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Philosophy and Religion

The Logic Book
5th Edition

Bergmann–Moor–Nelson

McGraw-Hill
A Division of The McGraw-Hill Companies

McGraw–Hill Primis

ISBN–10: 0–39–046233–0
ISBN–13: 978–0–39–046233–6

Text:

The Logic Book, Fifth Edition
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This book was printed on recycled paper.

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http://www.primisonline.com

111 PHILGEN ISBN-10: 0-39-046233-0 ISBN-13: 978-0-39-046233-6

Philosophy and Religion

Contents

PREFACE

In the fifth edition of *The Logic Book* we retain our overall goal: to present symbolic logic in an accessible yet formally rigorous manner. This involved a major overhaul of several chapters, along with some terminological and notational changes.

Most of the material in Chapters 5 and 10 is new or extensively rewritten. We have tried to present the derivation systems we develop in ways that make them more transparent. We emphasize the need for and use of specific strategies in constructing derivations, and in each chapter we explicitly list those strategies. We have also introduced new annotations for the assumptions that begin subderivations, annotations that specify the reason these assumptions are being made. We use the notation "A /⊃I", for example, to indicate that an auxiliary assumption has been made to introduce a Conditional Introduction subderivation. We believe that these annotations underscore the point that auxiliary assumptions should always be made with clear strategies in mind. We have significantly expanded the number and variety of exercises in these chapters as well.

There are also significant changes to Chapters 4 and 9. The latter sections of Chapter 9 have been reorganized so that systematic trees for *PL* are presented prior to, and independently of, trees for *PLE*, and the sections motivating the rules for *PLE* are less circuitous. In Chapter 4 we dispense with talk of fragments of truth-value assignments and instead adopt the convention that a display such as

A	B	C	D
T	F	T	T

specifies the infinitely many truth-value assignments that each assign the specified values to 'A', 'B', 'C', and 'D'. In both chapters we have also introduced

a new annotation for trees to indicate completed open branches ('o') as well as closed ones ('x').

Chapter 8 now introduces the concept of a *model*, to be used there and in subsequent chapters.

As always, we have corrected known errors and typos from previous editions.

The Logic Book presupposes no previous training in logic, and because it covers sentential logic through the metatheory of first-order predicate logic, it is suitable for both introductory and intermediate courses in symbolic logic. There is sufficient material in the text for a one- or two-semester course. There are several sequences of chapters that would form good syllabi for courses in symbolic logic. If an instructor has two semesters (or one semester with advanced students), it is possible to work through the entire book.

The instructor who does not want to emphasize metatheory can simply omit Chapters 6 and 11. The chapters on truth-trees and the chapters on derivations are independent, so it is possible to cover truth-trees but not derivations, and vice versa. The chapters on truth-trees do depend on the chapters presenting semantics; that is, Chapter 4 depends on Chapter 3, and Chapter 9 depends on Chapter 8. And although most instructors will want to cover semantics before derivations, the opposite order is possible. Finally, in covering predicate logic, the instructor has the option in each chapter of skipping material on identity and functions, as well as the option of including the material on identity but omitting that on functions.

The Logic Book includes large numbers of exercises in all chapters. Answers to the starred exercises appear in the *Instructor's Manual*; answers to the unstarred exercises appear in the *Student Solutions Manual*.

SOFTWARE

Two software packages, *BERTIE and TWOOTIE*, are available for use with *The Logic Book*. *BERTIE* is a program that allows students to construct derivations online and checks those derivations for accuracy. *TWOOTIE* allows students to construct truth-trees online (and checks those trees for accuracy) and also produces trees for specified sets of sentences. Both programs were written by Austen Clarke; *BERTIE* is based on an earlier program by James Moor and Jack Nelson. Both programs run in a DOS environment. Information on the software can be found at

http://selfpace.uconn.edu/BertieTwootie/software.htm

Both software packages can also be downloaded from this site.

ACKNOWLEDGMENTS

We are grateful to Lorne Falkenstein, Richard Grandy, Richard Shedenhelm and his students at the University of Georgia, and Takashi Yagisawa for valuable comments and suggestions on the previous edition.

M.B.
J.M.
J.N.


Notes

Chapter 1

BASIC NOTIONS OF LOGIC

1.1 BACKGROUND

This is a text in deductive logic—more specifically, in symbolic deductive logic. Chapters 1–5 are devoted to sentential logic, that branch of symbolic deductive logic that takes sentences as the fundamental units of logical analysis. Chapters 7–10 are devoted to predicate logic, that branch of symbolic deductive logic that takes predicates and individual terms as the fundamental units of logical analysis. Chapter 6 is devoted to the metatheory of sentential logic; Chapter 11, to the metatheory of predicate logic.

In the following chapters we will explore sentential and predicate logic in considerable detail. Here, we try to place that material in a larger context. Historically two overlapping concerns have driven research in deductive logic and the development of specific formal systems of deductive logic: the desire to formulate canons or principles of good reasoning in everyday life, as well as in science and mathematics, and the desire to formalize and systematize existing and emerging work in mathematics and science. Common to these concerns is the view that what distinguishes good reasoning from bad reasoning, and what makes good deductive reasoning "logical" as opposed to "illogical", is truth preservation.

A method or pattern of reasoning is truth-preserving if it never takes one from truths to a falsehood. The hallmark of good deductive reasoning is that it is truth-preserving. If one starts from truths and uses good deductive reasoning, the results one arrives at will also be true. Because we are all interested, as students and scholars, in everyday life and in our careers, in gaining truths and avoiding falsehoods, we all have reason to be interested in reasoning that is truth-preserving.

Most of the deductive systems of reasoning that have been developed for geometry, mathematics, and selected areas of science have been axiomatic systems. And most of us are familiar with at least one axiomatic system—that of Euclidean plane geometry. Euclid, a Greek scholar of the third century B.C., may have been the first person to develop a reasonably complete axiomatic system. Axiomatic systems start with a relatively small number of basic principles, referred to variously as axioms, definitions, postulates, and assumptions, and provide a way of deducing or deriving from them the rest of the claims or assertions of the discipline being axiomatized (in Euclid's case plane geometry). If the starting principles are significantly altered, a new theory may emerge. For example, when Euclid's fifth postulate (the parallel postulate) is modified, theorems of non-Euclidean geometry can be deduced.

Through the centuries scholars have attempted to produce axiomatic systems for a wide variety of disciplines, ranging from plane and solid geometry, to arithmetic (which was successfully axiomatized by Giuseppe Peano in 1889), to parts of the natural and social sciences. Since successful axiomatic systems use only rules of reasoning that are truth-preserving, that never take one from truths to a falsehood, the advantage of successfully axiomatizing a body of knowledge is that it makes all the claims of that body of knowledge as certain as are the starting principles and the rules of reasoning used.

At about the same time that Euclid was developing his axiomatic treatment of plane geometry, another Greek scholar, Aristotle (384–322 B.C.), was developing a general system of logic intended to incorporate the basic principles of good reasoning and to provide a way of evaluating specific cases of reasoning. The system Aristotle produced is variously known as syllogistic, traditional, or Aristotelian logic. Predecessors of Aristotle, in the Greek world and elsewhere, were interested in reasoning well—in offering cogent arguments for their theses and theories and in identifying flaws and fallacies in their own and others' reasoning. But Aristotle was apparently the first person in the Western world to offer at least the outlines of a comprehensive system for codifying and evaluating a wide range of arguments and reasoning.

The following is an argument that has the form of an Aristotelian syllogism:

All mammals are vertebrates.
Some sea creatures are mammals.

Some sea creatures are vertebrates.

The horizontal line separates the two premises of this syllogistic argument from the conclusion. This syllogism is an example of good reasoning—it constitutes a good argument—because it is truth-preserving. If the first two sentences (the premises) of the syllogism are true, the third sentence (the conclusion) must also be true. Aristotle's achievement was not in identifying this particular argument about vertebrates, mammals, and sea creatures as a good or truth-preserving argument, but rather in providing an explanation of why this and all reasoning of this form are instances of good reasoning. Aristotle would classify the preceding syllogism as being of the form

All As are Bs.
Some Cs are As.

Some Cs are Bs.

And this form or schema produces truth-preserving reasoning whenever 'A', 'B', and 'C' are uniformly replaced by general terms, as in

All cardiologists are wealthy individuals.
Some doctors are cardiologists.

Some doctors are wealthy individuals.

Aristotelian logic is a variety of deductive symbolic logic. It is symbolic because it analyzes reasoning by identifying the form or structure of good reasoning, independent of the specific content of particular instances of such reasoning. It is deductive because the requirement it lays down for good reasoning is full truth-preservation. Argument forms all of whose instances are truth-preserving, as well as the arguments that are of those forms, are traditionally termed **valid.** The syllogistic form just displayed is a valid form; that is, no syllogism of this form has true premises and a false conclusion. All actual arguments that can be cast in this syllogistic form are therefore valid arguments.

An example of an **invalid** syllogistic form is

Some As are Bs.
All Cs are As.

All Cs are Bs.

There are, to be sure, actual arguments that are of this form and have true premises and a true conclusion—for example,

Some birds are hawks.
All osprey are birds.

All osprey are hawks.

But there are also arguments of this form that have true premises and a false conclusion—for example,

> Some positive numbers are even numbers.
>
> All numbers greater than zero are positive numbers.
>
> ---
>
> All numbers greater than zero are even numbers.

The two premises of this syllogism are true, but the conclusion, 'All numbers greater than zero are even', is false. The syllogistic form just displayed is an invalid form precisely because there are instances of it that have true premises and a false conclusion.

Aristotelian logic is very powerful. During the centuries following Aristotle, the rules and techniques associated with syllogistic logic were refined, and various test procedures developed, by Roman, Arabic, medieval, and modern logicians. Until the late nineteenth century Aristotelian logic remained the predominant system for formalizing and evaluating reasoning. It is still taught today in many introductory courses.

Nonetheless, there are important drawbacks to Aristotelian logic. Syllogisms are at the heart of Aristotelian logic, and each syllogism must have exactly two premises and a conclusion. Moreover, every sentence of a syllogism must be of one of the four following forms:

> All As are Bs.
> No As are Bs.
> Some As are Bs.
> Some As are not Bs.

Aristotelian logic is thus best suited to reasoning about relations among groups: 'All members of this group are members of that group', 'Some members of this group are members of that group', and so on. Aristotelian logic thus strains to handle reasoning about individuals. For example, 'Socrates is human' must be recast as something like 'All things that are Socrates [there is, we here assume, only one] are things that are human'.

The Aristotelian requirement that every conclusion be drawn from exactly two premises is unduly restrictive and does not mirror the complexity of actual reasoning and argumentation, a single instance of which may make use of a very large number of premises. Consider, for example, the following reasoning:

> Sarah and Hank are the only finalists for a position with Bowles, Blithers, and Blimy, an accounting firm. Whoever is hired will have a baccalaureate degree in accounting. Hank will get his baccalaureate in accounting only if he passes all the business courses he is taking this semester and completes the general education requirements.

> Sarah will get her baccalaureate only if she passes all her courses and raises her grade point average to 2.5. Hank will fail logic and so will not complete the general education requirements. Sarah will pass all her courses, but her grade point average will not reach 2.5. Therefore Bowles, Blithers, and Blimy will hire neither of the finalists.

The above reasoning is truth-preserving. That is, if the premises are all true, then the conclusion, the last sentence of the paragraph, must also be true. But it would be extremely difficult to recast this chain of reasoning in syllogistic terms.

Finally reasoning that relies on relations[1] cannot readily be accommodated within Aristotelian logic. For example, the reasoning 'Sarah is taller than Tom, and Tom is taller than Betty; therefore Sarah is taller than Betty' presupposes the transitivity of the taller-than relation, that is, presupposes the following truth:

> For any three things, if the first is taller than the second, and the second is taller than the third, then the first is taller than the third.

Principles such as the above and arguments relying on them cannot be incorporated within the Aristotelian framework in any intuitive way.

For these and other reasons, logicians in the mid-to-late 1800s looked for alternatives to Aristotelian logic. This work involved the development of systems of sentential logic, that is, systems based on the way sentences of natural languages can be generated from other sentences by the use of such expressions as 'or', 'and', 'if . . . then . . .', and 'not'. Consider this example:

> Karen is either in Paris or in Nairobi. She is not in Nairobi. So Karen is in Paris.

Simple arguments such as this one are not readily represented within syllogistic logic. Yet the argument is clearly an example of good reasoning. Whenever the first two sentences are true, the last sentence is also true. Reasoning of this sort can readily be symbolized in systems of sentential logic.

On the other hand, sentential logic cannot easily deal with reasoning that rests on claims about all, some, or none of this sort of thing being of that sort—the sort of claims Aristotelian logic can often handle. Predicate logic incorporates sentential logic and is also able to handle all the kinds of sentences that are expressible in Aristotelian logic, as well as many of those that pose difficulties for Aristotelian logic.

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[1] See Chapter 7 for an explication of relations.

1.2 WHY STUDY LOGIC?

There are a variety of reasons for studying logic. It is a well-developed discipline that many find interesting in its own right, a discipline that has a rich history and important current research programs and practical applications. Certainly, anyone who plans to major or do graduate work in areas such as philosophy, mathematics, computer science, or linguistics should have a solid grounding in symbolic logic. In general, the study of formal logic also helps develop the skills needed to present and evaluate arguments in any discipline.

Another reason for studying symbolic logic is that, in learning to symbolize natural language sentences (in our case English sentences) in a formal language, students become more aware and more appreciative of the importance of the structure and complexities of natural languages. Precisely what words are used often has a major bearing on whether an argument is valid or invalid, a piece of reasoning convincing or unconvincing. For example, distinguishing between 'Roberta will pass if she completes all the homework' and 'Roberta will pass only if she completes all the homework' is essential to anyone who wants to reason well about the prospects for Roberta's passing.

However, the focus of this text is not primarily on sharpening the critical and evaluative skills readers bring to bear on everyday discourse, newspaper columns, and the rhetoric of politicians. Inculcating these skills is the goal of texts on "critical thinking" or "informal logic", where the primary emphasis is on nonformal techniques for identifying fallacies, figuring out puzzles, and constructing persuasive arguments. Formal or symbolic logic, which is the domain of this book, is a discipline with its own body of theory and results, just as are mathematics and physics. This text is an introduction to that discipline, a discipline whose principles underlie the techniques presented in informal logic texts. This text will help readers not only identify good and bad arguments but also understand why arguments are good arguments or bad arguments. Even though only the most avid devotees of formal systems will be constructing truth-tables, truth-trees, or derivations after completing this text, mastering these formal techniques is a way of coming to understand the principles underlying reasoning and the relations among sentences and sets of sentences.

There is another, quite practical, reason for studying symbolic logic. In most of the chapters that follow, the discussion will center on seven or fewer central concepts. These concepts are related, from chapter to chapter. For example, the concept of truth-functional validity developed in Chapter 3 is one way of refining the concept of logical validity laid out in this chapter. All these concepts are abstract. They cannot be touched or weighed or examined under a microscope. Mastering these concepts and the relations among them is an exercise in abstract thinking. The skills involved are, we think, important and will be useful in a wide variety of theoretical and applied fields. For these reasons the "theory questions" found at the end of most exercise sets are in many ways the most important part of the exercise sets.

1.3 SENTENCES, TRUTH-VALUES, AND ARGUMENTS

'True' and 'false' are properties of sentences. That is, it is sentences that are either true or false.[2] Throughout this text we will use the notion of a *truth-value*. We will say that true sentences have the truth-value **T**, and false sentences the truth-value **F**. 'Washington, DC, is the capital of the United States' and 'The volume of a gas is directly proportional to its temperature and inversely proportional to its pressure' are both true, and so both have the truth-value **T**. The truth of the first derives from the political organization of the United States, and the truth of the second from the fundamentals of physics and chemistry. 'Toronto is the capital of Canada' and 'Atoms are indivisible' are both false, so both have the truth-value **F**—the first for reasons having to do with the political organization of Canada, and the second for reasons having to do with the existence and behavior of subatomic particles. Although it is only sentences that are either true or false, not every sentence of English is one or the other. Sentences that are obviously neither true nor false include questions ('Where is Kansas City?'), requests ('Please shut the door when you leave'), commands ('Don't darken my door again'), and exclamations ('Ouch!'). The formal systems we develop in this text are intended to deal only with sentences that are either true or false as asserted on a particular occasion in a particular context. To say that the sentences we will be dealing with are those that are either true or false is not, of course, to say that for any given sentence we know which it is, true or false, but only that it is one or the other.

Much of this text is devoted to the study of arguments. Previously, in discussing syllogistic arguments, we presented them by listing the premises followed by the conclusion, with a horizontal line separating the premises from the conclusion. Arguments so displayed are presented in **standard form.** Of course, in natural languages, whether in spoken discourse or in writing, arguments are rarely presented in standard form. Indeed, in English and other natural languages, arguments, or bits of reasoning that can be reconstructed as one or more arguments, generally neither occur in what we call standard form nor are set off from preceding and following discourse. Moreover, the premises are not always given first and the conclusion last. Consider

> Michael will not get the job, for whoever gets the job will have strong references, and Michael's references are not strong.

---

[2]Many philosophical disputes arise about such basic concepts as sentences, meaning, content, and truth. For example, some philosophers argue that propositions are the kinds of entities that are really true or false, propositions being taken to be the meanings of sentences and to exist independently of any particular language. Other philosophers eschew propositions as unnecessary metaphysical baggage. Because we have to adopt some terminology and because all philosophers agree that sentences play some role in language, we shall talk of sentences as being true or false. But we consider such talk to be shorthand for talk of sentences that have certain meanings as used in particular contexts.

This single sentence can be recast as the following explicit argument in standard form:

Whoever gets the job will have strong references.
Michael's references are not strong.

Michael will not get the job.

As we just saw, in everyday discourse the conclusion is sometimes given before the premises. The conclusion can also come between premises, with the whole argument being buried in an ongoing text:

> I've got more relatives than I know what to do with. I've got relatives in Idaho and in New Jersey, in Ireland and in Israel. Among them are a couple of cousins, Tom and Fred Culverson. Both Tom and Fred are hard working, and Tom is as tenacious as a bulldog. So Tom is sure to be a success, for if there is one thing I have learned in life, it is that everyone who is both hard working and tenacious succeeds. But I'm sure success won't change Tom. He'll work just as hard after he makes his first million as he does now. He is, after all, a Culverson. And no one is as predictable as a Culverson, unless it's a Hutchings. There are lots of Hutchings on my mother's side, but I haven't had much to do with them. . . .

The following explicit argument can be extracted from this passage and placed in standard form:

Tom and Fred are hard working.
Tom is tenacious.
Everyone who is both hard working and tenacious succeeds.

Tom will succeed.

There is a lot of information in this passage that is not relevant to the specific argument we have extracted. This is frequently the case.

The first step in analyzing arguments is to extract them from the discourse within which they are embedded and present them in standard form. Doing so requires practice. In natural language the presence of an argument is often signaled by the use of premise and/or conclusion indicator words. *Conclusion indicator words*—that is, words indicating that what follows is intended as the conclusion of an argument—include

therefore
thus
it follows that

so
hence
consequently
as a result

Premise indicator words—that is, words whose use signals that what follows is intended as a premise of an argument—include

since
for
because
on account of
inasmuch as
for the reason that

Not every piece of discourse is intended as an argument. Consider

> Our galaxy is made up of our sun and billions of other stars. The galaxy is a huge, flat spiral system that rotates like a wheel, and the myriad of stars move around its center somewhat as the planets revolve around our sun. There are millions of other galaxies in addition to our galaxy.

Each of the sentences in this passage is either true or false, but there is no good reason to treat one of the sentences as a conclusion and the others as premises.

So far we have given examples of arguments, talked about them, and presented several of them in standard form. But we have not defined 'argument', and it is time do so. In our definition we make use, as we frequently will throughout the rest of this text, of the notion of a set of sentences. Sets are abstract objects that have members (no members, one member, two members, . . . an infinite number of members). The identity of a set is determined by its members. That is, if set A and set B have exactly the same members, then they are the same set; if they do not, they are different sets.

> An *argument* is a set of two or more sentences, one of which is designated as the conclusion and the others as the premises.

This is a very broad notion of an argument. For example, it allows us to count the following as an argument:

> Herbert is four years old.
>
> ---
>
> The sun will shine tomorrow.

This is, of course, a paradigm of a bad argument. The one premise supplies no support whatsoever for the conclusion. The advantage of this broad

definition is that it sidesteps the problem of having to give an account of how plausible a line of reasoning has to be to count as an argument or of how likely it is that a given group of sentences will be taken to support a designated sentence in order for those sentences together to count as an argument.

Given our broad definition, we could recast the passage about our galaxy as an argument for one or another of the constituent claims. But there is no reason to do so except to demonstrate that, by our account, any set of two or more sentences can be taken as an argument and evaluated as such. It is, in fact, our contention that it is the job of the logician, not to set limits on what counts as an argument, but rather to provide means of distinguishing good arguments from bad ones. And this we will do. That said, most (but not all) of the examples of arguments we use in this text will be ones in which someone might think that the premises do support the conclusion.

Sometimes an argument turns out to be a good argument even when the premises do not appear, on first review, to support the conclusion. This is another reason for not appealing to some level of apparent support in the definition of an argument. Consider, for example, this passage:

> Everyone loves a lover. Tom loves Alice. Everyone loves everyone.

If by 'lover' we mean 'someone who loves someone', and if we take the last sentence as the conclusion of an argument of which the first two sentences are premises, we have a valid argument. The conclusion does follow, though not obviously, from the premises. The missing reasoning is this: If Tom loves Alice, then Tom is a lover. It follows from the first premise, 'Everyone loves a lover', that everyone loves Tom. And if a lover is someone who loves someone, it further follows that everyone is a lover (because everyone loves Tom). And if everyone is a lover and everyone loves a lover, it follows, finally, that everyone loves everyone. Of course, this reasoning does not work if 'loves' is not being used in the same way in all its occurrences in the original argument, and it may not be ('Everyone loves a lover' may be being used, for example, in the sense of 'Everyone is fond of a person who is in love').

1.3E EXERCISES

(*Note:* The accompanying *Solutions Manual* contains answers to all unstarred exercises.)

1. For each of the following, indicate whether it is the kind of sentence that falls within the scope of this text—that is, is either true or false. If it is not, explain why not.

a. George Washington was the second president of the United States.
*b. The next president of the United States will be a Republican.
c. Turn in your homework on time or not at all.
*d. Would that John Kennedy had not been assassinated.
e. Two is the smallest prime number.

*f. One is the smallest prime number.
g. George Bush senior was the immediate predecessor to George W. as president.
*h. On January 15, 1134, there was a snowstorm in what is now Manhattan, at 3:00 p.m. EST.
i. Sentence m below is true.
*j. May you live long and prosper.
k. Never look a gift horse in the mouth.
*l. Who created these screwy examples?
m. This sentence is false.
*n. Beware of Greeks bearing gifts.

2. For each of the following passages, specify what argument, if any, is being advanced. Where the intent is probably not to express an argument, explain why this is so. Where an argument is probably being expressed, restate the argument in standard form.

a. When Mike, Sharon, Sandy, and Vicky are all out of the office, no important decisions get made. Mike is off skiing, Sharon is in Spokane, Vicky is in Olympia, and Sandy is in Seattle. So no decisions will be made today.
*b. Our press releases are always crisp and upbeat. That's because, though Jack doesn't like sound bytes, Mike does. And Mike is the press officer.
c. Shelby and Noreen are wonderful in dealing with irate students and faculty. Stephanie is wonderful at managing the chancellor's very demanding schedule, and Tina keeps everything moving and cheers everyone up.
*d. This is a great office to work in. Shelby and Noreen are wonderful in dealing with irate students and faculty. Stephanie is wonderful at managing the chancellor's very demanding schedule, and Tina keeps everything moving and cheers everyone up.
e. The galvanized nails, both common and finishing, are in the first drawer. The plain nails are in the second drawer. The third drawer contains Sheetrock screws of various sizes, and the fourth drawer contains wood screws. The bottom drawer contains miscellaneous hardware.
*f. The galvanized nails, both common and finishing, are in the first drawer. The plain nails are in the second drawer. The third drawer contains Sheetrock screws of various sizes, and the fourth drawer contains wood screws. The bottom drawer contains miscellaneous hardware. So we should have everything we need to repair the broken deck chair.
g. The weather is perfect; the view is wonderful; and we're on vacation. So why are you unhappy?
*h. The new kitchen cabinets are done, and the installers are scheduled to come Monday. But there will probably be a delay of at least a week, for the old cabinets haven't been removed, and the carpenter who is to do the removal is off for a week of duck hunting in North Dakota.
i. Wood boats are beautiful, but they require too much maintenance. Fiberglass boats require far less maintenance, but they tend to be more floating bathtubs than real sailing craft. Steel boats are hard to find, and concrete boats never caught on. So there's no boat that will please me.
*j. Sarah, John, Rita, and Bob have all worked hard and all deserve promotion. But the company is having a cash flow problem and is offering those over 55 a $50,000 bonus if they will retire at the end of this year. Sarah, John, and Bob are all over 55 and will take early retirement. So Rita will be promoted.

k. Everyone from anywhere who's anyone knows Barrett. All those who know her respect her and like her. Friedman is from Minneapolis and Barrett is from Duluth. Friedman doesn't like anyone from Duluth. Therefore, either Friedman is a nobody or Minneapolis is a nowhere.

*l. I'm not going to die today. I didn't die yesterday, and I didn't die the day before that, or the day before that, and so on back some fifty years.

m. Having cancer is a good, for whatever is required by something that is a good is itself a good. Being cured of cancer is a good, and being cured of cancer requires having cancer.

*n. The Soviet Union disintegrated because the perceived need for the military security offered by the union disappeared with the end of the cold war and because over 70 years of union had produced few economic benefits. Moreover the Soviet Union never successfully addressed the problem of how to inspire loyalty to a single state by peoples with vastly different cultures and histories.

o. Only the two-party system is compatible both with effective governance and with the presenting and contesting of dissenting views, for when there are more than two political parties, support tends to split among the parties, with no party receiving the support of a majority of voters. And no party can govern effectively without majority support. When there is only one political party, dissenting views are neither presented nor contested. When there are two or more viable parties, dissenting views are presented and contested.

*p. Humpty Dumpty sat on a wall. Humpty Dumpty had a great fall. All the king's horses and all the king's men couldn't put Humpty together again. So they made him into an omelet and had a great lunch.[3]

1.4 DEDUCTIVE VALIDITY AND SOUNDNESS

We have already noted that truth-preservation is what distinguishes good reasoning from bad reasoning. A deductively valid argument is one whose form or structure is fully truth-preserving—that is, whose form or structure is such that instances of it never proceed from true premises to a false conclusion.[4] A deductively invalid argument is one whose form or structure is such that instances of it do, on occasion, proceed from true premises to a false conclusion.

An example of a **valid deductive argument** is

There are three, and only three, people in the room: Juarez, Sloan, and Wang.

Juarez is left-handed.

Sloan is left-handed.

Wang is left-handed.

All the people in the room are left-handed.

[3] With apologies to Lewis Carroll.

[4] There are good arguments that are not valid deductive arguments. See Section 1.5.

This argument is truth-preserving. That is, if the premises ('There are three, and only three, people in the room: Juarez, Sloan, and Wang', 'Juarez is left-handed', 'Sloan is left-handed', and 'Wang is left-handed') are all true, then the conclusion ('All the people in the room are left-handed') must also be true. Arguments that are truth-preserving in this strong sense, where it is not possible at the same time for all the premises to be true and the conclusion false, are said to be deductively valid. Such arguments never have true premises and a false conclusion.

> An argument is *deductively valid* if and only if it is not possible for the premises to be true and the conclusion false. An argument is *deductively invalid* if and only if it is not deductively valid.

Consider this example of an invalid deductive argument:

> Sloan is left-handed.
>
> Wang is left-handed.
>
> ---
>
> Everyone is left-handed.

It is invalid because, whereas the premises may well be true, the conclusion is false. Not everyone is left-handed.

Logic is about the relations among sentences and groups of sentences. For example, if we are told that a given argument is deductively valid, we can conclude that if the premises are true the conclusion must also be true. But we cannot conclude, given only that the argument is valid, that the premises are true or that the conclusion is true. Consider, for example, the following argument:

> The corner grocery store was burglarized, and whoever did it both knew the combination to the safe and was in town the night of the burglary.
>
> Carolyn, Albert, and Barbara are the only ones who knew the combination to the safe.
>
> Albert and Barbara were out of town the night of the burglary.
>
> ---
>
> Carolyn committed the burglary.

This argument is deductively valid; that is, if the premises are true, the conclusion must also be true. But it does not follow that the premises are true, and hence it does not follow that the conclusion is true. If we have good reason to believe each of the premises, then we also have good reason to believe the conclusion. But note that it is also the case that, if we have good reason to doubt the conclusion (suppose, for example, that we know Carolyn and also know that burglary is just not her style), then we have good reason to believe that at least one of the premises is false.

The important point to note here is that, given only that an argument is deductively valid, it may still be reasonable to doubt the conclusion or to doubt one or more premises. But what is not reasonable is to accept the premises and doubt or reject the conclusion. One who accepts the premises of a deductively valid argument ought, on pain of irrationality, also accept the conclusion. Correspondingly one who denies the conclusion of a deductively valid argument ought, again on pain of irrationality, reject at least one of the premises of that argument. (In the previous case, no police detective would be impressed by a "defense" of Carolyn that consisted of accepting the premises of the argument but steadfastly denying that Carolyn committed the burglary. If Carolyn did not commit the burglary, then either there was no burglary, or the burglar was not someone who knew the combination, or more people than those mentioned knew the combination, or Carolyn did not know the combination, or Albert and Barbara were not both out of town the night of the burglary, or Carolyn was out of town the night of the burglary, or the person who committed the burglary was not in town the night of the burglary.)

It follows from the definition of deductive validity that if an argument is deductively valid then it does not have all true premises and a false conclusion. But every other combination is possible. For example, a deductively valid argument may have all true premises and a true conclusion. The following is such as argument:

> In 2000 Bush and Gore were the only major party candidates in the presidential election.
>
> A major party candidate won.
>
> Gore did not win.
>
> ---
>
> Bush won the presidential election in 2000.

Deductively valid arguments all of whose premises are true are said to be **deductively sound.**

> An argument is *deductively sound* if and only if it is deductively valid and its premises are true. An argument is *deductively unsound* if and only if it is not deductively sound.

The foregoing argument concerning Bush and Gore is both deductively valid and deductively sound.

A deductively valid argument may also have one or more false premises and a conclusion that is false. Here is such an argument:

> France and Great Britain were the major powers in the Napoleonic Wars.
>
> France had the largest army, Great Britain the largest navy.

The power with the largest army won in the end.

France won in the end.

The third premise, 'The power with largest army won in the end', is false (and the argument is, for this reason, deductively unsound). The conclusion is also false. (Great Britain won the Napoleonic Wars when Wellington defeated Napoleon at the Battle of Waterloo in 1815.)

Finally a deductively valid argument may have a true conclusion and one or more false premises. An example is

Chicago is the capital of the United States.

The capital of the United States is in Illinois.

Chicago is in Illinois.

Both premises of this argument are false (and the argument is therefore deductively unsound); the conclusion is true. This illustrates that good reasoning can move from one or more false premises to a true conclusion.

A deductively invalid argument may have any combination of truths and falsehoods as premises and conclusion. That is, such an argument may have all true premises and a true conclusion, or all true premises and a false conclusion, or one or more false premises and a true conclusion, or one or more false premises and a false conclusion. Here are some examples:

Albany is the capital of New York State.

Annapolis is the capital of Maryland.

Columbus is the capital of Ohio.

Denver is the capital of Colorado.

The three premises and the conclusion are all true. But the argument is obviously deductively invalid. Were the legislature of Colorado to vote to move the capital to Boulder, the three premises of this argument would be true and the conclusion false. So it is possible for the premises to be true and the conclusion false. Consider

Albany is the capital of New York State.

Annapolis is the capital of Maryland.

Columbus is the capital of Ohio.

Minneapolis is the capital of Minnesota.

This deductively invalid argument has true premises and a false conclusion. (St. Paul, not Minneapolis, is the capital of Minnesota.) It is also easy to

produce a deductively invalid argument with at least one false premise (two in the following case) and a true conclusion:

Albany is the capital of New York State.
Minneapolis is the capital of Minnesota.
Annapolis is the capital of Maryland.
Boulder is the capital of Colorado.

Columbus is the capital of Ohio.

The conclusion of this argument is true; the second and fourth premises are false. By changing the conclusion to 'Dayton is the capital of Ohio', we produce a deductively invalid argument with at least one (here two) false premise and a false conclusion.

The point to remember is that the only time we can determine whether an argument is deductively valid, given only the truth-values of the premises and conclusion, is when the premises are all true and the conclusion false. We know, again, that such an argument is deductively invalid. In all other cases, to determine whether an argument is deductively valid, we have to consider not what the actual truth-values of the premises and conclusion are, but whether it is possible for the premises all to be true and the conclusion false. For example, consider this argument:

No sea creatures are mammals.
Dolphins are sea creatures.

Dolphins are not mammals.

It has one false premise (the first), one true premise (the second), and a false conclusion. This information does not determine whether the argument is deductively valid or deductively invalid. Rather, we come to see that the argument is deductively valid when we realize that if both premises were true then the conclusion would have to be true as well—that is, that dolphins, being sea creatures, would have to be nonmammals. This argument is valid because it is not possible for the premises to be true and the conclusion false.

1.4E EXERCISES

1. Which of the following are true and which are false? Explain your answers, giving examples as appropriate.

a. If an argument is valid, all the premises of that argument are true.
*b. If all the premises of an argument are true, the argument is valid.
c. All sound arguments are valid.
*d. All valid arguments are sound.

e. No argument with a false conclusion is valid.
*f. Every argument with a true conclusion is valid.
g. If all the premises of an argument are true and the conclusion is true, then the argument is valid.
*h. If all the premises of an argument are true and the conclusion is false, the argument is invalid.
i. There are sound arguments with false conclusions.
*j. There are sound arguments with at least one false premise.

2. Give arguments with the following characteristics:
a. A valid argument with true premises and a true conclusion.
*b. A valid argument with at least one false premise and a true conclusion.
c. A valid argument with a false conclusion.
*d. An invalid argument all of whose premises are true and whose conclusion is true.
e. An invalid argument all of whose premises are true and whose conclusion is false.
*f. An invalid argument with at least one false premise and a false conclusion.

1.5 INDUCTIVE ARGUMENTS

There are good arguments that are not deductively valid—that is, whose use involves some acceptable risk of proceeding from true premises to a false conclusion. Consider the following example:

> Juarez, Sloan, and Wang are all left-handed.
>
> Juarez and Sloan both have trouble using can openers made for right-handed people.
>
> ---
>
> Wang also has trouble using can openers made for right-handed people.

This argument is not deductively valid. But the conclusion, 'Wang also has trouble using can openers made for right-handed people', is to some extent probable given the fact that Wang is left-handed and that Juarez and Sloan, who are also left-handed, have trouble using can openers made for right-handed people. Nonetheless, the premises could be true and the conclusion false. This might be the case if, for example, Wang is especially adroit with kitchen implements or if the trouble the other two have derives from their having arthritis rather than from their being left-handed.

An argument that is not deductively valid can still be a useful argument—the premises can, as in the prior case, make the conclusion likely even though not certain. Such arguments are said to have **inductive strength,** the strength being proportional to the degree of probability the premises lend to the conclusion.

An argument has *inductive strength* to the extent that the conclusion is probable given the premises.

Inductive strength is thus a matter of degree.

Inductive reasoning is extremely common both in science and in everyday life. Walter Reed's hypothesizing that mosquitoes spread yellow fever is an example of inductive reasoning. While serving in Cuba after the Spanish-American War, Reed, a physician, noticed that those stricken with yellow fever always had recent mosquito bites and that those not stricken tended to work in areas not infested by mosquitoes. Based on these observations, Reed hypothesized that mosquitoes spread yellow fever. He then asked volunteers not infected with yellow fever to allow themselves to be bitten by mosquitoes that had recently bitten yellow-fever-infected patients. The volunteers quickly contracted yellow fever. Reed concluded that mosquitoes transmit yellow fever. His reasoning can be represented as follows:

> Individuals stricken with yellow fever have recent mosquito bites.
>
> Individuals not stricken with yellow fever tend to work in areas not infested with mosquitoes.
>
> Most individuals bitten by mosquitoes that have recently bitten yellow fever patients soon contract yellow fever.
>
> ---
>
> Mosquitoes transmit yellow fever.

Reed's observations made his conclusion probable but not certain. The mechanism of transmission might have turned out to be an airborne bacterium that survives only under conditions that also encourage a high density of mosquitoes. It might have been a coincidence that the volunteers contracted yellow fever soon after being bitten by the test mosquitoes. So we can say of this argument that it is inductively strong but deductively invalid.

Deductive logic, which is the province of this text, and inductive logic, which lies beyond the scope of this text, both provide methods for evaluating arguments, and methods of both sorts can be applied to the same argument, as above. Which methods are most appropriately applied depends on the context. If an argument is given with the assumption that if the premises are true the conclusion must also be true, then the argument should be evaluated by the standards of deductive logic. However, if an argument is given with the weaker assumption that if the premises are true the conclusion is probable, then the argument should be evaluated by the standards of inductive logic.

1.5E EXERCISES

Evaluate the passages in Exercise 2 in Section 1.5 that contain arguments. In each case say whether deductive or inductive standards are most appropriate. If the former, state whether the argument is deductively valid. If the latter, state to what extent the argument is inductively strong.

1.6 LOGICAL CONSISTENCY, TRUTH, FALSITY, AND EQUIVALENCE

One of the important relations that can hold among a set of sentences is **consistency.**

> A set of sentences is *logically consistent* if and only if it is possible for all the members of that set to be true. A set of sentences is *logically inconsistent* if and only if it is not logically consistent.

We will indicate that we are talking about a set of sentences by enclosing the component sentences within curly braces— '{' and '}'. The following set is logically consistent:

> {Texas is larger than Oklahoma. The Phlogiston theory of heat has been disproven. The United States Congress consists of the Senate and the House of Representatives.}[3]

Note that there is no requirement that the members of a set have "something to do with each other". The three sentences listed are largely if not entirely unrelated. Together they constitute a consistent set because it is possible that all three are true at the same time. (In fact, all three are true.) We obtain a different but also consistent set by replacing the second sentence, 'The Phlogiston theory of heat has been disproven', with 'The Phlogiston theory of heat has been proven'.

> {Texas is larger than Oklahoma. The Phlogiston theory of heat has been proven. The United States Congress consists of the Senate and the House of Representatives.}

Someone who believes all the members of this new set has, to be sure, at least one false belief ('The Phlogiston theory of heat has been proven'), but this does not make the set inconsistent. There is nothing in the nature of the three sentences and their relations to one another that keeps all three from being true. What keeps the second listed sentence from being true is the nature of heat, that it does not behave the way the Phlogiston theory says it behaves.

The following set of sentences is inconsistent:

> {Michael and Benjamin both applied for positions at the local fast-food outlet, and at least one of them will be hired. No one who applied for a position will get it.}

[3] Technically what should be listed between the curly braces are the names of the members of the set—here the names of sentences. These are formed by placing *single* quotation marks around the sentences. See Chapter 2, Section 2.4.

If the first listed sentence is true, then the second sentence, 'No one who applied for a position will get it', is false. In contrast, if the second sentence is true, then it cannot be (as the first sentence claims it is) that Michael and Benjamin both applied and that at least one of them will be hired. That is, if the second sentence is true, then the first sentence is false. So it is not possible for both members of this set to be true. We are able to figure this out without knowing who Michael and Benjamin are. The relationships between the members of the set make it impossible for all the members to be true.

The following set is also inconsistent:

> [Anyone who takes astrology seriously is foolish. Alice is my sister, and no sister of mine has a husband who is foolish. Horace is Alice's husband, and he reads the horoscope column every morning. Anyone who reads the horoscope column every morning takes astrology seriously.]

A little reflection shows that not all the members of the foregoing set can be true. If the first, third, and fourth sentences are true, the second cannot be. Alternatively, if the second, third, and fourth are true, the first cannot be. And so on.

Logic cannot normally tell us whether a given sentence is true or false, but we can use logic to discover whether a set is consistent or inconsistent. And if a set is inconsistent, we know that at least one member of it is false, and hence that believing all the members of the set would involve believing at least one false sentence, something we don't want to do. Establishing that a set is consistent does not establish that all, or even any, of its members are true; but it does establish that it is possible for all the members to be true.

Although logic cannot normally be used to determine whether a particular sentence is true or false, there are two special cases. Some sentences are true because of their form or structure. For example, 'Either Cynthia will get a job or she will not get a job' is true no matter how Cynthia fares vis-à-vis her job-seeking activities. Indeed, every sentence that is of the form 'either . . . or . . .' and is such that what comes after the 'or' is the denial of what comes after the 'either' is true. Sentences such as these do not give us any new information. They do not "tell us anything about the world". Whoever Cynthia is, we all know that she either will or will not get a job, and so being told this does not convey any new information. Other sentences of this type include 'If Henry gets fired, he gets fired', 'If everyone passes, Denise will pass', and 'If Sarah will go mountain climbing if and only if Marjorie does, then if Marjorie does not go, neither will Sarah'. Sentences of this sort are said to be **logically true.**

> A sentence is *logically true* if and only if it is not possible for the sentence to be false.

Just as some sentences are true by virtue of their form or structure, so too some sentences are false by virtue of their form or structure. These include 'Sarah is an A student and Sarah is not an A student', 'All lions are ferocious but

there are lions in zoos that are not ferocious', 'I'm here and nobody is here', and 'Some dollar bills are not dollar bills'. Such sentences are said to be **logically false.**

> A sentence is *logically false* if and only if it is not possible for the sentence to be true.

Logically false sentences, like logically true sentences, give us no information about the world.[6]

Sentences that purport to give us information about the world—and these constitute most of the sentences we encounter outside logic and mathematics—are neither logically true nor logically false. They include 'Ivan is driving from Boston to New Orleans', 'Anyone who takes astrology seriously is foolish', and 'Perkins advocates the relaxation of air pollution standards because he owns a lot of stock in a company producing coal with a high sulfur content'. Such sentences claim that the world, or some part of it, is a certain way, and to determine whether they are true we have to gather information about the world, and not merely about how those sentences are constructed. Such sentences are said to be **logically indeterminate.**

> A sentence is *logically indeterminate* if and only if it is neither logically true nor logically false.

The final concept we introduce in this chapter is that of logical equivalence. Sentences are sometimes related in such a way that, because of their structure or form, if one is true the other is as well, and vice versa. Examples of such pairs of sentences include these:

Henry loves Sarah.
Sarah is loved by Henry.

Both Sarah and Henry will pass.
Both Henry and Sarah will pass.

Not all tumors are cancerous.
Some tumors are not cancerous.

Of course, the members of this pair, perhaps to Henry's dismay, are not **logically equivalent:**

Henry loves Sarah.
Sarah loves Henry.

[6] But logically true sentences are useful in ways that logically false sentences are not. By some accounts mathematics consists exclusively of logical truths.

The members of a pair of sentences are *logically equivalent* if and only if it is not possible for one of the sentences to be true while the other sentence is false.

Note that we allow a sentence to be equivalent to itself, by counting, for example, 'Sarah is very bright' and 'Sarah is very bright' as constituting a pair of (identical) sentences. On this definition of logical equivalence, it also follows that all logically true sentences are logically equivalent and that all logically false sentences are logically equivalent. But it does not follow that all logically indeterminate sentences are logically equivalent. Clearly, logically indeterminate sentences with different truth-values—for example, 'Philadelphia is in Pennsylvania' (true) and 'Denver is in Wyoming' (false)—are not logically equivalent. Moreover, not all logically indeterminate sentences with the same truth-value are logically equivalent. 'California produces red wine' and 'California produces white wine' are both true, but these claims are not logically equivalent. The test for logical equivalence is not sameness of truth-value, but rather whether the sentences in question *must* have the same truth-value—whether it is impossible for them to have different truth-values. Since it is possible for 'California produces white wine' to be true but 'California produces red wine' to be false (California vintners might decide that all the money is to be made in white wine and stop producing red wine), these sentences are not logically equivalent.

1.6E EXERCISES

1. Where possible, give an example of each of the following. Where not possible, explain why no example can be given.

a. A consistent set all of whose members are true.

*b. A consistent set with at least one true member and at least one false member.

c. An inconsistent set all of whose members are true.

*d. A consistent set all of whose members are false.

2. For each of the following sets of sentences, indicate whether the set is consistent or inconsistent, and why.

a. [Good vegetables are hard to find. The Dodgers are no longer in Brooklyn. Today is hotter than yesterday.]

*b. [Henry likes real ice cream. Real ice cream is a dairy product. There isn't a dairy product Henry likes.]

c. [Washington, D.C., is the capital of the United States. Paris is the capital of France. Ottawa is the capital of Canada.]

*d. [Washington, D.C., is the capital of the United States. Paris is the capital of France. Toronto is the capital of Canada.]

e. [The weather is fine. Tomorrow is Tuesday. Two plus two equals four. We're almost out of gas.]

*f. [Sue is taller than Tom. Tom is taller than Henry. Henry is just as tall as Sue.]

g. [Tom, Sue, and Robin are all bright. No one who fails "Poetry for Scientists" is bright. Tom failed "Poetry for Scientists".]

*h. [The United States does not support dictatorships. In the 1980s the United State supported Iraq. Iraq has been a dictatorship since 1979.]
i. [Roosevelt was a better president than Truman, as was Eisenhower. Eisenhower was also a better president than his successor, Kennedy. Kennedy was the best president we ever had.]
*j. [Jones and his relatives own all the land in Gaylord, Minnesota. Smith is no relation to Jones. Smith owns land in Gaylord, Minnesota.]
k. [Everyone who likes film classics likes *Casablanca*. Everyone who likes Humphrey Bogart likes *Casablanca*. Sarah likes *Casablanca*, but she doesn't like most film classics and she doesn't like Humphrey Bogart.]
*l. [Everyone who likes film classics likes *Casablanca*. Everyone who likes Humphrey Bogart likes *Casablanca*. Sarah likes film classics and she likes Humphrey Bogart, but she can't stand *Casablanca*.]

3. Give an example of each of the following. Explain, in each case, why the given example is of the sort requested.
a. A logically true sentence
*b. A logically false sentence
c. A logically indeterminate sentence

4. For each of the following, indicate whether it is logically true, logically false, or logically indeterminate, and why.
a. Sarah passed the bar exam but she never went to law school.
*b. Helen is a doctor but not an MD.
c. Helen is an MD but not a doctor.
*d. Bob is in London but his heart is in Texas.
e. Robin will either make it to class by starting time or she won't.
*f. Robin will either make it to class by starting time or she will be late.
g. Bob knows everyone in the class, which includes Robin, whom he doesn't know.
*h. Sarah likes all kinds of fish but she doesn't like ocean fish.
i. If Sarah likes all kinds of fish, then she likes ocean fish.
*j. Anyone who likes rare beef likes rare emu.
k. Anyone who loves everyone is lacking in discrimination.
*l. Anyone who loves everyone loves a lot of people.

5. Where possible, give examples of the following. Where not possible, explain why not.
a. A pair of sentences, both of which are logically indeterminate and are logically equivalent.
*b. A pair of sentences that are not logically equivalent but that are both true.
c. A pair of sentences that are logically equivalent, one of which is logically true and one of which is not.
*d. A pair of sentences that are logically equivalent and both false.
e. A pair of sentences, at least one of which is logically true, that are logically equivalent.
*f. A pair of sentences that are logically equivalent, one of which is logically false and the other of which is logically true.

6. For each of the following pairs of sentences, indicate whether the sentences are logically equivalent, and explain why.
a. Henry is in love with Sue.
Sue is in love with Henry.

*b. Sue married Barbara
Barbara married Sue.
c. Tom likes all kinds of fish.
Tom claims to like all kinds of fish.
*d. Bill and Mary were both admitted to the Golden Key Honor Society.
Bill was admitted to the Golden Key Honor Society and Mary was admitted to the Golden Key Honor Society.
e. Neither Bill nor Mary will get into law school.
Bill will not get into law school or Mary will get into law school.
*f. The judge pronounced Bill and Mary husband and wife.
Bill and Mary got married.
g. Only Mariner fans came to the rally.
All Mariner fans came to the rally.
*h. I know there are people who are starving in every large city in America.
In every large city in America I know people who are starving.
i. Every newscast reported that a strike is imminent.
A strike is imminent.
*j. A bad day of sailing is better than a good day at work.
A good day at work isn't as good as a bad day of sailing.
k. Sarah and Anna won't both be elected president of the senior class.
Either Sarah will be elected president of the senior class or Anna will be elected president of the senior class.
*l. Sarah and Anna won't both be elected president of the senior class.
Either Sarah will not be elected president of the senior class or Anna will not be elected president of the senior class.
m. Everyone dislikes someone.
There is someone whom everyone dislikes.
*n. Everyone dislikes someone.
There is no universally liked person.
o. Everyone likes someone.
Someone is liked by everyone.
*p. Not everyone likes someone.
There is someone who doesn't like anyone.

1.7 SPECIAL CASES OF VALIDITY

Having introduced the notions of logical consistency, logical truth, and logical falsity, we are now in a position to consider some special and rather counterintuitive cases of validity. We have defined a deductively valid argument to be one in which it is impossible for the premises to be true and the conclusion false. Such an argument, we have said, is truth-preserving in that it never takes us from true premises to a false conclusion. Here we consider two special cases of validity. Consider first an argument whose conclusion is logically true. An example is

The Philadelphia Phillies are the best team in the National League.

Either the next president will be a woman or the next president will not be a woman.

The conclusion of this argument is clearly logically true. No matter who wins the next presidential election, that person either will or will not be a woman. The premise is, at the moment of this writing, anything but true. Note that the premise is utterly unconnected with the conclusion. For the latter reason one might very well be tempted to say that this is an invalid argument, for surely the premises of a valid argument must be relevant to (have some connection with) the conclusion. But recall our definition of validity: An argument is deductively valid if and only if it is not possible for the premises to be true and the conclusion false. The above argument does satisfy this requirement. It is not possible for the conclusion, a logical truth, to be false. Therefore it is not possible for the premises to be true and the conclusion false—again, because the conclusion cannot be false.[7]

To put the point another way, this argument is truth-preserving. It will never lead us from truths to a falsehood because it will never lead us to a falsehood—because the conclusion is logically true. There is no risk of reaching a false conclusion here precisely because there is no risk that the conclusion is false. All arguments whose conclusions are logically true are deductively valid for this reason.[8]

Consider next arguments whose premises form logically inconsistent sets. This may be because one or more of an argument's premises are logically false (in which case it is impossible for those premises to be true, and hence impossible for all the premises to be true), or it may be because, while no single premise is logically false, the premises taken together are nonetheless logically inconsistent. The following is a case of the latter sort:

> Albert is brighter than all his sisters.
>
> Albert and Sally are brother and sister.
>
> Sally is brighter than all her brothers.
>
> ---
>
> *Tyrannosaurus rex* was the fiercest of all dinosaurs.

In this case, if the first and second premises are both true, the third premise cannot be true. And, if the second and third premises are both true, the first premise cannot be true. So not all the premises can be true. The set consisting of the premises is logically inconsistent. Here, as in the preceding case, there is no obvious connection between the premises and the conclusion. Yet the argument does satisfy our definition of deductive validity because it is impossible for all the premises of this argument to be true and therefore

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[7] Arguments whose conclusions are logically true are deductively valid whether or not their conclusions are related to their premises. For example,

> The Philadelphia Phillies are the best team in the National League; therefore the Phillies either will or will not win the National League pennant

is a deductively valid argument, not because the premise and conclusion both concern the Phillies but because the conclusion is logically true and it is therefore impossible for the premise to be true *and* the conclusion false.

[8] One way to think of such an argument is that, since the conclusion is logically true, it requires *no* support. Hence, whatever support the premises provide (even if it is none at all—even if the premises are utterly unrelated to the conclusion) is enough.

impossible for all the premises to be true and the conclusion false. The argument is truth-preserving because it will never take us from truths to a falsehood. It will not do so because the premises cannot all be true, and hence there is no possibility of going from truths to a falsehood. Arguments whose premises are inconsistent, while valid, are of course never sound.

Every argument whose premises constitute a logically inconsistent set is thus deductively valid. As a further example, consider

> Sandra will get an A in the course and Sandra will not get an A in the course.
>
> ---
>
> Sandra will graduate.

The one premise of this argument is logically false. Therefore that premise cannot be true. And so it is impossible for every premise of this argument (there is only one) to be true and the conclusion false. The conclusion may be false, but not while the premise is true.

Arguments of the sort we are discussing here are sometimes dismissed as not being arguments at all, precisely because their validity does not depend on a relation between the premises and conclusion. There are, however, systematic reasons for allowing these cases to constitute arguments and thus for recognizing them as valid deductive arguments. It is important to remember that such arguments are valid because they meet the requirement of truth preservation—they will never take us from truths to a falsehood—not because the premises support the conclusion in any intuitive way.

1.7E EXERCISES

1. Which of the following are true, and which are false? Explain your answers giving examples where appropriate.

a. If at least one member of a set of sentences is logically false, then the set is logically inconsistent.

*b. No two false sentences are logically equivalent.

c. Every argument whose conclusion is logically equivalent to one of its premises is valid.

*d. Any argument that includes among its premises 'Everyone is a scoundrel' and 'I'm no scoundrel' is deductively valid.

e. Every argument that has 'Whatever will be, will be' as a conclusion is deductively valid.

*f. Every argument that has 'Everyone is a scoundrel and I'm no scoundrel' as a conclusion is deductively invalid.

g. Every argument all of whose premises are logically true is deductively valid.

2. Answer each of the following:

a. Does every person who believes that New York City is the capital of the United States have inconsistent beliefs?

*b. Need one be engaged in a disagreement or dispute to have use for an argument as we have been using the term 'argument'? Explain.

c. Explain why logic cannot normally tell us whether a valid argument is sound. Under what conditions could we decide, on logical grounds alone, that a valid argument is sound?

*d. Suppose an argument is valid but has a false conclusion. What can we conclude about the premises? Explain.

e. Explain why an argument with at least one logically false premise must be valid no matter what the other premises are and no matter what the conclusion is.

*f. Suppose an argument has a premise that is logically equivalent to a logical falsehood. Must the argument be valid? Explain.

g. Suppose an argument has a logical truth as its conclusion. Explain why the argument must be valid no matter what its premises are. Explain why some such arguments are sound and some are not.

*h. Suppose the premises of an argument form an inconsistent set of sentences. Explain why the argument must be valid but unsound.

i. Suppose a set of a million sentences is consistent. Now suppose a new set of sentences is constructed so that every sentence in the new set is logically equivalent to at least one of the sentences in the old set. Must the new set be consistent? Explain.

GLOSSARY

ARGUMENT: An argument is a set of two or more sentences, one of which is designated as the conclusion and the others as the premises.

DEDUCTIVE VALIDITY: An argument is *deductively valid* if and only if it is not possible for the premises to be true and the conclusion false. An argument is *deductively invalid* if and only if it is not deductively valid.

DEDUCTIVE SOUNDNESS: An argument is *deductively sound* if and only if it is deductively valid and all its premises are true. An argument is *deductively unsound* if and only if it is not deductively sound.

INDUCTIVE STRENGTH: An argument has *inductive strength* to the extent that the conclusion is probable given the premises.

LOGICAL CONSISTENCY: A set of sentences is *logically consistent* if and only if it is possible for all the members of that set to be true. A set of sentences is *logically inconsistent* if and only if it is not logically consistent.

LOGICAL TRUTH: A sentence is *logically true* if and only if it is not possible for the sentence to be false.

LOGICAL FALSITY: A sentence is *logically false* if and only if it is not possible for the sentence to be true.

LOGICAL INDETERMINACY: A sentence is *logically indeterminate* if and only if it is neither logically true nor logically false.

LOGICAL EQUIVALENCE: The members of a pair of sentences are *logically equivalent* if and only if it is not possible for one of the sentences to be true while the other sentence is false.

Chapter 2

SENTENTIAL LOGIC: SYMBOLIZATION AND SYNTAX

2.1 SYMBOLIZATION AND TRUTH-FUNCTIONAL CONNECTIVES

Sentential logic, as the name suggests, is a branch of formal logic in which sentences are the basic units. In this chapter we shall introduce *SL*, a symbolic language for sentential logic, which will facilitate our development of formal techniques for assessing the logical relations among sentences and groups of sentences. The sentences of English that can be symbolized in *SL* are those that are either true or false, that is, have truth-values.

In English there are various ways of generating sentences from other sentences. One way is to place a linking term such as 'and' between them. The result, allowing for appropriate adjustments in capitalization and punctuation, will itself be a sentence of English. In this way we can generate

Socrates is wise and Aristotle is crafty

by writing 'and' between 'Socrates is wise' and 'Aristotle is crafty'. Some other linking terms of English are 'or', 'although', 'unless', 'before', and 'if and only if'. As used to generate sentences from other sentences, these terms are called

sentential connectives (they connect or join sentences to produce further sentences).

Some sentence-generating words and expressions do not join two sentences together but rather work on a single sentence. Examples are 'it is not the case that' and 'it is alleged that'. Prefacing a sentence with either of these expressions generates a further sentence. Since these expressions do not literally connect two sentences, the term "sentential connective" is perhaps a little misleading. Nonetheless, such sentence-generating devices as these are commonly classified as sentential connectives, and we shall follow this usage.

Sentences generated from other sentences by means of sentential connectives are **compound sentences.** All other sentences are **simple sentences.** In developing sentential logic we shall be especially interested in the **truth-functional** use of sentential connectives. Intuitively a compound sentence generated by a truth-functional connective is one in which the truth-value of the compound is a function of, or is fixed by, the truth-values of its components.

> A sentential connective is used *truth-functionally* if and only if it is used to generate a compound sentence from one or more sentences in such a way that the truth-value of the generated compound is wholly determined by the truth-values of those one or more sentences from which the compound is generated, no matter what those truth-values may be.

Few, if any, connectives of English are always used truth-functionally. However, many connectives of English are often so used. We shall call these connectives, as so used, **truth-functional connectives.** A **truth-functionally compound sentence** is a compound sentence generated by a truth-functional connective.

In English 'and' is often used truth-functionally. Consider the compound sentence

Alice is in England and Bertram is in France.

Suppose that Alice is in Belgium, not England. Then 'Alice is in England' is false. The compound sentence is then clearly also false. Similarly, if 'Bertram is in France' is false, the compound 'Alice is in England and Bertram is in France' is false as well. In fact, this compound will be true if and only if both of the sentences from which it is generated are true. Hence the truth-value of this compound is wholly determined by the truth-values of the component sentences from which it is generated. Given their truth-values, whatever they may be, we can always compute the truth-value of the compound in question. This is just what we mean when we say that 'and' functions as a truth-functional connective.

SENTENCES OF SENTENTIAL LOGIC

In *SL* capital Roman letters are used to abbreviate individual sentences of English. Thus

Socrates is wise

can be abbreviated as

W

Of course, we could have chosen any capital letter for the abbreviation, but it is common practice to select a letter that reminds us of the sentence being abbreviated. In this case 'W' reminds us of the word 'wise'. But it is essential to remember that the capital letters of *SL* abbreviate entire sentences and *not* individual words within sentences.

To ensure that we have enough sentences in our symbolic language to represent any number of English sentences, we shall also count capital Roman letters with positive-integer subscripts as sentences of *SL*. Thus all the following are sentences of *SL*:

A, B, Z, T_{25}, Q_6

In *SL* capital letters with or without subscripts are **atomic sentences.** Sentences of *SL* that are made up of one or more atomic sentences and one or more sentential connectives of *SL* are *molecular sentences.*

CONJUNCTION

We could abbreviate

Socrates is wise and Aristotle is crafty

in our symbolic language as 'A', but in doing so we would bury important information about this English sentence. This sentence is a compound made up of two simple sentences: 'Socrates is wise' and 'Aristotle is crafty'. Furthermore, in this case the word 'and', which connects the two sentences, is serving as a truth-functional connective. This compound sentence is true if both of its component sentences are true and is false otherwise. We shall use '&' (ampersand) as the sentential connective of *SL* that captures the force of this truth-functional use of 'and' in English. Instead of symbolizing 'Socrates is wise and Aristotle is crafty' as 'A', we can now symbolize it as

W & C

where 'W' abbreviates 'Socrates is wise' and 'C' abbreviates 'Aristotle is crafty'. Remember that the letters abbreviate entire sentences, not merely specific words like the words 'wise' and 'crafty'. The compound sentence 'W & C' is an example of a molecular sentence of *SL*.

A sentence of the form

P & Q

where **P** and **Q** are sentences of *SL*, is a **conjunction.**[1] **P** and **Q** are the **conjuncts** of the conjunction. Informally we shall use the terms "conjunction" and "conjunct" in talking of English sentences that can be symbolized as conjunctions of *SL*. The relation between the truth or falsity of a conjunction and the truth or falsity of its conjuncts can be simply put: A conjunction is true if and only if both of its conjuncts are true. This is summarized by the following table:

P	Q	P & Q
T	T	T
T	F	F
F	T	F
F	F	F

Such a table is called a *characteristic truth-table* because it defines the use of '&' in *SL*. The table is read horizontally, row by row. The first row contains three **T**'s. The first two indicate that we are considering the case in which **P** has the truth-value **T** and **Q** has the truth-value **T**. The last item in the first row is a **T**, indicating that the conjunction has the truth-value **T** under these conditions. The second row indicates that, when **P** has the truth-value **T** and **Q** has the truth-value **F**, the conjunction has the truth-value **F**. The third row shows that, when **P** has the truth-value **F** and **Q** has the truth-value **T**, the conjunction has the truth-value **F**. The last row indicates that when both **P** and **Q** have the truth-value **F**, the conjunction has the truth-value **F** as well.

Sometimes an English sentence that is not itself a compound sentence can be paraphrased as a compound sentence. The sentence

Fred and Nancy passed their driving examinations

can be paraphrased as

Both Fred passed his driving examination and Nancy passed her driving examination.

We underscore the connectives in paraphrases to emphasize that we are using those connectives truth-functionally. We use '<u>both</u> . . . <u>and</u> . . .',

[1] Our use of boldface letters to talk generally about the sentences of *SL* is explained in Section 2.4.

rather than just 'and', to mark off the conjuncts unambiguously. Where the example being paraphrased is complex, we shall sometimes also use parentheses—'(' and ')'—and brackets—'[' and ']'—to indicate grouping. The foregoing paraphrase is an adequate paraphrase of the original sentence inasmuch as both the original sentence and the paraphrase are true if and only if 'Fred passed his driving examination' and 'Nancy passed her driving examination' are both true. The paraphrase is a conjunction and can be symbolized as

F & N

where 'F' abbreviates 'Fred passed his driving examination' and 'N' abbreviates 'Nancy passed her driving examination'.

Symbolizing English sentences in *SL* should be thought of as a two-step process. First, we construct in English a truth-functional paraphrase of the original English sentence; next, we symbolize that paraphrase in *SL*. The paraphrase stage serves to remind us that the compounds symbolized as molecular sentences of *SL* are always truth-functional compounds.

The preceding example illustrates that the grammatical structure of an English sentence is not a completely reliable indication of its logical structure. Key words like 'and' serve as clues but are not infallible guides to symbolization. The sentence

Two jiggers of gin and a few drops of dry vermouth make a great martini

cannot be fairly paraphrased as

Both two jiggers of gin make a great martini and a few drops of dry vermouth make a great martini.

Together these ingredients may make a great martini, but separately they make no martini at all. Such a paraphrase completely distorts the sense of the original sentence. Thus the original sentence must be regarded as a simple sentence and symbolized in *SL* as an atomic sentence, say

M

Many sentences generated by such other connectives of English as 'but', 'however', 'although', 'nevertheless', 'nonetheless', and 'moreover' can be closely paraphrased using 'and' in its truth-functional sense. Consider some examples:

Susan loves country music, but she hates opera

can be paraphrased as

Both Susan loves country music and Susan hates opera.

The paraphrase can be symbolized as 'L & H', where 'L' abbreviates 'Susan loves country music' and 'H' abbreviates 'Susan hates opera'.

The members came today; however, the meeting is tomorrow

can be paraphrased as

Both the members came today and the meeting is tomorrow

which can be symbolized as 'C & M', where 'C' abbreviates 'The members came today' and 'M' abbreviates 'The meeting is tomorrow'.

Although George purchased a thousand raffle tickets, he lost

can be paraphrased as

Both George purchased a thousand raffle tickets and George lost

which can be symbolized as 'P & L', where 'P' abbreviates 'George purchased a thousand raffle tickets' and 'L' abbreviates 'George lost'.

In each of these cases, the paraphrase perhaps misses part of the sense of the original English sentence. In the last example, for instance, there is the suggestion that it is surprising that George could have purchased a thousand raffle tickets and still have lost the raffle. Truth-functional paraphrases often fail to capture all the nuances present in the sentences of which they are paraphrases. This loss is usually not important for the purposes of logical analysis.

In symbolizing sentences of a natural language—in our case English—grammatical structure and key words provide important clues, but they are not infallible guides to correct symbolizations. Ultimately we have to ask ourselves, as speakers of English, whether the sentence can be reasonably paraphrased as a truth-functional compound. If so, we can symbolize it as a molecular sentence of *SL*. If not, we have to symbolize it as an atomic sentence of *SL*.

DISJUNCTION

Another sentential connective of English is 'or', used in such sentences as

Henry James was a psychologist or William James was a psychologist.

This English sentence contains two simple sentences as components: 'Henry James was a psychologist' and 'William James was a psychologist'. The truth-value of the compound wholly depends upon the truth-values of the component sentences. As long as at least one of the component sentences is true, the compound is true; but if both the components are false, then the compound is false. When used in this way, 'or' serves as a truth-functional

connective of English. In *SL* '∨' (wedge) is the symbol that expresses this truth-functional relation. Thus the sentence about Henry and William James can be symbolized as

H ∨ W

where 'H' abbreviates 'Henry James was a psychologist' and 'W' abbreviates 'William James was a psychologist'. 'H ∨ W' is true if 'H' is true or 'W' is true, and it is false only when both 'H' and 'W' are false.

A sentence of the form

P ∨ Q

where **P** and **Q** are sentences of *SL*, is a **disjunction. P** and **Q** are the **disjuncts** of the sentence. Informally we shall use the terms "disjunction" and "disjunct" in talking of English sentences that can be symbolized as disjunctions of *SL*. A disjunction is true if and only if at least one of its disjuncts is true. This is summarized by the following characteristic truth-table:

P	Q	P ∨ Q
T	T	T
T	F	T
F	T	T
F	F	F

The only case in which a disjunction has the truth-value **F** is when both disjuncts have the truth-value **F**.

Some sentences of English that do not contain the word 'or' can be paraphrased as a disjunction. For instance,

> At least one of the two hikers, Jerry and Amy, will get to the top of the mountain

can adequately be paraphrased as

> Either Jerry will get to the top of the mountain or Amy will get to the top of the mountain.

This paraphrase can be symbolized as 'J ∨ A', where 'J' abbreviates 'Jerry will get to the top of the mountain' and 'A' abbreviates 'Amy will get to the top of the mountain'. Remember, the letters abbreviate the entire sentences, not just the words 'Jerry' and 'Amy'. In paraphrasing English sentences as disjunctions of *SL*, we use the 'either . . . or . . .' construction to mark off the two disjuncts unambiguously.

In English sentences that can be paraphrased as disjunctions, 'or' does not always occur between full sentences. For example,

Nietzsche is either a philosopher or a mathematician

can be paraphrased as

Either Nietzsche is a philosopher or Nietzsche is a mathematician.

This truth-functional paraphrase can be symbolized as 'P ∨ M', where 'P' abbreviates 'Nietzsche is a philosopher' and 'M' abbreviates 'Nietzsche is a mathematician'.

We use the wedge to symbolize disjunctions in the *inclusive* sense. Suppose the following appears on a menu:

With your meal you get apple pie or chocolate cake.

We might try to paraphrase this as

Either with your meal you get apple pie or with your meal you get chocolate cake.

Since we use 'or' only in the inclusive sense in paraphrases, this paraphrase is true if either or both of the disjuncts are true. In ordinary English, on the other hand, 'or' is sometimes used in a more restrictive sense. In the present example, if someone orders both pie and cake, the waiter is likely to point out that either cake or pie, but *not* both, comes with the dinner. This is the exclusive sense of 'or'—either one or the other but not both. Although this sense of 'or' cannot be captured by '∨' alone, there is, as we shall soon see, a combination of connectives of *SL* that will allow us to express the exclusive sense of 'or'.

NEGATION

'It is not the case that' is a sentential connective of English. Consider the following compound generated by this connective:

It is not the case that Franklin Pierce was president.

This sentence is true if its component sentence, 'Franklin Pierce was president', is false, and it is false if that component sentence is true. 'It is not the case that' is a truth-functional connective because the truth-value of the generated sentence is wholly determined by the truth-value of the component sentence. In *SL*, '~' (tilde) is the sentential connective that captures this

truth-functional relationship. Thus the sentence in question can be symbolized as

$$\sim F$$

where 'F' abbreviates 'Franklin Pierce was president'. The tilde is a **unary connective,** because it "connects" only one sentence. On the other hand, '&' and '∨' are **binary connectives** since each connects two sentences. When '~ ' is placed in front of a sentence, the truth-value of the generated sentence is the opposite of the truth-value of the original sentence. So the characteristic truth-table for negation is this:

P	~ P
T	F
F	T

Notice that, because '~ ' is a unary connective, we need a truth-table of only two rows to represent all the possible "combinations" of truth-values that a single sentence to which '~ ' is attached might have.

Putting a '~ ' in front of a sentence forms the negation of that sentence. Hence '~ A' is the negation of 'A' (though 'A' is *not* the negation of '~ A'), '~ ~ A' is the negation of '~ A' (though '~ A' is not the negation of '~ ~ A'), and so forth. Informally we shall use the term "negation" in talking about sentences of English that can be symbolized as negations in *SL*. Thus

It is not the case that Franklin Pierce was president

is the negation of

Franklin Pierce was president.

Whether an English sentence should be symbolized as a negation depends on the context. As before, grammar and key words give us clues. Consider some examples:

Not all sailors are good swimmers

is readily paraphrased as

It is not the case that all sailors are good swimmers.

This paraphrase can be symbolized as '~ G', where 'G' abbreviates 'All sailors are good swimmers'. But the following example is not as straightforward:

No doctors are rich.

One might be tempted to paraphrase this sentence as 'It is not the case that all doctors are rich', but to do so is to treat 'No doctors are rich' as the negation of 'All doctors are rich'. This is a mistake because a sentence and its negation are so related that, if one is true, the other is false, and vice versa. In fact, since some doctors are rich and some doctors are not rich, both 'All doctors are rich' and 'No doctors are rich' are false. Hence the latter cannot be the negation of the former. Rather, 'No doctors are rich' is the negation of 'Some doctors are rich'. 'No doctors are rich' is true if and only if 'Some doctors are rich' is false, so the former sentence can be paraphrased as

It is not the case that some doctors are rich.

This can be symbolized as '~ D', where 'D' abbreviates 'Some doctors are rich'.

Some further examples will be helpful:

Chlorine is not a metal

can plausibly be understood as

It is not the case that chlorine is a metal.

This paraphrase can be symbolized as '~ C', where 'C' abbreviates 'Chlorine is a metal'. Notice that 'Chlorine is a metal' and 'Chlorine is not a metal' are such that if either is true the other is false, which must be the case if the latter is to be the negation of the former. But now consider an apparently similar case:

Some humans are not male.

This sentence should not be paraphrased as 'It is not the case that some humans are male'. The latter sentence is true if and only if *no* humans are male, which is not the claim made by the original sentence. The proper paraphrase is

It is not the case that all humans are male

which can be symbolized as '~ H', where 'H' abbreviates 'All humans are male'. Often sentences containing words with such prefixes as 'un-', 'in-', and 'non-' are best paraphrased as negations. But we must be careful here.

Kant was unmarried

can be understood as

It is not the case that Kant was married

and then symbolized as '~ K', where 'K' abbreviates 'Kant was married'. 'Kant was unmarried' is the negation of 'Kant was married'. But

> Some people are unmarried

should not be paraphrased as 'It is not the case that some people are married'. 'Some people are married' and 'Some people are unmarried' are both true. A proper paraphrase in this case is

> It is not the case that all people are married

which can be symbolized as '~ M', where 'M' abbreviates 'All people are married'.

COMBINATIONS OF SENTENTIAL CONNECTIVES

So far we have discussed three types of truth-functional compounds—conjunctions, disjunctions, and negations—and the corresponding sentential connectives of *SL*— '&', '∨', and '~ '. These connectives can be used in combination to symbolize complex passages. Suppose we wish to symbolize the following:

> Either the steam engine or the computer was the greatest modern invention, but the zipper, although not the greatest modern invention, has made life much easier.

The main connective in this sentence is 'but', and the sentence can be paraphrased as a conjunction. The left conjunct can be paraphrased as a disjunction, and the right can be paraphrased as a conjunction making the claim that the zipper was not the greatest modern invention and the claim that the zipper has made life much easier. Finally the claim that the zipper was not the greatest modern invention can be paraphrased as a negation. The resulting truth-functional paraphrase is

> Both (either the steam engine was the greatest modern invention or the computer was the greatest modern invention) and (both it is not the case that the zipper was the greatest modern invention and the zipper has made life much easier).

For clarity we have inserted some parentheses in the paraphrase to emphasize the grouping of the components. The order of placement of 'both' and 'either' is important. In this case 'both' occurring before 'either' at the beginning shows that the overall sentence is a conjunction, not a disjunction. The paraphrase can be symbolized as

> (S ∨ C) & (~ Z & E)

where 'S' abbreviates 'The steam engine was the greatest modern invention', 'C' abbreviates 'The computer was the greatest modern invention', 'Z' abbreviates 'The zipper was the greatest modern invention', and 'E' abbreviates 'The zipper has made life much easier'.

The connectives '&', '∨', and '~ ' can be used in combination to symbolize English sentential connectives such as 'neither . . . nor . . .'. The sentence

> Neither Sherlock Holmes nor Watson is fond of criminals

can be paraphrased as

> Both it is not the case that Sherlock Holmes is fond of criminals and it is not the case that Watson is fond of criminals.

This can be symbolized as

> ~ H & ~ W

where 'H' abbreviates 'Sherlock Holmes is fond of criminals' and 'W' abbreviates 'Watson is fond of criminals'.

Another equally good paraphrase of the original sentence is

> It is not the case that either Sherlock Holmes is fond of criminals or Watson is fond of criminals.

This paraphrase can be symbolized using the above abbreviations as

> ~ (H ∨ W)

Note that the original sentence, the paraphrases, and the symbolic sentences are all true if Sherlock Holmes is not fond of criminals and Watson is not fond of criminals, and they are all false otherwise.

A similar, but nonequivalent, connective is 'not both . . . and . . .'. Consider this claim:

> A Republican and a Democrat will not both become president.

Truth-functionally paraphrased this becomes

> It is not the case that both a Republican will become president and a Democrat will become president

which is symbolized as

> ~ (R & D)

This sentence does not maintain that neither a Republican nor a Democrat will become president but only that not both of them will become president. '~ (R ∨ D)' is not an acceptable symbolization, but '~ (R & D)' is. Another possible and acceptable paraphrase of this particular 'not both . . . and . . .' claim is

> Either it is not the case that a Republican will become president or it is not the case that a Democrat will become president

which when symbolized becomes

> ~ R ∨ ~ D

Here is a table summarizing the truth conditions for 'neither . . . nor . . .'. Notice that a 'neither . . . nor . . .' expression is true only when both of its components, **P** and **Q**, are false.

Truth Conditions for
'Neither . . . nor . . .'

P	Q	~ P & ~ Q	~ (P ∨ Q)
T	T	F	F
T	F	F	F
F	T	F	F
F	F	T	T

Compare this table with the next table, which shows the truth conditions for 'not both . . . and . . .':

Truth Conditions for
'Not both . . . and . . .'

P	Q	~ (P & Q)	~ P ∨ ~ Q
T	T	F	F
T	F	T	T
F	T	T	T
F	F	T	T

A 'not both . . . and . . .' expression is false only when both of its components, **P** and **Q**, are true.

A combination of the sentential connectives of *SL* can also be used to capture the exclusive sense of 'or' discussed earlier. Recall that the sentence

> With your meal you get apple pie or chocolate cake

is true in the exclusive sense of 'or' if with your meal you get apple pie or chocolate cake but not both apple pie and chocolate cake. We now know how to paraphrase the 'not both . . . and . . .' portion of the sentence. The paraphrase of the whole sentence is

> Both (either with your meal you get apple pie or with your meal you get chocolate cake) and it is not the case that (both with your meal you get apple pie and with your meal you get chocolate cake).

This can be symbolized as

$$(A \vee C) \ \& \ - \ (A \ \& \ C)$$

where 'A' abbreviates 'With your meal you get apple pie' and 'C' abbreviates 'With your meal you get chocolate cake'. Here is a table showing the truth conditions for exclusive 'or':

Truth Conditions for Exclusive 'Or'

P	Q	(P ∨ Q) & − (P & Q)
T	T	F
T	F	T
F	T	T
F	F	F

MATERIAL CONDITIONAL

One of the most common sentential connectives of English is 'if . . . then . . .'. A simple example is

> If Jones got the job then he applied for it.

This can be paraphrased as

> Either it is not the case that Jones got the job or Jones applied for the job

which can be symbolized as

$$- G \vee A$$

where 'G' abbreviates 'Jones got the job' and 'A' abbreviates 'Jones applied for the job'. It will be convenient to have a symbol in *SL* that expresses the truth-functional sense of 'if . . . then . . .'; we introduce '⊃' (horseshoe) for this

purpose. The sentence 'If Jones got the job then Jones applied for the job' can then be symbolized as

$$G \supset A$$

A sentence of the form $\mathbf{P} \supset \mathbf{Q}$, where **P** and **Q** are sentences of SL, is a **material conditional. P**, the sentence on the left of the '$\supset$', is the **antecedent,** and **Q**, the sentence on the right of the '$\supset$', is the **consequent** of the conditional. It is important to remember that, whenever we write a sentence of the form $\mathbf{P} \supset \mathbf{Q}$, we could express it as $\sim \mathbf{P} \vee \mathbf{Q}$. A sentence of the form $\sim \mathbf{P} \vee \mathbf{Q}$ is a disjunction, and a disjunction is false in only one case—when both disjuncts are false. Thus a sentence of the form $\sim \mathbf{P} \vee \mathbf{Q}$ is false when $\sim \mathbf{P}$ is false and **Q** is false, that is, when **P** is true and **Q** is false. This is also the only case in which a sentence of the form $\mathbf{P} \supset \mathbf{Q}$ is false, that is, when the antecedent is true and the consequent is false. The characteristic truth-table is shown here:

P	Q	P ⊃ Q
T	T	T
T	F	F
F	T	T
F	F	T

Informally we can regard the 'if' clause of an English conditional as the antecedent of that conditional and the 'then' clause as the consequent. Here is an example of an English conditional converted to a truth-functional paraphrase that is symbolized by the material conditional:

If Michelle is in Paris then she is in France.

Expressed in a truth-functional paraphrase this becomes

If Michelle is in Paris then Michelle is in France.

The truth-functional paraphrase can be symbolized as a material conditional

$$P \supset F$$

Notice that the truth-functional paraphrase is false if Michelle is in Paris but is not in France—that is, if the antecedent is true and the consequent is false. But the truth-functional paraphrase is true under all other conditions. Thus, if Michelle is in Paris and in France, the paraphrase is true. If Michelle is not in Paris but is somewhere else in France, the paraphrase is true. If Michelle is not in Paris and not in France, the paraphrase is true.

However, the material conditional is not adequate as a complete treatment of conditional sentences in English. Material conditionals are truth-functional, but conditionals in English frequently convey information that exceeds

a truth-functional analysis. For instance, 'if . . . then . . .' constructions sometimes have a causal force that is lost in a truth-functional paraphrase. Consider:

1. If this rod is made of metal then it will expand when heated.
2. If this rod is made of metal then it will contract when heated.

Each of these sentences can be used to make a causal claim, to assert a causal relation between the substance of which the rod in question is composed and the reaction of the rod to heat. But sentence 1 is in accord with the laws of nature, and sentence 2 is not. So, as used to make causal claims, sentence 1 is true and sentence 2 is false, even if it is false that the rod is made of metal.

Now suppose we paraphrase these two sentences as material conditionals:

1a. If this rod is made of metal then this rod will expand when heated.
2a. If this rod is made of metal then this rod will contract when heated.

These paraphrases can be symbolized as

1b. $M \supset E$
2b. $M \supset C$

where 'M' abbreviates 'The rod is made of metal', 'E' abbreviates 'This rod will expand when heated', and 'C' abbreviates 'This rod will contract when heated'. Remember that a material conditional is true if the antecedent is false. If the rod in the example is not made of metal, then both sentences 1a and 2a, and consequently their symbolizations 1b and 2b, are true. Sentence 1 says more than either 1a or 1b, and sentence 2 says more than either 2a or 2b. The fact that sentence 2 is false, whereas 2a and 2b are both true, shows this. It follows that when they are used to assert a causal relation, sentences 1 and 2, like many other English conditionals, are not truth-functional compounds. When it is and when it is not appropriate to paraphrase such sentences as material conditionals will be discussed further in Section 2.3.

Here are further examples of English sentences that can be paraphrased by using 'if . . . then . . .', but here and elsewhere we must keep in mind that sometimes information contained in the English conditionals will be lost in truth-functional paraphrasing.

Larry will become wealthy provided that he inherits the family fortune

can be paraphrased as

If Larry inherits the family fortune then Larry will become wealthy

which can be symbolized as

$F \supset W$

where 'F' abbreviates 'Larry inherits the family fortune' and 'W' abbreviates 'Larry will become wealthy'.

The Democratic candidate will win the election if he wins in the big cities

can be paraphrased as

If the Democratic candidate wins in the big cities then the Democratic candidate will win the election

which can be symbolized as '$C \supset E$', where 'C' abbreviates 'The Democratic candidate wins in the big cities' and 'E' abbreviates 'The Democratic candidate will win the election'.

Betty is in London only if Betty is in England

can be paraphrased as

If Betty is in London then Betty is in England

which can be symbolized as '$L \supset E$', where 'L' abbreviates 'Betty is in London' and 'E' abbreviates 'Betty is in England'. In this case be sure to notice the order in which the sentences are paraphrased. A common mistake in paraphrasing the sentential connective 'only if' is to ignore the word 'only' and reverse the order of the sentences. It is *incorrect* to paraphrase the original as 'If Betty is in England then Betty is in London'.

A connective that can be paraphrased either as a disjunction or as a conditional is 'unless'. Consider the sentence

This plant will die unless it is watered.

The only circumstance under which this sentence is false is the situation in which this plant does not die and is not watered. If either of the sentences that 'unless' connects is true, then the whole sentence is true. The simplest paraphrase is to treat the sentence as the disjunction

Either this plant will die or it is watered

which can be symbolized as

$D \vee W$

We can also understand the sentence 'This plant will die unless it is watered' as expressing a conditional:

If it is not the case that it is watered, then this plant will die

which can be symbolized as

$$\sim W \supset D$$

Equally well, we can understand the sentence as expressing the equivalent conditional:

> If it is not the case that this plant will die, then it is watered

which when symbolized is

$$\sim D \supset W$$

The two conditional paraphrases look different from each other and from the disjunction, but they make identical truth-functional claims. The disjunction claims that at least one of its component sentences is true. Each of the conditionals claims that, if one of two component sentences is not true, the other one is true. Here is a table that shows the truth-functional equivalence of the symbolizations for 'unless':

Truth Conditions for 'Unless'

P	Q	P ∨ Q	~P ⊃ Q	~Q ⊃ P
T	T	T	T	T
T	F	T	T	T
F	T	T	T	T
F	F	F	F	F

MATERIAL BICONDITIONAL

In English the connective 'if and only if' is used to express more than either the connective 'if' or the connective 'only if'. For example

> John will get an A in the course if and only if he does well on the final examination

can be paraphrased as

> Both (if John will get an A in the course then John does well on the final examination) and (if John does well on the final examination then John will get an A in the course).

We can symbolize the paraphrase as

$$(C \supset E) \ \& \ (E \supset C)$$

where 'C' abbreviates 'John will get an A in the course' and 'E' abbreviates 'John does well on the final examination'. The original sentence can also be paraphrased as

> Either (both John will get an A in the course and John does well on the final examination) or (both it is not the case that John will get an A in the course and it is not the case that John does well on the final examination).

Using the same abbreviations, this paraphrase is symbolized as

$$(C \mathbin{\&} E) \vee (\sim C \mathbin{\&} \sim E)$$

Both of these paraphrases and their corresponding symbolizations are truth-functional compounds. Each is true just in case either both atomic sentences are true or both atomic sentences are false. We introduce the connective '≡' (triple bar) to capture the truth-functional use of the connective 'if and only if'. The original English sentence can be symbolized as

$$C \equiv E$$

A sentence of the form

$$\mathbf{P} \equiv \mathbf{Q}$$

where **P** and **Q** are sentences of *SL*, is a **material biconditional.** Informally we shall use the term "material biconditional" when describing English sentences that can be symbolized as material biconditionals in *SL*. Here is the characteristic truth-table for '≡':

P	Q	P ≡ Q
T	T	T
T	F	F
F	T	F
F	F	T

The connective 'just in case' is sometimes used in English as an equivalent to 'if and only if'.

> Andy will win the lottery just in case Andy has the winning ticket

can be properly paraphrased as

> Andy will win the lottery if and only if Andy has the winning ticket

and symbolized as

$$W \equiv T$$

However, care must be taken when paraphrasing 'just in case' because this connective sometimes is used in ways *not* equivalent to 'if and only if'. Consider

Marty takes her umbrella to work just in case it rains.

This does not mean 'Marty takes her umbrella to work if and only if it rains'. Rather, the sentence means

Marty takes her umbrella to work because it may rain.

SUMMARY OF SOME COMMON CONNECTIVES

Note that we use lowercase boldface '**p**' and '**q**' to designate sentences of English and uppercase boldface '**P**' and '**Q**' to designate sentences of *SL*.

English Connectives	*Paraphrases*	*Symbolizations*
not **p**	it is not the case that **p**	$\sim \mathbf{P}$
p and **q** **p** but **q** **p** however **q** **p** although **q** **p** nevertheless **q** **p** nonetheless **q** **p** moreover **q**	both **p** and **q**	$\mathbf{P} \,\&\, \mathbf{Q}$
p or **q**	either **p** or **q**	$\mathbf{P} \vee \mathbf{Q}$
p or **q** [*exclusive*]	both either **p** or **q** and it is not the case that both **p** and **q**	$(\mathbf{P} \vee \mathbf{Q}) \,\&\, \sim (\mathbf{P} \,\&\, \mathbf{Q})$
if **p** then **q** **p** only if **q** **q** if **p** **q** provided that **p** **q** given **p**	if **p** then **q**	$\mathbf{P} \supset \mathbf{Q}$
p if and only if **q** **p** if but only if **q** **p** just in case **q**	**p** if and only if **q**	$\mathbf{P} \equiv \mathbf{Q}$
neither **p** nor **q**	both it is not the case that **p** and it is not the case that **q** it is not the case that either **p** or **q**	$\sim \mathbf{P} \,\&\, \sim \mathbf{Q}$ $\sim (\mathbf{P} \vee \mathbf{Q})$
not both **p** and **q**	it is not the case that both **p** and **q** either it is not the case that **p** or it is not the case that **q**	$\sim (\mathbf{P} \,\&\, \mathbf{Q})$ $\sim \mathbf{P} \vee \sim \mathbf{Q}$
p unless **q**	either **p** or **q** if it is not the case that **p** then **q** if it is not the case that **q** then **p**	$\mathbf{P} \vee \mathbf{Q}$ $\sim \mathbf{P} \supset \mathbf{Q}$ $\sim \mathbf{Q} \supset \mathbf{P}$

The connective 'because' is not truth-functional. ('Because' can join two true sentences resulting in a true sentence and 'because' can join two true sentences resulting in a false sentence.) Hence 'Marty takes her umbrella to work just in case it rains' should be symbolized by a single sentence letter such as 'M'.

In our discussion of the material conditional and the material biconditional, we have been careful to distinguish among connectives such as 'if', 'only if', and 'if and only if'. These distinctions are very important in logic, philosophy, and mathematics. However, in everyday discourse people speak casually. For example, people may use 'if' or 'only if' when they mean 'if and only if'. Our general policy in this book is to take disjunctions and conditionals in their weaker rather than their stronger senses. That is, normally 'or' will be read in the inclusive sense, and 'if . . . then . . .' (and other conditional connectives) will be taken in the material conditional sense (not the biconditional sense). When stronger readings are intended, we will indicate that by explicitly using expressions such as 'either . . . or . . . but not both' and 'if and only if'.

2.1E EXERCISES

1. For each of the following sentences, construct a truth-functional paraphrase and symbolize the paraphrase in *SL*. Use these abbreviations:

A: Albert jogs regularly.
B: Bob jogs regularly.
C: Carol jogs regularly.

a. Bob and Carol jog regularly.
*b. Bob does not jog regularly, but Carol does.
c. Either Bob jogs regularly or Carol jogs regularly.
*d. Albert jogs regularly and so does Carol.
e. Neither Bob nor Carol jogs regularly.
*f. Bob does jog regularly; however, Albert doesn't.
g. Bob doesn't jog regularly unless Carol jogs regularly.
*h. Albert and Bob and also Carol do not jog regularly.
i. Either Bob jogs regularly or Albert jogs regularly, but they don't both jog regularly.
*j. Although Carol doesn't jog regularly, either Bob or Albert does.
k. It is not the case that Carol or Bob jogs regularly; moreover Albert doesn't jog regularly either.
*l. It is not the case that Albert, Bob, or Carol jogs regularly.
m. Either Albert jogs regularly or he doesn't.
*n. Neither Albert nor Carol nor Bob jogs regularly.

2. Using the abbreviations given in Exercise 1, construct idiomatic English sentences from the following sentences of *SL*:
a. A & B
*b. A ∨ ~ A

c. $A \vee C$
*d. $\sim (A \vee C)$
e. $\sim A \,\&\, \sim C$
*f. $\sim\sim B$
g. $B \,\&\, (A \vee C)$
*h. $(A \vee C) \,\&\, \sim (A \,\&\, C)$
i. $(A \,\&\, C) \,\&\, B$
*j. $\sim A \vee (\sim B \vee \sim C)$
k. $(B \vee C) \vee \sim (B \vee C)$

3. Assuming that 'Albert jogs regularly' is true, 'Bob jogs regularly' is false, and 'Carol jogs regularly' is true, which of the symbolic sentences in Exercise 2 are true and which are false? Use your knowledge of the characteristic truth-tables in answering.

4. Paraphrase each of the following using the phrase 'it is not the case that'. Symbolize the results, indicating what your abbreviations are.
a. Some joggers are not marathon runners.
*b. Bob is not a marathon runner.
c. Each and every marathon runner is not lazy.
*d. Some joggers are unhealthy.
e. Nobody is perfect.

5. For each of the following sentences, construct a truth-functional paraphrase and symbolize the paraphrase in *SL*. Use these abbreviations:

A: Albert jogs regularly.
B: Bob jogs regularly.
C: Carol jogs regularly.
L: Bob is lazy.
M: Carol is a marathon runner.
H: Albert is healthy.

a. If Bob jogs regularly he is not lazy.
*b. If Bob is not lazy he jogs regularly.
c. Bob jogs regularly if and only if he is not lazy.
*d. Carol is a marathon runner only if she jogs regularly.
e. Carol is a marathon runner if and only if she jogs regularly.
*f. If Carol jogs regularly, then if Bob is not lazy he jogs regularly.
g. If both Carol and Bob jog regularly, then Albert does too.
*h. If either Carol or Bob jogs regularly, then Albert does too.
i. If either Carol or Bob does not jog regularly, then Albert doesn't either.
*j. If neither Carol nor Bob jogs regularly, then Albert doesn't either.
k. If Albert is healthy and Bob is not lazy, then both jog regularly.
*l. If Albert is healthy, he jogs regularly if and only if Bob does.
m. Assuming Carol is not a marathon runner, she jogs regularly if and only if Albert and Bob both jog regularly.
*n. Although Albert is healthy he does not jog regularly, but Carol does jog regularly if Bob does.
o. If Carol is a marathon runner and Bob is not lazy and Albert is healthy, then they all jog regularly.

*p. If Albert jogs regularly, then Carol does provided Bob does.
q. If Albert jogs regularly if Carol does, then Albert is healthy and Carol is a marathon runner.
*r. If Albert is healthy if he jogs regularly, then if Bob is lazy he doesn't jog regularly.
s. If Albert jogs regularly if either Carol or Bob does, then Albert is healthy and Bob isn't lazy.
*t. If Albert is not healthy, then Bob and Albert do not both jog regularly.

6. Using the abbreviations given in Exercise 5, construct idiomatic English sentences from the following sentences of *SL*.

a. L ∨ ~ L
*b. M ⊃ C
c. A ≡ H
*d. C & ~ B
e. ~ B & ~ C
*f. [A ∨ (B ∨ C)] ⊃ [A & (B & C)]
g. (~ A ∨ ~ C) ⊃ B
*h. ~ (A ∨ C) ⊃ B
i. C ⊃ (A & ~ B)
*j. B ≡ (~ L & A)
k. C & ~ C
*l. A & (C ≡ B)
m. (L ⊃ L) & B
*n. ~ ~ H & ~ A
o. ~ A ⊃ (~ B ⊃ ~ C)
*p. (C ⊃ A) & (A ⊃ B)
q. ~ A & (B ≡ ~ L)
*r. (H ⊃ A) ⊃ (~ L ⊃ B)

7. Give a truth-functional paraphrase for each of the following, and symbolize the paraphrase in *SL*.

a. Neither men nor women are from Mars or Venus.
*b. This dog won't hunt; moreover he is not even a good pet.
c. Not both Butch Cassidy and the Sundance Kid escaped.
*d. The tea will not taste robust unless it steeps for a while.
e. That lady was both cut in half and torn asunder unless it was a magic trick.
*f. Neither wind nor rain nor dark of night will stop the mail.
g. The prisoner will receive either a life sentence or the death penalty.
*h. Unless snowstorms arrive, skiing and snowboarding will be impossible.

8. What are the truth-conditions for the exclusive 'or'? How might the exclusive 'or' be expressed as a biconditional?

2.2 COMPLEX SYMBOLIZATIONS

Going through the paraphrase stage is useful when learning how to symbolize sentences. The paraphrases serve as reminders of exactly what is being symbolized in *SL*. Each sentence of a paraphrase will be either a simple sentence, a truth-functionally compound sentence, or a non-truth-functionally compound

sentence. The simple sentences and non-truth-functionally compound sentences are to be symbolized as atomic sentences of *SL*. The truth-functionally compound sentences are to be symbolized as molecular sentences of *SL*. In constructing a paraphrase, we must be alert to the grammar, wording, and context of the original passage. Sometimes there will be a loss of information in moving from the original passage to the paraphrase, but often the loss of information will not matter.

GUIDELINES FOR PARAPHRASING

In paraphrasing sentences, following several guidelines will be useful:

1. Any sentence of the original passage that is going to be treated as a simple sentence, that will eventually be abbreviated as an atomic sentence in *SL*, can be copied as its own paraphrase.
2. Any sentence of the original passage that is going to be paraphrased as a truth-functionally compound sentence can be paraphrased using one or more of the connectives 'both . . . and . . .', 'either . . . or . . .', 'it is not the case that', 'if . . . then . . .', and 'if and only if'. We underscore these connectives in the paraphrases to emphasize their truth-functional usage.
3. Ambiguities should be eliminated in the paraphrase. For instance, sometimes it may be clearer to insert parentheses in the paraphrase to establish how sentences are to be grouped. If the connective 'it is not the case that' is applied to an entire material biconditional, rather than just to the first component, parentheses will show this, as in 'It is not the case that (the Republican candidate will win if and only if he is supported by big business).'
4. If the passage is an argument, put the paraphrased argument in standard form. That is, list the paraphrased premises first, draw a line, and then list the paraphrased conclusion.
5. Where an English passage contains two or more different wordings of the same claim, use just one wording in constructing a paraphrase of that passage.

The intent of the last of these guidelines can be made clear through the use of examples. Suppose someone offers the following rather trivial argument:

> If Sue and Bill got married yesterday, they are honeymooning today. They did get married yesterday. So they are honeymooning today.

The sentence 'They did get married yesterday' is not the antecedent of 'If Sue and Bill got married yesterday, they are honeymooning today'. Yet, in the context of this passage, 'they' refers to Sue and Bill. So the second premise of our

paraphrase should be 'Sue and Bill got married yesterday', *not* 'They did get married yesterday'. The full paraphrase will be

> If Sue and Bill got married yesterday, then Sue and Bill are honeymooning today.
>
> Sue and Bill got married yesterday.
>
> ---
>
> Sue and Bill are honeymooning today.

Note that we have replaced 'they' with 'Sue and Bill' throughout. Here is another example in which rewording is necessary in constructing a paraphrase:

> Either Jim will not pass the test or Jim spent last night studying logic. Jim's night was not spent poring over his logic text. Hence Jim will fail the test.

In constructing a paraphrase of this argument, it is important to word the premises and conclusion so that we can use a minimum number of sentential letters to symbolize the paraphrase. Suppose someone gives the following paraphrase:

> Either it is not the case that Jim will pass the test or Jim spent last night studying logic.
>
> It is not the case that Jim's night was spent poring over his logic text.
>
> ---
>
> Jim will fail the test.

To symbolize this argument, we need four sentence letters: 'J', 'S', 'O', and 'F'.

> $\sim J \vee S$
>
> $\sim O$
>
> ---
>
> F

Here 'J' abbreviates 'Jim will pass the test', 'S' abbreviates 'Jim spent last night studying logic', 'O' abbreviates 'Jim's night was spent poring over his logic text', and 'F' abbreviates 'Jim will fail the test'. Symbolized in this way our argument is invalid. But the original English argument is valid. The following is a far better paraphrase:

> Either it is not the case that Jim will pass the test or Jim spent last night studying logic.
>
> It is not the case that Jim spent last night studying logic.
>
> ---
>
> It is not the case that Jim will pass the test.

'Jim will not pass the test' and 'Jim will fail the test' express the same claim in this context. So do 'He spent last night studying logic' and 'Jim's night was spent poring over his logic text'. Our second paraphrase reflects this and allows us to give the following symbolization:

$$\begin{array}{l} \sim J \vee S \\ \sim S \\ \hline \sim J \end{array}$$

This symbolic argument is valid, as our formal techniques will show.

We shall now present and symbolize more complex sentences and groups of sentences. In our first series of examples, we shall consider sentences about an international yacht race in which there are just three major competitors: the Americans, the British, and the Canadians. In symbolizing these sentences we shall make use of the following abbreviations:

M: The Americans win the race.
R: The British win the race.
N: The Canadians win the race.
A: The Americans have good luck.
B: The British have good luck.
C: The Canadians have good luck.
E: Everyone is surprised.
T: A major tradition is broken.

Our first two examples illustrate the important difference between sentences compounded by 'if . . . then . . .' and those compounded by 'only if':

1. The British will win if neither of the other two major competitors wins.
2. The British will win only if neither of the other two major competitors wins.

The first of these sentences tells us, in effect, that the British do not have to worry about the minor competitors. According to sentence 1, for the British to win all that is needed is that the Americans and Canadians not win. The second sentence makes a more modest claim—it expresses only the truism that for the British to win the other major competitors must not win. Here are our truth-functional paraphrases of these sentences:

1a. If both it is not the case that the Americans win the race and it is not the case that the Canadians win the race, then the British win the race.
2a. If the British win the race, then both it is not the case that the Americans win the race and it is not the case that the Canadians win the race.

The rule to remember here is that a sentence compounded by 'if' rather than by 'only if' should be paraphrased as a conditional whose *antecedent* is the sentence following 'if' in the original compound. A sentence compounded by 'only if' should be paraphrased as a conditional whose *consequent* is the sentence following 'only if' in the original compound. The symbolizations of these paraphrases will be

1b. $(\sim M \,\&\, \sim N) \supset R$
2b. $R \supset (\sim M \,\&\, \sim N)$

In sentences 1 and 2 some of the verbs are in the present tense and some are in the future tense. But in these particular examples, the difference in tense does not reflect a difference in the temporal order of the events under discussion. ('The British will win if neither of the other two major competitors wins' does *not* mean that if neither of the other two major competitors wins *now*, then the British will win *later*.) Accordingly in our paraphrases we have made all the verbs present tense. We could, alternatively, have made them all future tense. In giving paraphrases it is often useful to make as many of the verbs as possible the same tense; but this should be done *only* when doing so does not distort the truth-functional connections between the sentences in the passage.

Often there is more than one correct paraphrase of a sentence. For example, in paraphrasing both sentence 1 and sentence 2, we could have used 'or' instead of 'and'. For sentence 1 we would then have

If it is not the case that either the Americans win the race or the Canadians win the race then the British win the race.

Here the symbolization is

$\sim (M \vee N) \supset R$

Recall that 'neither . . . nor . . .' after paraphrasing will be symbolized by a sentence of the form

$\sim \mathbf{P} \,\&\, \sim \mathbf{Q}$ or $\sim (\mathbf{P} \vee \mathbf{Q})$

and 'not both . . . and . . .' by a sentence of the form

$\sim (\mathbf{P} \,\&\, \mathbf{Q})$ or $\sim \mathbf{P} \vee \sim \mathbf{Q}$

Further examples will help illustrate.

3. The Canadians will win if both the other major competitors do not have good luck.

4. The Canadians will win if either of the other major competitors does not have good luck.
5. The Canadians will win if not both of the other major competitors have good luck.

These are best paraphrased as

3a. If (both it is not the case that the Americans have good luck and it is not the case that the British have good luck) then the Canadians win the race.
4a. If (either it is not the case that the Americans have good luck or it is not the case that the British have good luck) then the Canadians win the race.
5a. If it is not the case that (both the Americans have good luck and the British have good luck) then the Canadians win the race.

In *SL* these become

3b. $(\sim A \,\&\, \sim B) \supset N$
4b. $(\sim A \vee \sim B) \supset N$
5b. $\sim (A \,\&\, B) \supset N$

Sentences 4a and 5a are equivalent, as are 4b and 5b. To say that either one or the other of the major competitors does not have good luck is to say only that they will *not both* have good luck. Where 'not' goes in relation to 'both' is important, as we shall see if we compare sentences 3 and 5. The phrase 'both . . . not . . .' means that each of the two things in question does not have the property in question. But the phrase 'not both' means only that at least one of those two things does not have the property in question.

Here are two more examples:

6. The Americans will win unless the British have good luck, in which case the British will win.
7. A major tradition will be broken if but only if no major competitor wins.

In sentence 6 the phrase 'in which case' is to be understood as 'in case the British have good luck'. The proper paraphrase is thus

6a. Both if it is not the case that the British have good luck then the Americans win the race and if the British have good luck then the British win the race.

This is symbolized as

6b. $(\sim B \supset M) \ \& \ (B \supset R)$

In paraphrasing sentence 7 we need only remember that there are exactly three major competitors: the Americans, the British, and the Canadians.

7a. A major tradition will be broken if and only if it is not the case that [either (either the Americans win the race or the British win the race) or the Canadians win the race].

In symbols this becomes

7b. $T \equiv \sim [(M \vee R) \vee N]$

Sometimes sentences containing such quantity terms as 'at least', 'at most', and 'all' can be paraphrased as truth-functional compounds. This will be the case when the number of things or events or cases we are talking about is finite. All the following can be given truth-functional paraphrases:

8. At least one of the major competitors will have good luck.
9. Exactly one of the major competitors will have good luck.
10. At least two of the major competitors will have good luck.
11. Exactly two of the major competitors will have good luck.

Since there are three major competitors, to say that at least one of them will have good luck is equivalent to saying that either the first, the second, or the third will have good luck. So

8a. Either the Americans have good luck or (either the British have good luck or the Canadians have good luck).

And in symbols we have

8b. $A \vee (B \vee C)$

The grouping here is arbitrary. We could just as well have written '$(A \vee B) \vee C$'. But grouping is necessary, since '$A \vee B \vee C$' is not a sentence of *SL*. (The connectives of *SL* are all, except for '$\sim$', *binary* connectives; that is, each connects *two* sentences. When the parentheses are removed from sentence 8b, it is unclear which sentences the first '$\vee$' connects. So the expression is not well formed; that is, it is not a sentence of *SL*.)

Since '$\vee$' is used to capture the inclusive sense of disjunction, we have to work some to say that one and only one of the three major competitors will have good luck. One way of doing it is this:

9a. Either [both the Americans have good luck and it is not the case that (either the British have good luck or the Canadians have good luck)] or (either [both the British have good luck and it is not the case that (either the Americans have good luck or the Canadians have good luck)] or [both the Canadians have good luck and it is not the case that (either the Americans have good luck or the British have good luck)]).

The symbolic version of sentence 9 is a good deal more perspicuous than is the paraphrase:

9b. $[A \mathbin{\&} \sim (B \vee C)] \vee ([B \mathbin{\&} \sim (A \vee C)] \vee [C \mathbin{\&} \sim (A \vee B)])$

As sentence 9a illustrates, truth-functional paraphrases of complex English passages can themselves become very complex. Constructing truth-functional paraphrases is of most value when one is first learning to symbolize English sentences in *SL*. After some facility with the techniques of symbolization has been gained, the paraphrase stage can be skipped, except when there is something especially difficult or interesting about the passage being symbolized. Hence hereafter we shall sometimes omit the paraphrase stage. Sentence 10 is fairly readily symbolized as

10b. $(A \mathbin{\&} B) \vee [(A \mathbin{\&} C) \vee (B \mathbin{\&} C)]$

Sentence 11 is a repeat of sentence 10 with the additional proviso that not all the teams have good luck. One appropriate symbolization is

11b. $[(A \mathbin{\&} B) \vee [(A \mathbin{\&} C) \vee (B \mathbin{\&} C)]] \mathbin{\&} \sim [A \mathbin{\&} (B \mathbin{\&} C)]$

We can symbolize an argument using the following abbreviations:

R: The Australians raise their spinnaker.
I: The wind increases.
A: The Australians win the race.
C: The Australians capsize.
L: The Australians look foolish.
J: The Australians strike their jib.
M: The Australians reef their main.

If the Australians raise their spinnaker then if the wind doesn't increase they will win the race, but if they raise their spinnaker and the wind does increase they will lose the race and look foolish. The wind will increase and the Australians will reef their main and strike their jib, and will not raise their spinnaker. So if they don't capsize the Australians will win the race.

In symbolizing this argument we shall identify losing the race with not winning the race. In the context this is surely permissible. Here is our symbolization of the argument:

$$\frac{[R \supset (\sim I \supset A)] \ \& \ [(R \ \& \ I) \supset (\sim A \ \& \ L)] \quad [I \ \& \ (M \ \& \ J)] \ \& \ \sim R}{\sim C \supset A}$$

Our formal techniques will reveal that this argument is truth-functionally invalid.

2.2E EXERCISES

1. Paraphrase the following sentences about the performance of the French, German, and Danish teams in the next Olympics, and symbolize the paraphrases as sentences in *SL* using these abbreviations:

F: The French team will win at least one gold medal.
G: The German team will win at least one gold medal.
D: The Danish team will win at least one gold medal.
P: The French team is plagued with injuries.
S: The star German runner is disqualified.
R: It rains during most of the competition.

a. At least one of the French, German, or Danish teams will win a gold medal.
*b. At most one of them will win a gold medal.
c. Exactly one of them will win a gold medal.
*d. They will not all win gold medals.
e. At least two of them will win gold medals.
*f. At most two of them will win gold medals.
g. Exactly two of them will win gold medals.
*h. They will all win gold medals.

2. Using the abbreviations given in Exercise 1, construct idiomatic English sentences from the following sentences of *SL*.

a. $\sim F \ \& \ (\sim G \ \& \sim D)$
*b. $\sim (F \ \& \ (G \ \& \ D))$
c. $\sim (F \vee (G \vee D))$
*d. $\sim (F \vee G) \vee (\sim (G \vee D) \vee \sim (F \vee D))$
e. $(F \vee G) \vee ((G \vee D) \vee (F \vee D))$
*f. $(F \ \& \ G) \vee ((G \ \& \ D) \vee (F \ \& \ D))$
g. $F \ \& \ ((G \vee D) \ \& \sim (G \ \& \ D))$
*h. $(F \ \& \ G) \vee (F \ \& \ D)$

3. Paraphrase the following and, using the abbreviations given in Exercise 1, symbolize the resulting paraphrases as sentences in *SL*.
   a. If any of them wins a gold medal so will the other two.
   *b. The French will win a gold medal only if they are not plagued with injuries, in which case they won't win.
   c. If the star German runner is disqualified, the Germans will win a gold medal only if neither of the other two teams does.
   *d. Provided it doesn't rain during most of the competition and their star runner isn't disqualified, the Germans will win a gold medal if either of the other teams does.
   e. The Danes will win a gold medal if and only if the French are plagued with injuries and the star German runner is disqualified.
   *f. The Germans will win a gold medal only if it doesn't rain during most of the competition and their star runner is not disqualified.
   g. If the French are plagued with injuries, they will win a gold medal only if neither of the other teams does and it rains during most of the competition.
   *h. The Danes will win a gold medal unless it rains during most of the competition, in which case they won't but the other two teams will win gold medals.

4. Using the abbreviations given in Exercise 1, construct idiomatic English sentences from the following sentences of *SL*.
   a. $(S \supset \sim G) \ \& \ S$
   *b. $\sim (F \vee G) \supset D$
   c. $\sim G \equiv (D \ \& \ F)$
   *d. $(P \ \& \ S) \supset D$
   e. $[(G \supset F) \ \& \ (F \supset D)] \supset (G \supset D)$
   *f. $R \supset [(\sim F \ \& \ \sim G) \ \& \ \sim D]$
   g. $[F \vee (G \vee D)] \vee [P \vee (S \vee R)]$
   *h. $D \vee [(F \ \& \ \sim P) \vee (G \ \& \ \sim S)]$

5. Paraphrase and then symbolize the following passages, being careful to indicate the abbreviations you are using.
   a. *Robert's Rules of Order* was written by an engineer or a clergyman if it was not written by a politician. The author of *Robert's Rules of Order* was motivated to write the book by an unruly church meeting but was not a clergyman. The book's author was not a politician and could not persuade a publisher that the book would make money, forcing him to publish the book himself. *Robert's Rules of Order* was written by an engineer.
   *b. Either George doesn't have a high cholesterol level or cholesterol is trapped in the walls of his arteries. If cholesterol is trapped in his arteries, then plaque will build up and block his arteries, and with such a buildup and blockage, he is a candidate for a heart attack. Hence George is a candidate for a heart attack.
   c. Either the maid or the butler committed the murder unless the cook did it. The cook did it only if a knife was the murder weapon; moreover, if a knife was used, neither the butler nor the maid did it. The murder weapon was a knife. Therefore the cook did it.
   *d. If neither Henry nor Fred will play the lawyer, then Morris will not be upset; and moreover, if Morris will not be upset the drama will be successful. Thus the drama will get good reviews. After all, both Henry and Fred will not play

the part of the lawyer, and the drama will get good reviews if and only if the drama will be a success.

e. The candidate will win at least two of three states—California, New York, and Texas—for if the candidate is perceived as conservative, she will not win New York but will win the other two. She is perceived as conservative if her advertising campaign is effective; and she has an effective advertising campaign.

*f. Assuming Betty is the judge, Peter won't get a suspended sentence. The trial will be long unless the district attorney is brief, but the district attorney is not brief. Fred is the defense lawyer. However, if Fred is the defense lawyer, Peter will be found guilty; and if Peter will be found guilty, he will be given a sentence. Consequently after a long trial Peter will be given a sentence, which won't be suspended by the judge.

2.3 NON-TRUTH-FUNCTIONAL CONNECTIVES

As stated in Section 2.1,

> A sentential connective is used *truth-functionally* if and only if it is used to generate a compound sentence from one or more sentences in such a way that the truth-value of the generated compound is wholly determined by the truth-values of those one or more sentences from which the compound is generated, no matter what those truth-values may be.

The sentential connectives of *SL* have only truth-functional uses. Many sentential connectives of English have truth-functional uses, but many do not. And many of those that do are not always used truth-functionally.

Determining whether a particular connective is or is not being used in a truth-functional sense is a complex matter. But a good rule of thumb is this: If the connective is being used truth-functionally, we should be able to construct a truth-table that adequately characterizes that use. (This is just what we did for standard uses of the English connectives introduced in Section 2.1.) If a truth-table that adequately characterizes the use of a connective in a particular sentence cannot be constructed, then that connective is not being used truth-functionally in the sentence in question.

To see how this rule of thumb operates, consider the use of 'if . . . then . . .' in the following sentence:

> If Germany's U-boats had been able to shut off the flow of supplies to Great Britain, then Germany would have won the war.

If 'if . . . then . . .' is being used truth-functionally in this conditional, it is probably being used in the sense captured by the horseshoe of *SL*, in the sense characterized by this table:

P	Q	P ⊃ Q
T	T	T
T	F	F
F	T	T
F	F	T

The truth-functional paraphrase of the sentence would be

> If Germany's U-boats were able to shut off the flow of supplies to Great Britain then Germany won the war.

In fact, Germany's U-boats were not able to shut off the flow of supplies to Great Britain; that is, the antecedent of this material conditional is false. The material conditional is therefore true. But historians do not all think the original conditional is true. Some think it true, and some false, depending upon their appraisal of the historical evidence.

One might still argue that in the example 'if . . . then . . .' is being used in some truth-functional sense. If so, we should be able to construct a paraphrase and a truth-table that express that sense. But a little reflection will show that no rearrangement of the **T**s and **F**s in the final column will produce such a table. This is because such conditionals are claims about what would happen in certain situations, regardless of whether those specified situations actually obtain. That is, knowledge of whether the situation described by the antecedent and consequent obtain is not sufficient to determine the truth-value of such conditionals. Some of these conditionals are true when the situations described do not hold ('If Germany had won World War II, Britain would have lost' is one), and some are false ('If Germany had invaded Spain, Germany would have won World War II').

Conditionals such as we have just been discussing are called *subjunctive conditionals* (because they are in the subjunctive, rather than the indicative, mood), and 'if . . . then . . .' as used in subjunctive conditionals is not truth-functional. In this case and others in which connectives are not being used truth-functionally, the safest course is to abbreviate the compounds generated by the connectives as atomic sentences of *SL*.

But being safe has it costs. Many arguments do make use of subjunctive conditionals, and we do want to evaluate the validity of these arguments whenever it is possible to do so. Consider the case of a doctor testifying at an inquest. He claims that the deceased did not die of strychnine poisoning and, when asked by the coroner to support his claim, argues as follows:

> Had the deceased died of strychnine poisoning, there would have been traces of that poison in the body. The autopsy would have found those traces had they been there. The autopsy did not reveal any traces of strychnine. Hence the deceased did not die of strychnine poisoning.

Here the following truth-functional paraphrase seems appropriate:

> If the deceased died of strychnine poisoning, then there were traces of strychnine in the body.
>
> If there were traces of strychnine in the body, then the autopsy found traces of strychnine in the body.
>
> It is not the case that the autopsy found traces of strychnine in the body.
>
> ---
>
> It is not the case that the deceased died of strychnine poisoning.

Symbolizing this argument yields

$$\begin{array}{l} S \supset T \\ T \supset R \\ \sim R \\ \hline \sim S \end{array}$$

This symbolic argument is valid, and so is the English paraphrase of our original argument. In constructing that paraphrase, we weakened the premises but not the conclusion. A sentence **p** is *weaker* than a sentence **q** if and only if the truth of **q** guarantees the truth of **p**, but not vice versa. If **p** is weaker than **q**, **q** is *stronger* than **p**. Sentences **p** and **q** are *equivalent* if and only if **p** guarantees the truth of **q** and **q** guarantees the truth of **p**. Consequently, if the premises of the original argument are true, then so are those of the paraphrase. And, since the paraphrase is valid, its conclusion is true if its premises are. The conclusion of the paraphrase is merely a rewording of the conclusion of the original argument. Hence, if the premises of the original argument are true, the conclusion of that argument is also true. That is, the original argument is also valid.

Here is another argument using subjunctive conditionals:

> If Hitler had kept his treaty with Stalin, he would have defeated England. Hitler did not keep his treaty with Stalin. Therefore, if Hitler had kept his treaty with Stalin, he would have freed all the Jews and disbanded the SS.

Suppose we construct the following truth-functional paraphrase:

> If Hitler kept his treaty with Stalin then Hitler defeated England.
> It is not the case that Hitler kept his treaty with Stalin.
>
> ---
>
> If Hitler kept his treaty with Stalin, then both Hitler freed all the Jews and Hitler disbanded the SS.

This paraphrase can be readily symbolized as

$$\begin{array}{l} K \supset E \\ \sim K \\ \hline K \supset (F \,\&\, D) \end{array}$$

Here the conclusion is equivalent to '$\sim K \vee (F \,\&\, D)$' and hence accurately symbolizes only 'Either Hitler did not keep his treaty with Stalin or Hitler did free all the Jews and Hitler disbanded the SS'. This claim does validly follow from the second premise of the argument. So our paraphrase is valid (as, of course, is the symbolic version of it). But the original English argument is clearly invalid. What has happened is that in paraphrasing that argument we made the conclusion, which is a subjunctive conditional, a material conditional. And if we weaken a conclusion in constructing a truth-functional paraphrase, there can be no guarantee that the symbolic argument we obtain by symbolizing that paraphrase will be valid *only if* the original English argument is valid. It may well be impossible for certain premises to be true and a weak conclusion false, where it is *not* impossible for those same premises to be true and a stronger conclusion false.

Although it may appear from the examples so far that truth-functional paraphrases will always be equivalent to or weaker than the original passages, this is not always the case. Sometimes the truth-functional paraphrase will be stronger. Consider this argument:

> It is not the case that if astronauts were to travel to Venus, they would find the surface of the planet hospitable. Hence astronauts travel to Venus.

Here the premise is the negation of a subjunctive conditional. Because the surface conditions of Venus are most unpleasant, astronauts would not find the planet hospitable if they traveled there. Therefore the premise is true. The conclusion of the argument is false because astronauts do not travel to Venus. The argument is invalid. But now consider a truth-functional paraphrase of the argument:

> It is not the case that if astronauts travel to Venus, then astronauts find the surface of the planet hospitable.
>
> ---
>
> Astronauts travel to Venus.

This paraphrased argument is valid. Suppose the conclusion is false. The conclusion, 'Astronauts travel to Venus', is also the antecedent of the embedded conditional in the premise. If the antecedent of a truth-functional conditional is false, then the conditional is true. And, if the conditional is true, its negation is false. Therefore, whenever the conclusion is false, the

premise will be false as well. Hence the paraphrased argument and its symbolization

$$\frac{\sim (V \supset H)}{V}$$

are valid!

Why is the original argument invalid but its truth-functional paraphrase and symbolization valid? The answer is that, because subjunctive conditionals are usually stronger than material conditionals, negated subjunctive conditionals are generally weaker than their truth-functional counterparts. Because the truth-functional premise is stronger, it can support a conclusion the original premise cannot. In such cases the validity of a truth-functionally paraphrased argument and its symbolization will not establish the validity of the original argument containing the negated subjective conditional.

In view of these examples a further guideline for paraphrasing and symbolizing non-truth-functional compounds is in order:

> 6. The safest policy in dealing with non-truth-functional compounds is to paraphrase them as single sentences. In constructing a paraphrase for an argument, if non-truth-functional compounds are paraphrased as truth-functional compounds, be sure that the paraphrased premises are equivalent to or weaker than the original premises and that the paraphrased conclusion is equivalent to or stronger than the original conclusion.[2]

There are many connectives of English that have no truth-functional senses. One such connective is 'before'. When placed between two English sentences, 'before' does generate a further sentence (though sometimes an awkward one). From the sentences 'Nixon was elected president' and 'Bush's son was elected president' we can in this way obtain

> Nixon was elected president before Bush's son was elected president.

This compound is true. But writing 'before' between two true sentences does not always produce a true sentence. A case in point is

> Nixon was elected president before Kennedy was elected president.

This compound is false though the sentences from which it is generated are both true. Reflection should show that there is no truth-functional use of

[2]For this policy, which differs from the one we proposed in earlier editions, we are indebted to David Sherry. In his article "Note on the Scope of Truth-Functional Logic," (*Journal of Philosophical Logic*, 28 [1999], 527–528). Sherry explains why our earlier policy was inadequate.

'before' because there is no use of 'before' in which the truth-value of

p before **q**

is determined, given only that **p** and **q** are both true. Similar considerations will show that 'after', 'when', and 'because' lack truth-functional senses in English.

There are also unary connectives of English that operate only non-truth-functionally. 'It is well known that' is one such connective. There is no use of this connective in which knowing only the truth-value of **p** always allows one to calculate the truth-value of

It is well known that **p**.

For example, both 'Cleveland is a city in Ohio' and 'Arcadia is a town in Ohio' are true. And, though 'It is well known that Cleveland is a city in Ohio' is true, 'It is well known that Arcadia is a town in Ohio' is false. Such considerations show that this unary connective has no truth-functional use. Similar reasoning will show that such other unary connectives as 'necessarily', 'probably', 'possibly', 'it is alleged that', and 'many people fear that' have no truth-functional senses.

Such expressions as 'Tom believes that', 'Tom knows that', and 'Tom hopes that' can be attached to sentences to generate further sentences. But sentences generated in this way are not truth-functionally compound sentences. For example, 'Paris is in France' is true, but knowing this does not allow us to calculate the truth-value of

Tom believes that Paris is in France.

For all we know, Tom may believe that Paris is in Belgium, not France. Tom, like most of us, has some true beliefs and some false beliefs.

We have yet to consider the rather special case of a non-truth-functional connective generating a compound sentence from sentences that are themselves truth-functionally compound. For example,

Either Mary is late or the clock is wrong

is clearly a truth-functionally compound sentence. But

Tom believes that either Mary is late or the clock is wrong

is not. Nor should it be paraphrased as a truth-functional compound—for example,

Either Tom believes that Mary is late or Tom believes that the clock is wrong.

Tom may well believe the disjunction about Mary and the clock without believing either disjunct, just as one might believe that one will either pass or fail a given course, without either believing that one will pass or believing that one will fail. (We can often reasonably predict that one or the other of two events will happen, without being able to predict *which* one will happen.)

Similarly

> Probably the coin will come up heads or tails

cannot fairly be paraphrased as

> Either the coin will probably come up heads or the coin will probably come up tails.

In fact, if the coin is a fair coin the odds are very high that it will come up either heads or tails (that it will not stand on edge). But the odds that it will come up heads are slightly less than one in two, as are the odds that it will come up tails (it just might stand on edge). So the truth-functional paraphrase is false, even though the claim it allegedly paraphrases is true.

But now consider

> Probably Alice and Tom will both pass the course.

It certainly seems appropriate to paraphrase this sentence as

> Both probably Alice will pass the course and probably Tom will pass the course.

If a conjunction is probable, then each conjunct is probable. But a disjunction can be probable without either disjunct alone being probable. The same reasoning holds for such sentence-compounding expressions as 'necessarily' and 'certainly'. Roughly speaking, when either 'necessary' or 'probably' is attached to a sentence that can be paraphrased as a conjunction, the result can itself be paraphrased as a conjunction; but this is not the case when one of these terms is attached to a sentence that is a disjunction or to other kinds of truth-functional compounds.

2.3E EXERCISES

1. Decide which of the following sentences are truth-functional compounds, and explain why the remaining sentences are not. Symbolize all the sentences in *SL*.
   a. It's possible that every family on this continent owns a television set.
   *b. Rocky knows who will arrive on the train or George knows.
   c. Necessarily, the coin will come up heads or tails.
   *d. Tamara won't be visiting tonight because she is working late.

e. Although Tamara won't stop by, she has promised to phone early in the evening.
*f. If the defendant had originally pleaded guilty, the trial would have lasted twice as long.
g. John believes that our manuscript has been either lost or stolen.
*h. John believes that our manuscript has been stolen, and Howard believes that it has been lost.
i. The defendant relented only after much testimony was discredited.

2. Symbolize the following arguments in *SL*, being sure to state the abbreviations you are using.

a. The murder was committed by the maid only if she believed her life was in danger. Had the butler done it, it would have been done silently and the body would not have been mutilated. As a matter of fact it was done silently; however, the maid's life was not in danger. The butler did it if and only if the maid failed to do it. Hence the maid did it.
*b. If this piece of metal is gold, then it has atomic number 79. Nordvik believes this piece of metal is gold. Therefore Nordvik believes this piece of metal has atomic number 79.
c. If Charles Babbage had had the theory of the modern computer and had had modern electronic parts, then the modern computer would have been developed before the beginning of the twentieth century. In fact, although he lived in the early nineteenth century, Babbage had the theory of the modern computer. But he did not have access to modern electronic parts, and he was forced to construct his computers out of mechanical gears and levers. Therefore, if Charles Babbage had had modern electronic parts available to him, the modern computer would have been developed before the beginning of the twentieth century.

2.4 THE SYNTAX OF *SL*

Symbolic languages have a precision that everyday languages lack and that facilitates examination of the logical properties of sentences and arguments. We have already seen a large sample of sentences of *SL*. In this section a precise specification of the expressions of *SL* will be given. To ensure that our discussion of *SL* is as clear as possible, it will be helpful to draw some distinctions that are usually neither formulated nor observed in everyday language.

OBJECT LANGUAGE AND METALANGUAGE, USE AND MENTION

We have been talking about the language *SL* in this chapter. When we talk about a language, we call that language the **object language.** In this text *SL* is an object language, and English is the metalanguage used to discuss it. A **metalanguage** is a language used to discuss or describe some object language. The distinction between object language and metalanguage is a relative one. If we talk about the German language in English, German is the object language and English the metalanguage; if we talk about the English language in German, then English is the object language and German the metalanguage.

Ordinarily we employ words and expressions to talk about something other than those words themselves. But occasionally we do want to talk about expressions themselves, and we must use words to do so. For instance, in the sentence

Minnesota was the thirty-second state admitted to the Union

the word 'Minnesota' is being used to designate a political subdivision of the United States. On the other hand, in the sentence

'Minnesota' is an Indian word

the word 'Minnesota' itself is under discussion. When a word or expression is being talked about, we say that that word or expression is being *mentioned* rather than *used*. One way to mention an expression is to use a name of that expression, and the standard way to form the name of an expression is to enclose that expression in single quotation marks. Throughout this text we use this method of forming the names of expressions. Thus in

'Saratoga' contains four syllables

the word 'Saratoga' is mentioned, not used. Omitting the single quotation marks produces a false sentence:

Saratoga contains four syllables.

Saratoga is a city, not a word; it contains buildings and people, not syllables.

In discussing the object language SL, we often need to refer to, that is, to mention, specific expressions. We do so by using names of those expressions. One way to form the name of an expression is to enclose that expression in single quotation marks. The sentence

'$\sim$ B' is a negation

is about the expression of SL enclosed within single quotation marks. We also mention expressions by displaying them. The expression

$(A \vee B)$

is a sentence of SL and is mentioned here by being displayed.

Note that the names of expressions in the language we are talking about do not themselves have to be part of that language. In fact, the *names* of expressions of SL are not part of the language SL. So an expression like

'$(A \vee B)$'

is not an expression of SL although the expression named

$(A \vee B)$

is an expression *SL*. This is because single quotation marks are not part of the vocabulary of the language *SL*. We use names of expressions of *SL* in order to talk about those expressions; hence in this text these names are part of the metalanguage that we are using.

METAVARIABLES

Besides naming specific expressions of *SL*, we sometimes want to talk about these expressions more generally. For this purpose we use **metalinguistic variables,** or **metavariables** for short. A metavariable is an expression in the metalanguage that is used to talk generally about expressions of the object language. In this text we use the boldface letters '**P**', '**Q**', '**R**', and '**S**' as metavariables that range over the expressions of our symbolic languages. (We used these metavariables in Section 2.1 in giving the characteristic truth-tables for the truth-functional connectives of *SL*.)

When we say

> If '~ (H ∨ I)' is an expression of *SL* consisting of a tilde followed by a sentence of *SL*, then '~ (H ∨ I)' is a negation

we are making a claim about a specific sentence of *SL*. But by using metavariables we can talk generally about expressions of *SL*. Thus we may write

> If **P** is an expression of *SL* consisting of a tilde followed by a sentence of *SL*, then **P** is a negation.

Here '**P**' is a metavariable that ranges over (is used to talk about) expressions of the object language. The displayed sentence means: Every expression of *SL* that consists of a tilde followed by a sentence is a *negation*. The displayed sentence is not about the metavariable '**P**', for '**P**' is not an expression of *SL*. Rather, the sentence is about all the *values* of **P**, that is, all those expressions that *are* expressions of *SL*. (When we want to talk about a metavariable, that is, to mention a metavariable, we place that metavariable in single quotation marks.)

THE LANGUAGE SL

We are now in a position to provide a rigorous definition of the sentences of the language *SL*. This is done in two steps: The vocabulary of *SL* is specified, and then the grammar is specified. The specification of the vocabulary involves stating what the basic expressions of *SL* are. These are like the words and punctuation marks of English, in the sense that the items in the vocabulary of basic expressions of *SL* are the building blocks from which all sentences of *SL* are generated. The difference is that in *SL* we do not have words and punctuation marks; rather, we have sentence letters, truth-functional connectives, and punctuation marks. The sentence letters are capitalized Roman letters

(nonboldface) with or without positive-integer subscripts:

$$A, B, C, \ldots, A_1, B_1, C_1, \ldots, A_2, B_2, C_2, \ldots$$

Note that a capitalized Roman letter with a numerical subscript counts as *one* sentence letter, so 'A_1' is a *single* sentence letter. The connectives of *SL* are the five truth-functional connectives:

$$\sim \; \& \; \vee \; \supset \; \equiv$$

The connective '~' is a *unary* connective; the others are *binary* connectives. The punctuation marks consist of the left and right parentheses:

$$(\quad)$$

Other expressions of *SL* are formed by writing one basic expression after another. But, just as the expression 'Some and vanity men will left' is not a sentence of the English language even though it is formed entirely of English words, so there are expressions that consist entirely of basic expressions of *SL* but are not themselves sentences of *SL*. We specify the grammar of *SL* by specifying what expressions of *SL* count as sentences of *SL*. The sentences of *SL* are defined as follows:

1. Every sentence letter is a sentence.
2. If **P** is a sentence, then $\sim$ **P** is a sentence.[5]
3. If **P** and **Q** are sentences, then (**P** & **Q**) is a sentence.
4. If **P** and **Q** are sentences, then (**P** $\vee$ **Q**) is a sentence.

---

[5]The expression '$\sim$ **P**' is a hybrid insofar as the connective '$\sim$' belongs to the object language *SL*, whereas the metavariable '**P**' does not. We use the expression

$\sim$ **P**

as an expression of our metalanguage to stand for any sentence of *SL* that consists of a tilde followed by a sentence of *SL*. Similarly

(**P** $\vee$ **Q**)

is a metalinguistic expression that we use to stand for any sentence of *SL* that consists of the following sequence of expressions: a left parenthesis, a sentence of *SL*, a wedge, a sentence of *SL*, a right parenthesis.

In such contexts we do not place single quotes around these metalinguistic expressions because we want to talk about sentences of *SL* rather than about the metalinguistic expressions. That is,

'$\sim$ **P**' is a sentence of *SL*

says *falsely* that the metavariable '**P**' preceded by a tilde is a sentence of *SL*. When we do place single quotes around an expression containing a metavariable, it is because we want to talk about that expression, not about a sentence of *SL*.

We adopt the following conventions: Whenever we use expressions consisting of both metavariables and expressions of *SL*, we let the expressions of *SL* occurring therein function as their own names, while the metavariables continue to function as metavariables (not as their own names). Thus each symbol that occurs in such an expression is being used to designate some expression(s) of *SL*. Moreover the order of the symbols in such an expression indicates the order of the symbols in the object language sentences that the expression stands for. (We shall observe the same conventions in the second half of this book when we discuss the language *PL*.)

5. If **P** and **Q** are sentences, then (**P** ⊃ **Q**) is a sentence.
6. If **P** and **Q** are sentences, then (**P** ≡ **Q**) is a sentence.
7. Nothing is a sentence unless it can be formed by repeated application of clauses 1–6.

The sentences specified by the first clause—the sentence letters of *SL*—are the **atomic sentences** of *SL*. Clauses 1–6 specify how sentences are built up from shorter sentences. The final clause specifies that only expressions that can be formed in accordance with clauses 1–6 are sentences. This is an example of a *recursive* definition, in which complex cases are defined in terms of simpler ones.

The definition provides the basis for an **effective** method of determining whether an expression is a sentence. This means that we can determine in a finite number of mechanical steps whether an expression is a sentence. We may show that an expression is a sentence by beginning with the sentence letters that occur in the expression and continually using the clauses of the definition until we have generated the sentence in question. To illustrate this, we shall use this definition to show that '(~ B & (~ B ∨ A))' is a sentence. By clause 1, 'A' and 'B' are sentences. By clause 2, '~ B' is a sentence. By clause 4, '(~ B ∨ A)' is a sentence. Finally, by clause 3, '(~ B & (~ B ∨ A))' is a sentence.

The following expressions are not sentences of *SL*:

(B ∨ C ∨ D)
~ & A
(BC ⊃ D)
(B ⊂ (C ∨ D))
(p ≡ q)
(((A & B) & (C ∨ D))

The reasons in each case are as follows:

'(B ∨ C ∨ D)' needs another pair of parentheses because it contains two binary connectives.

'& A' is not a sentence since '&' is a *binary* connective; so '~ & A' cannot be a sentence.

'(BC ⊃ D)' contains two consecutive sentence letters, but no rule allows us to form a sentence in which sentence letters appear consecutively.

'(B ⊂ (C ∨ D))' is not a sentence because '⊂' is not an expression of *SL*.

'(p ≡ q)' contains symbols that are not expressions of *SL*—the two lowercase letters.

'(((A & B) & (C ∨ D))' contains more left parentheses than right, and the clauses that introduce parentheses introduce them in pairs.

We adopt the convention that the *outermost* parentheses of a sentence may be dropped whenever that sentence occurs by itself (when it is not part of another sentence). We followed this convention in earlier sections of this chapter. So we may write 'A ⊃ (B & C)' instead of '(A ⊃ (B & C))', but we may *not* write '~ B ∨ C' instead of '~ (B ∨ C)'. The second sentence is a negation; the first is not. Our convention also covers the *outermost* parentheses of metalinguistic expressions ranging over sentences of *SL*: for example, we write '**P** ∨ **Q**' instead of '(**P** ∨ **Q**)'. Finally we adopt the convention, for both sentences of *SL* and metalinguistic expressions, that brackets may be used in place of parentheses. Thus '(A ∨ B) & C' may be written as '[A ∨ B] & C'.

