Topology Course Lecture Notes

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Chapter 1

Fundamental Concepts

In the study of metric spaces, we observed that:

(i) many of the concepts can be described purely in terms of open sets,

(ii) open-set descriptions are sometimes simpler than metric descriptions, e.g. continuity,

(iii) many results about these concepts can be proved using *only* the basic properties of open sets (namely, that both the empty set and the underlying set X are open, that the intersection of any two open sets is again open and that the union of arbitrarily many open sets is open).

This prompts the question: How far would we get if we started with a collection of subsets possessing these above-mentioned properties and proceeded to define everything in terms of them?

1.1 Describing Topological Spaces

We noted above that many important results in metric spaces can be proved using only the basic properties of open sets that

- the empty set and underlying set X are both open,
- the intersection of any two open sets is open, and
- unions of arbitrarily many open sets are open.

We will call any collection of sets on X satisfying these properties a topology. In the following section, we also seek to give alternative ways of describing this important collection of sets.

1.1.1 Defining Topological Spaces

Definition 1.1 *A* **topological space** *is a pair* $(X, \mathcal{T})$ *consisting of a set* X *and a family* $\mathcal{T}$ *of subsets of* X *satisfying the following conditions:*

(T1) $\emptyset \in \mathcal{T}$ *and* $X \in \mathcal{T}$

(T2) $\mathcal{T}$ *is closed under arbitrary union*

(T3) $\mathcal{T}$ *is closed under finite intersection.*

The set X is called a *space*, the elements of X are called *points* of the space and the subsets of X belonging to $\mathcal{T}$ are called **open** in the space; the family $\mathcal{T}$ of open subsets of X is also called a **topology** for X.

Examples

(i) Any metric space (X, d) is a topological space where $\mathcal{T}_d$, the topology for X induced by the metric d, is defined by agreeing that G shall be declared as open whenever each x in G is contained in an open ball entirely in G, i.e.

$$\emptyset \subset G \subseteq X \text{ is open in } (X, \mathcal{T}_d) \Leftrightarrow$$
$$\forall x \in G,\ \exists r_x > 0 \text{ such that } x \in B_{r_x}(x) \subseteq G.$$

(ii) The following is a special case of (i), above. Let R be the set of real numbers and let $\mathcal{I}$ be the usual (metric) topology defined by agreeing that

$$\emptyset \subset G \subseteq X \text{ is open in } (R, \mathcal{I}) \text{ (alternatively, } \mathcal{I}\text{-open)} \ \Leftrightarrow$$
$$\forall x \in G,\ \exists r_x > 0 \text{ such that } (x - r_x, x + r_x) \subset G.$$

(iii) Define $\mathcal{T}_0 = \{\emptyset, X\}$ for any set X — known as the *trivial* or *anti-discrete* topology.

(iv) Define $\mathcal{D} = \{G \subseteq X : G \subseteq X\}$ — known as the *discrete* topology.

(v) For any non-empty set X, the family $\mathcal{C} = \{G \subseteq X : G = \emptyset \text{ or } X \setminus G \text{ is finite}\}$ is a topology for X called the *cofinite* topology.

(vi) For any non-empty set X, the family $\mathcal{L} = \{G \subseteq X : G = \emptyset \text{ or } X \setminus G \text{ is countable}\}$ is a topology for X called the *cocountable* topology.

1.1.2 Neighbourhoods

Occasionally, arguments can be simplified when the sets involved are not "over-described". In particular, it is sometimes suffices to use sets which contain open sets but are not necessarily open. We call such sets neighborhoods.

Definition 1.2 *Given a topological space $(X, \mathcal{T})$ with $x \in X$, then $N \subseteq X$ is said to be a ($\mathcal{T}$)-***neighbourhood** *of $x \Leftrightarrow \exists$ open set G with $x \in G \subseteq N$.*

It follows then that a set $U \subseteq X$ is open iff for every $x \in U$, there exists a neighbourhood N_x of x contained in U. (Check this!)

Lemma 1.1 *Let $(X, \mathcal{T})$ be a topological space and, for each $x \in X$, let $\mathcal{N}(x)$ be the family of neighbourhoods of x. Then*

(i) $U \in \mathcal{N}(x) \Rightarrow x \in U$.

(ii) $\mathcal{N}(x)$ *is closed under finite intersections.*

(iii) $U \in \mathcal{N}(x)$ *and* $U \subseteq V \Rightarrow V \in \mathcal{N}(x)$.

(iv) $U \in \mathcal{N}(x) \Rightarrow \exists W \in \mathcal{N}(x)$ *such that* $W \subseteq U$ *and* $W \in \mathcal{N}(y)$ *for each* $y \in W$.

Proof Exercise!

Examples

(i) Let $x \in X$, and define $\mathcal{T}_x = \{\emptyset, \{x\}, X\}$. Then $\mathcal{T}_x$ is a topology for X and $V \subseteq X$ is a neighbourhood of x iff $x \in V$. However, the only nhd of $y \in X$ where $y \neq x$ is X itself

(ii) Let $x \in X$ and define a topology $\mathcal{I}(x)$ for X as follows:

$$\mathcal{I}(x) = \{G \subseteq X : x \in G\} \cup \{\emptyset\}.$$

Note here that *every* nhd of a point in X is open.

(iii) Let $x \in X$ and define a topology $\mathcal{E}(x)$ for X as follows:

$$\mathcal{E}(x) = \{G \subseteq X : x \notin G\} \cup \{X\}.$$

Note here that $\{y\}$ is open for every $y \neq x$ in X, that $\{x, y\}$ is *not* open, is not a nhd of x yet *is* a nhd of y.

In fact, the only nhd of x is X.

1.1.3 Bases and Subbases

It often happens that the open sets of a space can be very complicated and *yet* they can all be described using a selection of fairly simple special ones. When this happens, the set of simple open sets is called a **base** or **subbase** (depending on how the description is to be done). In addition, it is fortunate that many topological concepts can be characterized in terms of these simpler base or subbase elements.

Definition 1.3 *Let $(X, \mathcal{T})$ be a topological space. A family $\mathcal{B} \subseteq \mathcal{T}$ is called a* **base for** *$(X, \mathcal{T})$ if and only if every non-empty open subset of X can be represented as a union of a subfamily of $\mathcal{B}$.*

It is easily verified that $\mathcal{B} \subseteq \mathcal{T}$ is a base for $(X, \mathcal{T})$ if and only if whenever $x \in G \in \mathcal{T}, \exists B \in \mathcal{B}$ such that $x \in B \subseteq G$.
Clearly, a topological space can have many bases.

Lemma 1.2 *If $\mathcal{B}$ is a family of subsets of a set X such that*

(B1) for any $B_1, B_2 \in \mathcal{B}$ and every point $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ with $x \in B_3 \subseteq B_1 \cap B_2$, and

(B2) for every $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$,

then $\mathcal{B}$ is a base for a unique topology on X.
Conversely, any base $\mathcal{B}$ for a topological space $(X, \mathcal{T})$ satisfies (B1) and (B2).

Proof (Exercise!)

Definition 1.4 *Let $(X, \mathcal{T})$ be a topological space. A family $\mathcal{S} \subseteq \mathcal{T}$ is called a* **subbase for** *$(X, \mathcal{T})$ if and only if the family of all finite intersections $\cap_{i=1}^{k} U_i$, where $U_i \in \mathcal{S}$ for $i = 1, 2, \ldots, k$ is a base for $(X, \mathcal{T})$.*

Examples

(i) In any metric space (X, d), $\{B_r(x) : x \in X, r > 0\}$ forms a base for the induced metric topology $\mathcal{T}_d$ on X.

(ii) For the real line R with its usual (Euclidean) topology, the family $\{(a, b) : a, b \in Q, a < b\}$ is a base.

(iii) For an arbitrary set X, the family $\{\{x\} : x \in X\}$ is a base for $(X, \mathcal{D})$.

(iv) The family of all 'semi-infinite' open intervals (a, ∞) and $(-\infty, b)$ in R is a subbase for $(R, \mathcal{I})$.

1.1.4 Generating Topologies

From the above examples, it follows that for a set X one can select in many different ways a family $\mathcal{T}$ such that $(X, \mathcal{T})$ is a topological space. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are two topologies for X and $\mathcal{T}_2 \subseteq \mathcal{T}_1$, then we say that the topology $\mathcal{T}_1$ is finer than the topology $\mathcal{T}_2$, or that $\mathcal{T}_2$ is coarser than the topology $\mathcal{T}_1$. The discrete topology for X is the finest one; the trivial topology is the coarsest. If X is an arbitrary infinite set with distinct points x and y, then one can readily verify that the topologies $\mathcal{I}(x)$ and $\mathcal{I}(y)$ are incomparable i.e. neither is finer than the other.

By *generating a topology* for X, we mean selecting a family $\mathcal{T}$ of subsets of X which satisfies conditions (T1)–(T3). Often it is more convenient *not* to describe the family $\mathcal{T}$ of open sets directly. The concept of a base offers an alternative method of generating topologies.

Examples

- [Sorgenfrey line] Given the real numbers R, let $\mathcal{B}$ be the family of all intervals $[x, r)$ where $x, r \in R$, $x < r$ and r is rational. One can readily check that $\mathcal{B}$ has properties (B1)–(B2). The space R_s, generated by $\mathcal{B}$, is called the *Sorgenfrey line* and has many interesting properties. Note that the Sorgenfrey topology is finer than the Euclidean topology on R. (Check!)

- [Niemytzki plane] Let L denote the closed upper half-plane. We define a topology for L by declaring the basic open sets to be the following:

(I) the (Euclidean) open discs in the upper half-plane;

(II) the (Euclidean) open discs tangent to the 'edge' of the L, together with the point of tangency.

Note If $y_n \to y$ in L, then

(i) y not on 'edge': same as Euclidean convergence.

(ii) y on the 'edge': same as Euclidean, but y_n must approach y from 'inside'. Thus, for example, $y_n = (\frac{1}{n}, 0) \not\to (0, 0)$!

1.1.5 New Spaces from Old

A subset of a topological space inherits a topology of its own, in an obvious way:

Definition 1.5 *Given a topological space* $(X, \mathcal{T})$ *with* $A \subseteq X$*, then the family* $\mathcal{T}_A = \{A \cap G : G \in \mathcal{T}\}$ *is a topology for* A*, called the* **subspace** *(or* **relative** *or* **induced***) topology for* A*.* $(A, \mathcal{T}_A)$ *is called a subspace of* $(X, \mathcal{T})$*.*

Example
The interval $I = [0, 1]$ with its natural (Euclidean) topology is a (closed) subspace of $(R, \mathcal{I})$.
Warning: Although this definition, and several of the results which flow from it, may suggest that subspaces in general topology are going to be 'easy' in the sense that a lot of the structure just gets traced onto the subset, there is unfortunately a rich source of mistakes here also: because we are handling two topologies at once. When we inspect a subset B of A, and refer to it as 'open' (or 'closed', or a 'neighbourhood' of some point p ....) we must be exceedingly careful as to *which* topology is intended. For instance, in the previous example, $[0, 1]$ itself is open in the subspace topology on I but, of course, not in the 'background' topology of R. In such circumstances, it is advisable to *specify* the topology being used each time by saying $\mathcal{T}$-open, $\mathcal{T}_A$-open, and so on.

1.2 Closed sets and Closure

Just as many concepts in metric spaces were described in terms of basic open sets, yet others were characterized in terms of closed sets. In this section we

- define closed sets in a general topological space and
- examine the related notion of the closure of a given set.

1.2.1 Closed Sets

Definition 1.6 *Given a topological space* $(X, \mathcal{T})$ *with* $F \subseteq X$*, then* F *is said to be* $\mathcal{T}$*-closed iff its complement* $X \setminus F$ *is* $\mathcal{T}$*-open.*

From De Morgan's Laws and properties (T1)–(T3) of open sets, we infer that the family $\mathcal{F}$ of closed sets of a space has the following properties:

(F1) $X \in \mathcal{F}$ and $\emptyset \in \mathcal{F}$

(F2) $\mathcal{F}$ is closed under finite union

(F3) $\mathcal{F}$ is closed under arbitrary intersection.

Sets which are simultaneously closed *and* open in a topological space are sometimes referred to as **clopen** sets. For example, members of the base $\mathcal{B} = \{[x, r) : x,\ r \in R,\ x < r,\ r \text{ rational }\}$ for the Sorgenfrey line are clopen with respect to the topology generated by $\mathcal{B}$. Indeed, for the discrete space $(X, \mathcal{D})$, *every* subset is clopen.

1.2.2 Closure of Sets

Definition 1.7 *If* $(X, \mathcal{T})$ *is a topological space and* $A \subseteq X$*, then*

$$\overline{A}^{\mathcal{T}} = \cap\{F \subseteq X : A \subseteq F \text{ and } F \text{ is closed}\}$$

is called the $\mathcal{T}$**-closure** *of* A*.*

Evidently, $\overline{A}^{\mathcal{T}}$ (or $\overline{A}$ when there is no danger of ambiguity) is the smallest closed subset of X which contains A. Note that A is closed $\Leftrightarrow A = \overline{A}$.

