Lecture notes for complex variables

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1 Topological preliminaries

This chapter is a brief introduction to general topology. Topological spaces consist of a set and a subset of the set of all subsets of this set called the open sets or topology which satisfy certain axioms. Like other areas in mathematics the abstraction inherent in this approach is an attempt to unify many different useful examples into one general theory.

For example, consider $\mathbb{R}^n$ with the usual norm given by

$$|\mathbf{x}| \equiv \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2}.$$

We say a set U in $\mathbb{R}^n$ is an open set if every point of U is an "interior" point which means that if $\mathbf{x} \in U$, there exists $\delta > 0$ such that if $|\mathbf{y} - \mathbf{x}| < \delta$, then $\mathbf{y} \in U$. It is easy to see that with this definition of open sets, the axioms 1.1 - 1.2 given below are satisfied if τ is the collection of open sets as just described. There are many other sets of interest besides $\mathbb{R}^n$ however, and the appropriate definition of "open set" may be very different and yet the collection of open sets may still satisfy these axioms. By abstracting the concept of open sets, we can unify many different examples. Here is the definition of a general topological space.

Let X be a set and let τ be a collection of subsets of X satisfying

$$\emptyset \in \tau,\ X \in \tau, \tag{1.1}$$

$$\text{If } \mathcal{C} \subseteq \tau, \text{ then } \cup\mathcal{C} \in \tau$$

$$\text{If } A, B \in \tau, \text{ then } A \cap B \in \tau. \tag{1.2}$$

Definition 1.1 *A set X together with such a collection of its subsets satisfying 1.1-1.2 is called a topological space. τ is called the topology or set of open sets of X. Note $\tau \subseteq \mathcal{P}(X)$, the set of all subsets of X, also called the power set.*

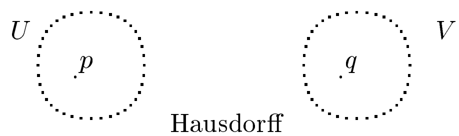
Definition 1.2 *A subset $\mathcal{B}$ of τ is called a basis for τ if whenever $p \in U \in \tau$, there exists a set $B \in \mathcal{B}$ such that $p \in B \subseteq U$. The elements of $\mathcal{B}$ are called basic open sets.*

The preceding definition implies that every open set (element of τ) may be written as a union of basic open sets (elements of $\mathcal{B}$). This brings up an interesting and important question. If a collection of subsets $\mathcal{B}$ of a set X is specified, does there exist a topology τ for X satisfying 1.1-1.2 such that $\mathcal{B}$ is a basis for τ?

Theorem 1.3 *Let X be a set and let $\mathcal{B}$ be a set of subsets of X. Then $\mathcal{B}$ is a basis for a topology τ if and only if whenever $p \in B \cap C$ for $B, C \in \mathcal{B}$, there exists $D \in \mathcal{B}$ such that $p \in D \subseteq C \cap B$ and $\cup \mathcal{B} = X$. In this case τ consists of all unions of subsets of $\mathcal{B}$.*

Proof: The only if part is left to the reader. Let τ consist of all unions of sets of $\mathcal{B}$ and suppose $\mathcal{B}$ satisfies the conditions of the proposition. Then $\emptyset \in \tau$ because $\emptyset \subseteq \mathcal{B}$. $X \in \tau$ because $\cup \mathcal{B} = X$ by assumption. If $\mathcal{C} \subseteq \tau$ then clearly $\cup \mathcal{C} \in \tau$. Now suppose $A, B \in \tau$, $A = \cup \mathcal{S}$, $B = \cup \mathcal{R}$, $\mathcal{S}, \mathcal{R} \subseteq \mathcal{B}$. We need to show $A \cap B \in \tau$. If $A \cap B = \emptyset$, we are done. Suppose $p \in A \cap B$. Then $p \in S \cap R$ where $S \in \mathcal{S}$, $R \in \mathcal{R}$. Hence there exists $U \in \mathcal{B}$ such that $p \in U \subseteq S \cap R$. It follows, since $p \in A \cap B$ was arbitrary, that $A \cap B =$ union of sets of $\mathcal{B}$. Thus $A \cap B \in \tau$. Hence τ satisfies 1.1-1.2.

Definition 1.4 *A topological space is said to be Hausdorff if whenever p and q are distinct points of X, there exist disjoint open sets U, V such that $p \in U, q \in V$.*



Hausdorff

Definition 1.5 *A subset of a topological space is said to be closed if its complement is open. Let p be a point of X and let $E \subseteq X$. Then p is said to be a limit point of E if every open set containing p contains a point of E distinct from p.*

Theorem 1.6 *A subset, E, of X is closed if and only if it contains all its limit points.*

Proof: Suppose first that E is closed and let x be a limit point of E. We need to show $x \in E$. If $x \notin E$, then E^C is an open set containing x which contains no points of E, a contradiction. Thus $x \in E$. Now suppose E contains all its limit points. We need to show the complement of E is open. But if $x \in E^C$, then x is not a limit point of E and so there exists an open set, U containing x such that U contains no point of E other than x. Since $x \notin E$, it follows that $x \in U \subseteq E^C$ which implies E^C is an open set.

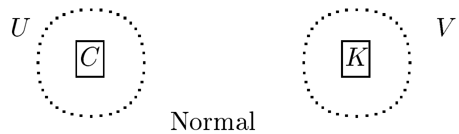
Theorem 1.7 *If (X, τ) is a Hausdorff space and if $p \in X$, then $\{p\}$ is a closed set.*

Proof: If $x \neq p$, there exist open sets U and V such that $x \in U, p \in V$ and $U \cap V = \emptyset$. Therefore, $\{p\}^C$ is an open set so $\{p\}$ is closed.

Note that the Hausdorff axiom was stronger than needed in order to draw the conclusion of the last theorem. In fact it would have been enough to assume that if $x \neq y$, then there exists an open set containing x which does not intersect y.

Definition 1.8 *A topological space (X, τ) is said to be regular if whenever C is a closed set and p is a point not in C, then there exist disjoint open sets U and V such that $p \in U$, $C \subseteq V$. The topological space, (X, τ) is said to be normal if whenever C and K are disjoint closed sets, there exist disjoint open sets U and V such that $C \subseteq U$, $K \subseteq V$.*

Regular



Normal

Definition 1.9 *Let E be a subset of X. $\bar{E}$ is defined to be the smallest closed set containing E. Note that this is well defined since X is closed and the intersection of any collection of closed sets is closed.*

Theorem 1.10 $\overline{E} = E \cup \{$*limit points of* $E\}$.

Proof: Let $x \in \overline{E}$ and suppose that $x \notin E$. If x is not a limit point either, then there exists an open set, U,containing x which does not intersect E. But then U^C is a closed set which contains E which does not contain x, contrary to the definition that $\overline{E}$ is the intersection of all closed sets containing E. Therefore, x must be a limit point of E after all.

Now $E \subseteq \overline{E}$ so suppose x is a limit point of E. We need to show $x \in \overline{E}$. If H is a closed set containing E, which does not contain x, then H^C is an open set containing x which contains no points of E other than x negating the assumption that x is a limit point of E.

Definition 1.11 *Let X be a set and let $d : X \times X \to [0, \infty)$ satisfy*

$$d(x, y) = d(y, x), \tag{1.3}$$

$$d(x, y) + d(y, z) \geq d(x, z), \ (\textit{triangle inequality})$$

$$d(x, y) = 0 \textit{ if and only if } x = y. \tag{1.4}$$

Such a function is called a metric. For $r \in [0, \infty)$ and $x \in X$, define

$$B(x, r) = \{y \in X : d(x, y) < r\}$$

This may also be denoted by $N(x, r)$.

Definition 1.12 *A topological space (X, τ) is called a metric space if there exists a metric, d, such that the sets $\{B(x, r), x \in X, \ r > 0\}$ form a basis for τ. We write (X, d) for the metric space.*

Theorem 1.13 *Suppose X is a set and d satisfies 1.3-1.4. Then the sets $\{B(x, r) : r > 0, \ x \in X\}$ form a basis for a topology on X.*

Proof: We observe that the union of these balls includes the whole space, X. We need to verify the condition concerning the intersection of two basic sets. Let $p \in B(x, r_1) \cap B(z, r_2)$. Consider

$$r \equiv \min(r_1 - d(x, p), r_2 - d(z, p))$$

and suppose $y \in B(p,r)$. Then

$$d(y,x) \leq d(y,p) + d(p,x) < r_1 - d(x,p) + d(x,p) = r_1$$

and so $B(p,r) \subseteq B(x,r_1)$. By similar reasoning, $B(p,r) \subseteq B(z,r_2)$. This verifies the conditions for this set of balls to be the basis for some topology.

Theorem 1.14 *If (X,τ) is a metric space, then (X,τ) is Hausdorff, regular, and normal.*

Proof: It is obvious that any metric space is Hausdorff. Since each point is a closed set, it suffices to verify any metric space is normal. Let H and K be two disjoint closed nonempty sets. For each $h \in H$, there exists $r_h > 0$ such that $B(h,r_h) \cap K = \emptyset$ because K is closed. Similarly, for each $k \in K$ there exists $r_k > 0$ such that $B(k,r_k) \cap H = \emptyset$. Now let

$$U \equiv \cup \{B(h,r_h/2) : h \in H\},\ V \equiv \cup \{B(k,r_k/2) : k \in K\}.$$

then these open sets contain H and K respectively and have empty intersection for if $x \in U \cap V$, then $x \in B(h,r_h/2) \cap B(k,r_k/2)$ for some $h \in H$ and $k \in K$. Suppose $r_h \geq r_k$. Then

$$d(h,k) \leq d(h,x) + d(x,k) < r_h,$$

a contradiction to $B(h,r_h) \cap K = \emptyset$. If $r_k \geq r_h$, the argument is similar. This proves the theorem.

Definition 1.15 *A metric space is said to be separable if there is a countable dense subset of the space. This means there exists $D = \{p_i\}_{i=1}^{\infty}$ such that for all x and $r > 0$, $B(x,r) \cap D \neq \emptyset$.*

Definition 1.16 *A topological space is said to be completely separable if it has a countable basis for the topology.*

Theorem 1.17 *A metric space is separable if and only if it is completely separable.*

Proof: If the metric space has a countable basis for the topology, pick a point from each of the basic open sets to get a countable dense subset of the metric space.

Now suppose the metric space, (X,d), has a countable dense subset, D. Let $\mathcal{B}$ denote all balls having centers in D which have positive rational radii. We will show this is a basis for the topology. It is clear it is a countable set. Let U be any open set and let $z \in U$. Then there exists $r > 0$ such that $B(z,r) \subseteq U$. In $B(z,r/3)$ pick a point from D, x. Now let r_1 be a positive rational number in the interval $(r/3, 2r/3)$ and consider the set from $\mathcal{B}$, $B(x,r_1)$. If $y \in B(x,r_1)$ then

$$d(y,z) \leq d(y,x) + d(x,z) < r_1 + r/3 < 2r/3 + r/3 = r.$$

Thus $B(x,r_1)$ contains z and is contained in U. This shows, since z is an arbitrary point of U that U is the union of a subset of $\mathcal{B}$.

The concept of a Cauchy sequence is very important. This is defined next.

Definition 1.18 *A sequence $\{p_n\}_{n=1}^{\infty}$ in a metric space is called a Cauchy sequence if for every $\varepsilon > 0$ there exists N such that $d(p_n,p_m) < \varepsilon$ whenever $n,m > N$. A metric space is called complete if every Cauchy sequence converges to some element of the metric space.*

Example 1.19 *$\mathbb{R}^n$ and $\mathbb{C}^n$ are complete metric spaces for the metric defined by $d(\mathbf{x},\mathbf{y}) \equiv |\mathbf{x}-\mathbf{y}| \equiv (\sum_{i=1}^n |x_i - y_i|^2)^{1/2}$.*

Not all topological spaces are metric spaces and so the traditional $\epsilon - \delta$ definition of continuity must be modified for more general settings. The following definition does this for general topological spaces.

Definition 1.20 *Let (X, τ) and (Y, η) be two topological spaces and let $f : X \to Y$. We say f is continuous at $x \in X$ if whenever V is an open set of Y containing $f(x)$, there exists an open set $U \in \tau$ such that $x \in U$ and $f(U) \subseteq V$. We say that f is continuous if $f^{-1}(V) \in \tau$ whenever $V \in \eta$.*

Definition 1.21 *Let (X, τ) and (Y, η) be two topological spaces. $X \times Y$ is the Cartesian product. ($X \times Y = \{(x, y) : x \in X,\ y \in Y\}$). We can define a product topology as follows. Let $\mathcal{B} = \{(A \times B) : A \in \tau,\ B \in \eta\}$. $\mathcal{B}$ is a basis for the product topology.*

Theorem 1.22 *$\mathcal{B}$ defined above is a basis satisfying the conditions of Theorem 1.3.*

More generally we have the following definition which considers any finite Cartesian product of topological spaces.

Definition 1.23 *If (X_i, τ_i) is a topological space, we make $\prod_{i=1}^n X_i$ into a topological space by letting a basis be $\prod_{i=1}^n A_i$ where $A_i \in \tau_i$.*

Theorem 1.24 *Definition 1.23 yields a basis for a topology.*

The proof of this theorem is almost immediate from the definition and is left for the reader.

The definition of compactness is also considered for a general topological space. This is given next.

Definition 1.25 *A subset, E, of a topological space (X, τ) is said to be compact if whenever $\mathcal{C} \subseteq \tau$ and $E \subseteq \cup\mathcal{C}$, there exists a finite subset of $\mathcal{C}, \{U_1 \cdots U_n\}$, such that $E \subseteq \cup_{i=1}^n U_i$. (Every open covering admits a finite subcovering.) A topological space is called Locally Compact if it has a basis $\mathcal{B}$, with the property that $\bar{B}$ is compact for each $B \in \mathcal{B}$.*

Examples of locally compact topological spaces are $\mathbb{R}$ or $\mathbb{C}$.

In general topological spaces there may be no concept of "bounded". Even if there is, closed and bounded is not necessarily the same as compactness. However, we can say that in any Hausdorff space every compact set must be a closed set.

Theorem 1.26 *If (X, τ) is a Hausdorff space, then every compact subset must also be a closed set.*

Proof: Suppose $p \notin K$. For each $x \in X$, there exist open sets, U_x and V_x such that

$$x \in U_x,\ p \in V_x,$$

and

$$U_x \cap V_x = \emptyset.$$

Since K is assumed to be compact, there are finitely many of these sets, $U_{x_1}, \cdots, U_{x_m}$ which cover K. Then let $V \equiv \cap_{i=1}^m V_{x_i}$. It follows that V is an open set containing p which has empty intersection with each of the U_{x_i}. Consequently, V contains no points of K and is therefore not a limit point. This proves the theorem.

Lemma 1.27 *Let (X, τ) be a topological space and let $\mathcal{B}$ be a basis for τ. Then K is compact if and only if every open cover of basic open sets admits a finite subcover.*

The proof follows directly from the definition and is left to the reader. A very important property enjoyed by a collection of compact sets is the property that if it can be shown that any finite intersection of this collection has non empty intersection, then it can be concluded that the intersection of the whole collection has non empty intersection. If every finite subset of a collection of sets has nonempty intersection, we say the collection has the finite intersection property.

Theorem 1.28 *Let $\mathcal{K}$ be a set whose elements are compact subsets of a Hausdorff topological space, (X, τ). Suppose $\mathcal{K}$ has the finite intersection property. Then $\emptyset \neq \cap \mathcal{K}$.*

Proof: Suppose to the contrary that $\emptyset = \cap \mathcal{K}$. Then consider

$$\mathcal{C} \equiv \left\{K^C : K \in \mathcal{K}\right\}.$$

It follows $\mathcal{C}$ is an open cover of K_0 where K_0 is any particular element of $\mathcal{K}$. But then there are finitely many $K \in \mathcal{K}$, $K_1, \cdots, K_r$ such that $K_0 \subseteq \cup_{i=1}^r K_i^C$ implying that $\cap_{i=0}^r K_i = \emptyset$, contradicting the finite intersection property.

2 Compactness in metric space

Many existence theorems in analysis depend on some set being compact. Therefore, it is important to be able to identify compact sets. The purpose of this section is to describe compact sets in a metric space. The most famous description is that contained in the Heine Borel theorem which states that the compact sets in $\mathbb{R}^n$ are those which are closed and bounded. However, this result is certainly not true in general metric space. For example, let X be any infinite set and let

$$d(x, y) = \begin{cases} 1 \text{ if } x \neq y \\ 0 \text{ if } x = y \end{cases}. \tag{2.1}$$

We leave it as an exercise to verify that d is a metric and that "closed and bounded" is not the same as compact for this example. We will have need of theorems which will give compactness for spaces of continuous functions later on. For these spaces, compactness is also not equivalent to closed and bounded. In fact, it can be shown that the two concepts are the same for a normed linear space exactly when the normed linear space is finite dimensional. Thus the Heine Borel theorem is a very specialized result and must not be confused with the general topological concept of compactness. What follows is a general result for compactness in an arbitrary metric space. We will prove this result, use it to get a proof of the Heine Borel theorem and then apply it to an infinite dimensional space to obtain the very significant Arzela Ascoli theorem.

Definition 2.1 *In any metric space, we say a set E is totally bounded if for every $\epsilon > 0$ there exists a finite set of points $\{x_1, \cdots, x_n\}$ such that*

$$E \subseteq \cup_{i=1}^n B(x_i, \epsilon).$$

This finite set of points is called an ϵ net.

The following proposition tells which sets in a metric space are compact.

Proposition 2.2 *Let (X, d) be a metric space. Then the following are equivalent.*

$$(X, d) \text{ is compact,} \tag{2.2}$$

$$(X, d) \text{ is sequentially compact,} \tag{2.3}$$

$$(X, d) \text{ is complete and totally bounded.} \tag{2.4}$$

Recall that X is "sequentially compact" means every sequence has a convergent subsequence converging so an element of X.

Proof: Suppose 2.2 and let $\{x_k\}$ be a sequence. Suppose $\{x_k\}$ has no convergent subsequence. If this is so, then $\{x_k\}$ has no limit point and no value of the sequence is repeated more than finitely many times. Thus the set

$$C_n = \cup\{x_k : k \geq n\}$$

is a closed set and if

$$U_n = C_n^C,$$

then

$$X = \cup_{n=1}^{\infty} U_n$$

but there is no finite subcovering, contradicting compactness of (X, d).

Now suppose Formula 2.3 and let $\{x_n\}$ be a Cauchy sequence. Then $x_{n_k} \to x$ for some subsequence. Let $\epsilon > 0$ be given. Let n_0 be such that if $m, n \geq n_0$, then $d(x_n, x_m) < \frac{\epsilon}{2}$ and let l be such that if $k \geq l$ then $d(x_{n_k}, x) < \frac{\epsilon}{2}$. Let $n_1 > \max(n_l, n_0)$. If $n > n_1$, let $k > l$ and $n_k > n_0$.

$$\begin{aligned} d(x_n, x) &\leq d(x_n, x_{n_k}) + d(x_{n_k}, x) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Thus $\{x_n\}$ converges to x and this shows (X, d) is complete. If (X, d) is not totally bounded, then there exists $\epsilon > 0$ for which there is no ϵ net. Hence there exists a sequence $\{x_k\}$ with $d(x_k, x_l) \geq \epsilon$ for all $l \neq k$. This contradicts Formula 2.3 because this is a sequence having no convergent subsequence. This shows Formula 2.3 implies Formula 2.4.

Now suppose Formula 2.4. We show this implies Formula 2.3. Let $\{p_n\}$ be a sequence and let $\{x_i^n\}_{i=1}^{m_n}$ be a 2^{-n} net for $n = 1, 2, \cdots$. Let

$$B_n \equiv B\left(x_{i_n}^n, 2^{-n}\right)$$

be such that B_n contains p_k for infinitely many values of k and $B_n \cap B_{n+1} \neq \emptyset$. Let p_{n_k} be a subsequence having

$$p_{n_k} \in B_k.$$

Then if $k \geq l$,

$$\begin{aligned} d(p_{n_k}, p_{n_l}) &\leq \sum_{i=l}^{k-1} d\left(p_{n_{i+1}}, p_{n_i}\right) \\ &< \sum_{i=l}^{k-1} 2^{-(i-1)} < 2^{-(l-2)}. \end{aligned}$$

Consequently $\{p_{n_k}\}$ is a Cauchy sequence. Hence it converges. This proves Formula 2.3.

Now suppose Formula 2.3 and Formula 2.4. Let D_n be a n^{-1} net for $n = 1, 2, \cdots$ and let

$$D = \cup_{n=1}^{\infty} D_n.$$

Thus D is a countable dense subset of (X, d). The set of balls

$$\mathcal{B} = \{B(q, r) : q \in D,\ r \in Q \cap (0, \infty)\}$$

is a countable basis for (X, d). To see this, let $p \in B(x, \epsilon)$ and choose $r \in Q \cap (0, \infty)$ such that

$$\epsilon - d(p, x) > 2r.$$

Let $q \in B(p, r) \cap D$. If $y \in B(q, r)$, then

$$\begin{aligned} d(y, x) &\leq d(y, q) + d(q, p) + d(p, x) \\ &< r + r + \epsilon - 2r = \epsilon. \end{aligned}$$

Hence $p \in B(q, r) \subseteq B(x, \epsilon)$ and this shows each ball is the union of balls of $\mathcal{B}$. Now suppose $\mathcal{C}$ is any open cover of X. Let $\widetilde{\mathcal{B}}$ denote the balls of $\mathcal{B}$ which are contained in some set of $\mathcal{C}$. Thus

$$\cup \widetilde{\mathcal{B}} = X.$$

For each $B \in \widetilde{\mathcal{B}}$, pick $U \in \mathcal{C}$ such that $U \supseteq B$. Let $\widetilde{\mathcal{C}}$ be the resulting countable collection of sets. Then $\widetilde{\mathcal{C}}$ is a countable open cover of X. Say $\widetilde{\mathcal{C}} = \{U_n\}_{n=1}^{\infty}$. If $\mathcal{C}$ admits no finite subcover, then neither does $\widetilde{\mathcal{C}}$ and we can pick $p_n \in X \setminus \cup_{k=1}^{n} U_k$. Then since X is sequentially compact, there is a subsequence $\{p_{n_k}\}$ such that $\{p_{n_k}\}$ converges. Say

$$p = \lim_{k \to \infty} p_{n_k}.$$

All but finitely many points of $\{p_{n_k}\}$ are in $X \setminus \cup_{k=1}^{n} U_k$. Therefore $p \in X \setminus \cup_{k=1}^{n} U_k$ for each n. Hence

$$p \notin \cup_{k=1}^{\infty} U_k$$

contradicting the construction of $\{U_n\}_{n=1}^{\infty}$. Hence X is compact. This proves the proposition.

Next we apply this very general result to a familiar example, $\mathbb{R}^n$. In this setting totally bounded and bounded are the same. This will yield another proof of the Heine Borel theorem.

Lemma 2.3 *A subset of $\mathbb{R}^n$ is totally bounded if and only if it is bounded.*

Proof: Let A be totally bounded. We need to show it is bounded. Let $\mathbf{x}_1, \cdots, \mathbf{x}_p$ be a 1 net for A. Now consider the ball $B(\mathbf{0}, r+1)$ where $r > \max(||\mathbf{x}_i|| : i = 1, \cdots, p)$. If $\mathbf{z} \in A$, then $\mathbf{z} \in B(\mathbf{x}_j, 1)$ for some j and so by the triangle inequality,

$$||\mathbf{z} - \mathbf{0}|| \leq ||\mathbf{z} - \mathbf{x}_j|| + ||\mathbf{x}_j|| < 1 + r.$$

Thus $A \subseteq B(\mathbf{0}, r+1)$ and so A is bounded.

Now suppose A is bounded and suppose A is not totally bounded. Then there exists $\epsilon > 0$ such that there is no ϵ net for A. Therefore, there exists a sequence of points $\{a_i\}$ with $||a_i - a_j|| \geq \epsilon$ if $i \neq j$. Since A is bounded, there exists $r > 0$ such that

$$A \subseteq [-r, r)^n.$$

($\mathbf{x} \in [-r, r)^n$ means $x_i \in [-r, r)$ for each i.) Now define $\mathcal{S}$ to be all cubes of the form

$$\prod_{k=1}^{n} [a_k, b_k)$$

where

$$a_k = -r + i2^{-p} r, \; b_k = -r + (i+1) 2^{-p} r,$$

for $i \in \{0, 1, \cdots, 2^{p+1} - 1\}$. Thus $\mathcal{S}$ is a collection of $\left(2^{p+1}\right)^n$ nonoverlapping cubes whose union equals $[-r, r)^n$ and whose diameters are all equal to $2^{-p} r \sqrt{n}$. Now choose p large enough that the diameter of these cubes is less than ϵ. This yields a contradiction because one of the cubes must contain infinitely many points of $\{a_i\}$. This proves the lemma.

The next theorem is called the Heine Borel theorem and it characterizes the compact sets in $\mathbb{R}^n$.

Theorem 2.4 *A subset of* $\mathbb{R}^n$ *is compact if and only if it is closed and bounded.*

Proof: Since a set in $\mathbb{R}^n$ is totally bounded if and only if it is bounded, this theorem follows from Proposition 2.2 and the observation that a subset of $\mathbb{R}^n$ is closed if and only if it is complete. This proves the theorem.

The following corollary is an important existence theorem which depends on compactness.

Corollary 2.5 *Let* (X, τ) *be a compact topological space and let* $f : X \to \mathbb{R}$ *be continuous. Then* $\max\{f(x) : x \in X\}$ *and* $\min\{f(x) : x \in X\}$ *both exist.*

Proof: Since X is compact, it follows that $f(X)$ is compact. (See Problem 4 in the next exercise set.) From Theorem 2.4 $f(X)$ is closed and bounded. This implies it has a largest and a smallest value. This proves the corollary.

2.1 Compactness in spaces of continuous functions

Let (X, τ) be a compact space and let $C(X; \mathbb{R}^n)$ denote the space of continuous $\mathbb{R}^n$ valued functions. For $f \in C(X; \mathbb{R}^n)$ let

$$||f||_\infty \equiv \sup\{|f(x)| : x \in X\}$$

where the norm in the parenthesis refers to the usual norm in $\mathbb{R}^n$.

The following proposition shows that $C(X; \mathbb{R}^n)$ is an example of a Banach space.

Proposition 2.6 $(C(X; \mathbb{R}^n), ||\ ||_\infty)$ *is a Banach space.*

Proof: It is obvious $||\ ||_\infty$ is a norm because (X, τ) is compact. Also it is clear that $C(X; \mathbb{R}^n)$ is a linear space. Suppose $\{f_r\}$ is a Cauchy sequence in $C(X; \mathbb{R}^n)$. Then for each $x \in X$, $\{f_r(x)\}$ is a Cauchy sequence in $\mathbb{R}^n$. Let

$$f(x) \equiv \lim_{k \to \infty} f_k(x).$$

Therefore,

$$\sup_{x \in X} |f(x) - f_k(x)| = \sup_{x \in X} \lim_{m \to \infty} |f_m(x) - f_k(x)|$$

$$\leq \lim \sup_{m \to \infty} ||f_m - f_k||_\infty < \epsilon$$

for all k large enough. Thus,

$$\lim_{k \to \infty} \sup_{x \in X} |f(x) - f_k(x)| = 0.$$

It only remains to show that f is continuous. Let

$$\sup_{x \in X} |f(x) - f_k(x)| < \epsilon/3$$

whenever $k \geq k_0$ and pick $k \geq k_0$.

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_k(x)| + |f_k(x) - f_k(y)| + |f_k(y) - f(y)| \\ &< 2\epsilon/3 + |f_k(x) - f_k(y)| \end{aligned}$$

Now f_k is continuous and so there exists U an open set containing x such that if $y \in U$, then

$$|f_k(x) - f_k(y)| < \epsilon/3.$$

Thus, for all $y \in U$, $|f(x) - f(y)| < \epsilon$ and this shows that f is continuous and proves the proposition.

This space is a normed linear space and so it is a metric space with the distance given by $d(f,g) \equiv ||f-g||_\infty$. The next task is to find the compact subsets of this metric space. We know these are the subsets which are complete and totally bounded by Proposition 2.2, but which sets are those? We need another way to identify them which is more convenient. This is the extremely important Ascoli Arzela theorem which is the next big theorem.

Definition 2.7 *We say $\mathcal{F} \subseteq C(X;\mathbb{R}^n)$ is equicontinuous at x_0 if for all $\epsilon > 0$ there exists $U \in \tau$, $x_0 \in U$, such that if $x \in U$, then for all $f \in \mathcal{F}$,*

$$|f(x) - f(x_0)| < \epsilon.$$

If $\mathcal{F}$ is equicontinuous at every point of X, we say $\mathcal{F}$ is equicontinuous. We say $\mathcal{F}$ is bounded if there exists a constant, M, such that $||f||_\infty < M$ for all $f \in \mathcal{F}$.

Lemma 2.8 *Let $\mathcal{F} \subseteq C(X;\mathbb{R}^n)$ be equicontinuous and bounded and let $\epsilon > 0$ be given. Then if $\{f_r\} \subseteq \mathcal{F}$, there exists a subsequence $\{g_k\}$, depending on ϵ, such that*

$$||g_k - g_m||_\infty < \epsilon$$

whenever k, m are large enough.

Proof: If $x \in X$ there exists an open set U_x containing x such that for all $f \in \mathcal{F}$ and $y \in U_x$,

$$|f(x) - f(y)| < \epsilon/4. \tag{2.5}$$

Since X is compact, finitely many of these sets, $U_{x_1}, \cdots, U_{x_p}$, cover X. Let $\{f_{1k}\}$ be a subsequence of $\{f_k\}$ such that $\{f_{1k}(x_1)\}$ converges. Such a subsequence exists because $\mathcal{F}$ is bounded. Let $\{f_{2k}\}$ be a subsequence of $\{f_{1k}\}$ such that $\{f_{2k}(x_i)\}$ converges for $i = 1, 2$. Continue in this way and let $\{g_k\} = \{f_{pk}\}$. Thus $\{g_k(x_i)\}$ converges for each x_i. Therefore, if $\epsilon > 0$ is given, there exists m_ϵ such that for $k, m > m_\epsilon$,

$$\max\{|g_k(x_i) - g_m(x_i)| : i = 1, \cdots, p\} < \frac{\epsilon}{2}.$$

Now if $y \in X$, then $y \in U_{x_i}$ for some x_i. Denote this x_i by x_y. Now let $y \in X$ and $k, m > m_\epsilon$. Then by 2.5,

$$|g_k(y) - g_m(y)| \leq |g_k(y) - g_k(x_y)| + |g_k(x_y) - g_m(x_y)| + |g_m(x_y) - g_m(y)|$$

$$< \frac{\epsilon}{4} + \max\{|g_k(x_i) - g_m(x_i)| : i = 1, \cdots, p\} + \frac{\epsilon}{4} < \varepsilon.$$

It follows that for such k, m,

$$||g_k - g_m||_\infty < \epsilon$$

and this proves the lemma.

Theorem 2.9 *(Ascoli Arzela) Let $\mathcal{F} \subseteq C(X;\mathbb{R}^n)$. Then $\mathcal{F}$ is compact if and only if $\mathcal{F}$ is closed, bounded, and equicontinuous.*

Proof: Suppose $\mathcal{F}$ is closed, bounded, and equicontinuous. We will show this implies $\mathcal{F}$ is totally bounded. Then since $\mathcal{F}$ is closed, it follows that $\mathcal{F}$ is complete and will therefore be compact by Proposition 2.2. Suppose $\mathcal{F}$ is not totally bounded. Then there exists $\epsilon > 0$ such that there is no ϵ net. Hence there exists a sequence $\{f_k\} \subseteq \mathcal{F}$ such that

$$||f_k - f_l|| \geq \epsilon$$

for all $k \neq l$. This contradicts Lemma 2.8. Thus $\mathcal{F}$ must be totally bounded and this proves half of the theorem.

Now suppose $\mathcal{F}$ is compact. Then it must be closed and totally bounded. This implies $\mathcal{F}$ is bounded. It remains to show $\mathcal{F}$ is equicontinuous. Suppose not. Then there exists $x \in X$ such that $\mathcal{F}$ is not equicontinuous at x. Thus there exists $\epsilon > 0$ such that for every open U containing x, there exists $f \in \mathcal{F}$ such that $|f(x) - f(y)| \geq \epsilon$ for some $y \in U$.

Let $\{h_1, \cdots, h_p\}$ be an $\epsilon/4$ net for $\mathcal{F}$. For each z, let U_z be an open set containing z such that for all $y \in U_z$,

$$|h_i(z) - h_i(y)| < \epsilon/8$$

for all $i = 1, \cdots, p$. Let $U_{x_1}, \cdots, U_{x_m}$ cover X. Then $x \in U_{x_i}$ for some x_i and so, for some $y \in U_{x_i}$,there exists $f \in \mathcal{F}$ such that $|f(x) - f(y)| \geq \epsilon$. Since $\{h_1, \cdots, h_p\}$ is an $\epsilon/4$ net, it follows that for some j, $||f - h_j||_\infty < \frac{\epsilon}{4}$ and so

$$\epsilon \leq |f(x) - f(y)| \leq |f(x) - h_j(x)| + |h_j(x) - h_j(y)| +$$

$$|h_i(y) - f(y)| \leq \epsilon/2 + |h_j(x) - h_j(y)| \leq \epsilon/2 +$$

$$|h_j(x) - h_j(x_i)| + |h_j(x_i) - h_j(y)| \leq 3\epsilon/4,$$

a contradiction. This proves the theorem.

2.2 Exercises

1. Let $(X, \tau), (Y, \eta)$ be topological spaces and let $A \subseteq X$ be compact. Then if $f : X \to Y$ is continuous, show that $f(A)$ is also compact.

2. $\uparrow$ In the context of Problem 1, suppose $\mathbb{R} = Y$ where the usual topology is placed on $\mathbb{R}$. Show f achieves its maximum and minimum on A.

3. Let V be an open set in $\mathbb{R}^n$. Show there is an increasing sequence of compact sets, K_m, such that $V = \cup_{m=1}^\infty K_m$. **Hint:** Let

$$C_m \equiv \left\{ \mathbf{x} \in \mathbb{R}^n : dist\left(\mathbf{x}, V^C\right) \geq \frac{1}{m} \right\}$$

where

$$dist(\mathbf{x}, S) \equiv \inf\{|\mathbf{y} - \mathbf{x}| \text{ such that } \mathbf{y} \in S\}.$$

Consider $K_m \equiv C_m \cap \overline{B(\mathbf{0}, m)}$.

4. Show that if X is compact and $f : X \to Y$ is continuous, then $f(X)$ is also compact.

5. Let $B(X;\mathbb{R}^n)$ be the space of functions $\mathbf{f}$, mapping X to $\mathbb{R}^n$ such that

$$\sup\{|\mathbf{f}(\mathbf{x})| : \mathbf{x} \in X\} < \infty.$$

Show $B(X;\mathbb{R}^n)$ is a complete normed linear space if

$$||\mathbf{f}|| \equiv \sup\{|\mathbf{f}(\mathbf{x})| : \mathbf{x} \in X\}.$$

6. Let H and K be disjoint closed sets in a metric space, (X,d), and let

$$g(x) \equiv \frac{2}{3}h(x) - \frac{1}{3}$$

where

$$h(x) \equiv \frac{dist(x,H)}{dist(x,H) + dist(x,K)}.$$

Show $g(x) \in \left[-\frac{1}{3}, \frac{1}{3}\right]$ for all $x \in X$, g is continuous, and g equals $\frac{-1}{3}$ on H while g equals $\frac{1}{3}$ on K.

7. $\uparrow$ Suppose M is a closed set in X where X is the metric space of problem 6 and suppose $f : M \to [-1,1]$ is continuous. Show there exists $g : X \to [-1,1]$ such that g is continuous and $g = f$ on M. **Hint:** Show there exists

$$g_1 \in C(X),\ g_1(x) \in \left[\frac{-1}{3}, \frac{1}{3}\right],$$

and $|f(x) - g_1(x)| \le \frac{2}{3}$ for all $x \in H$. To do this, consider the disjoint closed sets

$$H \equiv f^{-1}\left(\left[-1, \frac{-1}{3}\right]\right),\ K \equiv f^{-1}\left(\left[\frac{1}{3}, 1\right]\right)$$

and use Problem 6 if the two sets are nonempty. When this has been done, let

$$\frac{3}{2}(f(x) - g_1(x))$$

play the role of f and let g_2 be like g_1. Obtain

$$\left| f(x) - \sum_{i=1}^{n} \left(\frac{2}{3}\right)^{i-1} g_i(x) \right| \le \left(\frac{2}{3}\right)^n$$

and consider

$$g(x) \equiv \sum_{i=1}^{\infty} \left(\frac{2}{3}\right)^{i-1} g_i(x).$$

8. $\uparrow$ Let M be a closed set in a metric space (X,d) and suppose $f \in C(M)$. Show there exists $g \in C(X)$ such that $g(x) = f(x)$ for all $x \in M$ and if $f(M) \subseteq [a,b]$, then $g(X) \subseteq [a,b]$. This is a version of the Tietze extension theorem. Is it necessary to be in a metric space for this to work?

9. Let X be a compact topological space and suppose $\{f_n\}$ is a sequence of functions continuous on X having values in $\mathbb{R}^n$. Show there exists a countable dense subset of X, $\{x_i\}$ and a subsequence of $\{f_n\}$, $\{f_{n_k}\}$, such that $\{f_{n_k}(x_i)\}$ converges for each x_i. **Hint:** First get a subsequence which converges at x_1, then a subsequence of this subsequence which converges at x_2 and a subsequence of this one which converges at x_3 and so forth. Thus the second of these subsequences converges at both x_1 and x_2 while the third converges at these two points and also at x_3 and so forth. List them so the second is under the first and the third is under the second and so forth thus obtaining an infinite matrix of entries. Now consider the diagonal sequence and argue it is ultimately a subsequence of every one of these subsequences described earlier and so it must converge at each x_i. This procedure is called the Cantor diagonal process.

10. $\uparrow$ Use the Cantor diagonal process to give a different proof of the Ascoli Arzela theorem than that presented in this chapter. **Hint:** Start with a sequence of functions in $C(X;\mathbb{R}^n)$ and use the Cantor diagonal process to produce a subsequence which converges at each point of a countable dense subset of X. Then show this sequence is a Cauchy sequence in $C(X;\mathbb{R}^n)$.

11. Let (X,d) be a metric space where d is a bounded metric. Let $\mathcal{C}$ denote the collection of closed subsets of X. For $A,B\in\mathcal{C}$, define

$$\rho(A,B)\equiv\inf\{\delta>0:A_\delta\supseteq B\text{ and }B_\delta\supseteq A\}$$

where for a set S,

$$S_\delta\equiv\{x:dist(x,S)\equiv\inf\{d(x,s):s\in S\}\leq\delta\}.$$

Show $x\to dist(x,S)$ is continuous and that therefore, S_δ is a closed set containing S. Also show that ρ is a metric on $\mathcal{C}$. This is called the Hausdorff metric.