In this section we have been discussing *SL* syntactically. The **syntactical** study of a language is the study of the expressions of the language and the relations among them, without regard to possible interpretations of these expressions. Thus, for example, we have defined sentences of *SL* only in terms of expressions of *SL*; nowhere in the definition are the possible interpretations of the expressions mentioned. Of course, we have certain interpretations of these expressions in mind. We intend the connective '&' to symbolize the English connective 'and' in its truth-functional sense, the sentence letters to abbreviate sentences of English, and so on. But we could have presented this syntactic discussion of *SL* without regard to the possible interpretations of the expressions. When we specify and investigate interpretations of the expressions of a language, we are looking at the **semantics** of the language. For instance, the specification of the characteristic truth-tables for the truth-functional connectives is part of the specification of a *semantics* for *SL*.

Before closing this section, we shall introduce four more syntactic concepts: the **main connective** of a sentence and the **immediate sentential components, sentential components,** and **atomic components** of a sentence. These are defined in terms of the specification of sentences as follows:

1. If **P** is an atomic sentence, **P** contains no connectives and hence does not have a main connective, **P** has no immediate sentential components.
2. If **P** is of the form ~ **Q**, where **Q** is a sentence, then the main connective of **P** is the tilde that occurs before **Q**, and **Q** is the immediate sentential component of **P**.
3. If **P** is of the form **Q** & **R**, **Q** ∨ **R**, **Q** ⊃ **R**, or **Q** ≡ **R**, where **Q** and **R** are sentences, then the main connective of **P** is the connective that occurs between **Q** and **R**, and **Q** and **R** are the immediate sentential components of **P**.

The *sentential components* of a sentence include the sentence itself, its immediate sentential components, and the sentential components of its immediate sentential components. The *atomic components* of a sentence are all the sentential components that are atomic sentences.

2.4E EXERCISES

1. Which of the following are true and which are false?
a. Copper is copper.
*b. 'Copper' is the name of copper.
c. The chemical symbol 'Cu' names 'copper'.
*d. 'Copper' is copper.
e. Copper is the name of copper.
*f. Some coins are made of copper.
g. 'Copper' is a metal.

2. In each of the following sentences, 'Deutschland' is either used or mentioned. Indicate where that word is being used or mentioned and explain how this is being done.
a. The only German word mentioned in the instructions to these exercises contains eleven letters.
*b. Some people think Deutschland and Germany are two different countries, but actually 'Deutschland' is the German name of Germany.
c. The German name of Germany is mentioned several times in these examples but only is used once.
*d. 'Deutschland' is 'Deutschland'.
e. The word 'Deutschland' is not being used in this sentence.
*f. Deutschland is the German name of Germany.

3. Which of the following are sentences of *SL* and which are not? For those that are not explain why they are not.

a. B & Z
*b. & H
c. ~ O
*d. M ~ N
e. J ⊃ (K ⊃ (A ∨ N))
*f. **P** ∨ **Q**
g. (I ∨ [T & E])
*h. (U & C & ~ L)
i. (F ≡ K) ⊃ [M ∨ K]
*j. [(G ∨ E) ⊃ (~ H & (K ∨ B)]

4. For each of the following sentences, specify the main connective and the immediate sentential components. Then list all the sentential components, indicating which ones are atomic.

a. ~ A & H
*b. ~ (A & H)
c. ~ (S & G) ∨ B
*d. K ⊃ (~ K ⊃ K)
e. (C ≡ K) ∨ (~ H ∨ (M & N))
*f. M ⊃ [~ N ⊃ ((B & C) ≡ ~ [(L ⊃ J) ∨ X])]

5. Which of the following sentences are of the form ~ **P** ⊃ **Q**? In each case justify your answer.

a. A ⊃ B
*b. ~ A ⊃ B
c. ~ A ⊃ ~ B
*d. ~ ~ A ⊃ B
e. ~ (A ⊃ B)
*f. ~ ~ A ⊃ ~ B
g. ~ (~ A ⊃ B)
*h. ~ ~ (A ⊃ B) ⊃ (C ⊃ D)
i. ~ (A ∨ ~ B) ⊃ ~ (C & ~ D)
*j. ~ (A ≡ B) & (~ C ⊃ D)

6. Which of the following characters can occur immediately to the left of '~ ' in a sentence of *SL*? When one can so occur, give a sentence of *SL* in which it does; when it cannot so occur, explain why. Which of these characters could occur immediately to the right of 'A' in a sentence of *SL*? When one can so

occur, give a sentence of *SL* in which it does; when it cannot so occur, explain why.

a. H
*b. &
c. (
*d.)
e. [
*f. ~

GLOSSARY

TRUTH-FUNCTIONAL USE OF A CONNECTIVE: A sentential connective is used truth-functionally if and only if it is used to generate a compound sentence from one or more sentences in such a way that the truth-value of the generated compound is wholly determined by the truth-values of those one or more sentences from which the compound is generated, no matter what those truth-values may be.

Chapter 3

SENTENTIAL LOGIC: SEMANTICS

3.1 TRUTH-VALUE ASSIGNMENTS AND TRUTH-TABLES FOR SENTENCES

In Chapter 1 we introduced logical concepts such as logical truth and deductive validity and used them to evaluate sentences and arguments stated in English. In this chapter we shall develop formal tests for truth-functional versions of the concepts introduced in Section 1.4—specifically truth-functional truth, falsity, and indeterminacy; truth-functional consistency; truth-functional entailment; and truth-functional validity. All these concepts fall within the realm of *semantics:* They concern the truth-values and truth-conditions of sentences. Before defining these truth-functional concepts for sentences and arguments of *SL*, our first task is to specify how truth-values and truth-conditions for sentences of *SL* are determined.

Every sentence of *SL* can be built up from its atomic components in accordance with the definition of sentences. Similarly the truth-value of a sentence of *SL* is determined completely by the truth-values of its atomic components in accordance with the characteristic truth-tables for the connectives. We repeat the characteristic truth-tables here:

P	~ P
T	F
F	T

P	Q	P & Q
T	T	T
T	F	F
F	T	F
F	F	F

P	Q	P ∨ Q
T	T	T
T	F	T
F	T	T
F	F	F

P	Q	P ⊃ Q
T	T	T
T	F	F
F	T	T
F	F	T

P	Q	P ≡ Q
T	T	T
T	F	F
F	T	F
F	F	T

These tables tell us how to determine the truth-value of a truth-functionally compound sentence given the truth-values of its immediate sentential components. And, if the immediate sentences of a truth-functionally compound sentence are themselves truth-functionally compound, we can use the information in the characteristic truth-tables to determine how the truth-value of each immediate component depends on the truth-values of *its* immediate components, and so on until we arrive at atomic components.

The truth-values of atomic sentences are fixed by **truth-value assignments**:

> A truth-value assignment is an assignment of truth-values (**T**s or **F**s) to the atomic sentences of *SL*.

Truth-value assignment is the basic semantic concept of *SL*. Intuitively each truth-value assignment gives us a description of a way the world *might* be, for in each we consider a combination of truth-values that atomic sentences might have. We assume that the atomic sentences of *SL* are truth-functionally independent—that is, that the truth-value assigned to one does not affect the truth-value assigned to any other. For generality we stipulate that a truth-value assignment must assign a truth-value to every atomic sentence of *SL*. Thus a truth-value assignment gives a *complete* description of a way the world might be. It tells us of each atomic sentence of *SL* whether or not that sentence is true. The truth-values of truth-functionally compound sentences of *SL* are determined uniquely and completely by the truth-values of their atomic components. Because every atomic sentence of *SL* is assigned a truth-value by a truth-value assignment, it follows that every truth-functionally compound sentence also has a truth-value on each truth-value assignment.

A truth-table for a sentence of *SL* is used to record its truth-value on each truth-value assignment. Because each truth-value assignment assigns truth-values to an infinite number of atomic sentences (*SL* has infinitely many atomic sentences), we cannot list an entire truth-value assignment in a truth-table. Instead, we list all the possible combinations of truth-values that the sentence's atomic components may have on a truth-value assignment. As an example here is the beginning of a truth-table for '~ B ⊃ C':

B	C	~ B ⊃ C
T	T	
T	F	
F	T	
F	F	

The atomic components of the sentence are 'B' and 'C', and there are four combinations of truth-values that these components might have, as indicated in the four rows of the table. (Rows in truth-tables go from left to right; columns go from top to bottom.) Each row represents an infinite number of truth-value assignments, namely, all the truth-value assignments that assign to 'B' and 'C' the values indicated in that row. Since the truth-value of '~ B ⊃ C' on a truth-value assignment depends upon only the truth-values that its atomic components have on that assignment (and not, say, on the truth-value of 'D'), the four combinations that we have displayed will allow us to determine the truth-value of '~ B ⊃ C' on any truth-value assignment. That is, no matter which of the infinitely many truth-value assignments we might select, that truth-value assignment will assign one of the four pairs of truth-values displayed in the table to 'B' and 'C'.

The first step in constructing a truth-table for a sentence **P** of *SL* is to determine the number of different combinations of truth-values that its atomic components might have. There is a simple way to do this. Consider first the case in which **P** has one atomic component. There are two different combinations of truth-values that the single atomic component may have: **T** and **F**. Now suppose that **P** is a sentence with two atomic components. In this case there are four combinations of truth-values that the atomic components of **P** might have, as we have seen in the case of '~ B ⊃ C' above.

If **P** has three atomic components, there are eight combinations of truth-values that its atomic components might have. To see this, suppose we were to add a third sentence letter to the truth-table for '~ B ⊃ C':

A	B	C	(~ B ⊃ C) & (A ≡ B)
	T	T	
	T	F	
	F	T	
	F	F	

What truth-tables do we enter in the first row under 'A'? The combination of truth-values that would be displayed by entering **T** there is different from the combination that would be displayed by entering **F**. And we see that the same holds for each row. So we need to list each of the four combinations of truth-values that 'B' and 'C' may have *twice* in order to represent all combinations of truth-values for the three atomic components.

A	B	C	(~ B ⊃ C) & (A ≡ B)
T	T	T	
T	T	F	
T	F	T	
T	F	F	
F	T	T	
F	T	F	
F	F	T	
F	F	F	

Extending this reasoning, we find that every time we add a new atomic sentence to the list the number of rows in the truth-table doubles. If **P** has **n** atomic components, there are 2^n different combinations of truth-values for its atomic components.[1] (If the same sentence letter occurs more than once in **P**, we do not count each occurrence as a different atomic component of **P**. To determine the number of atomic components, we count the number of *different* sentence letters that occur in **P**.)

In constructing a truth-table, we adopt a systematic method of listing the combinations of truth-values that the atomic components of a sentence **P** might have. We first list the atomic components of **P** to the left of the vertical line at the top of the truth-table, in alphabetical order.[2]

Under the first sentence letter listed, write a column of 2^n entries, the first half of which are **T**s and the second half of which are **F**s. In the second column the number of **T**s and **F**s being alternated is half the number alternated in the first column. In the column under the third sentence letter listed, the number of **T**s and **F**s being alternated will again be half the number in the second column. We repeat this process until a column has been entered under each sentence letter to the left of the vertical line. The column under the last sentence letter in this list will then consist of single **T**s alternating with single **F**s. Thus, for a truth-table with **n** sentence letters, the first column consists of 2^{n-1} **T**s alternating with 2^{n-1} **F**s, the second of 2^{n-2} **T**s alternating with 2^{n-2} **F**s, and in general the ith column consists of 2^{n-i} **T**s alternating with 2^{n-i} **F**s. (Note that $2^0 = 1$.)

Now we can complete the rest of the truth-table for '(~ B ⊃ C) & (A ≡ B)'. We first repeat under 'A', 'B', and 'C', wherever these occur, the columns we have already entered to the left of the vertical line:

A	B	C	(~ B	⊃	C)	&	(A	≡	B)
T	**T**	**T**		**T**			**T**		**T**
T	**T**	**F**		**T**			**T**		**T**
T	**F**	**T**		**F**			**T**		**F**
T	**F**	**F**		**F**			**T**		**F**
F	**T**	**T**		**T**			**F**		**T**
F	**T**	**F**		**T**			**F**		**T**
F	**F**	**T**		**F**			**F**		**F**
F	**F**	**F**		**F**			**F**		**F**

Next we may enter the column for the component '~ B' under its main connective, the tilde. In each row in which 'B' has the truth-value **T**, '~ B' has the

[1] '2^n' is 2 if $n = 1$, 2×2 if $n = 2$, $2 \times 2 \times 2$ if $n = 3$, and so on.

[2] This is an extended sense of 'alphabetical order' since some sentence letters have subscripts. In this order all the nonsubscripted letters appear first, then all letters subscripted with '1', then all letters subscripted with '2', and so on.

truth-value **F**, and in each row in which 'B' has the truth-value **F**, '~ B' has the truth-value **T**:

A	B	C	(~ B	⊃	C)	&	(A	=	B)
T	T	T	F	T	T		T		T
T	T	F	F	T	F		T		T
T	F	T	T	F	T		T		F
T	F	F	T	F	F		T		F
F	T	T	F	T	T		F		T
F	T	F	F	T	F		F		T
F	F	T	T	F	T		F		F
F	F	F	T	F	F		F		F

The column for '~ B ⊃ C' is entered under the horseshoe:

A	B	C	(~ B	⊃	C)	&	(A	=	B)
T	T	T	F	T	T		T		T
T	T	F	F	T	F		T		T
T	F	T	T	F	T		T		F
T	F	F	T	F	F		T		F
F	T	T	F	T	T		F		T
F	T	F	F	T	F		F		T
F	F	T	T	F	T		F		F
F	F	F	T	F	F		F		F

The truth-values of the immediate components of 'A = B' for each row have been recorded, so we can now complete the column for 'A = B' in accordance with the characteristic truth-table for '=':

A	B	C	(~ B	⊃	C)	&	(A	=	B)
T	T	T	F	T	T		T	T	T
T	T	F	F	T	F		T	T	T
T	F	T	T	F	T		T	F	F
T	F	F	T	F	F		T	F	F
F	T	T	F	T	T		F	F	T
F	T	F	F	T	F		F	F	T
F	F	T	T	F	T		F	T	F
F	F	F	T	F	F		F	T	F

Remember that a material biconditional has the truth-value **T** on all truth-value assignments on which its immediate components have the same truth-value,

and the truth-value **F** on all other truth-value assignments. Finally we enter the column for '(~ B ⊃ C) & (A ≡ B)' under its main connective, the ampersand:

A	B	C	(~ B	⊃	C)	&	(A	≡	B)
						↓			
T	T	T	F	T	T	T	T	T	T
T	T	F	F	T	F	T	T	T	T
T	F	T	T	F	T	F	T	F	F
T	F	F	T	F	F	F	T	F	F
F	T	T	F	T	T	F	F	F	T
F	T	F	F	T	F	F	F	F	T
F	F	T	T	F	T	T	F	T	F
F	F	F	T	F	F	F	F	T	F

We use arrows to indicate the main connective of the sentence for which a truth-table has been constructed. Each row of the truth-table displays, underneath the arrow, the truth-value that the sentence has on every truth-value assignment that assigns to the atomic components of that sentence the truth-values displayed to the left of the vertical line.

Here is the truth-table for the sentence '[A ≡ (B ≡ A)] ∨ ~ C':

A	B	C	[A	≡	(B	≡	A)]	∨	~ C
								↓	
T	T	T	T	T	T	T	T	T	F T
T	T	F	T	T	T	T	T	T	T F
T	F	T	T	F	F	F	T	F	F T
T	F	F	T	F	F	F	T	T	T F
F	T	T	F	T	T	F	F	T	F T
F	T	F	F	T	T	F	F	T	T F
F	F	T	F	F	F	T	F	F	F T
F	F	F	F	F	F	T	F	T	T F

The column for '~ C' is constructed in accordance with the characteristic truth-table for the tilde. '~ C' has the truth-value **T** on all and only those truth-value assignments on which 'C' has the truth-value **F**, and '~ C' has the truth-value **F** on every assignment on which 'C' has the truth-value **T**. The column for '~ C' appears directly underneath the tilde. The immediate components of '(B ≡ A)' are 'B' and 'A'. The characteristic truth-value for '≡' tells us that a material biconditional has the truth-value **T** on all and only those truth-value assignments on which both of its immediate sentential components have the same truth-value (both have the truth-value **T** or both have the truth-value **F**). Thus '(B ≡ A)' has the truth-value **T** for the combinations of truth-values displayed in the first two and last two rows of the truth-table and the truth-value **F** for the other combinations.

Similarly '[A ≡ (B ≡ A)]' has the truth-value **T** on exactly those truth-value assignments on which 'A' and '(B ≡ A)' have the same truth-value. The column for '[A ≡ (B ≡ A)]' appears directly underneath its main connective, which is the first occurrence of the triple bar. '[A ≡ (B ≡ A)] ∨ – C' has the truth-value **T** on exactly those truth-value assignments on which at least one disjunct has the truth-value **T**. The disjuncts are '[A ≡ (B ≡ A)]' and '– C'. So [A ≡ (B ≡ A)] ∨ – C' has the truth-value **T** on every truth-value assignment on which either '[A ≡ (B ≡ A)]' or '– C' has the truth-value **T**. Where both disjuncts have the truth-value **F**, so does '[A ≡ (B ≡ A)] ∨ – C'. The truth-value of the entire sentence for each combination of truth-values assigned to its atomic components is written in the column directly underneath the wedge, the sentence's main connective.

Here is the truth-table for the sentence '– [(U ∨ (W ⊃ – U)) ≡ W]':

			↓								
U	W		–	[(U	∨	(W	⊃	–	U))	≡	W]
T	T		F	T	T	T	F	F	T	T	T
T	F		T	T	T	F	T	F	T	F	F
F	T		F	F	T	T	T	T	F	T	T
F	F		T	F	T	F	T	T	F	F	F

The column under the first occurrence of the tilde represents the truth-value of the entire sentence '– [(U ∨ (W ⊃ – U)) ≡ W]' for each combination of truth-values that its atomic components might have. The truth-table tells us that '– [(U ∨ (W ⊃ – U)) ≡ W]' has the truth-value **T** on those truth-value assignments on which either 'U' is assigned the truth-value **T** and 'W' is assigned the truth-value **F** or both 'U' and 'W' are assigned the truth-value **F**; the sentence is false on every other truth-value assignment.

Sometimes we are not interested in determining the truth-values of a sentence **P** for every truth-value assignment but are interested only in the truth-value of **P** on a particular truth-value assignment. In that case we may construct a shortened truth-table for **P** that records only the truth-values that its atomic components are assigned by that truth-value assignment. For example, suppose we want to know the truth-value of '(A & B) ⊃ B' on a truth-value assignment that assigns **F** to 'A' and **T** to 'B' and all the other atomic sentences of *SL*. We head the shortened truth-table as before, with the atomic components of the sentence to the left of the vertical line and '(A & B) ⊃ B' itself to the right. We list only one combination of truth-values for 'A' and 'B', namely, the truth-values they have on the assignment we are interested in:

				↓	
A	B	(A	&	B)	⊃	B
F	T	F	F	T	T	T

The truth-values of '(A & B)' and '(A & B) ⊃ B' are determined in accordance with the characteristic truth-tables, as before. Thus '(A & B)' has the truth-value **F** on this truth-value assignment, for 'A' has the truth-value **F**. Since the antecedent of '(A & B) ⊃ B' has the truth-value **F** and the consequent the truth-value **T**, '(A & B) ⊃ B' has the truth-value **T**.

We emphasize that, when we want to determine the truth-value of a sentence on a particular truth-value assignment, we do not display the full truth-value assignment in question. Truth-value assignments assign truth-values to *every* atomic sentence of *SL*. Rather, we display only the combinations of truth-values that the atomic components of the sentence in question have on the assignment. There is no loss here because the truth-value of a sentence on a truth-value assignment depends *only* upon the truth-values of its atomic components on that assignment. Conversely each row of a truth-table for a sentence gives information about infinitely many truth-value assignments. It tells us the truth-value of the sentence on every truth-value assignment that assigns to the atomic components of the sentence the combination of truth-values displayed in that row (there are infinitely many such assignments).

To review: The truth-value of a sentence **P** on a truth-value assignment is determined by starting with the truth-values of the atomic components of **P** on the truth-value assignment and then using the characteristic truth-tables for the connectives of *SL* to compute the truth-values of larger and larger sentential components of **P** on the truth-value assignment. Ultimately we determine the truth-value of the largest sentential component of **P**, namely, **P** itself. This procedure is used in the construction of a truth-table for **P**, where each row displays a different combination of truth-values for the atomic components of **P**. The truth-value of **P** for each such combination is recorded directly underneath the main connective of **P** in the row representing that combination. (If **P** is atomic, the truth-value is recorded under **P**.)

We also define the notions of being **true on a truth-value assignment** and **false on a truth-value assignment:**

> A sentence is *true on a truth-value assignment* if and only if it has the truth-value **T** on the truth-value assignment.
>
> A sentence is *false on a truth-value assignment* if and only if it has the truth-value **F** on the truth-value assignment.

3.1E EXERCISES

1. How many rows will be in the truth-table for each of the following sentences?
   a. A ≡ (~ A ≡ A)
   *b.]~ D & (B ∨ G)[⊃ [~ (H & A) ∨ ~ D]
   c. (B & C) ⊃ [B ∨ (C & ~ C)]

2. Construct truth-tables for the following sentences.

a. ~ ~ (E & ~ E)
*b. (A & B) ≡ ~ B
c. A ≡ [J ≡ (A ≡ J)]
*d. [A ⊃ (B ⊃ C)] & [(A ⊃ B) ⊃ C]
e. [~ A ∨ (H ⊃ J)] ⊃ (A ∨ J)
*f. (~ ~ A & ~ B) ⊃ (~ A ≡ B)
g. ~ (A ∨ B) ⊃ (~ A ∨ ~ B)
*h. ~ D & [~ H ∨ (D & E)]
i. ~ (E & [H ⊃ (B & E)])
*j. ~ (D ≡ (~ A & B)) ∨ (~ D ∨ ~ B)
k. ~ [D & (E ∨ F)] ≡ [~ D & (E & F)]
*l. (J & [(E ∨ F) & (~ E & ~ F)]) ⊃ ~ J
m. (A ∨ (~ A & (H ⊃ J))) ⊃ (J ⊃ H)

3. Construct shortened truth-tables to determine the truth-value of each of the following sentences on the truth-value assignment that assigns **T** to 'B' and 'C', and **F** to 'A' and to every other atomic sentence of *SL*.

a. ~ [~ A ∨ (~ C ∨ ~ B)]
*b. ~ [A ∨ (~ C & ~ B)]
c. (A ⊃ B) ∨ (B ⊃ C)
*d. (A ⊃ B) ⊃ (B ⊃ C)
e. (A ≡ B) ∨ (B ≡ C)
*f. ~ A ⊃ (B ≡ C)
g. ~ [B ⊃ (A ∨ C)] & ~ ~ B
*h. ~ [~ A ≡ ~ (B ≡ ~ [A ≡ (B & C)])]
i. ~ [~ (A ≡ ~ B) ≡ ~ A] ≡ (B ∨ C)
*j. ~ (B ⊃ ~ A) & [C ≡ (A & B)]

4. Construct a truth-table for each of the sentences in Exercise 1 in Section 2.2E.

5. Construct a truth-table for each of the sentences in Exercise 3 in Section 2.2E.

3.2 TRUTH-FUNCTIONAL TRUTH, FALSITY, AND INDETERMINACY

In Chapter 1 we introduced the concepts of logical truth, logical falsity, and logical indeterminacy. A logically true sentence of English, it will be remembered, is one that cannot possibly be false. A sentence that is logically true (or logically false) may be so on purely truth-functional grounds. For example, we may symbolize 'Either Cynthia will get a job or Cynthia will not get a job' as 'C ∨ ~ C', and the truth-table for this sentence shows that it is true on every truth-value assignment:

		↓		
C	C	∨	~	C
T	**T**	**T**	**F**	**T**
F	**F**	**T**	**T**	**F**

Thus the sentence cannot possibly be false. A sentence that is logically true on truth-functional grounds is a **truth-functionally true** sentence.

> A sentence **P** of *SL* is *truth-functionally true* if and only if **P** is true on every truth-value assignment.[5]

Since every sentence of *SL* has exactly one of the two truth-values on any truth-value assignment, it follows that a sentence **P** is truth-functionally true if and only if there is no truth-value assignment on which **P** is false.

Once the truth-table for a sentence has been constructed, it is a simple matter to determine whether the sentence is truth-functionally true. Simply examine the column of truth-values under its main connective. The sentence is truth-functionally true if and only if that column consists solely of **T**s. Since the rows of the truth-table represent *all* combinations of truth-values that may be assigned to the atomic components of the sentence by any truth-value assignment, the absence of **F**s under the sentence's main connective shows that there is no truth-value assignment on which the sentence is false.

Here is the truth-table for another truth-functionally true sentence:

				↓		
X	Z	Z	⊃	(X	∨	Z)
T	**T**	**T**	**T**	**T**	**T**	**T**
T	**F**	**F**	**T**	**T**	**T**	**F**
F	**T**	**T**	**T**	**F**	**T**	**T**
F	**F**	**F**	**T**	**F**	**F**	**F**

The column under the main connective of '$Z \supset (X \vee Z)$' contains only **T**s. Note that the immediate sentential components of a truth-functionally true sentence need not themselves be truth-functionally true.

Truth-functional falsity is also defined in terms of truth-value assignments.

> A sentence **P** of *SL* is *truth-functionally false* if and only if **P** is false on every truth-value assignment.

It follows that if **P** is truth-functionally false then there is no truth-value assignment on which **P** is true. We can show that a sentence of *SL* is truth-functionally false by constructing a truth-table for the sentence; if the column of truth-values under the sentence's main connective contains only **F**s, then the sentence is

[5] Truth-functionally true sentences are sometimes called *tautologies* or *truth-functionally valid* sentences. Truth-functionally false sentences (introduced shortly) are sometimes called *contradictions*, or *self-contradictory* sentences. Truth-functionally indeterminate sentences (also to be introduced) are sometimes called *contingent* sentences.

truth-functionally false. Here are truth-tables for two truth-functionally false sentences:

A	A	&	~ A
		↓	
T	T	F	F T
F	F	F	T F

H	K	[(H	∨	K)	⊃	~	(H	∨	K)]	&	H
										↓	
T	T	T	T	T	F	F	T	T	T	F	T
T	F	T	T	F	F	F	T	T	F	F	T
F	T	F	T	T	F	F	F	T	T	F	F
F	F	F	F	F	T	T	F	F	F	F	F

Note that the immediate sentential components of a truth-functionally false sentence need not themselves be truth-functionally false. When we negate a truth-functionally true sentence, we end up with a truth-functionally false sentence:

A	~	(A	∨	~ A)
	↓			
T	F	T	T	F T
F	F	F	T	T F

If we add another tilde to obtain '~ ~ (A ∨ ~ A)', we will have a truth-functionally true sentence again.

Although the two sentences 'A ⊃ (B ⊃ A)' and '(A ⊃ B) ⊃ A' look very much alike, one is truth-functionally true and the other is not:

A	B	A	⊃	(B	⊃	A)
		↓				
T	T	T	T	T	T	T
T	F	T	T	F	T	T
F	T	F	T	T	F	F
F	F	F	T	F	T	F

A	B	(A	⊃	B)	⊃	A
					↓	
T	T	T	T	T	T	T
T	F	T	F	F	T	T
F	T	F	T	T	F	F
F	F	F	T	F	F	F

'A ⊃ (B ⊃ A)' is true on every truth-value assignment, whereas '(A ⊃ B) ⊃ A' is not. The latter sentence is **truth-functionally indeterminate.**

> A sentence **P** of *SL* is *truth-functionally indeterminate* if and only if **P** is neither truth-functionally true nor truth-functionally false.

A truth-functionally indeterminate sentence is true on at least one truth-value assignment and false on at least one truth-value assignment. We can use a truth-table to show that a truth-functionally compound sentence is truth-functionally indeterminate by showing that the column under its main connective contains at least one **T** and at least one **F**. Every atomic sentence of *SL* is truth-functionally indeterminate. For example, the truth-table for 'H' is

H	H
T	**T**
F	**F**

'H' is true on every truth-value assignment on which it is assigned the truth-value **T**, and false on every other truth-value assignment. Truth-tables for several truth-functionally indeterminate sentences appeared in Section 3.1. Every sentence of *SL* is either truth-functionally true, truth-functionally false, or truth-functionally indeterminate.

Sometimes we can show that a sentence is not truth-functionally true or is not truth-functionally false by displaying only one row of the sentence's truth-table—that is, by constructing a shortened truth-table. Consider the sentence '(A & ~ A) ∨ ~ A'. If this sentence is truth-functionally true, then there is no truth-value assignment on which it is false. So, if we can show that the sentence is false for some combination of truth-values its atomic components might have, then we can conclude that it is not truth-functionally true. The following shortened truth-table represents such a combination:

A	(A & ~ A) ∨ ~ A
T	**T F F T F F T**

This shortened truth-table shows that the sentence '(A & ~ A) ∨ ~ A' is false on every truth-value assignment that assigns the truth-value **T** to 'A'. Note that the shortened table shows only that '(A & ~ A) ∨ ~ A' is not truth-functionally true. The table does not show whether the sentence is true on those truth-value assignments on which 'A' is assigned the truth-value **F**. If it is, then the

sentence is truth-functionally indeterminate; if not, the sentence is truth-functionally false.

Similarly we may construct a shortened truth-table in order to show that '$J \,\&\, (\sim K \vee \sim J)$' is not truth-functionally false:

J	K	J	&	(~	K	∨	~	J)
			↓					
T	F	T	T	T	F	T	F	T

This truth-table shows that the sentence is true on every truth-value assignment that assigns **T** to 'J' and **F** to 'K'. We thus know that the sentence is either truth-functionally indeterminate or truth-functionally true.

There is a systematic way to develop a shortened truth-table that shows that a sentence is true on at least one truth-value assignment or false on at least one truth-value assignment. Let's first consider the previous example, in which we wanted to show that '$J \,\&\, (\sim K \vee \sim J)$' is true on at least one truth-value assignment. We start by placing a **T** under the main connective:

J	K	J	&	(~	K	∨	~	J)
			↓					
			T					

Because the main connective is an ampersand, we know that each conjunct must be true as well:

J	K	J	&	(~	K	∨	~	J)
			↓					
		T	T			T		

Whenever we place a **T** or **F** under a sentence letter, we repeat it under all occurrences of that sentence letter:

J	K	J	&	(~	K	∨	~	J)
			↓					
T		T	T			T		T

Once we have placed a **T** under 'J', we know that we must fill in an **F** under the last tilde, since a negation is false if the negated sentence is true:

J	K	J	&	(~	K	∨	~	J)
			↓					
T		T	T			T		F T

Now we have a true disjunction with one false disjunct, so we know that the other disjunct must be true (otherwise the disjunction could not be true):

J	K	J	&	(~	K	∨	~	J)
			↓					
T		**T**	**T**	**T**		**T**	**FT**	

And if '~ K' is true, then 'K' must be false:

J	K	J	&	(~	K	∨	~	J)
			↓					
T	**F**	**T**	**T**	**T**	**F**	**T**	**FT**	

Note that we also placed an **F** under the occurrence of 'K' to the left of the vertical bar. This completes our shortened truth-table.

Now consider the earlier example, in which we wanted to show that '(A & ~ A) ∨ ~ A' is false on at least one truth-value assignment. We begin by placing an **F** under the sentence's main connective:

A	(A	&	~	A)	∨	~	A
					↓		
					F		

If a disjunction is false, both of its disjuncts must be false:

A	(A	&	~	A)	∨	~	A
					↓		
	F				**F**	**F**	

We have just recorded an **F** for '~ A', and since '~ A' occurs elsewhere in the sentence, we repeat the **F** there:

A	(A	&	~	A)	∨	~	A
					↓		
	F	**F**			**F**	**F**	

Note that we have now assigned the value **F** to one of the conjuncts of '(A & ~ A)', thus ensuring that the conjunction is false, so it won't matter if

we end up assigning the value **T** to the other conjunct. Next we note that if '~ A' is false then 'A' must be true:

A	(A & ~ A) ∨ ~ A
T	**T F F T F F T**

And this completes the shortened truth-value.

In these two examples every addition to the table is dictated by some previous truth-value that had been entered: If a conjunction is true, both conjuncts must be true; if a disjunction is false, both disjuncts must be false; a negation is true if and only if the negated sentence is false; and a component of a sentence must have the same truth-value for each of its occurrences. But sometimes we have examples where choices need to be made. For example, suppose we want to show that the sentence '(A ⊃ B) ≡ (B ⊃ A)' is not truth-functionally true. We can begin constructing a shortened truth-table with an **F** under the sentence's main connective as follows:

A B	(A ⊃ B) ≡ (B ⊃ A)
	F

At this point we have to make a choice, because there are two ways that a biconditional can be false. Either the first immediate component is true and the second false, or the first immediate component is false and the second true. There is no simple rule of thumb to follow in this case. So we'll try one of the possibilities and see where it leads:

A B	(A ⊃ B) ≡ (B ⊃ A)
	T F F

Since '(B ⊃ A)' is false, we know that 'B' must be true and 'A' false. We'll add these values in two steps. First, we have

A B	(A ⊃ B) ≡ (B ⊃ A)
F T	**T F T F F**

We also need to add the values under the other occurrences of 'A' and 'B'—but in doing so we must make sure that these values are consistent with the

assignment of **T** to the conditional '(A ⊃ B)':

A	B	(A	⊃	B)	↓ =	(B	⊃	A)
F	**T**	**F**	**T**	**T**	**F**	**T**	**F**	**F**

Fortunately they are: A conditional with a false antecedent and a true consequent is itself true. So we have successfully completed the shortened table.

It turns out that we could have assigned **F** to the first immediate component of the biconditional and **T** to the second and produced another shortened truth-table representing a different set of truth-value assignments on which the biconditional is false. But sometimes, when we have a choice, one possible way of assigning truth-values won't work while another one will. Suppose, for example, that we want to show that the sentence '(A ⊃ B) ⊃ (B ⊃ ~ A)' is not truth-functionally false—that is, that there is at least one truth-value assignment on which it is true. We start with

A	B	(A	⊃	B)	↓ ⊃	(B	⊃	~ A)
					T			

There are three ways in which a conditional can be true: Both the antecedent and consequent are true, or the antecedent is false and the consequent is true, or the antecedent is false and the consequent is false. We might try the first case first:

A	B	(A	⊃	B)	↓ ⊃	(B	⊃	~ A)
		T			**T**			**T**

We now have two true conditionals whose immediate components do not have truth-values. We'll work with the first one, and again, let's make its antecedent true and its consequent true:

A	B	(A	⊃	B)	↓ ⊃	(B	⊃	~ A)
		T	**T**	**T**	**T**			**T**

Filling in **T** under each 'A' and 'B'—because 'A' and 'B' have each been assigned the truth-value **T**—we get

A	B	(A	⊃	B)	↓ ⊃	(B	⊃	~ A)
T	**T**	**T**	**T**	**T**	**T**	**T**	**T**	**T**

Now we must put **F** under the tilde:

A	B	(A	⊃	B)	⊃	(B	⊃	~	A)
T	T	T	T	T	T	T	T	F	T

FAILURE!

The problem is that the conditional '(B ⊃ ~ A)' cannot be true if 'B' is true and '~ A' is false—there is no such truth-value assignment.

But we must not conclude that the sentence *cannot* be true. All we conclude is that we haven't come up with a way of assigning truth-values that will make it true. We can go back to

A	B	(A	⊃	B)	⊃	(B	⊃	~	A)
			T		T		T		

and try another way to make the conditional '(A ⊃ B)' true—say, by making 'A' false and 'B' true. This yields

A	B	(A	⊃	B)	⊃	(B	⊃	~	A)
F	T	F	T	T	T	T	T		F

and we can fill in a **T** under the tilde:

A	B	(A	⊃	B)	⊃	(B	⊃	~	A)
F	T	F	T	T	T	T	T	T	F

Note that this time the conditional '(B ⊃ ~ A)' will be true since both of its immediate components are, so we have correctly produced a shortened truth-table. But even if this hadn't worked, there are still other possibilities, including trying to make the entire sentence true by a different assignment of truth-values to its immediate components.[4]

Of course, we may fail even when we try all the possibilities—which means that, although we thought a sentence might be true (or false) on some truth-value assignment, we were incorrect. The sentence is, in fact, truth-functionally false (or true), so there is no such assignment. Here's a simple

[4] Sometimes we have to try every possibility before coming up with a correct shortened truth-table (or concluding that there is no such table). The problem in constructing a shortened truth-table to show that a sentence can be true or that it can be false is one of a class of problems known to theoreticians as "NP-complete problems." These are problems for which the only known solutions guaranteed to produce a correct result are solutions that require us, in the worst case, to try every possibility.

example: We'll try to produce a shortened truth-table with an assignment of truth-values that makes the sentence 'A ⊃ A' false:

A	A ⊃ A
	F

If the conditional is false, the antecedent must be true and the consequent false:

A	A ⊃ A
	T F F

FAILURE!

We failed because 'A' cannot have two different truth-values on the same truth-value assignment. Here we have, in fact, tried all the possibilities for making the conditional false (the antecedent must be true and the conclusion must be false)—unsuccessfully. That's as it should be, since the sentence is truth-functionally true.

3.2E EXERCISES

1. Determine whether each of the following sentences is truth-functionally true, truth-functionally false, or truth-functionally indeterminate by constructing truth-tables.

a. ~ A ⊃ A
*b. J ⊃ (K ⊃ J)
c. (A ≡ ~ A) ⊃ ~ (A ≡ ~ A)
*d. (E ≡ H) ⊃ (~ E ⊃ ~ H)
e. (~ B & ~ D) ∨ ~ (B ∨ D)
*f. ([(C ⊃ D) & (D ⊃ E)] & C) & ~ E
g. [(A ∨ B) & (A ∨ C)] ⊃ ~ (B & C)
*h. ~ [[(A ∨ B) & (B ∨ B)] & (~ A & ~ B)]
i. (J ∨ ~ K) ≡ ~ ~ (K ⊃ J)
*j. ~ B ⊃ [(B ∨ D) ⊃ D]
k. [(A ∨ ~ D) & ~ (A & D)] ⊃ ~ D
*l. (M ≡ ~ N) & (M ≡ N)

2. For each of the following sentences, either show that the sentence is truth-functionally true by constructing a full truth-table or show that the sentence is not truth-functionally true by constructing an appropriate shortened truth-table.

a. (F ∨ H) ∨ (~ F ≡ H)
*b. (F ∨ H) ∨ ~ (~ F ⊃ H)
c. ~ A ⊃ [(B & A) ⊃ C]
*d. A ≡ (B ≡ A)
e. [(C ∨ ~ C) ⊃ C] ⊃ C
*f. [C ⊃ (C ∨ ~ D)] ⊃ (C ∨ D)

3. For each of the following sentences, either show that the sentence is truth-functionally false by constructing a full truth-table or show that the sentence is

not truth-functionally false by constructing an appropriate shortened truth-table.

a. $(B \equiv D) \;\&\; (B \equiv \sim D)$
*b. $(B \supset H) \;\&\; (B \supset \sim H)$
c. $A \equiv (B \equiv A)$
*d. $[(F \;\&\; G) \supset (C \;\&\; \sim C)] \;\&\; F$
e. $[(C \vee D) \equiv C] \supset \sim C$
*f. $[\sim (A \;\&\; F) \supset (B \vee A)] \;\&\; \sim [\sim B \supset \sim (F \vee A)]$

4. Which of the following are true? Explain.

a. A conjunction with one truth-functionally true conjunct must itself be truth-functionally true.
*b. A disjunction with one truth-functionally true disjunct must itself be truth-functionally true.
c. A material conditional with a truth-functionally true consequent must itself be truth-functionally true.
*d. A conjunction with one truth-functionally false conjunct must itself be truth-functionally false.
e. A disjunction with one truth-functionally false disjunct must itself be truth-functionally false.
*f. A material conditional with a truth-functionally false consequent must itself be truth-functionally false.
g. A sentence is truth-functionally true if and only if its negation is truth-functionally false.
*h. A sentence is truth-functionally indeterminate if and only if its negation is truth-functionally indeterminate.
i. A material conditional with a truth-functionally true antecedent must itself be truth-functionally true.
*j. A material conditional with a truth-functionally false antecedent must itself be truth-functionally false.

5. Answer the following questions; explain your answers.

a. Suppose that **P** is a truth-functionally true sentence and **Q** is a truth-functionally false sentence. On the basis of this information, can you determine whether $\mathbf{P} \equiv \mathbf{Q}$ is truth-functionally true, false, or indeterminate? If so, which is it?
*b. Suppose that **P** and **Q** are truth-functionally indeterminate sentences. Does it follow that $\mathbf{P} \;\&\; \mathbf{Q}$ is truth-functionally indeterminate?
c. Suppose that **P** and **Q** are truth-functionally indeterminate. Does it follow that $\mathbf{P} \vee \mathbf{Q}$ is truth-functionally indeterminate?
*d. Suppose that **P** is a truth-functionally true sentence and that **Q** is truth-functionally indeterminate. On the basis of this information, can you determine whether $\mathbf{P} \supset \mathbf{Q}$ is truth-functionally true, false, or indeterminate? If so, which is it?

3.3 TRUTH-FUNCTIONAL EQUIVALENCE

We now introduce the concept of **truth-functional equivalence.**

> Sentences **P** and **Q** of *SL* are *truth-functionally equivalent* if and only if there is no truth-value assignment on which **P** and **Q** have different truth-values.

Hence, to show that **P** and **Q** are truth-functionally equivalent, we construct a single truth-table for both **P** and **Q** and show that in each row the two sentences have the same truth-value. The columns under the *main* connectives must be identical.

The sentences 'A & A' and 'A ∨ A' are truth-functionally equivalent, as shown by the following truth-table:

A	A & A	A ∨ A
T	T T T	T T T
F	F F F	F F F

On any truth-value assignment that assigns **T** to 'A', both 'A & A' and 'A ∨ A' are true. On any truth-value assignment that assigns **F** to 'A', both 'A & A' and 'A ∨ A' are false. The sentences '(W & Y) ⊃ H' and 'W ⊃ (Y ⊃ H)' are also truth-functionally equivalent:

H	W	Y	(W & Y) ⊃ H	W ⊃ (Y ⊃ H)
T	T	T	T T T T T	T T T T T
T	T	F	T F F T T	T T F T T
T	F	T	F F T T T	F T T T T
T	F	F	F F F T T	F T F T T
F	T	T	T T T F F	T F T F F
F	T	F	T F F T F	T T F T F
F	F	T	F F T T F	F T T F F
F	F	F	F F F T F	F T F T F

The columns under the main connectives of '(W & Y) ⊃ H' and 'W ⊃ (Y ⊃ H)' are identical, which shows that the two sentences have the same truth-value on every truth-value assignment.

It is important to remember that two sentences are truth-functionally equivalent only if they have the same truth-value on every truth-value assignment. That is, their truth-table columns (in the same truth-table) must be identical. Consider the following truth-table:

E	H	J	E ∨ H	(H ∨ J) ∨ E
T	T	T	T T T	T T T T T
T	T	F	T T T	T T F T T
T	F	T	T T F	F T T T T
T	F	F	T T F	F F F T T
F	T	T	F T T	T T T T F
F	T	F	F T T	T T F T F
F	F	T	F F F	F T T T F
F	F	F	F F F	F F F F F

The table shows that the sentences 'E ∨ H' and '(H ∨ J) ∨ E' are not truth-functionally equivalent, for they have different truth-values on any truth-value assignment that assigns **F** to 'E' and 'H' and **T** to 'J'. The fact that 'E ∨ H' and '(H ∨ J) ∨ E' have the same truth-value for all *other* truth-value assignments is irrelevant to the question of whether the two sentences are truth-functionally equivalent. When we want to show that two sentences are not truth-functionally equivalent, we will circle at least one row of the truth-table in which the sentences do not have the same truth-value.

All truth-functionally true sentences are truth-functionally equivalent. This is because every truth-functionally true sentence has the truth-value **T** on every truth-value assignment. In a table for two truth-functionally true sentences, the columns under the main connectives of those sentences are always identical. For example, '~ (C & ~ C)' and 'A ⊃ (B ⊃ A)' are truth-functionally equivalent:

				↓							↓			
A	B	C		~	(	C	&	~	C	)	A	⊃	(	B	⊃	A	)
T	T	T		T	T	F	F	F	T		T	T	T	T	T	
T	T	F		T	F	F	F	T	F		T	T	T	T	T	
T	F	T		T	T	F	F	F	T		T	T	F	T	T	
T	F	F		T	F	F	F	T	F		T	T	F	T	T	
F	T	T		T	T	F	F	F	T		F	T	T	F	F	
F	T	F		T	F	F	F	T	F		F	T	T	F	F	
F	F	T		T	T	F	F	F	T		F	T	F	T	F	
F	F	F		T	F	F	F	T	F		F	T	F	T	F	

Likewise all truth-functionally false sentences are truth-functionally equivalent.

But not all truth-functionally indeterminate sentences are truth-functionally equivalent—for example,

			↓			↓		
B	D		B	&	D	~	B	&	D
T	T		T	T	T	F	T	F	T
T	F		T	F	F	F	T	F	F
F	T		F	F	T	T	F	T	T
F	F		F	F	F	T	F	F	F

On any truth-value assignment on which 'B' and 'D' are both true, or 'B' is false and 'D' is true, the sentences 'B & D' and '~ B & D' have different truth-values. Hence they are not truth-functionally equivalent.

If **P** and **Q** are not truth-functionally equivalent, we can construct a shortened truth-table to show this. The shortened truth-table will display a combination of truth-values for which one sentence is true and the other false. For example, the following shortened truth-table shows that 'A' and 'A ∨ B' are not truth-functionally equivalent:

A	B	A	A	∨	B
F	**T**	**F**	**F**	**T**	**T**

The shortened truth-table shows that, on any truth-value assignment that assigns **F** to 'A' and **T** to 'B', 'A' is false and 'A ∨ B' is true. Hence the sentences are not truth-functionally equivalent. Note that, if we construct a shortened truth-table that displays a row in which both sentences have the same truth-value, this is not sufficient to show that they are truth-functionally equivalent. This is because they are truth-functionally equivalent if and only if they have the same truth-value on *every* truth-value assignment. To show this, we must consider every combination of truth-values that their atomic components might have.

We may construct our shortened truth-tables for two (or more) sentences in a systematic way, just as we did for single sentences in Section 3.2. For example, we could begin constructing the previous table by assigning the sentence 'A' the truth-value **F** and 'B' the truth-value **T**:

A	B	A	A	∨	B
		F		**T**	

(We might first have tried to make 'A' true and 'A ∨ B' false, but this would not lead to a correct truth-table since we would have a false disjunction with a true disjunct.) Filling in **F** under all the other occurrences of 'A' yields

A	B	A	A	∨	B
F		**F**	**F**	**T**	

Now we can make 'B' true, which will secure the truth of the disjunction:

A	B	A	A	∨	B
F	**T**	**F**	**F**	**T**	**T**

3.3E EXERCISES

1. Decide, by constructing truth-tables, in which of the following pairs the sentences are truth-functionally equivalent.

a.	~ (A & B)	~ (A ∨ B)
*b.	A ⊃ (B ⊃ A)	(C & ~ C) ∨ (A ⊃ A)
c.	K ≡ H	~ K ≡ ~ H
*d.	C & (B ∨ A)	(C & B) ∨ A

e.	$(G \supset F) \supset (F \supset G)$	$(G \equiv F) \vee (\sim F \vee G)$
*f.	$\sim C \supset \sim B$	$B \supset C$
g.	$\sim (H \,\&\, J) \equiv (J \equiv \sim K)$	$(H \,\&\, J) \supset \sim K$
*h.	$\sim (D \vee B) \supset (C \supset B)$	$C \supset (D \,\&\, B)$
i.	$[A \vee \sim (D \,\&\, C)] \supset \sim D$	$[D \vee \sim (A \,\&\, C)] \supset \sim A$
*j.	$A \supset [B \supset (A \supset B)]$	$B \supset [A \supset (B \supset A)]$
k.	$F \vee \sim (G \vee \sim H)$	$(H \equiv \sim F) \vee G$

2. For each of the following pairs of sentences, either show that the sentences are truth-functionally equivalent by constructing a full truth-table or show that they are not truth-functionally equivalent by constructing an appropriate shortened truth-table.

a.	$G \vee H$	$\sim G \supset H$
*b.	$\sim (B \,\&\, \sim A)$	$A \vee B$
c.	$(D \equiv A) \,\&\, D$	$D \,\&\, A$
*d.	$F \,\&\, (J \vee H)$	$(F \,\&\, J) \vee H$
e.	$A \equiv (\sim A \equiv A)$	$\sim (A \supset \sim A)$
*f.	$\sim (\sim B \vee (\sim C \vee \sim D))$	$(D \vee C) \,\&\, \sim B$

3. Symbolize each of the following pairs of sentences and determine whether the sentences are truth-functionally equivalent by constructing truth-tables.

a. Unless the sky clouds over, the night will be clear and the moon will shine brightly.
The moon will shine brightly if and only if the night is clear and the sky doesn't cloud over.

*b. Although the new play at the Roxy is a flop, critics won't ignore it unless it is canceled.
The new play at the Roxy is a flop, and if it is canceled critics will ignore it.

c. If the *Daily Herald* reports on our antics, then the antics are effective.
If our antics aren't effective, then the *Daily Herald* won't report on them.

*d. The year 1972 wasn't a good vintage year, 1973 was, and neither 1974 nor 1975 was.
Neither 1974 nor 1972 was a good vintage year, and not both 1973 and 1975 were.

e. If Mary met Tom and she liked him, then Mary didn't ask George to the movies.
If Mary met Tom and she didn't like him, then Mary asked George to the movies.

*f. Either the blue team or the red team will win the tournament, and they won't both win.
The red team will win the tournament if and only if the blue team won't win the tournament.

4. Answer the following questions; explain your answers.

a. Suppose that two sentences **P** and **Q** are truth-functionally equivalent. Are $\sim$ **P** and $\sim$ **Q** truth-functionally equivalent as well?

*b. Suppose that two sentences **P** and **Q** are truth-functionally equivalent. Show that it follows that **P** and **P** & **Q** are truth-functionally equivalent as well.

c. Suppose that two sentences **P** and **Q** are truth-functionally equivalent. Show that it follows that $\sim$ **P** $\vee$ **Q** is truth-functionally true.

3.4 TRUTH-FUNCTIONAL CONSISTENCY

To define truth-functional consistency, we need the notion of a *set* of sentences, informally introduced in Chapter 1. A set of sentences of *SL* is a group, or collection, of sentences of *SL*. We have special notation for representing finite sets of sentences (sets consisting of a finite number of sentences): We write the names of the sentences, separated by commas, and enclose the whole list in braces. Thus {'A', 'B ⊃ H', 'C ∨ A'} is the set of sentences consisting of 'A', 'B ⊃ H', and 'C ∨ A'. We say that these three sentences are *members* of the set. For convenience we will drop the single quotes from names of sentences when they are written between the braces; our convention is that this is merely a way of abbreviating the set notation. So we may write

{A, B ⊃ H, C ∨ A}

instead of

{'A', 'B ⊃ H', 'C ∨ A'}

All sets of sentences that have at least one member are nonempty sets of sentences. '∅' is the name of the empty set; the empty set of sentences of *SL* is the set that contains no members at all. In what follows we shall use the variable 'Γ' (gamma), with or without a subscript, to range over sets of sentences of *SL*.

Truth-functional consistency may now be introduced.

> A set of sentences of *SL* is *truth-functionally consistent* if and only if there is at least one truth-value assignment on which all the members of the set are true. A set of sentences of *SL* is *truth-functionally inconsistent* if and only if it is not truth-functionally consistent.

The set {A, B ⊃ H, B} is truth-functionally consistent, as is shown by the following truth-table:

			↓		↓		↓
A	B	H	A	B	⊃	H	B
T	T	T	T	T	T	T	T
T	T	F	T	T	F	F	T
T	F	T	T	F	T	T	F
T	F	F	T	F	T	F	F
F	T	T	F	T	T	T	T
F	T	F	F	T	F	F	T
F	F	T	F	F	T	T	F
F	F	F	F	F	T	F	F

The truth-table shows that, on any truth-value assignment on which 'A', 'B', and 'H' are all true, all three set members are true. So the set is truth-functionally consistent. We have circled the row of the truth-table that shows this. (Sometimes when we construct a truth-table to test a set of sentences for truth-functional consistency, we will find that there is more than one row in which all the members of the set are true. In such cases we shall circle only one of these rows of the truth-table.)

The set of sentences {L, L ⊃ J, ~ J} is truth-functionally inconsistent:

			↓			↓		↓	
J	L		L		L	⊃	J	~	J
T	T		T		T	T	T	F	T
T	F		F		F	T	T	F	T
F	T		T		T	F	F	T	F
F	F		F		F	T	F	T	F

In each row at least one of the three sentences has the truth-value **F** in the column under its main connective. Hence there is no single truth-value assignment on which all three set members are true. The following set of sentences is also truth-functionally inconsistent: {C ∨ ~ C, ~ C & D, ~ D}.

				↓				↓			↓
C	D		C	∨	~	C	~	C	&	D	~ D
T	T		T	T	F	T	F	T	F	T	F T
T	F		T	T	F	T	F	T	F	F	T F
F	T		F	T	T	F	T	F	T	T	F T
F	F		F	T	T	F	T	F	F	F	T F

In this case it does not matter that one of the sentences, 'C ∨ ~ C', is true on every truth-value assignment. All that matters for establishing truth-functional inconsistency is that there is no single truth-value assignment on which all three members are true.

We can show that a set of sentences is truth-functionally consistent by constructing a shortened truth-table that lists one row in which all the set members are true. For instance, the following shortened truth-table shows that the set {(E ≡ H) ≡ E, H & ~ E} is truth-functionally consistent:

					↓			↓		
E	H		(E	≡	H)	≡	E	H	&	~ E
F	T		F	F	T	T	F	T	T	T F

The table shows that, on any truth-value assignment on which 'E' is false and 'H' is true, the set members will all be true. Note that if we construct a shortened table that lists a row in which not all the members of the set are true, this is not sufficient to show that the set is truth-functionally inconsistent. This

is because a set of sentences is truth-functionally inconsistent if and only if there is *no* truth-value assignment on which every member of the set is true. To show this, we would have to consider every combination of truth-values that the atomic components of the set members might have.

3.4E EXERCISES

1. Using truth-tables, determine which of the following sets are truth-functionally consistent.

a. {A ⊃ B, B ⊃ C, A ⊃ C}
*b. {B ≡ (J & K), ~ J, ~ B ⊃ B}
c. {~ [J ∨ (H ⊃ L)], L ≡ (~ J ∨ ~ H), H ≡ (J ∨ L)}
*d. {(A & B) & C, C ∨ (B ∨ A), A ≡ (B ⊃ C)}
e. {(J ⊃ J) ⊃ H, ~ J, ~ H}
*f. {U ∨ (W & H), W ≡ (U ∨ H), H ∨ ~ H}
g. {A, B, C}
*h. {~ (A & B), ~ (B & C), ~ (A & C), A ∨ (B & C)}
i. {(A & B) ∨ (C ⊃ B), ~ A, ~ B}
*j. {A ⊃ (B ⊃ (C ⊃ A)), B ⊃ ~ A}

2. For each of the following sets of sentences, either show that the set is truth-functionally consistent by constructing an appropriate shortened truth-table or show that the set is truth-functionally inconsistent by constructing a full truth-table.

a. {B ⊃ (D ⊃ E), ~ D & B}
*b. {H ≡ (~ H ⊃ H)}
c. {F ⊃ (J ∨ K), F ≡ ~ J}
*d. {~ (~ C ∨ ~ B) & A, A ≡ ~ C}
e. {(A ⊃ B) ≡ (~ B ∨ B), A}
*f. {H ⊃ J, J ⊃ K, K ⊃ ~ H}

3. Symbolize each of the following passages and determine whether the set consisting of those sentences is truth-functionally consistent by constructing a truth-table.

a. If space is infinitely divisible, then Zeno's paradoxes are compelling. Zeno's paradoxes are neither convincing nor compelling. Space is infinitely divisible.
*b. Newtonian mechanics can't be right if Einsteinian mechanics is. But Einsteinian mechanics is right if and only if space is non-Euclidean. Space is non-Euclidean, or Newtonian mechanics is correct.
c. Eugene O'Neil was an alcoholic. His plays show it. But *The Iceman Cometh* must have been written by a teetotaler. O'Neill was an alcoholic unless he was a fake.
*d. Neither sugar nor saccharin is desirable if and only if both are lethal. Sugar is lethal if and only if saccharin is desirable. Sugar is undesirable if and only if saccharin isn't lethal.
e. If the Red Sox win next Sunday, then if Joan bet $5 against them she'll buy Ed a hamburger. The Red Sox won't win, and Joan won't buy Ed a hamburger.
*f. Either Johnson or Hartshorne pleaded guilty, or neither did. If Johnson pleaded guilty, then the newspaper story is incorrect. The newspaper story is correct, and Hartshorne pleaded guilty.

4.a. Prove that {**P**} is truth-functionally inconsistent if and only if ~ **P** is truth-functionally true.

*b. If [**P**] is truth-functionally consistent, must [~ **P**] be truth-functionally consistent as well? Show that you are right.
c. If **P** and **Q** are truth-functionally indeterminate, does it follow that [**P**, **Q**] is truth-functionally consistent? Explain your answer.
*d. Prove that if **P** ≡ **Q** is truth-functionally true then [**P**, ~ **Q**] is truth-functionally inconsistent.

3.5 TRUTH-FUNCTIONAL ENTAILMENT AND TRUTH-FUNCTIONAL VALIDITY

Truth-functional entailment is a relation that may hold between a sentence of *SL* and a set of sentences of *SL*.

> A set Γ of sentences of *SL truth-functionally entails* a sentence **P** if and only if there is no truth-value assignment on which every member of Γ is true and **P** is false.

In other words Γ truth-functionally entails **P** just in case **P** is true on every truth-value assignment on which every member of Γ is true. We have a special symbol for truth-functional entailment: the double turnstile '⊨'. The expression

Γ ⊨ **P**

is read

Γ truth-functionally entails **P**.

To indicate that Γ does not truth-functionally entail **P**, we write

Γ ⊭ **P**

Thus

[A, B & C] ⊨ 'B'

and

[A, B ∨ C] ⊭ 'B'

mean, respectively,

[A, B & C] truth-functionally entails 'B'

and

[A, B ∨ C] does not truth-functionally entail 'B'.

Henceforth we adopt the convention that, when using the turnstile notation, we drop the single quotation marks around the sentence following the turnstile. We also have a special abbreviation to indicate that a sentence is truth-functionally entailed by the empty set of sentences:

$\models$ **P**

The expression '$\models$ **P**' is an abbreviation for '$\varnothing \models$ **P**'. All and only truth-functionally true sentences are truth-functionally entailed by the empty set of sentences; the proof of this is left as an exercise in Section 3.6.

If Γ is a finite set, we can determine whether Γ truth-functionally entails **P** by constructing a truth-table for the members of Γ and for **P**. If there is a row in the truth-table in which all the members of Γ have the truth-value **T** and **P** has the truth-value **F**, then Γ does not truth-functionally entail **P**. If there is no such row, then Γ truth-functionally entails **P**. We can see that {A, B & C} $\models$ B by checking the following truth-table:

					↓			↓	
A	B	C		A		B	&	C	B
T	T	T		T		T	T	T	T
T	T	F		T		T	F	F	T
T	F	T		T		F	F	T	F
T	F	F		T		F	F	F	F
F	T	T		F		T	T	T	T
F	T	F		F		T	F	F	T
F	F	T		F		F	F	T	F
F	F	F		F		F	F	F	F

There is only one row in which both members of {A, B & C} are true, namely, the row in which 'A', 'B', and 'C' all have the truth-value **T**. But since 'B' is true in this row, it follows that there is no combination of truth-values for the atomic components of all these sentences that will make both 'A' and 'B & C' true and 'B' false. Hence there is no truth-value assignment on which 'A' and 'B & C' are true and 'B' is false: {A, B & C} $\models$ B.

In the same way we can show that {W $\vee$ J, (W $\supset$ Z) $\vee$ (J $\supset$ Z), ~ Z} $\models$ ~ (W & J):

					↓						↓				↓		↓		
J	W	Z		W	∨	J		(W	⊃	Z)	∨	(J	⊃	Z)	~	Z	~	(W	&	J)
T	T	T		T	T	T		T	T	T	T	T	T	T	F	T	F	T	T	T
T	T	F		T	T	T		T	F	F	F	T	F	F	T	F	F	T	T	T
T	F	T		F	T	T		F	T	T	T	T	T	T	F	T	T	F	F	T
T	F	F		F	T	T		F	T	F	T	T	F	F	T	F	T	F	F	T
F	T	T		T	T	F		T	T	T	T	F	T	T	F	T	T	T	F	F
F	T	F		T	T	F		T	F	F	T	F	T	F	T	F	T	T	F	F
F	F	T		F	F	F		F	T	T	T	F	T	T	F	T	T	F	F	F
F	F	F		F	F	F		F	T	F	T	F	T	F	T	F	T	F	F	F

The fourth and sixth rows are the only ones in which all the set members are true; '~ (W & J)' is true in these rows as well. The following truth-table shows that {K ∨ J, ~ (K ∨ J)} ⊨ K:

		↓			↓				↓
J	K	K	∨	J	~	(	K	∨	J)	K
T	T	T	T	T	F	T	T	T	T	T
T	F	F	T	T	F	F	T	T	T	F
F	T	T	T	F	F	T	T	F	F	T
F	F	F	F	F	T	F	F	F	F	F

There is no row in which 'K ∨ J' and '~ (K ∨ J)' are both true, and hence no truth-value assignment on which the set members are both true. Consequently there is no truth-value assignment on which the members of the set are both true and 'K' is false; so the set truth-functionally entails 'K'.

On the other hand, {A, B ∨ C} does *not* truth-functionally entail 'B'. The following truth-table shows this:

			↓		↓		↓
A	B	C	A	B	∨	C	B
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	F	T	T	F
T	F	F	T	F	F	F	F
F	T	T	F	T	T	T	T
F	T	F	F	T	T	F	T
F	F	T	F	F	T	T	F
F	F	F	F	F	F	F	F

The circled row shows that 'A' and 'B ∨ C' are both true and 'B' is false on any truth-value assignment that assigns **T** to 'A' and 'C' and **F** to 'B'.

An **argument** of *SL* is a group of two or more sentences of *SL*, one of which is designated as the conclusion and the others as the premises.

> An argument of *SL* is *truth-functionally valid* if and only if there is no truth-value assignment on which all the premises are true and the conclusion is false. An argument of *SL* is *truth-functionally invalid* if and only if it is not truth-functionally valid.

Thus an argument of *SL* is truth-functionally valid just in case on every truth-value assignment on which the premises are true the conclusion is true as well. This means that an argument is truth-functionally valid if and only if the set consisting of the premises of the argument truth-functionally entails the conclusion.

The argument

$$\frac{\begin{array}{l} F \equiv G \\ F \vee G \end{array}}{F \mathbin{\&} G}$$

is truth-functionally valid, as the following truth-table shows:

F	G	F	≡	G	F	∨	G	F	&	G
			↓			↓			↓	
T	T	T	T	T	T	T	T	T	T	T
T	F	T	F	F	T	T	F	T	F	F
F	T	F	F	T	F	T	T	F	F	T
F	F	F	T	F	F	F	F	F	F	F

The first row lists the only combination of truth-values for the atomic components of these sentences for which the premises, 'F ≡ G' and 'F ∨ G', are both true; the conclusion, 'F & G', is true in this row as well. Similarly the argument

$$\frac{\begin{array}{l} (A \mathbin{\&} G) \vee (B \supset G) \\ \sim G \vee B \end{array}}{\sim B \vee G}$$

is truth-functionally valid:

A	B	G	(A	&	G)	∨	(B	⊃	G)	~	G	∨	B	~	B	∨	G
						↓						↓				↓	
T	T	T	T	T	T	T	T	T	T	F	T	T	T	F	T	T	T
T	T	F	T	F	F	F	T	F	F	T	F	T	T	F	T	F	F
T	F	T	T	T	T	T	F	T	T	F	T	F	F	T	F	T	T
T	F	F	T	F	F	T	F	T	F	T	F	T	F	T	F	T	F
F	T	T	F	F	T	T	T	T	T	F	T	T	T	F	T	T	T
F	T	F	F	F	F	F	T	F	F	T	F	T	T	F	T	F	F
F	F	T	F	F	T	T	F	T	T	F	T	F	F	T	F	T	T
F	F	F	F	F	F	T	F	T	F	T	F	T	F	T	F	T	F

The conclusion, '~ B ∨ G', is true on every truth-value assignment on which the premises are true.

The following argument is truth-functionally invalid:

$$\frac{\begin{array}{l} D \equiv (\sim W \vee G) \\ G \equiv \sim D \end{array}}{\sim D}$$

This is shown by the following truth-table:

				↓							↓			↓	
D	G	W	D	=	(~	W	∨	G)	G	=	~	D	~	D
T	T	T	T	T	F	T	T	T	T	F	F	T	F	T
T	T	F	T	T	T	F	T	T	T	F	F	T	F	T
T	F	T	T	F	F	T	F	F	F	T	F	T	F	T
T	F	F	T	T	T	F	T	F	F	T	F	T	F	T
F	T	T	F	F	F	T	T	T	T	T	T	F	T	F
F	T	F	F	F	T	F	T	T	T	T	T	F	T	F
F	F	T	F	T	F	T	F	F	F	F	T	F	T	F
F	F	F	F	F	T	F	T	F	F	F	T	F	T	F

The premises, 'D ≡ (~ W ∨ G)' and 'G ≡ ~ D', are both true on every truth-value assignment that assigns **T** to 'D' and **F** to 'G' and 'W', and the conclusion, '~ D', is false on these truth-value assignments.

Where an argument is truth-functionally invalid, we can show this by constructing a shortened truth-table that displays a row in which the premises are true and the conclusion false. The argument

~ (B ∨ D)
~ H

B

is truth-functionally invalid, as the following shortened truth-table shows:

			↓				↓	↓
B	D	H	~	(B	∨	D)	~ H	B
F	F	F	T	F	F	F	T F	F

For any argument of *SL* that has a finite number of premises, we may form a sentence called the *corresponding material conditional*, and that sentence is truth-functionally true if and only if the argument is truth-functionally valid. First, we may form an *iterated conjunction* $(\ldots (\mathbf{P}_1 \,\&\, \mathbf{P}_2) \,\&\, \ldots \,\&\, \mathbf{P}_n)$ from the sentences $\mathbf{P}_1, \ldots, \mathbf{P}_n$. The iterated conjunction for the sentences '~ (A ⊃ B)', 'D', and 'J ∨ H' is '((~ (A ⊃ B) & D) & (J ∨ H))'. The corresponding material conditional for an argument is then formed by constructing a material conditional with the iterated conjunction of the premises as antecedent and the conclusion of the argument as consequent. The corresponding material conditional for the argument

~ (A ⊃ B)
D
J ∨ H

~ H ∨ ~ A

is '[[~ (A ⊃ B) & D] & (J ∨ H)] ⊃ (~ H ∨ ~ A)', and the corresponding material conditional for the argument

A
A ⊃ B

B

is '[A & (A ⊃ B)] ⊃ B'.[5]

An argument with a finite number of premises is truth-functionally valid if and only if its corresponding material conditional is truth-functionally true (see Exercise 5). We can show that the argument

A
A ⊃ B

B

is truth-functionally valid by showing that the corresponding material conditional '[A & (A ⊃ B)] ⊃ B' is truth-functionally true:

							↓	
A	B	[A	&	(A	⊃	B)]	⊃	B
T	T	T	T	T	T	T	T	T
T	F	T	F	T	F	F	T	F
F	T	F	F	F	T	T	T	T
F	F	F	F	F	T	F	T	F

There is no truth-value assignment on which 'A & (A ⊃ B)' is true and 'B' is false, which means that there is no truth-value assignment on which 'A' and 'A ⊃ B' are both true and 'B' is false. And we can show that the argument

~ A ≡ ~ B
B ∨ A

~ A

[5]Strictly speaking, an argument with more than one premise will have more than one corresponding material conditional. This is because the premises of an argument can be conjoined in more than one order. But all the corresponding material conditionals for any one argument are truth-functionally equivalent, and so we speak loosely of *the* corresponding material conditional for a given argument.

is truth-functionally invalid by showing that the corresponding material conditional is not truth-functionally true:

												↓	
A	B	[(~	A	≡	~	B)	&	(B	∨	A)]	⊃	~	A
T	T		F	T	T	F	T	T	T	T		F	F T
T	F		F	T	F	T	F	F	T	T		T	F T
F	T		T	F	F	F	T	T	T	F		T	T F
F	F		T	F	T	T	F	F	F	F		T	T F

The first row represents truth-value assignments on which the antecedent is true and the consequent false. On these truth-value assignments the premises of the argument, '~ A ≡ ~ B' and 'B ∨ A', are both true and the conclusion, '~ A', is false. Hence the argument is truth-functionally invalid.

3.5E EXERCISES

1. Use truth-tables to determine whether the following arguments are truth-functionally valid.

a. A ⊃ (H & J)
J ≡ H
~ J

~ A

*b. B ∨ (A & ~ C)
(C ⊃ A) ≡ B
~ B ∨ A

~ (A ∨ C)

c. (D ≡ ~ G) & G
(G ∨ [(A ⊃ D) & A]) ⊃ ~ D

G ⊃ ~ D

*d. ~ (Y ≡ A)
~ Y
~ A

W & ~ W

e. (C ⊃ D) ⊃ (D ⊃ E)
D

C ⊃ E

*f. $B \vee B$
$[\sim B \supset (\sim D \vee \sim C)] \;\&\; [(\sim D \vee C) \vee (\sim B \vee C)]$
C

g. $(G \equiv H) \vee (\sim G \equiv H)$
$(\sim G \equiv \sim H) \vee \sim (G \equiv H)$

*h. $[(J \;\&\; T) \;\&\; Y] \vee (\sim J \supset \sim Y)$
$J \supset T$
$T \supset Y$
$Y \equiv T$

i. $\sim\sim F \supset \sim\sim G$
$\sim G \supset \sim F$
$G \supset F$

*j. $[A \;\&\; (B \vee C)] \equiv (A \vee B)$
$B \supset \sim B$
$C \vee A$

2. For each of the following arguments, either show that the argument is truth-functionally invalid by constructing an appropriate shortened truth-table or show that the argument is truth-functionally valid by constructing a full truth-table.

a. $(J \vee M) \supset \sim (J \;\&\; M)$
$M \equiv (M \supset J)$
$M \supset J$

*b. $B \;\&\; F$
$\sim (B \;\&\; G)$
G

c. $A \supset \sim A$
$(B \supset A) \supset B$
$A \equiv \sim B$

*d. $J \vee [M \supset (T \equiv J)]$
$(M \supset J) \;\&\; (T \supset M)$
$T \;\&\; \sim M$

e. $A \;\&\; \sim [(B \;\&\; C) \equiv (C \supset A)]$
$B \supset \sim B$
$\sim C \supset C$

3. Construct the corresponding material conditional for each of the following arguments. For each of the arguments, either show that the argument is truth-functionally invalid by constructing an appropriate shortened truth-table for the corresponding material conditional or show that the argument is truth-functionally valid by constructing a full truth-table for the corresponding material conditional.

a. $\dfrac{B \,\&\, C}{B \vee C}$

*b. $\dfrac{K \equiv L \quad L \supset J \quad \sim J}{\sim K \vee L}$

c. $\dfrac{(J \supset T) \supset J \quad (T \supset J) \supset T}{\sim J \vee \sim T}$

*d. $\dfrac{(A \vee C) \,\&\, \sim H}{\sim C}$

e. $\dfrac{B \,\&\, C \quad B \vee D}{D}$

*f. $\dfrac{\sim [A \vee \sim (B \vee \sim C)] \quad B \supset (A \supset C)}{\sim A \equiv \sim B}$

4. Symbolize each of the following arguments and use truth-tables to test for truth-functional validity.

a. 'Stern' means the same as 'star' if 'Nacht' means the same as 'day'. But 'Nacht' doesn't mean the same as 'day'; therefore 'Stern' means something different from 'star'.

*b. Many people believe that war is inevitable. But war is inevitable if and only if our planet's natural resources are nonrenewable. So many people believe that our natural resources are nonrenewable.

c. Thirty days hath September, April, and November. But February has forty days, since April has thirty days if and only if May doesn't, and May has thirty days if November does.

*d. The town hall is now a grocery store, and, unless I'm mistaken, the little red schoolhouse is a movie theater. No, I'm not mistaken. The old schoolhouse is a boutique, and the old theater is an elementary school if the little red schoolhouse is a movie theater. So the little red schoolhouse is a movie theater.

e. Computers can think if and only if they can have emotions. If computers can have emotions, then they can have desires as well. But computers can't think if they have desires. Therefore computers can't think.

*f. If the butler murdered Devon, then the maid is lying, and if the gardener murdered Devon, then the weapon was a slingshot. The maid is lying if and only if the weapon wasn't a slingshot, and if the weapon wasn't a slingshot, then the butler murdered Devon. Therefore the butler murdered Devon.

5.a. Show that $(\ldots(\mathbf{P}_1 \,\&\, \mathbf{P}_2) \,\&\, \ldots \,\&\, \mathbf{P}_n) \supset \mathbf{Q}$ is truth-functionally true if and only if

$$\begin{array}{l}\mathbf{P}_1\\ \vdots\\ \underline{\mathbf{P}_n}\\ \mathbf{Q}\end{array}$$

is truth-functionally valid.

*b. Show that $\{\mathbf{P}\} \models \mathbf{Q}$ and $\{\mathbf{Q}\} \models \mathbf{P}$ if and only if $\mathbf{P}$ and $\mathbf{Q}$ are truth-functionally equivalent.

c. Suppose that $\{\mathbf{P}\} \models \mathbf{Q} \vee \mathbf{R}$. Does it follow that either $\{\mathbf{P}\} \models \mathbf{Q}$ or $\{\mathbf{P}\} \models \mathbf{R}$? Show that you are right.

*d. Show that if $\{\mathbf{P}\} \models \mathbf{Q}$ and $\{\mathbf{Q}\} \models \mathbf{R}$, then $\{\mathbf{P}\} \models \mathbf{R}$.

3.6 TRUTH-FUNCTIONAL PROPERTIES AND TRUTH-FUNCTIONAL CONSISTENCY

In this section we show that the truth-functional concepts of truth-functional truth, truth-functional falsehood, truth-functional indeterminacy, truth-functional entailment, and truth-functional validity can all be explicated in terms of truth-functional consistency. This is important because in Chapter 4 we shall introduce an alternative test for truth-functional consistency, and the possibility of explicating the other concepts in terms of truth-functional consistency means that we shall be able to use the test to determine other truth-functional properties of sentences and sets of sentences as well.

> A sentence $\mathbf{P}$ is *truth-functionally false* if and only if $\{\mathbf{P}\}$ is *truth-functionally inconsistent.*

(We call $\{\mathbf{P}\}$ the *unit set* of $\mathbf{P}$.) To prove that this is so, we first assume that $\mathbf{P}$ is truth-functionally false. Then, by definition, there is no truth-value assignment on which $\mathbf{P}$ is true. Consequently, as $\mathbf{P}$ is the only member of the unit set $\{\mathbf{P}\}$, there is no truth-value assignment on which every member of that set is true. So $\{\mathbf{P}\}$ is truth-functionally inconsistent. Now assume that $\{\mathbf{P}\}$ is truth-functionally inconsistent. Then, by definition, there is no truth-value assignment on which every member of $\{\mathbf{P}\}$ is true. Since $\mathbf{P}$ is the only member of

its unit set, there is no truth-value assignment on which **P** is true. Hence **P** is truth-functionally false.

The corresponding relation for truth-functionally true sentences is more complicated:

> A sentence **P** is *truth-functionally true* if and only if {~ **P**} is *truth-functionally inconsistent.*

We first assume that **P** is truth-functionally true. Then, by definition, **P** is true on every truth-value assignment. We know that a sentence is true on a truth-value assignment if and only if the negation of the sentence is false on that truth-value assignment. So it follows from our assumption that ~ **P** is false on every truth-value assignment; that is, there is no truth-value assignment on which ~ **P** is true. But then there is no truth-value assignment on which every member of {~ **P**} is true, which means that {~ **P**} is truth-functionally inconsistent. The proof of the converse, that if {~ **P**} is truth-functionally inconsistent then **P** is truth-functionally true, is left as an exercise.

Since a sentence **P** is truth-functionally true if and only if {~ **P**} is truth-functionally inconsistent and **P** is truth-functionally false if and only if {**P**} is truth-functionally inconsistent, it follows that

> A sentence **P** is *truth-functionally indeterminate* if and only if both {~ **P**} and {**P**} are *truth-functionally consistent.*

Now we turn to truth-functional equivalence. Where **P** and **Q** are sentences of *SL*, **P** ≡ **Q** is their *corresponding material biconditional.* **P** and **Q** are truth-functionally equivalent if and only if their corresponding material biconditional **P** ≡ **Q** is truth-functionally true. If we assume that **P** and **Q** are truth-functionally equivalent, then, by definition, **P** and **Q** have the same truth-value on every truth-value assignment. But we know that a material biconditional has the truth-value **T** on every truth-value assignment on which its immediate sentential components have the same truth-value. So, on our assumption, **P** ≡ **Q** is true on every truth-value assignment and hence is truth-functionally true. The converse of this, that if **P** ≡ **Q** is truth-functionally true then **P** and **Q** are truth-functionally equivalent, is left as an exercise. It follows from these results that

> Sentences **P** and **Q** are *truth-functionally equivalent* if and only if {~ (**P** ≡ **Q**)} is *truth-functionally inconsistent.*

Consider: **P** ≡ **Q** is truth-functionally true if and only if {~ (**P** ≡ **Q**)} is truth-functionally inconsistent, by our previous result concerning truth-functional truths. Moreover we have just shown that **P** and **Q** are truth-functionally equivalent if and only if **P** ≡ **Q** is truth-functionally true.

To make these results more concrete, we shall consider an example. The set {~ [(A ∨ B) ≡ (~ A ⊃ B)]} is truth-functionally inconsistent, as shown by the following truth-table:

				↓						
A	B		~	[(A	∨	B)	≡	(~ A	⊃	B)]
T	T		F	T	T	T	T	F T	T	T
T	F		F	T	T	F	T	F T	T	F
F	T		F	F	T	T	T	T F	T	T
F	F		F	F	F	F	T	T F	F	F

The set is truth-functionally inconsistent because there is no truth-value assignment on which every member of the set (in this case there is just one member) is true. From this we know the following:

1. '~ [(A ∨ B) ≡ (~ A ⊃ B)]' is truth-functionally false. (**P** is truth-functionally false if and only if {**P**} is truth-functionally inconsistent. Here {~ [(A ∨ B) ≡ (~ A ⊃ B)]} is truth-functionally inconsistent. Hence there is no truth-value assignment on which the only member of that set, '~ [(A ∨ B) ≡ (~ A ⊃ B)]', is true. That one member is thus truth-functionally false.)
2. '(A ∨ B) ≡ (~ A ⊃ B)' is truth-functionally true. (**P** is truth-functionally true if and only if {~ **P**} is truth-functionally inconsistent. We have just reasoned that '~ [(A ∨ B) ≡ (~ A ⊃ B)]' is truth-functionally false. Hence the sentence of which it is the negation, '(A ∨ B) ≡ (~ A ⊃ B)', is true on every truth-value assignment—it is a truth-functionally true sentence.)
3. 'A ∨ B' and '~ A ⊃ B' are truth-functionally equivalent. (**P** and **Q** are truth-functionally equivalent if and only if {~ (**P** ≡ **Q**)} is truth-functionally inconsistent. Since '(A ∨ B) ≡ (~ A ⊃ B)' is truth-functionally true, 'A ∨ B' and '~ A ⊃ B' have the same truth-value on every truth-value assignment—they are truth-functionally equivalent.)

Of course, each of these claims can be directly verified by examining the truth-table, but our general proofs show that this is not necessary.

Next we relate the concepts of truth-functional entailment and truth-functional consistency. Where Γ is a set of sentences of *SL* and **P** is any sentence of *SL*, we may form a set that contains **P** and all the members of Γ. This set is represented as

$$\Gamma \cup \{\mathbf{P}\}$$

which is read as

the union of gamma and the unit set of **P**

Thus, if Γ is {A, A ⊃ B} and **P** is 'J', then $\Gamma \cup$ {**P**}—that is, {A, A ⊃ B} ∪ {J}—is {A, A ⊃ B, J}. Of course, if **P** is a member of Γ, then $\Gamma \cup$ {**P**} is identical with Γ. So {A, A ⊃ B} ∪ {A ⊃ B} is simply {A, A ⊃ B}. In the case where Γ is ∅ (the empty set), $\Gamma \cup$ {**P**} is simply {**P**}. This follows because ∅ contains no members.

We may now prove that, if $\Gamma \models$ **P**, for some sentence **P** and set of sentences Γ, then $\Gamma \cup$ {~ **P**} is truth-functionally inconsistent. Suppose that $\Gamma \models$ **P**. Then, by definition, there is no truth-value assignment on which every member of Γ is true and **P** is false. But we know that ~ **P** is true on a truth-value assignment if and only if **P** is false on that truth-value assignment. So it follows from our assumption that there is no truth-value on which every member of Γ is true and ~ **P** is true. But then there is no truth-value assignment on which every member of the set $\Gamma \cup$ {~ **P**} is true—so the set is truth-functionally inconsistent. It follows from this proof that since {J ∨ C} ⊨ ~ (~ J & ~ C) the set {J ∨ C, ~ ~ (~ J & ~ C)} is truth-functionally inconsistent. The converse, that if $\Gamma \cup$ {~ **P**} is truth-functionally inconsistent then $\Gamma \models$ **P**, holds as well. The proof is left as an exercise.

It follows from this result, as well as the fact that an argument is truth-functionally valid if and only if the set consisting of the argument's premises truth-functionally entails the conclusion, that

> An argument of *SL* is *truth-functionally valid* if and only if the set containing as its only members the premises of the argument and the negation of the conclusion is *truth-functionally inconsistent.*

So the argument

(A ⊃ D) & H
F ∨ H

D

is truth-functionally valid if and only if {(A ⊃ D) & H, F ∨ H, ~ D} is truth-functionally inconsistent.

3.6E EXERCISES

1. Prove each of the following:
   a. If {~ **P**} is truth-functionally inconsistent, where **P** is a sentence of *SL*, then **P** is truth-functionally true.
   *b. If **P** ≡ **Q** is truth-functionally true, where **P** and **Q** are sentences of *SL*, then **P** and **Q** are truth-functionally equivalent.
   c. If $\Gamma \cup$ {~ **P**} is truth-functionally inconsistent, where Γ is a set of sentences of *SL* and **P** is a sentence of *SL*, then $\Gamma \models$ **P**.

2. Prove each of the following:

a. A sentence **P** is truth-functionally true if and only if $\varnothing \models \mathbf{P}$.

*b. $\Gamma \models \mathbf{P} \supset \mathbf{Q}$, where Γ is a set of sentences of *SL* and **P** and **Q** are sentences of *SL*, if and only if $\Gamma \cup \{\mathbf{P}\} \models \mathbf{Q}$.

c. If Γ is truth-functionally inconsistent, where Γ is a set of sentences of *SL*, then Γ truth-functionally entails every sentence of *SL*.

*d. For any set Γ of sentences of *SL* and any truth-functionally false sentence **P** of *SL*, $\Gamma \cup \{\mathbf{P}\}$ is truth-functionally inconsistent.

3. Prove each of the following:

a. If a set Γ of sentences of *SL* is truth-functionally consistent and **P** is a truth-functionally true sentence of *SL*, then $\Gamma \cup \{\mathbf{P}\}$ is truth-functionally consistent.

*b. If $\Gamma \models \mathbf{P}$ and $\Gamma \models \sim \mathbf{P}$, for some sentence **P** and set Γ of sentences of *SL*, then Γ is truth-functionally inconsistent.

4. Prove each of the following:

a. If $\{\mathbf{P}\} \models \mathbf{Q}$ and $\{\sim \mathbf{P}\} \models \mathbf{R}$, where **P**, **Q**, and **R** are sentences of *SL*, then $\mathbf{Q} \vee \mathbf{R}$ is truth-functionally true.

*b. If **P** and **Q** are truth-functionally equivalent, where **P** and **Q** are sentences of *SL*, then for any sentence **R** of *SL*, $\{\mathbf{P}\} \models \mathbf{R}$ if and only if $\{\mathbf{Q}\} \models \mathbf{R}$.

c. If $\Gamma \models \mathbf{P}$ and $\Gamma' \models \mathbf{Q}$, where Γ and Γ' are sets of sentences of *SL* and **P** and **Q** are sentences of *SL*, then $\Gamma \cup \Gamma' \models \mathbf{P} \,\&\, \mathbf{Q}$, where $\Gamma \cup \Gamma'$ is the set that contains all the sentences in Γ and all the sentences in Γ'.

GLOSSARY

TRUTH-FUNCTIONAL TRUTH: A sentence **P** of *SL* is *truth-functionally true* if and only if **P** is true on every truth-value assignment.

TRUTH-FUNCTIONAL FALSITY: A sentence **P** of *SL* is *truth-functionally false* if and only if **P** is false on every truth-value assignment.

TRUTH-FUNCTIONAL INDETERMINACY: A sentence **P** of *SL* is *truth-functionally indeterminate* if and only if **P** is neither truth-functionally true nor truth-functionally false.

TRUTH-FUNCTIONAL EQUIVALENCE: Sentences **P** and **Q** of *SL* are *truth-functionally equivalent* if and only if there is no truth-value assignment on which **P** and **Q** have different truth-values.

TRUTH-FUNCTIONAL CONSISTENCY: A set of sentences of *SL* is *truth-functionally consistent* if and only if there is at least one truth-value assignment on which all the members of the set are true. A set of sentences of *SL* is *truth-functionally inconsistent* if and only if the set is not truth-functionally consistent.

TRUTH-FUNCTIONAL ENTAILMENT: A set Γ of sentences of *SL truth-functionally entails* a sentence **P** of *SL* if and only if there is no truth-value assignment on which every member of Γ is true and **P** is false.

TRUTH-FUNCTIONAL VALIDITY: An argument of *SL* is *truth-functionally valid* if and only if there is no truth-value assignment on which all the premises are true and the conclusion is false. An argument of *SL* is *truth-functionally invalid* if and only if it is not truth-functionally valid.

Chapter 4

SENTENTIAL LOGIC: TRUTH-TREES

4.1 THE TRUTH-TREE METHOD

In Chapter 3 we used the notion of a truth-value assignment to give formal accounts of the important semantic concepts of truth-functional logic. At the end of Chapter 3, we saw that, once truth-functional consistency has been defined based on the concept of a truth-value assignment, the remaining semantic concepts of sentential logic can be explicated in terms of truth-functional consistency. In this chapter we make use of this fact to provide an additional method, the **truth-tree** method, of determining whether truth-functional properties hold for sentences and sets of sentences of *SL*. Truth-trees provide a systematic method of searching for truth-value assignments that are of special interest—for example, a truth-value assignment on which a given sentence of *SL* is false, or a truth-value assignment on which the premises of a given argument of *SL* are true and the conclusion false. The truth-tree method also reveals when no such truth-value assignments exist.

The truth-table method is mechanical. And the truth-tree method we develop in this chapter can easily be made so. The advantage of truth-tables is that they graphically display how the truth-values of truth-functionally compound sentences are generated from the truth-values of their components. The

disadvantage of truth-tables is that they become unwieldy when the number of distinct atomic components of the sentence or sentences being tested is much greater than 3. Truth-trees, it must be admitted, can also become unwieldy. However, the size and complexity of truth-trees are not as direct a function of the number of distinct atomic components of the sentences being tested as are the size and complexity of truth-tables. Sets of sentences with a large number of distinct atomic components frequently have reasonably concise truth-trees. What is of theoretical importance here, as with truth-tables, is that the truth-tree system can be used, for any finite set of sentences of *SL*, to yield, in a finite number of steps, an answer to the question 'Is this set truth-functionally consistent?' We establish this claim in Chapter 11.

The rules we will use in constructing truth-trees are derived directly from the characteristic truth-tables for the five truth-functional connectives. For this reason, and because truth-value assignments on which all the members of the set being tested are true can readily be recovered from truth-trees for consistent sets, we take truth-trees to constitute a second semantic method of determining whether the truth-functional properties defined in Chapter 3 hold for sentences and finite sets of sentences of *SL*.

4.2 TRUTH-TREE RULES FOR SENTENCES CONTAINING '~', '∨', AND '&'

Recall that a set of sentences is truth-functionally consistent if and only if there is at least one truth-value assignment on which all the members of that set are true. Sometimes we can tell at a glance that a set is truth-functionally consistent or that it is truth-functionally inconsistent. For example, {A & ~ B, C} is fairly obviously a consistent set, and {A & ~ B, ~ A} is fairly obviously an inconsistent set. But most of us cannot tell immediately whether {(~ B & C) & (A ∨ B), A & C} is consistent or inconsistent. Truth-trees provide us with a systematic method for determining, for any finite set of sentences of *SL*, whether that set is truth-functionally consistent.

We begin with some easy examples. First, we show that {A & ~ B, C} is indeed truth-functionally consistent. In constructing a truth-tree, the first step is to display the members of the set being tested, one above another, in a column:

A & ~ B
C

What we want to know is whether there is a truth-value assignment on which all the sentences in the column are true. The first sentence in the column is a conjunction, and we know that a conjunction is true on a truth-value assignment if and only if both its conjuncts are true on that assignment. So we can break down or decompose 'A & ~ B' into its conjuncts, adding these conjuncts, one below the other, to our column:

A & – B✓
C
A
– B

We put a check mark after 'A & – B' to indicate that we are finished with it. We have, in effect, replaced it in our list of sentences with two simpler sentences. This replacement is appropriate inasmuch as 'A & – B' is true if and only if both 'A' and '– B' are true. All the sentences on the tree either have been decomposed (and checked off) or are atomic sentences or negations of atomic sentences. We shall call a sentence that is either an atomic sentence or the negation of an atomic sentence a *literal*. Once we have a tree on which the only undecomposed sentences are literals, it is easy to determine whether there is a truth-value assignment on which all the members of the set we are testing are true—that is, whether the set is truth-functionally consistent. We try to generate the desired assignment by reading up the column of sentences, starting at the bottom. We pay attention only to the literals. If a literal is an atomic sentence, we assign the truth-value **T** to that atomic sentence. If the literal is the *negation* of an atomic sentence, we assign the truth-value **F** to the atomic sentence (not to the literal). Applying this procedure to the tree, we generate the following assignments:

A	B	C
T	**F**	**T**

Clearly every member of the set we are testing, {A & – B, C}, is true on every truth-value assignment that assigns these truth-values to 'A', 'B', and 'C'. Therefore the set we are testing is truth-functionally consistent.

Next we shall use the truth-tree method to show that {A & – B, – A} is truth-functionally inconsistent. We begin, as before, by listing the members of the set in a column. Then we decompose the conjunction 'A & – B'.

A & – B✓
– A
A
– B

All the literals on this tree must be true for the members of our set to be true. '– B' occurs on the tree. To make it true we must assign 'B' the truth-value **F**. 'A' and '– A' both occur on the tree. To make the former true, we must assign 'A' the truth-value **T**. To make the latter true, we must assign 'A' the truth-value **F**. But clearly there is no truth-value assignment on which 'A' is assigned both the truth-value **T** and the truth-value **F**. Hence there is no truth-value assignment on which all the literals on this tree are true. We entered the literals

'A' and '~ B' because they must be true if the first sentence, 'A & ~ B', is to be true, so it follows that there is no truth-value assignment on which the members of the set {A & ~ B, ~A} that we are testing are both true. The set is therefore truth-functionally inconsistent.

The truth-trees we have constructed so far both consist of single branches, that is, of single columns of sentences of *SL*. However, many trees are more complex. We can illustrate how multiple branches are formed by constructing a tree for {A & ~ B, C, ~ A ∨ ~ C}. We formed this set by adding one more sentence, '~ A ∨ ~ C', to the first set we tested. Thus we can use our tree for the first set as a model for the first part of our tree for this set, adding the additional sentence in the set being tested after the first two:

A & ~ B✓
C
~ A ∨ ~ C
A
~ B

This tree is not yet complete; it contains a truth-functionally compound sentence, '~ A ∨ ~ C', that is not a literal and that has not been decomposed. We must decompose this sentence in such a way as to show what sentences must be true for this disjunction to be true. For a disjunction to be true, only one of its disjuncts need be true (though both may be true). If we add both disjuncts to our list, one below the other, we would incorrectly be *requiring* that both those disjuncts be true. But we can represent the fact that there are *alternative* ways in which a disjunction can be made true by *branching* our tree:

A & ~ B✓
C
~ A ∨ ~ C✓
A
~ B
~ A | ~ C

By displaying '~ A' and '~ C' on a single line, rather than one below the other, we show that making either of them true is sufficient to make the sentence we are decomposing true. We now have two branches, and we have to inspect each to see whether there is a way of making all the sentences we are testing true. If there is such a way, it will involve either making all the literals on the left branch true or making all the literals on the right branch true (or both, since '∨' is inclusive). A **branch,** in our sense, consists of all the sentences that can be reached by starting with a sentence at the bottom of the tree and tracing a

path upward through the tree, never moving down or horizontally, and ending with the sentence at the top of the tree. A sentence may thus occur just once on a tree but still be on several branches. In our present example the members of the set we are testing, along with the literals 'A' and '~ B', occur on both branches of the tree, the branch ending in '~ A' and the branch ending in '~ C'.

Inspecting the branches of this tree, we can see that neither branch will yield a truth-value assignment on which all the members of the set we are testing are true. The left-hand branch shows us that to obtain such an assignment we would have to assign 'A' the truth-value **F** and also the truth-value **T**, since both '~ A' and 'A' occur on that branch. Similarly the right-hand branch shows us that to obtain such an assignment we would have to assign 'C' both the truth-value **F** and the truth-value **T** since both '~ C' and 'C' occur on that branch. Neither alternative can produce a truth-value assignment. So there is no truth-value assignment on which all the members of the set we are testing are true; that set is truth-functionally inconsistent.

In constructing truth-trees we decompose truth-functionally compound sentences that are not literals in such a way as to display the truth-conditions for those compounds. Truth-trees are, in effect, ways of searching for truth-value assignments on which the sentences in the set being tested are true. A branch that contains both an atomic sentence and the negation of that sentence represents a failure to find such an assignment, for no truth-value assignment assigns any atomic sentence both the truth-value **T** and the truth-value **F**. A branch that does contain both an atomic sentence and the negation of that sentence is a **closed branch.** A branch that is not closed is **open.** Eventually each branch will either close or become a **completed open branch,** that is, an open branch such that every sentence on it either is a literal or has been decomposed. Note that since a completed open branch is a kind of open branch it will not contain an atomic sentence and the negation of that sentence. Any truth-value assignment on which all the literals on a completed open branch are true will be, owing to the tree rules, an assignment on which the members of the set being tested are also all true.[1] To generate such an assignment from an open branch, we assign **T** to every atomic sentence occurring on that branch, **F** to every atomic sentence whose negation occurs on that branch, and either **T** or **F** (it does not matter which) to every other atomic sentence in the language *SL*. Accordingly:

> A finite set Γ of sentences of *SL* is *truth-functionally consistent* if and only if Γ has a truth-tree with at least one completed open branch.

[1] These results are proven in Chapter 11.

We will call a truth-tree each of whose branches is closed a **closed truth-tree.** Accordingly we can also say this:

> A finite set Γ of sentences of *SL* is *truth-functionally inconsistent* if and only if Γ has a closed truth-tree.

We note a special case. The truth-tree for the empty set is the null tree, that is, the truth-tree that has no sentences on its single null branch. The null branch therefore is not closed and trivially counts as a completed open branch, so the empty set is truth-functionally consistent by our account, a desirable result.[2]

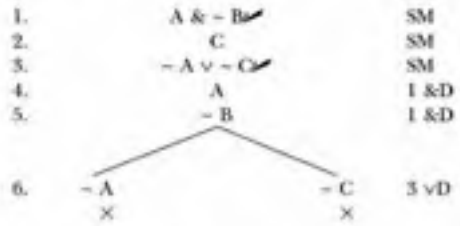
A completed tree is a tree each of whose branches is either closed or is a completed open branch. An **open tree** is a tree that is not closed. (Note that an open tree need not be a completed tree, and that an open tree that is not completed may become a closed tree.) In summary, we shall use the following vocabulary for truth-trees:

Branch:	All the sentences that can be reached by starting with a sentence at the bottom of a tree and tracing an upward path through the tree, ending with the first sentence listed at the top of the tree
Closed branch:	A branch containing both an atomic sentence and the negation of that sentence
Closed truth-tree:	A truth-tree each of whose branches is closed
Open branch:	A branch that is not closed
Completed open branch:	An open branch such that every sentence on it is either a literal or has been decomposed
Completed truth-tree:	A truth-tree each of whose branches is either closed or is a completed open branch
Open truth-tree:	A truth-tree that is not closed

For the sake of clarity, we adopt the convention of numbering the lines of our truth-trees in a column on the left. We also include a justification column

[2]Our truth-trees can only be used to test finite sets of sentences, hence the restrictions in the boxes. But we know that infinite sets of sentences of *SL* are also either consistent or inconsistent. In Chapter 6 we shall prove that an infinite set of sentences of *SL* is truth-functionally consistent if and only if every finite subset of that set is truth-functionally consistent. Therefore, we can also say that an infinite set Γ of sentences of *SL* is truth-functionally consistent if and only if every finite subset of Γ has a truth-tree with at least one completed open branch, and an infinite set Γ of sentences of *SL* is truth-functionally inconsistent if and only if at least one finite subset of Γ has a closed truth-tree.

on the right. The column of line numbers and the column of justifications are not, strictly speaking, parts of truth-trees. They are notational devices we use to make the structure and logic of trees more transparent. The lines containing the members of the set we are testing for truth-functional consistency will all be justified by entering the abbreviation 'SM' for 'set member'. Later lines will be justified by entering a line number and a rule abbreviation. The two rules we have presented so far are *Conjunction Decomposition* and *Disjunction Decomposition*, abbreviated '&D' and '∨D', respectively. When these conventions are followed, our last tree appears as follows:



For the sake of clarity, no line of the justification column is allowed to contain more than one rule abbreviation or reference to more than one previous line. An '×' under a branch of a tree indicates that that branch is closed. We use 'O' to indicate that a branch is a completed open branch, so our first tree will now look like this:

1.	A & ~ B ✓	SM
2.	C	SM
3.	A	1 &D
4.	~ B	1 &D
	O	

As with line numbers and justifications, check marks after decomposed sentences as well as closed branch and completed open branch indicators ('×' and 'O') are notational conveniences and are not literally parts of truth-trees.

We noted earlier that the set [(~ B & C) & (A ∨ B), A & C] is neither obviously consistent nor obviously inconsistent. We can now use the truth-tree method to test this set for truth-functional consistency. We begin our tree in the usual way:

1.	(~ B & C) & (A ∨ B) ✓	SM
2.	A & C	SM
3.	~ B & C	1 &D
4.	A ∨ B	1 &D

Here the results of using the rule Conjunction Decomposition are themselves truth-functionally compound sentences that will have to be decomposed.

First we decompose '(– B & C) & (A ∨ B)', and then we decompose '– B & C' and 'A ∨ B'.

1.	(– B & C) & (A ∨ B) ✓	SM
2.	A & C ✓	SM
3.	– B & C ✓	1 &D
4.	A ∨ B ✓	1 &D
5.	A	2 &D
6.	C	2 &D
7.	– B	3 &D
8.	C	3 &D
9.	A ○ | B ×	4 ∨D

The only sentences on this tree that have not been decomposed are literals, so there are no more sentences that can be decomposed. The branch on the left is therefore a completed open branch. This tree is the first one we have encountered that has at least one completed open branch *and* at least one closed branch. A completed open truth-tree often has one or more closed branches.

The literals on the completed open branch of the preceding tree can be used to describe truth-value assignments on which every member of the set {(– B & C) & (A ∨ B), A & C} will be true. To make all of those literals true, we must assign 'A' and 'C' the truth-value **T** and 'B' the truth-value **F** (since '– B' occurs on the open branch). The literals, and therefore the set members being tested, will therefore all be true on every truth-value assignment that assigns the following values to 'A', 'B', and 'C':

A	B	C
T	**F**	**T**

That 'A' and 'C' each occur twice on the open branch has no special significance for recovering truth-value assignments; they so occur because their truth is required by two different members of the set we are testing.

The process of describing those truth-value assignments that will make all of the literals on a completed open branch true is called *recovering a set of truth-value assignments*. In recovering truth-value assignments we write down only the relevant (finite) parts of those truth-value assignments: we specify only the assignments made to the atomic components of the sentence or set of sentences for which the tree was constructed. However, it should be remembered that a truth-value assignment assigns values to *every* atomic sentence of *SL*. So when we specify the assignments that are made to several atomic sentences of *SL*, as we did in the previous paragraph, we have not thereby specified a single truth-value assignment. Rather, we have described a *set* of truth-value assignments, those that assign

the specified values to the relevant atomic sentences and that additionally assign truth-values to all other atomic sentences of the language (and there are infinitely many of these). For example, the set of truth-value assignments that we recovered in the previous paragraph will all assign **T** to 'A', **F** to 'B', and **T** to 'C'. One of these truth-value assignments will assign **T** to all other atomic sentences, one will assign **F** to all other atomic sentences, and the rest (there are infinitely many) will assign **T** to some of the remaining atomic sentences and **F** to the others. So although we will only display truth-values assigned to a small number of atomic sentences, we will always be recovering *sets* of truth-value assignments, those that agree on the truth-values assigned to the relevant atomic sentences.

In the previous example we decomposed the two conjunctions '$B \& C$' and '$A \& C$' before we decomposed the disjunction '$A \vee B$'. There is no rule stating that we must do this. Any order of decomposition would have led to a tree with a completed open branch. Altering the order of decomposition sometimes makes a tree more or less complex, but it never alters the final result. For a given set of sentences, if one order of decomposition generates a completed open branch, all orders of decomposition generate completed open branches. And if one order of decomposition generates a closed tree, all orders of decomposition generate closed trees. In the present case decomposing '$A \vee B$' earlier would produce the following tree:

1.	$(\sim B \& C) \& (A \vee B)$		SM
2.	$A \& C$		SM
3.	$\sim B \& C$		1 &D
4.	$A \vee B$		1 &D
5.	A	B	4 ∨D
6.	A	A	2 &D
7.	C	C	2 &D
8.	$\sim B$	$\sim B$	3 &D
9.	C	C	3 &D
	o	×	

In decomposing '$A \vee B$' before '$A \& C$ and '$\sim B \& C$', we produced a branching early in the tree. Consequently we had to enter the results of decomposing the remaining sentences on both open branches, thus considerably complicating the tree. (It is important to remember that when a sentence is decomposed the results of that decomposition must be entered *on every open branch passing through the sentence being decomposed.*) But this more complex tree also has a completed open branch, so the end result is the same. Note also that the '×' was not placed under the right-hand branch until both '$\sim B$' and 'C' had been entered on that branch, even though the closure here occurs because the branch contains the pair of literals 'B' and '$\sim B$'. We do not stop halfway through the application of a decomposition rule to mark closures.

So far we have presented the rules for decomposing conjunctions and disjunctions but not those for decomposing conditionals, biconditionals, and

negations. To be able to construct truth-trees for all finite sets of sentences of *SL*, we need to know how to decompose these truth-functionally compound sentences as well. How a negation is to be decomposed depends upon the kind of sentence being negated, and we shall have separate rules for decomposing negated conjunctions, negated disjunctions, negated material conditionals, negated material biconditionals, and negated negations. Literals—that is, atomic sentences and negations of atomic sentences—are not decomposed.

We have already used the rules for decomposing conjunctions and disjunctions. Schematically these rules are

Conjunction Decomposition (&D)

P & Q ✓
P
Q

Disjunction Decomposition (∨D)

P ∨ Q ✓
P | **Q**

The rule for decomposing negated negations is also obvious. A sentence of the form **~ ~ P** is true if and only if **P** is true. Hence we have the rule

Negated Negation Decomposition (~ ~ D)

~ ~ P ✓
P

Turning to negated conjunctions, a sentence of the form **~ (P & Q)** is truth-functionally equivalent to the corresponding sentence of the form **~ P ∨ ~ Q**, and sentences of this latter form are disjunctions. The rule for decomposing disjunctions is a branching rule, that is, a rule that increases the number of branches on a tree, so the rule for decomposing negated conjunctions will be a branching rule as well:

Negated Conjunction Decomposition (~ &D)

~ (P & Q) ✓
~ P | **~ Q**

Similarly we know that, for any sentences **P** and **Q**, **~ (P ∨ Q)** and **~ P & ~ Q** are truth-functionally equivalent sentences, and we already know how to decompose conjunctions. So we have

Negated Disjunction Decomposition (~ ∨D)

~ (P ∨ Q) ✓
~ P
~ Q

We now have all the rules we need to decompose sentences that contain only the connectives '&', '∨', and '~'. It will be useful to pause here to construct a few truth-trees for sets consisting of sentences whose only connectives are those just mentioned. Consider first the set [A & ~ B, ~ (B ∨ ~ A), (B & A) ∨ (B & ~ A)]. This set is truth-functionally inconsistent, as the following truth-tree shows:

1.	A & ~ B ✓		SM
2.	~ (B ∨ ~ A) ✓		SM
3.	(B & A) ∨ (B & ~ A) ✓		SM
4.	A		1 &D
5.	~ B		1 &D
6.	~ B		2 ~ ∨D
7.	~ ~ A ✓		2 ~ ∨D
8.	A		7 ~ ~ D
9.	B & A ✓	B & ~ A ✓	3 ∨D
10.	B	B	9 &D
11.	A	~ A	9 &D
	×	×	

Both branches of this tree are closed, so the tree is closed, and no truth-value assignment can be recovered from it. Hence there is no truth-value assignment on which every member of the set being tested is true. Therefore that set is truth-functionally inconsistent. Note that decomposing the sentences on lines 1 and 2 does not increase the number of branches, whereas decomposing the sentence on line 3 does. Since branching makes for complexity, we decomposed the sentences on lines 1 and 2 before we decomposed the sentence on line 3. This is a use of the first of several strategies we will develop for keeping trees simple.

Strategy 1: Give priority to decomposing sentences whose decompositions do not require branching.

Note also that the justifications given for lines 10 and 11 actually apply to the decomposition of two sentences, 'B & A' and 'B & ~ A'. No confusion results here because both sentences occur on line 9, both are conjunctions, and both are therefore decomposed by the rule Conjunction Decomposition. For the sake of expository clarity, we shall avoid writing the products of multiple decompositions on the same line except where those products are, as here, obtained by applying the same decomposition rule multiple times to sentences occurring on the same earlier line.

Consider next a set whose members contain three distinct atomic sentences: {G ∨ (H ∨ I), ~ (G ∨ H), ~ (H ∨ I), ~ (I ∨ G)}. Here is a tree for this set:

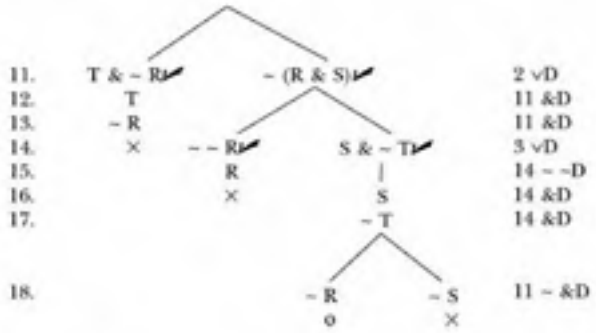
1.	G ∨ (H ∨ I)✓	SM
2.	~ (G ∨ H)✓	SM
3.	~ (H ∨ I)✓	SM
4.	~ (I ∨ G)✓	SM
5.	~ G	2 ~ ∨D
6.	~ H	2 ~ ∨D
7.	~ H	3 ~ ∨D
8.	~ I	3 ~ ∨D
9.	~ I	4 ~ ∨D
10.	~ G	4 ~ ∨D
11.	G × | H ∨ I✓	1 ∨D
12.	H × | I ×	11 ∨D

This tree has three branches, all of which are closed. Hence the set being tested is truth-functionally inconsistent.

Next we construct a truth-tree for the set {~ (~ S ∨ T) & ~ (T ∨ R), (T & ~ R) ∨ ~ (R & S), ~ ~ R ∨ (S & ~ T)}. Following our maxim of not using rules that produce branching until forced to do so, we obtain the following as the first part of our truth-tree for this set:

1.	~ (~ S ∨ T) & ~ (T ∨ R)✓	SM
2.	(T & ~ R) ∨ ~ (R & S)	SM
3.	~ ~ R ∨ (S & ~ T)	SM
4.	~ (~ S ∨ T)✓	1 &D
5.	~ (T ∨ R)✓	1 &D
6.	~ ~ S✓	4 ~ ∨D
7.	~ T	4 ~ ∨D
8.	~ T	5 ~ ∨D
9.	~ R	5 ~ ∨D
10.	S	6 ~ ~ D

Note that decomposing '~ (~ S ∨ T)' (using the rule Negated Disjunction Decomposition) yields '~ ~ S' and '~ T' on lines 6 and 7, respectively, *not* 'S' and '~ T'. To get 'S' we had to apply Negated Negation Decomposition to the sentence '~ ~ S'. The tree can be completed as follows:

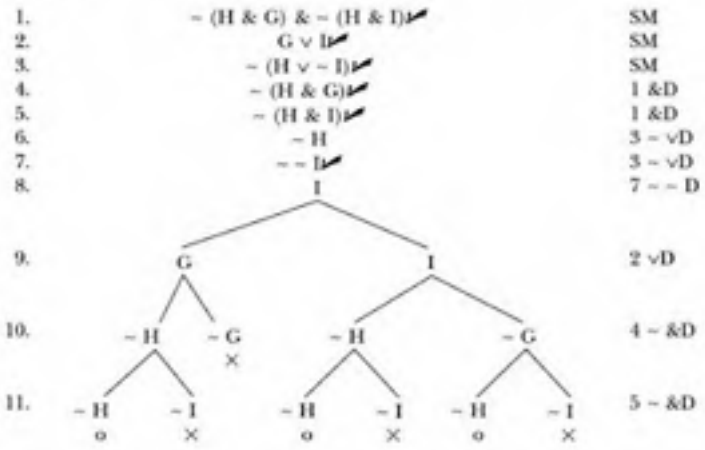


This tree has three closed branches and one completed open branch. The set we are testing is therefore truth-functionally consistent. The truth-value assignments that can be recovered from the open branch must all assign **F** to 'R', **F** to 'T', and **T** to 'S':

R	S	T
F	**T**	**F**

The three set members will all be true on every truth-value assignment that assigns these values to 'R', 'S', and 'T', no matter what values are assigned to the other atomic sentences of *SL*.

Completed truth-trees frequently contain more than one completed open branch. Consider this tree for the set {~ (H & G) & ~ (H & I), G ∨ I, ~ (H ∨ ~ I)}:



The tree has three open branches. The truth-value assignments that we can recover from the leftmost branch make the following assignments to 'H', 'G', and I:

G	H	I
T	F	T

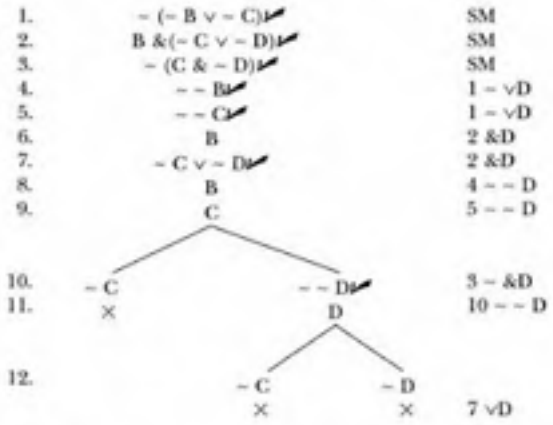
The truth-value assignments that we can recover from the middle open branch must assign the following values to 'H' and 'T', but that branch is mute concerning 'G':

G	H	I
	F	T

The significance of this is that the members of the set being tested are all true on every truth-value assignment that assigns **F** to 'H' and **T** to 'I'. What is assigned to 'G', given these assignments to 'H' and 'I', does not matter. Hence we can recover *two* sets of truth-value assignments from this branch, that is, those that assign the values in the first row to 'G', 'H', and T, and those that assign the values in the second row:

G	H	I
T	F	T
F	F	T

We complete this discussion by giving two additional sample trees. Both are closed, so in each case the set being tested is truth-functionally inconsistent.



1.	~ [(A ∨ B)∨ ~ C]	SM
2.	C & ~ (A & B)	SM
3.	(A ∨ C) & (A ∨ B)	SM
4.	A	SM
5.	~ (A ∨ B)	1 ~ ∨D
6.	~ ~ C	1 ~ ∨D
7.	C	2 &D
8.	~ (A & B)	2 &D
9.	A ∨ C	3 &D
10.	A ∨ B	3 &D
11.	~ A	5 ~ ∨D
12.	~ B	5 ~ ∨D
	×	

4.2E EXERCISES

1. Use the truth-tree method to test each of the following sets of sentences for truth-functional consistency. If a set is consistent, recover from one completed open branch the set of truth-value assignments on which every member of the set is true.

a. {A & ~ (B ∨ A)}
*b. {A & ~ (B & A)}
c. {~ (A ∨ B) & (A ∨ ~ B)}
*d. {~ A & ~ (A & B), B}
e. {(A ∨ B) & (A ∨ ~ B)}
*f. {(J & ~ K) & I, ~ I ∨ K}
g. {(J ∨ ~ K) & I, ~ I ∨ K}
*h. {(H & ~ I) & (I ∨ ~ H)}
i. {(H ∨ ~ I) & I, ~ (H & I)}
*j. {(A & B) ∨ (A & C), ~ (A & B)}
k. {~ (A & B), ~ (~ C ∨ B), ~ (A & C)}
*l. {(A & B) ∨ (A & C), ~ (A ∨ B)}
m. {(A ∨ B) & (A ∨ C), ~ C & ~ A}
*n. {(A ∨ B) & (A ∨ C), C & ~ A, ~ B ∨ ~ A}
o. {(H & ~ I) ∨ (I ∨ ~ H), J ∨ I, ~ J}

2. Use the truth-tree method to test each of the following sets of sentences for truth-functional consistency. If a set is consistent, recover from one completed open branch the set of truth-value assignments on which every member of the set is true.

a. {~ (H & I), H ∨ I}
*b. {~ [(F ∨ ~ F) & G]}
c. {~ (H & I) ∨ J, ~ (J ∨ ~ I)}
*d. {~ [(A ∨ B) ∨ C], ~ D ∨ C, D}
e. {A & (B & C), ~ [A & (B & C)]}
*f. {A & (B & ~ C), ~ [A & (B & C)]}
g. {~ C ∨ (A & B), C, ~ (A & B)}
*h. {~ (~ A ∨ B), A ∨ ~ B, ~ (~ B & ~ A)}
i. {(~ F & ~ G) & [(G ∨ ~ I) & (I ∨ ~ H)]}

*j. {~ (~ A ∨ ~ B), ~ [A & ~ (B & C)], A ∨ (B ∨ C)}
k. {(F ∨ ~ G) & [(G ∨ ~ I) & (I ∨ ~ H)]}
*l. {~ [A & (~ B & ~ C)], ~ A ∨ ~ C, ~ (~ B ∨ ~ ~ C)}
m. {A ∨ (B ∨ C), ~ (A ∨ B), ~ (B & C), ~ (A & C)}
*n. {H ∨ ~ I) & (I ∨ ~ G), ~ (H & G), H ∨ (~ I & ~ G)}

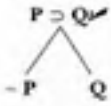
4.3 RULES FOR SENTENCES CONTAINING '⊃' AND '≡'

We now need to develop decomposition rules for material conditionals, for material biconditionals, and for the negations of each of these. We know that, for any sentences **P** and **Q** of *SL*, **P** ⊃ **Q** is equivalent to ~ **P** ∨ **Q**. And we have already developed a rule for decomposing disjunctions:

Disjunction Decomposition (∨D)

Given this, and the fact that **P** ⊃ **Q** is equivalent to ~ **P** ∨ **Q**, the appropriate decomposition rule for **P** ⊃ **Q** is

Conditional Decomposition (⊃D)

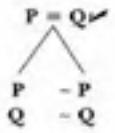


The negation of a material conditional, ~ (**P** ⊃ **Q**), is true if and only if that conditional is false, and a conditional is false if and only if its antecedent is true and its consequent is false. In other words, for any sentences **P** and **Q** of *SL*, ~ (**P** ⊃ **Q**) and **P** & ~ **Q** are truth-functionally equivalent sentences. Given the rule already developed for decomposing conjunctions, the appropriate rule for decomposing negated conditionals is clearly

Negated Conditional Decomposition (~ ⊃D)

The only rules we have left to present are those for material biconditionals and negated material biconditionals. A material biconditional is true if and only if either its immediate components are both true or its immediate components are both false. Alternatively, for any sentences **P** and **Q**, **P** ≡ **Q** is truth-functionally equivalent to (**P** & **Q**) ∨ (~ **P** & ~ **Q**). The rule for material biconditionals can thus be thought of as a combination of the rule for disjunctions and the rule for conjunctions. Decomposing a material biconditional splits every open branch running through that material biconditional into two branches, and on each new branch we enter two sentences:

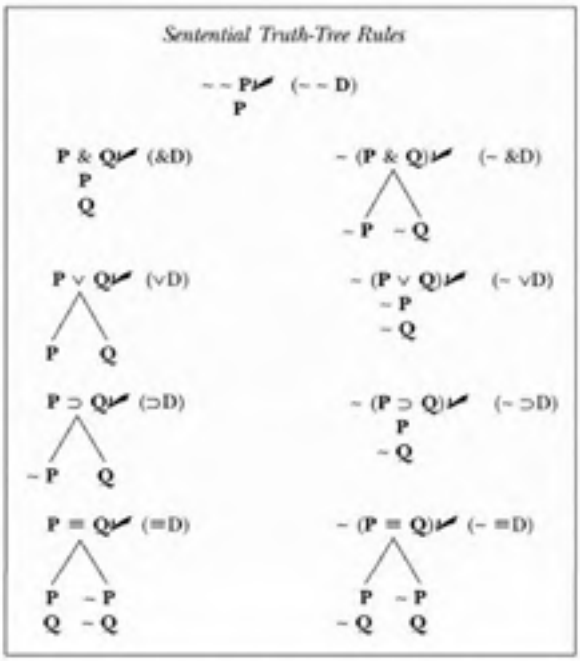
Biconditional Decomposition (≡D)



The rule for decomposing negations of material biconditionals is also a branching rule. Since a material biconditional is true if and only if its immediate components have the same truth-value, the negation of a material biconditional will be true if and only if the immediate components of the material biconditional have different truth-values. In other words, for ~ (**P** ≡ **Q**) to be true, either **P** must be true and **Q** false, or **P** must be false and **Q** true. So, for any sentences **P** and **Q**, ~ (**P** ≡ **Q**) is truth-functionally equivalent to (**P** & ~ **Q**) ∨ (~ **P** & **Q**). Hence

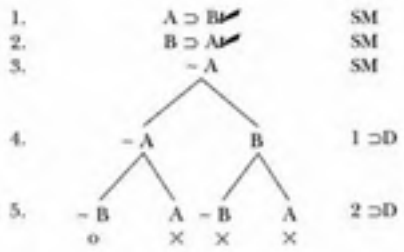
Negated Biconditional Decomposition (~ ≡D)

We present on the following page our complete set of rules for decomposing sentences of *SL*:



In learning how to construct trees, the best procedure is not simply to memorize the rules. Instead, try to grasp the rationale for the rules. One way to do this, as we have tried to show, is first to understand the bases for the simple rules for conjunctions, disjunctions, and negated negations and then to see how the other rules can be viewed as applications of these three simple rules.

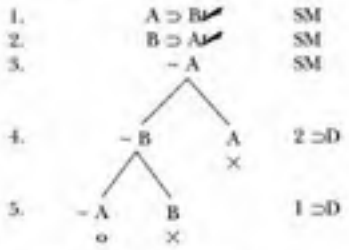
We begin with a few straightforward examples. Here is a tree for the set {A ⊃ B, B ⊃ A, ~ A}:



Earlier we noted that, when given a choice of decomposing a sentence that will produce branching or a sentence that will not, decomposing the latter generally

produces a simpler tree. In this case we had no such choice, for the set consists of a literal and two conditionals, and the rule for decomposing conditionals is a branching rule, that is, a rule that produces branches. After decomposing the first member of the set, we have two open branches, one ending in '~ A', the other in 'B'. Decomposing the second member of the set yields four branches, but three of them close, leaving us a completed tree with one open branch. The set is therefore truth-functionally consistent. All of the set members will be true on the recoverable truth-value assignments, those that assign **F** both to 'A' and to 'B'.

Here is another tree for the same set:



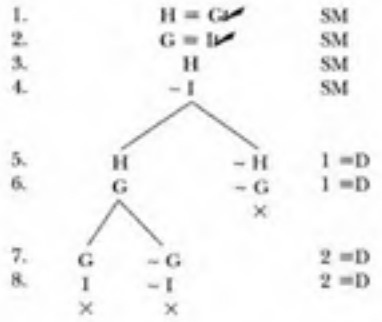
This tree is at least marginally simpler than the first; it has a total of three branches, whereas the first one had four branches. Although the sentences on lines 1 and 2 both produce multiple branches when decomposed, one of the branches produced by decomposing the sentence on line 2, the one ending in 'A', closes immediately. We can now formulate a second strategy for keeping trees simple:

Strategy 2: Give priority to decomposing sentences whose decompositions result in the closing of one or more branches.

Here is a tree for the set {~ (A ⊃ B), B ⊃ A}:

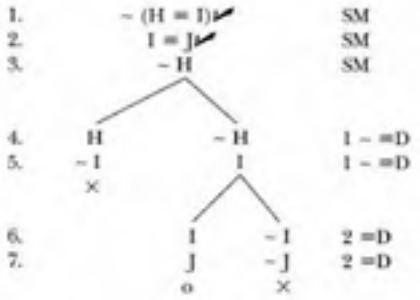
We decomposed the sentence on line 1 first because the rule Negated Conditional Decomposition does not branch. The tree has two completed open branches, so the set is truth-functionally consistent. The members of the set are all true on every truth-value assignment that assigns **T** to 'A' and **F** to 'B'.

Next we construct a tree for {H ≡ G, G ≡ I, H, ~ I}:



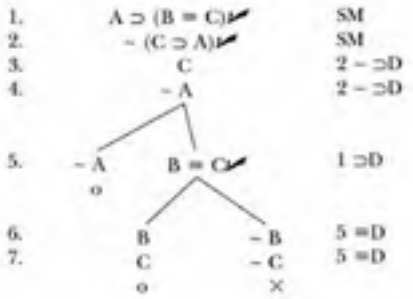
The tree is closed, so the set is truth-functionally inconsistent. Here the order of decomposition makes no difference. Both the sentence on line 1 and the sentence on line 2 branch when decomposed; whichever is decomposed first produces one closed branch.

The set {~ (H ≡ I), I ≡ J, ~ H} yields a tree with a completed open branch:



Here the order of decomposition does matter. Decomposing the sentence on line 1 first produces two branches, one of which immediately closes. Decomposing the sentence on line 2 first would produce two branches, neither of which would close immediately. From the one completed open branch we know that the set members will be true on every truth-value assignment that assigns **F** to 'H' and **T** to 'I' and 'J', and that the set is therefore truth-functionally consistent. It is important to remember, as illustrated here, that both the rule for decomposing material biconditionals and the rule for decomposing negated material biconditionals branch, and both introduce tildes.

Finally consider the set $\{A \supset (B \equiv C), \sim (C \supset A)\}$. Here is a tree:



The tree has two completed open branches, so the set is truth-functionally consistent. We can recover two sets of truth-value assignments from the completed open branches. The set members will be true on every truth-value assignment that assigns one of the following combinations of values to 'A', 'B', and 'C':

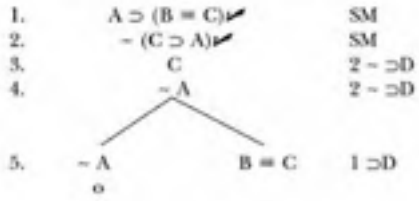
A	B	C
F	**T**	**T**
F	**F**	**T**

What is of interest here is that the left-hand open branch becomes a completed open branch at line 5. At this point we know the set we are testing is truth-functionally consistent because we know the tree we are constructing has, and will continue to have, at least one completed branch, no matter what happens to the other open branch of the tree (the branch containing 'B ≡ C' on line 5). This suggests a third strategy:

Strategy 3: Stop when a tree yields an answer to the question being asked.

Of course, nothing less than a completed tree, with every branch closed, shows that a set is truth-functionally *inconsistent*. But if our only interest is in determining whether a set is consistent, and an incomplete tree for it has a completed open branch, there is no virtue in completing the tree. As soon as a branch becomes a completed open branch, we know the answer to the question we are asking: The set is truth-functionaly consistent. We can recover truth-value

assignments demonstrating the set's consistency from the completed open branch.[3] So in the present case we could just as well have stopped after line 5, that is, with the open but incomplete tree:



4.3E EXERCISES

1. Use the truth-tree method to test each of the following sets of sentences for truth-functional consistency. If a set is consistent, recover one set of truth-value assignments on which every member of the set of sentences is true.

a. {~ (A ⊃ B), ~ (B ⊃ A)}
*b. {~ [~ A ⊃ (B ⊃ C)], A ⊃ C}
c. {B ⊃ (D ⊃ E), D & B}
*d. {H ≡ G, ~ H, G}
e. {H ≡ G, ~ G}
*f. {(H ≡ G) ≡ ~ H}
g. {H ≡ G, ~ (H ⊃ G)}
*h. {H ≡ ~ G, H ⊃ G}
i. {H ≡ G, G ≡ I, ~ (H ⊃ I)}
*j. {A ⊃ ~ (A ≡ B), ~ (A ⊃ B)}
k. {L ≡ (J & K), ~ J, ~ L ⊃ L}
*l. {B ≡ D, B ≡ ~ D}
m. {~ [A ≡ (B ≡ A)]}
*n. {[B ⊃ (A ∨ C)] & ~ ~ B, A ≡ ~ B}
o. {H ⊃ J, J ⊃ K, K ⊃ ~ H}
*p. {A ⊃ (B ⊃ (C ⊃ A)), ~ (B ⊃ ~ A)}

[3]We will continue, however, to show *completed* trees in our solutions to exercise problems. We do so because alternative orders of decomposition produce alternative trees, and the truth-value assignments that are recoverable from the first completed open branch on one tree may not coincide with those that can be recovered from the first complete open branch on another tree.

There is another way in which truth-trees could be shortened, namely, by defining a closed branch to be a branch that contains, for *any* sentence **P**, literal or not, both **P** and ~ **P**. Exactly the same sets would have closed trees, given this revised system, as they do in the present system, and exactly the same sets would have trees with at least one completed open branch as they do in the present system.

2. Use the truth-tree method to test each of the following sets of sentences for truth-functional consistency. If a set is consistent, recover one set of truth-value assignments on which every member of the set of sentences is true.

a. {~ [(A ⊃ ~ B) ⊃ (B ⊃ A)], ~ (~ A ⊃ ~ B)}
*b. {~ [(A ⊃ ~ B) ⊃ (B ⊃ A)], ~ (~ A ⊃ B)}
c. {(A & ~ C) ∨ (A & ~ B), A ⊃ B, C}
*d. {J ⊃ K, K ⊃ J, ~ (J ≡ K)}
e. {~ [A ⊃ (B ≡ C)], A ≡ ~ C, A ≡ B}
*f. {~ (A & ~ B) ⊃ ~ A, ~ (~ A & B) ⊃ ~ B, B & ~ A}
g. {~ (A ∨ B) ⊃ ~ A, ~ (A ∨ B) ⊃ ~ B, A}
*h. {~ (A ∨ B) ⊃ A, ~ (A ∨ B) ⊃ B, ~ B}
i. {A ≡ (B & ~ C), ~ A ∨ ~ B, ~ (~ B ≡ C)}
*j. {A ≡ (B & ~ C), ~ A ∨ ~ B, B ⊃ ~ C}
k. {A ≡ (~ B ≡ C), ~ A ⊃ (B ⊃ ~ C), ~ (A ⊃ ~ C)}
*l. {~ [A ⊃ (B ≡ C)], A ≡ ~ C, A ≡ B}
m. {J ⊃ (H ≡ ~ I), ~ (J ≡ H)}
*n. {(A & ~ B) ≡ (C & ~ E), ~ A ≡ E, B ≡ C}

4.4 MORE COMPLEX TRUTH-TREES

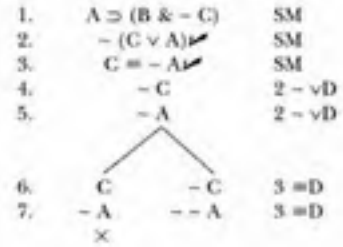
Consider the set {A ⊃ (B & ~ C), ~ (C ∨ A), C ≡ ~ A}. In constructing a truth-tree for this set, we start, as always, by listing the members of the set, one below the other:

1.	A ⊃ (B & ~ C)	SM
2.	~ (C ∨ A)	SM
3.	C ≡ ~ A	SM

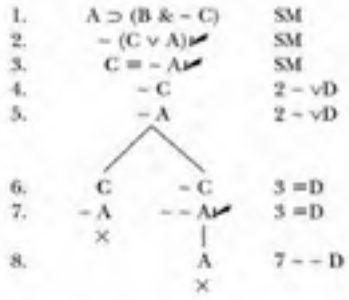
No member of the set is a literal; hence all are candidates for decomposition. Which sentence should we decompose first? In one sense it does not matter: If any order of decomposition yields a completed open branch, all will; and, if any order of decomposition yields a closed tree, all will. However, as noted earlier, it is desirable to keep truth-trees as concise as possible, and one strategy for doing this is to decompose sentences that do not branch before decomposing those that do.

The sentence on line 1, a material conditional, will branch when decomposed. So will the sentence on line 3, a material biconditional. But the sentence on line 2, a negated disjunction, will not branch, so we decompose it first. This leaves two undecomposed sentences on the tree that are not literals, 'A ⊃ (B & ~ C)' and 'C ≡ ~ A', both of which will branch when decomposed. Decomposing the material conditional will yield two open branches, one ending in '~ A' and the other in 'B & ~ C'. Neither of these branches will close

immediately. However, decomposing the material biconditional yields an immediate branch closure:



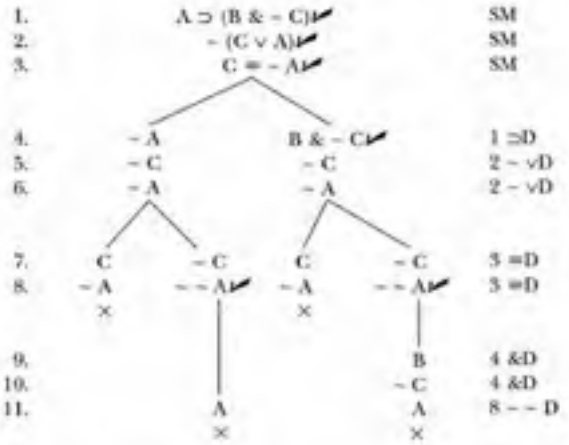
We are now left with just one open branch. There are two undecomposed nonliterals on that branch, 'A ⊃ (B & ~ C)' and '~ ~ A'. We decompose '~ ~ A' first, since negated negations do not branch when decomposed. Moreover, when we decompose '~ ~ A', we add 'A' to the one open branch of our tree and thus close that branch and the tree:



Several aspects of this tree are of interest. First, the tree is closed, and the set we are testing, [A ⊃ (B & ~ C), ~ (C ∨ A), C ≡ ~ A], is therefore truth-functionally inconsistent. Every attempt to find a truth-value assignment on which every member of that set is true ended in failure. Second, we have shown that the set is inconsistent without decomposing every nonliteral on the tree. The sentence on line 1 was never decomposed, since decomposing the other nonliterals on the tree generated a closed tree. What this shows is that the set we are testing would be inconsistent even without its first member, 'A ⊃ (B & ~ C)'. Whenever a branch closes we are through with that branch, even if it contains one or more undecomposed nonliterals.

This fairly concise tree was generated by following our strategies of giving priority to sentences whose decomposition does not require branching and

to sentences whose decomposition generates one or more closed branches. Had we ignored these strategies and simply worked straight down the tree, always decomposing every nonliteral on a given level before moving to a lower level, the result would have been the following more complex tree:



Here we have four branches, whereas in the earlier tree we had only two. Moreover this tree takes eleven lines to construct; the earlier one took only eight. But the difference between the trees is only one of complexity. Each tree shows equally well that the set of sentences we are testing is truth-functionally inconsistent, for each tree is closed.

The last of the preceding trees can be used to illustrate two important aspects of tree construction. Note that when we decompose '~ (C ∨ A)' at lines 5 and 6, the tree already has two open branches. Hence the results of decomposing this sentence must be entered on each of these open branches. The results of decomposing a sentence must always be entered on *every open branch that runs through the sentence being decomposed.*

Consider the tree after line 8 is completed: Two branches are closed, but two are open. We next decompose 'B & ~ C' on the right-hand branch only, at lines 9 and 10 (not on the left-hand branch because, although it is open, it does not go through 'B & ~ C'). We then decompose each occurrence of '~ ~ A' on line 8 by writing 'A' on line 11, at the end of each branch (since each branch does go through '~ ~ A'). Why didn't we put 'A' on the left-hand branch at line 9 at the same time that we put 'B' on the right-hand branch at line 9? It is because the policy we follow is this: *Trees are to be so constructed that every line of the tree is fully justified either by writing 'SM' in the justification column or by entering the number of one and only one earlier line and one and only one rule abbreviation in the justification column.* All the entries made on line 7 come from

line 3, and they are all obtained by one rule, Material Biconditional Decomposition. Had we tried to write 'A' at line 9 on the second branch from the left, we would have had two entries on that line coming from two different lines, by the use of two different rules, and thus would have been forced to enter both '8 ~ ~ D' and '4 &D' in the justification column for line 9, in clear violation of this policy.