Lemma 1.3 *If* $(X, \mathcal{T})$ *is a topological space with* A*,* $B \subseteq X$*, then*

(i) $\bar{\emptyset} = \emptyset$

(ii) $A \subseteq \bar{A}$

(iii) $\bar{\bar{A}} = \bar{A}$

(iv) $\overline{A \cup B} = \bar{A} \cup \bar{B}$.

Proof Exercise!

Theorem 1.1 *Given a topological space with* $A \subseteq X$*, then* $x \in \bar{A}$ *iff for* each *nhd* U *of* x*,* $U \cap A \neq \emptyset$*.*

Proof

$\Rightarrow$: Let $x \in \bar{A}$ and let U be a nhd of x; then there exists open G with $x \in G \subseteq U$. If $U \cap A = \emptyset$, then $G \cap A = \emptyset$ and so $A \subseteq X \backslash G \Rightarrow \bar{A} \subseteq X \backslash G$ whence $x \in X \backslash G$, thereby contradicting the assumption that $U \cap A = \emptyset$.

$\Leftarrow$: If $x \in X \setminus \bar{A}$, then $X \setminus \bar{A}$ is an open nhd of x so that, by hypothesis, $(X \setminus \bar{A}) \cap A \neq \emptyset$, which is a contradiction (i.e., a false statement).

Examples

(i) For an arbitrary infinite set X with the cofinite topology $\mathcal{C}$, the closed sets are just the finite ones together with X. So for any $A \subseteq X$,

$$\bar{A} = \begin{cases} A & \text{if } A \text{ is finite} \\ X & \text{if } A \text{ is infinite} \end{cases}$$

Note that any two non-empty open subsets of X have non-empty intersection.

(ii) For an arbitrary uncountable set X with the cocountable topology $\mathcal{L}$, the closed sets are the countable ones and X itself. Note that if we let $X = R$, then $\overline{[0,1]} = R$! (In the usual Euclidean topology, $\overline{[0,1]} = [0,1]$.)

(iii) For the space $(X, \mathcal{T}_x)$ defined earlier, if $\emptyset \subset A \subseteq X$, then

$$\bar{A} = \begin{cases} X & \text{if } x \in A \\ X \setminus \{x\} & \text{if } x \notin A \end{cases}$$

(iv) For $(X, \mathcal{I}(x))$ with $A \subseteq X$,

$$\bar{A} = \begin{cases} A & \text{if } x \notin A \\ X & \text{if } x \in A \end{cases}$$

(v) For $(X, \mathcal{E}(x))$ with $\emptyset \subset A \subseteq X$,

$$\bar{A} = \begin{cases} A & \text{if } x \in A \\ A \cup \{x\} & \text{if } x \notin A \end{cases}$$

(vi) In $(X, \mathcal{D})$, every subset equals its own closure.

1.3 Continuity and Homeomorphism

The central notion of continuity of functions is extended in this section to general topological spaces. The useful characterization of continuous functions in metric spaces as those functions where the inverse image of every open set is open is used as a definition in the general setting.
Because many properties of spaces are preserved by continuous functions, spaces related by a bijection (one-to-one and onto function) which is continuous in both directions will have many properties in common. These properties are identified as *topological properties*. Spaces so related are called *homeomorphic*.

1.3.1 Continuity

The primitive intuition of a continuous process is that of one in which small changes in the input produce small, 'non-catastrophic' changes in the corresponding output. This idea formalizes easily and naturally for mappings from one *metric* space to another: f is continuous at a point p in such a setting whenever we can force the distance between $f(x)$ and $f(p)$ to be as small as is desired, merely by taking the distance between x and p to be small enough. That form of definition is useless in the absence of a properly defined 'distance' function but, fortunately, it is equivalent to the demand that the preimage of each open subset of the target metric space shall be open in the domain. Thus expressed, the idea is immediately transferrable to general topology:

Definition 1.8 *Let $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ be topological spaces; a mapping $f : X \to Y$ is called* **continuous** *iff $f^{-1}(U) \in \mathcal{T}$ for each $U \in \mathcal{S}$ i.e. the inverse image of any open subset of Y is open in X.*

Examples

(i) If $(X, \mathcal{D})$ is discrete and $(Y, \mathcal{S})$ is an arbitrary topological space, then *any* function $f : X \to Y$ is continuous!

Again, if $(X, \mathcal{T})$ is an arbitrary topological space and $(Y, \mathcal{T}_0)$ is trivial, *any* mapping $g : X \to Y$ is continuous.

(ii) If $(X, \mathcal{T})$, $(Y, \mathcal{S})$ are arbitrary topological spaces and $f : X \to Y$ is a constant map, then f is continuous.

(iii) Let X be an arbitrary set having more than two elements, with $x \in X$. Let $\mathcal{T} = \mathcal{I}(x)$, $\mathcal{S} = \mathcal{T}_x$ in the definition of continuity; then the identity map $id_X : X \to X$ is continuous. However, if we interchange $\mathcal{T}$ with $\mathcal{S}$ so that $\mathcal{T} = \mathcal{T}_x$ and $\mathcal{S} = \mathcal{I}(x)$, then $id_X : X \to X$ is *not* continuous! Note that $id_X : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ is continuous if and only if $\mathcal{T}_1$ is finer than $\mathcal{T}_2$.

Theorem 1.2 *If $(X_1, \mathcal{T}_1)$, $(X_2, \mathcal{T}_2)$ and $(X_3, \mathcal{T}_3)$ are topological spaces and $h : X_1 \to X_2$ and $g : X_2 \to X_3$ are continuous, then $g \circ h : X_1 \to X_3$ is continuous.*

Proof Immediate.

There are several different ways to 'recognise' continuity for a mapping between topological spaces, of which the next theorem indicates two of the most useful apart from the definition itself:

Theorem 1.3 *Let f be a mapping from a topological space $(X_1, \mathcal{T}_1)$ to a topological space $(X_2, \mathcal{T}_2)$. The following statements are equivalent:*

(i) f is continuous,

(ii) the preimage under f of each closed subset of X_2 is closed in X_1,

(iii) for every subset A of X_1, $f(\bar{A}) \subseteq \overline{f(A)}$.

Proof It is easy to see that (i) implies (ii). Assuming that (ii) holds, apply it to the closed set $\overline{f(A)}$ and (iii) readily follows. Now if (iii) is assumed and G is a given open subset of X_2, use (iii) on the set $A = X_1 \setminus f^{-1}(G)$ and verify that it follows that $f^{-1}(G)$ must be open.

1.3.2 Homeomorphism

Definition 1.9 *Let* $(X, \mathcal{T})$, $(Y, \mathcal{S})$ *be topological spaces and let* $h : X \to Y$ *be bijective. Then* h *is a* **homeomorphism** *iff* h *is continuous* and h^{-1} *is continuous. If such a map exists,* $(X, \mathcal{T})$ *and* $(Y, \mathcal{S})$ *are called* **homeomorphic**.

Such a map has the property that

$$G \in \mathcal{T} \Leftrightarrow f(G) \in \mathcal{S}.$$

It follows that any statement about a topological space which is ultimately expressible solely in terms of the open sets (together with set-theoretic relations and operations) will be true for both $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ if it is true for either. In other words, $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ are indistinguishable as topological spaces. The reader who has had abstract algebra will note that homeomorphism is the analogy in the setting of topological spaces and continuous functions to the notion of isomorphism in the setting of groups (or rings) and homomorphisms, and to that of linear isomorphism in the context of vector spaces and linear maps.

Example

For every space $(X, \mathcal{T})$, the identity mapping $id_X : X \to X$ is a homeomorphism.

A property of topological spaces which when possessed by a space is also possessed by every space homeomorphic to it is called a **topological invariant**. We shall meet some examples of such properties later.

One can readily verify that if f is a homeomorphism, then the inverse mapping f^{-1} is also a homeomorphism and that the composition $g \circ f$ of two homeomorphisms f and g is again a homeomorphism. Thus, the relation 'X and Y are homeomorphic' is an equivalence relation.

In general, it may be quite difficult to demonstrate that two spaces are homeomorphic (unless a homeomorphism is obvious or can easily be discovered). For example, to verify that $(R, \mathcal{I})$ is homeomorphic to $(0, 1)$ with its induced metric topology, it is necessary to demonstrate, for instance, that $h : (0, 1) \to R$ where $h(x) = \frac{2x-1}{x(x-1)}$ is a homeomorphism.

It is often easier to show that two spaces are *not* homeomorphic: simply exhibit an invariant which is possessed by one space and not the other.

Example

The spaces $(X, \mathcal{I}(x))$ and $(X, \mathcal{E}(x))$ are *not* homeomorphic since, for example,

$(X, \mathcal{I}(x))$ has the topological invariant 'each nhd is open' while $(X, \mathcal{E}(x))$ does not.

1.4 Additional Observations

Definition 1.10 *A* sequence (x_n) *in a topological space* $(X, \mathcal{T})$ *is said to converge to a point* $x \in X$ *iff* (x_n) *eventually belongs to every nhd of* x *i.e. iff for every nhd* U *of* x*, there exists* $n_0 \in N$ *such that* $x_n \in U$ *for all* $n \geq n_0$.

Caution
We learnt that, for metric spaces, sequential convergence was adequate to describe the topology of such spaces (in the sense that the basic primitives of 'open set', 'neighbourhood', 'closure' etc. could be fully characterised in terms of sequential convergence). However, for general topological spaces, sequential convergence fails. We illustrate:

(i) Limits are *not* always unique. For example, in $(X, \mathcal{T}_0)$, *each* sequence (x_n) converges to *every* $x \in X$.

(ii) In R with the cocountable topology $\mathcal{L}$, $[0, 1]$ is *not* closed and so $G = (-\infty, 0) \cup (1, \infty)$ is not open — yet if $x_n \to x$ where $x \in G$, then Assignment 1 shows that $x_n \in G$ for all sufficiently large n. Further, $2 \in \overline{[0, 1]}^{\mathcal{L}}$, yet no sequence in $[0, 1]$ can approach 2. So another characterisation fails to carry over from metric space theory.

Finally, every $\mathcal{L}$-convergent sequence of points in $[0, 1]$ must have its limit *in* $[0, 1]$ — but $[0, 1]$ is *not* closed (in $\mathcal{L}$)!

Hence, to discuss topological convergence thoroughly, we need to develop a new basic set-theoretic tool which generalises the notion of sequence. It is called a **net** — we shall return to this later.

Definition 1.11 *A topological space* $(X, \mathcal{T})$ *is called* **metrizable** *iff there exists a metric* d *on* X *such that the topology* $\mathcal{T}_d$ *induced by* d *coincides with the original topology* $\mathcal{T}$ *on* X.

The investigations above show that $(X, \mathcal{T}_0)$ and $(R, \mathcal{L})$ are examples of non-metrizable spaces. However, the discrete space $(X, \mathcal{D})$ *is* metrizable, being induced by the discrete metric

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

Chapter 2

Topological Properties

We explained in the previous chapter what a topological property (homeomorphic invariant) is but gave few good examples. We now explore some of the most important ones. Recurring themes will be:

- When do subspaces inherit the property?
- How do continuous maps relate to the property?
- Does the property behave specially in metric spaces?

2.1 Compactness

We all recall the important and useful theorem from calculus, that functions which are continuous on a closed and bounded interval take on a maximum and minimum value on that interval. The classic theorem of Heine-Borel-Lebesgue asserts that every covering of such an interval by open sets has a finite subcover. In this section, we use this feature of closed and bounded subsets to define the corresponding notion, *compactness*, in a general topological space. In addition, we consider important variants of this notion: sequential compactness and local compactness.

2.1.1 Compactness Defined

Given a set X with $A \subseteq X$, a *cover* for A is a family of subsets $\mathcal{U} = \{U_i :\in I\}$ of X such that $A \subseteq \cup_{i \in I} U_i$. A *subcover* of a given cover $\mathcal{U}$ for A is a subfamily $\mathcal{V} \subset \mathcal{U}$ which still forms a cover for A.

If A is a subspace of a space $(X, \mathcal{T})$, $\mathcal{U}$ is an *open cover* for A iff $\mathcal{U}$ is a cover for A such that each member of $\mathcal{U}$ is open in X.
The classic theorem of Heine-Borel-Lebesgue asserts that, in R, every open cover of a closed bounded subset has a finite subcover. This theorem has extraordinarily profound consequences and like most good theorems, its conclusion has become a definition.

Definition 2.1 *$(X, \mathcal{T})$ is said to be* **compact** *iff every open cover of X has a finite subcover.*

Theorem 2.1 (Alexander's Subbase Theorem) *Let $\mathcal{S}$ be any subbase for $(X, \mathcal{T})$. If every open cover of X by members of $\mathcal{S}$ has a finite subcover, then X is compact.*

The proof of this deep result is an application of Zorn's lemma, and is not an exercise for the faint-hearted!