12. $\uparrow$Suppose (X,d) is a compact metric space. Show $(\mathcal{C},\rho)$ is a complete metric space. **Hint:** Show first that if $W_n\downarrow W$ where W_n is closed, then $\rho(W_n,W)\to 0$. Now let $\{A_n\}$ be a Cauchy sequence in $\mathcal{C}$. Then if $\epsilon>0$ there exists N such that when $m,n\geq N$, then $\rho(A_n,A_m)<\epsilon$. Therefore, for each $n\geq N$,

$$(A_n)_\epsilon\supseteq\overline{\cup_{k=n}^\infty A_k}.$$

Let $A\equiv\cap_{n=1}^\infty\overline{\cup_{k=n}^\infty A_k}$. By the first part, there exists $N_1>N$ such that for $n\geq N_1$,

$$\rho\left(\overline{\cup_{k=n}^\infty A_k},A\right)<\epsilon,\text{ and }(A_n)_\epsilon\supseteq\overline{\cup_{k=n}^\infty A_k}.$$

Therefore, for such n, $A_\epsilon\supseteq W_n\supseteq A_n$ and $(W_n)_\epsilon\supseteq(A_n)_\epsilon\supseteq A$ because

$$(A_n)_\epsilon\supseteq\overline{\cup_{k=n}^\infty A_k}\supseteq A.$$

13. $\uparrow$ Let X be a compact metric space. Show $(\mathcal{C},\rho)$ is compact. **Hint:** Let $\mathcal{D}_n$ be a 2^{-n} net for X. Let $\mathcal{K}_n$ denote finite unions of sets of the form $\overline{B(p,2^{-n})}$ where $p\in\mathcal{D}_n$. Show $\mathcal{K}_n$ is a $2^{-(n-1)}$ net for $(\mathcal{C},\rho)$.

2.3 Connected sets

Stated informally, connected sets are those which are in one piece. More precisely, we give the following definition.

Definition 2.10 *We say a set, S in a general topological space is separated if there exist sets, A,B such that*

$$S=A\cup B,\ A,B\neq\emptyset,\ \textit{and }\overline{A}\cap B=\overline{B}\cap A=\emptyset.$$

In this case, the sets A and B are said to separate S. We say a set is connected if it is not separated.

One of the most important theorems about connected sets is the following.

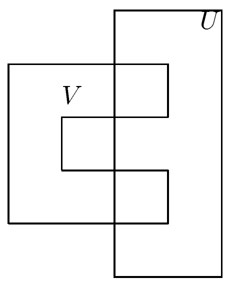
Theorem 2.11 *Suppose U and V are connected sets having nonempty intersection. Then $U\cup V$ is also connected.*

Proof: Suppose $U \cup V = A \cup B$ where $\overline{A} \cap B = \overline{B} \cap A = \emptyset$. Consider the sets, $A \cap U$ and $B \cup U$. Since

$$\overline{(A \cap U)} \cap (B \cap U) = (A \cap U) \cap \left(\overline{B \cap U}\right) = \emptyset,$$

It follows one of these sets must be empty since otherwise, U would be separated. It follows that U is contained in either A or B. Similarly, V must be contained in either A or B. Since U and V have nonempty intersection, it follows that both V and U are contained in one of the sets, A, B. Therefore, the other must be empty and this shows $U \cup V$ cannot be separated and is therefore, connected.

The intersection of connected sets is not necessarily connected as is shown by the following picture.



Theorem 2.12 *Let $f : X \to Y$ be continuous where X and Y are topological spaces and X is connected. Then $f(X)$ is also connected.*

Proof: We show $f(X)$ is not separated. Suppose to the contrary that $f(X) = A \cup B$ where A and B separate $f(X)$. Then consider the sets, $f^{-1}(A)$ and $f^{-1}(B)$. If $z \in f^{-1}(B)$, then $f(z) \in B$ and so $f(z)$ is not a limit point of A. Therefore, there exists an open set, U containing $f(z)$ such that $U \cap A = \emptyset$. But then, the continuity of f implies that $f^{-1}(U)$ is an open set containing z such that $f^{-1}(U) \cap f^{-1}(A) = \emptyset$. Therefore, $f^{-1}(B)$ contains no limit points of $f^{-1}(A)$. Similar reasoning implies $f^{-1}(A)$ contains no limit points of $f^{-1}(B)$. It follows that X is separated by $f^{-1}(A)$ and $f^{-1}(B)$, contradicting the assumption that X was connected.

An arbitrary set can be written as a union of maximal connected sets called connected components. This is the concept of the next definition.

Definition 2.13 *Let S be a set and let $p \in S$. Denote by C_p the union of all connected subsets of S which contain p. This is called the connected component determined by p.*

Theorem 2.14 *Let C_p be a connected component of a set S in a general topological space. Then C_p is a connected set and if $C_p \cap C_q \neq \emptyset$, then $C_p = C_q$.*

Proof: Let $\mathcal{C}$ denote the connected subsets of S which contain p. If $C_p = A \cup B$ where

$$\overline{A} \cap B = \overline{B} \cap A = \emptyset,$$

then p is in one of A or B. Suppose without loss of generality $p \in A$. Then every set of $\mathcal{C}$ must also be contained in A also since otherwise, as in Theorem 2.11, the set would be separated. But this implies B is empty. Therefore, C_p is connected. From this, and Theorem 2.11, the second assertion of the theorem is proved.

This shows the connected components of a set are equivalence classes and partition the set.

A set, I is an interval in $\mathbb{R}$ if and only if whenever $x, y \in I$ then $(x, y) \subseteq I$. The following theorem is about the connected sets in $\mathbb{R}$.

Theorem 2.15 *A set, C in $\mathbb{R}$ is connected if and only if C is an interval.*

Proof: Let C be connected. If C consists of a single point, p, there is nothing to prove. The interval is just $[p,p]$. Suppose $p<q$ and $p,q\in C$. We need to show $(p,q)\subseteq C$. If

$$x\in(p,q)\setminus C$$

let $C\cap(-\infty,x)\equiv A$, and $C\cap(x,\infty)\equiv B$. Then $C=A\cup B$ and the sets, A and B separate C contrary to the assumption that C is connected.

Conversely, let I be an interval. Suppose I is separated by A and B. Pick $x\in A$ and $y\in B$. Suppose without loss of generality that $x<y$. Now define the set,

$$S\equiv\{t\in[x,y]:[x,t]\subseteq A\}$$

and let l be the least upper bound of S. Then $l\in\overline{A}$ so $l\notin B$ which implies $l\in A$. But if $l\notin\overline{B}$, then for some $\delta>0$,

$$(l,l+\delta)\cap B=\emptyset$$

contradicting the definition of l as an upper bound for S. Therefore, $l\in\overline{B}$ which implies $l\notin A$ after all, a contradiction. It follows I must be connected.

The following theorem is a very useful description of the open sets in $\mathbb{R}$.

Theorem 2.16 *Let U be an open set in $\mathbb{R}$. Then there exist countably many disjoint open sets, $\{(a_i,b_i)\}_{i=1}^{\infty}$ such that $U=\cup_{i=1}^{\infty}(a_i,b_i)$.*

Proof: Let $p\in U$ and let $z\in C_p$, the connected component determined by p. Since U is open, there exists, $\delta>0$ such that $(z-\delta,z+\delta)\subseteq U$. It follows from Theorem 2.11 that

$$(z-\delta,z+\delta)\subseteq C_p.$$

This shows C_p is open. By Theorem 2.15, this shows C_p is an open interval, (a,b) where $a,b\in[-\infty,\infty]$. There are therefore at most countably many of these connected components because each must contain a rational number and the rational numbers are countable. Denote by $\{(a_i,b_i)\}_{i=1}^{\infty}$ the set of these connected components. This proves the theorem.

Definition 2.17 *We say a topological space, E is arcwise connected if for any two points, $p,q\in E$, there exists a closed interval, $[a,b]$ and a continuous function, $\gamma:[a,b]\to E$ such that $\gamma(a)=p$ and $\gamma(b)=q$. We say E is locally connected if it has a basis of connected open sets. We say E is locally arcwise connected if it has a basis of arcwise connected open sets.*

An example of an arcwise connected topological space would be the any subset of $\mathbb{R}^n$ which is the continuous image of an interval. Locally connected is not the same as connected. A well known example is the following.

$$\left\{\left(x,\sin\frac{1}{x}\right):x\in(0,1]\right\}\cup\{(0,y):y\in[-1,1]\} \tag{2.6}$$

We leave it as an exercise to verify that this set of points considered as a metric space with the metric from $\mathbb{R}^2$ is not locally connected or arcwise connected but is connected.

Proposition 2.18 *If a topological space is arcwise connected, then it is connected.*

Proof: Let X be an arcwise connected space and suppose it is separated. Then $X=A\cup B$ where A,B are two separated sets. Pick $p\in A$ and $q\in B$. Since X is given to be arcwise connected, there must exist a continuous function $\gamma:[a,b]\to X$ such that $\gamma(a)=p$ and $\gamma(b)=q$. But then we would have $\gamma([a,b])=(\gamma([a,b])\cap A)\cup(\gamma([a,b])\cap B)$ and the two sets, $\gamma([a,b])\cap A$ and $\gamma([a,b])\cap B$ are separated thus showing that $\gamma([a,b])$ is separated and contradicting Theorem 2.15 and Theorem 2.12. It follows that X must be connected as claimed.

Theorem 2.19 *Let U be an open subset of a locally arcwise connected topological space, X. Then U is arcwise connected if and only if U is connected. Also the connected components of an open set in such a space are open sets, hence arcwise connected.*

Proof: By Proposition 2.18 we only need to verify that if U is connected and open in the context of this theorem, then U is arcwise connected. Pick $p \in U$. We will say $x \in U$ satisfies $\mathcal{P}$ if there exists a continuous function, $\gamma : [a,b] \to U$ such that $\gamma(a) = p$ and $\gamma(b) = x$.

$$A \equiv \{x \in U \text{ such that } x \text{ satisfies } \mathcal{P}.\}$$

If $x \in A$, there exists, according to the assumption that X is locally arcwise connected, an open set, V, containing x and contained in U which is arcwise connected. Thus letting $y \in V$, there exist intervals, $[a,b]$ and $[c,d]$ and continuous functions having values in U, γ, η such that $\gamma(a) = p, \gamma(b) = x, \eta(c) = x$, and $\eta(d) = y$. Then let $\gamma_1 : [a, b+d-c] \to U$ be defined as

$$\gamma_1(t) \equiv \begin{cases} \gamma(t) & \text{if } t \in [a,b] \\ \eta(t) & \text{if } t \in [b, b+d-c] \end{cases}$$

Then it is clear that γ_1 is a continuous function mapping p to y and showing that $V \subseteq A$. Therefore, A is open. We also know that $A \neq \emptyset$ because there is an open set, V containing p which is contained in U and is arcwise connected.

Now consider $B \equiv U \setminus A$. We will verify that this is also open. If B is not open, there exists a point $z \in B$ such that every open set containing z is not contained in B. Therefore, letting V be one of the basic open sets chosen such that $z \in V \subseteq U$, we must have points of A contained in V. But then, a repeat of the above argument shows $z \in A$ also. Hence B is open and so if $B \neq \emptyset$, then $U = B \cup A$ and so U is separated by the two sets, B and A contradicting the assumption that U is connected.

We need to verify the connected components are open. Let $z \in C_p$ where C_p is the connected component determined by p. Then picking V an arcwise connected open set which contains z and is contained in U, $C_p \cup V$ is connected and contained in U and so it must also be contained in C_p. This proves the theorem.

Corollary 2.20 *Let U be an open subset of $\mathbb{R}^n$. Then U is arcwise connected if and only if U is connected. Also the connected components of an open set in $\mathbb{R}^n$ are open sets, hence arcwise connected. Also, there are at most countably many components of U.*

Proof: This follows from Theorem 2.19 by observing that $\mathbb{R}^n$ is locally arcwise connected due to the convexity of balls. Since $\mathbb{Q}^n$ is a dense countable subset of $\mathbb{R}^n$ it follows that there are at most countably many components because these components are disjoint and each, being an open set, contains a point of the countable dense open set, $\mathbb{Q}^n$.

2.4 Exercises

1. Prove the definition of distance in $\mathbb{R}^n$ or $\mathbb{C}^n$ satisfies 1.3-1.4. In addition to this, prove that $||\cdot||$ given by $||\mathbf{x}|| = (\sum_{i=1}^n |x_i|^2)^{1/2}$ is a norm. This means it satisfies the following.

$$||\mathbf{x}|| \geq 0, \; ||\mathbf{x}|| = 0 \text{ if and only if } \mathbf{x} = \mathbf{0}.$$

$$||\alpha \mathbf{x}|| = |\alpha| ||\mathbf{x}|| \text{ for } \alpha \text{ a number.}$$

$$||\mathbf{x} + \mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}||.$$

2. Completeness of $\mathbb{R}$ is an axiom. Using this, show $\mathbb{R}^n$ and $\mathbb{C}^n$ are complete metric spaces with respect to the distance given by the usual norm.

3. Verify that for the metric given in 2.1 defined on any infinite set, closed and bounded does not imply compact.

4. Verify that $\mathbb{Q}^n \equiv \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \{x_1, \cdots, x_n\}, x_k \in \mathbb{Q}\}$ is a countable and dense subset of $\mathbb{R}^n$.

5. Verify that $\mathbb{R}^n$ is locally arcwise connected. Show, in particular, that the balls are all arcwise connected.

6. Prove Urysohn's lemma. A Hausdorff space, X, is normal if and only if whenever K and H are disjoint nonempty closed sets, there exists a continuous function $f : X \to [0,1]$ such that $f(k) = 0$ for all $k \in K$ and $f(h) = 1$ for all $h \in H$.

7. Prove that $f : X \to Y$ is continuous if and only if f is continuous at every point of X.

8. Suppose (X, d), and (Y, ρ) are metric spaces and let $f : X \to Y$. Show f is continuous at $x \in X$ if and only if whenever $x_n \to x$, $f(x_n) \to f(x)$. (Recall that $x_n \to x$ means that for all $\epsilon > 0$, there exists n_ϵ such that $d(x_n, x) < \epsilon$ whenever $n > n_\epsilon$.)

9. If (X, d) is a metric space, give an easy proof independent of Problem 6 that whenever K, H are disjoint non empty closed sets, there exists $f : X \to [0,1]$ such that f is continuous, $f(K) = \{0\}$, and $f(H) = \{1\}$.

10. Let (X, τ) (Y, η)be topological spaces with (X, τ) compact and let $f : X \to Y$ be continuous. Show $f(X)$ is compact.

11. (An example) Let $X = [-\infty, \infty]$ and consider $\mathcal{B}$ defined by sets of the form (a, b), $[-\infty, b)$, and $(a, \infty]$. Show $\mathcal{B}$ is the basis for a topology on X.

12. $\uparrow$ Show (X, τ) defined in Problem 11 is a compact Hausdorff space.

13. $\uparrow$ Show (X, τ) defined in Problem 11 is completely separable.

14. $\uparrow$ In Problem 11, show sets of the form $[-\infty, b)$ and $(a, \infty]$ form a subbasis for the topology described in Problem 11.

15. Let (X, τ) and (Y, η) be topological spaces and let $f : X \to Y$. Also let $\mathcal{S}$ be a subbasis for η. Show f is continuous if and only if $f^{-1}(V) \in \tau$ for all $V \in \mathcal{S}$. Thus, it suffices to check inverse images of subbasic sets in checking for continuity.

16. Show the usual topology of $\mathbb{R}^n$ is the same as the product topology of

$$\prod_{i=1}^{n} \mathbb{R} \equiv \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}.$$

    Do the same for $\mathbb{C}^n$.

17. If M is a separable metric space and $T \subseteq M$, then T is separable also.

18. Show the rational numbers, $\mathbb{Q}$, are countable.

19. Verify that the set of 2.6 is connected but not locally connected or arcwise connected.

3 The complex numbers

In this chapter we consider the complex numbers, $\mathbb{C}$ and a few basic topics such as the roots of a complex number. Just as a real number should be considered as a point on the line, a complex number is considered a point in the plane. We can identify a point in the plane in the usual way using the Cartesian coordinates of the point. Thus (a, b) identifies a point whose x coordinate is a and whose y coordinate is b. In dealing with complex numbers, we write such a point as $a + ib$ and multiplication and addition are defined in the most obvious way subject to the convention that $i^2 = -1$. Thus,

$$(a + ib) + (c + id) = (a + c) + i(b + d)$$

and

$$(a + ib)(c + id) = (ac - bd) + i(bc + ad).$$

We can also verify that every non zero complex number, $a+ib$, with $a^2 + b^2 \neq 0$, has a unique multiplicative inverse.

$$\frac{1}{a + ib} = \frac{a - ib}{a^2 + b^2} = \frac{a}{a^2 + b^2} - i\frac{b}{a^2 + b^2}.$$

Theorem 3.1 *The complex numbers with multiplication and addition defined as above form a field.*

The field of complex numbers is denoted as $\mathbb{C}$. An important construction regarding complex numbers is the complex conjugate denoted by a horizontal line above the number. It is defined as follows.

$$\overline{a + ib} = a - ib.$$

What it does is reflect a given complex number across the x axis. Algebraically, the following formula is easy to obtain.

$$\left(\overline{a + ib}\right)(a + ib) = a^2 + b^2.$$

The length of a complex number, refered to as the modulus of z and denoted by $|z|$ is given by

$$|z| \equiv \left(x^2 + y^2\right)^{1/2} = (z\overline{z})^{1/2},$$

and we make $\mathbb{C}$ into a metric space by defining the distance between two complex numbers, z and w as

$$d(z, w) \equiv |z - w|.$$

We see therefore, that this metric on $\mathbb{C}$ is the same as the usual metric of $\mathbb{R}^2$. A sequence, $z_n \to z$ if and only if $x_n \to x$ in $\mathbb{R}$ and $y_n \to y$ in $\mathbb{R}$ where $z = x + iy$ and $z_n = x_n + iy_n$. For example if $z_n = \frac{n}{n+1} + i\frac{1}{n}$, then $z_n \to 1 + 0i = 1$.

Definition 3.2 *A sequence of complex numbers, $\{z_n\}$ is a Cauchy sequence if for every $\varepsilon > 0$ there exists N such that $n, m > N$ implies $|z_n - z_m| < \varepsilon$.*

This is the usual definition of Cauchy sequence. There are no new ideas here.

Proposition 3.3 *The complex numbers with the norm just mentioned forms a complete normed linear space.*

Proof: Let $\{z_n\}$ be a Cauchy sequence of complex numbers with $z_n = x_n + iy_n$. Then $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences of real numbers and so they converge to real numbers, x and y respectively. Thus

$z_n = x_n + iy_n \to x + iy$. By Theorem 3.1 $\mathbb{C}$ is a linear space with the field of scalars equal to $\mathbb{C}$. It only remains to verify that $|\ |$ satisfies the axioms of a norm which are:

$$|z + w| \leq |z| + |w|$$

$$|z| \geq 0 \text{ for all } z$$

$$|z| = 0 \text{ if and only if } z = 0$$

$$|\alpha z| = |\alpha| \, |z| \, .$$

We leave this as an exercise.

Definition 3.4 *An infinite sum of complex numbers is defined as the limit of the sequence of partial sums. Thus,*

$$\sum_{k=1}^{\infty} a_k \equiv \lim_{n \to \infty} \sum_{k=1}^{n} a_k .$$

Just as in the case of sums of real numbers, we see that an infinite sum converges if and only if the sequence of partial sums is a Cauchy sequence.

Definition 3.5 *We say a sequence of functions of a complex variable, $\{f_n\}$ converges uniformly to a function, g for $z \in S$ if for every $\varepsilon > 0$ there exists N_ε such that if $n > N_\varepsilon$, then*

$$|f_n(z) - g(z)| < \varepsilon$$

for all $z \in S$. The infinite sum $\sum_{k=1}^{\infty} f_n$ converges uniformly on S if the partial sums converge uniformly on S.

Proposition 3.6 *A sequence of functions, $\{f_n\}$ defined on a set S, converges uniformly to some function, g if and only if for all $\varepsilon > 0$ there exists N_ε such that whenever $m, n > N_\varepsilon$,*

$$||f_n - f_m||_\infty < \varepsilon.$$

Here $||f||_\infty \equiv \sup \{|f(z)| : z \in S\}$.

Just as in the case of functions of a real variable, we have the Weierstrass M test.

Proposition 3.7 *Let $\{f_n\}$ be a sequence of complex valued functions defined on $S \subseteq \mathbb{C}$. Suppose there exists M_n such that $||f_n||_\infty < M_n$ and $\sum M_n$ converges. Then $\sum f_n$ converges uniformly on S.*

Since every complex number can be considered a point in $\mathbb{R}^2$, we define the polar form of a complex number as follows. If $z = x + iy$ then $\left(\frac{x}{|z|}, \frac{y}{|z|}\right)$ is a point on the unit circle because

$$\left(\frac{x}{|z|}\right)^2 + \left(\frac{y}{|z|}\right)^2 = 1.$$

Therefore, there is an angle θ such that

$$\left(\frac{x}{|z|}, \frac{y}{|z|}\right) = (\cos\theta, \sin\theta) \, .$$

It follows that

$$z = x + iy = |z| \left(\cos\theta + i \sin\theta\right).$$

This is the polar form of the complex number, $z = x + iy$.

One of the most important features of the complex numbers is that you can always obtain n *nth* roots of any complex number. To begin with we need a fundamental result known as De Moivre's theorem.

Theorem 3.8 *Let $r > 0$ be given. Then if n is a positive integer,*

$$[r(\cos t + i\sin t)]^n = r^n(\cos nt + i\sin nt).$$

Proof: It is clear the formula holds if $n = 1$. Suppose it is true for n.

$$[r(\cos t + i\sin t)]^{n+1} = [r(\cos t + i\sin t)]^n [r(\cos t + i\sin t)]$$

which by induction equals

$$= r^{n+1}(\cos nt + i\sin nt)(\cos t + i\sin t)$$

$$= r^{n+1}((\cos nt\cos t - \sin nt\sin t) + i(\sin nt\cos t + \cos nt\sin t))$$

$$= r^{n+1}(\cos(n+1)t + i\sin(n+1)t)$$

by standard trig. identities.

Corollary 3.9 *Let z be a non zero complex number. Then there are always exactly k kth roots of z in $\mathbb{C}$.*

Proof: Let $z = x + iy$. Then

$$z = |z|\left(\frac{x}{|z|} + i\frac{y}{|z|}\right)$$

and from the definition of $|z|$,

$$\left(\frac{x}{|z|}\right)^2 + \left(\frac{y}{|z|}\right)^2 = 1.$$

Thus $\left(\frac{x}{|z|}, \frac{y}{|z|}\right)$ is a point on the unit circle and so

$$\frac{y}{|z|} = \sin t, \ \frac{x}{|z|} = \cos t$$

for a unique $t \in [0, 2\pi)$. By De Moivre's theorem, a number is a kth root of z if and only if it is of the form

$$|z|^{1/k}\left(\cos\left(\frac{t + 2l\pi}{k}\right) + i\sin\left(\frac{t + 2l\pi}{k}\right)\right)$$

for l an integer. By the fact that the cos and sin are 2π periodic, if $l = k$ in the above formula the same complex number is obtained as if $l = 0$. Thus there are exactly k of these numbers.

If $S \subseteq \mathbb{C}$ and $f : S \to \mathbb{C}$, we say f is continuous if whenever $z_n \to z \in S$, it follows that $f(z_n) \to f(z)$. Thus f is continuous if it takes converging sequences to converging sequences.

3.1 Exercises

1. Let $z = 3 + 4i$. Find the polar form of z and obtain all cube roots of z.
2. Prove Propositions 3.6 and 3.7.
3. Verify the complex numbers form a field.

4. Prove that $\overline{\prod_{k=1}^{n} z_k} = \prod_{k=1}^{n} \overline{z}_k$. In words, show the conjugate of a product is equal to the product of the conjugates.

5. Prove that $\overline{\sum_{k=1}^{n} z_k} = \sum_{k=1}^{n} \overline{z}_k$. In words, show the conjugate of a sum equals the sum of the conjugates.

6. Let $P(z)$ be a polynomial having real coefficients. Show the zeros of $P(z)$ occur in conjugate pairs.

7. If A is a real $n \times n$ matrix and $A\mathbf{x} = \lambda \mathbf{x}$, show that $A\overline{\mathbf{x}} = \overline{\lambda}\overline{\mathbf{x}}$.

8. Tell what is wrong with the following proof that $-1 = 1$.

$$-1 = i^2 = \sqrt{-1}\sqrt{-1} = \sqrt{(-1)^2} = \sqrt{1} = 1.$$

9. If $z = |z|(\cos\theta + i\sin\theta)$ and $w = |w|(\cos\alpha + i\sin\alpha)$, show

$$zw = |z|\,|w|\left(\cos(\theta+\alpha) + i\sin(\theta+\alpha)\right).$$

10. Since each complex number, $z = x + iy$ can be considered a vector in $\mathbb{R}^2$, we can also consider it a vector in $\mathbb{R}^3$ and consider the cross product of two complex numbers. Recall from calculus that for $\mathbf{x} \equiv (a, b, c)$ and $\mathbf{y} \equiv (d, e, f)$, two vectors in $\mathbb{R}^3$,

$$\mathbf{x} \times \mathbf{y} \equiv \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ d & e & f \end{pmatrix}$$

and that geometrically $|\mathbf{x} \times \mathbf{y}| = |\mathbf{x}|\,|\mathbf{y}|\sin\theta$, the area of the parallelogram spanned by the two vectors, $\mathbf{x}, \mathbf{y}$ and the triple, $\mathbf{x}, \mathbf{y}, \mathbf{x} \times \mathbf{y}$ forms a right handed system. Show

$$z_1 \times z_2 = \operatorname{Im}(\overline{z}_1 z_2)\,\mathbf{k}.$$

Thus the area of the parallelogram spanned by z_1 and z_2 equals $|\operatorname{Im}(\overline{z}_1 z_2)|$.

11. Prove that $f : S \subseteq \mathbb{C} \to \mathbb{C}$ is continuous at $z \in S$ if and only if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that whenever $w \in S$ and $|w - z| < \delta$, it follows that $|f(w) - f(z)| < \varepsilon$.

12. Verify that every polynomial $p(z)$ is continuous on $\mathbb{C}$.

13. Show that if $\{f_n\}$ is a sequence of functions converging uniformly to a function, f on $S \subseteq \mathbb{C}$ and if f_n is continuous on S, then so is f.

14. Show that if $|z| < 1$, then $\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$.

15. Show that whenever $\sum a_n$ converges it follows that $\lim_{n\to\infty} a_n = 0$. Give an example in which $\lim_{n\to\infty} a_n = 0$, $a_n \geq a_{n+1}$ and yet $\sum a_n$ fails to converge to a number.

16. Prove the root test for series of complex numbers. If $a_k \in \mathbb{C}$ and $r \equiv \limsup_{n\to\infty} |a_n|^{1/n}$ then

$$\sum_{k=0}^{\infty} a_k \begin{cases} \text{converges absolutely if } r < 1 \\ \text{diverges if } r > 1 \\ \text{test fails if } r = 1. \end{cases}$$

17. Does $\lim_{n\to\infty} n\left(\frac{2+i}{3}\right)^n$ exist? Tell why and find the limit if it does exist.

18. Let $A_0 = 0$ and let $A_n \equiv \sum_{k=1}^{n} a_k$ if $n > 0$. Prove the partial summation formula,

$$\sum_{k=p}^{q} a_k b_k = A_q b_q - A_{p-1} b_p + \sum_{k=p}^{q-1} A_k \left(b_k - b_{k+1}\right).$$

Now using this formula, suppose $\{b_n\}$ is a sequence of real numbers which converges to 0 and is decreasing. Determine those values of ω such that $|\omega| = 1$ and $\sum_{k=1}^{\infty} b_k \omega^k$ converges. **Hint:** From Problem 15 you have an example of a sequence $\{b_n\}$ which shows that $\omega = 1$ is not one of those values of ω.

19. Let $f : U \subseteq \mathbb{C} \to \mathbb{C}$ be given by $f(x+iy) = u(x,y) + iv(x,y)$. Show f is continuous on U if and only if $u : U \to \mathbb{R}$ and $v : U \to \mathbb{R}$ are both continuous.

3.2 The extended complex plane

The set of complex numbers has already been considered along with the topology of $\mathbb{C}$ which is nothing but the topology of $\mathbb{R}^2$. Thus, for $z_n = x_n + iy_n$ we say $z_n \to z \equiv x + iy$ if and only if $x_n \to x$ and $y_n \to y$. The norm in $\mathbb{C}$ is given by

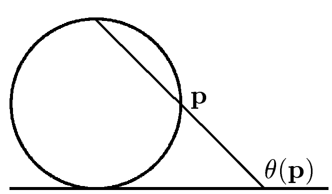
$$|x+iy| \equiv ((x+iy)(x-iy))^{1/2} = \left(x^2+y^2\right)^{1/2}$$

which is just the usual norm in $\mathbb{R}^2$ identifying (x,y) with $x+iy$. Therefore, $\mathbb{C}$ is a complete metric space and we have the Heine Borel theorem that compact sets are those which are closed and bounded. Thus, as far as topology is concerned, there is nothing new about $\mathbb{C}$.

We need to consider another general topological space which is related to $\mathbb{C}$. It is called the extended complex plane, denoted by $\widehat{\mathbb{C}}$ and consisting of the complex plane, $\mathbb{C}$ along with another point not in $\mathbb{C}$ known as ∞. For example, ∞ could be any point in $\mathbb{R}^3$. We say a sequence of complex numbers, z_n, converges to ∞ if, whenever K is a compact set in $\mathbb{C}$, there exists a number, N such that for all $n > N$, $z_n \notin K$. Since compact sets in $\mathbb{C}$ are closed and bounded, this is equivalent to saying that for all $R > 0$, there exists N such that if $n > N$, then $z_n \notin B(0,R)$ which is the same as saying $\lim_{n\to\infty} |z_n| = \infty$ where this last symbol has the same meaning as it does in calculus.

A geometric way of understanding this in terms of more familiar objects involves a concept known as the Riemann sphere.

Consider the unit sphere, S^2 given by $(z-1)^2 + y^2 + x^2 = 1$. We define a map from the unit sphere with the point, $(0,0,2)$ left out which is one to one onto $\mathbb{R}^2$ as follows.



We extend a line from the north pole of the sphere, the point $(0,0,2)$, through the point on the sphere, $\mathbf{p}$, until it intersects a unique point on $\mathbb{R}^2$. This mapping, known as stereographic projection, which we will denote for now by θ, is clearly continuous because it takes converging sequences, to converging sequences. Furthermore, it is clear that θ^{-1} is also continuous. In terms of the extended complex plane, $\widehat{\mathbb{C}}$, we see a sequence, z_n converges to ∞ if and only if $\theta^{-1} z_n$ converges to $(0,0,2)$ and a sequence, z_n converges to $z \in \mathbb{C}$ if and only if $\theta^{-1}(z_n) \to \theta^{-1}(z)$.

3.3 Exercises

1. Try to find an explicit formula for θ and θ^{-1}.
2. What does the mapping θ^{-1} do to lines and circles?
3. Show that S^2 is compact but $\mathbb{C}$ is not. Thus $\mathbb{C} \neq S^2$. Show that a set, K is compact (connected) in $\mathbb{C}$ if and only if $\theta^{-1}(K)$ is compact (connected) in $S^2 \setminus \{(0,0,2)\}$.
4. Let K be a compact set in $\mathbb{C}$. Show that $\mathbb{C} \setminus K$ has exactly one unbounded component and that this component is the one which is a subset of the component of $S^2 \setminus K$ which contains ∞. If you need to rewrite using the mapping, θ to make sense of this, it is fine to do so.
5. Make $\widehat{\mathbb{C}}$ into a topological space as follows. We define a basis for a topology on $\widehat{\mathbb{C}}$ to be all open sets and all complements of compact sets, the latter type being those which are said to contain the point ∞. Show this is a basis for a topology which makes $\widehat{\mathbb{C}}$ into a compact Hausdorff space. Also verify that $\widehat{\mathbb{C}}$ with this topology is homeomorphic to the sphere, S^2.

4 Riemann Stieltjes integrals

In the theory of functions of a complex variable, the most important results are those involving contour integration. Before we define what we mean by contour integration, it is necessary to define the notion of a Riemann Steiltjes integral, a generalization of the usual Riemann integral and the notion of a function of bounded variation.

Definition 4.1 *Let $\gamma : [a,b] \to \mathbb{C}$ be a function. We say γ is of bounded variation if*

$$\sup\left\{\sum_{i=1}^{n} |\gamma(t_i) - \gamma(t_{i-1})| : a = t_0 < \cdots < t_n = b\right\} \equiv V(\gamma, [a,b]) < \infty$$

where the sums are taken over all possible lists, $\{a = t_0 < \cdots < t_n = b\}$.

The idea is that it makes sense to talk of the length of the curve $\gamma([a,b])$, defined as $V(\gamma, [a,b])$. For this reason, in the case that γ is continuous, such an image of a bounded variation function is called a rectifiable curve.

Definition 4.2 *Let $\gamma : [a,b] \to \mathbb{C}$ be of bounded variation and let $f : [a,b] \to \mathbb{C}$. Letting $\mathcal{P} \equiv \{t_0, \cdots, t_n\}$ where $a = t_0 < t_1 < \cdots < t_n = b$, we define*

$$||\mathcal{P}|| \equiv \max\{|t_j - t_{j-1}| : j = 1, \cdots, n\}$$

and the Riemann Steiltjes sum by

$$S(\mathcal{P}) \equiv \sum_{j=1}^{n} f(\tau_j)(\gamma(t_j) - \gamma(t_{j-1}))$$

where $\tau_j \in [t_{j-1}, t_j]$. (Note this notation is a little sloppy because it does not identify the specific point, τ_j used. It is understood that this point is arbitrary.) We define $\int_\gamma f(t)\, d\gamma(t)$ as the unique number which satisfies the following condition. For all $\varepsilon > 0$ there exists a $\delta > 0$ such that if $||\mathcal{P}|| \leq \delta$, then

$$\left|\int_\gamma f(t)\, d\gamma(t) - S(\mathcal{P})\right| < \varepsilon.$$

Sometimes this is written as

$$\int_\gamma f(t)\, d\gamma(t) \equiv \lim_{||\mathcal{P}|| \to 0} S(\mathcal{P}).$$

The function, $\gamma([a,b])$ is a set of points in $\mathbb{C}$ and as t moves from a to b, $\gamma(t)$ moves from $\gamma(a)$ to $\gamma(b)$. Thus $\gamma([a,b])$ has a first point and a last point. If $\phi : [c,d] \to [a,b]$ is a continuous nondecreasing function, then $\gamma \circ \phi : [c,d] \to \mathbb{C}$ is also of bounded variation and yields the same set of points in $\mathbb{C}$ with the same first and last points. In the case where the values of the function, f, which are of interest are those on $\gamma([a,b])$, we have the following important theorem on change of parameters.

Theorem 4.3 *Let ϕ and γ be as just described. Then assuming that*

$$\int_\gamma f(\gamma(t))\, d\gamma(t)$$

exists, so does

$$\int_{\gamma\circ\phi} f(\gamma(\phi(s)))\, d(\gamma\circ\phi)(s)$$

and

$$\int_\gamma f(\gamma(t))\, d\gamma(t) = \int_{\gamma\circ\phi} f(\gamma(\phi(s)))\, d(\gamma\circ\phi)(s). \tag{4.1}$$

Proof: There exists $\delta > 0$ such that if $\mathcal{P}$ is a partition of $[a,b]$ such that $||\mathcal{P}|| < \delta$, then

$$\left|\int_\gamma f(\gamma(t))\, d\gamma(t) - S(\mathcal{P})\right| < \varepsilon.$$

By continuity of ϕ, there exists $\sigma > 0$ such that if $\mathcal{Q}$ is a partition of $[c,d]$ with $||\mathcal{Q}|| < \sigma$, $\mathcal{Q} = \{s_0, \cdots, s_n\}$, then $|\phi(s_j) - \phi(s_{j-1})| < \delta$. Thus letting $\mathcal{P}$ denote the points in $[a,b]$ given by $\phi(s_j)$ for $s_j \in \mathcal{Q}$, it follows that $||\mathcal{P}|| < \delta$ and so

$$\left|\int_\gamma f(\gamma(t))\, d\gamma(t) - \sum_{j=1}^{n} f(\gamma(\phi(\tau_j)))(\gamma(\phi(s_j)) - \gamma(\phi(s_{j-1})))\right| < \varepsilon$$

where $\tau_j \in [s_{j-1}, s_j]$. Therefore, from the definition we see that 4.1 holds and that

$$\int_{\gamma\circ\phi} f(\gamma(\phi(s)))\, d(\gamma\circ\phi)(s)$$

exists.

This theorem shows that $\int_\gamma f(\gamma(t))\, d\gamma(t)$ is independent of the particular γ used in its computation to the extent that if ϕ is any nondecreasing function from another interval, $[c,d]$, mapping to $[a,b]$, then the same value is obtained by replacing γ with $\gamma \circ \phi$.

The fundamental result in this subject is the following theorem.

Theorem 4.4 *Let $f : [a,b] \to \mathbb{C}$ be continuous and let $\gamma : [a,b] \to \mathbb{C}$ be of bounded variation. Then $\int_\gamma f(t)\, d\gamma(t)$ exists. Also if $\delta_m > 0$ is such that $|t-s| < \delta_m$ implies $|f(t) - f(s)| < \frac{1}{m}$, then*

$$\left|\int_\gamma f(t)\, d\gamma(t) - S(\mathcal{P})\right| \leq \frac{2V(\gamma,[a,b])}{m}$$

whenever $||\mathcal{P}|| < \delta_m$.