We have so far specified three strategies for keeping truth-trees concise. We repeat them here and add a fourth:

Strategies for Constructing Truth-Trees

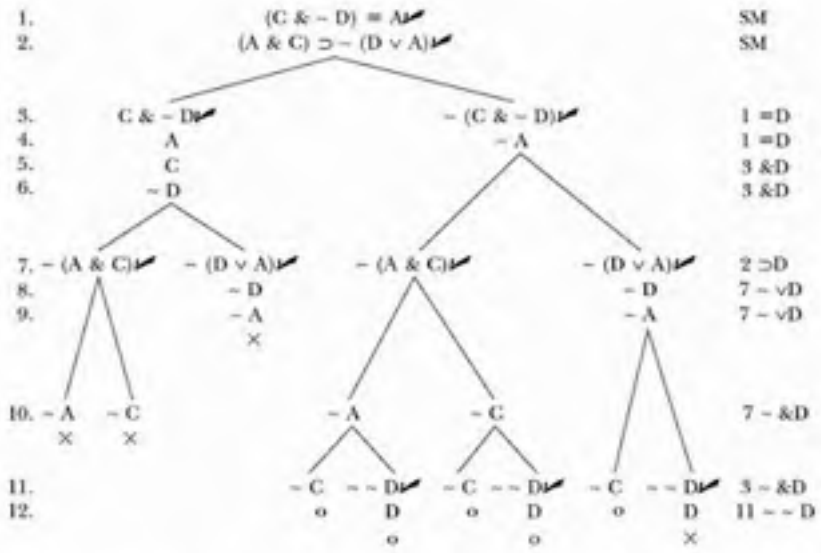
1. Give priority to decomposing sentences whose decompositions do not require branching.
2. Give priority to decomposing sentences whose decompositions result in the closing of one or more branches.
3. Stop when a tree yields an answer to the question being asked.
4. Where strategies 1–3 are not applicable, decompose the more complex sentences first.

The rationale for the first three strategies should be clear by now. The fourth strategy is designed to save tedious work, for a complex sentence takes more work to decompose than does a less complex one. Moreover, if a complex sentence is decomposed early in a tree, chances are there will be only a few open branches on which the results must be entered. If complex sentences are left until the end, it is likely that the results of decomposing them will have to be entered on many open branches. Roughly speaking, a sentence **P** is more complex than a sentence **Q** if decomposing **P** requires entering more sentences or longer sentences on a tree than does decomposing **Q**. In this sense longer sentences are generally more complex than shorter ones, and material biconditionals and negations of material biconditionals are more complex than other sentences of approximately the same length.

The strategies we have presented provide guidelines, not rules, for constructing truth-trees.[4] Disregarding one or more of them may produce a more complex tree than is necessary but will never yield a completed open branch where following them would yield a closed tree, or vice versa.

As a final example we construct a truth-tree for {(C & ~ D) ≡ A, (A & C) ⊃ ~ (D ∨ A)}:

[4]We said earlier that the truth-tree method could easily be made mechanical. We can do so by replacing our guidelines for decomposing sentences with some mandatory order of decomposition.



This tree has five completed open branches. The literals occurring on the left-most completed open branch are '~ C' and '~ A'. Hence this branch tells us that, to make all the sentences in the set we are testing true, it is sufficient to make '~ C' and '~ A' both true, and, to do this, we must assign the truth-value **F** to both 'A' and 'C'. But note that 'D' is also an atomic component of both members of that set. No assignment has yet been made to 'D' because neither 'D' nor '~ D' occurs on the open branch just examined. The significance of the nonoccurrence of both 'D' and '~ D' is this: It does not matter which truth-value we assign to 'D'; the sentences we are testing will both be true as long as we assign the truth-value **F** to 'A' and to 'C', no matter what we assign to 'D'. Thus we can recover *two* sets of truth-value assignments from the left-hand open branch, those that assign the values in the first row and those that assign the values in the second row:

A	C	D
F	**F**	**T**
F	**F**	**F**

The next open branch we come to contains the literals 'D' and '~ A'. Neither 'C' nor '~ C' occurs on this second open branch. Hence we can expect to recover two sets of truth-value assignments from this branch as well, those

that assign the values indicated in the first row below and those that assign the values indicated in the second row:

A	C	D
F	T	T
F	F	T

In fact, only the first of these rows specifies a new set of truth-value assignments; the second set we also obtained from the first open branch. The third open branch contains the same literals as the first, so here there are no new truth-value assignments to be recovered. The fourth open branch contains the literals 'D', '~ C', and '~ A'; from this branch we can recover the set of truth-value assignments that assign the following values to 'A', 'C', and 'D':

A	C	D
F	F	T

This set is also yielded by the other three open branches we have examined. The last open branch contains the literals '~ C', '~ A', and '~ D' and so yields the set of truth-value assignments that assign the following values to 'A', 'C', and 'D':

A	C	D
F	F	F

This is also not a new set of truth-value assignments; we can recover this set from the first and third open branches as well. In sum, we have five completed open branches on our tree, and these branches yield three distinct sets of truth-value assignments. The number of open branches on a completed truth-tree is, again, no guide to the number of distinct sets of truth-value assignments that can be recovered from that tree. Of course, to show that a set of sentences is truth-functionally consistent, we need only show that there is at least one truth-value assignment on which all the members of the set are true. And, to show that there *is* such an assignment, it suffices to recover one set of truth-value assignments.

Truth-value assignments can be recovered only from completed open branches. Closed branches represent unsuccessful attempts to find such assignments. Thus the branches of a truth-tree should not be thought of as corresponding to the rows of a truth-table. They do not. However, constructing a truth-tree for a set of sentences does tell us a lot about what the truth-table for the same set of sentences would be like. If the tree is closed, we know that there is no row in the corresponding truth-table in which every member of the set in question has a **T** under its main connective. If the tree has a

completed open branch, we know that there is at least one row in that table in which every member of the set in question has a **T** under its main connective. And, if we count the number of distinct sets of truth-value assignments we can recover, we know that there will be exactly that many rows in the corresponding truth-table such that every member of the set in question has a **T** under its main connective.

4.4E EXERCISES

1. Construct truth-trees to test each of the following sets of sentences for truth-functional consistency. If a set is consistent, recover one set of truth-value assignments from your tree that shows this.

a. {H ∨ G, ~ G & ~ H}
*b. {K ∨ (M & ~ M), J & ~ C}
c. {~ ~ C, C & [U ∨ (~ C & B)]}
*d. {~ (M & ~ N), ~ (K ∨ M) & ~ ~ M, ~ ~ ~ K}
e. {~ [~ (E ∨ ~ C) & A], ~ (E ∨ ~ C) & A}
*f. {~ [~ (L ∨ ~ L) & (N ∨ ~ N)]}
g. {~ A ∨ ~ ~ [~ (K & ~ A) ∨ R], ~ [D ∨ (A & ~ K)], A & (R ∨ K)}
*h. {~ [~ (J ⊃ M) ≡ (J & ~ M)]}
i. {B ⊃ J, H ≡ J, ~ H ∨ B}
*j. {H & (~ K ⊃ M), ~ K, ~ (H ⊃ M)}
k. {~ [(B & J) ≡ ~ (W ∨ Z)], ~ (J & W)}
*l. {~ ~ ~ [(K ∨ M) ⊃ ~ G], G ≡ (J & U), U ⊃ (~ G & K), K & ~ U)}

2. Which of the following claims are true? Explain your reasoning.

a. If a completed truth-tree contains at least one open branch, then at least one set of truth-value assignments on which all the members of the set being tested are true can be recovered from that open branch.
*b. A completed open branch of a truth-tree yields at most one set of truth-value assignments on which every member of the set being tested is true.
c. If a set of sentences has a truth-tree with a completed open branch, then that set is truth-functionally consistent.
*d. If a truth-tree is closed, there are no open branches on the tree.
e. If a truth-tree is closed, the set of sentences being tested is truth-functionally inconsistent.
*f. If a truth-tree is closed, every sentence on the tree either has been decomposed or is a literal.
g. If there are eight distinct atomic components of the members of a set Γ of sentences of *SL*, then a completed tree for Γ will have eight branches.
*h. A completed truth-tree with at least one open branch and at least one closed branch is an open tree.
i. If a tree has a closed branch, then there is a truth-value assignment on which all the members of the set being tested are false.
*j. If a set Γ of sentences of *SL* has a tree with a completed open branch, then every nonempty subset of Γ also has a tree with a completed open branch.
k. If no member of a set Γ of sentences of *SL* contains a tilde, then no tree for Γ will have a closed branch.

3. Use the truth-tree method to test symbolizations of the following passages for truth-functional consistency. If your symbolization is truth-functionally consistent, recover one set of truth-value assignments from your tree that shows this.

a. Poison caused the victim's death if and only if there was a change in his blood chemistry or a residue of poison in the stomach. There was neither a change in blood chemistry nor a residue of poison in his stomach, but there were puncture marks on the body. Poison was injected by a needle only if there were puncture marks on the body. Either poison was the cause of the victim's death or there were no puncture marks on the body.

*b. Either the bullet was fired from an intermediate distance or it wasn't. If it wasn't, there are powder burns on the body (provided the bullet was fired at close range) or signs of a rifle bullet (provided the bullet was fired at a great distance). Although there are no powder burns on the body, there are signs of a rifle bullet. Unless the angle at which the bullet entered the body was elevated, the bullet wasn't fired from an intermediate distance, and the angle wasn't elevated.

c. The murder was committed by at least one of the staff—the butler, the maid, and the gardener—but not by all three. The butler did it only if the crime was committed indoors; and if it was not committed indoors, the gardener didn't do it. If poison was used, then the butler did it unless the maid did; but the maid didn't do it. Poison was used; moreover the crime was not committed indoors.

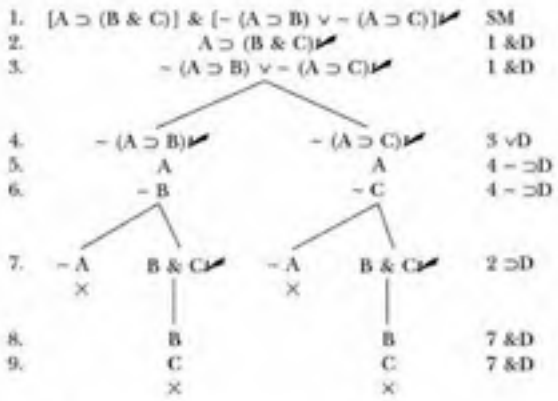
*d. Exactly two of Albert, Betty, and Christine will find employment when they graduate from law school. If Albert gets a job, Betty and Christine surely will too. Betty will not get a job unless Albert does. Christine is a first-rate lawyer and will certainly be hired by a good law firm.

4.5 USING TRUTH-TREES TO TEST FOR TRUTH-FUNCTIONAL TRUTH, FALSITY, AND INDETERMINACY

We know that each sentence of *SL* is either truth-functionally true, truth-functionally false, or truth-functionally indeterminate. Truth-trees can be used to determine into which of these categories a particular sentence of *SL* falls. Truth-trees test for the consistency of finite sets of sentences of *SL*. Suppose that we want to know whether a sentence **P** is truth-functionally false. Remember that, if **P** is not truth-functionally false, there is some truth-value assignment on which it is true; hence the unit set {**P**} will be truth-functionally consistent. However, if **P** is truth-functionally false, there is no truth-value assignment on which it is true; hence there is no assignment on which every member of {**P**} is true, and so {**P**} is truth-functionally inconsistent.

> A sentence **P** of *SL* is *truth-functionally false* if and only if the set {**P**} has a closed truth-tree.

Here is a tree for the set {[A ⊃ (B & C] & [~ (A ⊃ B) ∨ ~ (A ⊃ C)]}:



All the branches of this tree do close, so there is no truth-value assignment on which the one member of the set we are testing is true. Hence the set is truth-functionally inconsistent, and its single member is truth-functionally false. In constructing this tree we were able to save work at lines 5 and 6 by decomposing two sentences, '~ (A ⊃ B)' and '~ (A ⊃ C)', in one step. We could do so because these sentences occur on the same line, line 4, and are decomposed by the same rule, Negated Conditional Decomposition. Of course, we also could have done them separately.

Next we use the tree method to determine whether 'A ⊃ [B ⊃ (A ⊃ B)]' is truth-functionally false:

This tree obviously has a completed open branch (in fact it has four), so the unit set we are testing is truth-functionally consistent. Hence there is at least one truth-value assignment on which the one member of that set is true, and that sentence is thus not truth-functionally false. (Note that we could have stopped at line 2, where the first completed open branch ends.)

Although we know that 'A ⊃ [B ⊃ (A ⊃ B)]' is not truth-functionally false, we do not yet know whether this sentence is truth-functionally indeterminate or

truth-functionally true. We can find out by constructing another tree. Suppose that the sentence we are concerned with, '$A \supset [B \supset (A \supset B)]$', is truth-functionally true. Then its negation, '$\sim (A \supset [B \supset (A \supset B)])$', must be truth-functionally false. So we can determine whether the sentence is truth-functionally true by testing whether its negation is truth-functionally false, that is, by seeing whether the unit set of its negation has a closed tree. Here is a tree for that set:

1.	$\sim (A \supset [B \supset (A \supset B)])$ ✓	SM
2.	A	1 $\sim\supset$D
3.	$\sim [B \supset (A \supset B)]$ ✓	1 $\sim\supset$D
4.	B	3 $\sim\supset$D
5.	$\sim (A \supset B)$ ✓	3 $\sim\supset$D
6.	A	5 $\sim\supset$D
7.	$\sim B$	5 $\sim\supset$D
	×	

The tree is closed. So there is no truth-value assignment on which the sentence '$\sim (A \supset [B \supset (A \supset B)])$' is true. Since that sentence is a negation, there is no truth-value assignment on which the sentence of which it is a negation, '$A \supset [B \supset (A \supset B)]$', is false. That sentence is therefore truth-functionally true.

> A sentence **P** of *SL* is *truth-functionally true* if and only if the set {$\sim$ **P**} has a closed tree.

A sentence is truth-functionally indeterminate if and only if it is neither truth-functionally true nor truth-functionally false. Therefore

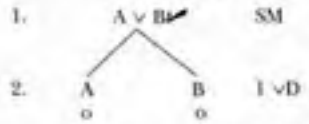
> A sentence **P** of *SL* is *truth-functionally indeterminate* if and only if neither the set {**P**} nor the set {$\sim$ **P**} has a closed tree.

When we are interested in determining the truth-functional status of a sentence, the trees we construct will be trees for unit sets of sentences. However, we shall allow ourselves to talk informally of constructing a tree for **P** or for $\sim$ **P**. Such talk is to be understood as shorthand for talk of trees for unit sets.

When determining the truth-functional status of a sentence **P**, we shall sometimes end up constructing two trees, one for **P** and one for $\sim$ **P**. Of course, if we suspect that **P** is truth-functionally true, we should first do a truth-tree for $\sim$ **P**; if we suspect that **P** is truth-functionally false, we should first do a truth-tree for **P** itself.

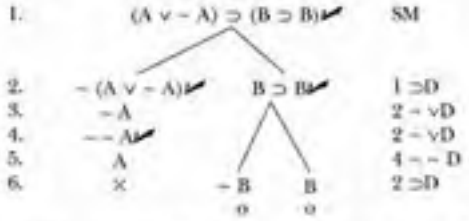
Recall that all of the branches of our tree for '$A \supset [B \supset (A \supset B)]$' were completed open branches. One might think that it follows from this alone that '$A \supset [B \supset (A \supset B)]$' is truth-functionally true, for surely, if that sentence were not truth-functionally true, a tree for that sentence would have at least one closed branch. But this reasoning is mistaken. Many sentences that are not truth-functional truths have trees all of whose branches are completed open

branches, and many truth-functional truths have trees with some closed branches. Consider the truth-tree for the simple disjunction 'A ∨ B':



Both branches of this tree are completed open branches. Yet we know that 'A ∨ B' is not a truth-functional truth. Its truth-table will mirror the characteristic truth-table for disjunctions; that is, the first three rows under its main connective will contain **T**, and the fourth row will contain **F**.

To see that not all truth-functional truths have completed truth-trees all of whose branches are open, consider the sentence '(A ∨ ~ A) ⊃ (B ⊃ B)'. This sentence is a truth-functional truth inasmuch as its consequent is a truth-functional truth (its antecedent is as well), and for this reason there is no truth-value assignment on which '(A ∨ ~ A) ⊃ (B ⊃ B)' is false. But this tree for the sentence does have one closed branch:



There is a way we can avoid constructing two truth-trees for one sentence. Suppose that we construct a tree for a sentence **P**, thinking it may be truth-functionally false, but the tree does not close. We now know that **P** is either truth-functionally true or truth-functionally indeterminate. If it is true on all truth-value assignments, it is truth-functionally true; if it is true on only some assignments, it is truth-functionally indeterminate. We can find out which is the case by counting the number of distinct sets of truth-value assignments that are recoverable from the completed open tree—for these sets correspond to the rows of the truth-table for the sentence being tested in which there is a **T** under that sentence's main connective. If **P** has **n** atomic components, we shall recover 2^n distinct sets of truth-value assignments from our tree if and only if **P** is truth-functionally true.

Recall our tree for 'A ∨ B', which has two open and no closed branches. The only literal occurring on the left-hand branch is 'A', so from that branch we can recover two sets of truth-value assignments, one set assigning the truth-value **T** to 'B' and one set assigning the truth-value **F** to 'B':

A	B
T	**T**
T	**F**

We can also recover two sets of truth-value assignments from the right-hand open branch. But only one of these is a new set:

A	B
F	**T**

From neither branch can we recover the set of truth-value assignments that assign the truth-value **F** to both 'A' and 'B', and this is just what we expected, for a disjunction is false when (and only when) both its disjuncts are false. By identifying all of the recoverable sets of truth-value assignments—and finding that there are only three such sets—we have shown that 'A ∨ B' is truth-functionally indeterminate, without having to construct two trees for that sentence.

We can use this same procedure with our last truth-tree to verify that '(A ∨ ~ A) ⊃ (B ⊃ B)' is indeed truth-functionally true. This sentence has two atomic components, so we can expect to recover four distinct sets of truth-value assignments from the tree for this sentence, each set representing one combination of values that the atomic components 'A' and 'B' can have. The tree has two completed open branches. The only literal on the left-hand branch is '~ B', so this branch yields two sets of truth-value assignments:

A	B
T	**F**
F	**F**

The only literal occurring on the right-hand branch is 'B', so this branch yields two new fragments:

A	B
T	**T**
F	**T**

We have recovered four distinct sets of truth-value assignments, thus showing that the sentence being tested is true on every truth-value assignment. We have verified that it is truth-functionally true, even though the tree for that sentence has one closed branch.

Suppose we suspect that a sentence **P** is truth-functionally true and accordingly construct a tree for the negation of that sentence, ~ **P**. Suppose also that our tree has at least one completed open branch, and thus that in this case our suspicions were wrong: **P** is not truth-functionally true. The standard procedure would now be to construct a tree for **P** to see whether that sentence is truth-functionally false or truth-functionally indeterminate. Instead, we can see which distinct sets of truth-value assignments can be recovered from the tree we have already constructed for ~ **P**. The sets of truth-value assignments we can recover are those on which ~ **P** is true. If we can recover all sets of truth-value assignments, each set assigning a distinct combination of values to the atomic components of **P**, then we know that ~ **P** is true on every truth-value assignment and is thus truth-functionally true. And if ~ **P** is truth-functionally

true, **P** is truth-functionally false. If we cannot recover all sets of truth-value assignments from our tree, we know that there is at least one set of truth-value assignments on which ~ **P** is false, and hence on which **P** is true. In this case **P** is truth-functionally indeterminate.

The method of recovering truth-value assignments always allows us to avoid constructing a second tree. However, to use the method, we must complete the tree we are working with (rather than stopping when we have one completed open branch). As a result, when the tree is complex and the number of distinct combinations of truth-values that can be assigned to the atomic components of a sentence is relatively large—eight, sixteen, thirty-two, or more—it is often easier to construct a second tree than to recover and count distinct sets of truth-value assignments.

4.5E EXERCISES

1. For each of the following sentences, use the truth-tree method to determine its truth-functional status—that is, whether it is truth-functionally true, truth-functionally false, or truth-functionally indeterminate.

a. M & ~ M
*b. M ∨ ~ M
c. ~ M ∨ ~ M
*d. (C ⊃ R) ⊃ [~ R ⊃ ~ (C & J)]
e. (G ⊃ R) & [(G ⊃ ~ R) & ~ (~ C ∨ R)]
*f. (K ≡ W) ∨ (A & W)
g. (~ A ≡ ~Z) & (A & ~ Z)
*h. [L ∨ (J ∨ ~ K)] & (K & [(J ∨ L) ⊃ ~ K])
i. (A ∨ B) & ~ (A ∨ B)
*j. (A ∨ B) ⊃ ~ (A ∨ B)
k. (A ∨ B) ≡ ~ (A ∨ B)
*l. ~ (D ∨ F) ≡ ~ (D & F)
m. ~ (D ∨ F) ≡ (~ D ∨ ~ F)
*n. ~ (D ∨ F) ≡ (~ D & ~ F)

2. For each of the following sentences, use the truth-tree method to determine whether the sentence is truth-functionally true. Where appropriate, recover a set of truth-value assignments that supports your answer.

a. (B ⊃ L) ∨ (L ⊃ B)
*b. (B ⊃ L) & (L ⊃ B)
c. (A ≡ K) ⊃ (A ∨ K)
*d. (A ≡ K) ⊃ (~ A ∨ K)
e. [(J ⊃ Z) & ~ Z] ⊃ ~ J
*f. [(J ⊃ Z) & ~ J] ⊃ ~ Z
g. (B ⊃ (M ⊃ H)) ≡ [(B ⊃ M) ⊃ (B ⊃ H)]
*h. M ⊃ [L ≡ (~ M ≡ ~ L)]
i. (A & ~ B) ⊃ ~ (A ∨ B)
*j. (A & ~ B) ⊃ ~ (A & B)
k. [(A & B) ⊃ C] ≡ [(A ⊃ ~ B) ∨ C]
*l. (D ≡ ~ E) ≡ ~ (D ≡ E)
m. [A ⊃ (B & C)] ⊃ [A ⊃ (B ⊃ C)]

*n. $[A \supset (B \,\&\, C)] \equiv [A \supset (B \supset C)]$
o. $[(A \,\&\, B) \supset C] \equiv [A \supset (B \supset C)]$

3. For each of the following sentences, use the truth-tree method to determine its truth-functional status—that is, whether it is truth-functionally true, truth-functionally false, or truth-functionally indeterminate. In each case construct a tree only for the given sentence. If the tree does not close, determine the truth-functional status of the sentence by recovering and counting distinct sets of truth-value assignments.

a. $\sim (\sim A \supset A)$
*b. $J \supset (K \supset L)$
c. $(A \equiv \sim A) \supset \sim (A \equiv \sim A)$
*d. $(E \equiv H) \supset (\sim E \supset \sim H)$
e. $(\sim B \,\&\, \sim D) \vee \sim (B \vee D)$
*f. $([(C \supset D) \,\&\, (D \supset E)] \,\&\, C) \,\&\, \sim E$
g. $[(A \vee B) \,\&\, (A \vee C)] \supset \sim (B \,\&\, C)$
*h. $\sim ([(A \vee B) \,\&\, (B \vee C)] \,\&\, (\sim A \,\&\, \sim B))$
i. $(J \vee \sim K) \equiv \sim\sim (K \supset J)$
*j. $\sim B \supset [(B \vee D) \supset D]$

4. Decide which of the following claims are true and which are false. In each case explain and defend your reasoning. Use examples where appropriate.

a. If a completed tree for the unit set of **P**, {**P**} has at least one open branch and at least one closed branch, then **P** is truth-functionally indeterminate.
*b. If **P** is truth-functionally true and has four atomic components, then a completed tree for {**P**} will have four open branches.
c. If a completed tree for {**P**} has all open branches, then **P** is truth-functionally true.
*d. If {**P**} has a closed tree and {**Q**} has a closed tree, then the unit set of every truth-functionally compound sentence whose immediate components are **P** and **Q** will also have a closed tree.
e. If {**P**} and {**Q**} each has a tree with at least one completed open branch, then the unit set of every truth-functionally compound sentence that has **P** and **Q** as its immediate components will have a completed tree with an open branch.
*f. If a completed truth-tree for {**P**} has exactly one open branch, then ~ **P** is truth-functionally indeterminate.
g. If **P** and **Q** are both truth-functionally true, then **P** & **Q**, **P** ∨ **Q**, **P** ⊃ **Q**, and **P** ≡ **Q** will each have a completed tree all of whose branches are open.
*h. If **P** and **Q** are both truth-functionally true, then **P** & **Q**, **P** ∨ **Q**, **P** ⊃ **Q**, and **P** ≡ **Q** will each have a tree with at least two completed open branches.
i. If **P** and **Q** are both truth-functionally false, then **P** & **Q**, **P** ∨ **Q**, **P** ⊃ **Q**, and **P** ≡ **Q** will each have a closed tree.
*j. If **P** and **Q** are both truth-functionally false, then **P** & **Q**, **P** ∨ **Q**, **P** ⊃ **Q**, and **P** ≡ **Q** will each have a completed tree with at least one closed branch.
k. If **P** is truth-functionally true and **Q** is truth-functionally false, then **P** & **Q**, **P** ∨ **Q**, **P** ⊃ **Q**, and **P** ≡ **Q** will each have a completed tree with at least one open branch and one closed branch.

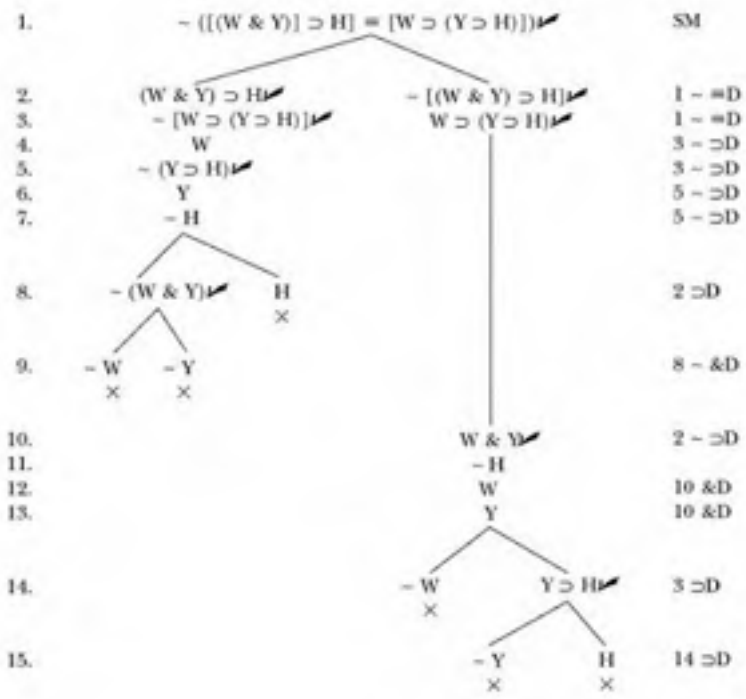
4.6 TRUTH-FUNCTIONAL EQUIVALENCE

Sentences **P** and **Q** of *SL* are truth-functionally equivalent if and only if there is no truth-value assignment on which **P** and **Q** have different truth-values. It

follows that sentences **P** and **Q** are truth-functionally equivalent if and only if their corresponding material biconditional, **P** ≡ **Q**, is truth-functionally true. And a material biconditional **P** ≡ **Q** is truth-functionally true if and only if the tree for the negation of that biconditional is closed. That is, to determine whether a biconditional is truth-functionally true, we simply apply the test for truth-functional truth developed in the previous section.

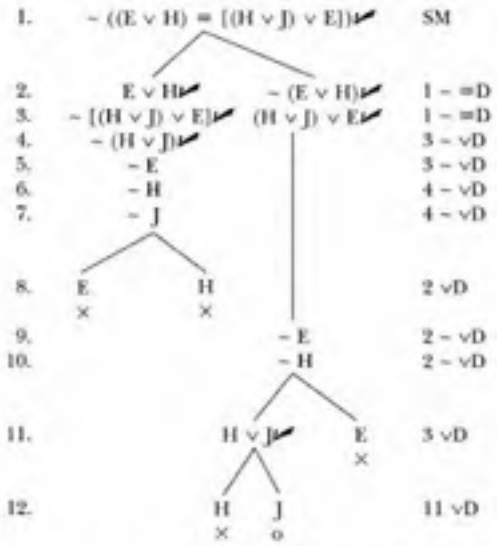
> Sentences **P** and **Q** of *SL* are *truth-functionally equivalent* if and only if the set {~ (**P** ≡ **Q**)} has a closed tree.

In Chapter 3, we showed that '(W & Y) ⊃ H' is truth-functionally equivalent to 'W ⊃ (Y ⊃ H)' by producing a truth-table revealing that these two sentences have the same truth-value on every truth-value assignment. We can now use the truth-tree method to show the same result. To show that these sentences are equivalent, we need show only that their corresponding material biconditional is truth-functionally true:



This tree is closed. The sentence at the top of the tree is therefore false on every truth-value assignment, and the biconditional of which it is the negation is therefore true on every truth-value assignment. So the immediate components of that biconditional, '(W & Y) ⊃ H' and 'W ⊃ (Y ⊃ H)', are truth-functionally equivalent.

In Chapter 3 we also showed that 'E ∨ H' and '(H ∨ J) ∨ E' are not truth-functionally equivalent. We can now show this by using the truth-tree method. These sentences are truth-functionally equivalent if and only if their corresponding material biconditional, '(E ∨ H) ≡ [(H ∨ J) ∨ E]', is truth-functionally true. And that biconditional is truth-functionally true if and only if the tree for its negation closes:



Since this truth-tree has a completed open branch, there is at least one truth-value assignment on which the sentence at the top of the tree is true. That sentence is therefore not truth-functionally false, and the biconditional of which it is the negation is thus not truth-functionally true. It follows that the sentences that are the immediate components of that biconditional, 'E ∨ H' and '(H ∨ J) ∨ E', are not truth-functionally equivalent. They have different truth-values on every truth-value assignment that assigns the following values to 'E', 'H', and 'J':

E	H	J
F	F	T

4.6E EXERCISES

1. Use the truth-tree method to determine whether the following pairs of sentences are truth-functionally equivalent. For those pairs that are not truth-functionally equivalent, recover a set of truth-value assignments that shows this.

a.	$\sim (Z \vee K)$	$\sim Z \,\&\, \sim K$
*b.	$\sim (Z \vee K)$	$\sim Z \vee \sim K$
c.	$(B \,\&\, C) \supset R$	$(B \supset R) \,\&\, (C \supset R)$
*d.	$(B \vee C) \supset R$	$(B \supset R) \,\&\, (C \supset R)$
e.	$A \,\&\, (B \vee C)$	$(A \,\&\, B) \vee (A \,\&\, C)$
*f.	$A \vee (B \,\&\, C)$	$(A \vee B) \,\&\, (A \vee C)$
g.	$D \supset (L \supset M)$	$(D \supset L) \supset M$
*h.	$J \supset (K \equiv L)$	$(J \supset K) \equiv (J \supset L)$
i.	$A \supset A$	$B \supset B$
*j.	$A \,\&\, \sim A$	$B \,\&\, \sim B$
k.	$A \,\&\, \sim B$	$\sim A \vee B$
*l.	$\sim (A \vee B)$	$\sim (A \,\&\, B)$
m.	$\sim (A \equiv B)$	$\sim A \equiv \sim B$
*n.	$A \supset (B \supset C)$	$(A \supset B) \supset C$
o.	$A \,\&\, (B \vee C)$	$(A \,\&\, B) \vee (A \,\&\, C)$
*p.	$A \supset (B \supset C)$	$A \supset (B \,\&\, C)$

2. Decide which of the following claims are true and which are false. In each case explain and defend your reasoning. If **P** and **Q** are truth-functionally equivalent, then

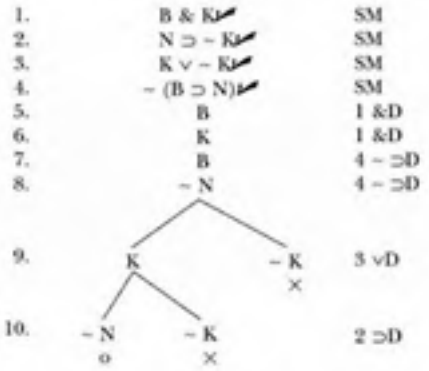
a. A completed truth-tree for $\{\mathbf{P} \equiv \mathbf{Q}\}$ will be open.
*b. A completed truth-tree for $\{\mathbf{P} \equiv \sim \mathbf{Q}\}$ will be open.
c. A completed truth-tree for the set $\{\mathbf{P}, \mathbf{Q}\}$ will be open.
*d. A completed truth-tree for $\{\sim \mathbf{P} \equiv \sim \mathbf{Q}\}$ will be open.

4.7 TRUTH-FUNCTIONAL ENTAILMENT AND TRUTH-FUNCTIONAL VALIDITY

We can use truth-trees to test for truth-functional entailment. Recall that, where **P** is a sentence and Γ is a set of sentences, Γ truth-functionally entails **P**—that is, $\Gamma \models \mathbf{P}$ if and only if there is no truth-value assignment on which every member of Γ is true and **P** is false. Put another way, a set Γ of sentences truth-functionally entails a sentence **P** if and only if the set of sentences $\Gamma \cup \{\sim \mathbf{P}\}$ is truth-functionally inconsistent. Hence, to see if a finite set Γ truth-functionally entails **P**, we construct a tree for the members of $\Gamma \cup \{\sim \mathbf{P}\}$. Here we have to be careful to negate the allegedly entailed sentence before constructing the tree.

> A finite set Γ of sentences of *SL truth-functionally entails* a sentence **P** of *SL* if and only if the set $\Gamma \cup \{\sim \mathbf{P}\}$ has a closed truth-tree.

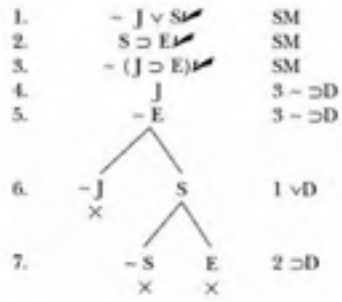
Does the set [B & K, N ⊃ ~ K, K ∨ ~ K] truth-functionally entail 'B ⊃ N'? We can find out by constructing a tree for [B & K, N ⊃ ~ K, K ∨ ~ K, ~ (B ⊃ N)]:



Since this truth-tree has a completed open branch, there is a truth-value assignment on which all the sentences we are testing are true. Hence there is an assignment on which the members of the set [B & K, N ⊃ ~ K, K ∨ ~ K] are all true and the sentence 'B ⊃ N' is false. So the entailment does not in fact hold. The set members are true while 'B ⊃ N' is false on every truth-value assignment that assigns the following values to 'B', 'K', and 'N':

B	K	N
T	T	F

On the other hand, [~ J ∨ S, S ⊃ E] does truth-functionally entail 'J ⊃ E', as the following closed truth-tree shows:



An argument is truth-functionally valid if and only if there is no truth-value assignment on which the premises are true and the conclusion false.

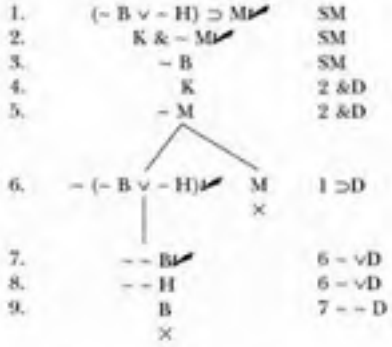
Alternatively, an argument is truth-functionally valid if and only if there is no truth-value assignment on which both the premises and the *negation* of the conclusion are true. Hence an argument is truth-functionally valid if and only if the set consisting of the premises and the *negation* of the conclusion is truth-functionally inconsistent:

> An argument of *SL* with a finite number of premises is *truth-functionally valid* if and only if the set consisting of the premises and the negation of the conclusion has a closed truth-tree.

In our next example we use the tree method to determine whether the following argument is truth-functionally valid:

$$\begin{array}{l} (\sim B \vee \sim H) \supset M \\ K \,\&\, \sim M \\ \hline B \end{array}$$

Trees here are no different from the trees we have already constructed, but we must remember to construct a tree for the premises and the *negation* of the conclusion, rather than for the premises and the conclusion:

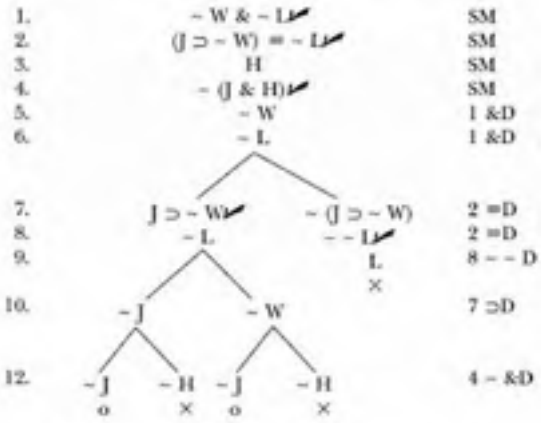


This truth-tree is closed. So we know that the set consisting of the sentences we are testing is truth-functionally inconsistent, and hence that the argument from which the set was formed is truth-functionally valid. Our reasoning is this: The closed tree shows that there is no truth-value assignment on which the premises of our argument are all true and the negation of the conclusion is also true. Therefore there is no truth-value assignment on which those premises are true and the conclusion false, so the argument is truth-functionally valid.

As another example, we'll construct a truth-tree to test the following argument:

$$\begin{array}{l} \sim W \,\&\, \sim L \\ (J \supset \sim W) \equiv \sim L \\ H \\ \hline J \,\&\, H \end{array}$$

Our tree for this argument follows. Again, it is the negation of the conclusion that we use along with the premises, not the conclusion itself:



Because this tree has at least one completed open branch, we can recover a set of truth-value assignments on which the premises and the negation of the conclusion are true, and hence on which the premises are true and the conclusion false. So the argument we are testing is truth-functionally invalid. The recoverable truth-value assignments assign the following values to the four atomic sentences that occur in the premises and conclusion:

H	J	L	W
T	**F**	**F**	**F**

Because an argument is truth-functionally valid if and only if the set consisting of the premises of that argument truth-functionally entails the conclusion of that argument, the procedures for constructing truth-trees to test for truth-functional validity and for truth-functional entailment are similar. In the case of testing for truth-functional validity, the conclusion is negated; in the case of testing for truth-functional entailment, the allegedly entailed sentence is negated.

4.7E EXERCISES

1. Use the truth-tree method to determine which of the following claims are true and which are false. For those that are false, recover a set of truth- value assignments that shows this.

a. $[A \supset (B \,\&\, C),\ C \equiv B,\ \sim C] \models \sim A$
*b. $[K \supset H,\ H \supset L,\ L \supset M] \models K \supset M$
c. $[\sim (A \equiv B),\ \sim A,\ \sim B] \models C \,\&\, \sim C$
*d. $[\sim (\sim A \,\&\, \sim B)] \models A \,\&\, B$
e. $[\sim \sim F \supset \sim \sim G,\ \sim G \supset \sim F] \models G \supset F$
*f. $[A \,\&\, (B \supset C)] \models (A \,\&\, C) \vee (A \,\&\, B)$
g. $[[(C \vee D) \,\&\, H] \supset A,\ D] \models H \supset A$
*h. $[(G \equiv H) \vee (\sim G \equiv H)] \models (\sim G \equiv \sim H) \vee \sim (G \equiv H)$
i. $[(J \vee M) \supset \sim (J \,\&\, M),\ M \equiv (M \supset J)] \models M \supset J$
*j. $\models [A \vee ((K \supset \sim H) \,\&\, \sim A)] \vee \sim A$
k. $\models \sim (A \equiv B) \supset (\sim A \equiv \sim B)$
*l. $\models \sim (C \equiv C) \equiv (C \vee \sim C)$
m. $\models [(A \supset B) \supset (C \supset D)] \supset [C \supset (B \supset D)]$

2. Use the truth-tree method to determine which of the following arguments are truth-functionally valid and which are truth-functionally invalid. For those that are truth-functionally invalid, recover a set of a truth-value assignments that show this.

a. $M \supset (K \supset B)$
$\sim K \supset \sim M$
$L \,\&\, M$
$\therefore B$

*b. $(\sim J \vee K) \supset (L \,\&\, M)$
$\sim (\sim J \vee K)$
$\therefore \sim (L \,\&\, M)$

c. $A \,\&\, (B \vee C)$
$(\sim C \vee H) \,\&\, (H \supset \sim H)$
$\therefore A \,\&\, B$

*d. $(D \equiv \sim G) \,\&\, G$
$[G \vee ((A \supset D) \,\&\, A)] \supset \sim D$
$\therefore G \supset \sim D$

e. $(M \equiv K) \vee \sim (K \,\&\, D)$
$\sim M \supset \sim K$
$\sim D \supset \sim (K \,\&\, D)$
$\therefore M$

*f. $(J \supset T) \supset J$
$(T \supset J) \supset T$
$\therefore \sim J \vee \sim T$

g. $B \,\&\, (H \vee Z)$
$\sim Z \supset K$
$(B \equiv Z) \supset \sim Z$
$\sim K$
$\therefore M \,\&\, N$

*h. $A \vee \sim (B \,\&\, C)$
$\sim B$
$\sim (A \vee C)$
$\therefore A$

i. $A \,\&\, (B \supset C)$
$\therefore (A \,\&\, C) \vee (A \,\&\, \sim B)$

*j. $(G \equiv H) \vee (\sim G \equiv H)$
$\therefore (\sim G \equiv \sim H) \vee \sim (G \equiv H)$

k. $A \supset \sim A$
$(B \supset A) \supset B$
$A \equiv \sim B$

*l. $B \vee (A \,\&\, \sim C)$
$(C \vee A) \equiv B$
$\sim B \vee A$
$\sim (A \vee C)$

3. Symbolize each of the following arguments and then use the truth-tree method to determine whether the symbolized argument is truth-functionally valid. If an argument is not truth-functionally valid, recover a set of truth-value assignments that show this.

a. The social security system will succeed if and only if more money is collected through social security taxes. Unless the social security system succeeds, many senior citizens will be in trouble. Although members of Congress claim to be sympathetic to senior citizens, more money won't be collected through social security taxes. Hence the social security system will not succeed.

*b. Either the president and the senators will support the legislation, or the president and the representatives will support it. Moreover, the representatives will support the legislation, provided that a majority of the people support it. The people don't support it. Thus the senators will support the legislation.

c. If the president acts quickly the social security system will be saved, and if the social security system is saved, senior citizens will be delighted. If the president is pressured by members of the Senate, by members of the House of Representatives, or by senior citizens, he will act quickly. However, neither members of the Senate nor members of the House will pressure the president, but senior citizens will. Therefore senior citizens will be delighted.

*d. The president won't veto the bill if Congress passes it, and Congress will pass it if and only if both the Senate passes it and the House of Representatives passes it. But the House of Representatives will pass it only if a majority of Democrats will vote for it; and indeed, a majority of Democrats will vote for it. Therefore the president won't veto the bill.

e. At most, one of the two houses of Congress will pass the bill. If either the House of Representatives or the Senate passes it, the voters will be pleased; but if both houses of Congress pass the bill, the president will not be pleased. If the president is not pleased, not all the members of the White House will be happy. Hence some members of the White House will not be happy.

4. Show that constructing a tree for the premises and conclusion (not the negation of the conclusion) of an argument of *SL* yields no useful information concerning the validity of the argument by completing the following exercises.

a. Give two arguments of *SL*, one valid and the other invalid, such that the trees for the premises and conclusion of these arguments both have at least one completed open branch. Construct the trees and explain why they are not useful in determining whether the arguments in question are truth-functionally valid.

*b. Give two arguments of *SL*, one valid and the other invalid, such that the trees for the premises and conclusion of these arguments are both closed. Construct the trees and explain why they are not useful in determining whether the arguments in question are truth-functionally valid.

c. Explain why (a) and (b) together constitute a proof that there is no useful information concerning the validity of an argument to be obtained by doing a tree for the premises and conclusion of the argument.

5. Suppose we define a new connective, '/', thus:

P	Q	P/Q
T	T	F
T	F	T
F	T	T
F	F	T

To accommodate this new connective, we have to add two new rules to our truth-tree system, one for decomposing sentences of the form **P/Q** and one for decomposing sentences of the form ~ (**P/Q**).

a. Give the rules needed for sentences of these two forms.
b. Use the new rules to test the sentences 'A/B' and '(A/A) ∨ (B/B)' for truth-functional equivalence, using the truth-tree method. State your result.

SUMMARY

Key Semantic Properties

TRUTH-FUNCTIONAL CONSISTENCY: A finite set Γ of sentences of *SL* is *truth-functionally consistent* if and only if Γ has a truth-tree with at least one completed open branch.

TRUTH-FUNCTIONAL INCONSISTENCY: A finite set Γ of sentences of *SL* is *truth-functionally inconsistent* if and only Γ has a closed truth-tree.

TRUTH-FUNCTIONAL FALSITY: A sentence **P** of *SL* is *truth-functionally false* if and only if the set {**P**} has a closed truth-tree.

TRUTH-FUNCTIONAL TRUTH: A sentence **P** of *SL* is *truth-functionally true* if and only if the set {~ **P**} has a closed truth-tree.

TRUTH-FUNCTIONAL INDETERMINACY: A sentence **P** of *SL* is *truth-functionally indeterminate* if and only if neither the set {**P**} nor the set {~ **P**} has a closed truth-tree.

TRUTH-FUNCTIONAL EQUIVALENCE: Sentences **P** and **Q** of *SL* are *truth-functionally equivalent* if and only if the set {~ (**P** ≡ **Q**)} has a closed truth-tree.

TRUTH-FUNCTIONAL ENTAILMENT: A finite set Γ of sentences of *SL truth-functionally entails* a sentence **P** of *SL* if and only if the set Γ ∪ {~ **P**} has a closed truth-tree.

TRUTH-FUNCTIONAL VALIDITY: An argument of *SL* with a finite number of premises is *truth-functionally valid* if and only if the set consisting of the premises and the negation of the conclusion has a closed truth-tree.

Key Truth-Tree Concepts

CLOSED BRANCH: A branch containing both an atomic sentence and the negation of that sentence.

CLOSED TRUTH-TREE: A truth-tree each of whose branches is closed.

OPEN BRANCH: A truth-tree branch that is not closed.

COMPLETED OPEN BRANCH: An open truth-tree branch on which every sentence either is a literal or has been decomposed.

COMPLETED TRUTH-TREE: A truth-tree each of whose branches either is closed or is a completed open branch.

OPEN TRUTH-TREE: A truth-tree that is not closed.

Chapter 5

SENTENTIAL LOGIC: DERIVATIONS

5.1 THE DERIVATION SYSTEM *SD*

In Chapters 3 and 4 we gave semantic accounts of consistency, validity, equivalence, entailment, and of the status of individual sentences of *SL* (truth-functional truth, truth-functional falsity, truth-functional indeterminacy). The semantic truth-table and truth-tree tests we developed show whether there is or is not a truth-value assignment of a particular kind for a particular sentence or group of sentences. These test procedures can hardly be said to reflect the reasoning we do in everyday discourse when we are trying to show, for example, that an argument is valid or that a set of sentences is consistent. In this chapter we develop techniques that do parallel to a considerable extent the kind of reasoning we do make use of in everyday discourse. These techniques rely on the form or structure of sentences of *SL* and are not intended to reveal whether there is or is not a truth-value assignment of a certain sort. These are therefore syntactic techniques.

How might we, in everyday discourse, convince ourselves that the following argument is valid?

> If Marshall survives the current scandal and if her opponent doesn't outspend her then Marshall will be reelected. If it continues to be politics as usual Marshall will survive the latest scandal. The scandal is no longer front page news, so it is going to be politics as usual. Marshall's opponent will not outspend her. So Marshall will be reelected.

Here, as in earlier chapters, it is useful to start by constructing truth-functional paraphrases of the premises and conclusion:

> If both Marshall survives the current scandal and it is not the case that Marshall's opponent outspends Marshall, then Marshall will be reelected.
>
> If it continues to be politics as usual, then Marshall will survive the current scandal.
>
> Both it is not the case that the scandal is still front page news and it is going to be politics as usual.
>
> It is not the case that Marshall's opponent outspends Marshall.
>
> ---
>
> Marshall will be reelected.

Note that we paraphrased the third premise as a conjunction. The task before us is to show that starting with the premises as assumptions we can, by a series of obvious inferences, reach the conclusion. We can do this as follows.

1.	If both Marshall survives the current scandal and it is not the case that Marshall's opponent outspends Marshall, then Marshall will be reelected.	Assumption
2.	If it continues to be politics as usual, then Marshall will survive the current scandal.	Assumption
3.	Both it is not the case that the scandal is still front page news and it will continue to be politics as usual.	Assumption
4.	It is not the case that Marshall's opponent will outspend Marshall.	Assumption
5.	It will continue to be politics as usual.	From 3
6.	Marshall will survive the current scandal.	From 2 and 5
7.	Marshall will survive the current scandal and it is not the case that Marshall's opponent will outspend Marshall.	From 6 and 4
8.	Marshall will be reelected.	From 1 and 7

The structure of our reasoning may be more apparent when we symbolize these steps in *SL*:

1	$(S \mathbin{\&} \sim O) \supset R$	Assumption
2	$C \supset S$	Assumption
3	$\sim F \mathbin{\&} C$	Assumption

4 ~ O	Assumption
5 C	From 3
6 S	From 2 and 5
7 S & ~ O	From 6 and 4
8 R	From 1 and 7

In this chapter we will develop two systems of syntactic rules, *SD* and *SD+* (*SD+* will include all the rules of *SD* and additional rules). Each of the inferences represented by lines 5 through 8 will be justified by a syntactic rule that tells us, roughly, that if we have a sentence or sentences of such-and-such forms or structures, then we may infer a sentence of such-and-such a form or structure. We will call these rules derivation rules and the structures we construct using them derivations.

5.1.1 REITERATION AND INTRODUCTION AND ELIMINATION RULES FOR '&' AND '⊃'

REITERATION

The simplest of the derivation rules we will present is Reiteration. This rule simply allows one to enter on a line of a derivation a sentence that occurs on an earlier line of the derivation. Schematically:

Reiteration (R)

	P
▷	P

Here, and in the other rule schema presented below, the '▷' sign indicates the sentence that can be derived using the rule in question. Here is a simple and admittedly uninteresting use of Reiteration:

1	A		Assumption
2	A		1 R

As we will see later in this chapter, Reiteration is often used in strategies that involve subderivations.

INTRODUCTION AND ELIMINATION RULES FOR '&'

The derivation system *SD* includes two rules for each connective, one for deriving a sentence *from* a sentence with a specified connective as its main connective, and one for *introducing* the connective (deriving a sentence whose main connective is the connective after which the rule is named). The former are

called **elimination** rules because they take us from a sentence whose main connective is the connective after which the rule is named to one that may have a different—or no—main connective. The latter are, analogously, called **introduction** rules, because the sentence introduced has as its main connective the connective for which the rule is named. Here is the elimination rule for the ampersand (&):

Conjunction Elimination (&E)

	P & Q		**P & Q**
		or	
▷	**P**	▷	**Q**

This rule specifies that from a conjunction one can infer or derive either the left conjunct or the right conjunct (or both, in two steps). We implicitly used this rule at line 5 earlier, where we inferred 'C' from '~ F & C'. The introduction rule for the ampersand is

Conjunction Introduction (&I)

	P
	Q
▷	**P & Q**

We used this rule at line 7 when we derived 'S & ~ O' from 'S' on line 6 and '~ O' on line 4. This schema should be interpreted as allowing the derivation of a conjunction when each of its conjuncts appears earlier in the derivation in any order. That is, the left conjunct may occur on an earlier line, or on a later line, than the line on which the right conjunct appears.

INTRODUCTION AND ELIMINATION RULES FOR '⊃'

In our first example we also used the rule Conditional Elimination:

Conditional Elimination (⊃E)

	P ⊃ Q
	P
▷	**Q**

This rule specifies that if, in a derivation, we have already obtained both a material conditional and the sentence that is the antecedent of that conditional (in either order), then we may enter the consequent of that conditional. We used this rule at line 6 where we derived 'S' from 'C ⊃ S' (line 2) and 'C' (line 5),

and again at line 8, where we derived 'R' from '(S & ~ O) ⊃ R' (line 1) and 'S & ~ O'(line 7).

Hereafter we will adopt the convention of writing the name of the rule we use to the right of each sentence entered in a derivation (or, where the sentence is an assumption, the word 'Assumption'). We will also specify the line or lines from which the sentence we have entered is derived. Finally, we will draw a horizontal line to separate the initial assumptions of the derivation (which we will call the primary assumptions) from subsequent lines, and a vertical line to the left of the column of derived sentences. Using these notational conventions, our first derivation becomes:

Derive: R

1	(S & ~ O) ⊃ R	Assumption
2	C ⊃ S	Assumption
3	~ F & C	Assumption
4	~ O	Assumption
5	C	3 &E
6	S	2, 5 ⊃E
7	S & ~ O	4, 6 &I
8	R	1, 7 ⊃E

Here is another derivation that uses just the three rules already introduced:

Derive: A ⊃ B

1	C ⊃ [(C & D) ⊃ (A ⊃ B)]	Assumption
2	D ⊃ C	Assumption
3	D & B	Assumption
4	D	4 &E
5	C	2, 4 ⊃E
6	(C & D) ⊃ (A ⊃ B)	1, 5 ⊃E
7	C & D	4, 5 &I
8	A ⊃ B	6, 7 ⊃D

It is worth pausing here to discuss how the derivation rules of *SD* and *SD+* are selected. Derivation rules are syntactic templates. They specify that if, in a derivation, we have a sentence or sentences of such-and-such form we may enter a sentence of such-and-such form. Consider the following template, which *is not* a derivation rule of *SD*.

Conditional Elimination2 (⊃E2) (A very bad rule!)

	P ⊃ Q
	Q
▷	**P**

This unacceptable rule specifies that if, in a derivation, we have obtained a conditional and the consequent of that conditional we may enter the antecedent of the conditional. Why do we include Conditional Elimination and not Conditional

Elimination2 among the rules of *SD*? The answer is that the rules of *SD* (and every acceptable derivation system) are selected on a semantic basis. The rules we do include in *SD* will never take us from truths to a falsehood: they are truth-preserving. That is, there will be no truth-value assignment on which the sentence or sentences the rule cites in a derivation are true and the derived sentence false. Recall the characteristic truth-table for the material conditional:

P	Q	P ⊃ Q
T	T	T
T	F	F
F	T	T
F	F	T

This table demonstrates that the rule Conditional Elimination is truth-preserving: there is no row in which the material conditional and the antecedent of that conditional both have the truth-value **T** and the consequent has the truth-value **F**. (Wherever the conditional and the antecedent of the conditional have a value of **T** the consequent also has the value **T**.) The same characteristic truth-table shows that Conditional Elimination2 is not truth-preserving. Consider the third row of the table. In this row the conditional has the value **T** as does the consequent of the conditional, but the antecedent has the value **F**. So Conditional Elimination2 does sometimes take us from truths to a falsehood. Conditional Elimination meets the semantic requirement of truth-preservation whereas Conditional Elimination2 does not. All of the rules of *SD* and *SD+* are truth-preserving. We prove this in Chapter 6.

Here is the introduction rule for '⊃':

Conditional Introduction (⊃I)

	P
	⋮
	Q
▷	**P ⊃ Q**

This rule makes use of a new structure, that of a **subderivation,** the idea of which is this. We want to derive a conditional, a sentence of the form **P ⊃ Q**. To do so we take **P**, the antecedent of the desired conditional, as a new assumption. We then show that from that new assumption, and all other available assumptions, we can derive **Q**, the consequent of the desired material conditional. This amounts to showing that *if **P** then **Q***, or **P ⊃ Q**, follows from the assumptions that are were available before we assumed **P**.

To illustrate, consider the argument

If Wendy is on the Eiffel Tower, then she is in Paris.

If she is in Paris, then she is in France.

If Wendy is on the Eiffel Tower, then she is in France.

To show that the conclusion follows from the premises, we might reason as follows: We take the premises of the argument as our assumptions, and temporarily add a further assumption, namely that Wendy is on the Eiffel Tower. On the basis of this assumption and the first premise, we can infer that Wendy is in Paris. And from 'Wendy is in Paris' and the second premise, we can infer that Wendy is in France. Of course, we have not shown that from the premises alone it follows that Wendy is in France. Rather we have shown that from those premises and an additional assumption, that Wendy is on the Eiffel Tower, it follows that Wendy is in France. But this amounts to showing that from the premises alone it follows that *if* Wendy is on the Eiffel Tower *then* Wendy is in France.

Here is a derivation for a symbolic version of this argument:

Derive: $E \supset F$

1	$E \supset P$	Assumption
2	$P \supset F$	Assumption
3	$\quad E$	A / $\supset$I
4	$\quad P$	1, 3 $\supset$E
5	$\quad F$	2, 4 $\supset$E
6	$E \supset F$	3-5 $\supset$I

There are several important points to note here. The vertical lines in a derivation are called **scope lines.** Assumptions with just one scope line to their left are the **primary assumptions** of the derivation—they are the assumptions we are given at the beginning of our work, for example the premises of an argument whose validity we are seeking to establish. The scope line to the left of primary assumptions continues to the end of the derivation and indicates that the primary assumptions are in force—are being assumed—for the entire derivation. Each subderivation begins with an **auxiliary assumption** whose scope is indicated by the scope line immediately to its left. An auxiliary assumption is in force, is available for use, only as long as the vertical line immediately to its left continues. In the above example there is one subderivation, occupying lines 3 through 5. The assumption of that subderivation is in force only through line 5. Subderivations are constructed so that we can use a rule that requires that there be a subderivation of a certain sort. In the above example the rule we intend to use is Conditional Introduction, which calls for assuming, as an auxiliary assumption, the antecedent of the material conditional we wish to obtain. In the justification column for a sentence entered as an auxiliary assumption we enter 'A' (for 'Assumption'), and the abbreviation for the rule that calls a subderivation of the sort we are constructing (here '$\supset$I'), with the two notations separated by a slash ('/').

We end a successful subderivation by using the rule indicated on the assumption line of the subderivation to derive a sentence *outside* of the subderivation, citing the entire subderivation. When we do so we of course also terminate the scope line of the subderivation. It is the entire subderivation that justifies applying a subderivation rule. In our last example, it is the subderivation occurring on lines 3 thorough 5 that justifies entering the sentence

on line 6. Note that the notation in the justification column for line 6 is '3–5 ⊃I' not '3, 5 ⊃I'. This is because we are citing not the individual lines 3 and 5. Rather, we are citing the subderivation that occurs on lines 3 through 5. When a subderivation is ended we say that the assumption of that subderivation has been **discharged;** it is no longer in force and may not be cited on subsequent lines. We will also refer to assumptions that have not been discharged as being **open,** and to those that have been discharged as being **closed.** In our example, the scope of the assumption on line 3 ends after line 5; the assumption was only made to license deriving the conditional 'E ⊃ F' on line 6.

We can now explain the concept of **accessibility:** A sentence is accessible at a given line of a derivation if and only if it is either an open assumption (one that has not been discharged) or falls within the scope of an open assumption. Scope lines, the vertical lines to the left of the sentences of a derivation, provide a visual way of telling when a sentence is accessible. The leftmost vertical line is the scope line of the entire derivation. Primary assumptions, if any, appear to the immediate right of this scope line at the top of the derivation. Every auxiliary assumption has its own scope line, a line that continues only so long as that assumption remains open. A sentence is accessible only as long as the scope line to its immediate left continues. Primary assumptions, of course, are never discharged. If a sentence is accessible at a given line of a derivation then it can be appealed to in justifying the sentence entered on that line. In the preceding example, the assumption on line 3 is accessible through line 5 but is not accessible after line 5. (In justifying a step of a derivation we cannot cite an earlier line or subderivation that falls within the scope of an assumption that is no longer accessible. For example, if we were to continue the preceding derivation beyond line 6 the sentences on lines 1 and 2 (the primary assumptions) would continue to be accessible, as would the sentence on line 6. But the sentences on lines 3 through 5 would not be accessible because they fall within the scope of the assumption on line 3, which is no longer accessible.

A subderivation is accessible so long as every scope line to the left of that subderivation continues. Note that we take the scope line of the subderivation, the one that begins when the auxiliary assumption of the subderivation is entered, as being part of the subderivation and therefore not to the left of the subderivation. In the preceding derivation, the subderivation occurring on lines 3 through 5 is accessible at line 6 and is cited at line 6.

Here is an example that violates the accessibility requirement:

Derive: A & B

1	C ⊃ A	Assumption
2	A ⊃ B	Assumption
3	C	A / ⊃I
4	A	1, 3 ⊃E
5	B	4, 2 ⊃E
6	C ⊃ B	3–5 ⊃I
7	A & B	4, 5 &I **MISTAKE!**

At line 7 it is a mistake to cite lines 4 and 5 because not all of the assumptions within whose scope the sentences on those lines, 'A' and 'B', occur are still accessible, and this is not so. The sentences on lines 4 and 5 fall within the scope of the assumption at line 3, which is not accessible at line 7. At line 7 the assumption at line 3 is closed, not open. Note again that this restriction does not prevent us from citing, at line 6, the entire subderivation occurring on lines 3 through 5. For all the assumptions within whose scope that entire subderivation falls, namely the primary assumptions of the derivation, are still open.

Subderivations can be nested inside other subderivations. Here is an example:

Derive: $G \supset (H \supset K)$

1	$(G \ \& \ H) \supset K$	Assumption
2	$\quad G$	A / $\supset$I
3	$\quad\quad H$	A / $\supset$I
4	$\quad\quad G \ \& \ H$	2, 3 &I
5	$\quad\quad K$	1, 4 $\supset$E
6	$\quad H \supset K$	3–5 $\supset$I
7	$G \supset (H \supset K)$	2–6 $\supset$I

Line 7 cites the subderivation occurring on lines 2 through 6. This is permissible because the only assumption within which that subderivation falls, the primary assumption on line 1, is still accessible at line 7. Line 6 cites the subderivation occurring on lines 3 through 5. This is also permissible because the assumptions within whose scope that subderivation falls, those on lines 1 and 2, are both accessible at line 6.

5.1.1E EXERCISES

1. Complete the following derivations.

a. Derive: A & B

1	A	Assumption
2	$A \supset B$	Assumption

*b. Derive: ~ C

1	$A \supset (B \ \& \sim C)$	Assumption
2	A & B	Assumption

c. Derive: $A \supset (\sim C \ \& \sim B)$

1	$A \supset (\sim B \ \& \sim C)$	Assumption

*d. Derive: (E & D) & (~ B & C)

1	~ B ⊃ (D & E)	Assumption
2	(A & ~ B) & C	Assumption

e. Derive: ~ A ⊃ [B & (D & C)]

1	~ A ⊃ B	Assumption
2	B ⊃ D	Assumption
3	~ A ⊃ C	Assumption

*f. Derive: (H & ~ J) ⊃ (~ I & K)

1	H ⊃ (~ I & ~ L)	Assumption
2	~ J ⊃ (K & M)	Assumption

g. Derive: [(K ∨ L) ⊃ I] & [(K ∨ L) ⊃ ~ J]

1	(K ∨ L) ⊃ (I & ~ J)	Assumption

*h. Derive: M & ~ N

1	(K & ~ L) & (~ I & J)	Assumption
2	~ L ⊃ M	Assumption
3	(K & ~ I) ⊃ ~ N	Assumption

i. Derive: A ⊃ (B ⊃ C)

1	(A & B) ⊃ C	Assumption

*j. Derive: (A & B) ⊃ C

1	A ⊃ (B & C)	Assumption

k. Derive: (A & B) ⊃ (C & D)

1	(B & A) ⊃ (D & C)	Assumption

*l. Derive: M ⊃ (L & ~ L)

1	(M ⊃ ~ L) & (M ⊃ L)	Assumption

m. Derive: (A & B) ⊃ E

1	A ⊃ C	Assumption
2	B ⊃ D	Assumption
3	(C & D) ⊃ E	Assumption

5.1.2 INTRODUCTION AND ELIMINATION RULES FOR '~'

A strategy that is probably as old as is intelligent thought is that of establishing a thesis by assuming its opposite and showing that this assumption leads to clearly unacceptable results. By 'assuming its opposite' we mean that if **P** is the thesis to be established, then *not* **P** is assumed, and if *not* **P** is the thesis to be established, **P** is assumed. If such an assumption leads to an unacceptable result, this constitutes grounds for rejecting the assumption and accepting the original thesis. This general strategy is commonly known as the '*reductio ad absurdum*' strategy.

Consider the argument:

> If management will not negotiate the union will strike. If the union strikes, production will cease. If production ceases, management will negotiate. Therefore, management will negotiate.

We want to show that management will negotiate. So we assume the opposite, that management will not negotiate. From this and the first premise it follows that the union will strike. And from this result and the second premise it follows that production will cease. And from 'Production will cease' and the last premise it follows that management will negotiate. We now have the "unacceptable" result, namely the first premise, which says management will not negotiate, and the result of our last inference, which is that management will negotiate. Both cannot be the case. Since we got to this unacceptable result by assuming management will not negotiate, we reject that assumption and conclude that management will negotiate. In *SD* the rule that parallels the reasoning we have just completed is Negation Elimination:

Negation Elimination (~ E)

~ **P**
Q
~ **Q**
▷ **P**

This rule calls for us to assume the negation of the sentence we want to establish and then derive a sentence, any sentence, and its negation from the accessible assumptions. The "absurdity," of course, is in having reached both a sentence and its negation. Here is a derivation for a symbolized version of the above argument:

Derive: N

1	~ N ⊃ S	Assumption
2	S ⊃ C	Assumption
3	C ⊃ N	Assumption
4	| ~ N	A / ~ E
5	| S	1, 4 ⊃E
6	| C	2, 5 ⊃E
7	| N	3, 6 ⊃E
8	| ~ N	4 R
9	N	5–8 ~ E

We used Reiteration to obtain '~ N' at line 8, and this gives us both 'N' and '~ N' in the scope of the assumption on line 4. In the present case it happens that the sentence we want to derive, 'N', also serves as one member of the pair **Q** and ~ **Q** we derive within the scope of the subderivation starting at line 4. That the sentence to be derived be used as one member of the required contradictory pair is allowed but not required. Any sentence and its negation will do.

Note that Reiteration is frequently useful in derivations employing either Negation Elimination or Negation Introduction, for both rules require the derivation of a sentence and its negation, meaning the sentence and its negation must occur below the horizontal line marking the auxiliary assumption with which the negation subderivation begins.

The introduction rule for '~' is

Negation Introduction (~ I)

	| **P**	
	| **Q**	
	| ~ **Q**	
▷	~ **P**	

Here is a very simple derivation that uses this rule:

Derive: ~ H

1	H ⊃ F	Assumption
2	~ F	Assumption
3	| H	A / ~ I
4	| F	1, 3 ⊃E
5	| ~ F	2 R
6	~ H	3–5 ~I

Notice that in *SD* given a material conditional and its antecedent we can obtain the consequent of the material conditional in one step, by Conditional Elimination. But there is no rule in *SD* that takes us from a material conditional and the negation of its consequent to the negation of its antecedent. Rather, in such a case a negation subderivation such as we've used is appropriate.

It is important to remember that both negation rules call for deriving a sentence, any sentence, and its negation. An assumption, primary or auxiliary, often serves as one of these sentences. And a truth-functional compound sentence and its negation, as well as an atomic sentence and its negation, can serve as the pair of derived sentences both rules call for. Suppose we are given the following derivation to complete:

Derive: ~ B

1	~ (A ⊃ B)	Assumption
	~ B	

No rule of *SD* that we have introduced to this point (other than Reiteration), nor any that remain to be introduced, allows us to draw an inference directly from the negation of a material conditional. Realizing this, we might decide that the best way to obtain '~ B' is through Negation Introduction. So we fill in the next step:

Derive: ~ B

1	~ (A ⊃ B)	Assumption
2	B	A / ~ E
	~B	2–__ ~ E

Here and throughout the rest of this chapter we fill in as much of the justification column as we can as we construct the derivation. In this instance we use the notation '2–__' to indicate that we expect '~ B' to be obtained by Negation Elimination using a subderivation beginning on line 2. We do not yet know what the line number of the last line will be, hence the underscore.

We know we need to derive a sentence and its negation. A negation, '~ (A ⊃ B)', is already available, so we decide to use it and make our other goal 'A ⊃ B'. We can now fill in two more lines of our derivation:

Derive: ~ B

1	~ (A ⊃ B)	Assumption
2	B	A / ~ E
	A ⊃ B	
	~ (A ⊃ B)	1 R
	~ B	2–__ ~ E

We can complete the derivation if we can just find a way to derive 'A ⊃ B' from our accessible assumptions on lines 1 and 2. Our goal 'A ⊃ B' is a material conditional, and the rule for introducing material conditionals is Conditional Introduction. So we might try this:

Derive: ~ B

1	~ (A ⊃ B)	Assumption
2	B	A / ~ E
3	A	A / ⊃I
	B	
	A ⊃ B	3–__ ⊃I
	~ (A ⊃ B)	1 R
	~ B	2–__ ~ E

Note that the new subderivation is constructed within the subderivation beginning at line 2; this is necessary if we want to derive 'A ⊃ B' within the subderivation beginning at line 2.

At this point our task is to find a way to get from our auxiliary assumption on line 3 to 'B'. And here is where it is important to learn to see what we have available to us. We want 'B'. What lines are accessible at this point? The answer is lines 1 through 3. And the sentence we want, 'B', does occur on one of those lines—line 2. We can get it at line 4 by Reiteration and complete the derivation as follows:

Derive: ~ B

1	~ (A ⊃ B)	Assumption
2	B	A / ~ E
3	A	A / ⊃I
4	B	2 R
5	A ⊃ B	3–4 ⊃I
6	~ (A ⊃ B)	1 R
7	~B	2–6 ~ I

The sentence '~ (A ⊃ B)' that served as the negation ~ **Q** in this use of Negation Introduction is the negation of a compound sentence and is one of our primary assumptions. Notice also that in this example we constructed our derivation by working from our final goal, which we entered some distance below our primary assumption, backwards. We filled in the missing steps by working from the bottom up. This bottom up strategy is almost always the preferred strategy for constructing derivations.

Notice also that in this example the assumption we made in order to use Negation Elimination does not, by itself, lead to an absurdity. Rather, it is that assumption *in combination with the other accessible assumption*, the sentence on line 1, that gives us the absurdity. What we know, when we get to line 6, is that the sentences on lines 1 and 2 cannot both hold, for together they lead to a contradiction (the sentences on lines 5 and 6). Since the point of derivations is to show what can be derived from given primary assumptions, we hold on to our primary assumption and reject the auxiliary assumption which, together with the primary assumption, led to the absurdity.

Suppose we are next asked to complete the following derivation:

Derive: C

1	A & B	Assumption
2	~ (A & B)	Assumption
	C	

'C' is not a component of either primary assumption, so how can it possibly be derived from those assumptions? It is actually quite easy to derive 'C'. The "trick", again, is in learning to see what is available to us. We have a sentence 'A & B' and the negation of that sentence, '~ (A & B)'. Negation Elimination allows us to derive any sentence we want so long as we can assume its negation and derive some sentence **Q** and its negation. We can easily do that here:

Derive: C

1	A & B	Assumption
2	~ (A & B)	Assumption
3	~ C	A / ~ E
4	A & B	1 R
5	~ (A & B)	2 R
6	C	3–5 ~ E

Here, having assumed '~ C' and derived a sentence and its negation, we use Negation Elimination to derive 'C'. What this shows is not that 'C' has been

established on its own, but rather that 'C' follows from our primary assumptions. The primary assumptions themselves are inconsistent. And what this example shows is that any sentence can be derived from inconsistent assumptions, because we could use any sentence in place of 'C' in this derivation.

5.1.2E EXERCISES

1. Complete the following derivations.

a. Derive: ~ G

1	(G ⊃ I) & ~ I	Assumption

*b. Derive: K

1	M & ~ M	Assumption

c. Derive: ~ ~ B

1	~ B ⊃ A	Assumption
2	~ B ⊃ ~ A	Assumption

*d. Derive: I & M

1	~(I & M) ⊃ (L & ~ N)	Assumption
2	N	Assumption

e. Derive: A

1	(~ A ⊃ ~ B) & (~ B ⊃ B)	Assumption

5.1.3 INTRODUCTION AND ELIMINATION RULES FOR '∨'

The introduction rule for ∨ is Disjunction Introduction:

Disjunction Introduction (∨I)

▷	**P**		▷	**P**
	P ∨ Q	or		**Q ∨ P**

A common reaction to this rule is that it gives us something for nothing—by allowing us to derive **P ∨ Q**, where **Q** is any sentence whatsoever, from **P**. But we are not really getting something for nothing. To see this we need only remember that a disjunction is true if at least one of its disjuncts is true.

So given that **P** is true, it follows that **P** ∨ **Q** is also true. Disjunction Introduction is a useful rule. Consider the argument:

> Business is booming although the cost of energy is up.
>
> If either the cost of energy is up or food prices are increasing, inflation will continue.
>
> ---
>
> Inflation will continue.

Derive: I

1	B & E	Assumption
2	(E ∨ F) ⊃ I	Assumption
3	E	1 &E
4	E ∨ F	3 ∨I
5	I	2, 4 ⊃E

At line 3 we obtain 'E'. But what we need to get 'I' by Conditional Elimination is 'E ∨ F', the antecedent of the conditional on line 2. And we obtain 'E ∨ F' at line 4 by Disjunction Introduction.

The elimination rule for '∨' is Disjunction Elimination:

Disjunction Elimination (∨E)

P ∨ **Q**

P	
	R
Q	
	R

▷ **R**

This rule specifies that if we have a disjunction, **P** ∨ **Q**, and can, through two subderivations, derive a sentence **R** from each disjunct then we can infer **R**. Disjunction Elimination parallels a pattern of reasoning we often use in ordinary discourse. We know that if a disjunction is true then at least one of its disjuncts is true. Even if we don't know which disjunct is true, if we can derive the claim we are interested in from each disjunct then we know it follows from the disjunction, for it follows no matter which disjunct of that disjunction is true. We can use Disjunction Elimination to derive the conclusion of the following argument from its premises:

> The mill manager will either resign or be fired. If she resigns she will keep her pension and move back east. If she is fired she will lose her pension and move back east. So the mill manager will move back east.

Derive: E

1	R ∨ F	Assumption
2	R ⊃ (K & E)	Assumption
3	F ⊃ (~ K & E)	Assumption
4	R	A / ∨E
5	K & E	2, 4 ⊃E
6	E	5 &E
7	F	A / ∨E
8	~ K & E	3, 7 ⊃E
9	E	8 &E
10	E	1, 4–6, 7–9 ∨E

If the primary assumptions hold, then since 'R ∨ F' is one of those assumptions either 'R' holds or 'F' holds as well. In the two subderivations we show that 'E' follows from the primary assumptions and 'R' and that it follows from the primary assumptions and 'F'. Therefore 'E' follows from the primary assumptions alone.

Note that the justification for line 10 cites the line on which the disjunction occurs (line 1), and the two subderivations beginning with the individual disjunctions (the subderivation occurring on lines 4 through 6 and that occurring on lines 7 through 9).

5.1.3E EXERCISES

1. Complete the following derivations.

a. Derive B ∨ (K ∨ G)

1	K	Assumption

*b. Derive: A

1	B ∨ C	Assumption
2	B ⊃ A	Assumption
3	C ⊃ A	Assumption

c. Derive: D ∨ E

1	E ∨ D	Assumption

*d. Derive: I ∨ J

1	K ∨ G	Assumption
2	K ⊃ I	Assumption
3	G ⊃ J	Assumption

e. Derive: F

1	~ E ∨ F	Assumption
2	~ E ⊃ F	Assumption

5.1.4 INTRODUCTION AND ELIMINATION RULES FOR '≡'

Sentences of the form **P ≡ Q** are called biconditionals for good reason. They are equivalent to the conjunction of two conditionals. That is, a sentence of the form **P ≡ Q** is equivalent to a sentence of the form **(P ⊃ Q) & (Q ⊃ P)**. Bearing this in mind, it should not be surprising that the introduction rule for '≡' involves two subderivations:

Biconditional Introduction (≡I)

	P
	Q
	Q
	P
▷	**P ≡ Q**

As this schema indicates, to derive **P ≡ Q** it is sufficient to derive **Q** from **P** and **P** from **Q**.

We can use Biconditional Introduction to derive the conclusion of the following simple argument from its premises.

> If Alice will get into law school, then Betty will also. If Betty will get into law school, then both Charles and Alice will get into law school. So Betty and Charles will both get into law school if and only if Alice will get into law school.

Derive: (B & C) ≡ A

1	A ⊃ B	Assumption
2	B ⊃ (C & A)	Assumption
3	B & C	A / ≡I
4	B	3 &E
5	C & A	2, 4 ⊃E
6	A	5 &E
7	A	A / ≡I
8	B	7, 1 ⊃E
9	C & A	2, 8 ⊃E
10	C	9 &E
11	B & C	8, 10 &I
12	(B & C) ≡ A	3–6, 7–11 ≡I

Note that while the sentences on lines 8 and 9 do occur earlier in the derivation, on lines 4 and 5, they cannot be obtained in their second occurrences by Reiteration, for lines 4 and 5 are not accessible after line 6.

Biconditional Elimination is straightforward:

Biconditional Elimination (≡E)

	P ≡ Q			**P ≡ Q**
	P	or		**Q**
▷	**Q**		▷	**P**

From a material biconditional and one of its immediate components we can derive the other immediate component.

We use Biconditional Elimination twice in deriving the conclusion of the following argument from the argument's premises.

> Alex will graduate if and only if she passes both logic and physics. Alex will pass physics but she will pass logic if and only if she aces the final and does all the remaining homework assignments. Alex never does homework. So Alex won't graduate.

Derive: ~ G

1	G ≡ (L & P)	Assumption
2	P & [L ≡ (A & H)]	Assumption
3	~ H	Assumption
4	G	A / ~ I
5	L & P	1, 4 ≡E
6	L	5 &E
7	L ≡ (A & H)	2 &E
8	A & H	6, 7 ≡E
9	H	8 &E
10	~ H	3 R
11	~ G	4–10 ~ I

In this derivation we selected 'H' and '~ H' as the **Q** and ~ **Q** of our subderivation largely because '~ H' is readily available—it can be gotten by Reiteration on line 3.

5.1.4E EXERCISES

1. Complete the following derivations.

a. Derive: L

1	K ≡ (~ E & L)	Assumption
2	K	Assumption

*b. Derive: ~ D ≡ E

1	(~ D ⊃ E) & (E ⊃ ~ D)	Assumption

c. Derive: S & ~ A

1	(S ≡ ~ I) & N	Assumption
2	(N ≡ ~ I) & ~ A	Assumption

*d. Derive: N

1	A ∨ L	Assumption
2	A ≡ N	Assumption
3	L ⊃ N	Assumption

e. Derive: E ≡ O

1	(E ⊃ T) & (T ⊃ O)	Assumption
2	O ⊃ E	Assumption

5.1.5 RULE SUMMARY

All the derivation rules of *SD* have been introduced. We repeat them here for easy reference. They can also be found on the inside front cover of this text.

Reiteration (R)

	P
▷	P

Conjunction Introduction (&I)

	P
	Q
▷	P & Q

Conjunction Elimination (&E)

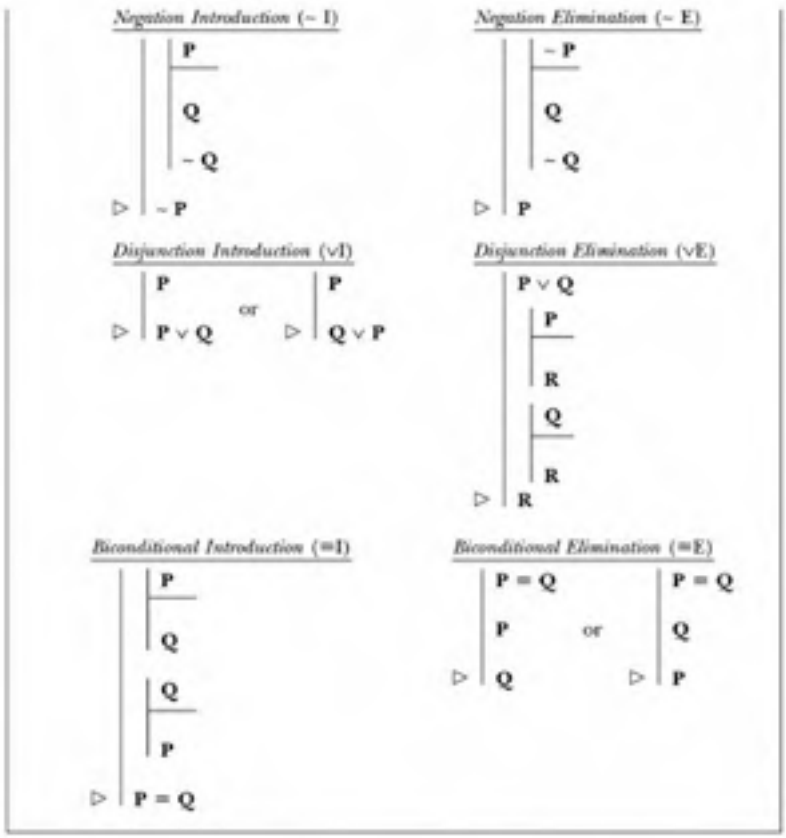
	P & Q	or		P & Q
▷	P		▷	Q

Conditional Introduction (⊃I)

	P	
	Q	
▷	P ⊃ Q	

Conditional Elimination2 (⊃E2)

	P ⊃ Q
	Q
▷	P



We have presented the derivation rules of *SD* and constructed a fair number of derivations. But we haven't actually defined the term 'derivation in *SD*'. We do so now:

> A *derivation in SD* is a series of sentences of *SL*, each of which is either an assumption or is obtained from previous sentences by one of the rules of *SD*.

We will continue to annotate our derivations with line numbers, scope and assumption lines, and line justifications. However, these annotations are not, as the above definition makes clear, officially parts of derivations.

There are many truth-preserving templates we do not include as rules of either *SD* or *SD+*. Why are some included and others not? For *SD* the answer

is fairly simple. We want a derivation system to be truth-preserving (include no rule that ever takes us from truths to a falsehood). A system that has this property, never takes us from truths to a falsehood, is said to be **sound.** We also want our derivation systems to be **complete.** A derivation system is complete if and only if every sentence that is truth-functionally entailed by a set of sentences can be derived from that set. *SD* is complete in this sense and it is a fairly minimalist derivation system—it includes only two rules for each connective.[1] *SD+* will also be complete but includes additional derivation rules, some because they mirror reasoning patterns that are common in everyday discourse, some because they have historically been included in derivation systems. We prove that both *SD* and *SD+* are complete in Chapter 6.

Before ending this section we will take time to caution against some mistakes that are commonly made while constructing derivations. First, the derivation rules of *SD* are rules of inference, which is to say that when they appeal to a line earlier in the derivation they appeal to the entire sentence on that line, not to a sentence that is a component of a longer sentence. Here is an attempt at a derivation that misuses Conjunction Elimination by appealing to a component of a longer sentence.

Derive A ⊃ C

1	A ⊃ (B & C)	Assumption	
2	| A	A / ⊃I	
3	| C	1 &E	**MISTAKE!**
4	A ⊃ C	2–3 ⊃I	

The mistake at line 3 results from trying to apply Conjunction Elimination to a component of a longer sentence. The sentence on line 1 is *not* of the form **P** & **Q**, and while a component of that sentence, 'B & C', is of that form, rules of inference work, again, on sentences that are not themselves parts of longer sentences. A correct derivation for this problem is

Derive A ⊃ C

1	A ⊃ (B & C)	Assumption
2	| A	A / ⊃I
3	| B & C	1, 2 ⊃E
4	| C	3 &E
5	A ⊃ C	2–4 ⊃I

The sentence on line 3 is of the form **P** & **Q**. It is not part of a longer sentence on that line. So we can apply Conjunction Elimination to it and obtain 'C' at line 4.

[1] Two rules of *SD*, Reiteration and Negation Introduction, could be dropped without making the system incomplete. This is not true of any of the other rules of *SD*.

Here is a similar misuse of a derivation rule.

Derive: C

1	B ⊃ (A ⊃ C)	Assumption	
2	A	Assumption	
3	C	1, 2 ⊃E	**MISTAKE!**

Here an attempt has been made to apply Conditional Elimination to a component, 'A ⊃ C' of the longer sentence 'B ⊃ (A ⊃ C)' and this cannot be done. In this case there is no correct derivation. 'C' does not follow from the assumptions on lines 1 and 2.

Another common mistake is to appeal to lines or subderivations that are not accessible. In a derivation a sentence or subderivation is *accessible* at line **n** (it can be appealed to when justifying a sentence on line **n**) if and only if that sentence or subderivation does not lie within the scope of a closed assumption, that is, an assumption that has been discharged prior to line **n**. Here is an attempt at a derivation that twice violates the accessibility requirement:

Derive: B

1	B	A / ⊃I	
2	A	A / ⊃I	
3	B	1 R	
4	A ⊃ B	2-3 ⊃I	
5	B ⊃ (A ⊃ B)	1-4 ⊃I	
6	A ⊃ B	2-3 ⊃I	**MISTAKE!**
7	B	2, 6 ⊃E	**MISTAKE!**

Line 6 is a mistake because it appeals to a subderivation, that occurring on lines 2 through 3, that is no longer accessible. It is not accessible at line 6, because not every scope line to the left of that subderivation (there are two) continues to line 6. The auxiliary assumption occurring on line 2 was discharged at line 4, when Conditional Introduction was used. (We also cannot use Reiteration to obtain A ⊃ B on line 6, because the sentence on line 4 is inaccessible at that point.) Line 7 is a mistake because it appeals to a line, line 2, which is no longer accessible. Of course, it is also a mistake because it appeals to a line, line 6, which is itself a mistake. In fact, neither line 6 nor line 7 can be derived in the main derivation without primary assumptions. On the other hand, part of the above attempt, namely the part consisting of lines 1 through 5, is correct, demonstrating that some sentences can be derived starting from no primary assumptions. 'B ⊃ (A ⊃ B)' is one such sentence.

The following derivation is correctly done.

Derive: ~ U ⊃ ~ S

1	~ U ⊃ ~ W	Assumption
2	~ W ⊃ ~ S	Assumption
3	| ~ U	A / ⊃I
4	| ~W	1, 3 ⊃E
5	| ~ S	2, 4 ⊃E
6	~ U ⊃ ~S	3–5 ⊃I

Line 4 cites lines 1 and 3, which are both accessible at line 4. The sentences on lines 1 and 3 do not lie within the scope of an assumption that has been discharged prior to line 4. (Neither the sentence on line 1 nor the sentence on line 3 has a scope line to its left that is not also to the left of the sentence on line 4.) Similarly line 5 cites lines 2 and 4, which are both accessible at line 5. Line 6 cites the subderivation from lines 3–5. This subderivation is accessible at line 6 because the subderivation does not lie within the scope of an assumption that has been closed prior to line 6.

To summarize, once an assumption has been discharged or closed, none of the lines or subderivations within the subderivation that began with the now closed assumption is accessible for justifying sentences on later lines. In the last example, once the assumption on line 3 is closed, none of the lines within the scope of that assumption is accessible. This as is as it should be, for the sentences within the closed subderivation may have been derived using the assumption of the subderivation, an assumption that has been discharged. There is no guarantee that sentences derived using an assumption can be derived without it. So it would be incorrect to continue the derivation as follows:

Derive: ~ U ⊃ ~ S

1	~ U ⊃ ~ W	Assumption	
2	~ W ⊃ ~ S	Assumption	
3	| ~ U	A / ⊃I	
4	| ~ W	1, 3 ⊃E	
5	| ~ S	2, 4 ⊃E	
6	~ U ⊃ ~ S	3–5 ⊃I	
7	~ S	2, 4 ⊃E	**MISTAKE!**

The mistake at line 7 is citing line 4, which is not accessible at line 7. This is because the sentence on line 4 does lie within the scope of an assumption (the one on line 3) that has been discharged before line 7. (There is a scope line to the left of the sentence on line 4 that does not appear to the left of the sentence on line 7.)

Here is another example in which an inaccessible subderivation is cited:

Derive A ≡ C

1	~ C	Assumption	
2	B ⊃ C	Assumption	
3	~ A & ~ B	Assumption	
4	A	A / ≡I	
5	~ B	A / ~ E	
6	~ A	3 &E	
7	A	4 R	
8	B	5–7 ~E	
9	C	2, 8 ⊃E	
10	C	A / ≡I	
11	~ B ⊃ A	5–7 ⊃I	**MISTAKE!**
12	~ B	3 &E	
13	A	11, 12 ⊃E	
14	A ≡ C	4–9, 10–13 ≡I	

The mistake at line 11 is that of citing a subderivation that is not available at line 11. That it is not available is indicated by there being a scope line to the left of the subderivation, the scope line running from line 4 through line 9, that is not to the left of the sentence entered at line 11. More substantively, 'A' was derived at line 7 by Reiteration on line 4. The assumption at line 4 is not available at line 11, and neither are results obtained while it was available.

In fact, it is possible to derive 'A ≡ C' from the above primary assumptions. Here is a derivation that does so.

Derive A ≡ C

1	~ C	Assumption
2	B ⊃ C	Assumption
3	~ A & ~ B	Assumption
4	A	A / ≡I
5	~ C	A / ~ E
6	~ A	3 &E
7	A	4 R
8	C	2, 7 ⊃E
9	C	A / ≡I
10	~ A	A / ~ E
11	C	9 R
12	~ C	1 R
13	A	10–12
14	A ≡ C	4–8, 9–13 ≡I

It is possible to use a single auxiliary assumption to generate a subderivation that allows the use of two different subderivation rules. Here is such a case:

Derive: C & (A ⊃ C)

1	A ∨ B	Assumption
2	A ⊃ D	Assumption
3	B ⊃ D	Assumption
4	~ C ⊃ ~ D	Assumption
5	A	A / ∨E / ⊃I
6	~ C	A – E
7	~ D	4, 6 ⊃E
8	D	2, 5 ⊃E
9	C	6–8 ~ E
10	B	A / ∨E
11	~ C	A – E
12	~ D	11, 4 ⊃E
13	D	3, 10 ⊃E
14	C	11–13 ~ E
15	C	1, 5-9, 10–14 ∨E
16	A ⊃ C	5–9 ⊃I
17	C & (A ⊃ C)	15, 16 &I

Notice that the subderivation occupying lines 5 through 9 is cited twice, once as part of an application of the rule Disjunction Elimination (at line 15) and once as the basis for entering a conditional at line 16. In the present case it is unlikely that when the assumption at line 5 is made it was foreseen that the subderivation to be constructed would be used in both of the above indicated ways. So most likely at the time the assumption was made the only notation entered in the justification column was 'A / ∨E'. It is only after reaching 'C' at line 15 and wondering how 'A ⊃ C' can be obtained that it became apparent that work already done, the subderivation on lines 5 through 9, could be reused. So the extra notation '/ ⊃I' was added to line 5 when line 16 was entered.

In the above example identical subderivations occur on lines 6 through 8 and lines 11 through 13. We had to do this work twice because when trying to get from 'B' at line 10 to 'C' on a subsequent line the subderivation occupying lines 6 through 8 is no longer available.

Finally, it is possible to end a subderivation at any time, without using one of the introduction rules that requires a subderivation. This is likely to occur when one decides the strategy being pursued is unproductive and simply abandons the work done within the subderivation. Here is an example:

Derive: A ⊃ (B ⊃ C)

1	A	A / ⊃I
2	~ (B ⊃ A)	A / ~ I
3	A	1 R
4	B	A / ⊃I
5	A	1 R
6	B ⊃ A	4–5 ⊃I
7	A ⊃ (B ⊃ A)	1–6 ⊃I

Here the subderivation on lines 2–3 is in effect wasted work, work we have thrown away. It does no harm, but neither does it do any good.

5.1.5E EXERCISES

1. Complete each of the following derivations by entering the appropriate justifications.

a. Derive: (A & C) ∨ (B & C)

1	(A ∨ B) & C	
2	A ∨ B	
3	C	
4	A	
5	A & C	
6	(A & C) ∨ (B & C)	
7	B	
8	B & C	
9	(A & C) ∨ (B & C)	
10	(A & C) ∨ (B & C)	

c. Derive: ~ B

1	B ⊃ (A & ~ B)	
2	B	
3	A & ~ B	
4	~ B	
5	B	
6	~ B	

*b. Derive: A ⊃ (B ⊃ C)

1	(A & B) ⊃ C	
2	A	
3	B	
4	A & B	
5	C	
6	B ⊃ C	
7	A ⊃ (B ⊃ C)	

*d. Derive: A ⊃ B

1	(A & ~ B) ⊃ (~ B & C)	
2	C ⊃ ~ A	
3	A	
4	~ B	
5	A & ~ B	
6	~ B & C	
7	C	
8	~ A	
9	A	
10	B	
11	A ⊃ B	

e. Derive: $C \supset (\sim A \ \& \ B)$

1	$\sim D$	
2	$C \supset (A \equiv B)$	
3	$(D \vee B) \supset \sim A$	
4	$(A \equiv B) \supset (D \ \& \ E)$	
5	$\sim B \supset D$	
6	C	
7	$A \equiv B$	
8	$D \ \& \ E$	
9	D	
10	$D \vee B$	
11	$\sim A$	
12	$\sim B$	
13	D	
14	$\sim D$	
15	B	
16	$\sim A \ \& \ B$	
17	$C \supset (\sim A \ \& \ B)$	

*f. Derive: $A \supset (B \vee C)$

1	$(\sim B \ \& \sim C) \supset \sim A$	
2	A	
3	$\sim (B \vee C)$	
4	B	
5	$B \vee C$	
6	$\sim (B \vee C)$	
7	$\sim B$	
8	C	
9	$B \vee C$	
10	$\sim (B \vee C)$	
12	$\sim C$	
13	$\sim B \ \& \sim C$	
14	$\sim A$	
15	A	
16	$B \vee C$	
17	$A \supset (B \vee C)$	

g. Derive: $A \equiv B$

1	$\sim A \ \& \sim B$	
2	A	
3	$\sim B$	
4	$\sim A$	
5	A	
6	B	
7	B	
8	$\sim A$	
9	B	
10	$\sim B$	
11	A	
12	$A \equiv B$	

*h. Derive: $A \equiv (B \vee C)$

1	$(A \equiv B) \ \& \ (A \equiv C)$	
2	A	
3	$A \equiv B$	
4	B	
5	$B \vee C$	
6	$B \vee C$	
7	B	
8	$A \equiv B$	
9	A	
10	C	
11	$A \equiv C$	
12	A	
13	A	
14	$A \equiv (B \vee C)$	

2. Find and explain each mistake in the following attempted derivations.

a. Derive: $\sim D$

1	$\sim \sim A \supset (B \ \& \sim D)$	Assumption
2	$\sim A$	Assumption
3	$B \ \& \sim D$	1, 2 $\supset$E
4	$\sim D$	3 &E

*b. Derive: B

1	C ⊃ [D ≡ (A & B)]	Assumption
2	C & D	Assumption
3	C	2 &E
4	A & B	1, 3 ≡E
5	B	4 &E

c. Derive: H & A

1	B ⊃ A	Assumption
2	H	Assumption
3	B	Assumption
4	A	1, 3 ⊃E
5	A & A	4, 4 &I
6	B ⊃ (A & A)	3–5 ⊃I
7	H & A	2, 4 &I

*d. Derive: M

1	(G ⊃ ~ ~ M) & G	Assumption
2	G	1 &E
3	G ⊃ ~ ~ M	1 &E
4	~ ~ M	2, 3 ⊃E
5	M	4 ~ E

5.2 BASIC CONCEPTS OF *SD*

We now define the key concepts of *SD*. These are all syntactical concepts as each is defined by reference to there being a derivation of a certain sort—no reference is made in any of these definitions either to truth-values or to truth-value assignments.

Derivability: A sentence **P** of *SL* is *derivable in SD* from a set Γ of sentences of *SL* if and only if there is a derivation in *SD* in which all the primary assumptions are members of Γ and **P** occurs within the scope of only the primary assumptions.

Valid in SD: An argument of *SL* is *valid in SD* if and only if the conclusion of the argument is derivable in *SD* from the set consisting of the premises. An argument of *SL* is *invalid in SD* if and only if it is not valid in *SD*.

Theorem in SD: A sentence **P** of *SL* is a *theorem in SD* if and only if **P** is derivable in *SD* from the empty set.

Equivalence in SD: Sentences **P** and **Q** are *equivalent in SD* if and only if **Q** is derivable in *SD* from {**P**} and **P** is derivable in *SD* from {**Q**}.

Inconsistent in SD: A set Γ of sentences of *SL* is *inconsistent in SD* if and only if there is a sentence **P** such that both **P** and $-$ **P** are derivable in *SD* from Γ. A set Γ is *consistent in SD* if and only if it is not inconsistent in *SD*.

A few additional notational conventions will be useful. We will use the single turnstile, '$\vdash$' to assert derivability, and will read

$$\Gamma \vdash \mathbf{P}$$

as '**P** is derivable from Γ'. We will read '$\Gamma \nvdash \mathbf{P}$' as '**P** is not derivable from Γ'. This parallels our use of the double turnstile in previous chapters, where we read

$$\Gamma \models \mathbf{P}$$

as 'Γ truth-functionally entails **P**' and '$\Gamma \not\models \mathbf{P}$' as '$\Gamma$ does not truth-functionally entail **P**'. The parallelism is for good reason. It will turn out that for any finite set Γ of sentences of *SL* and any sentence **P** of *SL*,

$\Gamma \vdash \mathbf{P}$ in *SD* if and only if $\Gamma \models \mathbf{P}$.

This is a key claim of metatheory that we prove in Chapter 6. Finally, we will read

$$\vdash \mathbf{P}$$

as '**P** is a theorem'. This notation derives from '$\varnothing \vdash \mathbf{P}$', which is read '**P** is derivable from the empty set'. And of course a sentence of *SL* is a theorem of *SD* if and only if it is derivable in *SD* from the empty set. We will also refer to a derivation of a sentence of *SL* from no primary assumptions as a **proof** of the theorem that is the last line of that derivation.

The careful reader will recall that there are seven key semantical concepts of *SL*: Truth-functional consistency, truth-functional truth, truth-functional falsity, truth-functional indeterminacy, truth-functional equivalence, truth-functional validity, and truth-functional entailment. We have syntactic parallels for only five of those concepts. These pair up as follows:

Truth-functional consistency	Consistency in *SD*
Truth-functional truth	Theorem in *SD*
Truth-functional equivalence	Equivalence in *SD*
Truth-functional validity	Valid in *SD*
Truth-functional entailment	Derivability in *SD*

There is no syntactic counterpart to either truth-functional falsity or truth-functional indeterminacy. Introducing such counterparts is easy enough—we

could define an *anti-theorem* of *SD* as a sentence **P** of *SL* whose negation, ~ **P**, is a theorem of *SD*. And we could take a sentence **P** of *SL* to be *syntactically undetermined* in *SD* if and only if neither it nor its negation is a theorem of *SD*. We would then have syntactic counterparts to all seven central semantic concepts, but historically logicians have never felt the need to add these or equivalent definitions. We will follow their lead.

Below we construct a derivation that establishes that the following simple argument is valid in *SD*:

$$\begin{array}{l} A \supset B \\ \sim B \\ \hline \sim A \end{array}$$

Derive: ~ A

1	A ⊃ B	Assumption
2	~ B	Assumption
3	| A	A /∴ I
4	| B	1, 3 ⊃E
5	| ~ B	2 R
6	~ A	3–5 ~ I

This derivation establishes that the above argument is valid in *SD*. (The conclusion of the argument has been derived from the set consisting of the premises of the argument.)

On the other hand, the following *does not* establish the validity of the above argument:

Derive: ~ A

1	A ⊃ B	Assumption
2	~ B	Assumption
3	| ~ A	A
4	| ~ A	3 R

Here '~ A', the conclusion of the argument, has not been derived from the set consisting of the premises of the argument. Rather, it has been derived from those sentences and '~ A'—that is, from the primary assumptions and an auxiliary assumption. We have not shown that '~ A' is derivable from the set consisting of the premises A ⊃ B and ~ A.

Note that no notation has been made on line 3 as to the reason for assuming '~ A'. Someone constructing a derivation such as this may well have reasoned "I want to obtain '~ A'. Since I can assume anything, I will assume what I want, namely '~ A', and then use Reiteration to derive my goal, '~ A'." It is true that any sentence of *SL* can be assumed at any time. But there is no point to

assuming a sentence unless one has a rule in mind for discharging that assumption. This is why we require the justification column for auxiliary assumptions to include both the indication that the sentence just entered is an assumption ('A') and an indication of what rule will be used to discharge the assumption. There are only five rules (Conditional Introduction, Disjunction Elimination, Negation Introduction, Negation Elimination, and Biconditional Introduction) that require an assumption be made. Hence there are only five rules for discharging an assumption. Requiring a notation that indicates what rule will be used to discharge an assumption largely prevents the making of assumptions that do not serve a strategic purpose.

A theorem of *SD* is a sentence of *SL* that can be derived from no primary assumptions. A derivation of such a sentence is said to be a **proof** of that sentence. Here is a proof of the theorem '$[A \supset (B \supset C)] \supset [(A \,\&\, B) \supset C]$':

Derive: $[A \supset (B \supset C)] \supset [(A \,\&\, B) \supset C]$

1	$A \supset (B \supset C)$	A / ⊃I
2	$A \,\&\, B$	A / ⊃I
3	A	2 &E
4	$B \supset C$	3, 1 ⊃E
5	B	2 &E
6	C	5, 4 ⊃E
7	$(A \,\&\, B) \supset C$	2–6 ⊃I
8	$[A \supset (B \supset C)] \supset [(A \,\&\, B) \supset C]$	1–7 ⊃I

There are no primary assumptions in this derivation, and every auxiliary assumption has been closed. The sentence '$[A \supset (B \supset C)] \supset [(A \,\&\, B) \supset C]$' on the last line does not lie within the scope of any assumption. Hence it has been derived from the empty set and is a theorem of *SD*.

The sentences '$A \supset B$' and '$\sim B \supset \sim A$' are equivalent in *SD*, as the following two derivations show. (Establishing equivalence in *SD* of two distinct sentences of *SL* requires *two* derivations because we must show that each sentence is derivable from the unit set of the other.)[2]

Derive: $\sim B \supset \sim A$

1	$A \supset B$	Assumption
2	$\sim B$	A / ⊃I
3	A	A / ~ I
4	B	1, 3 ⊃E
5	$\sim B$	2 R
6	$\sim A$	3–5 ~ I
7	$\sim B \supset \sim A$	2–6 ⊃I

[2]Each sentence of *SL* is equivalent in *SD* to itself. And to show this we need only one derivation, a derivation of the sentence in question from itself.

Having derived '~ B ⊃ ~ A' from 'A ⊃ B', we now derive 'A ⊃ B' from '~ B ⊃ ~ A'. *Talk of reversing the process makes it sound like the derivations will be one the reverse of the other . . . .*

Derive: A ⊃ B

1	~ B ⊃ ~ A	Assumption
2	A	A / ⊃I
3	~ B	A / ~ E
4	~ A	3, 1 ⊃E
5	A	2 R
6	B	3-5 ~I
7	A ⊃ B	2-6 ⊃I

5.3 STRATEGIES FOR CONSTRUCTING DERIVATIONS IN *SD*

Derivations are unlike truth-tables and truth-trees in two important respects. First, when one of the syntactic properties we have defined holds (for a sentence, a pair of sentences, an argument, etc.) there is a derivation that demonstrates that this property holds. For example, if an argument is valid in *SD* it is the existence of a derivation of the conclusion of the argument from the set consisting of the argument's premises that makes this so. But if an argument is invalid in *SD* there is no derivation that demonstrates this. Rather, it is the *absence* of a derivation that makes an argument invalid in *SD*. But while one can use the derivation system *SD* to show that there is a derivation of a certain sort (by producing such a derivation), one cannot use it to *show* that there is no derivation of a certain sort. No number of unsuccessful attempts to construct a derivation of a certain sort proves that there is no such derivation. Hence, the system *SD* can be used to establish validity in *SD*, but not invalidity in *SD*. So too for equivalence in *SD*, inconsistency in *SD*, and theoremhood in *SD*. That is, one cannot use the system *SD* to prove that the members of a pair of sentences are not equivalent in *SD*, that a set is consistent in *SD*, or that a sentence is not a theorem in *SD*. In this way the derivation system is unlike truth-tables and truth-trees, for those procedures are able to establish, for each key semantic concept of *SL*, whether that concept holds or does not hold for a sentence or set of sentences of *SL*.

A second important difference between truth-tables and truth-trees and derivations is that while it is fairly easy to see how an explicit procedure can be developed for constructing truth-tables and truth-trees such that following the procedure does not call for making any choices and always results in a truth-table or truth-tree that yields an answer to the question being asked (*e.g.*, is this set truth-functionally consistent), it is considerably harder to specify such an explicit procedure for constructing derivations. Procedures that do determine every step of the construction process, whether for truth-tables, trees, or derivations, are said to be **mechanical** procedures. While mechanical

procedures for constructing derivations in systems like *SD* (derivation systems for sentential logic)—procedures that will always produce a derivation of a certain sort when one does exist—have been formulated, they are very complex and we will make no attempt to present such a procedure here.[3] There are thus two ways in which one's efforts to construct a derivation of a certain sort might end in frustration—where there is no such derivation and where there is one but all attempts one makes to find it fail. Of course these are very different situations; the first results from trying to do what is impossible, the second from failing to find a solution that does exist.

While we will not present a mechanical procedure for constructing derivations we will provide some useful strategies, strategies that can help avoid frustration of the second sort just alluded to. The overarching strategy is that of goal analysis. In every derivation the goal is to derive a sentence, or sentences, from primary assumptions where there are such, otherwise from no assumptions. Goal analysis is the process of determining how a goal sentence can be derived, and involves working backward from the intended last line of the derivation as well as forward from the primary assumptions, if any, of the derivation.

No matter what the goal sentence is, the derivation step that produces that sentence might be the application of any of the elimination rules. To see this one need only remember that the elimination rules tell us nothing about the derived sentence—in each case it might be an atomic sentence, a conjunction, a disjunction, a conditional, a negation, or a biconditional. On the other hand, the introduction rules do tell us a lot about the sentence derived by using one of these rules. First, atomic sentences cannot be derived by using an introduction rule, for all such rules produce truth-functionally compound sentences. Second, we know, for each introduction rule, what the main connective is of a sentence obtained by that rule. Conjunction Introduction produces conjunctions, Disjunction Introduction disjunctions, and so on.

The first step in goal analysis is therefore to determine what kind of a sentence the goal sentence is. If it is an atomic sentence it must be obtained by one of the elimination rules (or by Reiteration). If it is a truth-functional compound sentence it might be obtained by any of the elimination rules or by the appropriate introduction rules, namely the introduction rule that produces sentences whose main connective is the main connective of the goal sentence. The bottom line, of course, is that there will always be multiple ways in which the goal sentence might be derived. But some ways will generally be more plausible than others, as we will soon see.

Having picked one way in which a goal sentence can be obtained, the next step is to determine whether this way of obtaining the goal sentence generates one or more new goal sentences, and then to ask of each of these how they might be obtained. The idea is that eventually the rule picked as a way of

[3] These procedures are generally called decision procedures because what the procedure does, in the first instance, is give mechanical instructions for constructing a proof of a theorem. These procedures are very complicated. It is also important to note that such procedures, when applied to a sentence that is not a theorem of the system, will produce no result that shows the sentence in question is not a theorem.

obtaining the current goal can be applied directly to currently available sentences, thus completing the derivation. Multiple examples will, we hope, make all of this much clearer.

We here enumerate the strategies we will use throughout the rest of this chapter:

- If the sentence that is the current goal can be derived by using an elimination rule or some sequence of elimination rules to accessible sentences, then that is the strategy to follow.
- If the current goal can be obtained by an introduction rule, that is the strategy to follow.
- In most cases the successful strategy will make use of several of these approaches, working from the "bottom up" and from the "top down" as the occasion indicates.
- When using a negation rule try to use a negation that is readily available as the ~ **Q** that the rule requires within the negation subderivation.
- If a sentence is derivable from a set of sentences, then it is derivable using a negation rule as the primary strategy. So if no other strategy suggests itself it is useful to consider a negation strategy. But like all strategies, just because a negation strategy is available doesn't mean it is always the best choice.
- There will often be more than one plausible strategy, and often more than one will lead to success. Rather than trying to figure out which of these is the most promising it is often wise to just pick one and pursue it.

Suppose we are trying to derive '(A & B) ⊃ C' from {A ⊃ C}. The derivation will obviously have just one primary assumption. So we start work as follows:

Derive: (A & B) ⊃ C

1	A ⊃ C	Assumption
G	(A & B) ⊃ C	

Our current goal is the sentence '(A & B) ⊃ C'. We have indicated this by writing 'G' where a line number will eventually be placed. We will follow this convention, of indicating goal sentences by writing 'G' where the number of the line will eventually be, throughout the rest of this section. Readers should

follow this convention when constructing their own derivations only if they are working in pencil and can erase these goal sentence markers and replace them with line numbers as appropriate. We write this goal sentence a substantial distance below the primary assumptions, because we do not know, at this stage, how many steps it will take to derive this sentence. At this early stage we know neither the line number nor the justification for the final line of the derivation. We note that the goal sentence is a material conditional. Hence, in principle it could come by any one of the elimination rules, by Reiteration, or by Conditional Introduction. Reiteration is not plausible, as the goal sentence is not among the primary assumptions (there is only one). An elimination rule is not a likely way of generating the goal sentence because the only accessible sentence is the conditional on line 1 and Conditional Elimination requires that we have both a conditional and the antecedent of that conditional. In this case we do not have the antecedent of '$A \supset C$', and even if we did the result of applying Conditional Elimination would be 'C', not '$(A \& B) \supset C$'. So Conditional Introduction seems to be the most likely rule to have produced our goal sentence. We now note that to use Conditional Introduction we need a subderivation whose assumption is the antecedent of our goal sentence, namely '$A \& B$', and we need to derive the consequent of our goal sentence, 'C', within the scope of that assumption. That is, we know our derivation will look like this:

Derive: $(A \& B) \supset C$

1	$A \supset C$	Assumption
2	$A \& B$	A / $\supset$I
G	C	
G	$(A \& B) \supset C$	2–__ $\supset$I

We still do not know the line number of the last line of our derivation, but we do know we will use Conditional Introduction to obtain it and that we will cite a subderivation that begins on line 2. We note this in the justification column for the last line by entering '2–__ $\supset$I' where the underscore marks the space where we will subsequently enter the number of the preceding line. We also know that line 2 will be an auxiliary assumption made for the purpose of doing Conditional Introduction. We are now in a position to stop wondering how '$(A \& B) \supset C$' will be obtained. We have a strategy for obtaining that sentence, Conditional Introduction. Accordingly we now switch our focus to how we can complete the subderivation we have started. That is, how can we get from our two assumptions, one primary and one auxiliary, to 'C'? 'C' is an atomic sentence, so we know we will not use an introduction rule to obtain this sentence. Nor will Reiteration generate 'C'. So we are left with the elimination rules. Which elimination rule seems promising? Here it is important to learn to "see" what is available to us at this point in our work. We have two sentences to work

from, 'A ⊃ C' and 'A & B'. We want 'C'. We know that 'C' can be obtained from 'A ⊃ C' by Conditional Elimination *if* we have 'A'. We do not currently have 'A'. But we do have 'A & B', and 'A' can be obtained from 'A & B' by Conjunction Elimination. So we now see a path to the completion of our derivation:

Derive: (A & B) ⊃ C

1	A ⊃ C	Assumption
2	| A & B	A / ⊃I
3	| A	2 &E
4	| C	1, 3 ⊃E
5	(A & B) ⊃ C	2–4 ⊃I

ARGUMENTS

Consider next the following argument.

~ N
(~ N ⊃ L) & [D ≡ (~ N ∨ A)]

L & D

To show that this argument is valid in *SD* we need to derive the conclusion from the set consisting of the premises. So we start as follows:

Derive: L & D

1	~ N	Assumption
2	(~ N ⊃ L) & [D ≡ (~ N ∨ A)]	Assumption
	|	
	|	
	|	
G	| L & D	—, — &I

Our goal is a conjunction. It seems unlikely that it will be obtained by an elimination rule, in part because 'L & D' does not occur as a component of any accessible sentence. An introduction rule seems more promising, and since the main connective of our goal sentence is '&' it is Conjunction Introduction that seems most promising. We have noted this by writing '&I' in the justification column for our goal sentence, and we have indicated with two underscores that two line numbers will need to be supplied later. If we are to use Conjunction Introduction we will need to have the two conjuncts 'L'

and 'D' available on accessible earlier lines. So we now add two **subgoals** to our derivation structure:

Derive: L & D

1	~ N	Assumption
2	(~ N ⊃ L) & [D ≡ (~ N ∨ A)]	Assumption
G	L	
G	D	
G	L & D	—, — &I

If we can obtain both 'L' and 'D' we can use Conjunction Introduction to obtain 'L & D'. Our new goal sentences, 'L' and 'D' are both atomic sentences, so neither will come by an introduction rule. We note that 'L' occurs as the consequent of a conditional embedded in our second primary assumption. If we could get that conditional, '~ N ⊃ L', out of line 2 we could obtain 'L' by Conditional Elimination, as we do have the antecedent of that conditional '~ N' at line 1. Conjunction Elimination does allow us to extract '~ N ⊃ L' from line 2:

Derive: L & D

1	~ N	Assumption
2	(~ N ⊃ L) & [D ≡ (~ N ∨ A)]	Assumption
3	~ N ⊃ L	2 &E
4	L	1, 3 ⊃E
G	D	
G	L & D	4, — &I

The remaining task, then, is to obtain 'D'. We note that this sentence occurs in the biconditional embedded in line 2. Since the main connective of the sentence on line 2 is '&', we can obtain the biconditional by Conjunction Elimination. To get 'D' from that biconditional we can use Biconditional Elimination, if we have '~ N ∨ A'. This reasoning allows us to add the following steps to our derivation:

Derive: L & D

1	~ N	Assumption
2	(~ N ⊃ L) & [D ≡ (~ N ∨ A)]	Assumption
3	~ N ⊃ L	2 &E
4	L	1, 3 ⊃E
5	D ≡ (~ N ∨ L)	2 &E
G	~ N ∨ L	
G	D	5, __ ≡E
G	L & D	4, __ &I

Note that we have added '~ N ∨ L' as a new goal sentence. The main connective of this sentence is '∨', so if we had either '~ N' or 'L' we could obtain our current goal by Disjunction Introduction. As it happens, we do have '~ N'—it occurs as a primary assumption on line 1. So we can now complete our derivation.

Derive: L & D

1	~ N	Assumption
2	(~ N ⊃ L) & [D ≡ (~ N ∨ A)]	Assumption
3	~ N ⊃ L	2 &E
4	L	1, 3 ⊃E
5	D ≡ (~ N ∨ L)	2 &E
6	~ N ∨ L	1 ∨I
7	D	5, 6 ≡E
8	L & D	4, 7, &I

We will next show that the following argument is valid in *SD* by deriving its conclusion from the set consisting of its premises.

~ A ∨ B
~ A ⊃ B
B ≡ C

C

We begin as always, by taking the premises as primary assumptions and making the conclusion our primary goal.

Derive: C

1	~ A ∨ B	Assumption
2	~ A ⊃ B	Assumption
3	B ≡ C	Assumption
G	C	

After some reflection, two strategies suggest themselves: using Negation Elimination to obtain 'C' and using Disjunction Elimination to obtain 'C'. Both will, in the end, work. We choose to use Disjunction Elimination.

Derive: C

1	~ A ∨ B	Assumption
2	~ A ⊃ B	Assumption
3	B ≡ C	Assumption
4	~ A	A / ∨E
G	C	
	B	A / ∨E
G	C	
G	C	1, 4–__, __–__ ∨E

Our strategy, as the above schema indicates, is to show that the conclusion of the argument, 'C', can be derived from each disjunct of '~ A ∨ B', and hence that 'C' itself can be obtained by Disjunction Elimination. Completing the second subderivation is trivial, for 'C' can be obtained from line 3 and our second auxiliary assumption by Biconditional Elimination.

Derive: C

1	~ A ∨ B	Assumption
2	~ A ⊃ B	Assumption
3	B ≡ C	Assumption
4	~ A	A / ∨E
G	C	
	B	A / ∨E
G	C	3, __ ≡E
G	C	1, 4–__, __–__ ∨E

Completing the first subderivation is only slightly more challenging. From lines 4 and 2 we can obtain 'B' by Conditional Elimination. And we can then use Biconditional Elimination to obtain 'C'.

Derive: C

1	~ A ∨ B	Assumption
2	~ A ⊃ B	Assumption
3	B ≡ C	Assumption
4	~ A	A / ∨E
5	B	2, 4 ⊃E
6	C	3, 5 ≡E
7	B	A / ∨E
8	C	3, 7 ≡E
9	C	1, 4–6, 7–8 ∨E

THEOREMS

Next we will construct proofs of several theorems. We start with a very obvious theorem, 'A ∨ ~ A', whose proof is not obvious. Our task is to derive this sentence using no primary assumptions.

Derive: A ∨ ~ A

1		
G	A ∨ ~ A	

Our goal is 'A ∨ ~ A' and here it should be obvious that though this sentence is a disjunction we will not be able to obtain it by Disjunction Introduction. Neither 'A' nor '~ A' is a theorem, and neither can be derived given no primary assumptions. So the only sensible strategy is to use Negation Elimination.

Derive: A ∨ ~ A

1	~ (A ∨ ~ A)	A / ~ E
G	A ∨ ~ A	1–__ ~ E

Note that the only accessible sentence, the sentence on line 1, is a negation. There is no rule of *SD* that allows us to "take apart" a negation. In the present context, we can use Reiteration on line 1, but there is little else we can do with it. Fortunately, this will be useful. Our current strategy is to use Negation Elimination and to do so we need to derive a sentence and its negation. So we will use the assumption on line 1 as the negation and make '$A \vee \sim A$' our new goal.

Derive: $A \vee \sim A$

1	$\sim (A \vee \sim A)$	A/ ~ E
G	$A \vee \sim A$	1 R
G	$A \vee \sim A$	1–__ ~ E

We noted above that obtaining the last line of our derivation by Disjunction Introduction will not work because neither 'A' nor '$\sim A$' is a theorem. But our current goal, which is the same sentence as that occurring on the last line of the derivation, is to be obtained with the help of the auxiliary assumption '$\sim (A \vee \sim A)$', and here it is reasonable to hope to use Disjunction Introduction. We will make 'A' our new goal and try to derive it by Negation Elimination.

Derive: $A \vee \sim A$

1	$\sim (A \vee \sim A)$	A / ~ E
2	$\sim A$	A / ~ E
G	A	__–__ ~E
G	$A \vee \sim A$	__ vI
G	$\sim (A \vee \sim A)$	1 R
G	$A \vee \sim A$	1–__ ~ E

One of the points we have emphasized is that when using a Negation Elimination subderivation it is wise to use as the ~ **Q** we need to derive a negation that is readily available. In the present instance two negations are readily available, '$\sim A$' and '$\sim (A \vee \sim A)$'. There may be a temptation to select '$\sim A$' as ~ **Q**. But this would be a mistake, for doing so would require that **Q** be 'A' and that sentence is not readily derived from the available assumptions. (We should take a hint from the fact that the point of our current subderivation is to obtain 'A'. If there were an easy way to obtain it we would not be involved in the current

Negation Elimination subderivation.) But if we take – **Q** to be '– (A ∨ – A)' then our new goal becomes 'A ∨ – A' and this sentence is readily derived—by applying Disjunction Introduction to line 2. We are now able to complete the derivation.

Derive: A ∨ ~ A

1		~ (A ∨ ~ A)	A / ~ E
2		~ A	A / ~ E
3		A ∨ ~ A	2 ∨I
4		~ (A ∨ ~ A)	1 R
5		A	2–4 ~ E
6		A ∨ ~ A	5 ∨I
7		~ (A ∨ ~ A)	1 R
8	A ∨ ~ A		1–7 ~ E

Next we will prove the theorem '– (A ∨ B) ≡ (– A & – B)'. This theorem is a biconditional, so it is plausible the last line will come from Biconditional Introduction, and that rule requires two subderivations, one in which we derive '– A & – B' from {– (A ∨ B)} and the other in which we derive '– (A ∨ B)' from {– A & – B}.

Derive: – (A ∨ B) ≡ (– A & – B)

1		~ (A ∨ B)	A / ≡I
G		~ A & ~ B	
		~ A & ~ B	A / ≡I
G		~ (A ∨ B)	
G	~(A ∨ B) ≡ (~ A & ~ B)		1–__, __–__ ≡I

We now have two goals, '– A & – B' in the first subderivation and '– (A ∨ B)' in the second subderivation. We will work on the upper subderivation first. Since our goal is a conjunction, we will take as new subgoals the two conjuncts of that conjunction, '– A' and '– B', and attempt to derive each by Negation Introduction.

Derive: ~ (A ∨ B) ≡ (~ A & ~ B)

1	~ (A ∨ B)	A / ≡I
2	A	A / ~ I
3	A ∨ B	2 ∨I
4	~ (A ∨ B)	1 R
5	~ A	2–4 ~ I
6	B	A / ~ I
7	A ∨ B	6 ∨I
8	~ (A ∨ B)	1 R
9	~ B	6–8 ~ I
10	~ A & ~ B	5, 9 &I
11	~ A & ~ B	A / ≡I
G	~ (A ∨ B)	
G	~ (A ∨ B) ≡ (~ A & ~ B)	1–10, 11–__ ≡I

Note that within the first of our two main subderivations we twice use Negation Introduction, and in each case use '$A \vee B$' and '$\sim (A \vee B)$' as **Q** and $\sim$ **Q**.

Completing our second main subderivation requires deriving '$\sim (A \vee B)$', and this invites a Negation Introduction subderivation, giving us a new assumption, '$A \vee B$', which in turn invites a Disjunction Elimination strategy:

11	~ A & ~ B	A / ≡I
12	A ∨ B	A / ~ I
13	A	A / ∨E
	B	
G	~ (A ∨ B)	12–__ ~ I
G	~ (A ∨ B) ≡ (~ A & ~ B)	1–10, 11–__ ≡I

The question now is what sentence we want to play the role of '**R**' in our Disjunction Elimination subderivation. We need a sentence and its negation to make our Negation Elimination subderivation, began at line 12, work. Two negations are readily available, '~ A' and '~ B'. So we will arbitrarily select one of these, say '~ B' and then make 'B' the sentence we try to obtain by Disjunction Elimination:

11	~ A & ~ B	A / ≡I
12	A ∨ B	A / ~ I
13	A	A / ∨E
G	B	
	B	A / ∨E
G	B	
G	B	2, 3–__, __–__ ∨E
G	~ (A ∨ B)	12–__ ~ I
G	~ (A ∨ B) ≡ (~ A & ~ B)	1–10, 11–__ ≡I

We now have two subderivations to complete. The second is, in fact, already complete, for it involves deriving 'B' from an auxiliary assumption of 'B', so Reiteration will accomplish the task. The first involves deriving 'B' from the assumptions on lines 11 through 13. Fortunately a sentence, 'A', and its negation, '~ A' are both readily available. So Negation Elimination will yield the desired result:

11	~ A & ~ B	A / ≡I
12	A ∨ B	A /~ I
13	A	A / ∨E
14	~ B	A / ~ E
15	~ A	11 &E
16	A	13 R
17	B	14–16 ~ E
18	B	A / ∨E
19	B	18 R
20	B	12, 13–17, 18–19 ∨E
21	~ B	11 &E
22	~ (A ∨ B)	12–21 ~ I
23	~ (A ∨ B) ≡ (~ A & ~ B)	1–10, 11–22 ≡I

This completes our proof of the theorem '~ (A ∨ B) ≡ (~ A & ~ B)'.

We will conclude our discussion of theorems by constructing a proof of what has become known as Peirce's Law.[4]

$$[(A \supset B) \supset A] \supset A$$

Since the theorem is a conditional it is plausible that we will be using Conditional Introduction as our primary strategy.

Derive: $[(A \supset B) \supset A] \supset A$

1	$(A \supset B) \supset A$	A / ⊃I
G	A	
G	$[(A \supset B) \supset A] \supset A$	1–__ ⊃I

But how we should proceed next may not be obvious. We could derive our current goal, 'A', from line 1 by Conditional Elimination *if* we also had 'A ⊃ B', but we do not. So perhaps we should take the sentence 'A ⊃ B' as our new goal, and try to obtain it by Conditional Introduction.

Derive: $[(A \supset B) \supset A] \supset A$

1	$(A \supset B) \supset A$	A / ⊃I
2	A	A / ⊃I
G	B	
G	$A \supset B$	2–__ ⊃I
G	A	1, __ ⊃E
G	$[(A \supset B) \supset A] \supset A$	1–__ ⊃I

So far, one might think, so good. But how are we to obtain 'B' from the sentences on lines 1 and 2? We could assume '~ B' and hope to use Negation Elimination.

Derive: $[(A \supset B) \supset A] \supset A$

1	$(A \supset B) \supset A$	A / ⊃I
2	A	A / ⊃I
3	~ B	A / ~ E
G	B	
G	$A \supset B$	2–__ ⊃I
G	A	1, __ ⊃E
G	$[(A \supset B) \supset A] \supset A$	1–__ ⊃I

[4]The first proof of this theorem was given by Charles Peirce, a nineteenth century American philosopher.

Unfortunately, the only negation now available is '~ B', so it appears that to make Negation Elimination work we will have to derive '~ B' (by Reiteration) and 'B'. But how do we derive 'B'? We seem to be back where we were before we assumed '~ B'. That is, 'B' is again our goal sentence.

We appear to be on the wrong track. Suppose that when we had 'A' as our goal, instead of planning on deriving 'A' by Conditional Elimination we try to derive it by Negation Elimination.

Derive: $[(A \supset B) \supset A] \supset A$

1	$(A \supset B) \supset A$	A / ⊃I
2	$\neg A$	A / ~ E
G	A	2–__ ~ E
G	$[(A \supset B) \supset A] \supset A$	1–__ ⊃I

Since we have a negation available, '~ A', perhaps we should take 'A' and '~ A' as the sentences **Q** and ~ **Q** we need to use Negation Elimination and accordingly make 'A' our new goal. This may seem no more promising than was the line of reasoning recently abandoned, since deriving 'A' was our goal before assuming '~ A'. But we are, in fact, making progress.

Derive: $[(A \supset B) \supset A] \supset A$

1	$(A \supset B) \supset A$	A / ⊃I
2	~ A	A / ~ E
G	A	
	~ A	2 R
G	A	2–__ ~ E
G	$[(A \supset B) \supset A] \supset A$	1–__ ⊃I

We can obtain 'A' from line 1 by Conditional Elimination if we can first obtain 'A ⊃ B'. This is, of course, the position we were in at the start of our work. But now we have an additional assumption available to us, namely '~ A'.

Derive: $[(A \supset B) \supset A] \supset A$

1	$(A \supset B) \supset A$	A / ⊃I
2	~ A	A / ~ E
3	A	A / ⊃I
G	B	
G	A ⊃ B	3–G ⊃I
G	A	1–G ⊃E
	~ A	2 R
G	A	2–G ~ E
G	$[(A \supset B) \supset A] \supset A$	1–G ⊃I

And now we can see our way to the end. We need 'B' and we have a sentence and its negation readily available ('A' and '~ A'), so we can assume '~ B' and use Negation Elimination. Here is the completed derivation.

Derive: $[(A \supset B) \supset A] \supset A$

1	$(A \supset B) \supset A$	A / ⊃I
2	~ A	A / ~ E
3	A	A / ⊃I
4	~ B	A / ~ E
5	A	3 R
6	~A	2 R
7	B	4–6 ~ E
8	A ⊃ B	3–7 ⊃I
9	A	1, 8 ⊃E
10	~A	2 R
11	A	2–10 ~ E
12	$[(A \supset B) \supset A] \supset A$	1–11 ⊃I

It is worth noting that in this example, as is frequently the case, a strategy that at first seems obvious (using Conditional Elimination to obtain 'A' as the penultimate line of the derivation) but proves problematic can successfully be used as a secondary strategy inside an alternative strategy (here Negation Elimination).

EQUIVALENCE

Suppose we want to establish that 'A ≡ ~ B' and '~ A ≡ B' are equivalent in *SD* (they are). Two derivations are required, one deriving '~ A ≡ B' from

[A ≡ ~ B] and one deriving 'A ≡ ~ B' from [~ A ≡ B]. Here is a start for the first of these:

Derive: ~ A ≡ B

1	A ≡ ~ B	Assumption
G	~ A ≡ B	

It should be apparent that our goal, '~ A ≡ B' is not going to be obtained by an elimination rule. We have too little to work with by way of primary assumptions for that to be a viable strategy. Since the main connective of our goal sentence is '≡', Biconditional Introduction may be a viable strategy. So we continue our derivation thus:

Derive: ~ A ≡ B

1	A ≡ ~ B	Assumption
2	~ A	A / ≡I
G	B	
	B	A / ≡I
G	~ A	
G	~ A ≡ B	2–, –2 ≡I

We now have two subderivations to complete. The goal of the first is 'B', and it can be obtained by Negation Elimination. The goal of the second, '~ A', can be obtained by Negation Introduction:

Derive: ~ A = B

1	A = ~ B	Assumption
2	| ~ A	A / =I
3	| | ~ B	A / ~ E
4	| | A	1, 3 =E
5	| | ~ A	2 R
6	| B	3–5 ~ E
7	| B	A / =I
8	| | A	A / ~ I
9	| | ~ B	1, 8 =E
10	| | B	7 R
11	| ~ A	8–10 ~ I
12	~ A = B	2–6, 7–11 =I

The second half of our current task is to derive 'A = ~ B' from {~A = B}.

Derive: A = ~ B

1	~ A = B	Assumption
	|	
	|	
G	A = ~ B	

Biconditional Introduction is also a good strategy in this case.

Derive: A = ~ B

1	~ A = B	Assumption
2	| A	A / =I
	|	
G	| ~ B	A / =I
	| ~ B	
	|	
G	| A	
G	A = ~ B	2–__, __–__ =I

Here, too, negation strategies will yield the desired results:

Derive: A ≡ ~ B

1	~ A ≡ B	Assumption
2	A	A / ≡I
3	B	A / ~ I
4	~ A	1, 2 ≡E
5	A	2 R
6	~ B	3–5 ~ I
7	~ B	A / ≡I
8	~ A	A / ~ E
9	B	1, 7 ≡E
10	~ B	7 R
11	A	8–10 ~ E
12	A ≡ ~ B	2–6, 7–11 ≡I

We next show that 'A ⊃ B' and '~ A ∨ B' are equivalent in *SD*. To do so will require deriving each sentence from the unit set of the other. So we will be doing two derivations. Both of these derivations are rather difficult but also highly instructive as they will allow us to illustrate strategies associated with a number of introduction and elimination rules. We set up our first derivation as follows:

Derive: ~ A ∨ B

1	A ⊃ B	Assumption
G	~ A ∨ B	

Our goal sentence is '~ A ∨ B', a disjunction. So we might be tempted to try to obtain our goal by Disjunction Introduction. While this strategy will not work, we will explore it anyway to illustrate how one can fall into unproductive strategies. If we are to use Disjunction Introduction we will need to first obtain either '~ A' or 'B'. We will take 'B' as our new goal 'B'. (In fact, neither 'B' nor '~ A' is obtainable given just 'A ⊃ B'.)

Derive: ~ A ∨ B

1	A ⊃ B	Assumption
G	B	
G	~ A ∨ B	___ ∨I

Since our goal is now 'B', and we have 'A ⊃ B' at line 1, it might seem like a good idea to assume 'A' and then use Conditional Elimination to obtain 'B'.

Derive: ~ A ∨ B

1	A ⊃ B	Assumption	
2	| A	A	
3	| B	1, 2 ⊃E	
4	B	3 R	**MISTAKE!**
5	~ A ∨ B	4 ∨ I	

Line 4 is a mistake because it appeals to a sentence, 'B', on line 3 that is not accessible at line 4. There is a scope line to the left of 'B' at line 3 that does not continue through line 4. We had two chances to avoid going down this path to a mistake. First, thinking we could get '~ A ∨ B' by first deriving 'B' from the assumption on line 1 was a bad idea. That assumption is 'A ⊃ B'. We are trying to show that 'A ⊃ B' and '~ A ∨ B' are equivalent in *SD*, as indeed they are. Although we are here concerned with syntactic properties of sentences and sets of sentences of *SL*, it is well to remember that for any set **Γ** of sentences of *SL* and any sentence **P** of *SL*,

Γ ⊢ **P** in *SD* if and only if **Γ** ⊨ **P**.

Our ill-advised strategy involved trying to show that

{A ⊃ B} ⊢ B

where in fact 'B' is not derivable from {A ⊃ B}. For if this derivability claim did hold then it would also have to be the case that

{A ⊃ B} ⊨ B

and this claim is false. There are truth-value assignments on which 'A ⊃ B' is true and 'B' false, namely every truth-value assignment on which 'A' and 'B' are both assigned **F**.

We had a second chance to avoid going down an unpromising road when we assumed 'A' at line 2. Note that there is nothing in the justification column for line 2 indicating why we are making this assumption. Had we been paying attention at that time we would have realized that we have no good reason for assuming 'A'. There is no subderivation strategy that will allow us to assume 'A', derive some sentence or sentences, and then end the subderivation and enter 'B' as the next line.

A more promising strategy for completing our first derivation, though one that does not initially come to mind when one is first learning to do derivations, is to use Negation Elimination to obtain '~ A ∨ B'.

Derive: ~ A ∨ B

1	A ⊃ B	A / ⊃I
2	| ~ (~ A ∨ B)	A / ~ E
G	~ A ∨ B	2-__ ~ E

This strategy will seem unpromising if one thinks either that the **Q** and ~ **Q** that need to be derived to use a negation rule must be an atomic sentence and its negation, or that a negation must be among or easily obtained from the sentences that are accessible before one makes the auxiliary assumption that begins a negation subderivation. Neither is the case. The **Q** and ~ **Q** that both negation rules require deriving can be a compound sentence and its negation as well as an atomic sentence and its negation. And the ~ **Q** that is derived can occur as the auxiliary assumption that initiates the negation subderivation. Keeping this in mind we proceed as follows:

Derive: ~ A ∨ B

1	A ⊃ B	Assumption
2	| ~ (~ A ∨ B)	A / ~ E
G	| ~ A ∨ B	
	| ~ (~ A ∨ B)	2 R
G	~ A ∨ B	2-__ ~ E

It certainly might appear that we are making no progress. The goal of this derivation is '~ A ∨ B'. And this same sentence is now our goal within the subderivation begun at line 2. But in fact we are making progress. We noted earlier that we cannot derive '~ A ∨ B' by Disjunction Introduction when the only accessible sentence is 'A ⊃ B'. But we now have two accessible sentences to appeal to, those at lines 1 and 2. If we can use these two assumptions to derive '~ A', we can obtain our current goal, '~ A ∨ B' by Disjunction Introduction. This suggests we try to obtain '~ A' by Negation Introduction.