Examples

(i) $(R, \mathcal{I})$ is *not* compact, for consider $\mathcal{U} = \{(-n, n) : n \in N\}$. Similarly, $(C, \mathcal{T}_{usual})$ is not compact.

(ii) $(0, 1)$ is *not* compact, for consider $\mathcal{U} = \{(\frac{1}{n}, 1) : n \geq 2\}$.

(iii) $(X, \mathcal{C})$ *is* compact, for any X.

(iv) Given $x \in X$, $(X, \mathcal{E}(x))$ is compact; $(X, \mathcal{I}(x))$ is *not* compact unless X is finite.

(v) $\mathcal{T}$ finite for *any* $X \Rightarrow (X, \mathcal{T})$ compact.

(vi) X finite, $\mathcal{T}$ any topology for $X \Rightarrow (X, \mathcal{T})$ compact.

(vii) X infinite $\Rightarrow (X, \mathcal{D})$ not compact.

(viii) Given $(X, \mathcal{T})$, if (x_n) is a sequence in X convergent to x, then $\{x_n : n \in N\} \cup \{x\}$ is compact.

2.1.2 Compactness for Subspaces

We call a subset A of $(X, \mathcal{T})$ a *compact subset* when the subspace $(A, \mathcal{T}_A)$ is a compact space. It's a nuisance to have to look at $\mathcal{T}_A$ in order to decide on this. It would be easier to use the original $\mathcal{T}$. Thankfully, we can!

Lemma 2.1 *A is a compact subset of* $(X, \mathcal{T})$ *iff every* $\mathcal{T}$-open *cover of A has a finite subcover.*

Proof Exercise.

Lemma 2.2 *Compactness is closed-hereditary and preserved by continuous maps.*

Proof Exercise.
Example
The unit circle in R^2 is compact; indeed, paths in any space are compact.

2.1.3 Compactness in Metric Spaces

In any metric space (M, d), every compact subset K is closed and bounded: (bounded, since given any $x_0 \in M$,

$$\begin{aligned} K &\subseteq B(x_0, 1) \cup B(x_0, 2) \cup B(x_0, 3) \cup \cdots \\ \Rightarrow K &\subseteq \cup_{i=1}^{j} B(x_0, n_i) \end{aligned}$$

where we can arrange $n_1 < n_2 < \ldots < n_j$. Thus $K \subseteq B(x_0, n_j)$ and so any two points of K lie within n_j of x_0 and hence within $2n_j$ of each other i.e. K is bounded.
K is closed, since if $x \in \bar{K}$ and $x \notin K$, then for each $y \in K$, $d_y = \frac{1}{2} d(x, y) > 0$ so we may form the (open) cover of K as follows: $\{B(y, d_y) : y \in K\}$ which reduces to a finite subcover $\{B(y_i, d_{y_i}) : y_i \in K, i = 1, \ldots, n\}$. The corresponding neighbourhoods of x, namely $B(x, d_{y_i})$, $i = 1, \ldots, n$, may be intersected giving a neighbourhood of x which misses K —contradiction!)
Neither half is valid in *all* topological spaces;

- 'compact $\Rightarrow$ bounded' doesn't even make sense since 'bounded' depends on the metric.

- 'compact $\Rightarrow$ closed' makes sense but is not always true. For example, in $(R, \mathcal{C})$, $(0,1)$ is not closed yet it *is* compact (since its topology is the cofinite topology!)

Further, in a metric space, a closed bounded subset needn't be compact (e.g. consider M with the discrete metric and let $A \subseteq M$ be infinite; then A is closed, bounded (since $A \subseteq B(x,2) = M$ for any $x \in M$), yet it is certainly not compact! Alternatively, the subspace $(0,1)$ is closed (in itself), bounded, but not compact.)
However, the Heine-Borel theorem asserts that such is the case for R and R^n; the following is a special case of the theorem:

Theorem 2.2 *Every closed, bounded interval* $[a,b]$ *in* R *is compact.*

Proof Let $\mathcal{U}$ be any open cover of $[a,b]$ and let $K = \{x \in [a,b] : [a,x]$ is covered by a finite subfamily of $\mathcal{U}\}$. Note that if $x \in K$ and $a \leq y \leq x$, then $y \in K$. Clearly, $K \neq \emptyset$ since $a \in K$. Moreover, given $x \in K$, there exists $\delta_x > 0$ such that $[x, x+\delta_x) \subseteq K$ (since $x \in$ some open $U \in$ chosen finite subcover of $\mathcal{U}$). Since K is bounded, $k^* = \sup K$ exists.

(i) $k^* \in K$: Choose $U \in \mathcal{U}$ such that $k^* \in U$; then there exists $\epsilon > 0$ such that $(k^* - \epsilon, k^*] \subseteq U$. Since there exists $x \in K$ such that $k^* - \epsilon < x < k^*$, $k^* \in K$. *Note*

(ii) $k^* = b$: If $k^* < b$, choose $U \in \mathcal{U}$ with $k^* \in U$ and note that $[k^*, k^* + \delta) \subseteq U$ for some $\delta > 0$ —contradiction!

An alternative proof [Willard, Page 116] is to invoke the connected nature of $[a,b]$ by showing K is clopen in $[a,b]$.

Theorem 2.3 *Any continuous map from a compact space into a metric space is bounded.*

Proof Immediate.

Corollary 2.1 *If* $(X, \mathcal{T})$ *is compact and* $f : X \to R$ *is continuous, then* f *is bounded and attains its bounds.*

Proof Clearly, f is bounded. Let $m = \sup f(X)$ and $l = \inf f(X)$; we must prove that $m \in f(X)$ and $l \in f(X)$. Suppose that $m \notin f(X)$. Since

$\overline{f(X)} = f(X)$, then there exists $\epsilon > 0$ such that $(m-\epsilon, m+\epsilon) \cap f(X) = \emptyset$ i.e. for all $x \in X$, $f(x) \le m - \epsilon \ldots$contra!
Similarly, if $l \notin f(X)$, then there exists $\epsilon > 0$ such that $[l, l+\epsilon) \cap f(X) = \emptyset$ whence $l + \epsilon$ is a lower bound for $f(X)$!

2.1.4 Sequential Compactness

Definition 2.2 *A topological space $(X, \mathcal{T})$ is said to be* **sequentially compact** *if and only if every sequence in X has a convergent subsequence.*

Recall from Chapter 1 the definition of convergence of sequences in topological spaces and the cautionary remarks accompanying it. There we noted that, contrary to the metric space situation, sequences in topology can have *several different* limits! Consider, for example, $(X, \mathcal{T}_0)$ and $(R, \mathcal{L})$. In the latter space, if $x_n \to l$, then $x_n = l$ for all $n \geq$ some n_0. Thus the sequence $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ does *not* converge in $(R, \mathcal{L})$!

Lemma 2.3 *Sequential compactness is closed-hereditary and preserved by continuous maps.*

Proof Exercise.
We shall prove in the next section that in metric spaces, sequential compactness and compactness are equivalent!

Definition 2.3 *Given a topological space $(X, \mathcal{T})$, a subset A of X and $x \in X$, x is said to be an* **accumulation point of A** *iff every neighbourhood of x contains infinitely many points of A.*

Lemma 2.4 *Given a compact space $(X, \mathcal{T})$ with an infinite subset A of X, then A* has *an accumulation point.*

Proof Suppose not; then for each $x \in X$, there exists a neighbourhood N_x of x such that $N_x \cap A$ is (at most) finite; the family $\{N_x : x \in X\}$ is an open cover of X and so has a finite subcover $\{N_{x_i} : i = 1, \ldots, n\}$. But $A \subseteq X$ and A is infinite, whence

$$A = A \cap X = A \cap (\cup N_{x_i}) = \cup_{i=1}^{n} (A \cap N_{x_i})$$

is finite!

Lemma 2.5 *Given a sequentially compact* metric *space* (M, d) *and* $\epsilon > 0$, *there is a* finite *number of open balls, radius* ϵ, *which cover* M.

Proof Suppose not and that for some $\epsilon > 0$, there exists *no* finite family of open balls, radius ϵ, covering M. We derive a contradiction by constructing a sequence (x_n) inductively such that $d(x_m, x_n) \geq \epsilon$ for all n, m $(n \neq m)$, whence no subsequence is even Cauchy!

Let $x_1 \in M$ and suppose inductively that $x_1, \ldots, x_k$ have been chosen in M such that $d(x_i, x_j) \geq \epsilon$ for all $i, j \leq k, i \neq j$.By hypothesis, $\{B(x_i, \epsilon) : i = 1, \ldots, k\}$ is not an (open) cover of M and so there exists $x_{k+1} \in M$ such that $d(x_{k+1}, x_i) \geq \epsilon$ for $1 \leq i \leq k$. We thus construct the required sequence (x_n), which clearly has no convergent subsequence.

Theorem 2.4 *A metric space is compact iff it is sequentially compact.*

Proof

$\Rightarrow$: Suppose (M, d) is compact. Given any sequence (x_n) in M, either $A = \{x_1, x_2, \ldots\}$ is finite or it is infinite. If A is finite, there must be at least one point l in A which occurs infinitely often in the sequence and its occurrences form a subsequence converging to l. If A is infinite, then by the previous lemma there exists $x \in X$ such that every neighbourhood of x contains infinitely many points of A.

For each $k \in \omega$, $B(x, \frac{1}{k})$ contains infinitely many x_n's: select one, call it x_{n_k}, making sure that $n_k > n_{k-1} > n_{k-2} \ldots$. We have a subsequence $(x_{n_1}, x_{n_2}, \ldots, x_{n_k}, \ldots)$ so that $d(x, x_{n_k}) < \frac{1}{k} \to 0$ i.e. $x_{n_k} \to x$. Thus in either case there exists a convergent subsequence and so (M, d) is sequentially compact.

$\Leftarrow$: Conversely, suppose (M, d) is sequentially compact and *not* compact. Then there exists some open cover $\{G_i : i \in I\}$ of M having *no* finite subcover. By Lemma 2.5, with $\epsilon = \frac{1}{n}$ $(n \in \omega)$, we can cover M by a *finite* number of balls of radius $\frac{1}{n}$. For each n, there has to be one of these, say $B(x_n, \frac{1}{n})$, which cannot be covered by any finite number of the sets G_i. The sequence (x_n) must have a convergent subsequence (x_{n_k}) which converges to a limit l. Yet $\{G_i : i \in I\}$ covers M, so $l \in$ some G_{i_0}, say.

As $k \to \infty$, $x_{n_k} \to l$; but also $\frac{1}{n_k} \to 0$ and $1/n_k$ is the radius of the ball centred on x_{n_k}. So eventually $B(x_{n_k}, \frac{1}{n_k})$ is inside G_{i_0}, contradictory to their choice! (More rigorously, there exists $m \in \omega$ such that $B(l, \frac{2}{m}) \subseteq G_{i_0}$. Now $B(l, \frac{1}{m})$ contains x_{n_k} for all $k \geq k_0$ say, so choose $k \geq k_0$ such that $n_k \geq m$. Then $B(x_{n_k}, \frac{1}{n_k}) \subseteq B(l, \frac{2}{m}) \subseteq G_{i_0}$.) Hence, M is compact.

2.1.5 Compactness and Uniform Continuity

Recall that a map $f : (X_1, d_1) \to (X_2, d_2)$, where (X_i, d_i) is a metric space for each i, is *uniformly continuous on* X_i if given any $\epsilon > 0, \exists \delta > 0$ such that $d_1(x, y) < \delta$ for $x, y \in X_1 \Rightarrow d_2(f(x), f(y)) < \epsilon$.
Ordinary continuity of f is a local property, while uniform continuity is a global property since it says something about the behaviour of f over the whole space X_1. Since compactness allows us to pass from the local to the global, the next result is not surprising:

Theorem 2.5 *If (X, d) is a compact metric space and $f : X \to R$ is continuous, then f is uniformly continuous on X.*

Note Result holds for any metric space codomain.
Proof Let $\epsilon > 0$; since f is continuous, for each $x \in X$, $\exists \delta_x > 0$ such that $d(x, y) < 2\delta_x \Rightarrow |f(x) - f(y)| < \frac{\epsilon}{2}$. The family $\{B_{\delta_x}(x) : x \in X\}$ is an open cover of X and so has a finite subcover $\{B_{\delta_{x_i}}(x_i) : i = 1, \ldots, n\}$ of X. Let $\delta = \min\{\delta_{x_i} : i = 1, \ldots, n\}$; then, given $x, y \in X$ such that $d(x, y) < \delta$, it follows that $|f(x) - f(y)| < \epsilon$
(for $x \in B_{\delta_{x_i}}(x_i)$ for some i, whence $d(x, x_i) < \delta_{x_i}$ and so $d(y, x_i) \leq d(y, x) + d(x, x_i) < \delta + \delta_{x_i} \leq 2\delta_{x_i} \Rightarrow |f(y) - f(x_i)| < \frac{\epsilon}{2}$.
Thus $|f(x) - f(y)| \leq |f(x) - f(x_i)| + |f(x_i) - f(y)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$).
Note Compactness is not a necessary condition on the domain for uniform continuity. For example, for *any* metric space (X, d), let $f : X \to X$ be the identity map. Then f is easily seen to be uniformly continuous on X.

