus, for any neighbourhood U of x , we have $U \cap (\mathbb{R} \setminus A) \neq \emptyset$, showing that $x \in \text{cl}(\mathbb{R} \setminus A)$.

Now let $x \in \text{cl}(\mathbb{R} \setminus A)$. Then for any neighbourhood U of x we have $U \cap (\mathbb{R} \setminus A) \neq \emptyset$. Thus $x \notin \text{int}(A)$, so $x \in \mathbb{R} \setminus A$.

(ii) The proof here strongly resembles that for part (i), and we encourage the reader to provide the explicit arguments.

(iii) This follows from part (v).

(iv) Clearly $\text{int}(A) \subset A$. Suppose that $x \in A \cap \text{bd}(A)$. Then, for any neighbourhood U of x , the set $U \cap (\mathbb{R} \setminus A)$ is nonempty. Therefore, no neighbourhood of x is a subset of A , and so $x \notin \text{int}(A)$. Conversely, if $x \in \text{int}(A)$ then there is a neighbourhood U of x such that $U \subset A$. This precludes the set $U \cap (\mathbb{R} \setminus A)$ from being nonempty, and so we must have $x \notin \text{bd}(A)$.

(v) Let $x \in \text{cl}(A)$. For a neighbourhood U of x it then holds that $U \cap A \neq \emptyset$. If there exists a neighbourhood V of x such that $V \subset A$, then $x \in \text{int}(A)$. If there exists *no* neighbourhood V of x such that $V \subset A$, then for every neighbourhood V of x we have $V \cap (\mathbb{R} \setminus A) \neq \emptyset$, and so $x \in \text{bd}(A)$.

Now let $x \in \text{int}(A) \cup \text{bd}(A)$. If $x \in \text{int}(A)$ then $x \in A$ and so $x \in \text{cl}(A)$. If $x \in \text{bd}(A)$ then it follows immediately from Proposition 2.5.18 that $x \in \text{cl}(A)$.

(vi) Let $x \in \text{cl}(A)$. If $x \notin A$ then, for every neighbourhood U of x , $U \cap A = U \cap (A \setminus \{x\}) \neq \emptyset$, and so $x \in \text{der}(A)$.

If $x \in A \cup \text{der}(A)$ then either $x \in A \subset \text{cl}(A)$, or $x \notin A$. In this latter case, $x \in \text{der}(A)$ and so the set $U \cap (A \setminus \{x\})$ is nonempty for each neighbourhood U of x , and we again conclude that $x \in \text{cl}(A)$.

(vii) Clearly $\text{int}(A) \cap \text{int}(\mathbb{R} \setminus A) = \emptyset$ since $A \cap (\mathbb{R} \setminus A) = \emptyset$. Now let $x \in \mathbb{R} \setminus (\text{int}(A) \cup \text{int}(\mathbb{R} \setminus A))$. Then, for any neighbourhood U of x , we have $U \not\subset A$ and $U \not\subset (\mathbb{R} \setminus A)$. Thus the sets $U \cap (\mathbb{R} \setminus A)$ and $U \cap A$ must both be nonempty, from which we conclude that $x \in \text{bd}(A)$. ■

An interesting class of subset of \mathbb{R} is the following.

2.5.22 Definition (Discrete subset of \mathbb{R}) A subset $A \subset \mathbb{R}$ is *discrete* if there exists $\epsilon > 0$ such that, for each $x, y \in A$, $|x - y| \geq \epsilon$. •

Let us give a characterisation of discrete sets.

2.5.23 Proposition (Characterisation of discrete sets in \mathbb{R}) A discrete subset $A \subset \mathbb{R}$ is countable and has no accumulation points.

Proof It is easy to show (Exercise 2.5.6) that if A is discrete and if $N \in \mathbb{N}$, then the set $A \cap [-N, N]$ is finite. Therefore

$$A = \bigcup_{N \in \mathbb{N}} A \cap [-N, N],$$

which gives A as a countable union of finite sets, implying that A is countable by Proposition 1.7.16. Now let $\epsilon > 0$ satisfy $|x - y| \geq \epsilon$ for $x, y \in A$. Then, if $x \in A$ then the set $A \cap \mathcal{B}(\frac{\epsilon}{2}, x)$ is empty, implying that x is not an accumulation point. If $x \notin A$ then $\mathcal{B}(\frac{\epsilon}{2}, x)$ can contain at most one point from A , which again prohibits x from being an accumulation point. ■

The notion of a discrete set is actually a more general one having to do with what is known as the discrete topology (cf. Example II-2.2.2-3). The reader can explore some facts about discrete subsets of \mathbb{R} in Exercise 2.5.6.

2.5.4 Compactness

The idea of compactness is absolutely fundamental in much of mathematics. The reasons for this are not at all clear to a newcomer to analysis. Indeed, the definition we give for compactness comes across as extremely unmotivated. This might be particularly since for \mathbb{R} (or more generally, in \mathbb{R}^n) compact sets have a fairly banal characterisation as sets that are closed and bounded (Theorem 2.5.27). However, the original definition we give for a compact set is the most useful one. The main reason it is useful is that it allows for certain pointwise properties to be automatically extended to the entire set. A good example of this is Theorem 3.1.25, where continuity of a function on a compact set is extended to uniform continuity on the set. This idea of uniformity is an important one, and accounts for much of the value of the notion of compactness. But we are getting ahead of ourselves.

As indicated in the above paragraph, we shall give a rather strange seeming definition of compactness. Readers looking for a quick and dirty definition of compactness, valid for subsets of \mathbb{R} , can refer ahead to Theorem 2.5.27. Our construction relies on the following idea.

2.5.24 Definition (Open cover of a subset of \mathbb{R}) Let $A \subset \mathbb{R}$.

- (i) An *open cover* for A is a family $\{U_i\}_{i \in I}$ of open subsets of \mathbb{R} having the property that $A \subset \bigcup_{i \in I} U_i$.

- (ii) A **subcover** of an open cover $\{U_i\}_{i \in I}$ of A is an open cover $\{V_j\}_{j \in J}$ of A having the property that $\{V_j\}_{j \in J} \subset \{U_i\}_{i \in I}$. •

The following property of open covers of subsets of \mathbb{R} is useful.

2.5.25 Lemma (Lindelöf⁷ Lemma for \mathbb{R}) *If $\{U_i\}_{i \in I}$ is an open cover of $A \subset \mathbb{R}$, then there exists a countable subcover of A .*

Proof Let $\mathcal{B} = \{B(r, x) \mid x, r \in \mathbb{Q}\}$. Note that \mathcal{B} is a countable union of countable sets, and so is countable by Proposition 1.7.16. Therefore, we can write $\mathcal{B} = \{B(r_j, x_j)\}_{j \in \mathbb{N}}$. Now define

$$\mathcal{B}' = \{B(r_j, x_j) \mid B(r_j, x_j) \subset U_i \text{ for some } i \in I\}.$$

Let us write $\mathcal{B}' = \{B(r_{j_k}, x_{j_k})\}_{k \in \mathbb{N}}$. We claim that \mathcal{B}' covers A . Indeed, if $x \in A$ then there exists $k \in \mathbb{N}$ and $i \in I$ such that $x \in B(r_{j_k}, x_{j_k}) \subset U_i$. Now, for each $k \in \mathbb{N}$, let $i_k \in I$ satisfy $B(r_{j_k}, x_{j_k}) \subset U_{i_k}$. Then the countable collection of open sets $\{U_{i_k}\}_{k \in \mathbb{N}}$ clearly covers A since \mathcal{B}' covers A . ■

2.5.26 Definition (Bounded, compact, and totally bounded in \mathbb{R}) A subset $A \subset \mathbb{R}$ is:

- (i) **bounded** if there exists $M > 0$ such that $A \subset \overline{B}(M, 0)$;
- (ii) **compact** if every open cover $\{U_i\}_{i \in I}$ of A possesses a finite subcover;
- (iii) **relatively compact** if $\text{cl}(A)$ is compact;
- (iv) **totally bounded** if, for every $\epsilon > 0$ there exists $x_1, \dots, x_k \in \mathbb{R}$ such that $A \subset \bigcup_{j=1}^k B(\epsilon, x_j)$. •

The simplest characterisation of compact subsets of \mathbb{R} is the following. We shall freely interchange our use of the word compact between the definition given in Definition 2.5.26 and the conclusions of the following theorem.