Derive: ~ A ∨ B

1	A ⊃ B	Assumption
2	| ~ (~ A ∨ B)	A / ~ E
3	| | A	A / ~ I
	| |	
	| |	
G	| | ~ A	3–__ ~ I
G	| | ~ A ∨ B	
	| | ~ (~ A ∨ B)	2 R
G	~ A ∨ B	2–__ ~ E

We are again at the point where it is essential to be able to "see" what we can obtain from the sentences that are accessible at the point where we are working (inside the subderivation that we began at line 3). The accessible sentences are those on lines 1–3. At line 3 we have 'A'. At line 1 we have 'A ⊃ B'. From these two sentences we can obtain 'B' by Conditional Elimination. From 'B' we can obtain '~ A ∨ B' by Disjunction Introduction, and we can derive the negation of this sentence, '~ (~ A ∨ B)' by Reiteration on line 2. These steps will complete the first half of our current task, that of showing that 'A ⊃ B' and '~ A ∨ B' are equivalent in *SD*.

Derive: ~ A ∨ B

1	A ⊃ B	Assumption
2	| ~ (~ A ∨ B)	A / ~ E
3	| | A	A / ~ I
4	| | B	3, 1 ⊃E
5	| | ~ A ∨ B	4 ∨I
6	| | ~ (~ A ∨ B)	2 R
7	| ~ A	3–6 ~ I
8	| ~ A ∨ B	7 ∨I
9	| ~ (~ A ∨ B)	2 R
10	~ A ∨ B	2–9 ~ E

This derivation of '~ A ∨ B' from 'A ⊃ B' is instructive in several ways. First, given that a disjunction is derivable, it does not follow that the last step in that derivation is Disjunction Introduction. Second, in picking a goal sentence it is wise to consider whether it is plausible that the selected sentence is derivable from the currently accessible sentences. Third, when using a negation rule the **Q** and ~ **Q** to be derived within the scope of the assumption called for by the rule may well both be compound sentences. Fourth, it does sometimes happen that one sentence is a goal in multiple parts of a derivation. Fifth, in using a negation rule it

is advisable to use as – **Q** a sentence that is readily available, and it may be available as the assumption of the very subderivation in which we are working. Finally, there is nothing wrong with using two or more instances of negation rules within which the same sentences (on different lines) play the roles of **Q** and – **Q**.

The second part of our proof that '$A \supset B$' and '$\sim A \vee B$' are equivalent in *SD*, a derivation of '$A \supset B$' from {$\sim A \vee B$}, is also instructive.

Derive: $A \supset B$

1	$\sim A \vee B$		Assumption
G	$A \supset B$		

We now need a strategy for getting from '$\sim A \vee B$' to '$A \supset B$'. A little reflection suggests two alternative strategies. Since the goal sentence is a material conditional, we could use Conditional Introduction, and accordingly assume 'A' at line 2 for the purpose of using Conditional Introduction. Alternatively, since the only accessible sentence, the one at line 1, is a disjunction, we could plan to work to the conditional we want by using Disjunction Elimination. That is, in this case we can either let our goal sentence drive our strategy, working from the bottom up, or we can let our one accessible sentence drive our strategy, working from the top down. Here, as is often the case, both strategies will work. Moreover, whichever strategy we pick as our primary strategy we will end up using the other strategy within the first strategy. This is also often the case. Picking Disjunction Elimination as our primary strategy yields the following:

Derive: $A \supset B$

1	$\sim A \vee B$		Assumption
2		$\sim A$	A / $\vee$E
G		$A \supset B$	
		B	A / $\vee$E
G		$A \supset B$	
G	$A \supset B$		1, 2~~, ~~$\vee$E

Lines 1 and 2, by themselves, don't suggest a strategy for deriving '$A \supset B$'. But '$A \supset B$' is a material conditional and this suggests we use Conditional Introduction to obtain it.

Derive: A ⊃ B

1	~ A ∨ B	Assumption
2	~ A	A / ∨E
3	A	A / ⊃I
G	B	
G	A ⊃ B	3–__ ⊃I
	B	A / ∨E
G	A ⊃ B	
G	A ⊃ B	1, 2–__, __–__ ∨E

Our goal within the subderivation beginning on line 3 is 'B'. We now note that the three accessible sentences include both 'A' and '~ A'. Their availability invites a negation strategy. To obtain 'B' we thus assume '~ B' and derive 'A' and '~ A', both by Reiteration.

Derive: A ⊃ B

1	~ A ∨ B	Assumption
2	~ A	A / ∨E
3	A	A / ⊃I
4	~ B	A / ~ E
5	A	3 R
6	~ A	2 R
7	B	4–6 ~ E
8	A ⊃ B	3–7 ⊃I
9	B	A / ∨E
G	A ⊃ B	
G	A ⊃ B	1, 2–8, 9–__ ∨E

What remains is to derive 'A ⊃ B' from 'B'. This is actually quite easy. We can use Conditional Introduction, assuming 'A' and deriving 'B' by Reiteration on line 9.

Derive: A ⊃ B

1	~ A ∨ B	Assumption
2	~ A	A / ∨E
3	A	A / ⊃I
4	~B	A / ~ E
5	A	3 R
6	~ A	2 R
7	B	4–6 ~ E
8	A ⊃ B	3–7 ⊃I
9	B	A / ∨E
10	A	A / ⊃I
11	B	9 R
12	A ⊃ B	10–11 ⊃I
13	A ⊃ B	1, 2–8, 9–12 ∨E

We have derived '~ A ∨ B' from {A ⊃ B} and 'A ⊃ B' from {~ A ∨ B}, thus demonstrating that these sentences are equivalent in *SD*. Two important lessons about material conditionals are illustrated in our last derivation. The first is that a conditional can be derived from the negation of its antecedent, as we did in lines 2 through 8 above. The second is that a material conditional can be derived from its consequent as we did in lines 9–12 above.

In our last derivation we used Disjunction Elimination as our primary strategy. Using Conditional Introduction as the primary strategy works just as well:

Derive: A ⊃ B

1	~ A ∨ B	Assumption
2	A	A / ⊃I
3	~ A	A / ∨E
4	~ B	A / ~ E
5	A	2 R
6	~ A	3 R
7	B	4–6 ~ E
8	B	A / ∨E
9	B	8 R
10	B	1, 3–7, 8–9 ∨E
11	A ⊃ B	2–10 ⊃I

INCONSISTENCY

We will conclude our illustration of strategies for constructing derivations in *SD* by doing several derivations that demonstrate the inconsistency of given sets.

Consider first the set {~ (A ⊃ B), B}. To show this set is inconsistent in *SD* we need to derive from it some sentence **Q** and its negation ~ **Q**. In planning a strategy it helps to remember that **Q** need not be an atomic sentence, and that it is often useful to use as ~ **Q** a sentence that is readily available. In the present case the only readily available negation is '~ (A ⊃ B)'. This suggests the following strategy:

Derive: A ⊃ B, ~ (A ⊃ B)

1	~ (A ⊃ B)	Assumption
2	B	Assumption
G	A ⊃ B	
G	~ (A ⊃ B)	1 R

Our goal is now to derive 'A ⊃ B' from our two assumptions. Since this goal sentence is a conditional, we will plan on using Conditional Introduction:

Derive: A ⊃ B, ~ (A ⊃ B)

1	~ (A ⊃ B)	Assumption
2	B	Assumption
3	A	A / ⊃I
G	B	
G	A ⊃ B	3–__ ⊃I
G	~ (A ⊃ B)	1 R

It is now apparent that our derivation is effectively done. Our only remaining goal, 'B', can be obtained by Reiteration on line 2:

Derive: A ⊃ B, ~ (A ⊃ B)

1	~ (A ⊃ B)	Assumption
2	B	Assumption
3	A	A / ⊃I
4	B	2 R
5	A ⊃ B	3–4 ⊃I
6	~ (A ⊃ B)	1 R

Establishing that the following set is inconsistent in *SD* is only modestly more challenging:

{A ≡ ~ B, B ≡ C, A ≡ C}

In this example the only negation that occurs as a component of any of the members of the set is '~ B'. So perhaps our goal should be to derive both 'B' and '~ B', even though neither can be derived by Reiteration or by any other rule in a single step.

Derive: B, ~ B

1	A ∨ ~ B	Assumption
2	B ∨ C	Assumption
3	A ∨ C	Assumption
G	B	
G	~ B	

To obtain our first goal, 'B', we might try using Negation Elimination:

Derive: B, ~ B

1	A ∨ ~ B	Assumption
2	B ∨ C	Assumption
3	A ∨ C	Assumption
4	~ B	A / ~ I
G	B	4–__ ~ E
G	~ B	4–__ ~ I

A cursory inspection of the sentences on lines 1–4 reveals that we can obtain '~ B' by Reiteration and 'B' by repeated uses of Biconditional Elimination:

Derive: B, ~ B

1	A ≡ ~ B	Assumption
2	B ≡ C	Assumption
3	A ≡ C	Assumption
4	| ~ B	A / ~ I
5	| A	4, 1 ≡E
6	| C	5, 3 ≡E
7	| B	6, 2 ≡E
8	| ~ B	4 R
9	B	4–8 ~ I
G	~ B	

The remaining task is to derive '~ B', and this too can be accomplished by repeated applications of Biconditional Elimination:

Derive: B, ~ B

1	A ≡ ~ B	Assumption
2	B ≡ C	Assumption
3	A ≡ C	Assumption
4	| ~ B	A / ~ I
5	| A	4, 1 ≡E
6	| C	5, 3 ≡E
7	| B	6, 2 ≡E
8	| ~ B	4 R
9	B	4–8 ~ I
10	C	9, 2 ≡E
11	A	10, 3 ≡E
12	~ B	11, 1 ≡E

Finally, we will show that the set {~ (A ⊃ B), ~ (B ⊃ C)} is inconsistent in *SD*. This is a challenging exercise. We do have two negations immediately available, so we will probably use one of them as ~ **Q**; which one makes no difference. So we set up our derivation this way:

Derive: A ⊃ B, ~ (A ⊃ B)

1	~ (A ⊃ B)	Assumption
2	~ (B ⊃ C)	Assumption
G	A ⊃ B	
	~ (A ⊃ B)	1 R

We cannot apply any elimination rule to either assumption since they are both negations. So we proceed by asking how our current goal, 'A ⊃ B', could be obtained by an introduction rule, and the answer is of course by Conditional Introduction:

Derive: A ⊃ B, ~ (A ⊃ B)

1	~ (A ⊃ B)	Assumption
2	~ (B ⊃ C)	Assumption
3	A	A / ⊃I
G	B	
G	A ⊃ B	3–__ ⊃I
	~ (A ⊃ B)	1 R

Our new goal is 'B'. The only strategy for obtaining 'B' that seems remotely promising is that of Negation Elimination:

Derive: A ⊃ B, ~ (A ⊃ B)

1	~ (A ⊃ B)	Assumption
2	~ (B ⊃ C)	Assumption
3	A	A / ⊃I
4	~ B	A / ~ E
G	B	4–__ ~ E
G	A ⊃ B	3–__ ⊃I
	~ (A ⊃ B)	1 R

We need to derive, within the subderivation beginning on line 4, a sentence **Q** and its negation ~ **Q**. Three negations, '~ (A ⊃ B)', '~ (B ⊃ C)', and '~B' are readily available. Since the presumed inconsistency of the set we are testing fairly clearly derives from the interplay of those two assumptions—that is, neither assumption by itself is problematic—we will eventually have to appeal to both assumptions. And we are already using '~ (A ⊃ B)' (as the last line of our derivation), so perhaps it is time to find a role for '~ (B ⊃ C)'. Accordingly we will try to obtain 'B ⊃ C' and '~ (B ⊃ C)'.

Derive: A ⊃ B, ~ (A ⊃ B)

1	~ (A ⊃ B)	Assumption
2	~ (B ⊃ C)	Assumption
3	A	A / ⊃I
4	~ B	A / ~ E
	B ⊃ C	
	~ (B ⊃ C)	2 R
	B	4–__ ~ E
	A ⊃ B	3–__ ⊃I
	~ (A ⊃ B)	1 R

Our new goal, 'B ⊃ C', is a conditional, so Conditional Introduction seems appropriate:

Derive: A ⊃ B, ~ (A ⊃ B)

1	~ (A ⊃ B)	Assumption
2	~ (B ⊃ C)	Assumption
3	A	A / ⊃I
4	~B	A / ~ E
5	B	A / ⊃I
	C	
	B ⊃ C	5–__ ⊃I
	~ (B ⊃ C)	2 R
	B	4–__ ~ E
	A ⊃ B	3–__ ⊃I
	~ (A ⊃ B)	1 R

At this point, as is often the case, the "trick" is to be aware of what sentences are available to us—in this case the sentences on lines 1–5—and what we can do with those sentences. Note that we have both 'B' (at line 5) and '~ B' (at line 4),

and we know that whenever we can obtain a sentence and its negation we can obtain any sentence whatsoever by the appropriate negation strategy. We want 'C', so we will obtain it by Negation Elimination.

Derive: $A \supset B, \sim (A \supset B)$

1	$\sim (A \supset B)$	Assumption
2	$\sim (B \supset C)$	Assumption
3	$\quad A$	A / $\supset$I
4	$\quad\quad \sim B$	A / $\sim$E
5	$\quad\quad\quad B$	A / $\supset$I
6	$\quad\quad\quad\quad \sim C$	A / $\sim$E
7	$\quad\quad\quad\quad\quad B$	5 R
8	$\quad\quad\quad\quad\quad \sim B$	4 R
9	$\quad\quad\quad C$	6–8 $\sim$E
10	$\quad\quad B \supset C$	5–9 $\supset$I
11	$\quad\quad \sim (B \supset C)$	2 R
12	$\quad B$	4–11 $\sim$E
13	$A \supset B$	3–12 $\supset$I
14	$\sim (A \supset B)$	1 R

5.3E EXERCISES

1. Construct derivations that establish the following derivability claims. In each case start by setting up the main structure of the derivation—with the primary assumption or assumptions at the top and the sentence to be derived at the bottom, and then identify the initial subgoal or goals. Complete the derivation, remembering to consider both the form of the current goal sentence and the content of the accessible sentences in selecting appropriate subgoals.

a. $\{A \supset B\} \vdash A \supset (A \,\&\, B)$
*b. $\{\sim B \equiv A\} \vdash A \supset \sim B$
c. $\{(K \supset L) \,\&\, (L \supset K)\} \vdash L \equiv K$
*d. $\{M \equiv P, \sim P\} \vdash \sim M$
e. $\{B \,\&\, \sim B\} \vdash C$
*f. $\{D\} \vdash A \supset (B \supset D)$
g. $\{A \supset C, (\sim A \vee C) \supset (D \supset B)\} \vdash D \supset B$
*h. $\{\sim A \supset \sim B, A \supset C, B \vee D, D \supset E\} \vdash E \vee C$
i. $\{A \supset B, \sim (B \,\&\, \sim C) \supset A\} \vdash B$
*j. $\{\sim A \supset B, C \supset \sim B, \sim (\sim C \,\&\, \sim A)\} \vdash A$
k. $\{A \vee (B \,\&\, C), C \supset \sim A\} \vdash B \vee \sim C$
*l. $\{(A \supset B) \supset \sim B\} \vdash \sim B$
m. $\{A \vee B) \supset C, (D \vee E) \supset [(F \vee G) \supset A]\} \vdash D \supset (F \supset C)$
*n. $\{(F \vee G) \supset (H \,\&\, I)\} \vdash \sim F \vee H$
o. $\{A \supset \sim (B \vee C), (C \vee D) \supset A, \sim F \supset (D \,\&\, \sim E)\} \vdash B \supset F$
*p. $\{(A \,\&\, B \equiv (A \vee B), C \,\&\, (C \equiv \sim \sim A)\} \vdash B$
q. $\{F \supset (G \vee H), \sim (\sim F \vee H), \sim G\} \vdash H$
*r. $\{\sim (A \supset B) \,\&\, (C \,\&\, \sim D), (B \vee \sim A) \vee [(C \,\&\, E) \supset D]\} \vdash \sim E$

2. Show that each of the following arguments is valid in *SD*.

a. $A \supset \sim B$
$\sim B \supset C$
$\therefore A \supset C$

*b. $B \supset (A \,\&\, \sim B)$
$\therefore \sim B$

c. $A \equiv B$
$\sim A$
$\therefore \sim B$

*d. $A \supset (B \,\&\, C)$
$\sim C$
$\therefore \sim A$

e. D
$\therefore A \supset [B \supset (C \supset D)]$

*f. $A \equiv B$
$B \equiv C$
$\therefore A \equiv C$

g. $A \supset (B \supset C)$
$D \supset B$
$\therefore A \supset (D \supset C)$

*h. $\sim B \supset A$
$C \vee \sim B$
$\sim C$
$\therefore A$

i. $\sim A \vee B$
$B \supset C$
$\therefore A \supset C$

*j. $(E \supset T) \,\&\, (T \supset O)$
$O \supset E$
$\therefore (E \equiv O) \,\&\, (O \equiv E)$

k. $A \supset (C \supset B)$
$\sim C \supset \sim A$
A
$\therefore B$

*l. $\sim F$
$\sim G$
$\therefore \sim (F \vee G)$

m. $F \equiv G$
$F \vee G$
$\therefore F \,\&\, G$

3. Prove that each of the following is a theorem in *SD*.

a. $A \supset (A \vee B)$
*b. $A \supset (B \supset A)$
c. $A \supset [B \supset (A \,\&\, B)]$
*d. $(A \,\&\, B) \supset [(A \vee C) \,\&\, (B \vee C)]$
e. $(A \equiv B) \supset (A \supset B)$
*f. $(A \,\&\, \sim A) \supset (B \,\&\, \sim B)$
g. $(A \supset B) \supset [(C \supset A) \supset (C \supset B)]$
*h. $A \vee \sim A$
i. $[(A \supset B) \,\&\, \sim B] \supset \sim A$
*j. $(A \,\&\, A) \equiv A$
k. $A \supset [B \supset (A \supset B)]$
*l. $\sim A \supset [(B \,\&\, A) \supset C]$
m. $(A \supset B) \supset [\sim B \supset \sim (A \,\&\, D)]$
*n. $[(A \supset B) \supset A] \supset A$

4. Show that the members of each of the following pairs of sentences are equivalent in *SD*.

a. $A \,\&\, \sim A$ — $B \,\&\, \sim B$
*b. $A \,\&\, A$ — $A \vee A$
c. $(A \vee B) \supset A$ — $B \supset A$
*d. $\sim (A \supset B)$ — $A \,\&\, \sim B$
e. $\sim (A \equiv B)$ — $(A \,\&\, \sim B) \vee (B \,\&\, \sim A)$
*f. $A \equiv \sim B$ — $\sim (A \equiv B)$

5. Show that each of the following sets of sentences is inconsistent in *SD*.

a. $\{\sim (A \supset A)\}$
*b. $\{A \supset (B \,\&\, \sim B), A\}$
c. $\{A \equiv B, B \supset \sim A, A\}$
*d. $\{A \equiv \sim (A \equiv A), A\}$
e. $\{A \supset \sim A, \sim A \supset A\}$
*f. $\{A \supset (C \supset B), \sim C \supset B, A \,\&\, \sim B\}$
g. $\{\sim (A \vee B), C \supset A, \sim C \supset B\}$
*h. $\{\sim (B \equiv A), \sim B, \sim A\}$
i. $\{\sim (F \vee G) \equiv (A \supset A), H \supset F, \sim H \supset F\}$

6. Show that the following derivability claims hold in *SD*.

a. $\{A \supset B, \sim A \supset \sim B)\} \vdash A \equiv B$
*b. $\{F \equiv \sim (G \equiv \sim H), \sim (F \vee G)\} \vdash H$
c. $\{A \equiv \sim (B \vee C), B \supset C\} \vdash A$
*d. $\{G \vee \sim H, \sim G \vee \sim H\} \vdash \sim H$
e. $\{B \vee (C \vee D), C \supset A, A \supset \sim C\} \vdash B \vee D$
*f. $\{(A \supset B) \supset C, (A \supset B) \vee \sim C \vdash \sim C \equiv \sim (A \supset B)$
g. $\{(A \supset (D \,\&\, B), (\sim D \equiv B) \,\&\, (C \supset A)\} \vdash (A \supset B) \supset \sim C$
*h. $\{\sim (A \equiv B) \vdash (A \,\&\, \sim B) \vee (B \,\&\, \sim A)$

7. Show that each of the following arguments is valid in *SD*.

a.
$\sim (C \vee A)$
$\therefore \sim (C \equiv \sim A)$

*b.
$C \vee \sim D$
$C \supset E$
D
$\therefore E$

c.
$\sim A \,\&\, \sim B$
$\therefore A \equiv B$

*d.
$\sim (F \vee \sim G) \equiv (\sim (H \vee I)$
$F \vee I$
$\therefore H \vee I$

e.
$H \equiv \sim (I \,\&\, \sim J)$
$\sim I \equiv \sim H$
$J \supset \sim I$
$\therefore \sim H$

*f.
$\sim (F \supset G)$
$\sim (G \supset H)$
$\therefore I$

g.
$(F \vee G) \vee (H \vee \sim I)$
$F \supset H$
$I \supset \sim G$
$\therefore H \vee \sim I$

*h. $\sim D$
$C \supset (A \equiv B)$
$(D \vee B) \supset \sim A$
$(A \equiv B) \supset (D \mathbin{\&} E)$
$\sim B \supset D$
$\therefore C \supset (\sim A \mathbin{\&} B)$

i. $\sim (F \vee \sim G) \equiv \sim (H \vee I)$
$F \vee I$
$\therefore F \vee (I \mathbin{\&} \sim G)$

*j. $(A \vee \sim B) \supset (C \mathbin{\&} D)$
$A \equiv \sim D$
$\sim B \equiv \sim C$
$\therefore \sim (A \vee B)$

k. $(\sim A \equiv \sim C) \equiv (B \equiv \sim D)$
$\sim A \supset \sim B$
$C \supset \sim D$
$\therefore (\sim A \equiv \sim C) \supset (\sim A \equiv D)$

*l. $F \supset (G \vee H)$
$\sim (\sim F \vee H)$
$\sim G$
$\therefore H$

m. $\sim (A \supset B) \mathbin{\&} (C \mathbin{\&} \sim D)$
$(B \vee \sim A) \vee [(C \mathbin{\&} E) \supset D]$
$\therefore \sim E$

8. Prove that each of the following is a theorem in *SD*.

a. $\sim (A \supset B) \supset \sim (A \equiv B)$
*b. $\sim (A \equiv B) \supset \sim (A \mathbin{\&} B)$
c. $(A \supset B) \vee (B \supset A)$
*d. $[A \supset (B \supset C)] \equiv [(A \supset B) \supset (A \supset C)]$
e. $[(A \vee B) \supset C] \equiv [(A \supset C) \mathbin{\&} (B \supset C)]$
*f. $[A \vee (B \vee C)] \supset [(D \supset A) \vee ((D \supset B) \vee (D \supset C))]$
g. $\sim (A \equiv B) \equiv (A \equiv \sim B)$

9. Show that the members of each of the following pairs of sentences are equivalent in *SD*.

a.	A	$\sim \sim A$	Double Negation
*b.	A	$A \mathbin{\&} A$	Idempotence
c.	A	$A \vee A$	Idempotence
*d.	$A \mathbin{\&} B$	$B \mathbin{\&} A$	Commutation
e.	$A \vee B$	$B \vee A$	Commutation
*f.	$A \mathbin{\&} (B \mathbin{\&} C)$	$(A \mathbin{\&} B) \mathbin{\&} C$	Association
g.	$A \vee (B \vee C)$	$(A \vee B) \vee C$	Association
*h.	$A \supset (B \supset C)$	$(A \mathbin{\&} B) \supset C$	Exportation
i.	$A \supset B$	$\sim B \supset \sim A$	Transposition
*j.	$A \equiv B$	$(A \supset B) \mathbin{\&} (B \supset A)$	Equivalence
k.	$A \equiv B$	$(A \mathbin{\&} B) \vee (\sim A \mathbin{\&} \sim B)$	Equivalence
*l.	$A \mathbin{\&} (B \vee C)$	$(A \mathbin{\&} B) \vee (A \mathbin{\&} C)$	Distribution
m.	$A \vee (B \mathbin{\&} C)$	$(A \vee B) \mathbin{\&} (A \vee C)$	Distribution
*n.	$\sim (A \vee B)$	$\sim A \mathbin{\&} \sim B$	De Morgan
o.	$\sim (A \mathbin{\&} B)$	$\sim A \vee \sim B$	De Morgan
*p.	$A \supset B$	$\sim A \vee B$	Implication

10. Show that each of the following sets of sentences of *SL* is inconsistent in *SD*.

a. $[(A \supset B) \mathbin{\&} (A \supset \sim B), (C \supset A) \mathbin{\&} (\sim C \supset A)]$
*b. $[B \equiv (A \mathbin{\&} \sim A), \sim B \supset (A \mathbin{\&} \sim A)]$

c. [C ⊃ ~ A, C ⊃ A]
*d. [~ (F ∨ G) ⊃ (~ F ⊃ ~ F), ~ G ⊃ F]
e. [~ [A ∨ (B ∨ C)], A ⊃ ~ C]
*f. [F ∨ (G ⊃ H), ~ H & ~ (F ∨ ~ G)]
g. [A & (B ∨ C), (~ C ∨ H) & (H ⊃ ~ H), ~ B]
*h. [[(A ⊃ B) ⊃ (D & ~ D)] ⊃ B, A]

11. Symbolize the following arguments in *SL*. Then show that the symbolized arguments are valid in *SD*.

a. Spring has sprung, and the flowers are blooming. If the flowers are blooming, the bees are happy. If the bees are happy but aren't making honey, then spring hasn't sprung. So the bees are making honey.

*b. If Luscious Food Industries goes out of business, then food processing won't be improved. And if they go out of business, canned beans will be available if and only if Brockport Company stays in business. But Brockport Company is going out of business, and canned beans will be available. Hence Luscious Food Industries is staying in business unless food processing is improved.

c. If civil disobedience is moral, then not all resistance to the law is morally prohibited, although our legal code is correct if all resistance to the law is morally prohibited. But civil disobedience is moral if and only if either civil disobedience is moral or our legal code is correct. Our judges have acted well only if all resistance to the law is morally prohibited. So our judges haven't acted well.

*d. If oranges contain citric acid so do lemons, or if lemons don't contain citric acid neither do grapefruit. Thus, if oranges and grapefruit contain citric acid, so do lemons.

e. Neither rubber nor wood is a good conductor of electricity. But either rubber is a good conductor if and only if metal is, or if metal or glass is a good conductor then wood is a good conductor if and only if metal is. So metal isn't a good conductor of electricity.

*f. If the trains stop running then airline prices will increase, and buses will reduce their fares provided that trains don't stop running. If airline prices increase, then buses won't lose their customers. Hence buses will lose their customers only if they reduce their fares.

g. If the house is built and taxes increase, Jones will go bankrupt. If Smith becomes mayor, then the tax director will quit; and Smith will become mayor unless the tax director quits. But taxes won't increase if but only if the tax director doesn't quit and Smith becomes mayor. So if the house is built, Jones will go bankrupt.

*h. Jim is a Democrat only if Howard or Rhoda is. If Howard is a Democrat, so are Barbara and Allen. If Barbara is a Democrat, then Allen is a Democrat only if Freda is. But not both Freda and Jim are Democrats. Therefore Jim is a Democrat only if Rhoda is too.

i. If life is a carnival, then I'm a clown or a trapeze artist. But either life isn't a carnival or there are balloons, and either there aren't any balloons or I'm not a clown. So, if life is a carnival, then I'm a trapeze artist.

12. Symbolize the following passages in *SL* and show that the resulting sets of sentences are inconsistent in *SD*.

a. If motorcycling is dangerous sailboating is also dangerous, and if sailboating is dangerous parachuting is dangerous. Motorcycling is dangerous but parachuting is not.

*b. If the recipe doesn't call for flavoring or it doesn't call for eggs, it's not a recipe for tapioca. If the recipe calls for eggs, then it's a tapioca recipe and it doesn't call for flavoring. But this recipe calls for eggs.

c. Bach is popular only if Beethoven is ignored. If Bach is unpopular and Beethoven isn't ignored, then current musical tastes are hopeless. Current musical tastes aren't hopeless, and Beethoven isn't ignored.

*d. Historians are right just in case theologians are mistaken, if and only if Darwin's theory is correct. And if historians or philosophers are right, then Darwinian theory is correct and theologians are mistaken. Historians are right if and only if philosophers are wrong. But if Darwinian theory is correct, then historians are mistaken.

e. Either Martha was commissioned to write the ballet or, if the fund-raising sale was a failure, Tony was commissioned. Nancy will dance if and only if Tony wasn't commissioned. But the fund-raiser was a failure. Nancy will dance, and Martha wasn't commissioned.

13. Explain:

a. Why we would not want to include the following derivation rule in *SD*.

$$\begin{array}{|l} \mathbf{P} \vee \mathbf{Q} \\ \\ \mathbf{P} \end{array}$$

*b. Why Negation Introduction is a dispensable rule in *SD*. We take a rule to be dispensible in *SD* if and only if the last line of every derivation that makes use of the rule in question can also be derived from the given assumptions without using that rule.

c. Why Reiteration is a dispensable rule in *SD*.

*d. Why deriving a sentence and its negation within the scope of an auxilliary assumption *does not* show that the primary assumptions constitute an inconsistent set but *does* show that the set that consists of the primary assumptions and the assumptions of all open subderivations is inconsistent.

e. Why an argument of *SL* that has as one of its premises the negation of a theorem is valid in *SD*.

14. In Chapter 6 (see Sections 6.3 and 6.4) we prove that, for any sentence **P** and set Γ of sentences of *SL*,

$\Gamma \vdash \mathbf{P}$ in *SD* if and only if $\Gamma \models \mathbf{P}$.

Show that a-c below follow from this result.

a. An argument of *SL* is valid in *SD* if and only if the argument is truth-functionally valid.

*b. A sentence **P** of *SL* is a theorem in *SD* if and only if **P** is truth-functionally true.

c. Sentences **P** and **Q** of *SL* are equivalent in *SD* if and only if **P** and **Q** are truth-functionally equivalent.

5.4 THE DERIVATION SYSTEM *SD+*

In this section we introduce a new natural deduction system, *SD+*, which contains all the derivation rules of *SD* plus some more. However, *SD+* is not a stronger system than *SD* in the sense that more arguments of *SL* can be shown

to be valid or that more sentences of *SL* are theorems in *SD* than are in *SD+*. That is

$$\Gamma \vdash \mathbf{P} \text{ in } SD$$

if and only if

$$\Gamma \vdash \mathbf{P} \text{ in } SD+$$

However, historically a larger set of rules, such as those constituting *SD+*, have been used in many derivation systems. This larger set contains some rules absent from *SD* that do correspond to reasoning patterns commonly used in ordinary discourse, and often derivations in *SD+* are shorter than corresponding derivations in *SD*.

RULES OF INFERENCE

Suppose that prior to line **n** of a derivation two accessible lines, **i** and **j**, contain $\mathbf{P} \supset \mathbf{Q}$ and $\sim \mathbf{Q}$, respectively. In *SD* we can derive $\sim \mathbf{P}$ as follows:

i	$\mathbf{P} \supset \mathbf{Q}$	
j	$\sim \mathbf{Q}$	
n	$\mathbf{P}$	A / $\sim$ I
n + 1	$\mathbf{Q}$	i, j $\supset$E
n + 2	$\sim \mathbf{Q}$	j R
n + 3	$\sim \mathbf{P}$	n – n + 2 – I

To avoid going through this routine every time such a situation arises, we introduce the rule **Modus Tollens:**

Modus Tollens (MT)

	$\mathbf{P} \supset \mathbf{Q}$
	$\sim \mathbf{Q}$
▷	$\sim \mathbf{Q}$

Now suppose that prior to line **n** of a derivation two accessible lines, **i** and **j**, contain $\mathbf{P} \supset \mathbf{Q}$ and $\mathbf{Q} \supset \mathbf{R}$. A routine to derive $\mathbf{P} \supset \mathbf{R}$ in *SD* beginning at line **i** is as follows:

i	$\mathbf{P} \supset \mathbf{Q}$	
j	$\mathbf{Q} \supset \mathbf{R}$	
n	$\mathbf{P}$	A / $\sim$ I
n + 1	$\mathbf{Q}$	i, j $\supset$E
n + 2	$\mathbf{R}$	j, n + 2 $\supset$E
n + 3	$\mathbf{P} \supset \mathbf{R}$	n – n + 2 $\supset$I

To avoid this routine, we introduce the rule **Hypothetical Syllogism:**

Hypothetical Syllogism (HS)

	$\mathbf{P \supset Q}$
	$\mathbf{Q \supset R}$
▷	$\mathbf{P \supset R}$

Finally suppose that prior to the line **n** of a derivation two accessible lines, **i** and **j**, contain $\mathbf{P \vee Q}$ and $\mathbf{\sim P}$ and that we wish to derive **Q**. A routine for accomplishing this in *SD* is as follows:

i	$\mathbf{P \vee Q}$	
j	$\mathbf{\sim P}$	
n	**P**	A / ∨E
n + 1	$\mathbf{\sim Q}$	A / ~ E
n + 2	**P**	**n** R
n + 3	$\mathbf{\sim P}$	**j** R
n + 4	**Q**	**n + 1 – n + 3** ~ E
n + 5	**Q**	A / ∨E
n + 6	**Q**	**n + 5** R
n + 7	**Q**	**i, n – n + 4, n + 5 – n + 6** ∨E

The rule of *Disjunctive Syllogism* allows us to avoid going through this routine for this and similar cases.

Disjunctive Syllogism (DS)

$\mathbf{P \vee Q}$		$\mathbf{P \vee Q}$
$\mathbf{\sim P}$	or	$\mathbf{\sim Q}$
Q		**P**

The three rules of inference just introduced can be thought of as derived rules. They are added for convenience only; whatever we can derive with them, we can derive without them, using only the rules of *SD*.

RULES OF REPLACEMENT

In addition to rules of inference, there are also derivation rules known as *rules of replacement*. Rules of replacement, as their name suggests, allow us to derive

some sentences from other sentences by replacing sentential components. For example, from the sentence

G ∨ (H & K)

we can certainly infer

G ∨ (~ ~ H & K)

In this instance the sentential component 'H' has been replaced with '~ ~ H'. Similarly from

G ∨ (~ ~ H & K)

we can certainly infer

G ∨ (H & K)

Double Negation is the rule of replacement that licenses such moves within a derivation.

Double Negation (DN)
P ◁ ▷ ~ ~ **P**

That is, by using Double Negation, we can derive from a sentence **Q** that contains **P** as a sentential component another sentence that is like **Q**, except that one occurrence of the sentential component **P** has been replaced with ~ ~ **P**. And, by using Double Negation, we can derive from a sentence **Q** that contains ~ ~ **P** as a sentential component another sentence that is like **Q**, except that one occurrence of the sentential component ~ ~ **P** has been replaced with **P**.

Double Negation can be applied to any of the sentential components of a sentence. For instance, from

G ∨ (H & K)

Double Negation permits us to derive

G ∨ ~ ~ (H & K)

And from

G ∨ ~ ~ (H & K)

Double Negation allows us to derive

$$G \vee (H \,\&\, K)$$

Since every sentence is a sentential component of itself, Double Negation applies to the entire sentence as well. In a derivation Double Negation permits us to go from

$$G \vee (H \,\&\, K)$$

to

$$\sim\sim[G \vee (H \,\&\, K)]$$

and from

$$\sim\sim[G \vee (H \,\&\, K)]$$

to

$$G \vee (H \,\&\, K)$$

Here are the rules of replacement for *SD+*:

Commutation (Com)
$\mathbf{P \,\&\, Q \lhd\rhd Q \,\&\, P}$
$\mathbf{P \vee Q \lhd\rhd Q \vee P}$

Association (Assoc)
$\mathbf{P \,\&\, (Q \,\&\, R) \lhd\rhd (P \,\&\, Q) \,\&\, R}$
$\mathbf{P \vee (Q \vee R) \lhd\rhd (P \vee Q) \vee R}$

Implication (Impl)
$\mathbf{P \supset Q \lhd\rhd \sim P \vee Q}$

Double Negation (DN)
$\mathbf{P \lhd\rhd \sim\sim P}$

De Morgan (DeM)
$\mathbf{\sim (P \,\&\, Q) \lhd\rhd \sim P \vee \sim Q}$
$\mathbf{\sim (P \vee Q) \lhd\rhd \sim P \,\&\, \sim Q}$

Idempotence (Idem)
$\mathbf{P \lhd\rhd P \,\&\, P}$
$\mathbf{P \lhd\rhd P \vee P}$

Transposition (Trans)
$\mathbf{P \supset Q \lhd\rhd \sim Q \supset \sim P}$

Exportation (Exp)
$\mathbf{P \supset (Q \supset R) \lhd\rhd (P \,\&\, Q) \supset R}$

Distribution (Dist)
$\mathbf{P \,\&\, (Q \vee R) \lhd\rhd (P \,\&\, Q) \vee (P \,\&\, R)}$
$\mathbf{P \vee (Q \,\&\, R) \lhd\rhd (P \vee Q) \,\&\, (P \vee R)}$

Equivalence (Equiv)
$\mathbf{P \equiv Q \lhd\rhd (P \supset Q) \,\&\, (Q \supset P)}$
$\mathbf{P \equiv Q \lhd\rhd (P \,\&\, Q) \vee (\sim P \,\&\, \sim Q)}$

Rules of replacement always allow the replacement of sentential components. In addition, all these rules of replacement are two-way rules; that is, a sentential component that has the form of the sentence on the left of '◁ ▷' can be replaced with a sentential component that has the form of the sentence on the right of '◁ ▷', and vice versa.

Consider the following derivation:

Derive: $J \supset [M \vee (G \vee I)]$

1	$J \supset [K \vee (L \vee H)]$	Assumption
2	$[(K \vee L) \vee H] \supset [(M \vee G) \vee I]$	Assumption
3	$J \supset [(K \vee L) \vee H]$	1 Assoc
4	$J \supset [(M \vee G) \vee I]$	2, 3 HS
5	$J \supset [M \vee (G \vee I)]$	4 Assoc

Here the replacement rule Association has been used twice—first to replace a sentential component of the form $\mathbf{P \vee (Q \vee R)}$ with a sentential component of the form $\mathbf{(P \vee Q) \vee R}$ and then to replace a sentential component of the form $\mathbf{(P \vee Q) \vee R}$ with a sentential component of the form $\mathbf{P \vee (Q \vee R)}$.

Since all the derivation rules of *SD* are derivation rules of *SD+*, the procedures for properly applying the rules of *SD* apply to *SD+* as well. The rules of inference of *SD+*, including Modus Tollens, Hypothetical Syllogism, and Disjunctive Syllogism, must be applied to entire sentences on a line. Rules of replacement, on the other hand, can be applied to all sentential components. The following derivation illustrates the proper use of several of the rules of replacement:

Derive: $\sim C \equiv E$

1	$(D \vee B) \vee (E \supset \sim C)$	Assumption
2	$\sim B \,\&\, [\sim D \,\&\, (\sim E \supset C)]$	Assumption
3	$(\sim B \,\&\, \sim D) \,\&\, (\sim E \supset C)$	2 Assoc
4	$\sim (B \vee D) \,\&\, (\sim E \supset C)$	3 DeM
5	$\sim (B \vee D)$	4 &E
6	$\sim (D \vee B)$	5 Com
7	$E \supset \sim C$	1, 6 DS
8	$\sim E \supset C$	3 &E
9	$\sim C \supset \sim\sim E$	8 Trans
10	$\sim C \supset E$	9 DN
11	$(\sim C \supset E) \,\&\, (E \supset \sim C)$	10, 7 &I
12	$\sim C \equiv E$	11 Equiv

Notice that each application of a derivation rule requires a separate line. Moreover care must be taken to apply each derivation rule only to sentences that

have the proper form (or, in the case of rules of replacement, sentences that have components that have the proper form).

Here is an example in which these points are ignored:

Derive: ~ A ⊃ [B ⊃ (G ∨ D)]

1	(A ∨ ~ B) ∨ ~ C	Assumption	
2	(D ∨ G) ∨ C	Assumption	
3	~ (~ A & B) ∨ ~ C	1 DeM	**MISTAKE!**
4	(~ A & B) ⊃ ~ C	3 Impl	
5	C ∨ (G ∨ D)	2 Com	**MISTAKE!**
6	~ C ⊃ (G ∨ D)	5 Impl	**MISTAKE!**
7	(~ A & B) ⊃ (G ∨ D)	4, 6 HS	
8	~ A ⊃ [B ⊃ (G ∨ D)]	7 Exp	

De Morgan does not license entering the sentence on line 3. What De Morgan does allow is the replacement of a sentential component of the form ~ **P** ∨ ~ **Q** with a sentential component of the form ~ (**P** & **Q**), but the sentential component 'A ∨ ~ B' does not have the form ~ **P** ∨ ~ **Q**. However, by applying Double Negation to the first assumption, we can obtain '(~ ~ A ∨ ~ B) ∨ ~ C'. And this latter sentence does have a sentential component of the form ~ **P** ∨ ~ **Q**, namely, '~ ~ A ∨ ~ B'. Here **P** is '~ A', and **Q** is 'B'. Hence the derivation should begin this way:

Derive: ~ A ⊃ [B ⊃ (G ∨ D)]

1	(A ∨ ~ B) ∨ ~ C	Assumption
2	(D ∨ G) ∨ C	Assumption
3	(~ ~ A ∨ ~ B) ∨ ~ C	1 DN
4	~ (~ A & B) ∨ ~ C	3 DeM

The second mistake in our example, in line 5, is that Commutation is applied twice within the same line. Each application of a rule, even if it is the same rule, requires a separate line. Correctly done, the derivation proceeds:

5	(~ A & B) ⊃ ~ C	4 Impl
6	C ∨ (D ∨ G)	2 Com
7	C ∨ (G ∨ D)	6 Com

The third mistake, in line 6 of the example, also stems from our trying to apply a rule of replacement to a sentential component that does not have the form required by the rule. Implication permits the replacement of a sentential component of the form ~ **P** ∨ **Q** with a sentential component of the form **P** ⊃ **Q**, but 'C ∨ (G ∨ D)' does not have the form ~ **P** ∨ **Q**. However, applying Double Negation to 'C', a sentential component of 'C ∨ (G ∨ D)', generates '~ ~ C ∨ (G ∨ D)'. This latter sentence does have the form ~ **P** ∨ **Q**, where P is '~ C' and **Q** is 'G ∨ D'. Here is the entire derivation done correctly:

Derive: ~ A ⊃ [B ⊃ (G ∨ D)]

1	(A ∨ ~ B) ∨ ~ C	Assumption
2	(D ∨ G) ∨ C	Assumption
3	(~ ~ A ∨ ~ B) ∨ ~ C	1 DN
4	~ (~ A & B) ∨ ~ C	3 DeM
5	(~ A & B) ⊃ ~ C	4 Impl
6	C ∨ (D ∨ G)	2 Com
7	C ∨ (G ∨ D)	6 Com
8	~ ~ C ∨ (G ∨ D)	7 DN
9	~ C ⊃ (G ∨ D)	8 Impl
10	(~ A & B) ⊃ (G ∨ D)	5, 9 HS
11	~ A ⊃ [B ⊃ (G ∨ D)]	10 Exp

The definitions of the basic concepts of *SD+* parallel the definitions for the basic concepts of *SD*, except that '*SD*' is replaced with '*SD+*'. For example, the concept of derivability is defined as follows:

A sentence **P** of *SL* is *derivable in SD+* from a set **Γ** of sentence of *SL* if and only if there is a derivation in *SD+* in which all the primary assumptions are members of **Γ** and **P** occurs within the scope of only those assumptions.

Consequently tests for the various syntactic properties in *SD+* are analogous to those of *SD*. To show that an argument is valid in *SD+*, we construct a derivation in *SD+* showing that the conclusion of the argument is derivable in *SD+* from the set all of whose members are premises of the argument. To show that a sentence **P** of *SL* is a theorem in *SD+*, we show that **P** is derivable in *SD+* from the empty set. And so on. Remember that, although *SD* and *SD+* are different syntactic systems, whatever can be derived in one can be derived in the other.

The Derivation Rules of SD+

All the Derivation Rules of *SD* and Rules of Inference

Modus Tollens (MT)

P ⊃ Q
~ Q
∴ **~ P**

Hypothetical Syllogism (HS)

P ⊃ Q
Q ⊃ R
∴ **P ⊃ R**

Disjunctive Syllogism (DS)

P ∨ Q **~ P** ∴ **Q**

or

P ∨ Q **~ Q** ∴ **P**

Rules of Replacement

Commutation (Com)
P & **Q** ◁▷ **Q** & **P**
P ∨ **Q** ◁▷ **Q** ∨ **P**

Association (Assoc)
P & (**Q** & **R**) ◁▷ (**P** & **Q**) & **R**
P ∨ (**Q** ∨ **R**) ◁▷ (**P** ∨ **Q**) ∨ **R**

Implication (Impl)
P ⊃ **Q** ◁▷ ~ **P** ∨ **Q**

Double Negation (DN)
P ◁▷ ~ ~ **P**

De Morgan (DeM)
~ (**P** & **Q**) ◁▷ ~ **P** ∨ ~ **Q**
~ (**P** ∨ **Q**) ◁▷ ~ **P** & ~ **Q**

Idempotence (Idem)
P ◁▷ **P** & **P**
P ◁▷ **P** ∨ **P**

Transposition (Trans)
P ⊃ **Q** ◁▷ ~ **Q** ⊃ ~ **P**

Exportation (Exp)
P ⊃ (**Q** ⊃ **R**) ◁▷ (**P** & **Q**) ⊃ **R**

Distribution (Dist)
P & (**Q** ∨ **R**) ◁▷ (**P** & **Q**) ∨ (**P** & **R**)
P ∨ (**Q** & **R**) ◁▷ (**P** ∨ **Q**) & (**P** ∨ **R**)

Equivalence (Equiv)
P ≡ **Q** ◁▷ (**P** ⊃ **Q**) & (**Q** ⊃ **P**)
P ≡ **Q** ◁▷ (**P** & **Q**) ∨ (~ **P** & ~ **Q**)

5.4E EXERCISES

1. Show that the following derivability claims hold in *SD+*.

a. {D ⊃ E, E ⊃ (Z & W), ~ Z ∨ ~ W} ⊢ ~ D
*b. {(H & G) ⊃ (L ∨ K), G & H} ⊢ K ∨ L
c. {(W ⊃ S) & ~ M, (~ W ⊃ H) ∨ M, (~ S ⊃ H) ⊃ K} ⊢ K
*d. {[(K & J) ∨ I] ∨ ~ Y, Y & [(I ∨ K) ⊃ F]} ⊢ F ∨ N
e. {(M ∨ B) ∨ (C ∨ G), ~ B & (~ G & ~ M)} ⊢ C
*f. {~ L ∨ (~ Z ∨ ~ U), (U & G) ∨ H, Z} ⊢ L ⊃ H

2. Show that each of the following is valid in *SD+*.

a.
~ Y ⊃ ~ Z
~ Z ⊃ ~ X
~ X ⊃ ~ Y

Y ≡ Z

*b.
(~ A & ~ B) ∨ (~ A & ~ C)
(E & D) ⊃ A

~ E ∨ ~ D

c.
(F & G) ∨ (H & ~ I)
I ⊃ ~ (F & D)

I ⊃ ~ D

*d.
F ⊃ (~ G ∨ H)
F ⊃ G
~ (H ∨ I)

F ⊃ J

e. $F \supset (G \supset H)$
$\sim I \supset (F \vee H)$
$F \supset G$
$\therefore I \vee H$

g. $[(X \ \& \ Z) \ \& \ Y] \vee (\sim X \supset \sim Y)$
$X \supset Z$
$Z \supset Y$
$\therefore X \equiv Y$

*f. $G \supset (H \ \& \sim K)$
$H \equiv (L \ \& \ I)$
$\sim I \vee K$
$\therefore \sim G$

3. Show that each of the following is a theorem in *SD+*.
a. $A \vee \sim A$
*b. $\sim \sim \sim \sim \sim (A \ \& \sim A)$
c. $A \vee [(\sim A \vee B) \ \& \ (\sim A \vee C)]$
*d. $[(A \ \& \ B) \supset (B \ \& \ A)] \ \& \ [\sim (A \ \& \ B) \supset \sim (B \ \& \ A)]$
e. $[A \supset (B \ \& \ C)] \equiv [(\sim B \vee \sim C) \supset \sim A]$
*f. $[A \vee (B \vee C)] \equiv [C \vee (B \vee A)]$
g. $[A \supset (B \equiv C)] \equiv (A \supset [(\sim B \vee C) \ \& \ (\sim C \vee B)])$
*h. $(A \vee [B \supset (A \supset B)]) \equiv (A \vee [(\sim A \vee \sim B) \vee B])$
i. $[\sim A \supset (\sim B \supset C)] \supset [(A \vee B) \vee (\sim \sim B \vee C)]$
*j. $(\sim A \equiv \sim A) \equiv [\sim (\sim A \supset A) \equiv (A \supset \sim A)]$

4. Show that the members of each of the following pairs of sentences are equivalent in *SD+*.
a. $A \vee B$
$\sim (\sim A \ \& \sim B)$
*b. $A \ \& \ (B \vee C)$
$(B \ \& \ A) \vee (C \ \& \ A)$
c. $(A \ \& \ B) \supset C$
$\sim (A \supset C) \supset \sim B$
*d. $(A \vee B) \vee C$
$\sim A \supset (\sim B \supset C)$
e. $A \vee (B \equiv C)$
$A \vee (\sim B \equiv \sim C)$
*f. $(A \ \& \ B) \vee [(C \ \& \ D) \vee A]$
$([(C \vee A) \ \& \ (C \vee B)] \ \& \ [(D \vee A) \ \& \ (D \vee B)]) \vee A$

5. Show that the following sets of sentences are inconsistent in *SD+*.
a. $\{[(E \ \& \ F) \vee \sim \sim G] \supset M, \sim [[(G \vee E) \ \& \ (F \vee G)] \supset (M \ \& \ M)]\}$
*b. $\{\sim [(\sim C \vee \sim \sim C) \vee \sim \sim C]\}$
c. $\{M \ \& \ L, [L \ \& \ (M \ \& \sim S)] \supset K, \sim K \vee \sim S, \sim (K \equiv \sim S)\}$
*d. $\{B \ \& \ (H \vee Z), \sim Z \supset K, (B \equiv Z) \supset \sim Z, \sim K\}$
e. $\{\sim [W \ \& \ (Z \vee Y)], (Z \supset Y) \supset Z, (Y \supset Z) \supset W\}$
*f. $\{[(F \supset G) \vee (\sim F \supset G)] \supset H, (A \ \& \ H) \supset \sim A, A \vee \sim H\}$

6. Symbolize the following arguments in *SL*, and show that they are valid in *SD+*.
a. If the phone rings Ed is calling, or if the beeper beeps Ed is calling. If not both Ed and Agnes are at home today, then it's not the case that if the phone rings, Ed is calling. Ed isn't home today, and he isn't calling. So the beeper won't beep.

*b. If Monday is a bad day, then I'll lose my job provided the boss doesn't call in sick. The boss won't call in sick. So I'll lose my job—since either Monday will be a bad day, or the boss won't call in sick only if I lose my job.

c. Army coats are warm only if they're either made of wool or not made of cotton or rayon. If army coats are not made of rayon, then they're made of cotton. Hence, if they're not made of wool, army coats aren't warm.

*d. If either the greenhouse is dry or the greenhouse is sunny if and only if it's not raining, the violets will wither. But if the violets wither the greenhouse is sunny, or if the violets wither the greenhouse isn't dry. It's raining, and the greenhouse isn't sunny. So the greenhouse is dry only if the violets won't wither.

e. It's not the case that John is rich and Hugo isn't. In fact, Hugo isn't rich, unless Moe is. And if Moe just emptied his bank account, then he isn't rich. Thus, if John is rich, then it's not the case that either Moe emptied his bank account or Moe isn't rich.

*f. Neither aspirin nor gin will ease my headache, unless it's psychosomatic. If it's psychosomatic and I'm really not ill, then I'll go out to a party and drink some martinis. So, if I'm not ill and don't drink any martinis, then aspirin won't ease my headache.

g. If I stay on this highway and don't slow down, I'll arrive in Montreal by 5:00. If I don't put my foot on the brake, I won't slow down. Either I won't slow down or I'll stop for a cup of coffee at the next exit. I'll stop for a cup of coffee at the next exit only if I'm falling asleep. So, if I don't arrive in Montreal by 5:00, then I'll stay on this highway only if I'm falling asleep and I put my foot on the brake.

*h. The weather is fine if and only if it's not snowing, and it's not snowing if and only if the sky is clear. So, either the weather is fine, the sky is clear, and it's not snowing; or it's snowing, the sky isn't clear, and the weather is lousy.

7. Symbolize the following passages in *SL*, and show that the resulting sets of sentences of *SL* are inconsistent in *SD+*.

a. Unless Stowe believes that all liberals are atheists, his claims about current politics are unintelligible. But if liberals are atheists only if they're not churchgoers, then Stowe's claims about current politics are nevertheless intelligible. Liberals are, in fact, churchgoers if and only if Stowe doesn't believe that they're all atheists, and if liberals aren't atheists, then Stowe doesn't believe that they are atheists. Liberals aren't atheists.

*b. Either Congress won't cut taxes or the elderly and the poor will riot, if but only if big business prospers. If the elderly don't riot, then Congress won't cut taxes. It won't happen that both the poor will riot and big business will prosper, and it won't happen that the poor don't riot and big business doesn't prosper. But if big business prospers, then Congress will cut taxes.

8. Answer the following.

a. Suppose we can derive **Q** from **P** by using only the rules of replacement. Why can we be sure that we can derive **P** from **Q**?

*b. Why must all arguments that are valid in *SD* be valid in *SD+* as well?

c. Suppose we develop a new natural deduction system *SD**. Let *SD** contain all the derivation rules of *SD* and in addition the derivation rule Absorption.

Absorption

$$\triangleright \begin{array}{|l} \mathbf{P} \supset \mathbf{Q} \\ \mathbf{P} \supset (\mathbf{P} \,\&\, \mathbf{Q}) \end{array}$$

Using only the derivation rules of *SD*, develop a routine showing that any sentence derived by using Absorption could be derived in *SD* without using it.

GLOSSARY[5]

DERIVABILITY IN *SD*: A sentence **P** of *SL* is *derivable in SD* from a set Γ of sentences of *SL* if and only if there is a derivation in *SD* in which all the primary assumptions are members of Γ and **P** occurs in the scope of only those assumptions.

VALIDITY IN *SD*: An argument of *SL* is *valid in SD* if and only if the conclusion of the argument is derivable in *SD* from the set consisting of the premises. An argument of *SL* is *invalid in SD* if and only if it is not valid in *SD*.

THEOREM IN *SD*: A sentence **P** of *SL* is a *theorem in SD* if and only if **P** is derivable in *SD* from the empty set.

EQUIVALENCE IN *SD*: Sentences **P** and **Q** of *SL* are *equivalent in SD* if and only if **Q** is derivable in *SD* from {**P**} and **P** is derivable in *SD* from {**Q**}.

INCONSISTENCY IN *SD*: A set Γ of sentences of *SL* is *inconsistent in SD* if and only if both a sentence **P** of *SL* and its negation ~ **P** are derivable in *SD* from Γ. A set Γ of sentences of *SL* is *consistent in SD* if and only if it is not inconsistent in *SD*.

[5]Similar definitions hold for the derivation system *SD+*.

Chapter 6

SENTENTIAL LOGIC: METATHEORY

6.1 MATHEMATICAL INDUCTION

In the three previous chapters we concentrated on developing and using techniques of sentential logic, both semantic and syntactic. In this chapter we step back to prove some claims *about* the semantics and syntax of sentential logic. Such results constitute the **metatheory** of sentential logic.

For the language *SL* the semantic accounts of such logical properties of sentences and sets of sentences of *SL* as validity, consistency, and equivalence given in Chapter 3 are fundamental in the sense that they are the standards by which other accounts of these properties are judged. For instance, although the techniques of Chapter 5 are purely syntactical—all the derivation rules appeal to the structures or forms of sentences, not to their truth-conditions—those techniques are intended to yield results paralleling those yielded by the semantic techniques of Chapter 3. One of the important metatheoretic results that we shall prove in this chapter is that this parallel does hold. We shall prove this by proving that the natural deduction system *SD* allows us to construct all and only the derivations we want to be able to construct, given the semantics of Chapter 3. Specifically we shall prove that, given any set Γ of sentences and any sentence **P** of *SL*, **P** is derivable from Γ in *SD* if and only if **P** is truth-functionally entailed by Γ. The results mentioned at the end of Section 5.3 follow from this. For example, all and only the truth-functionally valid arguments of *SL* are valid in *SD*, and all and only the truth-functionally true sentences of *SL* are theorems in *SD*.

Before establishing the foregoing results, we introduce the method of proof known as **mathematical induction** and use that method to establish some other interesting results in the metatheory of sentential logic. We use mathematical induction in later sections to prove the claims made in the previous paragraph. Mathematical induction is an extremely powerful method in that it allows us to establish results holding for an infinite number of items.

We introduce mathematical induction with an example. It seems obvious that in each sentence of *SL* the number of left parentheses equals the number of right parentheses. How might we *prove* that this claim is true for every sentence of *SL*? We cannot show that it is true by considering the sentences of *SL* one at a time; there are infinitely many sentences of *SL*, and so we would never get through all of them. Rather, we shall reason more generally about the sentences of *SL*, using the recursive definition of those sentences that was presented in Chapter 2:

1. Every sentence letter is a sentence.
2. If **P** is a sentence, then $\sim$ **P** is a sentence.
3. If **P** and **Q** are sentences, then (**P** & **Q**) is a sentence.
4. If **P** and **Q** are sentences, then (**P** $\vee$ **Q**) is a sentence.
5. If **P** and **Q** are sentences, then (**P** $\supset$ **Q**) is a sentence.
6. If **P** and **Q** are sentences, then (**P** $\equiv$ **Q**) is a sentence.
7. Nothing is a sentence unless it can be formed by repeated application of clauses 1–6.

It is trivial to show that every atomic sentence—that is, every sentence formed in accordance with clause 1—has an equal number of left and right parentheses (namely, zero), because atomic sentences contain no parentheses. All other sentences of *SL* are formed in accordance with clauses 2–6. We note that in each of these cases an equal number of outermost left and right parentheses are added to those already occurring in the sentence's immediate components to form the new sentence (zero of each in clause 2, one of each in clauses 3–6). Therefore, if we can be sure that the immediate components **P** and **Q** of sentences formed in accordance with clauses 2–6 themselves contain an equal number of parentheses, then we may conclude that the application of one of these clauses will result in a new sentence that also contains an equal number of left and right parentheses.

How can we be sure, though, that each of the immediate components of a molecular sentence *does* contain an equal number of left and right parentheses? Start with molecular sentences that contain one occurrence of a connective—sentences like '$\sim$ A', '(A $\supset$ B)', and '(A & B)'. Every sentence that contains one occurrence of a connective has one of the forms $\sim$ **P**, (**P** & **Q**), (**P** $\vee$ **Q**), (**P** $\supset$ **Q**), or (**P** $\equiv$ **Q**), in accordance with clauses 2–6. Moreover in each case the immediate components **P** and **Q** are atomic. We have already noted that every atomic sentence contains an equal number of left and right parentheses (namely, zero), and so, because clauses 2–6 each add an equal

number of left and right parentheses to the ones already occurring in its immediate components, every molecular sentence with one occurrence of a connective must also have an equal number of left and right parentheses.

Now consider molecular sentences that contain two occurrences of connectives—sentences like '~ ~ A', '~ (A ∨ B)', '(A ∨ ~ B)', '((A ≡ B) ⊃ C)', and '(A ∨ (B & C))'. We may reason as we did in the previous paragraph. That is, every sentence that contains two occurrences of connectives has one of the forms ~ **P**, (**P** & **Q**), (**P** ∨ **Q**), (**P** ⊃ **Q**), or (**P** ≡ **Q**), in accordance with clauses 2–6. And in each case the immediate components **P** and **Q** each contain fewer than two occurrences of connectives. We have already found that, in all sentences containing fewer than two occurrences of connectives (atomic sentences or sentences containing one occurrence of a connective), the number of left parentheses equals the number of right parentheses. Therefore, because clauses 2–6 each add an equal number of left and right parentheses to those already occurring in its immediate components, we may conclude that every molecular sentence with two occurrences of connectives also has an equal number of left and right parentheses.

And in sentences containing *three* occurrences of connectives—sentences like '~ ~ ~ A', '~ (~ A ∨ B)', '((A ⊃ B) & (A ∨ C))', and '(~ (A ≡ B) ≡ C)'—the same pattern of reasoning emerges. In every sentence that contains three occurrences of connectives, the immediate components each contain fewer than three occurrences of connectives—either zero, one, or two occurrences. We have already shown that, in any sentence of *SL* that contains either zero, one, or two occurrences of connectives, the number of left parentheses equals the number of right parentheses. Therefore, because clauses 2–6 each add an equal number of left and right parentheses, we may conclude that the number of left parentheses in a sentence that contains three occurrences of connectives is equal to the number of right parentheses. Having established that the claim holds true for every sentence with three or fewer occurrences of connectives, we may show that it also holds for every sentence with four occurrences, then for every sentence with five, and so on—in each case using the same reasoning that we used for earlier cases. Generally, as soon as we have established that the claim holds for every sentence with **k** or fewer occurrences of connectives, the same pattern of reasoning shows that the claim also holds for every sentence that contains **k** + 1 occurrences of connectives.

We shall now present an argument by mathematical induction establishing that our claim is true of *every* sentence of *SL*:

> Every sentence of *SL* containing zero occurrences of connectives—that is, every atomic sentence of *SL*—is such that the number of left parentheses in that sentence equals the number of right parentheses.
>
> If every sentence of *SL* with **k** or fewer occurrences of connectives is such that the number of left parentheses in that sentence equals the number of right parentheses, then every sentence of *SL* with **k** + 1

occurrences of connectives is also such that the number of left parentheses in that sentence equals the number of right parentheses.

Therefore every sentence of *SL* is such that the number of left parentheses in that sentence equals the number of right parentheses.

(Here we use '**k**' as a variable ranging over the nonnegative integers, that is, the positive integers plus zero.) This argument is deductively valid—if the premises are true, then the conclusion is true as well. The first premise is our claim about parentheses for sentences with no connectives, and the second premise says that it follows that the claim also holds for sentences containing one occurrence of a connective. Having concluded that the claim holds for all sentences containing zero or one occurrences of connectives, we are assured by the second premise that the claim must also hold for sentences containing two occurrences of connectives. Having concluded that the claim holds for all sentences containing zero, one, or two occurrences of connectives, we are assured by the second premise that the claim also holds for sentences containing three occurrences of connectives, and so on for any number of occurrences of connectives that a sentence may contain. Because the argument is deductively valid, we can establish that its conclusion is true by showing that both premises are true.

We have already shown that the first premise is true. Sentences that contain zero occurrences of connectives are atomic sentences, and atomic sentences are simply sentence letters. The first premise is called the *basis clause* of the argument.

The second premise of the argument is called the *inductive step*. We shall prove that the inductive step is true by generalizing on the reasoning that we have already used. The antecedent of the inductive step is called the *inductive hypothesis*. We shall assume that the inductive hypothesis is true—that is, that every sentence of *SL* containing **k** or fewer occurrences of connectives contains an equal number of left and right parentheses—and we must show that on this assumption it follows that any sentence **P** that has **k** + 1 occurrences of connectives also contains an equal number of left and right parentheses. Since **k** is nonnegative, **k** + 1 is positive, and hence such a sentence **P** contains at least one occurrence of a connective. So **P** will be a molecular sentence, having one of the forms ~ **Q**, (**Q** & **R**), (**Q** ∨ **R**), (**Q** ⊃ **R**), or (**Q** ≡ **R**). We divide these forms into two cases.

Case 1: **P** has the form ~ **Q**. If ~ **Q** contains **k** + 1 occurrences of connectives, then **Q** contains **k** occurrences of connectives. By the inductive hypothesis (that every sentence containing **k** or fewer connectives has an equal number of left and right parentheses), the number of left parentheses in **Q** equals the number of right parentheses in **Q**. But ~ **Q** contains all the parentheses occurring in **Q** and no others. So ~ **Q** contains an equal number of left and right parentheses as well.

Case 2: **P** has one of the forms (**Q** & **R**), (**Q** ∨ **R**), (**Q** ⊃ **R**), or (**Q** ≡ **R**). In each instance, if **P** contains **k** + 1 occurrences of connectives, then each of its immediate components, **Q** and **R**, must

contain **k** or fewer occurrences of connectives. By the inductive hypothesis, then, we have the following:

a. The number of left parentheses in **Q** equals the number of right parentheses in **Q**.
b. The number of left parentheses in **R** equals the number of right parentheses in **R**.

We also have this:

c. The number of left parentheses in **P** is the number of left parentheses in **Q** plus the number of left parentheses in **R** plus 1—the one for the outermost left parenthesis in **P**.
d. The number of right parentheses in **P** is the number of right parentheses in **Q** plus the number of right parentheses in **R** plus 1.

By simple arithmetic, using (a) and (b), it follows that (c) equals (d)—**P** therefore has an equal number of left and right parentheses as well.

This completes our proof that the second premise, the inductive step, is true. Having established that both premises are true, we may conclude that the conclusion is true as well. Every sentence of *SL* contains an equal number of left and right parentheses.

We may now generally characterize arguments by mathematical induction. In such an argument, we arrange the items about which we wish to prove some thesis in a series of groups. In our example, we arranged the sentences of *SL* into the series: all sentences containing zero occurrences of connectives, all sentences containing one occurrence of connectives, all sentences containing two occurrences of connectives, and so on. Every sentence of *SL* appears in some group in this series—any sentence with **k** occurrences of connectives is part of group **k** + 1 in the series. Having arranged the items in such a series, an argument by mathematical induction then takes the following form.[1]

The thesis holds for every member of the first group in the series.

For each group in the series, if the thesis holds of every member of every prior group then the thesis holds for every member of that group as well.

The thesis holds for every member of every group of the series.

All arguments of this form are valid. Of course, only those with true premises are sound. Hence, to establish that the thesis holds for every member of every

[1] Strictly speaking, this is the form for arguments by strong mathematical induction. There is another type of mathematical induction, known as weak induction. We shall use only the strong variety of mathematical induction in this text. There is no loss here, for every claim that can be proved by weak mathematical induction can also be proved by strong mathematical induction.

group in the series, we must show first that the thesis does hold for every member of the first group and then that, no matter what group in the series we consider, the thesis holds for every member of that group if it holds for every member of every prior group. The first premise of such arguments is called the **basis clause,** and the second premise is called the **inductive step.** The antecedent of the second premise is called the **inductive hypothesis.**

We further illustrate mathematical induction with another example. Let **P** be a sentence that contains only '~ ', '∨', and '&' as connectives, and let **P'** be the sentence that results from doing this:

a. Replacing each occurrence of '∨' in **P** with '&'
b. Replacing each occurrence of '&' in **P** with '∨'
c. Adding a '~ ' in front of each atomic component of **P**

We shall call a sentence that contains only '~ ', '∨', and '&' as connectives a **TWA** sentence (short for 'tilde, wedge, and ampersand'), and we shall call the sentence **P'** that results from **P** by (a), (b), and (c) the **dual** of **P**. Here are some examples of duals for TWA sentences:

P	*Dual of* **P**
A	~A
((A ∨ F) & G)	((~A & ~F) ∨ ~G)
(((B & C) & C) ∨ D)	(((~B ∨ ~C) ∨ ~C) & ~D)
~((A ∨ ~B) ∨ (~A & ~B))	~((~A & ~~B) & (~~A ∨ ~~B))

We shall use mathematical induction to establish the following thesis:

Every TWA sentence **P** is such that **P** and its dual **P'** have opposite truth-values on each truth-value assignment (that is, if **P** is true then **P'** is false, and if **P** is false then **P'** is true).

As in the previous example, our series will classify sentences by the number of occurrences of connectives that they contain:

Basis clause: Every TWA sentence **P** of *SL* that contains zero occurrences of connectives is such that **P** and its dual **P'** have opposite truth-values on each truth-value assignment.

Inductive step: If every TWA sentence **P** of *SL* with **k** or fewer occurrences of connectives is such that **P** and its dual **P'** have opposite truth-values on each truth-value assignment, then every TWA sentence **P** of *SL* with **k** + 1 occurrences of connectives is such that **P** and its dual **P'** have opposite truth-values on each truth-value assignment.

Conclusion: Every TWA sentence **P** of *SL* is such that **P** and its dual **P'** have opposite truth-values on each truth-value assignment.

To show that the conclusion of this argument is true, we must show that the first premise, the basis clause, is true and also that the second premise, the inductive step, is true.

Proof of basis clause: A TWA sentence **P** that contains zero occurrences of connectives must be an atomic sentence, and its dual is ~ **P**—because there are no connectives to replace, we simply place a tilde in front of the atomic sentence. If **P** is true on a truth-value assignment, then according to the characteristic truth-table for the tilde, ~ **P** must be false. And if **P** is false on a truth-value assignment, then ~ **P** is true. We conclude that **P** and its dual have opposite truth-values on each truth-value assignment.

Proof of inductive step: We assume that the inductive hypothesis is true for all sentences that contain fewer than **k** + 1 connectives—that is, that every TWA sentence that contains fewer than **k** + 1 occurrences of connectives is such that it and its dual have opposite truth-values on each truth-value assignment. We must show that it follows from this assumption that the claim is also true of all TWA sentences that contain **k** + 1 occurrences of connectives. A TWA sentence **P** that contains **k** + 1 occurrences of connectives must be molecular, and because it is TWA, it has one of the three forms ~ **Q**, (**Q** ∨ **R**), or (**Q** & **R**). We will consider each form.

Case 1: **P** has the form ~ **Q**. If **P** contains **k** + 1 occurrences of connectives, then **Q** contains **k** occurrences of connectives, and **Q** is a TWA sentence (if it were not—if it contained a horseshoe or triple bar—then **P** would not be a TWA sentence either). Let **Q**′ be the dual of **Q**. Then the dual of **P** is ~ **Q**′, the sentence that results from ~ **Q** by making the changes (a), (b), and (c) of our definition of dual sentences within **Q** and leaving the initial tilde of ~ **Q** intact.

If **P**, that is, ~ **Q**, is true on a truth-value assignment, then **Q** is false. Because **Q** is a TWA sentence with fewer than **k** + 1 occurrences of connectives, it follows from the inductive hypothesis that **Q**′ is true. Therefore ~ **Q**′—the dual of **P**—is false. So, if **P** is true then its dual is false, and if **P** is false on a truth-value assignment then **Q** is true. It follows from the inductive hypothesis that **Q**′ is false, and therefore ~ **Q**′ is true. So, if **P** is false then its dual is true. We conclude that **P** and its dual have opposite truth-values on each truth-value assignment.

Case 2: **P** has the form (**Q** ∨ **R**). If **P** contains **k** + 1 occurrences of connectives, then **Q** and **R** each contain **k** or fewer occurrences of connectives. **Q** and **R** are also TWA sentences. Let **Q**′ be the dual of **Q** and **R**′ be the dual of **R**. Then the dual of **P** is (**Q**′ & **R**′)—the changes specified by (a), (b), and (c) must be made within **Q**, yielding its dual, and within **R**, yielding *its* dual, and the main connective '∨' of **P** must be replaced with '&'.

If **P** is true on a truth-value assignment, then by the characteristic truth-table for the wedge, either **Q** is true or **R** is true. Because **Q** and **R** each contain **k** or fewer occurrences of connectives, it follows from the inductive hypothesis that either **Q'** is false or **R'** is false. Either way, (**Q'** & **R'**), the dual of **P**, must be false as well. But if **P** is false on a truth-value assignment, then both **Q** and **R** must be false. By the inductive hypothesis both **Q'** and **R'** are true. So (**Q'** & **R'**) is true as well. We conclude that **P** and its dual have opposite truth-values on each truth-value assignment.

Case 3: **P** has the form (**Q** & **R**). If **P** contains **k** + 1 occurrences of connectives, then **Q** and **R** each contain **k** or fewer occurrences of connectives. And they are also TWA sentences. Let **Q'** be the dual of **Q** and **R'** be the dual of **R**. Then the dual of **P** is (**Q'** ∨ **R'**); changes (a), (b), and (c) have to be made within each of **Q** and **R**, producing their duals, and the main connective '&' has to be replaced with '∨'.

If **P** is true on a truth-value assignment, then, by the characteristic truth-table for the ampersand, both **Q** and **R** are true. Because **Q** and **R** each contain **k** or fewer occurrences of connectives, it follows from the inductive hypothesis that **Q'** and **R'** are both false, and therefore that the dual of **P**, (**Q'** ∨ **R'**), is false. If **P** is false on a truth-value assignment, then either **Q** is false or **R** is false. If **Q** is false, then it follows by the inductive hypothesis that **Q'** is true. If **R** is false, then it follows by the inductive hypothesis that **R'** is true. So at least one of **Q'** and **R'** is true, and (**Q'** ∨ **R'**), the dual of **P**, must be true as well. We conclude that **P** and its dual have opposite truth-values on each truth-value assignment.

These three cases establish the inductive step of the argument by mathematical induction, and we may now conclude that its conclusion is true as well. Our argument shows that the thesis about duals is true of every TWA sentence of *SL*. The basis clause shows that the thesis is true of every TWA sentence with zero occurrences of connectives. It follows, from the inductive step, that the thesis is also true of every TWA sentence with one connective. Because the thesis holds for all TWA sentences with zero or one occurrences of connectives, it follows from the inductive step that the thesis is also true of every TWA sentence with two occurrences of connectives. And so on, for any number of occurrences of connectives that a TWA sentence may have. Together the basis clause and the inductive step take every TWA sentence into account.

6.1E EXERCISES

1. Prove the following theses by mathematical induction.
   a. No sentence of *SL* that contains only binary connectives, if any, is truth-functionally false (that is, every truth-functionally false sentence of *SL* contains at least one '~').

b. Every sentence of *SL* that contains no binary connectives is truth-functionally indeterminate.
c. If two truth-value assignments $\mathbf{A}'$ and $\mathbf{A}''$ assign the same truth-values to the atomic components of a sentence $\mathbf{P}$, then $\mathbf{P}$ has the same truth-value on $\mathbf{A}'$ and $\mathbf{A}''$.
d. An iterated conjunction $(\ldots(\mathbf{P}_1 \,\&\, \mathbf{P}_2) \,\&\, \ldots \,\&\, \mathbf{P}_n)$ of sentences of *SL* is true on a truth-value assignment if and only if $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$ are all true on that assignment.
e. Where $\mathbf{P}$ is a sentence of *SL* and $\mathbf{Q}$ is a sentential component of $\mathbf{P}$, let $[\mathbf{P}]\ (\mathbf{Q}_1//\mathbf{Q})$ be a sentence that is the result of replacing at least one occurrence of $\mathbf{Q}$ in $\mathbf{P}$ with the sentence $\mathbf{Q}_1$. If $\mathbf{Q}$ and $\mathbf{Q}_1$ are truth-functionally equivalent, then $\mathbf{P}$ and $[\mathbf{P}]\ (\mathbf{Q}_1//\mathbf{Q})$ are truth-functionally equivalent.

2. Consider this thesis:

> No sentence of *SL* that contains only binary connectives is truth-functionally true.

Show that this thesis is false by producing a sentence that contains only binary connectives and that is truth-functionally true. Explain where an attempt to prove the thesis by mathematical induction (in the manner of the answer to Exercise 1.a) would fail.

6.2 TRUTH-FUNCTIONAL COMPLETENESS

In Chapter 2 we defined the truth-functional use of sentential connectives as follows:

> A sentential connective is used *truth-functionally* if and only if it is used to generate a compound sentence from one or more sentences in such a way that the truth-value of the generated compound is wholly determined by the truth-values of those one or more sentences from which the compound is generated, no matter what those truth-values may be.

The connectives of *SL* are used only truth-functionally since their intended interpretations are given wholly by their characteristic truth-tables. In Chapter 2 we constructed truth-functional paraphrases of many English sentences and showed how to symbolize these paraphrases in *SL*. Although *SL* contains only five sentential connectives, we found that a great variety of English compounds can nevertheless be adequately symbolized by using various combinations of these connectives. For instance, an English sentence of the form

> Neither **p** nor **q**

can be appropriately symbolized either by a sentence of the form

$$\sim(\mathbf{P} \vee \mathbf{Q})$$

or by a sentence of the form

$$\sim \mathbf{P} \,\&\, \sim \mathbf{Q}$$

An interesting question now arises: Is *SL* capable of representing all the ways in which sentences can be built up from other sentences by truth-functional means? We want the answer to this question to be 'yes' because we want *SL* to be an adequate vehicle for all of truth-functional logic. If there is some way of truth-functionally compounding sentences that cannot be represented in *SL*, then there may be some truth-functionally valid arguments that do not have valid symbolizations in *SL* simply because they cannot be adequately symbolized in *SL*. Similarly there may be sets of sentences that are truth-functionally inconsistent but that cannot be shown to be inconsistent by the truth-table method, again because these sentences cannot be adequately symbolized in *SL*. And so on.

To settle this question, we might try to produce complicated examples of truth-functionally compound sentences of English and then show that each can be adequately symbolized in *SL*. But obviously we cannot in this way prove that every truth-functionally compound sentence can be adequately symbolized in *SL*. Rather, we must show that all the possible ways of truth-functionally compounding sentences—of building up sentences from sentences by truth-functional connectives—yield sentences that can be adequately symbolized in *SL*.

We must first formulate our question somewhat more precisely: Can every truth-function be expressed by a sentence of *SL*? A *truth-function* is a mapping, for some positive integer **n**, of each combination of truth-values that **n** sentences of *SL* may have to a truth-value. Functions are most familiar in mathematics. Addition and multiplication are, for example, both functions that map each pair of numbers to a unique number. Addition maps each pair of numbers to the sum of those numbers. Multiplication maps each pair of numbers to the product of those numbers. The members of the pairs of numbers that are mapped are the *arguments* of the function, and the number to which a pair is mapped is the *value* of the function for that pair of arguments. (Arguments in the sense of arguments of functions are not to be confused with arguments consisting of premises and conclusions.) Thus the addition function maps the pair of arguments 3 and 4 to the value 7, and the multiplication function maps that pair of arguments to the value 12.

Instead of mapping combinations of numbers to numbers, a truth-function maps each combination of truth-values that **n** atomic sentences of *SL* may have to a truth-value. The characteristic truth-table for '$\supset$' defines the material conditional truth-function:

P	Q	P $\supset$ Q
T	T	T
T	F	F
F	T	T
F	F	T

This truth-function is a truth-function of *two* arguments. There are four distinct combinations of truth-values that two sentences may have, and the table defining the truth-function accordingly contains four rows. Each distinct combination of arguments is listed to the left of the vertical line, and the truth-value to which that combination of arguments is mapped is listed to the right of the vertical line.

The characteristic truth-table for '– ' defines the negation truth-function:

P	– **P**
T	**F**
F	**T**

The negation truth-function is a truth-function of *one* argument since it maps each combination of truth-values that one sentence of *SL* may have to a truth-value. There are only two such combinations; each consists of a single truth-value. The truth-value to which each combination is mapped is listed in the same row to the right of the vertical line.

A truth-function is said to be **expressed** in *SL* by any sentence whose truth-table contains (in the column under its main connective) exactly the column of **T**s and **F**s that occurs on the right-hand side of the characteristic truth-table for the truth-function in question. For example, each sentence of the form – **P**, where **P** is an atomic sentence of *SL*, expresses the negation truth-function—for every such sentence has a two-row truth-table in which the column under the main connective contains an **F** in the first row and a **T** in the second row. This truth-function is also expressed by other sentences of *SL*—for example, by all sentences of the form – **P** & – **P**, where **P** is an atomic sentence. Every such sentence has a two-row truth-table in which the column under the main connective is

F
T

The important question for us is not how many sentences of *SL* express the same truth-function but rather whether for each truth-function there is *at least one* sentence of *SL* that expresses that truth-function. There are an infinite number of truth-functions. This is most easily seen by considering that for every positive integer **n** there are truth-functions of **n** arguments (truth-functions that map each combination of truth-values that **n** sentences of *SL* may have to a truth-value), and there are infinitely many positive integers. In Chapter 2 we defined one truth-function of one argument and four truth-functions of two arguments via the five characteristic truth-tables for the connectives of *SL*. There are three other truth-functions of one argument:

P	
T	**T**
F	**F**

P	
T	**T**
F	**T**

P	
T	**F**
F	**F**

And there are twelve other truth-functions of two arguments (because there are sixteen different ways of arranging **T**s and **F**s in a column of a four-row truth-table). Generally, where **n** is any positive integer, there are $2^{(2^n)}$ truth-functions of **n** arguments. So there are 256 truth-functions of three arguments, 65,536 truth-functions of four arguments, and so on. What we want to show is that, given any truth-function of any finite number of arguments, there is at least one sentence of *SL* that expresses that truth-function. In fact, we shall prove something even stronger:

Metatheorem 6.2.1:[7] Every truth-function can be expressed by a sentence of *SL* that contains no sentential connectives other than '~', '∨', and '&'.

The connectives of a language in which every truth-function can be expressed form a **truth-functionally complete** set of connectives. In proving Metatheorem 6.2.1 we shall be proving that the set that contains the connectives '~', '&', and '∨', defined as they are defined in *SL*, is truth-functionally complete.

Characteristic truth-tables define truth-functions by giving an exhaustive list of the combinations of arguments that each truth-function takes and displaying the value to which each such combination is mapped. That is, it is the rows of **T**s and **F**s that serve to define truth-functions in characteristic truth-tables. It should now be clear that the following schema also specifies a truth-function:

T	**T**	**F**
T	**F**	**F**
F	**T**	**F**
F	**F**	**T**

To the left of the vertical line, the four distinct combinations of truth-values that two sentences of *SL* may have are displayed. The specified truth-function is thus a function of two arguments. The value of the function for each combination of arguments is displayed to the right of the vertical line. Since every truth-function maps only a finite number of combinations of arguments, every truth-function can be specified in a table like the previous one. We call such a table a **truth-function schema**. A truth-function schema is simply a truncated truth-table.

We shall now show that the set of connectives {'~', '&', '∨'} is truth-functionally complete by producing an **algorithm** for constructing, given any possible truth-function schema, a sentence of *SL* that contains no connectives other than '~', '&', and '∨' and that expresses the truth-function specified by the schema. An algorithm is an **effective** procedure for producing a desired result—that is, a mechanical procedure that, when correctly followed, yields the desired result in a finite number of steps. Given a truth-function schema, our algorithm will produce a sentence whose truth-table contains, under its

[7] We number our metatheoretic results in a way that makes clear where to find them in the text. The first two digits, '6.2', refer to the chapter and section. The third digit, '1', means that this is the first numbered metatheoretic result in this section.

main connective, exactly the same column of **T**s and **F**s as occurs to the right of the vertical line in the truth-function schema. Once we produce the algorithm, Metatheorem 6.2.1 will be proved; the construction of such an algorithm will show that every truth-function can be expressed by a sentence of *SL* containing no connectives other than '–', '&', and '∨'.

To begin, we need a stock of atomic sentences. If the truth-function is a function of **n** arguments, we use the alphabetically first **n** atomic sentences of *SL*. So for the truth-function schema

T	**T**	**F**
T	**F**	**F**
F	**T**	**F**
F	**F**	**T**

we start with the atomic sentences 'A' and 'B'. Next we form, for each row of the truth-table, a sentence that is true if and only if its atomic components have the truth-values indicated in that row. This sentence is called the **characteristic sentence** for the row in question. The characteristic sentence for row **i** is the iterated conjunction

$$(\ldots(\mathbf{P}_1 \,\&\, \mathbf{P}_2) \,\&\, \ldots \,\&\, \mathbf{P}_n)$$

where $\mathbf{P}_j$ is the **j**th atomic sentence if the **j**th value in row **i** (to the left of the vertical bar) is **T**, and $\mathbf{P}_j$ is the negation of the **j**th atomic sentence if the **j**th value in row **i** is **F**. Thus the characteristic sentences for the four rows in our sample truth-function schema are 'A & B', 'A & – B', '– A & B', and '– A & – B', respectively. The first sentence is true if and only if both 'A' and 'B' are true; the second sentence is true if and only if 'A' is true and 'B' is false; the third sentence is true if and only if 'A' is false and 'B' is true; and the fourth sentence is true if and only if both 'A' and 'B' are false. We leave it as an exercise to prove that the characteristic sentence for each row of a truth-function schema is true if and only if its atomic components have the truth-values presented in that row.

Finally we identify the rows in the truth-function schema that have a **T** to the right of the vertical bar. If there is only one such row, then the characteristic sentence for that row is a sentence that expresses the truth-function specified in the schema. In our example the fourth row is the only row that has a **T** to the right of the vertical bar, and the characteristic sentence for that row is '– A & – B'. This sentence is true if and only if both 'A' and 'B' are false, and therefore this sentence expresses the truth-function specified by the truth-function schema:

A	B	– A	&	– B
T	**T**	**F**	**F**	**F**
T	**F**	**F**	**F**	**T**
F	**T**	**T**	**F**	**F**
F	**F**	**T**	**T**	**T**

If the truth-function schema has more than one **T** to the right of the vertical bar, as does the following,

T	**T**	**F**
T	**F**	**T**
F	**T**	**F**
F	**F**	**T**

then we form an iterated disjunction of the characteristic sentences for the rows that have a **T** to the right of the vertical bar. In the present case the disjunction is '(A & ~ B) ∨ (~ A & ~ B)'—the disjunction of the characteristic sentences for the second and fourth rows. This sentence is true if and only if either 'A' is true and 'B' is false or both 'A' and 'B' are false, and it therefore expresses the truth-function specified in this schema:

A	B	(A	&	~	B)	∨	(~	A	&	~	B)
T	**T**	**T**	**F**	**F**	**T**	**F**	**F**	**T**	**F**	**F**	**T**
T	**F**	**T**	**T**	**T**	**F**	**T**	**F**	**T**	**F**	**T**	**F**
F	**T**	**F**	**F**	**F**	**T**	**F**	**T**	**F**	**F**	**F**	**T**
F	**F**	**F**	**F**	**T**	**F**	**T**	**F**	**F**	**T**	**T**	**F**

And if the schema is

T	**T**	**F**
T	**F**	**T**
F	**T**	**T**
F	**F**	**T**

then the disjunction of the characteristic sentences for the last three rows, '((A & ~ B) ∨ (~ A & B)) ∨ (~ A & ~ B)', expresses the truth-function in the schema.

In general, in the case where there is more than one **T** to the right of the vertical bar in a truth-function schema, the iterated disjunction that we form from the characteristic sentences for those rows will be true if and only if at least one of its disjuncts is true, and each disjunct is true only in the row for which it is a characteristic sentence. Therefore the iterated disjunction is true if and only if its atomic components have the truth-values specified by one of the rows that have a **T** to the right of the vertical bar, and so the disjunction expresses the truth-function specified by that schema.

If there are no **T**s in the column to the right of the vertical bar, then we conjoin the characteristic sentence for the first row of the truth-function schema with its negation. (Any other row's characteristic sentence would have done as well.) The result will be a sentence of the form **P** & ~ **P**, which is false on every truth-value assignment and hence expresses a truth-function

that maps every combination of **n** truth-values into **F**. For example, if our schema is

T	**T**	**F**
T	**F**	**F**
F	**T**	**F**
F	**F**	**F**

then the sentence '(A & B) & ~ (A & B)' expresses the truth-function specified in the schema.

In sum, we have three cases. If a truth-function schema has exactly one row with a **T** to the right of the vertical bar, then the characteristic sentence for that row expresses the truth-function specified in the schema. If a truth-function schema has more than one row with a **T** to the right of the vertical bar, then an iterated disjunction of the characteristic sentences for all such rows will express the truth-function specified in the schema. If a truth-function schema has no **T**s to the right of the vertical bar, then the conjunction of the characteristic sentence for the first row and its negation will express the truth-function specified by the schema.

The algorithm tells us how to construct a sentence that expresses the truth-function indicated in a given truth-function schema, and we may use it for any truth-function schema. As a final example consider the schema

T	**T**	**T**	**F**
T	**T**	**F**	**F**
T	**F**	**T**	**T**
T	**F**	**F**	**T**
F	**T**	**T**	**F**
F	**T**	**F**	**F**
F	**F**	**T**	**T**
F	**F**	**F**	**F**

This falls under our second case; there is more than one **T** to the right of the vertical line. We shall use the first three sentence letters of *SL*, because the truth-function is a truth-function of three arguments. We form the characteristic sentences for rows 3, 4, and 7 and then disjoin those characteristic sentences to produce

$$(((A \mathbin{\&} \sim B) \mathbin{\&} C) \vee ((A \mathbin{\&} \sim B) \mathbin{\&} \sim C)) \vee ((\sim A \mathbin{\&} \sim B) \mathbin{\&} C)$$

This sentence is true if and only if 'A', 'B', and 'C' have one of the combinations of truth-values represented in the third, fourth, and seventh rows of the schema.

Our algorithm shows how to construct, for any truth-function, a sentence of *SL* that expresses that truth-function. It therefore shows that for each truth-function there is *at least one* sentence of *SL* that expresses that truth-function. Moreover, because we have used only the three connectives '~', '&', and '∨', we have shown that the set of connectives {'~', '&', '∨'} is truth-functionally complete. This completes the proof of Metatheorem 6.2.1.

There is a consequence of the theorem that follows almost immediately: The smaller set {'~ ', '∨'} is also truth-functionally complete. Every conjunction **P** & **Q** is truth-functionally equivalent to ~ (~ **P** ∨ ~ **Q**), and so we may rewrite each sentence produced by the algorithm using only '~ ' and '∨'. For example, the sentence

(~ (~ A ∨ ~ ~ B) ∨ ~ (~ ~ A ∨ ~ ~ B))

expresses the same truth-function as

((A & ~ B) ∨ (~ A & ~ B))

Therefore every truth-function can be expressed by a sentence that contains only '~ ' and '∨' as connectives. It is also a consequence of Metatheorem 6.2.1 that the sets of connectives {'~ ', '&'} and {'~ ', '⊃'} are truth-functionally complete; we leave the proofs as an exercise.

On the other hand, the set of connectives {'∨', '&'} is *not* truth-functionally complete. To prove this, we must show that there is at least one truth-function that cannot be expressed by any sentence that contains at most the connectives '∨' and '&'. We call such a sentence a **W-A** sentence (short for 'wedge and ampersand'). A little reflection suggests that, no matter how many times we conjoin and disjoin, if we do not have the tilde available we can never produce a false sentence from atomic components that are all true. That is, every W-A sentence is true whenever its atomic components are all true. And if this is the case, then there are many truth-functions that cannot be expressed by any W-A sentence. Take the negation truth-function as an example. This truth-function maps the argument **T** into the value **F**. If our reflection is correct, there is no false W-A sentence with a single atomic component when that atomic component is true.

We shall therefore show that the set of connectives {'∨', '&'} is not truth-functionally complete by proving the following thesis:

> Every W-A sentence has the truth-value **T** on every truth-value assignment on which its atomic components all have the truth-value **T**.

This is a general claim about *all* W-A sentences, and so it cannot be proved by examining W-A sentences one by one (there are infinitely many). Instead, we shall prove the thesis by mathematical induction.

The shortest W-A sentences—that is, those with zero occurrences of connectives, are simply the atomic sentences of *SL*.

> *Basis clause:* Every atomic sentence of *SL* has the truth-value **T** on every truth-value assignment on which its atomic components all have the truth-value **T**.
>
> **Proof of basis clause:** The basis clause is obviously true, since an atomic sentence is itself its only component.

Inductive step: If every W-A sentence of *SL* with **k** or fewer occurrences of connectives is such that it has the truth-value **T** on every truth-value assignment on which its atomic components all have the truth-value **T**, then every W-A sentence with **k** + 1 occurrences of connectives has the truth-value **T** on every truth-value assignment on which its atomic components all have the truth-value **T**.

Proof of inductive step: We now assume that the inductive hypothesis is true for an arbitrary nonnegative integer **k**; that is, we assume that every W-A sentence with **k** or fewer occurrences of connectives is true whenever all its atomic components are true. We must show that it follows that the thesis also holds for any W-A sentence **P** with **k** + 1 occurrences of connectives. Since these sentences contain only '∨' and '&' as connectives, there are two cases.

Case 1: **P** has the form **Q** ∨ **R**. Then **Q** and **R** each contain fewer than **k** + 1 occurrences of connectives. They are also W-A sentences. So, by the inductive hypothesis, each disjunct is true on every truth-value assignment on which each of its atomic components is true. So, if all the atomic components of **Q** ∨ **R** are true, then both **Q** and **R** are true, and hence **Q** ∨ **R** is itself true.

Case 2: **P** has the form **Q** & **R**. Then each of **Q** and **R** is a W-A sentence with **k** or fewer occurrences of connectives. Hence the inductive hypothesis holds for both **Q** and **R**. Each conjunct is true on every truth-value assignment on which all its atomic components are true. So, if all the atomic components of **Q** & **R** are true, then both **Q** and **R** are true, and hence **Q** & **R** itself is true.

This proves the inductive step, and we can conclude that the thesis holds for every W-A sentence:

Conclusion: Every W-A sentence has the truth-value **T** on every truth-value assignment on which its atomic components all have the truth-value **T**.

It follows that no W-A sentence can express the negation truth-function as defined in the characteristic truth-table for the tilde since no W-A sentence can express a truth-function that maps the truth-value **T** to the truth-value **F**. (Whenever all the atomic components of a W-A sentence are true, the W-A sentence itself is true.)

6.2E EXERCISES

1. Show that a sentence constructed in accordance with our characteristic sentence algorithm is indeed a characteristic sentence for the row of the truth-function schema in question.
2. Using the algorithm in the proof of Metatheorem 6.2.1, construct a sentence containing at most '~', '&', and '∨' that expresses the truth-function defined in each of the following truth-function schemata.

a.
T	T	F
T	F	T
F	T	F
F	F	T

b.
T	F
F	F

*c.
T	T	F
T	F	T
F	T	T
F	F	F

d.
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	F
F	T	T	F
F	T	F	F
F	F	T	T
F	F	F	F

3. Give an algorithm analogous to that in Metatheorem 6.2.1 for constructing a characteristic sentence containing only '~ ' and '∨' for each row of a truth-function schema.

4. Using Metatheorem 6.2.1, prove that the sets {'~ ', '&'} and {'~ ', '⊃'} are truth-functionally complete.

5. Prove that the set consisting of the dagger '↓' is truth-functionally complete, where the dagger has the following characteristic truth-table:

P	Q	P ↓ Q
T	T	F
T	F	F
F	T	F
F	F	T

*6. Prove that the set consisting of the stroke '|' is truth-functionally complete, where the stroke has the following characteristic truth-table:

P	Q	P | Q
T	T	F
T	F	T
F	T	T
F	F	T

7. Using the results of Exercises 1.a and 1.b in Section 6.1E, prove that the following sets of connectives are not truth-functionally complete: {'~ '}, {'&', '∨', '⊃', '≡'}.

8. Prove that the set ['~', '∗'] is not truth-functionally complete. *Hint:* Show that the truth-table for any sentence **P** that contains only these two connectives and just two atomic components will have, in the column under the main connective, an even number of **T**s and an even number of **F**s.

9. Prove that if a truth-functionally complete set of connectives consists of exactly one binary connective, then that connective has either the characteristic truth-table for '↓' or the characteristic truth-table for '|'. (That is, show that the connective must be either '↓' or '|', though possibly under a different name.) (*Hint:* In the proofs for Exercises 7 and 8 above, it became apparent that characteristic truth-tables for truth-functionally complete sets of connectives must have certain properties. Show that only two characteristic truth-tables with just four rows have these properties.)

6.3 THE SOUNDNESS OF *SD* AND *SD+*

We now turn to the results announced at the beginning of this chapter. In this section we shall prove that, if a sentence **P** is derivable in *SD* from a set of sentences Γ, then Γ truth-functionally entails **P**. A natural deduction system for which this result holds is said to be **sound** for sentential logic. In the next section we shall prove the converse—that if a set of sentences Γ truth-functionally entails a sentence **P**, then **P** is derivable in *SD* from Γ. A natural deduction system for which this second result holds is said to be **complete** for sentential logic. Soundness and completeness are important properties for natural deduction systems. A natural deduction system that is not sound will sometimes lead us from true sentences to false ones, and a natural deduction system that is not complete will not allow us to construct all the derivations that we want to construct. In either case the natural deduction system would not be adequate for the purposes of sentential logic.

Metatheorem 6.3.1 is the *Soundness Metatheorem* for *SD*. That is, for any set Γ of sentences of *SL* and any sentence **P** of *SL*, we have this:

Metatheorem 6.3.1: If $\Gamma \vdash \mathbf{P}$ in *SD*, then $\Gamma \models \mathbf{P}$.[3]

Recall that $\Gamma \models \mathbf{P}$ if and only if there is no truth-value assignment on which all the members of Γ are true and **P** is false. Metatheorem 6.3.1 therefore says that the derivation rules of *SD* are *truth-preserving;* that is, when correctly applied, they will never take us from true sentences to a false sentence. When we constructed *SD,* our intent was to pick out truth-preserving derivation rules, and we shall now prove that we were successful.

Our proof will use mathematical induction to establish that each sentence in a derivation is true if all the open assumptions in whose scope the sentence lies are true. The basis clause will show that this claim is true of the

[3] In what follows we shall abbreviate '$\Gamma \vdash \mathbf{P}$ in *SD*' as '$\Gamma \vdash \mathbf{P}$'.

first sentence in a derivation. And the inductive step will show that, if the claim is true for the first **k** sentences in a derivation, then the claim is also true for the (**k** + 1)th sentence—that is, each time we apply another derivation rule in the derivation, that application is truth-preserving. We will then be able to conclude that the last sentence in any derivation, no matter how long the derivation is, is true if all the open assumptions in whose scope the sentence lies are true. And this conclusion is just what Metatheorem 6.3.1 says.

In the course of the proof, we shall use some set-theoretic terminology, which we here explain: Let Γ and Γ' be sets. If every member of Γ is also a member of Γ', then Γ is said to be a **subset** of Γ'. Note that every set is a subset of itself, and the empty set is trivially a subset of every set (because the empty set has no members, it has no members that are not members of every set). As an example, the set of sentences

{A, B, C}

has eight subsets: {A, B, C}, {A, B}, {B, C}, {A, C}, {A}, {B}, {C}, and ∅. If a set Γ is a subset of a set Γ', then Γ' is said to be a **superset** of Γ. Thus {A, B, C} is a superset of each of its eight subsets.

We will also make use of several semantic results, which we gather together here. First, if **P** is truth-functionally entailed by a set of sentences Γ, then **P** is truth-functionally entailed by every superset of Γ:

6.3.2: If $\Gamma \models \mathbf{P}$, then for every superset Γ' of Γ, $\Gamma' \models \mathbf{P}$.

Proof: Assume that $\Gamma \models \mathbf{P}$ and let Γ' be any superset of Γ. If every member of Γ' is true, then every member of its subset Γ is true, and so, because $\Gamma \models \mathbf{P}$, **P** is also true. Therefore $\Gamma' \models \mathbf{P}$.

Second, we have two results that were proved in the exercises for Chapter 3:

6.3.3: If $\Gamma \cup \{\mathbf{Q}\} \models \mathbf{R}$, then $\Gamma \models \mathbf{Q} \supset \mathbf{R}$ (see Exercise 2.b in Section 3.6E).

6.3.4: If $\Gamma \models \mathbf{Q}$ and $\Gamma \models \sim \mathbf{Q}$ for some sentence **Q**, then Γ is truth-functionally inconsistent (see Exercise 3.b in Section 3.6E).

Finally, if a set of sentences is truth-functionally inconsistent, then, for any sentence **Q** in the set, the set consisting of all the *other* sentences in the set truth-functionally entails ~ **Q**:

6.3.5: If $\Gamma \cup \{\mathbf{Q}\}$ is truth-functionally inconsistent, then $\Gamma \models \sim \mathbf{Q}$.

Proof: Assume that $\Gamma \cup \{\mathbf{Q}\}$ is truth-functionally inconsistent. Then there is no truth-value assignment on which every member of $\Gamma \cup \{\mathbf{Q}\}$ is true. Therefore, if every member of Γ is true on some truth-value assignment, **Q** must be false on that assignment, and ~ **Q** will be true. So $\Gamma \models \sim \mathbf{Q}$.

We are now prepared to prove that each sentence in a derivation is truth-functionally entailed by the set of the open assumptions in whose scope the sentence lies. We introduce the following notation: For any derivation, let $\mathbf{P_k}$ be the **k**th sentence in the derivation, and let $\Gamma_\mathbf{k}$ be the set of open assumptions in whose scope $\mathbf{P_k}$ lies. Here is our argument by mathematical induction on the position **k** in a derivation:

Basis clause: $\Gamma_1 \models \mathbf{P}_1$.

Inductive step: If $\Gamma_\mathbf{i} \models \mathbf{P_i}$ for every positive integer $\mathbf{i} \leq \mathbf{k}$, then $\Gamma_{\mathbf{k}+1} \models \mathbf{P}_{\mathbf{k}+1}$.

Conclusion: For every positive integer **k**, $\Gamma_\mathbf{k} \models \mathbf{P_k}$.

Once we have established that the premises of this argument are true, and hence that the conclusion is true as well, we may then conclude that in every derivation the last sentence—which is $\mathbf{P_k}$ for some positive integer **k**—is truth-functionally entailed by the primary assumptions of the derivation; these are the open assumptions in whose scope the last sentence lies.

Proof of basis clause: $\mathbf{P}_1$ is the first sentence in a derivation. Moreover, because every derivation in *SD* begins with one or more assumptions, $\mathbf{P}_1$ is an open assumption that lies in its own scope. (We remind the reader that, by definition, every assumption of a derivation lies within its own scope.) That is, Γ_1, the set of open assumptions in whose scope $\mathbf{P}_1$ lies, is $\{\mathbf{P}_1\}$. Because $\{\mathbf{P}_1\} \models \mathbf{P}_1$, we conclude that the basis clause is true.

Proof of inductive step: Let **k** be an arbitrary positive integer and assume the inductive hypothesis: for every positive integer $\mathbf{i} \leq \mathbf{k}$, $\Gamma_\mathbf{i} \models \mathbf{P_i}$. We must show that on this assumption it follows that $\Gamma_{\mathbf{k}+1} \models \mathbf{P}_{\mathbf{k}+1}$. We shall consider each way in which $\mathbf{P}_{\mathbf{k}+1}$ might be justified and show that our thesis holds whichever justification is used. We now turn to cases.

Case 1: $\mathbf{P}_{\mathbf{k}+1}$ is an Assumption. Then $\mathbf{P}_{\mathbf{k}+1}$ is a member of $\Gamma_{\mathbf{k}+1}$, the set of open assumptions in whose scope $\mathbf{P}_{\mathbf{k}+1}$ lies. Therefore, if every member of $\Gamma_{\mathbf{k}+1}$ is true, $\mathbf{P}_{\mathbf{k}+1}$, being a member of the set, is true as well. So $\Gamma_{\mathbf{k}+1} \models \mathbf{P}_{\mathbf{k}+1}$.

Case 2: $\mathbf{P}_{\mathbf{k}+1}$ is justified by Reiteration. Then $\mathbf{P}_{\mathbf{k}+1}$ occurs earlier in the derivation as sentence $\mathbf{P_i}$ at some position **i**. Moreover every assumption that is open at position **i** must remain open at position $\mathbf{k} + 1$—for if even one assumption in whose scope $\mathbf{P_i}$ lies were closed before position $\mathbf{k} + 1$, then $\mathbf{P_i}$ would not be accessible at position $\mathbf{k} + 1$. Therefore $\Gamma_\mathbf{i}$ is a subset of $\Gamma_{\mathbf{k}+1}$; every member of $\Gamma_\mathbf{i}$ is still an open assumption at position $\mathbf{k} + 1$. By our inductive hypothesis $\Gamma_\mathbf{i} \models \mathbf{P_i}$. Because $\Gamma_\mathbf{i}$ is a subset of $\Gamma_{\mathbf{k}+1}$, it follows, by 6.3.2, that $\Gamma_{\mathbf{k}+1} \models \mathbf{P_i}$. And because $\mathbf{P}_{\mathbf{k}+1}$ is the same sentence as $\mathbf{P_i}$, $\Gamma_{\mathbf{k}+1} \models \mathbf{P}_{\mathbf{k}+1}$.

Case 3: $\mathbf{P_{k+1}}$ is justified by Conjunction Introduction. The conjuncts of $\mathbf{P_{k+1}}$ occur earlier in the derivation, say at positions **h** and **j**:

h	**Q**	
j	**R**	
k + 1	**Q & R** (= $\mathbf{P_{k+1}}$)	**h, j** &I

(There may be open assumptions between positions **h** and **j** and between positions **j** and **k** + 1. Moreover it may be that **R** occurs earlier in the derivation than **Q** does—the order is immaterial.) By the inductive hypothesis, $\Gamma_{\mathbf{h}} \models \mathbf{Q}$ and $\Gamma_{\mathbf{j}} \models \mathbf{R}$. Moreover every member of $\Gamma_{\mathbf{h}}$ is a member of $\Gamma_{\mathbf{k+1}}$ and every member of $\Gamma_{\mathbf{j}}$ is a member of $\Gamma_{\mathbf{k+1}}$—for if this were not the case, then either **Q** or **R** would not be accessible at position **k** + 1. $\Gamma_{\mathbf{h}}$ and $\Gamma_{\mathbf{j}}$ are therefore subsets of $\Gamma_{\mathbf{k+1}}$ and so, by 6.3.2, $\Gamma_{\mathbf{k+1}} \models \mathbf{Q}$ and $\Gamma_{\mathbf{k+1}} \models \mathbf{R}$. But whenever both **Q** and **R** are true, $\mathbf{P_{k+1}}$, which is **Q** & **R**, is also true. So $\Gamma_{\mathbf{k+1}} \models \mathbf{P_{k+1}}$ as well.

Case 4: $\mathbf{P_{k+1}}$ is justified by Conjunction Elimination:

h	**Q** & $\mathbf{P_{k+1}}$		or	**h**	$\mathbf{P_{k+1}}$ & **Q**		
k + 1	$\mathbf{P_{k+1}}$	**h** &E		**k + 1**	$\mathbf{P_{k+1}}$	**h** &E	

By the inductive hypothesis, $\Gamma_{\mathbf{h}}$ truth-functionally entails the conjunction at position **h**. And whenever the conjunction is true, both conjuncts must be true. So $\Gamma_{\mathbf{h}} \models \mathbf{P_{k+1}}$. $\Gamma_{\mathbf{h}}$ is a subset of $\Gamma_{\mathbf{k+1}}$—all assumptions that have not been closed by position **h** must remain open at position **k** + 1. It follows, by 6.3.2, that $\Gamma_{\mathbf{k+1}} \models \mathbf{P_{k+1}}$.

Case 5: $\mathbf{P_{k+1}}$ is justified by Disjunction Introduction:

h	**Q**		or	**h**	**R**		
k + 1	**Q** ∨ **R** (= $\mathbf{P_{k+1}}$)	**h** ∨I		**k + 1**	**Q** ∨ **R** (= $\mathbf{P_{k+1}}$)	**h** ∨I	

By the inductive hypothesis, $\Gamma_{\mathbf{h}}$ truth-functionally entails the sentence at position **h**. That sentence is one of the disjuncts of **Q** ∨ **R**, so whenever it is true, so is **Q** ∨ **R**. Thus $\Gamma_{\mathbf{h}} \models \mathbf{P_{k+1}}$. $\Gamma_{\mathbf{h}}$ must be a subset of $\Gamma_{\mathbf{k+1}}$ if the sentence at position **h** is accessible at position **k** + 1, and so, by 6.3.2, $\Gamma_{\mathbf{k+1}} \models \mathbf{P_{k+1}}$.

Case 6: $\mathbf{P_{k+1}}$ is justified by Conditional Elimination:

h	**Q**	
j	**Q** ⊃ $\mathbf{P_{k+1}}$	
k + 1	$\mathbf{P_{k+1}}$	**h, j** ⊃E

By the inductive hypothesis, $\Gamma_h \models \mathbf{Q}$ and $\Gamma_j \models \mathbf{Q} \supset \mathbf{P}_{k+1}$. Both Γ_h and Γ_j must be subsets of Γ_{k+1} if the sentences at positions **h** and **j** are accessible at position $\mathbf{k} + 1$. By 6.3.2, then, $\Gamma_{k+1} \models \mathbf{Q}$ and $\Gamma_{k+1} \models \mathbf{Q} \supset \mathbf{P}_{k+1}$. Because $\mathbf{P}_{k+1}$ must be true whenever both **Q** and $\mathbf{Q} \supset \mathbf{P}_{k+1}$ are true, $\Gamma_{k+1} \models \mathbf{P}_{k+1}$ as well.

Case 7: $\mathbf{P}_{k+1}$ is justified by Biconditional Elimination:

h	**Q**			h	**Q**		
j	$\mathbf{Q} = \mathbf{P}_{k+1}$		or	j	$\mathbf{P}_{k+1} = \mathbf{Q}$		
$k + 1$	$\mathbf{P}_{k+1}$	h, j =E		$k + 1$	$\mathbf{P}_{k+1}$	h, j =E	

By the inductive hypothesis, $\Gamma_h \models \mathbf{Q}$ and Γ_j truth-functionally entails the biconditional at position **j**. Γ_h and Γ_j must be subsets of Γ_{k+1} if the sentences at positions **h** and **j** are accessible at position $\mathbf{k} + 1$. By 6.3.2, then, Γ_{k+1} truth-functionally entails both **Q** and the biconditional at position **j**. Because the sentence $\mathbf{P}_{k+1}$ must be true whenever both **Q** and the biconditional at position **j** are true, $\Gamma_{k+1} \models \mathbf{P}_{k+1}$ as well.

Case 8: $\mathbf{P}_{k+1}$ is justified by Conditional Introduction:

h	**Q**	
j	**R**	
$k + 1$	$\mathbf{Q} \supset \mathbf{R}$ $(= \mathbf{P}_{k+1})$	h-j $\supset$I

By the inductive hypothesis, $\Gamma_j \models \mathbf{R}$. Because the subderivation in which **R** is derived from **Q** is accessible at position $\mathbf{k} + 1$, every assumption that is open at position **j** is open at position $\mathbf{k} + 1$, except for the assumption **Q** that begins the subderivation. So the set of open assumptions Γ_j is a subset of $\Gamma_{k+1} \cup \{\mathbf{Q}\}$. Because $\Gamma_j \models \mathbf{R}$, it follows, by 6.3.2, that $\Gamma_{k+1} \cup \{\mathbf{Q}\} \models \mathbf{R}$. And from this it follows, by 6.3.3, that $\Gamma_{k+1} \models \mathbf{Q} \supset \mathbf{R}$.

Case 9: $\mathbf{P}_{k+1}$ is justified by Negation Introduction:

h	**Q**	
j	**R**	
m	$\sim \mathbf{R}$	
$k + 1$	$\sim \mathbf{Q}$ $(= \mathbf{P}_{k+1})$	h-m $\sim$I

By the inductive hypothesis, $\Gamma_j \models \mathbf{R}$ and $\Gamma_m \models \sim \mathbf{R}$. Because the subderivation that derives **R** from **Q** is accessible at position $\mathbf{k} + 1$, every

assumption that is open at position **j** is open at position **k** + 1 except for the assumption **Q** that begins the subderivation. That is, the set of open assumptions Γ_j is a subset of $\Gamma_{k+1} \cup \{\mathbf{Q}\}$. By similar reasoning Γ_m must be a subset of $\Gamma_{k+1} \cup \{\mathbf{Q}\}$. Therefore, by 6.3.2, $\Gamma_{k+1} \cup \{\mathbf{Q}\} \vDash \mathbf{R}$ and $\Gamma_{k+1} \cup \{\mathbf{Q}\} \vDash \sim \mathbf{R}$. From this it follows, by 6.3.4, that $\Gamma_{k+1} \cup \{\mathbf{Q}\}$ is truth-functionally inconsistent and then, by 6.3.5, that $\Gamma_{k+1} \vDash \sim \mathbf{Q}$.

Case 10: $\mathbf{P}_{k+1}$ is justified by Negation Elimination. See Exercise 3.

Case 11: $\mathbf{P}_{k+1}$ is justified by Disjunction Elimination:

h	**Q** ∨ **R**	
j	| **Q**	
	|----	
m	| $\mathbf{P}_{k+1}$	
n	| **R**	
	|----	
p	| $\mathbf{P}_{k+1}$	
k + 1	$\mathbf{P}_{k+1}$	**h, j–m, n–p** ∨E

By the inductive hypothesis, $\Gamma_h \vDash \mathbf{Q} \vee \mathbf{R}$, $\Gamma_m \vDash \mathbf{P}_{k+1}$, and $\Gamma_p \vDash \mathbf{P}_{k+1}$. Because the two subderivations are accessible at position **k** + 1, the open assumptions Γ_m form a subset of $\Gamma_{k+1} \cup \{\mathbf{Q}\}$ and the open assumptions Γ_p form a subset of $\Gamma_{k+1} \cup \{\mathbf{R}\}$. By 6.3.2, then, $\Gamma_{k+1} \cup \{\mathbf{Q}\} \vDash \mathbf{P}_{k+1}$ and $\Gamma_{k+1} \cup \{\mathbf{R}\} \vDash \mathbf{P}_{k+1}$. Moreover, because $\mathbf{Q} \vee \mathbf{R}$ at position **h** is accessible at position **k** + 1, Γ_h is a subset of Γ_{k+1}. So, because $\Gamma_h \vDash \mathbf{Q} \vee \mathbf{R}$, it follows, by 6.3.2, that $\Gamma_{k+1} \vDash \mathbf{Q} \vee \mathbf{R}$. Now consider any truth-value assignment on which every member of Γ_{k+1} is true. Because $\Gamma_{k+1} \vDash \mathbf{Q} \vee \mathbf{R}$, $\mathbf{Q} \vee \mathbf{R}$ is also true on this assignment. So either **Q** or **R** is true. If **Q** is true, then every member of $\Gamma_{k+1} \cup \{\mathbf{Q}\}$ is true and hence $\mathbf{P}_{k+1}$ is true as well because $\Gamma_{k+1} \cup \{\mathbf{Q}\} \vDash \mathbf{P}_{k+1}$. Similarly, if **R** is true, then every member of $\Gamma_{k+1} \cup \{\mathbf{R}\}$ is true, and hence $\mathbf{P}_{k+1}$ is true as well because $\Gamma_{k+1} \cup \{\mathbf{R}\} \vDash \mathbf{P}_{k+1}$. Either way, it follows that $\mathbf{P}_{k+1}$ must be true on any truth-value assignment on which every member of Γ_{k+1} is true. So $\Gamma_{k+1} \vDash \mathbf{P}_{k+1}$.

Case 12: $\mathbf{P}_{k+1}$ is justified by Biconditional Introduction:

h	| **Q**	
	|----	
j	| **R**	
m	| **R**	
	|----	
n	| **Q**	
k + 1	**Q** ≡ **R** (≡ $\mathbf{P}_{k+1}$)	**h–j, m–n** ≡I

By the inductive hypothesis $\Gamma_j \models \mathbf{R}$ and $\Gamma_n \models \mathbf{Q}$. Because the two subderivations are accessible at position $\mathbf{k} + 1$, Γ_j is a subset of $\Gamma_{k+1} \cup \{\mathbf{Q}\}$ and Γ_n is a subset of $\Gamma_{k+1} \cup \{\mathbf{R}\}$. By 6.3.2, then, $\Gamma_{k+1} \cup \{\mathbf{Q}\} \models \mathbf{R}$ and $\Gamma_{k+1} \cup \{\mathbf{R}\} \models \mathbf{Q}$. Now consider any truth-value assignment on which every member of Γ_{k+1} is true. If $\mathbf{R}$ is also true on that assignment, then so is $\mathbf{Q}$ because $\Gamma_{k+1} \cup \{\mathbf{R}\} \models \mathbf{Q}$. If $\mathbf{R}$ is false on that assignment, then $\mathbf{Q}$ must also be false—if $\mathbf{Q}$ were true, then $\mathbf{R}$ would have to be true as well because $\Gamma_{k+1} \cup \{\mathbf{Q}\} \models \mathbf{R}$. Either way, $\mathbf{Q}$ and $\mathbf{R}$ have the same truth-value, and so $\mathbf{Q} \equiv \mathbf{R}$ is true on every truth-value assignment on which every member of Γ_{k+1} is true. So $\Gamma_{k+1} \models \mathbf{P}_{k+1}$.

This completes the proof of the inductive step; we have considered every way in which the sentence at position $\mathbf{k} + 1$ of a derivation might be justified and have shown that in each case $\Gamma_{k+1} \models \mathbf{P}_{k+1}$ if the same is true of all earlier positions in the derivation. We have therefore established that the conclusion of the argument by mathematical induction is also true: For any position in a derivation, the sentence at that position is truth-functionally entailed by the set of open assumptions in whose scope it lies. In particular, this thesis is true of the last position in any derivation—the sentence that has been derived is truth-functionally entailed by the open assumptions in whose scope it lies, and these are the primary assumptions of the derivation. So the soundness metatheorem for *SD* has been established: If $\Gamma \vdash \mathbf{P}$ in *SD*, then $\Gamma \models \mathbf{P}$. It follows from Metatheorem 6.3.1 that every sentence of *SL* that is a theorem in *SD* is truth-functionally true and that every argument that is valid in *SD* is truth-functionally valid (see Exercise 20 in Section 5.4E).

6.3E EXERCISES

1. List all the subsets of each of the following sets:
a. {A ⊃ B, C ⊃ D}
b. {C ∨ ~ D, ~ D ∨ C, C ∨ C}
c. {(B & A) ≡ K}
d. ∅

2. Of which of the following sets is {A ⊃ B, C & D, D ⊃ A} a superset?
a. {A ⊃ B}
b. {D ⊃ A, A ⊃ B}
c. {A ⊃ D, C & D}
d. ∅
e. {C & D, D ⊃ A, A ⊃ B}

***3.** Prove Case 10 of the inductive step in the proof of Metatheorem 6.3.1.

4.

a. Suppose that system SD^a is just like SD except that it also contains a new rule of inference:

Negated Biconditional Introduction (~≡I)

$$\begin{array}{|l} \mathbf{P} \\ \\ \sim \mathbf{Q} \\ \\ \sim (\mathbf{P} \equiv \mathbf{Q}) \end{array}$$

Prove that system SD^a is a sound system for sentential logic; that is, prove that if $\Gamma \vdash \mathbf{P}$ in SD^a then $\Gamma \models \mathbf{P}$. (You may use Metatheorem 6.3.1.)

*b. Suppose that system SD^b is just like SD except that it also contains a new rule of inference:

Backward Conditional Introduction (B⊃I)

$$\begin{array}{|l} \quad \sim \mathbf{Q} \\ \quad \overline{\phantom{\sim \mathbf{Q}}} \\ \quad \sim \mathbf{P} \\ \mathbf{P} \supset \mathbf{Q} \end{array}$$

Prove that system SD^b is sound for sentential logic.

c. Suppose that system SD^c is just like SD except that it also contains a new rule of inference:

Crazy Disjunction Elimination (C∨E)

$$\begin{array}{|l} \mathbf{P} \vee \mathbf{Q} \\ \\ \mathbf{P} \\ \mathbf{Q} \end{array}$$

Prove that SD^c is not a sound system for sentential logic.

*d. Suppose that system SD^d is just like SD except that it also contains a new rule of inference:

Crazy Conditional Introduction (C⊃I)

$$\begin{array}{|l} \quad \sim \mathbf{P} \\ \quad \overline{\phantom{\sim \mathbf{P}}} \\ \quad \mathbf{Q} \\ \mathbf{P} \supset \mathbf{Q} \end{array}$$

Prove that SD^d is not a sound system for sentential logic.

e. Suppose that the rules of a system SD^e form a subset of the rules of SD. Is SD^e a sound system for sentential logic? Explain.

5. Suppose that in our semantics for *SL* the characteristic truth-table for '&' is

P	Q	P & Q
T	T	T
T	F	T
F	T	F
F	F	F

while the characteristic truth-tables for the other sentential connectives remain the same. Would *SD* still be a sound system for sentential logic? Explain.

6. Using Metatheorem 6.3.1 and Exercise 1.e in Section 6.1E, prove that *SD+* is sound for sentential logic.

6.4 THE COMPLETENESS OF *SD* AND *SD+*

We proved in the last section that derivations in *SD* never lead from true premises to a false conclusion, and so every derivation in *SD* is semantically acceptable. This fact alone does not establish that *SD* is an adequate natural deduction system for sentential logic. In addition, it is to be hoped that if an argument is truth-functionally valid then we can derive its conclusion from its premises in *SD* and that every sentence that is truth-functionally true can be derived in *SD* from the empty set—in short, that everything that we want to derive in *SD can* be derived in *SD*. If there is even one argument that is truth-functionally valid but for which no derivation can be constructed in *SD*, then *SD* is not adequate to sentential logic. Our final metatheorem assures us that we can derive all that we want to derive in *SD*; it is called the *Completeness Metatheorem:*

Metatheorem 6.4.1: If $\Gamma \models \mathbf{P}$, then $\Gamma \vdash \mathbf{P}$ in *SD*.

That is, if a set Γ truth-functionally entails a sentence **P**, then **P** may be derived from Γ in *SD*. It follows from this metatheorem that every argument of *SL* that is truth-functionally valid is valid in *SD* and that every sentence of *SL* that is truth-functionally true is a theorem in *SD* (see Exercise 20 in Section 5.4E). A system for which Metatheorem 6.4.1 holds is said to be *complete* for sentential logic.

There are several well-known ways to establish a completeness metatheorem. Some of these are said to be *constructive*—they show, for any set Γ and sentence **P** such that $\Gamma \models \mathbf{P}$, how to construct a derivation of **P** from Γ. We shall present a *nonconstructive* proof of completeness.[4] The proof will establish that, for every truth-functional entailment, there is at least one corresponding derivation in *SD*. It will not, however, show how to construct such a derivation.

[4] The method that we use to prove completeness is due to Leon Henkin, "The Completeness of the First-Order Functional Calculus," *Journal of Symbolic Logic*, 14 (1949), 159–166.

Before diving into the details of the proof, we shall give an overview of the proof's structure. The proof of Metatheorem 6.4.1 relies on several results that we present here; we defer the lengthy proof of 6.4.3 to the following pages.

> **6.4.2:** For any set of sentences Γ and any sentence $\mathbf{P}$, $\Gamma \models \mathbf{P}$ if and only if $\Gamma \cup \{\sim \mathbf{P}\}$ is truth-functionally inconsistent. (This follows from result 6.3.5 and Exercise 1.c in Section 3.6E.)
>
> **6.4.3:** (The *Inconsistency Lemma*): If a set Γ of sentences of *SL* is truth-functionally inconsistent, then Γ is also inconsistent in *SD*.
>
> **6.4.4:** For any set of sentences Γ and any sentence $\mathbf{P}$, $\Gamma \vdash \mathbf{P}$ in *SD* if and only if $\Gamma \cup \{\sim \mathbf{P}\}$ is inconsistent in *SD* (see Exercise 1).

Here is how the Completeness Theorem follows from these results. If the antecedent of Metatheorem 6.4.1, $\Gamma \models \mathbf{P}$, is true, it follows from 6.4.2 that $\Gamma \cup \{\sim \mathbf{P}\}$ is truth-functionally inconsistent, from the Inconsistency Lemma (6.4.3) that $\Gamma \cup \{\sim \mathbf{P}\}$ is inconsistent in *SD*, and from 6.4.4 that $\Gamma \vdash \mathbf{P}$ in *SD*.

We shall prove 6.4.3, the Inconsistency Lemma, by proving the following equivalent claim:

> If a set Γ of sentences of *SL* is consistent in *SD*, then Γ is also truth-functionally consistent.

(The claim is equivalent because any sentence of the form $\sim \mathbf{P} \supset \sim \mathbf{Q}$ is equivalent to the corresponding sentence of the form $\mathbf{Q} \supset \mathbf{P}$.) Our strategy will be to show, for any set Γ that is consistent in *SD*, how to construct a truth-value assignment on which every member of Γ is true. We shall construct the truth-value assignment in two steps. First, we shall form a superset of Γ (a set that includes all the members of Γ and possibly other sentences) that is *maximally consistent in SD*. A maximally consistent set is, intuitively, a consistent set that contains as many sentences as it can without being inconsistent in *SD*:

> A set Γ of sentences of *SL* is **maximally consistent in *SD*** if and only if Γ is consistent in *SD* and, for every sentence $\mathbf{P}$ of *SL* that is not a member of Γ, $\Gamma \cup \{\mathbf{P}\}$ is inconsistent in *SD*.

If a set is maximally consistent in *SD*, then if we add to the set any sentence that is not already a member, it will be possible to derive some sentence and its negation from the augmented set.

Having constructed a maximally consistent superset of Γ, we shall next construct a model for that maximally consistent superset, that is, a truth-value assignment on which every member of the maximally consistent superset is true. We use a superset of Γ that is maximally consistent in *SD*, rather than simply using the original set Γ, because there is a straightforward way to construct a model for a maximally consistent set. Of course, because every member of Γ will be in the maximally consistent superset, it follows that every member of Γ

will be true on the model that we have constructed and therefore that Γ is truth-functionally consistent.

So, to begin with, we need to establish that we can form a maximally consistent superset of any set of sentences Γ that is consistent in *SD*; that is, we need to prove this:

> **6.4.5** (The *Maximal Consistency Lemma*): If Γ is a set of sentences of *SL* that is consistent in *SD*, then Γ is a subset of at least one set of sentences that is maximally consistent in *SD*.

This lemma is important, for we are then going to show how to construct the desired truth-value assignment for a maximally consistent set. If there were any sets of sentences that were consistent in *SD* but that were not subsets of any set that is *maximally* consistent in *SD*, our construction would fail to show that the Inconsistency Lemma (6.4.3) is true of these sets. At most, we would be able to conclude that some sets of sentences that are consistent in *SD*—those that can be expanded to maximally consistent supersets—are also truth-functionally consistent.

In proving the Maximal Consistency Lemma (6.4.5), we shall make use of the fact that the sentences of *SL* can be *enumerated*, that is, placed in a definite order in one-to-one correspondence with the positive integers so that there is a sentence for each positive integer. Here is one method of enumerating the sentences of *SL*. First, we associate with each symbol of *SL* the two-digit numeral occurring to its right:

Symbol	*Numeral*	*Symbol*	*Numeral*
~	10	A	30
∨	11	B	31
&	12	C	32
⊃	13	D	33
≡	14	E	34
(	15	F	35
)	16	G	36
0	20	H	37
1	21	I	38
⋮	⋮	⋮	⋮
9	29	Z	55

(The ellipses mean that the next two-digit numeral is assigned to the next digit or letter of the alphabet.) Next we associate with each sentence of *SL*, atomic or molecular, the number designated by the numeral that consists of the numerals associated with the symbols in the sentence, in the order in which those symbols occur. For example, the numbers associated with the sentences

$$(A \vee C_2) \qquad \sim\sim(A \supset (B \,\&\, \sim C))$$

are, respectively,

153011322216 10101530131531121032l616

It is obvious that each sentence of *SL* will thus have a distinct number associated with it. Finally we enumerate all the sentences of *SL* by taking them in the order of their associated numbers: The first sentence in the enumeration is the sentence with the smallest associated number, the second sentence is the one with the next smallest associated number, and so on. In effect, we have imposed an alphabetical order on the sentences of *SL* so that we may freely talk of the first sentence of *SL* (which turns out to be 'A'—because only atomic sentences will have two-digit associated numbers, and the number for 'A' is the smallest of these), the second sentence of *SL* (which turns out to be 'B'), and so on.

We shall start with a set Γ of sentences that is consistent in *SD* (as provided for in the antecedent of the Maximal Consistency Lemma) and use our enumeration to construct a superset of Γ that is maximally consistent in *SD*. The construction considers in turn each sentence in the enumeration we have just described and adds it to the set if and only if the resulting set is consistent in *SD*. In the end the construction will have added as many sentences as can be added to the original set without producing a set that is inconsistent in *SD*. As the construction goes through the sentences of *SL*, deciding whether to add each sentence, it produces an infinite sequence $\Gamma_1, \Gamma_2, \Gamma_3, \ldots$ of sets of sentences of *SL*:

1. Γ_1 is Γ, the original set.
2. If $\mathbf{P}_i$ is the ith sentence in the enumeration, then Γ_{i+1} is $\Gamma_i \cup \{\mathbf{P}_i\}$ if $\Gamma_i \cup \{\mathbf{P}_i\}$ is consistent in *SD*; otherwise Γ_{i+1} is Γ_i.

As an example, if Γ_i is $\{\sim \mathrm{B}, \sim \mathrm{C} \vee \sim \mathrm{B}\}$ and $\mathbf{P}_i$ is 'A', then $\Gamma_i \cup \{\mathbf{P}_i\}$, which is $\{\sim \mathrm{B}, \sim \mathrm{C} \vee \sim \mathrm{B}, \mathrm{A}\}$, is consistent in *SD*. In this case Γ_{i+1} will be the expanded set $\Gamma_i \cup \{\mathbf{P}_i\}$. If Γ_i is $\{\mathrm{A}, \sim \mathrm{B}, \sim \mathrm{C} \vee \sim \mathrm{B}\}$ and $\mathbf{P}_i$ is 'B', then $\Gamma_i \cup \{\mathbf{P}_i\}$, which is $\{\mathrm{A}, \sim \mathrm{B}, \sim \mathrm{C} \vee \sim \mathrm{B}, \mathrm{B}\}$, is inconsistent in *SD* (this is readily verified). In this case $\mathbf{P}_i$ is not added to the set—Γ_{i+1} is merely the set $\{\mathrm{A}, \sim \mathrm{B}, \sim \mathrm{C} \vee \sim \mathrm{B}\}$.

Because we have an infinite sequence of sets, we cannot take the last member of the series as the maximally consistent set desired—there is no last member. Instead, we form a set Γ^* that is the union of all the sets in the series: Γ^* is defined to contain every sentence that is a member of at least one set in the series and no other sentences. Γ^* is a superset of Γ because it follows from the definition of Γ^* that every sentence in Γ_1 (as well as $\Gamma_2, \Gamma_3, \ldots$) is a member of Γ^*, and Γ_1 is the original set Γ.

Having formed the set Γ^*, it remains to be proved that Γ^* is consistent in *SD* and that it is *maximally* consistent in *SD*. To prove the first claim, we note that every set in the sequence $\Gamma_1, \Gamma_2, \Gamma_3, \ldots$ is consistent in *SD*. This is easily

established by mathematical induction:

> *Basis clause:* The first member of the sequence, Γ_1, is consistent in *SD*.
>
> **Proof:** Γ_1 is defined to be the original set Γ, which is consistent in *SD*.
>
> *Inductive step:* If every set in the sequence prior to Γ_{k+1} is consistent in *SD*, then Γ_{k+1} is consistent in *SD*.
>
> **Proof:** Γ_{k+1} was defined to be $\Gamma_k \cup \{\mathbf{P}_k\}$ if the latter set is consistent in *SD* and to be Γ_k otherwise. In the first case Γ_{k+1} is obviously consistent in *SD*. In the second case Γ_{k+1} is consistent because, by the inductive hypothesis, Γ_k is consistent in *SD*.
>
> ---
>
> *Conclusion:* Every member of the series $\Gamma_1, \Gamma_2, \Gamma_3, \ldots$ is consistent in *SD*.

Now suppose, contrary to what we wish to prove, that Γ^* is *inconsistent* in *SD*.

> **6.4.6:** If Γ is inconsistent in *SD*, then some finite subset of Γ is inconsistent in *SD* (see Exercise 2).

It follows that there is a finite subset Γ' of Γ^* that is inconsistent in *SD*. Γ' must be nonempty, for the empty set is consistent in *SD* (see Exercise 3). Moreover, because Γ' is finite, there is a sentence in Γ' that comes after all the other members of Γ' in our enumeration—call this sentence $\mathbf{P}_j$. (That is, any other member *of* Γ' is $\mathbf{P}_h$ for some $h < j$.) Then every member *of* Γ' is a member of Γ_{j+1}, by the way we constructed the series $\Gamma_1, \Gamma_2, \Gamma_3, \ldots$ (This is because we have constructed the sets in such a way that if a sentence that is the ith sentence in our enumeration is a member of *any* set in the sequence—and hence of Γ^*—it must be in the set Γ_{i+1} and every set thereafter. After the construction of Γ_{i+1}, the only sentences that are added are sentences at position $i + 1$ in the enumeration or later.) But if Γ' is inconsistent in *SD*, and every member of Γ' is a member of Γ_{j+1}, then Γ_{j+1} is inconsistent in *SD* as well, by 6.4.7:

> **6.4.7:** If Γ is inconsistent in *SD*, then every superset of Γ is inconsistent in *SD*.
>
> **Proof:** Assume that Γ is inconsistent in *SD*. Then for some sentence $\mathbf{P}$ there is a derivation of $\mathbf{P}$ in which all the primary assumptions are members of Γ, and also a derivation of $\sim \mathbf{P}$ in which all the primary assumptions are members of Γ. The primary assumptions of both derivations are members of every superset of Γ, so $\mathbf{P}$ and $\sim \mathbf{P}$ are both derivable from every superset of Γ. Therefore every superset of Γ is inconsistent in *SD*.

But we have already proved by mathematical induction that every set in the infinite sequence is consistent in *SD*. So Γ_{j+1} *cannot* be inconsistent in *SD*, and our supposition that led to this conclusion is wrong—we may conclude that Γ^* is consistent in *SD*.

It remains to be proved not only that Γ^* is consistent in *SD* but that it is, in addition, *maximally* consistent. Suppose that Γ^* is not maximally consistent in *SD*. Then there is at least one sentence $\mathbf{P}_k$ of *SL* that is not a member of Γ^* and is such that $\Gamma^* \cup \{\mathbf{P}_k\}$ is consistent in *SD*. We showed, in 6.4.7, that every superset of a set that is inconsistent in *SD* is itself inconsistent, so every subset of a set that is *consistent* in *SD* must itself be consistent in *SD*. In particular, the subset $\Gamma_k \cup \{\mathbf{P}_k\}$ of $\Gamma^* \cup \{\mathbf{P}_k\}$ must be consistent in *SD*. But then, by step 2 of the construction of the sequence of sets, Γ_{k+1} is defined to be $\Gamma_k \cup \{\mathbf{P}_k\}$—$\mathbf{P}_k$ is a member of Γ_{k+1}. $\mathbf{P}_k$ is therefore a member of Γ^*, contradicting our supposition that it is not a member of Γ^*. Therefore Γ^* must be maximally consistent in *SD*—every sentence that can be consistently added to Γ^* is already a member of Γ^*. This and the result of the previous paragraph establish the Maximal Consistency Lemma (6.4.5); we have shown that, given any set of sentences that is consistent in *SD*, we can construct a superset that is maximally consistent in *SD*.

Finally, we will show that we can construct a model for every set that is maximally consistent in *SD*. From this we will have the following:

> **6.4.8** (the *Consistency Lemma*): Every set of sentences of *SL* that is maximally consistent in *SD* is truth-functionally consistent.