2.1.6 Local Compactness

Definition 2.4 *A topological space $(X, \mathcal{T})$ is* **locally compact** *iff each point of X has a compact neighbourhood.*

Clearly, every compact space is locally compact. However, the converse is *not* true.
Examples

(i) With X infinite, the discrete space $(X, \mathcal{D})$ is clearly locally compact (for each $x \in X$, $\{x\}$ is a compact neighbourhood of x!) but *not* compact.

(ii) With X infinite and $x \in X$, $(X, \mathcal{I}(x))$ is locally compact (but not compact).

(iii) $(R, \mathcal{I})$ is locally compact ($x \in R \Rightarrow [x-1, x+1]$ is a compact neighbourhood of x).

(iv) The set of rational numbers Q with its usual topology is *not* a locally compact space, for suppose otherwise; then 0 has a compact neighbourhood C in Q so we can choose $\epsilon > 0$ such that $J = Q \cap [-\epsilon, \epsilon] \subseteq C$. Now J is closed in (compact) C and is therefore compact in R. Thus, J must be closed in R—but $\bar{J}^R = [-\epsilon, \epsilon]$!

Lemma 2.6 *(i) Local compactness is closed-hereditary.*

(ii) Local compactness is preserved by continuous open *maps — it is not preserved by continuous maps in general. Consider $id_Q : (Q, \mathcal{D}) \to (Q, \mathcal{I}_Q)$ which is continuous and onto; $(Q, \mathcal{D})$ is locally compact while $(Q, \mathcal{I}_Q)$ isn't!*

Proof Exercise.

2.2 Other Covering Conditions

Definition 2.5 *A topological space $(X, \mathcal{T})$ is said to be*

(i) **Lindelöf** *iff every open cover of X has a countable subcover*

(ii) **countably compact** *iff every countable open cover of X has a finite subcover.*

Thus, a space is compact precisely when it is both Lindelöf and countably compact. Further, every sequentially compact space is countably compact, although the converse is not true. Moreover, sequential compactness neither implies nor is implied by compactness.
However, for metric spaces, or more generally, metrizable spaces, the conditions compact, countably compact and sequentially compact are equivalent.
Note Second countable $\Rightarrow$ separable; separable + metrizable $\Rightarrow$ second countable ... and so in metrizable spaces, second countability and separability are equivalent.

2.3 Connectedness

It is not terribly hard to know when a set on the real line is connected, or 'of just one piece.' This notion is extended to general topological spaces in this section and alternative characterizations of the notion are given. In addition the relationship between continuous maps and and connectedness is given. This provides an elegant restatement of the familiar Intermediate Value Theorem from first term calculus.

2.3.1 Definition of Connectedness

A *partition* of $(X, \mathcal{T})$ means a pair of disjoint, non-empty, $\mathcal{T}$-open subsets whose union is X. Notice that, since these sets are complements of one another, they are both closed as well as both open. Indeed, the definition of 'partition' is not affected by replacing the term 'open' by 'closed'.

Definition 2.6 *A* **connected** *space* $(X, \mathcal{T})$ *is one which has* no *partition. (Otherwise,* $(X, \mathcal{T})$ *is said to be* **disconnected***.)*
If $\emptyset \neq A \subseteq (X, \mathcal{T})$*, we call* A *a connected set in* X *whenever* $(A, \mathcal{T}_A)$ *is a connected space.*

Lemma 2.7 $(X, \mathcal{T})$ *is connected iff* X *and* $\emptyset$ *are the only subsets which are clopen.*

Examples

(i) $(X, \mathcal{T}_0)$ is connected.

(ii) $(X, \mathcal{D})$ cannot be connected unless $|X| = 1$. (Indeed the only connected subsets are the singletons!)

(iii) The Sorgenfrey line R_s is disconnected (for $[x, \infty)$ is clopen!).

(iv) The subspace Q of $(R, \mathcal{I})$ is not connected because

$$\underbrace{Q \cap [-\sqrt{2}, \sqrt{2}]}_{\text{closed in } Q} = \underbrace{Q \cap (-\sqrt{2}, \sqrt{2})}_{\text{open in } Q}$$

is clopen and is neither universal nor empty.

(v) $(X, \mathcal{C})$ is connected except when X is finite; indeed, every infinite subset of X is connected.

(vi) $(X, \mathcal{L})$ is connected except when X is countable; indeed, every uncountable subset of X is connected.

(vii) In $(R, \mathcal{I})$, $A \subseteq R$ is connected iff A is an interval. (Thus, subspaces of connected spaces are *not* usually connected — examples abound in $(R, \mathcal{I})$.)

2.3.2 Characterizations of Connectedness

Lemma 2.8 *$\emptyset \subset A \subseteq (X, \mathcal{T})$ is* not *connected iff there exist $\mathcal{T}$-open sets G, H such that $A \subseteq G \cup H$, $A \cap G \neq \emptyset$, $A \cap H \neq \emptyset$ and $A \cap G \cap H = \emptyset$. (Again, we can replace 'open' by 'closed' here.)*

Proof Exercise.
Note By an interval in R, we mean any subset I such that whenever $a < b < c$ and whenever $a \in I$ and $c \in I$ then $b \in I$. It is routine to check that the only ones are (a, b), $[a, b]$, $[a, b)$, $(a, b]$, $[a, \infty)$, (a, ∞), $(-\infty, b)$, $(-\infty, b]$, $(-\infty, \infty) = R$ and $\{a\}$ for real a, b, $a < b$ where appropriate.
It turns out that these are exactly the connected subsets of $(R, \mathcal{I})$:-

Lemma 2.9 *In R, if $[a, b] = F_1 \cup F_2$ where F_1, F_2 are both closed and $a \in F_1$, $b \in F_2$ then $F_1 \cap F_2 \neq \emptyset$.*

Proof Exercise.

Theorem 2.6 *Let $\emptyset \subset I \subseteq (R, \mathcal{I})$. Then I is connected iff I is an interval.*

Proof

$\Rightarrow$: If I is *not* an interval, then there exist $a < b < c$ with $a \in I$, $b \notin I$ and $c \in I$. Take $A = I \cap (-\infty, b)$ and $B = I \cap (b, \infty)$. Then $A \cup B = I$, $A \cap B = \emptyset$, $A \neq \emptyset$, $B \neq \emptyset$, $A \subset I$, $B \subset I$ and A, B are both open in I i.e. A and B partition I and so I is not connected.

$\Leftarrow$: Suppose I is not connected and that I *is* an interval. By the 'closed' version of Lemma 2.8, there exist closed subsets K_1, K_2 of R such that $I \subseteq K_1 \cup K_2$, $I \cap K_1 \neq \emptyset$, $I \cap K_2 \neq \emptyset$ and $I \cap K_1 \cap K_2 = \emptyset$. Select $a \in I \cap K_1$, $b \in I \cap K_2$; without loss of generality, $a < b$. Then $[a, b] \subseteq I$ so that $[a, b] = ([a, b] \cap K_1) \cup ([a, b] \cap K_2)$, whence by Lemma 2.9, $\emptyset \neq [a, b] \cap K_1 \cap K_2 \subseteq I \cap K_1 \cap K_2 = \emptyset$!

2.3.3 Connectedness and Continuous Maps

Lemma 2.10 *Connectedness is preserved by continuous maps.*

Proof Exercise.

Corollary 2.2 (Intermediate Value Theorem) *If $f : [a, b] \to R$ is continuous and $f(a) < y < f(b)$, then y must be a value of f.*

Proof Exercise.

Corollary 2.3 (Fixed point theorem for $[0, 1]$) *If $f : [0, 1] \to [0, 1]$ is continuous, then it has a 'fixed point' i.e. there exists some $x \in [0, 1]$ such that $f(x) = x$.*

Proof Consider $g(x) = f(x) - x$. Then $g : [0, 1] \to R$ is continuous. Further, $g(0) = f(0) \geq 0$ and $g(1) = f(1) - 1 \leq 0$ so that 0 is intermediate between $g(0)$ and $g(1)$. Thus, by the Intermediate Value Theorem, there exists $x \in [0, 1]$ such that $0 = g(x) = f(x) - x$ i.e. such that $f(x) = x$.
Note Given continuous $h : [a, b] \to [a, b]$, it follows that h has a fixed point since $[a, b] \cong [0, 1]$ and 'every continuous function has a fixed point' is a homeomorphic invariant.

Lemma 2.11 *Let $(X, \mathcal{T})$ be disconnected with $\emptyset \subset Y \subset X$, Y clopen. If A is any connected subset of X, then $A \subseteq Y$ or $A \subseteq X \setminus Y$.*

Proof If $A \cap Y \neq \emptyset \neq A \cap (X \setminus Y)$, then $\emptyset \subset A \cap Y \subset A$ and $A \cap Y$ is clopen *in* A. Thus, A is *not* connected! It follows that either $A \cap Y = \emptyset$ or $A \cap X \setminus Y = \emptyset$ i.e. either $A \subseteq X \setminus Y$ or $A \subseteq Y$.

Lemma 2.12 *If the family $\{A_i : i \in I\}$ of connected subsets of a space $(X, \mathcal{T})$ has a non-empty intersection, then its union $\cup_{i \in I} A_i$ is connected.*

Proof Suppose not and that there exists a non-empty proper clopen subset Y of $\cup_{i\in I}A_i$. Then for each $i \in I$, either $A_i \subseteq Y$ or $A_i \subseteq \cup_{i\in I}A_i \setminus Y$. However if for some j, $A_j \subseteq Y$, then $A_i \subseteq Y$ for each $i \in I$ (since $\cap_{i\in I}A_i \neq \emptyset$) which implies that $\cup_{i\in I}A_i \subseteq Y$!
Similarly, if for some $k \in I$, $A_k \subseteq \cup_{i\in I}A_i \setminus Y$, then $\cup_{i\in I}A_i \subseteq \cup_{i\in I}A_i \setminus Y$!

Corollary 2.4 *Given a family $\{C_i : i \in I\}$ of connected subsets of a space $(X, \mathcal{T})$, if $B \subseteq X$ is also connected and $B \cap C_i \neq \emptyset$ for all $i \in I$, then $B \cup (\cup_{i\in I}C_i)$ is connected.*

Proof Take $A_i = B \cup C_i$ in Lemma 2.12.

Lemma 2.13 *If A is a connected subset of a space $(X, \mathcal{T})$ and $A \subseteq B \subseteq \bar{A}^{\mathcal{T}}$, then B is a connected subset.*

Proof If B is *not* connected, then there exists $\emptyset \subset Y \subset B$ which is clopen *in* B. By Lemma 2.11, either $A \subseteq Y$ or $A \subseteq B \setminus Y$. Suppose $A \subseteq Y$ (a similar argument suffices for $A \subseteq B \setminus Y$); then $\bar{A}^{\mathcal{T}} \subseteq \bar{Y}^{\mathcal{T}}$ and so $B \setminus Y = B \cap (B \setminus Y) = \bar{A}^{\mathcal{T}_B} \cap (B \setminus Y) \subseteq \bar{Y}^{\mathcal{T}_B} \cap (B \setminus Y) = Y \cap (B \setminus Y) = \emptyset$ — a contradiction!

Definition 2.7 *Let $(X, \mathcal{T})$ be a topological space with $x \in X$; we define the* **component of** *x, C_x, in $(X, \mathcal{T})$ to be the union of all connected subsets of X which contain x i.e.*

$$C_x = \cup\{A \subseteq X : x \in A \text{ and } A \text{ is connected}\}.$$

For each $x \in X$, it follows from Lemma 2.12 that C_x is the *maximum* connected subset of X which contains x. Also it is clear that if x, $y \in X$, either $C_x = C_y$ or $C_x \cap C_y = \emptyset$ (for if $z \in C_x \cap C_y$, then $C_x \cup C_y \subseteq C_z \subseteq C_x \cap C_y$ whence $C_x = C_y (= C_z)$). Thus we may speak of *the* components of a space $(X, \mathcal{T})$ (without reference to specific points of X): they partition the space into connected *closed* subsets (by Lemma 2.13) and are precisely the *maximal* connected subsets of X.
Examples
(i) If $(X, \mathcal{T})$ is connected, $(X, \mathcal{T})$ has only one component, namely X!
(ii) For any discrete space, the components are the singletons.
(iii) In Q (with its usual topology), the components are the singletons. (Thus, components need not be open.)

Definition 2.8 *A space* $(X, \mathcal{T})$ *is* **totally disconnected** *iff the only connected subsets of* X *are the singletons (equivalently, the components of* $(X, \mathcal{T})$ *are the singletons).*

Thus, by the previous examples, we see that the space Q of rationals, the space $R \setminus Q$ of irrationals and any discrete space are all totally disconnected. Further, the Sorgenfrey line R_s is totally disconnected.