2.5.27 Theorem (Heine–Borel⁸ Theorem in \mathbb{R}) *A subset $K \subset \mathbb{R}$ is compact if and only if K is closed and bounded.*

Proof Suppose that K is closed and bounded. We first consider the case when $K = [a, b]$. Let $\mathcal{O} = \{U_i\}_{i \in I}$ be an open cover for $[a, b]$ and let

$$S_{[a,b]} = \{x \in \mathbb{R} \mid x \leq b \text{ and } [a, x] \text{ has a finite subcover in } \mathcal{O}\}.$$

Note that $S_{[a,b]} \neq \emptyset$ since $a \in S_{[a,b]}$. Let $c = \sup S_{[a,b]}$. We claim that $c = b$. Suppose that $c < b$. Since $c \in [a, b]$ there is some $\bar{i} \in I$ such that $c \in U_{\bar{i}}$. As $U_{\bar{i}}$ is open, there is some $\epsilon > 0$ sufficiently small that $B(\epsilon, c) \subset U_{\bar{i}}$. By definition of c , there exists some $x \in (c - \epsilon, c)$ for which $x \in S_{[a,b]}$. By definition of $S_{[a,b]}$ there is a finite collection of open sets U_{i_1}, \dots, U_{i_m} from \mathcal{O} which cover $[a, x]$. Therefore, the finite collection $U_{i_1}, \dots, U_{i_m}, U_{\bar{i}}$ of open sets covers $[a, c + \epsilon]$. This then contradicts the fact that $c = \sup S_{[a,b]}$, so showing that $b = \sup S_{[a,b]}$. The result follows by definition of $S_{[a,b]}$.

Now suppose that K is a general closed and bounded set. Then $K \subset [a, b]$ for some suitable $a, b \in \mathbb{R}$. Suppose that $\mathcal{O} = \{U_i\}_{i \in I}$ is an open cover of K , and define a new open cover $\tilde{\mathcal{O}} = \mathcal{O} \cup (\mathbb{R} \setminus K)$. Note that $\bigcup_{i \in I} U_i \cup (\mathbb{R} \setminus K) = \mathbb{R}$ showing that $\tilde{\mathcal{O}}$ is an open cover for \mathbb{R} , and therefore also is an open cover for $[a, b]$. By the first part of the proof, there exists a finite subset of $\tilde{\mathcal{O}}$ which covers $[a, b]$, and therefore also covers K . We must show that this finite cover

⁷Ernst Leonard Lindelöf (1870–1946) was a Finnish mathematician who worked in the areas of differential equations and complex analysis.

⁸Heinrich Eduard Heine (1821–1881) was a German mathematician who worked mainly with special functions. Félix Edouard Justin Emile Borel (1871–1956) was a French mathematician, and he worked mainly in the area of analysis.

can be chosen so as not to include the set $\mathbb{R} \setminus K$ as this set is not necessarily in \mathcal{O} . However, if $[a, b]$ is covered by $U_{i_1}, \dots, U_{i_k}, \mathbb{R} \setminus K$, then one sees that K is covered by U_{i_1}, \dots, U_{i_k} , since $K \cap (\mathbb{R} \setminus K) = \emptyset$. Thus we have arrived at a finite subset of \mathcal{O} covering K , as desired.

Now suppose that K is compact. Consider the following collection of open subsets: $\mathcal{O}_K = \{\mathbf{B}(\epsilon, x)\}_{x \in K}$. Clearly this is an open cover of K . Thus there exists a finite collection of point $x_1, \dots, x_k \in K$ such that $\{\mathbf{B}(\epsilon, x_j)\}_{j \in \{1, \dots, k\}}$ covers K . If we take

$$M = \max\{|x_1|, \dots, |x_k|\} + 2$$

then we easily see that $K \subset \overline{\mathbf{B}}(M, 0)$, so that K is bounded. Now suppose that K is not closed. Then $K \subsetneq \text{cl}(K)$. By part (vi) of Proposition 2.5.21 there exists an accumulation point x_0 of K that is not in K . Then, for any $j \in \mathbb{N}$ there exists a point $x_j \in K$ such that $|x_0 - x_j| < \frac{1}{j}$. Define

$$U_j = (-\infty, x_0 - \frac{1}{j}) \cup (x_0 + \frac{1}{j}, \infty),$$

noting that U_j is open, since it is the union of open sets (see Exercise 2.5.1). We claim that $\{U_j\}_{j \in \mathbb{N}}$ is an open cover of K . Indeed, we will show that $\cup_{j \in \mathbb{N}} U_j = \mathbb{R} \setminus \{x_0\}$. To see this, let $x \in \mathbb{R} \setminus \{x_0\}$ and choose $k \in \mathbb{N}$ such that $\frac{1}{k} < |x - x_0|$. Then it follows by definition of U_k that $x \in U_k$. Since $x_0 \notin K$, we then have $K \subset \cup_{j \in \mathbb{N}} U_j$. Next we show that there is no finite subset of $\{U_j\}_{j \in \mathbb{N}}$ that covers K . Indeed, consider a finite set $j_1, \dots, j_k \in \mathbb{N}$, and suppose without loss of generality that $j_1 < \dots < j_k$. Then the point $x_{j_{k+1}}$ satisfies $|x_0 - x_{j_{k+1}}| < \frac{1}{j_{k+1}} < \frac{1}{j_k}$, implying that $x_{j_{k+1}} \notin U_{j_k} \supset \dots \supset U_{j_1}$. Thus, if K is not closed, we have constructed an open cover of K having no finite subcover. From this we conclude that if K is compact, then it is closed. ■

The Heine–Borel Theorem has the following useful corollary.

2.5.28 Corollary (Closed subsets of compact sets in \mathbb{R} are compact) *If $A \subset \mathbb{R}$ is compact and if $B \subset A$ is closed, then B is compact.*

Proof Since A is bounded by the Heine–Borel Theorem, B is also bounded. Thus B is also compact, again by the Heine–Borel Theorem. ■

In Chapter II-2 we shall encounter many of the ideas in this section in the more general setting of topological spaces. Many of the ideas for \mathbb{R} transfer directly to this more general setting. However, with compactness, some care must be exercised. In particular, it is *not* true that, in a general topological space, a subset is compact if and only if it is closed and bounded. Indeed, in a general topological space, the notion of bounded is not defined. It is not an uncommon error for newcomers to confuse “compact” with “closed and bounded” in situations where this is not the case.

The following result is another equivalent characterisation of compact subsets of \mathbb{R} , and is often useful.