In establishing the Consistency Lemma, we shall appeal to the following important facts about sets that are maximally consistent in *SD*:

> **6.4.9:** If $\Gamma \vdash \mathbf{P}$ and Γ^* is a maximally consistent superset of Γ, then $\mathbf{P}$ is a member of Γ^*.
>
> **Proof:** Assume that $\Gamma \vdash \mathbf{P}$ and let Γ^* be a maximally consistent superset of Γ. By the definition of derivability in *SD*, $\Gamma^* \vdash \mathbf{P}$ as well. Now suppose, contrary to what we wish to prove, that $\mathbf{P}$ is *not* a member of Γ^*. Then, by the definition of maximal consistency, $\Gamma^* \cup \{\mathbf{P}\}$ is inconsistent in *SD*. Therefore by
>
> **6.4.10:** If $\Gamma \cup \{\mathbf{P}\}$ is inconsistent in *SD*, then $\Gamma \vdash \sim \mathbf{P}$ (see Exercise 1)
>
> it follows that $\Gamma^* \vdash \sim \mathbf{P}$. But then, because both $\mathbf{P}$ and $\sim \mathbf{P}$ are derivable in *SD* from Γ^*, it follows that Γ^* is inconsistent in *SD*. But this is impossible if Γ is maximally consistent in *SD*. We conclude that our supposition about $\mathbf{P}$, that it is not a member of Γ^*, is wrong—$\mathbf{P}$ *is* a member of Γ^*.

In what follows, we will use the standard notation

$$\mathbf{P} \notin \Gamma$$

to mean

P is a member of Γ

and the standard notation

$\mathbf{P} \notin \Gamma$

to mean

P is not a member of Γ.

The next result concerns the composition of the membership of any set that is maximally consistent in *SD*:

6.4.11: If Γ^* is maximally consistent in *SD* and **P** and **Q** are sentences of *SL*, then:

a. $\sim \mathbf{P} \in \Gamma^*$ if and only if $\mathbf{P} \notin \Gamma^*$.
b. $\mathbf{P} \,\&\, \mathbf{Q} \in \Gamma^*$ if and only if both $\mathbf{P} \in \Gamma^*$ and $\mathbf{Q} \in \Gamma^*$.
c. $\mathbf{P} \vee \mathbf{Q} \in \Gamma^*$ if and only if either $\mathbf{P} \in \Gamma^*$ or $\mathbf{Q} \in \Gamma^*$.
d. $\mathbf{P} \supset \mathbf{Q} \in \Gamma^*$ if and only if either $\mathbf{P} \notin \Gamma^*$ or $\mathbf{Q} \in \Gamma^*$.
e. $\mathbf{P} \equiv \mathbf{Q} \in \Gamma^*$ if and only if either $\mathbf{P} \in \Gamma^*$ and $\mathbf{Q} \in \Gamma^*$, or $\mathbf{P} \notin \Gamma^*$ and $\mathbf{Q} \notin \Gamma^*$.

Proof of (a): Assume that $\sim \mathbf{P} \in \Gamma^*$. Then $\mathbf{P} \notin \Gamma^*$ for, if it were a member, then Γ^* would have a finite subset that is inconsistent in *SD*, namely, $\{\mathbf{P}, \sim \mathbf{P}\}$, and according to 6.4.7 this is impossible if Γ^* is consistent in *SD*. Now assume that $\mathbf{P} \notin \Gamma^*$. Then, by the definition of maximal consistency in *SD*, $\Gamma^* \cup \{\mathbf{P}\}$ is inconsistent in *SD*. So, by reasoning similar to that used in proving 6.4.9, some finite subset Γ' of Γ^* is such that $\Gamma' \cup \{\mathbf{P}\}$ is inconsistent in *SD*, and therefore such that $\Gamma' \cup \{\sim \sim \mathbf{P}\}$ is inconsistent in *SD* and hence that $\Gamma' \vdash \sim \mathbf{P}$, by 6.4.4. It follows, by 6.4.9, that $\sim \mathbf{P} \in \Gamma^*$.

Proof of (b): Assume that $\mathbf{P} \,\&\, \mathbf{Q} \in \Gamma^*$. Then $\{\mathbf{P} \,\&\, \mathbf{Q}\}$ is a subset of Γ^*. Because $\{\mathbf{P} \,\&\, \mathbf{Q}\} \vdash \mathbf{P}$ and $\{\mathbf{P} \,\&\, \mathbf{Q}\} \vdash \mathbf{Q}$ (both by Conjunction Elimination), it follows, by 6.4.9, that $\mathbf{P} \in \Gamma^*$ and $\mathbf{Q} \in \Gamma^*$. Now suppose that $\mathbf{P} \in \Gamma^*$ and $\mathbf{Q} \in \Gamma^*$. Then $\{\mathbf{P}, \mathbf{Q}\}$ is a subset of Γ^* and, because $\{\mathbf{P}, \mathbf{Q}\} \vdash \mathbf{P} \,\&\, \mathbf{Q}$ (by Conjunction Introduction), it follows, by 6.4.9, that $\mathbf{P} \,\&\, \mathbf{Q} \in \Gamma^*$.

Proof of (c): See Exercise 5.

Proof of (d): Assume that $\mathbf{P} \supset \mathbf{Q} \in \Gamma^*$. If $\mathbf{P} \notin \Gamma^*$, then it follows trivially that either $\mathbf{P} \notin \Gamma^*$ or $\mathbf{Q} \in \Gamma^*$. If $\mathbf{P} \in \Gamma^*$, then $\{\mathbf{P}, \mathbf{P} \supset \mathbf{Q}\}$ is a subset of Γ^*. Because $\{\mathbf{P}, \mathbf{P} \supset \mathbf{Q}\} \vdash \mathbf{Q}$ (by Conditional Elimination), it follows,

by 6.4.9, that $\mathbf{Q} \in \Gamma^*$. So, if $\mathbf{P} \supset \mathbf{Q} \in \Gamma^*$, then either $\mathbf{P} \notin \Gamma^*$ or $\mathbf{Q} \in \Gamma^*$. Now assume that either $\mathbf{P} \notin \Gamma^*$ or $\mathbf{Q} \in \Gamma^*$. In the former case, by (a), $\sim \mathbf{P} \in \Gamma^*$. So either $\{\sim \mathbf{P}\}$ is a subset of Γ^* or $\{\mathbf{Q}\}$ is a subset of Γ^*. $\mathbf{P} \supset \mathbf{Q}$ is derivable from either subset:

1	$\sim \mathbf{P}$	Assumption
2	$\quad \mathbf{P}$	A / $\sim$ E
3	$\quad\quad \sim \mathbf{Q}$	A / $\sim$ E
4	$\quad\quad \mathbf{P}$	2 R
5	$\quad\quad \sim \mathbf{P}$	1 R
6	$\quad \mathbf{Q}$	3–5 $\sim$ E
7	$\mathbf{P} \supset \mathbf{Q}$	2–6 $\supset$I

1	$\mathbf{Q}$	Assumption
2	$\quad \mathbf{P}$	A / $\supset$I
3	$\quad \mathbf{Q}$	1 R
4	$\mathbf{P} \supset \mathbf{Q}$	2–3 $\supset$I

Either way, there is a finite subset of Γ^* from which $\mathbf{P} \supset \mathbf{Q}$ is derivable; so, by 6.4.9, it follows that $\mathbf{P} \supset \mathbf{Q} \in \Gamma^*$.
Proof of (e): See Exercise 5.

Turning now to the Consistency Lemma (6.4.8), let Γ be a set of sentences that is maximally consistent in *SD*. We said earlier that it is easy to construct a model for a maximally consistent set, and it is; we need only consider the atomic sentences in the set. Let $\mathbf{A}^*$ be the truth-value assignment that assigns the truth-value **T** to every atomic sentence of *SL* that is a member of Γ^* and assigns the truth-value **F** to every other atomic sentence of *SL*. We shall prove by mathematical induction that each sentence of *SL* is true on the truth-value assignment $\mathbf{A}^*$ if and only if it is a member of Γ^*—from which it follows that every member of Γ^* is true on $\mathbf{A}^*$, thus establishing truth-functional consistency. The induction will be based on the number of occurrences of connectives in the sentences of *SL*:

Basis clause: Each atomic sentence of *SL* is true on $\mathbf{A}^*$ if and only if it is a member of Γ^*.

Inductive step: If every sentence of *SL* with $\mathbf{k}$ or fewer occurrences of connectives is such that it is true on $\mathbf{A}^*$ if and only if it is a member of Γ^*, then every sentence of *SL* with $\mathbf{k} + 1$ occurrences of connectives is such that it is true on $\mathbf{A}^*$ if and only if it is a member of Γ^*.

Conclusion: Every sentence of *SL* is such that it is true on $\mathbf{A}^*$ if and only if it is a member of Γ^*.

The basis clause is obviously true; we defined $\mathbf{A}^*$ to be an assignment that assigns **T** to all and only the atomic sentences of *SL* that are members of Γ^*. To prove the inductive step, we will assume that the inductive hypothesis holds for an arbitrary integer $\mathbf{k}$: That each sentence containing $\mathbf{k}$ or fewer occurrences of connectives is true on $\mathbf{A}^*$ if and only if it is a member of Γ^*.

We must now show that the same holds true for every sentence **P** containing **k** + 1 occurrences of connectives. We consider five cases, reflecting the five forms that a molecular sentence of *SL* might have.

Case 1: **P** has the form $\sim$**Q**. If $\sim$**Q** is true on $\mathbf{A}^*$, then **Q** is false on $\mathbf{A}^*$. Because **Q** contains fewer than **k** + 1 occurrences of connectives, it follows by the inductive hypothesis that $\mathbf{Q} \notin \Gamma^*$. Therefore, by 6.4.11(a), $\sim\mathbf{Q} \in \Gamma^*$. If $\sim$**Q** is false on $\mathbf{A}^*$, then **Q** is true on $\mathbf{A}^*$. It follows by the inductive hypothesis that $\mathbf{Q} \in \Gamma^*$. Therefore, by 6.4.11(a), $\sim\mathbf{Q} \notin \Gamma^*$.

Case 2: **P** has the form **Q** & **R**. If **Q** & **R** is true on $\mathbf{A}^*$, then both **Q** and **R** are true on $\mathbf{A}^*$. Because **Q** and **R** each contain fewer than **k** + 1 occurrences of connectives, it follows by the inductive hypothesis that $\mathbf{Q} \in \Gamma^*$ and $\mathbf{R} \in \Gamma^*$. Therefore, by 6.4.11(b), $\mathbf{Q} \,\&\, \mathbf{R} \in \Gamma^*$. If **Q** & **R** is false on $\mathbf{A}^*$, then either **Q** is false on $\mathbf{A}^*$ or **R** is false on $\mathbf{A}^*$. Therefore, by the inductive hypothesis, either $\mathbf{Q} \notin \Gamma^*$ or $\mathbf{R} \notin \Gamma^*$ and so, by 6.4.11(b), $\mathbf{Q} \,\&\, \mathbf{R} \notin \Gamma^*$.

Case 3: **P** has the form $\mathbf{Q} \vee \mathbf{R}$. See Exercise 6.

Case 4: **P** has the form $\mathbf{Q} \supset \mathbf{R}$. If $\mathbf{Q} \supset \mathbf{R}$ is true on $\mathbf{A}^*$, then either **Q** is false on $\mathbf{A}^*$ or **R** is true on $\mathbf{A}^*$. Because **Q** and **R** each contain fewer than **k** + 1 occurrences of connectives, it follows from the inductive hypothesis that either $\mathbf{Q} \notin \Gamma^*$ or $\mathbf{R} \in \Gamma^*$. By 6.4.11(d), then, $\mathbf{Q} \supset \mathbf{R} \in \Gamma^*$. If $\mathbf{Q} \supset \mathbf{R}$ is false on $\mathbf{A}^*$, then **Q** is true on $\mathbf{A}^*$ and **R** is false on $\mathbf{A}^*$. By the inductive hypothesis, then, $\mathbf{Q} \in \Gamma^*$ and $\mathbf{R} \notin \Gamma^*$. And by 6.4.11(d), it follows that $\mathbf{Q} \supset \mathbf{R} \notin \Gamma^*$.

Case 5: See Exercise 6.

This completes the proof of the inductive step. Hence we may conclude that each sentence of *SL* is such that it is a member of Γ^* if and only if it is true on $\mathbf{A}^*$. So every member of a set Γ^* that is maximally consistent in *SD* is true on $\mathbf{A}^*$, and the set Γ^* is therefore truth-functionally consistent. This establishes the Consistency Lemma (6.4.8).

We now know that the Inconsistency Lemma (6.4.3) is true. Because every set of sentences Γ that is consistent in *SD* is a subset of a set of sentences that is maximally consistent in *SD* (the Maximal Consistency Lemma (6.4.5)), and because every set of sentences that is maximally consistent in *SD* is truth-functionally consistent (the Consistency Lemma (6.4.8)), it follows that every set of sentences that is consistent in *SD* is a subset of a truth-functionally consistent set and is therefore itself truth-functionally consistent. So, if a set is truth-functionally *inconsistent*, it must be inconsistent in *SD*.

It now follows that Metatheorem 6.4.1:

If $\Gamma \models \mathbf{P}$, then $\Gamma \vdash \mathbf{P}$

is true. For if $\Gamma \models \mathbf{P}$, then, by 6.4.2, $\Gamma \cup \{\sim\mathbf{P}\}$ is truth-functionally inconsistent. Then, by the Inconsistency Lemma (6.4.3), $\Gamma \cup \{\sim\mathbf{P}\}$ is inconsistent in *SD*. And

if $\Gamma \cup \{\sim \mathbf{P}\}$ is inconsistent in *SD*, then, by 6.4.4, $\Gamma \vdash \mathbf{P}$ in *SD*. So *SD* is complete for sentential logic—for every truth-functional entailment there is at least one corresponding derivation that can be constructed in *SD*. This, together with the proof of the Soundness Metatheorem in Section 6.3, shows that *SD* is an adequate system for sentential logic.

We conclude by noting that another important result, the Compactness Theorem for sentential logic, follows from the Inconsistency Lemma (6.4.3) and Metatheorem 6.3.1:

Metatheorem 6.4.12: A set Γ of sentences of *SL* is truth-functionally consistent if and only if every finite subset of Γ is truth-functionally consistent.

And, as a consequence, a set of sentences of *SL* is truth-functionally inconsistent if and only if at least one finite subset of Γ is inconsistent.

6.4E EXERCISES

1. Prove 6.4.4 and 6.4.10.

2. Prove 6.4.6.

***3.** Prove that the empty set is consistent in *SD*.

4. Using Metatheorem 6.4.1, prove that *SD*+ is complete for sentential logic.

***5.** Prove that every set that is maximally consistent in *SD* has the following properties:
c. $\mathbf{P} \vee \mathbf{Q} \in \Gamma^*$ if and only if either $\mathbf{P} \in \Gamma^*$ or $\mathbf{Q} \in \Gamma^*$.
e. $\mathbf{P} \equiv \mathbf{Q} \in \Gamma^*$ if and only if either $\mathbf{P} \in \Gamma^*$ and $\mathbf{Q} \in \Gamma^*$, or $\mathbf{P} \notin \Gamma^*$ and $\mathbf{Q} \notin \Gamma^*$.

***6.** Establish Cases 3 and 5 of the inductive step in the proof of the Consistency Lemma 6.4.8.

7.a. Suppose that SD^a is like *SD* except that it lacks Reiteration. Show that SD^a is complete for sentential logic.
***b.** Suppose that SD^b is like *SD* except that it lacks Negation Introduction. Show that SD^b is complete for sentential logic.

8. Suppose that SD^c is like *SD* except that it lacks Conjunction Elimination. Show where our completeness proof for *SD* will fail as a completeness proof for SD^c.

9. Using the Inconsistency Lemma 6.4.3 and Metatheorem 6.3.1, prove Metatheorem 6.4.12.

Chapter 7

PREDICATE LOGIC: SYMBOLIZATION AND SYNTAX

7.1 THE LIMITATIONS OF *SL*

In Chapter 2 we introduced the language *SL* and techniques for symbolizing sentences of English in *SL*. In subsequent chapters we presented various semantic and syntactic methods for testing sentences and sets of sentences of *SL* for the logical properties defined for that language, including both semantic and syntactic versions of deductive validity, logical consistency, and logical truth (the former defined in terms of truth-value assignments, the latter in terms of derivability). *SL* has a number of virtues. For example, any English language argument that has an acceptable symbolization in *SL* that is truth-functionally valid is itself deductively valid. So, too, a sentence of English that can be fairly symbolized as a truth-functionally true (or truth-functionally false) sentence of *SL* is itself logically true (or logically false); similarly for equivalence and entailment. And if a set of English sentences can be fairly symbolized as a set of sentences of *SL* that is truth-functionally *inconsistent*, then that set of English sentences is itself logically *inconsistent*.

A further advantage is that two of the test procedures we developed in conjunction with the language *SL*—that based on truth-tables and that based on truth-trees—can readily be made into mechanical procedures. (A procedure is "mechanical" in this sense if each step is dictated by some rule, given prior

steps. Thus a procedure for which a computer program can be written is a mechanical procedure.) Some mechanical procedures have no stopping point: that is, once started, they run on indefinitely. For example, if someone is hired to scrape and paint a bridge, told to start at the west end of the bridge and, upon reaching the east bank, to go back and start over on the west bank, we have a procedure that will go on indefinitely. The continuous scraping and painting will stop when the painter quits or retires, the bridge is removed or abandoned, the supply of paint runs out, or whatever. But the parameters established by the instructions do not determine or even envision a stopping point. The mechanical procedures based on truth-tables and truth-trees for *SL* are not of this sort—they always come to a stop after a finite number of steps. Moreover, when they stop, they always provide either a "yes" or a "no" answer to the question being asked (for instance, 'Is this argument truth-functionally valid?' or 'Is this sentence truth-functionally true?'). The semantic properties of consistency, truth-functional truth, truth-functional falsity, truth-functional indeterminacy, truth-functional equivalence, truth-functional validity, and truth-functional entailment are termed *decidable* properties precisely because there are mechanical test procedures for these properties, procedures that always terminate after a finite number of steps and always yield either a "yes" or "no" answer to the question being asked ('Does this semantic property hold of this sentence, or argument, or pair of sentences, or set of sentences?').

One of the goals of formal logic is to develop tools that allow us to understand (and test for the holding of) various logical properties of sentences and sets of sentences *of natural languages*. Until well into the twentieth century many, if not most, logicians assumed that the way to meet this goal was to develop formal languages in which all natural language discourse, or at least all "important" discourse (for example, mathematics and physics), could be represented and then to develop test procedures for these formal languages. It was expected that the test procedures would be such that each of the logical properties defined for a formal system would be decidable in the above sense.

So *SL* has at least these two advantages: There are decidable test procedures associated with it, and at least some of the results of these tests can be carried over to English. (For example, again, if a fair symbolization of an English language argument is found to be truth-functionally valid, we may conclude that the English language argument is deductively valid.) But not all test results obtained for arguments, sentences, and sets of sentences of *SL* can be carried back to the English arguments, sentences, and sets of sentences from which they were derived. Specifically *it does not follow* from the fact that the most appropriate symbolization of an English language argument is truth-functionally invalid that the original English argument is invalid. If a sentence of *SL* is not truth-functionally true, *it does not follow* that the English sentence it symbolizes is not logically true. If a sentence of *SL* is not truth-functionally false, *it does not follow* that the English sentence it symbolizes is not logically false; so, too, for equivalence and entailment. And if a set of sentences of *SL* is truth-functionally consistent, *it does not follow* that the set of English sentences we are trying to evaluate is logically consistent.

The problem is not that we have no test for determining when a sentence of *SL* constitutes a "fair" or "most appropriate" symbolization of a sentence of English, although it is true that we do not have such a test. The problem is rather that the language *SL* is itself not sophisticated enough to allow adequate symbolization of a great deal of natural language discourse. Put another way, even the most appropriate symbolization of an English sentence by a sentence of *SL* frequently fails to capture much of the content of the English sentence. This is so because the syntactic structure of English, and of every natural language, is much more complex than the structure of purely truth-functional languages such as *SL*. No truth-functional language can mirror all or even all the important semantic relationships that hold among sentences and parts of sentences of natural languages. For example, while the sentence

> Each citizen will either vote or pay a fine

might form part of a recommendation for rather dramatic reforms in our political system, the sentence

> Each citizen either will vote or will not vote

is not similarly controversial. Rather, it smacks of being a logical truth. Each citizen—for example, Cynthia—obviously either will vote or will not vote. Indeed, the claim about Cynthia, or any other specified citizen, can be symbolized as a truth-functional truth of *SL*. Where 'C' abbreviates 'Cynthia will vote', 'C ∨ – C' says of Cynthia what the general claim says of each citizen. But there is, barring heroic measures, no symbolization of the general claim in *SL* that is truth-functionally true.[1]

Similarly the following argument should strike the reader as being deductively valid, although it has no symbolization in *SL* that is truth-functionally valid:

> None of David's friends supports Republicans. Sarah supports Breitlow, and Breitlow is a Republican. So Sarah is no friend of David's.

One attempt at a symbolization of this argument in *SL* is

N
S & B
———
– F

Here 'S' abbreviates 'Sarah supports Breitlow', 'B' abbreviates 'Breitlow is a Republican', and 'F' abbreviates 'Sarah is a friend of David's'. 'None of David's friends supports Republicans' is treated as an atomic sentence and symbolized

[1] Since there are presumably only finitely many citizens, we could construct a very extended conjunction with as many conjuncts of the sort 'C ∨ – C' as there are citizens. But even such heroic measures fail when the items about which we wish to talk (for example, the positive integers) constitute an infinite, and not just an exceedingly large, set. See Section 7.4.

as 'N'. This argument of *SL* is truth-functionally invalid. We could have treated 'None of David's friends supports Republicans' as the negation of 'Some of David's friends support Republicans' and symbolized it as '~ D', but the result would still be truth-functionally invalid.

The problem is that we cannot show, via the syntax of *SL*, that there is a relation between supporting Breitlow, Breitlow's being a Republican, and supporting Republicans. This is because *SL*, in taking sentences to be the smallest linguistic units (other than sentential connectives), makes all *subsentential* relationships invisible.

In this chapter we shall develop a new language, *PL* (for predicate logic) that will allow us to express many subsentential relationships.[2] It will turn out that the preceding argument has a valid symbolization in *PL* and that 'Each citizen will either vote or not vote' has a symbolization in *PL* that is logically true. However, it will also turn out that *PL* and its associated test procedures do not constitute a *decidable* system. That is, there is no mechanical test procedure that always yields, in a finite number of steps, a "yes" or a "no" answer to such a question as 'Is this argument of *PL* valid?' In fact, we now know that there can be no formal system that is both decidable and powerful enough to allow the expression of even moderately complex natural language discourse, including the claims of mathematics and physics.[3] So, while in moving from *SL* to *PL* we gain expressive power and are able, for example, to demonstrate the validity of a wider range of English arguments, we lose decidability.

7.2 PREDICATES, INDIVIDUAL CONSTANTS, AND QUANTITY TERMS OF ENGLISH

A distinction between singular terms and predicates is central to understanding the subsentential structure of English discourse. A **singular term** is any word or phrase that designates or purports to designate (or denote or refer to) some one thing. Singular terms are of two sorts: proper names and definite descriptions. Examples of proper names include 'George Washington', 'Marie Curie', 'Sherlock Holmes', 'Rhoda', and 'Henry'. Generally speaking, proper names are attached to the things they name by simple convention. Definite descriptions—for example, 'the discoverer of radium', 'the person Henry is talking to', 'Mary's best friend', and 'James' only brother'—on the other hand, pick out or purport to pick out a thing by providing a unique description of that thing. A definite description is a description that, by its grammatical structure, purports to describe exactly one thing. Thus 'James' only brother' is a definite description whereas 'James' brother' is not—the latter could accurately describe many persons whereas the former can describe at most one.

[2] There are, as one might expect, arguments that are deductively valid but whose symbolizations in *PL* are not valid, sentences that are logically true but whose symbolizations in *PL* do not exhibit this, and so on. To deal with natural language discourse that cannot be represented in *PL*, even more powerful formal systems are available—for example, tense logic and modal logic. A discussion of these systems is beyond the scope of this text.

[3] See Section 8.2 for further discussion of this point.

In English not every singular term designates. For example, in its normal use 'Sherlock Holmes' fails to designate because there is no such person as Sherlock Holmes. Similarly a definite description fails to designate if nothing satisfies—that is, if nothing is uniquely described by—that description. Both 'the largest prime number' and 'the present prime minister of the United States' are definite descriptions that for this reason fail to designate.

What thing, if any, a name or definite description designates clearly depends upon the context of use. In its most familiar use 'George Washington' designates the first president of the United States. But a historian may have named her dog after the first president, and if so there will be contexts in which she and her friends use the term 'George Washington' to designate a dog, not a figure from American history. In the same way 'the person Henry is talking to' may designate one person on one occasion, another on another occasion, and (Henry being a taciturn fellow) very often no one at all. Hereafter, when we use a sentence of English as an example or in an exercise set, we shall, unless otherwise noted, be assuming that a context is available for that sentence such that in that context all the singular terms in the sentence do designate. Moreover, when we are working with a group of sentences, the context that is assumed must be the same for all the sentences in the group. That is, we assume that a singular term that occurs several times in the piece of English discourse under discussion designates the same thing in each of its occurrences.

In English pronouns are often used in place of proper names and definite descriptions. When they are so used, their references are determined by the proper names or definite descriptions for which they substitute. For example, in the most straightforward reading of the conditional

> If Sue has read Darwin, then she's no creationist

the reference of 'she' is established by the use of 'Sue' in the antecedent of that conditional. So it is clearly appropriate to paraphrase this sentence as

> If Sue has read Darwin, then Sue's no creationist.

But not every pronoun can be replaced by a singular term. Replacing 'her or his' in

> This test is so easy that if anyone fails the test, then it's her or his own fault

with a singular term, *any* singular term, creates a nonequivalent sentence, as in this example:

> This test is so easy that if anyone fails the test, then it's Cynthia's own fault.

The former claim places responsibility for failure on the test taker, the latter places it, no matter who the test taker is, on Cynthia (suggesting, perhaps, that

Cynthia is the instructor). (We shall return to uses of pronouns that cannot be replaced with singular terms shortly.)

Obviously a sentence can contain more than one singular term. For example,

New York is between Philadelphia and Boston

contains three singular terms: 'New York', 'Philadelphia', and 'Boston'.[4] **Predicates** of English are parts of English sentences that can be obtained by deleting one or more singular terms from an English sentence. Alternatively a predicate is a string of words with one or more holes or blanks in it such that when the holes are filled with singular terms, a sentence of English results. From the preceding example all the following predicates can be obtained:

___ is between Philadelphia and Boston.
New York is between ___ and Boston.
New York is between Philadelphia and ___.
___ is between ___ and Boston.
___ is between Philadelphia and ___.
New York is between ___ and ___.
___ is between ___ and ___.

A predicate with just one blank is a one-place predicate. A predicate with more than one blank is a many-place predicate. (A predicate with exactly two blanks is a two-place predicate, a predicate with exactly three blanks is a three-place predicate, and so on. Generally, where **n** is a positive integer, a predicate with **n** blanks is an **n-place predicate.**)

One way of generating a sentence from a predicate is to fill the blanks in the predicate with singular terms. Any singular term can be put in any blank, and the same singular term can be put in more than one blank. So, from the two-place predicate '___ works for ___' and the singular terms 'Pat', 'Tom', '3M', 'IBM', and 'the smallest prime number', we can generate the following sentences:

Tom works for 3M.
Pat works for 3M.
Tom works for Pat.
Pat works for Tom.
Pat works for Pat.
3M works for Tom.
IBM works for 3M.
The smallest prime number works for IBM.

---

[4] We are here concerned only with isolating singular terms that do not occur as constituents of other singular terms. That is, we here take

The Roman general who defeated Pompey invaded both Gaul and Germany

to contain just three singular terms: 'The Roman general who defeated Pompey', 'Gaul', and 'Germany'. In Section 7.9 we shall introduce techniques that allow us to recognize and symbolize singular terms that are themselves constituents of singular terms—for example, 'Pompey' as it occurs in 'The Roman general who defeated Pompey'.

And so on. Note that all of these are sentences of English by the standard grammatical rules of English. When a sentence that consists of an **n**-place predicate with the blanks filled with **n** singular terms is true, we say that that predicate is true of the **n** things designated by those **n** singular terms. As it happens, '___ works for ___' is true of the pair consisting of Tom and 3M but false of the pair consisting of 3M and Tom (pairs, and triples, and so on, have an order built in). That is, the Tom we have in mind does work for 3M, but 3M does not work for Tom.

It may be objected that not all the preceding sentences "make sense"—what would it mean for the smallest prime number—2—to work for anything or anyone? One approach here would be to declare such sentences as the last listed to be semantically deviant and therefore not candidates for truth, that is, neither true nor false. We, however, take the simpler approach of counting such sentences as meaningful but false. After all, on any normal understanding, the smallest prime number is not the sort of thing that works for anyone or anything, so the claim that it works for IBM is false. (The predicate '___ works for ___' is not true of the pair consisting of the smallest prime number and IBM.) By this move we will gain an overall simplicity and generality when we come to develop the formal syntax and semantics for *PL*.

So far, in displaying predicates, we have been marking the blanks into which singular terms can be placed with underscores. It is time to adopt a more standard notation. Hereafter, in displaying predicates, we shall use the lowercase letters 'w', 'x', 'y', and 'z' (with numerical subscripts where necessary) to mark the blanks in those predicates. (These are, as we shall see, the **variables** of *PL*.) Using this convention, the three-place predicate of English discussed earlier can be displayed as

x is between y and z

(or as 'w is between x and y', or 'z is between x and y', and so on). We must use distinct variables to replace the different occurrences of singular terms, but which variables are used is immaterial.

Given a stock of predicates, singular terms, and the sentential connectives 'and', 'or', 'if . . . then . . .', 'if and only if', and 'not', we can generate a wide variety of sentences of English. For example, from the just enumerated sentential connectives, the singular terms 'Henry', 'Sue', 'Rita', and 'Michael' and the predicates 'x is easygoing', 'x likes y', and 'x is taller than y,' we can generate

Michael is easygoing.
Sue is easygoing.
Michael is taller than Sue and Sue is taller than Henry.
Sue likes Henry and Michael likes Rita.
If Rita likes Henry, then Rita is taller than Henry.
If Michael is easygoing, then Rita isn't easygoing.

But we cannot, with these limited resources, generate such simple but powerful claims as

Everyone is easygoing.
No one is easygoing.
Someone is easygoing.
Someone is not easygoing.
Michael likes everyone.
Michael does not like anyone.
Michael doesn't like everyone.
Someone likes Sue.
No one is taller than her or himself.

What is missing is an account of such "quantity" terms as 'every', 'all', 'each', 'some', and 'none'.

The first thing to note is that quantity terms are not singular terms. 'Everyone' is neither a proper name nor a definite description—there is no thing that is either named or described by the term 'everyone'. So, too, for 'everything', 'no one', 'nothing', 'anyone', 'anything', 'someone', and 'something'. These and other quantity terms serve to indicate *how many* of the persons or things under discussion are thus-and-so, not to name or refer to some single entity.

Consider the simple claim 'Someone is easygoing'. We can see this sentence as being composed of the one-place predicate 'x is easygoing' and the expression 'someone'. If this claim is true, then there is some person who is easygoing, that is, someone of whom the predicate 'x is easygoing' is true. But his or her name is not 'someone', nor is 'someone' a description of that person.

Similarly,

Everyone is easygoing

is true if and only if 'x is easygoing' is true of each and every person,

No one is easygoing

is true if and only if there is no person of whom the predicate 'x is easygoing' is true, and

Someone is not easygoing

is true if and only if there is at least one person of whom 'x is easygoing' is not true.[5]

[5]Instead of talking of a predicate's being true or false of a thing or an ordered collection of things, we shall hereafter frequently talk instead of a thing or ordered collection of things *satisfying* or *failing to satisfy* a predicate. Thus all and only red things *satisfy* the predicate 'x is red'. This notion of *satisfaction* will be used in the semantics for *PL*.

7.2E EXERCISES

1. Identify the singular terms in the following sentences, and then specify all the one or more place predicates that can be obtained from each sentence by deleting one or more singular terms.

a. The president is a Democrat.
*b. The speaker of the house is a Republican.
c. Sarah attends Smith College.
*d. Bob flunked out of U Mass.
e. Charles and Rita are brother and sister.
*f. Oregon is south of Washington and north of California.
g. 2 times 4 is 8.
*h. 3 times 4 equals 2 times 6.
i. 0 plus 0 equals 0.

2. List all the distinct sentences of English that can be generated using the following predicates and singular terms.

Singular terms:
Herman
Juan
Antonio

Predicates:
x is larger than y
x is to the right of y
x is larger than y but smaller than z

7.3 INTRODUCTION TO *PL*

It is time to introduce the basic elements of the formal language *PL*. We will need the sentential connectives of *SL* and analogs to the singular terms, predicates, and quantity terms of English. The sentential connectives are, to review, the five truth-functional connectives '&', '∨', '⊃', '≡', and '–'. As analogs to denoting singular terms of English—that is, singular terms that actually do, on the occasion of use in question, denote—*PL* contains **individual constants.** These are the lowercase Roman letters 'a' through 'v', with or without numerical subscripts. The **predicates** of *PL* are the uppercase Roman letters 'A' through 'Z', with or without numerical subscripts, followed by one or more primes. Predicates of *PL*, like predicates of English, come with holes or blanks, with the number of holes indicated by the number of primes. A predicate with one hole is called a 'one-place predicate', a predicate with two holes a 'two-place predicate', and so on. Hence

F'
G'
H'

are all one-place predicates and

$$F'$$
$$G'$$
$$H'$$

are all two-place predicates of *PL*. In specifying predicates, we shall, in practice, generally omit the primes and indicate that the predicate in question is an **n**-place predicate by writing **n** of the letters 'w', 'x', 'y', and 'z' (with subscripts if necessary) after the predicate letter. (For example, the predicate in 'Fx' is a one-place predicate and the predicate in 'Fxy' is a two-place predicate.) The letters 'w' through 'z', with and without subscripts, are called the **variables** of *PL* and have more than a hole-marking use.

In *SL* a single sentence letter can be used to symbolize or abbreviate different English sentences on different occasions. Analogously in *PL* we can use the two-place predicate 'Lxy' to symbolize, on different occasions, a variety of two-place predicates of English, including 'x likes y', 'x loves y', 'x loathes y', and 'x is less than y'. Of course, we could use 'Txy' to symbolize 'x likes y', but that would be harder to remember. Similarly on one occasion we might use the individual constant 'a' to designate Adriana, on another Alfred, and on another the number 1.

It will be useful to have a way of specifying how predicates and constants of *PL* are being used on a particular occasion, as well as what things are being talked about on that occasion. We call the set of things being talked about on a given occasion the **universe of discourse** for that occasion and use the abbreviation 'UD' in specifying a universe of discourse.[6] For this purpose we introduce the notion of a **symbolization key.** The following is an example of a symbolization key. We shall use it in symbolizing the English sentences discussed previously concerning Henry, Michael, Rita, and Sue.

UD: People in Michael's office
Lxy: x likes y
Ex: x is easygoing
Txy: x is taller than y
h: Henry
m: Michael
r: Rita
s: Sue

Note that, whereas in English proper names are capitalized and predicates written with lowercase letters, in *PL* lowercase letters are used to symbolize singular terms of English, including proper names, and uppercase letters are

[6] By stipulation, in *PL* universes of discourse must be nonempty; that is, discourse must always be about at least one thing. This is not a very restrictive stipulation because if the universe of discourse is the empty set, then there is nothing in that universe to say anything about.

used to symbolize predicates. In English sentences can be generated from predicates by filling the holes with singular terms. Similarly in *PL* sentences can be generated from predicates by filling the holes (replacing the variables that mark the holes) with individual constants. For example, 'Lsh' symbolizes, given the preceding symbolization key, 'Sue likes Henry'. 'Henry likes Sue' is symbolized as 'Lhs'. And 'Michael is easygoing' is symbolized as 'Em'. Still using the above symbolization key, the sentences

Sue is easygoing.
Michael is taller than Sue and Sue is taller than Henry.
Sue likes Henry and Michael likes Rita.
If Rita likes Henry, then Rita is taller than Henry.
If Michael is easygoing, then Rita is not easygoing.

can be symbolized as follows in *PL*:

Es
Tms & Tsh
Lsh & Lmr
Lrh ⊃ Trh
Em ⊃ ~ Er

In $PL_=$ as in $SL_=$ when a binary connective is used to join sentences, the result must be enclosed within parentheses. So, for example, the official versions of 'Lsh & Lmr' and 'Em ⊃ ~ Er' are '(Lsh & Lmr)' and '(Em ⊃ ~ Er)'. But with $PL_=$ as with $SL_=$ we shall informally omit the outermost parentheses of a sentence whose main logical operator (what in *SL* is termed the 'main connective') is a binary connective. (Also, as in $SL_=$ we shall informally allow the use of square brackets in place of parentheses.)

We can use our present symbolization key to give English readings for the following sentences of *PL*:

Lhr & ~ Lrh
Lrh ⊃ Lrm
Trh & ~ Trs
Tsh ⊃ Lhs
(Lmh ∨ Lms) ⊃ (Lmh & Lms)

In English these become, respectively,

Henry likes Rita and Rita does not like Henry.
If Rita likes Henry, then Rita likes Michael.
Rita is taller than Henry and Rita is not taller than Sue.

If Sue is taller than Henry, then Henry likes Sue.
If Michael likes Henry or Michael likes Sue, then Michael likes Henry and Michael likes Sue.

We can, of course, improve on the English. For example, the last sentence can be more colloquially paraphrased as

If Michael likes either Henry or Sue he likes both of them.

We can symbolize some English sentences involving quantity terms using only the resources of *PL* so far available to us. If we are talking just about the people in Michael's office, that is, just about Michael, Sue, Rita, and Henry, then one way to symbolize 'Everyone is easygoing' in *PL* is

(Es & Eh) & (Er & Em)

Note that we are here taking the scope or range of application of 'Everyone' in 'Everyone is easygoing' to be all and only the people in Michael's office. We could use the same strategy to symbolize 'Michael likes someone' as

(Lms ∨ Lmh) ∨ (Lmr ∨ Lmm)

and 'Michael likes everyone' as

(Lms & Lmh) & (Lmr & Lmm)

Note that, since we are talking about *everyone* in Michael's office, and Michael is one of those persons, we have to include 'Lmm' in our symbolization; that is, we take 'Michael likes everyone' to mean, in part, that Michael likes himself.

7.3E EXERCISES

1. Use the following symbolization key to symbolize the English sentences given as answers to Exercise 2 in Section 7.2E.

UD: Herman, Juan, and Antonio
Sxyz: x is larger than y but smaller than z
Lxy: x is larger than y
Rxy: x is to the right of y
a: Antonio
h: Herman
m: Juan

2. Symbolize the following sentences in *PL* using the given symbolization key.

UD: Alfy, Barbara, Clarence, Dawn, Ellis, and the cities Houston, Indianapolis, Kalamazoo, Newark, Philadelphia, San Francisco, and Tulsa
Bxy: x was born in y
Lxy: x lives in y
Axy: x is larger than y
Txy: x is taller than y
a: Alfy
b: Barbara
c: Clarence
d: Dawn
e: Ellis
h: Houston
i: Indianapolis
k: Kalamazoo
n: Newark
p: Philadelphia
s: San Francisco
t: Tulsa

a. Alfy was born in Indianapolis.
*b. Clarence was born in Tulsa.
c. Barbara was born in Newark.
*d. Dawn was born in San Francisco.
e. Ellis was born in Houston.
*f. No one was born in Kalamazoo.
g. Philadelphia is larger than Houston, Houston is larger than Newark, and Newark is larger than Kalamazoo.
*h. Tulsa isn't larger than either Philadelphia or Houston.
i. Indianapolis is larger than Houston if and only if it is larger than Philadelphia.
*j. Barbara lives in Philadelphia only if Dawn does.
k. Everyone lives in Philadelphia, but no one was born there.
*l. Barbara is taller than Clarence and Clarence is taller than Alfy, but neither Barbara nor Clarence is taller than Ellis.
m. Dawn is the tallest person in the office.
*n. Alfy isn't taller than everyone else in the office.
o. Alfy isn't taller than anyone in the office, but he is larger than everyone else in the office.
*p. If Clarence is taller than Barbara, he's also larger than Alfy.

3. Symbolize the following sentences in *PL* using the given symbolization key.

UD: Andrea, Bentley, Charles, and Deirdre
Bx: x is beautiful
Ix: x is intelligent
Rx: x is rich
Axy: x is attracted to y
Dxy: x despises y
Lxy: x loves y
Sxy: x is shorter than y

a: Andrea
b: Bentley
c: Charles
d: Deirdre

a. Andrea is both intelligent and beautiful, but she is not rich.
*b. Charles is rich and beautiful but not intelligent.
c. Deirdre is beautiful, rich, and intelligent.
*d. Bentley is neither rich, nor beautiful, nor intelligent.
e. If Bentley is intelligent, so are both Deirdre and Andrea.
*f. Andrea is beautiful and intelligent, Bentley is intelligent but not beautiful, and neither is rich.
g. Andrea loves Bentley but despises Charles.
*h. Andrea loves both herself and Charles and despises both Bentley and Deirdre.
i. Charles neither loves nor despises Andrea but both loves and despises Deirdre.
*j. Neither Deirdre nor Bentley is attracted to Charles, but Charles is attracted to both of them.
k. Charles is attracted to Bentley if and only if Bentley both is shorter than Charles and is rich.
*l. Andrea is attracted to both Bentley and Deirdre but doesn't love either of them.
m. If Deirdre is shorter than Charles and Charles is shorter than Andrea, then Deirdre is shorter than Andrea.
*n. If Bentley is attracted to Deirdre and she is attracted to him, then they love each other.
o. If Charles loves Bentley and Bentley loves Andrea, then Charles both despises and is shorter than Andrea.
*p. If Charles is neither rich nor beautiful nor intelligent, then no one loves him.
q. Only Deirdre is rich.
*r. Only Deirdre is both rich and intelligent.

4. For each of the following passages, provide a symbolization key and then use it to symbolize the passage in *PL*.

a. Margaret and Todd both like skateboarding, but neither is good at it. Charles is good at skateboarding but doesn't like it. Sarah is both good at skateboarding and likes it. All of them wear headgear, but Charles and Sarah are the only ones who wear knee pads. Sarah is more reckless than the rest, and Charles is more skillful than the rest.
*b. Charles is a sailor but not a tennis player, while Linda is both. Linda is a yuppie, and while Charles wants to be one, he isn't. Everyone likes Charles, but everyone also likes someone else more. Stan is a yuppie, and although Linda likes Charles, she likes Stan more. Stan is a sailor, a tennis player, and a squash player, and he likes himself more than he likes either of the other two. (*Hint:* Take the universe of discourse to consist of just Charles, Linda, and Stan.)
c. Andrew and Christopher are both hikers, but neither is a mountain climber. Amanda is a hiker and a mountain climber and also a kayaker. One, but not both, of Andrew and Christopher is also a kayaker. None of them is a swimmer. Andrew, Christopher, and Amanda all like each other, and Amanda is nuts about Andrew, and vice versa.

*d. Joan, Mark, Alice, and Randy are all in law school. Joan and Randy are studying tax law; Mark and Alice medical malpractice law. Alice gets better grades than Randy, and Mark gets better grades than Joan. They will all finish in three years, and everyone but Mark will pass the bar exam. At least two of the three who pass the bar exam will get jobs as attorneys.

7.4 QUANTIFIERS INTRODUCED

In the preceding section we saw how quantity claims can sometimes be symbolized using conjunctions and/or disjunctions of sentences. For example, we symbolize 'Michael likes someone in his office' as '(Lms ∨ Lmh) ∨ (Lmr ∨ Lmm)'. But this is a bit awkward. This strategy will not, in practice, work if we want to symbolize claims such as 'Michael likes everyone' where the number of people encompassed by 'everyone' is even modestly large—for example, the several hundred people Michael has met in the last five years. Worse still, suppose Michael is a mathematician and likes the positive integers, all infinitely many of them. On the present strategy we would need an infinitely long sentence, and we require the sentences of both *SL* and *PL* to be finitely long.[7] We need, within *PL*, analogs to the quantity terms of English; that is, we need the quantifier symbols and variables of *PL*. There are two **quantifier symbols**: '∀' and '∃'. The variables of *PL* are the letters 'w' through 'z', with or without subscripts. A **quantifier** of *PL* consists of a quantifier symbol followed by a variable, both enclosed in parentheses. Thus '(∀x)' and '(∃y)' are both quantifiers.

Recalling that our universe of discourse consists of the people in Michael's office, we can now abandon the use of iterated conjunctions and disjunctions. For example, we can symbolize 'Michael likes everyone' as

(∀x)Lmx

Here we have a predicate with one hole filled by an individual constant and one filled by a variable. The predicate is prefaced with a quantifier built from the same variable that fills the second hole. Quantifiers formed from '∀' are called **universal quantifiers** and are used to claim that each of the things being talked about is of the sort specified by the expression following the quantifier. The things being talked about, the members of the current universe of discourse, are called the *values* of the variables—because they are the things the variables are used to talk about. Here the claim being made is that each thing being talked about, that is, each person in Michael's office (each value of the variable 'x'), is of the sort Lmx, that is, is such that Michael likes it.

To symbolize 'Michael likes someone', still talking exclusively about the people in Michael's office, we can use an **existential quantifier**, that is, a quantifier built from '∃'.

(∃x)Lmx

[7] The problem is even more serious than suggested here, for Michael might like all real numbers, and there are more real numbers than there are individual constants of *SL*.

says there is at least one x (at least one value of the variable 'x') such that Michael likes that x. Since we are talking exclusively of the people in Michael's office, this amounts to

There is at least one person Michael likes

or

Michael likes someone.

Note that we interpret 'some' to mean 'at least one'.[8]

Variables of *PL* serve some of the functions of English pronouns and of such place-holder terms as 'thing', 'body', and 'one'—as in, for example, 'something', 'somebody', and 'someone'. 'Something is out of place' means that at least one of the things, whatever they may be, we are currently discussing is out of place. 'Everything is out of place' means that each of the things we are currently discussing is out of place. It is more stilted, but still acceptable English, to paraphrase these claims as, respectively,

At least one of the things under discussion is such that it is out of place

and

Each of the things under discussion is such that it is out of place.[9]

These paraphrases have a syntax that closely mirrors that of *PL*. Using 'Ox' to express 'x is out of place', we can symbolize the foregoing sentences in *PL* as

$(\exists x)Ox$

and

$(\forall x)Ox$

Note that in each the variable 'x' occurs twice; the first occurrence corresponds to 'thing' in the stilted English version, the second to 'it'. We can paraphrase these sentences of *PL* in quasi-English as 'At least one x is such that x is out of place' and 'Each x is such that x is out of place', respectively.

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[8] This is certainly appropriate for such English locutions as 'Someone is in the house' and 'John knows someone in the bursar's office'. But 'There are some cookies in the cookie jar' suggests to many that there are *at least two* cookies in that vessel. Nonetheless, we will always take the existential quantifier to mean 'at least one'. We will later introduce a way of saying 'at least two' when it is important to distinguish between 'at least one' and 'at least two'.

[9] In this chapter we frequently underline quantity expressions and truth-functional expressions in our paraphrases of sentences.

We now return to our example of Michael and his co-workers. This time we provide symbolizations that make use of quantifiers:

UD: People in Michael's office
Lxy: x likes y
Ex: x is easygoing
Txy: x is taller than y
h: Henry
m: Michael
r: Rita
s: Sue

The sentences to be symbolized are

Everyone is easygoing.
No one likes Michael.
Michael likes everyone.
Michael doesn't like anyone.
Michael doesn't like everyone.
Someone likes Sue.
No one is taller than herself or himself.

These can be symbolized in *PL* as, respectively,

(∀x)Ex
~ (∃x)Lxm
(∀x)Lmx
~ (∃x)Lmx
~ (∀x)Lmx
(∃x)Lxs
~ (∃x)Txx

Points to Note

1. There is nothing sacrosanct about our choice of variables. In each of the preceding cases, every occurrence of 'x' can be replaced with any other variable. '(∃y)Lys' says 'Someone likes Sue' just as well as '(∃x)Lxs' does.
2. There is nothing sacrosanct about the variables used in specifying predicates in symbolization keys. Our symbolization key includes

   Ex: x is easygoing

   Any other variable would have done as well in specifying how the one-place predicate in question is to be interpreted. (For example, we could have used 'Ez: z is easygoing'.)
3. The variables used in specifying predicates in symbolization keys need not be used in constructing symbolizations based on those keys. Given the previous symbolization key, we are perfectly free to symbolize 'Everyone is easygoing' as '(∀y)Ey'.

We are treating claims such as 'Everyone is easygoing' and 'Michael likes everyone' as ways of saying, in shorthand fashion, of each thing under discussion, in the first instance that it is easygoing and, in the second instance that Michael likes it. That is, we are *not* treating 'Everyone is easygoing' as being a sentence that has a subject term, 'Everyone', that denotes something (the collection of people being talked about) of which a property, easygoingness, is being predicated.

Note also that 'every' and 'all' function somewhat differently grammatically. Grammatically 'All people are mortal' has a plural subject term and thus requires a plural verb, 'are'. 'Everyone is mortal' has a singular term in subject position and thus requires a singular verb, 'is'.

Some uses of 'all' and 'every' can be confusing. Consider 'All the bricks are too heavy' uttered in response to the question 'Can you take all the bricks in one load?' Here 'All the bricks are too heavy' probably means, not that each brick, by itself, is too heavy to take in one load, but rather that the bricks taken collectively, all at once, are too heavy. Note that this ambiguity is absent from 'Every brick is too heavy'. Here there is no chance we mean anything other than that each brick, by itself, is too heavy. At any rate, in this text, we treat 'All such-and-such are thus-and-so', 'Every such-and-such is thus-and-so', and 'Each such-and-such is thus-and-so', as various ways of saying of each thing that is such-and-such that it is thus-and-so—that is, as claims about individual things, not as claims about collections of things (all the bricks taken together).

There is a very important difference between 'Michael doesn't like everyone' and 'Michael doesn't like anyone'. If Michael is like most of us, he likes some people and not others, and so it is true that he doesn't like everyone (he doesn't, for example, like Rita at all) but false that he doesn't like anyone (he likes Sue very much). The difference between 'doesn't like every' and 'doesn't like any' is very clearly marked by the syntax of *PL*: The first can be expressed by a '~' followed by a universal quantifier, and the second by a '~' followed by an existential quantifier.

So far the sentences of *PL* we have dealt with have contained only a single quantifier, one predicate, and in some cases a tilde. It is an easy next step to form truth-functional compounds of such sentences. Here are some examples:

1. Either everyone is easygoing or no one is.
2. If Rita is easygoing, everyone is.
3. Rita likes Sue if and only if everyone does.
4. Henry likes everyone but Sue doesn't.
5. Henry likes everyone and Sue doesn't like anyone.
6. Not everyone is easygoing, but everyone is ambitious.
7. If anyone is ambitious, Michael is.
8. Everyone is ambitious if and only if no one is easygoing.

We can symbolize these English sentences in *PL* using the symbolization key introduced previously, with the addition of

Ax: x is ambitious

The first sentence is a disjunction. We can symbolize the left disjunct, 'Everyone is easygoing' as '(∀y)Ey' (remember, the fact that we specified the predicate for 'is easygoing' as 'Ex' in the symbolization key does *not* mean we can only use the variable 'x' when working with that predicate). We must be careful in symbolizing the second half of our disjunction—'no one is easygoing' is *not* the negation of 'Everyone is easygoing'. That is, the symbolization '~ (∀y)Ey' says not that *no one is* easygoing', but rather that *not everyone is* easygoing. For the second disjunct of the sentence of *PL* we are constructing, we can use either '~ (∃w)Ew' or '(∀w) ~ Ew'. The first says it is not the case that there is even one person who is easygoing (hence no one is), and the second that each person is not easygoing (again, hence that no one is). So for a complete symbolization of our first example we might pick

(∀y)Ey ∨ ~ (∃w)Ew

Note that in the left disjunct we used the variable 'y', and in the right disjunct the variable 'w'. This was an arbitrary selection; we could have used the same variable in both. But we cannot use two different variables within either disjunct. That is, '(∀y)Ex' is not allowed since the purpose of the quantifier is to indicate how many things are of the sort specified by the predicate that follows it, and to do this the variable used in forming the quantifier must match the variable used with the predicate following it.

Sentence 2, 'If Rita is easygoing, everyone is', is a conditional whose antecedent is 'Rita is easygoing', or 'Er' in *PL*, and whose consequent is short for 'everyone is easygoing'. The latter can be symbolized as '(∀w)Ew', and the whole sentence as

Er ⊃ (∀w)Ew

The third sentence, 'Rita likes Sue if and only if everyone does', can be symbolized as a material biconditional

Lrs ≡ (∀z)Lzs

Note that, while 'Lrs ≡ (∀z)Lsz' looks a lot like the above sentence of *PL*, it says something very different, namely, that Rita likes Sue if and only if Sue likes everyone.

Sentence 5, 'Henry likes everyone but Sue doesn't', becomes a conjunction of *PL*. '(∀y)Lhy' is an appropriate left conjunct, and '~ (∀x)Lsx' an appropriate right conjunct, yielding the conjunction in *PL*:

(∀y)Lhy & ~ (∀x)Lsx

The second conjunct says of Sue that she doesn't like everyone, not that she doesn't like *anyone*, which is what the next example says of Sue. To symbolize the fifth example, we can replace the right conjunct of the preceding sentence of *PL* with either '(∀x) ~ Lsx' or '~ (∃x)Lsx'. The first of the foregoing can be paraphrased as 'each person is such that it is not the case that Sue likes that person', and the second as 'it is not the case that there is at least one person that Sue likes'. These amount to the same thing—that Sue doesn't like anyone. So we might pick for our symbolization of the fifth example

(∀y)Lhy & ~ (∃x)Lsx

'Not everyone is easygoing, but everyone is ambitious', our sixth example, can be symbolized as the conjunction of '~ (∀z)Ez' and '(∀y)Ay', or as

~ (∀z)Ez & (∀y)Ay

Sentence 7, 'If anyone is ambitious, Michael is', is a conditional. Here the 'anyone' of the antecedent means 'at least one', for the claim is that if anyone, anyone at all—that is, at least one person—is ambitious, Michael is. So an appropriate symbolization is

(∃w)Aw ⊃ Am

The last of our examples can be symbolized as a material biconditional linking 'Everyone is ambitious', or '(∀w)Aw', and 'no one is easygoing', or '~ (∃w)Ew'. So an appropriate symbolization in *PL* is

(∀w)Aw ≡ ~ (∃w)Ew

For 'no one is easygoing' we could also have used '(∀w) ~ Ew'; that is, each person is not easygoing.

Finally we do not really need both existential and universal quantifiers. Instead of saying 'Everything is thus-and-so' ('Everyone likes Michael', or '(∀x)Lxm'), we can say 'It is not the case that something is not thus-and-so' ('It is not the case that someone does not like Michael', or '~ (∃x) ~ Lxm'). And instead of saying 'Something is thus-and-so' ('Someone likes Michael', or '(∃x)Lxm'), we can say 'It is not the case that everything is not thus-and-so' ('It is not the case that everyone does not like Michael', or '~ (∀x) ~ Lxm'). However, having both quantifiers available does make symbolization somewhat easier and more natural.

7.4E EXERCISES

1. Symbolize the following sentences in *PL* using the given symbolization key.

UD: The jellybeans in the jar on the coffee table
Bx: x is black
Rx: x is red

a. All the jellybeans are black.
*b. Some of the jellybeans are black.
c. None of the jellybeans is black.
*d. Some of the jellybeans are not black.
e. Some of the jellybeans are black and some are red.
*f. If all the jellybeans are black then none is red.
g. If some are red some are black.
*h. If none is black all are red.
i. All are black if and only if none is red.
*j. Either all are black or all are red.

2. Symbolize the following sentences in *PL*. (*Note*: Not all of these sentences are true.)

UD: The integers 1–100
Ex: x is even
Ox: x is odd
Lxy: x is less than y
Px: x is prime
Gx: x is greater than 0
a: 1
b: 2
c: 4
d: 100

a. Some integers are odd and some are even.
*b. Some integers are both odd and even.
c. No integer is less than 1.
*d. No integer is greater than 100.
e. Every integer is greater than 0.
*f. 100 is greater than every odd integer.
g. There is a prime that is even.
*h. Some primes are not even.
i. All prime integers greater than 2 are odd.
*j. 1 is not prime and is not greater than any integer.
k. There is an integer that is greater than 2 and less than 4.

3. Symbolize the following sentences in *PL* using the given symbolization key.

UD: The students in a logic class
Px: x will pass
Sx: x will study
j: Jamie
r: Rhoda

a. If Jamie will pass everyone will pass.
*b. Either no one will pass or Jamie will pass.
c. If anyone passes both Jamie and Rhoda will.
*d. Not everyone will pass, but Rhoda will.
e. If Rhoda doesn't pass no one will.
*f. Some, but not all, of the students will pass.
g. Rhoda will pass if Jamie does, and if Rhoda passes everyone will pass.
*h. No one will pass if Jamie doesn't pass, and if she does everyone will.
i. Everyone will study but not everyone will pass.
*j. If everyone studies everyone will pass.
k. If everyone studies some will pass.

7.5 THE FORMAL SYNTAX OF *PL*

Before attempting more complex symbolizations, it will be useful to acquire a fuller understanding of the syntax of *PL*. To this end we now pause to present the formal syntax for the language of *PL* and to introduce some important syntactical concepts. While this material may at first seem difficult, it can be readily mastered, and doing so will make mastering the rest of this chapter much easier.

The vocabulary of *PL* consists of the following:

Sentence letters of PL: The capital Roman letters 'A' through 'Z', with or without positive-integer subscripts (These are just the sentence letters of *SL*.)	$A, B, C, \ldots, Z,$ $A_1, B_1, C_1, \ldots, Z_1, \ldots$
Predicates of PL: The capital Roman letters 'A' through 'Z', with or without positive-integer subscripts, followed by one or more primes (An **n**-place predicate is indicated by the presence of exactly **n** primes.)	$A', B', C' \ldots, Z',$ $A_1', B_1', C_1', \ldots, Z_1',$ $\ldots$
Individual terms of PL:	
Individual constants of PL: The lowercase Roman letters 'a' through 'v', with or without positive-integer subscripts	$a, b, c, \ldots, v,$ $a_1, b_1, c_1, \ldots, v_1, \ldots$
Individual variables of PL: The lowercase Roman letters 'w' through 'z', with or without positive-integer subscripts.	$w, x, y, z, w_1, x_1, y_1, z_1, \ldots$
Truth-functional connectives:	$\sim \; \& \; \vee \; \supset \; \equiv$
Quantifier symbols:	$\forall \; \exists$
Punctuation marks:	( )

By informal convention we will continue to omit the primes from predicates of *PL* where no confusion results from doing so. By including the sentence letters of *SL* as sentence letters of *PL*, we make every sentence of *SL* a sentence of *PL*. Thus every English sentence that can be symbolized in *SL* can also be symbolized in *PL*. However, the content of English sentences is generally better captured by using predicates rather than sentence letters. Hence we shall rarely, if ever, have use for sentence letters in the rest of this text.[10]

We define an **expression of PL** to be a sequence of not necessarily distinct elements of the vocabulary of *PL*. The following are expressions of *PL*:

((((()(((a ⊃ bba)
(A ⊃ Bab))
(∀x)(∃x)Fxx

but

((|ABA)
(A ⊃ 3)
A # Bab
(∀ @)(Cab)

are not since '|', '3', '#', and '@' are not elements of the vocabulary of *PL*.

In what follows we will use the bold letters

P Q R

as metavariables ranging over expressions of *PL*. We will use a bold '**a**' as a metavariable ranging over individual constants of *PL* and a bold '**x**' as a metavariable ranging over individual variables of *PL*.

Quantifier of PL: An expression of *PL* of the form (∀**x**) or (∃**x**). An expression of the first form is a universal quantifier, and one of the second form is an existential quantifier.

We will say that a quantifier *contains* a variable. Thus '(∀y)' and '(∃y)' both contain the variable 'y' (and are 'y-quantifiers'); '(∀z)' and '(∃z)' both contain the variable 'z' (and are 'z-quantifiers').

Atomic formulas of PL: Every expression of *PL* that is either a sentence letter of *PL* or an **n**-place predicate of *PL* followed by **n** individual terms of *PL*.

[10]We could include among the predicates of *PL* zero-place predicates. Doing so would make the sentence letters of *SL* zero-place predicates and would obviate the need for the separate category 'Sentence letters of *PL*'.

We are now ready to give a recursive definition of 'formula of *PL*':

1. Every atomic formula of *PL* is a formula of *PL*.
2. If **P** is a formula of *PL*, so is − **P**.
3. If **P** and **Q** are formulas of *PL*, so are (**P** & **Q**), (**P** ∨ **Q**), (**P** ⊃ **Q**), and (**P** ≡ **Q**).
4. If **P** is a formula of *PL* that contains at least one occurrence of **x** and no **x**-quantifier, then (∀**x**)**P** and (∃**x**)**P** are both formulas of *PL*.
5. Nothing is a formula of *PL* unless it can be formed by repeated applications of clauses 1–4.

Last, we specify the **logical operators** of *PL*:

> *Logical operator of PL:* An expression of *PL* that is either a quantifier or a truth-functional connective

Consider the following expressions of *PL*:

Rabz
− (Rabz & Hxy)
(~ Rabz & Hxy)
(Hab ⊃ (∀z)(Fz ⊃ Gza))
(Haz ⊃ − (∀z)(Fz ⊃ Gza))
(∀z)(Haz ⊃ (∀z)(Fz ⊃ Gza))
(∀x)(Haz ⊃ (∀z)(Fz ⊃ Gza))
(∀y)(Hay ⊃ (Fy ⊃ Gya))

The first expression consists of a three-place predicate followed by three individual terms, the first two being individual constants and the third an individual variable. Hence it is an atomic formula of *PL* and, by clause 1 of the recursive definition of 'formula of *PL*', a formula of *PL*.

The second expression consists of a tilde, '~', followed by '(Rabz & Hxy)', and so it is a formula of *PL* by clause 2 if '(Rabz & Hxy)' is a formula of *PL*. And since 'Rabz' and 'Hxy' are both atomic formulas of *PL*, and hence formulas of *PL*, '(Rabz & Hxy)' is a formula of *PL* by clause 3 of the recursive definition. The third expression, '(− Rabz & Hxy)', is a formula of *PL* by clause 3 if '− Rabz' and 'Hxy' are both formulas of *PL*. They are: 'Hxy' is an atomic formula and hence a formula, and 'Rabz' is an atomic formula and hence a formula; and so '− Rabz' is a formula by clause 2.

The fourth expression, '(Hab ⊃ (∀z)(Fz ⊃ Gza))', is a formula of *PL* by clause 3 if 'Hab' and '(∀z)(Fz ⊃ Gza)' are both formulas of *PL*. The first is an atomic formula, a two-place predicate followed by two individual terms (both constants), and hence a formula of *PL* by clause 1. The second is a formula of

PL by clause 4 if '(Fz ⊃ Gza)' is a formula containing at least one occurrence of 'z' and no z-quantifier. It clearly satisfies the last two conditions, and since 'Fz' and 'Gza' are both atomic formulas of *PL* and hence formulas of *PL*, '(Fz ⊃ Gza)' is a formula of *PL* by clause 3 of the recursive definition. So the whole expression is a formula of *PL*. For reasons parallel to those outlined previously, the fifth expression, '(Haz ⊃ ~ (∀z)(Fz ⊃ Gza))', is also a formula of *PL*. The differences are that the antecedent of the conditional, 'Haz', is an atomic formula containing one constant and one variable instead of two constants, and the consequent is a negation, '~ (∀z)(Fz ⊃ Gza)'. Since '(∀z)(Fz ⊃ Gza)' is a formula, so is '~ (∀z)(Fz ⊃ Gza)' by clause 2 of the recursive definition.

The sixth expression, '(∀z)(Haz ⊃ (∀z)(Fz ⊃ Gza))', is not a formula of *PL*. It would be a formula, by clause 4, if '(Haz ⊃ (∀z)(Fz ⊃ Gza))' were a formula containing at least one occurrence of 'z' and no z-quantifier. The first two conditions are satisfied, but the third is not. '(Haz ⊃ (∀z)(Fz ⊃ Gza))' does contain a z-quantifier in '(∀z)(Fz ⊃ Gza)'. The seventh expression, '(∀x)(Haz ⊃ (∀z)(Fz ⊃ Gza))', is also not a formula. As we saw, '(Haz ⊃ (∀z)(Fz ⊃ Gza))' is a formula. But since it contains no occurrence of the variable 'x', prefixing it with an x-quantifier does not produce a formula of *PL*.

The last expression, '(∀y)(Hay ⊃ (Fy ⊃ Gya))', is a formula. While it looks rather similar to the two expressions just considered, it is built up in rather different ways. Note first that 'Fy' and 'Gya' are formulas of *PL*, so '(Fy ⊃ Gya)' is also a formula of *PL*. And since 'Hay' is an atomic formula, and therefore a formula, of *PL*, '(Hay ⊃ (Fy ⊃ Gya))' is also a formula of *PL*. Since this formula contains at least one occurrence of the variable 'y' and no y-quantifier, prefixing it with a y-quantifier, here '(∀y)', produces a formula of *PL*—that is, '(∀y)(Hay ⊃ (Fy ⊃ Gya))'.

Not all formulas of *PL* qualify as sentences of *PL*. But before we can explicitly state the relationship between formulas and sentences, we need to introduce the concepts of **subformula** and **main logical operator.** We do so by cases:

1. If **P** is an atomic formula of *PL*, then **P** contains no logical operator, and hence no main logical operator, and **P** is the only subformula of **P**.
2. If **P** is a formula of *PL* of the form ~ **Q**, then the tilde ('~') that precedes **Q** is the main logical operator of **P**, and **Q** is the immediate subformula of **P**.
3. If **P** is a formula of *PL* of the form (**Q** & **R**), (**Q** ∨ **R**), (**Q** ⊃ **R**), or (**Q** ≡ **R**), then the binary connective between **Q** and **R** is the main logical operator of **P**, and **Q** and **R** are the immediate subformulas of **P**.
4. If **P** is a formula of *PL* of the form (∀**x**)**Q** or of the form (∃**x**)**Q**, then the quantifier that occurs before **Q** is the main logical operator of **P**, and **Q** is the immediate subformula of **P**.
5. If **P** is a formula of *PL*, then every subformula (immediate or not) of a subformula of **P** is a subformula of **P**, and **P** is a subformula of itself.

We can classify formulas of *PL* (and later sentences) by their main logical operator. Atomic formulas have no main logical operator. Quantified formulas have a quantifier as their main logical operator. Truth-functional compounds have a truth-functional connective as their main logical operator. Consider again the eight expressions of *PL* displayed previously. The sixth and seventh are not formulas of *PL*, and hence the notions of main logical operator and subformula do not apply to them. For each of the rest we display its subformulas, identify the main logical operator (if any), and classify its subformula as either atomic, quantified, or a truth-functional compound.

Formula	*Subformula*	*Main Logical Operator*	*Type*
Rabz	Rabz	None	Atomic
~ (Rabz & Hxy)	~ (Rabz & Hxy)	~	Truth-functional
	(Rabz & Hxy)	&	Truth-functional
	Rabz	None	Atomic
	Hxy	None	Atomic
(~ Rabz & Hxy)	(~ Rabz & Hxy)	&	Truth-functional
	~ Rabz	~	Truth-functional
	Hxy	None	Atomic
	Rabz	None	Atomic
(Hab ⊃ (∀z)(Fz ⊃ Gza))	(Hab ⊃ (∀z)(Fz ⊃ Gza)	⊃	Truth-functional
	Hab	None	Atomic
	(∀z)(Fz ⊃ Gza)	(∀z)	Quantified
	(Fz ⊃ Gza)	⊃	Truth-functional
	Fz	None	Atomic
	Gza	None	Atomic
(Haz ⊃ ~ (∀z)(Fz ⊃ Gza))	(Haz ⊃ ~ (∀z)(Fz ⊃ Gza))	⊃	Truth-functional
	Haz	None	Atomic
	~ (∀z)(Fz ⊃ Gza)	~	Truth-functional
	(∀z)(Fz ⊃ Gza)	(∀z)	Quantified
	(Fz ⊃ Gza)	⊃	Truth-functional
	Fz	None	Atomic
	Gza	None	Atomic
(∀y)(Hay ⊃ (Fy ⊃ Gya))	(∀y)(Hay ⊃ (Fy ⊃ Gya))	(∀y)	Quantified
	(Hay ⊃ (Fy ⊃ (Gya))	⊃	Truth-functional
	Hay	None	Atomic
	(Fy ⊃ Gya)	⊃	Truth-functional
	Fy	None	Atomic
	Gya	None	Atomic

Earlier we talked informally of quantifiers serving to interpret variables. We can now make that notion explicit. The interpretive range of a quantifier is its **scope.**

> *Scope of a quantifier*: The scope of a quantifier in a formula **P** of *PL* is the subformula **Q** of **P** of which that quantifier is the main logical operator.

Recall, from the recursive definition of 'formula of *PL*', that the only way quantifiers get into formulas is by clause 4, which specifies the conditions under which a quantifier may be attached to a formula. So attaching a quantifier to a formula produces a new formula, of which the quantifier is the main logical operator. The scope of that quantifier is all of the new formula; that is, it is the quantifier itself and the formula to which it is being attached. For example, '(∀x)Fxy' is a quantified formula of which '(∀x)' is the main logical operator. The scope of that quantifier is all of '(∀x)Fxy'; that is, the scope includes the quantifier '(∀x)' and the formula immediately following the quantifier, namely, 'Fxy'.

Consider the formula '(Hx ⊃ (∀y)Fxy)'. This expression is a formula (by clause 3 of the recursive definition of 'formula of *PL*') inasmuch as 'Hx' is a formula (an atomic formula) and '(∀y)Fxy' is a formula by clause 4 ('Fxy' is a formula of *PL* in which x occurs and in which no x-quantifier occurs). The formula contains two distinct variables, 'x' and 'y', and a total of four occurrences of variables ('x' and 'y' each occur twice). The scope of '(∀y)' includes the occurrence of 'y' from which it is formed and the occurrences of 'x' and 'y' in 'Fxy', for the subformula of which '(∀y)' is the main logical operator is '(∀y)Fxy'. But the first occurrence of 'x', that in 'Hx', does not fall within the scope of '(∀y)', for it is not in the subformula '(∀y)Fxy'. In '((∀z)Gz ⊃ ~ Hz)' the scope of the quantifier '(∀z)' is '(∀z)Gz; hence the first two occurrences of 'z' in this formula fall within its scope, but the last occurrence, that in '~ Hz', does not. We can now introduce the notions of **free** and **bound variables** of *PL*.

> *Bound variable:* An occurrence of a variable **x** in a formula **P** of *PL* that is within the scope of an **x**-quantifier
>
> *Free variable:* An occurrence of a variable **x** in a formula **P** of *PL* that is not bound

At long last we are ready to formally introduce the notion of a **sentence of *PL***:

> *Sentence of PL:* A formula **P** of *PL* is a sentence of *PL* if and only if no occurrence of a variable in **P** is free.

We shall speak of a formula of *PL* that is not a sentence of *PL* as an **open sentence of *PL***.

We can now see that '(Hx ⊃ (∀y)Fxy)' is not a sentence of *PL* for two reasons: The first occurrence of 'x' does not fall within the scope of any quantifier and is therefore free, and the second occurrence of 'x', while falling within the scope of a quantifier, does not fall within the scope of an x-quantifier. And '((∀z)Gz ⊃ ~ Hz)' is not a sentence because the third occurrence of 'z' does not fall within the scope of a z-quantifier. The scope of '(∀z)' is limited to the subformula of which it is the main logical operator—that is, to '(∀z)Gz'.

Earlier we considered the following eight expressions of *PL*:

Rabz
~ (Rabz & Hxy)
(~ Rabz & Hxy)
(Hab ⊃ (∀z)(Fz ⊃ Gza))
(Haz ⊃ ~ (∀z)(Fz ⊃ Gza))
(∀z)(Haz ⊃ (∀z)(Fz ⊃ Gza))
(∀x)(Haz ⊃ (∀z)(Fz ⊃ Gza))
(∀y)(Hay ⊃ (Fy ⊃ Gya))

The first is not a sentence because it contains a free occurrence of 'z'. However, this formula can be made into a sentence by prefacing it with a z-quantifier; that is, both '(∀z)Rabz' and '(∃z)Rabz' are sentences of *PL*. Note that formulas that contain no variables—for example, 'Rabc', 'Hab', and '(Gd & Fab)'—are sentences of *PL*; they contain no occurrences of variables and hence no free occurrences of variables. It is individual variables, *not individual constants*, that need to be interpreted by quantifiers.

The second formula contains three free occurrences of variables—one each of 'z', 'x', and 'y'—and so is not a sentence of *PL*. We would have to add three quantifiers to this formula to make it a sentence: a z-quantifier, an x-quantifier, and a y-quantifier, in any order. So, too, for the third formula. The fourth formula is a sentence since the only variable it contains is 'z', and all occurrences of 'z' fall within the scope of '(∀z)'. (The scope of that quantifier is '(∀z)(Fz ⊃ Gza)'.)

The fifth formula is not a sentence of *PL* since it does contain a free variable, the first occurrence of 'z' (in 'Haz'). The sixth expression, '(∀z)(Haz ⊃ (∀z)(Fz ⊃ Gza))', is, as noted earlier, not a formula of *PL* because the initial z-quantifier is attached to an expression that is a formula that already contains a z-quantifier. Since it is not a formula of *PL*, it is not a sentence of *PL*. We can now see why the fourth clause of the recursive definition of 'formula of *PL*',

> 4. If **P** is a formula of *PL* that contains at least one occurrence of **x** and no **x**-quantifier, then (∀**x**)**P** and (∃**x**)**P** are both formulas of *PL*

is as complicated as it is. If we dropped the restriction 'and no **x**-quantifier' from clause 4, the expression '(∀z)(Haz ⊃ (∀z)(Fz ⊃ Gza))' would be a formula with two z-quantifiers with overlapping scopes. We would then need some further rule to determine which quantifier interprets the last two occurrences of 'z' for those occurrences of 'z' that fall within the scope of *both* quantifiers.

The seventh expression, '(∀x)(Haz ⊃ (∀z)(Fz ⊃ Gza))', is also not a formula, and hence not a sentence. It would be a formula if clause 4 of the

recursive definition did not include the requirement that the formula to which a quantifier is added—here '(Haz ⊃ (∀z)(Fz ⊃ Gza))'—contain at least one occurrence of the variable from which the added quantifier—here '(∀x)'—is formed. Clause 4 is intentionally written so as to disallow the use of quantifiers that do no work, that is, quantifiers that bind no variables in the formula to which they are attached.

Since sentences of *PL* are formulas of *PL*, we can speak of sentences as being either quantified (sentences whose main logical operator is a quantifier), truth-functional (sentences whose main logical operator is a truth-functional connective), or atomic (sentences that have no main logical operator).

We have been omitting the primes that, by the formal requirements of *PL*, are parts of the predicates of *PL*, and we will continue to do so. We will also frequently omit the outermost parentheses of a formula of *PL*. In our usage outermost parentheses are a pair of left and right parentheses that are added, as a pair, when a binary connective is inserted between two formulas of *PL*. Thus we may write 'Fa & ~ (∀x)Fx' instead of '(Fa & ~ (∀x)Fx)'. Note that, while '~ (Fa & (∃x) ~ Fx)' is a truth-functionally compound formula (and sentence), it has no outermost parentheses. So, too, '(∀x)(Fx ⊃ Gx)' has as its first symbol a left parentheses and as its last a right parentheses, but these are not 'outermost parentheses', for the first and last symbols of this sentence were not added as a pair when formulas were joined by a binary connective.

The omission of outermost parentheses should cause no confusion. Note, however, that when outer parentheses are customarily dropped, it is not safe to assume that every sentence that begins with a quantifier is a quantified sentence. Consider

$$(\forall x)(Fx \supset Ga)$$

and

$$(\forall x)Fx \supset Ga$$

Both begin with quantifiers, but only the first is a quantified sentence. The scope of the x-quantifier in this sentence is the whole formula. The second sentence is a truth-functional compound; the scope of the x-quantifier is just '(∀x)Fx'. It turns out that the two sentences are not only syntactically distinct but also that they say very different things.

To make complicated formulas of *PL* easier to read, we also allow the use of square brackets, '[' and ']', in place of the parentheses required by clause 3 of the recursive definition of 'formula of *PL*', that is, by the use of truth-functional connectives. But we will not allow square brackets in place of parentheses in quantifiers. So, instead of

$$\sim(\forall y)[(\exists z)Fzy \supset (\exists x)Gxy]$$

we can write

$$\sim(\forall y)[(\exists z)Fzy \supset (\exists x)Gxy]$$

In later chapters we shall require one further syntactic concept, that of a **substitution instance** of a quantified sentence. We use the notation

$$\mathbf{P}(\mathbf{a}/\mathbf{x})$$

to specify the formula of *PL* that is like **P** except that it contains the individual constant **a** wherever **P** contains the individual variable **x**. Thus if **P** is

$$(Fza \vee \sim Gz)$$

$\mathbf{P}(c/z)$ is

$$(Fca \vee \sim Gc)$$

Substitution instance of **P**: If **P** is a sentence of *PL* of the form $(\forall \mathbf{x})\mathbf{Q}$ or $(\exists \mathbf{x})\mathbf{Q}$, and **a** is an individual constant, then $\mathbf{Q}(\mathbf{a}/\mathbf{x})$ is a substitution instance of **P**. The constant **a** is the *instantiating constant*.

For example, 'Fab', 'Fbb', and 'Fcb' are all substitution instances of '$(\forall z)Fzb$'. In the first case 'a' has been substituted for 'z' in 'Fzb'; in the second case 'b' has been substituted for 'z'; and in the third case 'c' has been substituted for 'z'.

In forming a substitution instance of a quantified sentence, we drop the initial quantifier and replace all remaining occurrences of the variable that that quantifier contains with some one constant. Thus '$(\exists y)Hay$' and '$(\exists y)Hgy$' are both substitution instances of '$(\forall x)(\exists y)Hxy$', but 'Hab' is not. (In forming substitution instances *only* the initial quantifier is dropped, and every occurrence of the variable that becomes free when that quantifier is dropped is replaced by the *same* constant.) All the following are substitution instances of '$(\exists w)[Fw \supset (\forall y)(\sim Dwy \equiv Ry)]$':

$$Fd \supset (\forall y)(\sim Ddy \equiv Ry)$$
$$Fa \supset (\forall y)(\sim Day \equiv Ry)$$
$$Fn \supset (\forall y)(\sim Dny \equiv Ry)$$

but

$$Fd \supset (\forall y)(\sim Duy \equiv Ry)$$

is not—for here we have used one constant to replace the first occurrence of 'w' and a different constant to replace the second occurrence of 'w'. Again, in

generating substitution instances, each occurrence of the variable being replaced must be replaced by the same individual constant.

Only quantified sentences have substitution instances, and those instances are formed by dropping the initial quantifier. Thus '~ Fa' is *not* a substitution instance of '~ (∀x)Fx'. '~ (∀x)Fx' is a truth-functional compound, not a quantified sentence, and hence has no substitution instances. And '(∀x)Fxb' is not a substitution instance of '(∀x)(∀y)Fxy' because, while the latter is a quantified sentence, only the initial quantifier can be dropped in forming substitution instances, and here the initial quantifier is '(∀x)', not '(∀y)'.

7.5E EXERCISES

1. Which of the following are formulas of *PL*? (Here we allow the deletion of outer parentheses and the use of square brackets in place of parentheses.) For those that are not, explain why they are not. For those that are, state whether they are sentences or open sentences.

a. Ba & Zz
*b. (x)Px ∨ Py
c. (∃y) ~ Hyy & Ga
*d. (∀z)(Ex)(Fzx & Fxz)
e. (∀z)((∃x)Fzx & Fxz)
*f. (∀x)Faa
g. (∃z)(Fz & Bgz) ≡ (∃z)Gzb
*h. (∃x)[Fx & (∀x)(Px ⊃ Gx)]
i. (~ ∃x)(Fx ∨ Gx)
*j. ~ (∀x)(Gx ≡ (∃z)Fzx)
k. (∃x)(∃y)Lxx
*l. (∀x)[(∃y)Fyx ⊃ (∃y)Fxy]
m. (Bu & ~ Faa) ⊃ (∀w) ~ Fww
*n. (∃a)Fa
o. Fw ⊃ (∃w)Gww
*p. (∀z)(Hza ⊃ (∃z)Gaz)

2. For each of the following formulas, indicate whether it is a sentence of *PL*. If it is not a sentence, explain why it is not. Also list all its subformulas, identifying the main logical operator of each.

a. (∃x)(∀y)Byx
*b. (∃x) ~ (∀y)Byx
c. (∀x)(~ Fx & Gx) ≡ (Bg ⊃ Fx)
*d. (∀y)[(∀z) ~ Byz ∨ Byy]
e. ~ (∃x)Px & Rab
*f. Rax ⊃ ~ (∀y)Ryx
g. ~ [~ (∀x)Fx ≡ (∃w) ~ Gw] ⊃ Maa
*h. (∀x)(∀y)(∀z)Mxyz & (∀z)(∀x)(∀y)Myzx
i. ~ ~ ~ (∃x)(∀z)(Gxaz ∨ ~ Hazb)
*j. (∀z)[Fz ⊃ (∃w)(~ Fw & Gwaz)]
k. (∃x)[Fx ⊃ (∀w)(~ Gx ⊃ ~ Hwx)]
*l. ~ [(∀x)Fx ∨ (∀x) ~ Fx]

m. $(Hb \vee Fa) \equiv (\exists z)(\sim Fz \ \& \ Gza)$
*n. $(\exists w)(Fw \ \& \sim Fw) \equiv (He \ \& \sim He)$

3. Indicate, for each of the following sentences, whether it is an atomic sentence, a truth-functional compound, or a quantified sentence.

a. $(\forall x)(Fx \supset Ga)$
*b. $(\forall x) \sim (Fx \supset Ga)$
c. $\sim (\forall x)(Fx \supset Ga)$
*d. $(\exists w)Raw \vee (\exists w)Rwa$
e. $\sim (\exists x)Hx$
*f. Habc
g. $(\forall x)(Fx \equiv (\exists w)Gw)$
*h. $(\forall x)Fx \equiv (\exists w)Gw$
i. $(\exists w)(Pw \supset (\forall y)(Hy \equiv \sim Kyw))$
*j. $\sim (\exists w)(Jw \vee Nw) \vee (\exists w)(Mw \vee Lw)$
k. $\sim [(\exists w)(Jw \vee Nw) \vee (\exists w)(Mw \vee Lw)]$
*l. Da
m. $(\forall z)Gza \supset (\exists z)Fz$
*n. $\sim (\exists x)(Fx \ \& \sim Gxa)$
o. $(\exists z) \sim Hza$
*p. $(\forall w)(\sim Hw \supset (\exists y)Gwy)$
q. $(\forall x) \sim Fx \equiv (\forall z) \sim Hza$

4. For each of the following sentences, give the substitution instance in which 'a' is the instantiating term.

a. $(\forall w)(Mww \ \& \ Fw)$
*b. $(\exists y)(Mby \supset Mya)$
c. $(\exists z) \sim (Cz \equiv \sim Cz)$
*d. $(\forall x)[(Laa \ \& \ Lab) \supset Lax]$
e. $(\exists z)[Fz \ \& \sim Gb) \supset (Bzb \vee Bbz)]$
*f. $(\exists w)[Fw \ \& \ (\forall y)(Cyw \supset Cwa)]$
g. $(\forall y)[\sim (\exists z)Nyz \equiv (\forall w)(Mww \ \& \ Nyw)]$
*h. $(\forall y)[(Fy \ \& \ Hy) \supset [(\exists z)(Fz \ \& \ Gz) \supset Gy]]$
i. $(\exists x)(Fxb \equiv Gbx)$
*j. $(\forall x)(\forall y)[(\exists z)Hzx \supset (\exists z)Hzy]$
k. $(\forall x) \sim (\exists y)(Hxy \ \& \ Hyx)$
*l. $(\forall z)[Fz \supset (\exists w)(\sim Fw \ \& \ Gwaz)]$
m. $(\forall w)(\forall y)[(Hwy \ \& \ Hyw) \supset (\exists z)Gzw]$
*n. $(\exists z)(\exists w)(\exists y)[(Fzwy \equiv Fwzy) \equiv Fyzw]$

5. Which of the following examples are substitution instances of the sentence '$(\exists w)(\forall y)(Rwy \supset Byy)$'?

a. $(\forall y)Ray \supset Byy$
*b. $(\forall y)(Ray \supset Byy)$
c. $(\forall y)(Rwy \supset Byy)$
*d. $(\forall y)(Rcy \supset Byy)$
e. $(\forall y)(Ryy \supset Byy)$
*f. $(\exists y)(Ray \supset Byy)$
g. $(Ray \supset Byy)$
*h. $(\forall y)(Ray \supset Baa)$
i. $Rab \supset Bbb$

6. Which of the following examples are substitution instances of the sentence '(∀x)[(∀y) – Rxy ≡ Pxa]'?

a. (∀y) – Ray ≡ Paa
*b. (∀y) – Raa ≡ Paa
c. (∀y) – Ray ≡ Pba
*d. (∀y) – Rpy ≡ Ppa
e. (∀y)(– Ryy ≡ Paa)
*f. (∀y) – Ray ≡ Pya
g. (∀y) – Rax ≡ Paa
*h. (∀y) – Rcy ≡ Pca

7.6 A-, E-, I-, AND O-SENTENCES

In Section 7.4 we symbolized fairly simple sentences of English in *PL*. The quantified sentences of *PL* that we used each had as its immediate subformula either an atomic formula or the negation of an atomic formula. That is, we produced such sentences as '(∀x)Lmx', '(∃x)Txx', and '(∀y)Lhy & – (∃x)Lsx', but *not* such sentences as '(∀z)(Fz ⊃ Gzz)' or '(∀x)(∀y)[Fxy ≡ (∃z)Gzx]'. In Section 7.5 we presented the syntax of *PL* and became familiar with the syntactic properties of complex sentences of *PL*, including sentences containing multiple quantifiers. Some of these contained quantifiers with *overlapping scope*: that is, some had a quantifier falling within the scope of another quantifier—for example, '(∀y)' within the scope of '(∀x)' in '(∀x)(Fx ⊃ (∀y)Gxy)'. In this and the following sections we shall learn to use the resources of *PL* to express a rich variety of English claims. In this section and the next we limit ourselves to sentences of *PL* *without* quantifiers with overlapping scope, though we will work with sentences having multiple quantifiers and many-place predicates.

Sentences of *PL* such as those we produced in Section 7.4 express English claims to the effect that it is, or is not, the case that everything, or something, is, or is not, of the sort such-and-such, where we capture the 'such-and-such' with a single predicate. We also produced truth-functional compounds of such sentences and of atomic sentences of *PL*. Such sentences allow us to express a substantial variety of English claims within *PL*. We can, for example, say that all bears are dangerous—by making our universe of discourse (UD) bears and using 'Dx' for 'x is dangerous', '(∀w)Dw' will do the job. But with the resources so far used and a UD of *all* bears, we cannot say that grizzly bears are dangerous and black bears are not. The limitation is substantial, for our UD is frequently diverse and it is rare that we want to say that everything, or nothing, in such a UD is of the sort specified by an atomic formula. What is needed is a way of saying, not that *everything*, or *something*, or *nothing* is of the sort specified by a given atomic formula, but rather that *everything*, or *something*, or *nothing* of the sort specified by a given formula is (or is not) of

the sort specified by a second formula. We want, for example, to be able to symbolize sentences such as the following in *PL*:

All dolphins are mammals.
All reptiles are cold-blooded.
Every cheese is a dairy product.
Every logic text bores Michael.

We also want to be able to symbolize such claims as these:

No fatty foods are conducive to good health.
No government is unbureaucratic.
No zebra is unicolored.
Some automobiles are Fords.
Some apples are Granny Smiths.
Some instructors are without a sense of humor.
Some horses are not racehorses.
Some lawyers are not rich.

The mechanism for symbolizing claims such as these is to let a quantifier's scope or interpretive range extend not just into an atomic formula or the negation of an atomic formula but also to more complex formulas. Consider the first of the examples, 'All dolphins are mammals'. Suppose our UD is all living things, 'Dy' symbolizes 'y is a dolphin', and 'My' symbolizes 'y is a mammal'. Here is an *unsuccessful* attempt at symbolizing 'All dolphins are mammals':

$$(\forall y)Dy \ \& \ (\forall y)My$$

This sentence, a conjunction, asserts both that all things are dolphins and that all things are mammals. This is patently false where the UD is *all* living things.

What we want is a way of saying, not that each thing is both a dolphin and a mammal, but rather that each thing that is a dolphin is also a mammal. The universal quantifier is, however, used to make a claim about each thing in the UD. So the trick is to figure out what claim we can make about each thing that will amount to our attributing being a mammal to each dolphin but not to everything. The puzzle is solved when we recall that the material conditional, a sentence of the form $\mathbf{P} \supset \mathbf{Q}$, asserts neither **P** nor **Q** but rather asserts **Q** on the condition that **P**. What we can say of each thing, dolphins and nondolphins alike, is that if the thing in question is a dolphin then it is a mammal. This claim is as true of rattlesnakes, bumblebees, and bacteria as it is of dolphins. So an appropriate sentence of *PL* is

$$(\forall z)(Dz \supset Mz)$$

Here we are saying, again, neither that each living thing is a dolphin nor that each living thing is a mammal, but rather that each living thing is such that *if* it is a dolphin *then* it is a mammal. So this claim applied to rattlesnakes comes to naught, but when applied to dolphins, it commits us to dolphins being mammals.

It is also important to note that '(∀y)(Dy & My)' is not an appropriate symbolization of 'All dolphins are mammals'. Given our UD and specification of predicates, this sentence of *PL* says that each living thing is of the sort specified by the conjunction 'Dy & My'—that is, that each living things is both a dolphin and a mammal. And this is equivalent to '(∀y)Dy & (∀y)My', which is, as we pointed out, an incorrect symbolization of our English sentence.

With this model in hand, it is easy to symbolize the next three of our examples. Again taking our UD to be all living things and using 'Rx' for 'x is a reptile' and 'Cx' for 'x is cold-blooded', we can symbolize 'All reptiles are cold-blooded' as

$$(\forall w)(Rw \supset Cw)$$

The foregoing is as true of dolphins as it is of rattlesnakes, for what it says of a given dolphin is that *if* that thing is a reptile (which it is not) *then* it is cold-blooded. Changing our UD to foods and using 'Cy' for 'y is a cheese' and 'Dz' for 'z is a dairy product', we can symbolize 'Every cheese is a dairy product' as

$$(\forall x)(Cx \supset Dx)$$

And if we take our UD to be books and people, 'Lz' to represent 'z is a logic text', 'Bzw' to represent 'z bores w', and 'm' to stand for Michael, we can symbolize 'Every logic text bores Michael' as

$$(\forall y)(Ly \supset Bym)$$

In sentences of *PL* such as those we have just presented, it is essential that the quantifier's scope extend over the entire sentence. For example, removing the parentheses around '(Ly ⊃ Bym)' produces '(∀y)Ly ⊃ Bym', which is a formula but not a sentence of *PL*. In the present examples it is also important that we use one and the same variable in the quantifier and in the atomic formulas it interprets. Doing otherwise also produces a nonsentence—for example, '(∀y)(Lx ⊃ Bym)'. In the foregoing discussion we purposely used different variables in the specification of predicates of *PL* and in the symbolizations we gave using those predicates. We did so to illustrate that the variables used in specifying predicates, whether informally as above or in symbolization keys, are only place-holders, marking holes in predicates. In actual symbolizations using those predicates, other variables and/or individual constants may be used.

The claim, 'No fatty foods are conducive to good health', asserts that anything that is a fatty food is not conducive to good health. Taking foods as our UD and using 'Fz' for 'z is fatty' and 'Cw' as 'w is conducive to good health', we can symbolize this claim in *PL* as

$$(\forall x)(Fx \supset \sim Cx)$$

Alternatively, but equivalently, we can see 'No fatty foods are conducive to good health' as denying that there is a fatty food that is conducive to good health.

This suggests the symbolization

$$\sim (\exists x)(Fx \ \& \ Cx)$$

These two sentences of $PL_=$ as one would expect, turn out to be equivalent. The latter claim of PL is very different from '$\sim (\exists x)Fx \ \& \ \sim (\exists x)Cx$', which asserts both that nothing is fatty and that nothing is conducive to good health.

Next we can symbolize 'No government is unbureaucratic' by taking our UD to be organizations, 'Gx' to represent 'x is a government', and 'Bx' 'x is bureaucratic'. Since to say no government is unbureaucratic is just to say that every government is bureaucratic, an appropriate symbolization in PL is

$$(\forall y)(Gy \supset By)$$

An equally appropriate symbolization is

$$\sim (\exists y)(Gy \ \& \sim By)$$

Returning to a UD of living things and taking 'Zy' to represent 'y is a zebra' and 'Uw' to represent 'w is unicolored', we can symbolize the claim 'No zebra is unicolored' either as

$$(\forall z)(Zz \supset \sim Uz)$$

or as

$$\sim (\exists w)(Zw \ \& \ Uw)$$

The first says of each thing in the UD that if it is a zebra then it is not unicolored; the second says that it is not the case that there is something in the UD that is a zebra and unicolored. These are equivalent.

Now take our UD to be vehicles (including automobiles, bicycles, boats, airplanes, and so on). The claim 'Some automobiles are Fords' can be construed, not as making a conditional claim about each thing in the UD, but rather as saying that in the UD there is at least one thing that both is a car and is a Ford (remember, we take 'some' to mean 'at least one'). Where 'Ax' stands for 'x is an automobile' and 'Fx' for 'x is a Ford', an appropriate symbolization is

$$(\exists y)(Ay \ \& \ Fy)$$

Similarly, if we take our UD to be all fruits and vegetables, 'Ay' to represent 'y is an apple', and 'Gy' to represent 'y is a Granny Smith', we can symbolize the claim 'Some apples are Granny Smiths' as

$$(\exists z)(Az \ \& \ Gz)$$

Taking our UD to be people, 'Ix' to represent 'x is an instructor', and 'Hx' to represent 'x has a sense of humor', 'Some instructors are without a sense of humor' can be symbolized as

$$(\exists y)(Iy \ \& \sim Hy)$$

And the claim 'Some horses are not racehorses' can be symbolized as

$$(\exists w)(Hw \ \& \sim Rw)$$

if we take our UD to be living things, 'Hx' to represent 'x is a horse', and 'Rx' to represent 'x is a racehorse'. Finally 'Some lawyers are not rich' can be symbolized as

$$(\exists y)(Ly \ \& \sim Ry)$$

with a UD of living things and 'Lw' and 'Rw' representing, respectively, 'w is a lawyer' and 'w is rich'.

We noted in Chapter 1 that Aristotelian logic holds that syllogistic arguments are composed of sentences of the following sorts:

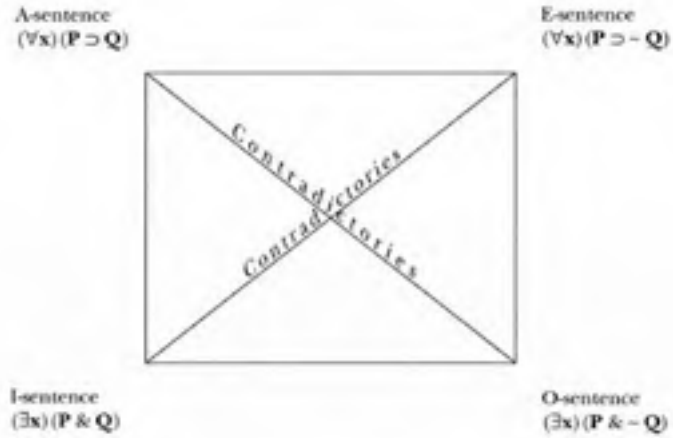
All As are Bs.
No As are Bs.
Some As are Bs.
Some As are not Bs.

where 'A' and 'B' are used as variables for general terms. Sentences of these types are traditionally classified as **A-, E-, I-, and O-sentences,** respectively. Using '**P**' and '**Q**' as variables for formulas containing the variable **x**, we can employ the following schema to present this traditional classification as it applies to sentences of *PL*:

A: $(\forall \mathbf{x})(\mathbf{P} \supset \mathbf{Q})$
E: $(\forall \mathbf{x})(\mathbf{P} \supset \sim \mathbf{Q})$
I: $(\exists \mathbf{x})(\mathbf{P} \ \& \ \mathbf{Q})$
O: $(\exists \mathbf{x})(\mathbf{P} \ \& \sim \mathbf{Q})$

The sentences of *PL* we have so far considered in this section can, in fact, each be seen as being of one of these four sorts, as can a very large number of other sentences. For example, we have symbolized 'All dolphins are mammals' as '$(\forall z)(Dz \supset Mz)$', an A-sentence; 'No fatty foods are conducive to good health' as '$(\forall x)(Fx \supset \sim Cx)$', an E-sentence; 'Some automobiles are Fords' as '$(\exists y)(Ay \ \& \ Fy)$', an I-sentence; and 'Some horses are not racehorses' as '$(\exists w)(Hw \ \& \sim Rw)$', an O-sentence.

Aristotle believed that there are important logical relations among A-, E-, I-, and O-sentences. These are usually presented using a "square of opposition":



The following relationships hold: If an A-sentence is true, then the corresponding O-sentence is false, and vice versa; and if an E-sentence is true, the corresponding I-sentence is false, and vice versa. This is sometimes expressed by saying that the sentences connected by diagonal lines are **contradictories**—if either is true the other is false. It follows that each sentence on the square of opposition is equivalent to the negation of the sentence at the other end of the diagonal. Thus A-sentences are equivalent to the negation of O-sentences, and vice versa; and E-sentences are equivalent to the negation of I-sentences, and vice versa. This gives us four pairs of equivalences:

(∀x)(P ⊃ Q)	and	~ (∃x)(P & ~ Q)
(∀x)(P ⊃ ~ Q)	and	~ (∃x)(P & Q)
(∃x)(P & Q)	and	~ (∀x)(P ⊃ ~ Q)
(∃x)(P & ~ Q)	and	~ (∀x)(P ⊃ Q)

An example will help here. We will use the following symbolization key:

UD: The jawbreakers (hard, round candies) in a large glass jar
Yz: z is yellow
Sz: z is sweet

The following are examples of A-, E-, I-, and O-sentences of English and their symbolizations in *PL*:

All yellow jawbreakers are sweet.	$(\forall w)(Yw \supset Sw)$
No yellow jawbreakers are sweet.	$(\forall w)(Yw \supset \sim Sw)$
Some yellow jawbreakers are sweet.	$(\exists w)(Yw \ \& \ Sw)$
Some yellow jawbreakers are not sweet.	$(\exists w)(Yw \ \& \sim Sw)$

The A-sentence 'All yellow jawbreakers are sweet' is the contradictory of the O-sentence 'Some yellow jawbreakers are not sweet'. If it is true that all yellow jawbreakers are sweet, then it is clearly false that some yellow jawbreakers are not sweet, and vice versa. So '$(\forall w)(Yw \supset Sw)$' is equivalent to the negation of the corresponding O-sentence, that is, equivalent to '$\sim (\exists w)(Yw \ \& \sim Sw)$'. And the O-sentence '$(\exists w)(Yw \ \& \sim Sw)$' is equivalent to the negation of the corresponding A-sentence, that is, to '$\sim (\forall w)(Yw \supset Sw)$'. So, too, if the E-sentence 'No yellow jawbreakers are sweet' is true, then its contradictory, the I-sentence 'Some yellow jawbreakers are sweet', is false, and vice versa. So the E-sentence '$(\forall w)(Yw \supset \sim Sw)$' is equivalent to '$\sim (\exists)(Yw \ \& \ Sw)$'. And the I-sentence '$(\exists w)(Yw \ \& \ Sw)$' is equivalent to the negation of the corresponding E-sentence, that is, to '$\sim (\forall w)(Yw \supset \sim Sw)$'. These relationships explain why, in symbolizing the examples at the beginning of this section, we often came up with two alternative, equivalent symbolizations.

In some systems of logic, if an A-sentence is true then so is the corresponding I-sentence, and if an E-sentence is true then so is the corresponding O-sentence. These relationships do not hold in *PL*. This may *seem* counterintuitive, for it is very tempting to believe that if 'All rabbits are mammals' is true, then so must be 'Some rabbits are mammals', and that if 'No rabbits are cold-blooded' is true, then so must be 'Some rabbits are not cold-blooded'. The reason these relationships do not hold for A- and I- and for E- and O-sentences is that in *PL* we allow predicates that are not satisfied by any member of the universe of discourse.

Given that we treat universal claims as saying of each thing that if it is of this sort it is also of that sort, this is obviously a sensible policy. For example, when working with a culture of unknown bacteria, we might reasonably say, after having placed the culture in a hermetically sealed container for an appropriate length of time,

All the aerobic (air-dependent) bacteria in the culture are dead.

Our claim will be true even if there are no such bacteria in the culture, for we will paraphrase it as saying of each bacterium, of whatever sort, that *if* it is aerobic *then* it is dead. Since there may be no aerobic bacteria in the culture, dead or alive, we clearly do not want the corresponding I-sentence

Some aerobic bacteria in the culture are dead.

to follow, for it says that there are bacteria in the culture that are both aerobic and dead. Again, in the system we are developing, A-sentences do not entail I-sentences and E-sentences do not entail O-sentences.

Aristotle also believed that A- and E-sentences are contraries and that I- and O-sentences are subcontraries. That is, an A- and the corresponding E-sentence cannot both be true, and an I- and the corresponding O-sentence cannot both be false. Neither of these relationships holds in *PL*, again because we allow predicates that are not satisfied by any member of the universe of discourse. Hence, if there are no things in the universe of discourse that are of the sort specified by the one-place predicate 'F', then both the A-sentence '$(\forall x)(Fx \supset Gx)$' and the corresponding E-sentence '$(\forall x)(Fx \supset \sim Gx)$' will be true, and both the I-sentence '$(\exists x)(Fx \,\&\, Gx)$' and the corresponding O-sentence '$(\exists x)(Fx \,\&\, \sim Gx)$' will be false.

Neither every sentence of English nor every sentence of *PL* can reasonably be construed as being of one of the four sorts of sentences we have been discussing. However, many can, and it is frequently helpful in symbolizing sentences of English to keep these four types of sentences in mind.

Not every sentence of English that can be symbolized as an A-sentence uses the quantity term 'all'. For example, in an introductory biology class an instructor might assert any one of the following sentences:

> All mammals are warm-blooded.
> Every mammal is warm-blooded.
> Each mammal is warm-blooded.
> Mammals are warm-blooded.
> A mammal is warm-blooded.

In the envisioned context these sentences are interchangeable. They can all be symbolized as the A-sentence '$(\forall y)(My \supset Wy)$' given a UD of living things and using 'Mz' for 'z is a mammal' and 'Wz' for 'z is warm-blooded'. This is true of the fourth example, 'Mammals are warm-blooded', even though this sentence contains no quantity term. The context makes it clear that it is all, not just some, mammals that are under discussion. The fifth sentence, 'A mammal is warm-blooded', is about all mammals even though it uses the singular article 'a'. This is not uncommon. Consider 'A mind is a terrible thing to waste', 'An accident incurred while at work is covered by the employee insurance plan', or 'An unexcused absence on an examination day will result in a failure'. The claims just mentioned concern, respectively, *all* minds, *all* accidents incurred while at work, and *all* unexcused absences on examination days.

'Any' can also be used to make a claim about all things of the specified sort. The cynic's crack about the findings of modern medicine, "Anything that tastes good is bad," is equivalent to 'All things that taste good are bad'. Both can be symbolized as '$(\forall x)(Tx \supset Bx)$' given a UD of all foods and drinks and taking 'Tx' to represent 'x tastes good' and 'Bx' to represent 'x is bad'.

It is important to remember that claims of the sort we have been discussing, claims to the effect that all things of this sort (for example, mammals) are also of that sort (for example, warm-blooded), do not pick out or refer to a group of things and then predicate something of that group. It is not the group or set of mammals that is warm-blooded but each individual mammal. ('Supertankers are very large ships' means that each supertanker is a very large ship, not that the set of supertankers is very large, that is, not that there are a very large number of supertankers.) The claims we have been discussing are better analyzed as universal claims deriving from conditionals that apply to each thing in the given universe of discourse. They say of each thing under discussion that if it is of such-and-such a sort then it is also thus-and-so. Each living thing is such that if it is a mammal then it is warm-blooded. This is, again, true as much of a given reptile as it is of a given koala bear.[11]

E-sentences, which can be symbolized as sentences of *PL* of the form $(\forall \mathbf{x})(\mathbf{P} \supset \sim \mathbf{Q})$, are also universal claims. They say of each thing under discussion that if it is of the sort **P** then it is not of the sort **Q**. Again assuming the context of an introductory biology class, all of the following can be used to make the point that reptiles are not warm-blooded:

No reptile is warm-blooded.
Reptiles are not warm-blooded.
A reptile is not warm-blooded.

Taking our universe of discourse to be living things and using 'Rw' to represent 'w is a reptile' and 'Ww' to represent 'w is warm-blooded', we can symbolize all these claims as

$$(\forall y)(Ry \supset \sim Wy)$$

We claimed earlier that 'Mammals are warm-blooded' and 'All mammals are warm-blooded' make the same claim. So, too, one would expect 'Reptiles are not warm-blooded' and 'All reptiles are not warm-blooded' to make the same claim. And, literally speaking, they do. 'All reptiles are not warm-blooded' says, literally, that each thing that is a reptile is not a thing that is warm-blooded, and thus can be symbolized as above.

But there is a complication here. Imagine a conversation about future careers. Someone says, "I want to become a lawyer because lawyers are all rich"; someone more thoughtful replies, "While the stereotype of lawyers may be that they are rich, the fact is that all lawyers are not rich". The claims we want to contrast are 'All lawyers are rich' and 'All lawyers are not rich'. The first clearly

[11] But consider 'Insects are more numerous than mammals'. This is not a disguised conditional claim about each living thing—its force is not 'Each living thing is such that if it is an insect it is more numerous than . . .' The correct analysis here is rather something like 'The set consisting of all insects is larger than the set consisting of all mammals'—that is, a claim about a relation between two things, the set of insects and the set of mammals.

means that each lawyer is rich and can be symbolized, given a UD of people and using 'Ly' as 'y is a lawyer' and 'Ry' as 'y is rich', as

$$(\forall z)(Lz \supset Rz)$$

The second claim, 'All lawyers are not rich', literally means that each lawyer is not rich, a claim that can be symbolized as an E-sentence:

$$(\forall z)(Lz \supset \sim Rz)$$

But in the envisioned conversation 'All lawyers are not rich' is clearly intended, not as the claim that there are no rich lawyers, but rather merely as a denial of 'All lawyers are rich', that is, as equivalent to 'It is not the case that all lawyers are rich'. Thus it can reasonably be symbolized as

$$\sim (\forall z)(Lz \supset Rz)$$

And this sentence of *PL*, the denial of an A-sentence, is equivalent to the O-sentence '$(\exists z)(Lz \mathbin{\&} \sim Rz)$', which says 'There is something that is a lawyer and is not rich'.

In practice we must rely on the context to determine whether what is literally an E-sentence—our example was 'All lawyers are not rich', and another is 'Every politician is not a scoundrel'—is being used as an E-sentence or as the negation of an A-sentence. In this text, whenever we do use a sentence of the sort 'All **P**s are not **Q**s' or 'Each **P** is not **Q**', and no context is provided, we mean the sentence to be interpreted literally as saying each and every **P** thing is not a **Q** thing.

I- and O-sentences of *PL* are existential claims. They do not make a claim about each thing in the universe of discourse; rather, they say that included within that universe is at least one thing of a specified sort, either the sort **P** & **Q** (an I-sentence) or the sort (**P** & ~ **Q**) (an O-sentence). Both

Some mammals are carnivorous

and

There are carnivorous mammals

can be symbolized as the O-sentence '$(\exists y)(My \mathbin{\&} Cy)$', where our UD is living things, 'Mx' represents 'x is a mammal', and 'Cx' represents 'x is carnivorous'. Similarly 'Some mammals are not carnivorous' and 'There are mammals that are not carnivorous' can both be symbolized as

$$(\exists z)(Mz \mathbin{\&} \sim Cz)$$

Note that 'Some mammals are carnivorous' does not identify a particular mammal and say of it that it is carnivorous. It says that there are carnivorous

mammals but not which ones they are. Note also that the following are not symbolizations of 'Some mammals are carnivorous':

$(\exists x)Mx \ \& \ (\exists x)Cx$
$(\exists x)(Mx \supset Cx)$

The first is a truth-functional compound. It says that there is something that is a mammal and there is something (not necessarily the same thing) that is carnivorous. The second says that there is some living thing such that if it is a mammal then it is carnivorous. While this is true, it is a much weaker claim than is intended. It would, for example, be true even if the UD were limited to reptiles. For each reptile (and hence at least one) is such that *if* it is a mammal (which it is not) *then* it is carnivorous. Remember that *material conditionals with false antecedents are true.*

Note that neither '$(\forall x)(Mx \supset Wx)$' nor '$(\exists x)(Mx \ \& \ Cx)$' contains outermost parentheses. The formulas to which the quantifiers attach, '$(Mx \supset Wx)$' and '$(Mx \ \& \ Cx)$', respectively, do contain outer parentheses, but these are not "outer" once the quantifiers are attached. '$(\forall x)Mx \supset Wx$' is therefore not an informal version of '$(\forall x)(Mx \supset Wx)$', and '$(\exists x)Mx \ \& \ Cx$' is not an informal version of '$(\exists x)(Mx \ \& \ Cx)$'. Rather, both are formulas that are not sentences of *PL*. The main logical operator of the first is '$\supset$', not '$(\forall x)$', and so the occurrence of 'x' in 'Wx' is not bound; and the main logical operator of the second is '&', not '$(\exists x)$', and so the occurrence of 'x' in 'Cx' is not bound.

We next symbolize a further group of sentences about the people in Michael's office:

1. Everyone whom Michael likes is easygoing.
2. Everyone who is taller than Rita is taller than Henry.
3. No one who likes Michael likes Henry.
4. Some of those whom Michael likes like Rita.
5. Some of those whom Michael likes don't like Rita.

We will use the symbolization key given in Section 7.3:

UD: People in Michael's office
Lxy: x likes y
Ex: x is easygoing
Txy: x is taller than y
h: Henry
m: Michael
r: Rita
s: Sue

The first of these claims is a straightforward A-sentence and can be symbolized as

$(\forall x)(Lmx \supset Ex)$

The only interesting difference between this and the A-sentences we symbolized earlier (for example, 'All mammals are warm-blooded') is that here the antecedent of the immediate subformula, 'Lmx', of '(Lmx ⊃ Ex)' is an atomic formula formed from a two-place rather than a one-place predicate. The second sentence is also an A-sentence, and here both the antecedent and the consequent of the immediate subformula are formed from two-place predicates:

(∀z)(Tzr ⊃ Tzh)

The third English sentence—'No one who likes Michael likes Henry'—can be parsed as 'Each thing is such that if it likes Michael then it is not the case that it likes Henry', an E-sentence, and symbolized as

(∀x)(Lxm ⊃ ~ Lxh)

This English sentence can also be symbolized as the negation of an I-sentence:

~ (∃x)(Lxm & Lxh)

which can be read as 'It is not the case that there is something that likes Michael and likes Henry'. The fourth and fifth sentences can be treated as I- and O-sentences, respectively. Appropriate symbolizations are

(∃w)(Lmw & Lwr)
(∃w)(Lmw & ~ Lwr)

Next we work through a series of symbolizations concerning the marbles being used in a marble game. We will symbolize these sentences:

1. All the marbles are blue.
2. None of the marbles is blue.
3. Some of the marbles are blue.
4. Some of the marbles are not blue.
5. Some but not all of the marbles are blue.
6. All the marbles are blue or all the marbles are green.
7. Some of the marbles are blue and some are green, but none is red.
8. If any marble is blue they all are.
9. If any marble is blue it's a cat's-eye.
10. All the shooters are red.

To illustrate how the choice of a UD affects the symbolizations required, we give two symbolizations for each of the above sentences, the first using the

symbolization key:

UD: The marbles being used by Ashley, Clarence, Rhoda, and Terry
Bx: x is blue
Gx: x is green
Rx: x is red
Sx: x is a shooter
Cx: x is a cat's-eye

The second symbolization key we will use is like the above but with a UD of the marbles being used and the players, and the additional predicate 'Mx' for 'x is a marble'. Thus we have

UD: Marbles	UD: Marbles and marble players
1′. $(\forall y)By$	1″. $(\forall y)(My \supset By)$
2′. $\sim (\exists y)By$	2″. $\sim (\exists y)(My \ \& \ By)$
3′. $(\exists y)By$	3″. $(\exists y)(My \ \& \ By)$
4′. $(\exists y) \sim By$	4″. $(\exists y)(My \ \& \sim By)$
5′. $(\exists y)By \ \& \sim (\forall y)By$	5″. $(\exists y)(My \ \& \ By) \ \& \sim (\forall y)(My \supset By)$
6′. $(\forall z)Bz \vee (\forall y)Gy$	6″. $(\forall z)(Mz \supset Bz) \vee (\forall y)(My \supset Gy)$
7′. $[(\exists x)Bx \ \& \ (\exists x)Gx] \ \& \sim (\exists x)Rx$	7″. $[(\exists x)(Mx \ \& \ Bx) \ \& \ (\exists x)(Mx \ \& \ Gx)] \ \& \sim (\exists x)(Mx \ \& \ Rx)$
8′. $(\exists w)Bw \supset (\forall x)Bx$	8″. $(\exists w)(Mw \ \& \ Bw) \supset (\forall x)(Mx \supset Bx)$
9′. $(\forall x)(Bx \supset Cx)$	9″. $(\forall x)[(Mx \ \& \ Bx) \supset Cx]$
10′. $(\forall y)(Sy \supset Ry)$	10″. $(\forall y)[(My \ \& \ Sy) \supset Ry]$

If the universe of discourse is just marbles, then 1′, '$(\forall y)By$', is an appropriate symbolization of 'All the marbles are blue'. But 1′ will not suffice if the UD is marbles and marble players, for we want to say that the marbles are blue but not that the players are. So 1″ is called for, '$(\forall y)(My \supset By)$'. Generally, where the universe of discourse is severely restricted and we do want to attribute some property to, or deny a property of, every or at least one member of that universe, A-, E-, I-, and O-sentences can be specified as follows, where **P** may be an atomic formula:

A: $(\forall \mathbf{x})\mathbf{P}$
E: $(\forall \mathbf{x}) \sim \mathbf{P}$
I: $(\exists \mathbf{x})\mathbf{P}$
O: $(\exists \mathbf{x}) \sim \mathbf{P}$

Our 1′ and 1″ are thus both A-sentences, and 6′ and 6″ are both disjunctions of A-sentences; 2′ and 2″ are both O-sentences; 3′ and 3″ are both I-sentences, and 7′ and 7″ are both conjunctions of the conjunction of two I-sentences and an E-sentence; 4′ and 4″ are both E-sentences; 5′ and 5″ are both conjunctions of an I-sentence and the negation of an A-sentence; 6′ and 6″ are both disjunctions of A-sentences; and 8′ and 8″ are both material conditionals whose antecedents are I-sentences and whose consequents are A-sentences. The 'any' of sentence 9 has the force of 'every', for 'If any marble is blue it's a cat's-eye' says the same thing as 'Every blue marble is a cat's-eye'. Hence 9′ is an A-sentence, as is 10′. In addition, 9″ and 10″ are A-sentences of the form $(\forall \mathbf{x})(\mathbf{P} \supset \mathbf{Q})$, where **P** is itself a conjunction. (We shall discuss this sort of complexity further in Section 7.7.)

While 2′ and 2″ are negations of I-sentences, we could also have used E-sentences, '$(\forall y) - By$' and '$(\forall y)(My \supset - By)$', respectively. For 3′ and 3″ we could have used the negation of E-sentences instead of I-sentences, and so on. The notion of there being a single correct, or even "most intuitive", symbolization for each English sentence is even more inappropriate here than it was in *SL*.

7.6E EXERCISES

1. Identify each of the following sentences as either an A-, E-, I-, or O-sentence and symbolize each in *PL* using the given symbolization key.

UD: A pile of coins consisting of quarters, dimes, nickels, and pennies
Qx: x is a quarter
Dx: x is a dime
Nx: x is a nickel
Cx: x contains copper
Px: x is a penny
Sx: x contains silver
Kx: x contains nickel
Zx: x contains zinc
Bx: x is a buffalo head coin
Ix: x is an Indian head coin
Mx: x was minted before 1965

a. All the pennies contain copper.
*b. Some of the dimes contain silver.
c. Some of the dimes do not contain silver.
*d. None of the quarters contains silver.
e. Some of the nickels are buffalo heads.
*f. All the nickels contain nickel.
g. No penny contains silver.
*h. Some of the nickels are not buffalo heads.
i. Every penny was minted before 1965.
*j. Some quarters were not minted before 1965.

k. Every coin containing silver contains copper.
*l. No penny contains nickel.
m. No coin that contains nickel contains silver.
*n. Every coin minted during or after 1965 contains zinc.
o. None of the quarters contains zinc.
*p. Some of the pennies are not Indian heads.

2. Symbolize the following sentences in *PL* using the given symbolization key.

UD: The jellybeans in a larger glass jar
By: y is black
Ry: y is red
Gy: y is green
Ly: y is licorice-flavored
Cy: y is cherry-flavored
Sy: y is sweet
Oy: y is sour

a. All the black jellybeans are licorice-flavored.
*b. All the red jellybeans are sweet.
c. None of the red jellybeans is licorice-flavored.
*d. Some red jellybeans are cherry-flavored.
e. Some jellybeans are black and some are red.
*f. Some jellybeans are sour and some are not.
g. Some jellybeans are black and some are red, but none is both.
*h. The red jellybeans are sweet, and the green jellybeans are sour.
i. Some jellybeans are black, some are sweet, and some are licorice-flavored.
*j. No jellybeans are red and licorice-flavored.
k. All the cherry-flavored jellybeans are red, but not all the red jellybeans are cherry-flavored.
*l. Every jellybean is red, and some are cherry-flavored and some are not cherry-flavored.
m. Every jellybean is red or every jellybean is black or every jellybean is green.
*n. Not all the jellybeans are licorice-flavored, but all those that are, are black.
o. Some red jellybeans are sweet and some are not.
*p. Some jellybeans are sweet and some are sour, but none is sweet and sour.
q. Some of the jellybeans are sour, but none of the licorice ones is.

3. With respect to the square of opposition, answer the following:
a. Can an I-sentence of *PL* and the corresponding O-sentence both be true? Can two such sentences both be false? Explain.
*b. Can an A-sentence of *PL* and the corresponding E-sentence both be false? Can two such sentences both be true? Explain.

7.7 SYMBOLIZATION TECHNIQUES

So far most of the A- and E-sentences we have considered have had as immediate subformulas atomic formulas, negations of atomic formulas, or material conditionals (whose immediate subformulas have themselves been

either atomic formulas or the negations of atomic formulas). Similarly most of the I- and O-sentences we have considered have had as immediate subformulas atomic formulas, the negations of atomic formulas, or conjunctions (whose immediate subformulas have been either atomic formulas or the negations of atomic formulas). But we need not restrict ourselves to these simple combinations. The **P** and **Q** of sentences of the forms

$$(\forall \mathbf{x})(\mathbf{P} \supset \mathbf{Q})$$
$$(\forall \mathbf{x})(\mathbf{P} \supset \sim \mathbf{Q})$$
$$(\exists \mathbf{x})(\mathbf{P} \,\&\, \mathbf{Q})$$
$$(\exists \mathbf{x})(\mathbf{P} \,\&\, \sim \mathbf{Q})$$

can themselves be any formulas of *PL*, including negations, conjunctions, disjunctions, material conditionals, and material biconditionals. Consider

> Everyone that Michael likes likes either Henry or Sue.

Here, and in other, more complicated examples to come, it may help to first paraphrase the English sentence into a more explicit quasi-English sentence:

> Each thing is such that if Michael likes it, then either it likes Henry or it likes Sue.

This sort of paraphrase uses 'it' where the symbolization in *PL* uses a variable. The paraphrase makes it clear that this sentence can be symbolized as a universally quantified sentence, the immediate subformula of which will be a material conditional, with the consequent of that conditional being a disjunction. Using a universe of discourse of people in Michael's office and the rest of the familiar symbolization key already specified, an appropriate symbolization is

$$(\forall y)[Lmy \supset (Lyh \vee Lys)]$$

This is an A-sentence, where **P** is 'Lmy' and **Q** is the disjunction '(Lyh ∨ Lys)'.

Here are some further sentences about Michael and his co-workers:

> Michael likes everyone that both Sue and Rita like.
> Michael likes everyone that either Sue or Rita likes.
> Rita doesn't like Michael but she likes everyone that Michael likes.

The first can be paraphrased as

> Each thing is such that if both Sue likes it and Rita likes it, then Michael likes it.

The paraphrase makes it clear that the *PL* symbolization of this sentence will be a universally quantified sentence whose immediate subformula is a material conditional, the antecedent of which will be a conjunction:

$$(\forall z)[(Lsz \,\&\, Lrz) \supset Lmz]$$

This is also an A-sentence, where **P** is '(Lsz & Lrz)' and **Q** is 'Lmz'.

The second sentence can be paraphrased as

> Each thing is such that if either Sue likes it or Rita likes it, then Michael likes it.

An appropriate symbolization is

$$(\forall w)[(Lsw \vee Lrw) \supset Lmw]$$

This is an A-sentence, where **P** is '(Lsw ∨ Lrw)', a disjunction, and **Q** is 'Lmw'.

The last of the sentences we are considering can be paraphrased as

> Both it is not the case that Rita likes Michael and each thing is such that if Michael likes it then Rita likes it.

An appropriate symbolization here is

$$\sim Lrm \,\&\, (\forall x)(Lmx \supset Lrx)$$

This sentence of *PL* is a conjunction of the negation of an atomic sentence and an A-sentence. We note that, if the foregoing sentence is true, if follows that Michael does not like himself, for if he did, Rita, who likes everyone Michael likes, would also like Michael, but she doesn't like Michael, so Michael must not like himself.

Lest we neglect bears, who figured briefly in the beginning of Section 7.6, we now consider

> Grizzly bears are dangerous but black bears are not.

The grammatical structure of English sentences is often a good guide to what the structure of symbolizations in *PL* should be, and this is so in the present case. The foregoing sentence can be paraphrased as 'Both grizzly bears are dangerous and it is not the case that black bears are dangerous' and symbolized, as one would expect, as the conjunction of an A- and an E-sentence. When we take our UD to be living things, 'Gw' to represent 'w is a grizzly bear', 'Bw' to represent 'w is a black bear', and 'Dw' to represent 'w is dangerous', an appropriate symbolization is

$$(\forall y)(Gy \supset Dy) \,\&\, (\forall z)(Bz \supset \sim Dz)$$

But consider next

> Grizzly bears and polar bears are dangerous, but black bears are not.

The 'black bears are not' clearly becomes, as above, '$(\forall z)(Bz \supset \sim Dz)$'. But what of 'Grizzly bears and polar bears are dangerous'? If we add 'Pw' for 'w is a polar bear' to our symbolization key, we might, as a first attempt, try

$$(\forall x)[(Gx \ \& \ Px) \supset Dx]$$

But we can see that this first attempt misses the mark as soon as we read it back into quasi-English, for it says 'Each thing is such that if it both is a grizzly bear and is a polar bear, then it is dangerous'. But there are no things that are both grizzly bears and polar bears, and our intent was not to make a vacuous claim.

In a second attempt we might realize that the original sentence is a shortened form of the fuller claim

> Grizzly bears are dangerous and polar bears are dangerous, but black bears are not

and then realize that an adequate symbolization is

$$[(\forall w)(Gw \supset Dw) \ \& \ (\forall w)(Pw \supset Dw)] \ \& \ (\forall w)(Bw \supset \sim Dw)$$

We have here the conjunction of a conjunction of A-sentences and an E-sentence. But there is also a shorter symbolization:

$$(\forall w)[(Gw \vee Pw) \supset Dw] \ \& \ (\forall w)(Bw \supset \sim Dw)$$

This sentence of *Pl.* is the conjunction of an A-sentence and an E-sentence. To say that grizzly bears *and* polar bears are dangerous is to say that everything in the group consisting of all grizzly bears *and* all polar bears is dangerous. And to be a member of that group, a creature need only be one or the other, a grizzly or a polar bear, not both.

Consider now 'Every self-respecting polar bear is a good swimmer'. Still taking our universe of discourse to be bears and adding as predicates 'Rxy' for 'x respects y' and 'Sx' for 'x is a good swimmer', we can symbolize this sentence as

$$(\forall z)[(Pz \ \& \ Rzz) \supset Sz]$$

We here illustrate that a thing can bear a relation to itself. (To be self-respecting is just to respect oneself.)

Here are two examples concerning the positive integers:

> Every integer is either odd or even

and

> Every integer is odd or every integer is even.

In the second sentence there are two quantity terms. Each falls within a disjunct of the overall sentence, which is clearly a disjunction. But in the first sentence the disjunction-indicating terms 'either' and 'or' both fall within the scope of the quantity term 'every'. This suggests, correctly, that the first sentence can be symbolized as a quantified sentence, and the second as a truth-functional compound of quantified sentences. If we restrict our universe of discourse to integers, appropriate paraphrases are

> Each integer y is such that either y is odd or y is even

and

> Either each integer y is such that y is odd or each integer y is such that y is even.

Using obvious predicates, we can now produce *PL* symbolizations:

$$(\forall y)(Oy \vee Ey)$$
$$(\forall y)Oy \vee (\forall y)Ey$$

There is a world of difference here. The first sentence of *PL* is clearly true: Each integer is either odd or even—one is odd, two is even, three is odd, four is even, and so on. The second sentence is just as clearly false: It is not the case that all integers are odd, and it is not the case that all integers are even. The great importance of the placement of quantifiers in relation to truth-functional connectives is here illustrated. (The first of the two sentences of *PL* is an A-sentence—it says each thing is of the sort specified by '$(Oy \vee Ey)$'; the second is a disjunction of A-sentences.)

Care must be taken when symbolizing English sentences using the quantity term 'any'. Consider these examples:

1. Anyone who likes Sue likes Rita.
2. Everyone who likes Sue likes Rita.
3. If anyone likes Sue, Michael does.
4. If everyone likes Sue, Michael does.
5. If anyone likes Sue, he or she likes Rita.

Assuming we restrict our UD to people, it is probably apparent that the first two of these sentences can each appropriately be symbolized as the A-sentence '$(\forall x)(Lxs \supset Lxr)$'. So in the first sentence the 'any' of 'anyone' has the force of 'every'. But in the third sentence 'anyone' does not have the force of

'everyone', for the third and fourth sentences clearly make different claims. (Sentence 3 may be very informative—we might suspect that someone likes Sue but have no idea that Michael does. But sentence 4 is not at all informative. Of course, Michael likes Sue if everyone does, for Michael is one of *everyone*.) Appropriate symbolizations for sentences 3 and 4 are, respectively,

$$(\exists x)Lxs \supset Lms$$

and

$$(\forall x)Lxs \supset Lms$$

Both of these sentences of *PL* are truth-functional compounds (material conditionals). It might be tempting to conclude that 'any' means 'every' except when it is used in the antecedent of an explicit conditional, in which case it means 'at least one'. But this rule is too simplistic, as sentence 5 makes clear. In 'If anyone likes Sue, he or she likes Rita', 'any' appears in the antecedent of an explicit English conditional. But here the force of 'any' cannot be captured by an existential quantifier, nor can the English sentence be symbolized as a conditional sentence of *PL*. Attempting to do so is likely to generate

$$(\exists x)Lxs \supset Lxr$$

which is a formula but not a sentence of *PL* (the third occurrence of 'x' is free). Changing the scope of the existential quantifier will not help either, for while

$$(\exists x)(Lxs \supset Lxr)$$

is a sentence of *PL* (albeit not a conditional), it says that there is someone such that if that person likes Sue then that person likes Rita. It is sufficient for the truth of this claim that there be someone who does not like Sue, for if a person does not like Sue, then '(Lxs ⊃ Lxr)' is true of that person—remember the weakness of the material conditional. To say what we want to say, we need a universally quantified sentence, that is, we need the symbolization we used for sentences 1 and 2:

$$(\forall x)(Lxs \supset Lxr)$$

This sentence of *PL* will be true if and only if each person who likes Sue also likes Rita. That we end up with this symbolization should not be surprising, for the force of sentence 5 ('If anyone likes Sue, he or she likes Rita') is, upon reflection, clearly the same as that of sentence 1 ('Anyone who likes Sue likes Rita').

A better rule can be formulated by appealing to the notion of *pronominal cross-reference*. In 'Sarah will deliver the lumber if she gets her truck fixed', the reference of 'she' is established by the earlier use of the noun 'Sarah'—there is pronominal cross-reference from the pronoun 'she' (as well as from 'her') back to the noun 'Sarah'. Pronominal cross-reference can be both to quantity terms and to nouns. In sentence 5, 'If anyone likes Sue, he or she likes Rita', the reference of 'he or she' is fixed by 'anyone who likes Sue'.

In sentence 3, 'If anyone likes Sue, Michael does', there is no pronominal cross-reference from the consequent of the English conditional back to the 'any' term in the antecedent. 'Michael does' in the consequent can be expanded to 'Michael likes Sue', a complete sentence that does not need further interpretation. But in 'If anyone likes Sue, he or she likes Rita', the consequent, 'he or she likes Rita', cannot be understood in isolation. This allows us to state the following rule:

> Where a quantity term is used in the antecedent of an English conditional and there is, in the consequent of that conditional, pronominal cross-reference to that quantity term, a universal quantifier is called for.

We can, with a little stretching, use this rule in dealing with such sentences as

> Anyone who fails the final examination flunks the course.

This sentence is not, on grammatical grounds, a conditional, and there is no obvious pronominal cross-reference. But since the person who flunks the course is the one who fails the final examination, we can paraphrase this sentence as a conditional in which there is pronominal cross-reference:

> If a person fails the final examination, then he or she flunks the course.

This is a sentence to which our new rule applies. Taking our universe of discourse to be students in the class and using 'Fx' for 'x fails the final examination' and 'Cx' for 'x flunks the course', we can, following this rule, offer

$$(\forall x)(Fx \supset Cx)$$

as an appropriate symbolization.

'Any' also functions differently from 'all', 'every', and 'each' when combined with a negation. For example, as noted in Section 7.4,

> Michael doesn't like everyone

and

> Michael doesn't like anyone

are very different claims. In general, 'not any' can be symbolized as the negation of an existential quantification ('not at least one'), whereas 'not every', 'not all', and 'not each' call for the negation of a universal quantification. In the present case, taking the people in Michael's office as our UD, 'm' as designating Michael, and 'Lxy' as 'x likes y', we can use '~ (∀x)Lmx' as a symbolization of the first sentence and '~ (∃x)Lmx' as a symbolization of the second.

Quantity constructions built from 'some' usually call for an existential quantifier. But some uses of 'some' constructions call for universal quantifiers, and the rule just developed helps in identifying them. Consider these two sentences:

If someone likes Sue, then he or she likes Rita.
If someone likes Sue, then someone likes Rita.

The first of these is a conditional with a quantity construction in the antecedent to which the 'he or she' in the consequent bears pronominal cross-reference. So a universal quantifier is called for, even though 'someone' usually signals an existential quantifier. A correct symbolization is '(∀x)(Lxs ⊃ Lxr)'. That this symbolization is correct becomes apparent when we reflect that the force of 'someone' in the first sentence is clearly that of 'anyone'. There is, in the second sentence, no pronominal cross-reference from the consequent back to the antecedent. The claim is not that if someone likes Sue then that very person likes Rita, but rather that if someone likes Sue then *someone*, quite possibly someone different, likes Rita. Here two existential quantifiers are called for:

(∃x)Lxs ⊃ (∃x)Lxr

Consider now some examples concerning runners:

UD: Runners
Bxy: x can beat y
Ay: y is on the American team
Sy: y is on the South African team
My: y is a marathon runner
Py: y is a sprinter
Dy: y has determination
Ey: y has endurance
Oy: y is over 50
Uy: y is under 20
j: Jim
k: Kerry
n: Noah
s: Seth
h: Shelly

Consider

> Marathon runners have both endurance and determination.
> Marathon runners are both over 50 and under 20.

The first example clearly attributes to marathon runners, and presumably to all marathon runners, two properties: endurance and determination. So an appropriate symbolization is

$$(\forall w)[Mw \supset (Ew \ \& \ Dw)]$$

Our symbolization is an A-sentence; it says that all things of this sort (marathon runners) are also of that sort (having endurance and determination). The second example should not be similarly taken to attribute two properties—being over 50 and being under 20—to all or even some marathon runners. Rather, this example has the force of 'Among marathon runners there are runners over 50 and runners under 20' and can be paraphrased as

> There are marathon runners that are over 50 and there are marathon runners that are under 20.

One appropriate symbolization is thus a conjunction of two existentially quantified sentences (each of which is an I-sentence):

$$(\exists x)(Mx \ \& \ Ox) \ \& \ (\exists x)(Mx \ \& \ Ux)$$

For our next two examples we consider

> There are no American sprinters over 50, but there are American marathon runners over 50.
>
> There are sprinters under 20 on both the American team and the South African team.

Our symbolization of the first of these examples is a conjunction:

$$\sim (\exists y)[(Py \ \& \ Oy) \ \& \ Ay] \ \& \ (\exists y)[(My \ \& \ Oy) \ \& \ Ay]$$

Note that the first conjunct of this sentence of *PL* is the negation of an I-sentence, and the second is an I-sentence. I-sentences say there are members of the UD that are such-and-such, and sometimes it takes several predicates to capture the content of 'such-and-such'. The intent of the second example is clearly not that there are sprinters under 20 who are on both the American and the South African teams, but rather that on each team there are sprinters under 20. So an appropriate symbolization is

$$(\exists z)[(Pz \ \& \ Uz) \ \& \ Az] \ \& \ (\exists w)[(Pw \ \& \ Uw) \ \& \ Sw]$$

Consider next

> Kerry and Shelly are both South African sprinters, and Shelly can beat every American sprinter Kerry can beat.

This example can be paraphrased, somewhat laboriously, as

> Kerry is a sprinter and Kerry is on the South African team, and Shelly is a sprinter and Shelly is on the South African Team; and every runner is such that, if she or he is a sprinter and is on the American team and Kerry can beat her or him, then Shelly can beat her or him.

An appropriate symbolization is

$$[(Pk \ \& \ Sk) \ \& \ (Ph \ \& \ Sh)] \ \& \ (\forall z)([(Pz \ \& \ Az) \ \& \ Bkz] \supset Bhz)$$

Next we consider

> If there is any marathon runner over 50 who can beat Seth, Jim can. Every South African sprinter can beat Jim, but they cannot all beat Seth. Noah is an American sprinter and marathon runner, and he can beat every sprinter, but not every marathon runner, on the South African team.