Proof: The function, f , is uniformly continuous because it is defined on a compact set. Therefore, there exists a decreasing sequence of positive numbers, $\{\delta_m\}$ such that if $|s-t|<\delta_m$, then

$$|f(t)-f(s)|<\frac{1}{m}.$$

Let

$$F_m \equiv \overline{\{S(\mathcal{P}) : ||\mathcal{P}||<\delta_m\}}.$$

Thus F_m is a closed set. (When we write $S(\mathcal{P})$ in the above definition, we mean to include all sums corresponding to $\mathcal{P}$ for any choice of τ_j.) We wish to show that

$$diam\,(F_m) \leq \frac{2V(\gamma,[a,b])}{m} \tag{4.2}$$

because then there will exist a unique point, $I \in \cap_{m=1}^{\infty} F_m$. It will then follow that $I = \int_\gamma f(t)\,d\gamma(t)$. To verify 4.2, it suffices to verify that whenever $\mathcal{P}$ and $\mathcal{Q}$ are partitions satisfying $||\mathcal{P}||<\delta_m$ and $||\mathcal{Q}||<\delta_m$,

$$|S(\mathcal{P})-S(\mathcal{Q})| \leq \frac{2}{m}V(\gamma,[a,b]). \tag{4.3}$$

Suppose $||\mathcal{P}||<\delta_m$ and $\mathcal{Q}\supseteq\mathcal{P}$. Then also $||\mathcal{Q}||<\delta_m$. To begin with, suppose that $\mathcal{P}\equiv\{t_0,\cdots,t_p,\cdots,t_n\}$ and $\mathcal{Q}\equiv\{t_0,\cdots,t_{p-1},t^*,t_p,\cdots,t_n\}$. Thus $\mathcal{Q}$ contains only one more point than $\mathcal{P}$. Letting $S(\mathcal{Q})$ and $S(\mathcal{P})$ be Riemann Steiltjes sums,

$$S(\mathcal{Q}) \equiv \sum_{j=1}^{p-1} f(\sigma_j)(\gamma(t_j)-\gamma(t_{j-1})) + f(\sigma_*)(\gamma(t^*)-\gamma(t_{p-1}))$$

$$+f(\sigma^*)(\gamma(t_p)-\gamma(t^*)) + \sum_{j=p+1}^{n} f(\sigma_j)(\gamma(t_j)-\gamma(t_{j-1})),$$

$$S(\mathcal{P}) \equiv \sum_{j=1}^{p-1} f(\tau_j)(\gamma(t_j)-\gamma(t_{j-1})) + f(\tau_p)(\gamma(t^*)-\gamma(t_{p-1}))$$

$$+f(\tau_p)(\gamma(t_p)-\gamma(t^*)) + \sum_{j=p+1}^{n} f(\tau_j)(\gamma(t_j)-\gamma(t_{j-1})).$$

Therefore,

$$|S(\mathcal{P})-S(\mathcal{Q})| \leq \sum_{j=1}^{p-1}\frac{1}{m}|\gamma(t_j)-\gamma(t_{j-1})| + \frac{1}{m}|\gamma(t^*)-\gamma(t_{p-1})| +$$

$$\frac{1}{m}|\gamma(t_p)-\gamma(t^*)| + \sum_{j=p+1}^{n}\frac{1}{m}|\gamma(t_j)-\gamma(t_{j-1})| \leq \frac{1}{m}V(\gamma,[a,b]). \tag{4.4}$$

Clearly the extreme inequalities would be valid in 4.4 if $\mathcal{Q}$ had more than one extra point. Let $S(\mathcal{P})$ and $S(\mathcal{Q})$ be Riemann Steiltjes sums for which $||\mathcal{P}||$ and $||\mathcal{Q}||$ are less than δ_m and let $\mathcal{R}\equiv\mathcal{P}\cup\mathcal{Q}$. Then

$$|S(\mathcal{P})-S(\mathcal{Q})| \leq |S(\mathcal{P})-S(\mathcal{R})| + |S(\mathcal{R})-S(\mathcal{Q})| \leq \frac{2}{m}V(\gamma,[a,b]).$$

and this shows 4.3 which proves 4.2. Therefore, there exists a unique complex number, $I\in\cap_{m=1}^{\infty}F_m$ which satisfies the definition of $\int_\gamma f(t)\,d\gamma(t)$. This proves the theorem.

The following theorem follows easily from the above definitions and theorem.

Theorem 4.5 *Let $f \in C([a,b])$ and let $\gamma : [a,b] \to \mathbb{C}$ be of bounded variation. Let*

$$M \geq \max\{|f(t)| : t \in [a,b]\}. \tag{4.5}$$

Then

$$\left|\int_\gamma f(t)\, d\gamma(t)\right| \leq MV(\gamma, [a,b]). \tag{4.6}$$

Also if $\{f_n\}$ is a sequence of functions of $C([a,b])$ which is converging uniformly to the function, f, then

$$\lim_{n\to\infty} \int_\gamma f_n(t)\, d\gamma(t) = \int_\gamma f(t)\, d\gamma(t). \tag{4.7}$$

Proof: Let 4.5 hold. From the proof of the above theorem we know that when $||\mathcal{P}|| < \delta_m$,

$$\left|\int_\gamma f(t)\, d\gamma(t) - S(\mathcal{P})\right| \leq \frac{2}{m} V(\gamma, [a,b])$$

and so

$$\left|\int_\gamma f(t)\, d\gamma(t)\right| \leq |S(\mathcal{P})| + \frac{2}{m} V(\gamma, [a,b])$$

$$\begin{aligned} &\leq \sum_{j=1}^{n} M\,|\gamma(t_j) - \gamma(t_{j-1})| + \frac{2}{m} V(\gamma, [a,b]) \\ &\leq MV(\gamma, [a,b]) + \frac{2}{m} V(\gamma, [a,b]). \end{aligned}$$

This proves 4.6 since m is arbitrary. To verify 4.7 we use the above inequality to write

$$\left|\int_\gamma f(t)\, d\gamma(t) - \int_\gamma f_n(t)\, d\gamma(t)\right| = \left|\int_\gamma (f(t) - f_n(t))\, d\gamma(t)\right|$$

$$\leq \max\{|f(t) - f_n(t)| : t \in [a,b]\}\, V(\gamma, [a,b]).$$

Since the convergence is assumed to be uniform, this proves 4.7.

It turns out that we will be mainly interested in the case where γ is also continuous in addition to being of bounded variation. Also, it turns out to be much easier to evaluate such integrals in the case where γ is also $C^1([a,b])$. The following theorem about approximation will be very useful.

Theorem 4.6 *Let $\gamma : [a,b] \to \mathbb{C}$ be continuous and of bounded variation, let $f : [a,b] \times K \to \mathbb{C}$ be continuous for K a compact set in $\mathbb{C}$, and let $\varepsilon > 0$ be given. Then there exists $\eta : [a,b] \to \mathbb{C}$ such that $\eta(a) = \gamma(a)$, $\gamma(b) = \eta(b)$, $\eta \in C^1([a,b])$, and*

$$||\gamma - \eta|| < \varepsilon, \tag{4.8}$$

$$\left|\int_\gamma f(t,z)\, d\gamma(t) - \int_\eta f(t,z)\, d\eta(t)\right| < \varepsilon, \tag{4.9}$$

$$V(\eta, [a,b]) \leq V(\gamma, [a,b]), \tag{4.10}$$

where $||\gamma - \eta|| \equiv \max\{|\gamma(t) - \eta(t)| : t \in [a,b]\}$.

Proof: We extend γ to be defined on all $\mathbb{R}$ according to $\gamma(t) = \gamma(a)$ if $t < a$ and $\gamma(t) = \gamma(b)$ if $t > b$. Now we define

$$\gamma_h(t) \equiv \frac{1}{2h} \int_{-2h+t+\frac{2h}{(b-a)}(t-a)}^{t+\frac{2h}{(b-a)}(t-a)} \gamma(s)\, ds.$$

where the integral is defined in the obvious way. That is,

$$\int_a^b \alpha(t) + i\beta(t)\, dt \equiv \int_a^b \alpha(t)\, dt + i \int_a^b \beta(t)\, dt.$$

Therefore,

$$\gamma_h(b) = \frac{1}{2h} \int_b^{b+2h} \gamma(s)\, ds = \gamma(b),$$

$$\gamma_h(a) = \frac{1}{2h} \int_{a-2h}^{a} \gamma(s)\, ds = \gamma(a).$$

Also, because of continuity of γ and the fundamental theorem of calculus,

$$\gamma_h'(t) = \frac{1}{2h} \left\{ \gamma\left(t + \frac{2h}{b-a}(t-a)\right)\left(1 + \frac{2h}{b-a}\right) - \right.$$

$$\left. \gamma\left(-2h + t + \frac{2h}{b-a}(t-a)\right)\left(1 + \frac{2h}{b-a}\right) \right\}$$

and so $\gamma_h \in C^1([a,b])$. The following lemma is significant.

Lemma 4.7 $V(\gamma_h, [a,b]) \leq V(\gamma, [a,b])$.

Proof: Let $a = t_0 < t_1 < \cdots < t_n = b$. Then using the definition of γ_h and changing the variables to make all integrals over $[0, 2h]$,

$$\sum_{j=1}^{n} |\gamma_h(t_j) - \gamma_h(t_{j-1})| =$$

$$\sum_{j=1}^{n} \left| \frac{1}{2h} \int_0^{2h} \left[\gamma\left(s - 2h + t_j + \frac{2h}{b-a}(t_j - a)\right) - \right.\right.$$

$$\left.\left. \gamma\left(s - 2h + t_{j-1} + \frac{2h}{b-a}(t_{j-1} - a)\right) \right] \right|$$

$$\leq \frac{1}{2h} \int_0^{2h} \sum_{j=1}^{n} \left| \gamma\left(s - 2h + t_j + \frac{2h}{b-a}(t_j - a)\right) - \right.$$

$$\left. \gamma\left(s - 2h + t_{j-1} + \frac{2h}{b-a}(t_{j-1} - a)\right) \right| ds.$$

For a given $s \in [0, 2h]$, the points, $s - 2h + t_j + \frac{2h}{b-a}(t_j - a)$ for $j = 1, \cdots, n$ form an increasing list of points in the interval $[a - 2h, b + 2h]$ and so the integrand is bounded above by $V(\gamma, [a - 2h, b + 2h]) = V(\gamma, [a, b])$. It follows

$$\sum_{j=1}^{n} |\gamma_h(t_j) - \gamma_h(t_{j-1})| \leq V(\gamma, [a, b])$$

which proves the lemma.

With this lemma the proof of the theorem can be completed without too much trouble. First of all, if $\varepsilon > 0$ is given, there exists δ_1 such that if $h < \delta_1$, then for all t,

$$\begin{aligned} |\gamma(t) - \gamma_h(t)| &\leq \frac{1}{2h} \int_{-2h+t+\frac{2h}{(b-a)}(t-a)}^{t+\frac{2h}{(b-a)}(t-a)} |\gamma(s) - \gamma(t)| \, ds \\ &< \frac{1}{2h} \int_{-2h+t+\frac{2h}{(b-a)}(t-a)}^{t+\frac{2h}{(b-a)}(t-a)} \varepsilon ds = \varepsilon \end{aligned} \tag{4.11}$$

due to the uniform continuity of γ. This proves 4.8. From 4.2 there exists δ_2 such that if $||\mathcal{P}|| < \delta_2$, then for all $z \in K$,

$$\left| \int_{\gamma} f(t, z) \, d\gamma(t) - S(\mathcal{P}) \right| < \frac{\varepsilon}{3}, \left| \int_{\gamma_h} f(t, z) \, d\gamma_h(t) - S_h(\mathcal{P}) \right| < \frac{\varepsilon}{3}$$

for all h. Here $S(\mathcal{P})$ is a Riemann Steiltjes sum of the form

$$\sum_{i=1}^{n} f(\tau_i, z)(\gamma(t_i) - \gamma(t_{i-1}))$$

and $S_h(\mathcal{P})$ is a similar Riemann Steiltjes sum taken with respect to γ_h instead of γ. Therefore, fix the partition, $\mathcal{P}$, and choose h small enough that in addition to this, we have the following inequality valid for all $z \in K$.

$$|S(\mathcal{P}) - S_h(\mathcal{P})| < \frac{\varepsilon}{3}$$

We can do this thanks to 4.11 and the uniform continuity of f on $[a, b] \times K$. It follows

$$\left| \int_{\gamma} f(t, z) \, d\gamma(t) - \int_{\gamma_h} f(t, z) \, d\gamma_h(t) \right| \leq$$

$$\left| \int_{\gamma} f(t, z) \, d\gamma(t) - S(\mathcal{P}) \right| + |S(\mathcal{P}) - S_h(\mathcal{P})|$$

$$+ \left| S_h(\mathcal{P}) - \int_{\gamma_h} f(t, z) \, d\gamma_h(t) \right| < \varepsilon.$$

Formula 4.10 follows from the lemma. This proves the theorem.

Of course the same result is obtained without the explicit dependence of f on z.

This is a very useful theorem because if γ is $C^1([a, b])$, it is easy to calculate $\int_{\gamma} f(t) \, d\gamma(t)$. We will typically reduce to the case where γ is C^1 by using the above theorem. The next theorem shows how easy it is to compute these integrals in the case where γ is C^1.

Theorem 4.8 *If $f : [a,b] \to \mathbb{C}$ and $\gamma : [a,b] \to \mathbb{C}$ is in $C^1([a,b])$, then*

$$\int_\gamma f(t)\, d\gamma(t) = \int_a^b f(t)\, \gamma'(t)\, dt. \tag{4.12}$$

Proof: Let $\mathcal{P}$ be a partition of $[a,b]$, $\mathcal{P} = \{t_0, \cdots, t_n\}$ and $||\mathcal{P}||$ is small enough that whenever $|t-s| < ||\mathcal{P}||$,

$$|f(t) - f(s)| < \varepsilon \tag{4.13}$$

and

$$\left| \int_\gamma f(t)\, d\gamma(t) - \sum_{j=1}^n f(\tau_j)(\gamma(t_j) - \gamma(t_{j-1})) \right| < \varepsilon.$$

Now

$$\sum_{j=1}^n f(\tau_j)(\gamma(t_j) - \gamma(t_{j-1})) = \int_a^b \sum_{j=1}^n f(\tau_j)\, \mathcal{X}_{(t_{j-1},t_j]}(s)\, \gamma'(s)\, ds$$

and thanks to 4.13,

$$\left| \int_a^b \sum_{j=1}^n f(\tau_j)\, \mathcal{X}_{(t_{j-1},t_j]}(s)\, \gamma'(s)\, ds - \int_a^b f(s)\, \gamma'(s)\, ds \right|$$

$$< \int_a^b \varepsilon\, |\gamma'(s)|\, ds.$$

It follows that

$$\left| \int_\gamma f(t)\, d\gamma(t) - \int_a^b f(s)\, \gamma'(s)\, ds \right| < \varepsilon \int_a^b |\gamma'(s)|\, ds + \varepsilon.$$

Since ε is arbitrary, this verifies 4.12.

Definition 4.9 *Let $\gamma : [a,b] \to U$ be a continuous function with bounded variation and let $f : U \to \mathbb{C}$ be a continuous function. Then we define,*

$$\int_\gamma f(z)\, dz \equiv \int_\gamma f(\gamma(t))\, d\gamma(t).$$

The expression, $\int_\gamma f(z)\, dz$, is called a contour integral and γ is referred to as the contour. We also say that a function $f : U \to \mathbb{C}$ for U an open set in $\mathbb{C}$ has a primitive if there exists a function, F, the primitive, such that $F'(z) = f(z)$. Thus F is just an antiderivative. Also if $\gamma_k : [a_k, b_k] \to \mathbb{C}$ is continuous and of bounded variation, for $k = 1, \cdots, m$ and $\gamma_k(b_k) = \gamma_{k+1}(a_k)$, we define

$$\int_{\sum_{k=1}^m \gamma_k} f(z)\, dz \equiv \sum_{k=1}^m \int_{\gamma_k} f(z)\, dz. \tag{4.14}$$

In addition to this, for $\gamma : [a,b] \to \mathbb{C}$, we define $-\gamma : [a,b] \to \mathbb{C}$ by $-\gamma(t) \equiv \gamma(b + a - t)$. Thus γ simply traces out the points of $\gamma([a,b])$ in the opposite order.

The following lemma is useful and follows quickly from Theorem 4.3.

Lemma 4.10 *In the above definition, there exists a continuous bounded variation function, γ defined on some closed interval, $[c,d]$, such that $\gamma([c,d]) = \cup_{k=1}^{m} \gamma_k([a_k,b_k])$ and $\gamma(c) = \gamma_1(a_1)$ while $\gamma(d) = \gamma_m(b_m)$. Furthermore,*

$$\int_{\gamma} f(z)\, dz = \sum_{k=1}^{m} \int_{\gamma_k} f(z)\, dz.$$

If $\gamma : [a,b] \to \mathbb{C}$ is of bounded variation and continuous, then

$$\int_{\gamma} f(z)\, dz = -\int_{-\gamma} f(z)\, dz.$$

Theorem 4.11 *Let K be a compact set in $\mathbb{C}$ and let $f : U \times K \to \mathbb{C}$ be continuous for U an open set in $\mathbb{C}$. Also let $\gamma : [a,b] \to U$ be continuous with bounded variation. Then if $r > 0$ is given, there exists $\eta : [a,b] \to U$ such that $\eta(a) = \gamma(a)$, $\eta(b) = \gamma(b)$, η is $C^1([a,b])$, and*

$$\left| \int_{\gamma} f(z,w)\, dz - \int_{\eta} f(z,w)\, dz \right| < r, \ ||\eta - \gamma|| < r.$$

Proof: Let $\varepsilon > 0$ be given and let H be an open set containing $\gamma([a,b])$ such that $\overline{H}$ is compact. Then f is uniformly continuous on $\overline{H} \times K$ and so there exists a $\delta > 0$ such that if $z_j \in H, j = 1,2$ and $w_j \in K$ for $j = 1,2$ such that if

$$|z_1 - z_2| + |w_1 - w_2| < \delta,$$

then

$$|f(z_1, w_1) - f(z_2, w_2)| < \varepsilon.$$

By Theorem 4.6, let $\eta : [a,b] \to \mathbb{C}$ be such that $\eta([a,b]) \subseteq H$, $\eta(x) = \gamma(x)$ for $x = a, b$, $\eta \in C^1([a,b])$, $||\eta - \gamma|| < \min(\delta, r)$, $V(\eta, [a,b]) < V(\gamma, [a,b])$, and

$$\left| \int_{\eta} f(\gamma(t), w)\, d\eta(t) - \int_{\gamma} f(\gamma(t), w)\, d\gamma(t) \right| < \varepsilon$$

for all $w \in K$. Then, since $|f(\gamma(t), w) - f(\eta(t), w)| < \varepsilon$ for all $t \in [a,b]$,

$$\left| \int_{\eta} f(\gamma(t), w)\, d\eta(t) - \int_{\eta} f(\eta(t), w)\, d\eta(t) \right| < \varepsilon V(\eta, [a,b]) \leq \varepsilon V(\gamma, [a,b]).$$

Therefore,

$$\left| \int_{\eta} f(z,w)\, dz - \int_{\gamma} f(z,w)\, dz \right| =$$

$$\left| \int_{\eta} f(\eta(t), w)\, d\eta(t) - \int_{\gamma} f(\gamma(t), w)\, d\gamma(t) \right| < \varepsilon + \varepsilon V(\gamma, [a,b]).$$

Since $\varepsilon > 0$ is arbitrary, this proves the theorem.

We will be very interested in the functions which have primitives. It turns out, it is not enough for f to be continuous in order to possess a primitive. This is in stark contrast to the situation for functions of a real variable in which the fundamental theorem of calculus will deliver a primitive for any continuous function. The reason for our interest in such functions is the following theorem and its corollary.

Theorem 4.12 *Let $\gamma : [a,b] \to \mathbb{C}$ be continuous and of bounded variation. Also suppose $F'(z) = f(z)$ for all $z \in U$, an open set containing $\gamma([a,b])$ and f is continuous on U. Then*

$$\int_\gamma f(z)\,dz = F(\gamma(b)) - F(\gamma(a)).$$

Proof: By Theorem 4.11 there exists $\eta \in C^1([a,b])$ such that $\gamma(a) = \eta(a)$, and $\gamma(b) = \eta(b)$ such that

$$\left|\int_\gamma f(z)\,dz - \int_\eta f(z)\,dz\right| < \varepsilon.$$

Then since η is in $C^1([a,b])$, we may write

$$\begin{aligned}\int_\eta f(z)\,dz &= \int_a^b f(\eta(t))\,\eta'(t)\,dt = \int_a^b \frac{dF(\eta(t))}{dt}dt \\ &= F(\eta(b)) - F(\eta(a)) = F(\gamma(b)) - F(\gamma(a)).\end{aligned}$$

Therefore,

$$\left|(F(\gamma(b)) - F(\gamma(a))) - \int_\gamma f(z)\,dz\right| < \varepsilon$$

and since $\varepsilon > 0$ is arbitrary, this proves the theorem.

Corollary 4.13 *If $\gamma : [a,b] \to \mathbb{C}$ is continuous, has bounded variation, is a closed curve, $\gamma(a) = \gamma(b)$, and $\gamma([a,b]) \subseteq U$ where U is an open set on which $F'(z) = f(z)$, then*

$$\int_\gamma f(z)\,dz = 0.$$

4.1 Exercises

1. Let $\gamma : [a,b] \to \mathbb{R}$ be increasing. Show $V(\gamma, [a,b]) = \gamma(b) - \gamma(a)$.

2. Suppose $\gamma : [a,b] \to \mathbb{C}$ satisfies a Lipschitz condition, $|\gamma(t) - \gamma(s)| \leq K|s-t|$. Show γ is of bounded variation and that $V(\gamma, [a,b]) \leq K|b-a|$.

3. We say $\gamma : [c_0, c_m] \to \mathbb{C}$ is piecewise smooth if there exist numbers, $c_k, k = 1, \cdots, m$ such that $c_0 < c_1 < \cdots < c_{m-1} < c_m$ such that γ is continuous and $\gamma : [c_k, c_{k+1}] \to \mathbb{C}$ is C^1. Show that such piecewise smooth functions are of bounded variation and give an estimate for $V(\gamma, [c_0, c_m])$.

4. Let $\gamma : [0, 2\pi] \to \mathbb{C}$ be given by $\gamma(t) = r(\cos mt + i \sin mt)$ for m an integer. Find $\int_\gamma \frac{dz}{z}$.

5. Show that if $\gamma : [a,b] \to \mathbb{C}$ then there exists an increasing function $h : [0,1] \to [a,b]$ such that $\gamma \circ h([0,1]) = \gamma([a,b])$.

6. Let $\gamma : [a,b] \to \mathbb{C}$ be an arbitrary continuous curve having bounded variation and let f, g have continuous derivatives on some open set containing $\gamma([a,b])$. Prove the usual integration by parts formula.

$$\int_\gamma fg'dz = f(\gamma(b))\,g(\gamma(b)) - f(\gamma(a))\,g(\gamma(a)) - \int_\gamma f'gdz.$$

7. Let $f(z) \equiv |z|^{-(1/2)} e^{-i\frac{\theta}{2}}$ where $z = |z|e^{i\theta}$. This function is called the principle branch of $z^{-(1/2)}$. Find $\int_\gamma f(z)\,dz$ where γ is the semicircle in the upper half plane which goes from $(1,0)$ to $(-1,0)$ in the counter clockwise direction. Next do the integral in which γ goes in the clockwise direction along the semicircle in the lower half plane.

8. Prove an open set, U is connected if and only if for every two points in U, there exists a C^1 curve having values in U which joins them.

9. Let $\mathcal{P}, \mathcal{Q}$ be two partitions of $[a,b]$ with $\mathcal{P} \subseteq \mathcal{Q}$. Each of these partitions can be used to form an approximation to $V(\gamma, [a,b])$ as described above. Recall the total variation was the supremum of sums of a certain form determined by a partition. How is the sum associated with $\mathcal{P}$ related to the sum associated with $\mathcal{Q}$? Explain.

10. Consider the curve,

$$\gamma(t) = \begin{cases} t + it^2 \sin\left(\frac{1}{t}\right) \text{ if } t \in (0,1] \\ 0 \text{ if } t = 0 \end{cases}.$$

Is γ a continuous curve having bounded variation? What if the t^2 is replaced with t? Is the resulting curve continuous? Is it a bounded variation curve?

11. Suppose $\gamma : [a,b] \to \mathbb{R}$ is given by $\gamma(t) = t$. What is $\int_\gamma f(t)\, d\gamma$? Explain.

5 Analytic functions

In this chapter we define what we mean by an analytic function and give a few important examples of functions which are analytic.

Definition 5.1 *Let U be an open set in $\mathbb{C}$ and let $f : U \to \mathbb{C}$. We say f is analytic on U if for every $z \in U$,*

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \equiv f'(z)$$

exists and is a continuous function of $z \in U$. Here $h \in \mathbb{C}$.

Note that if f is analytic, it must be the case that f is continuous. It is more common to not include the requirement that f' is continuous but we will show later that the continuity of f' follows.

What are some examples of analytic functions? The simplest example is any polynomial. Thus

$$p(z) \equiv \sum_{k=0}^{n} a_k z^k$$

is an analytic function and

$$p'(z) = \sum_{k=1}^{n} a_k k z^{k-1}.$$

We leave the verification of this as an exercise. More generally, power series are analytic. We will show this later. For now, we consider the very important Cauchy Riemann equations which give conditions under which complex valued functions of a complex variable are analytic.

Theorem 5.2 *Let U be an open subset of $\mathbb{C}$ and let $f : U \to \mathbb{C}$ be a function, such that for $z = x + iy \in U$,*

$$f(z) = u(x,y) + iv(x,y).$$

Then f is analytic if and only if u, v are $C^1(U)$ and

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Furthermore, we have the formula,

$$f'(z) = \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y).$$

Proof: Suppose f is analytic first. Then letting $t \in \mathbb{R}$,

$$f'(z) = \lim_{t \to 0} \frac{f(z+t) - f(z)}{t} =$$

$$\lim_{t \to 0} \left(\frac{u(x+t,y) + iv(x+t,y)}{t} - \frac{u(x,y) + iv(x,y)}{t} \right)$$

$$= \frac{\partial u(x,y)}{\partial x} + i \frac{\partial v(x,y)}{\partial x}.$$

But also

$$f'(z) = \lim_{t \to 0} \frac{f(z+it) - f(z)}{it} =$$

$$\lim_{t \to 0} \left(\frac{u(x,y+t) + iv(x,y+t)}{it} - \frac{u(x,y) + iv(x,y)}{it} \right)$$

$$\frac{1}{i} \left(\frac{\partial u(x,y)}{\partial y} + i \frac{\partial v(x,y)}{\partial y} \right)$$

$$= \frac{\partial v(x,y)}{\partial y} - i \frac{\partial u(x,y)}{\partial y}.$$

This verifies the Cauchy Riemann equations. We are assuming that $z \to f'(z)$ is continuous. Therefore, the partial derivatives of u and v are also continuous. To see this, note that from the formulas for $f'(z)$ given above, and letting $z_1 = x_1 + iy_1$

$$\left| \frac{\partial v(x,y)}{\partial y} - \frac{\partial v(x_1,y_1)}{\partial y} \right| \leq |f'(z) - f'(z_1)|,$$

showing that $(x,y) \to \frac{\partial v(x,y)}{\partial y}$ is continuous since $(x_1,y_1) \to (x,y)$ if and only if $z_1 \to z$. The other cases are similar.

Now suppose the Cauchy Riemann equations hold and the functions, u and v are $C^1(U)$. Then letting $h = h_1 + ih_2$,

$$f(z+h) - f(z) = u(x+h_1, y+h_2)$$

$$+ iv(x+h_1, y+h_2) - (u(x,y) + iv(x,y))$$

We know u and v are both differentiable and so

$$f(z+h) - f(z) = \frac{\partial u}{\partial x}(x,y) h_1 + \frac{\partial u}{\partial y}(x,y) h_2 +$$

$$i \left(\frac{\partial v}{\partial x}(x,y) h_1 + \frac{\partial v}{\partial y}(x,y) h_2 \right) + o(h).$$

Dividing by h and using the Cauchy Riemann equations,

$$\frac{f(z+h)-f(z)}{h} = \frac{\frac{\partial u}{\partial x}(x,y)\,h_1 + i\frac{\partial v}{\partial y}(x,y)\,h_2}{h} +$$

$$\frac{i\frac{\partial v}{\partial x}(x,y)\,h_1 + \frac{\partial u}{\partial y}(x,y)\,h_2}{h} + \frac{o(h)}{h}$$

$$= \frac{\partial u}{\partial x}(x,y)\,\frac{h_1 + ih_2}{h} + i\frac{\partial v}{\partial x}(x,y)\,\frac{h_1+ih_2}{h} + \frac{o(h)}{h}$$

Taking the limit as $h \to 0$, we obtain

$$f'(z) = \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y).$$

It follows from this formula and the assumption that u, v are $C^1(U)$ that f' is continuous.

It is routine to verify that all the usual rules of derivatives hold for analytic functions. In particular, we have the product rule, the chain rule, and quotient rule.

5.1 Exercises

1. Verify all the usual rules of differentiation including the product and chain rules.

2. Suppose f and $f' : U \to \mathbb{C}$ are analytic and $f(z) = u(x,y) + iv(x,y)$. Verify $u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$. This partial differential equation satisfied by the real and imaginary parts of an analytic function is called Laplace's equation. We say these functions satisfying Laplace's equation are harmonic functions. If u is a harmonic function defined on $B(0,r)$ show that $v(x,y) \equiv \int_0^y u_x(x,t)\,dt - \int_0^x u_y(t,0)\,dt$ is such that $u + iv$ is analytic.

3. Define a function $f(z) \equiv \overline{z} \equiv x - iy$ where $z = x + iy$. Is f analytic?

4. If $f(z) = u(x,y) + iv(x,y)$ and f is analytic, verify that
$$\det\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = |f'(z)|^2.$$

5. Show that if $u(x,y) + iv(x,y) = f(z)$ is analytic, then $\nabla u \cdot \nabla v = 0$. Recall
$$\nabla u(x,y) = \langle u_x(x,y), u_y(x,y)\rangle.$$

6. Show that every polynomial is analytic.

7. If $\gamma(t) = x(t) + iy(t)$ is a C^1 curve having values in U, an open set of $\mathbb{C}$, and if $f : U \to \mathbb{C}$ is analytic, we can consider $f \circ \gamma$, another C^1 curve having values in $\mathbb{C}$. Also, $\gamma'(t)$ and $(f \circ \gamma)'(t)$ are complex numbers so these can be considered as vectors in $\mathbb{R}^2$ as follows. The complex number, $x+iy$ corresponds to the vector, $\langle x, y\rangle$. Suppose that γ and η are two such C^1 curves having values in U and that $\gamma(t_0) = \eta(s_0) = z$ and suppose that $f : U \to \mathbb{C}$ is analytic. Show that the angle between $(f \circ \gamma)'(t_0)$ and $(f \circ \eta)'(s_0)$ is the same as the angle between $\gamma'(t_0)$ and $\eta'(s_0)$ assuming that $f'(z) \neq 0$. Thus analytic mappings preserve angles at points where the derivative is nonzero. Such mappings are called isogonal. . **Hint:** To make this easy to show, first observe that $\langle x, y\rangle \cdot \langle a, b\rangle = \frac{1}{2}(z\overline{w} + \overline{z}w)$ where $z = x + iy$ and $w = a + ib$.

8. Analytic functions are even better than what is described in Problem 7. In addition to preserving angles, they also preserve orientation. To verify this show that if $z = x + iy$ and $w = a + ib$ are two complex numbers, then $\langle x, y, 0\rangle$ and $\langle a, b, 0\rangle$ are two vectors in $\mathbb{R}^3$. Recall that the cross product, $\langle x, y, 0\rangle \times \langle a, b, 0\rangle$, yields a vector normal to the two given vectors such that the triple, $\langle x, y, 0\rangle, \langle a, b, 0\rangle$, and $\langle x, y, 0\rangle \times \langle a, b, 0\rangle$ satisfies the right hand rule and has magnitude equal to the product of the sine of the included angle times the product of the two norms of the vectors. In this case, the cross product either points in the direction of the positive z axis or in the direction of the negative z axis. Thus, either the vectors $\langle x, y, 0\rangle, \langle a, b, 0\rangle, \mathbf{k}$ form a right handed system or the vectors $\langle a, b, 0\rangle, \langle x, y, 0\rangle, \mathbf{k}$ form a right handed system. These are the two possible orientations. Show that in the situation of Problem 7 the orientation of $\gamma'(t_0), \eta'(s_0), \mathbf{k}$ is the same as the orientation of the vectors $(f \circ \gamma)'(t_0), (f \circ \eta)'(s_0), \mathbf{k}$. Such mappings are called conformal. **Hint:** You can do this by verifying that $(f \circ \gamma)'(t_0) \times (f \circ \eta)'(s_0) = \gamma'(t_0) \times \eta'(s_0)$. To make the verification easier, you might first establish the following simple formula for the cross product where here $x + iy = z$ and $a + ib = w$.

$$\langle x, y, 0\rangle \times \langle a, b, 0\rangle = \operatorname{Re}(zi\overline{w})\,\mathbf{k}.$$

9. Write the Cauchy Riemann equations in terms of polar coordinates. Recall the polar coordinates are given by

$$x = r\cos\theta,\ y = r\sin\theta.$$

5.2 Examples of analytic functions

A very important example of an analytic function is $e^z \equiv e^x(\cos y + i\sin y) \equiv \exp(z)$. We can verify this is an analytic function by considering the Cauchy Riemann equations. Here $u(x, y) = e^x \cos y$ and $v(x, y) = e^x \sin y$. The Cauchy Riemann equations hold and the two functions u and v are $C^1(\mathbb{C})$. Therefore, $z \to e^z$ is an analytic function on all of $\mathbb{C}$. Also from the formula for $f'(z)$ given above for an analytic function,

$$\frac{d}{dz}e^z = e^x(\cos y + i\sin y) = e^z.$$

We also see that $e^z = 1$ if and only if $z = 2\pi k$ for k an integer. Other properties of e^z follow from the formula for it. For example, let $z_j = x_j + iy_j$ where $j = 1, 2$.

$$\begin{aligned} e^{z_1}e^{z_2} &\equiv e^{x_1}(\cos y_1 + i\sin y_1)\, e^{x_2}(\cos y_2 + i\sin y_2) \\ &= e^{x_1+x_2}(\cos y_1 \cos y_2 - \sin y_1 \sin y_2) + \\ &\quad ie^{x_1+x_2}(\sin y_1 \cos y_2 + \sin y_2 \cos y_1) \end{aligned}$$

$$= e^{x_1+x_2}(\cos(y_1 + y_2) + i\sin(y_1 + y_2)) = e^{z_1+z_2}.$$

Another example of an analytic function is any polynomial. We can also define the functions $\cos z$ and $\sin z$ by the usual formulas.

$$\sin z \equiv \frac{e^{iz} - e^{-iz}}{2i}, \cos z \equiv \frac{e^{iz} + e^{-iz}}{2}.$$

By the rules of differentiation, it is clear these are analytic functions which agree with the usual functions in the case where z is real. Also the usual differentiation formulas hold. However,

$$\cos ix = \frac{e^{-x} + e^x}{2} = \cosh x$$

and so $\cos z$ is not bounded. Similarly $\sin z$ is not bounded.

A more interesting example is the log function. We cannot define the log for all values of z but if we leave out the ray, $(-\infty, 0]$, then it turns out we can do so. On $\mathbb{R} + i(-\pi, \pi)$ it is easy to see that e^z is one to one, mapping onto $\mathbb{C} \setminus (-\infty, 0]$. Therefore, we can define the log on $\mathbb{C} \setminus (-\infty, 0]$ in the usual way,

$$e^{\log z} \equiv z = e^{\ln|z|} e^{i \arg(z)},$$

where $\arg(z)$ is the unique angle in $(-\pi, \pi)$ for which the equal sign in the above holds. Thus we need

$$\log z = \ln|z| + i \arg(z). \tag{5.1}$$

There are many other ways to define a logarithm. In fact, we could take any ray from 0 and define a logarithm on what is left. It turns out that all these logarithm functions are analytic. This will be clear from the open mapping theorem presented later but for now you may verify by brute force that the usual definition of the logarithm, given in 5.1 and referred to as the principle branch of the logarithm is analytic. This can be done by verifying the Cauchy Riemann equations in the following.

$$\log z = \ln\left(x^2 + y^2\right)^{1/2} + i\left(-\arccos\left(\frac{x}{\sqrt{x^2 + y^2}}\right)\right) \text{ if } y < 0,$$

$$\log z = \ln\left(x^2 + y^2\right)^{1/2} + i\left(\arccos\left(\frac{x}{\sqrt{x^2 + y^2}}\right)\right) \text{ if } y > 0,$$

$$\log z = \ln\left(x^2 + y^2\right)^{1/2} + i\left(\arctan\left(\frac{y}{x}\right)\right) \text{ if } x > 0.$$

With the principle branch of the logarithm defined, we may define the principle branch of z^α for any $\alpha \in \mathbb{C}$. We define

$$z^\alpha \equiv e^{\alpha \log(z)}.$$

5.3 Exercises

1. Verify the principle branch of the logarithm is an analytic function.
2. Find i^i corresponding to the principle branch of the logarithm.
3. Show that $\sin(z + w) = \sin z \cos w + \cos z \sin w$.
4. If f is analytic on U, an open set in $\mathbb{C}$, when can it be concluded that $|f|$ is analytic? When can it be concluded that $|f|$ is continuous? Prove your assertions.
5. Let $f(z) = \overline{z}$ where $\overline{z} \equiv x - iy$ for $z = x + iy$. Describe geometrically what f does and discuss whether f is analytic.
6. A fractional linear transformation is a function of the form

$$f(z) = \frac{az + b}{cz + d}$$

where $ad - bc \neq 0$. Note that if $c = 0$, this reduces to a linear transformation $(a/d)z + (b/d)$. Special cases of these are given defined as follows.

$$\text{dilations: } z \to \delta z,\ \delta \neq 0,\ \text{inversions: } z \to \frac{1}{z},$$

$$\text{translations: } z \to z + \rho.$$

In the case where $c \neq 0$, let $S_1(z) = z + \frac{d}{c}$, $S_2(z) = \frac{1}{z}$, $S_3(z) = \frac{(bc-ad)}{c^2} z$ and $S_4(z) = z + \frac{a}{c}$. Verify that $f(z) = S_4 \circ S_3 \circ S_2 \circ S_1$. Now show that in the case where $c = 0$, f is still a finite composition of dilations, inversions, and translations.

7. Show that for a fractional linear transformation described in Problem 6 circles and lines are mapped to circles or lines. **Hint:** This is obvious for dilations, and translations. It only remains to verify this for inversions. Note that all circles and lines may be put in the form

$$\alpha\left(x^2 + y^2\right) - 2ax - 2by = r^2 - \left(a^2 + b^2\right)$$

where $\alpha = 1$ gives a circle centered at (a, b) with radius r and $\alpha = 0$ gives a line. In terms of complex variables we may consider all possible circles and lines in the form

$$\alpha z\overline{z} + \beta z + \overline{\beta}\overline{z} + \gamma = 0,$$

Verify every circle or line is of this form and that conversely, every expression of this form yields either a circle or a line. Then verify that inversions do what is claimed.

8. It is desired to find an analytic function, $L(z)$ defined for all $z \in \mathbb{C} \setminus \{0\}$ such that $e^{L(z)} = z$. Is this possible? Explain why or why not.

9. If f is analytic, show that $z \to \overline{f(\overline{z})}$ is also analytic.

10. Find the real and imaginary parts of the principle branch of $z^{1/2}$.

6 Cauchy's formula for a disk

In this chapter we prove the Cauchy formula for a disk. Later we will generalize this formula to much more general situations but the version given here will suffice to prove many interesting theorems needed in the later development of the theory. First we give a few preliminary results from advanced calculus.

Lemma 6.1 *Let $f : [a, b] \to \mathbb{C}$. Then $f'(t)$ exists if and only if $\operatorname{Re} f'(t)$ and $\operatorname{Im} f'(t)$ exist. Furthermore,*

$$f'(t) = \operatorname{Re} f'(t) + i \operatorname{Im} f'(t).$$

Proof: The if part of the equivalence is obvious.