2.5.29 Theorem (Bolzano–Weierstrass⁹ Theorem in \mathbb{R}) *A subset $K \subset \mathbb{R}$ is compact if and only if every sequence in K has a subsequence which converges in K .*

Proof Let $\{x_j\}_{j \in \mathbb{N}}$ be a sequence in K . Since K is bounded by Theorem 2.5.27, the sequence $\{x_j\}_{j \in \mathbb{N}}$ is bounded. We next show that there exists either a monotonically increasing, or a monotonically decreasing, subsequence of $\{x_j\}_{j \in \mathbb{N}}$. Define

$$D = \{j \in \mathbb{N} \mid x_k > x_j, k > j\}$$

⁹Bernard Placidus Johann Nepomuk Bolzano (1781–1848) was a Czechoslovakian philosopher, mathematician, and theologian who made mathematical contributions to the field of analysis. Karl Theodor Wilhelm Weierstrass (1815–1897) is one of the greatest of all mathematicians. He made significant contributions to the fields of analysis, complex function theory, and the calculus of variations.

If the set D is infinite, then we can write $D = \{j_k\}_{k \in \mathbb{N}}$. By definition of D , it follows that $x_{j_{k+1}} > x_{j_k}$ for each $k \in \mathbb{N}$. Thus the subsequence $\{x_{j_k}\}_{k \in \mathbb{N}}$ is monotonically increasing. If the set D is finite choose $j_1 > \sup D$. Then there exists $j_2 > j_1$ such that $x_{j_2} \leq x_{j_1}$. Since $j_2 > \sup D$, there then exists $j_3 > j_2$ such that $x_{j_3} \leq x_{j_2}$. By definition of D , this process can be repeated inductively to yield a monotonically decreasing subsequence $\{x_{j_k}\}_{k \in \mathbb{N}}$. It now follows from Theorem 2.3.7 that the sequence $\{x_{j_k}\}_{k \in \mathbb{N}}$, be it monotonically increasing or monotonically decreasing, converges.

Next suppose that every sequence $\{x_j\}_{j \in \mathbb{N}}$ in K possesses a convergent subsequence. Let $\{U_i\}_{i \in I}$ be an open cover of K , and by Lemma 2.5.25 choose a countable subcover which we denote by $\{U_j\}_{j \in \mathbb{N}}$. Now suppose that every finite subcover of $\{U_j\}_{j \in \mathbb{N}}$ does not cover K . This means that, for every $k \in \mathbb{N}$, the set $C_k = K \setminus (\cup_{j=1}^k U_j)$ is nonempty. Thus we may define a sequence $\{x_k\}_{k \in \mathbb{N}}$ in \mathbb{R} such that $x_k \in C_k$. Since the sequence $\{x_k\}_{k \in \mathbb{N}}$ is in K , it possesses a convergent subsequence $\{x_{k_m}\}_{m \in \mathbb{N}}$, by hypotheses. Let x be the limit of this subsequence. Since $x \in K$ and since $K = \cup_{j \in \mathbb{N}} U_j$, $x \in U_l$ for some $l \in \mathbb{N}$. Since the sequence $\{x_{k_m}\}_{m \in \mathbb{N}}$ converges to x , it follows that there exists $N \in \mathbb{N}$ such that $x_{k_m} \in U_l$ for $m \geq N$. But this contradicts the definition of the sequence $\{x_k\}_{k \in \mathbb{N}}$, forcing us to conclude that our assumption is wrong that there is no finite subcover of K from the collection $\{U_j\}_{j \in \mathbb{N}}$. ■

The following property of compact intervals of \mathbb{R} is useful.

2.5.30 Theorem (Lebesgue¹⁰ number for compact intervals) *Let $I = [a, b]$ be a compact interval. Then for any open cover $\{U_a\}_{a \in \mathbb{N}}$ of $[a, b]$, there exists $\delta > 0$, called the **Lebesgue number** of I , such that, for each $x \in [a, b]$, there exists $a \in A$ such that $B(\delta, x) \cap I \subset U_a$.*

Proof Suppose there exists an open cover $\{U_a\}_{a \in A}$ such that, for all $\delta > 0$, there exists $x \in [a, b]$ such that none of the sets U_a , $a \in A$, contains $B(\delta, x) \cap I$. Then there exists a sequence $\{x_j\}_{j \in \mathbb{N}}$ in I such that

$$\left\{ a \in A \mid B\left(\frac{1}{j}, x_j\right) \subset U_a \right\} = \emptyset$$

for each $j \in \mathbb{N}$. By the Bolzano–Weierstrass Theorem there exists a subsequence $\{x_{j_k}\}_{k \in \mathbb{N}}$ that converges to a point, say x , in $[a, b]$. Then there exists $\epsilon > 0$ and $a \in A$ such that $B(\epsilon, y) \subset U_a$. Now let $N \in \mathbb{N}$ be sufficiently large that $|x_{j_k} - x| < \frac{\epsilon}{2}$ for $k \geq N$ and such that $\frac{1}{j_N} < \frac{\epsilon}{2}$. Now let $k \geq N$. Then, if $y \in B(\frac{1}{j_k}, x_{j_k})$ we have

$$|y - x| = |y - x_{j_k} + x_{j_k} - x| \leq |y - x_{j_k}| + |x_{j_k} - x| < \epsilon.$$

Thus we arrive at the contradiction that $B(\frac{1}{j_k}, x_{j_k}) \subset U_a$. ■

The following result is sometimes useful.

2.5.31 Proposition (Countable intersections of nested compact sets are nonempty) *Let $\{K_j\}_{j \in \mathbb{N}}$ be a collection of compact subsets of \mathbb{R} satisfying $K_{j+1} \subset K_j$. Then $\cap_{j \in \mathbb{N}} K_j$ is nonempty.*

Proof It is clear that $K = \cap_{j \in \mathbb{N}} K_j$ is bounded, and moreover it is closed by Exercise 2.5.1. Thus K is compact by the Heine–Borel Theorem. Let $\{x_j\}_{j \in \mathbb{N}}$ be a sequence for which $x_j \in K_j$ for $j \in \mathbb{N}$. This sequence is thus a sequence in K_1 and so, by the Bolzano–Weierstrass Theorem, has a subsequence $\{x_{j_k}\}_{k \in \mathbb{N}}$ converging to $x \in K_1$. The sequence $\{x_{j_{k+1}}\}_{k \in \mathbb{N}}$ is then a sequence in K_2 which is convergent, so showing that $x \in K_2$. Similarly, one shows that $x \in K_j$ for all $j \in \mathbb{N}$, giving the result. ■

Finally, let us indicate the relationship between the notions of relative compactness and total boundedness. We see that for \mathbb{R} these concepts are the same. This may not be true in general.

¹⁰Henri Léon Lebesgue (1875–1941) was a French mathematician. His work was in the area of analysis. The Lebesgue integral is considered to be one of the most significant contributions to mathematics in the past century or so.

2.5.32 Proposition (“Relatively compact” equals “totally bounded” in \mathbb{R}) *A subset of \mathbb{R} is relatively compact if and only if it is totally bounded.*

Proof Let $A \subset \mathbb{R}$.

First suppose that A is relatively compact. Since $A \subset \text{cl}(A)$ and since $\text{cl}(A)$ is bounded by the Heine–Borel Theorem, it follows that A is bounded. It is then easy to see that A is totally bounded.