The first of these three sentences is fairly straightforward and can be symbolized either as

$$(\exists w)[(Mw \ \& \ Ow) \ \& \ Bws] \supset Bjs$$

or as

$$(\forall w)([(Mw \ \& \ Ow) \ \& \ Bws] \supset Bjs)$$

The second example *cannot* be symbolized as

$$(\forall x)[(Px \ \& \ Sx) \supset (Bxj \ \& \sim Bxs)]$$

for this sentence of *PL* says that all the South African sprinters are able to beat Jim and that they are all unable to beat Seth, whereas the original said merely that not all of them can beat Seth. The original contained two quantity expressions—'every' and 'all'—and we need a sentence of *PL* with two quantifiers:

$$(\forall x)[(Px \ \& \ Sx) \supset Bxj] \ \& \sim (\forall x)[(Px \ \& \ Sx) \supset Bxs]$$

This says that all the South African sprinters can beat Jim and that not all the South African sprinters can beat Seth (allowing that some may be able to do so), which is what was intended. The third example is a conjunction and can be paraphrased as

> Noah is on the American team and is a sprinter and a marathon runner, and Noah can beat every runner who is a sprinter and on the South African team, and Noah cannot beat every runner who is a marathon runner and on the South African team.

This can be symbolized as a conjunction, with the left conjunct being '[An & (Pn & Mn)]' and the right conjunct itself being a conjunction of an A-sentence and the negation of an A-sentence:

$$[An \ \& \ (Pn \ \& \ Mn)] \ \& \ ((\forall y)[(Py \ \& \ Sy) \supset Bny] \ \&$$
$$\sim (\forall y)[(My \ \& \ Sy) \supset Bny])$$

For our next set of examples, we expand the symbolization key used at the end of Section 7.6 as follows. The UD includes marbles and people. Also, we encounter a three-place predicate for the first time.

UD: Ashley, Clarence, Rhoda, Terry, and their marbles
a: Ashley
c: Clarence
r: Rhoda
t: Terry
Bx: x is blue
Gx: x is green
Rx: x is red
Sx: x is a shooter
Cx: x is a cat's-eye
Tx: x is a steely
Mx: x is a marble
Bxy: x belongs to y
Wxy: x wins y
Gxyz: x gives y to z

Here are our examples concerning an old-fashioned marble game:

1. All the cat's-eyes belong to Rhoda.
2. All the marbles but the shooters are cat's-eyes.
3. Some, but not all, of the cat's-eyes are green.
4. None of the steelies is red, green, or blue.
5. All of the shooters that are steelies belong to Terry.

6. Some green marbles and some blue marbles but no red ones belong to Clarence.
7. Ashley wins all Clarence's marbles.
8. Rhoda wins all Terry's cat's-eyes and shooters.
9. Terry doesn't have any marbles.
10. Rhoda gives all the red marbles she wins to Clarence.
11. Clarence gives all his green marbles to Ashley and all his blue marbles to Terry.

We now give one correct symbolization of each of these sentences. Then we shall discuss some of the noteworthy aspects of these examples.

1′. $(\forall y)(Cy \supset Byr)$
2′. $(\forall x)[(Mx \ \& \ \sim Sx) \supset Cx]$
3′. $(\exists x)(Cx \ \& \ Gx) \ \& \ \sim (\forall x)(Cx \supset Gx)$
4′. $(\forall w)[Tw \supset \sim (Rw \vee (Gw \vee Bw))]$
5′. $(\forall z)[(Sz \ \& \ Tz) \supset Bzt]$
6′. $[(\exists y)((My \ \& \ Gy) \ \& \ Byc) \ \& \ (\exists y)((My \ \& \ By) \ \& \ Byc)] \ \& \ \sim (\exists y)((My \ \& \ Ry) \ \& \ Byc)$
7′. $(\forall x)[(Mx \ \& \ Bxc) \supset Wax]$
8′. $(\forall x)([(Cx \vee Sx) \ \& \ Bxt] \supset Wrx)$
9′. $\sim (\exists z)(Mz \ \& \ Bzt)$
10′. $(\forall x)[((Mx \ \& \ Rx) \ \& \ Wrx) \supset Grxc]$
11′. $(\forall z)[(Mz \ \& \ Bzc) \supset ((Gz \supset Gcza) \ \& \ (Bz \supset Gczt))]$

Sentence 1 is unproblematic, and 1′ an obvious symbolization. Sentence 2 is not quite so straightforward. It does not claim that all the marbles are cat's-eyes—that can be symbolized as '$(\forall x)(Mx \supset Cx)$'—but that all the marbles but the *shooters* are cat's-eyes. Up to this point we have most commonly seen 'but' in contexts where it functions as a surrogate for 'and'. This is not the case here, where 'but' signals that the shooters are being exempted from the claim being made. Note that literally speaking no claim is being made about the shooters—either that they are or that they are not cat's-eyes. What is being said is merely that when the shooters are excluded the rest are cat's-eyes. (The context, for example, may be that someone asks whether all the marbles are cat's-eyes, and someone else replies as in example 2 and adds, when asked about the shooters, that she has examined all the marbles that are not shooters and found them all to be cat's-eyes but has not yet examined the shooters and hence has excluded them from consideration.) Analogously 'Everyone except Tom passed the test' does not *mean*—though it may *suggest*—that Tom did not pass. Tom's test may not yet be graded, or the speaker may not know how Tom fared or may simply not want to reveal whether Tom passed. In

general 'All but such-and-such' and 'All except such-and-such' do not mean 'All and not such-and-such'. Rather, they mean 'All excluding such-and-such', to be followed or not by a separate comment about such-and-such.

Example 3 is also straightforward. An alternative and perhaps more intuitive, although longer, symbolization for example 4 is '$(\forall x)(Tx \supset - Rx)$ & $[(\forall x)(Tx \supset - Gx)$ & $(\forall x)(Tx \supset - Bx)]$'. But note that we really do not need three quantifiers. We can, as in 4', single out the members of the UD we are concerned with (steelies) just once and then in one swoop deny that any such member is either red, or green, or yellow. Example 5 asserts, not that all the shooters belong to Terry, but that all those that are steelies do. So the group we need to single out consists of those things that are both shooters and steelies, and this is what we do in the antecedent of the conditional in 5.

We needed only one quantifier to symbolize example 4, but this is not so with example 6. Here even two quantifiers are not enough. For example, the force of

$$(\exists x)[(Mx \,\&\, Gx) \,\&\, (Bx \,\&\, Bxc)] \,\&\, - (\exists x)[(Mx \,\&\, Rx) \,\&\, Bxc]$$

is that Clarence possesses at least one marble that is both green and blue and that he possesses no red marbles, and this is not what example 6 claims.

Example 7 is easy enough once we realize that Clarence's marbles are just the marbles that belong to Clarence. What is of interest in example 8 is that it can be symbolized using only one quantifier, although Terry's cat's-eyes and her shooters may constitute mutually exclusive groups. For while we could use a conjunction, for example,

$$(\forall x)[(Cx \,\&\, Bxt) \supset Wrx] \,\&\, (\forall x)[(Sx \,\&\, Bxt) \supset Wrx]$$

doing so is being more verbose than we need to be. 'Cx & Bxt' applies to things that are cat's-eyes and belong to Terry. '$(Cx \vee Sx)$ & Bxt' picks out those things that are either cat's-eyes or shooters and that belong to Terry; that is, it *picks* out all the cat's-eyes and all the shooters that belong to Terry.

Examples 9 and 10 are straightforward. Example 11 is interesting in that it, like example 4, can be symbolized using just one quantifier. We could also have used two quantifiers:

$$(\forall x)[((Mx \,\&\, Gx) \,\&\, Bxc) \supset Gcxa] \,\&\, (\forall x)[((Mx \,\&\, Bx) \,\&\, Bxc) \supset Gcxt]$$

But if we first single out those things that are marbles and belong to Clarence, as we do in 11', and then say that if such a thing is green, then Clarence gives it to Ashley, and that if it is blue, then Clarence gives it to Terry, we can get by with one quantifier.

Before ending this section we issue some cautionary notes about symbolizing sentences in *PL*. The first concerns the selection of predicates of *PL* for use in symbolizing English sentences. Frequently, but not always, English descriptions that consist of "stacked-up" adjectives, as in 'A second-hand,

broken-down, uncomfortable, tan recliner is in the corner", can be captured by conjoining appropriate predicates of *PL*. Taking the furniture in the room to constitute the universe of discourse and using obvious predicates, we can symbolize the foregoing as

$$(\exists z)([(Sz \ \& \ Bz) \ \& \ (Uz \ \& \ Tz)] \ \& \ (Rz \ \& \ Cz))$$

This symbolization is appropriate because the recliner in question is second-hand, is broken-down, is uncomfortable, is tan, is a recliner, and is in the corner. In contrast, a bloody fool is presumably a very foolish person but not necessarily a person covered with blood. So, too, a counterfeit dollar is not something that both is counterfeit and is a dollar (because it is not a dollar). Similarly, while the animal in the corner may be a large mouse, it is not clear that there is something in the corner that is large, is an animal, and is a mouse—even large mice are not large as animals go. And a second-rate mathematician who is also a first-rate drama critic is not a second-rate person and a first-rate person. Rather, 'second-rate mathematician' and 'first-rate drama critic' should each normally be symbolized by a single predicate of *PL*, as should 'bloody fool', 'counterfeit dollar', and 'large mouse'.

This practice will cause problems in some contexts. For example, from 'Sue is a first-rate drama critic' we will not be able to infer 'Sue is a drama critic'. We can save such inferences by the admittedly ad hoc device of using one predicate for 'first-rate drama critic' and another for 'drama critic'. That is, using the symbolization key

UD: People
Fx: x is a first-rate drama critic
Dx: x is a drama critic
s: Sue

we can symbolize 'Sue is a first-rate drama critic' as

Fs & Ds

and 'Sue is a drama critic' as

Ds

And we can show that the second of these *PL* sentences follows from the first.

As this discussion illustrates, the appropriate selection of predicates commonly depends upon the context. For example, given just that the UD is animals and the sentence

Rabid bats are dangerous

and no context, we might decide to treat being a rabid bat as having a single property, use 'Rx' for 'x is a rabid bat', use 'Dx' for 'x is dangerous', and symbolize the example as

$$(\forall y)(Ry \supset Dy)$$

Alternatively we could treat being a rabid bat as having two properties: that of being a bat and that of being rabid (rabid bats are things that are both rabid and bats). Now, using 'Rx' for 'x is rabid' and 'Bx' for 'x is a bat', we could symbolize the given sentence as

$$(\forall y)[(Ry \ \& \ By) \supset Dy]$$

Taken in isolation, neither symbolization is preferable to the other. But suppose that, instead of the foregoing single sentence, we are given a complete argument:

> Some bats are rabid. Rabid animals are dangerous. Therefore some bats are dangerous.

Here we want our symbolization to reveal as much as possible of what is common to the premises and the conclusion. To do this we clearly need to use separate predicates of *PL* for 'x is a bat' and 'x is rabid'. Where animals constitute the universe of discourse, an appropriate symbolization is

$$\begin{array}{l} (\exists y)(Ry \ \& \ By) \\ (\forall z)(Rz \supset Dz) \\ \hline (\exists y)(By \ \& \ Dy) \end{array}$$

We can show that this is a valid argument of *PL*. But had we chosen to use a single predicate, say, 'Rx', to symbolize being a rabid bat, we would have had to use a different predicate to symbolize being a rabid animal, say, 'Ax':

$$\begin{array}{l} (\exists x)Rx \\ (\forall y)(Ay \supset Dy) \\ \hline (\exists y)(By \ \& \ Dy) \end{array}$$

In this second symbolization we have made opaque the obvious fact that rabid bats are rabid animals and the obvious fact that rabid bats are bats. As a result, although the English language argument is valid, as is our first symbolization of it, the second symbolization is not valid.

There is a further complication in the selection of predicates. Suppose that the Spanish explorer Ponce de Leon did, as legend has it, spend a lot of time searching for the fountain of youth. How would we symbolize the following?

Ponce de Leon is searching for the fountain of youth.

We cannot use

Spf

where 'Sxy' is interpreted as 'x is searching for y', 'p' designates Ponce de Leon, and 'f' the fountain of youth, for while Ponce de Leon might believe there is a fountain of youth, there is, in fact, no such thing. We can interpret 'Yx' as 'x is searching for the fountain of youth' and symbolize the sentence as

Yp

Although things that do not exist cannot be found, it is unfortunately all too easy to search for them. For this reason

Ponce de Leon is searching for mermaids

also cannot be symbolized using the two-place 'Sxy' for 'x is searching for y'. We might indeed be tempted, using 'Mx' for 'x is a mermaid', to offer the following as possible symbolizations of 'Ponce de Leon is searching for mermaids':

$(\exists y)(My \ \& \ Spy)$
$(\forall y)(My \supset Spy)$

But neither of these is adequate to the task. The problem is not with using 'Mx' for 'x is a mermaid' when there are no mermaids. As noted earlier, we do not presuppose that every predicate of *PL* we use is true of at least one member of the selected universe of discourse. Rather, one problem with the previous existentially quantified sentence is that it commits us, by its use of the existential quantifier, to there being at least one mermaid, whereas the sentence being symbolized does not. (One can search for what does not exist.) The universally quantified sentence of *PL* given earlier says not that there are mermaids—so here we escape a commitment to the existence of mermaids—but rather that anything that is a mermaid is such that Ponce de Leon is searching for it. This is too weak for, given the nonexistence of mermaids, it is true no matter what Ponce de Leon is doing, for it says only that if a thing is a mermaid (and nothing is) then Ponce de Leon is searching for it.

The way out of the present difficulty is to use a one-place predicate—for example, to interpret 'Mx' as 'x is searching for mermaids'—and to symbolize

Ponce de Leon is searching for mermaids

as

Mp

Difficulties arise in symbolizing sentences concerned with such activities as searching for, hunting, looking for, and . . . , even when what is being sought, hunted, desired, . . . , does exist. Suppose the sentence we want to symbolize is

Ponce de Leon is searching for a good harbor.

It might be thought that, if we interpret 'Sxy' as 'x is searching for y' and 'Gx' as 'x is a good harbor', a proper symbolization of this sentence would be

$(\exists x)(Gx \,\&\, Spx)$

This symbolization might be acceptable if Ponce de Leon is looking for a particular harbor, say, the harbor at Vera Cruz. But, if he is prowling the Florida coast and merely wants a haven from an impending storm, any good harbor, it is false to say that there is a good harbor such that he is looking for that harbor. Nor is '$(\forall x)(Gx \supset Spx)$' a proper symbolization. Imagine that there are three good harbors in his vicinity. Ponce de Leon will be glad to reach any one of them, and he is not interested in reaching all of them. So neither '$(\exists x)(Gx \,\&\, Spx)$' nor '$(\forall x)(Gx \supset Spx)$' is an acceptable symbolization—the first because it would be false to say of a good harbor our hero finds that he was searching for *that* harbor all along, and the second because he wants only one harbor and not all good harbors. So here, too, we should use a one-place predicate. If we interpret 'Hx' as 'x is searching for a good harbor', the proper symbolization is

Hp

On the other hand, if Ponce de Leon's ship got separated from an accompanying ship during the night, and Ponce de Leon is searching for *that* ship, a proper symbolization, using 'Sx' for 'x is a ship', would be

$(\exists x)(Sx \,\&\, Spx)$

Generally, unless what is being sought, hunted, searched for, hoped for, or desired is a particular thing, rather than a kind of thing, a one-place rather than a two-place predicate of *PL* should be used.[12]

[12] There are logics known as *intensional logics* in which problematic sentences of the sort just discussed can be further analyzed.

7.7E EXERCISES

1. Symbolize the following sentences in *PL* using the given symbolization key.

UD: Persons
Dx: x is at the door
Hx: x is honest
Ix: x is an influence peddler
Lx: x is likeable
Px: x is a politician
Rx: x is a registered lobbyist
h: Harrington

a. All politicians are honest.
*b. No politicians are honest.
c. Some politicians are honest.
*d. Some politicians are not honest.
e. An honest politician is not an influence peddler.
*f. An honest politician is at the door.
g. Politicians and influence peddlers are not all honest.
*h. Honest influence peddlers are nonexistent.
i. An influence peddler is honest only if he or she is a registered lobbyist.
*j. Some but not all registered lobbyists are honest.
k. If anyone is an influence peddler Harrington is.
*l. If anyone is an influence peddler, he or she is either a politician or a registered lobbyist.
m. If anyone is an influence peddler every registered lobbyist is.
*n. Harrington is no influence peddler but he is an honest politician.
o. No one is honest, a politician, and an influence peddler.
*p. Everyone is a politician but not everyone is honest.
q. If every politician is an influence peddler, then no politician is honest.
*r. Some politicians who are influence peddlers are honest, but none is likeable.
s. Registered lobbyists are likeable influence peddlers, but they are not honest.

2. Symbolize the following sentences in *PL* using the given symbolization key.

UD: Mammals
Cxy: x is chasing y
Lx: x is a lion
Ax: x is a formidable animal
Fx: is ferocious
Tx: x is a tiger
Bx: x is best avoided
b: Bruce Willis
d: Danny DeVito

a. A lion is a formidable animal.
*b. Lions are ferocious.
c. Lions are ferocious, but tigers are not.
*d. A lion is chasing Danny DeVito.
e. Danny DeVito is chasing a ferocious lion.

*f. Ferocious lions are best avoided.
g. Lions and tigers are ferocious.
*h. Lions and tigers are chasing Danny DeVito.
i. Some, but not all, tigers are ferocious.
*j. Ferocious lions and tigers are best avoided.
k. Any lion Bruce Willis is chasing is a formidable animal but is not ferocious.
*l. Danny DeVito and ferocious lions and tigers are all best avoided.
m. If any lion is ferocious, all tigers are.
*n. A lion is ferocious if and only if Danny DeVito is chasing it.
o. Bruce Willis is not ferocious, but he is best avoided.
*p. If Danny DeVito is ferocious, all lions are tigers.

3. Symbolize the following sentences in *PL* using the given symbolization key.

UD: Persons
Ex: x is a real estate agent
Lx: x is a lawyer
Px: x is a professor
Nx: x lives next door
Ix: x is rich
Sx: x can sell to yuppies
Yx: x is a yuppie
Rxy: x respects y
f: Fred

a. All real estate agents are yuppies.
*b. No real estate agents are yuppies.
c. Some but not all real estate agents are yuppies.
*d. Some real estate agents are yuppies and some are not.
e. If any real estate agent is a yuppie, all lawyers are.
*f. Any real estate agent who isn't a yuppie isn't rich.
g. If any real estate agent can sell to yuppies, he or she is a yuppie.
*h. If any real estate agent can sell to yuppies, Fred can.
i. Anyone who is a lawyer and a real estate agent is a yuppie and rich.
*j. Yuppies who aren't rich don't exist.
k. Real estate agents and lawyers are rich if they are yuppies.
*l. If Fred is a yuppie he's not a professor, and if he's a professor he's not rich.
m. No professor who isn't rich is a yuppie.
*n. No professor who is self-respecting is a yuppie.
o. Every self-respecting real estate agent is a yuppie.
*p. Real estate agents and lawyers who are rich are self-respecting.
q. Real estate agents and lawyers who are either rich or yuppies are self-respecting.
*r. A yuppie who is either a real estate agent or a lawyer is self-respecting.
s. A yuppie who is both a lawyer and a real estate agent is self-respecting.
*t. A yuppie who is both a lawyer and a real estate agent lives next door.

4. Symbolize the following sentences in *PL* using the given symbolization key.

UD: Persons
Ax: x is an administrator
Px: x is a professor

Ux: x is underpaid
Ox: x is overworked
Sx: x is a secretary

a. Professors are underpaid and overworked.
*b. Overworked professors are underpaid.
c. Administrators are neither overworked nor underpaid.
*d. Administrators are neither overworked nor underpaid, but professors are both.
e. A person is overworked if and only if he or she is underpaid.
*f. If any administrator is underpaid, all professors are; and if any professor is underpaid, all secretaries are.
g. Some professors are underpaid, but those who are administrators are not.
*h. Administrators are overworked but not underpaid; secretaries are underpaid but not overworked; and professors are both overworked and underpaid.
i. Some professors are overworked and underpaid, and all secretaries are.
*j. Some underpaid professors are also secretaries, and some overworked administrators are also professors, but no administrator is a secretary.
k. Some secretaries and some professors are underpaid, but no administrator is.

5. Use the following symbolization key to translate these sentences into fluent English. (*Note:* Not all of the following claims are true.)

UD: Positive integers
Lxy: x is larger than y
Dxy: x is evenly divisible by y
Ex: x is even
Ox: x is odd
Px: x is prime
a: 1
b: 2
c: 3
d: 4

a. Pb & Pc
*b. ~ (Pa ∨ Pd)
c. (∃x)Ex & (∃x)Ox
*d. ~ (∃y)(Ey & Oy)
e. (∀y)(Ey ∨ Oy)
*f. ~ (∃y)Lay
g. (∃x) ~ Lxa
*h. (∀z)(Pz ⊃ Lza)
i. (∀x)(Ex ⊃ Dxb)
*j. ~ (∃y)(Oy & Dyb)
k. (∀y)Dya
*l. ~ (∀x)Dxb
m. (∀y)(Dyb ≡ Ey)
*n. (∀x)(Dxb ⊃ ~ Dxc)
o. (∃y)Lay ⊃ (∀y)Lay
*p. (∃x)(Px & Dxb)
q. ~ (∃y)(Py & Dyd)
*r. (∀x)(Px ⊃ Lxa)

7.8 MULTIPLE QUANTIFIERS WITH OVERLAPPING SCOPE

In symbolizing sentences of English as sentences of *PL*, we have frequently encountered sentences of *PL* that contain more than one quantifier. But these have all been truth-functional compounds. In none of these sentences has one quantifier fallen within the scope of another quantifier. It is time to consider sentences of English whose *PL* symbolizations contain multiple quantifiers with *overlapping scope*.

Consider again the people in Michael's office. We symbolized 'Michael likes everyone' as '$(\forall x)Lmx$', but we did not attempt a symbolization of

Everyone likes everyone.

To symbolize 'Everyone likes everyone', we need to say of each person what '$(\forall x)Lmx$' says of Michael. To accomplish this we replace the constant 'm' with a second variable and add a second universal quantifier:

$$(\forall y)(\forall x)Lyx$$

In quasi-English this says 'Each person y is such that for each person x, y likes x' or 'Each person y is such that y likes everyone' or 'Everyone likes everyone'.

Similarly, just as '$(\exists x)Lmx$' symbolizes 'Michael likes someone',

$$(\exists y)(\exists x)Lyx$$

symbolizes 'Someone likes someone'. In each of these sentences of *PL*, the scope of the second quantifier falls within that of the first quantifier. It is also possible to mix universal and existential quantifiers. Consider

$$(\forall x)(\exists y)Lxy$$

and

$$(\exists y)(\forall x)Lxy$$

The first of the example sentences can be paraphrased as 'Each person x is such that x likes at least one person y' or 'Everyone likes someone'. For this claim to be true, it is sufficient that each person like at least one person. Perhaps Michael likes Rita, Rita likes Henry, Henry likes Sue, and Sue likes Michael. The second of the sentences looks very much like the first—only the order of the quantifiers is different. The second sentence says, however, not that everyone likes someone, but that someone is liked by everyone. If we limit our universe of discourse to the people in Michael's office or to any other reasonably small group, there may be such a lucky person. But if the UD is all people, there is no such person, for there is no person who is even known to

everyone, let alone liked by everyone. The moral here is that we need to pay attention to the order of quantifiers in mixed quantification.

Generally, when we have two universal quantifiers, the order in which they occur does not matter. Similarly, when we have two existential quantifiers, the order in which they occur does not matter. More generally, when we have a series of quantifiers, all existential or all universal, the order in which they occur does not matter. But this is, again, not in general true where we have mixed quantification—that is, at least one universal and at least one existential quantifier.

There are four combinations in which pairs of quantifiers can occur. We display them here along with useful quasi-English paraphrases:

$(\exists x)(\exists y)$	There is an x and there is a y such that . . . [or] There is a pair x and y such that . . .
$(\forall x)(\forall y)$	For each x and for each y . . . [or] For each pair x and y . . .
$(\forall x)(\exists y)$	For each x there is a y such that . . .
$(\exists x)(\forall y)$	There is an x such that for each y . . .

So far we have been assuming that our universe of discourse is limited to persons, either just the persons in Michael's office or all persons. Suppose we now allow our UD to be more heterogeneous, including, say, all living things. To be able to say that, for example, everyone (as opposed to everything) likes someone (as opposed to something) we need a predicate that singles out persons. We will use 'Px', here interpreted as 'x is a person'. Appropriate symbolizations of

1. Everyone likes everyone
2. Someone likes someone
3. Everyone likes someone
4. Someone likes everyone
5. Everyone is liked by someone
6. Someone is liked by everyone

are, respectively,

1′. $(\forall x)(\forall y)[(Px \ \& \ Py) \supset Lxy]$
2′. $(\exists x)(\exists y)[(Px \ \& \ Py) \ \& \ Lxy]$
3′. $(\forall x)[Px \supset (\exists y)(Py \ \& \ Lxy)]$
4′. $(\exists x)[Px \ \& \ (\forall y)(Py \supset Lxy)]$
5′. $(\forall x)[Px \supset (\exists y)(Py \ \& \ Lyx)]$
6′. $(\exists x)[Px \ \& \ (\forall y)(Py \supset Lyx)]$

Note that in 1′ and 2′ both quantifiers occur at the beginning of the sentence, whereas in 3′–6′ the second quantifier occurs later in the sentence. We can

move the y-quantifier closer to the first predicate containing 'y' in symbolizing sentences 1 and 2. That is, '$(\forall x)[Px \supset (\forall y)(Py \supset Lxy)]$' is also an appropriate symbolization of 1, as is '$(\exists x)[Px \,\&\, (\exists y)(Py \,\&\, Lxy)]$' of 2. Where quantifiers are placed, and when a quantifier can and cannot be moved, is a complicated issue, and we return to it later.

In symbolizing sentences of English that call for sentences of *PL* with multiple quantifiers with overlapping scope, it is especially important to learn to "read" the sentences of *PL* into quasi-English in order to check one's symbolization. In doing so, it is crucial that the role of each logical operator be identified—that is, that one identify the formula of which each logical operator is the main logical operator. In 1' '$(\forall x)$' is the main logical operator and '$(\forall y)$' is the main logical operator of that sentence's immediate subformula, '$(\forall y)[(Px \,\&\, Py) \supset Lxy]$'. So we read '$(\forall x)$' first and '$(\forall y)$' second. The reading begins either 'Every x is such that every y is such that' or, perhaps more insightfully, 'Every pair x and y is such that'. The horseshoe is the main logical operator of '$(Px \,\&\, Py) \supset Lxy$', so we read it next, then the antecedent of the conditional, and finally the consequent. The full quasi-English reading is

> Every pair x and y is such that if x is a person and y is a person then x likes y.

The main logical operator of 4' is an existential quantifier, '$(\exists x)$', and the immediate subformula of 4' is a conjunction whose right conjunct is a universally quantified formula. So we read the existential quantifier first, and then the conjunction. The quasi-English reading is

> There is at least one thing x such that both x is a person and each thing y is such that if y is a person then x likes y.

In 5' the main logical operator is again a universal quantifier, '$(\forall x)$'. Here the main logical operator of the immediate subformula, '$Px \supset (\exists y)(Py \,\&\, Lyx)$', is the horseshoe, so we read the universal quantifier first, and then the conditional, the consequent of which is itself an existentially quantified formula. The quasi-English reading is

> Each thing x is such that if x is a person then there is a y such that both y is a person and y likes x.

We next symbolize a series of claims concerning the positive integers, which we met briefly in Exercise Set 7.7. We pick positive integers as our UD because the relations among positive integers are very clear and easily stated and because a familiarity with positive integers and claims regarding them will be useful in Chapter 8. The positive integers are the numbers 1, 2, 3, 4, . . . (note that 0 is not a positive integer).

For our symbolization key we use

UD: Positive integers
Ex: x is even
Dx: x is odd
Px: x is prime
Lxy: x is larger than y
Exy: x times y is even
Oxy: x times y is odd
Pxy: x times y is prime
a: 1
b: 2

The claim 'Every positive integer is either odd or even and no positive integer is both' can be symbolized without using quantifiers with overlapping scope:

$$(\forall y)(Dy \vee Ey) \ \& \sim (\exists y)(Dy \ \& \ Ey)$$

But the claim 'There is no largest positive integer' does require use of quantifiers with overlapping scope. It says that each positive integer is such that there is a larger positive integer. A start at an appropriate symbolization is

$$(\forall x)(\text{there is an integer larger than } x)$$

and a full symbolization is

$$(\forall x)(\exists y)Lyx$$

The sentence '$\sim (\exists y)Lay$' says that it is not the case that there is a positive integer such that 1 is larger than it. From here it is a short step to '$(\exists x) \sim (\exists y)Lxy$', which says that there is a positive integer x such that there is no positive integer y that x is larger than—that is, that there is a positive integer that is not larger than any positive integer or that there is a lower bound to the positive integers. This is true.

The sentence '2 is prime and there is no smaller prime' is equivalent to '2 is prime and 2 is not larger than any prime', which can be symbolized as

$$Pb \ \& \sim (\exists y)(Py \ \& \ Lby)$$

'An odd number times an odd number is odd' is clearly a claim about all positive integers—no matter what positive integers we select, if both are odd their product is odd. An appropriate paraphrase is

> Each x and each y are such that if x is odd and y is odd then x times y is odd

or, alternatively,

> Each pair of integers, x and y, is such that if x is odd and y is odd, then x times y is odd.

An appropriate *PL* symbolization is

$$(\forall w)(\forall x)[(Dw \ \& \ Dx) \supset Owx]$$

Similarly 'An even number times an even number is even' becomes '$(\forall x)(\forall y)[(Ex \ \& \ Ey) \supset Exy]$'. And 'An even number times an odd number is even' becomes '$(\forall z)(\forall y)[(Ez \ \& \ Dy) \supset Ezy]$'.

'No product of prime numbers is prime' means that there is no pair of positive integers, each of which is prime, whose product is also prime. An appropriate paraphrase is

> There is no w and z such that w is prime, z is prime, and w times z is prime.

In *PL* we have

$$\sim (\exists w)(\exists z)[(Pw \ \& \ Pz) \ \& \ Pwz]$$

Now consider

$$(\forall x)(\forall y)[Exy \supset (Ex \vee Ey)]$$

This sentence of *PL* says that for any pair of positive integers, if the first times the second is even, then at least one of the integers is even. This is true, for if neither integer were even, their product would be odd. Similarly

$$(\forall x)(\forall y)[Oxy \supset (Dx \ \& \ Dy)]$$

says that for any pair of positive integers, if the first times the second is odd, then both of those integers are odd. The sentence

$$\sim (\exists z)Ozb$$

says, truly, that there is no positive integer such that it times 2 is odd. And

$$(\forall x)(\forall y)(\forall z)[(Lxy \ \& \ Lyz) \supset Lxz]$$

says that for any triplet of positive integers, if the first is larger than the second and the second is larger than the third, then the first is larger than the third. This claim is true (see Section 7.9). The sentence

$$(\forall x)(Dx \equiv (\exists y)Oyx) \ \& \ (\forall x)(Ex \equiv \sim (\exists y)Oyx)$$

says that a positive integer is odd if and only if there is some positive integer such that it times the first integer is odd and that a positive integer is even if and only if there is no positive integer such that it times that integer is odd.

In Section 7.1 we presented a valid English language argument that cannot be shown to be valid by the techniques associated with *SL*:

> None of David's friends supports Republicans. Sarah supports Breitlow, and Breitlow is a Republican. So Sarah is no friend of David's.

We can *now* symbolize this argument in *PL*. An appropriate symbolization key is

UD: People
Fxy: x is a friend of y
Sxy: x supports y
Rx: x is a Republican
d: David
b: Breitlow
s: Sarah

The second premise, a conjunction, is readily symbolized as 'Ssb & Rb'. The conclusion is also easy to symbolize once we see that it simply amounts to the claim that Sarah is not a friend of David's: '~ Fsd'. It is only the first premise that seems to pose difficulties. That premise is of the general form

> No thing of such-and-such a sort is a thing of such-and-such a sort.

That is, it is an E-sentence. In Section 7.6 we saw that such sentences can be symbolized either as universally quantified sentences or as negations of existentially quantified sentences. If we opt for the former, an appropriate first step toward a symbolization is

> Each x is such that if x is a friend of David's then x does not support Republicans.

This quasi-English locution readily becomes

$$(\forall x)(Fxd \supset \text{it is not the case that x supports Republicans})$$

What remains is to find a symbolization for 'It is not the case that x supports Republicans'. A quasi-English first step is

> It is not the case that there is a y such that y is a Republican and x supports y.

This can be symbolized as '~ (∃y)(Ry & Sxy)'. The full symbolization of the first premise is thus

(∀x)(Fxd ⊃ ~ (∃y)(Ry & Sxy))

The resulting argument of *PL* is

(∀x)[Fxd ⊃ ~ (∃y)(Ry & Sxy)]
Ssb & Rb

~ Fsd

This argument is, as we shall see in later chapters, valid.

Note that, while we chose to treat the embedded clause 'It is not the case that x supports Republicans' as the negation of an I-claim, we could equally well have treated it as an E-claim, symbolizing it as '(∀y)(Ry ⊃ ~ Sxy)'. Doing so would yield the following alternative symbolization of the first premise:

(∀x)[Fxd ⊃ (∀y)(Ry ⊃ ~ Sxy)]

Both of these symbolizations of the first premise, and many others we have not given, are equally acceptable. In constructing symbolizations it is often useful to start, as we did here, by determining whether the sentence to be symbolized fits one of the four patterns provided by the A-, E-, I-, O-sentence classification. If it does, the next step is to pick the overall structure to be used (for example, universal quantification of a conditional formula). Finally we fill in the missing pieces—successively replacing bits of English with formulas of *PL*.

Here is a somewhat more interesting argument:

> Anyone who is proud of anyone is proud of Samantha. Rhoda isn't proud of anyone who's proud of himself or herself, but she is proud of everyone who has mastered calculus. Therefore if Art has mastered calculus Samantha isn't proud of herself.

We will use the following symbolization key:

UD: People in Samantha's class
Pxy: x is proud of y
Mx: x has mastered calculus
a: Art
r: Rhoda
s: Samantha

The first occurrence of 'anyone' in the first premise clearly goes over to a universal quantifier in *PL*, as becomes apparent when we try to paraphrase the sentence

(y is proud of anyone ⊃ y is proud of Samantha)

Here there is clear pronominal cross-reference—the y that is proud of Samantha is the y that is proud of anyone. So as a next step we have

(∀y)(y is proud of anyone ⊃ Pys)

an A-sentence. What remains is to determine whether the second 'anyone' should go over to a universal or an existential quantifier in *PL*. Note that there is no pronominal cross-reference from the consequent of 'y is proud of anyone ⊃ Pys' back to 'anyone'. So we can use an existential quantifier. That a universal quantifier is not called for is also apparent when we consider that

(∀y)(y is proud of everyone ⊃ Pys)

is clearly an inappropriate paraphrase of the first premise, while

(∀y)(y is proud of someone ⊃ Pys)

is an appropriate paraphrase. To be proud of someone is for there to be someone of whom one is proud. So the missing formula is '(∃x)Pyx'. The complete symbolization of the first premise is thus

(∀y)[(∃x)Pyx ⊃ Pys]

The second premise is a conjunction and should be symbolized as a conjunction of *PL*. The left conjunct will be a symbolization of 'Rhoda isn't proud of anyone who is proud of himself or herself', which can be treated as an E-sentence (as 'No person who is proud of himself or herself is a person of whom Rhoda is proud'). So an appropriate left conjunct for our *PL* symbolization is '(∀z)(Pzz ⊃ ~ Prz)'. The right conjunct of the second premise can be treated as an A-sentence (as 'Everyone who has mastered calculus is a person of whom Rhoda is proud') and symbolized as '(∀z)(Mz ⊃ Prz)'. The second premise of our symbolized argument is thus

(∀z)(Pzz ⊃ ~ Prz) & (∀z)(Mz ⊃ Prz)

The conclusion of our English language argument is a conditional and can be symbolized as 'Ma ⊃ ~ Pss'. The complete argument of *PL* is

(∀y)[(∃x)Pyx ⊃ Pys]
(∀z)(Pzz ⊃ ~ Prz) & (∀z)(Mz ⊃ Prz)

Ma ⊃ ~ Pss

This is also a valid argument of *PL*.

We just symbolized 'Anyone who is proud of anyone is proud of Samantha' as '$(\forall y)[(\exists x)Pyx \supset Pys]$'. An alternative symbolization is

$$(\forall y)(\forall x)(Pyx \supset Pys)$$

A quasi-English reading of this second symbolization is

> Each y and each x is such that if y is proud of x then y is proud of Samantha.

The obvious difference between these two sentences of *PL* is that in the second the x-quantifier is a universal quantifier whose scope extends to the end of the sentence. A simpler example may be helpful here. Consider these sentences:

> If any student passes, Donna will pass.
> Each student is such that if that student passes Donna will pass.

If we restrict our UD to students in the class in question, interpret 'Px' as 'x will pass', and let 'd' designate Donna, these sentences can be symbolized as

$$(\exists x)Px \supset Pd$$

and

$$(\forall x)(Px \supset Pd)$$

respectively. The first of these sentences of *PL* can be read

> If there is at least one student x such that x passes, then Donna passes.

Now suppose that some student, say, Art, does pass. Then, according to the first of the above sentences of *PL*, Donna also passes. The second sentence of *PL* can be read

> Each student x is such that if x passes then Donna passes.

Now, if each student is of this sort, then Art is of this sort. Therefore, if '$(\forall x)(Px \supset Pd)$' holds and Art passes, Donna passes. So '$(\exists x)Px \supset Pd$' and '$(\forall x)(Px \supset Pd)$' both commit us to Donna's passing if at least one student passes. These sentences are also false under just the same circumstances. The first will be false only if some student passes and Donna does not. Suppose, for example, that Bud passes but that Donna does not. Then '$(\exists x)Px \supset Pd$' is false and so is '$(\forall x)(Px \supset Pd)$', for the latter says that each student, including Bud, is such that if he or she passes Donna passes. And this is false if Bud passes but Donna does not.

The general rule is this: When an existential quantifier has only the antecedent of a material conditional within its scope and its scope is broadened to include the consequent of that conditional, the existential quantifier must

be replaced with a universal quantifier. That is, where **P** is a formula in which x does not occur and **Ax** is a formula containing **x**,

$$(\exists \mathbf{x})\mathbf{Ax} \supset \mathbf{P}$$

and

$$(\forall \mathbf{x})(\mathbf{Ax} \supset \mathbf{P})$$

are equivalent sentence forms.

An analogous though less common case occurs when a universal quantifier has only the antecedent of a material conditional within its scope and its scope is broadened to include the entire conditional. When this happens, the universal quantifier must be replaced with an existential quantifier. That is, where **x** does not occur in **P** the following sentence forms are equivalent:

$$(\forall \mathbf{x})\mathbf{Ax} \supset \mathbf{P}$$

and

$$(\exists \mathbf{x})(\mathbf{Ax} \supset \mathbf{P})$$

The cases to watch out for, then, are cases where the consequent of a material conditional does not lie within the scope of a quantifier and is then brought within that scope, or vice versa. In these cases the quantifier in question must be replaced with a universal quantifier if it was an existential and with an existential quantifier if it was a universal.

Fortunately there are many cases in which quantifiers do not have to be changed when scopes are broadened or narrowed. If the scope of a quantifier extends over only one disjunct of a disjunction or over only one conjunct of a conjunction and that scope is broadened to include the entire disjunction or conjunction, the quantifier does not change. Similarly, when a quantifier has scope over only the consequent of a material conditional and its scope is broadened by relocating the quantifier so as to have scope over the entire conditional, the quantifier does not change. So where **x** does not occur in **P** the following are all pairs of equivalent sentence forms:

$(\exists \mathbf{x})\mathbf{Ax} \supset \mathbf{P}$	$(\forall \mathbf{x})(\mathbf{Ax} \supset \mathbf{P})$
$(\forall \mathbf{x})\mathbf{Ax} \supset \mathbf{P}$	$(\exists \mathbf{x})(\mathbf{Ax} \supset \mathbf{P})$
$\mathbf{P} \supset (\exists \mathbf{x})\mathbf{Ax}$	$(\exists \mathbf{x})(\mathbf{P} \supset \mathbf{Ax})$
$\mathbf{P} \supset (\forall \mathbf{x})\mathbf{Ax}$	$(\forall \mathbf{x})(\mathbf{P} \supset \mathbf{Ax})$
$(\exists \mathbf{x})\mathbf{Ax} \vee \mathbf{P}$	$(\exists \mathbf{x})(\mathbf{Ax} \vee \mathbf{P})$
$(\forall \mathbf{x})\mathbf{Ax} \vee \mathbf{P}$	$(\forall \mathbf{x})(\mathbf{Ax} \vee \mathbf{P})$
$\mathbf{P} \vee (\exists \mathbf{x})\mathbf{Ax}$	$(\exists \mathbf{x})(\mathbf{P} \vee \mathbf{Ax})$
$\mathbf{P} \vee (\forall \mathbf{x})\mathbf{Ax}$	$(\forall \mathbf{x})(\mathbf{P} \vee \mathbf{Ax})$
$(\exists \mathbf{x})\mathbf{Ax} \,\&\, \mathbf{P}$	$(\exists \mathbf{x})(\mathbf{Ax} \,\&\, \mathbf{P})$
$(\forall \mathbf{x})\mathbf{Ax} \,\&\, \mathbf{P}$	$(\forall \mathbf{x})(\mathbf{Ax} \,\&\, \mathbf{P})$
$\mathbf{P} \,\&\, (\exists \mathbf{x})\mathbf{Ax}$	$(\exists \mathbf{x})(\mathbf{P} \,\&\, \mathbf{Ax})$
$\mathbf{P} \,\&\, (\forall \mathbf{x})\mathbf{Ax}$	$(\forall \mathbf{x})(\mathbf{P} \,\&\, \mathbf{Ax})$

Material biconditionals are a special case. $(\forall \mathbf{x})\mathbf{Ax} \equiv \mathbf{P}$ is equivalent neither to $(\forall \mathbf{x})(\mathbf{Ax} \equiv \mathbf{P})$ nor to $(\exists \mathbf{x})(\mathbf{Ax} \equiv \mathbf{P})$. That is, the scope of a quantifier that does not extend over both sides of a material biconditional *cannot* be broadened to cover both sides, nor can the scope of a quantifier that does cover both sides of a material biconditional be narrowed to cover only one side.

We conclude this section by symbolizing a series of increasingly complex sentences in *PL*. The first three are as follows:

1. Everyone who understands either Bertrand Russell's *Principia Mathematica* or Lewis Carroll's *Alice in Wonderland* understands this text.
2. No one understands everything.
3. No one understands anything.

For these and subsequent sentences we will use the following symbolization key:

UD: Everything
Exy: x envies y
Uxy: x understands y
Px: x is a person
a: Lewis Carroll's *Alice in Wonderland*
p: Bertrand Russell's *Principia Mathematica*
t: this text

In symbolizing these sentences we shall again use the procedure of moving gradually from English to symbols.

Sentence 1 is an A-sentence, so it will be symbolized as a universally quantified sentence. We can start with

> Each x is such that, if x is a person and x understands either Bertrand Russell's *Principia Mathematica* or x understands Lewis Carroll's *Alice in Wonderland*, then x understands this text

and move to

> $(\forall x)$(if Px and x understands either Bertrand Russell's *Principia Mathematica* or Lewis Carroll's *Alice in Wonderland*, then x understands this text)

We can now see that we can complete our symbolization without using any more quantifiers:

$$(\forall x)([Px \ \& \ (Uxp \vee Uxa)] \supset Uxt)$$

Sentence 2 is an E-sentence. So we symbolize it as a universal quantification that says of each thing that if it is a person then it doesn't understand everything. That is,

> Each y is such that if y is a person then y does not understand everything.

Next we move to

> (∀y)(Py ⊃ it is not the case that y understands everything)

The remaining bit of English obviously goes over to '~ (∀z)Uyz', and so the entire sentence of *PL* is

> (∀y)(Py ⊃ ~ (∀z)Uyz)

Sentence 2, 'No one understands everything', and sentence 3, 'No one understands anything', are very different claims. The former is certainly true and the latter certainly false. Sentence 3 can, however, also be paraphrased and symbolized as an E-sentence:

> Each x is such that if x is a person, then it is not the case that x understands anything.

This gives way to

> (∀x)(Px ⊃ it is not the case that there is something x understands)

for to not understand anything is for there not to be something one understands. So a full symbolization is

> (∀x)(Px ⊃ ~ (∃y)Uxy)

An alternative symbolization is '(∀x)(Px ⊃ (∀y) ~ Uxy)', for to not understand anything is for each thing to be such that one does not understand it.

Now consider this sentence:

> 4. If someone understands Bertrand Russell's *Principia Mathematica*, then that person understands Lewis Carroll's *Alice in Wonderland*.

We here have one of the rare uses of 'someone' that goes over to a universal quantifier. This becomes apparent when we realize that there is pronominal cross-reference from the consequent of this English conditional (from the phrase 'that

person') back to the quantity term in the antecedent ('someone'). Seeing this, it becomes clear that an appropriate paraphrase and symbolization are

> Each x is such that if x is a person and x understands Bertrand Russell's *Principia Mathematica*, then x understands Lewis Carroll's *Alice in Wonderland*

and

> $(\forall x)[(Px \ \& \ Uxp) \supset Uxa]$

Sentence 5 is somewhat more complex:

> 5. Only people who understand either Bertrand Russell's *Principia Mathematica* or Lewis Carroll's *Alice in Wonderland* understand this text.

We have here a quantificational analog of an 'only if' claim of sentential logic. That is, we are told, not that all those persons who understand either of the works in question understand this text, but rather that those who do understand this text also understand one of the other cited works. An appropriate paraphrase is thus

> Each y is such that if y is a person and y understand this text then y understands either Bertrand Russell's *Principia Mathematica* or Lewis Carroll's *Alice in Wonderland*.

And a correct symbolization is

> $(\forall y)([(Py \ \& \ Uyt) \supset (Uyp \vee Uya)]$

In subsequent chapters we shall establish that this is equivalent to

> $(\forall y)[Py \ \& \sim (Uyp \vee Uya)] \supset \sim Uyt)$

but not to

> $(\forall y)([Py \ \& \ (Uyp \vee Uya)] \supset Uyt)$

Symbolizing our sixth example requires the use of three quantifiers:

> 6. Anyone who understands anything is envied by someone.

The first occurrence of 'anyone' yields a universal quantifier because 'is envied by someone' refers back to it; that is, the person who is envied by someone is the person who understands anything. So a paraphrase is

> Each x is such that if x is a person and x understands anything then x is envied by someone.

In this context to understand anything is to understand at least one thing, so a fuller paraphrase is

> Each x is such that if x is a person and there is at least one y such that x understands y then there is some z such that z is a person and z envies x.

An appropriate symbolization is

$$(\forall x)[(Px \ \& \ (\exists y)Uxy) \supset (\exists z)(Pz \ \& \ Ezx)]$$

Consider, finally,

> 7. Anyone who understands everything is envied by everyone.

We will use three universal quantifiers in symbolizing this sentence. An appropriate paraphrase is

> Each x is such that if x is a person and every y is such that x understands y then every z is such that if z is a person, then z envies x.

This yields the following sentence of *PL*:

$$(\forall x)[(Px \ \& \ (\forall y)Uxy) \supset (\forall z)(Pz \supset Ezx)]$$

7.8E EXERCISES

1. Symbolize the following sentences in *PL* using the given symbolization key.

UD: People
Sx: x is a sailor
Lx: x is lucky
Cx: x is careless
Yx: x dies young
Sxy: x is a son of y
Dxy: x is a daughter of y
Wx: x is Wilcox
d: Daniel Wilcox
j: Jacob Wilcox
r: Rebecca Wilcox

a. Some sailors are both careless and lucky.
*b. Some careless sailors aren't lucky.
c. Not all lucky sailors are careless.
*d. All careless sailors, except the lucky ones, die young.

e. Not all sons of sailors are sailors.
*f. Not all daughters of sailors are sailors.
g. Not all sons and daughters of sailors are sailors.
*h. Sailors who aren't lucky and are careless have neither daughters nor sons.
i. Sailors who have either sons or daughters are lucky.
*j. Sailors who have both daughters and sons are lucky.
k. Rebecca Wilcox is either a sailor or the daughter of a sailor.
*l. Every Wilcox is either a sailor or the offspring of a sailor.
m. Either Rebecca Wilcox and all her children are sailors or Jacob Wilcox and all his children are sailors.

2. Symbolize the following sentences in *PL* using the given symbolization key.

UD: The employees of this college
Exy: x earns more than y
Dxy: x distrusts y
Fx: x is a faculty member
Ax: x is an administrator
Cx: x is a coach
Ux: x is a union member
Rx: x should be fired
Mx: x is an MD
Px: x is paranoid
Ox: x is a union officer
p: the president
j: Jones

a. Every administrator earns more than some faculty member, and every faculty member earns more than some administrator.
*b. If any administrator earns more than every faculty member, Jones does.
c. No faculty member earns more than the president.
*d. Any administrator who earns more than every faculty member should be fired.
e. No faculty member earns more than the president, but some coaches do.
*f. Not all faculty members are union members, but all union members are faculty members.
g. No administrator is a union member, but some are faculty members.
*h. Every faculty member who is an administrator earns more than some faculty members who are not administrators.
i. At least one administrator who is not a faculty member earns more than every faculty member who is an administrator.
*j. Every faculty member who is an MD earns more than every faculty member who is not an MD.
k. Some faculty members distrust every administrator, and some administrators distrust every faculty member.
*l. There is an administrator who is a faculty member and distrusts all administrators who are not faculty members.
m. Anyone who distrusts everyone is either paranoid or an administrator or a union officer.
*n. Everyone distrusts someone, but only administrators who are not faculty members distrust everyone.

3. Symbolize the following sentences in *PL* using the given symbolization key.

UD: Everything
Axyz: x understands y as well as does z
Bxy: x bores y
Gxy: x gives a low grade to y
Lxy: x listens to y
Sxy: x is a student of y
Nxy: x understands y
Dx: x deserves to be fired
Px: x is a professor
Ux: x is unpopular
Wx: x is wasting x's time
t: this text

a. All professors bore some of their students.
*b. All professors who bore all their students deserve to be fired.
c. Any professor who is bored by everything bores all his or her students.
*d. Professors bore all and only those of their students they are bored by.
e. If all professors bore all their students, then all professors are wasting their time.
*f. If a professor bores a student, then both are wasting their time.
g. Professors don't understand the students they bore, and students don't listen to the professors they are bored by.
*h. No professor understands everything.
i. Some professors bore all professors.
*j. An unpopular professor either bores or gives a low grade to each of his or her students.
k. Unpopular professors either bore all of their students or give all of their students low grades.
*l. If a professor doesn't listen to a student, then that student is wasting his or her time.
m. If a student and his or her professor bore each other, then both are wasting their time.
*n. Some professors don't understand this text.
o. Some professors don't understand this text as well as some of their students do.
*p. No professor who understands this text bores any of his or her students.
q. Any student who doesn't listen to his or her professor doesn't understand that professor and bores that professor.

4. Construct fluent English readings for the following sentences of *PL* using the given symbolization key.

UD: Everything (including times)
Lxyz: x loves y at z
Px: x is a person
Tx: x is a time
h: Hildegard
m: Manfred
s: Siegfried

a. $(\exists x)(Tx \ \& \ Lhmx)$
*b. $(\forall y)[(Ty \ \& \ Lmhy) \supset Lhmy]$
c. $(\exists w)(Tw \ \& \ Lmhw) \ \& \ (\forall z)(Tz \supset Lmsz)$
*d. $(\forall x)(Tx \supset Lshx)$
e. $(\exists x)(Tx \ \& \ Lmmx) \supset (\forall x)[(Tx \ \& \ Lhmx) \supset Lmmx]$
*f. $(\forall x)[Px \supset (\exists y)(Ty \ \& \ \sim (\exists z)(Pz \ \& \ Lzxy))]$
g. $(\exists x)[Px \ \& \ \sim (\exists y)(\exists z)(Ty \ \& \ (Pz \ \& \ Lzxy))]$
*h. $(\forall x)[Tx \supset (\exists y)(Py \ \& \ \sim (\exists z)(Pz \ \& \ Lzyx))]$
i. $(\exists x)[Tx \ \& \ (\exists y)(Py \ \& \ (\forall z)(Pz \supset Lyzx))]$
*j. $(\forall x)[Tx \supset (\exists y)(\exists z)((Py \ \& \ Pz) \ \& \ Lyzx)]$
k. $(\forall x)[Tx \supset (\exists y)(Py \ \& \ (\forall z)(Pz \supset Lyzx))]$
*l. $(\forall x)[Px \supset (\exists y)(\exists z)((Py \ \& \ Tz) \ \& \ Lxyz)]$
m. $\sim (\exists x)(\exists y)[(Px \ \& \ Py) \ \& \ (\forall z)(Tz \supset Lxyz)]$
*n. $(\exists x)[Px \ \& \ (\forall y)(\forall z)((Ty \ \& \ Pz) \supset Lxzy)]$
o. $(\forall x)[Px \supset (\exists y)(Ty \ \& \ Lxxy)]$

5. Use the following symbolization key to translate sentences a–r into fluent English. (*Note:* All of the following claims are true.)

UD: Positive integers
Dxy: the sum of x and y is odd
Exy: x times y is even
Lxy: x is larger than y
Oxy: x times y is odd
Sxy: x plus y is even
Ex: x is even
Ox: x is odd
Px: x is prime
Pxy: x times y is prime
a: 1
b: 2
c: 3

a. $(\forall x)[Ex \supset (\forall y)Exy]$
*b. $(\forall x)(\forall y)[(Ox \ \& \ Oy) \supset Oxy]$
c. $(\forall x)(\forall y)[Sxy \supset [(Ex \ \& \ Ey) \vee (Ox \ \& \ Oy)]]$
*d. $(\forall x)[(Px \ \& \ (\exists y)(Py \ \& \ Lxy)) \supset Ox]$
e. $\sim (\exists y)[Py \ \& \ (\forall x)(Px \supset Lyx)]$
*f. $(\forall y)(\forall z)([(Py \ \& \ Pz) \ \& \ (Lyb \ \& \ Lzb)] \supset Oyz)$
g. $\sim (\exists x)(\exists y)[(Px \ \& \ Py) \ \& \ Pxy]$
*h. $(\exists x)(Px \ \& \ Ex)$
i. $(\exists x)[Px \ \& \ (\forall y)Eyx]$
*j. $\sim (\forall x)(\exists y)Lxy \ \& \ (\forall x)(\exists y)Lyx$
k. $(\forall x)(\forall y)[Oxy \equiv (Ox \ \& \ Oy)]$
*l. $(\forall x)(\forall y)[Exy \equiv (Ex \vee Ey)]$
m. $(\forall x)(\forall y)[(Ox \ \& \ Oy) \supset (Oxy \ \& \ Sxy)]$
*n. $(\forall x)(\forall y)(Lxy \supset \sim Lyx)$
o. $(\forall x)(\forall y)[(Ox \ \& \ Ey) \supset (Dxy \ \& \ Exy)]$
*p. $(\forall x)(\forall y)[[(Px \ \& \ Py) \ \& \ Lcx] \supset Exy]$
q. $(\exists y)[(Lya \ \& \ Lcy) \ \& \ (Py \ \& \ Ey)]$
*r. $(\exists x)[(Px \ \& \ Ex) \ \& \ (\forall y)((Py \ \& \ Lyx) \supset Oy)]$

7.9 IDENTITY, DEFINITE DESCRIPTIONS, PROPERTIES OF RELATIONS, AND FUNCTIONS

Our standard reading of 'some' is 'at least one'. Some may object that this is not an accurate reading, that 'some' sometimes means something like 'at least two'. It is alleged, for example, that to say

There are still some apples in the basket

when there is only one apple in the basket is at best misleading and at worst false. In any event we clearly do want a means of symbolizing such claims as

There are at least two apples in the basket.

We can do this by interpreting one of the two-place predicates of *PL* as expressing the identity relation. For example, we could interpret 'Ixy' as 'x is identical with y'. Given the symbolization key

UD: Everything
Nxy: x is in y
Ixy: x is identical with y
Ax: x is an apple
b: the basket

both

$$(\exists x)(Ax \ \& \ Nxb)$$

and

$$(\exists x)[(Ax \ \& \ Nxb) \ \& \ (\exists y)(Ay \ \& \ Nyb)]$$

say 'There is at least one apple in the basket'. The latter merely says it twice, so to speak. But

$$(\exists x)(\exists y)([(Ax \ \& \ Ay) \ \& \ (Nxb \ \& \ Nyb)] \ \& \ -\ Ixy)$$

does say 'There are at least two apples in the basket'. This sentence of *PL* says, in quasi-English, 'There is an x and there is a y such that both x and y are apples, both x and y are in the basket, and x and y are not identical'. This last clause is not redundant because using different variables does not commit us to there being more than one thing of the specified sort.

THE IDENTITY PREDICATE

An alternative to interpreting one of the two-place predicates of *PL* as expressing identity is to introduce a special two-place predicate and specify that it *always* be interpreted as expressing the identity relation. This is the course we shall follow. In adding this predicate to $PL_=$ we generate a new language, *PLE*. As an extension of $PL_=$ it includes all the vocabulary of *PL* and an additional two-place predicate. *PLE* also includes, as we detail later in this section, functors (used to express functions). The formulas and sentences of *PL* are also formulas and sentences of *PLE*.

The new two-place predicate that is distinctive of *PLE* is the **identity predicate,**

$=''$

When using this predicate we shall, as we have been doing with other predicates, omit the two primes as the number of individual terms used (two) will show that this is a two-place predicate. This predicate is always interpreted as the identity predicate. For example, '= ab' says that a is identical to b. However, it is customary to write, informally, 'a = b', rather than '= ab'—that is to place one individual term before the predicate and one after it—and we shall follow this custom.

So, instead of '= ab', ' = xy', and ' = aa', we write 'a = b', 'x = y', and 'a = a'. And in place of, for example, '~ = ab', we write '~ a = b'. Since the interpretation of '=' is fixed, we never have to include an interpretation of this predicate in a symbolization key.

We can now symbolize 'There are at least two apples in the basket' in *PLE*, using the preceding symbolization key (but dispensing with the now superfluous 'Ixy'), as

$$(\exists x)(\exists y)[[(Ax \ \& \ Ay) \ \& \ (Nxb \ \& \ Nyb)] \ \& \sim x = y)$$

In *PLE* we can also say that there are just so many apples in the basket and no more—for example, that there is exactly one apple in the basket. An appropriate paraphrase is

> There is a y such that y is an apple and y is in the basket, and each thing z is such that if z is an apple and is in the basket then z is identical with y.

A full symbolization is

$$(\exists y)[(Ay \ \& \ Nyb) \ \& \ (\forall z)[(Az \ \& \ Nzb) \supset z = y]]$$

What we are saying is that there is at least one apple in the basket and that anything that is an apple and is in the basket is *that very apple*.

Consider next

Henry hasn't read *Alice in Wonderland* but everyone else in the class has.

If we limit our universe of discourse to the students in the class in question, let 'h' designate Henry, and interpret 'Ax' as 'x has read *Alice in Wonderland*', we can symbolize this claim as

$$\sim Ah \;\&\; (\forall y)[\sim y = h \supset Ay]$$

And, using 'b' to designate Bob, we can symbolize 'Only Henry and Bob have not read *Alice in Wonderland*', as

$$\sim (Ah \vee Ab) \;\&\; (\forall x)[\sim (x = h \vee x = b) \supset Ax]$$

This says that neither Henry nor Bob has read *Alice in Wonderland* and that everyone else—that is, each person in the class who is neither identical to Henry nor identical to Bob—has read it.

We can also use the identity predicate to symbolize the following sentences of *PLE*:

1. There are apples and pears in the basket.
2. The only pear in the basket is rotten.
3. There are at least two apples in the basket.
4. There are two (and only two) apples in the basket.
5. There are no more than two pears in the basket.
6. There are at least three apples in the basket.

UD: Everything
Ax: x is an apple
Nxy: x is in y
Px: x is a pear
Rx: x is rotten
b: the basket

If we paraphrase sentence 1 as 'There is at least one apple and at least one pear in the basket', we can symbolize it without using the identity predicate:

$$(\exists x)(\exists y)[(Ax \;\&\; Py) \;\&\; (Nxb \;\&\; Nyb)]$$

However, if we take sentence 1 to assert that there are at least two apples and at least two pears in the basket, we do need the identity predicate:

$$(\exists x)(\exists y)[((Ax \;\&\; Ay) \;\&\; (Nxb \;\&\; Nyb)) \;\&\; \sim x = y] \;\&$$
$$(\exists x)(\exists y)[((Px \;\&\; Py) \;\&\; (Nxb \;\&\; Nyb)) \;\&\; \sim x = y]$$

Sentence 2 says that there is one and only one pear in the basket and that that one pear is rotten:

$$(\exists x)[((Px \ \& \ Nxb) \ \& \ Rx) \ \& \ (\forall y)[(Py \ \& \ Nyb) \supset y = x]]$$

Sentence 3 says only that there are *at least* two apples in the basket, not that there are *exactly* two. Hence

$$(\exists x)(\exists y)[((Ax \ \& \ Ay) \ \& \ (Nxb \ \& \ Nyb)) \ \& \sim x = y]$$

To symbolize sentence 4 we start with the symbolization for sentence 3 and add a clause saying there are no additional apples in the basket:

$$(\exists x)(\exists y)([((Ax \ \& \ Ay) \ \& \ (Nxb \ \& \ Nyb)) \ \& \sim x = y] \ \&$$
$$(\forall z)[(Az \ \& \ Nzb) \supset (z = x \vee z = y)])$$

The added clause says, in effect, 'and anything that is an apple and is in the basket is either x or y'. Sentence 5 does not say that there are two pears in the basket; rather, it says that there are *at most* two pears in the basket. We can express this in *PLE* by saying that of any pears, x, y, and z that are in the basket these are really at most two; that is, either x is identical to y, or x is identical to z, or y is identical to z. In other words

$$(\forall x)(\forall y)(\forall z)[([(Px \ \& \ Py) \ \& \ Pz] \ \& \ [(Nxb \ \& \ Nyb) \ \& \ Nzb]) \supset$$
$$((x = y \vee x = z) \vee y = z)]$$

A shorter version is

$$(\forall x)(\forall y)(\forall z)[([(Px \ \& \ Py) \ \& \ Pz] \ \& \ [(Nxb \ \& \ Nyb) \ \& \ Nzb]) \supset$$
$$(z = x \vee z = y)]$$

This says, in effect, that any alleged third pear, z, is not a third pear but is the very same pear as either x or y. Finally sentence 6 can be symbolized by building on the symbolization for sentence 3:

$$(\exists x)(\exists y)(\exists z)(([(Ax \ \& \ Ay) \ \& \ Az] \ \& \ [(Nxb \ \& \ Nyb) \ \& \ Nzb]) \ \&$$
$$[(\sim x = y \ \& \sim y = z) \ \& \sim x = z)$$

We now return to our discussion of positive integers. This time we will use this symbolization key for the sentences that follow.

UD: Positive integers
Bxyz: x is between y and z
Lxy: x is larger than y
Sxy: x is a successor of y
Ex: x is even

Px: x is prime
a: 1
b: 2
c: 10
d: 14

1. There is no largest positive integer.
2. There is a unique smallest positive integer.
3. 2 is the only even prime.
4. There are exactly two primes between 10 and 14.
5. Every positive integer has exactly one successor.
6. 2 is the only prime whose successor is prime.

As we saw in our earlier discussion, we can symbolize sentence 1 without using the identity predicate, for to say that there is no largest positive integer it suffices to say that for every integer there is a larger integer (no matter what integer one might pick, there is an integer larger than it):

$$(\forall x)(\exists y)Lyx$$

It is also tempting to symbolize sentence 2 without using the identity predicate, for to say that there is a smallest positive integer seems to be to say that there is an integer that is not larger than any integer:

$$(\exists x) \sim (\exists y)Lxy$$

But while the foregoing does say that there is a smallest positive integer, it does not say that there is a unique such integer. So a better symbolization is

$$(\exists x)(\forall y)(\sim y = x \supset Lyx)$$

This sentence of *PL* says that there is an integer such that every integer not identical to it is larger than it. This does imply uniqueness.

Sentence 3, '2 is the only even prime', says that 2 is prime and is even and that all other primes are not even:

> 2 is prime and 2 is even, and each z is such that if z is prime and z is not identical with 2 then z is not even.

In *PLE*

$$(Pb \,\&\, Eb) \,\&\, (\forall z)[(Pz \,\&\, \sim z = b) \supset \sim Ez]$$

This is equivalent to

$$(Pb \,\&\, Eb) \,\&\, (\forall z)[(Pz \,\&\, Ez) \supset z = b]$$

Notice that we could equally well have paraphrased and symbolized sentence 3 as

> 2 is prime and 2 is even, and it is not the case that there is a z such that z is prime and z is even, and z is not identical with 2

and symbolized this claim as

$$(Pb \ \& \ Eb) \ \& \ \sim (\exists z)[(Pz \ \& \ Ez) \ \& \ \sim z = b]$$

Notice, too, that all three symbolic versions of sentence 3 are truth-functional compounds, not quantified sentences.

To symbolize sentence 4, 'There are exactly two primes between 10 and 14', we must say that there are at least two such primes and that there are no additional ones. So our paraphrase starts

> There is an x and there is a y such that x is prime and y is prime, x is between 10 and 14 and y is between 10 and 14, and x is not identical with y, . . .

This much can be symbolized as

$$(\exists x)(\exists y)((Px \ \& \ Py) \ \& \ [(Bxcd \ \& \ Bycd) \ \& \sim x = y])$$

What we now need to add is that any prime that is between 10 and 14 is one of these two primes:

> Each z is such that if z is prime and z is between 10 and 14 then z is either x or y.

That is,

$$(\forall z)[(Pz \ \& \ Bzcd) \supset (z = x \vee z = y)]$$

In joining the two fragments of our symbolization, we must be sure to extend the scope of our two existential quantifiers over the entire sentence, for we want to bind the occurrences of 'x' and 'y' in the last half of the sentence:

$$(\exists x)(\exists y)[((Px \ \& \ Py) \ \& \ [(Bxcd \ \& \ Bycd) \ \& \sim x = y]) \ \&$$
$$(\forall z)[(Pz \ \& \ Bzcd) \supset (z = x \vee z = y)]]$$

It is perhaps worth noting here that we could have symbolized sentence 4 without using the three-place predicate 'Bxyz'. To see this, note that to say a positive integer x is between 10 and 14 is just to say that x is larger than 10 and that 14 is larger than x. An appropriate symbolization is

$$(\exists x)(\exists y)[((Px \ \& \ Py) \ \& \ [((Lxc \ \& \ Ldx) \ \& \ (Lyc \ \& \ Ldy)) \ \& \sim x = y]) \ \&$$
$$(\forall z)([Pz \ \& \ (Lzc \ \& \ Ldz)] \supset (z = x \vee z = y))]$$

A successor of an integer is the sum of that integer and 1. Sentence 5, 'Every positive integer has exactly one successor', can be symbolized as

$$(\forall x)(\exists y)[Syx \ \& \ (\forall z)(Szx \supset z = y)]$$

This says that each positive integer x has a successor y and that any integer that is a successor of x is identical to y—that is, that each positive integer has exactly one successor.

Sentence 6, '2 is the only prime whose (only) successor is prime', can be paraphrased as a conjunction:

> 2 is prime and its only successor is prime, and any successor of any prime other than 2 is not prime.

The first conjunct can be symbolized as

$$Pb \ \& \ (\exists x)[(Sxb \ \& \ (\forall y)(Syb \supset y = x)) \ \& \ Px]$$

The second conjunct can be symbolized as

$$(\forall x)(\forall y)[(Sxy \ \& \ (Py \ \& \sim y = b)) \supset \sim Px]$$

Putting these together we obtain

$$(Pb \ \& \ (\exists x)[(Sxb \ \& \ (\forall y)(Syb \supset y = x)) \ \& \ Px]) \ \& \ (\forall x)(\forall y)[(Sxy \ \& \ (Py \ \& \sim y = b)) \supset \sim Px]$$

DEFINITE DESCRIPTIONS

In Section 7.1 we noted that there are two kinds of singular terms in English: proper names and **definite descriptions.** We subsequently noted that individual constants of *PL* can be used to symbolize both kinds of singular terms of English. But following this practice means that the internal structure of definite descriptions is not represented in *PL*. Consider, by way of illustration, this argument:

> The Roman general who defeated Pompey invaded both Gaul and Germany. Therefore Pompey was defeated by someone who invaded both Gaul and Germany.

This is fairly obviously a valid argument. But its symbolization in *PL* is not valid:

UD: Persons and countries
Ixy: x invaded y
Dxy: x defeated y
r: The Roman general who defeated Pompey
p: Pompey
g: Gaul
e: Germany

Treating 'The Roman general who defeated Pompey' as an unanalyzable unit, to be symbolized by 'r', and paraphrasing the conclusion as 'There is an x such that x defeated Pompey and invaded Gaul and invaded Germany' yields this argument:

Irg & Ire

(∃x)[Dxp & (Ixg & Ixe)]

The techniques we develop for testing arguments of *PL* will show that this argument of *PL* is invalid. This should not be surprising, for the premise tells us only that the thing designated by 'r' invaded both Gaul and Germany; it does not tell us that that thing is a thing that defeated Pompey, as the conclusion claims.

By using the identity predicate we can capture the structure of definite descriptions within *PLE*. Suppose we paraphrase the first premise of the preceding argument as

> There is exactly one thing that is a Roman general and defeated Pompey, and that thing invaded both Gaul and Germany.

Definite descriptions are, after all, descriptions that purport to specify conditions that are satisfied by exactly one thing. Using the symbolization key, plus 'Rx' for 'x is a Roman general', we can symbolize the first premise as

(∃x)[[(Rx & Dxp) & (∀y)[(Ry & Dyp) ⊃ y = x]] & (Ixg & Ixe)]

We shall later show that in *PLE* the conclusion '(∃x)[Dxp & (Ixg & Ixe)]' does follow from this premise.

By transforming definite descriptions into unique existence claims, that is, claims that there is exactly one object of such-and-such a sort, we gain the further benefit of being able to symbolize English language definite descriptions that may, in fact, not designate anything. For example, taking the UD to be persons and using 'Dxy' for 'x is a daughter of y', 'Bx' for 'x is a biochemist', and 'j' to designate John, we might symbolize 'John's only daughter is a biochemist' as

(∃x)[(Dxj & (∀y)(Dyj ⊃ y = x)) & Bx]

If it turns out that John has no, or more than one, daughter, or that his only daughter is not a biochemist, the above sentence of *PLE* will be false, not meaningless or truth-valueless. This is an acceptable result.

PROPERTIES OF RELATIONS

Identity is a relation with three rather special properties. First, identity is a **transitive** relation. That is, if an object x is identical with an object y, and y is

identical with an object z, then x is identical with z. The following sentence of *PLE* says, in effect, that identity is transitive:

$$(\forall x)(\forall y)(\forall z)[(x = y \ \& \ y = z) \supset x = z]$$

Many relations other than identity are also transitive relations. The predicates

x is larger than y
x is taller than y
x is an ancestor of y
x is heavier than y
x occurs before y

all express transitive relations. But, 'x is a friend of y' does not represent a transitive relation. That is, 'Any friend of a friend of mine is a friend of mine' is a substantive claim, and one that is generally false. Where **x**, **y**, and **z** are all variables of *PL* or *PLE* and **A** is a two-place predicate of *PL* or *PLE*, the following says that **A** expresses a transitive relation:

$$(\forall \mathbf{x})(\forall \mathbf{y})(\forall \mathbf{z})[(\mathbf{Axy} \ \& \ \mathbf{Ayz}) \supset \mathbf{Axz}]$$

Identity is also a **symmetric** relation; that is, if an object x is identical with an object y, then y is identical with x. The following says that identity is a symmetric relation:

$$(\forall x)(\forall y)(Axy \supset Ayx)$$

The following predicates also express symmetric relations:

x is a sibling of y
x is a classmate of y
x is a relative of y
x has the same father as does y

Note that neither 'x is a sister of y' nor 'x loves y' expresses a symmetric relation. Jane Fonda is a sister of Peter Fonda, but Peter Fonda is not a sister of Jane Fonda. And, alas, it may be that Manfred loves Hildegard even though Hildegard does not love Manfred.

A relation is **reflexive** if and only if each object stands in that relation to itself. In *PL* and *PLE* the following says that **A** expresses a reflexive relation:

$$(\forall \mathbf{x})\mathbf{Axx}$$

Identity is a reflexive relation. In an unrestricted UD it is rather hard to find other reflexive relations. For example, a little thought should show that none

of the following expresses a reflexive relation in an unrestricted universe of discourse:

x is the same age as y
x is the same height as y
x is in the same place as y

Since the number 48 is not of any age, it is not the same age as itself nor the same height as itself. Numbers have neither age nor height, though inscriptions of numerals usually have both. So, too, neither the number 93 nor the set of human beings is in any place. Numbers and sets do not have spatial positions; hence neither is in the same place as itself. However, the relations just discussed are reflexive relations in suitably restricted universes of discourse. For example, if the universe of discourse consists exclusively of people, then

x is the same age as y

expresses a reflexive relation (it is also transitive and symmetric). Every person is the same age as him- or herself. In this restricted universe 'x is the same height as y' and 'x is in the same place as y' also represent reflexive relations. Each person is the same height as him- or herself and is in the same place as him- or herself. And, if the universe of discourse is restricted to the positive integers, then

x is evenly divisible by y

expresses a reflexive relation, for every positive integer is evenly divisible by itself. This relation is not symmetric (not every positive integer evenly divides all the positive integers it is evenly divisible by). However, 'x is evenly divisible by y' does express a transitive relation.

FUNCTIONS

A **function** is an operation that takes one or more element of a set as arguments and returns a single value. *Addition*, *subtraction*, *multiplication*, *square*, and *successor* are all common functions of arithmetic. Each returns, for each number or pair of numbers, a single value. *Addition* takes a pair of numbers as arguments and returns their sum; *multiplication* takes a pair of numbers and returns the product of those numbers; *subtraction* returns, for each pair of numbers, the first number minus the second. The *square* function returns, for each number, the result of multiplying that number by itself; the *successor* function returns, for any positive integer **n**, the integer **n** + 1.

Not all functions are arithmetic functions. We have already encountered truth-functions—functions that map values from the set consisting of the truth-values (the set {**T**, **F**}) to truth-values. *Negation* is a function of one

argument that returns **F** when given **T** as an argument and returns **T** when given **F** as an argument. *Conjunction, disjunction,* the *material conditional,* and the *material biconditional* are all functions that take two arguments (two truth-values) and return a single truth-value. Characteristic truth-tables display the value of each of these functions for each pair of truth-values.

Functions are also found outside of formal logic and mathematics. Consider a set of monogamously married individuals.[13] Here *spouse* is a function that takes a single member of the set as an argument and returns that person's spouse as its value. For the set of all twins, the function *twin* returns, for each member of the set, that member's twin. In *PLE* we shall use lowercase italicized Roman letters *a*–*z*, with or without a positive-integer subscript, followed by one or more prime marks to symbolize functions. We call these symbols **functors.** Where **n** is the number of prime marks after the functor, the function assigned to the functor takes **n** arguments. For example, in talking about the set of positive integers, we might assign the successor function to the functor f'.[14] We specify this assignment in a symbolization key much the way we have been assigning interpretations to predicates. The following symbolization key assigns the successor function to f':

UD: Positive integers
f'(x): the successor of x
Ex: x is even
Ox: x is odd
a: 2
b: 3

The variable x in parentheses indicates that we are assigning to f' a function that takes a single argument. The expression to the right of the colon assigns the successor function to f'. Given the above symbolization key,

Ob

says 3 is odd. The sentence

Of'(a)

says the successor of 2, which is 3, is odd. Both claims are, of course, true. And

f'(a) = b

says the successor of 2 is 3, which it is. Similarly,

$(\exists x)$Of'(x) & $(\exists x)$Ef'(x)

[13] The example is from Geoffrey Hunter, *Metalogic: An Introduction to the Metatheory of Standard First Order Logic* (Berkeley: University of California Press, 1973).

[14] It is customary to use, where only a few functors are needed, the letters 'f', 'g', 'h', . . . We will follow this custom.

says there is a positive integer whose successor is odd and there is a positive integer whose successor is even. We can also use the symbolization key to symbolize 'The successor of an even number is odd'. A first step is the quasi-English

$$(\forall x)(Ex \supset \text{the successor of x is odd})$$

The successor of x is $f'(x)$, so the full symbolization is

$$(\forall x)(Ex \supset Of'(x))$$

We can add the following to our symbolization key

$h''(x,y)$: the sum of x and y

and symbolize 'The sum of an even number and an odd number is odd' as

$$(\forall x)(\forall y)[(Ex \mathbin{\&} Oy) \supset Oh''(x,y)]$$

Since the number of distinct individual terms occurring within the parentheses after a functor indicates how many arguments the function assigned to that functor takes, we can informally omit the primes that officially follow functors, just as we do for predicate letters. Hereafter we will do so.

Returning to our example of the set of twins, we can use the following symbolization key

UD: Set of twins
$f(x)$: the twin of x
c: Cathy
h: Henry
j: Jose
s: Simone

to symbolize

Simone is Henry's twin

as

$$s = f(h)$$

and

Jose is Cathy's twin

as

$$j = f(c)$$

Using 'Bx' for 'x is bald', we can symbolize 'A twin is bald if and only if her or his twin is bald' as

$$(\forall x)[Bx \equiv Bf(x)]$$

and 'Some bald twins have twins that are not bald' as

$$(\exists x)Bx \ \& \ \sim Bf(x)$$

The symbolization

$$(\forall x)(\forall y)[(\exists z)(z = f(x) \ \& \ z = f(y)) \supset x = y]$$

says, in quasi-English, 'Any members of the UD x and y who are such that there is a z who is both a twin of x and a twin of y are in fact the same member of the UD', or 'No one is a twin of two different twins'.

We require that the functions we symbolize with functors have the following characteristics:

1. An **n**-place function must yield one and only one value for each **n**-tuple of arguments.[15]
2. The value of a function for an **n**-tuple of members of a UD must be a member of that UD.

If the UD is the set of integers, the square root operation does not meet condition 1 because it can yield more than one value for its arguments (there are two square roots of 4—2 and −2). (It also fails to meet condition 2 because not all square roots of integers are integers.) If the UD is the set of positive integers, the subtraction function does not meet condition 2, because when y is greater than x, x minus y yields a value that is not a positive integer (3 minus 9 is −6, and −6 is not a positive integer). Subtraction *does* meet condition 2 when the UD is the set of all integers—positive, zero, and negative. If the UD is the set of positive integers, division also fails to meet condition 2 (3 divided by 9 yields $\frac{1}{3}$, which is not a positive integer). Division *does* meet condition 2 when the UD is the set of positive rational numbers (positive integers plus numbers expressible as the ratio between positive integers). Finally division does not meet condition 1 when the UD is the set of *all* integers because it is undefined when the divisor is zero.

As we have just seen, functors can be used to generate a new kind of individual term (in addition to the individual constants and variables of *PL*). We call these new terms **complex terms.** Complex terms are of the form

$$f(\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_n)$$

[15] An **n**-tuple is an ordered set containing **n** members.

where f is an **n**-place functor and $\mathbf{t}_1, \mathbf{t}_2, \ldots \mathbf{t}_n$ are individual terms. Further examples of complex terms include

f(a,b)
h(a,b,c)
g(a)
f(b,b)
f(x,y)
f(a,y)
f(y,a)
g(x)
f(g(a),b)
f(a,g(x))

Complex terms are complex in that they are always formed from a functor and at least one individual term. Some complex terms contain variables, and some do not. We call individual terms that do not contain variables **closed terms,** and those that do **open terms.** This makes *both* individual constants and complex terms that contain no variables closed terms. Complex terms that do contain at least one variable, as well as variables themselves, are open terms. Individual terms that are not complex terms (the individual constants and individual variables) are **simple individual terms.** In the above list, the first four complex terms are closed, the next four open, the ninth closed, and the last open. Note the last two examples. In each, one of the individual terms from which the example is built is itself a complex term. This is wholly in order, as complex terms are individual terms and can occur anywhere a constant can occur. The kinds of individual terms included in *PLE* are summarized in the following table:

INDIVIDUAL TERMS OF *PLE*

	Open	*Closed*
Simple	Individual variables	Individual constants
Complex	Individual term formed from a functor and *at least one* individual variable—for example, f(x), f(a,x), $g(f$(a),y), $g(h$(x,y),a)	Individual term formed from a functor and containing *no* individual variable—for example, f(a), g(a,b), $f(g$(a,f(a,c)))

All of the following are formulas of *PLE*:

F$a f$(x)
Ff(x)a
Ff(a)b
(∀x)F$a f$(x)
(∀x)(∃y)F$x f$(y)

In each of these examples 'F' is a two-place predicate. The first and second are formulas of *PLE* but are not sentences (because the x in 'f(x)' is not bound). The third, fourth, and fifth examples are all both formulas and sentences of *PLE*. The third says that f(a) bears the relation F to b. The fourth says that each thing x in the UD is such that a bears the relation F to f(x), that is, to the value of the function f as applied to x. The fifth says that each thing x in the UD is such that there is a thing y such that x bears the relation F to f(y). Every example contains a complex individual term, and all but the third an open complex individual term.

Consider this symbolization key:

UD:	Positive integers
Ox:	x is odd
Ex:	x is even
Px:	x is prime
Gxy:	x is greater than y
h(x,y):	the sum of x and y
f(x):	the successor of x
a:	1
b:	2

The sentence

$$(\forall x)[Ex \supset Of(x)]$$

says, truly, that each positive integer is such that if it is even then its successor is odd. And

$$(\forall x)[Ex \supset Ef(f(x))]$$

says, truly, that each positive integer is such that if it is even then the successor of its successor is also even. The sentence

$$(\forall x)(\forall y)[(Ex \,\&\, Ey) \supset Eh(x,y)]$$

can be read in quasi-English as 'For each x and each y, if both x and y are even, then the sum of x and y is even'. This is, of course, true.

Here are further sentences of *PLE* that can be read in English using the above symbolization key. The sentence

$$(\forall x)(\forall y)[Gh(x,y)x \,\&\, Gh(x,y)y]$$

says that for any positive integers x and y the sum of x and y is greater than x, and the sum of x and y is greater than y. This is true. The sentence

$$(\exists x)Gxh(a,b)$$

says that there is a positive integer, x, that is greater than the sum of 1 and 2—that is, there is a positive integer that is greater than 3. This is also true. The sentence

$$(\forall x)(\forall y)[(Ex \ \& \ Oy) \supset Odd(x,y)]$$

says that, for any pair of positive integers x and y, if the first is even and the second is odd, then their sum is odd. This is true as well. Finally the sentence

$$(\forall x)(\forall y)[Prime(x,y) \supset \sim (Px \ \& \ Py)]$$

says that, for any pair of positive integers, if their sum is prime then it is not the case that they are both prime, or, in other words, that there are no prime numbers x and y such that their sum is also prime. This sentence is false; 2 and 3 are both prime, and so is their sum, 5.

THE SYNTAX OF PLE

The language of *PLE* is an expansion of *PL* and as such includes all the vocabulary of *PL*. Every formula of *PL* is a formula of *PLE*, and every sentence of *PL* is a sentence of *PLE*. The vocabulary of *PLE* also includes

"=": The two-place identity predicate (fixed interpretation)

Functors of PLE: Lowercase italicized Roman letters a, b, c, . . . , with or without a numeric subscript, followed by **n** primes.

Individual terms of PLE:

Individual constants are individual terms of *PLE*

Individual variables are individual terms of *PLE*

Expressions of the form $f(\mathbf{t}_1, \mathbf{t}_2, \ldots \mathbf{t}_n)$, where f is an **n**-place functor and $\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_n$) are individual terms of *PLE*, are individual terms of *PLE*

We can classify the individual terms of *PLE* as follows:

Simple terms of PLE: The individual constants and individual variables of *PLE*

Complex terms of PLE: Individual terms of the form $f(\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_n)$, where f is an **n**-place functor

Closed individual term: An individual term in which no variable occurs

Open individual term: An individual term in which at least one variable occurs

Individual variables and functors that contain at least one individual variable are thus open terms. Individual constants and functors that contain no variables are thus closed terms.

In *PLE* a substitution instance is defined as follows:

> *Substitution instance of* **P**: If **P** is a sentence of *PLE* of the form $(\forall \mathbf{x})\mathbf{Q}$ or $(\exists \mathbf{x})\mathbf{Q}$ and **t** is a closed individual term, then $\mathbf{Q}(\mathbf{t}/\mathbf{x})$ is a substitution instance of **P**. The individual term **t** is the *instantiating individual term*.

Note that every substitution instance of a sentence of *PL* is also a substitution instance of that same sentence in *PLE*.

7.9E EXERCISES

1. Symbolize the following sentences in *PLE* using the symbolization key given in Exercise 1 in Section 7.8E.

a. Every Wilcox except Daniel is a sailor.
*b. Every Wilcox except Daniel is the offspring of a sailor.
c. Every Wilcox except Daniel is either a sailor or the offspring of sailor.
*d. Daniel is the only son of Jacob.
e. Daniel is the only child of Jacob.
*f. All the Wilcoxes except Daniel are sailors.
g. Rebecca's only son is Jacob's only son.
*h. Rebecca Wilcox has only one son who is a sailor.
i. Rebecca Wilcox has at least two daughters who are sailors.
*j. There are two and only two sailors in the Wilcox family.
k. Jacob Wilcox has one son and two daughters, and they are all sailors.

2. Give fluent English readings for the following sentences of *PLE* using the given symbolization key.

UD: Positive integers
Lxy: x is less than y
Gxy: x is greater than y
Ex: x is even
Ox: x is odd
Px: x is prime
$f(x,y)$: the product of x and y
t: 2
f: 5
n: 9

a. $(\forall x)(\exists y)Lxy$
*b. $(\exists x)(\forall y)(\sim x = y \supset Lxy)$
c. $(\exists x)(\forall y) \sim Lyx$
*d. $\sim (\exists x)(Ex \,\&\, Lxt)$

e. $(Pt \ \& \ Et) \ \& \ (\forall x)[(Px \ \& \ Ex) \supset x = t]$
*f. $\sim (\exists x)(\exists y)[(Px \ \& \ Py) \ \& \ Pf(x,y)]$
g. $(\forall y)(\forall z)[(Oy \ \& \ Oz) \supset Of(y,z)]$
*h. $(\forall y)(\forall z)[(Ey \ \& \ Ez) \supset Ef(z,y)]$
i. $(\forall y)(\forall z)[(Ey \vee Ez) \supset Ef(y,z)]$
*j. $(\forall x)[Ex \supset (\exists y)(Oy \ \& \ Gxy)] \ \& \sim (\forall x)[Ox \supset (\exists y)(Ey \ \& \ Gxy)]$
k. $(\exists x)[[Px \ \& \ (Gxf \ \& \ Lxn)] \ \& \ (\forall y)([Py \ \& \ (Gyf \ \& \ Lyn)] \supset y = x)]$

3. For a–p, decide whether the specified relation is reflexive, whether it is symmetric, and whether it is transitive (in suitably restricted universes of discourse). In each case give the sentences of *PL* that assert the appropriate properties of the relation in question. If the relation is reflexive, symmetric, or transitive in a restricted universe of discourse, specify such a universe of discourse.

a. Nxy: x is a neighbor of y
*b. Mxy: x is married to y
c. Axy: x admires y
*d. Nxy: x is north of y
e. Rxy: x is a relative of y
*f. Sxy: x is the same size as y
g. Txy: x is at least as tall as y
*h. Cxy: x coauthors a book with y
i. Exy: x enrolls in the same course as y
*j. Fxy: x fights y
k. Wxy: x weighs the same as y
*l. Cxy: x contracts with y
m. Axy: x is an ancestor of y
*n. Cxy: x is a cousin of y
o. Lxy: x and y have the same taste in food
*p. Rxy: x respects y

4. Symbolize the following sentences in *PLE* using the given symbolization key.

UD: People in Doreen's hometown
Dxy: x is a daughter of y
Sxy: x is a son of y
Bxy: x is a brother of y
Oxy: x is older than y
Mxy: x is married to y
Txy: x is taller than y
Px: x is a physician
Bx: x is a baseball player
Mx: x is a marine biologist
d: Doreen
c: Cory
j: Jeremy
h: Hal

a. Jeremy is Cory's son.
*b. Jeremy is Cory's only son.
c. Jeremy is Cory's oldest son.
*d. Doreen's only daughter is a physician.

e. Doreen's eldest daughter is a physician.
*f. Doreen is a physician and so is her eldest daughter.
g. Cory is Doreen's eldest daughter.
*h. Cory is married to Hal's only son.
i. Cory is married to Hal's tallest son.
*j. Doreen's eldest daughter is married to Hal's only son.
k. The only baseball player in town is the only marine biologist in town.
*l. The only baseball player in town is married to one of Jeremy's daughters.
m. Cory's husband is Jeremy's only brother.

5. Symbolize the following sentences in *PLE* using the given symbolization key.

UD: Positive integers
Ox: x is odd
Ex: x is even
Px: x is prime
a: 1
b: 2
f(x): the successor of x
g(x): x squared
t(x,y): the product of x and y
s(x,y): the sum of x and y

a. One is not the successor of any integer.
*b. One is not prime but its successor is.
c. There is a prime that is even.
*d. There is one and only one even prime.
e. Every integer has a successor.
*f. The square of a prime is not prime.
g. The successor of an odd integer is even.
*h. The successor of an even integer is odd.
i. If the product of a pair of positive integers is odd, then the product of the successors of those integers is even.
*j. If the product of a pair of positive integers is even, then one of those integers is even.
k. If the sum of a pair of positive integers is odd, then one member of the pair is odd and the other member is even.
*l. If the sum of a pair of positive integers is even, then either both members of the pair are even or both members are odd.
m. The product of a pair of prime integers is not prime.
*n. There are no primes such that their product is prime.
o. The square of an even number is even and the square of an odd number is odd.
*p. The successor of the square of an even number is odd.
q. The successor of the square of an odd number is even.
*r. 2 is the only even prime.
s. The sum of 2 and a prime other than 2 is odd.
*t. There is exactly one integer that is prime and is the successor of a prime.
u. There is a pair of primes such that their product is the successor of their sum.

Chapter 8
PREDICATE LOGIC: SEMANTICS

8.1 INFORMAL SEMANTICS FOR *PL*

The basic semantic concept for the language of sentential logic, SL, is that of a truth-value assignment. The semantics for PL is more complex than truth-functional semantics. One source of the added complexity is this: Whereas the atomic sentences of SL are not analyzable in terms of more basic linguistic units of SL, the same does not hold for all atomic sentences of PL. Some atomic sentences of PL, such as 'Fa', are themselves complex expressions composed of predicates and individual constants. Consequently we do not directly assign truth-values to all the atomic sentences of PL; only the sentence letters are directly assigned truth-values. The truth-values of complex atomic sentences like 'Fa' depend on the interpretations of the predicates and individual constants that constitute such sentences. The basic semantic concept of PL, in terms of which other semantic concepts are defined, is that of an *interpretation*. Just as truth-value assignments for SL assign truth-values to every sentence of SL, an interpretation interprets *every* individual constant, predicate, and sentence letter of PL. Usually, however, we shall be interested only in that part of an interpretation that affects the truth-value of a particular sentence or set of sentences that we are looking at.

We can view the symbolization keys for sentences presented in Chapter 7 as embodying interpretations for those sentences. That is, the truth-conditions of sentences of *PL* are dependent upon the choice of universe of discourse and upon how each of the predicates and individual constants in the sentences is interpreted. In this section we shall discuss in an informal manner how interpretations determine the truth-conditions of sentences, appealing to the readings of sentences of *PL* that were used in Chapter 7.

Let us start with an example of an atomic sentence of *PL*: 'Fa'. Whether this sentence is true depends on how we interpret the predicate 'F' and the individual constant 'a'. If we interpret them as follows:

Fx: x is human
a: Socrates

then 'Fa' is true, for Socrates was human. But if we interpret them as

Fx: x is a potato
a: Socrates

then 'Fa' is false, for Socrates was not a potato. Similarly the truth-value of the sentence 'Bdc' depends upon the interpretation of the expressions that constitute the sentence. If we interpret them as

Bxy: x is bigger than y
c: the Statue of Liberty
d: the Empire State Building

then 'Bdc', which may be read as 'The Empire State Building is bigger than the Statue of Liberty', is true. But with the following interpretations:

Bxy: x is bigger than y
c: the moon
d: the Empire State Building

'Bdc' is false. The Empire State Building is not bigger than the moon.

Predicates are interpreted relative to a *universe of discourse*. Recall that a universe of discourse (UD) is simply a nonempty set. We may choose the set of natural numbers, the set of all people, the set of words in this chapter, the set of all the objects in the world, the set containing only Mark Twain, or any other nonempty set as the UD when we specify an interpretation. The UD that we choose includes all and only those things that we want to interpret sentences of *PL* as being about. Once we specify the UD, our interpretations of predicates are interpretations relative to that UD. For example, if an interpretation includes

UD: Set of living creatures
Fx: x is human

then 'F' picks out all the living creatures in the UD that are human. We call the set of those things that the predicate picks out the *extension* of the predicate for the interpretation. If an interpretation includes

UD: Set of living creatures in San Francisco
Fx: x is human

then the predicate 'F' picks out all those living creatures in San Francisco that are human. The set of such creatures is the extension of the predicate 'F'. And if an interpretation includes

UD: Set of living creatures
Fx: x is an automobile

then the predicate 'F' picks out nothing—no member of the UD is an automobile. In this case the extension of the predicate 'F' is the empty set.

Now let us consider two-place predicates. Suppose that an interpretation includes the following:

UD: Set of positive integers
Gxy: x is greater than y

In this case we cannot simply say that the predicate 'G' picks out members of the UD. We are interpreting 'G' as a *relational* predicate, so here the extension of the predicate is a set of *pairs* of objects rather than simply a set of objects. One of the pairs of positive integers that is in the extension of the predicate is the pair consisting of the number 5 and the number 2, in that order, since 5 is greater than 2. We must think of these pairs as ordered because, although the predicate picks out the pair whose first member is 5 and whose second member is 2, it does not pick out the pair whose first member is 2 and whose second member is 5—2 is not greater than 5. The extension of 'G' includes all and only those pairs of objects in the UD (pairs of positive integers) of which the first member is greater than the second.

On some interpretations the extension of a two-place predicate includes pairs in which the first and second members are the same. For example, on an interpretation that includes

UD: Set of positive integers
Lxy: x is less than or equal to y

the extension of 'L' includes the pair consisting of 1 and 1, the pair consisting of 2 and 2, the pair consisting of 3 and 3, and so on—because each positive integer is less than or equal to itself. (The extension includes other pairs as well—any pair of positive integers in which the first member is less than or equal to the second.)

Three-place predicates, four-place predicates, and all other many-place predicates are interpreted similarly. A three-place predicate has as its extension

a set of ordered triples of objects in the UD; a four-place predicate has as its extension a set of ordered quadruples of members of the UD; and so on.

An individual constant is interpreted by assigning to the constant some member of the selected UD. So, if we choose as our UD the set of living creatures, then we may assign to 'a', as its interpretation, some specific living creature. Here are two examples of interpretations for the sentence 'Fa':

1. UD: Set of positive integers
   Fx: x is a prime number
   a: 4
2. UD: Set of animals in the Bronx Zoo
   Fx: x is a giraffe
   a: the oldest giraffe in the Bronx Zoo

Once we have given interpretations of the expressions in the sentence 'Fa', we may determine the truth-value of 'Fa' on that interpretation. The sentence 'Fa' is true on an interpretation just in case the object that the constant 'a' designates is a member of the set that is the extension of the predicate 'F' for that interpretation. The sentence 'Fa' is false on interpretation 1 and true on interpretation 2. (Actually neither 1 nor 2 is a full interpretation; a full interpretation interprets every constant, predicate, and sentence letter of *PL*. For example, 1 represents infinitely many interpretations. It represents all interpretations that have the specified UD, that interpret 'F' and 'a' as indicated, and that interpret all other predicates, constants, and sentence letters as they please. However, we shall continue to talk informally of our partial interpretations simply as interpretations.)

Here is an interpretation for 'Gab':

3. UD: Set of positive integers
   Gxy: x is greater than y
   a: 2
   b: 16

The sentence 'Gab' is false on interpretation 3, for on this interpretation 'Gab' says that 2 is greater than 16, which is not the case. But 'Gba', which may be read as 'The number 16 is greater than the number 2', is true on this interpretation because the pair of numbers whose first member is 16 and whose second member is 2 *is* in the extension of the predicate 'G'.

An interpretation may assign the same member of the UD to more than one constant. The following is a legitimate interpretation for 'Jln':

4. UD: Set of planets in the solar system
   Jxy: x is closer to the sun than is y
   l: Jupiter
   n: Jupiter

Here both 'l' and 'n' have been interpreted as designating Jupiter. The sentence is false on this interpretation since Jupiter is not closer to the sun than itself. Usually, when we *symbolize* English sentences in *PL*, we use different constants to designate different individuals. But, if we want to use different constants to designate the same individual, we may do so. This is similar to the case in which an object is referred to by more than one expression in the English language. For instance, 'the first U.S. president' and 'George Washington' designate the same person. But note that, whereas in English one name may stand for more than one object (in which case it is *ambiguous*), we do not allow this in *PL*. Two names may designate the same object, but *no one name may designate more than one object.*

The truth-conditions for compound sentences of *PL* that do not contain quantifiers are determined in accordance with the truth-functional reading of the connectives, so we use the information in the characteristic truth-tables for the truth-functional connectives to determine the truth-values of such sentences. Consider the sentence '(Bs ∨ ~ Fh) & Gsh'. Here is an interpretation for the sentence:

5. UD: Set of all things
Bx: x is an author
Fx: x is an animal
Gxy: x owns y
h: The Liberty Bell
s: Stephen King

The sentence 'Bs' is true on this interpretation since Stephen King is an author. The sentence 'Fh' is false on this interpretation since the Liberty Bell is not an animal, so '~ Fh' is true. Since 'Bs' and '~ Fh' are both true, the sentence 'Bs ∨ ~ Fh' is true. Stephen King does not own the Liberty Bell, so 'Gsh' is false on this interpretation. Consequently '(Bs ∨ ~ Fh) & Gsh' has one false conjunct and is false on interpretation 5. Another interpretation may make the same sentence true:

6. UD: Set of people
Bx: x is male
Fx: x is a negative integer
Gxy: x is the mother of y
h: Jay Doe
s: Jane Doe (who is the mother of Jay Doe)

On this interpretation 'Fh' is false. As we have interpreted the predicate 'F', its extension is the empty set. The predicate picks out nothing in the UD since no person is a negative integer, so '~ Fh' is true. 'Bs' is false since Jane Doe is not male, but 'Bs ∨ ~ Fh' is true—a disjunction with one true disjunct is itself

true. Since Jane Doe is Jay Doe's mother, 'Gsh' is true. Thus the sentence '(Bs ∨ ~ Fh) & Gsh' has two true conjuncts and is true on interpretation 6.

We have yet to consider interpretations for the *quantified* sentences of *PL*. Quantified sentences are not atomic, and they are not truth-functions of smaller sentences of *PL* either. The quantified sentence

$$(\forall x)(Fx \supset Gx)$$

is not truth-functionally compound; indeed, it contains no proper subformula that is itself a sentence. To give an interpretation for this sentence, we must specify a UD and interpret the predicate letters 'F' and 'G'. *We do not interpret individual variables.* As noted in Chapter 7, individual variables function in *PL* much as pronouns do in English. They are not names, so interpretations do not assign to them members of the UD.

We may read '$(\forall x)(Fx \supset Gx)$' as 'Each x is such that if x is F then x is G' or 'Everything that is F is G'. When we specify a UD for interpreting the sentence, we thereby specify what 'everything' is, namely, everything in the UD. Here is an interpretation for '$(\forall x)(Fx \supset Gx)$':

7. UD: Set of people
   Fx: x is a politician
   Gx: x is honest

With this interpretation we may read the sentence as 'Every person who is a politician is honest'. Unfortunately some politicians are not honest; the sentence is therefore false on interpretation 7. The part of the sentence that follows the universal quantifier, the open sentence 'Fx ⊃ Gx', specifies a condition that may or may not hold for the individual members of the UD; the function of the universal quantifier is to state that this condition holds for *each* member of the UD. Consequently the sentence is true if and only if every member of the UD meets that condition.

Consider the following interpretation for the sentence

$$(\forall x)(Ax \equiv \sim Bx)$$

8. UD: Set of positive integers
   Ax: x is evenly divisible by 2
   Bx: x is an odd number

The universal quantifier states that the condition specified by 'Ax ≡ ~ Bx' holds for every member of the UD. The sentence, which we may read as 'Each positive integer is evenly divisible by 2 if and only if it is not an odd number', is true on this interpretation. But the interpretation

9. UD: Set of positive integers
   Ax: x is evenly divisible by 4
   Bx: x is an odd number

makes the same sentence false. A universally quantified sentence is false if there is at least one member of the UD for which the condition specified after the quantifier does not hold. The number 6 is one member for which the condition specified by '$Ax \equiv -Bx$' does not hold. The number 6 is not evenly divisible by 4 and is not an odd number—so 6 does not meet the condition that it *is* evenly divisible by 4 if and only if it is *not* an odd number.

In determining the truth-conditions for universally quantified sentences, then, we should keep two points in mind. First, individual variables are *not* interpreted. The function of these variables is to range over the members of the UD; consequently it is the specification of the UD for an interpretation that is relevant in determining the contribution that individual variables and quantifiers make to the truth-conditions of sentences of *PL*. Second, the role of the universal quantifier in a sentence of *PL* is to indicate that *every* member of the UD satisfies a certain condition. The condition is specified by that part of the sentence that lies within the scope of the quantifier.

Existential quantifiers function in sentences of *PL* to indicate that at least one member of the UD satisfies a certain condition. Here is an interpretation for the existentially quantified sentence

$(\exists y)(Cy \ \& \ By)$

10. UD: Set of all things
    Cx: x is a car
    Bx: x is brown

The sentence is true on this interpretation. It may be read as 'At least one object (in the universe) is a brown car'. Because it is existentially quantified, the sentence is true just in case at least one object in the universe is a car and is brown, that is, just in case at least one object in the universe satisfies the condition specified by '$Cy \ \& \ By$'. Since there is at least one such object, '$(\exists y)(Cy \ \& \ By)$' is true on this interpretation. However, the same sentence is false on the following interpretation:

11. UD: Set of all things
    Cx: x is a car
    Bx: x has a brain

Marvelous as technology is, it has not yet produced cars with brains. Hence no object satisfies the condition specified by '$Cy \ \& \ By$', and so '$(\exists y)(Cy \ \& \ By)$' is false on the present interpretation.

The different sentence

$(\exists y)Cy \ \& \ (\exists y)By$

is true on interpretation 11. This is because both '$(\exists y)Cy$' and '$(\exists y)By$' are true—there is an object that is a car and there is an object that has a brain. It

is not one and the same object that has both these properties, however. That is why the sentence '(∃y)(Cy & By)' is false on this interpretation. Although 'y' occurs in both 'Cy' and 'By' of '(∃y)Cy & (∃y)By', these two occurrences of the variable are not in the scope of a single occurrence of the quantifier '(∃y)'. So the predicates 'is a car' and 'has a brain' do not have to hold for the same object for the sentence '(∃y)Cy & (∃y)By' to be true. In '(∃y)(Cy & By)', however, all occurrences of 'y' *are* within the scope of one occurrence of '(∃y)'; so, for '(∃y)(Cy & By)' to be true on interpretation 11, a single member of the UD must both be a car and have a brain.

Now we shall consider an interpretation for a sentence containing a two-place predicate and two quantifiers:

$(\exists x)(\forall y)Fxy$

12. UD: Set of people
    Fxy: x is acquainted with y

Since the whole sentence is existentially quantified, it is true on interpretation 12 if at least one person satisfies the condition specified by the rest of the sentence, that is, if at least one person is acquainted with every member of the UD. Obviously there is no such person, so '(∃x)(∀y)Fxy' is false on interpretation 12. Using the same interpretation, let us look at the sentence that is formed by reversing the order of the quantifiers:

$(\forall y)(\exists x)Fxy$

Prefixed with the universal quantifier, the sentence says that *each* member of the UD satisfies a certain condition. For this sentence to be true, each person y must be such that at least one person is acquainted with y. This is true (since every person is at least self-acquainted), so '(∀y)(∃x)Fxy' is true on interpretation 12. Note that the difference that accounts for the diverging truth-values of '(∃x)(∀y)Fxy' and '(∀y)(∃x)Fxy' is that in the first case it is one and the same person who must be acquainted with everyone. Another sentence that we may interpret using interpretation 12 is

$(\exists x)(\exists y)Fxy$

This sentence is true on interpretation 12 since there is at least one person who is acquainted with at least one person.

The following is an interpretation for the sentence

$(\forall x)(Fx \supset (\exists y) \sim Gyy)$

13. UD: Set of houses in the world
    Fx: x is made of brick
    Gxy: x is larger than y

Given this interpretation, the sentence may be read as 'For any brick house, there is at least one house that is not larger than itself'. This sentence is universally quantified and hence is true just in case every house x in the world satisfies the condition that if x is made of brick then at least one house is not larger than itself. Any house that is not made of brick trivially satisfies the condition since it does not fulfill the condition of being made of brick. That is, a house x that is not made of brick is such that if x is made of brick (which it is not) then some house is not larger than itself. Brick houses also satisfy the condition specified in the sentence. For since it is true that at least one house is not larger than itself, it is true of any brick house x that if x is made of brick (which it is) then some house is not larger than itself (which is true).

The sentence

$$(\forall y)(\forall x)(Fyx \supset Fxy)$$

is false on the following interpretation:

14. UD: Set of integers
Fxy: x is smaller than y

No integer y satisfies the condition specified by '$(\forall x)(Fyx \supset Fxy)$', which is that every integer that y is smaller than is, in turn, smaller than y. For any integer y there are (infinitely many) integers x that y is smaller than, but not even one of these integers is, in turn, smaller than y. But the sentence

$$(\forall y)(\forall x)(Fyy \supset Fxy)$$

is true on interpretation 14. Every integer y trivially satisfies the condition specified by '$(\forall x)(Fyy \supset Fxy)$', which is that every integer x is such that if y is smaller than itself (it is not) then x is also smaller than y.

Now we shall consider sentences that contain individual constants and sentence letters, as well as quantifiers. Consider

$$(\exists x)(Fx \ \& \ (\forall y)Gxy) \supset \sim Gsl$$

and this interpretation:

15. UD: Set of people
Fx: x is female
Gxy: x is the sister of y
l: Michael Jackson
s: Janet Jackson

On this interpretation the sentence may be read as 'If some person is a female and is everybody's sister (including her own) then Janet Jackson is not Michael Jackson's sister'. The consequent of this sentence, '~ Gsl', is false because Janet

is Michael's sister. But the antecedent is also false; there is no female person who is everybody's sister. So the sentence '$(\exists x)(Fx \ \& \ (\forall y)Gxy) \supset \sim Gsl$' is true on this interpretation.

Here are two interpretations for the sentence

$$\sim Dm \vee (\forall x)(\exists y)(Bx \supset Cyx)$$

16. UD: Set of people
    Dx: x is a golf pro
    Bx: x is bald
    Cxy: x has seen y on television
    m: Tiger Woods

17. UD: Set of people
    Dx: x is a politician
    Bx: x is a banana
    Cxy: x votes for y
    m: Madonna

The sentence is false on interpretation 16 since both disjuncts are false on that interpretation. The disjunct '$\sim Dm$' is false since 'Dm' is true—Tiger Woods is a golf pro. '$(\forall x)(\exists y)(Bx \supset Cyx)$', which may be read as 'Every bald person has been seen on television by someone', is false. At least one bald person has not been seen on television by anyone. Interpretation 17 makes the sentence true because both disjuncts are true. Since 'Dm' is false—Madonna is not a politician—'$\sim Dm$' is true. And since no person is a banana, it follows trivially that each person x satisfies the condition that if x is a banana (which x is not) then someone votes for x. On interpretation 17 the sentence '$\sim Dm \vee (\forall x)(\exists y)(Bx \supset Cyx)$' may be read as 'Either Madonna is a politician or everybody who is a banana receives a vote from someone'.

As a final example here is an interpretation for the four sentences

$(\exists x)(Nx \supset (\exists y)Lyx)$
$(\forall x)(Nx \supset (\exists y)Lyx)$
$(\exists x)Nx \supset (\exists y)Lya$
$(\forall x)Nx \supset (\exists y)Lya$

18. UD: Set of positive integers
    Nx: x is odd
    Lxy: x is smaller than y
    a: 1

The first sentence, '$(\exists x)(Nx \supset (\exists y)Lyx)$', is true on this interpretation. It is existentially quantified, and there *is* at least one positive integer x such that if it is odd then some positive integer is smaller than x. Every even positive integer trivially satisfies this condition (because every even positive integer fails to satisfy the antecedent), and every odd positive integer except 1 satis-

fies it. Because the number 1 does not satisfy the condition specified by '$Nx \supset (\exists y)Lyx$', the second sentence is false. The positive integer 1 is odd, but there is no positive integer that is smaller than 1. So it is not true of every positive integer x that if x is odd then there is a positive integer that is smaller than x.

The third sentence, '$(\exists x)Nx \supset (\exists y)Lya$', is false on interpretation 18 because its antecedent, '$(\exists x)Nx$', is true, whereas the consequent, '$(\exists y)Lya$', is false. There is at least one odd positive integer, but there is no positive integer that is smaller than the integer 1. In contrast, '$(\forall x)Nx \supset (\exists y)Lya$' is true because its antecedent is false—some positive integers are not odd.

In summary, the truth-conditions for sentences of *PL* are determined by *interpretations*. Officially an interpretation consists of the specification of a UD and the interpretation of each sentence letter, predicate, and individual constant in the language *PL*. (This parallels the definition of truth-value assignments for *SL*, where a truth-value assignment assigns a truth-value to *every* atomic sentence of *SL*.) But for most purposes we can ignore most of the interpreting that each interpretation does. In *SL* we were able to determine the truth-value of a sentence on a truth-value assignment by considering only the relevant part of that assignment, that is, the truth-values assigned to the atomic components of the sentence in question. Similarly, in order to determine the truth-value of a sentence of *PL* on an interpretation, we need only consider the UD and the interpretation of those sentence letters, predicates, and constants that occur in the sentence in question. In what follows, we shall continue the practice of displaying only the relevant parts of interpretations and of informally referring to such partial interpretations simply as *interpretations*.

8.1E EXERCISES

1. Determine the truth-value of the following sentences on this interpretation:

UD: Set of integers
Ax: x is a positive number
Cx: x is a negative number
Bxy: x is a square root of y
a: 0
b: 39
c: −4

a. $Cc \ \& \ (Ac \vee Bca)$
*b. $Ab \supset Ab$
c. $\sim Bcb \supset (Bba \vee \sim Ac)$
*d. $Cb \equiv (\sim Ab \equiv Ac)$
e. $(Cb \ \& \ Cc) \ \& \sim Baa$
*f. $\sim (\sim Ab \vee Cb) \supset Baa$
g. $Baa \equiv [Bca \supset (Cb \vee \sim Ab)]$
*h. $\sim (Ab \vee Bcc) \ \& \ (Cc \supset \sim Ac)$

2. Determine the truth-value of the following sentences on this interpretation:

UD: Set of countries, cities, and people
Bxyz: x is between y and z
Dxy: x lives in y
Fx: x is a large city
a: West Germany
b: the United States
c: Italy
d: the U.S. president
e: Tokyo
f: Rome

a. Fa ⊃ Dda
*b. Ff ⊃ Ddb
c. (~ Babc ∨ ~ Bbac) ∨ ~ Bcab
*d. (Fa ≡ Fe) ⊃ Dde
e. (~ Fe ∨ Ddf) & (Fe ∨ Fb)
*f. Baaa ⊃ Bfff
g. (Dda ∨ Ddc) ∨ (Dde ∨ Ddf)
*h. (Fa ≡ Dda) & ~ (Ddb ⊃ Bccc)

3. For each of the following sentences, construct an interpretation on which the sentence is true.
a. Nad ⊃ ~ Nda
*b. Da ≡ ~ (Fb ∨ Gc)
c. (Lm & ~ Lm) ∨ Chm
*d. ~ (Wab ⊃ (Wbb & Eb))
e. (Ma ∨ Na) ∨ (Mb ∨ Nb)
*f. ~ Fc & [(Fa ⊃ Na) & (Fb ⊃ Nb)]

4. For each of the following sentences, construct an interpretation on which the sentence is false.
a. (Crs ∨ Csr) ∨ (Css ∨ Crr)
*b. (Ka ≡ ~ Ma) ≡ Gh
c. (Li ∨ Lj) ∨ Lm
*d. Iap ⊃ (Ipa ⊃ Iaa)
e. (~ Ja ≡ Jb) & (~ Jc ≡ ~ Jd)
*f. (Ha ∨ ~ Ha) ⊃ (Fbb ⊃ Fba)

5. For each pair construct an interpretation on which one sentence is true and the other false.
a. Fab ⊃ Fba, Fba ⊃ Fab
*b. (Caa & Cab) ∨ Da, ~ Da ≡ ~ (Caa & Cab)
c. ~ Ma ∨ Cpqr, Capq ∨ ~ Mr
*d. Kac ∨ Kad, Kac & Kad
e. ~ Ljk ≡ (Mjk ∨ Mkj), (Mjk & Mkj) & Ljk
*f. Fab ⊃ (Fbc ⊃ Fac), Fac ⊃ (Fcb ⊃ Fab)

6.a. Explain why there is no interpretation on which 'Ba ∨ ~ Ba' is false.
*b. Can the sentence 'Eab & ~ Eba' be true on an interpretation for which the UD contains exactly one member? Explain.

7. Determine the truth-value of the following sentences on this interpretation:

UD: Set of people
Bx: x is a child
Cx: x is over 40 years old
Dxy: x and y are sisters
Fxy: x and y are brothers

a. $(\forall w)(Cw \supset (\exists x)Dxx)$
*b. $(\exists x)(\exists y)(Fxy \ \& \ Cx)$
c. $(\exists x)(\forall y)(By \vee Fxy)$
*d. $(\forall x)(\forall y)(Dxy = Fxy)$
e. $(\exists x)Cx \supset ((\exists x)(\exists y)Fxy \supset (\exists y)By)$
*f. $\sim (\forall w)(Cw \vee Bw)$
g. $(\forall x)Bx \supset (\forall x)Cx$
*h. $(\forall x)[(\exists y)(Dxy \vee Fxy) \supset Bx]$
i. $(\exists x)[Cx \vee (\exists y)(Dxy \ \& \ Cy)]$
*j. $(\forall w)((Cw \vee Bw) \supset (\exists y)Fwy)$

8. Determine the truth-value of each of the following sentences on this interpretation:

UD: Set of U.S. presidents
Ax: x was the first U.S. president
Bx: x is a female
Ux: x is a U.S. citizen
Dxy: x held office after y's first term of office
g: George Washington

a. $(\forall w)Dwg$
*b. $(\forall x)(\forall y)((Bx \ \& \ Ay) \supset Dyx)$
c. $(\exists x)(Ax \ \& \ (\exists y)Dyx)$
*d. $((\exists x)Ax \ \& \sim (\exists z)Bz) \ \& \ (Ag \supset (\forall y)Uy)$
e. $(\forall y)(Uy \supset (\exists x)(Dyx \vee Dxy))$
*f. $(\forall w)(Bw = \sim Uw)$
g. $(\forall x)(Dxg \supset (\exists y)(\sim Uy \ \& \ Dxy))$
*h. $(\exists x)(Ax \ \& \ Bx) = (\forall y)(Ay \supset Uy)$
i. $\sim (Bg \vee (\exists x)(\forall y)Dxy)$
*j. $(\forall y)((By \ \& \ Ay) \supset Dgy)$

9. Determine the truth-value of each of the following sentences on this interpretation:

UD: Set of positive integers
Bx: x is an even number
Gxy: x is greater than y
Exy: x equals y
Mxyz: x minus y equals z
a: 1
b: 2
c: 3

a. Bb & (∀w)(Gwb ⊃ ~ Ewb)
*b. (∀x)(∀z)(~ Exz ≡ Gxz)
c. (∀x)(∀z)(Gxz ⊃ ~ Exz)
*d. (∀x)(∃w)(Gwx & (∃z)Mzxw)
e. ~ (∀w)(∀y)Gwy ⊃ Mcba
*f. (∀y)(Eya ∨ Gya)
g. (∀z)(Bz ⊃ ~ (∃y)(By & Mzzy))
*h. (∀y)[(Bb & (∃x)Exb) ⊃ Mcby]
i. (∀x)(Exx ≡ ~ (∃y)(∃z)Myzx)
*j. (∃x)((Bx & Gxc) & ~ (∃z)Mxcz)

8.2 QUANTIFICATIONAL TRUTH, FALSEHOOD, AND INDETERMINACY

Using the concept of an interpretation, we may now specify the quantificational counterparts of various truth-functional concepts. Here are the relevant properties that individual sentences of *PL* may have:

> A sentence **P** of *PL* is *quantificationally true* if and only if **P** is true on every interpretation.

> A sentence **P** of *PL* is *quantificationally false* if and only if **P** is false on every interpretation.

> A sentence **P** of *PL* is *quantificationally indeterminate* if and only if **P** is neither quantificationally true nor quantificationally false.

These are the quantificational analogs of truth-functional truth, falsehood, and indeterminacy. The definitions here, however, are stated in terms of interpretations rather than truth-value assignments.

A sentence **P** is quantificationally true if and only if it is true on every interpretation. The sentence '(∃x)(Gx ∨ ~ Gx)' is quantificationally true. We cannot hope to show this by going through each of the interpretations of the sentence since there are infinitely many. (To see this, it suffices to note that there are infinitely many possible universes of discourse for the sentence. We can, for instance, choose as our UD a set containing exactly one positive integer. Because there are an infinite number of positive integers, there are an infinite number of such universes of discourse.)

However, we may reason about the sentence as follows: Because the sentence is existentially quantified, it is true on an interpretation just in case at least one member of the UD satisfies the condition specified by 'Gx ∨ ~ Gx' —that is, just in case at least one member of the UD either is G or is not G. Without knowing what the interpretation of 'G' is, we know that every member of a UD satisfies this condition, for every member is either in or not in the extension of 'G'. And since by definition every interpretation has a nonempty

set as its UD, we know that the UD for any interpretation has at least one member and hence at least one member that satisfies the condition specified by the open sentence 'Gx ∨ – Gx'. Therefore '(∃x)(Gx ∨ – Gx)' is true on every interpretation.

In general, to show that a sentence of *PL* is quantificationally true, we must use reasoning showing that, no matter what the UD is and no matter how the sentence letters, predicates, and individual constants are interpreted, the sentence always turns out to be true. Here is another example. The sentence

$$(\exists x)(\exists y)(Gxy \supset (\forall z)(\forall w)Gzw)$$

is quantificationally true. That is, given any UD and any interpretation of 'G', there are always members x and y of the UD that satisfy the condition specified by '(Gxy ⊃ (∀z)(∀w)Gzw)'. The sentence claims that there is a pair of members of the UD such that if they stand in the relation G then all members of the UD stand in the relation G. We will consider two possibilities for the interpretation of the predicate: Either every pair of members of the UD is in its extension or not every pair is in its extension.

If every pair *is* in the extension of 'G', then every pair x and y (hence at least one pair) satisfies the condition specified by '(Gxy ⊃ (∀z)(∀w)Gzw)' because the consequent is true in this case. Now consider the other possibility—that some (at least one) pair is not in the extension of 'G'. In this case that pair satisfies the condition specified by '(Gxy ⊃ (∀z)(∀w)Gzw)' because that pair *fails* to satisfy the antecedent 'Gxy'. Because either the interpretation of 'G' includes every pair of members of the UD in its extension or it does not (there are no other possibilities), we have just shown that whatever the interpretation of 'G' may be there will always be at least one pair of members of the UD that satisfies '(Gxy ⊃ (∀z)(∀w)Gzw)'. This being so, the sentence '(∃x)(∃y)(Gxy ⊃ (∀z)(∀w)Gzw)' is true on every interpretation. The sentence is therefore quantificationally true.

The sentence

$$(\forall y)By \;\&\; (\exists z) - Bz$$

is quantificationally false. If an interpretation makes the first conjunct true, then every member of the UD will be in the extension of 'B'. But if this is so, then no member of the UD satisfies the condition specified by '– Bz', and so the existentially quantified second conjunct is false. So on any interpretation on which the first conjunct is true, the entire sentence is false. The sentence is also false on any interpretation on which the first conjunct is false, just because that conjunct is false. Since any interpretation either makes the first conjunct true or makes the first conjunct false, it follows that on every interpretation the sentence '(∀y)By & (∃z) – Bz' is false.

The sentence

$$(\forall x)(\exists y)(Fx \supset Gy) \equiv ((\exists x)Fx \;\&\; (\forall y) - Gy)$$

is also quantificationally false. Because the sentence is a biconditional, it is false on any interpretation on which its immediate components have different truth-values, and we can show that this is the case for every interpretation. Consider first an interpretation on which the immediate component, '$(\forall x)(\exists y)(Fx \supset Gy)$', is true. For this to be true, every member x of the UD must satisfy the condition specified by '$(\exists y)(Fx \supset Gy)$'. That is, every member x must be such that if it is in the extension of 'F' then there is some member y of the UD that is in the extension of 'G'. It follows that the second immediate component of the biconditional, '$((\exists x)Fx \,\&\, (\forall y) \sim Gy)$', cannot be true. If it *were* true, then some member of the UD would be in the extension of 'F' (to satisfy the first conjunct), and no member of the UD would be in the extension of 'G'. But the truth of '$(\forall x)(\exists y)(Fx \supset Gy)$', as we have seen, requires that if any object is in the extension of 'F' then at least one object must be in the extension of 'G'. It follows that if '$(\forall x)(\exists y)(Fx \supset Gy)$' is true on an interpretation then '$((\exists x)Fx \,\&\, (\forall y) \sim Gy)$' is false on that interpretation.

Now let us consider an interpretation on which '$(\forall x)(\exists y)(Fx \supset Gy)$' is false. In this case some member x of the UD must fail to satisfy the condition specified by '$(\exists y)(Fx \supset Gy)$'—x must be in the extension of 'F' (to satisfy the antecedent of the conditional), and the extension of 'G' must be empty (so the consequent is not satisfied). But in this case '$((\exists x)Fx \,\&\, (\forall y) \sim Gy)$' must be true because both conjuncts are true. '$(\exists x)Fx$' is true because some member of the UD is in the extension of 'F', and '$(\forall y) \sim Gy$' is true because the extension of 'G' is empty. So any interpretation that makes '$(\forall x)(\exists y)(Fx \supset Gy)$' false makes '$((\exists x)Fx \,\&\, (\forall y) \sim Gy)$' true. Combined with the results of the previous paragraph, this establishes that on any interpretation the immediate components of '$(\forall x)(\exists y)(Fx \supset Gy) \equiv ((\exists x)Fx \,\&\, (\forall y) \sim Gy)$' have different truth-values. So the biconditional must be false on every interpretation and therefore is quantificationally false.

Unfortunately it is not always so easy to show that a sentence is quantificationally true or that it is quantificationally false. However, because a quantificationally true sentence must be true on every interpretation, we can show that a sentence is *not* quantificationally true by showing that it is false on at least one interpretation. Take as an example the sentence

$$(Ga \,\&\, (\exists z)Bz) \supset (\forall x)Bx$$

This sentence is not quantificationally true. To show this, we shall construct an interpretation on which the sentence is false. The sentence is a material conditional, and so our interpretation must make its antecedent true and its consequent false. For the antecedent to be true, 'Ga' must be true and at least one member of the UD must be in the extension of 'B'. For the consequent to be false, at least one member of the UD must fail to be in the extension of 'B'. Using the set of positive integers as our UD, we shall interpret 'G' and 'a' so that 'Ga' comes out true, and we shall interpret 'B' so that at least one member of the UD, but not all, falls into the extension of 'B'. The following

interpretation will do the trick:

19. UD: Set of positive integers
Gx: x is odd
Bx: x is prime
a: 1

The antecedent '(Ga & (∃z)Bz)' is true because the number 1 is odd and at least one positive integer is prime, but '(∀x)Bx' is false because not all positive integers are prime.

As a second example '(∀x)[(Fx ∨ Gx) ∨ (∃y)Hxy]' is not quantificationally true. We shall show this by constructing an interpretation on which the sentence is false. Because the sentence is universally quantified, the UD must have at least one member that fails to satisfy the condition specified by '(Fx ∨ Gx) ∨ (∃y)Hxy'. We choose the set of positive integers as our UD and choose 2 as the member of the UD that does not satisfy the condition. (There is no particular reason for using 2, but choosing a number helps us develop the rest of the interpretation.) We interpret 'F' and 'G' so that the number 2 has neither property (otherwise it would satisfy either 'Fx' or 'Gx'). We must also interpret 'H' so that the number 2 does not stand in the relation H to any positive integer:

20. UD: Set of positive integers
Fx: x is odd
Gx: x is greater than 4
Hxy: x is equal to y squared

Because 2 is neither odd nor greater than 4, and it is not the square of any positive integer, it fails to satisfy the condition specified by '(Fx ∨ Gx) ∨ (∃y)Hxy'. Therefore the universally quantified sentence is false on interpretation 20. Having shown that there is at least one interpretation on which the sentence is false, we may conclude that it is not quantificationally true.

We may show that a sentence is not quantificationally *false* by constructing an interpretation on which it is true. The sentence

~ (~ Ga & (∃y)Gy)

is not quantificationally false. To construct an interpretation on which it is true, we must make '~ Ga & (∃y)Gy' false. To do so, we must make one or both conjuncts false. We choose the first and interpret 'G' and 'a' so that '~ Ga' is false:

21. UD: Set of positive integers
Gx: x is even
a: 2

Because the number 2 is even, 'Ga' is true. Hence '~ Ga' is false and so is '~ Ga & (∃y)Gy'. (The fact that the second conjunct turns out to be true on our interpretation is irrelevant—the conjunction as a whole is still false.) Therefore

the negation of the conjunction is true on interpretation 21, and we may conclude that the sentence is not quantificationally false.

Note that we cannot show that a sentence *is* quantificationally true or that it is quantificationally false by constructing a single interpretation. To show that a sentence is quantificationally true, we must demonstrate that it is true on every interpretation, and to show that a sentence is quantificationally false, we must show that it is false on every interpretation.

A quantificationally indeterminate sentence is one that is neither quantificationally true nor quantificationally false. We may show that a sentence is quantificationally indeterminate by constructing two interpretations: one on which it is true (to show that the sentence is not quantificationally false) and one on which it is false (to show that the sentence is not quantificationally true). The sentence

$$\sim (\sim Ga \ \& \ (\exists y)Gy)$$

is quantificationally indeterminate. We have already constructed an interpretation (interpretation 21) on which it is true; all that is left is to construct an interpretation on which it is false. For the sentence to be false, '$\sim Ga \ \& \ (\exists y)Gy$' must be true. To make '$\sim Ga$' true, our UD must contain at least one member that is not in the extension of 'G', and 'a' will designate this member. But the UD must also contain another member that is in the extension of 'G', to make '$(\exists y)Gy$' true:

22. UD: Set of positive integers
    Gx: x is odd
    a: 2

The number 2 is not odd, but at least one positive integer is, and so '$\sim Ga \ \& \ (\exists y)Gy$' is true and '$\sim (\sim Ga \ \& \ (\exists y)Gy)$' is false on interpretation 22. The sentence is therefore not quantificationally true. Having shown that the sentence is neither quantificationally true nor quantificationally false, we may conclude that it is quantificationally indeterminate.

Sometimes it takes ingenuity to find either an interpretation on which a sentence is true or an interpretation on which a sentence is false. Examine the sentence itself for guidelines, as we have just done. If it is a truth-functional compound, then use your knowledge of the truth-conditions for that type of compound. If the sentence is universally quantified, then the sentence will be true if and only if the condition specified after the quantifier is satisfied by all members of the UD you choose. If the sentence is existentially quantified, then it will be true if and only if the condition specified after the quantifier is satisfied by at least one member of the UD. As you examine the components of the sentence, you may reason in the same way—are they truth-functional compounds or quantified? Sometimes the desired interpretation cannot be obtained. For example, a quantificationally true sentence is not false on any interpretation; therefore any attempt to construct an interpretation that makes the sentence false fails.

Two theoretical points are of interest here. The first is that, if a sentence of predicate logic without identity is true on at least one interpretation, then it is true on some interpretation that has the set of positive integers as its UD. This result is known as the *Löwenheim Theorem* (it will be proved in the exercises in Chapter 11). It follows from this result that, if a sentence of *PL* is true on some interpretation with a finite UD, then it is true on some interpretation that has the set of positive integers as its UD. And if a sentence of *PL* is true on some interpretation for which the UD is *larger* than the set of positive integers (for example, the set of real numbers), then it is true on at least one interpretation that has the set of positive integers as its UD.

Note that this result means that the set of positive integers is always a good choice for your UD as you construct interpretations. In fact, there are sentences of *PL* that are not quantificationally true but that are nevertheless true on every interpretation with a finite UD, and there are sentences of *PL* that are not quantificationally false but are false on every interpretation with a finite UD. For instance, the following sentence is not quantificationally false:

$$(\forall x)(\forall y)(\forall z)[(Bxy \,\&\, Byz) \supset Bxz] \,\&\, [(\forall x)(\exists y)Bxy \,\&\, (\forall z) \sim Bzz]$$

But it is false on every interpretation with a finite UD. To show that it is not quantificationally false, then, you must choose a UD that has infinitely many members—and the set of positive integers is a good choice.

In fact, in this section all our interpretations have used the set of positive integers as the UD. Although this was not necessary—we could have constructed interpretations using the set of all people, the set of all countries in the world, the set consisting of the three authors of this book, or whatever—we now see why it is a good choice. We shall therefore continue to use this particular UD for our examples in the remainder of this chapter. In addition, we shall make repeated use of very simple interpretations of predicates for this UD—for example, the properties of being even and of being prime, the relation of being greater than, and so on. Again this is not necessary—other properties and relations could be used—but it is convenient to reuse the same interpretations for predicates.

The second point is that there is no decision procedure for deciding, for each sentence of *PL*, whether that sentence is quantificationally true, quantificationally false, or quantificationally indeterminate. (We shall not prove the result here.) This is a very important way in which the semantics for *PL* differs from the semantics for *SL*. For *SL* the construction of truth-tables gives a decision procedure for whether a sentence is truth-functionally true, false, or indeterminate. That is, in a finite number of mechanical steps, we can always correctly answer the questions 'Is this sentence truth-functionally true?', 'Is this sentence truth-functionally false?', and 'Is this sentence truth-functionally indeterminate?' The previous result mentioned in this paragraph, due to Church, is that there is no analogous method for predicate logic—we have no such general method now, and no such general method will ever be found. This result does not mean that we cannot ever show that some sentences of *PL* are

quantificationally true, false, or indeterminate; rather, it shows that there is no decision procedure (mechanical, certain, and requiring only a finite number of steps) for determining the quantificational status of *every* sentence of *PL*. However, it is interesting to note that there *is* such a procedure for determining the quantificational status of sentences of *PL* that contain no many-place predicates, that is, in which the predicates are all one-place predicates. This follows from a result by the logicians Bernays and Schönfinkel.[1]

8.2.E EXERCISES

1. Show that each of the following sentences is not quantificationally true by constructing an interpretation on which it is false.

a. $(\forall x)(Fx \supset Gx) \supset (\forall x)Gx$
*b. $(\exists x)(Fx \vee Gx) \supset ((\exists x)Fx \supset (\exists x) \sim Gx)$
c. $(\forall x)(\exists y)Bxy \supset (\exists y)(\forall x)Bxy$
*d. $(\forall x)(Fxb \vee Gx) \supset [(\forall x)Fxb \vee (\forall x)Gx]$
e. $[(\forall x)Fx \supset (\forall w)Gw] \supset (\forall z)(Fz \supset Gz)$
*f. $(\forall x)(Ax \supset (\forall y)By) \supset (\forall y)(By \supset (\forall x)Ax)$
g. $\sim (\exists x)Gx \supset (\forall y)(Fyy \supset Gy)$
*h. $(\forall x)(Bx \equiv Hx) \supset (\exists x)(Bx \,\&\, Hx)$

2. Show that each of the following sentences is not quantificationally false by constructing an interpretation on which it is true.

a. $\sim (\forall w)(\forall y)Bwy \equiv (\forall z)Bzz$
*b. $(\exists x)Fx \,\&\, (\exists x) \sim Fx$
c. $((\exists x)Fx \,\&\, (\exists x)Gx) \,\&\, \sim (\exists x)(Fx \,\&\, Gx)$
*d. $(\exists x)((\exists y)Fy \supset \sim Fx)$
e. $(\forall x)(Fx \supset Gx) \,\&\, (\forall x)(Gx \supset \sim Fx)$
*f. $(\exists x)(\forall y)(Dyx \supset \sim Dxy)$
g. $(\forall x)(Bx \equiv Hx) \supset (\exists x)(Bx \,\&\, Hx)$
*h. $(\exists x)(\forall y)Dxy \vee \sim (\forall y)(\exists x)Dxy$
i. $(\forall x)(\forall y)(\forall z)\,[(Bxy \,\&\, Byz) \supset Bxz] \,\&\, [(\forall x)(\exists y)Bxy \,\&\, (\forall z) \sim Bzz]$

3. Show that each of the following sentences is quantificationally indeterminate by constructing an interpretation on which it is true and an interpretation on which it is false.

a. $(\exists x)(Fx \,\&\, Gx) \supset (\exists x) \sim (Fx \vee Gx)$
*b. $(\exists x)Fx \supset (\forall w)(Cw \supset Fw)$
c. $(\forall x)Bnx \supset (\forall x) \sim Bnx$
*d. $(\exists x)(Fx \supset Gx) \supset (\exists x)(Fx \,\&\, Gx)$
e. $(\forall x)(\forall w)[(Nwx \vee Nxw) \supset Nww]$
*f. $(Ma \,\&\, Mb) \,\&\, (\exists x) \sim Mx$
g. $(\forall x)(Cx \vee Dx) \equiv (\exists y)(Cy \,\&\, Dy)$
*h. $[\sim (\exists x)Hx \vee \sim (\exists x)Gx] \vee (\forall x)(Hx \,\&\, Gx)$

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[1] "Zum Entscheidungsproblem der mathematischen Logik," *Mathematische Annalen*, 99 (1928), 342–372. The result mentioned in the previous paragraph is from Alonzo Church, "A Note on the Entscheidungsproblem," *Journal of Symbolic Logic*, 1 (1936), 40–41, 101–102.