Now suppose $f'(t)$ exists. Let both t and $t + h$ be contained in $[a, b]$

$$\left|\frac{\operatorname{Re} f(t+h) - \operatorname{Re} f(t)}{h} - \operatorname{Re}\left(f'(t)\right)\right| \leq \left|\frac{f(t+h) - f(t)}{h} - f'(t)\right|$$

and this converges to zero as $h \to 0$. Therefore, $\operatorname{Re} f'(t) = \operatorname{Re}\left(f'(t)\right)$. Similarly, $\operatorname{Im} f'(t) = \operatorname{Im}\left(f'(t)\right)$.

Lemma 6.2 *If $g : [a, b] \to \mathbb{C}$ and g is continuous on $[a, b]$ and differentiable on (a, b) with $g'(t) = 0$, then $g(t)$ is a constant.*

Proof: From the above lemma, we can apply the mean value theorem to the real and imaginary parts of g.

Lemma 6.3 *Let $\phi : [a, b] \times [c, d] \to \mathbb{R}$ be continuous and let*

$$g(t) \equiv \int_a^b \phi(s, t)\, ds. \tag{6.1}$$

Then g is continuous. If $\frac{\partial \phi}{\partial t}$ exists and is continuous on $[a,b] \times [c,d]$, then

$$g'(t) = \int_a^b \frac{\partial \phi(s,t)}{\partial t} ds. \tag{6.2}$$

Proof: The first claim follows from the uniform continuity of ϕ on $[a,b] \times [c,d]$, which uniform continuity results from the set being compact. To establish 6.2, let t and $t+h$ be contained in $[c,d]$ and form, using the mean value theorem,

$$\begin{aligned} \frac{g(t+h) - g(t)}{h} &= \frac{1}{h} \int_a^b [\phi(s,t+h) - \phi(s,t)] \, ds \\ &= \frac{1}{h} \int_a^b \frac{\partial \phi(s,t+\theta h)}{\partial t} h ds \\ &= \int_a^b \frac{\partial \phi(s,t+\theta h)}{\partial t} ds, \end{aligned}$$

where θ may depend on s but is some number between 0 and 1. Then by the uniform continuity of $\frac{\partial \phi}{\partial t}$, it follows that 6.2 holds.

Corollary 6.4 *Let $\phi : [a,b] \times [c,d] \to \mathbb{C}$ be continuous and let*

$$g(t) \equiv \int_a^b \phi(s,t) \, ds. \tag{6.3}$$

Then g is continuous. If $\frac{\partial \phi}{\partial t}$ exists and is continuous on $[a,b] \times [c,d]$, then

$$g'(t) = \int_a^b \frac{\partial \phi(s,t)}{\partial t} ds. \tag{6.4}$$

Proof: Apply Lemma 6.3 to the real and imaginary parts of ϕ.

With this preparation we are ready to prove Cauchy's formula for a disk.

Theorem 6.5 *Let $f : U \to \mathbb{C}$ be analytic on the open set, U and let*

$$\overline{B(z_0,r)} \subseteq U.$$

Let $\gamma(t) \equiv z_0 + re^{it}$ for $t \in [0, 2\pi]$. Then if $z \in B(z_0,r)$,

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-z} dw. \tag{6.5}$$

Proof: Consider for $\alpha \in [0,1]$,

$$g(\alpha) \equiv \int_0^{2\pi} \frac{f\left(z + \alpha\left(z_0 + re^{it} - z\right)\right)}{re^{it} + z_0 - z} rie^{it} dt.$$

If α equals one, this reduces to the integral in 6.5. We will show g is a constant and that $g(0) = f(z) 2\pi i$. First we consider the claim about $g(0)$.

$$\begin{aligned} g(0) &= \left(\int_0^{2\pi} \frac{re^{it}}{re^{it} + z_0 - z} dt\right) i f(z) \\ &= i f(z) \left(\int_0^{2\pi} \frac{1}{1 - \frac{z - z_0}{re^{it}}} dt\right) \\ &= i f(z) \int_0^{2\pi} \sum_{n=0}^{\infty} r^{-n} e^{-int} (z - z_0)^n \, dt \end{aligned}$$

because $\left|\frac{z-z_0}{re^{it}}\right| < 1$. Since this sum converges uniformly we may interchange the sum and the integral to obtain

$$\begin{aligned} g(0) &= if(z) \sum_{n=0}^{\infty} r^{-n} (z - z_0)^n \int_0^{2\pi} e^{-int} dt \\ &= 2\pi i f(z) \end{aligned}$$

because $\int_0^{2\pi} e^{-int} dt = 0$ if $n > 0$.

Next we show that g is constant. By Corollary 6.4, for $\alpha \in (0,1)$,

$$\begin{aligned} g'(\alpha) &= \int_0^{2\pi} \frac{f'\left(z + \alpha\left(z_0 + re^{it} - z\right)\right)\left(re^{it} + z_0 - z\right)}{re^{it} + z_0 - z} rie^{it} dt \\ &= \int_0^{2\pi} f'\left(z + \alpha\left(z_0 + re^{it} - z\right)\right) rie^{it} dt \\ &= \int_0^{2\pi} \frac{d}{dt}\left(f\left(z + \alpha\left(z_0 + re^{it} - z\right)\right)\frac{1}{\alpha}\right) dt \\ &= f\left(z + \alpha\left(z_0 + re^{i2\pi} - z\right)\right)\frac{1}{\alpha} - f\left(z + \alpha\left(z_0 + re^{0} - z\right)\right)\frac{1}{\alpha} = 0. \end{aligned}$$

Now g is continuous on $[0,1]$ and $g'(t) = 0$ on $(0,1)$ so by Lemma 6.2, g equals a constant. This constant can only be $g(0) = 2\pi i f(z)$. Thus,

$$g(1) = \int_\gamma \frac{f(w)}{w - z} dw = g(0) = 2\pi i f(z).$$

This proves the theorem.

This is a very significant theorem. We give a few applications next.

Theorem 6.6 *Let $f : U \to \mathbb{C}$ be analytic where U is an open set in $\mathbb{C}$. Then f has infinitely many derivatives on U. Furthermore, for all $z \in B(z_0, r)$,*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(w)}{(w - z)^{n+1}} dw \tag{6.6}$$

where $\gamma(t) \equiv z_0 + re^{it}, t \in [0, 2\pi]$ for r small enough that $B(z_0, r) \subseteq U$.

Proof: Let $z \in B(z_0, r) \subseteq U$ and let $\overline{B(z_0, r)} \subseteq U$. Then, letting $\gamma(t) \equiv z_0 + re^{it}, t \in [0, 2\pi]$, and h small enough,

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z} dw, \quad f(z + h) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z - h} dw$$

Now

$$\frac{1}{w - z - h} - \frac{1}{w - z} = \frac{h}{(-w + z + h)(-w + z)}$$

and so

$$\begin{aligned} \frac{f(z + h) - f(z)}{h} &= \frac{1}{2\pi hi} \int_\gamma \frac{hf(w)}{(-w + z + h)(-w + z)} dw \\ &= \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(-w + z + h)(-w + z)} dw. \end{aligned}$$

Now for all h sufficiently small, there exists a constant C independent of such h such that

$$\begin{aligned} & \left| \frac{1}{(-w+z+h)(-w+z)} - \frac{1}{(-w+z)(-w+z)} \right| \\ = & \left| \frac{h}{(w-z-h)(w-z)^2} \right| \leq C|h| \end{aligned}$$

and so, the integrand converges uniformly as $h \to 0$ to

$$= \frac{f(w)}{(w-z)^2}$$

Therefore, we may take the limit as $h \to 0$ inside the integral to obtain

$$f'(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w-z)^2} dw.$$

Continuing in this way, we obtain 6.6.

This is a very remarkable result. We just showed that the existence of one continuous derivative implies the existence of all derivatives, in contrast to the theory of functions of a real variable. Actually, we just showed a little more than what the theorem states. The above proof establishes the following corollary.

Corollary 6.7 *Suppose f is continuous on $\partial B(z_0, r)$ and suppose that for all $z \in B(z_0, r)$,*

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-z} dw,$$

where $\gamma(t) \equiv z + re^{it}, t \in [0, 2\pi]$. Then f is analytic on $B(z_0, r)$ and in fact has infinitely many derivatives on $B(z_0, r)$.

We also have the following simple lemma as an application of the above.

Lemma 6.8 *Let $\gamma(t) = z_0 + re^{it}$, for $t \in [0, 2\pi]$, suppose $f_n \to f$ uniformly on $\overline{B(z_0, r)}$, and suppose*

$$f_n(z) = \frac{1}{2\pi i} \int_\gamma \frac{f_n(w)}{w-z} dw \tag{6.7}$$

for $z \in B(z_0, r)$. Then

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-z} dw, \tag{6.8}$$

implying that f is analytic on $B(z_0, r)$.

Proof: From 6.7 and the uniform convergence of f_n to f on $\gamma([0, 2\pi])$, we have that the integrals in 6.7 converge to

$$\frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-z} dw.$$

Therefore, the formula 6.8 follows.

Proposition 6.9 *Let $\{a_n\}$ denote a sequence of complex numbers. Then there exists $R \in [0, \infty]$ such that*

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k$$

converges absolutely if $|z - z_0| < R$, diverges if $|z - z_0| > R$ and converges uniformly on $B(z_0, r)$ for all $r < R$. Furthermore, if $R > 0$, the function,

$$f(z) \equiv \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

is analytic on $B(z_0, R)$.

Proof: The assertions about absolute convergence are routine from the root test if we define

$$R \equiv \left(\limsup_{n \to \infty} |a_n|^{1/n} \right)^{-1}$$

with $R = \infty$ if the quantity in parenthesis equals zero. The assertion about uniform convergence follows from the Weierstrass M test if we use $M_n \equiv |a_n| r^n$. ($\sum_{n=0}^{\infty} |a_n| r^n < \infty$ by the root test). It only remains to verify the assertion about $f(z)$ being analytic in the case where $R > 0$. Let $0 < r < R$ and define $f_n(z) \equiv \sum_{k=0}^{n} a_k (z - z_0)^k$. Then f_n is a polynomial and so it is analytic. Thus, by the Cauchy integral formula above,

$$f_n(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f_n(w)}{w - z} dw$$

where $\gamma(t) = z_0 + re^{it}$, for $t \in [0, 2\pi]$. By Lemma 6.8 and the first part of this proposition involving uniform convergence, we obtain

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w - z} dw.$$

Therefore, f is analytic on $B(z_0, r)$ by Corollary 6.7. Since $r < R$ is arbitrary, this shows f is analytic on $B(z_0, R)$.

This proposition shows that all functions which are given as power series are analytic on their circle of convergence, the set of complex numbers, z, such that $|z - z_0| < R$. Next we show that every analytic function can be realized as a power series.

Theorem 6.10 *If $f : U \to \mathbb{C}$ is analytic and if $B(z_0, r) \subseteq U$, then*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{6.9}$$

for all $|z - z_0| < r$. Furthermore,

$$a_n = \frac{f^{(n)}(z_0)}{n!}. \tag{6.10}$$

Proof: Consider $|z - z_0| < r$ and let $\gamma(t) = z_0 + re^{it}$, $t \in [0, 2\pi]$. Then for $w \in \gamma([0, 2\pi])$,

$$\left| \frac{z - z_0}{w - z_0} \right| < 1$$

and so, by the Cauchy integral formula, we may write

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}dw \\ &= \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-z_0)\left(1-\frac{z-z_0}{w-z_0}\right)}dw \\ &= \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-z_0)}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^n dw. \end{aligned}$$

Since the series converges uniformly, we may interchange the integral and the sum to obtain

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-z_0)^{n+1}}\right)(z-z_0)^n \\ &\equiv \sum_{n=0}^{\infty} a_n (z-z_0)^n \end{aligned}$$

By Theorem 6.6 we see that 6.10 holds.

The following theorem pertains to functions which are analytic on all of $\mathbb{C}$, "entire" functions.

Theorem 6.11 *(Liouville's theorem) If f is a bounded entire function then f is a constant.*

Proof: Since f is entire, we can pick any $z \in \mathbb{C}$ and write

$$f'(z) = \frac{1}{2\pi i}\int_{\gamma_R} \frac{f(w)}{(w-z)^2}dw$$

where $\gamma_R(t) = z + Re^{it}$ for $t \in [0, 2\pi]$. Therefore,

$$|f'(z)| \leq C\frac{1}{R}$$

where C is some constant depending on the assumed bound on f. Since R is arbitrary, we can take $R \to \infty$ to obtain $f'(z) = 0$ for any $z \in \mathbb{C}$. It follows from this that f is constant for if z_j $j = 1, 2$ are two complex numbers, we can consider $h(t) = f(z_1 + t(z_2 - z_1))$ for $t \in [0,1]$. Then $h'(t) = f'(z_1 + t(z_2 - z_1))(z_2 - z_1) = 0$. By Lemma 6.2 h is a constant on $[0,1]$ which implies $f(z_1) = f(z_2)$.

With Liouville's theorem it becomes possible to give an easy proof of the fundamental theorem of algebra. It is ironic that all the best proofs of this theorem in algebra come from the subjects of analysis or topology. Out of all the proofs that have been given of this very important theorem, the following one based on Liouville's theorem is the easiest.

Theorem 6.12 *(Fundamental theorem of Algebra) Let*

$$p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$$

be a polynomial where $n \geq 1$ and each coefficient is a complex number. Then there exists $z_0 \in \mathbb{C}$ such that $p(z_0) = 0$.

Proof: Suppose not. Then $p(z)^{-1}$ is an entire function. Also

$$|p(z)| \geq |z|^n - \left(|a_{n-1}|\,|z|^{n-1} + \cdots + |a_1|\,|z| + |a_0|\right)$$

and so $\lim_{|z|\to\infty}|p(z)| = \infty$ which implies $\lim_{|z|\to\infty}\left|p(z)^{-1}\right| = 0$. It follows that, since $p(z)^{-1}$ is bounded for z in any bounded set, we must have that $p(z)^{-1}$ is a bounded entire function. But then it must be constant. However since $p(z)^{-1} \to 0$ as $|z| \to \infty$, this constant can only be 0. However, $\frac{1}{p(z)}$ is never equal to zero. This proves the theorem.

6.1 Exercises

1. Show that if $|e_k| \leq \varepsilon$, then $\left|\sum_{k=m}^{\infty} e_k \left(r^k - r^{k+1}\right)\right| < \varepsilon$ if $0 \leq r < 1$. **Hint:** Let $|\theta| = 1$ and verify that

$$\theta \sum_{k=m}^{\infty} e_k \left(r^k - r^{k+1}\right) = \left|\sum_{k=m}^{\infty} e_k \left(r^k - r^{k+1}\right)\right| = \sum_{k=m}^{\infty} \operatorname{Re}\left(\theta e_k\right) \left(r^k - r^{k+1}\right)$$

where $-\varepsilon < \operatorname{Re}(\theta e_k) < \varepsilon$.

2. Abel's theorem says that if $\sum_{n=0}^{\infty} a_n (z-a)^n$ has radius of convergence equal to 1 and if $A = \sum_{n=0}^{\infty} a_n$, then $\lim_{r \to 1-} \sum_{n=0}^{\infty} a_n r^n = A$. **Hint:** Show $\sum_{k=0}^{\infty} a_k r^k = \sum_{k=0}^{\infty} A_k \left(r^k - r^{k+1}\right)$ where A_k denotes the kth partial sum of $\sum a_j$. Thus

$$\sum_{k=0}^{\infty} a_k r^k = \sum_{k=m+1}^{\infty} A_k \left(r^k - r^{k+1}\right) + \sum_{k=0}^{m} A_k \left(r^k - r^{k+1}\right),$$

where $|A_k - A| < \varepsilon$ for all $k \leq m$. In the first sum, write $A_k = A + e_k$ and use Problem 1. Use this theorem to verify that $\arctan(1) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{2k+1}$.

3. Find the integrals using the Cauchy integral formula.

   (a) $\int_\gamma \frac{\sin z}{z-i} dz$ where $\gamma(t) = 2e^{it} : t \in [0, 2\pi]$.

   (b) $\int_\gamma \frac{1}{z-a} dz$ where $\gamma(t) = a + re^{it} : t \in [0, 2\pi]$

   (c) $\int_\gamma \frac{\cos z}{z^2} dz$ where $\gamma(t) = e^{it} : t \in [0, 2\pi]$

   (d) $\int_\gamma \frac{\log(z)}{z^n} dz$ where $\gamma(t) = 1 + \frac{1}{2} e^{it} : t \in [0, 2\pi]$ and $n = 0, 1, 2$.

4. Let $\gamma(t) = 4e^{it} : t \in [0, 2\pi]$ and find $\int_\gamma \frac{z^2+4}{z(z^2+1)} dz$.

5. Suppose $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for all $|z| < R$. Show that then

$$\frac{1}{2\pi} \int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^2 d\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}$$

for all $r \in [0, R)$. **Hint:** Let $f_n(z) \equiv \sum_{k=0}^{n} a_k z^k$, show $\frac{1}{2\pi} \int_0^{2\pi} \left|f_n\left(re^{i\theta}\right)\right|^2 d\theta = \sum_{k=0}^{n} |a_k|^2 r^{2k}$ and then take limits as $n \to \infty$ using uniform convergence.

6. The Cauchy integral formula, marvelous as it is, can actually be improved upon. The Cauchy integral formula involves representing f by the values of f on the boundary of the disk, $B(a, r)$. It is possible to represent f by using only the values of $\operatorname{Re} f$ on the boundary. This leads to the Schwarz formula . Supply the details in the following outline.

   Suppose f is analytic on $|z| < R$ and

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{6.11}$$

with the series converging uniformly on $|z| = R$. Then letting $|w| = R$,

$$2u(w) = f(w) + \overline{f(w)}$$

and so

$$2u(w) = \sum_{k=0}^{\infty} a_k w^k + \sum_{k=0}^{\infty} \overline{a_k} (\overline{w})^k. \tag{6.12}$$

Now letting $\gamma(t) = Re^{it}$, $t \in [0, 2\pi]$

$$\begin{aligned} \int_\gamma \frac{2u(w)}{w} dw &= (a_0 + \overline{a_0}) \int_\gamma \frac{1}{w} dw \\ &= 2\pi i (a_0 + \overline{a_0}). \end{aligned}$$

Thus, multiplying 6.12 by w^{-1},

$$\frac{1}{\pi i} \int_\gamma \frac{u(w)}{w} dw = a_0 + \overline{a_0}.$$

Now multiply 6.12 by $w^{-(n+1)}$ and integrate again to obtain

$$a_n = \frac{1}{\pi i} \int_\gamma \frac{u(w)}{w^{n+1}} dw.$$

Using these formulas for a_n in 6.11, we can interchange the sum and the integral (Why can we do this?) to write the following for $|z| < R$.

$$\begin{aligned} f(z) &= \frac{1}{\pi i} \int_\gamma \frac{1}{z} \sum_{k=0}^{\infty} \left(\frac{z}{w} \right)^{k+1} u(w)\, dw - \overline{a_0} \\ &= \frac{1}{\pi i} \int_\gamma \frac{u(w)}{w - z} dw - \overline{a_0}, \end{aligned}$$

which is the Schwarz formula. Now $\operatorname{Re} a_0 = \frac{1}{2\pi i} \int_\gamma \frac{u(w)}{w} dw$ and $\overline{a_0} = \operatorname{Re} a_0 - i \operatorname{Im} a_0$. Therefore, we can also write the Schwarz formula as

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{u(w)(w + z)}{(w - z) w} dw + i \operatorname{Im} a_0. \tag{6.13}$$

7. Take the real parts of the second form of the Schwarz formula to derive the Poisson formula for a disk,

$$u\left(re^{i\alpha}\right) = \frac{1}{2\pi} \int_0^{2\pi} \frac{u\left(Re^{i\theta}\right)\left(R^2 - r^2\right)}{R^2 + r^2 - 2Rr\cos(\theta - \alpha)} d\theta. \tag{6.14}$$

8. Suppose that $u(w)$ is a given real continuous function defined on $\partial B(0, R)$ and define $f(z)$ for $|z| < R$ by 6.13. Show that f, so defined is analytic. Explain why u given in 6.14 is harmonic. Show that

$$\lim_{r \to R-} u\left(re^{i\alpha}\right) = u\left(Re^{i\alpha}\right).$$

Thus u is a harmonic function which approaches a given function on the boundary and is therefore, a solution to the Dirichlet problem.

9. Suppose $f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$ for all $|z - z_0| < R$. Show that $f'(z) = \sum_{k=0}^{\infty} a_k k (z - z_0)^{k-1}$ for all $|z - z_0| < R$. **Hint:** Let $f_n(z)$ be a partial sum of f. Show that f_n' converges uniformly to some function, g on $|z - z_0| \le r$ for any $r < R$. Now use the Cauchy integral formula for a function and its derivative to identify g with f'.

10. Use Problem 9 to find the exact value of $\sum_{k=0}^{\infty} k^2 \left(\frac{1}{3}\right)^k$.

11. Prove the binomial formula,

$$(1+z)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} z^n$$

where

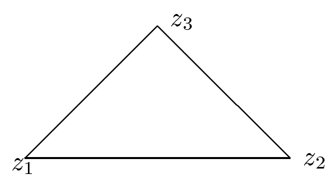
$$\binom{\alpha}{n} \equiv \frac{\alpha \cdots (\alpha - n + 1)}{n!}.$$

Can this be used to give a proof of the binomial formula, $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$? Explain.

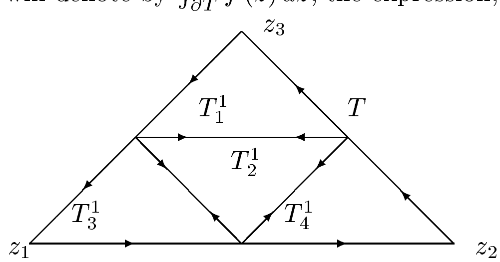
7 The general Cauchy integral formula

7.1 The Cauchy Goursat theorem

In this section we prove a fundamental theorem which is essential to the development which follows and is closely related to the question of when a function has a primitive. First of all, if we are given two points in $\mathbb{C}$, z_1 and z_2, we may consider $\gamma(t) \equiv z_1 + t(z_2 - z_1)$ for $t \in [0,1]$ to obtain a continuous bounded variation curve from z_1 to z_2. More generally, if $z_1, \cdots, z_m$ are points in $\mathbb{C}$ we can obtain a continuous bounded variation curve from z_1 to z_m which consists of first going from z_1 to z_2 and then from z_2 to z_3 and so on, till in the end one goes from z_{m-1} to z_m. We denote this piecewise linear curve as $\gamma(z_1, \cdots, z_m)$. Now let T be a triangle with vertices z_1, z_2 and z_3 encountered in the counter clockwise direction as shown.



Then we will denote by $\int_{\partial T} f(z)\, dz$, the expression, $\int_{\gamma(z_1,z_2,z_3)} f(z)\, dz$. Consider the following picture.



By Lemma 4.10 we may conclude that

$$\int_{\partial T} f(z)\, dz = \sum_{k=1}^{4} \int_{\partial T_k^1} f(z)\, dz. \tag{7.1}$$

On the "inside lines" the integrals cancel as claimed in Lemma 4.10 because there are two integrals going in opposite directions for each of these inside lines. Now we are ready to prove the Cauchy Goursat theorem.

Theorem 7.1 *(Cauchy Goursat) Let $f : U \to \mathbb{C}$ have the property that $f'(z)$ exists for all $z \in U$ and let T be a triangle contained in U. Then*

$$\int_{\partial T} f(w)\, dw = 0.$$

Proof: Suppose not. Then

$$\left|\int_{\partial T} f(w)\,dw\right| = \alpha \neq 0.$$

From 7.1 it follows

$$\alpha \leq \sum_{k=1}^{4} \left|\int_{\partial T_k^1} f(w)\,dw\right|$$

and so for at least one of these T_k^1, denoted from now on as T_1, we must have

$$\left|\int_{\partial T_1} f(w)\,dw\right| \geq \frac{\alpha}{4}.$$

Now let T_1 play the same role as T, subdivide as in the above picture, and obtain T_2 such that

$$\left|\int_{\partial T_2} f(w)\,dw\right| \geq \frac{\alpha}{4^2}.$$

Continue in this way, obtaining a sequence of triangles,

$$T_k \supseteq T_{k+1}, diam\,(T_k) \leq diam\,(T)\,2^{-k},$$

and

$$\left|\int_{\partial T_k} f(w)\,dw\right| \geq \frac{\alpha}{4^k}.$$

Then let $z \in \cap_{k=1}^{\infty} T_k$ and note that by assumption, $f'(z)$ exists. Therefore, for all k large enough,

$$\int_{\partial T_k} f(w)\,dw = \int_{\partial T_k} f(z) + f'(z)(w-z) + g(w)\,dw$$

where $|g(w)| < \varepsilon |w-z|$. Now observe that $w \to f(z) + f'(z)(w-z)$ has a primitive, namely,

$$F(w) = f(z)\,w + f'(z)(w-z)^2/2.$$

Therefore, by Corollary 4.13.

$$\int_{\partial T_k} f(w)\,dw = \int_{\partial T_k} g(w)\,dw.$$

From the definition, of the integral, we see

$$\begin{aligned} \frac{\alpha}{4^k} &\leq \left|\int_{\partial T_k} g(w)\,dw\right| \leq \varepsilon diam\,(T_k)\,(\text{length of } \partial T_k) \\ &\leq \varepsilon 2^{-k}\,(\text{length of } T)\,diam\,(T)\,2^{-k}, \end{aligned}$$

and so

$$\alpha \leq \varepsilon\,(\text{length of } T)\,diam\,(T).$$

Since ε is arbitrary, this shows $\alpha = 0$, a contradiction. Thus $\int_{\partial T} f(w)\,dw = 0$ as claimed.

This fundamental result yields the following important theorem.

Theorem 7.2 *(Morera) Let U be an open set and let $f\ '(z)$ exist for all $z \in U$. Let $D \equiv \overline{B(z_0, r)} \subseteq U$. Then there exists $\varepsilon > 0$ such that f has a primitive on $B(z_0, r+\varepsilon)$.*

Proof: Choose $\varepsilon > 0$ small enough that $B(z_0, r+\varepsilon) \subseteq U$. Then for $w \in B(z_0, r+\varepsilon)$, define

$$F(w) \equiv \int_{\gamma(z_0,w)} f(u)\, du.$$

Then by the Cauchy Goursat theorem, and $w \in B(z_0, r+\varepsilon)$, it follows that for $|h|$ small enough,

$$\frac{F(w+h) - F(w)}{h} = \frac{1}{h} \int_{\gamma(w,w+h)} f(u)\, du$$

$$= \frac{1}{h} \int_0^1 f(w+th)\, h dt = \int_0^1 f(w+th)\, dt$$

which converges to $f(w)$ due to the continuity of f at w. This proves the theorem.

We can also give the following corollary whose proof is similar to the proof of the above theorem.

Corollary 7.3 *Let U be an open set and suppose that whenever*

$$\gamma(z_1, z_2, z_3, z_1)$$

is a closed curve bounding a triangle T, which is contained in U, and f is a continuous function defined on U, it follows that

$$\int_{\gamma(z_1,z_2,z_3,z_1)} f(z)\, dz = 0,$$

Then f is analytic on U.

Proof: As in the proof of Morera's theorem, let $\overline{B(z_0, r)} \subseteq U$ and use the given condition to construct a primitive, F for f on $B(z_0, r)$. Then F is analytic and so by Theorem 6.6, it follows that F and hence f have infinitely many derivatives, implying that f is analytic on $B(z_0, r)$. Since z_0 is arbitrary, this shows f is analytic on U.

Theorem 7.4 *Let U be an open set in $\mathbb{C}$ and suppose $f : U \to \mathbb{C}$ has the property that $f'(z)$ exists for each $z \in U$. Then f is analytic on U.*

Proof: Let $z_0 \in U$ and let $B(z_0, r) \subseteq U$. By Morera's theorem f has a primitive, F on $B(z_0, r)$. It follows that F is analytic because it has a derivative, f, and this derivative is continuous. Therefore, by Theorem 6.6 F has infinitely many derivatives on $B(z_0, r)$ implying that f also has infinitely many derivatives on $B(z_0, r)$. Thus f is analytic as claimed.

It follows that we can say a function is analytic on an open set, U if and only if $f'(z)$ exists for $z \in U$. We just proved the derivative, if it exists, is automatically continuous.

The same proof used to prove Theorem 7.2 implies the following corollary.

Corollary 7.5 *Let U be a convex open set and suppose that $f'(z)$ exists for all $z \in U$. Then f has a primitive on U.*

Note that this implies that if U is a convex open set on which $f'(z)$ exists and if $\gamma : [a,b] \to U$ is a closed, continuous curve having bounded variation, then letting F be a primitive of f Theorem 4.12 implies

$$\int_\gamma f(z)\, dz = F(\gamma(b)) - F(\gamma(a)) = 0.$$

Notice how different this is from the situation of a function of a real variable. It is possible for a function of a real variable to have a derivative everywhere and yet the derivative can be discontinuous. A simple example is the following.

$$f(x) \equiv \begin{cases} x^2 \sin\left(\frac{1}{x}\right) \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}.$$

Then $f'(x)$ exists for all $x \in \mathbb{R}$. Indeed, if $x \neq 0$, the derivative equals $2x \sin \frac{1}{x} - \cos \frac{1}{x}$ which has no limit as $x \to 0$. However, from the definition of the derivative of a function of one variable, we see easily that $f'(0) = 0$.

7.2 The Cauchy integral formula

Here we develop the general version of the Cauchy integral formula valid for arbitrary closed rectifiable curves. The key idea in this development is the notion of the winding number. This is the number defined in the following theorem, also called the index

Theorem 7.6 *Let $\gamma : [a,b] \to \mathbb{C}$ be continuous and have bounded variation with $\gamma(a) = \gamma(b)$. Also suppose that $z \notin \gamma([a,b])$. We define*

$$n(\gamma, z) \equiv \frac{1}{2\pi i} \int_\gamma \frac{dw}{w - z}. \tag{7.2}$$

Then $n(\gamma, \cdot)$ is continuous and integer valued. Furthermore, there exists a sequence, $\eta_k : [a,b] \to \mathbb{C}$ such that η_k is $C^1([a,b])$,

$$||\eta_k - \gamma|| < \frac{1}{k}, \eta_k(a) = \eta_k(b) = \gamma(a) = \gamma(b),$$

and $n(\eta_k, z) = n(\gamma, z)$ for all k large enough. Also $n(\gamma, \cdot)$ is constant on every component of $\mathbb{C} \setminus \gamma([a,b])$ and equals zero on the unbounded component of $\mathbb{C} \setminus \gamma([a,b])$.

Proof: First we verify the assertion about continuity.

$$\begin{aligned} |n(\gamma, z) - n(\gamma, z_1)| &\leq C \left| \int_\gamma \left(\frac{1}{w - z} - \frac{1}{w - z_1} \right) dw \right| \\ &\leq \widetilde{C} \, (\text{Length of } \gamma) \, |z_1 - z| \end{aligned}$$

whenever z_1 is close enough to z. This proves the continuity assertion.

Next we need to show this equals an integer. To do so, use Theorem 4.11 to obtain η_k, a function in $C^1([a,b])$ such that $z \notin \eta_k([a,b])$ for all k large enough, $\eta_k(x) = \gamma(x)$ for $x = a, b$, and

$$\left| \frac{1}{2\pi i} \int_\gamma \frac{dw}{w - z} - \frac{1}{2\pi i} \int_{\eta_k} \frac{dw}{w - z} \right| < \frac{1}{k}, \; ||\eta_k - \gamma|| < \frac{1}{k}.$$

We will show each of $\frac{1}{2\pi i} \int_{\eta_k} \frac{dw}{w-z}$ is an integer. To simplify the notation, we write η instead of η_k.

$$\int_\eta \frac{dw}{w - z} = \int_a^b \frac{\eta'(s)\,ds}{\eta(s) - z}.$$

We define

$$g(t) \equiv \int_a^t \frac{\eta'(s)\,ds}{\eta(s) - z}. \tag{7.3}$$

Then

$$\left(e^{-g(t)}\left(\eta\left(t\right)-z\right)\right)' = e^{-g(t)}\eta'\left(t\right)-e^{-g(t)}g'\left(t\right)\left(\eta\left(t\right)-z\right)$$
$$= e^{-g(t)}\eta'\left(t\right)-e^{-g(t)}\eta'\left(t\right)=0.$$

It follows that $e^{-g(t)}\left(\eta\left(t\right)-z\right)$ equals a constant. In particular, using the fact that $\eta\left(a\right)=\eta\left(b\right),$

$$e^{-g(b)}\left(\eta\left(b\right)-z\right)=e^{-g(a)}\left(\eta\left(a\right)-z\right)=\left(\eta\left(a\right)-z\right)=\left(\eta\left(b\right)-z\right)$$

and so $e^{-g(b)}=1$. This happens if and only if $-g\left(b\right)=2m\pi i$ for some integer m. Therefore, 7.3 implies

$$2m\pi i=\int_a^b\frac{\eta'\left(s\right)ds}{\eta\left(s\right)-z}=\int_\eta\frac{dw}{w-z}.$$

Therefore, $\frac{1}{2\pi i}\int_{\eta_k}\frac{dw}{w-z}$ is a sequence of integers converging to $\frac{1}{2\pi i}\int_\gamma\frac{dw}{w-z}\equiv n\left(\gamma,z\right)$ and so $n\left(\gamma,z\right)$ must also be an integer and $n\left(\eta_k,z\right)=n\left(\gamma,z\right)$ for all k large enough.

Since $n\left(\gamma,\cdot\right)$ is continuous and integer valued, it follows that it must be constant on every connected component of $\mathbb{C}\setminus\gamma\left([a,b]\right)$. It is clear that $n\left(\gamma,z\right)$ equals zero on the unbounded component because from the formula,

$$\lim_{z\to\infty}\left|n\left(\gamma,z\right)\right|\leq\lim_{z\to\infty}V\left(\gamma,[a,b]\right)\left(\frac{1}{|z|-c}\right)$$

where $c\geq\max\left\{|w|:w\in\gamma\left([a,b]\right)\right\}$.This proves the theorem.

It is a good idea to consider a simple case to get an idea of what the winding number is measuring. To do so, consider $\gamma:[a,b]\to\mathbb{C}$ such that γ is continuous, closed and bounded variation. Suppose also that γ is one to one on (a,b). Such a curve is called a simple closed curve. It can be shown that such a simple closed curve divides the plane into exactly two components, an "inside" bounded component and an "outside" unbounded component. This is called the Jordan Curve theorem or the Jordan separation theorem. For a proof of this difficult result, see the chapter on degree theory. For now, it suffices to simply assume that γ is such that this result holds. This will usually be obvious anyway. We also suppose that it is possible to change the parameter to be in $[0,2\pi]$, in such a way that $\gamma\left(t\right)+\lambda\left(z+re^{it}-\gamma\left(t\right)\right)-z\neq0$ for all $t\in[0,2\pi]$ and $\lambda\in[0,1]$. (As t goes from 0 to 2π the point $\gamma\left(t\right)$ traces the curve $\gamma\left([0,2\pi]\right)$ in the counter clockwise direction.) Suppose $z\in D$, the inside of the simple closed curve and consider the curve $\delta\left(t\right)=z+re^{it}$ for $t\in[0,2\pi]$ where r is chosen small enough that $\overline{B\left(z,r\right)}\subseteq D$. Then we claim that $n\left(\delta,z\right)=n\left(\gamma,z\right)$.

Proposition 7.7 *Under the above conditions,* $n\left(\delta,z\right)=n\left(\gamma,z\right)$ *and* $n\left(\delta,z\right)=1$.

Proof: By changing the parameter, we may assume that $[a,b]=[0,2\pi]$. From Theorem 7.6 it suffices to assume also that γ is C^1. Define $h_\lambda\left(t\right)\equiv\gamma\left(t\right)+\lambda\left(z+re^{it}-\gamma\left(t\right)\right)$ for $\lambda\in[0,1]$. (This function is called a homotopy of the curves γ and δ.) Note that for each $\lambda\in[0,1]$, $t\to h_\lambda\left(t\right)$ is a closed C^1 curve. Also,

$$\frac{1}{2\pi i}\int_{h_\lambda}\frac{1}{w-z}dw=\frac{1}{2\pi i}\int_0^{2\pi}\frac{\gamma'\left(t\right)+\lambda\left(rie^{it}-\gamma'\left(t\right)\right)}{\gamma\left(t\right)+\lambda\left(z+re^{it}-\gamma\left(t\right)\right)-z}dt.$$

We know this number is an integer and it is routine to verify that it is a continuous function of λ. When $\lambda=0$ it equals $n\left(\gamma,z\right)$ and when $\lambda=1$ it equals $n\left(\delta,z\right)$. Therefore, $n\left(\delta,z\right)=n\left(\gamma,z\right)$. It only remains to compute $n\left(\delta,z\right)$.

$$n\left(\delta,z\right)=\frac{1}{2\pi i}\int_0^{2\pi}\frac{rie^{it}}{re^{it}}dt=1.$$

This proves the proposition.

Now if γ was not one to one but caused the point, $\gamma(t)$ to travel around $\gamma([a,b])$ twice, we could modify the above argument to have the parameter interval, $[0,4\pi]$ and still find $n(\delta,z) = n(\gamma,z)$ only this time, $n(\delta,z) = 2$. Thus the winding number is just what its name suggests. It measures the number of times the curve winds around the point. One might ask why bother with the winding number if this is all it does. The reason is that the notion of counting the number of times a curve winds around a point is rather vague. The winding number is precise. It is also the natural thing to consider in the general Cauchy integral formula presented below.

Theorem 7.8 *Let U be an open subset of the plane and let $f : U \to \mathbb{C}$ be analytic. If $\gamma_k : [a_k, b_k] \to U$, $k = 1, \cdots, m$ are continuous closed curves having bounded variation such that for all $z \notin U$,*

$$\sum_{k=1}^{m} n(\gamma_k, z) = 0,$$

then for all $z \in U \setminus \cup_{k=1}^{m} \gamma_k([a_k, b_k])$,

$$f(z) \sum_{k=1}^{m} n(\gamma_k, z) = \sum_{k=1}^{m} \frac{1}{2\pi i} \int_{\gamma_k} \frac{f(w)}{w - z} dw.$$

Proof: Let ϕ be defined on $U \times U$ by

$$\phi(z,w) \equiv \begin{cases} \frac{f(w)-f(z)}{w-z} \text{ if } w \neq z \\ f'(z) \text{ if } w = z \end{cases}.$$

Then ϕ is analytic as a function of both z and w and is continuous in $U \times U$. The claim that this function is analytic as a function of both z and w is obvious at points where $z \neq w$, and is most easily seen using Theorem 6.10 at points, where $z = w$. Indeed, if (z,z) is such a point, we need to verify that $w \to \phi(z,w)$ is analytic even at $w = z$. But by Theorem 6.10, for all h small enough,

$$\frac{\phi(z,z+h) - \phi(z,z)}{h} = \frac{1}{h}\left[\frac{f(z+h) - f(z)}{h} - f'(z)\right]$$

$$= \frac{1}{h}\left[\frac{1}{h}\sum_{k=1}^{\infty} \frac{f^{(k)}(z)}{k!} h^k - f'(z)\right]$$

$$= \left[\sum_{k=2}^{\infty} \frac{f^{(k)}(z)}{k!} h^{k-2}\right] \to \frac{f''(z)}{2!}.$$

Similarly, $z \to \phi(z,w)$ is analytic even if $z = w$.