Now suppose that A is totally bounded. For $\epsilon > 0$ let $x_1, \dots, x_k \in \mathbb{R}$ have the property that $A \subset \bigcup_{j=1}^k \mathbf{B}(\epsilon, x_j)$. If

$$M_0 = \max \{ |x_j - x_l| \mid j, l \in \{1, \dots, k\} \} + 2\epsilon,$$

then it is easy to see that $A \subset \mathbf{B}(M, 0)$ for any $M > M_0$. Then $\text{cl}(A) \subset \overline{\mathbf{B}}(M, 0)$ by part (iv) of Proposition 2.5.20, and so $\text{cl}(A)$ is bounded. Since $\text{cl}(A)$ is closed, it follows from the Heine–Borel Theorem that A is relatively compact. ■

an example
using convergent
subsequences

material on
directed sets in
old file

where

2.5.5 Connectedness

The idea of a connected set will come up occasionally in these volumes. Intuitively, a set is connected if it cannot be “broken in two.” We will study it more systematically in , and here we only give enough detail to effectively characterise connected subsets of \mathbb{R} .

2.5.33 Definition (Connected subset of \mathbb{R}) Subsets $A, B \subset \mathbb{R}$ are *separated* if $A \cap \text{cl}(B) = \emptyset$ and $\text{cl}(A) \cap B = \emptyset$. A subset $S \subset \mathbb{R}$ is *disconnected* if $S = A \cup B$ for nonempty separated subsets A and B . A subset $S \subset \mathbb{R}$ is *connected* if it is not disconnected. •

Rather than give examples, let us simply immediately characterise the connected subsets of \mathbb{R} , since this renders all examples trivial to understand.

2.5.34 Theorem (Connected subsets of \mathbb{R} are intervals and vice versa) *A subset $S \subset \mathbb{R}$ is connected if and only if S is an interval.*

Proof Suppose that S is not an interval. Then, by Proposition 2.5.4, there exists $a, b \in S$ with $a < b$ and $c \in (a, b)$ such that $c \notin S$. Let $A_c = S \cap (-\infty, c)$ and $B_c = S \cap (c, \infty)$, and note that both A_c and B_c are nonempty. Also, since $c \notin S$, $S = A_c \cup B_c$. Since $(-\infty, c) \cap [c, \infty) = \emptyset$ and $(-\infty, c] \cap (c, \infty) = \emptyset$, A_c and B_c are separated. That S is not connected.

Now suppose that S is not connected, and write $S = A \cup B$ for nonempty separated sets A and B . Without loss of generality, let $a \in A$ and $b \in B$ have the property that $a < b$. Note that $A \cap [a, b]$ is bounded so that $c = \sup A \cap [a, b]$ exists in \mathbb{R} . Then $c \in \text{cl}(A \cap [a, b]) \subset \text{cl}(A) \cap [a, b]$. In other words, $c \in \text{cl}(A)$. Since $\text{cl}(A) \cap B = \emptyset$, $c \notin B$. If $c \notin A$ then $c \notin S$, and so S is not connected by Proposition 2.5.4. If $c \in A$ then, since $A \cap \text{cl}(B) = \emptyset$, $c \notin \text{cl}(B)$. In this case there exists an open interval containing c that does not intersect $\text{cl}(B)$. In particular, there exists $d > c$ such that $d \notin B$. Since $d > c$ we also have $d \notin A$, and so $d \notin S$. Again we conclude that S is not an interval by Proposition 2.5.4. ■

2.5.6 Sets of measure zero

The topic of this section will receive a full treatment in the context of measure theory as presented in Chapter II-1. However, it is convenient here to talk about a simple concepts from measure theory, one which formalises the idea of a set being “small.” We shall only give here the definition and a few examples. The reader should look ahead to Chapter II-1 for more detail.

2.5.35 Definition (Set of measure zero in \mathbb{R}) A subset $A \subset \mathbb{R}$ has *measure zero*, or is *of measure zero*, if

$$\inf \left\{ \sum_{j=1}^{\infty} |b_j - a_j| \mid A \subset \bigcup_{j \in \mathbb{N}} (a_j, b_j) \right\} = 0.$$

(We allow the possibility of a finite sum in the equation above by allowing the possibility that $a_j = b_j$.) •

The idea, then, is that one can cover a set A with open intervals, each of which have some length. One can add all of these lengths to get a total length for the intervals used to cover A . Now, if one can make this total length arbitrarily small, then the set has measure zero.

2.5.36 Notation (“Almost everywhere” and “a.e.”) We give here an important piece of notation associated to the notion of a set of measure zero. Let $A \subset \mathbb{R}$ and let $P: A \rightarrow \{\text{true}, \text{false}\}$ be a property defined on A (see the prelude to the Principle of Transfinite Induction, Theorem 1.5.14). The property P holds *almost everywhere*, *a.e.*, or *for almost every* $x \in A$ if the set $\{x \in A \mid P(x) = \text{false}\}$ has measure zero. •

This is best illustrated with some examples.

2.5.37 Examples (Sets of measure zero)

1. Let $A = \{x_1, \dots, x_k\}$ for some distinct $x_1, \dots, x_k \in \mathbb{R}$. We claim that this set has measure zero. Note that for any $\epsilon > 0$ the intervals $(x_j - \epsilon, x_j + \epsilon)$, $j \in \{1, \dots, k\}$, clearly cover A . We also have

$$\sum_{j=1}^k |(x_j + \epsilon) - (x_j - \epsilon)| = 2k\epsilon.$$

Since $\inf \{2k\epsilon \mid \epsilon > 0\} = 0$, our claim that A has zero measure is validated.

2. Now let $A = \mathbb{Q}$ be the set of rational numbers. To show that A has measure zero, note that from Exercise 2.1.4 that A is countable. Thus we can write the elements of A as $\{q_j\}_{j \in \mathbb{N}}$. Now let $\epsilon > 0$ and for $j \in \mathbb{N}$ define $a_j = q_j - \frac{\epsilon}{2^j}$ and $b_j = q_j + \frac{\epsilon}{2^j}$. Then the collection (a_j, b_j) , $j \in \mathbb{N}$, covers A . Moreover,

$$\sum_{j=1}^{\infty} |b_j - a_j| = \sum_{j=1}^{\infty} \frac{2\epsilon}{2^j} = 2\epsilon,$$

using the fact, shown in Example 2.4.2-1, that the series $\sum_{j=1}^{\infty} \frac{1}{2^j}$ converges to 1. Now, since $\inf \{2\epsilon \mid \epsilon > 0\} = 0$, it follows that A indeed has measure zero.