2.3.4 Pathwise Connectedness

Definition 2.9 *A topological space* $(X, \mathcal{T})$ *is* **pathwise connected** *iff for any* $x, y \in X$*, there exists a continuous function* $f : [0, 1] \to X$ *such that* $f(0) = x$ *and* $f(1) = y$*. Such a function* f *is called a* **path** *from* x *to* y*.*

Theorem 2.7 *Every pathwise connected space is connected.*

Proof Let $(X, \mathcal{T})$ be pathwise connected and let $a \in X$; for every $x \in X$, there exists a path $p_x : [0, 1] \to X$ from a to x. Then, for each $x \in X$, $p_x([0, 1])$ is connected; moreover, $p_a(0) = a \in \cap_{x \in X} p_x([0, 1])$ so that by Lemma 2.12, $X = \cup_{x \in X} p_x([0, 1])$ is connected.
Note well The converse is false. Consider the following example, *the topologist's sine curve*:

$$V = \{(x, 0) : x \leq 0\} \cup \{(x, \sin \frac{1}{x}) : x > 0\}$$

is a connected space, but no path can be found from $(0, 0)$ to any point $(x, \sin \frac{1}{x})$ with $x > 0$
(for suppose, w.l.o.g., there exists a path $p : [0, 1] \to X$ with $p(0) = (\frac{1}{\pi}, 0)$ and $p(1) = (0, 0)$. Then $\pi_1 \circ p$, being continuous, must take all values between 0 and $\frac{1}{\pi}$, in particular $\frac{1}{(2n+\frac{1}{2})\pi}$ for each n i.e. there exists $t_n \in [0, 1]$ such that $\pi_1 \circ p(t_n) = \frac{1}{(2n+\frac{1}{2})\pi}$ for each n. Thus, $p(t_n) = (\frac{1}{(2n+\frac{1}{2})\pi}, 1) \to (0, 1)$ as $n \to \infty$. Now $t_n \in [0, 1]$ for all n which implies that there exists a subsequence (t_{n_k}) in $[0, 1]$ with $t_{n_k} \to \lambda$. Then $p(t_{n_k}) \to p(\lambda)$ and so $\pi_1 \circ p(t_{n_k}) \to 0$. Thus $p(\lambda) = (0, y)$ for some y, whence $y = 0$ (since $p(\lambda) \in X$)!)

2.4 Separability

Definition 2.10 *A topological space is said to be* *(i)* **separable** *iff it has a countable dense subset.*

(ii) **completely separable** *(equivalently,* **second countable***) iff it has a countable base.*

Examples

(i) $(R, \mathcal{I})$ is separable (since $\bar{Q} = R$).

(ii) $(X, \mathcal{C})$ is separable for any X.

(iii) $(R, \mathcal{L})$ is *not* separable.

Theorem 2.8 *(i) Complete separability implies separability.*

(ii) The converse is true in metric spaces.

Proof We prove only (ii). In metric space (M, d), let $D = \{x_1, x_2, \ldots\}$ be dense. Consider $\mathcal{B} = \{B(x_i, q) : i \in \omega, q \in Q, q > 0\}$, a countable collection of open sets. One can show that $\mathcal{B}$ is a base for $\mathcal{T}_d$ ... over to you!

Theorem 2.9 *(i) Complete separability is hereditary.*

(ii) Separability is not hereditary. (Consider the 'included point' topology $\mathcal{I}(0)$ on R; then $(R, \mathcal{I}(0))$ is *separable, since $\overline{\{0\}} = R$. However, $R \backslash \{0\}$* is not *separable because it is discrete.)*

Example
Separability does *not* imply complete separability since, for example, $(R, \mathcal{I}(0))$ is separable but *not* completely separable.(Suppose there exists a countable base $\mathcal{B}$ for its topology. Given $x \neq 0$, $\{0, x\}$ is an open neighbourhood of x and so there exists $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq \{0, x\}$.Thus $B_x = \{0, x\}$ i.e. $\mathcal{B}$ is uncountable ... contradiction!

Theorem 2.10 *Separability is preserved by continuous maps.*

Proof Exercise.
Note Complete separability is *not* preserved by continuous maps.

Chapter 3

Convergence

In Chapter 1, we defined limits of sequences in a topological space $(X, \mathcal{T})$ so as to assimilate the metric definition. We noted, however, that not everything we knew about this idea in metric spaces is valid in topological spaces.
We will examine two main ways around this difficulty:

- develop a kind of 'super-sequence' or *net* which does for general topology what ordinary sequences do for metric spaces.
- identify the class of topological spaces in which the old idea of sequential limit is good enough.

3.1 The Failure of Sequences

The following important results are probably familiar to us in the context of metric spaces, or at least in the setting of the real line, R.

Theorem 3.1 *Given $(X, \mathcal{T})$, $A \subseteq X$, $p \in X$: if there exists some sequence of points of A tending to p, then $p \in \bar{A}$.*

Theorem 3.2 *Given $(X, \mathcal{T})$, $A \subseteq X$: if A is closed, then A includes the limit of every convergent sequence of points of A.*

Theorem 3.3 *Given $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$: if f is continuous, then f 'preserves limits of sequences' i.e. whenever $x_n \to l$ in X, then $f(x_n) \to f(l)$ in Y.*

In each case above, it is routine to prove the statement true in a general topological space as asserted. We illustrate by proving Theorem 3.3:
Let f be continuous and $x_n \to l$ in X. We must show that $f(x_n) \to f(l)$. Given a neighbourhood N of $f(l)$, there exists open G such that $f(l) \in G \subseteq N$. Then $l \in f^{-1}(G) \subseteq f^{-1}(N)$ i.e. $f^{-1}(N)$ is a neighbourhood of l and so $x_n \in f^{-1}(N) \forall n \geq n_0$ say. Thus $f(x_n) \in N \forall n \geq n_0$, whence $f(x_n) \to f(l)$.
In metric spaces, the converses of these results are also true but our main point here is that in general topology, the converses are *not* valid.
Example
In $(R, \mathcal{L})$, $\overline{(0,1)} = R$. So, for example, $5 \in \overline{(0,1)}$ and yet the only way a sequence (x_n) converges to a limit l is for $x_n = l$ from some stage on. So *no* sequence in $(0,1)$ can converge to 5 proving that the converse of Theorem 3.1 is false.
Continuing, the limit of any convergent sequence in $(0,1)$ must belong to $(0,1)$ for the same reason and yet $(0,1)$ is not closed. Thus, Theorem 3.2's converse is false.
Further, $id_R : (R, \mathcal{L}) \to (R, \mathcal{I})$ is *not* continuous and yet it does preserve limits of sequences.
Now this is a great nuisance! Sequences are of immense usefulness in real analysis and in metric spaces and elsewhere — and their failure to describe general topology adequately is a technical handicap. What to do?

3.2 Nets - A Kind of 'Super-Sequence'

Recall that a sequence is just a function having the positive integers as domain. The set of positive integers, of course, possesses a particularly simple ordering; there is a first member, second member, third member, etc. Not all sets are naturally endowed with so simple an ordering. For example, dictionary (lexographical) ordering of words is more complex (though still relative nice as orderings go). By replacing the domain of positive integers with a set having a more complicated ordering we will:

- define a net (in analogy with sequence),
- identify an associated notion of convergence,
- show that net convergence is sufficient to characterize closure of sets,

- and that compactness can be characterized in terms of convergence of subnets.

Note that these last two items generalize the role of sequences in a metric space.

3.2.1 Definition of Net

Definition 3.1 *A binary relation $\leq$ on a set P is said to be a* **pre-order** *iff*

(i) $p \leq p \, \forall p \in P$

(ii) $p \leq q$ *and* $q \leq r$ *imply* $p \leq r \, \forall p, q, r \in P$.

We often refer to P as being a *pre-ordered set* when it is understood that $\leq$ is the pre-order in question.
If it is also true that for $p, q \in P$,

(iii) $p \leq q$ *and* $q \leq p$ *imply* $p = q$, *P is said to be a* **partially ordered set** *(or poset).*

Definition 3.2 *A pre-ordered set P is said to be* **directed** *(or updirected) iff each pair of members of P has an upperbound.*

(i.e. if $p, q \in P$, then there exists $s \in P$ such that $p \leq s$, $q \leq s$.)

Definition 3.3 *Let $(P, \leq)$ be a poset. Then if $x, y \in P$ with $x \not\leq y$ and $y \not\leq x$, we write $x \parallel y$ and say that x and y are* **incomparable***;*
If $E \subseteq P$, then E is said to be **totally unordered** *(or diverse) iff $x, y \in E$ implies $x = y$ or $x \parallel y$.*
If $C \subseteq P$, then C is said to be **linear** *(or a* **chain** *or a* **total order***) iff $x, y \in C$ implies $x < y$, $x = y$ or $y < x$.*
$(P, \leq)$ is said to be a **lattice** *iff each pair of members of P has a greatest lower bound and a least upper bound.*
A lattice $(P, \leq)$ is said to be **complete** *iff every non-empty subset Y of P has a greatest lower bound ($\wedge Y$) and a least upper bound ($\vee Y$).*
An element v of a poset $(P, \leq)$ is said to be **maximal** *(minimal) iff $v \leq x$ ($x \leq v$), $x \in P \Rightarrow v = x$.*

Definition 3.4 *A* **net** *in a (non-empty set) X is any function $x : A \to X$ whose domain A is a directed set.*

In imitation of the familiar notation in sequences, we usually write the net value $x(\alpha)$ as x_α. A typical net $x : A \to X$ will usually appear as $(x_\alpha, \alpha \in A)$ or $(x_\alpha)_{\alpha \in A}$ or some such notation.

Examples of Nets

(i) N, Z, $N \times N$ are all directed sets, where suitable pre-orders are respectively the usual magnitude ordering for N and Z, and $(i, j) \leq (m, n)$ iff $i \leq m$ and $j \leq n$, in $N \times N$. Thus, for example, a sequence is an example of a net.

(ii) The real function $f : R \setminus \{0\} \to R$ given by $f(x) = 3 - \frac{1}{x}$ is a net, since its domain is a chain. *Any* real function is a net.

(iii) Given $x \in (X, \mathcal{T})$, select in any fashion an element x_N from each neighbourhood N of x; then $(x_N)_{N \in \mathcal{N}_x}$ is a net in X (since it defines a mapping from $(\mathcal{N}_x, \leq)$ into X). Recall that $\mathcal{N}_x$ is ordered by *inverse* set inclusion!

3.2.2 Net Convergence

Definition 3.5 *A net $(x_\alpha)_{\alpha \in A}$ in $(X, \mathcal{T})$* **converges to a limit** *l if for each neighbourhood N of l, there exists some $\alpha_N \in A$ such that $x_\alpha \in N$ for all $\alpha \geq \alpha_N$.*

In such a case, we sometimes say that the net $(x_\alpha)_{\alpha \in A}$ *eventuates* N. Clearly, this definition incorporates the old definition of 'limit of a sequence'. The limit of the net f described in (ii) above is 3. In (iii), the net described converges to x no matter how the values x_N are chosen . . . prove!

3.2.3 Net Convergence and Closure

Our claim is that nets 'fully describe' the structure of a topological space. Our first piece of evidence to support this is that with nets, instead of sequences, Theorems 3.1, 3.2 and 3.3 have workable converses:

Theorem 3.4 *Given $(X, \mathcal{T})$, $A \subseteq X$, $p \in X$: $p \in \bar{A}$ iff there exists a net in A converging to p.*

Proof If some net of points of A converges to p, then every neighbourhood of p contains points of A (namely, values of the net) and so we get $p \in \bar{A}$. Conversely, if p is a closure point of A then, for each neighbourhood N of p, it will be possible to choose an element a_N of A that belongs also to N. The net which these choices constitute converges to p, as required.

Theorem 3.5 *Given $(X, \mathcal{T})$, $A \subseteq X$, A is closed iff it contains every limit of every (convergent) net of its own points.*

Proof This is really just a corollary of the preceding theorem.

Theorem 3.6 *Given $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$, f is continuous iff f preserves net convergence.*

Proof Exercise.

3.2.4 Nets and Compactness

Definition 3.6 *Let $(x_\alpha)_{\alpha \in A}$ be any net and let $\alpha_0 \in A$. The $\alpha_0^{\mathbf{th}}$* **tail** *of the net is the set $\{x_\alpha : \alpha \geq \alpha_0\} = x([\alpha_0,))$. We denote it by $x(\alpha_0 \to)$.*

Definition 3.7 *Let $(x_\alpha)_{\alpha \in A}$ and $(y_\beta)_{\beta \in B}$ be any two nets. We call $(y_\beta)_{\beta \in B}$ a* subnet *of $(x_\alpha)_{\alpha \in A}$ provided that every tail of (x_α) contains a tail of (y_β) i.e. provided:*

$$\forall \alpha_0 \in A \, \exists \beta_0 \in B \text{ such that } x(\alpha_0 \to) \supseteq y(\beta_0 \to).$$

We expected a definition like 'subsequence' to turn up here and we are disappointed that it has to be so complicated.
Net theory ceases to be a straightforward generalisation of sequence theory precisely when we have to take a subnet ... so we'll try to avoid this whenever possible! There is however one result certainly worth knowing:

Theorem 3.7 *$(X, \mathcal{T})$ is compact iff in X, every net has (at least one) convergent subnet.*

(So, for example, (n) is a net in R with *no* convergent subnet.)
Proof Not required.