We define

$$h(z) \equiv \frac{1}{2\pi i} \sum_{k=1}^{m} \int_{\gamma_k} \phi(z,w)\, dw.$$

We wish to show that h is analytic on U. To do so, we verify

$$\int_{\partial T} h(z)\, dz = 0$$

for every triangle, T, contained in U and apply Corollary 7.3. To do this we use Theorem 4.11 to obtain for each k, a sequence of functions, $\eta_{kn} \in C^1([a_k, b_k])$ such that

$$\eta_{kn}(x) = \gamma_k(x) \text{ for } x \in \{a_k, b_k\}$$

and

$$\eta_{kn}([a_k, b_k]) \subseteq U, \ \ ||\eta_{kn} - \gamma_k|| < \frac{1}{n},$$

$$\left| \int_{\eta_{kn}} \phi(z,w)\, dw - \int_{\gamma_k} \phi(z,w)\, dw \right| < \frac{1}{n}, \tag{7.4}$$

for all $z \in T$. Then applying Fubini's theorem, we can write

$$\int_{\partial T} \int_{\eta_{kn}} \phi(z,w)\, dwdz = \int_{\eta_{kn}} \int_{\partial T} \phi(z,w)\, dzdw = 0$$

because ϕ is given to be analytic. By 7.4,

$$\int_{\partial T} \int_{\gamma_k} \phi(z,w)\, dwdz = \lim_{n\to\infty} \int_{\partial T} \int_{\eta_{kn}} \phi(z,w)\, dwdz = 0$$

and so h is analytic on U as claimed.

Now let H denote the set,

$$H \equiv \left\{ z \in \mathbb{C} \setminus \cup_{k=1}^m \gamma_k([a_k, b_k]) : \sum_{k=1}^m n(\gamma_k, z) = 0 \right\}.$$

We know that H is an open set because $z \to \sum_{k=1}^m n(\gamma_k, z)$ is integer valued and continuous. Define

$$g(z) \equiv \begin{cases} h(z) \text{ if } z \in U \\ \frac{1}{2\pi i} \sum_{k=1}^m \int_{\gamma_k} \frac{f(w)}{w-z} dw \text{ if } z \in H \end{cases}. \tag{7.5}$$

We need to verify that $g(z)$ is well defined. For $z \in U \cap H$, we know $z \notin \cup_{k=1}^m \gamma_k([a_k, b_k])$ and so

$$\begin{aligned} g(z) &= \frac{1}{2\pi i} \sum_{k=1}^m \int_{\gamma_k} \frac{f(w) - f(z)}{w-z} dw \\ &= \frac{1}{2\pi i} \sum_{k=1}^m \int_{\gamma_k} \frac{f(w)}{w-z} dw - \frac{1}{2\pi i} \sum_{k=1}^m \int_{\gamma_k} \frac{f(z)}{w-z} dw \\ &= \frac{1}{2\pi i} \sum_{k=1}^m \int_{\gamma_k} \frac{f(w)}{w-z} dw \end{aligned}$$

because $z \in H$. This shows $g(z)$ is well defined. Also, g is analytic on U because it equals h there. It is routine to verify that g is analytic on H also. By assumption, $U^C \subseteq H$ and so $U \cup H = \mathbb{C}$ showing that g is an entire function.

Now note that $\sum_{k=1}^m n(\gamma_k, z) = 0$ for all z contained in the unbounded component of $\mathbb{C} \setminus \cup_{k=1}^m \gamma_k([a_k, b_k])$ which component contains $B(0,r)^C$ for r large enough. It follows that for $|z| > r$, it must be the case that $z \in H$ and so for such z, the bottom description of $g(z)$ found in 7.5 is valid. Therefore, it follows

$$\lim_{|z|\to\infty} |g(z)| = 0$$

and so g is bounded and entire. By Liouville's theorem, g is a constant. Hence, from the above equation, the constant can only equal zero.

For $z \in U \setminus \cup_{k=1}^{m} \gamma_k ([a_k, b_k])$,

$$0 = \frac{1}{2\pi i} \sum_{k=1}^{m} \int_{\gamma_k} \frac{f(w) - f(z)}{w - z} dw =$$

$$\frac{1}{2\pi i} \sum_{k=1}^{m} \int_{\gamma_k} \frac{f(w)}{w - z} dw - f(z) \sum_{k=1}^{m} n(\gamma_k, z).$$

This proves the theorem.

Corollary 7.9 *Let U be an open set and let $\gamma_k : [a_k, b_k] \to U$, $k = 1, \cdots, m$, be closed, continuous and of bounded variation. Suppose also that*

$$\sum_{k=1}^{m} n(\gamma_k, z) = 0$$

for all $z \notin U$. Then if $f : U \to \mathbb{C}$ is analytic, we have

$$\sum_{k=1}^{m} \int_{\gamma_k} f(w) \, dw = 0.$$

Proof: This follows from Theorem 7.8 as follows. Let

$$g(w) = f(w)(w - z)$$

where $z \in U \setminus \cup_{k=1}^{m} \gamma_k ([a_k, b_k])$. Then by this theorem,

$$0 = 0 \sum_{k=1}^{m} n(\gamma_k, z) = g(z) \sum_{k=1}^{m} n(\gamma_k, z) =$$

$$\sum_{k=1}^{m} \frac{1}{2\pi i} \int_{\gamma_k} \frac{g(w)}{w - z} dw = \frac{1}{2\pi i} \sum_{k=1}^{m} \int_{\gamma_k} f(w) \, dw.$$

Another simple corollary to the above theorem is Cauchy's theorem for a simply connected region.

Definition 7.10 *We say an open set, $U \subseteq \mathbb{C}$ is a region if it is open and connected. We say U is simply connected if $\widehat{\mathbb{C}} \setminus U$ is connected.*

Corollary 7.11 *Let $\gamma : [a, b] \to U$ be a continuous closed curve of bounded variation where U is a simply connected region in $\mathbb{C}$ and let $f : U \to \mathbb{C}$ be analytic. Then*

$$\int_{\gamma} f(w) \, dw = 0.$$

Proof: Let D denote the unbounded component of $\widehat{\mathbb{C}} \setminus \gamma([a, b])$. Thus $\infty \in \widehat{\mathbb{C}} \setminus \gamma([a, b])$.Then the connected set, $\widehat{\mathbb{C}} \setminus U$ is contained in D since every point of $\widehat{\mathbb{C}} \setminus U$ must be in some component of $\widehat{\mathbb{C}} \setminus \gamma([a, b])$ and ∞ is contained in both $\widehat{\mathbb{C}} \setminus U$ and D. Thus D must be the component that contains $\widehat{\mathbb{C}} \setminus U$. It follows that $n(\gamma, \cdot)$ must be constant on $\widehat{\mathbb{C}} \setminus U$, its value being its value on D. However, for $z \in D$,

$$n(\gamma, z) = \frac{1}{2\pi i} \int_{\gamma} \frac{1}{w - z} dw$$

and so $\lim_{|z|\to\infty} n(\gamma, z) = 0$ showing $n(\gamma, z) = 0$ on D. Therefore we have verified the hypothesis of Theorem 7.8. Let $z \in U \cap D$ and define

$$g(w) \equiv f(w)(w - z).$$

Thus g is analytic on U and by Theorem 7.8,

$$0 = n(z, \gamma) g(z) = \frac{1}{2\pi i} \int_\gamma \frac{g(w)}{w - z} dw = \frac{1}{2\pi i} \int_\gamma f(w) \, dw.$$

This proves the corollary.

The following is a very significant result which will be used later.

Corollary 7.12 *Suppose U is a simply connected open set and $f : U \to \mathbb{C}$ is analytic. Then f has a primitive, F, on U. Recall this means there exists F such that $F'(z) = f(z)$ for all $z \in U$.*

Proof: Pick a point, $z_0 \in U$ and let V denote those points, z of U for which there exists a curve, $\gamma : [a, b] \to U$ such that γ is continuous, of bounded variation, $\gamma(a) = z_0$, and $\gamma(b) = z$. Then it is easy to verify that V is both open and closed in U and therefore, $V = U$ because U is connected. Denote by $\gamma_{z_0,z}$ such a curve from z_0 to z and define

$$F(z) \equiv \int_{\gamma_{z_0,z}} f(w) \, dw.$$

Then F is well defined because if $\gamma_j, j = 1, 2$ are two such curves, it follows from Corollary 7.11 that

$$\int_{\gamma_1} f(w) \, dw + \int_{-\gamma_2} f(w) \, dw = 0,$$

implying that

$$\int_{\gamma_1} f(w) \, dw = \int_{\gamma_2} f(w) \, dw.$$

Now this function, F is a primitive because, thanks to Corollary 7.11

$$\begin{aligned} (F(z+h) - F(z)) h^{-1} &= \frac{1}{h} \int_{\gamma_{z,z+h}} f(w) \, dw \\ &= \frac{1}{h} \int_0^1 f(z + th) \, h dt \end{aligned}$$

and so, taking the limit as $h \to 0$, we see $F'(z) = f(z)$.

7.3 Exercises

1. If U is simply connected, f is analytic on U and f has no zeros in U, show there exists an analytic function, F, defined on U such that $e^F = f$.

2. Let U be an open set and let f be analytic on U. Show that if $a \in U$, then $f(z) = \sum_{k=0}^{\infty} b_k (z - a)^k$ whenever $|z - a| < R$ where R is the distance between a and the nearest point where f fails to have a derivative. The number R, is called the radius of convergence and the power series is said to be expanded about a.

3. Find the radius of convergence of the function $\frac{1}{1+z^2}$ expanded about $a = 2$. Note there is nothing wrong with the function, $\frac{1}{1+x^2}$ when considered as a function of a real variable, x for any value of x. However, if we insist on using power series, we find that there is a limitation on the values of x for which the power series converges due to the presence in the complex plane of a point, i, where the function fails to have a derivative.

4. What if we defined an open set, U to be simply connected if $\mathbb{C} \setminus U$ is connected. Would it amount to the same thing? **Hint:** Consider the outside of $B(0,1)$.

5. Let $\gamma(t) = e^{it} : t \in [0, 2\pi]$. Find $\int_\gamma \frac{1}{z^n} dz$ for $n = 1, 2, \cdots$.

6. Show $i \int_0^{2\pi} (2\cos\theta)^{2n} d\theta = \int_\gamma \left(z + \frac{1}{z}\right)^{2n} \left(\frac{1}{z}\right) dz$ where $\gamma(t) = e^{it} : t \in [0, 2\pi]$. Then evaluate this integral using the binomial theorem and the previous problem.

7. Let $f : U \to \mathbb{C}$ be analytic and $f(z) = u(x,y) + iv(x,y)$. Show u, v and uv are all harmonic although it can happen that u^2 is not. Recall that a function, w is harmonic if $w_{xx} + w_{yy} = 0$.

8. Suppose that for some constants $a, b \neq 0$, $a, b \in \mathbb{R}$, $f(z + ib) = f(z)$ for all $z \in \mathbb{C}$ and $f(z + a) = f(z)$ for all $z \in \mathbb{C}$. If f is analytic, show that f must be constant. Can you generalize this? **Hint:** This uses Liouville's theorem.

8 The open mapping theorem

In this chapter we present the open mapping theorem for analytic functions. This important result states that analytic functions map connected open sets to connected open sets or else to single points. It is very different than the situation for a function of a real variable.

8.1 Zeros of an analytic function

In this section we give a very surprising property of analytic functions which is in stark contrast to what takes place for functions of a real variable. It turns out the zeros of an analytic function which is not constant on some region cannot have a limit point.

Theorem 8.1 *Let U be a connected open set (region) and let $f : U \to \mathbb{C}$ be analytic. Then the following are equivalent.*

1. *$f(z) = 0$ for all $z \in U$*

2. *There exists $z_0 \in U$ such that $f^{(n)}(z_0) = 0$ for all n.*

3. *There exists $z_0 \in U$ which is a limit point of the set,*

$$Z \equiv \{z \in U : f(z) = 0\}.$$

Proof: It is clear the first condition implies the second two. Suppose the third holds. Then for z near z_0 we have

$$f(z) = \sum_{n=k}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

where $k \geq 1$ since z_0 is a zero of f. Suppose $k < \infty$. Then,

$$f(z) = (z - z_0)^k g(z)$$

where $g(z_0) \neq 0$. Letting $z_n \to z_0$ where $z_n \in Z, z_n \neq z_0$, it follows

$$0 = (z_n - z_0)^k g(z_n)$$

which implies $g(z_n) = 0$. Then by continuity of g, we see that $g(z_0) = 0$ also, contrary to the choice of k. Therefore, k cannot be less than ∞ and so z_0 is a point satisfying the second condition.

Now suppose the second condition and let

$$S \equiv \left\{ z \in U : f^{(n)}(z) = 0 \text{ for all } n \right\}.$$

It is clear that S is a closed set which by assumption is nonempty. However, this set is also open. To see this, let $z \in S$. Then for all w close enough to z,

$$f(w) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z)}{k!} (w - z)^k = 0.$$

Thus f is identically equal to zero near $z \in S$. Therefore, all points near z are contained in S also, showing that S is an open set. Now $U = S \cup (U \setminus S)$, the union of two disjoint open sets, S being nonempty. It follows the other open set, $U \setminus S$, must be empty because U is connected. Therefore, the first condition is verified. This proves the theorem. (See the following diagram.)

$$\begin{array}{ccccc} & & 1.) & & \\ & \swarrow\nearrow & & \searrow & \\ 2.) & & \longleftarrow & & 3.) \end{array}$$

Note how radically different this from the theory of functions of a real variable. Consider, for example the function

$$f(x) \equiv \begin{cases} x^2 \sin\left(\frac{1}{x}\right) \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$$

which has a derivative for all $x \in \mathbb{R}$ and for which 0 is a limit point of the set, Z, even though f is not identically equal to zero.

8.2 The open mapping theorem

With this preparation we are ready to prove the open mapping theorem, an even more surprising result than the theorem about the zeros of an analytic function.

Theorem 8.2 *(Open mapping theorem) Let U be a region in $\mathbb{C}$ and suppose $f : U \to \mathbb{C}$ is analytic. Then $f(U)$ is either a point or a region. In the case where $f(U)$ is a region, it follows that for each $z_0 \in U$, there exists an open set, V containing z_0 such that for all $z \in V$,*

$$f(z) = f(z_0) + \phi(z)^m \tag{8.1}$$

where $\phi : V \to B(0, \delta)$ is one to one, analytic and onto, $\phi(z_0) = 0$, $\phi'(z) \neq 0$ on V and ϕ^{-1} analytic on $B(0, \delta)$. If f is one to one, then $m = 1$ for each z_0 and $f^{-1} : f(U) \to U$ is analytic.

Proof: Suppose $f(U)$ is not a point. Then if $z_0 \in U$ it follows there exists $r > 0$ such that $f(z) \neq f(z_0)$ for all $z \in B(z_0, r) \setminus \{z_0\}$. Otherwise, z_0 would be a limit point of the set,

$$\{z \in U : f(z) - f(z_0) = 0\}$$

which would imply from Theorem 8.1 that $f(z) = f(z_0)$ for all $z \in U$. Therefore, making r smaller if necessary, we may write, using the power series of f,

$$f(z) = f(z_0) + (z - z_0)^m g(z)$$

for all $z \in B(z_0, r)$, where $g(z) \neq 0$ on $B(z_0, r)$. Then $\frac{g'}{g}$ is an analytic function on $B(z_0, r)$ and so by Corollary 7.5 it has a primitive on $B(z_0, r)$, h. Therefore, using the product rule and the chain rule, $(ge^{-h})' = 0$ and so there exists a constant, $C = e^{a+ib}$ such that on $B(z_0, r)$,

$$ge^{-h} = e^{a+ib}.$$

Therefore,

$$g(z) = e^{h(z)+a+ib}$$

and so, modifying h by adding in the constant, $a + ib$, we see $g(z) = e^{h(z)}$ where $h'(z) = \frac{g'(z)}{g(z)}$ on $B(z_0, r)$. Letting

$$\phi(z) = (z - z_0) e^{\frac{h(z)}{m}}$$

we obtain the formula 8.1 valid on $B(z_0, r)$. Now

$$\phi'(z_0) = e^{\frac{h(z_0)}{m}} \neq 0$$

and so, restricting r we may assume that $\phi'(z) \neq 0$ for all $z \in B(z_0, r)$. We need to verify that there is an open set, V contained in $B(z_0, r)$ such that ϕ maps V onto $B(0, \delta)$ for some $\delta > 0$.

Let $\phi(z) = u(x, y) + iv(x, y)$ where $z = x + iy$. Then

$$\begin{pmatrix} u(x_0, y_0) \\ v(x_0, y_0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

because for $z_0 = x_0 + iy_0$, $\phi(z_0) = 0$. In addition to this, the functions u and v are in $C^1(B(0, r))$ because ϕ is analytic. By the Cauchy Riemann equations,

$$\begin{vmatrix} u_x(x_0, y_0) & u_y(x_0, y_0) \\ v_x(x_0, y_0) & v_y(x_0, y_0) \end{vmatrix} = \begin{vmatrix} u_x(x_0, y_0) & -v_x(x_0, y_0) \\ v_x(x_0, y_0) & u_x(x_0, y_0) \end{vmatrix}$$

$$= u_x^2(x_0, y_0) + v_x^2(x_0, y_0) = |\phi'(z_0)|^2 \neq 0.$$

Therefore, by the inverse function theorem there exists an open set, V, containing z_0 and $\delta > 0$ such that $(u, v)^T$ maps V one to one onto $B(0, \delta)$. Thus ϕ is one to one onto $B(0, \delta)$ as claimed. It follows that ϕ^m maps V onto $B(0, \delta^m)$. Therefore, the formula 8.1 implies that f maps the open set, V, containing z_0 to an open set. This shows $f(U)$ is an open set. It is connected because f is continuous and U is connected. Thus $f(U)$ is a region. It only remains to verify that ϕ^{-1} is analytic on $B(0, \delta)$. We show this by verifying the Cauchy Riemann equations.

Let

$$\begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} \tag{8.2}$$

for $(u, v)^T \in B(0, \delta)$. Then, letting $w = u + iv$, it follows that $\phi^{-1}(w) = x(u, v) + iy(u, v)$. We need to verify that

$$x_u = y_v,\ x_v = -y_u. \tag{8.3}$$

The inverse function theorem has already given us the continuity of these partial derivatives. From the equations 8.2, we have the following systems of equations.

$$\begin{array}{c} u_x x_u + u_y y_u = 1 \\ v_x x_u + v_y y_u = 0 \end{array}, \quad \begin{array}{c} u_x x_v + u_y y_v = 0 \\ v_x x_v + v_y y_v = 1 \end{array}.$$

Solving these for x_u, y_v, x_v, and y_u, and using the Cauchy Riemann equations for u and v, yields 8.3.

It only remains to verify the assertion about the case where f is one to one. If $m > 1$, then $e^{\frac{2\pi i}{m}} \neq 1$ and so for $z_1 \in V$,

$$e^{\frac{2\pi i}{m}} \phi(z_1) \neq \phi(z_1).$$

But $e^{\frac{2\pi i}{m}} \phi(z_1) \in B(0, \delta)$ and so there exists $z_2 \neq z_1$(since ϕ is one to one) such that $\phi(z_2) = e^{\frac{2\pi i}{m}} \phi(z_1)$. But then

$$\phi(z_2)^m = \left(e^{\frac{2\pi i}{m}} \phi(z_1)\right)^m = \phi(z_1)^m$$

implying $f(z_2) = f(z_1)$ contradicting the assumption that f is one to one. Thus $m = 1$ and $f'(z) = \phi'(z) \neq 0$ on V. Since f maps open sets to open sets, it follows that f^{-1} is continuous and so we may write

$$\begin{aligned} \left(f^{-1}\right)'(f(z)) &= \lim_{f(z_1) \to f(z)} \frac{f^{-1}(f(z_1)) - f^{-1}(f(z))}{f(z_1) - f(z)} \\ &= \lim_{z_1 \to z} \frac{z_1 - z}{f(z_1) - f(z)} = \frac{1}{f'(z)}. \end{aligned}$$

This proves the theorem.

One does not have to look very far to find that this sort of thing does not hold for functions mapping $\mathbb{R}$ to $\mathbb{R}$. Take for example, the function $f(x) = x^2$. Then $f(\mathbb{R})$ is neither a point nor a region. In fact $f(\mathbb{R})$ fails to be open.

8.3 Applications of the open mapping theorem

Definition 8.3 *We will denote by ρ a ray starting at 0. Thus ρ is a straight line of infinite length extending in one direction with its initial point at 0.*

As a simple application of the open mapping theorem, we give the following theorem about branches of the logarithm.

Theorem 8.4 *Let ρ be a ray starting at 0. Then there exists an analytic function, $L(z)$ defined on $\mathbb{C} \setminus \rho$ such that*

$$e^{L(z)} = z.$$

We call L a branch of the logarithm.

Proof: Let θ be an angle of the ray, ρ. The function, e^z is a one to one and onto mapping from $\mathbb{R} + i(\theta, \theta + 2\pi)$ to $\mathbb{C} \setminus \rho$ and so we may define $L(z)$ for $z \in \mathbb{C} \setminus \rho$ such that $e^{L(z)} = z$ and we see that L defined in this way is analytic on $\mathbb{C} \setminus \rho$ because of the open mapping theorem. Note we could just as well have considered $\mathbb{R} + i(\theta - 2\pi, \theta)$. This would have given another branch of the logarithm valid on $\mathbb{C} \setminus \rho$. Also, there are infinitely many choices for θ, each of which leads to a branch of the logarithm by the process just described.

Here is another very significant theorem known as the maximum modulus theorem which follows immediately from the open mapping theorem.

Theorem 8.5 *(maximum modulus theorem) Let* U *be a bounded region and let* $f : U \to \mathbb{C}$ *be analytic and* $f : \overline{U} \to \mathbb{C}$ *continuous. Then if* $z \in U$,

$$|f(z)| \leq \max \{|f(w)| : w \in \partial U\}. \tag{8.4}$$

If equality is achieved for any $z \in U$, *then* f *is a constant.*

Proof: Suppose f is not a constant. Then $f(U)$ is a region and so if $z \in U$, there exists $r > 0$ such that $B(f(z), r) \subseteq f(U)$. It follows there exists $z_1 \in U$ with $|f(z_1)| > |f(z)|$. Hence $\max \{|f(w)| : w \in \overline{U}\}$ is not achieved at any interior point of U. Therefore, the point at which the maximum is achieved must lie on the boundary of U and so

$$\max \{|f(w)| : w \in \partial U\} = \max \{|f(w)| : w \in \overline{U}\} > |f(z)|$$

for all $z \in U$ or else f is a constant. This proves the theorem.

8.4 Counting zeros

The above proof of the open mapping theorem relies on the very important inverse function theorem from real analysis. The proof features this and the Cauchy Riemann equations to indicate how the assumption f is analytic is used. There are other approaches to this important theorem which do not rely on the big theorems from real analysis and are more oriented toward the use of the Cauchy integral formula and specialized techniques from complex analysis. We give one of these approaches next which involves the notion of "counting zeros". The next theorem is the one about counting zeros. We will use the theorem later in the proof of the Riemann mapping theorem.

Theorem 8.6 *Let* U *be a region and let* $\gamma : [a, b] \to U$ *be closed, continuous, bounded variation, and* $n(\gamma, z) = 0$ *for all* $z \notin U$. *Suppose also that* f *is analytic on* U *having zeros* $a_1, \cdots, a_m$ *where the zeros are repeated according to multiplicity, and suppose that none of these zeros are on* $\gamma([a, b])$. *Then*

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \sum_{k=1}^m n(\gamma, a_k).$$

Proof: We are given $f(z) = \prod_{j=1}^m (z - a_j) g(z)$ where $g(z) \neq 0$ on U. Hence

$$\frac{f'(z)}{f(z)} = \sum_{j=1}^m \frac{1}{z - a_j} + \frac{g'(z)}{g(z)}$$

and so

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \sum_{j=1}^m n(\gamma, a_j) + \frac{1}{2\pi i} \int_\gamma \frac{g'(z)}{g(z)} dz.$$

But the function, $z \to \frac{g'(z)}{g(z)}$ is analytic and so by Corollary 7.9, the last integral in the above expression equals 0. Therefore, this proves the theorem.

Theorem 8.7 *Let* U *be a region, let* $\gamma : [a, b] \to U$ *be continuous, closed and bounded variation such that* $n(\gamma, z) = 0$ *for all* $z \notin U$. *Also suppose* $f : U \to \mathbb{C}$ *be analytic and that* $\alpha \notin f(\gamma([a, b]))$. *Then* $f \circ \gamma : [a, b] \to \mathbb{C}$ *is continuous, closed, and bounded variation. Also suppose* $\{a_1, \cdots, a_m\} = f^{-1}(\alpha)$ *where these points are counted according to their multiplicities as zeros of the function* $f - \alpha$ *Then*

$$n(f \circ \gamma, \alpha) = \sum_{k=1}^m n(\gamma, a_k).$$

Proof: It is clear that $f \circ \gamma$ is closed and continuous. It only remains to verify that it is of bounded variation. Suppose first that $\gamma([a,b]) \subseteq B \subseteq \overline{B} \subseteq U$ where B is a ball. Then

$$\begin{aligned} & |f(\gamma(t)) - f(\gamma(s))| = \\ & \left| \int_0^1 f'(\gamma(s) + \lambda(\gamma(t) - \gamma(s)))(\gamma(t) - \gamma(s))\, d\lambda \right| \\ \leq \;& C|\gamma(t) - \gamma(s)| \end{aligned}$$

where $C \geq \max\left\{|f'(z)| : z \in \overline{B}\right\}$. Hence, in this case,

$$V(f \circ \gamma, [a,b]) \leq CV(\gamma, [a,b]).$$

Now let ε denote the distance between $\gamma([a,b])$ and $\mathbb{C} \setminus U$. Since $\gamma([a,b])$ is compact, $\varepsilon > 0$. By uniform continuity there exists $\delta = \frac{b-a}{p}$ for p a positive integer such that if $|s-t| < \delta$, then $|\gamma(s) - \gamma(t)| < \frac{\varepsilon}{2}$. Then

$$\gamma([t, t+\delta]) \subseteq \overline{B\left(\gamma(t), \frac{\varepsilon}{2}\right)} \subseteq U.$$

Let $C \geq \max\left\{|f'(z)| : z \in \cup_{j=1}^{p} \overline{B\left(\gamma(t_j), \frac{\varepsilon}{2}\right)}\right\}$ where $t_j \equiv \frac{j}{p}(b-a) + a$. Then from what was just shown,

$$\begin{aligned} V(f \circ \gamma, [a,b]) &\leq \sum_{j=0}^{p-1} V(f \circ \gamma, [t_j, t_{j+1}]) \\ &\leq C \sum_{j=0}^{p-1} V(\gamma, [t_j, t_{j+1}]) < \infty \end{aligned}$$

showing that $f \circ \gamma$ is bounded variation as claimed. Now from Theorem 7.6 there exists $\eta \in C^1([a,b])$ such that

$$\eta(a) = \gamma(a) = \gamma(b) = \eta(b), \; \eta([a,b]) \subseteq U,$$

and

$$n(\eta, a_k) = n(\gamma, a_k), \; n(f \circ \gamma, \alpha) = n(f \circ \eta, \alpha) \tag{8.5}$$

for $k = 1, \cdots, m$. Then

$$\begin{aligned} & n(f \circ \gamma, \alpha) = n(f \circ \eta, \alpha) \\ &= \frac{1}{2\pi i} \int_{f \circ \eta} \frac{dw}{w - \alpha} \\ &= \frac{1}{2\pi i} \int_a^b \frac{f'(\eta(t))}{f(\eta(t)) - \alpha} \eta'(t)\, dt \\ &= \frac{1}{2\pi i} \int_\eta \frac{f'(z)}{f(z) - \alpha} dz \\ &= \sum_{k=1}^{m} n(\eta, a_k) \end{aligned}$$

By Theorem 8.6. By 8.5, this equals $\sum_{k=1}^{m} n(\gamma, a_k)$ which proves the theorem.

The next theorem is very interesting for its own sake.

Theorem 8.8 *Let $f : B(a,R) \to \mathbb{C}$ be analytic and let*

$$f(z) - \alpha = (z-a)^m g(z), \ \infty > m \geq 1$$

where $g(z) \neq 0$ in $B(a,R)$. ($f(z) - \alpha$ has a zero of order m at $z = a$.) Then there exist $\varepsilon, \delta > 0$ with the property that for each z satisfying $0 < |z - \alpha| < \delta$, there exist points,

$$\{a_1, \cdots, a_m\} \subseteq B(a, \varepsilon),$$

such that

$$f^{-1}(z) \cap B(a,\varepsilon) = \{a_1, \cdots, a_m\}$$

and each a_k is a zero of order 1 for the function $f(\cdot) - z$.

Proof: By Theorem 8.1 f is not constant on $B(a,R)$ because it has a zero of order m. Therefore, using this theorem again, there exists $\varepsilon > 0$ such that $\overline{B(a, 2\varepsilon)} \subseteq B(a,R)$ and there are no solutions to the equation $f(z) - \alpha = 0$ for $z \in \overline{B(a, 2\varepsilon)}$ except a. Also we may assume ε is small enough that for $0 < |z-a| \leq 2\varepsilon$, $f'(z) \neq 0$. Otherwise, a would be a limit point of a sequence of points, z_n, having $f'(z_n) = 0$ which would imply, by Theorem 8.1 that $f' = 0$ on $B(0,R)$, contradicting the assumption that f has a zero of order m and is therefore not constant.

Now pick $\gamma(t) = a + \varepsilon e^{it}, t \in [0, 2\pi]$. Then $\alpha \notin f(\gamma([0,2\pi]))$ so there exists $\delta > 0$ with

$$B(\alpha, \delta) \cap f(\gamma([0,2\pi])) = \emptyset. \tag{8.6}$$

Therefore, $B(\alpha,\delta)$ is contained on one component of $\mathbb{C} \setminus f(\gamma([0,2\pi]))$. Therefore, $n(f \circ \gamma, \alpha) = n(f \circ \gamma, z)$ for all $z \in B(\alpha, \delta)$. Now consider f restricted to $\overline{B(a, 2\varepsilon)}$. For $z \in B(\alpha,\delta)$, $f^{-1}(z)$ must consist of a finite set of points because $f'(w) \neq 0$ for all w in $\overline{B(a, 2\varepsilon)} \setminus \{a\}$ implying that the zeros of $f(\cdot) - z$ in $\overline{B(a, 2\varepsilon)}$ are isolated. Since $\overline{B(a, 2\varepsilon)}$ is compact, this means there are only finitely many. By Theorem 8.7,

$$n(f \circ \gamma, z) = \sum_{k=1}^{p} n(\gamma, a_k) \tag{8.7}$$

where $\{a_1, \cdots, a_p\} = f^{-1}(z)$. Each point, a_k of $f^{-1}(z)$ is either inside the circle traced out by γ, yielding $n(\gamma, a_k) = 1$, or it is outside this circle yielding $n(\gamma, a_k) = 0$ because of 8.6. It follows the sum in 8.7 reduces to the number of points of $f^{-1}(z)$ which are contained in $B(a,\varepsilon)$. Thus, letting those points in $f^{-1}(z)$ which are contained in $B(a,\varepsilon)$ be denoted by $\{a_1, \cdots, a_r\}$

$$n(f \circ \gamma, \alpha) = n(f \circ \gamma, z) = r.$$

We need to verify that $r = m$. We do this by computing $n(f \circ \gamma, \alpha)$. However, this is easy to compute by Theorem 8.6 which states

$$n(f \circ \gamma, \alpha) = \sum_{k=1}^{m} n(\gamma, a) = m.$$

Therefore, $r = m$. Each of these a_k is a zero of order 1 of the function $f(\cdot) - z$ because $f'(a_k) \neq 0$. This proves the theorem.

This is a very fascinating result partly because it implies that for values of f near a value, α, at which $f(\cdot) - \alpha$ has a root of order m for $m > 1$, the inverse image of these values includes at least m points, not just one. Thus the topological properties of the inverse image changes radically. This theorem also shows that $f(B(a,\varepsilon)) \supseteq B(\alpha, \delta)$.

Theorem 8.9 *(open mapping theorem) Let U be a region and $f : U \to \mathbb{C}$ be analytic. Then $f(U)$ is either a point of a region. If f is one to one, then $f^{-1} : f(U) \to U$ is analytic.*

Proof: If f is not constant, then for every $\alpha \in f(U)$, it follows from Theorem 8.1 that $f(\cdot)-\alpha$ has a zero of order $m<\infty$ and so from Theorem 8.8 for each $a \in U$ there exist $\varepsilon,\delta>0$ such that $f(B(a,\varepsilon)) \supseteq B(\alpha,\delta)$ which clearly implies that f maps open sets to open sets. Therefore, $f(U)$ is open, connected because f is continuous. If f is one to one, Theorem 8.8 implies that for every $\alpha \in f(U)$ the zero of $f(\cdot)-\alpha$ is of order 1. Otherwise, that theorem implies that for z near α, there are m points which f maps to z contradicting the assumption that f is one to one. Therefore, $f'(z) \neq 0$ and since f^{-1} is continuous, due to f being an open map, it follows we may write

$$\begin{aligned}\left(f^{-1}\right)'(f(z)) &= \lim_{f(z_1)\to f(z)} \frac{f^{-1}(f(z_1)) - f^{-1}(f(z))}{f(z_1)-f(z)} \\ &= \lim_{z_1\to z} \frac{z_1-z}{f(z_1)-f(z)} = \frac{1}{f'(z)}.\end{aligned}$$

This proves the theorem.

8.5 Exercises

1. Use Theorem 8.6 to give an alternate proof of the fundamental theorem of algebra. **Hint:** Take a contour of the form $\gamma_r = re^{it}$ where $t \in [0,2\pi]$. Consider $\int_{\gamma_r} \frac{p'(z)}{p(z)}dz$ and consider the limit as $r\to\infty$.

2. Prove the following version of the maximum modulus theorem. Let $f: U \to \mathbb{C}$ be analytic where U is a region. Suppose there exists $a \in U$ such that $|f(a)| \geq |f(z)|$ for all $z \in U$. Then f is a constant.

3. Let M be an $n\times n$ matrix. Recall that the eigenvalues of M are given by the zeros of the polynomial, $p_M(z) = \det(M - zI)$ where I is the $n\times n$ identity. Formulate a theorem which describes how the eigenvalues depend on small changes in M. **Hint:** You could define a norm on the space of $n\times n$ matrices as $||M|| \equiv tr\ (MM^*)^{1/2}$ where M^* is the conjugate transpose of M. Thus

$$||M|| = \left(\sum_{j,k} |M_{jk}|^2\right)^{1/2}.$$

Argue that small changes will produce small changes in $p_M(z)$. Then apply Theorem 8.6 using γ_k a very small circle surrounding z_k, the kth eigenvalue.

4. Suppose that two analytic functions defined on a region are equal on some set, S which contains a limit point. (Recall p is a limit point of S if every open set which contains p, also contains infinitely many points of S.) Show the two functions coincide. We defined $e^z \equiv e^x(\cos y + i\sin y)$ earlier and we showed that e^z, defined this way was analytic on $\mathbb{C}$. Is there any other way to define e^z on all of $\mathbb{C}$ such that the function coincides with e^x on the real axis?

5. We know various identities for real valued functions. For example $\cosh^2 x - \sinh^2 x = 1$. If we define $\cosh z \equiv \frac{e^z+e^{-z}}{2}$ and $\sinh z \equiv \frac{e^z-e^{-z}}{2}$, does it follow that

$$\cosh^2 z - \sinh^2 z = 1$$

for all $z \in \mathbb{C}$? What about

$$\sin(z+w) = \sin z\cos w + \cos z \sin w?$$

Can you verify these sorts of identities just from your knowledge about what happens for real arguments?

6. Was it necessary that U be a region in Theorem 8.1? Would the same conclusion hold if U were only assumed to be an open set? Why? What about the open mapping theorem? Would it hold if U were not a region?

7. Let $f : U \to \mathbb{C}$ be analytic and one to one. Show that $f'(z) \neq 0$ for all $z \in U$. Does this hold for a function of a real variable?

8. We say a real valued function, u is subharmonic if $u_{xx} + u_{yy} \geq 0$. Show that if u is subharmonic on a bounded region, (open connected set) U, and continuous on $\overline{U}$ and $u \leq m$ on ∂U, then $u \leq m$ on U. **Hint:** If not, u achieves its maximum at $(x_0, y_0) \in U$. Let $u(x_0, y_0) > m + \delta$ where $\delta > 0$. Now consider $u_\varepsilon(x, y) = \varepsilon x^2 + u(x, y)$ where ε is small enough that $0 < \varepsilon x^2 < \delta$ for all $(x, y) \in U$. Show that u_ε also achieves its maximum at some point of U and that therefore, $u_{\varepsilon xx} + u_{\varepsilon yy} \leq 0$ at that point implying that $u_{xx} + u_{yy} \leq -\varepsilon$, a contradiction.

9. If u is harmonic on some region, U, show that u coincides locally with the real part of an analytic function and that therefore, u has infinitely many derivatives on U. **Hint:** Consider the case where $0 \in U$. You can always reduce to this case by a suitable translation. Now let $B(0, r) \subseteq U$ and use the Schwarz formula to obtain an analytic function whose real part coincides with u on $\partial B(0, r)$. Then use Problem 8.

10. Show the solution to the Dirichlet problem of Problem 8 in the section on the Cauchy integral formula for a disk is unique. You need to formulate this precisely and then prove uniqueness.

9 Singularities

9.1 The Laurent series

In this chapter we consider the functions which are analytic in some open set except at isolated points. The fundamental formula in this subject which is used to classify isolated singularities is the Laurent series.

Definition 9.1 *We define* $\operatorname{ann}(a, R_1, R_2) \equiv \{z : R_1 < |z - a| < R_2\}$.

Thus $\operatorname{ann}(a, 0, R)$ would denote the punctured ball, $B(a, R) \setminus \{0\}$. We now consider an important lemma which will be used in what follows.