3. Let $A = \mathbb{R} \setminus \mathbb{Q}$ be the set of irrational numbers. We claim that this set does not have measure zero. To see this, let $k \in \mathbb{N}$ and consider the set $A_k = A \cap [-k, k]$. Now let $\epsilon > 0$. We claim that if $\{(a_j, b_j)\}_{j \in \mathbb{N}}$ is a collection of open intervals for which $A_k \subset \bigcup_{j \in \mathbb{N}} (a_j, b_j)$, then

$$\sum_{j=1}^{\infty} |b_j - a_j| \geq 2k - \epsilon. \tag{2.8}$$

To see this, let $\{(c_l, d_l)\}_{l \in \mathbb{N}}$ be a collection of intervals such that $\mathbb{Q} \cap [-k, k] \subset \bigcup_{l \in \mathbb{N}} (c_l, d_l)$ and such that

$$\sum_{l=1}^{\infty} |d_l - c_l| < \epsilon.$$

Such a collection of intervals exists since we have already shown that \mathbb{Q} , and therefore $\mathbb{Q} \cap [-k, k]$, has measure zero (see Exercise 2.5.7). Now note that

$$[-k, k] \subset \left(\bigcup_{j \in \mathbb{N}} (a_j, b_j) \right) \cup \left(\bigcup_{l \in \mathbb{N}} (c_l, d_l) \right),$$

so that

$$\left(\sum_{j=1}^{\infty} |b_j - a_j| \right) + \left(\sum_{l=1}^{\infty} |d_l - c_l| \right) \geq 2k.$$

From this we immediately conclude that (2.8) does indeed hold. Moreover, (2.8) holds for every $k \in \mathbb{N}$, for every $\epsilon > 0$, and for every open cover $\{(a_j, b_j)\}_{j \in \mathbb{N}}$ of A_k . Thus,

$$\inf \left\{ \sum_{l=1}^{\infty} |\tilde{b}_l - \tilde{a}_l| \mid A \subset \bigcup_{l \in \mathbb{N}} (\tilde{a}_l, \tilde{b}_l) \right\} \geq \inf \left\{ \sum_{j=1}^{\infty} |b_j - a_j| \mid A_k \subset \bigcup_{j \in \mathbb{N}} (a_j, b_j) \right\} \geq 2k - \epsilon$$

for every $k \in \mathbb{N}$ and for every $\epsilon > 0$. This precludes A from having measure zero. •

The preceding examples suggest sets of measure zero are countable. This is not so, and the next famous example gives an example of an uncountable set with measure zero.

2.5.38 Example (An uncountable set of measure zero: the middle-thirds Cantor set) In this example we construct one of the standard “strange” sets used in real analysis to exhibit some of the characteristics that can possibly be attributed to subsets of \mathbb{R} . We shall also use this set in a construction in Example 3.3.4 to give an example of a continuous monotonically increasing function whose derivative is zero almost everywhere.

Let $C_0 = [0, 1]$. Then define

$$\begin{aligned} C_1 &= [0, \frac{1}{3}] \cup [\frac{2}{3}, 1], \\ C_2 &= [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1], \\ &\vdots \end{aligned}$$

so that C_k is a collection of 2^k disjoint closed intervals each of length 3^{-k} (see Figure 2.5). We define $C = \bigcap_{k \in \mathbb{N}} C_k$, which we call the *middle-thirds Cantor set*. Let us give some



Figure 2.5 The first few sets used in the construction of the middle-thirds Cantor set

of the properties of C .

1 Lemma C has the same cardinality as $[0, 1]$.

Proof Note that each of the sets C_k , $k \in \mathbb{N}_0$, is a collection of disjoint closed intervals. Let us write $C_k = \cup_{j=1}^{2^k} I_{k,j}$, supposing that the intervals $I_{k,j}$ are enumerated such that the right endpoint of $I_{k,j}$ lies to the left of the left endpoint of $I_{k,j+1}$ for each $k \in \mathbb{N}_0$ and $j \in \{1, \dots, 2^k\}$. Now note that each interval $I_{k+1,j}$, $k \in \mathbb{N}_0$, $j \in \{1, \dots, 2^{k+1}\}$ comes from assigning two intervals to each of the intervals $I_{k,j}$, $k \in \mathbb{N}_0$, $j \in \{1, \dots, 2^k\}$. Assign to an interval $I_{k+1,j}$, $k \in \mathbb{N}_0$, $j \in \{1, \dots, 2^k\}$, the number 0 (resp. 1) if it the left (resp. right) interval coming from an interval $I_{k,j'}$ of C_k . In this way, each interval in C_k , $k \in \mathbb{N}_0$, is assigned a 0 or a 1 in a unique manner. Since, for each point in $x \in C$, there is exactly one $j \in \{1, \dots, 2^k\}$ such that $x \in I_{k,j}$. Therefore, for each point in C there is a unique decimal expansion $0.n_1n_2n_3\dots$ where $n_k \in \{0, 1\}$. Moreover, for every such decimal expansion, there is a corresponding point in C . However, such decimal expansions are exactly binary decimal expansions for points in $[0, 1]$. In other words, there is a bijection from C to $[0, 1]$. ■

2 Lemma C is a set of measure zero.

Proof Let $\epsilon > 0$. Note that each of the sets C_k can be covered by a finite number of closed intervals whose lengths sum to $(\frac{2}{3})^k$. Therefore, each of the sets C_k can be covered by open intervals whose lengths sum to $(\frac{2}{3})^k + \frac{\epsilon}{2}$. Choosing k sufficiently large that $(\frac{2}{3})^k < \frac{\epsilon}{2}$ we see that C is contained in the union of a finite collection of open intervals whose lengths sum to ϵ . Since ϵ is arbitrary, it follows that C has measure zero. ■

This example thus shows that sets of measure zero, while “small” in some sense, can be “large” in terms of the number of elements they possess. Indeed, in terms of cardinality, C has the same size as $[0, 1]$, although their measures differ by as much as possible. •

2.5.7 Cantor sets

The remainder of this section is devoted to a characterisation of certain sorts of exotic sets, perhaps the simplest example of which is the middle-thirds Cantor set of Example 2.5.38. This material is only used occasionally, and so can be omitted until the reader feels they need/want to understand it.

The qualifier “middle-thirds” in Example 2.5.38 makes one believe that there might be a general notion of a “Cantor set.” This is indeed the case.

2.5.39 Definition (Cantor set) Let $I \subset \mathbb{R}$ be a closed interval. A subset $A \subset I$ is a *Cantor set* if

- (i) A is closed,
- (ii) $\text{int}(A) = \emptyset$, and
- (iii) every point of A is an accumulation point of A . •

We leave it to the reader to verify in Exercise 2.5.10 that the middle-thirds Cantor set is a Cantor set, according to the previous definition.

One might wonder whether all Cantor sets have the properties of having the cardinality of an interval and of having measure zero. To address this, we give a result and an example. The result shows that all Cantor sets are uncountable.

2.5.40 Proposition (Cantor sets are uncountable) If $A \subset \mathbb{R}$ is a nonempty set having the property that each of its points is an accumulation point, then A is uncountable. In particular, Cantor sets are uncountable.

Proof Any finite set has no accumulation points by Proposition 2.5.13. Therefore A must be either countably infinite or uncountable. Suppose that A is countable and write $A = \{x_j\}_{j \in \mathbb{N}}$. Let $y_1 \in A \setminus \{x_1\}$. For $r_1 < |x_1 - y_1|$ we have $x_1 \notin \overline{B}(r_1, y_1)$. We note that y_1 is an accumulation point for $A \setminus \{x_1, x_2\}$; this follows immediately from Proposition 2.5.13. Thus there exists $y_2 \in$

$A \setminus \{x_1, x_2\}$ such that $y_2 \in B(r_1, y_1)$ and such that $y_2 \neq y_1$. If $r_2 < \min\{|x_2 - y_2|, r_1 - |y_2 - y_1|\}$ then $x_2 \notin \overline{B}(r_2, y_2)$ and $\overline{B}(r_2, y_2) \subset B(r_1, y_1)$ by a simple application of the triangle inequality. Continuing in this way we define a sequence $\{\overline{B}(r_j, y_j)\}_{j \in \mathbb{N}}$ of closed balls having the following properties:

1. $\overline{B}(r_j, y_j) \subset \overline{B}(r_{j+1}, y_{j+1})$ for each $j \in \mathbb{N}$;
2. $x_j \notin \overline{B}(r_j, y_j)$ for each $j \in \mathbb{N}$.