4. Each of the following sentences is quantificationally true. Explain why.
a. $(\exists x)(\forall y)Bxy \supset (\forall y)(\exists x)Bxy$
*b. $[(\forall x)Fx \vee (\forall x)Gx] \supset (\forall x)(Fx \vee Gx)$
c. $Fa \vee [(\forall x)Fx \supset Ga]$
*d. $(\forall x)(\exists y)Mxy \supset (\exists x)(\exists y)Mxy$
e. $(\exists x)Hx \vee (\forall x)(Hx \supset Jx)$

5. Each of the following sentences is quantificationally false. Explain why.
a. $(\exists w)(Bw = \sim Bw)$
*b. $(\forall w)(Fw \supset Gw) \;\&\; [(\forall w)(Fw \supset \sim Gw) \;\&\; (\exists w)Fw]$
c. $[(\forall x)Fx \supset (\exists y)Gy] \;\&\; [\sim (\exists x)Gx \;\&\; \sim (\exists x) \sim Fx]$
*d. $(\exists x)(Fx \;\&\; \sim Gx) \;\&\; (\forall x)(Fx \supset Gx)$
e. $((\forall w)(Aw \supset Bw) \;\&\; (\forall w)(Bw \supset (Cw)) \;\&\; (\exists y)(Ay \;\&\; \sim Cy)$

6. For each of the following sentences, decide whether it is quantificationally true, quantificationally false, or quantificationally indeterminate. If the sentence is quantificationally true or quantificationally false, explain why. If it is quantificationally indeterminate, construct interpretations that establish this.
a. $((\exists x)Gx \;\&\; (\exists y)Hy) \;\&\; (\exists z) \sim (Gz \;\&\; Hz)$
*b. $((\exists x)Gx \;\&\; (\exists y)Hy) \;\&\; \sim (\exists z)(Gz \;\&\; Hz)$
c. $(\forall x)(Fx \supset Gx) \supset (\forall x)(\sim Gx \supset \sim Fx)$
*d. $(\forall x)Fx \supset \sim (\exists x) \sim (Fx \vee Gx)$
e. $(\forall x)(Dx \supset (\exists z)Hxz) \supset (\exists z)(\forall x)(Dx \supset Hxz)$
*f. $(\exists z)(\forall x)(Dx \supset Hxz) \supset (\forall x)(Dx \supset (\exists z)Hxz)$

8.3 QUANTIFICATIONAL EQUIVALENCE AND CONSISTENCY

The next concept to be introduced is that of quantificational equivalence.

> Sentences **P** and **Q** of *PL* are *quantificationally equivalent* if and only if there is no interpretation on which **P** and **Q** have different truth-values.

The sentences

$$(\exists x)Fx \supset Ga$$

and

$$(\forall x)(Fx \supset Ga)$$

are quantificationally equivalent. We may reason as follows: First suppose that '$(\exists x)Fx \supset Ga$' is true on some interpretation. Then '$(\exists x)Fx$' is either true or false on this interpretation. If it is true, then so is 'Ga' (by our assumption that '$(\exists x)Fx \supset Ga$' is true). But then, since 'Ga' is true, every object x in the UD is such that if x is F then a is G. So '$(\forall x)(Fx \supset Ga)$' is true. If '$(\exists x)Fx$' is false, however, then

every object x in the UD is such that if x is F (which, on our assumption, it is not) then a is G. Again '$(\forall x)(Fx \supset Ga)$' is true. Hence, if '$(\exists x)Fx \supset Ga$' is true on an interpretation, '$(\forall x)(Fx \supset Ga)$' is also true on that interpretation.

Now suppose that '$(\exists x)Fx \supset Ga$' is false on some interpretation. Since the sentence is a conditional, it follows that '$(\exists x)Fx$' is true and 'Ga' is false. But if '$(\exists x)Fx$' is true, then some object x in the UD is in the extension of 'F'. This object then does not satisfy the condition that if it is F (which it is) then a is G (which is false on our present assumption). So '$(\forall x)(Fx \supset Ga)$' is false if '$(\exists x)Fx \supset Ga$' is. Taken together with our previous result, this demonstrates that the two sentences are quantificationally equivalent.

The sentences

$$- (\exists x)(\forall y)(Gxy \vee Gyx)$$

and

$$(\forall x)(\exists y)(- Gxy \,\&\, - Gyx)$$

are also quantificationally equivalent. As in the previous example, we will show that if the first sentence is true on an interpretation then so is the second sentence, and that if the first sentence is false on an interpretation then so is the second sentence. First consider an interpretation on which '$- (\exists x)(\forall y)(Gxy \vee Gyx)$' is true. '$(\exists x)(\forall y)(Gxy \vee Gyx)$' must be false on this interpretation, so no member x of the UD satisfies the condition specified by '$(\forall y)(Gxy \vee Gyx)$'. That is, no member x of the UD is such that for every object y either the pair x and y or the pair y and x is in the extension of 'G'. Put another way, for each member x of the UD, there is at least one object y such that both '$- Gxy$' and '$- Gyx$' hold. And that is exactly what the second sentence says, so it is true as well.

Now consider an interpretation on which the first sentence is false; '$(\exists x)(\forall y)(Gxy \vee Gyx)$' is true on such an interpretation. So there is at least one member x of the UD such that for every object y, either 'Gxy' or 'Gyx' holds. Such a member x therefore does *not* satisfy the condition specified by '$(\exists y)(- Gxy \,\&\, - Gyx)$' (there is *no* y such that neither 'Gxy' nor 'Gyx' holds). And so the universally quantified sentence '$(\forall x)(\exists y)(- Gxy \,\&\, - Gyx)$' is also false. From this and the result of the preceding paragraph, we conclude that the two sentences are quantificationally equivalent.

If we want to establish that two sentences are *not* quantificationally equivalent, we can construct an interpretation to show this. The interpretation must make one of the two sentences true and the other sentence false. For example, the sentences

$$(\forall x)(Fx \supset Ga)$$

and

$$(\forall x)Fx \supset Ga$$

are not quantificationally equivalent. We shall construct an interpretation on which the first sentence is false and the second sentence is true. To make the first sentence false, 'Ga' has to be false, and there must be at least one object in the extension of 'F'—for then this object will fail to satisfy '(Fx ⊃ Ga)'. But we can still make '(∀x)Fx ⊃ Ga' true on our interpretation if the extension of 'F' does not include the entire UD—because then the antecedent '(∀x)Fx' will be false. Here is our interpretation:

23. UD: Set of positive integers
Fx: x is prime
Gx: x is even
a: 1

The number 3 (as one example) does not satisfy the condition that if it is prime (which it is) then the number 1 is even (which is false). So '(∀x)(Fx ⊃ Ga)' is false on the interpretation. But '(∀x)Fx ⊃ Ga' is true because its antecedent, '(∀x)Fx', is false—not every positive integer is prime. Once again we see that the scope of quantifiers is very important in determining the truth-conditions of sentences of *PL*.

The sentences

(∀x)(∃y)(Hy ⊃ Lx)

and

(∀x)[(∃y)Hy ⊃ Lx]

are also not quantificationally equivalent. We shall *show* this by constructing an interpretation on which the first sentence is true and the second sentence is false. To make '(∀x)[(∃y)Hy ⊃ Lx]' false, some member of the UD must fail to satisfy '(∃y)Hy ⊃ Lx'. Therefore the UD must contain at least one object in the extension of 'H' (so that '(∃y)Hy' is satisfied) and at least one object x that is not in the extension of 'L' (so that this object does not satisfy 'Lx'). To make '(∀x)(∃y)(Hy ⊃ Lx)' true, every member of the UD must satisfy '(∃y)(Hy ⊃ Lx)'—for every member x of the UD, there must be an object y such that if y is H then x is L. We have already decided that at least one object x will not be in the extension of 'L'. So, if x (along with all other members of the UD) is to satisfy '(∃y)(Hy ⊃ Lx)', then at least one member of y of the UD must not be in the extension of 'H'—for then y will be such that if it is H (it is not) then x is L.

To sum up, we need at least one object that is in the extension of 'L' and at least one object that is not in the extension of 'H'. Here is our interpretation:

24. UD: Set of positive integers
Hx: x is odd
Lx: x is prime

The sentence '$(\forall x)[(\exists y)Hy \supset Lx]$' is false—every positive integer x that is not prime fails to satisfy the condition that if some positive integer is odd (which at least one positive integer is) then x is prime. The sentence '$(\forall x)(\exists y)(Hy \supset Lx)$' is true because at least one positive integer is not odd. For any positive integer x there is at least one positive integer y that is not odd, and hence at least one positive integer y such that if y is odd (which y is not) then x is prime.

While we may construct single interpretations to show that two sentences are not quantificationally equivalent, we may not use the same method to show that sentences *are* quantificationally equivalent. In the latter case we must reason about every interpretation as we did in the examples at the beginning of this section.

Quantificational consistency is our next concept.

> A set of sentences of *PL* is *quantificationally consistent* if and only if there is at least one interpretation on which all the members of the set are true. A set of sentences of *PL* is *quantificationally inconsistent* if and only if the set is not quantificationally consistent.

The set of sentences

$$\{(\forall x)Gax, \sim Gba \vee (\exists x) \sim Gax\}$$

is quantificationally consistent. The following interpretation shows this:

25. UD: Set of positive integers
Gxy: x is less than or equal to y
a: 1
b: 2

On this interpretation '$(\forall x)Gax$' is true since 1 is less than or equal to every positive integer. '$\sim Gba$' is true since 2 is neither less than nor equal to 1; so '$\sim Gba \vee (\exists x) \sim Gax$' is true. Since both members of the set are true on this interpretation, the set is quantificationally consistent.

The set

$$\{(\forall w)(Fw \supset Gw), (\forall w)(Fw \supset \sim Gw)\}$$

is also quantificationally consistent. This may seem surprising since the first sentence says that everything that is F is G and the second sentence says that everything that is F is not G. But the set is consistent because, if no object in the UD of an interpretation is in the extension of 'F', then every object w in the UD will be such that if w is F (which w is not) then w is both G and not G. The following interpretation illustrates this.

26. UD: Set of positive integers
Fx: x is negative
Gx: x is even

No positive integer is negative, so each positive integer w is such that if w is negative (which w is not) then w is even, and each positive integer w is such that if w is negative (which w is not) then w is not even. Both '$(\forall w)(Fw \supset Gw)$' and '$(\forall w)(Fw \supset \sim Gw)$' are true on this interpretation.

Note that, while a single interpretation may be produced to show that a set of sentences is quantificationally consistent, a single interpretation *cannot* be used to show that a set of sentences is quantificationally inconsistent. To show that a set is quantificationally inconsistent, we must show that on every interpretation at least one sentence in the set is false. In some cases simple reasoning shows that a set of sentences is quantificationally inconsistent. The set

$$\{(\exists y)(Fy \ \& \sim Ny),\ (\forall y)(Fy \supset Ny)\}$$

is quantificationally inconsistent. For if '$(\exists y)(Fy \ \& \sim Ny)$' is true on some interpretation then some member y of the UD is F and is not N. But then that member is *not* such that if it is F (which it is) then it is N (which it is not). Hence the universally quantified sentence '$(\forall y)(Fy \supset Ny)$' is false on such an interpretation. So there is no interpretation on which both set members are true; the set is quantificationally inconsistent.

8.3E EXERCISES

1. Show that the sentences in each of the following pairs are not quantificationally equivalent by constructing an interpretation on which one of the sentences is true and the other is false.

a. $(\exists x)Fx \supset Ga$, $(\exists x)(Fx \supset Ga)$
*b. $(\exists x)Fx \ \& \ (\exists x)Gx$, $(\exists x)(Fx \ \& \ Gx)$
c. $(\forall x)Fx \vee (\forall x)Gx$, $(\forall x)(Fx \vee Gx)$
*d. $(\exists x)(Fx \vee Ga)$, $(\exists x)(Fx \vee Gb)$
e. $(\forall x)(Fx = Gx)$, $(\exists x)Fx = (\exists x)Gx$
*f. $(\forall x)(Fx \supset Gx)$, $(\forall y)((\forall x)Fx \supset Gy)$
g. $(\exists x)(Bx \ \& \ (\forall y)Dyx)$, $(\forall x)(Bx \supset (\forall y)Dyx)$
*h. $(\exists y)(My = Ny)$, $(\exists y)My = (\exists y)Ny$
i. $(\forall x)(\exists y)(Fx \supset Kyx)$, $(\exists x)(\exists y)(Fx \supset Kyx)$

2. In each of the following pairs the sentences are quantificationally equivalent. Explain why.

a. $(\forall x)Fx \supset Ga$, $(\exists x)(Fx \supset Ga)$
*b. $(\forall x)(Fx \supset Gx)$, $\sim (\exists x)(Fx \ \& \sim Gx)$
c. $(\exists x)(Fx \vee Gx)$, $\sim (\forall y)(\sim Fy \ \& \sim Gy)$
*d. $(\forall x)(\forall y)(Mxy \ \& \ Myx)$, $\sim (\exists x)(\exists y)(\sim Mxy \vee \sim Myx)$
e. $(\forall x)(\forall y)Gxy$, $(\forall y)(\forall x)Gxy$
*f. $(\forall x)(\forall y)(Fxy \supset Hyx)$, $\sim (\exists x)(\exists y)(Fxy \ \& \sim Hyx)$

3. Decide, for each of the following pairs of sentences, whether the sentences are quantificationally equivalent. If they are quantificationally equivalent, explain why. If they are not quantificationally equivalent, construct an interpretation that shows this.

a. $(\exists x)(Fx \vee Gx)$, $(\forall x) \sim (Fx \,\&\, Gx)$
*b. $(\exists x)(Fx \,\&\, Gx)$, $\sim (\forall x) \sim (Fx \vee Gx)$
c. $(\forall w)(\forall y)(Gyw \vee Gwy)$, $(\forall w)(\forall y)(Gww \vee Gwy)$
*d. $(\forall y)((\exists z)Hzy \supset Hyy)$, $(\forall y)((\exists z)(Hzz \supset Hzy)$

4. Show that each of the following sets of sentences is quantificationally consistent by constructing an interpretation on which every member of the set is true.

a. $\{(\exists x)Bx, (\exists x)Cx, \sim (\forall x)(Bx \vee Cx)\}$
*b. $\{(\exists x)Fx \vee (\exists x)Gx, (\exists x) \sim Fx, (\exists x) \sim Gx\}$
c. $\{(\forall x)(Fx \supset Gx), (\forall x)(Nx \supset Mx), (\forall x)(Gx \supset \sim Mx)\}$
*d. $\{(\forall x)(Dax \equiv Bax), \sim Dab, \sim Bba\}$
e. $\{(\forall w)(Nw \supset (\exists z)(Mz \,\&\, Cwz), (\forall z)(\forall w)(Mz \supset \sim Cwz)\}$
*f. $\{(\exists w)Fw, (\forall w)(Fw \supset (\exists x)Bxw), (\forall x) \sim Bxx\}$
g. $\{\sim (\forall y)(Ny \supset My), \sim (\forall y) \sim (Ny \supset My)\}$
*h. $\{(\forall x)(Bx \equiv (\forall y)Cxy), (\exists x) \sim Bx, (\exists x)(\exists y)Cxy\}$
i. $\{(\exists y)Fay, (\exists y) \sim Gay, (\forall y)(Fay \vee Gay)\}$

5. Each of the following sets of sentences is quantificationally inconsistent. Explain why.

a. $\{(\exists x)(Bx \,\&\, Cx), (\forall x) \sim (Bx \vee Cx)\}$
*b. $\{(\exists x)(\exists y)(Bxy \vee Byx), \sim (\exists x)(\exists y)Bxy\}$
c. $\{(\forall x)(\forall y)(Byx \vee Bxy), (\exists y) \sim Byy\}$
*d. $\{Ba, (\exists y)Day, (\forall x)(Bx \supset (\forall y) \sim Dxy)\}$
e. $\{(\exists x)(\forall y)Gxy, (\forall x)(\forall y) \sim Gxy\}$
*f. $\{(\forall x)Fx \vee (\forall x) \sim Fx, (\exists x)Fx \equiv (\exists x) \sim Fx\}$

6. Decide, for each of the following sets of sentences, whether the set is quantificationally consistent. If the set is quantificationally consistent, construct an interpretation that shows this. If it is quantificationally inconsistent, explain why.

a. $\{(\exists x)Fx \supset (\forall x)Fx, (\exists x) \sim Fx, (\exists x)\, Fx\}$
*b. $\{(\exists x)(\exists y)Gxy, (\forall y) \sim Gyy\}$
c. $\{(\forall x) \sim (\forall y)Gxy, (\forall x)Gxx\}$
*d. $\{(\exists x)Fx, (\forall y)(Fy \supset Hya), \sim (\forall x) \sim Hxa\}$

7. Explain why sentences **P** and **Q** of *PL* are quantificationally equivalent if and only if $\mathbf{P} \equiv \mathbf{Q}$ is quantificationally true.

8.4 QUANTIFICATIONAL ENTAILMENT AND VALIDITY

Our last two semantic concepts for the language *PL* are the concepts of quantificational entailment and quantificational validity.

> A set Γ of sentences of *PL quantificationally entails* a sentence **P** of *PL* if and only if there is no interpretation on which every member of Γ is true and **P** is false.

An argument of *PL* is *quantificationally valid* if and only if there is no interpretation on which every premise is true and the conclusion is false. An argument of *PL* is *quantificationally invalid* if and only if the argument is not quantificationally valid.

The set

$$\{(\forall x)(Bx \supset Ga),\ (\exists x)Bx\}$$

quantificationally entails the sentence 'Ga'. As in *SL* we may use the double turnstile and write this as

$$\{(\forall x)(Bx \supset Ga),\ (\exists x)Bx\} \models Ga$$

Suppose that '$(\forall x)(Bx \supset Ga)$' and '$(\exists x)Bx$' are both true on some interpretation. Since '$(\forall x)(Bx \supset Ga)$' is true, we know that every object x in the UD is such that if x is B then a is G. Since '$(\exists x)Bx$' is true, we know that at least one object x in the UD of the interpretation is in the extension of 'B'. Since it is true that, if that object is B (which it is) then a is G, 'Ga' must therefore be true. So, on any interpretation on which '$(\forall x)(Bx \supset Ga)$' and '$(\exists x)Bx$' are both true, 'Ga' is also true. So the entailment does hold.

The set

$$\{(\forall y)(\sim Jy \vee (\exists z)Kz),\ (\exists y)Jy\}$$

quantificationally entails the sentence

$$(\exists z)Kz$$

We shall show that any interpretation that makes the two sentences in the set true also makes '$(\exists z)Kz$' true. If an interpretation makes the first sentence in the set true, then every member y of the UD satisfies the condition specified by '$(\sim Jy \vee (\exists z)Kz)$'. Every member is such that either it is not J or some member of the UD is K. If the second sentence is also true on the interpretation, then some member of the UD is J. Because this member must satisfy the disjunction '$\sim Jy \vee (\exists z)Kz$' and, being J, it does not satisfy the disjunct '$\sim Jy$', the second disjunct must be true. And the second disjunct is '$(\exists z)Kz$', so it is true whenever the two set members are true.

The argument

$$\begin{array}{l}(\exists x)(Fx \vee Gx)\\(\forall x) \sim Fx\\\hline(\exists x)Gx\end{array}$$

is quantificationally valid. Suppose that on some interpretation both premises are true. If the first premise is true, then some member x of the UD is either F or G. If the premise '(∀x) – Fx' is true, then no member of the UD is F. Therefore, because the member that is either F or G is not F, it must be G. Thus '(∃x)Gx' will also be true on such an interpretation.

We can show that a set of sentences does *not* quantificationally entail a sentence by constructing an interpretation. For example, the set

$$\{\sim(\forall x)(Gx \equiv Fx), -Fb\}$$

does not quantificationally entail the sentence

$$(\forall x) - Gx$$

We will construct an interpretation on which the members of the set are true and '(∀x) – Gx' is false. For the sentence '~ (∀x)(Gx ≡ Fx)' to be true, the UD must contain at least one member that fails to satisfy 'Gx ≡ Fx'—the member must be in the extension of one of the two predicates but not in the extension of the other. For '– Fb' to be true, 'b' must designate an object that is not in the extension of 'F'. And '(∀x) – Gx', which claims that everything is not G, will be false if at least one object in the UD is in the extension of 'G'. Here is an interpretation that satisfies these conditions:

27. UD: Set of positive integers
Fx: x is greater than 5
Gx: x is prime
b: 3

Not all positive integers are prime if and only if they are greater than 5—take 2 as an example—and 3 is not greater than 5. Therefore the set members are both true on this interpretation. On the other hand, '(∀x) – Gx' is false, because some positive integers are prime.

To show that an argument is quantificationally invalid, we can construct an interpretation on which its premises are true and its conclusion is false. The argument

$$\begin{array}{l}(\exists x)[(\exists y)Fy \supset Fx] \\ (\exists y) - Fy \\ \hline - (\exists x)Fx\end{array}$$

is quantificationally invalid. We can make the first premise true by interpreting 'F' so that at least one member of the UD is in its extension—for then that object will satisfy the condition specified by '(∃y)Fy ⊃ Fx' because it will satisfy its consequent. The second premise will be true if at least one member of the

UD is not in the extension of 'F'. So 'F' will have some, but not all, of the members of the UD in its extension. Because some members will be in the extension, the conclusion will turn out to be false. Here is an interpretation:

28. UD: Set of positive integers
    Fx: x is prime

Some positive integer x is such that if there exists a prime positive integer then x is prime—for example, the integer 5 satisfies this condition—and some positive integer is not prime, but it is false that no positive integer is prime.

Note that we cannot prove that a quantificational entailment *does* hold or that an argument *is* quantificationally valid by constructing a single interpretation. Proving either of these involves proving something about the truth-value of sentences on every interpretation, not just a select few.

And, once again, there are limitations on deciding questions of quantificational equivalence, consistency, entailment, and validity. Owing to Church's result, mentioned at the end of Section 8.2, we know that there is no procedure for deciding these questions for every group of sentences of *PL*.[7] However, our method of producing interpretations to establish quantificational consistency, nonequivalence, nonentailment, and invalidity, although not a decision procedure, often produces the desired result. We have, for instance, just used this method to show that an argument is quantificationally invalid. Ingenuity in choosing an appropriate interpretation for sentences containing quantifiers is once again generally required.

8.4E EXERCISES

1. Establish each of the following by constructing an appropriate interpretation.

a. $\{(\forall x)(Fx \supset Gx),\ (\forall x)(Hx \supset Gx)\} \nvdash (\exists x)(Hx \ \&\ Fx)$

*b. $\{(\forall y)(Fy \equiv Fa),\ Fa\} \nvdash \sim Fb$

c. $\{(\exists x)Fx\} \nvdash Fa$

*d. $\{(\forall x)(Bx \supset Cx),\ (\exists x)Bx\} \nvdash (\forall x)Cx$

e. $\{(\exists x)(Bx \supset Cx),\ (\exists x)Cx\} \nvdash (\exists x)Bx$

*f. $\{(\forall x)(Fx \supset Gx),\ (\forall x)(Hx \supset \sim Fx)\} \nvdash (\forall x)(Hx \supset Gx)$

g. $\{(\forall x)(\exists y) \sim Lxy\} \nvdash (\forall x) \sim Lxx$

*h. $\{(\exists x)(\forall y)(Hxy \vee Jxy),\ (\exists x)(\forall y) \sim Hxy\} \nvdash (\exists x)(\forall y)Jxy$

2. Show that each of the following arguments is quantificationally invalid by constructing an appropriate interpretation.

[7] Moreover some arguments can be proved quantificationally invalid and some sets quantificationally consistent only by means of interpretations with universes of discourse containing an infinite number of members. However, there is a result for sets of sentences analogous to the Löwenheim Theorem (mentioned in Section 8.2), which says that if a set of sentences is quantificationally consistent—or an argument quantificationally invalid—then this can be shown by means of interpretations with the set of positive integers as the UD. It is not necessary in any case to check interpretations with larger universes of discourse. This result is known as the *Löwenheim-Skolem Theorem* and is assigned as an exercise in Chapter 11.

a. $(\forall x)(Fx \supset Gx) \supset (\exists x)Nx$
$(\forall x)(Nx \supset Gx)$
$\therefore (\forall x)(\sim Fx \vee Gx)$

*b. $(\sim (\exists y)Fy \supset (\exists y)Fy) \vee \sim Fa$
$\therefore (\exists z)Fz$

c. $(\exists x)(Fx \ \& \ Gx)$
$(\exists x)(Fx \ \& \ Hx)$
$\therefore (\exists x)(Gx \ \& \ Hx)$

*d. $(\forall x)(Fx \supset Gx)$
Ga
$\therefore Fa$

e. $(\forall x)(Fx \supset Gx)$
$\sim (\exists x)Fx$
$\therefore \sim (\exists x)Gx$

*f. $(\forall x)(\forall y)(Mxy \supset Nxy)$
$\therefore (\forall x)(\forall y)(Mxy \supset (Nxy \ \& \ Nyx))$

g. $(\exists x)Gx$
$(\forall x)(Gx \supset Dxx)$
$\therefore (\exists x)(\forall y)(Gx \ \& \ Dxy)$

*h. $Fa \vee (\exists y)Gya$
$Fb \vee (\exists y) \sim Gyb$
$\therefore (\exists y)Gya$

i. $(\forall x)(Fx \supset Gx)$
$(\forall x)(Hx \supset Gx)$
$\therefore (\forall x)(Fx \vee Hx)$

3. Using the given symbolization keys, symbolize the following arguments in *PL*. Then show that the first symbolized argument in each pair is quantificationally valid while the second is not.

a. UD: Set consisting of all things
Bx: x is beautiful
Px: x is a person

Everything is beautiful. Therefore something is beautiful.

Everyone is beautiful. Therefore someone is beautiful.

*b. UD: Set of people
Rx: x roller skates
Dx: x can dance

Not everyone can dance. Therefore someone can't dance.

No one who roller skates can dance. Therefore some roller skater can't dance.

c. UD: Set of people
Lxy: x loves y

There is a person who loves everyone. Therefore everyone is loved by someone.

Everyone is loved by someone. Therefore there is a person who loves everyone.

*d. UD: Set of numbers
Ex: x is even
Dx: x is divisible by 2

Some numbers are even and some numbers are divisible by 2. Therefore some numbers are even if and only if some numbers are divisible by 2.

A number is even if and only if it is divisible by 2. Therefore some number is even.

e. UD: Set of people
Tx: x is a student
Sx: x is smart
Hx: x is happy

Some students are smart and some students are not happy. Therefore there is a student who is smart or not happy.

All students are smart, and no student is happy. Therefore there is a student who is smart or not happy.

*f. UD: Set of people
Sx: x is a senator
Rx: x is a Republican
Dx: x is a Democrat

Any senator who is not a Republican is a Democrat. There is a senator who is not a Republican. Therefore some senator is a Democrat.

There is a senator who is not a Republican. Therefore some senator is a Democrat.

g. UD: Set of people
Ax: x likes asparagus
Sx: x likes spinach
Cx: x is crazy

Anyone who likes asparagus is crazy, and anyone who is crazy likes spinach. Therefore anyone who likes asparagus also likes spinach.

Anyone who likes spinach is crazy, and anyone who is crazy likes asparagus. Therefore anyone who likes asparagus also likes spinach.

4. Decide, for each of the following arguments, whether it is quantificationally valid. If the argument is quantificationally valid, explain why. If the argument is not quantificationally valid, construct an interpretation that shows this.

a. $(\forall x)((Lx \ \& \ Dx) \supset Fx)$
$(\exists x)(Dx \ \& \sim Fx)$
$\sim(\exists x)Lx$

*b. $(\forall x)(Sx \supset (Gx \vee Bx))$
$(\exists x)(Sx \ \& \sim Bx)$
$(\exists x)Gx$

c. $(\exists x)(Hx \equiv (Rx \vee Sx))$
$(\exists x)((Hx \ \& \ Rx) \vee (Hx \ \& \ Sx))$

*d. $(\exists x)(\exists y)((Rx \ \& \ Sy) \ \& \ Pxy)$
$(\forall x)(Rx \supset Tx)$
$(\exists x)(\exists y)(Tx \ \& \ Pxy)$

8.5 TRUTH-FUNCTIONAL EXPANSIONS

In the preceding sections we constructed interpretations for sentences of *PL* to establish various semantic results: A sentence is not quantificationally true, a set of sentences is quantificationally consistent, and so on. When we give an interpretation for a sentence or a set of sentences of *PL*, the UD we select may be very large or even infinite. However, when we ask whether certain sentences have various semantic properties, we can often find the answer by considering only interpretations with a relatively small UD. Truth-functional expansions enable us to reason about the truth-values of sentences for interpretations with small UDs.

We shall introduce truth-functional expansions with an example. Consider the sentence

$$(\forall x)(Wx \supset (\exists y)Cxy)$$

and the interpretation

29. UD: The set {1, 2}
Wx: x is even
Cxy: x is greater than y

The sentence is true on this interpretation; every even member of the UD (in this case the number 2) is greater than some member of the UD. If we designate each member of the UD with a constant, for example,

a: 1
b: 2

then we can use these constants to produce a sentence without quantifiers that says the same thing about our UD as the sentence above. We can eliminate the universal quantifier and use a conjunction instead to say that each member of the UD is such that if it is even then it is greater than some member of the UD:

$$(Wa \supset (\exists y)Cay) \ \& \ (Wb \supset (\exists y)Cby)$$

We can now eliminate the existential quantifier in '$(\exists y)Cay$' and use a disjunction instead to say that 1 is greater than some member of the UD:

$$(Wa \supset (Caa \vee Cab)) \ \& \ (Wb \supset (\exists y)Cby)$$

Because 'a' and 'b' designate the two objects in the UD, '$(Caa \vee Cab)$' makes the same claim about the UD as '$(\exists y)Cay$' does—the claim that 1 is greater than at least one member of the UD. We can eliminate the remaining existential quantifier in a similar way:

$$(Wa \supset (Caa \vee Cab)) \ \& \ (Wb \supset (Cba \vee Cbb))$$

The sentence that we have just produced says the same thing about our UD as the original sentence. It is called a *truth-functional expansion* of the original sentence for the set of constants {'a', 'b'}.

Although we introduced interpretation 29 for illustration, we may generalize what we have just said about the quantified sentence and its truth-functional expansion. On *any* interpretation in which each member of the UD is designated by one of the constants 'a' and 'b', the quantified sentence has the same truth-value as its truth-functional expansion using those constants.

The principles behind truth-functional expansions are simple. A universally quantified sentence says something about each member of the UD. If we have a finite UD and a set of constants such that each member of the UD is designated by at least one of these constants, then we can reexpress a universally quantified sentence as a conjunction of its substitution instances formed from the constants. As long as every member of the UD is designated by at least one of the constants, the conjunction ends up saying the same thing as the universally quantified sentence—that every member of the UD satisfies some condition. An existentially quantified sentence says that there is at least one member of the UD of which such-and-such is true and can be reexpressed as a disjunction of its substitution instances: The sentence says that such-and-such is true of *this* object or of *that* object or . . . As long as every member of

the UD is designated by at least one of the constants, the disjunction of substitution instances makes the same claim about the UD as did the existentially quantified sentence.

In constructing a truth-functional expansion, we first choose a set of individual constants. If the sentence contains any constants, they must be among the constants chosen. To expand a universally quantified sentence $(\forall \mathbf{x})\mathbf{P}$, we remove the initial quantifier from the sentence and replace the resulting open sentence with the iterated conjunction

$$(\ldots(\mathbf{P}(\mathbf{a}_1/\mathbf{x}) \,\&\, \mathbf{P}(\mathbf{a}_2/\mathbf{x})) \,\&\, \ldots \,\&\, \mathbf{P}(\mathbf{a}_n/\mathbf{x}))$$

where $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are the chosen constants and $\mathbf{P}(\mathbf{a}_i/\mathbf{x})$ is a substitution instance of $(\forall \mathbf{x})\mathbf{P}$. Each of the conjuncts is a substitution instance of $(\forall \mathbf{x})\mathbf{P}$, differing from one another only in that each is formed from a different constant, and there is one substitution instance for each of the individual constants.

We shall expand the sentence '$(\forall x)Nx$' for the set of constants {'a', 'b'}. Removing the quantifier gives us the open sentence 'Nx', and we replace 'Nx' with the conjunction '$Na \,\&\, Nb$'. We can expand '$(\forall y)(My \supset Jyy)$' for the same set of constants by first dropping the quantifier and then replacing '$My \supset Jyy$' with an iterated conjunction. In the first conjunct 'a' replaces the free variable 'y', and in the second conjunct 'b' replaces that variable. The truth-functional expansion

$$(Ma \supset Jaa) \,\&\, (Mb \supset Jbb)$$

has the same truth-value as the unexpanded sentence on every interpretation in which each member of the UD is named by at least one of the two constants. If we have an interpretation with a two-member UD, for example, in which 'a' designates one member and 'b' designates the other, then '$(Ma \supset Jaa) \,\&\, (Mb \supset Jbb)$' makes the same claim about the UD as '$(\forall y)(My \supset Jyy)$'—namely, that each of the two members is such that if it is M then it stands in the relation J to itself.

We have claimed that a truth-functional expansion has the same truth-value as the unexpanded sentence on any interpretation on which each member of the UD is named *by at least one* of the constants used in the expansion. We note two points about this claim, using the previous example to illustrate. The first is that the interpretations in question may assign the same object to several of the constants as long as each object in the UD is named by at least one of them. So, if we have an interpretation with a one-member UD, both 'a' and 'b' must refer to that one member. In this case every object in the UD is named by at least one of the two constants. Our expanded sentence says twice that the one member of the UD is such that if it is M then it stands in the relation J to itself, and this is equivalent to the universal claim that every member of the UD satisfies that condition.

The second point is that, if a UD for an interpretation has even one member that is not designated by one of the two constants, then the two

sentences may fail to have the same truth-value. The following interpretation shows this:

30. UD: The set {1, 2}
Mx: x is positive
Jxy: x equals y squared
a: 1
b: 1

The expanded sentence '(Ma ⊃ Jaa) & (Mb ⊃ Jbb)', which says twice that if 1 is positive then it equals itself squared, is true on this interpretation. But the universally quantified sentence '(∀y)(My ⊃ Jyy)' is false on this interpretation because 2, which was not mentioned in the expansion, does not satisfy the condition specified after the quantifier. If, however, interpretation 30 had interpreted 'b' to designate 2 (leaving 'a' to designate 1), our requirement that each member of the UD be designated by at least one of the constants would have been met. In this case the two sentences would have had the same truth-value (false).

Now we shall expand the sentence

(∀x)(Gac ∨ Fx)

We have stipulated that the set of constants we use for an expansion must include all the individual constants that occur in the sentence being expanded. So any set of constants for which we expand the sentence must include 'a' and 'c'. We can expand the sentence for the set containing just those constants, in which case removing the initial quantifier and replacing 'x' with each constant in turn results in the expansion

(Gac ∨ Fa) & (Gac ∨ Fc)

If we expand the sentence for the larger set {'a', 'c', 'e'} we obtain

((Gac ∨ Fa) & (Gac ∨ Fc)) & (Gac ∨ Fe)

If the sentence we want to expand contains more than one universal quantifier, we can start with the leftmost one and remove each in turn. To expand

(∀y)(Ly & (∀z)Bzy)

for the set of constants {'a', 'b'}, we first eliminate the quantifier '(∀y)' and expand the resulting open sentence, '(Ly & (∀z)Bzy)', to obtain

[La & (∀z)Bza] & [Lb & (∀z)Bzb]

The expanded sentence now contains two occurrences of the quantifier '(∀z)'; this is because '(∀z)Bzy' was part of the open sentence obtained by removing

the quantifier '(∀y)' and hence became part of each conjunct. We now expand each of the universally quantified sentences that are components of '[La & (∀z)Bza] & [Lb & (∀z)Bzb]' by eliminating each occurrence of '(∀z)' and expanding the resulting open sentences, to obtain first

[La & (Baa & Bba)] & [Lb & (∀z)Bzb]

and then

[La & (Baa & Bba)] & [Lb & (Bab & Bbb)]

Here we replaced '(∀z)Bza' with '(Baa & Bba)' and '(∀z)Bzb' with '(Bab & Bbb)'. Note that when we expand a quantified sentence that is a component of another sentence—as with '(∀z)Bza' and '(∀z)Bzb'—we replace that component exactly where it occurred in the sentence being expanded.

We may expand existentially quantified sentences just as we expand universally quantified sentences, except in this case we construct an iterated *disjunction* rather than an iterated conjunction. A sentence of the form $(\exists \mathbf{x})\mathbf{P}$ expands to the disjunction

$$(\ldots (\mathbf{P}(\mathbf{a}_1/\mathbf{x}) \vee \mathbf{P}(\mathbf{a}_2/\mathbf{x})) \vee \ldots \vee \mathbf{P}(\mathbf{a}_n/\mathbf{x}))$$

where $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are the constants in the chosen set and $\mathbf{P}(\mathbf{a}_i/\mathbf{x})$ is a substitution instance of $(\forall \mathbf{x})\mathbf{P}$. We construct an iterated disjunction because an existential quantification indicates that at least one member of the UD satisfies the specified condition: *This* member satisfies the condition, or *that* member satisfies the condition, and so on.

We can expand the sentence

$$(\exists x)(Fx \supset Gx)$$

for the set of constants {'a', 'b', 'c'} as

$$[(Fa \supset Ga) \vee (Fb \supset Gb)] \vee (Fc \supset Gc)$$

On any interpretation on which all the members of the UD are named by at least one of 'a', 'b', and 'c', the expanded sentence has the same truth-value as the existentially quantified sentence. If the existentially quantified sentence is true, for example, then some member of the UD is such that if it is F then it is G. As long as at least one of 'a', 'b', or 'c' designates this object, the disjunct that contains that constant is true as well. If the existentially quantified sentence is false, then no object in the UD satisfies the condition, and hence none of the disjuncts is true.

As with universally quantified sentences, our claims require that *each* member of the UD be designated by at least one of the constants. For example,

'$(\exists x)(Fx \supset Gx)$' is true but its expansion, '$[(Fa \supset Ga) \vee (Fb \supset Gb)] \vee (Fc \supset Gc)$', is false on interpretation 31:

31. UD: The set {1, 2}
Fx: x is prime
Gx: x is odd
a: 2
b: 2
c: 2

The number 1 satisfies the condition that if it is prime then it is odd, so the existentially quantified sentence is true. However, the expansion mentions the number 2 only on interpretation 31; it is false because 2 does not satisfy the condition that if it is prime (it is) then it is odd. The existentially quantified sentence and its expansion will have the same truth-value on an interpretation *only if* every member of the UD is named by at least one of the constants used in the expansion.

We expand the sentence

$$(\exists x)(\exists w)Zwx$$

for the set of constants {'a', 'b'} as follows: First, we eliminate '$(\exists x)$' and replace '$(\exists w)Zwx$' with an iterated disjunction:

$$(\exists w)Zwa \vee (\exists w)Zwb$$

Then we eliminate '$(\exists w)$' in each of its occurrences, first to obtain

$$(Zaa \vee Zba) \vee (\exists w)Zwb$$

and then to obtain

$$(Zaa \vee Zba) \vee (Zab \vee Zbb)$$

To expand the sentence

$$(\exists w)[Gw \supset \sim (Fw \vee (\exists z)Bz)]$$

for the set of constants {'a', 'b'}, we first eliminate '$(\exists w)$' to obtain

$$[Ga \supset \sim (Fa \vee (\exists z)Bz)] \vee [Gb \supset \sim (Fb \vee (\exists z)Bz)]$$

and then eliminate both occurrences of '$(\exists z)$' to obtain

$$[Ga \supset \sim (Fa \vee (Ba \vee Bb))] \vee [Gb \supset \sim (Fb \vee (Ba \vee Bb))]$$

The sentence

$$(\forall x)(Fx \vee (\exists z)[Fz \,\&\, \sim Izx])$$

can also be expanded by systematic elimination of its quantifiers. We shall expand it for the set of constants ['b', 'f']. First, the universal quantifier is eliminated to obtain the conjunction

$$[Fb \vee (\exists z)[Fz \,\&\, \sim Izb]] \,\&\, (Ff \vee (\exists z)[Fz \,\&\, \sim Izf])$$

Next we eliminate the first occurrence of '$(\exists z)$' to obtain

$$(Fb \vee [(Fb \,\&\, \sim Ibb) \vee (Ff \,\&\, \sim Ifb)] \,\&\, (Ff \vee (\exists z)[Fz \,\&\, \sim Izf])$$

Now we eliminate the second occurrence of '$(\exists z)$', again using a disjunction since we are eliminating an existential quantifier:

$$(Fb \vee [(Fb \,\&\, \sim Ibb) \vee (Ff \,\&\, \sim Ifb)]) \,\&\,$$
$$(Ff \vee [(Fb \,\&\, \sim Ibf) \vee (Ff \,\&\, \sim Iff)])$$

When we expand a sentence, we may choose a set containing only one constant for the expansion. In this case we simply remove the quantifier and replace the free variable in the resulting open sentence with that constant. '$(\forall x)Fx$' is expanded for the set of constants ['a'] as 'Fa', and '$(\exists x)Fx$' is also expanded as 'Fa'. With the same set of constants, '$(\exists y)Gyy$' is expanded as 'Gaa' and '$(\forall x)(Fx \vee (\exists y)Gyy)$' is expanded first to obtain '$(Fa \vee (\exists y)Gyy)$' and then to obtain '$(Fa \vee Gaa)$'.

As a final example we expand the sentence

$$Dg \vee (\forall y)(\exists x)Cyx$$

for the set of constants ['a', 'g'] (we must include 'g' in the set because it occurs in the sentence to be expanded). We first replace '$(\forall y)(\exists x)Cyx$' with its expansion to obtain

$$Dg \vee [(\exists x)Cax \,\&\, (\exists x)Cgx]$$

Then we replace '$(\exists x)Cax$' with its expansion to obtain

$$Dg \vee [(Caa \vee Cag) \,\&\, (\exists x)Cgx]$$

Finally we replace '$(\exists x)Cgx$' with its expansion to obtain

$$Dg \vee [(Caa \vee Cag) \,\&\, (Cga \vee Cgg)]$$

When we have expanded a sentence of *PL* to eliminate every quantifier, the truth-functional expansion that results is always an atomic sentence or a truth-functional compound of atomic sentences of *PL*. Because of this, we can construct truth-tables for truth-functional expansions. And the truth-tables, in turn, tell us something about the truth-conditions of the sentences that have been expanded. For example, the truth-functional expansion of the sentence '(∀x)(Fx & – Bx)' for the set of constants ['a', 'b'] is '(Fa & – Ba) & (Fb & – Bb)'. Here is a truth-table for the expansion:

Ba	Bb	Fa	Fb	(Fa	&	– Ba)	↓ &	(Fb	&	– Bb)
T	T	T	T	T	F	F T	F	T	F	F T
T	T	T	F	T	F	F T	F	F	F	F T
T	T	F	T	F	F	F T	F	T	F	F T
T	T	F	F	F	F	F T	F	F	F	F T
T	F	T	T	T	F	F T	F	T	T	T F
T	F	T	F	T	F	F T	F	F	F	T F
T	F	F	T	F	F	F T	F	T	T	T F
T	F	F	F	F	F	F T	F	F	F	T F
F	T	T	T	T	T	T F	F	T	F	F T
F	T	T	F	T	T	T F	F	F	F	F T
F	T	F	T	F	F	T F	F	T	F	F T
F	T	F	F	F	F	T F	F	F	F	F T
F	F	T	T	T	T	T F	T	T	T	T F
F	F	T	F	T	T	T F	F	F	F	T F
F	F	F	T	F	F	T F	F	T	T	T F
F	F	F	F	F	F	T F	F	F	F	T F

This truth-table tells us that the quantified sentence is true on some interpretations with one- or two-member UDs and false on some interpretations with one- or two-member UDs. We shall now explain why.

If each object in a UD is designated by either 'a' or 'b', then each of the combinations of truth-values to the left of the vertical line represents an interpretation of 'B' and 'F'. (This assumption means that we are restricting our attention to UDs with at most two members because the number of constants is not enough for naming more than two members.) For example, the first row represents interpretations with one- or two-member UDs in which all objects are in the extension of 'B' and also are in the extension of 'F'. If all objects are named by either 'a' or 'b', then the assignment of **T** to both 'Ba' and 'Bb' means that all objects are in the extension of 'B', and the assignment of **T** to both 'Fa' and 'Fb' means that all objects are in the extension of 'F'. The second row represents interpretations with two-member UDs in which both objects are in the extension of 'B' (because both 'Ba' and 'Bb' are true), one object is in the extension of 'F' (because 'Fa' is true), and one object is not in the extension of 'F' (because 'Fb' is false). Note that, because one object is in the extension of 'F' and one is not, the UD for any interpretation represented

by this row cannot have just one member—the single object in a one-member UD cannot both be in the extension of 'F' and not be in the extension of 'F'.

In fact, the sixteen rows between them represent all the combinations of extensions that the two predicates may have for a one- or two-member UD. For example, we have the following possibilities for a one-member UD: The one object is in the extension of both 'B' and 'F' (row 1), the one object is in the extension of 'B' but not of 'F' (row 4), the one object is in the extension of 'F' but not of 'B' (row 13), or the one object is not in the extension of either predicate (row 16). For a two-member UD we have the following possibilities: Both members are in the extensions of both 'B' and 'F' (row 1), both members are in the extension of 'B' but only one is in the extension of 'F' (rows 2 and 3), both members are in the extension of 'B' but neither is in the extension of 'F' (row 4), and so on.

The truth-value assigned to the truth-functional expansion in each row is the truth-value that '$(\forall x)(Fx \ \& \sim Bx)$' receives for the interpretations of 'B' and 'F' represented by that row. The expansion has the truth-value **F** in the first row, from which we may conclude that, on every interpretation with a one- or two-member UD in which every member is in the extension of 'B' and also in the extension of 'F', '$(\forall x)(Fx \ \& \sim Bx)$' is false. The expansion also has the truth-value **F** in the second row, from which we may conclude that, on every interpretation with a two-member UD (recall that this row cannot represent interpretations with one-member UDs) in which both members are in the extension of 'B' but only one member is in the extension of 'F', the sentence '$(\forall x)(Fx \ \& \sim Bx)$' is false.

The thirteenth row is the only one on which the expansion is true. From this we may conclude that every interpretation with a one- or two-member UD in which every member is in the extension of 'B' and no member is in the extension of 'F' makes '$(\forall x)(Fx \ \& \sim Bx)$' true, and that every other interpretation with a one- or two-member UD makes the sentence false.

We can use the information in the thirteenth row to construct an interpretation on which the unexpanded quantified sentence is true. Because neither 'a' nor 'b' appears in the quantified sentence, we need only specify a UD (we will choose one with two members rather than one), an interpretation of 'B' that holds for neither member of the UD (because 'Ba' and 'Bb' are both false), and an interpretation of 'F' that holds for both members of the UD (because 'Fa' and 'Fb' are both true). Here is a candidate:

32. UD: The set {3, 5}
    Bx: x is an even integer
    Fx: x is a positive integer

Both objects in the UD satisfy the condition of being positive and not even, so '$(\forall x)(Fx \ \& \sim Bx)$' is true on this interpretation.

We can use the information in the first row to construct an interpretation on which the sentence is false. For convenience we will choose the same UD. We shall interpret 'B' and 'F' so that both the number 3 and the number 5

are in the extension of both predicates—because the four atomic sentences in the first row are all true:

33. UD: The set {3, 5}
Bx: x is an odd integer
Fx: x is a positive integer

Any row in the truth-table in which 'Ba' has the same truth-value as 'Bb' and 'Fa' has the same truth-value as 'Fb' can be used to construct an interpretation with a one-member UD. For example, using the first row, we can construct an interpretation on which our quantified sentence is false by making sure that the one object in the UD is in the extension of both 'B' and 'F':

34. UD: The set {3}
Bx: x is an odd integer
Fx: x is a positive integer

A truth-functional expansion of the sentence '(∃x)(∀y)Nyx' for the set of constants {'a', 'b'} is '(Naa & Nba) ∨ (Nab & Nbb)'. We may show that the sentence '(∃x)(∀y)Nyx' is true on at least one interpretation with a two-member UD by producing a shortened truth-table in which the expansion is true:

								↓				
Naa	Nab	Nba	Nbb	(Naa	&	Nba)	∨	(Nab	&	Nbb)		
T	T	F	T	T	F	F	T	T	T	T		

(The table in this case gives us information only about two-member UDs, because if there were only one member in the UD then it would be named by both 'a' and 'b', and hence the four atomic sentences would have to have the same truth-value since each would in that case make the same claim—that the one object stands in the relation N to itself.) We do not have to give an actual interpretation on which the sentence is true; the shortened truth-table suffices to show that there is such an interpretation. It shows that the quantified sentence is true on any interpretation with a two-member UD in which both members stand in the relation N to themselves and one stands in the relation N to the other, but not vice versa. And the following shortened truth-table shows that the quantified sentence is false on at least one interpretation with a one- or two-member UD:

								↓				
Naa	Nab	Nba	Nbb	(Naa	&	Nba)	∨	(Nab	&	Nbb)		
F	F	F	F	F	F	F	F	F	F	F		

(Because the four atomic sentences have the same truth-value in this table, the row of assignments may represent an interpretation with a one-member UD.)

From these two shortened truth-tables we may conclude that '$(\exists x)(\forall y)Nyx$' is quantificationally indeterminate. The tables show that the sentence is true on at least one interpretation and false on at least one interpretation.

We may use truth-functional expansions and truth-tables to demonstrate that sentences of *PL* have, or fail to have, some other semantic properties as well. For example, to show that a sentence is not quantificationally true, we must show that the sentence is false on at least one interpretation. And we can show this by producing a shortened truth-table in which a truth-functional expansion of the sentence is false. We will have to choose a set of constants first—ideally a small set, to save us work. An expansion of the sentence '$(Ga \amp (\exists z)Bz) \supset (\forall x)Bx$' for the set of constants ['a', 'b'] is '$(Ga \amp (Ba \vee Bb) \supset (Ba \amp Bb)$', and the expansion is false in the following shortened truth-table:

							↓					
Ba	Bb	Ga	(Ga	&	(Ba	∨	Bb))	⊃	(Ba	&	Bb)
T	F	T	T	T	T	T	F	F	T	F	F

The table shows that there is at least one interpretation on which the sentence '$(Ga \amp (\exists z)Bz) \supset (\forall x)Bx$' is false. This sentence is therefore not quantificationally true.

Note that we cannot in general use truth-functional expansions to show that a sentence *is* quantificationally true. Even if we construct a full truth-table for a truth-functional expansion and find that the expansion is true in every row of the truth-table, all that we may generally conclude is that the sentence is true on every interpretation with a UD that is the same size as or smaller than the number of constants in the set that was used for the expansion. (An exception will be noted at the end of this section.)

The sentence '$\sim (\sim Ga \amp (\exists y)Gy)$' is not quantificationally false. The truth-functional expansion of this sentence for the set of constants ['a'] ('a' must be in this set because it occurs in the sentence) is '$\sim (\sim Ga \amp Ga)$', and this expansion is true in the following shortened truth-table:

	↓				
Ga	~	(~	Ga	&	Ga)
T	T	F	T	F	T

This shows that the sentence '$\sim (\sim Ga \amp (\exists y)Gy)$' is true on at least one interpretation and hence that the sentence is not quantificationally false. As with quantificational truth we cannot in general use truth-functional expansions to show that a sentence is quantificationally false, for that would involve showing that the sentence is false on every interpretation, not just those with a particular size UD.

The sentences

$(\forall x)(Fx \supset Ga)$

and

$$(\forall x)Fx \supset Ga$$

are not quantificationally equivalent. To show this, we shall expand both sentences for the same set of constants (which must include 'a') and produce a shortened truth-table in which the expansions have different truth-values. Expanding the sentences for the set ['a', 'b'], we obtain

$$(Fa \supset Ga) \ \& \ (Fb \supset Ga)$$

and

$$(Fa \ \& \ Fb) \supset Ga$$

The first sentence is false and the second is true in the following shortened truth-table:

						↓								↓	
Fa	Fb	Ga	(Fa	⊃	Ga)	&	(Fb	⊃	Ga)	(Fa	&	Fb)	⊃	Ga
T	F	F	T	F	F	F	F	T	F	T	F	F	T	F

This shows that there is at least one interpretation on which '$(\forall x)(Fx \supset Ga)$' is false and '$(\forall x)Fx \supset Ga$' is true.

The set of sentences

$$\{(\forall x)Gax, \sim Gba \vee (\exists x) \sim Gax\}$$

is quantificationally consistent. The truth-functional expansions of these sentences for the set ['a', 'b'] are 'Gaa & Gab' and '~ Gba ∨ (~ Gaa ∨ ~ Gab)'. Both expansions are true in the following shortened truth-table, and so we may conclude that there is at least one interpretation on which both members of the set are true:

				↓				↓				
Gaa	Gab	Gba	Gaa	&	Gab	~	Gba	∨	(~	Gaa	∨	~ Gab)
T	T	F	T	T	T	T	F	T	F	T	F	F T

The set of sentences

$$\{\sim (\forall x)(Ga \equiv Fx), \sim Fb\}$$

does not quantificationally entail the sentence

$$(\forall x) \sim Gx$$

We shall expand the sentences for the set of constants {'a', 'b'} to obtain

$$\sim [(Ga \equiv Fa) \;\&\; (Gb \equiv Fb)]$$

and

$$\sim Fb$$

for the set members ('~ Fb' expands to itself because it contains no quantifiers), and

$$\sim Ga \;\&\; \sim Gb$$

for the sentence '$(\forall x) \sim Gx$'. Here is a shortened truth-table in which the expanded set members are true and the expansion of '$(\forall x) \sim Gx$' is false:

					↓									↓			↓		
Fa	Fb	Ga	Gb		~	[(Ga	≡	Fa)	&	(Gb	≡	Fb)]		~ Fb		~ Ga	&	~ Gb	
F	F	F	T		T	F	T	F	F	T	F	F		TF		TF	F	FT	

We thus know that there is at least one interpretation on which the set members are both true and '$(\forall x) \sim Gx$' is false, so '$(\forall x) \sim Gx$' is not quantificationally entailed by the set.

Finally we may use truth-functional expansions to show that some arguments are not quantificationally valid. The expansions of the premises and conclusion of the argument

$$\begin{array}{l} (\exists x)[(\exists y)Fy \supset Fx] \\ (\exists y) \sim Fy \\ \hline \sim (\exists x)Fx \end{array}$$

for the set of constants {'a', 'b'} are

$$\begin{array}{l} [(Fa \vee Fb) \supset Fa] \vee [(Fa \vee Fb) \supset Fb] \\ \sim Fa \vee \sim Fb \\ \hline \sim (Fa \vee Fb) \end{array}$$

The premises of this expanded argument are true and the conclusion false in the following shortened truth-table:

							↓								↓			↓			
Fa	Fb	[(Fa	∨	Fb)	⊃	Fa]	∨	[(Fa	∨	Fb)	⊃	Fb]	~ Fa	∨	~ Fb	~ (Fa	∨	Fb)	
T	F	T	T	F	T	T	T	T	T	F	F	F	FT	T	TF	F	T	T	F		

There is thus an interpretation on which the premises of the original argument are true and the conclusion is false.

Note once again that truth-functional expansions cannot generally be used to show that a set of sentences of *PL* is quantificationally inconsistent, that a set of sentences *does* quantificationally entail some sentence, or that an argument of *PL* is quantificationally valid. In each of these cases we must prove something about every interpretation, not just those represented in the truth-table for a particular set of expansions.

However, there is an exception to our claims about the limitations of using truth-functional expansions to test for semantic properties. We noted at the end of Section 8.2 that there is a decision procedure (based on a result by Bernays and Schönfinkel) for determining the quantificational status of sentences of *PL* that contain no many-place predicates, that is, in which the predicates are all one-place predicates. A decision procedure allows us to answer correctly in a finite number of mechanical steps the question 'Is this sentence quantificationally true?' and hence also questions like 'Is this sentence quantificationally false?' (it is if its negation is quantificationally true) and 'Is this finite set of sentences quantificationally consistent?' (it is if the conjunction of the sentences in the set is not quantificationally false). It allows us to answer these questions correctly for sentences that do not contain many-place predicates.

Bernays and Schönfinkel's result is that a sentence that contains no many-place predicates and that contains **k** distinct one-place predicates is quantificationally true if and only if the sentence is true on every interpretation with a UD containing exactly $2^{\mathbf{k}}$ members. This being the case, we can truth-functionally expand the sentence for a set of at least $2^{\mathbf{k}}$ constants, produce a truth-table for the expanded sentence, and determine whether it is quantificationally true by examining the truth-table. If the expanded sentence is true in every row of the truth-table, we may conclude that the sentence is true on every interpretation with a UD that is the same size as the set of constants or smaller. In particular, we may conclude that the sentence is true on every interpretation with a UD that contains exactly $2^{\mathbf{k}}$ members. And, by Bernays and Schönfinkel's result, we may finally conclude that the sentence is quantificationally true.

8.5E EXERCISES

1. Give a truth-functional expansion of each of the sentences in Exercise 7 in Section 8.1E for a set containing one constant.
2. Give a truth-functional expansion of each of the sentences in Exercise 8 in Section 8.1E for a set containing two constants.
3. Give a truth-functional expansion of each of the sentences in Exercise 9 in Section 8.1E for a set containing two constants.

4. Give a truth-functional expansion of each of the following sentences for the set {'a', 'b', 'c'}.

a. $(\forall w)(Gw \supset Nww)$

*b. $(Na \vee (\exists z)Bz)$

c. $(\exists z)(Na \equiv Bz)$

*d. $(\forall w)Bw \vee \sim (\exists w)Bw$

5. Construct truth-functional expansions of the sentence

'$((\exists x)Fx \ \& \ (\exists y) \sim Fy) \supset (\forall x) \sim Fx$'

for the sets {'a'} and {'a', 'b'}. Construct a truth-table for each expansion. What information does the first truth-table give you about this sentence? What information does the second truth-table give you?

6. For each of the following sentences, construct a truth-functional expansion for the set of constants ('a', 'n'). Show that the expansion is true on at least one truth-value assignment. Then use the information in the truth-table to construct an interpretation on which the original sentence is true.

a. $(\forall x)(Nxx \vee (\exists y)Nxy)$

*b. $(\exists x)Fx \equiv (\forall x)Fx$

c. $(\forall y)Syyn$

7. Show that each of the sentences in Exercise 1 in Section 8.2E is not quantificationally true by producing a shortened truth-table in which a truth-functional expansion of the sentence is false.

8. Show that each of the sentences in Exercise 2 in Section 8.2E is not quantificationally false by producing a shortened truth-table in which a truth-functional expansion of the sentence is true.

9. Show that each of the sentences in Exercise 3 in Section 8.2E is quantificationally indeterminate by producing a shortened truth-table in which a truth-functional expansion of the sentence is true and a shortened truth-table in which a truth-functional expansion of the sentence is false.

***10.** In this section it was claimed that in general a sentence of *PL* that contains quantifiers cannot be shown to be quantificationally true by producing truth-tables for truth-functional expansions. Does the claim hold for sentences of *PL* that do not contain quantifiers, such as '$Fa \supset (Gb \supset Fa)$'? Explain.

11. The truth-functional expansion of the sentence '$(\exists y)Gy \ \& \ (\exists y) \sim Gy$' for the set {'a'} is '$Ga \ \& \sim Ga$'. The expanded sentence is quantificationally false. Explain this and then explain why this does *not* show that the original sentence '$(\exists y)Gy \ \& \ (\exists y) \sim Gy$' is quantificationally false.

12. Show that the sentences in each pair in Exercise 1 in Section 8.3E are not quantificationally equivalent by producing a shortened truth-table in which a truth-functional expansion of one sentence of the pair is true and a truth-functional expansion of the other sentence (for the same set of constants) is false.

13. Show that each set of sentences in Exercise 4 in Section 8.3E is quantificationally consistent by producing a shortened truth-table in which a truth-functional expansion of each sentence in the set (for the same set of constants) is true.

***14.** a. Is the set {Ba, Bb, Bc, Bd, Be, Bf, Bg, ~ (∀x)Bx} quantificationally consistent? Explain.
b. For the set in Exercise 14.a, what is the minimum size set of constants for which the sentences in the set must be expanded in order to show that the set is quantificationally consistent? Explain.
c. Can all the sentences in the set in Exercise 14.a be true on an interpretation with a UD smaller than the set of constants indicated in the answer to Exercise 14.b? Explain.

15. Show that each argument in Exercise 2 in Section 8.4E is quantificationally invalid by producing a shortened truth-table in which truth-functional expansions of the premises are true and a truth-functional expansion of the conclusion for the same set of constants is false.

8.6 SEMANTICS FOR PREDICATE LOGIC WITH IDENTITY AND FUNCTORS

In *PLE* interpretations for sentences containing the identity predicate but no functors are the same as interpretations for *PL*, because the *identity* predicate, '=', is *not* explicitly given an interpretation. This is because we always want its extension to be the set of ordered pairs of members of the UD in which the first member is identical to the second, no matter what the UD is. The extension of the identity predicate is determined once the UD has been determined. If the UD is the set of positive integers, for example, then the extension of the identity predicate will include the pair whose first and second members are 1, the pair whose first and second members are 2, and so on for each positive integer—and no other pairs.

Every atomic sentence of the form **a** = **a**, where **a** is an arbitrary individual constant, is true on every interpretation. This is because **a** designates exactly one member of the UD on a given interpretation, and the identity predicate must include the pair consisting of that object and itself in its extension. On the other hand, the truth-value of an atomic sentence of the form **a** = **b**, where **a** and **b** are different individual constants, depends on the interpretations of **a** and **b**. Interpretation 35 makes the sentence 'g = k' true, while interpretation 36 makes the sentence false:

35. UD: Set of positive integers
g: 1
k: 1

36. UD: Set of positive integers
g: 1
k: 2

The sentence '$(\forall x)(\forall y)(\sim x = y \supset Gxy)$' is true on interpretation 37 and false on interpretation 38:

37. UD: Set of positive integers
    Gxy: the sum of x and y is positive
38. UD: Set of positive integers
    Gxy: x is greater than y

On interpretation 37 the sentence may be read as 'The sum of any pair of nonidentical positive integers is a positive integer'—which is true. On interpretation 38 the sentence claims that for any pair of nonidentical positive integers the first is greater than the second. This is false—1 and 2 are nonidentical positive integers, for example, but 1 is not greater than 2.

We can show that various sentences and sets of sentences that contain the identity predicate have, or fail to have, semantic properties much as we did for *PL*. We shall give a few examples for the semantic properties of quantificational truth and quantificational validity. We can show that the sentence

$$(\forall x)(\forall y)(\sim x = y \vee (Fx \supset Fy))$$

is quantificationally true by reasoning generally about interpretations, showing that on every interpretation the sentence turns out to be true. The sentence is universally quantified and is true on an interpretation when, for every pair x and y of members of the UD, either they satisfy the condition specified by '$\sim x = y$' or they satisfy the condition specified by '$Fx \supset Fy$'. So let us consider two members x and y of an arbitrary UD. If x and y are not the same member, then the first disjunct '$\sim x = y$' is satisfied because the extension of the identity predicate includes only pairs in which the first and second members are the same. If, however, x and y *are* the same member of the UD (and hence do not satisfy the first disjunct), they satisfy the second disjunct. If x is in the extension of 'F', then so is y—because y is identical to x, and so x and y satisfy the condition '$Fx \supset Fy$'. Because x and y either are or are not the same member of the UD, we have shown that each pair of members of any UD satisfy the condition '$\sim x = y \vee (Fx \supset Fy)$' no matter what the interpretation of 'F' may be. Therefore the sentence '$(\forall x)(\forall y)(\sim x = y \vee (Fx \supset Fy))$' must be true on any interpretation; it is quantificationally true.

On the other hand, the sentence

$$(\forall x)(\forall y)(x = y \vee (Fx \supset Fy))$$

is not quantificationally true. To show this, we construct an interpretation on which the sentence is false. The sentence claims that every pair of members of the UD x and y satisfies '$x = y \vee (Fx \supset Fy)$'—that is, that either x and y are the same member or if x is F then so is y. If we choose a two-member UD, then

a pair consisting of the two members will not satisfy the condition 'x = y'. If the first member is F but the other is not, then this pair also will not satisfy 'Fx ⊃ Fy'. Here is our interpretation:

39. UD: The set {1, 2}
Fx: x is odd

The pair consisting of the numbers 1 and 2 does not satisfy 'x = y ∨ (Fx ⊃ Fy)'. The two numbers are not identical, and it is not true that if the number 1 is odd (which it is) then the number 2 is odd (it is not).

The argument

$$\frac{\begin{array}{l}(\forall x)(Fx \equiv Gx)\\ (\forall x)(\forall y)x = y\\ Ga\end{array}}{(\forall x)Fx}$$

is quantificationally valid. We shall show that any interpretation that makes the three premises true also makes '(∀x)Fx' true. If '(∀x)(Fx ≡ Gx)' is true, then every member of the UD that is F is also G, and every member of the UD that is G is also F. If '(∀x)(∀y)x = y' is also true, then there is exactly one object in the UD. The sentence says that, for any object x and any object y, x and y are identical—and this cannot be the case if there is more than one member of the UD. If 'Ga' is also true, then because there is exactly one object in the UD this object must be designated by 'a' and must therefore be in the extension of 'G'. It follows, from the truth of the first sentence, that this object is also in the extension of 'F'. Therefore it follows that '(∀x)Fx' is true—every object in our single-member UD is F.

On the other hand, the argument

$$\frac{(\forall x)(\exists y)x = y}{a = b}$$

is not quantificationally valid. The premise, it turns out, is quantificationally true—every member of any UD is identical to something (namely, itself). But the conclusion is false on any interpretation on which 'a' and 'b' designate different objects, such as the following:

40. UD: Set of positive integers
a: 6
b: 7

It is true that every positive integer is identical to some positive integer, but it is false that 6 is identical to 7.

Readers who have worked through the section on truth-functional expansions may wonder whether sentences containing the identity predicate may be expanded and truth-tables used to check for various semantic properties. The answer is yes, although we shall see that there is a complication. Sentences that contain the identity predicate are expanded in the same way as sentences without the identity predicate: Quantifiers are eliminated in favor of iterated conjunctions or disjunctions. The sentence '$(\forall x)(\exists y)x = y$' can be expanded for the set of constants {'a', 'b'} first to obtain

$$(\exists y)a = y \;\&\; (\exists y)b = y$$

and then to obtain

$$(a = a \vee a = b) \;\&\; (b = a \vee b = b)$$

But if we freely assign truth-values to the atomic components of this sentence, we end up with this truth-table:

a = a	a = b	b = a	b = b	(a = a	∨	a = b)	&	(b = a	∨	b = b)
T	T	T	T	T	T	T	T	T	T	T
T	T	T	F	T	T	T	T	T	T	F
T	T	F	T	T	T	T	T	F	T	T
T	T	F	F	T	T	T	F	F	F	F
T	F	T	T	T	T	F	T	T	T	T
T	F	T	F	T	T	F	T	T	T	F
T	F	F	T	T	T	F	T	F	T	T
T	F	F	F	T	T	F	F	F	F	F
F	T	T	T	F	T	T	T	T	T	T
F	T	T	F	F	T	T	T	T	T	F
F	T	F	T	F	T	T	T	F	T	T
F	T	F	F	F	T	T	F	F	F	F
F	F	T	T	F	F	F	F	T	T	T
F	F	T	F	F	F	F	F	T	T	F
F	F	F	T	F	F	F	F	F	T	T
F	F	F	F	F	F	F	F	F	F	F

MISTAKE!

There is something wrong with this truth-table! The sentence '$(\forall x)(\exists y)x = y$' is quantificationally true, and yet we have assigned its expansion the truth-value **F** in seven rows. Let us look at the first row where this happened: row 4. In this row we have assigned **T** to 'a = b' and **F** to 'b = a', and that is the problem. If an interpretation makes 'a = b' true then it must make 'b = a' true as well; a and b are the same object. So row 4 does not correspond to any interpretation at all. By the same reasoning we find that none of rows 3–6 or 11–14 correspond to interpretations, for each of these rows assigns different truth-values to 'a = b' and 'b = a'. However, this still leaves us with problematic rows 8, 15,

and 16—all of which make the expanded sentence false. The problem with each of these rows is that the truth-value **F** has been assigned to one or both of 'a = a' and 'b = b'—thus claiming that some object is not the same as itself. Because every interpretation makes every sentence of the form **a** = **a** true, rows 8, 15, and 16, as well as all other rows that make one or both of 'a = a' and 'b = b' false, do not correspond to interpretations. In fact, we have just ruled out all rows in the truth-table except rows 1 and 7. These are the only rows in which 'a = a' and 'b = b' are both true and in which 'a = b' and 'b = a' have the same truth-value—and, as we should have expected for a quantificationally true sentence, the expanded sentence is true in both rows.

The rows of a truth-table that do not correspond to any interpretation cannot be used to establish semantic properties of the sentence that has been expanded. We therefore require that each row in the truth-table we construct for an expansion of a sentence containing the identity predicate must meet two conditions:

1. Every sentence of the form **a** = **a** has the truth-value **T**.
2. If a sentence of the form **a** = **b** has the truth-value **T**, then for each atomic sentence **P** that contains **a**, every atomic sentence **P**(**b**//**a**) that results from replacing one or more occurrences of **a** in **P** with **b** must have the same truth-value as **P**.

If conditions 1 and 2 are met, then if a sentence containing **a** = **b** has the truth-value **T** in a row, **b** = **a** will also have the truth-value **T**. Condition 1 requires that **a** = **a** have the truth-value **T**, and because **b** = **a** can be obtained from **a** = **a** by replacing the first occurrence of **a** with **b**, condition 2 requires that **b** = **a** is true since **a** = **b** and **a** = **a** are. Condition 2 also rules out rows like the one in the following shortened truth-table for the expansion

$$[(a = a \supset (Fa \supset Fa)) \,\&\, (a = b \supset (Fa \supset Fb))] \,\&\, [(b = a \supset (Fb \supset Fa)) \,\&\, (b = b \supset (Fb \supset Fb))]$$

of the sentence

$$(\forall x)(\forall y)(x = y \supset (Fx \supset Fy))$$

for the set of constants {'a', 'b'}:

a = a	a = b	b = a	b = b	Fa	Fb	[(a = a ⊃ (Fa ⊃ Fa)) & (a = b ⊃ (Fa ⊃ Fb))]
T	T	T	T	T	F	T T T T T F T F T F F

↓

& [(b = a ⊃ (Fb ⊃ Fa)) & (b = b ⊃ (Fb ⊃ Fb))]
F T T F T T T T T F T F

Once again we have expanded a quantificationally true sentence and produced a row of a truth-table in which the truth-functional expansion is false. We have ensured that both sentences 'a = a' and 'b = b' are true and that 'a = b' and 'b = a' have the same truth-value. The problem is that we have assigned 'Fa' and 'Fb' different truth-values, although 'a = b' is true. Condition 2 rules out this combination: 'Fb' results from replacing 'a' in 'Fa' with 'b,' and so, because 'a = b' is true, 'Fb' must have the same truth-value as 'Fa'. Our second condition reflects the fact that when the identity predicate occurs in a truth-functional expansion the atomic sentences that are components of the expansion may not be truth-functionally independent. Once a sentence of the form **a** = **b**, where **a** and **b** are different constants, has been made true, certain other atomic sentences must agree in truth-value.

The following truth-table shows the only combinations of truth-values for the atomic components of our sentence that correspond to interpretations with one- or two-member UDs:

a = a	a = b	b = a	b = b	Fa	Fb	[(a = a ⊃ (Fa ⊃ Fa)) & (a = b ⊃ (Fa ⊃ Fb))]
T	T	T	T	T	T	T T T T T T T T T T
T	T	T	T	F	F	T T F T F T T T F T F
T	F	F	T	T	T	T T T T T F T T T T
T	F	F	T	T	F	T F T T T F T T F F
T	F	F	T	F	T	T T F T F T F T T T
T	F	F	T	F	F	T T F T F T F T F T F

↓

& [(b = a ⊃ (Fb ⊃ Fa)) & (b = b ⊃ (Fb ⊃ Fb))]
T T T T T T T T T T T
T T T F T F T T T F T F
T F T T T T T T T T T
T F T F T T T T T F T F
T F T T F F T T T T T
T F T F T F T T T F T F

All other rows are excluded by one or both of our conditions. And again we find that the expanded sentence is true in all six rows—we have shown that there are no interpretations with one- or two-member UDs on which the sentence is false.

Adhering to our two conditions, we now produce a shortened truth-table that shows that the sentence

$$(\forall z)((Fz \ \& \ (\exists y)z = y) \supset (\forall x)Fx)$$

is not quantificationally true. The sentence claims that, for each member of the UD, if it is F and is identical to something then everything is F. Certainly the sentence will be true if the UD contains exactly one object—but for larger

UDs it will be true only if either no member is F or all members are. We shall expand the sentence for the set of constants {'a', 'b'} and produce a shortened truth-table in which the expansion is false:

a = a	a = b	b = a	b = b	Fa	Fb	[(Fa & (a = a ∨ a = b)) ⊃ (Fa & Fb)]
T	F	F	T	T	F	T T T T F F T F F

↓

& [(Fb & (b = a ∨ b = b)) ⊃ (Fa & Fb)]
F F F F T T T T F F

Condition 1 has been met—both 'a = a' and 'b = b' are true. Condition 2 has also been met, trivially. The two identity statements that are true are 'a = a' and 'b = b', and the result of substituting 'a' for 'a' in any sentence is just that sentence itself and the same holds for 'b'. Here is an interpretation that has been constructed using the truth-values in the truth-table as a guide:

41. UD: The set {2, 3}
Fx: x is even

We have chosen a UD with two members because the identity statements 'a = b' and 'b = a' are false in the shortened table, and so 'a' and 'b' must designate different objects. We have interpreted 'F' so that one member of the UD, but not the other, is in its extension.

We now turn to the semantics for functors in *PLE*. A one-place functor is interpreted as a function that maps each ordered set of one member of the UD to a single member of the UD, namely, the member that is the value of the function for that ordered set. A two-place functor is interpreted as a function that maps each ordered pair of members of the UD to a single member of the UD, a three-place functor is interpreted as a function that maps each ordered triple of members of the UD to a single member of the UD, and so on. Here is an interpretation for the sentence '(∀x)(Px ⊃ H*f*(x))':

42. UD: Set of positive integers
Px: x is even
Hx: x is odd
f(x): the successor of x (the number that results from adding 1 to x)

On this interpretation the sentence may be read as 'The successor of any even positive integer is an odd positive integer', which is true. The sentence is false on the following interpretation:

43. UD: Set of positive integers
Px: x is even
Hx: x is odd
f(x): the square of x

On this interpretation the sentence may be read as 'The square of any even positive integer is an odd positive integer'.

Here is an interpretation for the sentence '$(\forall x)(\exists y)y = g(x) \supset (\forall x)(\exists y)Pf(x,y)$':

44. UD: Set of positive integers
Px: x is odd
g(x): the successor of x
f(x,y): the sum of x and y

On this interpretation the sentence may be read as 'If every positive integer has a successor, then for every positive integer x there is a positive integer y such that the sum of x and y is odd'. On this interpretation the sentence is true since both the antecedent and the consequent are true. The following interpretation makes the same sentence false:

45. UD: Set of positive integers
Px: x is prime
g(x): the successor of x
f(x,y): x raised to the power y

On this interpretation the sentence may be read as 'If every positive integer has a successor, then for every positive integer there is some positive power such that the integer raised to that power is prime'. While the antecedent remains true, the consequent in this case is false.

Here is an interpretation for the sentence '$(\forall x)(\forall y)f(x,y) = f(y,x)$':

46. UD: Set of positive integers:
f(x,y): the sum of x and y

On this interpretation the sentence may be read as 'The sum of any two positive integers x and y is equal to the sum of y and x', which is true. The same sentence is false on the following interpretation:

47. UD: Set of positive integers
f(x,y): x raised to the power y

It is not true that, for any two positive integers x and y, x raised to the power y equals y raised to the power x. For example, 2 cubed equals 8, but 3 squared equals 9.

The sentence '$(\forall x)Dh(x,f(x))$' is true on interpretation 48 and false on interpretation 49:

48. UD: Set of positive integers
Dx: x is even
$f(x)$: x cubed
$h(x,y)$: the sum of x and y

49. UD: Set of positive integers
Dx: x is even
$f(x)$: x doubled
$h(x,y)$: the sum of x and y

It is true that the sum of any positive integer and that same integer cubed is even, for the cube of an even integer is even and the cube of an odd integer is odd. But it is false that the sum of any positive integer and that same integer doubled is even, for the result of doubling an odd integer is even, and so the sum of an odd integer and its double is odd.

When we produce an interpretation for sentences containing functors, it is important that we really have interpreted the functors as functions. For example, it may be tempting to come up with an interpretation with the set of positive integers as the UD on which '$f(x)$' means 'the integer greater than x'. But this is not a function, for there are (infinitely!) many integers greater than any positive integer. A one-place function cannot map a member of the UD to more than one value. Similarly we cannot interpret '$h(x,y)$' (with the same UD) as 'the integer that is a factor of both x and y', because two positive integers can have more than one factor in common.

It is also important, when we produce an interpretation for sentences containing functors, that the interpretation assigns a function that meets the following two conditions. First, a one-place function that is used to interpret a one-place functor must be defined for every member of the UD, and an **n**-place function that is used to interpret an **n**-place functor must be defined for every ordered set of members of the UD. For example, with the set of positive integers as the UD, we cannot interpret '$f(x)$' to mean 'the integer that is the square root of x', since not every positive integer has an integral square root. Similarly we cannot interpret '$h(x,y)$' to mean 'the integer that is the result of dividing x by y', since, for example, no integer is the result of dividing 5 by 3.

Second, even when the function is defined for every member or ordered set of members of the UD, we also require that the value of the function in each case be a member of the UD. So, if our UD is the set of positive integers, we also cannot interpret '$h(x,y)$' to mean 'the *number* that is the result of dividing x by y'. Thus, although the division function is defined for every pair of positive integers, the resulting value is not in every case a positive integer. For example, the result of dividing 5 by 3, namely, $\frac{5}{3}$, is not a positive integer. Nor can we interpret '$h(x,y)$' to mean 'x minus y' if our UD is the set of positive integers, because, for example, 2 minus 3 is not a positive integer.

Similarly, with the UD of positive integers, we cannot interpret 'f(x)' to mean 'the predecessor of x'. Not every positive integer has a positive integer as its predecessor, for the predecessor of the positive integer 1 is 0.

We may show semantic results for sentences containing functors either by producing interpretations that are sufficient to prove the result or by arguing generally that the sentences will have certain truth-values on every interpretation. For example, interpretations 45 and 46, respectively, established that the sentence '$(\forall x)(\forall y)f(x,y) = f(y,x)$' is quantificationally indeterminate. On the other hand, the sentence '$(\forall x)(\exists y)y = f(x)$' is quantificationally true. The sentence is universally quantified and is true on an interpretation if for each x there is at least one y such that the pair x and y satisfies '$y = f(x)$'. This must be the case for any interpretation, since 'f' must be interpreted as a function that maps each member x of the UD to a member y of the UD.

The argument

$$\frac{(\forall x)\ Pf(x)}{Pf(f(a))}$$

is quantificationally valid. We must show that any interpretation that makes the premise true also makes the conclusion true. If '$(\forall x)\ Pf(x)$' is true, then every member x of the UD is such that f(x) has the property P. Now, f(a) is a member of the UD by our requirements for functor interpretations, so it follows from the universally quantified sentence that $f(f(a))$ must also have the property P, making the conclusion true as well.

The similar argument

$$\frac{(\forall x)\ Pf(f(x))}{Pf(a)}$$

is quantificationally invalid. Here is an interpretation on which the premise is true and the conclusion false:

50. UD: Set of positive integers
Px: x is greater than or equal to 3
f(x): the successor of x
a: 1

For any positive integer x the successor of the successor of x is greater than or equal to 3, but the successor of 1 is 2, which is not greater than or equal to 3.

We may also expand sentences containing functors in order to use truth-tables to check for various properties, although again there will be a complication. We first note that the rules for expanding sentences containing complex terms are the same as the rules for expanding sentences without complex terms. For example, the sentence '$(\forall x)(Px \supset Hf(x))$' is expanded for

the set of constants {'a'} by eliminating the universal quantifier and substituting 'a' for 'x' to obtain

$$Pa \supset Hf(a)$$

and expanding the same sentence for the set of constants {'a', 'b'} results in the conjunction

$$(Pa \supset Hf(a)) \;\&\; (Pb \supset Hf(b))$$

The expansion of the sentence

$$(\forall x)(\exists y)y = g(x) \supset (\forall x)(\exists y)Pf(x,y)$$

for the set of constants {'a'} results in

$$a = g(a) \supset Pf(a,a)$$

To expand the same sentence for the set of constants {'a', 'b'}, we expand the antecedent first to obtain

$$(\exists y)y = g(a) \;\&\; (\exists y)y = g(b)$$

and then to obtain

$$(a = g(a) \vee b = g(a)) \;\&\; (a = g(b) \vee b = g(b))$$

we expand the consequent first to obtain

$$(\exists y)Pf(a,y)) \;\&\; (\exists y)Pf(b,y)$$

and then to obtain

$$(Pf(a,a) \vee Pf(a,b)) \;\&\; (Pf(b,a) \vee Pf(b,b))$$

resulting in the expansion

$$((a = g(a) \vee b = g(a)) \;\&\; (a = g(b) \vee b = g(b))) \supset$$
$$((Pf(a,a) \vee Pf(a,b)) \;\&\; (Pf(b,a) \vee Pf(b,b)))$$

for the entire sentence.

Suppose now that we want to develop a truth-table for the expansion '$Pa \supset Pf(a)$' of the sentence '$(\forall x)(Px \supset Pf(x))$' for the set of constants {'a'}

and that, since 'Pa' and 'Pf(a)' are distinct sentences involving the distinct individual terms 'a' and 'f(a)', we decide that we can assign **T** to the antecedent and **F** to the consequent:

Pa	Pf(a)	Pa ⊃ Pf(a)
		↓
T	**F**	**T F F**

MISTAKE!

Something is wrong here—because the sentence '(∀x)(Px ⊃ Pf(x))' *cannot* be false on any interpretation with a one-member UD. If there is only one member of the UD, then the only candidate for the value of function f applied to that one member is that one member—since we require that the value of a function applied to any member of a UD must be a member of the UD.

Our method of using truth-functional expansions to determine possible truth-values assumes that every member of the UD is named by one of the constants used in the expansion. For this reason we cannot assume that terms containing functors might refer to individuals other than those referred to by the constants used in the expansion. To the contrary, we must assume that each term containing a functor refers to the same individual as at least one constant. Thus in the above example we must assume that 'a' refers to the same individual as 'f(a)', if the expansion is to tell us something about one-member UDs. We will make this explicit in our truth-table: The truth-value assignment must make the sentence 'a = f(a)' true.

Pa	Pf(a)	Pa ⊃ Pf(a)	a = f(a)
		↓	↓
			T

And now our conditions 1 and 2 for truth-tables containing expansions of sentences with the identity predicate must apply. In particular, condition 2 requires that, since 'a = f(a)' is true, the sentences 'Pa' and 'Pf(a)' must have the same truth-value. So the only shortened truth-tables we can obtain are

a = f(a)	Pa	Pf(a)	Pa ⊃ Pf(a)	a = f(a)
			↓	↓
T	**T**	**T**	**T T T**	**T**

and

a = f(a)	Pa	Pf(a)	Pa ⊃ Pf(a)	a = f(a)
			↓	↓
T	**F**	**F**	**F T F**	**T**

The shortened truth-tables show that the sentence '$(\forall x)(Px \supset Pf(x))$' must be true on any interpretation with a one-member UD.

Generalizing, when we construct a truth-table for the truth-functional expansion of a sentence or set of sentences containing functors, the following condition must be met in addition to those for sentences containing the identity predicate:

> 3. For each **n**-place functor f occurring in one or more of the sentences being expanded and each sequence of **n** constants $\mathbf{a}_1, \ldots, \mathbf{a}_n$ from the set of constants $[\mathbf{b}_1, \ldots, \mathbf{b}_m]$ for which the sentence(s) is (are) being expanded, the sentence $(\ldots (f(\mathbf{a}_1, \ldots, \mathbf{a}_n) = \mathbf{b}_1 \vee f(\mathbf{a}_1, \ldots, \mathbf{a}_n) = \mathbf{b}_2) \vee \ldots \vee f(\mathbf{a}_1, \ldots, \mathbf{a}_n) = \mathbf{b}_m)$ must be true.

That is, the value that the function produces when applied to $\mathbf{a}_1, \ldots, \mathbf{a}_n$ must be named by one of the constants in the set of constants for which we are producing an expansion.

Let us now construct a truth-table for the truth-functional expansion of '$(\forall x)(Px \supset Pf(x))$' for the set of constants ['a', 'b']. We begin by adding two sentences to the right of the vertical line in order to satisfy condition 3, and we add the atomic components of those sentences to the left of the vertical line:

$f(a) = a$	$f(a) = b$	$f(b) = a$	$f(b) = b$	Pa	Pb	$Pf(a)$	$Pf(b)$

$(Pa \supset Pf(a))$	&	$(Pb \supset Pf(b))$	$f(a) = a$	∨	$f(a) = b$	$f(b) = a$	∨	$f(b) = b$
				T			**T**	

Let us now assign truth-values to the four identity sentences:

$f(a) = a$	$f(a) = b$	$f(b) = a$	$f(b) = b$	Pa	Pb	$Pf(a)$	$Pf(b)$
T	**F**	**F**	**T**				

$(Pa \supset Pf(a))$	&	$(Pb \supset Pf(b))$	$f(a) = a$	∨	$f(a) = b$	$f(b) = a$	∨	$f(b) = b$
			T	**T**	**F**	**F**	**T**	**T**

By condition 2 for truth-tables for the expansions of sentences containing the identity predicate, 'Pa' and '$Pf(a)$' must have the same truth-value, because we have made '$f(a) = a$' true. And since we have made '$f(b) = b$' true, both 'Pb' and '$Pf(b)$' must have the same truth-value. Here, then, is one way of

completing the assignment of values:

$f(a) = a$	$f(a) = b$	$f(b) = a$	$f(b) = b$	Pa	Pb	$Pf(a)$	$Pf(b)$
T	F	F	T	T	F	T	F

$(Pa \supset Pf(a))$	&	$(Pb \supset Pf(b))$	$f(a) = a$	∨	$f(a) = b$	$f(b) = a$	∨	$f(b) = b$
T T T	T	F T F	T	T	F	F	T	T

And here is another (there are two additional ways, which we won't display):

$f(a) = a$	$f(a) = b$	$f(b) = a$	$f(b) = b$	Pa	Pb	$Pf(a)$	$Pf(b)$
T	F	F	T	F	F	F	F

$(Pa \supset Pf(a))$	&	$(Pb \supset Pf(b))$	$f(a) = a$	∨	$f(a) = b$	$f(b) = a$	∨	$f(b) = b$
F T F	T	F T F	T	T	F	F	T	T

Note that the expansion '$(Pa \supset Pf(a)) \,\&\, (Pb \supset Pf(b))$' had to come out true in both cases, since we have decided that a and $f(a)$ are the same member of the UD and that b and $f(b)$ are the same member of the UD.

Other ways of assigning truth-values to the identity sentences will make the expansion false—for example,

$f(a) = a$	$f(a) = b$	$f(b) = a$	$f(b) = b$	Pa	Pb	$Pf(a)$	$Pf(b)$
F	T	T	F	T	F	F	T

$(Pa \supset Pf(a))$	&	$(Pb \supset Pf(b))$	$f(a) = a$	∨	$f(a) = b$	$f(b) = a$	∨	$f(b) = b$
T F F	F	F T T	F	T	T	T	T	F

We may also choose to make '$f(a) = a$', '$f(a) = b$', '$f(b) = a$', and '$f(b) = b$' all true. In this case we are required also to make the sentence 'a = b' true, because of the former two identities and condition 2; to make '$f(a) = f(b)$' true, because of the latter two identities; and to make '$f(b) = a$' true, by virtue of the truth of '$f(b) = b$' and 'a = b'. As a consequence, 'Pa', 'Pb', '$Pf(a)$', and '$Pf(b)$' must all have the same truth-table, so in this case there are only two distinct shortened truth-tables:

$f(a) = a$	$f(a) = b$	$f(b) = a$	$f(b) = b$	Pa	Pb	$Pf(a)$	$Pf(b)$
T	T	T	T	T	T	T	T

$(Pa \supset Pf(a))$	&	$(Pb \supset Pf(b))$	$f(a) = a$	∨	$f(a) = b$	$f(b) = a$	∨	$f(b) = b$
T T T	T	T T T	T	T	T	T	T	T

and

$f(a) = a$	$f(a) = b$	$f(b) = a$	$f(b) = b$	Pa	Pb	Pf(a)	Pf(b)
T	T	T	T	F	F	F	F

(Pa ⊃ Pf(a))	&	(Pb ⊃ Pf(b))	$f(a) = a$	∨	$f(a) = b$	$f(b) = a$	∨	$f(b) = b$
F T F	T	F T F	T	T	T	T	T	T

The expanded sentence is true in both cases because the truth of 'a = b' means that our UD contains only one member, given the requirement that every member of the UD be named by one of the constants.

As a second and final example, we expand the sentence '(∀x)(∀y)(Dg(f(x), h(y)) ⊃ Dx)' for the set of constants ['a', 'b'] to obtain

((Dg(f(a), h(a)) ⊃ Da) & (Dg(f(a), h(b)) ⊃ Da)) &
((Dg(f(b), h(a)) ⊃ Db) & (Dg(f(b), h(b)) ⊃ Db))

Condition 3 requires us to make all of the following sentences true:

$f(a) = a \vee f(a) = b$
$f(b) = a \vee f(b) = b$
$h(a) = a \vee h(a) = b$
$h(b) = a \vee h(b) = b$
$g(a,a) = a \vee g(a,a) = b$
$g(a,b) = a \vee g(a,b) = b$
$g(b,a) = a \vee g(b,a) = b$
$g(b,b) = a \vee g(b,b) = b$

Let us suppose that we make all of these true by making the following identity sentences true:

1. $f(a) = a$
2. $f(b) = b$
3. $h(a) = b$
4. $h(b) = b$
5. $g(a,a) = a$
6. $g(b,a) = a$
7. $g(a,b) = b$
8. $g(b,b) = b$

and the rest of the atomic identity statements false. By conditions 1 and 2 we will then have the following true identities as well:

9. $g(f(a), h(a)) = g(a,b)$ from $g(a,b) = g(a,b)$ and 1 and 3
10. $g(f(a), h(b)) = g(a,b)$ from $g(a,b) = g(a,b)$ and 1 and 4
11. $g(f(b), h(a)) = g(b,b)$ from $g(b,b) = g(b,b)$ and 2 and 3
12. $g(f(b), h(b)) = g(b,b)$ from $g(b,b) = g(b,b)$ and 2 and 4
13. $g(f(a), h(a)) = g(f(a), h(b))$ from 9 and 10
14. $g(f(b), h(a)) = g(f(b), h(b))$ from 11 and 12

So '$Dg(f(a), h(a))$' and '$Dg(f(a), h(b))$' must have the same truth-value, and '$Dg(f(b), h(a))$' and '$Dg(f(b), h(b))$' must have the same truth-value. Here, then, is one shortened truth-table for the truth-functional expansion reflecting our choice of identities 1–8 and the consequences that follow by condition 2:

$f(a) = a$	$f(a) = b$	$f(b) = a$	$f(b) = b$	$h(a) = a$	$h(a) = b$	$h(b) = a$	$h(b) = b$
T	F	F	T	F	T	F	T

$g(a,a) = a$	$g(a,a) = b$	$g(a,b) = a$	$g(a,b) = b$	$g(b,a) = a$	$g(b,a) = b$	$g(b,b) = a$
T	F	F	T	T	F	F

$g(b,b) = b$	Da	Db	$Dg(f(a), h(a))$	$Dg(f(a), h(b))$	$Dg(f(b), h(a))$	$Dg(f(b), h(b))$
T	T	F	F	F	T	T

$(Dg(f(a), h(a)) \supset Da)$	&	$(Dg(f(a), h(b)) \supset Da)$	&	$(Dg(f(b), h(a)) \supset Db)$	&
F	T	T	F	T	F	F

$(Dg(f(b), h(b)) \supset Db)$	$f(a) = a$	$\vee$	$f(a) = b$	$f(b) = a$	$\vee$	$f(b) = b$
T	F	T	T	F	F	T	T

$h(a) = a$	$\vee$	$h(a) = b$	$h(b) = a$	$\vee$	$h(b) = b$	$g(a,a) = a$	$\vee$	$g(a,a) = b$
F	T	T	F	T	T	T	T	F

$g(a,b) = a$	$\vee$	$g(a,b) = b$	$g(b,a) = a$	$\vee$	$g(b,a) = b$	$g(b,b) = a$	$\vee$	$g(b,b) = b$
F	T	T	T	T	F	F	T	T

This shortened truth-table, albeit not very short, shows that the sentence '$(\forall x)(\forall y)(Dg(f(x), h(y)) \supset Dx)$' is false on at least one interpretation with a one- or two-member UD. There are other shortened truth-tables showing that the sentence is true on at least one interpretation with a one- or two-member UD; and producing one of those will suffice to establish that the sentence is truth-functionally indeterminate.

8.6E EXERCISES

1. Determine the truth-values of the following sentences on this interpretation:

UD: Set of positive integers
Ex: x is even
Gxy: x is greater than y
Ox: x is odd
Pxyz: x plus y equals z

a. $(\exists x)(\forall y)(x = y \supset Gxy)$
*b. $(\forall x)(\forall y) \sim x = y$
c. $(\forall x)(\exists y)(Oy \supset Gyx)$
*d. $(\forall x)(\forall y)(\forall z)[(Gxy \ \& \ Gyz) \supset \sim x = z]$
e. $(\exists w)[Ew \ \& \ (\forall y)(Oy \supset \sim w = y)]$
*f. $(\forall y)(\forall z)[(Oy \ \& \ y = z) \supset \sim Ez]$
g. $(\exists z)(\exists w)(z = w \ \& \ Gzw)$
*h. $(\forall x)(\forall y)(\exists z)[(Pxyz \ \& \sim x = z) \ \& \sim y = z]$
i. $(\forall x)(\forall y)(Pxyy \vee \sim x = y)$

2. Show that each of the following sentences is not quantificationally true by producing an interpretation on which it is false.

a. $(\exists x)(\forall y)x = y$
*b. $(\forall w)(w = b \supset Fw)$
c. $(\forall x)(\forall y)(\forall z)[(x = y \vee y = z) \vee x = z]$
*d. $(\exists w)[Gw \ \& \ (\forall z)(\sim Hzw \supset z = w)]$
e. $(\exists x)(\exists y)(\sim x = y \vee Gxy)$
*f. $(\forall x)(\forall y)(\exists z)(x = y \supset \sim x = z)$

3. Each of the following sentences is quantificationally true. Explain why.

a. $(\forall x)(\forall y)(\forall z)[(x = y \ \& \ y = z) \supset x = z]$
*b. $(\forall x)(\forall y)(\exists z)(x = z \vee y = z)$
c. $(\forall x)(\forall y)[x = y \supset (Gxy \equiv Gyx)]$

4. Show that the sentences in each of the following pairs are not quantificationally equivalent by constructing an interpretation on which one sentence is true and the other is false.

a. $(\forall x)(\exists y)\ x = y$, $(\forall x)(\forall y)x = y$
*b. $(\forall x)(\forall y)[x = y \supset (Fx \equiv Fy)]$, $(\forall x)(\forall y)[(Fx \equiv Fy) \supset x = y]$
c. $(a = b \vee a = c) \supset a = d$, $a = c \supset (a = b \vee a = d)$
*d. $(\exists x)(\forall y)(\sim x = y \supset Gy)$, $(\exists x)(\forall y)(Gy \supset \sim x = y)$

5. Show that each of the following sets of sentences is quantificationally consistent by constructing an interpretation on which each sentence in the set is true.

a. [a = b, a = c, ~ a = d]
*b. [(∀x)(∀y)x = y, (∃x)Fx, (∀y)Gy]
c. [(∃x)(∃z) ~ x = z, (∀x)(∃z)(∃w)(x = z ∨ x = w)]
*d. [(∀x)(Gx ⊃ (∀y)(~ y = x ⊃ Gy)), (∀x)(Hx ⊃ Gx), (∃z)Hz]

6. Establish each of the following by producing an interpretation on which the set members are true and the sentence after the double turnstile is false.
a. [(∀x)(∀y)(∀z)[(x = y ∨ x = z) ∨ y = z]] ⊮ (∀x)(∀y)(x = y)
*b. [(∃w)(∃z) ~ w = z, (∃w)Hw] ⊮ (∃w) ~ Hw
c. [(∃w)(∀y)Gwy, (∃w)(∀y)(~ w = y ⊃ ~ Gwy)] ⊮ (∃z) ~ Gzz
*d. [(∀x)(∀y)[(Fx ≡ Fy) ≡ x = y], (∃z)Fz] ⊮ (∃x)(∃y)[~ x = y & (Fx & ~ Fy)]

7. Using the given symbolization key, symbolize each of the following arguments in *PLE*. Then, for each symbolized argument, decide whether it is quantificationally valid and defend your answer.

UD: Set of all people
Fx: x is female
Mx: x is male
Lxy: x loves y
Pxy: x is a parent of y

a. Every male loves someone other than himself, and every male loves his children. Therefore no male is his own parent.
*b. Everyone loves her or his parents, and everyone has two parents. Therefore everyone loves at least two people.
c. A female who loves her children loves herself as well. Therefore every female loves at least two people.
*d. Everybody has exactly two parents. Therefore everybody has exactly four grandparents.
e. Nobody has three parents. Everybody has more than one parent. Therefore everybody has two parents.

8. Use truth-functional expansions to establish each of the following claims. Be sure that the truth-value assignments you produce meet the first two conditions discussed in this section.
a. The sentence '(∃x)(∃y) ~ x = y' is quantificationally indeterminate.
*b. The sentence '(∀w)(Fw ⊃ (∃y) ~ y = w) & (∃w)Fw' is quantificationally indeterminate.
c. The sentences '(∀y)(∀z)[(Gyz ∨ Gzy) ∨ y = z]' and '(∀y)(∃z)Gyz' are not quantificationally equivalent.
*d. The set of sentences {(∀x)(∀y)(∀z)[(Gxy ∨ Gyz) ∨ x = z], (∀y)(∃z)Gyz} is quantificationally consistent.
e. The set of sentences {(∀y)y = y, (∃z)(∃w) ~ w = z} does not quantificationally entail the sentence '(∃z)(∀w) ~ z = w'.
*f. The argument

$$\frac{(\forall y)(\forall z)(Gyz \supset y = z)}{(\forall y)(\forall z)(y = z \supset Gyz)}$$

is quantificationally invalid.

9. Determine the truth-values of the following sentences on this interpretation:

UD: Set of positive integers
Ex: x is even
Gxy: x is greater than y
f(x): the successor of x
g(x): x squared
h(x,y): the sum of x and y

a. $(\forall x)Gf(x)x$
*b. $(\forall x)Eg(x)$
c. $(\forall x)(\exists y)y = h(x,x)$
*d. $(\forall x)(\forall y)(y = h(x,x) \supset Ey)$
e. $(\exists x)(\exists y)((Ex \ \& \ \sim Ey) \ \& \ Eh(x,y))$
*f. $(\forall x)(\forall y)(\forall z)(Eh(h(x,y), z) \supset ((Ex \vee Ey) \vee Ez))$
g. $(\forall x)(\exists z)(Eh(g(x), z) \vee Eh(x,g(z)))$
*h. $(\forall x)(\forall y)Gh(f(x), f(y)), h(x,y)$

10. Show that each of the following sentences is not quantificationally true by producing an interpretation on which it is false.
a. $(\forall x)(Pf(x) \supset Px)$
*b. $(\forall x)(\forall y)(x = g(y) \vee y = g(x))$
c. $(\exists x)(\forall y)x = g(y)$
*d. $(\forall x)(\forall y)(\forall z)((x = f(y) \ \& \ y = f(z)) \supset x = f(z))$
e. $(\forall x) \sim x = f(x)$
*f. $(\forall x)(\forall y)(Dh(x,y) \supset Dh(y,x))$

11. Each of the following sentences is quantificationally true. Explain why.
a. $(\forall x)(\exists y)y = f(f(x))$
*b. $(\forall x)(\forall y)(\forall z)((y = f(x) \ \& \ z = f(x)) \supset y = z)$
c. $((\forall x)\ Hxf(x) \ \& \ (\forall x)(\forall y)(\forall z)((Hxy \ \& \ Hyz) \supset Hxz)) \supset (\forall x)Hxf(f(x))$

12. Show that the sentences in each of the following pairs are not quantificationally equivalent by constructing an interpretation on which one sentence is true and the other false.
a. $Labf(b)$, $Laf(b)b$
*b. $(\forall x)B(h(x), x)$, $(\forall x)B(x,h(x))$
c. $(\forall x)(\exists y)y = f(h(x))$, $(\exists z)z = f(h(z))$
*d. $(\exists x)(\exists y)(\exists z)(x = f(y) \ \& \ y = f(z))$, $(\forall x)(\exists y)(\exists z)(x = f(y) \ \& \ y = f(z))$

13. Show that each of the following sets of sentences is quantificationally consistent by constructing an interpretation on which each sentence in the set is true.
a. $\{a = f(b), b = f(c), c = f(a)\}$
*b. $\{(\forall x)Lxf(x), (\exists y) \sim Lf(y)y\}$
c. $\{(\exists x)(\forall y)x = f(y), (\exists x)(\forall y) \sim x = f(y)\}$
*d. $\{(\forall x)(Gx \supset \sim Gh(x)), (\exists x)(\sim Gx \ \& \ \sim Gh(x))\}$

14. For each of the following arguments, decide whether it is quantificationally valid. If it is quantificationally valid, explain why. If it is not quantificationally valid, construct an interpretation on which the premises are true and the conclusion false.

a. $\dfrac{(\forall x)(Fx \vee Fg(x))}{(\forall x)(Fx \vee Fg(g(x)))}$

*b. $\dfrac{(\forall x)(Fx \vee Fg(x))}{(\forall x)(Fg(x) \vee Fg(g(x)))}$

c. $\dfrac{(\forall x)(\exists y)(\exists z)Lf(x)yz}{(\exists x)(\forall y)(\forall z)Lxf(y)f(z)}$

*d. $\dfrac{(\forall x)(Lxf(x) \,\&\, \sim Lf(x)x)}{(\forall x)(\forall y)(y = f(x) \supset (Lxy \vee Lyx))}$

e. $\dfrac{(\forall x)(Bg(x) \supset (\forall y) \sim Hyg(x))}{(a = g(b) \,\&\, Hca) \supset \sim Ba}$

15. Use truth-functional expansions to establish each of the following. Be sure that the truth-value assignments you produce meet all three conditions discussed in this section.

a. The sentence '$(\forall x)(Fx \vee Fg(x))$' is quantificationally indeterminate.

*b. The sentences '$(\exists x)(\exists y)Hg(x,y)x$' and '$(\exists x)(\exists y)Hg(y,x)x$' are not quantificationally equivalent.

c. The set of sentences $\{(\forall x) \sim x = f(x), (\exists x)x = f(f(x))\}$ is quantificationally consistent.

*d. The argument

$$\frac{a = f(b) \,\&\, b = f(a)}{(\exists x)(\exists y) \sim x = y}$$

is quantificationally invalid.

8.7 FORMAL SEMANTICS OF *PL* AND *PLE*

The semantics for *PL* and *PLE* used so far in this chapter have been informal in the sense that strict definitions of truth on an interpretation and falsehood on an interpretation have not been specified. For example, we explained the truth-conditions for quantified sentences by saying that all or some of the members of the UD in question must satisfy the condition specified by that part of the sentence following the quantifier. But we have not as yet given a formalized account of *how* we determine what that condition is or how we determine whether it is satisfied as required. In this section we specify the formal semantics first for *PL* and then for *PLE*.

To do so, we first need to regiment the definition of an interpretation. Every interpretation must have a nonempty set as its UD. Interpreting

an individual constant consists in the assignment of a member of the UD to that constant, and interpreting a sentence letter consists in the assignment of a truth-value to that sentence letter.

In the first section of this chapter, we pointed out that, when we give an English reading for an **n**-place predicate of *PL*, that reading determines the *extension* of the predicate. The extension of an **n**-place predicate is a set of **n**-*tuples* of members of the UD that are picked out by the predicate. An **n**-tuple is an ordered set containing **n** members—it is ordered in the sense that one member is designated as the first, one as the second, and so on. A 2-tuple is an ordered pair, a 3-tuple is an ordered triple, a 4-tuple is an ordered quadruple, and so on.

For instance, if we take the set of positive integers as our UD and interpret 'Gxy' as 'x is greater than y', then the *extension* of 'G' is the set of ordered pairs (2-tuples) of positive integers such that the first member of each pair is greater than the second. A sentence containing the predicate 'G'—say, the sentence 'Gab'—is then true if and only if the ordered pair of positive integers whose first member is designated by 'a' and whose second member is designated by 'b' is a member of the extension of 'G'. It is possible to give 'G' a different English reading that determines the same extension; for example, we might interpret 'Gxy' to mean 'x plus 1 is greater than y plus 1'. Since, for any numbers x and y, x plus 1 is greater than y plus 1 if and only if x is greater than y, it follows that the extension of 'G' for these two English readings of the predicate is the same. And, since the *extension* is the same in each case, the truth-conditions of 'Gab' on the latter interpretation of 'G' coincide with the truth-conditions of 'Gab' on the former interpretation. It is thus the *extension* of the predicate, not the particular English reading we use to specify the extension, that is important in determining the truth-conditions of any sentence in which the predicate occurs. So, in our formal account, we take the interpretation of a predicate of *PL* to be the set that is the extension of that predicate.[3]

An interpretation of an **n**-place predicate in a UD thus consists in the assignment of a set of **n**-tuples (ordered sets containing **n** members) of members of the UD to that predicate. For example, in the UD that consists of all people, years, and cities, we may wish to interpret the four-place predicate 'D' so that 'Dwxyz' reads 'w marries x in the year y in the city z'. We would then have the interpretation assign to 'D' the set of 4-tuples of members of the UD such that the first two members are people who marry each other, the third member is the year in which they marry, and the fourth member is the city in which they marry. Thus, if John Doe and Jane Doe marry in 1975 in Kansas City, then one of the 4-tuples that the interpretation assigns to 'D' is

(John Doe, Jane Doe, 1975, Kansas City)

[3] And when we assign an extension, which is simply a set, to a predicate, we must not lose our natural English reading in mind. It is important to realize this. We when we say, for example, that a quantificationally true sentence is true on every interpretation, we include those interpretations that have no obvious English language renderings.

(the **n**-tuple whose first, second, third, and fourth members, respectively, are John Doe, Jane Doe, 1975, Kansas City).

Note that we designate an **n**-tuple by listing names of the members of the **n**-tuple, in the order in which the members occur in the **n**-tuple, between the angle brackets '⟨' and '⟩'. Thus we may designate the 3-tuple whose first member is the number 1, whose second member is Arthur Conan Doyle, and whose third member is Yankee Stadium as

⟨1, Arthur Conan Doyle, Yankee Stadium⟩

and the 5-tuple all of whose members are the fraction $\frac{1}{2}$ as

$\langle \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \rangle$

A special case of the interpretation of predicates deserves attention here. The interpretation of a one-place predicate assigns to that predicate a set of 1-tuples of members of the UD. A 1-tuple is an ordered set containing exactly one member. It might seem more natural simply to assign to a one-place predicate a set of members of the UD—namely, those members we want the predicate to pick out. But, for the sake of generality in our definition of truth on an interpretation, it is convenient to assign to a one-place predicate a set of 1-tuples of members of the UD. We may designate the 1-tuple of John Doe by

⟨John Doe⟩

We may now formally define the concept of an interpretation:

> An *interpretation* for *PL* consists in the specification of a UD and the assignment of a truth-value to each sentence letter of *PL*, a member of the UD to each individual constant of *PL*, and a set of **n**-tuples of members of the UD to each **n**-place predicate of *PL*.

The next concept we need is that of a variable assignment. This concept plays an important role in the specification of truth-conditions for the sentences of *PL*, and the idea is this: We are going to explain the truth-conditions of sentences like '(∀x)(Fx ∨ Gx)' in terms of the semantics of their subformulas. But 'Fx ∨ Gx' is an open sentence, and for us open sentences are neither true nor false—because free variables are not names. We have noted several times that variables function as pronouns do. Here we exploit that feature of variables in order to determine the truth-conditions of quantified sentences. That is, we determine the truth-conditions of quantified sentences by exploring whether the things to which a variable in its role as pronoun can refer satisfy the condition specified by the formula in which the variable occurs. Our concept of variable assignments will be used to regiment the informal notion of satisfaction that we have used throughout this chapter.

A *variable assignment for an interpretation* **I** assigns to each individual variable of *PL* a member of the UD. Intuitively a variable assignment captures one way in which the variables of *PL*, in their role as pronouns, can refer to objects in the UD. If an interpretation has a one-member UD, there is exactly one variable assignment for that interpretation, the variable assignment that assigns the one member of the UD to each individual variable of *PL*. If an interpretation has more than one member in its UD, there will be infinitely many different variable assignments. For a two-member UD consisting of the integers 1 and 2, for example, there is a variable assignment that assigns the integer 1 to every individual variable, and a variable assignment that assigns the integer 2 to every individual variable, and there are infinitely many variable assignments that assign the integer 1 to only some of the individual variables and the integer 2 to the remaining variables (there are infinitely many ways of choosing which of the infinite number of variables of *PL* designate 1 and which designate 2). Note that it is not required that distinct variables be assigned different members of the UD; some variable assignments assign the same member to two or more variables—in fact, every variable assignment for an interpretation with a finite UD must do so. Nor is it required that every member of the UD be assigned to a variable; some variable assignments leave some members unnamed.

We will use the letter '**d**' to range over variable assignments (think of '**d**' as shorthand for 'designates'). If **d** is a variable assignment and **x** is an individual variable of *PL*, then **d**(**x**) designates the member of the UD that **d** assigns to **x**. So **d**(**x**) is 2 if and only if **d** assigns 2 to **x**. We need some additional notation that will be used in regimenting our informal notion of satisfaction. If **d** is a variable assignment for an interpretation, **u** is a member of the interpretation's UD, and **x** is an individual variable of *PL*, then **d**[**u**/**x**] is a variable assignment that assigns the same value to each variable as **d** does *except that* it assigns **u** to **x**. **d**[**u**/**x**] is called a *variant* of the assignment **d**. More generally, if **d** is a variable assignment for an interpretation, $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ are (not necessarily distinct) members of the interpretation's UD, and $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are (not necessarily distinct) individual variables of *PL*, then $\mathbf{d}[\mathbf{u}_1/\mathbf{x}_1, \mathbf{u}_2/\mathbf{x}_2, \ldots, \mathbf{u}_n/\mathbf{x}_n]$ is shorthand for the variable assignment $\mathbf{d}[\mathbf{u}_1/\mathbf{x}_1][\mathbf{u}_2/\mathbf{x}_2]\ldots[\mathbf{u}_n/\mathbf{x}_n]$—the variable assignment that starts out like **d** and results from successive stipulations that $\mathbf{u}_1$ will be assigned to $\mathbf{x}_1$, $\mathbf{u}_2$ to $\mathbf{x}_2$, . . . , and $\mathbf{u}_n$ to $\mathbf{x}_n$.

As an example let us assume that an interpretation has the set of positive integers as its UD and that the variable assignment **d** assigns 1 to every individual variable of *PL*. Then **d**[5/'x', 8/'z'] is the variable assignment that assigns 5 to 'x' and 8 to 'z' and assigns 1 to all other individual variables of *PL*. It assigns 1 to all other individual variables because, aside from the assignments it makes to 'x' and 'z', our definition requires that **d**[5/'x', 8/'z'] assign the same values to variables as **d** does. Note that **d**[1/'x', 1/'y'] is the same variable assignment as **d** because the values that we specified for the variables 'x' and 'y' are the same values that **d** assigns to them. Also note that if a variable occurs more than once between the square brackets, the value it receives on

$\mathbf{d}[\mathbf{u}_1/\mathbf{x}_1, \mathbf{u}_2/\mathbf{x}_2, \ldots, \mathbf{u}_n/\mathbf{x}_n]$ is the last value that appears for the variable in that list. For example, where **d** is as above, $\mathbf{d}[1/\text{'x'}, 2/\text{'y'}, 3/\text{'x'}]$ assigns 3, not 1, to 'x'. For notational convenience we shall drop the single quotes around names of individual variables when they appear between the square brackets; thus $\mathbf{d}[5/\text{'x'}, 8/\text{'z'}]$ may be written as $\mathbf{d}[5/\text{x}, 8/\text{z}]$.

Relative to an interpretation and variable assignment, we define the *denotation of a term with respect to an interpretation* **I** *and variable assignment* **d**, symbolically $den_{\mathbf{I},\mathbf{d}}(\mathbf{t})$:

1. If **t** is a variable, then $\text{den}_{\mathbf{I},\mathbf{d}}(\mathbf{t}) = \mathbf{d}(\mathbf{t})$.
2. If **t** is an individual constant, then $\text{den}_{\mathbf{I},\mathbf{d}}(\mathbf{t}) = \mathbf{I}(\mathbf{t})$.

The denotation of a term is the member of the UD that the variable assignment or interpretation says it designates.

Truth and falsehood of sentences on an interpretation are not defined directly; rather, they are defined in terms of *satisfaction* by variable assignments. We shall first recursively define the concept of satisfaction, then define truth and falsehood, and afterward illustrate through examples the role of the intermediate step. Here, as in Chapter 7, we use '**P**', '**Q**' and '**R**' as metavariables ranging over formulas of *PL*, '**A**' as a metavariable ranging over predicates of *PL*, '**t**' with or without subscripts as a metavariable ranging over individual terms (individual constants and individual variables of *PL*) and '**x**' as a metavariable ranging over individual variables of *PL*. We shall use $\mathbf{I}(\mathbf{X})$ to mean the value that the interpretation **I** assigns to the symbol **X**.

Let **I** be an interpretation, **d** a variable assignment for **I**, and **P** a formula of *PL*. Then

1. If **P** is a sentence letter, then **d** satisfies **P** on interpretation **I** if and only if $\mathbf{I}(\mathbf{P}) = \mathbf{T}$.

Note that the values that **d** assigns to variables play no role in this case.

2. If **P** is an atomic formula of the form $\mathbf{At}_1 \ldots \mathbf{t}_n$ (where **A** is an **n**-place predicate), then **d** satisfies **P** on interpretation **I** if and only if $<\text{den}_{\mathbf{I},\mathbf{d}}(\mathbf{t}_1), \text{den}_{\mathbf{I},\mathbf{d}}(\mathbf{t}_2), \ldots, \text{den}_{\mathbf{I},\mathbf{d}}(\mathbf{t}_n)>$ is a member of $\mathbf{I}(\mathbf{A})$.

So **d** satisfies 'Gxa', for example, if $\langle \mathbf{d}(\text{'x'}), \mathbf{I}(\text{'a'})\rangle$—the 2-tuple whose first member is the object that **d** assigns to the variable 'x' and whose second member is the object that **I** assigns to the constant 'a'—is a member of the extension that **I** assigns to 'G'.

3. If **P** is of the form $\sim \mathbf{Q}$, then **d** satisfies **P** on interpretation **I** if and only if **d** does not satisfy **Q** on interpretation **I**.

4. If **P** is of the form **Q** & **R**, then **d** satisfies **P** on interpretation **I** if and only if **d** satisfies **Q** on interpretation **I** and **d** satisfies **R** on interpretation **I**.
5. If **P** is of the form **Q** ∨ **R**, then **d** satisfies **P** on interpretation **I** if and only if either **d** satisfies **Q** on interpretation **I** or **d** satisfies **R** on interpretation **I**.
6. If **P** is of the form **Q** ⊃ **R**, then **d** satisfies **P** on interpretation **I** if and only if either **d** does not satisfy **Q** on interpretation **I** or **d** satisfies **R** on interpretation **I**.
7. If **P** is of the form **Q** ≡ **R**, then **d** satisfies **P** on interpretation **I** if and only if either **d** satisfies **Q** on interpretation **I** and **d** satisfies **R** on interpretation **I**, or **d** does not satisfy **Q** on interpretation **I** and **d** does not satisfy **R** on interpretation **I**.
8. If **P** is of the form (∀**x**)**Q**, then **d** satisfies **P** on interpretation **I** if and only if for every member **u** of the UD, **d**[**u**/**x**] satisfies **Q** on interpretation **I**.
9. If **P** is of the form (∃**x**)**Q**, then **d** satisfies **P** on interpretation **I** if and only if there is at least one member **u** of the UD such that **d**[**u**/**x**] satisfies **Q** on interpretation **I**.

Finally, the definitions of truth and falsehood are

> A sentence **P** of *PL* is *true on an interpretation* **I** if and only if every variable assignment **d** (for **I**) satisfies **P** on **I**. A sentence **P** of *PL* is *false on an interpretation* **I** if and only if no variable assignment **d** (for **I**) satisfies **P** on **I**.

In Chapter 11 we shall prove that for each sentence **P** and interpretation **I** either all variable assignments for **I** satisfy **P** or none do. (This is not generally true for *open* formulas.) This being the case, each sentence is true or false on any interpretation according to our definitions.

Our definitions have been long, so we shall look at some examples. Consider the sentence

(∀y)(By ⊃ ~ (∃z)Dyz)

and an interpretation that makes the following assignments:

51. UD: Set of positive integers
    B: [<**u**>: **u** is prime]
    D: [<$\mathbf{u}_1$, $\mathbf{u}_2$>: $\mathbf{u}_1$ is greater than $\mathbf{u}_2$]

Note that we no longer write variables after predicates when indicating their interpretations. This is because an interpretation makes an assignment to the

predicate alone. Note also that we explicitly display the sets that are the interpretations of predicates. The notation used in displaying the interpretation of 'B' means: the set of 1-tuples of prime numbers. Because it would be cumbersome, we do not explicitly indicate that the prime numbers must be members of the UD—that is, the prime numbers in this set are all positive integers. The notation used in displaying the interpretation of 'D' means: the set of 2-tuples in which the first member is greater than the second. Again it is implicit that these 2-tuples contain only positive integers.

The interpretation must assign values to all other predicates, individual constants, and sentence letters of *PL* as well, but as we shall see, only the values assigned to 'B' and 'D' are used in determining the truth-value of the sentence. As we apply our definitions, it may be helpful to keep in mind the reading of the sentence on this interpretation: 'Every positive integer y is such that if it is prime then there is no positive integer than which it is greater'. The sentence is obviously false on this interpretation; what we shall now show is that we come to exactly this conclusion by using our definitions.

To show that the sentence is false, we must show that no variable assignment **d** (for **I**) satisfies the sentence. Let **d** be any variable assignment for **I**. According to clause 8 of the definition of satisfaction,

a. **d** satisfies '$(\forall y)(By \supset \sim (\exists z)Dyz)$' if and only if for every member **u** of the UD, **d**[**u**/y] satisfies '$(By \supset \sim (\exists z)Dyz)$'.

So **d**[1/y] must satisfy the open sentence, **d**[2/y] must satisfy the open sentence, **d**[3/y] must satisfy the open sentence, and so on for every positive integer. Intuitively this means that no matter what we may take y to be, y must satisfy the condition specified by that open sentence. We must consider all possible values that might be assigned to 'y' rather than just the value that **d** itself assigns to 'y'—that is, all values that variants of **d** might assign to 'y'—because the variable 'y' is universally quantified.

But not every member **u** of the UD is such that **d**[**u**/y] satisfies '$(By \supset \sim (\exists z)Dyz)$'—for example, 2 is not such that if it is prime then there is no positive integer than which it is greater. We shall show formally that **d**[2/y] does not satisfy '$(By \supset \sim (\exists z)Dyz)$'. According to clause 6,

b. **d**[2/y] satisfies '$(By \supset \sim (\exists z)Dyz)$' if and only if either **d**[2/y] does not satisfy 'By' or **d**[2/y] does satisfy '$\sim (\exists z)Dyz$'.

According to clause 2,

c. **d**[2/y] satisfies 'By' if and only if $\langle$**d**[2/y](y)$\rangle$ is a member of **I**(B).

The 1-tuple $\langle$**d**[2/y](y)$\rangle$ is $\langle 2 \rangle$—the set consisting of the integer that the variant assignment **d**[2/y] assigns to 'y', and **I**(B) is the set of 1-tuples of positive integers

that are prime. Because ⟨2⟩ is a member of this set, **d**[2/y] does satisfy 'By'. We now turn to '~ (∃z)Dyz'. According to clause 3,

d. **d**[2/y] satisfies '~ (∃z)Dyz' if and only if it does not satisfy '(∃z)Dyz'.

And clause 9 tells us that

e. **d**[2/y] satisfies '(∃z)Dyz' if and only if there is at least one member **u** of the UD such that **d**[2/y, **u**/z], the variable assignment that is just like **d**[2/y] except that it assigns **u** to 'z', satisfies 'Dyz'

that is, if and only if there is a member **u** of the UD that is smaller than 2. There is such a member—the integer 1. Let us consider the variable assignment **d**[2/y, 1/z]. By clause 2,

f. **d**[2/y, 1/z] satisfies 'Dyz' if and only if the 2-tuple ⟨2, 1⟩ (this is the pair ⟨**d**[2/y, 1/z](y), **d**[2/y, 1/z](z)⟩) is a member of **I**(D).

I(D) is the set of 2-tuples of positive integers in which the first member is greater than the second—and ⟨2, 1⟩ is indeed a member of this set. Thus **d**[2/y, 1/z] satisfies 'Dyz'. Returning to step e, this shows that **d**[2/y] satisfies '(∃z)Dyz', and so, by step d, **d**[2/y] does not satisfy '~ (∃z)Dyz'. We now have a variable assignment, **d**[2/y], that satisfies 'By' but does not satisfy '~ (∃z)Dyz'. Therefore, by step b, **d**[2/y] does not satisfy '(By ⊃ ~ (∃z)Dyz)'. The number 2 is *not* such that if it is prime then it is not larger than any positive integer. We may conclude from step a that **d** does not satisfy '(∀y)(By ⊃ ~ (∃z)Dyz)', because we have found a variant of **d** that does not satisfy the open sentence following the universal quantifier.

We may also conclude that the sentence is *false* on interpretation 51 because we have just shown that every variable assignment for **I** fails to satisfy '(∀y)(By ⊃ ~ (∃z)Dyz)'. The value that the original assignment **d** assigned to 'y' did not come into play in our proof, for when we removed the universal quantifier in step a we considered all values that might be assigned to 'y' by variants of **d**. The universally quantified sentence is satisfied by a variable assignment **d** if and only if the open sentence following the quantifier is satisfied no matter what value is assigned to 'y'. Similarly, when we removed the existential quantifier in step e, we considered all values that might be assigned to 'z'—not just the value assigned by **d** or its variant **d**[2/y]. The values that **d** itself assigned to the variables 'y' and 'z' therefore played no role in showing that **d** did not satisfy the sentence. Moreover, because no other individual variables appear in the sentence, the values that **d** assigned to the other individual variables of *PI*. also played no role. In sum, it does not matter which variable assignment we started with because, when we removed the quantifiers, we had to consider variants that explicitly assigned values to the variables thus freed.

We conclude that, no matter which variable assignment **d** we choose, **d** will fail to satisfy the sentence.

The sentence '$(\forall y)(By \supset \sim (\exists z)Dyz)$' is true, however, on interpretation 52:

52. UD: The set {2, 4}
B: {<**u**>: **u** is prime}
D: {<$\mathbf{u}_1$, $\mathbf{u}_2$>: $\mathbf{u}_1$ is greater than $\mathbf{u}_2$}

Consider any variable assignment **d** for this interpretation. By clause 8, **d** satisfies the sentence if and only if, for every member **u** of the UD, **d**[**u**/y] satisfies '$(By \supset \sim (\exists z)Dyz)$'—that is, if and only if both **d**[2/y] and **d**[4/y] satisfy the open sentence, because 2 and 4 are the only members of the UD. We shall examine each variant. **d**[2/y] satisfies the consequent of the open sentence (in addition to its antecedent) because it fails to satisfy '$(\exists z)Dyz$'. There is no member **u** of the set {2, 4} such that **d**[2/y, **u**/z] satisfies 'Dyz'. **d**[2/y, 2/z] does not satisfy 'Dyz' because $\langle \mathbf{d}[2/y, 2/z](y), \mathbf{d}[2/y, 2/z](z)\rangle$, which is ⟨2, 2⟩, is not in the extension of 'D'. **d**[2/y, 4/z] does not satisfy 'Dyz' because ⟨2, 4⟩ is not in the extension of 'D'. We have exhausted the possible values for 'z', so there is no member **u** of the UD such that **d**[2/y, **u**/z] satisfies 'Dyz'. Hence '$(\exists z)Dyz$' is not satisfied by **d**[2/y], and its negation is satisfied. This established that **d**[2/y] satisfies '$(By \supset \sim (\exists z)Dyz)$'. The variant **d**[4/y] also satisfies '$(By \supset \sim (\exists z)Dyz)$' because it fails to satisfy the antecedent (it also fails to satisfy the consequent). The 1-tuple $\langle \mathbf{d}[4/y](y)\rangle$ is just ⟨4⟩, and this 1-tuple is not a member of **I**(B), which is the set of 1-tuples of prime positive integers. So **d**[4/y] fails to satisfy 'By' and therefore does satisfy '$(By \supset \sim (\exists z)Dyz)$'. Having shown that both **d**[2/y] and **d**[4/y] satisfy '$(By \supset \sim (\exists z)Dyz)$', we have established that, for every member **u** of the UD, **d**[**u**/y] satisfies the conditional open sentence, and we may conclude that **d** satisfies '$(\forall y)(By \supset \sim (\exists z)Dyz)$'. Once again, because we used no specific values assigned to variables by our original variable assignment **d**, we have shown that every variable assignment for **I** satisfies the sentence. It is therefore true on interpretation 52.

As another example, we may use our definitions to prove that the sentence '$(\forall x)(Bx \equiv \sim Bx)$' is quantificationally false. We will begin by assuming that there is an interpretation on which the sentence is true and then show that this is impossible. Suppose that **I** is an interpretation on which '$(\forall x)(Bx \equiv \sim Bx)$' is true. By definition every variable assignment **d** for **I** must therefore satisfy '$(\forall x)(Bx \equiv \sim Bx)$'. And because the sentence is universally quantified, for every member **u** of **I**'s UD, the variant **d**[**u**/x] for each variable assignment **d** must satisfy '$(Bx \equiv \sim Bx)$' (by clause 8). We shall show that not even one variant of one variable assignment can do so. Suppose that **u** is a member of the UD such that ⟨**u**⟩ is in the extension of 'B'. Then, no matter which variable assignment **d** we started with, **d**[**u**/x] satisfies 'Bx' according to clause 2, because $\langle \mathbf{d}[\mathbf{u}/x](x)\rangle$ is a member of **I**(B). But then, by clause 3, **d**[**u**/x] does not satisfy '$\sim$ Bx', and hence, by clause 7, it does not satisfy '$(Bx \equiv \sim Bx)$'. Suppose, on the other hand, that **u** is a member of the UD such that ⟨**u**⟩ is not in

the extension of 'B'. In this case $\mathbf{d}[\mathbf{u}/x]$ does not satisfy 'Bx' because $\langle \mathbf{d}[\mathbf{u}/x](x) \rangle$ is not a member of $\mathbf{I}(B)$ and therefore does satisfy '~ Bx'. So, once again, '(Bx ≡ – Bx)' is not satisfied. Because each member of the UD either is or is not in the extension of 'B', we conclude that there is no member $\mathbf{u}$ such that $\mathbf{d}[\mathbf{u}/x]$ satisfies '(Bx ≡ – Bx)'. This being the case, no variable assignment for any interpretation $\mathbf{I}$ satisfies '$(\forall x)$(Bx ≡ – Bx)'. The sentence is false on every interpretation and is therefore quantificationally false.

It should now be clear why the definition of truth was given in terms of the concept of satisfaction rather than directly in terms of interpretations. For the language of *SL* we were able to define the truth-conditions of a sentence directly in terms of the truth-conditions of its atomic components, for the atomic components were themselves sentences and so had truth-values on truth-value assignments. But consider the sentence of *PL* '$(\forall y)$(By ⊃ – $(\exists z)$Dyz)', which we used as an example in this section. There is no proper subformula of this sentence that is itself a sentence. The largest proper subformula is '(By ⊃ – $(\exists z)$Dyz)', which is an *open sentence*. We do not consider open sentences to be true or false on interpretations because their free variables are not given values on interpretations. So we cannot define the truth-conditions of '$(\forall y)$(By ⊃ – $(\exists z)$Dyz)' in terms of the truth or falsehood of its subformula '(By ⊃ – $(\exists z)$Dyz)'. But variable assignments, which do assign values to variables, can satisfy or fail to satisfy open formulas for interpretations, and so we have defined the truth-conditions of sentences in terms of the satisfaction conditions of their subformulas.[5]

To accommodate sentences of *PLE* containing the identity predicate, we need to add the following clause to our definition of satisfaction:

10. If $\mathbf{P}$ is an atomic formula of the form $\mathbf{t}_1 = \mathbf{t}_2$, then $\mathbf{d}$ satisfies $\mathbf{P}$ on interpretation $\mathbf{I}$ if and only if $\text{den}_{\mathbf{I},\mathbf{d}}(\mathbf{t}_1) = \text{den}_{\mathbf{I},\mathbf{d}}(\mathbf{t}_2)$.

This clause implicitly defines an extension for the identity predicate: The extension includes $\langle \mathbf{u}, \mathbf{u} \rangle$ for each member $\mathbf{u}$ of the $\text{UD}_{\mathbf{I}}$ and that is all that it includes. We can use clause 10, along with others, to show that the sentence

$$(\exists x)(Fx \,\&\, (\forall y)(Fy \supset y = x))$$

is false on interpretation 53:

53. UD: Set of positive integers
F: {<**u**>: **u** is odd}

[5]Some authors allow all open sentences to be true or false on interpretations, so the concept of satisfaction is not needed in their truth-definitions. Other authors use a type of semantics for quantificational languages known as substitution semantics; in this type of semantics the concept of satisfaction is also unnecessary. For obvious reasons the semantics we have presented is known as *satisfaction semantics* (or sometimes as *referential* or *objectual semantics*). Satisfaction semantics was first presented by Alfred Tarski in "Der Wahrheitsbegriff in den formalisierten Sprachen," *Studia Philosophica*, 1 (1936), 261–405.

On this interpretation, the sentence may be read as 'Exactly one positive integer is odd'. To show that the sentence is false, we must show that no variable assignment for **I** satisfies it. Let **d** be a variable assignment. Then, by clause 9, **d** satisfies the sentence if and only if there is some member **u** of the UD such that **d**[**u**/x] satisfies '(Fx & (∀y)(Fy ⊃ y = x))'. Because at least one positive integer is odd, there is at least one member **u** such that **d**[**u**/x] does satisfy the first conjunct 'Fx'—for example, ⟨**d**[3/x](x)⟩, which is ⟨3⟩, is a member of **I**(F). However, no matter which odd integer **u** may be, **d**[**u**/x] cannot satisfy '(∀y)(Fy ⊃ y = x)'. If **d**[**u**/x] *did* satisfy this formula, then, according to clause 8, it would do so because, for every positive integer $\mathbf{u}_1$, **d**[**u**/x, $\mathbf{u}_1$/y] satisfies '(Fy ⊃ y = x)'. But for any odd positive integer **u**, there is at least one positive integer $\mathbf{u}_1$ such that **d**[**u**/x, $\mathbf{u}_1$/y] does *not* satisfy '(Fy ⊃ y = x)'—let $\mathbf{u}_1$ be any odd integer other than **u**. Here **d**[**u**/x, $\mathbf{u}_1$/y] satisfies 'Fy' because ⟨$\mathbf{u}_1$⟩ is in the extension of 'F', but according to clause 10 it does not satisfy 'y = x' because **d**[**u**/x, $\mathbf{u}_1$/y](x)—which is **u**—and **d**[**u**/x, $\mathbf{u}_1$/y](y)—which is $\mathbf{u}_1$—are different integers. It follows that no positive integer **u** is such that **d**[**u**/x] satisfies '(∀y)(Fy ⊃ y = x)', and so **d** does not satisfy the existentially quantified sentence '(∃x)(∀y)(Fy ⊃ y = x)'. The sentence is therefore false on interpretation 53.

Finally we shall complete our semantics for sentences of *PLE* that contain functors by amending our definition of an interpretation and of the denotation of a term with respect to an interpretation **I** and variable assignment **d**. We first define precisely the concept of an **n**-*place function on a UD*: An **n**-place function on a UD maps each **n**-tuple of members of the UD to a single member of the UD (not necessarily the same one in each case). So, if the UD consists of the integers 1 and 2, for example, there are four distinct 1-place functions: the function that maps each of 1 and 2 to itself, the function that maps both 1 and 2 to 1, the function that maps both 1 and 2 to 2, and the function that maps 1 to 2 and 2 to 1. We can represent each of these functions as a set of ordered pairs, where for each member **u** of the UD there is exactly one ordered pair with **u** as its first member, and the second member of that ordered pair is the value of the function for **u**. Here are the four sets of ordered pairs:

{<1, 1>, <2, 2>} (the function that maps each of 1 and 2 to itself)
{<1, 1>, <2, 1>} (the function that maps both 1 and 2 to 1)
{<1, 2>, <2, 2>} (the function that maps both 1 and 2 to 2)
{<1, 2>, <2, 1>} (the function that maps 1 to 2 and 2 to 1)

Similarly a 2-place function can be represented as a set of ordered triples, where for each pair of members $\mathbf{u}_1$ and $\mathbf{u}_2$ of the UD there is one ordered triple with $\mathbf{u}_1$ and $\mathbf{u}_2$ as the first two members, and the third member of that ordered triple is the value of the function for $\mathbf{u}_1$ and $\mathbf{u}_2$. So the multiplication function for the set of integers {0, 1} can be represented as the set {<0,0,0>, <0,1,0>, <1,0,0>, <1,1,1>}. (Note that multiplication is not a function on the set {1, 2} because we require that the value of the function for each pair of members of

the UD itself be a member of the UD, but 2×2 is not a member of {1, 2}. On the other hand, multiplication *is* a function on the set of positive integers.)

We may now amend our definition of an interpretation:

> An interpretation for *PLE* consists in the specification of a UD and the assignment of a truth-value to each sentence letter of *PLE*, a member of the UD to each individual constant of *PLE*, an **n**-place function on the UD to each **n**-place functor of *PLE*, and a set of **n**-tuples of members of the UD to each **n**-place predicate of *PLE*.