Corollary 3.1 *Compactness is closed-hereditary*

Proof (for if (x_α) is a net in a closed set $F \subseteq X$, then it has a convergent subnet (y_β) *in* X. Thus there exists a subnet (z_γ) of (y_β) *in* F which converges in X, whence its limit is in F).

Corollary 3.2 *Compactness is preserved by continuous maps*

Proof (for if X is compact and f continuous, let $(y_\alpha)_{\alpha\in A}$ be a net in $f(X)$. Then for each $\alpha \in A$, $y_\alpha = f(x_\alpha)$ for some $x_\alpha \in X$. The net $(x_\alpha)_{\alpha\in A}$ has a convergent subnet $(z_\beta)_{\beta\in B}$, say $z_\beta \to l$, whence $f(z_\beta) \to f(l)$. Then $(f(z_\beta))_{\beta\in B}$ is a convergent subnet of $(y_\alpha)_{\alpha\in A}$).
Example
If (x_{n_k}) is a subsequence of a sequence (x_n), then it is a subnet of it also; because the i_0^{th} tail of the sequence (x_n) is

$$\{x_{i_0}, x_{i_0+1}, x_{i_0+2}, \ldots\} \cdots (*)$$

while the i_0^{th} tail of the subsequence (x_{n_k}) is:

$$\{x_{n_{i_0}}, x_{n_{i_0+1}}, x_{i_{n_0+2}}, \ldots\} \cdots (**)$$

and we see that $(**) \subseteq (*)$ merely because $n_{i_0} \geq i_0$.

Lemma 3.1 *If a net (x_α) converges to a limit l, then so do all its subnets.*

Proof Let (y_β) be a subnet of (x_α); let N be a neighbourhood of l. Then there exists α_0 such that $x_\alpha \in N$ for all $\alpha \geq \alpha_0$. Further, there exists β_0 such that $\{y_\beta : \beta \geq \beta_0\} \subseteq \{x_\alpha : \alpha \geq \alpha_0\}$ and so $y_\beta \in N$ for all $\beta \geq \beta_0$.

3.3 First Countable Spaces - Where Sequences Suffice

Why do sequences suffice to describe structure in R, C and other metric spaces but not in many other topological spaces? The key here is recognizing that many proofs regarding convergence in metric spaces involve constructing sequences of nested open sets about a point. Sometimes these describe the topological structure near the point and other times not. In what follows we

- identify the local characteristic of topological space that makes these proofs work,
- and prove that sequences suffice to describe the topological structure of spaces with this characteristic.

3.3.1 First Countable Spaces

So what characteristic common to R, C and other metric spaces makes sequences so 'good' at describing their structure?

Definition 3.8 *Let $x \in (X, \mathcal{T})$. A* **countable neighbourhood base at** *x means: a sequence N_1, N_2, N_3, ... of particular neighbourhoods of x such that every neighbourhood of x shall contain one of the N_i's.*

Note that we may assume that $N_1 \supseteq N_2 \supseteq N_3 \supseteq \cdots$ because, if not, then we can work with N_1, $N_1 \cap N_2$, $N_1 \cap N_2 \cap N_3$, ...

Definition 3.9 *We call $(X, \mathcal{T})$* **first-countable** *when every point in X has a countable neighbourhood base.*

Example The classic example of a first-countable space is any metric (or metrizable) space because if $x \in (M, d)$, then $B(x, 1)$, $B(x, \frac{1}{2})$, $B(x, \frac{1}{3})$, ... is a countable neighbourhood base at x.

Theorem 3.8 *First-countability is hereditary and preserved by continuous open onto maps.*

Proof Left to the reader.

Theorem 3.9 *(i) Complete separability implies first countability.*

(ii) Converse not always true.

(iii) Converse valid on a countable underlying set.

Proof

(i) If $\mathcal{B}$ is a countable base for $(X, \mathcal{T})$ and $p \in X$, consider $\{B \in \mathcal{B} : p \in B\}$ which is a countable family of neighbourhoods of p. Moreover, they form a neighbourhood base at p.

(ii) An uncountable discrete space is first countable (since metrizable), yet is not completely separable.

(iii) Suppose X countable and $(X, \mathcal{T})$ first countable. For each $x \in X$, choose a countable neighbourhood base: $N(x, 1)$, $N(x, 2)$, $N(x, 3)$, ... Each is a neighbourhood of x and so contains an open neighbourhood of x: $G(x, 1)$, $G(x, 2)$, $G(x, 3)$, ...

Then $\mathcal{B} = \{G(x, n) : n \in N, x \in X\}$ is a countable family of open sets and is a base for $(X, \mathcal{T})$. Thus, $(X, \mathcal{T})$ is completely separable.

Example

The Arens-Fort space (see, for example, Steen and Seebach, *Counterexamples in Topology* is not first-countable because otherwise it would be completely separable which is false!

3.3.2 Power of Sequences in First Countable Spaces

The following three results illustrate that 'sequences suffice for first-countable spaces' in the sense that we don't need to use nets to describe their structure. This is why sequences are sufficiently general to describe, fully, metric and metrizable spaces.

Theorem 3.10 *Given a first-countable space* $(X, \mathcal{T})$

(i) $p \in X$, $A \subseteq X$, *then* $p \in \bar{A}$ *iff there exists a* <u>*sequence*</u> *of points of* A *converging to* p.

(ii) $A \subseteq X$ *is closed iff* A *contains every limit of every convergent* <u>*sequence*</u> *of its own points.*

(iii) $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ *is continuous iff it preserves limits of (convergent)* <u>*sequences*</u>.

<u>Proof</u>

(i) Theorem 3.1 said that if there exists a sequence in A converging to some $p \in X$, then $p \in \bar{A}$.

Conversely, if $p \in \bar{A}$, then p has a countable base of neighbourhoods $N_1 \supseteq N_2 \supseteq N_3 \supseteq \cdots$, each of which must intersect A. So choose $x_j \in N_j \cap A$ for all $j \geq 1$. Then (x_j) is a sequence in A and, given any neighbourhood H of p, H must contain one of the N_j's i.e. $H \supseteq N_{j_0} \supseteq N_{j_0+1} \supseteq \cdots$ so that $x_j \in H$ for all $j \geq j_0$. That is, $x_j \to p$.

(ii) Corollary of (i).

(iii) f continuous implies that it must preserve limits of sequences (by Theorem 3.3). Conversely, if f is not continuous, there exists $A \subseteq X$ such that $f(\bar{A}) \not\subseteq \overline{f(A)}$. Thus, there exists $p \in f(\bar{A}) \setminus \overline{f(A)}$ so $p = f(x)$, some $x \in \bar{A}$. So there exists a sequence (x_n) in A with $x_n \to x$.

Yet, if $f(x_n) \to f(x)(= p)$, p would be the limit of a sequence in $f(A)$ so that $p \in \overline{f(A)}$ —contradiction! Thus f fails to preserve convergence of this sequence.

Chapter 4

Product Spaces

A common task in topology is to construct new topological spaces from other spaces. One way of doing this is by taking products. All are familiar with identifying the plane or 3-dimensional Euclidean space with ordered pairs or triples of numbers each of which is a member of the real line. Fewer are probably familar with realizing the torus as ordered pairs of complex numbers of modulus one. In this chaper we answer two questions:

- How do the above product constructions generalize to topological spaces?
- What topological properties are preserved by this construction?

4.1 Constructing Products

The process of constructing a product falls naturally into two stages.

- The first stage, which is entirely set-theoretic, consists in describing an element of the underlying set of the product. This task is primarily one of generalizing the notion of ordered pair or triple.
- The second stage is describing what open sets look like. This will be done by describing a subbasis for the topology. The guiding goal is to provide just enough opens sets to guarantee the continuity of certain important functions.

4.1.1 Set-Theoretic Construction

Suppose throughout that we are given a family of topological spaces $\{(X_i, \mathcal{T}_i) : i \in I\}$ where I is some non-empty 'labelling' or index set.
Our first task is to get a clear mental picture of what we mean by the product of the *sets* X_i. Look again at the finite case where $I = \{1, 2, \ldots, n\}$. Here, the product set

$$X = X_1 \times X_2 \times X_3 \times \ldots \times X_n = \prod_{i=1}^{n} X_i = \{(p_1, p_2, \ldots, p_n) : p_i \in X_i,\ i \in I\}.$$

i.e. the elements of X are the functions $x : I \to \cup_{i=1}^{n} X_i$ such that $x(1) \in X_1$, $x(2) \in X_2, \ldots, x(n) \in X_n$ i.e. $x(i) \in X_i \forall i$ where, for convenience, we usually write x_i instead of $x(i)$. In this form, the definition extends immediately to any I, finite or infinite i.e. if $\{X_i : i \in I\}$ is *any* family of sets, then their product is

$$\{x : I \to \cup_{i \in I} X_i \text{ for which } x(i) \in X_i\, \forall i \in I\}$$

except that we normally write x_i rather than $x(i)$.
Then a typical element of $X = \prod X_i$ will look like: $(x_i)_{i \in I}$ or just (x_i). We will still call x_i the i^{th} coordinate of $(x_i)_{i \in I}$. (Note that the Axiom of Choice assures us that $\prod X_i$ is non-empty provided none of the X_i's are empty.)

4.1.2 Topologizing the Product

Of the many possible topologies that could be imposed on $X = \prod X_i$, we describe the most useful. This topology is 'just right' in the sense that it is barely fine enough to guarantee the continuity of the coordinate projection functions while being just course enough allow the important result of Theorem 4.1.

Definition 4.1 *For each $i \in I$, the i^{th}* **projection** *is the map $\pi_i : \prod X_i \to X_i$ which 'selects the i^{th} coordinate' i.e. $\pi_i((x_i)_{i \in I}) = x_i$.*

An *open cylinder* means the inverse projection of some non-empty $\mathcal{T}_i$-open set i.e. $\pi_i^{-1}(G_i)$ where $i \in I$, $G_i \neq \emptyset$, $G_i \in \mathcal{T}_i$.
An *open box* is the intersection of finitely many open cylinders $\cap_{j=1}^{n} \pi_{i_j}^{-1}(G_{i_j})$.
The only drawable case $I = \{1, 2\}$ may help explain:
(Here will be, eventually, a picture!)

We use these open cylinders and boxes to generate a topology with just enough open sets to guarantee that projection maps will be continuous. Note that the open cylinders form a subbase for a certain topology $\mathcal{T}$ on $X = \prod X_i$ and therefore the open boxes form a base for $\mathcal{T}$; $\mathcal{T}$ is called the [Tychonoff]**product topology** and $(X, \mathcal{T})$ is the product of the given family of spaces. We write $(X, \mathcal{T}) = \prod\{(X_i, \mathcal{T}_i) : i \in I\} = \prod_{i \in I}(X_i, \mathcal{T}_i)$ or even $\mathcal{T} = \prod_{i \in I} \mathcal{T}_i$.
Notice that if $\cap_{j=1}^{n} \pi_{i_j}^{-1}(G_{i_j})$ is any open box, then without loss of generality we can assume i_1, i_2, $\ldots i_n$ all different because if there were repetitions like

$$\ldots \cap \pi_{i_k}^{-1}(G) \cap \pi_{i_k}^{-1}(H) \ldots$$

we can replace each by

$$\ldots \cap \pi_{i_k}^{-1}(G \cap H) \cap \ldots$$

and thus eliminate all repetitions.
It is routine to check that if $\mathcal{T}_n$ is the usual topology on R^n, and $\mathcal{T}$ the usual topology on R, then

$$(R, \mathcal{T}) \times (R, \mathcal{T}) \times \ldots (R, \mathcal{T}) = (R^n, \mathcal{T}_n)$$

as one would hope!

Lemma 4.1 *In a product space $(X, \mathcal{T})$, N is a neighbourhood of $p \in X$ iff there exists some open box B such that $p \in B \subseteq N$.*

Lemma 4.2 *For each $i \in I$,*

(i) π_i is continuous

(ii) π_i is an open mapping.

Proof

(i) Immediate.

(ii) Given open $G \subseteq X$, then G is a union of basic open sets $\{B_k : k \in K\}$ in X, whence $\pi_i(G)$ is a union of open subsets $\{B_k^i : k \in K\}$ of X_i and is therefore open. (The notation here is intended to convey that B_k^i is the 'component along the i-th coordinate axis' of the open box B_k.)