Note that $\{\overline{B}(r_j, y_j) \cap A\}_{j \in \mathbb{N}}$ is a nested sequence of compact subsets of A , and so by Proposition 2.5.31, $\bigcap_{j \in \mathbb{N}} (\overline{B}(r_j, y_j) \cap A)$ is a nonempty subset of A . However, for any $j \in \mathbb{N}$, $x_j \notin \bigcap_{j \in \mathbb{N}} (\overline{B}(r_j, y_j) \cap A)$, and so we arrive, by contradiction, to the conclusion that A is not countable. ■

The following example shows that Cantor sets may not have measure zero.

2.5.41 Example (A Cantor set not having zero measure) We will define a subset of $[0, 1]$ that is a Cantor set, but does not have measure zero. The construction mirrors closely that of Example 2.5.38.

We let $\epsilon \in (0, \frac{1}{2})$. Let $C_{\epsilon,0} = [0, 1]$ and define $C_{\epsilon,1}$ by deleting from $C_{\epsilon,0}$ an open interval of length $\frac{\epsilon}{2}$ centered at the midpoint of $C_{\epsilon,0}$. Note that $C_{\epsilon,1}$ consists of two disjoint closed intervals whose lengths sum to $1 - \frac{\epsilon}{2}$. Next define $C_{\epsilon,2}$ by deleting from $C_{\epsilon,1}$ two open intervals, each of length $\frac{\epsilon}{8}$, centered at the midpoints of each of the intervals comprising $C_{\epsilon,1}$. Note that $C_{\epsilon,2}$ consists of four disjoint closed intervals whose lengths sum to $1 - \frac{\epsilon}{4}$. Proceed in this way, defining a sequence of sets $\{C_{\epsilon,k}\}_{k \in \mathbb{N}}$, where $C_{\epsilon,k}$ consists of 2^k disjoint closed intervals whose lengths sum to $1 - \sum_{j=1}^k \frac{\epsilon}{2^j} = 1 - \epsilon$. Take $C_\epsilon = \bigcap_{k \in \mathbb{N}} C_{\epsilon,k}$.

Let us give the properties of C_ϵ in a series of lemmata.

1 Lemma C_ϵ is a Cantor set.

Proof That C_ϵ is closed follows from Exercise 2.5.1 and the fact that it is the intersection of a collection of closed sets. To see that $\text{int}(C_\epsilon) = \emptyset$, let $I \subset [0, 1]$ be an open interval and suppose that $I \subset C_\epsilon$. This means that $I \subset C_{\epsilon,k}$ for each $k \in \mathbb{N}$. Note that the sets $C_{\epsilon,k}$, $k \in \mathbb{N}$, are unions of closed intervals, and that for any $\delta > 0$ there exists $N \in \mathbb{N}$ such that the lengths of the intervals comprising $C_{\epsilon,k}$ are less than δ for $k \geq N$. Thus the length of I must be zero, and so $I = \emptyset$. Thus C_ϵ contains no nonempty open intervals, and so must have an empty interior. To see that every point of C_ϵ is an accumulation point of C_ϵ , we note that all points in C_ϵ are endpoints for one of the closed intervals comprising $C_{\epsilon,k}$ for some $k \in \mathbb{N}$. Moreover, it is clear that every neighbourhood of a point in C_ϵ must contain another endpoint from one of the closed intervals comprising $C_{\epsilon,k}$ for some $k \in \mathbb{N}$. Indeed, were this not the case, this would imply the existence of a nonempty open interval contained in C_ϵ , and we have seen that there can be no such interval. ■

2 Lemma C_ϵ is uncountable.

Proof This can be proved in exactly the same manner as the middle-thirds Cantor set was shown to be uncountable. ■

3 Lemma C_ϵ does not have measure zero.

Proof Once one knows the basic properties of Lebesgue measure, it follows immediately that C_ϵ has, in fact, measure $1 - \epsilon$. However, since we have not yet defined measure, let us prove that C_ϵ does not have measure zero, using only the definition of a set of measure zero. Let $\{(a_j, b_j)\}_{j \in \mathbb{N}}$ be a countable collection of open intervals having the property that

$$C_\epsilon \subset \bigcup_{j \in \mathbb{N}} (a_j, b_j).$$

Since C_ϵ is closed, it is compact by Corollary 2.5.28. Therefore, there exists a finite collection $\{(a_{j_l}, b_{j_l})\}_{l \in \{1, \dots, m\}}$ of intervals having the property that

$$C_\epsilon \subset \bigcup_{l=1}^m (a_{j_l}, b_{j_l}). \quad (2.9)$$

We claim that there exists $k \in \mathbb{N}$ such that

$$C_{\epsilon, k} \subset \bigcup_{l=1}^m (a_{j_l}, b_{j_l}). \quad (2.10)$$

Indeed, suppose that, for each $k \in \mathbb{N}$ there exists $x_k \in C_{\epsilon, k}$ such that $x_k \notin \bigcup_{l=1}^m (a_{j_l}, b_{j_l})$. The sequence $\{x_k\}_{k \in \mathbb{N}}$ is then a sequence in the compact set $C_{\epsilon, 1}$, and so by the Bolzano–Weierstrass Theorem, possesses a subsequence $\{x_{k_r}\}_{r \in \mathbb{N}}$ converging to $x \in C_{\epsilon, 1}$. But the sequence $\{x_{k_{r+1}}\}_{r \in \mathbb{N}}$ is then a convergent sequence in $C_{\epsilon, 2}$, so $x \in C_{\epsilon, 2}$. Continuing in this way, $x \in \bigcap_{k \in \mathbb{N}} C_{\epsilon, k}$. Moreover, the sequence $\{x_k\}_{k \in \mathbb{N}}$ is also a sequence in the closed set $[0, 1] - \bigcup_{l=1}^m (a_{j_l}, b_{j_l})$, and so we conclude that $x \in [0, 1] - \bigcup_{l=1}^m (a_{j_l}, b_{j_l})$. Thus we contradict the condition (2.9), and so there indeed must be a $k \in \mathbb{N}$ such that (2.10) holds. However, this implies that any collection of open intervals covering C_ϵ must have lengths which sum to at least $1 - \epsilon$. Thus C_ϵ cannot have measure zero. ■

Cantor sets such as C_ϵ are sometimes called **fat Cantor sets**, reflecting the fact that they are not measure zero. Note, however, that they are not *that* fat, since they have an empty interior! •

Exercises

2.5.1 For an arbitrary collection $\{U_a\}_{a \in A}$ of open sets and an arbitrary collection $\{C_b\}_{b \in B}$ of closed sets, do the following:

- (a) show that $\bigcup_{a \in A} U_a$ is open;
- (b) show that $\bigcap_{b \in B} C_b$ is closed;

For open sets U_1 and U_2 and closed sets C_1 and C_2 , do the following:

- (c) show that $U_1 \cap U_2$ is open;
- (d) show that $C_1 \cup C_2$ is closed.