Lemma 9.2 *Let g be analytic on $\operatorname{ann}(a, R_1, R_2)$. Then if $\gamma_r(t) \equiv a + re^{it}$ for $t \in [0, 2\pi]$ and $r \in (R_1, R_2)$, then $\int_{\gamma_r} g(z)\, dz$ is independent of r.*

Proof: Let $R_1 < r_1 < r_2 < R_2$ and denote by $-\gamma_r(t)$ the curve, $-\gamma_r(t) \equiv a + re^{i(2\pi - t)}$ for $t \in [0, 2\pi]$. Then if $z \in B(a, R_1)$, we can apply Proposition 7.7 to conclude $n(-\gamma_{r_1}, z) + n(\gamma_{r_2}, z) = 0$. Also if $z \notin B(a, R_2)$, then by Corollary 7.11 we have $n\left(\gamma_{r_j}, z\right) = 0$ for $j = 1, 2$. Therefore, we can apply Theorem 7.8 and conclude that for all $z \in \operatorname{ann}(a, R_1, R_2) \setminus \cup_{j=1}^{2} \gamma_{r_j}([0, 2\pi])$,

$$0\left(n(\gamma_{r_2}, z) + n(-\gamma_{r_1}, z)\right) =$$

$$\frac{1}{2\pi i}\int_{\gamma_{r_2}} \frac{g(w)(w - z)}{w - z} dw - \frac{1}{2\pi i}\int_{\gamma_{r_1}} \frac{g(w)(w - z)}{w - z} dw$$

which proves the desired result.

With this preparation we are ready to discuss the Laurent series.

Theorem 9.3 *Let f be analytic on $\operatorname{ann}(a, R_1, R_2)$. Then there exist numbers, $a_n \in \mathbb{C}$ such that for all $z \in \operatorname{ann}(a, R_1, R_2)$,*

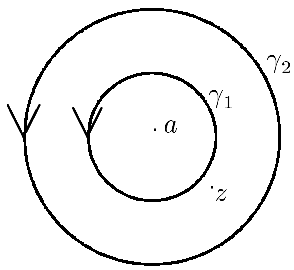
$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n, \tag{9.1}$$

where the series converges absolutely and uniformly on $\overline{\operatorname{ann}(a, r_1, r_2)}$ whenever $R_1 < r_1 < r_2 < R_2$. Also

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w-a)^{n+1}} dw \tag{9.2}$$

where $\gamma(t) = a + re^{it}, t \in [0, 2\pi]$ for any $r \in (R_1, R_2)$. Furthermore the series is unique in the sense that if 9.1 holds for $z \in \operatorname{ann}(a, R_1, R_2)$, then we obtain 9.2.

Proof: Let $R_1 < r_1 < r_2 < R_2$ and define $\gamma_1(t) \equiv a + (r_1 - \varepsilon) e^{it}$ and $\gamma_2(t) \equiv a + (r_2 + \varepsilon) e^{it}$ for $t \in [0, 2\pi]$ and ε chosen small enough that $R_1 < r_1 - \varepsilon < r_2 + \varepsilon < R_2$.



Then by Proposition 7.7 and Corollary 7.11, we see that

$$n(-\gamma_1, z) + n(\gamma_2, z) = 0$$

off $\operatorname{ann}(a, R_1, R_2)$ and that on $\operatorname{ann}(a, r_1, r_2)$,

$$n(-\gamma_1, z) + n(\gamma_2, z) = 1.$$

Therefore, by Theorem 7.8,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\left[\int_{-\gamma_1} \frac{f(w)}{w-z} dw + \int_{\gamma_2} \frac{f(w)}{w-z} dw\right] \\ &= \frac{1}{2\pi i}\left[\int_{\gamma_1} \frac{f(w)}{(z-a)\left[1 - \frac{w-a}{z-a}\right]} dw + \int_{\gamma_2} \frac{f(w)}{(w-a)\left[1 - \frac{z-a}{w-a}\right]} dw\right] \end{aligned}$$

$$= \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(w)}{w-a} \sum_{n=0}^{\infty} \left(\frac{z-a}{w-a}\right)^n dw +$$

$$\frac{1}{2\pi i} \int_{\gamma_1} \frac{f(w)}{(z-a)} \sum_{n=0}^{\infty} \left(\frac{w-a}{z-a}\right)^n dw. \tag{9.3}$$

From the formula 9.3, it follows that for $z \in \overline{\operatorname{ann}(a, r_1, r_2)}$, the terms in the first sum are bounded by an expression of the form $C\left(\frac{r_2}{r_2+\varepsilon}\right)^n$ while those in the second are bounded by one of the form $C\left(\frac{r_1-\varepsilon}{r_1}\right)^n$ and so by the Weierstrass M test, the convergence is uniform and so we may interchange the integrals and the sums in the above formula and rename the variable of summation to obtain

$$f(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{\gamma_2} \frac{f(w)}{(w-a)^{n+1}} dw \right) (z-a)^n +$$

$$\sum_{n=-\infty}^{-1} \left(\frac{1}{2\pi i} \int_{\gamma_1} \frac{f(w)}{(w-a)^{n+1}} \right) (z-a)^n . \tag{9.4}$$

By Lemma 9.2, we may write this as

$$f(z) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2\pi i} \int_{\gamma_r} \frac{f(w)}{(w-a)^{n+1}} dw \right) (z-a)^n .$$

where $r \in (R_1, R_2)$ is arbitrary.

If $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$ on $\text{ann}(a, R_1, R_2)$ let

$$f_n(z) \equiv \sum_{k=-n}^{n} a_k (z-a)^k \tag{9.5}$$

and verify from a repeat of the above argument that

$$f_n(z) = \sum_{k=-\infty}^{\infty} \left(\frac{1}{2\pi i} \int_{\gamma_r} \frac{f_n(w)}{(w-a)^{k+1}} dw \right) (z-a)^k . \tag{9.6}$$

Therefore, using 9.5 directly, we see

$$\frac{1}{2\pi i} \int_{\gamma_r} \frac{f_n(w)}{(w-a)^{k+1}} dw = a_k$$

for each $k \in [-n, n]$. However,

$$\frac{1}{2\pi i} \int_{\gamma_r} \frac{f_n(w)}{(w-a)^{k+1}} dw = \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(w)}{(w-a)^{k+1}} dw$$

because if $l > n$ or $l < -n$, then it is easy to verify that

$$\int_{\gamma_r} \frac{a_l (w-a)^l}{(w-a)^{k+1}} dw = 0$$

for all $k \in [-n, n]$. Therefore,

$$a_k = \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(w)}{(w-a)^{k+1}} dw$$

and so this establishes uniqueness. This proves the theorem.

Definition 9.4 *We say f has an isolated singularity at $a \in \mathbb{C}$ if there exists $R > 0$ such that f is analytic on $\text{ann}(a, 0, R)$. Such an isolated singularity is said to be a pole of order m if $a_{-m} \neq 0$ but $a_k = 0$ for all $k < m$. The singularity is said to be removable if $a_n = 0$ for all $n < 0$, and it is said to be essential if $a_m \neq 0$ for infinitely many $m < 0$.*

Note that thanks to the Laurent series, the possibilities enumerated in the above definition are the only ones possible. Also observe that a is removable if and only if $f(z) = g(z)$ for some g analytic near a. How can we recognize a removable singularity or a pole without computing the Laurent series? This is the content of the next theorem.

Theorem 9.5 *Let a be an isolated singularity of f. Then a is removable if and only if*

$$\lim_{z \to a} (z-a) f(z) = 0 \tag{9.7}$$

and a is a pole if and only if

$$\lim_{z \to a} |f(z)| = \infty. \tag{9.8}$$

The pole is of order m if

$$\lim_{z \to a} (z-a)^{m+1} f(z) = 0$$

but

$$\lim_{z \to a} (z-a)^m f(z) \neq 0.$$

Proof: First suppose a is a removable singularity. Then it is clear that 9.7 holds since $a_m = 0$ for all $m < 0$. Now suppose that 9.7 holds and f is analytic on $\text{ann}(a, 0, R)$. Then define

$$h(z) \equiv \begin{cases} (z-a) f(z) & \text{if } z \neq a \\ 0 \text{ if } z = a \end{cases}$$

We verify that h is analytic near a by using Morera's theorem. Let T be a triangle in $B(a, R)$. If T does not contain the point, a, then Corollary 7.11 implies $\int_{\partial T} h(z)\, dz = 0$. Therefore, we may assume $a \in T$. If a is a vertex, then, denoting by b and c the other two vertices, we pick p and q, points on the sides, ab and ac respectively which are close to a. Then by Corollary 7.11,

$$\int_{\gamma(q,c,b,p,q)} h(z)\, dz = 0.$$

But by continuity of h, it follows that as p and q are moved closer to a the above integral converges to $\int_{\partial T} h(z)\, dz$, showing that in this case, $\int_{\partial T} h(z)\, dz = 0$ also. It only remains to consider the case where a is not a vertex but is in T. In this case we subdivide the triangle T into either 3 or 2 subtriangles having a as one vertex, depending on whether a is in the interior or on an edge. Then, applying the above result to these triangles and noting that the integrals over the interior edges cancel out due to the integration being taken in opposite directions, we see that $\int_{\partial T} h(z)\, dz = 0$ in this case also.

Now we know h is analytic. Since h equals zero at a, we can conclude that

$$h(z) = (z-a) g(z)$$

where $g(z)$ is analytic in $B(a, R)$. Therefore, for all $z \neq a$,

$$(z-a) g(z) = (z-a) f(z)$$

showing that $f(z) = g(z)$ for all $z \neq a$ and g is analytic on $B(0, R)$. This proves the converse.

It is clear that if f has a pole at a, then 9.8 holds. Suppose conversely that 9.8 holds. Then we know from the first part of this theorem that $1/f(z)$ has a removable singularity at a. Also, if $g(z) = 1/f(z)$ for z near a, then $g(a) = 0$. Therefore, for $z \neq a$,

$$1/f(z) = (z-a)^m h(z)$$

for some analytic function, $h(z)$ for which $h(a) \neq 0$. It follows that $1/h \equiv r$ is analytic near a with $r(a) \neq 0$. Therefore, for z near a,

$$f(z) = (z-a)^{-m} \sum_{k=0}^{\infty} a_k (z-a)^k, \ a_0 \neq 0,$$

showing that f has a pole of order m. This proves the theorem.

Note that this is very different than what occurs for functions of a real variable. Consider for example, the function, $f(x) = x^{-1/2}$. We see $x\left(|x|^{-1/2}\right) \to 0$ but clearly $|x|^{-1/2}$ cannot equal a differentiable function near 0.

What about rational functions, those which are a quotient of two polynomials? It seems reasonable to suppose, since every finite partial sum of the Laurent series is a rational function just as every finite sum of a power series is a polynomial, it might be the case that something interesting can be said about rational functions in the context of Laurent series. In fact we will show the existence of the partial fraction expansion for rational functions. First we need the following simple lemma.

Lemma 9.6 *If f is a rational function which has no poles in $\mathbb{C}$ then f is a polynomial.*

Proof: We can write

$$f(z) = \frac{p_0 (z-b_1)^{l_1} \cdots (z-b_n)^{l_n}}{(z-a_1)^{r_1} \cdots (z-a_m)^{r_m}},$$

where we can assume the fraction has been reduced to lowest terms. Thus none of the b_j equal any of the a_k. But then, by Theorem 9.5 we would have poles at each a_k. Therefore, the denominator must reduce to 1 and so f is a polynomial.

Theorem 9.7 *Let $f(z)$ be a rational function,*

$$f(z) = \frac{p_0 (z-b_1)^{l_1} \cdots (z-b_n)^{l_n}}{(z-a_1)^{r_1} \cdots (z-a_m)^{r_m}}, \tag{9.9}$$

where the expression is in lowest terms. Then there exist numbers, b_j^k and a polynomial, $p(z)$, such that

$$f(z) = \sum_{l=1}^{m} \sum_{j=1}^{r_l} \frac{b_j^l}{(z-a_l)^j} + p(z). \tag{9.10}$$

Proof: We see that f has a pole at a_1 and it is clear this pole must be of order r_1 since otherwise we could not achieve equality between 9.9 and the Laurent series for f near a_1 due to different rates of growth. Therefore, for $z \in \operatorname{ann}(a_1, 0, R_1)$

$$f(z) = \sum_{j=1}^{r_1} \frac{b_j^1}{(z-a_1)^j} + p_1(z)$$

where p_1 is analytic in $B(a_1, R_1)$. Then define

$$f_1(z) \equiv f(z) - \sum_{j=1}^{r_1} \frac{b_j^1}{(z-a_1)^j}$$

so that f_1 is a rational function coinciding with p_1 near a_1 which has no pole at a_1. We see that f_1 has a pole at a_2 or order r_2 by the same reasoning. Therefore, we may subtract off the principle part of the Laurent series for f_1 near a_2 like we just did for f. This yields

$$f(z) = \sum_{j=1}^{r_1} \frac{b_j^1}{(z-a_1)^j} + \sum_{j=1}^{r_2} \frac{b_j^2}{(z-a_2)^j} + p_2(z).$$

Letting

$$f(z) - \left(\sum_{j=1}^{r_1} \frac{b_j^1}{(z-a_1)^j} + \sum_{j=1}^{r_2} \frac{b_j^2}{(z-a_2)^j} \right) = f_2(z),$$

and continuing in this way we finally obtain

$$f(z) - \sum_{l=1}^{m} \sum_{j=1}^{r_l} \frac{b_j^l}{(z-a_l)^j} = f_m(z)$$

where f_m is a rational function which has no poles. Therefore, it must be a polynomial. This proves the theorem.

How does this relate to the usual partial fractions routine of calculus? Recall in that case we had to consider irreducible quadratics and all the constants were real. In the case from calculus, since the coefficients of the polynomials were real, the roots of the denominator occurred in conjugate pairs. Thus we would have paired terms like

$$\frac{b}{(z-\overline{a})^j} + \frac{c}{(z-a)^j}$$

occurring in the sum. We leave it to the reader to verify this version of partial fractions does reduce to the version from calculus.

We have considered the case of a removable singularity or a pole and proved theorems about this case. What about the case where the singularity is essential? We give an interesting theorem about this case next.

Theorem 9.8 *(Casorati Weierstrass) If f has an essential singularity at a then for all $r > 0$,*

$$\overline{f(\text{ann}(a,0,r))} = \mathbb{C}$$

Proof: If not there exists $c \in \mathbb{C}$ and $r > 0$ such that $c \notin \overline{f(\text{ann}(a,0,r))}$. Therefore,there exists $\varepsilon > 0$ such that $B(c,\varepsilon) \cap f(\text{ann}(a,0,r)) = \emptyset$. It follows that

$$\lim_{z \to a} |z-a|^{-1} |f(z) - c| = \infty$$

and so by Theorem 9.5 $z \to (z-a)^{-1}(f(z)-c)$ has a pole at a. It follows that for m the order of the pole,

$$(z-a)^{-1}(f(z)-c) = \sum_{k=1}^{m} \frac{a_k}{(z-a)^k} + g(z)$$

where g is analytic near a. Therefore,

$$f(z) - c = \sum_{k=1}^{m} \frac{a_k}{(z-a)^{k-1}} + g(z)(z-a),$$

showing that f has a pole at a rather than an essential singularity. This proves the theorem.

This theorem is much weaker than the best result known, the Picard theorem which we state next. A proof of this famous theorem may be found in Conway [1].

Theorem 9.9 *If f is an analytic function having an essential singularity at z, then in every open set containing z the function f, assumes each complex number, with one possible exception, an infinite number of times.*

9.2 Exercises

1. Classify the singular points of the following functions according to whether they are poles or essential singularities. If poles, determine the order of the pole.

   (a) $\frac{\cos z}{z^2}$

   (b) $\frac{z^3+1}{z(z-1)}$

   (c) $\cos\left(\frac{1}{z}\right)$

2. Suppose f is defined on an open set, U, and it is known that f is analytic on $U \setminus \{z_0\}$ but continuous at z_0. Show that f is actually analytic on U.

3. A function defined on $\mathbb{C}$ has finitely many poles and $\lim_{|z|\to\infty} f(z)$ exists. Show f is a rational function. **Hint:** First show that if h has only one pole at 0 and if $\lim_{|z|\to\infty} h(z)$ exists, then h is a rational function. Now consider

$$h(z) \equiv \frac{\prod_{k=1}^{m}(z-z_k)^{r_k}}{\prod_{k=1}^{m} z^{r_k}} f(z)$$

   where z_k is a pole of order r_k.

9.3 Residues and evaluation of integrals

It turns out that the theory presented above about singularities and the Laurent series is very useful in computing the exact value of many hard integrals. First we define what we mean by a residue.

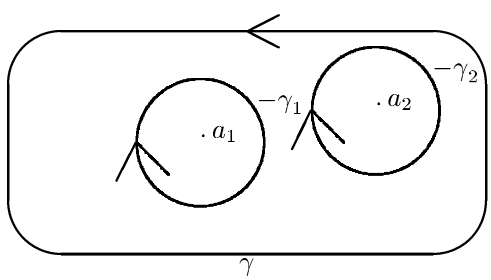
Definition 9.10 *Let a be an isolated singularity of f. Thus*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$$

for all z near a. Then we define the residue of f at a by

$$\text{Res}(f, a) = a_{-1}.$$

Now suppose that U is an open set and $f : U \setminus \{a_1, \cdots, a_m\} \to \mathbb{C}$ is analytic where the a_k are isolated singularities of f.



Let γ be a simple closed continuous, and bounded variation curve enclosing these isolated singularities such that $\gamma([a,b]) \subseteq U$ and $\{a_1, \cdots, a_m\} \subseteq D \subseteq U$, where D is the bounded component (inside) of $\mathbb{C} \setminus \gamma([a,b])$. Also assume $n(\gamma, z) = 1$ for all $z \in D$. As explained earlier, this would occur if $\gamma(t)$ traces out the curve in the counter clockwise direction. Choose r small enough that $B(a_j, r) \cap B(a_k, r) = \emptyset$ whenever $j \neq k$, $B(a_k, r) \subseteq U$ for all k, and define

$$-\gamma_k(t) \equiv a_k + re^{(2\pi - t)i}, \; t \in [0, 2\pi].$$

Thus $n(-\gamma_k, a_i) = -1$ and if z is in the unbounded component of $\mathbb{C}\setminus\gamma([a,b])$, $n(\gamma, z) = 0$ and $n(-\gamma_k, z) = 0$. If $z \notin U \setminus \{a_1, \cdots, a_m\}$, then z either equals one of the a_k or else z is in the unbounded component just described. Either way, $\sum_{k=1}^m n(\gamma_k, z) + \gamma(\gamma, z) = 0$. Therefore, by Theorem 7.8, if $z \notin D$,

$$\begin{aligned}
\sum_{j=1}^m \frac{1}{2\pi i}\int_{-\gamma_j} f(w)\frac{(w-z)}{(w-z)}dw + \frac{1}{2\pi i}\int_\gamma f(w)\frac{(w-z)}{(w-z)}dw &= \\
\sum_{j=1}^m \frac{1}{2\pi i}\int_{-\gamma_j} f(w)\,dw + \frac{1}{2\pi i}\int_\gamma f(w)\,dw &= \\
\left(\sum_{k=1}^m n(-\gamma_k, z) + \gamma(\gamma, z)\right) f(z)(z-z) &= 0.
\end{aligned}$$

and so, taking r small enough,

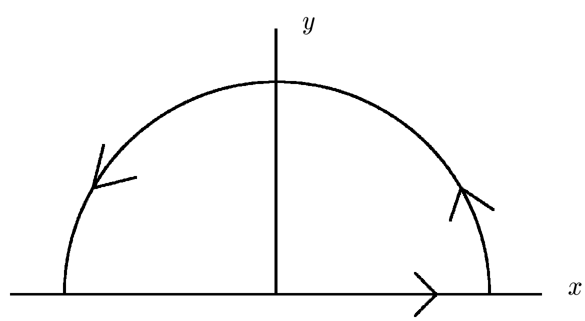
$$\begin{aligned}
\frac{1}{2\pi i}\int_\gamma f(w)\,dw &= \sum_{j=1}^m \frac{1}{2\pi i}\int_{\gamma_j} f(w)\,dw \\
&= \frac{1}{2\pi i}\sum_{k=1}^m \sum_{l=-\infty}^{\infty} a_l^k \int_{\gamma_k} (w-a_k)^l\,dw \\
&= \frac{1}{2\pi i}\sum_{k=1}^m a_{-1}^k \int_{\gamma_k} (w-a_k)^{-1}\,dw \\
&= \sum_{k=1}^m a_{-1}^k = \sum_{k=1}^m \operatorname{Res}(f, a_k).
\end{aligned}$$

Now we give some examples of hard integrals which can be evaluated by using this idea. This will be done by integrating over various closed curves having bounded variation.

Example 9.11 *The first example we consider is the following integral.*

$$\int_{-\infty}^{\infty} \frac{1}{1+x^4}dx$$

One could imagine evaluating this integral by the method of partial fractions and it should work out by that method. However, we will consider the evaluation of this integral by the method of residues instead. To do so, consider the following picture.



Let $\gamma_r(t) = re^{it}, t \in [0, \pi]$ and let $\sigma_r(t) = t : t \in [-r, r]$. Thus γ_r parameterizes the top curve and σ_r parameterizes the straight line from $-r$ to r along the x axis. Denoting by Γ_r the closed curve traced out by these two, we see from simple estimates that

$$\lim_{r\to\infty}\int_{\gamma_r}\frac{1}{1+z^4}dz=0.$$

This follows from the following estimate.

$$\left|\int_{\gamma_r}\frac{1}{1+z^4}dz\right|\leq\frac{1}{r^4-1}\pi r.$$

Therefore,

$$\int_{-\infty}^{\infty}\frac{1}{1+x^4}dx=\lim_{r\to\infty}\int_{\Gamma_r}\frac{1}{1+z^4}dz.$$

We compute $\int_{\Gamma_r}\frac{1}{1+z^4}dz$ using the method of residues. The only residues of the integrand are located at points, z where $1+z^4=0$. These points are

$$\begin{aligned} z &= -\frac{1}{2}\sqrt{2}-\frac{1}{2}i\sqrt{2}, z=\frac{1}{2}\sqrt{2}-\frac{1}{2}i\sqrt{2},\\ z &= \frac{1}{2}\sqrt{2}+\frac{1}{2}i\sqrt{2}, z=-\frac{1}{2}\sqrt{2}+\frac{1}{2}i\sqrt{2}\end{aligned}$$

and it is only the last two which are found in the inside of Γ_r. Therefore, we need to calculate the residues at these points. Clearly this function has a pole of order one at each of these points and so we may calculate the residue at α in this list by evaluating

$$\lim_{z\to\alpha}(z-\alpha)\frac{1}{1+z^4}$$

Thus

$$\begin{aligned} \operatorname{Res}\left(f,\frac{1}{2}\sqrt{2}+\frac{1}{2}i\sqrt{2}\right) &= \\ \lim_{z\to\frac{1}{2}\sqrt{2}+\frac{1}{2}i\sqrt{2}}\left(z-\left(\frac{1}{2}\sqrt{2}+\frac{1}{2}i\sqrt{2}\right)\right)\frac{1}{1+z^4} &= -\frac{1}{8}\sqrt{2}-\frac{1}{8}i\sqrt{2}\end{aligned}$$

Similarly we may find the other residue in the same way

$$\begin{aligned} \operatorname{Res}\left(f,-\frac{1}{2}\sqrt{2}+\frac{1}{2}i\sqrt{2}\right) &= \\ \lim_{z\to-\frac{1}{2}\sqrt{2}+\frac{1}{2}i\sqrt{2}}\left(z-\left(-\frac{1}{2}\sqrt{2}+\frac{1}{2}i\sqrt{2}\right)\right)\frac{1}{1+z^4} &= -\frac{1}{8}i\sqrt{2}+\frac{1}{8}\sqrt{2}.\end{aligned}$$

Therefore,

$$\int_{\Gamma_r}\frac{1}{1+z^4}dz=2\pi i\left(-\frac{1}{8}i\sqrt{2}+\frac{1}{8}\sqrt{2}+\left(-\frac{1}{8}\sqrt{2}-\frac{1}{8}i\sqrt{2}\right)\right)=\frac{1}{2}\pi\sqrt{2}.$$

Thus, taking the limit we obtain $\frac{1}{2}\pi\sqrt{2}=\int_{-\infty}^{\infty}\frac{1}{1+x^4}dx$.

Obviously many different variations of this are possible. The main idea being that the integral over the semicircle converges to zero as $r\to\infty$. Sometimes one must be fairly creative to determine the sort of curve to integrate over as well as the sort of function in the integrand and even the interpretation of the integral which results.

Example 9.12 *This example illustrates the comment about the integral.*

$$\int_0^\infty \frac{\sin x}{x} dx$$

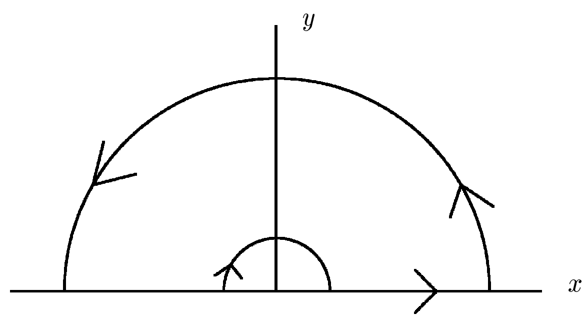
By this integral we mean $\lim_{r\to\infty} \int_0^r \frac{\sin x}{x} dx$. The function is not absolutely integrable so the meaning of the integral is in terms of the limit just described. To do this integral, we note the integrand is even and so it suffices to find

$$\lim_{R\to\infty} \int_{-R}^R \frac{e^{ix}}{x} dx$$

called the Cauchy principle value, take the imaginary part to get

$$\lim_{R\to\infty} \int_{-R}^R \frac{\sin x}{x} dx$$

and then divide by two. In order to do so, we let $R > r$ and consider the curve which goes along the x axis from $(-R, 0)$ to $(-r, 0)$, from $(-r, 0)$ to $(r, 0)$ along the semicircle in the upper half plane, from $(r, 0)$ to $(R, 0)$ along the x axis, and finally from $(R, 0)$ to $(-R, 0)$ along the semicircle in the upper half plane as shown in the following picture.



On the inside of this curve, the function, $\frac{e^{iz}}{z}$ has no singularities and so it has no residues. Pick R large and let $r \to 0+$. The integral along the small semicircle is

$$\int_\pi^0 \frac{e^{re^{it}} rie^{it}}{re^{it}} dt = \int_\pi^0 e^{\left(re^{it}\right)} dt.$$

and this clearly converges to $-\pi$ as $r \to 0$. Now we consider the top integral. For $z = Re^{it}$,

$$e^{iRe^{it}} = e^{-R\sin t} \cos(R\cos t) + ie^{-R\sin t} \sin(R\cos t)$$

and so

$$\left| e^{iRe^{it}} \right| \leq e^{-R\sin t}.$$

Therefore, along the top semicircle we get the absolute value of the integral along the top is,

$$\left| \int_0^\pi e^{iRe^{it}} dt \right| \leq \int_0^\pi e^{-R\sin t} dt$$

$$\leq \int_{\delta}^{\pi-\delta} e^{-R\sin\delta}dt + \int_{\pi-\delta}^{\pi} e^{-R\sin t}dt + \int_{0}^{\delta} e^{-R\sin t}dt$$
$$\leq e^{-R\sin\delta}\pi + \varepsilon$$

whenever δ is small enough. Letting δ be this small, it follows that

$$\lim_{R\to\infty}\left|\int_0^{\pi} e^{iRe^{it}}dt\right| \leq \varepsilon$$

and since ε is arbitrary, this shows the integral over the top semicircle converges to 0. Therefore, for some function $e(r)$ which converges to zero as $r \to 0$,

$$e(r) = \int_{\text{top semicircle}} \frac{e^{iz}}{z}dz - \pi + \int_r^R \frac{e^{ix}}{x}dx + \int_{-R}^{-r}\frac{e^{ix}}{x}dx$$

Letting $r \to 0$, we see

$$\pi = \int_{\text{top semicircle}} \frac{e^{iz}}{z}dz + \int_{-R}^{R}\frac{e^{ix}}{x}dx$$

and so, taking $R \to \infty$,

$$\pi = \lim_{R\to\infty}\int_{-R}^{R}\frac{e^{ix}}{x}dx = 2\lim_{R\to\infty}\int_0^R \frac{\sin x}{x},$$

showing that $\frac{\pi}{2} = \int_0^\infty \frac{\sin x}{x}dx$ with the above interpretation of the integral.

Sometimes we don't blow up the curves and take limits. Sometimes the problem of interest reduces directly to a complex integral over a closed curve. Here is an example of this.

Example 9.13 *The integral is*

$$\int_0^{\pi}\frac{\cos\theta}{2+\cos\theta}d\theta$$

This integrand is even and so we may write it as

$$\frac{1}{2}\int_{-\pi}^{\pi}\frac{\cos\theta}{2+\cos\theta}d\theta.$$

For z on the unit circle, $z = e^{i\theta}$, $\overline{z} = \frac{1}{z}$ and therefore, $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$. Thus $dz = ie^{i\theta}d\theta$ and so $d\theta = \frac{dz}{iz}$. Note that we are proceeding formally in order to get a complex integral which reduces to the one of interest. It follows that a complex integral which reduces to the one we want is

$$\frac{1}{2i}\int_\gamma \frac{\frac{1}{2}\left(z+\frac{1}{z}\right)}{2+\frac{1}{2}\left(z+\frac{1}{z}\right)}\frac{dz}{z} = \frac{1}{2i}\int_\gamma \frac{z^2+1}{z\left(4z+z^2+1\right)}dz$$

where γ is the unit circle. Now the integrand has poles of order 1 at those points where $z\left(4z+z^2+1\right) = 0$. These points are

$$0, -2+\sqrt{3}, -2-\sqrt{3}.$$

Only the first two are inside the unit circle. It is also clear the function has simple poles at these points. Therefore,

$$\operatorname{Res}(f, 0) = \lim_{z\to 0} z\left(\frac{z^2+1}{z\left(4z+z^2+1\right)}\right) = 1.$$

$$\text{Res}\left(f, -2+\sqrt{3}\right) =$$

$$\lim_{z \to -2+\sqrt{3}} \left(z - \left(-2+\sqrt{3}\right)\right) \frac{z^2+1}{z\left(4z+z^2+1\right)} = -\frac{2}{3}\sqrt{3}.$$

It follows

$$\begin{aligned} \int_0^\pi \frac{\cos\theta}{2+\cos\theta} d\theta &= \frac{1}{2i}\int_\gamma \frac{z^2+1}{z\left(4z+z^2+1\right)} dz \\ &= \frac{1}{2i} 2\pi i \left(1 - \frac{2}{3}\sqrt{3}\right) \\ &= \pi\left(1 - \frac{2}{3}\sqrt{3}\right). \end{aligned}$$

Other rational functions of the trig functions will work out by this method also.

Sometimes we have to be clever about which version of an analytic function that reduces to a real function we should use. The following is such an example.

Example 9.14 *The integral here is*

$$\int_0^\infty \frac{\ln x}{1+x^4} dx.$$

We would like to use the same curve we used in the integral involving $\frac{\sin x}{x}$ but this will create problems with the log since the usual version of the log is not defined on the negative real axis. This does not need to concern us however. We simply use another branch of the logarithm. We leave out the ray from 0 along the negative y axis and use Theorem 8.4 to define $L(z)$ on this set. Thus $L(z) = \ln|z| + i \arg_1(z)$ where $\arg_1(z)$ will be the angle, θ, between $-\frac{\pi}{2}$ and $\frac{3\pi}{2}$ such that $z = |z| e^{i\theta}$. Now the only singularities contained in this curve are

$$\frac{1}{2}\sqrt{2} + \frac{1}{2}i\sqrt{2}, -\frac{1}{2}\sqrt{2} + \frac{1}{2}i\sqrt{2}$$

and the integrand, f has simple poles at these points. Thus using the same procedure as in the other examples,

$$\text{Res}\left(f, \frac{1}{2}\sqrt{2} + \frac{1}{2}i\sqrt{2}\right) =$$

$$\frac{1}{32}\sqrt{2}\pi - \frac{1}{32}i\sqrt{2}\pi$$

and

$$\text{Res}\left(f, \frac{-1}{2}\sqrt{2} + \frac{1}{2}i\sqrt{2}\right) =$$

$$\frac{3}{32}\sqrt{2}\pi + \frac{3}{32}i\sqrt{2}\pi.$$

We need to consider the integral along the small semicircle of radius r. This reduces to

$$\int_\pi^0 \frac{\ln|r| + it}{1+\left(re^{it}\right)^4}\left(rie^{it}\right) dt$$

which clearly converges to zero as $r \to 0$ because $r \ln r \to 0$. Therefore, taking the limit as $r \to 0$,

$$\int_{\text{large semicircle}} \frac{L(z)}{1+z^4} dz + \lim_{r\to 0+} \int_{-R}^{-r} \frac{\ln(-t) + i\pi}{1+t^4} dt +$$

$$\lim_{r\to 0+} \int_r^R \frac{\ln t}{1+t^4} dt = 2\pi i \left(\frac{3}{32}\sqrt{2}\pi + \frac{3}{32} i\sqrt{2}\pi + \frac{1}{32}\sqrt{2}\pi - \frac{1}{32} i\sqrt{2}\pi \right).$$

Observing that $\int_{\text{large semicircle}} \frac{L(z)}{1+z^4} dz \to 0$ as $R \to \infty$, we may write

$$e(R) + 2 \lim_{r\to 0+} \int_r^R \frac{\ln t}{1+t^4} dt + i\pi \int_{-\infty}^0 \frac{1}{1+t^4} dt = \left(-\frac{1}{8} + \frac{1}{4} i \right) \pi^2 \sqrt{2}$$

where $e(R) \to 0$ as $R \to \infty$. From an earlier example this becomes

$$e(R) + 2 \lim_{r\to 0+} \int_r^R \frac{\ln t}{1+t^4} dt + i\pi \left(\frac{\sqrt{2}}{4} \pi \right) = \left(-\frac{1}{8} + \frac{1}{4} i \right) \pi^2 \sqrt{2}.$$

Now letting $r \to 0+$ and $R \to \infty$, we see

$$\begin{aligned} 2\int_0^\infty \frac{\ln t}{1+t^4} dt &= \left(-\frac{1}{8} + \frac{1}{4} i \right) \pi^2 \sqrt{2} - i\pi \left(\frac{\sqrt{2}}{4} \pi \right) \\ &= -\frac{1}{8}\sqrt{2}\pi^2, \end{aligned}$$

and so

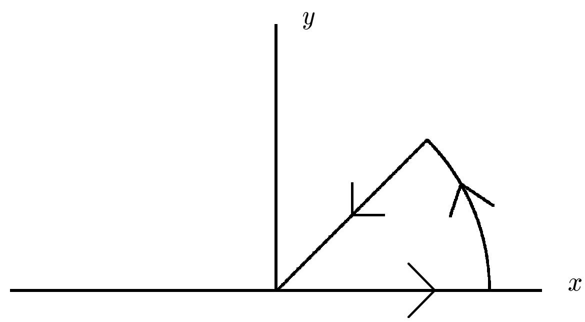
$$\int_0^\infty \frac{\ln t}{1+t^4} dt = -\frac{1}{16}\sqrt{2}\pi^2,$$

which is probably not the first thing you would thing of. You might try to imagine how this could be obtained using elementary techniques.

Example 9.15 *The Fresnel integrals are*

$$\int_0^\infty \cos x^2 dx, \int_0^\infty \sin x^2 dx.$$

To evaluate these integrals we will consider $f(z) = e^{iz^2}$ on the curve which goes from the origin to the point r on the x axis and from this point to the point $r\left(\frac{1+i}{\sqrt{2}}\right)$ along a circle of radius r, and from there back to the origin as illustrated in the following picture.



Thus the curve we integrate over is shaped like a slice of pie. Denote by γ_r the curved part. Since f is analytic,

$$\begin{aligned} 0 &= \int_{\gamma_r} e^{iz^2} dz + \int_0^r e^{ix^2} dx - \int_0^r e^{i\left(t\left(\frac{1+i}{\sqrt{2}}\right)\right)^2} \left(\frac{1+i}{\sqrt{2}}\right) dt \\ &= \int_{\gamma_r} e^{iz^2} dz + \int_0^r e^{ix^2} dx - \int_0^r e^{-t^2} \left(\frac{1+i}{\sqrt{2}}\right) dt \\ &= \int_{\gamma_r} e^{iz^2} dz + \int_0^r e^{ix^2} dx - \frac{\sqrt{\pi}}{2} \left(\frac{1+i}{\sqrt{2}}\right) + e(r) \end{aligned}$$

where $e(r) \to 0$ as $r \to \infty$. Here we used the fact that $\int_0^\infty e^{-t^2} dt = \frac{\sqrt{\pi}}{2}$. Now we need to examine the first of these integrals.

$$\begin{aligned} \left| \int_{\gamma_r} e^{iz^2} dz \right| &= \left| \int_0^{\frac{\pi}{4}} e^{i\left(re^{it}\right)^2} rie^{it} dt \right| \\ &\leq r \int_0^{\frac{\pi}{4}} e^{-r^2 \sin 2t} dt \\ &= \frac{r}{2} \int_0^1 \frac{e^{-r^2 u}}{\sqrt{1-u^2}} du \end{aligned}$$

$$\leq \frac{r}{2} \int_0^{r^{-(3/2)}} \frac{1}{\sqrt{1-u^2}} du + \frac{r}{2} \left(\int_0^1 \frac{1}{\sqrt{1-u^2}} \right) e^{-\left(r^{1/2}\right)}$$

which converges to zero as $r \to \infty$. Therefore, taking the limit as $r \to \infty$,

$$\frac{\sqrt{\pi}}{2} \left(\frac{1+i}{\sqrt{2}}\right) = \int_0^\infty e^{ix^2} dx$$

and so we can now find the Fresnel integrals

$$\int_0^\infty \sin x^2 dx = \frac{\sqrt{\pi}}{2\sqrt{2}} = \int_0^\infty \cos x^2 dx.$$

The next example illustrates the technique of integrating around a branch point.

Example 9.16 $\int_0^\infty \frac{x^{p-1}}{1+x} dx,\ p \in (0,1)$.

Since the exponent of x in the numerator is larger than -1. The integral does converge. However, the techniques of real analysis don't tell us what it converges to. The contour we will use is as follows: From $(\varepsilon, 0)$ to $(r, 0)$ along the x axis and then from $(r, 0)$ to $(r, 0)$ counter clockwise along the circle of radius r, then from $(r, 0)$ to $(\varepsilon, 0)$ along the x axis and from $(\varepsilon, 0)$ to $(\varepsilon, 0)$, clockwise along the circle of radius ε. You should draw a picture of this contour. The interesting thing about this is that we cannot define z^{p-1} all the way around 0. Therefore, we use a branch of z^{p-1} corresponding to the branch of the logarithm obtained by deleting the positive x axis. Thus

$$z^{p-1} = e^{(\ln|z| + iA(z))(p-1)}$$

where $z = |z| e^{iA(z)}$ and $A(z) \in (0, 2\pi)$. Along the integral which goes in the positive direction on the x axis, we will let $A(z) = 0$ while on the one which goes in the negative direction, we take $A(z) = 2\pi$. This is the appropriate choice obtained by replacing the line from $(\varepsilon, 0)$ to $(r, 0)$ with two lines having a small gap and then taking a limit as the gap closes. We leave it as an exercise to verify that the two integrals taken along the circles of radius ε and r converge to 0 as $\varepsilon \to 0$ and as $r \to \infty$. Therefore, taking the limit,

$$\int_0^\infty \frac{x^{p-1}}{1+x} dx + \int_\infty^0 \frac{x^{p-1}}{1+x} \left(e^{2\pi i(p-1)}\right) dx = 2\pi i \operatorname{Res}(f, -1).$$

Calculating the residue of the integrand at -1, and simplifying the above expression, we obtain

$$\left(1 - e^{2\pi i(p-1)}\right) \int_0^\infty \frac{x^{p-1}}{1+x} dx = 2\pi i e^{(p-1)i\pi}.$$

Upon simplification we see that

$$\int_0^\infty \frac{x^{p-1}}{1+x} dx = \frac{\pi}{\sin p\pi}.$$

The following example is one of the most interesting. By an auspicious choice of the contour it is possible to obtain a very interesting formula for $\cot \pi z$ known as the Mittag Leffler expansion of $\cot \pi z$.