Theorem 4.1 *A map into a product space is continuous iff its composite with each projection is continuous.*

Proof Since the projections are continuous, so must be their composites with any continuous map. To establish the converse, first show that if $\mathcal{S}$ is a subbase for the codomain (target) of a mapping f, then f will be continuous provided that the preimage of every member of $\mathcal{S}$ is open; now use the fact that the open cylinders constitute a subbase for the product topology.

Worked example Show that $(X, \mathcal{T}) \times (Y, \mathcal{S})$ is homeomorphic to $(Y, \mathcal{S}) \times (X, \mathcal{T})$.

Solution

Define $\left.\begin{array}{l} f : X \times Y \to Y \times X \\ g : Y \times X \to X \times Y \end{array}\right\}$ by $\begin{array}{l} f(x,y) = (y,x) \\ g(y,x) = (x,y). \end{array}$ Clearly these are one-one, onto and mutually inverse. It will suffice to show that both are continuous.

$\pi_1 \circ f = \pi_2^{'}$; $\pi_2 \circ f = \pi_1^{'}$. Now $\pi_i^{'}$ is continuous for $i = 1, 2$ and so f is continuous! Similarly, g is continuous.

Worked example Show that the product of infinitely many copies of $(N, \mathcal{D})$ is not locally compact.

Solution

We claim that *no* point has a compact neighbourhood. Suppose otherwise; then there exists $p \in X$, $C \subseteq X$ and $G \subseteq X$ with C compact, G open and $p \in G \subseteq C$. Pick an open box B such that $p \in B \subseteq G \subseteq C$. B looks like $\cap_{j=1}^{n} \pi_{i_j}^{-1}(G_{i_j})$. Choose $i_{n+1} \in I \setminus \{i_1, i_2, \ldots, i_n\}$; then $\pi_{i_{n+1}}(C)$ is compact (since compactness is preserved by continuous maps).

Thus, $p_{i_{n+1}} \in \pi_{i_{n+1}}(B) = X_{i_{n+1}} \subseteq \pi_{i_{n+1}}(C) \subseteq X_{i_{n+1}} = (N, \mathcal{D})$. Thus, $\pi_{i_{n+1}}(C) = (N, \mathcal{D}) \ldots$ which is *not* compact!

4.2 Products and Topological Properties

The topological properties possessed by a product depends, of course, on the properties possessed by the individual factors. There are several theorems which assert that certain topological properties are **productive** i.e. are possessed by the product if enjoyed by each factor. Several of these theorems are given below.

4.2.1 Products and Connectedness

Theorem 4.2 *Any product of connected spaces must be connected.*

Proof is left to the reader.

4.2.2 Products and Compactness

Theorem 4.3 (*Tychonoff's theorem*) *Any product of compact spaces is compact i.e. compactness is productive.*

Proof It suffices to prove that any covering of X by open cylinders has a finite subcover. Suppose not and let $\mathcal{C}$ be a family of open cylinders which covers X but for which no finite subcover exists. For each $i \in I$, consider

$$\{G_{i_j} : G_{i_j} \subseteq X_i \text{ and } \pi_i^{-1}(G_{i_j}) \in \mathcal{C}\}.$$

This cannot cover X_i (otherwise, X_i, being compact, would be covered by finitely many, say $X_i = G_{i_1} \cup G_{i_2} \cup \ldots \cup G_{i_n}$, whence

$$X = \pi_i^{-1}(X_i) = \underbrace{\pi_i^{-1}(G_{i_1} \cup \ldots \cup \pi_i^{-1}(G_{i_n}).}_{\text{all in } \mathcal{C}\text{, contrary to the choice of } \mathcal{C}}$$

Select, therefore, $z_i \in X_i \setminus \cup\{$ those G_{i_j}'s$\}$; consider $z = (z_i)_{i \in I} \in X$. Since $\mathcal{C}$ covered X, $z \in$ some $C \in \mathcal{C}$. Now $C = \pi_k^{-1}(G_k)$ for some $k \in I$ and so $\pi_k(z) = z_k \in G_k$, contradicting the choice of the z_i's.
To prove the above without Alexander's Subbase Theorem is very difficult in general, but it is fairly simple in the special case where I is finite. Several further results show that various topological properties are 'finitely productive' in this sense.

Theorem 4.4 *If* $(X_1, \mathcal{T}_1)$, $(X_2, \mathcal{T}_2)$, $\ldots$, $(X_n, \mathcal{T}_n)$ *are finitely many sequentially compact spaces, then their product is sequentially compact.*

Proof
Take any sequence $(x_n) \in X$. The sequence $(\pi_1(x_n))_{n\geq 1}$ in sequentially compact X_1 has a convergent subsequence $\pi_1(x_{n_k}) \to l_1 \in X_1$. The sequence $(\pi_2(x_{n_k}))_{k\geq 1}$ in sequentially compact X_2 has a convergent subsequence $(\pi_2(x_{n_{k_j}}))_{j\geq 1} \to l_2 \in X_2$ and $\pi_1(x_{n_{k_j}}) \to l_1$ also.
Do this n times! We get a subsequence $(y_p)_{p\geq 1}$ of the original sequence such that $\pi_i(y_p) \to l_i$ for $i = 1, 2, \ldots, n$. It's easy to check that $y_p \to (l_1, l_2, \ldots, l_n)$ so that X is sequentially compact, as required.

Lemma 4.3 *'The product of subspaces is a subspace of the product.'*

Proof
Let $(X, \mathcal{T}) = \prod_{i \in I}(X_i, \mathcal{T}_i)$; let $\emptyset \subset Y_i \subseteq X_i$ for each $i \in I$. There appear to be two different ways to topologise $\prod Y_i$:

either (i) give it the subspace topology induced by $\prod \mathcal{T}_i$

or (ii) give it the product of all the individual subspace topologies $(\mathcal{T}_i)_{Y_i}$.

The point is that these topologies coincide—if $G^*_{i_0}$ is open in $(\mathcal{T}_{i_0})_{Y_{i_0}}$ where $i_0 \in I$ i.e. $G^*_{i_0} = Y_{i_0} \cap G_{i_0}$ for some $G_{i_0} \in \mathcal{T}_{i_0}$, a typical subbasic open set for (ii) is

$$\{(y_i) \in \prod Y_i : y_{i_0} \in G^*_{i_0}\}$$

which equals

$$\prod Y_i \cap \{(x_i) \in \prod X_i : x_{i_0} \in G_{i_0} \in \mathcal{T}_{i_0}, i_0 \in I\}$$

$$= \prod Y_i \cap \{ \text{ a typical open cylinder in } \prod X_i\}$$

which is a typical subbasic open set in (i). Hence, (i) = (ii).

Theorem 4.5 *Local compactness is* finitely *productive.*

Proof
Given $x = (x_1, x_2, \ldots, x_n) \in (X, \mathcal{T}) = \prod_{i=1}^n (X_i, \mathcal{T}_i)$, we must show that x has a compact neighbourhood. Now, for all $i = 1, \ldots, n$, x_i has a compact neighbourhood C_i in $(X_i, \mathcal{T}_i)$ so we choose $\mathcal{T}_i$-open set G_i such that $x_i \in G_i \subseteq C_i$. Then

$$x \in \underbrace{G_1 \times G_2 \times \ldots \times G_n}_{\cap_1^n \pi_i^{-1}(G_i)} \subseteq \underbrace{C_1 \times C_2 \times \ldots \times C_n}_{\text{compact subset of } \prod X_i}$$

i.e. x has $C_1 \times C_2 \times \ldots \times C_n$ as a compact neighbourhood. (Note that the previous lemma is used here, to allow us to apply Tychonoff's theorem to the product of the compact subspaces C_i, and then to view this object as a subspace of the full product!) Thus, X is locally compact.

Lemma 4.4 $\overline{\prod Y_i}^{\mathcal{T}} = \prod \bar{Y_i}^{\mathcal{T}_i}$ *(in notation of previous lemma).*

Proof Do it yourself! ('The closure of a product is a product of the closures.')

4.2.3 Products and Separability

Theorem 4.6 *Separability is* finitely *productive.*

Proof
For $1 \leq i \leq n$, choose countable $D_i \subseteq X_i$ where $\bar{D_i}^{\mathcal{T}_i} = X_i$. Consider $D = D_1 \times D_2 \times \ldots \times D_n = \prod_1^n D_i$, again countable. Then $\bar{D} = \overline{\prod D_i}^{\mathcal{T}} = \prod \bar{D_i}^{\mathcal{T}_i} = \prod X_i = X$.
Notice that the converses of all such theorems are easily true. For example,

Theorem 4.7 *If* $(X, \mathcal{T}) = \prod_{i \in I}(X_i, \mathcal{T}_i)$ *is*

(i) compact

(ii) sequentially compact

(iii) locally compact

(iv) connected

(v) separable

(vi) completely separable

then so is every 'factor space' $(X_i, \mathcal{T}_i)$*.*

Proof For each $i \in I$, the projection mapping $\pi_i : X \to X_i$ is continuous, open and onto. Thus, by previous results, the result follows.

Chapter 5

Separation Axioms

We have observed instances of topological statements which, although true for all metric (and metrizable) spaces, fail for some other topological spaces. Frequently, the cause of failure can be traced to there being 'not enough open sets' (in senses to be made precise). For instance, in any metric space, compact subsets are always closed; but not in every topological space, for the proof ultimately depends on the observation

> 'given $x \neq y$, it is possible to find disjoint open sets G and H with $x \in G$ and $y \in H$'

which is true in a metric space (e.g. put $G = B(x, \epsilon)$, $H = B(y, \epsilon)$ where $\epsilon = \frac{1}{2}d(x, y)$) but fails in, for example, a trivial space $(X, \mathcal{T}_0)$.
What we do now is to see how 'demanding certain minimum levels-of-supply of open sets' gradually eliminates the more pathological topologies, leaving us with those which behave like metric spaces to a greater or lesser extent.

5.1 T_1 Spaces

Definition 5.1 *A topological space $(X, \mathcal{T})$ is T_1 if, for each x in X, $\{x\}$ is closed.*

Comment 5.1 *(i) Every metrizable space is T_1*

(ii) $(X, \mathcal{T}_0)$ isn't T_1 unless $|X| = 1$

Theorem 5.1 *(i) T_1 is hereditary*

(ii) *T_1 is productive*

(iii) *$T_1 \Rightarrow$ every finite set is closed. More precisely, $(X, \mathcal{T})$ is T_1 iff $\mathcal{T} \supseteq \mathcal{C}$, i.e. $\mathcal{C}$ is the weakest of all the T_1 topologies that can be defined on X.*

Proof is left to the reader.
The respects in which T_1-spaces are 'nicer' than others are mostly concerned with 'cluster point of a set' (an idea we have avoided!). We show the equivalence, in T_1 spaces, of the two forms of its definition used in analysis.

Theorem 5.2 *Given a T_1 space $(X, \mathcal{T})$, $p \in X$ and $A \subseteq X$, the following are equivalent:*

(i) *Every neighbourhood of p contains infinitely many points of A*

(ii) *Every neighbourhood of p contains at least one point of A different from p.*

Proof Obviously, (i) $\Rightarrow$ (ii); conversely, suppose (i) fails; so there exists a neighbourhood N of p such that $N \cap A$ is finite. Consider $H = [X \setminus (N \cap A)] \cup \{p\}$; it is cofinite and is thus an (open) neighbourhood of p. Hence $N \cap H$ is a neighbourhood of p which contains no points of A, except possibly p itself. Thus, (ii) fails also.
Hence, (i) $\Leftrightarrow$ (ii).

5.2 T_2 (Hausdorff) Spaces

Definition 5.2 *A topological space $(X, \mathcal{T})$ is T_2 (or* **Hausdorff***) iff given $x \neq y$ in X, $\exists$ disjoint neighbourhoods of x and y.*

Comment 5.2 (i) *Every metrizable space is T_2*

(ii) *$T_2 \Rightarrow T_1$ (i.e. any T_2 space is T_1, for if x, $y \in T_2 X$ and $y \in \overline{\{x\}}$, then every neighbourhood of y contains x, whence $x = y$.)*

(iii) *$(X, \mathcal{C})$, with X infinite, cannot be T_2*

Theorem 5.3 (i) *T_2 is hereditary*

(ii) *T_2 is productive.*

<u>Proof</u>

i The proof is left to the reader.

ii Let $(X,\mathcal{T}) = \prod_{i\in I}(X_i,\mathcal{T}_i)$ be any product of T_2 spaces. Let $x = (x_i)_{i\in I}$ and $y = (y_i)_{i\in I}$ be distinct elements of X. Then there exists $i_0 \in I$ such that $x_{i_0} \neq y_{i_0}$ in X_{i_0}. Choose disjoint open sets G, H in $(X_{i_0},\mathcal{T}_{i_0})$ so that $x_{i_0} \in G$, $y_{i_0} \in H$. Then $x \in \pi_{i_0}^{-1}(G) \in \mathcal{T}$, $y \in \pi_{i_0}^{-1}(H) \in \mathcal{T}$ and since $G \cap H = \emptyset$, $\pi_{i_0}^{-1}(G) \cap \pi_{i_0}^{-1}(H) = \emptyset$. Hence result.