2.5.2 Show that a set $A \subset \mathbb{R}$ is closed if and only if it contains all of its limit points.

2.5.3 For $A \subset \mathbb{R}$, show that $\text{bd}(A) = \text{bd}(\mathbb{R} \setminus A)$.

2.5.4 Find counterexamples to the following statements (cf. Propositions 2.5.15, 2.5.19, and 2.5.20):

- (a) $\text{int}(A \cup B) \subset \text{int}(A) \cup \text{int}(B)$;
- (b) $\text{int}(\bigcup_{i \in I} A_i) \subset \bigcup_{i \in I} \text{int}(A_i)$;
- (c) $\text{int}(\bigcap_{i \in I} A_i) \supset \bigcap_{i \in I} \text{int}(A_i)$;
- (d) $\text{cl}(A \cap B) \supset \text{cl}(A) \cap \text{cl}(B)$;
- (e) $\text{cl}(\bigcup_{i \in I} A_i) \subset \bigcup_{i \in I} \text{cl}(A_i)$;
- (f) $\text{cl}(\bigcap_{i \in I} A_i) \supset \bigcap_{i \in I} \text{cl}(A_i)$.

Hint: No fancy sets are required. Intervals will suffice in all cases.

2.5.5 For each of the following statements, prove the statement if it is true, and give a counterexample if it is not:

- (a) $\text{int}(A_1 \cup A_2) = \text{int}(A_1) \cup \text{int}(A_2)$;

- (b) $\text{int}(A_1 \cap A_2) = \text{int}(A_1) \cap \text{int}(A_2)$;
- (c) $\text{cl}(A_1 \cup A_2) = \text{cl}(A_1) \cup \text{cl}(A_2)$;
- (d) $\text{cl}(A_1 \cap A_2) = \text{cl}(A_1) \cap \text{cl}(A_2)$;
- (e) $\text{bd}(A_1 \cup A_2) = \text{bd}(A_1) \cup \text{bd}(A_2)$;
- (f) $\text{bd}(A_1 \cap A_2) = \text{bd}(A_1) \cap \text{bd}(A_2)$.

2.5.6 Do the following:

- (a) show that any finite subset of \mathbb{R} is discrete;
- (b) show that a discrete bounded set is finite;
- (c) find a set $A \subset \mathbb{R}$ that is countable and has no accumulation points, but that is not discrete.

2.5.7 Show that if $A \subset \mathbb{R}$ has measure zero and if $B \subset A$, then B has measure zero.

2.5.8 Show that any countable subset of \mathbb{R} has measure zero.

2.5.9 Let $\{Z_j\}_{j \in \mathbb{N}}$ be a collection of subsets of \mathbb{R} that each have measure zero. Show that $\cup_{j \in \mathbb{N}} Z_j$ also has measure zero.

2.5.10 Show that the set C constructed in Example 2.5.38 is a Cantor set.

Section 2.6

Extensions to \mathbb{R}^n

We shall have occasion in the series to consider the set \mathbb{R}^n , the n -fold Cartesian product of \mathbb{R} with itself, in various guises. Many of the necessary properties of \mathbb{R}^n will be developed in later chapters, particularly Chapters 5, II-2, and II-3. However, there is some benefit to outlining these results here for two reasons. First of all, the set \mathbb{R}^n is so fundamental that it is useful to have its properties summarised in one place. Second, by pointing out the necessary results now, we can use them freely, particularly in Chapter 6.

Thus, in this section, we merely indicate which of the properties of \mathbb{R} given in this chapter are transferable to \mathbb{R}^n , and point out the places in later chapters where the complete story is developed.

A typical element of \mathbb{R}^n will be denoted by $\mathbf{x} = (x_1, \dots, x_n)$.

Do I need to read this section? The material in this section is intended for organisation more than anything else. However, readers who wish to see what material from this chapter can be extended to higher-dimensions is encouraged to read through the material. Since there are no proofs, this is easily done. •

2.6.1 Algebra, etc. in \mathbb{R}^n

In \mathbb{R} , as we indicated in Section 2.2.1, we can perform familiar algebraic operations like addition, multiplication, and division. Not all of these operations generally carry over to \mathbb{R}^n . One *can* add elements of \mathbb{R}^n using the rule

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, \dots, x_n + y_n).$$

One can also multiply elements of \mathbb{R}^n by an element of \mathbb{R} using the rule

$$\alpha \mathbf{x} = (\alpha x_1, \dots, \alpha x_n).$$

These operations are discussed in their proper more general setting in Example 4.6.2–2. Generally, one cannot multiply or divide elements of \mathbb{R}^n together in a useful way. However, for $n = 2$ it turns out that multiplication and division *are* also possible, and this is described in Section 4.5.¹¹

Although Zermelo's Well Ordering Theorem tells us that \mathbb{R}^n possesses a well order, apart from $n = 1$ there is no useful (i.e., reacting well with the other structures of \mathbb{R}^n) partial order on \mathbb{R}^n . Thus any of the results about \mathbb{R} that relate to its natural total order \leq will not generally carry over to \mathbb{R}^n .

2.6.2 The generalisation of the absolute value function on \mathbb{R}^n

There is a generalisation to \mathbb{R}^n of the absolute value function on \mathbb{R} . Indeed, this is one of the more valuable features of \mathbb{R}^n . In fact, there are many generalisations of the absolute value function, but let us here give one of the more common ones. This assigns to an element

¹¹There are other values of n for which multiplication and division are possible, but this will not interest us here.

$\mathbf{x} \in \mathbb{R}^n$ the nonnegative number

$$\|\mathbf{x}\|_{\mathbb{R}^n} = \left(\sum_{j=1}^n x_j^2 \right)^{1/2}.$$

Note that when $n = 1$ we have $\|\cdot\|_{\mathbb{R}^1} = |\cdot|$. We refer the reader to Example II-2.1.2–3 for initial discussions of this structure, and to Section II-3.1 for more details and further generalisations. We shall make use of this structure of \mathbb{R}^n in Chapter 6 in the case where $n = 2$. Here let us merely note some properties of this operation:

1. $\|\alpha\mathbf{x}\|_{\mathbb{R}^n} = |\alpha| \|\mathbf{x}\|_{\mathbb{R}^n}$ for $\alpha \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$;
2. $\|\mathbf{x}\|_{\mathbb{R}^n} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$;
3. $\|\mathbf{x}\|_{\mathbb{R}^n} = 0$ only if $\mathbf{x} = \mathbf{0}$;
4. $\|\mathbf{x}_1 + \mathbf{x}_2\|_{\mathbb{R}^n} \leq \|\mathbf{x}_1\|_{\mathbb{R}^n} + \|\mathbf{x}_2\|_{\mathbb{R}^n}$.

Of these properties, perhaps only the last, the triangle inequality, is nontrivial. However, what is true is that all of these properties are sufficient to ensure that $\|\cdot\|_{\mathbb{R}^n}$ gives to \mathbb{R}^n sufficient structure to generalise many of the properties of \mathbb{R} , some of which we describe below.

2.6.3 Sequences and series in \mathbb{R}^n

Note that for \mathbb{R} the discussion of sequences and their convergence is reliant on the absolute value function. Since this can be generalised to \mathbb{R}^n , the ideas of Cauchy sequences and convergent sequences carries over to \mathbb{R}^n . Thus a sequence $\{\mathbf{x}_j\}_{j \in \mathbb{N}}$ in \mathbb{R}^n is a **Cauchy sequence** if, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|\mathbf{x}_j - \mathbf{x}_k\|_{\mathbb{R}^n} < \epsilon$ for $j, k \geq N$. The sequence **converges** to $\mathbf{s}_0 \in \mathbb{R}^n$ if, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|\mathbf{x}_j - \mathbf{s}_0\|_{\mathbb{R}^n} < \epsilon$ for every $j \in \mathbb{N}$. We also say that the sequence is **converges absolutely** if the sequence $\{\|\mathbf{x}_j\|_{\mathbb{R}^n}\}_{j \in \mathbb{N}}$ in \mathbb{R} converges. We shall show in Theorem II-2.1.25 that a sequence in \mathbb{R}^n is Cauchy if and only if it is convergent. In \mathbb{R} we can, by using language such as “diverges to $\pm\infty$,” be somewhat explicit about the nature in which is sequence does not converge. These ideas do not generalise to \mathbb{R}^n is a useful and general way, so we shall not attempt to refine the notion of divergence of sequences in \mathbb{R}^n .