Example 9.17 *We let γ_N be the contour which goes from $-N - \frac{1}{2} - Ni$ horizontally to $N + \frac{1}{2} - Ni$ and from there, vertically to $N + \frac{1}{2} + Ni$ and then horizontally to $-N - \frac{1}{2} + Ni$ and finally vertically to $-N - \frac{1}{2} - Ni$. Thus the contour is a large rectangle and the direction of integration is in the counter clockwise direction. We will look at the following integral.*

$$I_N \equiv \int_{\gamma_N} \frac{\pi \cos \pi z}{\sin \pi z \left(\alpha^2 - z^2\right)} dz$$

where $\alpha \in \mathbb{R}$ is not an integer. This will be used to verify the formula of Mittag Leffler,

$$\frac{1}{\alpha^2} + \sum_{n=1}^{\infty} \frac{2}{\alpha^2 - n^2} = \frac{\pi \cot \pi \alpha}{\alpha}. \tag{9.11}$$

We leave it as an exercise to verify that $\cot \pi z$ is bounded on this contour and that therefore, $I_N \to 0$ as $N \to \infty$. Now we compute the residues of the integrand at $\pm\alpha$ and at n where $|n| < N + \frac{1}{2}$ for n an integer. These are the only singularities of the integrand in this contour and therefore, we can evaluate I_N by using these. We leave it as an exercise to calculate these residues and find that the residue at $\pm\alpha$ is

$$\frac{-\pi \cos \pi \alpha}{2\alpha \sin \pi \alpha}$$

while the residue at n is

$$\frac{1}{\alpha^2 - n^2}.$$

Therefore,

$$0 = \lim_{N\to\infty} I_N = \lim_{N\to\infty} 2\pi i \left[\sum_{n=-N}^{N} \frac{1}{\alpha^2 - n^2} - \frac{\pi \cot \pi\alpha}{\alpha} \right]$$

which establishes the following formula of Mittag Leffler.

$$\lim_{N\to\infty} \sum_{n=-N}^{N} \frac{1}{\alpha^2 - n^2} = \frac{\pi \cot \pi\alpha}{\alpha}.$$

Writing this in a slightly nicer form, we obtain 9.11.

9.4 The argument principle and Rouche's theorem

This technique of evaluating integrals by computing the residues also leads to the proof of a theorem referred to as the argument principle.

Definition 9.18 *We say a function defined on U, an open set, is meromorphic if its only singularities are poles, isolated singularities, a, for which*

$$\lim_{z\to a} |f(z)| = \infty.$$

Theorem 9.19 *(argument principle) Let f be meromorphic in U and let its poles be $\{p_1, \cdots, p_m\}$ and its zeros be $\{z_1, \cdots, z_n\}$. Let z_k be a zero of order r_k and let p_k be a pole of order l_k. Let $\gamma : [a,b] \to U$ be a continuous simple closed curve having bounded variation for which the inside of $\gamma([a,b])$ contains all the poles and zeros of f and is contained in U. Also let $n(\gamma, z) = 1$ for all z contained in the inside of $\gamma([a,b])$. Then*

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \sum_{k=1}^{n} r_k - \sum_{k=1}^{m} l_k$$

Proof: This theorem follows from computing the residues of f'/f. It has residues at poles and zeros. See Problem 4.

With the argument principle, we can prove Rouche's theorem . In the argument principle, we will denote by Z_f the quantity $\sum_{k=1}^{m} r_k$ and by P_f the quantity $\sum_{k=1}^{n} l_k$. Thus Z_f is the number of zeros of f counted according to the order of the zero with a similar definition holding for P_f.

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = Z_f - P_f$$

Theorem 9.20 *(Rouche's theorem) Let f, g be meromorphic in U and let Z_f and P_f denote respectively the numbers of zeros and poles of f counted according to order. Let Z_g and P_g be defined similarly. Let $\gamma : [a,b] \to U$ be a simple closed continuous curve having bounded variation such that all poles and zeros of both f and g are inside $\gamma([a,b])$. Also let $n(\gamma, z) = 1$ for every z inside $\gamma([a,b])$. Also suppose that for $z \in \gamma([a,b])$*

$$|f(z) + g(z)| < |f(z)| + |g(z)|.$$

Then

$$Z_f - P_f = Z_g - P_g.$$

Proof: We see from the hypotheses that

$$\left|1+\frac{f(z)}{g(z)}\right|<1+\left|\frac{f(z)}{g(z)}\right|$$

which shows that for all $z\in\gamma([a,b])$,

$$\frac{f(z)}{g(z)}\in\mathbb{C}\setminus[0,\infty).$$

Letting l denote a branch of the logarithm defined on $\mathbb{C}\setminus[0,\infty)$, it follows that $l\left(\frac{f(z)}{g(z)}\right)$ is a primitive for the function, $\frac{(f/g)'}{(f/g)}$. Therefore, by the argument principle,

$$\begin{aligned} 0 &= \frac{1}{2\pi i}\int_\gamma \frac{(f/g)'}{(f/g)}dz=\frac{1}{2\pi i}\int_\gamma\left(\frac{f'}{f}-\frac{g'}{g}\right)dz \\ &= Z_f-P_f-(Z_g-P_g).\end{aligned}$$

This proves the theorem.

9.5 Exercises

1. In Example 9.11 we found the integral of a rational function of a certain sort. The technique used in this example typically works for rational functions of the form $\frac{f(x)}{g(x)}$ where $\deg(g(x))\geq\deg f(x)+2$ provided the rational function has no poles on the real axis. State and prove a theorem based on these observations.

2. Fill in the missing details of Example 9.17 about $I_N\to 0$. Note how important it was that the contour was chosen just right for this to happen. Also verify the claims about the residues.

3. Suppose f has a pole of order m at $z=a$. Define $g(z)$ by

$$g(z)=(z-a)^m f(z).$$

Show

$$\operatorname{Res}(f,a)=\frac{1}{(m-1)!}g^{(m-1)}(a).$$

Hint: Use the Laurent series.

4. Give a proof of Theorem 9.19. **Hint:** Let p be a pole. Show that near p, a pole of order m,

$$\frac{f'(z)}{f(z)}=\frac{-m+\sum_{k=1}^\infty b_k(z-p)^k}{(z-p)+\sum_{k=2}^\infty c_k(z-p)^k}$$

Show that $\operatorname{Res}(f,p)=-m$. Carry out a similar procedure for the zeros.

5. Use Rouche's theorem to prove the fundamental theorem of algebra which says that if $p(z)=z^n+a_{n-1}z^{n-1}\cdots+a_1z+a_0$, then p has n zeros in $\mathbb{C}$. **Hint:** Let $q(z)=-z^n$ and let γ be a large circle, $\gamma(t)=re^{it}$ for r sufficiently large.

6. Consider the two polynomials z^5+3z^2-1 and z^5+3z^2. Show that on $|z|=1$, we have the conditions for Rouche's theorem holding. Now use Rouche's theorem to verify that z^5+3z^2-1 must have two zeros in $|z|<1$.

7. Consider the polynomial, $z^{11} + 7z^5 + 3z^2 - 17$. Use Rouche's theorem to find a bound on the zeros of this polynomial. In other words, find r such that if z is a zero of the polynomial, $|z| < r$. Try to make r fairly small if possible.

8. Verify that $\int_0^\infty e^{-t^2} dt = \frac{\sqrt{\pi}}{2}$. **Hint:** Use polar coordinates.

9. Use the contour described in Example 9.11 to compute the exact values of the following improper integrals.

   (a) $\int_{-\infty}^{\infty} \frac{x}{(x^2+4x+13)^2} dx$

   (b) $\int_0^\infty \frac{x^2}{(x^2+a^2)^2} dx$

   (c) $\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}, a, b > 0$

10. Evaluate the following improper integrals.

    (a) $\int_0^\infty \frac{\cos ax}{(x^2+b^2)^2} dx$

    (b) $\int_0^\infty \frac{x \sin x}{(x^2+a^2)^2} dx$

11. Find the Cauchy principle value of the integral

$$\int_{-\infty}^{\infty} \frac{\sin x}{(x^2+1)(x-1)} dx$$

defined as

$$\lim_{\varepsilon \to 0+} \left(\int_{-\infty}^{1-\varepsilon} \frac{\sin x}{(x^2+1)(x-1)} dx + \int_{1+\varepsilon}^{\infty} \frac{\sin x}{(x^2+1)(x-1)} dx \right).$$

12. Find a formula for the integral $\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{n+1}}$ where n is a nonnegative integer.

13. Using the contour of Example 9.12 find $\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx$.

14. If $m < n$ for m and n integers, show

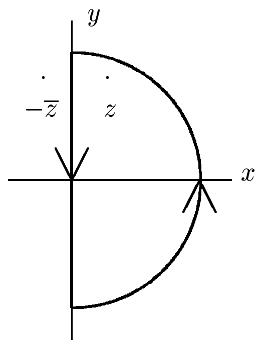
$$\int_0^\infty \frac{x^{2m}}{1+x^{2n}} dx = \frac{\pi}{n} \frac{1}{\sin\left(\frac{2m+1}{2n}\pi\right)}.$$

15. Find $\int_{-\infty}^{\infty} \frac{1}{(1+x^4)^2} dx$.

16. Find $\int_0^\infty \frac{\ln(x)}{1+x^2} dx = 0$

9.6 The Poisson formulas and the Hilbert transform

In this section we consider various applications of the above ideas by focussing on the contour, γ_R shown below, which represents a semicircle of radius R in the right half plane the direction of integration indicated by the arrows.



We will suppose that f is analytic in a region containing the right half plane and use the Cauchy integral formula to write

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_R} \frac{f(w)}{w-z}dw,\ 0 = \frac{1}{2\pi i}\int_{\gamma_R} \frac{f(w)}{w+\overline{z}}dw,$$

the second integral equaling zero because the integrand is analytic as indicated in the picture. Therefore, multiplying the second integral by α and subtracting from the first we obtain

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_R} f(w)\left(\frac{w+\overline{z}-\alpha w+\alpha z}{(w-z)(w+\overline{z})}\right)dw. \tag{9.12}$$

We would like to have the integrals over the semicircular part of the contour converge to zero as $R \to \infty$. This requires some sort of growth condition on f. Let

$$M(R) = \max\left\{\left|f\left(\mathrm{Re}^{it}\right)\right| : t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right\}.$$

We leave it as an exercise to verify that when

$$\lim_{R\to\infty} \frac{M(R)}{R} = 0 \text{ for } \alpha = 1 \tag{9.13}$$

and

$$\lim_{R\to\infty} M(R) = 0 \text{ for } \alpha \neq 1, \tag{9.14}$$

then this condition that the integrals over the curved part of γ_R converge to zero is satisfied. We assume this takes place in what follows. Taking the limit as $R \to \infty$

$$f(z) = \frac{-1}{2\pi}\int_{-\infty}^{\infty} f(i\xi)\left(\frac{i\xi+\overline{z}-\alpha i\xi+\alpha z}{(i\xi-z)(i\xi+\overline{z})}\right)d\xi \tag{9.15}$$

the negative sign occurring because the direction of integration along the y axis is negative. If $\alpha = 1$ and $z = x + iy$, this reduces to

$$f(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(i\xi)\left(\frac{x}{|z-i\xi|^2}\right)d\xi, \tag{9.16}$$

which is called the Poisson formula for a half plane.. If we assume $M(R) \to 0$, and take $\alpha = -1$, 9.15 reduces to

$$\frac{i}{\pi}\int_{-\infty}^{\infty} f(i\xi)\left(\frac{\xi-y}{|z-i\xi|^2}\right)d\xi. \tag{9.17}$$

Of course we can consider real and imaginary parts of f in these formulas. Let

$$f(i\xi) = u(\xi) + iv(\xi).$$

From 9.16 we obtain upon taking the real part,

$$u(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} u(\xi)\left(\frac{x}{|z-i\xi|^2}\right)d\xi. \tag{9.18}$$

Taking real and imaginary parts in 9.17 gives the following.

$$u(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} v(\xi)\left(\frac{y-\xi}{|z-i\xi|^2}\right)d\xi, \tag{9.19}$$

$$v(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} u(\xi)\left(\frac{\xi-y}{|z-i\xi|^2}\right)d\xi. \tag{9.20}$$

These are called the conjugate Poisson formulas because knowledge of the imaginary part on the y axis leads to knowledge of the real part for $\operatorname{Re} z > 0$ while knowledge of the real part on the imaginary axis leads to knowledge of the real part on $\operatorname{Re} z > 0$.

We obtain the Hilbert transform by formally letting $z = iy$ in the conjugate Poisson formulas and picking $x = 0$. Letting $u(0,y) = u(y)$ and $v(0,y) = v(y)$, we obtain, at least formally

$$\begin{aligned} u(y) &= \frac{1}{\pi}\int_{-\infty}^{\infty} v(\xi)\left(\frac{1}{y-\xi}\right)d\xi, \\ v(y) &= -\frac{1}{\pi}\int_{-\infty}^{\infty} u(\xi)\left(\frac{1}{y-\xi}\right)d\xi. \end{aligned}$$

Of course there are major problems in writing these integrals due to the integrand possessing a nonintegrable singularity at y. There is a large theory connected with the meaning of such integrals as these known as the theory of singular integrals. Here we evaluate these integrals by taking a contour which goes around the singularity and then taking a limit to obtain a principle value integral.

The case when $\alpha = 0$ in 9.15 yields

$$f(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{f(i\xi)}{(z-i\xi)}d\xi. \tag{9.21}$$

We will use this formula in considering the problem of finding the inverse Laplace transform.

We say a function, f, defined on $(0,\infty)$ is of exponential type if

$$|f(t)| < Ae^{at} \tag{9.22}$$

for some constants A and a. For such a function we can define the Laplace transform as follows.

$$F(s) \equiv \int_0^{\infty} f(t)\, e^{-st}dt \equiv Lf. \tag{9.23}$$

We leave it as an exercise to show that this integral makes sense for all $\operatorname{Re} s > a$ and that the function so defined is analytic on $\operatorname{Re} z > a$. Using the estimate, 9.22, we obtain that for $\operatorname{Re} s > a$,

$$|F(s)| \leq \left|\frac{A}{s-a}\right|. \tag{9.24}$$

We will show that if $f(t)$ is given by the formula,

$$e^{-(a+\varepsilon)t} f(t) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi t} F(i\xi + a + \varepsilon)\, d\xi,$$

then $Lf = F$ for all s large enough.

$$L\left(e^{-(a+\varepsilon)t} f(t)\right) = \frac{1}{2\pi}\int_{0}^{\infty} e^{-st}\int_{-\infty}^{\infty} e^{i\xi t} F(i\xi + a + \varepsilon)\, d\xi dt$$

Now if

$$\int_{-\infty}^{\infty} \left|F(i\xi + a + \varepsilon)\right| d\xi < \infty, \tag{9.25}$$

we can use Fubini's theorem to interchange the order of integration. Unfortunately, we do not know this. The best we have is the estimate 9.24. However, this is a very crude estimate and often 9.25 will hold. Therefore, we shall assume whatever we need in order to continue with the symbol pushing and interchange the order of integration to obtain with the aid of 9.21 the following:

$$\begin{aligned} L\left(e^{-(a+\varepsilon)t} f(t)\right) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{0}^{\infty} e^{-(s-i\xi)t} dt\right) F(i\xi + a + \varepsilon)\, d \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{F(i\xi + a + \varepsilon)}{s - i\xi} d\xi \\ &= F(s + a + \varepsilon) \end{aligned}$$

for all $s > 0$. (The reason for fussing with $\xi + a + \varepsilon$ rather than just ξ is so the function, $\xi \to F(\xi + a + \varepsilon)$ will be analytic on $\operatorname{Re}\xi > -\varepsilon$, a region containing the right half plane allowing us to use 9.21.) Now with this information, we may verify that $L(f)(s) = F(s)$ for all $s > a$. We just showed

$$\int_{0}^{\infty} e^{-wt} e^{-(a+\varepsilon)t} f(t)\, dt = F(w + a + \varepsilon)$$

whenever $\operatorname{Re} w > 0$. Let $s = w + a + \varepsilon$. Then $L(f)(s) = F(s)$ whenever $\operatorname{Re} s > a + \varepsilon$. Since ε is arbitrary, this verifies $L(f)(s) = F(s)$ for all $s > a$. It follows that if we are given $F(s)$ which is analytic for $\operatorname{Re} s > a$ and we want to find f such that $L(f) = F$, we should pick $c > a$ and define

$$e^{-ct} f(t) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi t} F(i\xi + c)\, d\xi.$$

Changing the variable, to let $s = i\xi + c$, we may write this as

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st} F(s)\, ds, \tag{9.26}$$

and we know from the above argument that we can expect this procedure to work if things are not too pathological. This integral is called the Bromwich integral for the inversion of the Laplace transform. The function $f(t)$ is the inverse Laplace transform.

We illustrate this procedure with a simple example. Suppose $F(s) = \frac{s}{(s^2+1)^2}$. In this case, F is analytic for $\operatorname{Re} s > 0$. Let $c = 1$ and integrate over a contour which goes from $c - iR$ vertically to $c + iR$ and then follows a semicircle in the counter clockwise direction back to $c - iR$. Clearly the integrals over the curved portion of the contour converge to 0 as $R \to \infty$. There are two residues of this function, one at i and one at $-i$. At both of these points the poles are of order two and so we find the residue at i by

$$\begin{aligned} \operatorname{Res}(f, i) &= \lim_{s \to i} \frac{d}{ds}\left(\frac{e^{ts} s (s-i)^2}{\left(s^2+1\right)^2}\right) \\ &= \frac{-ite^{it}}{4} \end{aligned}$$

and the residue at $-i$ is

$$\begin{aligned}\operatorname{Res}(f,-i) &= \lim_{s\to -i}\frac{d}{ds}\left(\frac{e^{ts}s(s+i)^2}{(s^2+1)^2}\right)\\ &= \frac{ite^{-it}}{4}\end{aligned}$$

Now evaluating the contour integral and taking $R\to\infty$, we find that the integral in 9.26 equals

$$2\pi i\left(\frac{ite^{-it}}{4}+\frac{-ite^{it}}{4}\right)=i\pi t\sin t$$

and therefore,

$$f(t)=\frac{1}{2}t\sin t.$$

You should verify that this actually works giving $L(f)=\frac{s}{(s^2+1)^2}$.

9.7 Exercises

1. Verify that the integrals over the curved part of γ_R in 9.12 converge to zero when 9.13 and 9.14 are satisfied.

2. Obtain similar formulas to 9.18 for the imaginary part in the case where $\alpha=1$ and formulas 9.19 - 9.20 in the case where $\alpha=-1$. Observe that these formulas give an explicit formula for $f(z)$ if either the real or the imaginary parts of f are known along the line $x=0$.

3. Verify that the formula for the Laplace transform, 9.23 makes sense for all $s>a$ and that F is analytic for $\operatorname{Re} z>a$.

4. Find inverse Laplace transforms for the functions,
$\frac{a}{s^2+a^2},\frac{a}{s^2(s^2+a^2)},\frac{1}{s^7},\frac{s}{(s^2+a^2)^2}$.

5. Consider the analytic function e^{-z}. Show it satisfies the necessary conditions in order to apply formula 9.16. Use this to verify the formulas,

$$\begin{aligned}e^{-x}\cos y &= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x\cos\xi}{x^2+(y-\xi)^2}d\xi,\\ e^{-x}\sin y &= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x\sin\xi}{x^2+(y-\xi)^2}d\xi.\end{aligned}$$

6. The Poisson formula gives

$$u(x,y)=\frac{1}{\pi}\int_{-\infty}^{\infty}u(0,\xi)\left(\frac{x}{x^2+(y-\xi)^2}\right)d\xi$$

whenever u is the real part of a function analytic in the right half plane which has a suitable growth condition. Show that this implies

$$1=\frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{x}{x^2+(y-\xi)^2}\right)d\xi.$$

7. Now consider an arbitrary continuous function, $u(\xi)$ and define

$$u(x,y) \equiv \frac{1}{\pi} \int_{-\infty}^{\infty} u(\xi) \left(\frac{x}{x^2 + (y-\xi)^2} \right) d\xi.$$

Verify that for $u(x,y)$ given by this formula,

$$\lim_{x \to 0+} |u(x,y) - u(y)| = 0,$$

and that u is a harmonic function, $u_{xx} + u_{yy} = 0$, on $x > 0$. Therefore, this integral yields a solution to the Dirichlet problem on the half plane which is to find a harmonic function which assumes given boundary values.

8. To what extent can we relax the assumption that $\xi \to u(\xi)$ is continuous?

9.8 Infinite products

In this section we give an introduction to the topic of infinite products and apply the theory to the Gamma function. To begin with we give a definition of what is meant by an infinite product.

Definition 9.21 $\prod_{n=1}^{\infty} (1 + u_n) \equiv \lim_{n\to\infty} \prod_{k=1}^{n} (1 + u_k)$ *whenever this limit exists. If* $u_n = u_n(z)$ *for* $z \in H$, *we say the infinite product converges uniformly on* H *if the partial products,* $\prod_{k=1}^{n} (1 + u_k(z))$ *converge uniformly on* H.

Lemma 9.22 *Let* $P_N \equiv \prod_{k=1}^{N} (1 + u_k)$ *and let* $Q_N \equiv \prod_{k=1}^{N} (1 + |u_k|)$. *Then*

$$Q_N \leq \exp\left(\sum_{k=1}^{N} |u_k| \right), \ |P_N - 1| \leq Q_N - 1$$

Proof: To verify the first inequality,

$$Q_N = \prod_{k=1}^{N} (1 + |u_k|) \leq \prod_{k=1}^{N} e^{|u_k|} = \exp\left(\sum_{k=1}^{N} |u_k| \right).$$

The second claim is obvious if $N = 1$. Consider $N = 2$.

$$\begin{aligned} |(1+u_1)(1+u_2) - 1| &= |u_2 + u_1 + u_1 u_2| \\ &\leq 1 + |u_1| + |u_2| + |u_1||u_2| - 1 \\ &= (1 + |u_1|)(1 + |u_2|) - 1 \end{aligned}$$

Continuing this way the desired inequality follows.

The main theorem is the following.

Theorem 9.23 *Let* $H \subseteq \mathbb{C}$ *and suppose that* $\sum_{n=1}^{\infty} |u_n(z)|$ *converges uniformly on* H. *Then*

$$P(z) \equiv \prod_{n=1}^{\infty} (1 + u_n(z))$$

converges uniformly on H. *If* $(n_1, n_2, \cdots)$ *is any permutation of* $(1, 2, \cdots)$, *then for all* $z \in H$,

$$P(z) = \prod_{k=1}^{\infty} (1 + u_{n_k}(z))$$

and P *has a zero at* z_0 *if and only if* $u_n(z_0) = -1$ *for some* n.

Proof: We use Lemma 9.22 to write for $m < n$, and all $z \in H$,

$$\begin{aligned}
&\left|\prod_{k=1}^{n}(1+u_k(z)) - \prod_{k=1}^{m}(1+u_k(z))\right| \\
\leq\ & \left|\prod_{k=1}^{m}(1+|u_k(z)|)\right| \left|\prod_{k=m+1}^{n}(1+u_k(z)) - 1\right| \\
\leq\ & \exp\left(\sum_{k=1}^{\infty}|u_k(z)|\right)\left|\prod_{k=m+1}^{n}(1+|u_k(z)|) - 1\right| \\
\leq\ & C\left(\exp\left(\sum_{k=m+1}^{\infty}|u_k(z)|\right) - 1\right) \\
\leq\ & C\left(e^{\varepsilon} - 1\right)
\end{aligned}$$

whenever m is large enough. This shows the partial products form a uniformly Cauchy sequence and hence converge uniformly on H. This verifies the first part of the theorem.

Next we need to verify the part about taking the product in different orders. Suppose then that $(n_1, n_2, \cdots)$ is a permutation of the list, $(1, 2, \cdots)$ and choose M large enough that for all $z \in H$,

$$\left|\prod_{k=1}^{\infty}(1+u_k(z)) - \prod_{k=1}^{M}(1+u_k(z))\right| < \varepsilon.$$

Then for all N sufficiently large, $\{n_1, n_2, \cdots, n_N\} \supseteq \{1, 2, \cdots, M\}$. Then for N this large, we use Lemma 9.22 to obtain

$$\begin{aligned}
&\left|\prod_{k=1}^{M}(1+u_k(z)) - \prod_{k=1}^{N}(1+u_{n_k}(z))\right| \leq \\
&\left|\prod_{k=1}^{M}(1+u_k(z))\right| \left|1 - \prod_{k\leq N, n_k > M}(1+u_{n_k}(z))\right| \\
\leq\ & \left|\prod_{k=1}^{M}(1+u_k(z))\right| \left|\prod_{k\leq N, n_k > M}(1+|u_{n_k}(z)|) - 1\right| \\
\leq\ & \left|\prod_{k=1}^{M}(1+u_k(z))\right| \left|\prod_{l=M}^{\infty}(1+|u_l(z)|) - 1\right| \\
\leq\ & \left|\prod_{k=1}^{M}(1+u_k(z))\right| \left(\exp\left(\sum_{l=M}^{\infty}|u_l(z)|\right) - 1\right) \\
\leq\ & \left|\prod_{k=1}^{M}(1+u_k(z))\right| (\exp\varepsilon - 1) \qquad (9.27) \\
\leq\ & \left|\prod_{k=1}^{\infty}(1+|u_k(z)|)\right| (\exp\varepsilon - 1) \qquad (9.28)
\end{aligned}$$

whenever M is large enough. Therefore, this shows, using 9.28 that

$$\left|\prod_{k=1}^{N}(1+u_{n_k}(z)) - \prod_{k=1}^{\infty}(1+u_k(z))\right| \leq$$

$$\left|\prod_{k=1}^{N}\left(1+u_{n_k}(z)\right)-\prod_{k=1}^{M}\left(1+u_k(z)\right)\right|+$$

$$\left|\prod_{k=1}^{M}\left(1+u_k(z)\right)-\prod_{k=1}^{\infty}\left(1+u_k(z)\right)\right|$$

$$\leq \varepsilon+\left(\left|\prod_{k=1}^{\infty}\left(1+\left|u_k(z)\right|\right)\right|+\varepsilon\right)(\exp\varepsilon-1)$$

which verifies the claim about convergence of the permuted products.

It remains to verify the assertion about the points, z_0, where $P(z_0)=0$. Obviously, if $u_n(z_0)=-1$, then $P(z_0)=0$. Suppose then that $P(z_0)=0$. Letting $n_k=k$ and using 9.27, we may take the limit as $N\to\infty$ to obtain

$$\left|\prod_{k=1}^{M}\left(1+u_k(z_0)\right)\right|=$$

$$\left|\prod_{k=1}^{M}\left(1+u_k(z_0)\right)-\prod_{k=1}^{\infty}\left(1+u_k(z_0)\right)\right|$$

$$\leq \left|\prod_{k=1}^{M}\left(1+u_k(z_0)\right)\right|(\exp\varepsilon-1).$$

If ε is chosen small enough in this inequality, we see this implies $\prod_{k=1}^{M}(1+u_k(z))=0$ and therefore, $u_k(z_0)=-1$ for some $k\leq M$. This proves the theorem.

Now we present the Weierstrass product formula. This formula tells how to factor analytic functions into an infinite product. It is a very interesting and useful theorem. First we need to give a definition of the elementary factors.

Definition 9.24 *Let $E_0(z)\equiv 1-z$ and for $p\geq 1$,*

$$E_p(z)\equiv(1-z)\exp\left(z+\frac{z^2}{2}+\cdots+\frac{z^p}{p}\right)$$

The fundamental factors satisfy an important estimate which is stated next.

Lemma 9.25 *For all $|z|\leq 1$ and $p=0,1,2,\cdots$,*

$$\left|1-E_p(z)\right|\leq|z|^{p+1}.$$

Proof: If $p=0$ this is obvious. Suppose therefore, that $p\geq 1$.

$$E_p'(z)=-\exp\left(z+\frac{z^2}{2}+\cdots+\frac{z^p}{p}\right)+$$

$$(1-z)\exp\left(z+\frac{z^2}{2}+\cdots+\frac{z^p}{p}\right)\left(1+z+\cdots+z^{p-1}\right)$$

and so, since $(1-z)\left(1+z+\cdots+z^{p-1}\right)=1-z^p$,

$$E_p'(z)=-z^p\exp\left(z+\frac{z^2}{2}+\cdots+\frac{z^p}{p}\right)$$

which shows that E_p' has a zero of order p at 0. Thus, from the equation just derived,

$$E_p'(z)=-z^p\sum_{k=0}^{\infty}a_kz^k$$

where each $a_k\geq 0$ and $a_0=1$. This last assertion about the sign of the a_k follows easily from differentiating the function $f(z)=\exp\left(z+\frac{z^2}{2}+\cdots+\frac{z^p}{p}\right)$ and evaluating the derivatives at $z=0$. A primitive for $E_p'(z)$ is of the form $-\sum_{k=0}^{\infty}a_k\frac{z^{k+1+p}}{k+p+1}$ and so integrating from 0 to z along $\gamma(0,z)$ we see that

$$E_p(z)-E_p(0)=$$

$$\begin{aligned}E_p(z)-1 &= -\sum_{k=0}^{\infty}a_k\frac{z^{k+p+1}}{k+p+1}\\ &= -z^{p+1}\sum_{k=0}^{\infty}a_k\frac{z^k}{k+p+1}\end{aligned}$$

which shows that $(E_p(z)-1)/z^{p+1}$ has a removable singularity at $z=0$.

Now from the formula for $E_p(z)$,

$$E_p(z)-1=(1-z)\exp\left(z+\frac{z^2}{2}+\cdots+\frac{z^p}{p}\right)-1$$

and so

$$E_p(1)-1=-1=-\sum_{k=0}^{\infty}a_k\frac{1}{k+p+1}$$

Since each $a_k\geq 0$, we see that for $|z|=1$,

$$\frac{|1-E_p(z)|}{|z^{p+1}|}\leq\sum_{k=1}^{\infty}a_k\frac{1}{k+p+1}=1.$$

Now by the maximum modulus theorem,

$$|1-E_p(z)|\leq|z|^{p+1}$$

for all $|z|\leq 1$. This proves the lemma.

Theorem 9.26 *Let z_n be a sequence of nonzero complex numbers which have no limit point in $\mathbb{C}$ and suppose there exist, p_n, nonnegative integers such that*

$$\sum_{n=1}^{\infty}\left(\frac{r}{|z_n|}\right)^{1+p_n}<\infty \tag{9.29}$$

for all $r \in \mathbb{R}$. Then

$$P(z) \equiv \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{z_n}\right)$$

is analytic on $\mathbb{C}$ and has a zero at each point, z_n and at no others. If w occurs m times in $\{z_n\}$, then P has a zero of order m at w.

Proof: The series

$$\sum_{n=1}^{\infty}\left|\frac{z}{z_n}\right|^{1+p_n}$$

converges uniformly on any compact set because if $|z| \leq r$, then

$$\left|\left(\frac{z}{z_n}\right)^{1+p_n}\right| \leq \left(\frac{r}{|z_n|}\right)^{1+p_n}$$

and so we may apply the Weierstrass M test to obtain the uniform convergence of $\sum_{n=1}^{\infty}\left(\frac{z}{z_n}\right)^{1+p_n}$ on $|z| < r$. Also,

$$\left|E_{p_n}\left(\frac{z}{z_n}\right) - 1\right| \leq \left(\frac{|z|}{|z_n|}\right)^{p_n+1}$$

by Lemma 9.25 whenever n is large enough because the hypothesis that $\{z_n\}$ has no limit point requires that $\lim_{n\to\infty}|z_n| = \infty$. Therefore, by Theorem 9.23,

$$P(z) \equiv \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{z_n}\right)$$

converges uniformly on compact subsets of $\mathbb{C}$. Letting $P_n(z)$ denote the nth partial product for $P(z)$, we have for $|z| < r$

$$P_n(z) = \frac{1}{2\pi i}\int_{\gamma_r}\frac{P_n(w)}{w - z}dw$$

where $\gamma_r(t) \equiv re^{it}$, $t \in [0, 2\pi]$. By the uniform convergence of P_n to P on compact sets, it follows the same formula holds for P in place of P_n showing that P is analytic in $B(0, r)$. Since r is arbitrary, we see that P is analytic on all of $\mathbb{C}$.

Now we ask where the zeros of P are. By Theorem 9.23, the zeros occur at exactly those points, z, where

$$E_{p_n}\left(\frac{z}{z_n}\right) - 1 = -1.$$

In that theorem $E_{p_n}\left(\frac{z}{z_n}\right) - 1$ plays the role of $u_n(z)$. Thus we need $E_{p_n}\left(\frac{z}{z_n}\right) = 0$ for some n. However, this occurs exactly when $\frac{z}{z_n} = 1$ so the zeros of P are the points $\{z_n\}$.

If w occurs m times in the sequence, $\{z_n\}$, we let $n_1, \cdots, n_m$ be those indices at which w occurs. Then we choose a permutation of $(1, 2, \cdots)$ which starts with the list $(n_1, \cdots, n_m)$. By Theorem 9.23,

$$P(z) = \prod_{k=1}^{\infty} E_{p_{n_k}}\left(\frac{z}{z_{n_k}}\right) = \left(1 - \frac{z}{w}\right)^m g(z)$$

where g is an analytic function which is not equal to zero at w. It follows from this that P has a zero of order m at w. This proves the theorem.

The next theorem is the Weierstrass factorization theorem which can be used to factor a given function, f, rather than only deciding convergence questions.

Theorem 9.27 *Let f be analytic on $\mathbb{C}$, $f(0) \neq 0$, and let the zeros of f be $\{z_k\}$, listed according to order. (Thus if z is a zero of order m, it will be listed m times in the list, $\{z_k\}$.) Then there exists an entire function, g and a sequence of nonnegative integers, p_n such that*

$$f(z) = e^{g(z)} \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{z_n}\right). \tag{9.30}$$

Note that $e^{g(z)} \neq 0$ for any z and this is the interesting thing about this function.

Proof: We know $\{z_n\}$ cannot have a limit point because if there were a limit point of this sequence, it would follow from Theorem 8.1 that $f(z) = 0$ for all z, contradicting the hypothesis that $f(0) \neq 0$. Hence $\lim_{n\to\infty} |z_n| = \infty$ and so

$$\sum_{n=1}^{\infty} \left(\frac{r}{|z_n|}\right)^{1+n-1} = \sum_{n=1}^{\infty} \left(\frac{r}{|z_n|}\right)^{n} < \infty$$

by the root test. Therefore, by Theorem 9.26 we may write

$$P(z) = \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{z_n}\right)$$

a function analytic on $\mathbb{C}$ by picking $p_n = n-1$ or perhaps some other choice. (We know $p_n = n-1$ works but we do not know this is the only choice that might work.) Then f/P has only removable singularities in $\mathbb{C}$ and no zeros thanks to Theorem 9.26. Thus, letting $h(z) = f(z)/P(z)$, we know from Corollary 7.12 that h'/h has a primitive, $\widetilde{g}$. Then

$$\left(he^{-\widetilde{g}}\right)' = 0$$

and so

$$h(z) = e^{a+ib} e^{\widetilde{g}(z)}$$

for some constants, a, b. Therefore, letting $g(z) = \widetilde{g}(z) + a + ib$, we see that $h(z) = e^{g(z)}$ and thus 9.30 holds. This proves the theorem.

Corollary 9.28 *Let f be analytic on $\mathbb{C}$, f has a zero of order m at 0, and let the other zeros of f be $\{z_k\}$, listed according to order. (Thus if z is a zero of order l, it will be listed l times in the list, $\{z_k\}$.) Then there exists an entire function, g and a sequence of nonnegative integers, p_n such that*

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{z_n}\right).$$

Proof: Since f has a zero of order m at 0, it follows from Theorem 8.1 that $\{z_k\}$ cannot have a limit point in $\mathbb{C}$ and so we may apply Theorem 9.27 to the function, $f(z)/z^m$ which has a removable singularity at 0. This proves the corollary.

9.9 Exercises

1. Show $\prod_{n=2}^{\infty}\left(1 - \frac{1}{n^2}\right) = \frac{1}{2}$. **Hint:** Take the ln of the partial product and then exploit the telescoping series.

2. Suppose $P(z) = \prod_{k=1}^{\infty} f_k(z) \neq 0$ for all $z \in U$, an open set, that convergence is uniform on compact subsets of U, and f_k is analytic on U. Show

$$P'(z) = \sum_{k=1}^{\infty} f_k'(z) \prod_{n \neq k} f_n(z).$$

Hint: Use a branch of the logarithm, defined near $P(z)$ and logarithmic differentiation.

3. Show that $\frac{\sin \pi z}{\pi z}$ has a removable singularity at $z = 0$ and so there exists an analytic function, q, defined on $\mathbb{C}$ such that $\frac{\sin \pi z}{\pi z} = q(z)$ and $q(0) = 1$. Using the Weierstrass product formula, show that

$$\begin{aligned} q(z) &= e^{g(z)} \prod_{k \in \mathbb{Z}, k \neq 0} \left(1 - \frac{z}{k}\right) e^{\frac{z}{k}} \\ &= e^{g(z)} \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right) \end{aligned}$$

for some analytic function, $g(z)$ and that we may take $g(0) = 0$.

4. $\uparrow$ Use Problem 2 along with Problem 3 to show that

$$\frac{\cos \pi z}{z} - \frac{\sin \pi z}{\pi z^2} = e^{g(z)} g'(z) \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right) -$$

$$2z e^{g(z)} \sum_{n=1}^{\infty} \frac{1}{n^2} \prod_{k \neq n} \left(1 - \frac{z^2}{k^2}\right).$$

Now divide this by $q(z)$ on both sides to show

$$\pi \cot \pi z - \frac{1}{z} = g'(z) + 2z \sum_{n=1}^{\infty} \frac{1}{z^2 - n^2}.$$

Use the Mittag Leffler expansion for the $\cot \pi z$ to conclude from this that $g'(z) = 0$ and hence, $g(z) = 0$ so that

$$\frac{\sin \pi z}{\pi z} = \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right).$$

5. $\uparrow$ In the formula for the product expansion of $\frac{\sin \pi z}{\pi z}$, let $z = \frac{1}{2}$ to obtain a formula for $\frac{\pi}{2}$ called Wallis's formula. Is this formula you have come up with a good way to calculate π?