The T_2 axiom is particularly valuable when exploring compactness. Part of the reason is that T_2 implies that points and compact sets can be 'separated off' by open sets and even implies that compact sets can be 'separated off' from other compact sets in the same way.

Theorem 5.4 *In a T_2-space $(X,\mathcal{T})$, if C is a compact set and $x \notin C$, then there exist $\mathcal{T}$-open sets G and H so that $x \in G$, $C \subseteq H$ and $G \cap H = \emptyset$.*

<u>Proof</u> A valuable exercise: separate each point of C from x using disjoint open sets, note that the open neighbourhoods of the various elements of C, thus obtained, make up an open covering of C, reduce it to a finite subcover by appealing to compactness ...

Corollary 5.1 *In a T_2-space, any compact set is closed.*

Corollary 5.2 *In a T_2-space, if C and K are non-empty compact and disjoint, then there exist open G, H such that $C \subseteq G$, $K \subseteq H$ and $G \cap H = \emptyset$.*

A basic formal distinction between algebra and topology is that although the inverse of a one-one, onto group homomorphism [etc!] is automatically a homomorphism again, the inverse of a one-one, onto continuous map can fail to be continuous. It is a consequence of Corollary 5.2 that, amongst compact T_2 spaces, this cannot happen.

Theorem 5.5 *Let $f : (X_1,\mathcal{T}_1) \to (X_2,\mathcal{T}_2)$ be one-one, onto and continuous, where X_1 is compact and X_2 is T_2. Then f is a homeomorphism.*

<u>Proof</u> It suffices to prove that f is closed. Given closed $K \subseteq X_1$, then K is compact whence $f(K)$ is compact and so $f(K)$ is closed. Thus f is a closed map.

Theorem 5.6 *$(X, \mathcal{T})$ is T_2 iff no net in X has more than one limit.*

Proof

(i) $\Rightarrow$ (ii): Let $x \neq y$ in X; by hypothesis, there exist disjoint neighbourhoods U of x, V of y. Since a net cannot eventually belong to each of two disjoint sets, it is clear that no net in X can converge to *both* x and y.

(ii) $\Rightarrow$ (i): Suppose that $(X, \mathcal{T})$ is not Hausdorff and that $x \neq y$ are points in X for which every neighbourhood of x intersects every neighbourhood of y. Let $\mathcal{N}_x$ $(\mathcal{N}_y)$ be the neighbourhood systems at x (y) respectively. Then both $\mathcal{N}_x$ and $\mathcal{N}_y$ are directed by reverse inclusion. We order the Cartesian product $\mathcal{N}_x \times \mathcal{N}_y$ by agreeing that

$$(U_x, U_y) \geq (V_x, V_y) \Leftrightarrow U_x \subseteq V_x \text{ and } U_y \subseteq V_y.$$

Evidently, this order is directed. For each $(U_x, U_y) \in \mathcal{N}_x \times \mathcal{N}_y$, $U_x \cap U_y \neq \emptyset$ and hence we may select a point $z_{(U_x,U_y)} \in U_x \cap U_y$. If W_x is *any* neighbourhood of x, W_y *any* neighbourhood of y and $(U_x, U_y) \geq (W_x, W_y)$, then

$$z_{(U_x,U_y)} \in U_x \cap U_y \subseteq W_x \cap W_y.$$

That is, the net $\{z_{(U_x,U_y)}, (U_x, U_y) \in \mathcal{N}_x \times \mathcal{N}_y\}$ eventually belongs to both W_x and W_y and consequently converges to *both* x and y!

Corollary 5.3 *Let $f : (X_1, \mathcal{T}_1) \to (X_2, \mathcal{T}_2)$, $g : (X_1, \mathcal{T}_1) \to (X_2, \mathcal{T}_2)$ be continuous where X_2 is T_2. Then their 'agreement set' is closed i.e. $A = \{x : f(x) = g(x)\}$ is closed.*

5.3 T_3 Spaces

Definition 5.3 *A space $(X, \mathcal{T})$ is called T_3 or* **regular** *provided :-*

(i) it is T_1, and

(ii) given $x \notin$ closed F, there exist disjoint open sets G and H so that $x \in G$, $F \subseteq H$.

Comment 5.3 *(i) Every metrizable space is T_3; for it is certainly T_1 and given $x \notin$ closed F, we have $x \in$ open $X \setminus F$ so there exists $\epsilon > 0$ so that $x \in B(x, \epsilon) \subseteq X \setminus F$. Put $G = B(x, \frac{\epsilon}{2})$ and $H = \{y : d(x, y) > \frac{\epsilon}{2}\}$; the result now follows.*

(ii) *Obviously* $T_3 \Rightarrow T_2$.

(iii) *One can devise examples of* T_2 *spaces which are not* T_3.

(iv) *It's fairly routine to check that* T_3 *is productive and hereditary.*

(v) *Warning: Some books take* T_3 *to mean Definition 5.3(ii) alone, and regular to mean Definition 5.3(i)* and *(ii); others do exactly the opposite!*

5.4 $T_{3\frac{1}{2}}$ Spaces

Definition 5.4 *A space* $(X, \mathcal{T})$ *is* $T_{3\frac{1}{2}}$ *or* **completely regular** *or* **Tychonoff** *iff*

(i) *it is* T_1, *and*

(ii) *given* $x \in X$, *closed non-empty* $F \subseteq X$ *such that* $x \notin F$, *there exists continuous* $f : X \to [0,1]$ *such that* $f(F) = \{0\}$ *and* $f(x) = 1$.

Comment 5.4 (i) *Every metrizable space is* $T_{3\frac{1}{2}}$

(ii) *Every* $T_{3\frac{1}{2}}$ *space is* T_3 *(such a space is certainly* T_1 *and given* $x \notin$ *closed* F, *choose* f *as in the definition; define* $G = f^{-1}([0, \frac{1}{3}))$, $H = f^{-1}((\frac{2}{3}, 1])$ *and observe that* T_3 *follows.)*

(iii) *Examples are known of* T_3 *spaces which fail to be Tychonoff*

(iv) $T_{3\frac{1}{2}}$ *is productive and hereditary.*

5.5 T_4 Spaces

Definition 5.5 *A space* $(X, \mathcal{T})$ *is* T_4 *or* **normal** *if*

(i) *it is* T_1, *and*

(ii) *given disjoint non-empty closed subsets* A, B *of* X, *there exist disjoint open sets* G, H *such that* $A \subseteq G$, $B \subseteq H$.

Theorem 5.7 *Every metrizable space* $(X, \mathcal{T})$ *is* T_4.

Proof Certainly, X is T_1; choose a metric d on X such that $\mathcal{T}$ is $\mathcal{T}_d$. The distance of a point p from a non-empty set A can be defined thus:

$$d(p, A) = \inf\{d(p, a) : a \in A\}$$

Given disjoint non-empty closed sets A, B, let

$$G = \{x : d(x, A) < d(x, B)\}$$

$$H = \{x : d(x, B) < d(x, A)\}.$$

Clearly, $G \cap H = \emptyset$. Also, each is open (if $x \in G$ and $\epsilon = \frac{1}{2}\{d(x, B) - d(x, A)\}$, then $B(x, \epsilon) \subseteq G$, by the triangle inequality.) Now, if $d(p, A) = 0$, then for all $n \in N$, there exists $x_n \in A$ such that $d(p, x_n) < \frac{1}{n}$. So $d(p, x_n) \to 0$ i.e. $x_n \to p$, whence $p \in \bar{A}$. Thus for each $x \in A$, $x \notin B = \bar{B}$ so that $d(x, B) > 0 = d(x, A)$ i.e. $x \in G$. Hence $A \subseteq G$. Similarly $B \subseteq H$.
It's true that $T_4 \Rightarrow T_{3\frac{1}{2}}$ but not very obvious. First note that if G_0, G_1 are open in a T_4 space with $\bar{G_0} \subseteq G_1$, then there exists open $G_{\frac{1}{2}}$ with $\bar{G_0} \subseteq G_{\frac{1}{2}}$ and $\bar{G_{\frac{1}{2}}} \subseteq G_1$ (because the given $\bar{G_0}$ and $X \setminus G_1$ are disjoint closed sets so that there exist disjoint open sets $G_{\frac{1}{2}}$, H such that $\bar{G_0} \subseteq G_{\frac{1}{2}}$, $X \setminus G_1 \subseteq H$ i.e. $G_1 \supseteq$ (closed) $X \setminus H \supseteq G_{\frac{1}{2}}$).

Lemma 5.1 (*Urysohn's Lemma*) *Let F_1, F_2 be disjoint non-empty closed subsets of a T_4 space; then there exists a continuous function $f : X \to [0, 1]$ such that $f(F_1) = \{0\}$, $f(F_2) = \{1\}$.*

Proof Given disjoint closed F_1 and F_2, choose disjoint open G_0 and H_0 so that $F_1 \subseteq G_0$, $F_2 \subseteq H_0$. Define $G_1 = X \setminus F_2$ (open). Since $G_0 \subseteq$ (closed) $X \setminus H_0 \subseteq X \setminus F_2 = G_1$, we have $\bar{G_0} \subseteq G_1$.
By the previous remark, we can now construct:

(i) $G_{\frac{1}{2}} \in \mathcal{T}$: $\bar{G_0} \subseteq G_{\frac{1}{2}}$, $\bar{G_{\frac{1}{2}}} \subseteq G_1$.

(ii) $G_{\frac{1}{4}}$, $G_{\frac{3}{4}} \in \mathcal{T}$: $\bar{G_0} \subseteq G_{\frac{1}{4}}$, $\bar{G_{\frac{1}{4}}} \subseteq G_{\frac{1}{2}}$, $\bar{G_{\frac{1}{2}}} \subseteq G_{\frac{3}{4}}$, $\bar{G_{\frac{3}{4}}} \subseteq G_1$.

(iii) …and so on!

Thus we get an indexed family of open sets

$$\{G_r : r = \frac{m}{2^n}, 0 \le m \le 2^n, n \ge 1\}$$

such that $r_1 \leq r_2 \Rightarrow \bar{G_{r_1}} \subseteq G_{r_2}$.
Observe that the index set is dense in $[0,1]$: if $s < t$ in $[0,1]$, there exists some $\frac{m}{2^n}$ such that $s < \frac{m}{2^n} < t$. Define

$$f(x) = \begin{cases} \inf\{r : x \in G_r\} & x \notin F_2 \\ 1 & x \in F_2. \end{cases}$$

Certainly $f : X \to [0,1]$, $f(F_2) = \{1\}$, $f(F_1) = \{0\}$. To show f continuous, it suffices to show that $f^{-1}([0,\alpha))$ and $f^{-1}((\alpha,1])$ are open for $0 < \alpha < 1$.
Well, $f(x) < \alpha$ iff there exists some $r = \frac{m}{2^n}$ such that $f(x) < r < \alpha$. It follows that $f^{-1}([0,\alpha)) = \cup_{r<\alpha} G_r$, a union of open sets.
Again, $f(x) > \alpha$ iff there exist r_1, r_2 such that $\alpha < r_1 < r_2 < f(x)$, implying that $x \notin G_{r_2}$ whence $x \notin \bar{G_{r_1}}$. It follows that $f^{-1}((\alpha,1]) = \cup_{r_1>\alpha}(X \setminus \bar{G_{r_1}})$, which is again open.

Corollary 5.4 *Every T_4 space is $T_{3\frac{1}{2}}$.*

Proof Immediate from Lemma 5.1. (Note that there exist spaces which are $T_{3\frac{1}{2}}$ but not T_4.)

Theorem 5.8 *Any compact T_2 space is T_4.*

Proof Use Corollary 5.2 to Theorem 5.3.
Note Unlike the previous axioms, T_4 is neither hereditary nor productive. The global view of the hierarchy can now be filled in as an exercise from data supplied above:-

METRIZABLE	*Hereditary?*	*Productive?*
T_4		
$T_{3\frac{1}{2}}$		
T_3		
T_2		
T_1		

The following is presented as an indication of how close we are to having 'come full circle'.

Theorem 5.9 *Any completely separable T_4 space is metrizable!*

Sketch Proof
Choose a countable base; list as $\{(G_n, H_n) : n \geq 1\}$ those pairs of elements

of the base for which $\bar{G_n} \subseteq H_n$. For each n, use Lemma 5.1 to get continuous $f_n : X \to [0,1]$ such that $f_n(\bar{G_n}) = \{0\}$, $f_n(X \setminus H_n) = \{1\}$. Define

$$d(x,y) = \sqrt{\sum_{n \geq 1} \{\frac{f_n(x) - f_n(y)}{2^n}\}^2}.$$

One confirms that d *is* a metric, *and* induces the original topology.