Many of the results in Section 2.3 concerning sequences in \mathbb{R} have analogues in \mathbb{R}^n , and indeed in more general settings than \mathbb{R}^n . We refer the reader to Section II-3.1 for the precise statements of these results in the more general setting. However, it should be understood that concepts for sequences in \mathbb{R} that are dependent on the total order in \mathbb{R} (e.g., monotonicity of sequences, lim sup, and lim inf) do not carry over to \mathbb{R}^n . Also, the convergence tests of Section 2.3.3 do not apply verbatim (although some might be modifiable to give results for sequences in \mathbb{R}^n). The results in Section 2.3.6 that make sense for \mathbb{R}^n also apply in that case.

The above comments also apply to series. Thus a **series** in \mathbb{R}^n is an expression of the form $\mathbf{S} = \sum_{j=1}^{\infty} \mathbf{x}_j$. For a series we define the sequence $\{\mathbf{S}_k\}_{k \in \mathbb{N}}$ of **partial sums** by

$$\mathbf{S}_k = \sum_{j=1}^k \mathbf{x}_j,$$

and a series **converges** to \mathbf{s}_0 if the sequence of partial sums converges to \mathbf{s}_0 . The series **converges absolutely** if the series $\sum_{j=1}^{\infty} \|\mathbf{x}_j\|_{\mathbb{R}^n}$ in \mathbb{R} converges. In Theorem II-3.1.1 (or more precisely, in Corollary 3.1.2) we shall show that an absolutely convergent sequence is necessarily convergent.

2.6.4 Subsets of \mathbb{R}^n

Note that the definition of open (and therefore closed) sets in \mathbb{R} relies on the absolute value function. Therefore, since the absolute value function has an appropriate generalisation to \mathbb{R}^n , the ideas of open and closed sets carry over to \mathbb{R}^n . The key idea is the generalisation of the notion of an open ball as seen in Example II-2.1.2–3. To be clear about this, the **open ball** of radius r about $\mathbf{x} \in \mathbb{R}^n$ is the set

$$\mathbf{B}(r, \mathbf{x}) = \{ \mathbf{y} \in \mathbb{R}^n \mid \|\mathbf{y} - \mathbf{x}\|_{\mathbb{R}^n} < r \}.$$

One similarly defines the **closed ball** in \mathbb{R}^n as

$$\overline{\mathbf{B}}(r, \mathbf{x}) = \{ \mathbf{y} \in \mathbb{R}^n \mid \|\mathbf{y} - \mathbf{x}\|_{\mathbb{R}^n} \leq r \}.$$

Using this notion of an open ball, the definition of open sets, closed sets, and neighbourhoods proceeds exactly as in the case where $n = 1$. Generalisations of intervals to \mathbb{R}^n in the form of “cubes” is also possible, although we shall not have occasion to use this notion.

The notions of accumulation point, cluster point, limit point, interior, closure, and boundary rely only on the notion of neighbourhoods, and so are generalisable to \mathbb{R}^n , and indeed well beyond. These ideas are discussed in Section II-2.2.3. There we shall see that the general results can be proved exactly as we have done in this chapter for \mathbb{R} . The same comments hold for the notions of interior, closure, and derived set. Moreover, Propositions 2.5.13, 2.5.15, 2.5.18, 2.5.19, 2.5.20, and 2.5.21 carry over directly to \mathbb{R}^n , and indeed to the more general setting in Section II-2.2.3.

The notion of compactness, relying as it does only on the idea of an open set, is transferable from \mathbb{R} to \mathbb{R}^n , and indeed to the general setting of Chapter II-2 (see Section II-2.4). That is to say, the idea of an open cover of a subset of \mathbb{R}^n transfers directly from \mathbb{R} , and, therefore, the definition of a compact set as being a set for which every open cover possesses a finite subcover also generalises. Note too that one can say that a subset of \mathbb{R}^n is **bounded** if it is contained in a ball of sufficiently large radius. We shall have occasion to use the fact that the Heine–Borel Theorem generalises from \mathbb{R} to \mathbb{R}^n . This is proved as Theorem II-2.4.2. Thus, a subset of \mathbb{R}^n is compact if and only if it is closed and bounded. The Bolzano–Weierstrass Theorem also holds in \mathbb{R}^n , and indeed also in more general settings. The general case is given as Theorem II-2.4.1. For convenience, let us summarise here the characterisation of compact subsets of \mathbb{R}^n .

Theorem For a subset $K \subset \mathbb{R}^n$, the following are equivalent:

- (i) K is compact;
- (ii) K is closed and bounded;
- (iii) every sequence $\{\mathbf{x}_j\}_{j \in \mathbb{N}}$ in K has a subsequence which converges in K .

One can also talk about subsets of \mathbb{R}^n which have measure zero. This is done in the obvious way using $\|\cdot\|_{\mathbb{R}^n}$ in place of $|\cdot|$. Thus a subset $A \subset \mathbb{R}^n$ has **measure zero** if

$$\inf \left\{ \sum_{j=1}^n \text{vol}(\mathbf{B}(r_j, \mathbf{x}_j)) \mid A \subset \bigcup_{j \in \mathbb{N}} \mathbf{B}(r_j, \mathbf{x}_j) \right\},$$

where $\text{vol}(\mathbf{B}(r, \mathbf{x}))$ is the volume of the open ball $\mathbf{B}(r, \mathbf{x})$, which we recall is Ideas concerning what? the generalisation to \mathbb{R}^n of sets of measure zero are discussed in Section II-1.4.

Exercises

2.6.1

Section 2.7

Notes

2.7.1 Properties of the set of real numbers

The Archimedean property of \mathbb{R} seems obvious. The lack of the Archimedean property would mean that there exists t for which $t > N$ for every natural number N . This property is actually possessed by certain fields used in so-called “nonstandard analysis,” and we refer the interested reader to [Robinson 1974].

2.7.2 e and π

The numbers e and π are not only irrational, but have the much stronger property of being *transcendental*. This means that they are not the roots of any polynomial having integer coefficients. That e is transcendental was proved by Hermite¹² in 1873, and that π is transcendental was proved by Lindemann¹³ in 1882.

The proof we give for the irrationality of π is essentially that of Niven [1947]; this is the most commonly encountered proof, and is simpler than the original proof of Lambert¹⁴ presented to the Berlin Academy in 1768.

¹²Charles Hermite (1822–1901) was a French mathematician who made contributions to the fields of number theory, algebra, differential equations, and analysis.

¹³Carl Louis Ferdinand von Lindemann (1852–1939) was born in what is now Germany. His mathematical contributions were in the areas of analysis and geometry. He also was interested in physics.

¹⁴Johann Heinrich Lambert (1728–1777) was born in France. His mathematical work included contributions to analysis, geometry, and probability. He also made contributions to astronomical theory.

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