6. This and the next collection of problems are dealing with the gamma function. Show that

$$\left| \left(1 + \frac{z}{n}\right) e^{\frac{-z}{n}} - 1 \right| \leq \frac{C(z)}{n^2}$$

and therefore,

$$\sum_{n=1}^{\infty} \left| \left(1 + \frac{z}{n}\right) e^{\frac{-z}{n}} - 1 \right| < \infty$$

with the convergence uniform on compact sets.

7. ↑ Show $\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{\frac{-z}{n}}$ converges to an analytic function on $\mathbb{C}$ which has zeros only at the negative integers and that therefore,

$$\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)^{-1}e^{\frac{z}{n}}$$

is a meromorphic function (Analytic except for poles) having simple poles at the negative integers.

8. ↑Show there exists γ such that if

$$\Gamma(z)\equiv\frac{e^{-\gamma z}}{z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)^{-1}e^{\frac{z}{n}},$$

then $\Gamma(1)=1$. **Hint:** $\prod_{n=1}^{\infty}(1+n)e^{-1/n}=c=e^{\gamma}$.

9. ↑Now show that

$$\gamma=\lim_{n\to\infty}\left[\sum_{k=1}^{n}\frac{1}{k}-\ln n\right]$$

Hint: Show $\gamma=\sum_{n=1}^{\infty}\left[\ln\left(1+\frac{1}{n}\right)-\frac{1}{n}\right]=\sum_{n=1}^{\infty}\left[\ln(1+n)-\ln n-\frac{1}{n}\right]$.

10. ↑Justify the following argument leading to Gauss's formula

$$\begin{aligned}\Gamma(z) &= \lim_{n\to\infty}\left(\prod_{k=1}^{n}\left(\frac{k}{k+z}\right)e^{\frac{z}{k}}\right)\frac{e^{-\gamma z}}{z}\\ &= \lim_{n\to\infty}\left(\frac{n!}{(1+z)(2+z)\cdots(n+z)}e^{z\left(\sum_{k=1}^{n}\frac{1}{k}\right)}\right)\frac{e^{-\gamma z}}{z}\\ &= \lim_{n\to\infty}\frac{n!}{(1+z)(2+z)\cdots(n+z)}e^{z\left(\sum_{k=1}^{n}\frac{1}{k}\right)}e^{-z\left[\sum_{k=1}^{n}\frac{1}{k}-\ln n\right]}\\ &= \lim_{n\to\infty}\frac{n!n^{z}}{(1+z)(2+z)\cdots(n+z)}.\end{aligned}$$

11. ↑ Verify from the Gauss formula above that $\Gamma(z+1)=\Gamma(z)z$ and that for n a nonnegative integer, $\Gamma(n+1)=n!$.

12. ↑ The usual definition of the gamma function for positive x is

$$\Gamma_1(x)\equiv\int_0^{\infty}e^{-t}t^{x-1}dt.$$

Show $\left(1-\frac{t}{n}\right)^n\leq e^{-t}$ for $t\in[0,n]$. Then show

$$\int_0^n\left(1-\frac{t}{n}\right)^n t^{x-1}dt=\frac{n!n^x}{x(x+1)\cdots(x+n)}.$$

Use the first part and the dominated convergence theorem or heuristics if you have not studied this theorem to conclude that

$$\Gamma_1(x)=\lim_{n\to\infty}\frac{n!n^x}{x(x+1)\cdots(x+n)}=\Gamma(x).$$

Hint: To show $\left(1-\frac{t}{n}\right)^n\leq e^{-t}$ for $t\in[0,n]$, verify this is equivalent to showing $(1-u)^n\leq e^{-nu}$ for $u\in[0,1]$.

13. ↑Show $\Gamma(z)=\int_0^{\infty}e^{-t}t^{z-1}dt$. whenever $\operatorname{Re}z>0$. **Hint:** You have already shown that this is true for positive real numbers. Verify this formula for $\operatorname{Re}z$ yields an analytic function.

14. ↑Show $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$. Then find $\Gamma\left(\frac{5}{2}\right)$.

10 The Riemann mapping theorem

We know from the open mapping theorem that analytic functions map regions to other regions or else to single points. In this chapter we prove the remarkable Riemann mapping theorem which states that for every simply connected region, U there exists an analytic function, f such that $f(U) = B(0,1)$ and in addition to this, f is one to one. The proof involves several ideas which have been developed up to now. We also need the following important theorem, a case of Montel's theorem.

Theorem 10.1 *Let U be an open set in $\mathbb{C}$ and let $\mathcal{F}$ denote a set of analytic functions mapping U to $B(0,M)$. Then there exists a sequence of functions from $\mathcal{F}$, $\{f_n\}_{n=1}^{\infty}$ and an analytic function, f such that $f_n^{(k)}$ converges uniformly to $f^{(k)}$ on every compact subset of U.*

Proof: First we note there exists a sequence of compact sets, K_n such that $K_n \subseteq \operatorname{int} K_{n+1} \subseteq U$ for all n where here $\operatorname{int} K$ denotes the interior of the set K, the union of all open sets contained in K and $\cup_{n=1}^{\infty} K_n = U$. We leave it as an exercise to verify that $\overline{B(0,n)} \cap \left\{z \in U : \operatorname{dist}\left(z, U^C\right) \leq \frac{1}{n}\right\}$ works for K_n. Then there exist positive numbers, δ_n such that if $z \in K_n$, then $\overline{B(z,\delta_n)} \subseteq \operatorname{int} K_{n+1}$. Now denote by $\mathcal{F}_n$ the set of restrictions of functions of $\mathcal{F}$ to K_n. Then let $z \in K_n$ and let $\gamma(t) \equiv z + \delta_n e^{it}, t \in [0, 2\pi]$. It follows that for $z_1 \in B(z, \delta_n)$, and $f \in \mathcal{F}$,

$$\begin{aligned} |f(z) - f(z_1)| &= \left| \frac{1}{2\pi i} \int_{\gamma} f(w) \left(\frac{1}{w-z} - \frac{1}{w-z_1} \right) dw \right| \\ &\leq \frac{1}{2\pi} \left| \int_{\gamma} f(w) \frac{z - z_1}{(w-z)(w-z_1)} dw \right| \end{aligned}$$

Letting $|z_1 - z| < \frac{\delta_n}{2}$, we can estimate this and write

$$\begin{aligned} |f(z) - f(z_1)| &\leq \frac{M}{2\pi} 2\pi \delta_n \frac{|z - z_1|}{\delta_n^2/2} \\ &\leq 2M \frac{|z - z_1|}{\delta_n}. \end{aligned}$$

It follows that $\mathcal{F}_n$ is equicontinuous and uniformly bounded so by the Arzela Ascoli theorem there exists a sequence, $\{f_{nk}\}_{k=1}^{\infty} \subseteq \mathcal{F}$ which converges uniformly on K_n. Let $\{f_{1k}\}_{k=1}^{\infty}$ converge uniformly on K_1. Then use the Arzela Ascoli theorem applied to this sequence to get a subsequence, denoted by $\{f_{2k}\}_{k=1}^{\infty}$ which also converges uniformly on K_2. Continue in this way to obtain $\{f_{nk}\}_{k=1}^{\infty}$ which converges uniformly on $K_1, \cdots, K_n$. Now the sequence $\{f_{nn}\}_{n=m}^{\infty}$ is a subsequence of $\{f_{mk}\}_{k=1}^{\infty}$ and so it converges uniformly on K_m for all m. Denoting f_{nn} by f_n for short, this is the sequence of functions promised by the theorem. It is clear $\{f_n\}_{n=1}^{\infty}$ converges uniformly on every compact subset of U because every such set is contained in K_m for all m large enough. Let $f(z)$ be the point to which $f_n(z)$ converges. Then f is a continuous function defined on U. We need to verify f is analytic. But, letting $T \subseteq U$,

$$\int_{\partial T} f(z)\, dz = \lim_{n \to \infty} \int_{\partial T} f_n(z)\, dz = 0.$$

Therefore, by Morera's theorem we see that f is analytic. As for the uniform convergence of the derivatives of f, this follows from the Cauchy integral formula. For $z \in K_n$, and $\gamma(t) \equiv z + \delta_n e^{it}, t \in [0, 2\pi]$,

$$\begin{aligned} |f'(z) - f_k'(z)| &\leq \frac{1}{2\pi} \left| \int_{\gamma} \frac{f_k(w) - f(w)}{(w-z)^2} dw \right| \\ &\leq ||f_k - f|| \frac{1}{2\pi} 2\pi \delta_n \frac{1}{\delta_n^2} \\ &= ||f_k - f|| \frac{1}{\delta_n}, \end{aligned}$$

where here $||f_k - f|| \equiv \sup \{|f_k(z) - f(z)| : z \in K_n\}$. Thus we get uniform convergence of the derivatives. The consideration of the other derivatives is similar.

Since the family, $\mathcal{F}$ satisfies the conclusion of Theorem 10.1 it is known as a normal family of functions.

The following result is about a certain class of so called fractional linear transformations,

Lemma 10.2 *For $\alpha \in B(0,1)$, let*

$$\phi_\alpha(z) \equiv \frac{z-\alpha}{1-\overline{\alpha}z}.$$

Then ϕ_α maps $B(0,1)$ one to one and onto $B(0,1)$, $\phi_\alpha^{-1} = \phi_{-\alpha}$, and

$$\phi_\alpha'(\alpha) = \frac{1}{1-|\alpha|^2}.$$

Proof: First we show $\phi_\alpha(z) \in B(0,1)$ whenever $z \in B(0,1)$. If this is not so, there exists $z \in B(0,1)$ such that

$$|z-\alpha|^2 \geq |1-\overline{\alpha}z|^2.$$

However, this requires

$$|z|^2 + |\alpha|^2 > 1 + |\alpha|^2 |z|^2$$

and so

$$|z|^2 \left(1-|\alpha|^2\right) > 1 - |\alpha|^2$$

contradicting $|z| < 1$.

It remains to verify ϕ_α is one to one and onto with the given formula for ϕ_α^{-1}. But it is easy to verify $\phi_\alpha\left(\phi_{-\alpha}(w)\right) = w$. Therefore, ϕ_α is onto and one to one. To verify the formula for ϕ_α', just differentiate the function. Thus,

$$\phi_\alpha'(z) = (z-\alpha)(-1)(1-\overline{\alpha}z)^{-2}(-\overline{\alpha}) + (1-\overline{\alpha}z)^{-1}$$

and the formula for the derivative follows.

The next lemma, known as Schwarz's lemma is interesting for its own sake but will be an important part of the proof of the Riemann mapping theorem.

Lemma 10.3 *Suppose $F : B(0,1) \to B(0,1)$, F is analytic, and $F(0) = 0$. Then for all $z \in B(0,1)$,*

$$|F(z)| \leq |z|, \tag{10.1}$$

and

$$|F'(0)| \leq 1. \tag{10.2}$$

If equality holds in 10.2 then there exists $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ and

$$F(z) = \lambda z. \tag{10.3}$$

Proof: We know $F(z) = zG(z)$ where G is analytic. Then letting $|z| < r < 1$, the maximum modulus theorem implies

$$|G(z)| \leq \sup \frac{\left|F\left(re^{it}\right)\right|}{r} \leq \frac{1}{r}.$$

Therefore, letting $r \to 1$ we get

$$|G(z)| \leq 1 \tag{10.4}$$

It follows that 10.1 holds. Since $F'(0) = G(0)$, 10.4 implies 10.2. If equality holds in 10.2, then from the maximum modulus theorem, we see that G achieves its maximum at an interior point and is consequently equal to a constant, $\lambda, |\lambda| = 1$. Thus $F(z) = z\lambda$ which shows 10.3. This proves the lemma.

Definition 10.4 *We say a region, U has the square root property if whenever $f, \frac{1}{f} : U \to \mathbb{C}$ are both analytic, it follows there exists $\phi : U \to \mathbb{C}$ such that ϕ is analytic and $f(z) = \phi^2(z)$.*

The next theorem will turn out to be equivalent to the Riemann mapping theorem.

Theorem 10.5 *Let $U \neq \mathbb{C}$ for U a region and suppose U has the square root property. Then there exists $h : U \to B(0,1)$ such that h is one to one, onto, and analytic.*

Proof: We define $\mathcal{F}$ to be the set of functions, f such that $f : U \to B(0,1)$ is one to one and analytic. We will show $\mathcal{F}$ is nonempty. Then we will show there is a function in $\mathcal{F}$, h, such that for some fixed $z_0 \in U$, $|h'(z_0)| \geq |\psi'(z_0)|$ for all $\psi \in \mathcal{F}$. When we have done this, we show h is actually onto. This will prove the theorem.

Now we begin by showing $\mathcal{F}$ is nonempty. Since $U \neq \mathbb{C}$ it follows there exists $\xi \notin U$. Then letting $f(z) \equiv z - \xi$, it follows f and $\frac{1}{f}$ are both analytic on U. Since U has the square root property, there exists $\phi : U \to \mathbb{C}$ such that $\phi^2(z) = f(z)$ for all $z \in U$. By the open mapping theorem, there exists a such that for some $r < |a|$,

$$B(a,r) \subseteq \phi(U).$$

It follows that if $z \in U$, then $\phi(z) \notin B(-a,r)$ because if this were to occur for some $z_1 \in U$, then $-\phi(z_1) \in B(a,r)$ and so there exists $z_2 \in B(a,r)$ such that

$$-\phi(z_1) = \phi(z_2).$$

Squaring both sides, it follows that $z_1 - \xi = z_2 - \xi$ and so $z_1 = z_2$. Therefore, we would have $\phi(z_2) = 0$ and so $0 \in B(a,r)$ contrary to the construction in which $r < |a|$. Now let

$$\psi(z) \equiv \frac{r}{\phi(z) + a}.$$

ψ is well defined because we just verified the denominator is nonzero. It also follows that $|\psi(z)| \leq 1$ because if not,

$$r > |\phi(z) + a|$$

for some $z \in U$, contradicting what was just shown about $\phi(U) \cap B(-a,r) = \emptyset$. Therefore, we have shown that $\mathcal{F} \neq \emptyset$.

For $z_0 \in U$ fixed, let

$$\eta \equiv \sup\left\{|\psi'(z_0)| : \psi \in \mathcal{F}\right\}.$$

Thus $\eta > 0$ because $\psi'(z_0) \neq 0$ for ψ defined above. By Theorem 10.1, there exists a sequence, $\{\psi_n\}$, of functions in $\mathcal{F}$ and an analytic function, h, such that

$$|\psi_n'(z_0)| \to \eta(z_0) \tag{10.5}$$

and

$$\psi_n \to h, \psi_n' \to h', \tag{10.6}$$

uniformly on all compact sets of U. It follows

$$|h'(z_0)| = \lim_{n\to\infty} |\psi_n'(z_0)| = \eta$$

and for all $z \in U$,

$$|h(z)| = \lim_{n\to\infty} |\psi_n(z)| \leq 1.$$

We need to verify that h is one to one. Suppose $h(z_1) = \alpha$ and $z_2 \in U$. We must verify that $h(z_2) \neq \alpha$. We choose $r > 0$ such that $h - \alpha$ has no zeros on $\partial B(z_2, r)$, $\overline{B(z_2, r)} \subseteq U$, and

$$\overline{B(z_2, r)} \cap \overline{B(z_1, r)} = \emptyset.$$

We can do this because, the zeros of $h - \alpha$ are isolated since h is not constant due to the fact that $h'(z_0) = \eta \neq 0$. Let $\psi_n(z_1) = \alpha_n$. Thus $\psi_n - \alpha_n$ has a zero at z_1 and since ψ_n is one to one, it has no zeros in $\overline{B(z_2, r)}$. Thus by Theorem 8.6, the theorem on counting zeros, for $\gamma(t) \equiv z_2 + re^{it}, t \in [0, 2\pi]$,

$$\begin{aligned} 0 &= \lim_{n\to\infty} \frac{1}{2\pi i} \int_\gamma \frac{\psi_n'(w)}{\psi_n(w) - \alpha_n} dw \\ &= \frac{1}{2\pi i} \int_\gamma \frac{h'(w)}{h(w) - \alpha} dw, \end{aligned}$$

which shows that $h - \alpha$ has no zeros in $B(z_2, r)$. This shows that h is one to one since $z_2 \neq z_1$ was arbitrary. Therefore, $h \in \mathcal{F}$. This completes the second step of the proof. It only remains to verify that h is onto.

To show h is onto, we use the fractional linear transformation of Lemma 10.2. Suppose h is not onto. Then there exists $\alpha \in B(0,1) \setminus h(U)$. Then $0 \notin \phi_\alpha \circ h$ because $\alpha \notin h(U)$. Therefore, since U has the square root property, there exists g, an analytic function defined on U such that

$$g^2 = \phi_\alpha \circ h.$$

The function g is one to one because if $g(z_1) = g(z_2)$, then we could square both sides and conclude that

$$\phi_\alpha \circ h(z_1) = \phi_\alpha \circ h(z_2)$$

and since ϕ_α and h are one to one, this shows $z_1 = z_2$. It follows that $g \in \mathcal{F}$ also. Now let $\psi \equiv \phi_{g(z_0)} \circ g$. Thus $\psi(z_0) = 0$. If we define $s(w) \equiv w^2$, then using Lemma 10.2, in particular, the description of $\phi_\alpha^{-1} = \phi_{-\alpha}$, we obtain

$$g = \phi_{-g(z_0)} \circ \psi$$

and therefore,

$$\begin{aligned} h(z) &= \phi_{-\alpha}\left(g^2(z)\right) \\ &= \left(\phi_{-\alpha} \circ s \circ \phi_{-g(z_0)} \circ \psi\right)(z) \\ &= (F \circ \psi)(z) \end{aligned}$$

Now $F(0) = \phi_\alpha^{-1}\left(\phi_{g(z_0)}^{-2}(0)\right) = \phi_\alpha^{-1}\left(g^2(z_0)\right) = h(z_0)$.

There are two cases to consider. First suppose that $h(z_0) \neq 0$. Then define

$$G \equiv \phi_{h(z_0)} \circ F.$$

Then $G : B(0,1) \to B(0,1)$ and $G(0) = 0$. Therefore by the Schwarz lemma, Lemma 10.3,

$$|G'(0)| = \left| \left(\frac{1}{1 - |h(z_0)|^2} \right) F'(0) \right| \leq 1$$

which implies $|F'(0)| < 1$. In the case where $h(z_0) = 0$, we note that because of the function, s, in the definition of F, F is not one to one and so we cannot have $F(z) = \lambda z$ for some $|\lambda| = 1$. Therefore, by the Schwarz lemma applied to F, we see $|F'(0)| < 1$. Therefore,

$$\begin{aligned} \eta &= |h'(z_0)| = |F'(\psi(z_0))| \, |\psi'(z_0)| \\ &= |F'(0)| \, |\psi'(z_0)| < |\psi'(z_0)|, \end{aligned}$$

contradicting the definition of η. Therefore, h must be onto and this proves the theorem.

We now give a simple lemma which will yield the usual form of the Riemann mapping theorem.

Lemma 10.6 *Let U be a simply connected region with $U \neq \mathbb{C}$. Then U has the square root property.*

Proof: Let f and $\frac{1}{f}$ both be analytic on U. Then $\frac{f'}{f}$ is analytic on U so by Corollary 7.12, there exists $\widetilde{F}$, analytic on U such that $\widetilde{F}' = \frac{f'}{f}$ on U. Then $\left(fe^{-\widetilde{F}}\right)' = 0$ and so $f(z) = Ce^{\widetilde{F}} = e^{a+ib}e^{\widetilde{F}}$. Now let $F = \widetilde{F} + a + ib$. Then F is still a primitive of f'/f and we have $f(z) = e^{F(z)}$. Now let $\phi(z) \equiv e^{\frac{1}{2}F(z)}$. Then ϕ is the desired square root and so U has the square root property.

Corollary 10.7 *(Riemann mapping theorem) Let U be a simply connected region with $U \neq \mathbb{C}$ and let $a \in U$. Then there exists a function, $f : U \to B(0,1)$ such that f is one to one, analytic, and onto with $f(a) = 0$. Furthermore, f^{-1} is also analytic.*

Proof: From Theorem 10.5 and Lemma 10.6 there exists a function, $g : U \to B(0,1)$ which is one to one, onto, and analytic. We need to show that there exists a function, f, which does what g does but in addition, $f(a) = 0$. We can do so by letting $f = \phi_{g(a)} \circ g$ if $g(a) \neq 0$. The assertion that f^{-1} is analytic follows from the open mapping theorem.

10.1 Exercises

1. Prove that in Theorem 10.1 it suffices to assume $\mathcal{F}$ is uniformly bounded on each compact subset of U.

2. Verify the conclusion of Theorem 10.1 involving the higher order derivatives.

3. What if $U = \mathbb{C}$? Does there exist an analytic function, f mapping U one to one and onto $B(0,1)$? Explain why or why not. Was $U \neq \mathbb{C}$ used in the proof of the Riemann mapping theorem?

4. Verify that $|\phi_\alpha(z)| = 1$ if $|z| = 1$. Apply the maximum modulus theorem to conclude that $|\phi_\alpha(z)| \leq 1$ for all $|z| < 1$.

5. Suppose that $|f(z)| \leq 1$ for $|z| = 1$ and $f(\alpha) = 0$ for $|\alpha| < 1$. Show that $|f(z)| \leq |\phi_\alpha(z)|$ for all $z \in B(0,1)$. **Hint:** Consider $\frac{f(z)(1-\overline{\alpha}z)}{z-\alpha}$ which has a removable singularity at α. Show the modulus of this function is bounded by 1 on $|z| = 1$. Then apply the maximum modulus theorem.

11 Approximation of analytic functions

Consider the function, $\frac{1}{z} = f(z)$ for z defined on $U \equiv B(0,1) \setminus \{0\}$. Clearly f is analytic on U. Suppose we could approximate f uniformly by polynomials on $\overline{\text{ann}\left(0, \frac{1}{2}, \frac{3}{4}\right)}$, a compact subset of U. Then, there would exist a suitable polynomial $p(z)$, such that $\left|\frac{1}{2\pi i}\int_\gamma f(z) - p(z)\, dz\right| < \frac{1}{10}$ where here γ is a circle of radius $\frac{2}{3}$. However, this is impossible because $\frac{1}{2\pi i}\int_\gamma f(z)\, dz = 1$ while $\frac{1}{2\pi i}\int_\gamma p(z)\, dz = 0$. This shows we cannot expect to be able to uniformly approximate analytic functions on compact sets using polynomials. It turns out we will be able to approximate by rational functions. The following lemma is the one of the key results which will allow us to verify a theorem on approximation. We will use the notation

$$||f - g||_{K,\infty} \equiv \sup\{|f(z) - g(z)| : z \in K\}$$

which describes the manner in which the approximation is measured.

Lemma 11.1 *Let R be a rational function which has a pole only at $a \in V$, a component of $\mathbb{C} \setminus K$ where K is a compact set. Suppose $b \in \overline{V}$. Then for $\varepsilon > 0$ given, there exists a rational function, Q, having a pole only at b such that*

$$||R - Q||_{K,\infty} < \varepsilon. \tag{11.1}$$

If it happens that V is unbounded, then there exists a polynomial, P such that

$$||R - P||_{K,\infty} < \varepsilon. \tag{11.2}$$

Proof: We say $b \in V$ satisfies P if for all $\varepsilon > 0$ there exists a rational function, Q_b, having a pole only at b such that

$$||R - Q_b||_{K,\infty}.$$

Now we define a set,

$$S \equiv \{b \in V : b \text{ satisfies } P\}.$$

We observe that $S \neq \emptyset$ because $a \in S$.

We now show that S is open. Suppose $b_1 \in S$. Then there exists a $\delta > 0$ such that

$$\left|\frac{b_1 - b}{z - b}\right| < \frac{1}{2} \tag{11.3}$$

for all $z \in K$ whenever $b \in B(b_1, \delta)$. If not, there would exist a sequence $b_n \to b$ for which $\left|\frac{b_1 - b_n}{\text{dist}(b_n, K)}\right| \geq \frac{1}{2}$. Then taking the limit and using the fact that $\text{dist}(b_n, K) \to \text{dist}(b, K) > 0$, (why?) we obtain a contradiction. Since b_1 satisfies P, there exists a rational function Q_{b_1} with the desired properties. We will show we can approximate Q_{b_1} with Q_b thus yielding an approximation to R by the use of the triangle inequality,

$$||R - Q_{b_1}||_{K,\infty} + ||Q_{b_1} - Q_b||_{K,\infty} \geq ||R - Q_b||_{K,\infty}.$$

Since Q_{b_1} has poles only at b_1, it follows it is a sum of functions of the form $\frac{\alpha_n}{(z-b_1)^n}$. Therefore, it suffices to assume Q_{b_1} is of the special form

$$Q_{b_1}(z) = \frac{1}{(z - b_1)^n}.$$

However,

$$\frac{1}{(z-b_1)^n} = \frac{1}{(z-b)^n\left(1-\frac{b_1-b}{z-b}\right)^n} = \frac{1}{(z-b)^n}\sum_{k=0}^{\infty} A_k\left(\frac{b_1-b}{z-b}\right)^k. \tag{11.4}$$

We leave it as an exercise to find A_k and to verify using the Weierstrass M test that this series converges absolutely and uniformly on K because of the estimate 11.3 which holds for all $z \in K$. Therefore, a suitable partial sum can be made as close as desired to $\frac{1}{(z-b_1)^n}$. This shows that b satisfies P whenever b is close enough to b_1 verifying that S is open.

Next we show that S is closed in V. Let $b_n \in S$ and suppose $b_n \to b \in V$. Then for all n large enough,

$$\frac{1}{2}\operatorname{dist}(b, K) \geq |b_n - b|$$

and so we obtain the following for all n large enough.

$$\left|\frac{b-b_n}{z-b_n}\right| < \frac{1}{2},$$

for all $z \in K$. Now a repeat of the above argument in 11.4 with b_n playing the role of b_1 shows that $b \in S$. Since S is both open and closed in V it follows that, since $S \neq \emptyset$, we must have $S = V$. Otherwise V would fail to be connected.

Now let $b \in \partial V$. Then a repeat of the argument that was just given to verify that S is closed shows that b satisfies P and proves 11.1.

It remains to consider the case where V is unbounded. Since $S = V$, pick $b \in V = S$ large enough that

$$\left|\frac{z}{b}\right| < \frac{1}{2} \tag{11.5}$$

for all $z \in K$. As before, it suffices to assume that Q_b is of the form

$$Q_b(z) = \frac{1}{(z-b)^n}$$

Then we leave it as an exercise to verify that, thanks to 11.5,

$$\frac{1}{(z-b)^n} = \frac{(-1)^n}{b^n}\sum_{k=0}^{\infty} A_k\left(\frac{z}{b}\right)^k \tag{11.6}$$

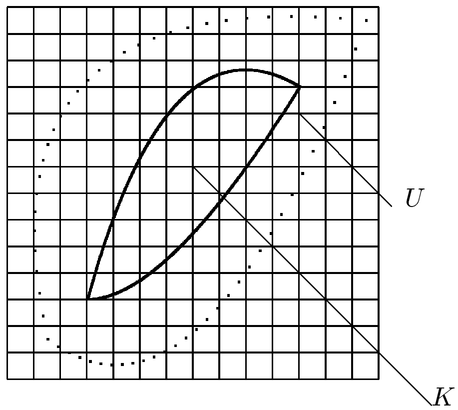
with the convergence uniform on K. Therefore, we may approximate R uniformly by a polynomial consisting of a partial sum of the above infinite sum.

The next theorem is interesting for its own sake. It gives the existence, under certain conditions, of a contour for which the Cauchy integral formula holds.

Theorem 11.2 *Let $K \subseteq U$ where K is compact and U is open. Then there exist linear mappings, $\gamma_k : [0,1] \to U \setminus K$ such that for all $z \in K$,*

$$f(z) = \frac{1}{2\pi i}\sum_{k=1}^{p}\int_{\gamma_k}\frac{f(w)}{w-z}dw. \tag{11.7}$$

Proof: Tile $\mathbb{R}^2 = \mathbb{C}$ with little squares having diameters less than δ where $0 < \delta \leq \text{dist}\left(K, U^C\right)$ (see Problem 3). Now let $\{R_j\}_{j=1}^m$ denote those squares that have nonempty intersection with K. For example, see the following picture.



Let $\left\{v_j^k\right\}_{k=1}^4$ denote the four vertices of R_j where v_j^1 is the lower left, v_j^2 the lower right, v_j^3 the upper right and v_j^4 the upper left. Let $\gamma_j^k : [0,1] \to U$ be defined as

$$\begin{aligned} \gamma_j^k(t) &\equiv v_j^k + t\left(v_j^{k+1} - v_j^k\right) \text{ if } k < 4, \\ \gamma_j^4(t) &\equiv v_j^4 + t\left(v_j^1 - v_j^4\right) \text{ if } k = 4. \end{aligned}$$

Define

$$\int_{\partial R_j} g(w)\, dw \equiv \sum_{k=1}^4 \int_{\gamma_j^k} g(w)\, dw.$$

Thus we integrate over the boundary of the square in the counter clockwise direction. Let $\left\{\gamma_j\right\}_{j=1}^p$ denote the curves, γ_j^k which have the property that $\gamma_j^k([0,1]) \cap K = \emptyset$.

Claim: $\sum_{j=1}^m \int_{\partial R_j} g(w)\, dw = \sum_{j=1}^p \int_{\gamma_j} g(w)\, dw$.

Proof of the claim: If $\gamma_j^k([0,1]) \cap K \neq \emptyset$, then for some $r \neq j$,

$$\gamma_r^l([0,1]) = \gamma_j^k([0,1])$$

but $\gamma_r^l = -\gamma_j^k$(The directions are opposite.). Hence, in the sum on the left, the only possibly nonzero contributions come from those curves, γ_j^k for which $\gamma_j^k([0,1]) \cap K = \emptyset$ and this proves the claim.

Now let $z \in K$ and suppose z is in the interior of R_s, one of these squares which intersect K. Then by the Cauchy integral formula,

$$f(z) = \frac{1}{2\pi i} \int_{\partial R_s} \frac{f(w)}{w - z} dw,$$

and if $j \neq s$,

$$0 = \frac{1}{2\pi i} \int_{\partial R_j} \frac{f(w)}{w - z} dw.$$

Therefore,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\sum_{j=1}^{m}\int_{\partial R_j}\frac{f(w)}{w-z}dw \\ &= \frac{1}{2\pi i}\sum_{j=1}^{p}\int_{\gamma_j}\frac{f(w)}{w-z}dw. \end{aligned}$$

This proves 11.7 in the case where z is in the interior of some R_s. The general case follows from using the continuity of the functions, $f(z)$ and

$$z \to \frac{1}{2\pi i}\sum_{j=1}^{p}\int_{\gamma_j}\frac{f(w)}{w-z}dw.$$

This proves the theorem.

11.1 Runge's theorem

With the above preparation we are ready to prove the very remarkable Runge theorem which says that we can approximate analytic functions on arbitrary compact sets with rational functions which have a certain nice form. Actually, the theorem we will present first is a variant of Runge's theorem because it focuses on a single compact set.

Theorem 11.3 *Let K be a compact subset of an open set, U and let $\{b_j\}$ be a set which consists of one point from the closure of each bounded component of $\mathbb{C}\setminus K$. Let f be analytic on U. Then for each $\varepsilon > 0$, there exists a rational function, Q whose poles are all contained in the set, $\{b_j\}$ such that*

$$||Q - f||_{K,\infty} < \varepsilon. \tag{11.8}$$

Proof: By Theorem 11.2 there are curves, γ_k described there such that for all $z \in K$,

$$f(z) = \frac{1}{2\pi i}\sum_{k=1}^{p}\int_{\gamma_k}\frac{f(w)}{w-z}dw. \tag{11.9}$$

Defining $g(w,z) \equiv \frac{f(w)}{w-z}$ for $(w,z) \in \cup_{k=1}^{p}\gamma_k([0,1]) \times K$, we see that g is uniformly continuous and so there exists a $\delta > 0$ such that if $||\mathcal{P}|| < \delta$, then for all $z \in K$,

$$\left| f(z) - \frac{1}{2\pi i}\sum_{k=1}^{p}\sum_{j=1}^{n}\frac{f(\gamma_k(\tau_j))(\gamma_k(t_i) - \gamma_k(t_{i-1}))}{\gamma_k(\tau_j) - z}\right| < \frac{\varepsilon}{2}.$$

The complicated expression is obtained by replacing each integral in 11.9 with a Riemann sum. Simplifying the appearance of this, it follows there exists a rational function of the form

$$R(z) = \sum_{k=1}^{M}\frac{A_k}{w_k - z}$$

where the w_k are elements of components of $\mathbb{C}\setminus K$ and A_k are complex numbers such that

$$||R - f||_{K,\infty} < \frac{\varepsilon}{2}.$$

Consider the rational function, $R_k(z) \equiv \frac{A_k}{w_k - z}$ where $w_k \in V_j$, one of the components of $\mathbb{C} \setminus K$, the given point of $\overline{V_j}$ being b_j or else V_j is unbounded. By Lemma 11.1, there exists a function, Q_k which is either a rational function having its only pole at b_j or a polynomial, depending on whether V_j is bounded, such that

$$||R_k - Q_k||_{K,\infty} < \frac{\varepsilon}{2M}.$$

Letting $Q(z) \equiv \sum_{k=1}^{M} Q_k(z)$,

$$||R - Q||_{K,\infty} < \frac{\varepsilon}{2}.$$

It follows

$$||f - Q||_{K,\infty} \leq ||f - R||_{K,\infty} + ||R - Q||_{K,\infty} < \varepsilon.$$

This proves the theorem.

Runge's theorem concerns the case where the given points are contained in $\mathbb{C} \setminus U$ for U an open set rather than a compact set. Note that here there could be uncountably many components of $\mathbb{C} \setminus U$ because the components are no longer open sets. An easy example of this phenomenon in one dimension is where $U = [0,1] \setminus P$ for P the Cantor set. Then you can show that $\mathbb{R} \setminus U$ has uncountably many components. Nevertheless, Runge's theorem will follow from Theorem 11.3 with the aid of the following interesting lemma.

Lemma 11.4 *Let U be an open set in $\mathbb{C}$. Then there exists a sequence of compact sets, $\{K_n\}$ such that*

$$U = \cup_{k=1}^{\infty} K_n, \ \cdots, K_n \subseteq \operatorname{int} K_{n+1} \cdots, \tag{11.10}$$

and for any $K \subseteq U$,

$$K \subseteq K_n, \tag{11.11}$$

for all n sufficiently large, and every component of $\widehat{\mathbb{C}} \setminus K_n$ contains a component of $\widehat{\mathbb{C}} \setminus U$.

Proof: Let

$$V_n \equiv \{z : |z| > n\} \cup \bigcup_{z \notin U} B\left(z, \frac{1}{n}\right).$$

Thus $\{z : |z| > n\}$ contains the point, ∞. Now let

$$K_n \equiv \widehat{\mathbb{C}} \setminus V_n = \mathbb{C} \setminus V_n \subseteq U.$$

We leave it as an exercise to verify that 11.10 and 11.11 hold. It remains to show that every component of $\widehat{\mathbb{C}} \setminus K_n$ contains a component of $\widehat{\mathbb{C}} \setminus U$. Let D be a component of $\widehat{\mathbb{C}} \setminus K_n \equiv V_n$.

If $\infty \notin D$, then D contains no point of $\{z : |z| > n\}$ because this set is connected and D is a component. (If it did contain a point of this set, it would have to contain the whole set..) Therefore, $D \subseteq \bigcup_{z \notin U} B\left(z, \frac{1}{n}\right)$ and so D contains some point of $B\left(z, \frac{1}{n}\right)$ for some $z \notin U$. Therefore, since this ball is connected, it follows D must contain the whole ball and consequently D contains some point of U^C. (The point z at the center of the ball will do.) Since D contains $z \notin U$, it must contain the component, H_z, determined by this point. The reason for this is that

$$H_z \subseteq \widehat{\mathbb{C}} \setminus U \subseteq \widehat{\mathbb{C}} \setminus K_n$$

and H_z is connected. Therefore, H_z can only have points in one component of $\widehat{\mathbb{C}} \setminus K_n$. Since it has a point in D, it must therefore, be totally contained in D. This verifies the desired condition in the case where $\infty \notin D$.

Now suppose that $\infty \in D$. We know that $\infty \notin U$ because U is given to be a set in $\mathbb{C}$. Letting H_∞ denote the component of $\widehat{\mathbb{C}} \setminus U$ determined by ∞, it follows from similar reasoning to the above that $H_\infty \subseteq D$ and this proves the lemma.

Theorem 11.5 *(Runge) Let U be an open set, and let A be a set which has one point in each bounded component of $\widehat{\mathbb{C}} \setminus U$ and let f be analytic on U. Then there exists a sequence of rational functions, $\{R_n\}$ having poles only in A such that R_n converges uniformly to f on compact subsets of U.*

Proof: Let K_n be the compact sets of Lemma 11.4 where each component of $\widehat{\mathbb{C}} \setminus K_n$ contains a component of $\widehat{\mathbb{C}} \setminus U$. It follows each bounded component of $\widehat{\mathbb{C}} \setminus K_n$ contains a point of A. Therefore, by Theorem 11.3 there exists R_n a rational function with poles only in A such that

$$||R_n - f||_{K_n,\infty} < \frac{1}{n}.$$

It follows, since a given compact set, K is a subset of K_n for all n large enough, that $R_n \to f$ uniformly on K. This proves the theorem.

Corollary 11.6 *Let U be simply connected and f is analytic on U. Then there exists a sequence of polynomials, $\{p_n\}$ such that $p_n \to f$ uniformly on compact sets of U.*

Proof: By definition of what is meant by simply connected, $\widehat{\mathbb{C}} \setminus U$ is connected and so there are no bounded components of $\widehat{\mathbb{C}} \setminus U$. Therefore, $A = \emptyset$ and it follows that R_n in the above theorem must be a polynomial since it is rational and has no poles.

11.2 Exercises

1. Let K be any nonempty set in $\mathbb{C}$ and define

$$\text{dist}(z, K) \equiv \inf\{|z - w| : w \in K\}.$$

Show that $z \to \text{dist}(z, K)$ is a continuous function.

2. Verify the series in 11.4 converges absolutely on K and find A_k. Also do the same for 11.6. **Hint:** You know that for $|z| < 1$, $\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k$. Differentiate both sides as many times as needed to obtain a formula for A_k. Then apply the Weierstrass M test and the ratio test.

3. In Theorem 11.2 we had a compact set, K contained in an open set U and we used the fact that

$$\text{dist}\left(K, U^C\right) \equiv \inf\left\{|z - w| : w \in U^C, z \in K\right\} > 0.$$

Prove this.

4. For $U = [0, 1] \setminus P$ for P the Cantor set, show that $\mathbb{R} \setminus U$ has uncountably many components. **Hint:** Show that the component of $\mathbb{R} \setminus U$ determined by $p \in P$, is just the single point, p and then show P is uncountable.

5. In the proof of Lemma 11.4, verify that 11.10 and 11.11 are satisfied for the given choice of K_n.

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