We extend the definition of the *denotation of a term with respect to an interpretation* $\mathcal{I}$ *and variable assignment* $\mathbf{d}$ as follows, to accommodate terms containing functors:

1. If $\mathbf{t}$ is a variable, then $\text{den}_{\mathcal{I},\mathbf{d}}(\mathbf{t}) = \mathbf{d}(\mathbf{t})$.
2. If $\mathbf{t}$ is an individual constant, then $\text{den}_{\mathcal{I},\mathbf{d}}(\mathbf{t}) = \mathbf{I}(\mathbf{t})$.
3. If $\mathbf{t}$ is a term $f(\mathbf{t}_1, \ldots, \mathbf{t}_n)$ where f is an **n**-place functor and $\mathbf{t}_1, \ldots, \mathbf{t}_n$ are terms, then if $\langle \text{den}_{\mathcal{I},\mathbf{d}}(\mathbf{t}_1), \ldots, \text{den}_{\mathcal{I},\mathbf{d}}(\mathbf{t}_n), \mathbf{u} \rangle$ is a member of $\mathbf{I}(f)$, $\text{den}_{\mathcal{I},\mathbf{d}}(\mathbf{t}) = \mathbf{u}$.

Recall that for any members $\mathbf{u}_1, \ldots, \mathbf{u}_n$ of the UD, there will be a unique **n**-tuple $\langle \mathbf{u}_1, \ldots, \mathbf{u}_n, \mathbf{u} \rangle$ that is a member of the function $\mathbf{I}(f)$, so clause 3 identifies exactly one member of the UD as $\text{den}_{\mathcal{I},\mathbf{d}}(\mathbf{t})$.

Using our new definitions, we can show that the sentence

$$(\forall x)(Gxa \supset Gf(x)a)$$

is true on interpretation 54:

54. UD: Set of positive integers
Gxy: x is greater than y
$f(x)$: the successor of x
a: 5

On this interpretation the sentence may be read as 'The successor of any positive integer that is greater than 5 is itself greater than 5'. To show that the sentence is true, we must show that it is satisfied by every variable assignment. Let $\mathbf{d}$ be a variable assignment. Then, by clause 8, $\mathbf{d}$ satisfies the sentence if and only if every member $\mathbf{u}$ of the UD is such that $\mathbf{d}[\mathbf{u}/x]$ satisfies '$Gxa \supset Gf(x)a$', and according to clause 6 this is the case if, whenever $\mathbf{d}[\mathbf{u}/x]$ satisfies 'Gxa', $\mathbf{d}[\mathbf{u}/x]$ also satisfies '$Gf(x)a$'. So assume that $\mathbf{u}$ is such that $\mathbf{d}[\mathbf{u}/x]$ satisfies 'Gxa'. It follows from clause 2 that $\langle \mathbf{d}[\mathbf{u}/x](x), \mathbf{I}(a) \rangle$, which is $\langle \mathbf{u}, 5 \rangle$, is a member of $\mathbf{I}(G)$—that is, $\mathbf{u}$ is greater than 5. But then $\text{den}_{\mathcal{I},\mathbf{d}}(f(x))$, which is the successor of $\mathbf{u}$, is also greater than 5, so $\langle \text{den}_{\mathcal{I},\mathbf{d}}(f(x)), 5 \rangle$, which is

$<den_{I,d}(f(x))$, $\mathbf{I}(a)>$, is also a member of $\mathbf{I}(G)$. It follows from clause 2, then, that $\mathbf{d}[\mathbf{u}/x]$ also satisfies '$Gf(x)a$'. Therefore every variable assignment $\mathbf{d}[\mathbf{u}/x]$ that satisfies the antecedent of '$Gxa \supset Gf(x)a$' also satisfies the consequent; so every variable assignment $\mathbf{d}[\mathbf{u}/x]$ satisfies the conditional, and hence every variable assignment $\mathbf{d}$ satisfies the universally quantified '$(\forall x)(Gxa \supset Gf(x)a)$'. This establishes that the sentence is therefore true on interpretation 54.

8.7E EXERCISES

1. Using the definitions in this section, determine the truth-value of each of the following sentences on an interpretation that makes these assignments:

UD: Set of positive integers
K: [$<\mathbf{u}>$: $\mathbf{u}$ is negative]
E: [$<\mathbf{u}>$: $\mathbf{u}$ is even]
L: [$<\mathbf{u}_1, \mathbf{u}_2>$: $\mathbf{u}_1$ is less than $\mathbf{u}_2$]
o: 1

a. $- (\forall x)Ex \supset (\exists y)Lyo$
*b. $- Loo \;\&\; - (\forall y) - Loy$
c. $(\exists x)(Ko \vee Ex)$
*d. $(\forall x)(Lox \supset (\forall y)Lxy)$
e. $(Ko \equiv (\forall x)Ex) \supset (\exists y)(\exists z)Lyz$
*f. $(\forall x)[Ex \supset (\exists y)(Lyx \vee Lyo)]$

2. Using the definitions in this section, determine the truth-value of each of the following sentences on an interpretation that makes these assignments:

UD: Set of positive integers
E: [$<\mathbf{u}>$: $\mathbf{u}$ is even]
G: [$<\mathbf{u}_1, \mathbf{u}_2>$: $\mathbf{u}_1$ is greater than $\mathbf{u}_2$]
T: [$<\mathbf{u}>$: $\mathbf{u}$ is less than 2]
c: 3

a. $(\exists x)(Ex \supset (\forall y)Ey)$
*b. $(\forall x)(\forall y)(Gxy \vee Gyx)$
c. $(\forall x)(Tx \supset (\exists y)Gyx)$
*d. $(\forall x)(Ec \supset Ex)$
e. $(\forall x)[(\forall y)Gxy \vee (\exists y)Gxy]$
*f. $(\forall y)[Ty \vee (\forall x)(Ex \supset Gxy)]$

3. Using the definitions in this section, determine the truth-value of each of the following sentences on an interpretation that makes these assignments:

UD: Set of positive integers
M: [$<\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3>$: $\mathbf{u}_1$ minus $\mathbf{u}_2$ equals $\mathbf{u}_3$]
P: [$<\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3>$: $\mathbf{u}_1$ plus $\mathbf{u}_2$ equals $\mathbf{u}_3$]
o: 1

a. Mooo = Pooo
*b. $(\forall x)(\forall y)(Mxyo = Pyox)$
c. $(\forall x)(\forall y)(\forall z)(Mxyz = Pxyz)$
*d. $(\exists x)(\forall y)(\forall z)(Mxyz \vee Pzyx)$
e. $(\forall y)(\exists z)(Pyoz \supset Pooo)$

***4.** Using the definitions in this section, explain why the following two sentences are quantificationally equivalent:

$(\forall x)Fx$
$\sim (\exists x) \sim Fx$

5. Using the definitions in this section, explain why the following sentence is quantificationally true:

$(\forall x)((\forall y)Fy \supset Fx)$

***6.** Using the definitions in this section, explain why $\{(\forall x)Fx\}$ quantificationally entails every substitution instance of '$(\forall x)Fx$'.

7. Using the definitions in this section, explain why 'Fa' quantificationally entails '$(\exists x)Fx$'.

***8.** Using the definitions in this section, explain why '$(\exists x)Fx \,\&\, (\forall x) \sim Fx$' is quantificationally false.

9. Using the definitions in this section, determine the truth-value of each of the following sentences on an interpretation that makes these assignments:

UD: Set of positive integers
Ex: x is even
Gxy: x is greater than y

a. $(\forall x)(\forall y)[\sim x = y \supset (Ex \supset Gxy)]$
*b. $(\forall x)(\forall y)(x = y \vee \sim Ey)$
c. $(\forall x)[Ex \supset (\exists y)(\sim x = y \,\&\, \sim Gxy)]$

10.a. Using the definitions in this section, explain why every sentence of the form $(\forall \mathbf{x})\mathbf{x} = \mathbf{x}$ is quantificationally true.
*b. Using the definitions in this section, explain why '$(\forall x)(\forall y)(x = y \supset Fxy) \supset (\forall x)Fxx$' is quantificationally true.

11. Using the definitions in this section, determine the truth-value of each of the following sentences on an interpretation that makes the following assignments:

UD: Set of positive integers
Ox: x is odd
h(x): x squared
g(x,y): the sum of x and y

a. $(\forall x)(Oh(x) \supset Og(x,x))$
*b. $(\forall x)(\forall y)(Og(x,y) \supset (Ox \vee Oy))$
c. $(\exists x)(\exists y)(Ox \,\&\, x = h(y))$

12.a. Using the definitions in this section, explain why every sentence of the form $(\forall \mathbf{x})(\exists \mathbf{y})\mathbf{y} = f(\mathbf{x})$ is quantificationally true.

*b. Using the definitions in this section, explain why '$(\forall x)Pf(x) \supset (\forall x)Pf(f(x))$' is quantificationally true.

GLOSSARY

QUANTIFICATIONAL TRUTH: A sentence **P** of *PL/PLE* is *quantificationally true* if and only if **P** is true on every interpretation.

QUANTIFICATIONAL FALSITY: A sentence **P** of *PL/PLE* is *quantificationally false* if and only if **P** is false on every interpretation.

QUANTIFICATIONAL INDETERMINACY: A sentence **P** of *PL/PLE* is *quantificationally indeterminate* if and only if **P** is neither quantificationally true nor quantificationally false.

QUANTIFICATIONAL EQUIVALENCE: Sentences **P** and **Q** of *PL/PLE* are *quantificationally equivalent* if and only if there is no interpretation on which **P** and **Q** have different truth-values.

QUANTIFICATIONAL CONSISTENCY: A set of sentences of *PL/PLE* is *quantificationally consistent* if and only if there is at least one interpretation on which all the members of the set are true. A set of sentences of *PL/PLE* is *quantificationally inconsistent* if and only if the set is not quantificationally consistent.

QUANTIFICATIONAL ENTAILMENT: A set Γ of sentences of *PL/PLE* *quantificationally entails* a sentence **P** of *PL/PLE* if and only if there is no interpretation on which every member of Γ is true and **P** is false.

QUANTIFICATIONAL VALIDITY: An argument of *PL/PLE* is *quantificationally valid* if and only if there is no interpretation on which all the premises are true and the conclusion is false. An argument of *PL/PLE* is *quantificationally invalid* if and only if the argument is not quantificationally valid.

Chapter 9

PREDICATE LOGIC: TRUTH-TREES

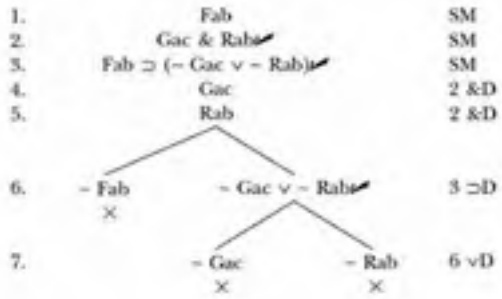
9.1 EXPANDING THE RULES FOR TRUTH-TREES FOR *PL*

Truth-trees, as developed in Chapter 4, provide the basis for an effective method of testing finite sets of sentences of *SL* for truth-functional consistency and thus for all the properties of sentences and finite groups of sentences that can be explicated in terms of truth-functional consistency (for example, truth-functional validity, truth-functional truth, and truth-functional equivalence). In this chapter we shall augment the truth-tree method to make it applicable to sets of sentences of *PL* and of *PLE*. The result will be a method of testing finite sets of sentences of *PL* and of *PLE* for quantificational consistency and thus for those properties of sentences and finite groups of sentences that can be explicated in terms of quantificational consistency.

Some sets of sentences of *PL* consist exclusively of sentences whose only logical operators are truth-functional connectives. We can test these sets for consistency by using the truth-tree rules already given in Chapter 4. For the set

{Fab, Gac & Rab, Fab ⊃ (~ Gac ∨ ~ Rab)}

we can construct the following tree:



The various branches represent failed attempts to find a way in which all the members of the set being tested might be true. If a set contains sentences of *PL* whose only logical operators are truth-functional connectives, then there will be an interpretation on which all the members of the set are true if and only if a tree for the set has at least one completed open branch. The above tree contains only closed branches; that is, each branch of this tree contains an atomic sentence and the negation of that sentence. We know that there is no interpretation on which both a sentence and its negation are true, so there is no interpretation on which the sentences in the set {Fab, Gac & Rab, Fab ⊃ (~ Gac ∨ ~ Rab)} are all true. We may conclude that this set is, on truth-functional grounds alone, quantificationally inconsistent.

However, many finite sets that are quantificationally inconsistent are not inconsistent on truth-functional grounds. The rules we presently have for constructing truth-trees do not allow us to construct closed trees for such sets. For example, using the decomposition rules we presently have, we can obtain only the following tree for the set {(∀x)(Fxc ⊃ Gxb), Fac & ~ Gab}:

1.	(∀x)(Fxc ⊃ Gxb)	SM
2.	Fac & ~ Gab	SM
3.	Fac	2 &D
4.	~ Gab	2 &D

We need a rule for decomposing the quantified sentence '(∀x)(Fxc ⊃ Gxb)' on line 1. More generally, we need rules for decomposing sentences of *PL* having any of the following four forms:

(∀x)P
(∃x)P
~ (∀x)P
~ (∃x)P

In this section we introduce one new tree rule for each of these kinds of sentences. We begin with the rules for negations of quantified sentences. Both are nonbranching rules:

Negated Existential Decomposition (~ ∃D)	*Negated Universal Decomposition* (~ ∀D)
~ (∃**x**)**P** ✓	~ (∀**x**)**P** ✓
(∀**x**) ~ **P**	(∃**x**) ~ **P**

In each case the sentence entered is equivalent to the sentence being decomposed. 'It is not the case that something is such-and-such' is equivalent to 'Each thing is such that it is not such-and-such', and 'It is not the case that each thing is such-and-such' is equivalent to 'Something is not such-and-such'.

If a universally quantified sentence (∀**x**)**P** is true, then so is each substitution instance **P**(**a**/**x**) of that sentence. We want a rule that allows us to "decompose" a universally quantified sentence to its substitution instances. So we add the following to our set of tree rules:

Universal Decomposition (∀D)

(∀**x**)**P**

P(**a**/**x**)

where **a** is any individual constant

At any point in the construction of a tree, a universally quantified sentence (∀**x**)**P** may be decomposed by entering any substitution instance **P**(**a**/**x**) of that sentence on *one or more* open branches passing through (∀**x**)**P**. Because a universally quantified sentence has an infinite number of substitution instances, we can never "finish" decomposing such a sentence. Consequently universally quantified sentences are never checked off.

Universal Decomposition does not require that a selected substitution instance be entered on *every* open branch passing through the universally quantified sentence being decomposed. A substitution instance is often of use on one open branch passing through the sentence being decomposed but not on another. And, because universally quantified sentences are never checked off, we can always later add more substitution instances of a universally quantified sentence to any open branch passing through that sentence.

The tree we started for the set {(∀x)(Fxc ⊃ Gxb), Fac & ~ Gab)} can now be completed:

1.	$(\forall x)(Fxc \supset Gxb)$	SM
2.	$Fac \ \& \sim Ga$ ✓	SM
3.	Fac	2 &D
4.	$\sim Gab$	2 &D
5.	$Fac \supset Gab$ ✓	1 ∀D
6.	$\sim Fac$ × | Gab ×	5 ⊃D

At line 5 we entered 'Fac ⊃ Gab' by Universal Decomposition. We could have entered any substitution instance of '(∀x)(Fxc ⊃ Gxb)', but only the one we did enter is of use in producing a closed tree. Recall that we do not check off the universally quantified sentence that is being decomposed.

The last tree rule to be added is for decomposing existentially quantified sentences:

Existential Decomposition (∃D)

$(\exists \mathbf{x})\mathbf{P}$ ✓

$\mathbf{P(a/x)}$

where **a** is a constant foreign to the branch

A constant is foreign to a branch of a tree if and only if it does not occur in any sentence on that branch. Existentially quantified sentences, unlike universally quantified sentences, are checked off when they are decomposed. This is because we know that if an existentially quantified sentence $(\exists \mathbf{x})\mathbf{P}$ is true then there is at least one thing that is of the sort specified by **P**, but there need not be more than one such thing. When we choose an individual constant for the substitution instance $\mathbf{P(a/x)}$, the constant **a** that we choose must be foreign to the branch because otherwise it would already have a role on that branch, and quite possibly a conflicting role. For example, the sentences 'Some cars are yellow' and 'Some cars are not yellow' are both true. Hence the set [(∃x)(Cx & Yx), (∃x) (Cx & ~ Yx)] consisting of symbolizations of these sentences should be quantificationally consistent and have only open truth-trees. However, if we were to drop the Existential Decomposition restriction that **a** be foreign to the branch on which the substitution instance is entered, we could produce a closed tree for this set:

1.	$(\exists x)(Cx \ \& \ Yx)$ ✓	SM	
2.	$(\exists x)(Cx \ \& \sim Yx)$ ✓	SM	
3.	$Ca \ \& \ Ya$ ✓	1 ∃D	
4.	Ca	3 &D	
5.	Ya	3 &D	
6.	$Ca \ \& \sim Ya$ ✓	2 ∃D	**MISTAKE!**
7.	Ca	6 &D	
8.	$\sim Ya$ ×	6 &D	

Line 6 is a mistake because the individual constant 'a' used in Existential Decomposition at line 6 was not foreign to the single branch of the tree prior to line 6. A correct tree uses an instantiating constant on line 6 that is different from that used on line 3:

1.	$(\exists x)(Cx \ \& \ Yx)$ ✓	SM
2.	$(\exists x)(Cx \ \& \sim Yx)$ ✓	SM
3.	$Ca \ \& \ Ya$ ✓	1 ∃D
4.	Ca	3 &D
5.	Ya	3 &D
6.	$Cb \ \& \sim Yb$ ✓	2 ∃D
7.	Cb	6 &D
8.	$\sim Yb$	6 &D
	o	

The single branch is completed and this shows that the set is indeed quantificationally consistent.

The following tree contains three uses of Existential Decomposition:

1.	$(\forall x)Fx \supset (\exists x) \sim Gx$ ✓		SM
2.	$(\exists x) \sim Fx$ ✓		SM
3.	$\sim Fa$		2 ∃D
4.	$\sim (\forall x)Fx$ ✓	$(\exists x) \sim Gx$ ✓	1 ⊃D
5.	$(\exists x) \sim Fx$ ✓	|	4 ~ ∀D
6.	$\sim Fb$	|	5 ∃D
7.	o	$\sim Gb$	4 ∃D
		o	

At line 3 Existential Decomposition is used for the first time. Since no constant occurs on the single branch that constitutes the tree at that point, 'a' is used as the instantiating constant. The next use of Existential Decomposition is at line 6 on the left-hand branch. At that point 'a' already occurs on the branch (at line 3—remember that the sentences on lines 1–3 occur on both branches of this tree). So a new instantiating constant, 'b', is used. The final use of Existential Decomposition is at line 7 on the right-hand branch. The constant 'a' cannot be used because it occurs on line 3. But 'b' can be used, for although it already occurs on the left-hand branch, it does not occur before line 7 on the right-hand branch.

The preceding tree has two open branches, each of which contains only literals and decomposed nonliterals. The complexities of predicate logic will force us to complicate the account of 'completed open branch' given in Chapter 4. However, an open branch that contains only literals and decomposed nonliterals that have been checked off will, as in Chapter 4, count as a completed open branch. So both branches of the tree are completed open branches.

Moreover, a completed open branch guarantees that we can construct an interpretation on which every member of the set being tested is true, that is, a model for that set, so this tree demonstrates that the set {(∀x)Fx ⊃ (∃x) ~ Gx, (∃x) ~ Fx} is quantificationally consistent. We'll show how interpretations can be constructed from each of the two completed open branches of the tree.[1] An interpretation that makes all of the literals on a completed open branch true will make all of the other sentences on that branch, including the members of the set being tested, true. Starting with the left branch, we see that the branch contains two literals, '~ Fa' and '~ Fb'. To make both of these true we will construct an interpretation with a two-member UD, letting the constant 'a' denote one member and 'b' the other, and we will interpret the predicate 'F' so that neither of these members is in its extension (recall that the extension of a predicate is the set of things to which it applies):

UD: The set {1, 3}
a: 1
b: 3
Fx: x is even

Both '~ Fa' and ~ Fb' are true on any interpretation with this UD that makes these assignments (there are infinitely many such interpretations, because an interpretation must interpret every constant and predicate of *PL*). The sentence '(∀x)Fx ⊃ (∃x) ~ Gx' is true on any interpretation that includes these assignments because the antecedent is false (so it doesn't matter how the predicate 'G' is interpreted). The sentence '(∃x) ~ Fx' is true because at least one member of the UD is excluded (in fact, both are) from the extension of the predicate 'F'.

Note that the truth-values of the sentences '(∀x)Fx ⊃ (∃x) ~ Gx' and '(∃x) ~ Fx' don't depend on the assignments made to 'a' and 'b', since those constants don't appear in these sentences. So more generally we can say that the set members will be true on any interpretation that includes the following assignments:

UD: The set {1, 3}
Fx: x is even

Interestingly, in this case we do not need a two-member UD either. This is because the literals on the complete open branch, '~ Fa' and '~ Fb', agree in what they say about the individual denoted by 'a' and the individual denoted by 'b'—so the UD for an interpretation that makes the literals on the open branch true need not contain more than one individual. But our purpose in this chapter is to show how completed open branches can be used as the basis

[1] In Chapter 11 we will prove that such an interpretation can always be found corresponding to a completed open branch of a tree for sentences of *PL*.

for constructing models, rather than to explore the finer points thereof, so while working with the language *PL* (rather than *PLE*) our practice will be to present a UD with exactly as many members as there are constants on the open branch, to assign distinct members of the UD to those constants, and to interpret predicates so as to make the literals occurring on the branch true.

The right-hand open branch of the previous tree contains two constants as well, and the literals '~ Fa' and '~ Gb', so the members of the set {(∀x)Fx ⊃ (∃x) ~ Gx, (∃x) ~ Fx} will both be true on any interpretation with a two-member UD such that 'a' and 'b' denote distinct individuals and the two literals are true. Any interpretation that includes the following assignments will satisfy these criteria:

UD: {1, 3}
a: 1
b: 3
Fx: x is even
Gx: x is greater than 4

The sentence '(∀x)Fx ⊃ (∃x) ~ Gx' will be true because the antecedent is false and the consequent is true, while '(∃x) ~ Fx' will be true because at least one member of the UD (in fact, both) is excluded from the extension of the predicate 'F'.

Except for Universal Decomposition, the truth-tree rules introduced in this section are like the tree rules of Chapter 4 in that the results of applying one of them must be entered on *every* open branch running through the sentence being decomposed. Also as in Chapter 4, it is generally wise to apply decomposition rules that do not produce new branches before applying those that do. In using Universal Decomposition it is a good idea to select substitution instances in which the instantiating constant already occurs on the open branch in question. It is also wise to try to use Existential Decomposition before using Universal Decomposition, for the former rule but not the latter places a restriction on the individual constant that can be used in the substitution instance that is added to the tree. We illustrate these last two points by constructing a tree for {(∀x)(∀y) ~ Mxy, (∃x)Mxb}:

1.	(∀x)(∀y) ~ Mxy	SM
2.	(∃x)Mxb ✓	SM
3.	Mab	2 ∃D
4.	(∀y) ~ May	1 ∀D
5.	~ Mab	4 ∀D
	×	

Note that we used Existential Decomposition before Universal Decomposition. At line 4 we entered '(∀y) ~ May' rather than, say, '(∀y) ~ Mgy', because 'a' occurs earlier on the tree on line 3. And although 'b' also occurs on line 3 we

entered '(∀y) ~ May' rather than '(∀y) ~ Mby' because the former but *not* the latter will, when appropriately decomposed, produce the negation of the sentence on line 3.

Using Universal Decomposition before Existential Decomposition—that is, decomposing the sentence on line 1 before the sentence on line 2—will also produce a closed tree, but a tree that is more complex:

1.	(∀x)(∀y) ~ Mxy	SM
2.	(∃x)Mxb	SM
3.	(∀y) ~ Mby	1 ∀D
4.	~ Mbb	3 ∀D
5.	Mab	2 ∃D
6.	(∀y) ~ May	1 ∀D
7.	~ Mab	6 ∀D
	×	

In this tree we had to enter 'Mab', rather than 'Mbb', at line 5 because Existential Decomposition requires that the instantiating constant be foreign to the branch. At line 5, the constant 'b' was not foreign to the branch. But now, having entered 'Mab' at line 5, we were able to close the tree only by reapplying Universal Decomposition to the sentence on line 1. Lines 3 and 4 of the tree are thus superfluous.

In Chapter 4 we developed four strategies for keeping truth-trees for sets of sentences of *SL* as concise as possible. Those strategies are also applicable here. We repeat them, along with the two new strategies just discussed (suitably rearranged):

Strategies for Constructing Truth-Trees

1. Give priority to decomposing sentences whose decomposition does not require branching.
2. Give priority to decomposing sentences whose decomposition results in the closing of one or more branches.
3. Give priority to decomposing existentially quantified sentences over universally quantified sentences.
4. When using Universal Decomposition, try to use a substitution instance in which the instantiating constant already occurs on the branch in question.
5. Stop when a tree yields an answer to the question being asked.
6. Where strategies 1-5 are not applicable, decompose the more complex sentences first.

Strategy 1 should be used with care when dealing with universally quantified sentences. Consider the following tree in which Universal Decomposition is used before Conditional Decomposition:

1.	$(\forall x)(Fxa \supset Fax)$	SM
2.	$(\forall y)(Hy \supset Fya)$	SM
3.	$(\exists x)(Hx \,\&\, \sim Fax)$	SM
4.	$Hb \,\&\, \sim Fab$ ✓	3 ∃D
5.	Hb	4 &D
6.	$\sim Fab$	4 &D
7.	$Faa \supset Faa$	1 ∀D
8.	$Fba \supset Fab$ ✓	1 ∀D
9.	$Ha \supset Faa$	2 ∀D
10.	$Hb \supset Fba$ ✓	2 ∀D
11.	$\sim Hb$ × | Fba	10 ⊃D
12.	$\sim Fba$ × | Fab ×	8 ⊃D

At line 7 we used Universal Decomposition and continued using it until each universally quantified sentence (there are two) was decomposed to every substitution instance that could be formed from a constant already on the branch. The idea is that these are the substitution instances that may be useful later on. As it turns out, lines 7 and 9 are unnecessary, but this was not completely obvious at the point where we had a choice between applying Universal Decomposition and Conditional Decomposition. An alternative strategy would be to use Universal Decomposition only when no other rule can be applied. But this strategy produces the following, considerably more complex, tree:

1.	$(\forall x)(Fxa \supset Fax)$	SM
2.	$(\forall y)(Hy \supset Fya)$	SM
3.	$(\exists x)(Hx \,\&\, \sim Fax)$ ✓	SM
4.	$Hb \,\&\, \sim Fab$ ✓	3 ∃D
5.	Hb	4 &D
6.	$\sim Fab$	4 &D
7.	$Faa \supset Faa$ ✓	1 ∀D
8.	$\sim Faa$ | Faa	7 ⊃D
9.	$Ha \supset Faa$ ✓ | $Ha \supset Faa$ ✓	2 ∀D
10.	$\sim Ha$ Faa × | $\sim Ha$ | Faa	9 ⊃D
11.	$Fba \supset Fab$ ✓ | $Fba \supset Fab$ ✓ | $Fba \supset Fab$ ✓	1 ∀D
12.	$\sim Fba$ Fab | $\sim Fba$ Fab | $\sim Fba$ Fab	11 ⊃D
13.	$Hb \supset Fba$ ✓ × | $Hb \supset Fba$ ✓ × | $Hb \supset Fba$ ✓ ×	2 ∀D
14.	$\sim Hb$ Fba × × | $\sim Hb$ Fba × × | $\sim Hb$ Fba × ×	13 ⊃D

The best policy appears to be to stick with strategy 1, but with the caveat that, when a shorter route to a closed tree is apparent, it should be pursued.[2]

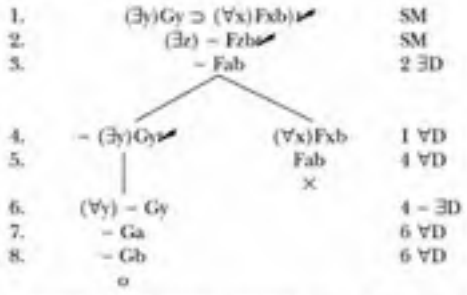
9.1E EXERCISES

Construct truth-trees for the following sets of sentences. For each, note whether the tree you construct has a completed open branch or is closed (by the accounts of 'completed open branch' and 'closed tree' given in Chapter 4).

a. {(∃x)Fx, (∃x) ~ Fx}
*b. {(∃x)Fx, (∀x) ~ Fx}
c. {(∃x)(Fx & ~ Gx), (∀x)(Fx ⊃ Gx)}
*d. {(∃x)(Fx & ~ Gx), (∀x)Fx ⊃ (∀x)Gx}
e. {~ (∀x)(Fx ⊃ Gx), ~ (∃x)Fx, ~ (∃x)Gx}
*f. {~ (∀x)(Fx & Gx), (∃y)(Fy & Gy)}
g. {(∃x)Fx, (∃y)Gy, (∃z)(Fz & Gz)}
*h. {(∀x)(Fx ⊃ Gx), (∀x)(Gx ⊃ Hx), (∃x)(Fx & ~ Hx)}
i. {(∀x)(∀y)(Fxy ⊃ Fyx), (∃x)(∃y)(Fxy & ~ Fyx)}
*j. {(∀x)(∃y)Lxy, Lta & ~ Lat, ~ (∃y)Lay}
k. {(∃x)Fx ⊃ (∀x)Fx, ~ (∀x)[Fx ⊃ (∀y)Fy]}
*l. {(∀x)(Fx ⊃ Gx), ~ (∀x) ~ Fx, (∀x) ~ Gx}
m. {(∀x)[Fx ⊃ (∃y)Gyx, ~ (∀x) ~ Fx, (∀x)(∀y) ~ Gxy}
*n. {(∃x)Gx ⊃ (∀x)Gx, (∃z)Gz & (∃y) ~ Gy}
o. {(∃x)Lxx, ~ (∃x)(∃y)(Lxy & Lyx)}
*p. {(∃y)(Fy ∨ Gy), ~ (∀y)Fy & ~ (∀y)Gy, ~ (∀x)(Fx & Gx)}
q. {(∃x)(Fx ∨ Gx), (∀x)(Fx ⊃ ~ Gx), (∀x)(Gx ⊃ ~ Fx), ~ (∃x)(~ Fx ∨ ~ Gx)}

9.2 TRUTH-TREES AND QUANTIFICATIONAL CONSISTENCY

In Chapter 4 we defined a *completed open branch* to be an open branch on which every sentence either is a literal or has been decomposed, so that no new sentence can be added to the branch. We will have to revise this definition for trees for sets of sentences of *PL*. Consider the following tree for the set {(∃y)Gy ⊃ (∀x)Fxb, (∃z) ~ Fzb}:



[2]A further caveat will be required when we introduce systematic trees in Section 9.4, for the routine for constructing such trees requires abandoning strategy 1 altogether as it applies to universally quantified sentences.

This tree has one closed branch and one open branch. Further sentences could be added to the open branch, for one of the sentences on that branch, '(∀y) ~ Gy', is a universally quantified sentence, and there is no limit to the number of times a universally quantified sentence can be decomposed (such sentences are never checked off). In the present example we added substitution instances of '(∀y) ~ Gy' on lines 7 and 8. While further substitution instances can be added, it is clear that, no matter what further substitution instances may be added, the branch will remain open. We have already added all the instances that can be formed from individual constants appearing earlier on the open branch. Substitution instances formed from individual constants not already on the open branch will be such that their truth or falsity does not bear on the truth of literals already on the branch, so there is no point in entering '~ Gh', for example. The leftmost branch is at this point sufficient for concluding that the set being tested is quantificationally consistent—we can use the literals '~ Fab', '~ Ga', and '~ Gb' that occur on this branch to construct an interpretation on which all of the set members are true. They'll be true on any interpretation that includes the following assignments:

UD: The set {1, 3}
a: 1
b: 3
Fxy: x is greater than y
Gx: x is even

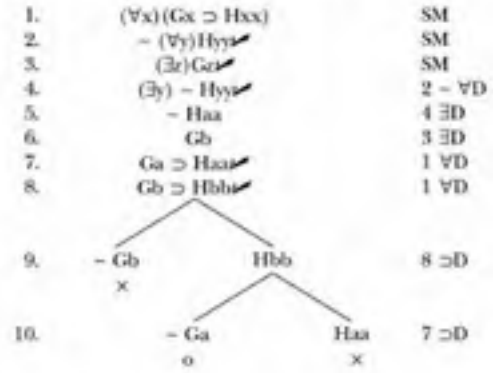
'(∃y)Gy ⊃ (∀x)Fxb' will be true because the antecedent is false—no member of the UD is even—while '(∃z) ~ Fzb' will be true because at least one member of the UD fails to be greater than 3.

We want open branches such as the left branch on the preceding tree—open branches that are such that no additional useful sentences can be added to them—to count as completed open branches. To accomplish this we modify our definition of a completed open branch as follows:

> A branch of a truth-tree for a set of sentences of *PL* is a **completed open branch** if and only if it is a finite open branch (that is, an open branch with a finite number of sentences) and each sentence occurring on the branch is one of the following:
>
> 1. A literal (an atomic sentence or the negation of an atomic sentence)
> 2. A compound sentence that is not a universally quantified sentence and is decomposed
> 3. A universally quantified sentence (∀**x**)**P** such that **P**(**a**/**x**) occurs on the branch *for each constant* **a** occurring on the branch and at least one substitution instance **P**(**a**/**x**) occurs on the branch

By this revised account the leftmost branch of the preceding tree is a completed open branch.

Here is another example, a tree for the set {(∀x)(Gx ⊃ Hxx), ~ (∀y)Hyy, (∃z)Gz} that contains a completed open branch:



The open branch is completed because each compound sentence that is not a universal quantification has been checked off, and the single universally quantified sentence has been decomposed using each of the two constants on the branch, at lines 7 and 8. The branch contains sufficient information for constructing a model of the set being tested. To make the literals on the completed open branch true, we can use the set {1, 2} as our UD and assign 1 to 'a' and 2 to 'b'. We need to interpret the predicates 'G' and 'H' in such a way that 'Gb' and 'Hbb' are true (since 'Gb' and 'Hbb' occur on the open branch) and 'Ga' and 'Haa' are false (since '~ Ga' and '~ Haa' occur on the open branch). The following assignments will do the trick:

UD: {1, 2}
a: 1
b: 2
Gx: x is even
Hxy: x squared is greater than y

Any interpretation that includes these assignments will make the three sentences in the set {(∀x)(Gx ⊃ Hxx), ~ (∀y)Hyy, (∃z)Gz} true, and this establishes that the set is quantificationally consistent. Since an interpretation on which all the members of the set being tested are true *can* always be constructed from a completed open branch, we shall take the presence of a completed open branch as a guarantee that the set being tested is quantificationally consistent.

To see why we require that a completed open branch on which a universally quantified sentence occurs contain *at least one* substitution instance of that sentence, consider the unit set {~ (∃x) (Fx ∨ ~ Fx)}. The sole member of this set says that it is not the case that there is an x such that either x is F or x is not F. But each thing x either is F or is not F. So this sentence is false—and indeed is quantificationally false (since every UD is nonempty). We therefore want every tree for the unit set of this sentence to close. One tree is as follows:

1.	~ (∃x) (Fx ∨ ~ Fx)✓	SM
2.	(∀x) ~ (Fx ∨ ~ Fx)	1 ~ ∃D
3.	~ (Fa ∨ ~ Fa)✓	2 ∀D
4.	~ Fa	3 ~ ∨D
5.	~ ~ Fa✓	3 ~ ∨D
6.	Fa	5 ~ ~ D
	×	

On line 2 we entered a universally quantified sentence by applying Negated Existential Decomposition to the sentence on line 1. If we did not require that a completed open branch contain at least one substitution instance of every universally quantified sentence occurring on that branch, we would have a completed open branch at line 2. A completed open branch is supposed to signal a consistent set, but the set we are testing is not consistent. Given the requirement that a completed open branch must have at least one substitution instance of each universally quantified sentence occurring on that branch, we entered such an instance on line 3 and doing so eventually yielded a closed tree. Note that the tree would close no matter what substitution instance of '(∀x) ~ (Fx ∨ ~ Fx)' is entered at line 3.

We summarize here the important properties of truth-trees for sets of sentences of *PL*. With the exception of the notion of a completed open branch, these definitions strictly parallel those given in Chapter 4:

Closed branch: A branch containing both an atomic sentence and the negation of that sentence

Closed truth-tree: A truth-tree each of whose branches is closed

Open branch: A branch that is not closed

Completed open branch: A finite open branch on which each sentence is one of the following:

1. A literal (an atomic sentence or the negation of an atomic sentence)
2. A compound sentence that is not a universally quantified sentence and is decomposed

3. A universally quantified sentence $(\forall \mathbf{x})\mathbf{P}$ such that $\mathbf{P}(\mathbf{a}/\mathbf{x})$ occurs on the branch *for each constant* **a** that occurs on the branch and at least one substitution instance $\mathbf{P}(\mathbf{a}/\mathbf{x})$ occurs on the branch

Completed truth-tree:	A truth-tree each of whose branches either is closed or is a completed open branch
Open truth-tree:	A truth-tree that is not closed

Note that a tree that has a completed open branch is an open tree, as is a tree that is still under construction—one that is not a completed truth-tree. So, while some open trees may become closed trees, those with a completed open branch will always be open trees.

As noted, we will prove the following claims in Chapter 11:

> A finite set Γ of sentences of *PL* is *quantificationally inconsistent* if and only if Γ has a closed truth-tree.

> A finite set Γ of sentences of *PL* is *quantificationally consistent* if and only if Γ is not quantificationally inconsistent, that is, if and only if Γ does not have a closed truth-tree.

If we can construct a closed tree for a finite set of sentences—that is, a closed tree that starts with the sentences in the set—then we can conclude that that finite set is quantificationally inconsistent. If we can construct a tree with a completed open branch for a finite set of sentences of *PL* (again, a tree that starts with the sentences in the set), we may conclude that the set is quantificationally consistent.[3] However, in *PL*, unlike *SL*, not all consistent finite sets have trees with completed open branches: some such sets have trees all of whose open branches are infinite (we require a completed open branch to be finite).[4] That is why our second box, characterizing consistency, does so in negative terms: a finite set is quantificationally consistent if and only if it does not have a closed truth-tree (rather than if and only if it has a tree with a completed open branch).

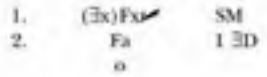
There is another importance difference between quantificational and sentential truth-trees. In the sentential case, we can say that the truth of all of

[3] As we noted in Chapter 4, truth-trees can only be used to test finite sets of sentences. In Chapter 11 we shall prove that an infinite set of sentences of *PL* is quantificationally consistent if and only if every finite subset of that set is quantificationally consistent. Therefore, we can also say that an infinite set Γ of sentences of *PL* is quantificationally inconsistent if and only if *at least one* finite subset of Γ has a closed truth-tree and that an infinite set Γ of sentences of *PL* is quantificationally consistent if and only if no finite subset of Γ has a closed truth-tree.

[4] We discuss such trees in detail in Section 9.5.

the members of the set being tested requires the truth of all of the sentences on at least one completed open branch. So, for example, the following tree for the set {A ⊃ B, B ⊃ A, ~ A} has exactly one completed open branch:

If all of the sentences in the set are true on some truth-value assignment, then the final two sentences, '~ B' and '~ A', must also be true on that assignment. (Obviously '~ A' will be true because it is a member of the original set; '~ B' must also be true given the truth of '~ A' and 'B ⊃ A'.) A similar claim (talking of interpretations rather than truth-value assignments) does not hold for quantificational trees. Consider the following simple tree for the set {(∃x)Fx}:



The tree for this unit set has only one completed open branch, and from this open branch we can conclude that there is a model for the set, for example, any interpretation that includes the following assignments:

UD: {1}
a: 1
Fx: x is odd

for such an interpretation will make the single literal on the branch true. But unlike the situation for *SL*, it is not the case that the truth of '(∃x)Fx' *requires* the truth of 'Fa'. For example, '(∃x)Fx' is also true on any interpretation that includes the following assignments:

UD: {1, 2}
a: 2
Fx: x is odd

While 'Fa' is false on this interpretation '(∃x)Fx' is true because there is a member of the UD, namely 1, that is odd. The completed open branch shows that the set {(∃x)Fx} is consistent not because the truth of '(∃x)Fx' requires the truth of 'Fa' but rather because the truth of 'Fa' is sufficient to guarantee the truth of '(∃x)Fx'—as illustrated by the first of these interpretations. If 'Fa' *is* true then it follows that '(∃x)Fx' is true as well.

Although the truth of '(∃x)Fx' does not require the truth of 'Fa', it *is* the case that if '(∃x)Fx' is true on some interpretation, then there must be an intrepretation on which 'Fa' is true. If there is an interpretion on which something is F then we can construct an interpretation in which 'a' designates that thing (leaving the interpretation of 'F' unchanged) so that 'Fa' will be true. It is for this reason that the following tree establishes the quantificational inconsistency of {(∃x)(Fx & ~ Fx), even though the truth of an existentially quantified formula does not require the truth of any particular one of its substitution instances:

1.	(∃x)(Fx & ~ Fx)✓	SM
2.	Fa & ~ Fa✓	1 ∃D
3.	Fa	2 &D
4.	~ Fa	2 &D
	×	

If there is indeed something that both is and is not F, then there is an interpretation that assigns that thing to 'a' and 'Fa & ~ Fa' will be true on that interpretation. But our tree shows that, if 'Fa & ~ Fa' is true on an interpretation, so are both 'Fa' and '~ Fa'. But we know that there is no interpretation on which an atomic sentence and its negation are both true, so we may conclude that there is no interpretation on which 'Fa & ~ Fa' is true and also that there is no interpretation on which '(∃x)(Fx & ~ Fx)' is true.

9.2E EXERCISES

Use the truth-tree method to test the following sets of sentences for quantificational consistency. State your result, and specify what it is about the tree that establishes this result. In addition, if your tree establishes consistency, show the relevant part of an interpretation that will make all of the literals on one completed branch, and therefore all of the members of the set being tested, true. (Be sure to list the literals that you are using in this case.)

a. {(∀x)Fx ∨ (∃y)Gy, (∃x)(Fx & Gb)}
*b. {(∀x)Fx ∨ (∃y)Gy, (∃x)(~ Fx & Gx)}
c. {(∀x)(Fx ⊃ Gxa), (∃x)Fx, (∀y) ~ Gya}
*d. {(∀x)(Fx ⊃ Gxa), (∃x)Fx}
e. {(∀x)(Fx ⊃ Gxa), (∃x)Fx, (∀y)Gya}
*f. {(∀x)(Fx ⊃ Gxa), (∃x)Fx, (∀x)(∀y)Gxy}
g. {(∀x)(Fx ∨ Gx), ~ (∃y)(Fy ∨ Gy)}
*h. {(∀x)(Fx ∨ Gx), ~ (∃y)(Fy ∨ Gy), Fa & ~ Gb}
i. {(∀z)Hz, (∃x)Hx ⊃ (∀y)Fy}
*j. {(∀z) ~ Hzb, (∃y)Fy ⊃ (∃x)Hxc}
k. {(∀x)(∀y)Lxy, (∃z) ~ Lza ⊃ (∀z) ~ Lza}
*l. {(∀x)(∀y)Lxy, (∃z) ~ Lza ⊃ (∀z) ~ Lzb}
m. {(∀x)(Rx ≡ ~ Hxa), ~ (∀y) ~ Hby, Ra}
*n. {(∀x)Fxa ≡ ~ (∀x)Gxb, (∃x)(Fxa & ~ Gxb)}

9.3 TRUTH-TREES AND OTHER SEMANTIC PROPERTIES

To facilitate the use of truth-trees to test sentences and sets of sentences for properties other than consistency, it will be useful to specify those other properties in terms of open and closed truth-trees. We begin with quantificational truth, quantificational falsity, and quantificational indeterminacy:

A sentence **P** of *PL* is *quantificationally true* if and only if the set {~ **P**} has a closed truth-tree.

A sentence **P** of *PL* is *quantificationally false* if and only if the set {**P**} has a closed truth-tree.

A sentence **P** of *PL* is *quantificationally indeterminate* if and only if neither the set {**P**} nor the set {~ **P**} has a closed truth-tree.

Quantificational equivalence, quantificational entailment, and quantificational validity are specified analogously:

Sentences **P** and **Q** of *PL* are *quantificationally equivalent* if and only if the set {~ (**P** ≡ **Q**)} has a closed truth-tree.

A finite set Γ of sentences of *PL quantificationally entails* a sentence **P** of *PL* if and only if $\Gamma \cup$ {~ **P**} has a closed truth-tree.

An argument of *PL* with a finite set of premises is *quantificationally valid* if and only if the set consisting of the premises and the negation of the conclusion has a closed truth-tree.

We shall illustrate how truth-trees can be used to test for each of these semantic properties. We begin with quantificational truth, quantificational falsity, and quantificational indeterminacy. Consider the sentence '(∀x)(Fx & (∃y) ~ Fy)'. It says, 'Each thing is F and at least one thing is not F', a claim for which we should not hold out much hope. To verify that this sentence is quantificationally false, we construct a tree for the unit set of this sentence, expecting the tree to close, which it does:

1.	(∀x)(Fx & (∃y) ~ Fy)	SM
2.	Fa & (∃y) ~ Fy	1 ∀D
3.	Fa	2 &D
4.	(∃y) ~ Fy	2 &D
5.	~ Fb	4 ∃D
6.	Fb & (∃y) ~ Fy	1 ∀D
7.	Fb	6 &D
8.	(∃y) ~ Fy	6 &D
	×	

Since the tree closes, the set being tested is quantificationally inconsistent. Therefore there is no interpretation on which every member of the set is true. Since there is only one sentence in the set, there is no interpretation on which that sentence, '(∀x)(Fx & (∃y) ~ Fy)', is true. Hence the sentence is indeed quantificationally false. Note that we used Universal Decomposition on the sentence on line 1 twice—once to obtain the sentence on line 2 and once to obtain the sentence on line 6. This was necessary because, by the time we reached line 5, we had introduced a new constant with which the universally quantified sentence on line 1 had not yet been decomposed.

Now consider the sentence '(∃x) ~ Fx ⊃ ~ (∀x)Fx', which says 'If there is something that is not F, then not everything is F' and is fairly obviously quantificationally true. To verify that this sentence is quantificationally true, we construct a tree for the unit set of its negation, that is, for {~ [(∃x) ~ Fx ⊃ ~ (∀x)Fx]} (note that in forming the negation of this truth-functionally compound sentence we were careful to reinstate the outer brackets that had been omitted):

1.	~ [(∃x) ~ Fx ⊃ ~ (∀x)Fx] ✓	SM
2.	(∃x) ~ Fx ✓	1 ~⊃D
3.	~ ~ (∀x)Fx ✓	1 ~⊃D
4.	(∀x)Fx	3 ~ ~ D
5.	~ Fa	2 ∃D
6.	Fa	4 ∀D
	×	

This tree is closed, so the set being tested is quantificationally inconsistent; there is no interpretation on which the one member of that set, '~ [(∃x) ~ Fx ⊃ ~ (∀x)Fx]', is true. Hence there is no interpretation on which the sentence of which it is the negation, '(∃x) ~ Fx ⊃ ~ (∀x)Fx', is false—and thus the latter sentence is quantificationally true.

One does not always have a clear intuition about a sentence's status, that is, about whether it is quantificationally true, quantificationally false, or quantificationally indeterminate. Consider, for example, the sentence '(∃x)(Fx ⊃ (∀y)Fy)', which may appear on first encounter to be quantificationally indeterminate. It is if and only if both the tree for '(∃x)(Fx ⊃ (∀y)Fy)' and the tree for its negation have at least one completed open branch. We begin with a tree for the unit set of the sentence:

1.	(∃x)(Fx ⊃ (∀y)Fy) ✓		SM
2.	Fa ⊃ (∀y)Fy ✓		1 ∃D
3.	~ Fa	(∀y)Fy	2 ⊃D
4.	o	Fa	3 ∀D
		o	

As expected, the tree is open (it has two completed open branches), so the sentence is not quantificationally false. We next construct a tree for the unit set of the negation of the sentence:

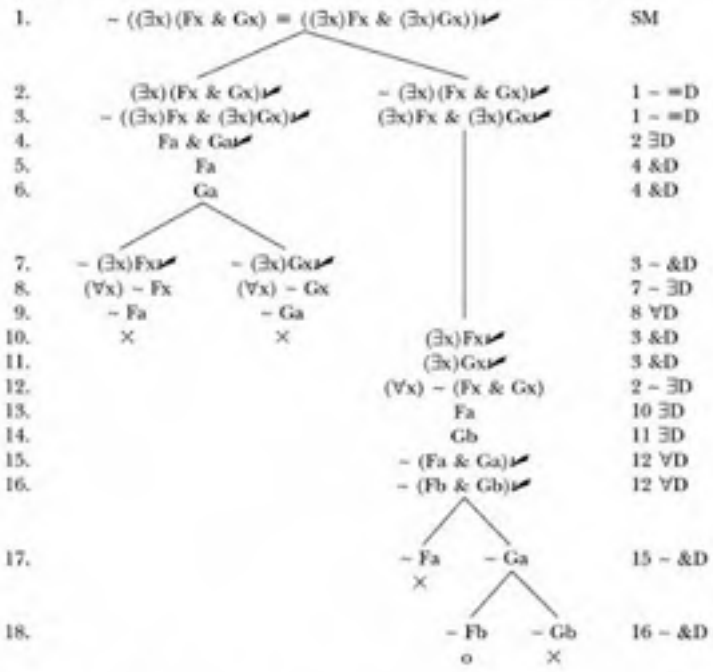
1.	~ (∃x)(Fx ⊃ (∀y)Fy)✓	SM
2.	(∀x) ~ (Fx ⊃ (∀y)Fy)	1 ~ ∃D
3.	~ (Fa ⊃ (∀y)Fy)✓	2 ∀D
4.	Fa	3 ~ ⊃D
5.	~ (∀y)Fy✓	3 ~ ⊃D
6.	(∃y) ~ Fy✓	5 ~ ∀D
7.	~ Fb	6 ∃D
8.	~ (Fb ⊃ (∀y)Fy)✓	2 ∀D
9.	Fb	8 ~ ⊃D
10.	~ (∀y)Fy	8 ~ ⊃D
	×	

Perhaps surprisingly, this tree is closed. So the negation being tested is quantificationally false, and the sentence of which it is a negation, '(∃x)(Fx ⊃ (∀y)Fy)', is in fact quantificationally true.

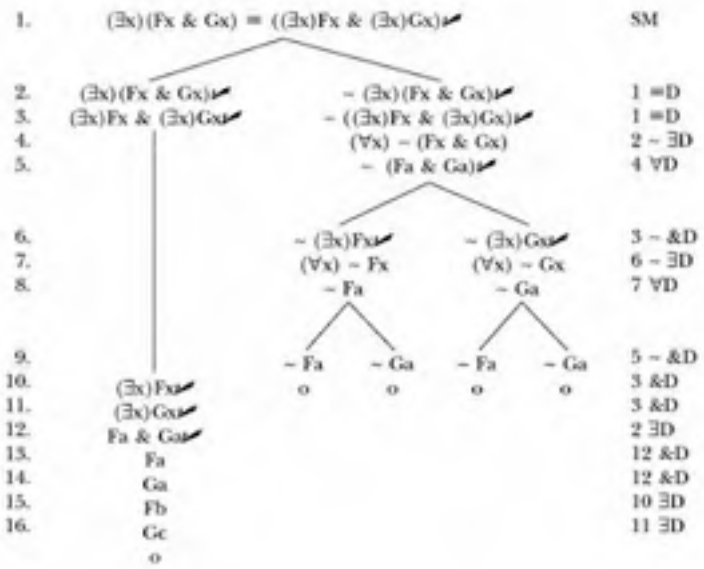
Insufficient attention to the importance of the scope of quantifiers might lead one to think that the sentences '(∃x)(Fx & Gx)' and '(∃x)Fx & (∃x)Gx' are quantificationally equivalent and hence that

(∃x)(Fx & Gx) ≡ ((∃x)Fx & (∃x)Gx)

is quantificationally true. To test this supposition, we construct a tree for the negation of the biconditional, for the tree for the negation of the biconditional will close if and only if the conditional itself is quantificationally true:



The tree has a completed open branch, so the negated biconditional we are testing is not quantificationally false. The biconditional itself is therefore not quantificationally true, and the immediate components '$(\exists x)(Fx \ \& \ Gx)$' and '$(\exists x)Fx \ \& \ (\exists x)Gx$' of the biconditional are not quantificationally equivalent. If we are interested in establishing, by the tree method, that the biconditional is quantificationally indeterminate (and not quantificationally false), we must construct a tree for the biconditional itself:



It is surely not surprising that this tree has at least one completed open branch, establishing that the biconditional being tested is not quantificationally false and is therefore, given the previous tree, which also had a completed open branch, quantificationally indeterminate.

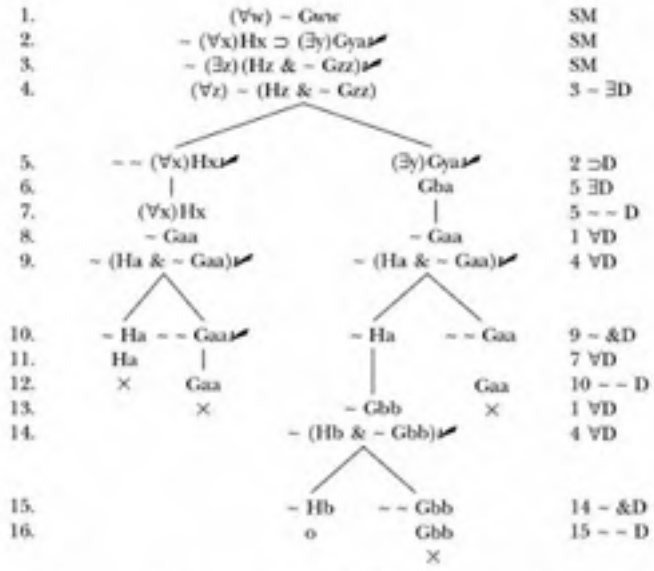
The sentences '$(\forall x)(Fx \supset (\exists y)Gya)$' and '$(\exists x)Fx \supset (\exists y)Gya$' are quantificationally equivalent, as the following closed tree for the negation of their

corresponding material biconditional establishes:

To use the tree method to test for quantificational validity, we construct a tree for the premises and the negation of the conclusion of the argument in question. A tree for the argument

$$\frac{(\forall w) \sim Gww \qquad \sim (\forall x)Hx \supset (\exists y)Gya}{(\exists z)(Hz \,\&\, \sim Gzz)}$$

follows:



The tree has a completed open branch, so the argument is quantificationally invalid. (There is an interpretation on which the premises and the negation of the conclusion are all true, that is, an interpretation on which the premises are true and the conclusion is false.)

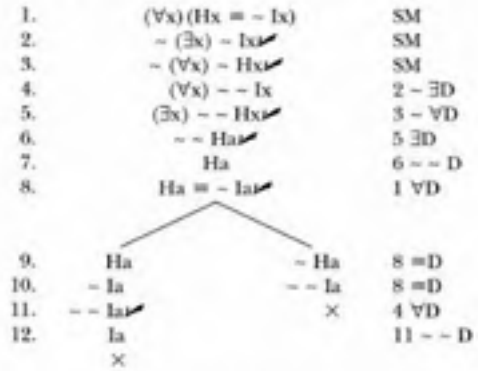
As with truth-trees for sentential logic, the procedure for testing alleged entailments parallels that for testing for validity. Consider the following entailment claim:

$$\{(\forall x)(Hx \equiv \sim Ix), \sim (\exists x) \sim Ix\} \models (\forall x) \sim Hx$$

If this claim is true, there is no interpretation on which the members of the above set are both true and the allegedly entailed sentence false; that is, there is no interpretation on which all the members of

$$\{(\forall x)(Hx \equiv \sim Ix), \sim (\exists x) \sim Ix, \sim (\forall x) \sim Hx\}$$

are true. So we shall test the latter set for quantificational consistency:



The tree is closed, so the set consisting of the members of the original set and the negation of the allegedly entailed sentence is quantificationally inconsistent. Therefore there is no interpretation on which all the members of that original set are true and the allegedly entailed sentence false, and so the entailment does hold.

9.3E EXERCISES

Construct truth-trees as necessary to provide the requested information. In each case state your result, and specify what it is about your tree that establishes this result.

1. Which of the following sentences are quantificationally true?

a. $(\exists x)Fx \vee \sim (\exists x)Fx$
*b. $(\exists x)Fx \vee (\exists x) \sim Fx$
c. $(\forall x)Fx \vee (\forall x) \sim Fx$
*d. $(\forall x)Fx \vee \sim (\forall x)Fx$
e. $(\forall x)Fx \vee (\exists x) \sim Fx$
*f. $(\forall x)(Fx \vee Gx) \supset [(\exists x)Fx \vee (\exists x)Gx]$
g. $(\forall x)(Fx \vee Gx) \supset [(\exists x) \sim Fx \supset (\exists x)Gx]$
*h. $(\forall x)(Fx \vee Gx) \supset [(\exists x)Fx \vee (\forall x)Gx]$
i. $[(\forall x)Fx \vee (\forall x)Gx] \supset (\forall x)(Fx \vee Gx)$
*j. $(\forall x)(Fx \vee Gx) \supset [(\forall x)Fx \vee (\forall x)Gx]$
k. $(\exists x)(Fx \,\&\, Gx) \supset [(\exists x)Fx \,\&\, (\exists x)Gx]$
*l. $[(\exists x)Fx \,\&\, (\exists x)Gx] \supset (\exists x)(Fx \,\&\, Gx)$
m. $\sim (\exists x)Fx \vee (\forall x) \sim Fx$
*n. $(\forall x)[Fx \supset (Gx \,\&\, Hx)] \supset (\forall x)[(Fx \,\&\, Gx) \supset Hx]$
o. $(\forall x)[(Fx \,\&\, Gx) \supset Hx] \supset (\forall x)[Fx \supset (Gx \,\&\, Hx)]$
*p. $(\forall x)(Fx \,\&\, \sim Gx) \vee (\exists x)(\sim Fx \vee Gx)$

q. $(\forall x)(Fx \supset Gx) \supset (\forall x)(Fx \supset (\forall y)Gy)$
*r. $(\forall x)(\forall y)Gxy \supset (\forall x)Gxx$
s. $(\forall x)Gxx \supset (\forall x)(\forall y)Gxy$
*t. $(\forall x)Fxx \supset (\forall x)(\exists y)Fxy$
u. $(\exists x)(\forall y)Gxy \supset (\forall x)(\exists y)Gyx$
*v. $(\exists x)(\exists y)(Lxy = Lyx)$
w. $((\exists x)Lxx \supset (\forall y)Lyy) \supset (Laa \supset Lgg)$

2. Which of the following sentences are quantificationally false?
a. $(\forall x)Fx \;\&\; (\exists x) \sim Fx$
*b. $(\forall x)Fx \;\&\; \sim (\exists x)Fx$
c. $(\exists x)Fx \;\&\; (\exists x) \sim Fx$
*d. $(\exists x)Fx \;\&\; \sim (\forall x)Fx$
e. $(\forall x)(Fx \supset (\forall y) \sim Fy)$
*f. $(\forall x)(Fx \supset \sim Fx)$
g. $(\forall x)(Fx = \sim Fx)$
*h. $(\exists x)Fx \supset (\forall x) \sim Fx$
i. $(\exists x)(\exists y)(Fxy \;\&\; \sim Fyx)$
*j. $(\exists x)Fx \;\&\; \sim (\exists y)Fy$
k. $(\forall x)(\forall y)(Fxy \supset \sim Fyx)$
*l. $(\forall x)(Gx = \sim Fx) \;\&\; \sim (\forall x) \sim (Gx = Fx)$
m. $(\exists x)(\forall y)Gxy \;\&\; \sim (\forall y)(\exists x)Gxy$

3. What is the quantificational status (quantificationally true, quantificationally false, or quantificationally indeterminate) of each of the following sentences?
a. $(\exists x)Fxx \supset (\exists x)(\exists y)Fxy$
*b. $(\exists x)(\exists y)Fxy \supset (\exists x)Fxx$
c. $(\exists x)(\forall y)Lxy \supset (\exists x)Lxx$
*d. $(\forall x)(Fx \supset (\exists y)Gyx) \supset ((\exists x)Fx \supset (\exists x)(\exists y)Gxy)$
e. $(\forall x)(Fx \supset (\exists y)Gya) \supset (Fb \supset (\exists y)Gya)$
*f. $((\exists x)Lxx \supset (\forall y)Lyy) \supset (Laa \supset Lgg)$
g. $(\forall x)(Fx \supset (\forall y)Gxy) \supset (\exists x)(Fx \supset \sim (\forall y)Gxy)$

4. Which of the following pairs of sentences are quantificationally equivalent?

a.	$(\forall x)Mxx$	$\sim (\exists x) \sim Mxx$
*b.	$(\exists x)(Fx \supset Ga)$	$(\exists x)Fx \supset Ga$
c.	$(\forall x)(Fa \supset Gx)$	$Fa \supset (\forall x)Gx$
*d.	$La = (\forall x)Lx$	$(\exists x)Lx$
e.	$(\exists x)Fx \supset Ga$	$(\exists x)(Fx \supset Ga)$
*f.	$(\forall x)(Fx \vee Gx)$	$(\forall x)Fx \vee (\forall x)Gx$
g.	$(\forall x)Fx \supset Ga$	$(\exists x)(Fx \supset Ga)$
*h.	$(\exists x)(Ax \;\&\; Bx)$	$(\exists x)Ax \;\&\; (\exists x)Bx$
i.	$(\forall x)(\forall y)(Fx \supset Gy)$	$(\forall x)(Fx \supset (\forall y)Gy)$
*j.	$(\forall x)(Fx = \sim Gx)$	$(\forall x) \sim (Fx = Gx)$
k.	$(\forall x)(Fx = Gx)$	$Fa = (\forall x)Gx$
*l.	$(\forall x)(Fx \vee (\exists y)Gy)$	$(\forall x)(\exists y)(Fx \vee Gy)$
m.	$(\forall x)(Fx \supset (\forall y)Gy)$	$(\forall x)(\forall y)(Fx \supset Gy)$

5. Which of the following arguments are quantificationally valid?

a. $(\forall x)(Fx \supset Gx)$
Ga
Fa

*b. $(\forall x)(Tx \supset Lx)$
$\sim Lb$
$\sim Tb$

c. $(\forall x)(Kx \supset Lx)$
$(\forall x)(Lx \supset Mx)$
$(\forall x)(Kx \supset Mx)$

*d. $(\forall x)(Fx \supset Gx)$
$(\forall x)(Hx \supset Gx)$
$(\forall x)((Fx \vee Hx) \supset Gx)$

e. $(\forall x)(Fx \supset Gx) \supset (\exists x)Nx$
$(\forall x)(Nx \supset Gx)$
$(\forall x)(\sim Fx \vee Gx)$

*f. $(\sim (\exists y)Fy \supset (\exists y)Fy) \vee \sim Fa$
$(\exists z)Fz$

g. $(\forall x)(\sim Ax \supset Kx)$
$(\exists y) \sim Ky$
$(\exists w)(Aw \vee \sim Lwf)$

*h. $(\forall y)(Hy \,\&\, (Jyy \,\&\, My))$
$(\exists x)Jxb \,\&\, (\forall x)Mx$

i. $(\forall x)(\forall y)Cxy$
$(Caa \,\&\, Cab) \,\&\, (Cba \,\&\, Cbb)$

*j. $(\exists x)(Fx \,\&\, Gx)$
$(\exists x)(Fx \,\&\, Hx)$
$(\exists x)(Gx \,\&\, Hx)$

k. $(\forall x)(Fx \supset Gx)$
$\sim (\exists x)Fx$
$\sim (\exists x)Gx$

*l. $(\exists z)Bzz$
$(\forall x)(Sx \supset Bxx)$
$\sim Sg$

m. $(\exists x)Cx \supset Ch$
$(\exists x)Cx \equiv Ch$

*n. $Fa \vee (\exists y)Gya$
$Fb \vee (\exists y) \sim Gyb$
$(\exists y)Gya$

6. Which of the following alleged entailments hold?

a. $\{(\forall x) \sim Jx,\ (\exists y)(Hby \vee Ryy) \supset (\exists x)Jx\} \models (\forall y) \sim (Hby \vee Ryy)$
*b. $\{(\forall x)(\forall y)(Mxy \supset Nxy)\} \models (\forall x)(\forall y)(Mxy \supset (Nxy \,\&\, Nyx))$
c. $\{(\forall y)((Hy \,\&\, Fy) \supset Gy),\ (\forall z)Fz \,\&\, \sim (\forall x)Kxb\} \models (\forall x)(Hx \supset Gx)$
*d. $\{(\forall x)(Fx \supset Gx),\ (\forall x)(Hx \supset Gx)\} \models (\forall x)(Fx \vee Hx)$
e. $\{(\forall z)(Lz \equiv Hz),\ (\forall x) \sim (Hx \vee \sim Bx)\} \models \sim Lb$

9.4 FINE-TUNING THE TREE METHOD FOR *PL*

In Chapter 8 we noted that there does not exist a decision procedure for deciding, for each sentence of *PL*, whether that sentence is quantificationally true, quantificationally false, or quantificationally indeterminate. That is, there is no mechanical test procedure that always yields, in a finite number of steps, a "yes"

or "no" answer to the question 'Is this sentence of *PL* quantificationally true, and if not is it quantificationally false, and if not is it quantificationally indeterminate?' Nor is there such a decision procedure for equivalence, consistency, validity, or entailment: the system *PL* is *undecidable*. In the current context this means that we cannot produce a mechanical method for constructing trees that will always give correct "yes" or "no" answers in a finite number of steps. The problem here is that not every finite set of sentences of *PL* has a finite truth-tree, where we define a **finite truth-tree** be a truth-tree that either is closed or has a completed open branch. It is an unavoidable result that not every finite set of sentences of *PL/PLE* has a finite tree. There are, however, two ways in which the tree method we have developed can be significantly improved, and our task in this section is to do so.

First, we would like our set of rules to be capable of producing a finite tree for any finite set that has a **finite model,** that is, any set for which there is an interpretation with a finite UD on which all of the members of the set are true. Our present tree rules do not ensure that there is finite tree for every finite set that has a finite model. Consider, for example, the start of a tree for {(∀y)(∃z)Fyz}:

1.	(∀y)(∃z)Fyz	SM
2.	(∃z)Faz✓	1 ∀D
3.	Fab	2 ∃D
4.	(∃z)Fbz✓	1 ∀D
5.	Fbc	4 ∃D
6.	(∃z)Fcz✓	1 ∀D
7.	Fcd	6 ∃D
	•	
	•	
	•	

The dots here indicate that the tree will continue indefinitely. There is no hope of closing the one open branch on this tree. At every other step after the first, a new atomic sentence is added to the open branch, using Existential Decomposition, and since every atomic sentence is quantificationally consistent with every other atomic sentence, continuing to add more atomic sentences will never close the tree. But this branch also can never become a *completed* open branch. Every time Universal Decomposition is applied to the sentence on line 1, a new existentially quantified sentence is added to the branch. And decomposing that sentence adds a new individual constant to the branch, necessitating a further application of Universal Decomposition to '(∀y)(∃z)Fyz', resuming the cycle. We call an open branch that cannot be completed—one that never closes and will never, in a finite number of steps, become a completed open branch—a ***nonterminating branch***.

Assuming we retain the requirement that every universally quantified sentence on a completed open branch must be decomposed using every constant on that branch, the only way to avoid the inevitability of a nonterminating branch in the preceding tree is to revise our Existential Decomposition rule. That rule currently stipulates that a sentence (∃**x**)**P** must be decomposed to a

substitution instance $\mathbf{P(a/x)}$ in which **a** is *foreign* to the branch in question. Let us recall the reason for this restriction: it is that using a constant that *already* occurs on the branch would be to make the unwarranted assumption that the thing that is of the sort **P** is also of the sort specified by the formulas in which it occurs elsewhere on the branch. The following tree illustrates this:

1.	$(\exists x)Fx$ ✓	SM	
2.	$(\exists x) \sim Fx$ ✓	SM	
3.	Fa	1 ∃D	
4.	~ Fa	2 ∃D	**MISTAKE!**
	×		

There is no interpretation on which something is of the sort F and that very same thing is of the sort not-F, but in using 'b' at line 4 as well as line 3 we are, in effect, looking for such an interpretation. It is no surprise that the search fails. Yet the set $\{(\exists x)Fx, (\exists x) \sim Fx\}$ is consistent, as the following correct tree verifies:

1.	$(\exists x)Fx$ ✓	SM
2.	$(\exists x) \sim Fx$ ✓	SM
3.	Fa	1 ∃D
4.	~ Fb	2 ∃D
	o	

However, as the example in the previous paragraph shows, the requirement that a new constant be used to instantiate an existentially quantified sentence sometimes leads to nonterminating branches.

Fortunately, there is another way to think about decomposing an existentially quantified sentence $(\exists \mathbf{x})\mathbf{P}$. Rather than avoid altogether the use of constants that already occur on the branch that contains $(\exists \mathbf{x})\mathbf{P}$, we shall introduce a second, *branching*, Existential Decomposition rule. The branching will allow us to consider substitution instances formed from constants already on the branch as well as a substitution instance formed from a new constant. We will call the new rule ***Existential Decomposition-2:***

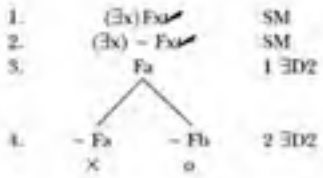
Existential Decomposition-2 (∃D2)

where $\mathbf{a_1}, \ldots, \mathbf{a_m}$ are the constants that already occur on the branch on which Existential Decomposition-2 is being applied to decompose $(\exists \mathbf{x})\mathbf{P}$ and $\mathbf{a_{m+1}}$ is a constant that is foreign to that branch.[5]

[5]This Existential Decomposition rule is due to George Boolos, "Trees and Finite Satisfiability: Proof of a Conjecture of Burgess," *Notre Dame Journal of Formal Logic*, 25(3) (1984), 193–197.

This rule requires that when we decompose an existentially quantified sentence $(\exists \mathbf{x})\mathbf{P}$ we must branch out to the relevant substitution instances. If $\mathbf{a}_1$ through $\mathbf{a}_m$ are the constants that occur on the branch that contains the sentence $(\exists \mathbf{x})\mathbf{P}$ that is being decomposed, then substitution instances formed from those constants are to be entered, each on a distinct branch, and $\mathbf{P}(\mathbf{a}_{m+1}/\mathbf{x})$ is to be entered on a further branch where $\mathbf{a}_{m+1}$ is any constant foreign to the branch in question. Thus Existential Decomposition-2 produces a varying number of new branches, depending on how many constants already occur on the branch to which it is applied.

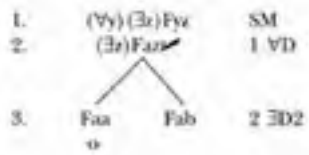
Here is a tree for the set {(∃x)Fx, (∃x) ~ Fx} in which Existential Decomposition-2 is used:



When we first used Existential Decomposition-2, on line 3, there were no constants already occurring on the open branch containing '(∃x)Fx'. In this case the rule requires adding only one substitution instance, formed from any constant. Note that in this case, where no constants yet occur on the branch, Existential Decomposition and Existential Decomposition-2 produce the same result. The second use of Existential Decompostion-2 required branching to two substitution instances of '(∃x) ~ Fx'. On the left branch we formed a substitution instance using the constant 'a' that already occured on the single branch in progress prior to line 4, as required by Existential Decomposition-2, and on the right branch we formed a substitution instance using a new constant, 'b'—again, as required by Existential Decomposition-2. The idea behind branching to these two possibilities is that the individual by virtue of which '(∃x) ~ Fx' is true might be the individual that we have already chosen to designate with the constant 'a', or it might be another individual. To allow for the latter possibility we form a substitution instance with a new constant. The left-hand branch closes because the individual by virtue of which '(∃x) ~ Fx' is true cannot be the individual denoted by 'a' in the sentence 'Fa' at line 3: There is no interpretation on which 'Fa' and '~ Fa' are both true. But the open right-hand branch is complete. This branch contains the literals 'Fa' and 'Fb' and shows that there is an interpretation on which 'Fa' and '~ Fb', and consequently both '(∃x)Fx' and '(∃x) ~ Fx', are true. Any interpretation that includes the following assignments will do:

UD: the set {1, 2}
Fx: x is odd
a: 1
b: 2

So far it may look like Existential Decomposition-2 makes for more work than is necessary, since using the earlier rule Existential Decomposition resulted in a completed open four-line tree for the set {(∃x)Fx, (∃x) ~ Fx} without any branching. But now consider again the set {(∀y)(∃z)Fyz}. We saw that with the rule Existential Decomposition we could only produce a tree with a nonterminating branch for this set, even though the set is quantificationally consistent and has a finite model. Using Existential Decomposition-2 we can produce a truth-tree with a completed open branch for this set:



At line 3 we branched to two substitution instances of the existentially quantified sentence on line 2. The instantiating constant in the substitution instance on the left-hand branch is 'a', which occurred earlier on that branch. The instantiating constant in the substitution instance on the right-hand branch, 'b', was chosen as it is foreign to the branch so far. The sentence at line 2 says that the individual designated by 'a' bears the relation F to something. The branching indicates that something might be the very same individual (this is the left-hand branch) or it might be something else (this is the right-hand branch) If both branches close, we will know there is neither sort of interpretation. But if *either* becomes a completed open branch, we will know that there is a model—indeed, a finite model—for the set being tested. Here the left-hand branch is completed and contains the single literal 'Faa'. So there is a finite model for the set being tested. Any interpretation that includes the following assignments will be such a model:

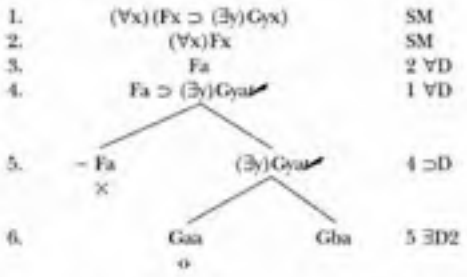
UD: {1}
Fxy: x = y

The right-hand branch is open but is *not* a completed open branch because the universally quantified sentence on line 1 has not been decomposed to a substitution instance formed from 'b'.

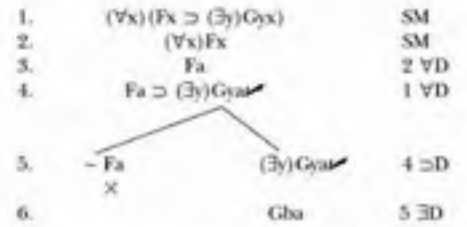
Note that if we were to continue the right-hand branch by instantiating the universal quantification on line 1 with 'b' we would need to branch to three substitution instances when decomposing the existentially quantified sentence '(∃z)Fbz', namely the substitution instances 'Fba' and 'Fbb' (because 'a' and 'b' already occur on that branch), and a substitution instance with a new constant such as 'Fbc'. The first two of these branches will also be completed

open branches, while the third will require us to once again decompose the universal quantification on line 1 using the new constant 'c'. It should be clear that we will never be able to complete the tree. But we don't need to, because the new rule Existential-Decomposition-2 produced a completed open branch at line 3, showing that the set is quantificationally consistent. Using Existential Decomposition-2 rather than Existential Decomposition ensures that the interplay between universal and existential quantifiers will not produce trees with only nonterminating open branches for sets that do have finite models. In fact, we shall prove in Chapter 11 that *using the rule ∃D2 in place of ∃D, every finite set of sentences of PL with a finite model will have a finite tree with a completed open branch.* This is clearly desirable.

Here is a tree using Existential Decomposition-2 for the set {(∀x)(Fx ⊃ (∃y)Gyx), (∀x)Fx}:

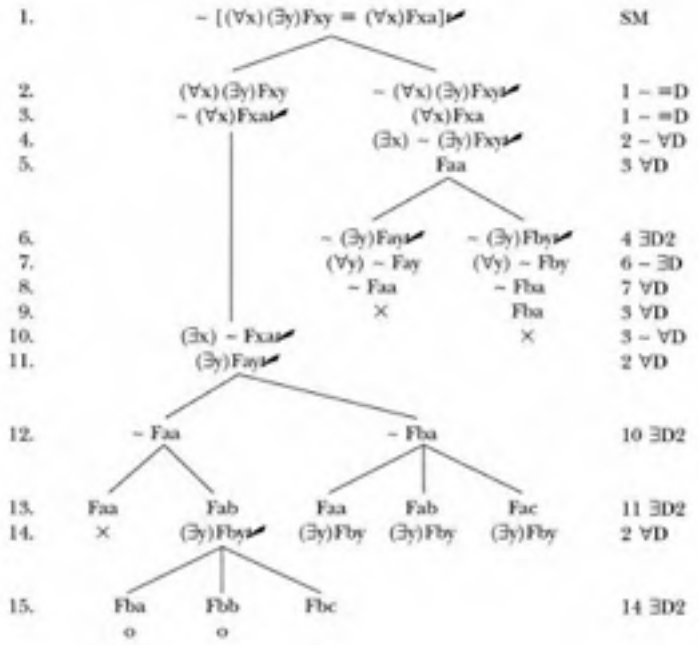


This tree has one completed open branch, the middle branch. The two universally quantified sentences on the branch have been decomposed to the single constant on the branch, 'a'. Every other sentence on the branch is either a literal or has been decomposed. So the set is quantificationally consistent. Had we used Existential Decomposition at line 6, the tree would have had only two branches at that point:



Of course, the left branch still closes, but repeated uses of ∃D on the right branch will only go on to produce more closed branches and a single nonterminating branch.

The following tree shows that the set {~ [(∀x)(∃y)Fxy ≡ (∀x)Fxa]} is quantificationally consistent:



Note that using Existential Decomposition-2 at line 13 resulted in adding two new branches to the existing leftmost branch (one with the constant 'a' that already occured on the leftmost branch and one with the constant 'b' that was foreign to that branch), and adding three new branches to the other existing open branch ending at line 12 (two with the constants 'a' and 'b' that already occurred on that branch and one with the constant 'c' that was foreign to that branch). At line 15, Existential Decomposition-2 produces three branches from

the leftmost open branch of line 14: two substitution instances use the constants 'a' and 'b' that already occur there, and a third uses the constant 'c' that was foreign to that branch. Although the tree is not now complete, it has two completed open branches, the one ending in 'Fba' and the one ending in 'Fbb'. The branch ending in 'Fbc' is not complete, as '$(\forall x)(\exists y)Fxy$' has not been decomposed using the constant 'c'. The branches to line 13 ending in undecomposed existentially quantified sentences are also, for that reason, not complete.

We turn now to the second improvement in our tree method for *PL*. We would like to be assured that when a set does have a finite tree we will eventually find it. The tree rules we have presented do not themselves guarantee this. For example, they allow the construction of trees such as the following one for the set $(\forall x)Fx$, $(\forall x) \sim Fx$:

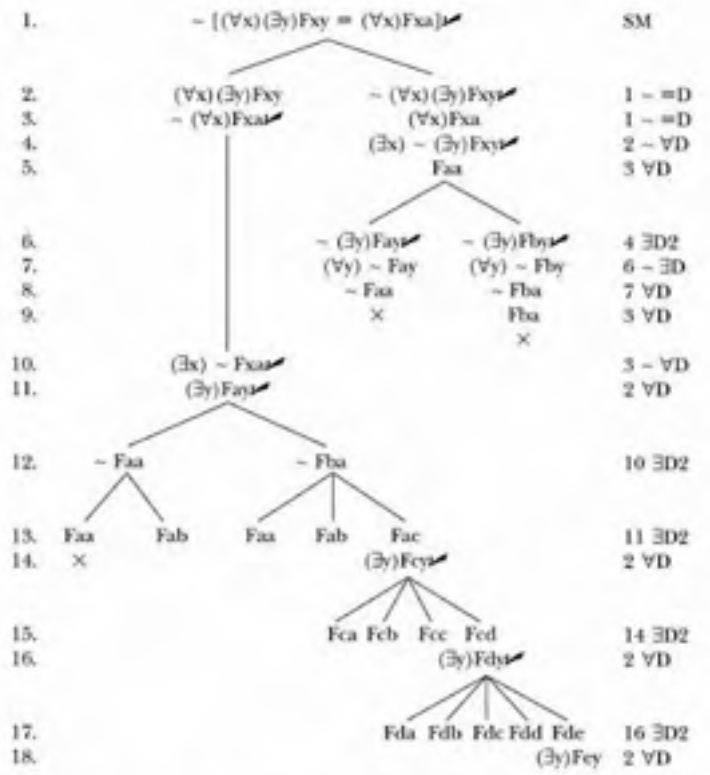
1.	$(\forall x)Fx$	SM
2.	$(\forall x) \sim Fx$	SM
3.	Fa	1 ∀D
4.	Fb	1 ∀D
5.	Fc	1 ∀D
6.	Fd	1 ∀D
	•	
	•	
	•	

Continuing in this way—adding substitution instances that result from applying ∀D to the sentence on line 1—does not involve misusing any tree rule but will never produce either a closed tree or a completed open branch. Yet a closed tree for the set can be produced in just four lines:

1.	$(\forall x)Fx$	SM
2.	$(\forall x) \sim Fx$	SM
3.	Fa	1 ∀D
4.	$\sim Fa$	2 ∀D
	×	

What we need is a procedure for applying the decomposition rules that is guaranteed to yield a finite tree where one exists, not only in this case but also in much more complicated ones. For example, above we produced a tree with a completed open branch for the set $[\sim [(\forall x)(\exists y)Fxy \equiv (\forall x)Fxa]]$, showing that

it is quantificationally consistent. But we could just as well have begun the tree for this set as follows:



In this case, after line 13 we have repeatedly added a substitution instance of the universal quantification on line 2 to the rightmost open branch, using the new constant just introduced by ∃D2, then applied ∃D2 again, generating a new instantiating constant on the rightmost branch, and so on. Clearly we can continue this process forever. So unless care is taken we can work continuously on a branch that won't terminate while ignoring a branch that, if continued, will become a completed open branch.

To guarantee that a finite branch will be found if it exists, we introduce a procedure for constructing trees for *PL* that pursues all possibilities in a systematic fashion, such that a completed open branch will be found if one exists and also such that a tree that can be closed will in fact close. The System is

The System for PL

List the members of the set to be tested.

Exit Conditions: Stop if

a. The tree closes.
b. An open branch becomes a completed open branch.

Construction Procedures:

Stage 1: Decompose all truth-functionally compound and existentially quantified sentences and each resulting sentence that is itself either a truth-functional compound or an existentially quantified sentence.

Stage 2: For each universally quantified sentence $(\forall \mathbf{x})\mathbf{P}$ on the tree, enter $\mathbf{P}(\mathbf{a}/\mathbf{x})$ on every open branch passing through $(\forall \mathbf{x})\mathbf{P}$ for every constant **a** on the branch. On each open branch passing through $(\forall \mathbf{x})\mathbf{P}$ on which no constant occurs, enter $\mathbf{P}(\mathbf{a}/\mathbf{x})$.

Repeat this process until every universally quantified sentence on the tree, including those added as a result of this process, has been so decomposed.

Return to Stage 1.

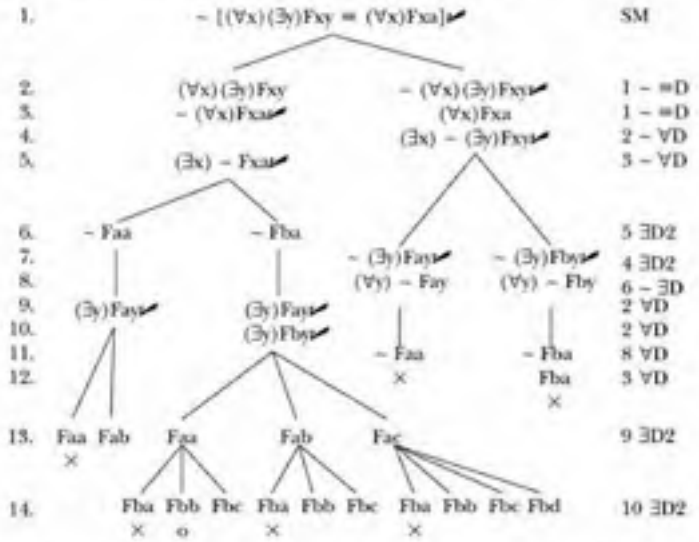
We call trees that have been constructed in accordance with the System ***systematic trees.*** In all systematic trees Existential Decomposition-2 is used rather than Existential Decomposition. To construct a systematic tree, proceed through the tree construction stages in the order specified. That is, first the members of the set being tested are listed. Next all truth-functionally compound and all existentially quantified sentences are decomposed. When decomposing these sentences it is wise to apply nonbranching rules before applying branching rules. When and only when the tree contains no undecomposed truth-functional compounds and no undecomposed existentially quantified sentences do we proceed to Stage 2. At Stage 2 each universally quantified sentence on the tree is decomposed as specified. It is important to note that work at Stage 2 is not complete until every universally quantified sentence on the tree has been decomposed in the required manner, including both those that were on the tree when we passed from Stage 1 to Stage 2 and those that are entered as a result of work at Stage 2. The first tree that we began to construct for the set $\{(\forall x)Fx, (\forall x) \sim Fx\}$ was not a systematic tree because it violated Stage 2. In that tree the universally quantified sentence on line 1 was decomposed at line 4 with the new constant 'b', violating the requirement that all universally quantified sentences must be decomposed using constants already on the tree, and only those (except

when there is no constant). Following The System produces a closed tree for this set:

1.	$(\forall x)Fx$	SM
2.	$(\forall x) \sim Fx$	SM
3.	Fa	1 $\forall$D
4.	$\sim Fa$	2 $\forall$D
	×	

(Note that this is the four-line tree that we earlier displayed for this set.) Here at line 4 we decomposed the universally quantified sentence from line 2 with the constant 'a' that already occurred on the branch. Doing so not only completes Stage 2 but also terminates construction because an Exit Condition has been met: the tree has closed.

Although our first tree for the set $[\sim [(\forall x)(\exists y)Fxy \equiv (\forall x)Fxa]]$ included a completed open branch, that tree is not a *systematic* tree. It first violates the method set out in The System at line 5, where it decomposes a universally quantified sentence before all of the truth-functionally compound and existentially quantified sentences have been decomposed. These latter include the negated universally quantified sentence on line 3 and the existentially quantified sentence on line 4. Here is a *systematic* tree for the same set:

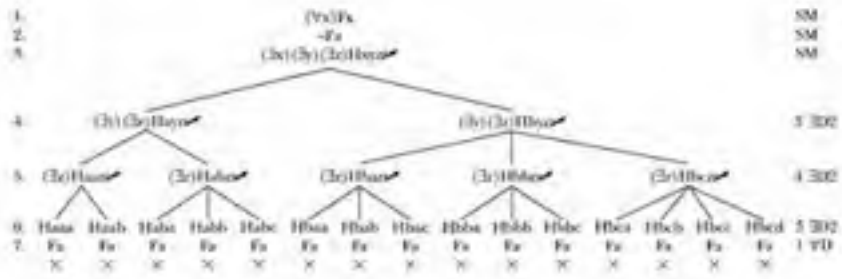


At line 8 all of the truth-functionally compound and existentially quantified sentences occurring on lines 1–7 have been decomposed, so Stage 1 has been completed and Stage 2 commences. Stage 2 continues through line 12, at which point all universally quantified sentences have been decomposed using all of

the relevant constants, so we return to Stage 1 to continue at line 13. Stage 1 produces a completed open branch on line 14, so the procedure terminates. Interestingly, this tree is one line shorter than our earlier tree with a completed open branch for the same set. Unfortunately, as we shall shortly see, systematic trees are not always so economical.

Systematic trees differ from nonsystematic trees in three important respects. First, in systematic trees Existential Decomposition-2 is always used to decompose existentially quantified sentences. In nonsystematic trees either Existential Decomposition, or Existential Decomposition-2 (or both) may be used. Second, The System does not allow work on one branch to be continued to the point of excluding all work on another open branch. Third, it does not allow us to ignore constants already occurring on a tree when we are using Universal Decomposition in Stage 2, so that if a substitution instance using such a constant can close a branch, we will be sure that the branch *does* close.

The advantage of constructing systematic trees is that doing so will always lead, in a finite number of steps, to a completed open branch when one exists, and, also in a finite number of steps, to a closed tree when one exists. The disadvantage is that systematic trees can frequently be much larger than are nonsystematic trees. For example, here is a systematic tree for the set {(∀x)Fx, ~ Fa, (∃x)(∃y)(∃z)Hxyz}:



Obviously, we could have produced a closed tree in four lines, by entering 'Fa' (obtained by decomposing the sentence on line 1) at line 4. But the result would not be a systematic tree. The System requires us first to decompose the sentence on line 3, then those on line 4, and then those on line 5, before decomposing the universally quantified sentence from line 1. So we do not claim that The System always produces the smallest trees possible, and wherever we see a more economical way to produce either a closed tree or a completed open branch we should do so. What The System *does* guarantee is that if there is a tree with a completed open branch, The System will generate such a tree, and if there is a closed tree, The System will generate a closed tree. Because The System is reliable in this sense, it should be used when one does

not see how to close a branch or produce a completed open branch without using The System.

This is not to say that the method of The System will always result either in a closed tree or in a tree with a completed open branch. Consider, for example, the set {(∀x)(∃y)Fxy, ~ (∃x)Fxx, (∀x)(∀y)(∀z)[(Fxy & Fyz ⊃ Fxz]}. This set is consistent, but every model for the set (every interpretation on which all of the set members are true) has an infinite UD. The System will always produce a completed open branch for a finite set that has a finite model, but it will not do so in the case of consistent sets that have only infinite models—for the very good reason that there are no trees with completed open branches for such sets. Here is the start of a systematic tree for this set:

After listing the set members, we move to Stage 1. Once '~ (∃x)Fxx' has been decomposed to '(∀x) ~ Fxx', every truth-functional compound (the sentence on line 2) and every existentially quantified sentence on the tree (there are none) has been decomposed. Proceeding to Stage 2 we decompose each universally quantified sentence on the tree to a substitution instance formed from 'a' (since there are at this point no constants occurring in the tree), taking us through line 9. Returning to Stage 1, there are two sentences to be decomposed: the existentially quantified sentence on line 5 and the truth-functional compound on line 9. We choose to decompose the latter first, as it yields one closed branch (the right branch), at line 10. Next we decompose the existentially quantified sentence from line 5. This yields one closed branch (the left branch) and one open branch.

At this point we still have one undecomposed truth-functional compound, on line 10. Decomposing this sentence yields two open branches. As it happens these are identical—exactly the same sentences occur on each branch—but The System requires us to pursue both branches. Here we proceed to Stage 2, adding lines 13–25c although all the universally quantified sentences on the tree have already been decomposed to substitution instances formed from 'a' they must now also all be decomposed to substitution instances formed from 'b' since 'Fab', which contains this constant, occurs on both branches. Back to Stage 1 at line 26, we decompose the existentially quantified sentences from line 13 first, splitting each of the existing two branches into three branches. Two of these branches close. At line 27 we branch again when we decompose the material conditional occurring on line 20. Four of the eight resulting branches close. Decomposing '~ (Fab & Fba)', which occurs four times on line 27, results in eight branches, six of which close.

Were we to continue, we would next decompose the remaining truth-functional compounds. Some, but not all branches, would close. Eventually we would return to Stage 1 and decompose all universally quantified sentences to substitution instances formed from 'c', since that now appears on the open branches (at line 26). This would produce a new existentially quantified sentence that would eventually be decomposed to substitution instances formed from 'a', 'b', 'c', and 'd', respectively. Branches containing the first three substitution instances would eventually close, but not all the branches containing the latter substitution instance would do so.

We have not *proved* that this tree will never close and will never have a completed open branch, but this is the case (the only way to demonstrate this is to show, independently of the tree method, that our set is quantificationally consistent, that it has only infinite models, and that no set with only infinite models has a finite tree). Here the point is that the tree method cannot be used to show that sets such as this one are quantificationally consistent. We abandon the tree; we do not complete it. However, having used The System, we can be sure that we have not, as far as we have gone, missed a completed open branch or a chance to close the tree.

While instructions for identifying (without fail) a systematic tree that is caught in an endless cycle of decompositions and is such that it has only

nonterminating branches would be desirable, there can be no such instructions because there is no decision procedure for quantificational consistency. We can only say that, if one has cycled through the stages of The System several times and there are still open branches, one should consider the possibility that the set has only infinite models and consider abandoning the tree. Abandoning a tree constitutes a failure to find an answer to the question being asked. Having abandoned a tree, one can try directly to establish the consistency of the set in question by trying to find an interpretation on which all the members of the set are true.

9.4E EXERCISES

1. Construct systematic trees to determine, for each of the following sets, whether that set is quantificationally consistent. State your result. If you abandon a tree, explain why.

a. {(∀x)Jx, (∀x)(Jx ≡ (∃y)(Gyx ∨ Ky))}
*b. {(∀x)(Fx ⊃ Cx), ~ (∀x)(Fx & Cx)}
c. {(∃x)Fx, (∃x) ~ Fx}
*d. {~ (∀x) ~ Hx, (∀x)(Hx ⊃ Kx), ~ (∃x)(Kx & Hx)}
e. {(∃x)Fx & (∃x) ~ Fx, (∃x)Fx ⊃ (∀x) ~ Fx}
*f. {(∃x)Fx & (∃x) ~ Fx, (∀x)Fx ⊃ (∀x) ~ Fx}
g. {(∀x)(∃y)Fxy, (∃y)(∀x) ~ Fyx}
*h. {(∀x)(~ Gx ⊃ Fx), (∃x)(Fx & ~ Gx), Fa ⊃ ~ Ga}
i. {(∃x)Hx, ~ (∀x)Hx, (∀x)(Hx ⊃ Kx), (∃x)(Kx & Hx)}
*j. {(∃x)(∀y)Lxy, (∃x)(∀y) ~ Lxy}
k. {(∀x)(∃y)Lxy, (∀x)(∃y) ~ Lxy}
*l. {(∀x) ~ (∃y)Lxy, (∀w)(∀y)(Swy ∨ ~ Lwy), ~ (∃x) ~ (∃z)Sxz}
m. {(∀x)(∃y)Fxy, (∃x)(∃y) ~ Fxy}
*n. {(∀x)(∀y)(∀z)((Hxy & Hyz) ⊃ Hxz), (∀x)(∀y)(Hxy ⊃ Hyx), (∃x) ~ Hxx}
o. {~ (∀x)(Kx ⊃ (∀y)(Ky ∨ Lxy)), (∀y)(Ky ⊃ (∀x)(Rx ⊃ Lyx)), (∀x)Rx}

2. Construct systematic trees to determine, for each of the following sentences, whether that sentence is quantificationally true, quantificationally false, or quantificationally indeterminate. In each case state your result. If you abandon a tree, explain why.

a. (∀x)(Fax ⊃ (∃y)Fya)
*b. (∃x) ~ Fx ⊃ (Fa ⊃ ~ Fb)
c. (∀x)[Fx ⊃ (∀y)(Hy ⊃ Fy)]
*d. (∃y)(∀x)Fxy ⊃ (∀x)(∃y)Fxy
e. (∃x)(Fx ∨ ~ Fx) ≡ ((∃x)Fx ∨ (∃x) ~ Fx)
*f. (∀x)(Fx ≡ [(∃y)Gyx ⊃ H]) ⊃ (∀x)[Fx ⊃ (∃y)(Gyx ⊃ H)]
g. (∀x)(Fx ⊃ [(∃y)Gyx ⊃ H]) ⊃ (∀x)[Fx ⊃ (∃y)(Gyx ⊃ H)]

3. Construct systematic trees to determine which of the following arguments are quantificationally valid. In each case state your result. If you abandon a tree, explain why.

a. Fa
$(\forall x)(Fx \supset Cx)$
$(\forall x)(Fx \mathbin{\&} Cx)$

*b. $(\forall x)(Jx \vee Ixb) \vee (\forall x)(\exists y)(Hxy \supset Mx)$
Iab

c. Fa
$(\forall x)(Fx \supset Cx)$
$(\exists x)(Fx \mathbin{\&} Cx)$

*d. $\sim (\forall y)Kyy \vee (\forall x)Hxx$
$(\exists x)(\sim Hxx \supset \sim Kxx)$

e. $(\forall x)(\forall y)(\forall z)[(Lxy \mathbin{\&} Lyz) \supset Lxz]$
$(\forall x)(\forall y)(Lxy \supset Lyx)$
$(\forall x)Lxx$

*f. $(\forall x)(\forall y)(Fx \vee Gxy)$
$(\exists x)Fx$
$(\exists x)(\exists y)Gxy$

g. $(\exists x)[(Lx \vee Sx) \vee Kx]$
$(\forall y) \sim (Ly \vee Ky)$
$(\exists x)Sx$

*h. $(\exists x)((Lx \vee Sx) \vee Kx)$
$(\forall y) \sim (Ly \vee Ky)$
$(\forall x)Sx$

i. $(\forall x)(Hx \supset Kcx)$
$(\forall x)(Lx \supset \sim Kcx)$
Ld
$(\exists y) \sim Hy$

4. Construct systematic trees to determine which of the following pairs of sentences are quantificationally equivalent. In each case state your result. If you abandon a tree, explain why.

a. $(\forall x)(\forall y) \sim Sxy$ — $\sim (\exists x)(\exists y)Sxy$
*b. $(\forall x)(\exists y)Lxy$ — $(\exists y)(\forall x)Lyx$
c. $(\exists x)(Ax \supset B)$ — $(\forall x)Ax \supset B$
*d. $(\forall x)(Ax \supset B)$ — $(\forall x)Ax \supset B$
e. $(\forall x)(Ax \supset B)$ — $(\exists x)Ax \supset B$
*f. $(\exists x)(Ax \supset B)$ — $(\exists x)Ax \supset B$
g. $(\exists x)(\exists y)Hxy$ — $(\exists y)(\exists x)Hxy$

5. Construct systematic trees to determine which of the following alleged entailments hold. In each case state your result. If you abandon a tree, explain why.

a. $[(\forall x)(Fax \supset Fxa)] \vDash Fab \vee Fba$
*b. $[(\forall x)(\forall y)(Fx \vee Gxy), (\exists x)Fx] \vDash (\exists x)(\exists y)Gxy$
c. $[\sim Fa, (\forall x)(Fa \supset (\exists y)Gxy)] \vDash \sim (\exists y)Gay$
*d. $[(\exists x)(\forall y)Gxy] \vDash (\forall y)(\exists x)Gxy$
e. $[(\exists x)Gx, (\forall x)(Gx \supset Dxx)) \vDash (\exists x)(Gx \mathbin{\&} (\forall y)Dxy)$
*f. $[(\forall y)(\exists x)Gxy \vDash (\exists x)(\forall y)Gxy$

***6.** Show that if the members of a set Γ of sentences of *PL* contain only '$\sim$' and universal and existential quantifiers as logical operators, then Γ has no tree with more than one branch if the rule $\exists$D is used but may have a tree with more than one branch if $\exists$D2 is used.

7. Show that no closed truth-tree can have an infinite branch.

*8. Could we replace Universal Decomposition and Existential Decomposition with the following two rules? Explain.

$(\forall \mathbf{x})\mathbf{P}$✓	$(\exists \mathbf{x})\mathbf{P}$✓
$-(\exists \mathbf{x})-\mathbf{P}$	$-(\forall \mathbf{x})-\mathbf{P}$

9. Let $\mathbf{P}(\mathbf{a}/\mathbf{x})$ be a substitution instance of some sentence $(\exists \mathbf{x})\mathbf{P}$ such that $\{\mathbf{P}(\mathbf{a}/\mathbf{x})\}$ has a closed tree. Does it follow that $\{(\exists \mathbf{x})\mathbf{P}\}$ has a closed tree? Explain.

*10. Let $(\forall \mathbf{x})\mathbf{P}$ be a sentence such that, for every substitution instance $\mathbf{P}(\mathbf{a}/\mathbf{x})$, $\{\mathbf{P}(\mathbf{a}/\mathbf{x})\}$ has a closed tree. Does it follow that a systematic tree for $\{(\forall \mathbf{x})\mathbf{P}\}$ will close? Explain.

11. What would have to be done to make The System a mechanical procedure?

*12. Suppose a tree for a set Γ of sentences of *PL* is abandoned without either closing or having a completed open branch. Suppose also that we find a model on which all the members of Γ are true. Suppose the model is an infinite model. Does it follow that all the open branches on the abandoned tree are nonterminating branches? Suppose the model is finite. Does anything follow regarding the abandoned tree?

9.5 TREES FOR *PLE*

To apply the tree method to sentences of the language *PLE*, we will modify the tree system developed in Sections 9.1–9.3 to accommodate the additional features of *PLE*: the identity predicate and complex terms. We shall introduce one new decomposition rule (Identity Decomposition), modify the definitions of a closed branch and of a completed open branch, and revise the Universal Decomposition rule to accommodate complex terms.[6]

We begin with the modification to Universal Decomposition, which is straightforward. The set $\{(\forall x) - Bx, Bf(c)\}$, which contains a closed complex term, '$f(c)$', is clearly quantificationally inconsistent and so we want it to have a closed truth-tree. To this end, we need to allow Universal Decomposition to yield substitution instances formed from *any* closed term, not just constants. For example, we want Universal Decomposition to license step 3 in the following tree:

1.	$(\forall x) - Bx$	SM
2.	$Bf(c)$	SM
3.	$-Bf(c)$	1 ∀D
	×	

[6] It is also possible to use the rule Existential Decomposition-2, developed in Section 9.4, for trees for *PLE*. We shall in fact do so in Section 9.6. But because Existential Decomposition is simpler in many cases (and because some readers may have chosen to skip 9.4), we revert to this rule for the purposes of this section.

We therefore revise Universal Decomposition as follows:

Universal Decomposition (∀D)

$$\frac{(\forall \mathbf{x})\mathbf{P}}{\mathbf{P}(\mathbf{t}/\mathbf{x})}$$

where **t** is a closed term

This change allows the use of Universal Decomposition at line 3 of the previous tree. And because the single branch of this tree contains the atomic sentence '$Bf(c)$' and its negation '$\sim Bf(c)$', the tree is closed and we may conclude that the set being tested is quantificationally inconsistent. We must also amend our definition of a completed open branch so as to require every universally quantified sentence to be decomposed to every substitution instance that can be formed from a *closed individual term* (individual constant or closed complex term) occurring on the branch in question. Previously we required only that universally quantified sentences be decomposed to every *individual constant* occurring on the branch in question.

We hasten to add that the rule Existential Decomposition remains the same for *PLE*; when we decompose an existentially quantified sentence $(\exists \mathbf{x})\mathbf{P}$ we will always use an individual constant **a** that is foreign to the branch on which the substitution instance $\mathbf{P}(\mathbf{a}/\mathbf{x})$ will be entered, just as we did for *PL*. We will *not* use complex terms in these instantiations, because a complex term such as '$h(a)$' carries information about the individual it denotes, namely, that the individual is related to some individual (that denoted by 'a') by the function h.

Here is the decomposition rule for identity sentences:

Identity Decomposition (=D)

$$\frac{\mathbf{t}_1 = \mathbf{t}_2 \qquad \mathbf{P}}{\mathbf{P}(\mathbf{t}_1//\mathbf{t}_2)}$$

where $\mathbf{t}_1$ and $\mathbf{t}_2$ are closed individual terms and **P** is a literal containing $\mathbf{t}_2$

This rule is to be understood as follows: If a branch contains both a sentence of the form $\mathbf{t}_1 = \mathbf{t}_2$ (where $\mathbf{t}_1$ and $\mathbf{t}_2$ are closed individual terms) and a literal **P** containing the term $\mathbf{t}_2$, $\mathbf{P}(\mathbf{t}_1//\mathbf{t}_2)$ may be entered on that branch where $\mathbf{P}(\mathbf{t}_1//\mathbf{t}_2)$ is like **P** except that it contains $\mathbf{t}_1$ *in at least one place* where **P** contains $\mathbf{t}_2$. The rationale behind this rule is that if $\mathbf{t}_1$ and $\mathbf{t}_2$ designate one and the same thing then whatever is true of the individual designated by $\mathbf{t}_1$ must

thereby be true of the individual designated by $\mathbf{t}_2$. Note that the identity sentence $\mathbf{t}_1 = \mathbf{t}_2$ is not checked off because it can be decomposed again and again.

Identity Decomposition is used at line 7 in the following tree:

1.	$(\forall x)(Fx \supset Gx)$		SM
2.	Fc		SM
3.	$\sim Gd$		SM
4.	$c = d$		SM
5.	$Fc \supset Gc$ ✓		1 ∀D
6.	$\sim Fc$	Gc	5 ⊃D
7.	×	$\sim Gc$	3, 4 =D
		×	

Here $\mathbf{t}_1 = \mathbf{t}_2$ is '$c = d$', $\mathbf{P}$ is '$\sim Gd$', and $\mathbf{P}(\mathbf{t}_1//\mathbf{t}_2)$ is '$\sim Gc$', the sentence entered on line 7, which is the result of substituting 'c' for 'd' in '$\sim Gd$'. Note that the justification column for line 7 contains two line numbers. This is because Identity Decomposition licenses the entry of a sentence on a branch based on the presence of two other sentences. In this respect it is unlike the other decomposition rules.

Now that we have added a rule for Identity Decomposition we will need to modify the definition of a closed branch for *PLE*. To see why, consider the sentence '$(\exists y) \sim y = y$'. This sentence says 'There is something that is not identical with itself' and is clearly quantificationally false. So we want the tree for the unit set of this sentence to close:

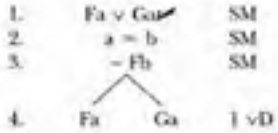
1.	$(\exists y) \sim y = y$ ✓	SM
2.	$\sim a = a$	1 ∃D

The one branch on this tree does not contain an atomic sentence and its negation. So it is not, by our present account, a closed branch. What is perhaps worse, the branch is, by the account given in Section 9.2, a *completed* open branch—the sentence on line 1 has been decomposed, the sentence on line 2 is a literal, and the branch is not closed. Recall that our reason for declaring a branch on which an atomic sentence and its negation both occur to be a closed branch was that there is no interpretation on which an atomic sentence and its negation are both true. But almost as obviously, there is no interpretation on which a sentence of the form $\sim \mathbf{t} = \mathbf{t}$ is true; this is a consequence of the fixed interpretation of the identity predicate. So we modify our definition of a closed branch for *PLE* as follows:

> **Closed branch:** A branch on which some atomic sentence and its negation both occur or on which a sentence of the form $\sim \mathbf{t} = \mathbf{t}$ occurs

By this revised account the single branch of the above tree is closed, so the tree itself is closed and we may conclude that the sentence '$(\exists y) \sim y = y$' is indeed quantificationally false.

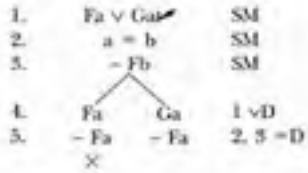
We noted earlier that we must amend our definition of a completed open branch so as to require every universally quantified sentence to be decomposed to every substitution instance that can be formed from a closed individual term (individual constant or closed complex term) occurring on the branch in question. But we also need to modify our definition of complete open branches in another way. To see why, consider the following tree for the set {Fa ∨ Ga, a = b, ~ Fb}, which is quantificationally consistent.



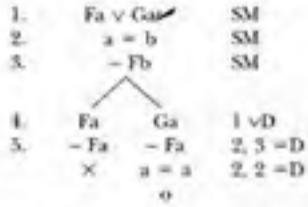
If we simply adopt the definition of completed open branches that we gave for trees for sentences of *PL*, both of the open branches on this tree will count as complete: on each branch every sentence is either a literal, or a sentence that is not universally quantified but is decomposed. (Here we ignore the requirement for universally quantified sentences since none appear in this tree.) But this result is not welcome, even though the set we are testing is quantificationally consistent, because Identity Decomposition does allow further lines to be added to the tree and doing so will produce one closed branch, for the three literals 'a = b', '~ Fb', and 'Fa' on the left branch cannot all be true on a single interpretation. If we apply Identity Decomposition to the sentences on lines 3 and 4 we will add '~ Fa' to both branches, causing the left branch to close. So we want to be certain that Identity Decomposition is applied exhaustively. We therefore modify our account of completed open branches for trees for *PLE* by qualifying the first and third clauses and adding a fourth:

> A branch of a truth-tree for a set of sentences of *PLE* is a **completed open branch** if and only if it is a finite open branch (that is, an open branch with a finite number of sentences) and each sentence occurring on that branch is either
>
> 1. A literal that is not an identity sentence
> 2. A compound sentence that is not a universally quantified sentence and is decomposed
> 3. A universally quantified sentence $(\forall \mathbf{x})\mathbf{P}$ such that $\mathbf{P}(\mathbf{t}/\mathbf{x})$ also occurs on that branch for *each closed individual term* $\mathbf{t}$ occurring on the branch and at least one substitution instance $\mathbf{P}(\mathbf{t}/\mathbf{x})$ occurs on the branch
> 4. A sentence of the form $\mathbf{t}_1 = \mathbf{t}_2$, where $\mathbf{t}_1$ and $\mathbf{t}_2$ are closed terms such that the branch also contains, for every literal $\mathbf{P}$ on that branch containing $\mathbf{t}_2$, every sentence $\mathbf{P}(\mathbf{t}_1//\mathbf{t}_2)$ that can be obtained from $\mathbf{P}$ by Identity Decomposition

Clause 4 requires that we continue work on our last tree, using Identity Decomposition to add '~ Fa' to both branches:



Here the left-hand branch has closed. The right-hand branch may appear to be a completed open branch, but it is not. This is because we must, by clause 4, replace 'b' with 'a' in *every* literal that occurs on the right branch, and 'a = b' is itself such a literal. Replacing 'b' with 'a' in this literal produces 'a = a'. This final application of Identity Decomposition yields the following completed tree, a tree with one closed branch and one completed open branch:



Adding a sentence of the form **t** = **t** will never, of course, bring about the closure of a branch, for if the negation of that sentence, ~ **t** = **t** were already on a branch, or were later added to a branch, the presence of that sentence by itself would close all the branches on which it occurs. For this reason we shall informally allow the omission of applications of Identity Decomposition that result in adding sentences of the form **t** = **t** to a branch. However, we will have to drop this informal practice when we develop *systematic trees* for *PLE* in Section 9.6, for the metatheory of Chapter 11 assumes that Identity Decomposition is rigorously applied in all such trees.

As we did for *PL*, we can once again use the literals on a completed open branch as a guide for constructing an interpretation on which all of the members of the set for which the tree was constructed are true. The open branch on the preceding tree has five literals: 'a = b', '~ Fb', 'Ga', '~ Fa', and 'a = a'. The last will be true on any interpretation, so it will not play a role in constructing a model for the set members. On the other hand, the identity 'a = b' tells us that 'a' and 'b' must designate the same individual. That suggests we try using a single-member UD, letting the constants 'a' and 'b' both designate that single member. To make the other tree literals true we need to intepret the predicate 'G' so that the single member is in its extension, and 'F' so that its extension *excludes* that single member. Any interpretation that includes the following assignments will therefore be a model for the set members:

UD: {1}
a: 1
b: 1
Fx: x is even
Gx: x is odd

All of the set {Fa ∨ Ga, a = b, ~ Fb} is true on any such interpretation. Because *PLE* includes identity sentences such as 'a = b', when we construct interpretations of sets of sentences of *PLE* (rather than of *PL*) from completed open branches we must sometimes assign the same member of the UD to distinct constants.

Given our expanded set of rules and revised definitions of closed and completed open branches, the explications developed in Sections 9.2 and 9.3 of semantic properties in terms of open and closed trees also hold for *PLE*. We therefore adopt them for *PLE* without repeating them here.

The set {a = b, (∀x)(Fbx & ~ Fxa)} is quantificationally inconsistent:

1.	a = b	SM
2.	(∀x)(Fbx & ~ Fax)	SM
3.	Fbb & ~ Fab ✓	2 ∀D
4.	Fbb	3 &D
5.	~ Fab	3 &D
6.	Fab	1, 4 =D
	×	

What is interesting here is the use of Identity Decomposition at line 6. We generated 'Fab' from 'a = b' and 'Fbb' by replacing only the first occurrence of 'b' in the latter with 'a'. (Recall that when generating $\mathbf{P}(\mathbf{t}_1//\mathbf{t}_2)$ from $\mathbf{P}$, given $\mathbf{t}_1 = \mathbf{t}_2$, it is not required that *every* occurrence of $\mathbf{t}_2$ in $\mathbf{P}$ be replaced with $\mathbf{t}_1$ but only that at least one occurrence be so replaced.) We could also have closed the tree by using Identity Decomposition to enter 'Faa' at line 6 (replacing both occurrences of 'b' in 'Fbb' with 'a') and then '~ Faa' at line 7 (replacing 'b' in '~ Fab' with 'a').

Consider *now* the quantificationally consistent set {c = b, (∀x)(Fxc ⊃ ~ Gxb), (∀x)Gxc}. Here is a tree for this set:

1.	c = b		SM
2.	(∀x)(Fxc ⊃ ~ Gxb)		SM
3.	(∀x)Gxc		SM
4.	Gcc		3 ∀D
5.	Gbc		3 ∀D
6.	Fcc ⊃ ~ Gcb ✓		2 ∀D
7.	Fbc ⊃ ~ Gbb ✓		2 ∀D
8.	~ Fcc	~ Gcb	6 ⊃D
9.		~ Gcc ×	1, 8 =D
10.	~ Fbc	~ Gbb	7 ⊃D
11.	o	~ Gbc ×	1, 10 =D

The left-hand branch of this tree is a completed open branch and hence establishes that the set is quantificationally consistent. The left-hand branch contains every required sentence that can be generated from the identity on line 1 and a literal containing 'b' (excepting the sentence 'c = c'). We could generate 'Gcc' by applying Identity Decomposition to the sentences at lines 1 and 5, but it already occurs on line 4 so there is need to do so. Similarly we could generate '~ Fcc' from lines 1 and 10, but it already occurs at line 8. Given the literals 'c = b', 'Gcc', 'Gbc', '~ Fcc', and '~ Fbc', we know that any interpretation that includes the following assignments will be a model for the set {c = b, (∀x)(Fxc ⊃ ~ Gxb), (∀x)Gxc}:

UD: {1}
b: 1
c: 1
Fxy: x is the successor of y
Gxy: x squared equals y

Note that while Identity Decomposition allows, given an identity sentence $\mathbf{t}_1 = \mathbf{t}_2$, the generation of literals in which one or more occurrences of $\mathbf{t}_2$ in an existing literal have been replaced with $\mathbf{t}_1$, it does not license the generation of literals in which one or more occurrences of $\mathbf{t}_1$ have been replaced with $\mathbf{t}_2$. We could rewrite Identity Decomposition so as to allow this, but it is not necessary to do so. That is, if a set is inconsistent it will have a closed tree given the rules as presently written. Rewriting Identity Decomposition in the suggested way would allow adding more literals to many trees, but for no useful purpose.

As explained in Chapter 7, identity is a reflexive, symmetric, and transitive relation. Accordingly, we expect the following sentences of *PLE*, which assert, respectively, the reflexivity, symmetry, and transitivity of identity to be quantificationally true:

(∀x)x = x
(∀x)(∀y)(x = y ⊃ y = x)
(∀x)(∀y)(∀z)[(x = y & y = z) ⊃ x = z]

The truth-tree method produces closed trees for the negations of these sentences. Here is the relevant tree for the claim that identity is reflexive:

1.	~ (∀x)x = x✓	SM
2.	(∃x) ~ x = x✓	1 ~ ∀D
3.	~ a = a	2 ∃D
	×	

The tree is closed, so the truth-tree method yields the desired result that the sentence '~ (∀x)x = x' is quantificationally false and '(∀x)x = x' is quantificationally true. It should be noted that, when we earlier modified the definition

of a closed branch so as to count every branch containing a sentence of the form '~ $\mathbf{t}_1 = \mathbf{t}_1$' as a closed branch, we were, in effect, presupposing the reflexivity of identity. The present result is therefore neither a surprising one nor an independent proof of the reflexivity of identity. The relevant tree for the claim that identity is symmeric is

1.	$\sim (\forall x)(\forall y)(x = y \supset y = x)$ ✓	SM
2.	$(\exists x) \sim (\forall y)(x = y \supset y = x)$ ✓	1 ~ ∀D
3.	$\sim (\forall y)(a = y \supset y = a)$ ✓	2 ∃D
4.	$(\exists y) \sim (a = y \supset y = a)$ ✓	3 ~ ∀D
5.	$\sim (a = b \supset b = a)$ ✓	4 ∃D
6.	$a = b$	5 ~ ⊃D
7.	$\sim b = a$	5 ~ ⊃D
8.	$\sim a = a$	6, 7 =D
	×	

The tree is closed, establishing that '~ $(\forall x)(\forall y)(x = y \supset y = x)$' is quantificationally false and '$(\forall x)(\forall y)(x = y \supset y = x)$' is quantificationally true.

Finally we consider transitivity. The relevant tree is

1.	$\sim (\forall x)(\forall y)(\forall z)[(x = y \,\&\, y = z) \supset x = z]$ ✓	SM
2.	$(\exists x) \sim (\forall y)(\forall z)[(x = y \,\&\, y = z) \supset x = z]$ ✓	1 ~ ∀D
3.	$\sim (\forall y)(\forall z)[(a = y \,\&\, y = z) \supset a = z]$ ✓	2 ∃D
4.	$(\exists y) \sim (\forall z)[(a = y \,\&\, y = z) \supset a = z]$ ✓	3 ~ ∀D
5.	$\sim (\forall z)[(a = b \,\&\, b = z) \supset a = z]$ ✓	4 ∃D
6.	$(\exists z) \sim [(a = b \,\&\, b = z) \supset a = z]$ ✓	5 ~ ∀D
7.	$\sim [(a = b \,\&\, b = c) \supset a = c]$ ✓	6 ∃D
8.	$(a = b \,\&\, b = c)$ ✓	7 ~ ⊃D
9.	$\sim a = c$	7 ~ ⊃D
10.	$a = b$	8 &D
11.	$b = c$	8 &D
12.	$a = c$	10, 11 =D
	×	

As expected, this tree is closed, reflecting the fact that the sentence on line 1 is quantificationally false and '$(\forall x)(\forall y)(\forall z)[(x = y \,\&\, y = z) \supset x = z]$' is quantificationally true; that is, that identity is transitive. Here we closed the tree by applying Identity Decomposition to lines 10 and 11, taking '$a = b$' as $\mathbf{t}_1 = \mathbf{t}_2$ and '$b = c$' as $\mathbf{P}$, producing '$a = c$' as $\mathbf{P}(\mathbf{t}_1//\mathbf{t}_2)$. At this point the one branch of the tree contains an atomic sentence, '$a = c$', and its negation, '~ $a = c$', and is therefore closed.

Consider now the sentence '$(\forall x)(\forall y)[(Fxx \,\&\, \sim Fyy) \supset \sim x = y]$'. We expect this sentence to be quantificationally true (if x but not y bears a

relation F to itself, then x and y are not identical). The following truth-tree confirms this expectation:

1.	$\sim(\forall x)(\forall y)[(Fxx \,\&\, \sim Fyy) \supset \sim x = y]$ ✓	SM
2.	$(\exists x)\sim(\forall y)[(Fxx \,\&\, \sim Fyy) \supset \sim x = y]$ ✓	1 ~ ∀D
3.	$\sim(\forall y)[(Faa \,\&\, \sim Fyy) \supset \sim a = y]$ ✓	2 ∃D
4.	$(\exists y)\sim[(Faa \,\&\, \sim Fyy) \supset \sim a = y]$ ✓	3 ~ ∀D
5.	$\sim[(Faa \,\&\, \sim Fbb) \supset \sim a = b]$ ✓	4 ∃D
6.	$Faa \,\&\, \sim Fbb$ ✓	5 ~ ⊃D
7.	$\sim\sim a = b$ ✓	5 ~ ⊃D
8.	$a = b$	7 ~ ~ D
9.	Faa	6 &D
10.	$\sim Fbb$	6 &D
11.	$\sim Faa$	8, 10 =D
	×	

At line 11 we replaced both occurrences of 'b' in '~ Fbb' with 'a' to generate '~ Faa'. Replacing just one occurrence, while allowed, would not have produced a closed tree.

We now test the argument

$$\frac{(\exists x)Gxa \,\&\, \sim(\exists x)Gax \qquad (\forall x)(Gxb \supset x = b)}{\sim a = b}$$

for quantificational validity by constructing a tree for the premises and the negation of the conclusion:

1.	$(\exists x)Gxa \,\&\, \sim(\exists x)Gax$ ✓		SM
2.	$(\forall x)(Gxb \supset x = b)$		SM
3.	$\sim\sim a = b$ ✓		SM
4.	$a = b$		3 ~ ~ D
5.	$(\exists x)Gxa$ ✓		1 &D
6.	$\sim(\exists x)Gax$ ✓		1 &D
7.	$(\forall x)\sim Gax$		6 ~ ∃D
8.	Gca		5 ∃D
9.	$Gab \supset a = b$		2 ∀D
10.	$Gbb \supset b = b$		2 ∀D
11.	$Gcb \supset c = b$ ✓		2 ∀D
12.	$\sim Gaa$		7 ∀D
13.	$\sim Gab$		7 ∀D
14.	$\sim Gac$		7 ∀D
15.	$\sim Gcb$	$c = b$	11 ⊃D
16.	$\sim Gca$	|	4, 15 =D
17.	×	$a = c$	4, 15 =D
18.		Gaa	8, 17 =D
		×	

This tree is closed. Therefore the argument we are testing is quantificationally valid. The secret to keeping this tree reasonably concise—for it could have grown quite large—came from carefully studying the sentences on lines 9–11 to determine which should be decomposed first. Line 11 yields '~ Gcb' on the left branch when decomposed, and replacing 'b' with 'a' in '~ Gcb' by virtue of the identity on line 4 then yields '~ Gca' at line 16, closing the left branch. At line 15 the right branch is still open and contains the identity 'c = b' in addition to 'a = b'. From these identities and the other literals on the branch ('Gca', '~ Gaa', '~ Gab', and '~ Gac') a host of sentences can be obtained by using Identity Decomposition, replacing 'b' with either 'a' or 'c'. Careful study reveals that any of 'Gac', 'Gab', 'Gaa', or '~ Gca' would close the branch. But none of these can be directly obtained from the existing literals by Identity Decomposition using the two identities 'a = b' and 'c = b'. However, we were able to obtain the additional identity 'a = c' on line 17 by applying Identity Decomposition to the two identities themselves, replacing 'b' with 'c' in 'a = b' (as licensed by 'c = b'). This identity allowed us to obtain 'Gaa' on line 18, which closed the branch and the tree.

In the remainder of this section, we shall work through a number of examples involving functors and identity. Consider first the sentence '~ $(\exists x)x = g(a)$'. This sentence is fairly obviously quantificationally false, for it says that there is nothing that is identical to $g(a)$; but, of course, we know that something *is* identical to $g(a)$, namely, $g(a)$ itself. Here is the start of a tree:

1.	$-(\exists x)x = g(a)$ ✓	SM
2.	$(\forall x) - x = g(a)$	1 – ∃D
3.	$-a = g(a)$	2 ∀D

As of line 3 this tree has one open branch. It might seem that this branch is a completed open branch, and hence that our intuitions about the sentence we are testing must have been misguided. The sentence on line 1 is checked off, the sentence on line 3 is a literal that is not an identity sentence (it is the *negation* of an identity sentence), and the sentence on line 2 is a universally quantified sentence that has been decomposed to a substitution instance formed from the constant 'a'. But the branch is not completed, for there is another closed individual term on the branch, '$g(a)$', and the sentence on line 2 has not been decomposed to a substitution instance formed from this latter term. Doing this decomposition results in the following *closed* tree:

1.	$-(\exists x)x = g(a)$ ✓	SM
2.	$(\forall x) - x = g(a)$	1 – ∃D
3.	$-a = g(a)$	2 ∀D
4.	$-g(a) = g(a)$	2 ∀D
	×	

It is now apparent that line 3 was unnecessary—the branch would close without that step. This example shows that it is important to remember that before a branch qualifies as a completed open branch all universally quantified sentences on that branch must be decomposed to *every* substitution instance that

can be formed from a closed individual term on that branch, and these terms consist of all the constants and all the closed complex terms on the branch.

Consider next the sentence '$(\forall x)f(x) = x$'. This sentence is clearly not quantificationally true—because a one-place function does not always return its argument as its value. For example, *successor* returns $x + 1$ for any value x, not x itself. The following tree establishes that this sentence is not quantificationally true:

1.	$\sim (\forall x)f(x) = x$ ✓	SM
2.	$(\exists x) \sim f(x) = x$ ✓	1 ~ ∀D
3.	$\sim f(a) = a$	2 ∃D

o

The one branch on this tree is a completed open branch. The sentences on lines 1 and 2 have been checked off, and the sentence on line 3 is a literal that is not an identity sentence. The sentence on line 1 is therefore not quantificationally false, and the sentence of which it is a negation, '$(\forall x)f(x) = x$', is not quantificationally true.[7] Moreover, we can use the literals on the branch as a guide to constructing a model for '$\sim (\forall x)f(x) = x$' (which will be an interpretation on which the unnegated sentence is true). There is one literal on the single open branch: '$\sim f(a) = a$'. Because '$f(a)$' and 'a' must denote distinct individuals for this literal to be true, we choose a two-member UD, let 'a' designate one member, and interpret 'f' so that '$f(a)$' designates the other. Any interpretation that includes the following assignments will be a model for '$\sim (\forall x)f(x) = x$':

UD: {1, 2}
a: 2
$f(x)$: $3 - x$

The sentence '$(\forall x)(\forall y)f(x,y) = f(y,x)$' is also not quantificationally true, as the following tree shows:

1.	$\sim (\forall x)(\forall y)f(x,y) = f(y,x)$ ✓	SM
2.	$(\exists x) \sim (\forall y)f(x,y) = f(y,x)$ ✓	1 ~ ∀D
3.	$\sim (\forall y)f(a,y) = f(y,a)$ ✓	2 ∃D
4.	$(\exists y) \sim f(a,y) = f(y,a)$ ✓	3 ~ ∃D
5.	$\sim f(a,b) = f(b,a)$	4 ∃D

o

The one branch on this tree is a completed open branch. The first four sentences are checked off, and the last one is a literal that is not an identity sentence. So '$\sim (\forall x)(\forall y)f(x,y) = f(y,x)$' is not quantificationally false, and '$(\forall x)(\forall y)f(x,y) = f(y,x)$' is not quantificationally true. The one branch contains the literal '$\sim f(a,b) = f(b,a)$', so any interpretation that makes this literal true will also make the sentence '$\sim (\forall x)(\forall y)f(x,y) = f(y,x)$' true. To find such

[7] We shall produce a tree in Section 9.6 that shows that this sentence is not quantificationally false.

an interpretation we'll choose a two-member UD—letting 'a' and 'b' denote the distinct members and making sure that 'f(a,b)' and 'f(b,a)' do *not* denote the same member. Any interpretation that includes the following assignments will be a model for the sentence '~ (∀x)(∀y)f(x,y) = f(y,x)' and hence an interpretation on which the unnegated sentence '(∀x)(∀y)f(x,y) = f(y,x)' is false:

UD: {1,2}
a: 1
b: 2
f(x,y): x raised to the power y

Next consider the sentence '(∀x)[Ex ⊃ (∃y)(Oy & y = f(x))]'. We can use the truth-tree method to show that this sentence is not quantificationally true:

1.	~ (∀x)[Ex ⊃ (∃y)(Oy & y = f(x))] ✓	SM
2.	(∃x) ~ [Ex ⊃ (∃y)(Oy & y = f(x))] ✓	1 ~ ∀D
3.	~ [Ea ⊃ (∃y)(Oy & y = f(a))] ✓	2 ∃D
4.	Ea	3 ~ ⊃D
5.	~ (∃y)(Oy & y = f(a)) ✓	3 ~ ⊃D
6.	(∀y) ~ (Oy & y = f(a))	5 ~ ∃D
7.	~ (Oa & a = f(a)) ✓	6 ∀D
8.	~ (Of(a) & f(a) = f(a)) ✓	6 ∀D
9.	~ Of(a) | ~ f(a) = f(a) ×	8 ~ &D
10.	~ Oa | ~ a = f(a)	7 ~ &D
	O O	

This tree has two completed open branches. The tree contains no (non-negated) identity sentences, and every sentence on each of these branches either is a literal, or has been checked off, or is a universally quantified sentence. There is only one of the latter, at line 6, and it has been decomposed to every closed term on the relevant branch (each branch contains only the closed terms 'a' and 'f(a)'). Because this tree has at least one completed open branch, the sentence '~ (∀x)[Ex ⊃ (∃y)(Oy & y = f(x))]' is not quantificationally false and therefore '(∀x)[Ex ⊃ (∃y)(Oy & y = f(x))]' is not quantificationally true. From the three literals 'Ea', '~ Of(a)', and '~ Oa' on the left open branch we know that '~ (∀x)[Ex ⊃ (∃y)(Oy & y = f(x))]' will be true on any interpretation that includes the following assignments:

UD: The set {1}
a: 1
f(x): x^2
Ex: x is odd
Ox: x is even

From the three literals 'Ea', 'Of(a)', and ~ a = f(a)' on the right open branch we also know that '~ (∀x)[Ex ⊃ (∃y)(Oy & y = f(x))]' will be true on any interpretation that includes the following assignments:

UD: {1, 2}
a: 2
f(x): 3 − x
Ex: x is even
Ox: x is odd

By definition a one-place function returns exactly one value for each argument. So the following sentence is quantificationally true:

(∀x)(∃y)[y = f(x) & (∀z)(z = f(x) ⊃ z = y)]

Here is a tree that establishes this:

1.	~ (∀x)(∃y)[y = f(x) & (∀z)(z = f(x) ⊃ z = y)] ✓		SM
2.	(∃x) ~ (∃y)[y = f(x) & (∀z)(z = f(x) ⊃ z = y)] ✓		1 ~ ∀D
3.	~ (∃y)[y = f(a) & (∀z)(z = f(a) ⊃ z = y)] ✓		2 ∃D
4.	(∀y) ~ [y = f(a) & (∀z)(z = f(a) ⊃ z = y)]		3 ~ ∃D
5.	~ [f(a) = f(a) & (∀z)(z = f(a) ⊃ z = f(a))] ✓		4 ∀D
6.	~ f(a) = f(a)	~ (∀z)(z = f(a) ⊃ z = f(a)) ✓	5 ~ &D
7.	×	(∃z) ~ (z = f(a) ⊃ z = f(a)) ✓	6 ~ ∀D
8.		~ (b = f(a) ⊃ b = f(a)) ✓	7 ∃D
9.		b = f(a)	8 ~ ⊃D
10.		~ b = f(a)	8 ~ ⊃D
		×	

Note that at line 5 we chose to replace 'y' with 'f(a)' rather than with 'a'. Both are individual terms already occurring on the branch, but using the former generates a closed branch on the left at line 6 and, a few steps later, a closed branch on the right.

Next consider the argument

(∀x)[Px ⊃ (Ex ∨ x = f(a)]
Pc & ~ c = f(a)

Ec

The following tree demonstrates that this argument is quantificationally valid:

1.	$(\forall x)[Px \supset (Ex \vee x = f(a)]$	SM
2.	$Pc \;\&\; \sim c = f(a)$ ✓	SM
3.	$\sim Ec$	SM
4.	Pc	2 &D
5.	$\sim c = f(a)$	2 &D
6.	$Pc \supset (Ec \vee c = f(a))$ ✓	1 ∀D
7.	$\sim Pc$ × | $Ec \vee c = f(a)$ ✓	6 ⊃D
8.	Ec × | $c = f(a)$ ×	7 ∨D

On the other hand, the following argument is quantificationally invalid:

$$\frac{(\forall y)y = f(y) \supset (\forall x)(\exists y)y = f(x)}{(\forall y)y = f(y)}$$

1.	$(\forall y)y = f(y) \supset (\forall x)(\exists y)y = f(x)$ ✓	SM
2.	$\sim (\forall y)y = f(y)$ ✓	SM
3.	$(\exists y) \sim y = f(y)$ ✓	2 ~ ∀D
4.	$\sim a = f(a)$	3 ∃D
5.	$\sim (\forall y)y = f(y)$ ✓ | $(\forall x)(\exists y)y = f(x)$	1 ⊃D
6.	$(\exists y) \sim y = f(y)$ ✓	5 ~ ∀D
7.	$\sim b = f(b)$	6 ∃D
	O	

At line 7 the left branch becomes a completed open branch, establishing that the argument is quantificationally invalid. Note that every sentence on that branch either is checked off (lines 1, 2, 3, 5, and 6), or is a negated identity sentence (thus a literal that is not itself an identity sentence). It is worth noting that the right branch of this tree is the beginning of an infinite branch because of the interplay of the existential quantifier within the scope of the universal quantifier of the sentence on line 5 of that branch. From the completed open branch, however, which contains the literals '~ a = f(a)' and '~ b = f(b)', we know that any interpretation that includes the following assignments is a model for the set {(∀y) y = f(y) ⊃ (∀x)(∃y) y = f(x), ~ (∀y) y = f(y)}:

UD: The set {1, 2}
a: 1
b: 2
f(b): 3 − x

9.5E EXERCISES

Construct truth-trees as necessary to provide the requested information. In each case state your result, and specify what it is about your tree that establishes this result.

1. Determine, for each of the following sets, whether the set is quantificationally consistent. In addition, if your tree establishes consistency, show the relevant part of an interpretation that will make all of the literals on one completed branch, and therefore all of the members of the set being tested, true. (Be sure to list the literals that you are using in this case.)

a. {(∀x)Fxx, (∃x)(∃y) ~ Fxy, (∀x)x = a}
*b. {(∀x)(Fxc ⊃ x = a), ~ c = a, (∃x)Fxc}
c. {(∀x)(x = a ⊃ Gxb), ~ (∃x)Gxx, a = b}
*d. {(∃x)(∃y) ~ x = y, (∀x)(Gxx ⊃ x = b), Gaa}
e. {(∀x)((Fx & ~ Gx) ⊃ ~ x = a), Fa & ~ Ga}
*f. {(∃y)(∀x)Fxy, ~ (∀x)(∀y)x = y, Fab & ~ Fba}
g. {(∀x)(x = a ⊃ Gxf(b)), ~ (∃x)Gxf(x), f(a) = f(b)}
*h. {(∀x)(Gxx ⊃ x = f(x,b)), Gaa, (∀x) ~ f(a,x) = a}
i. {(∃x) ~ x = g(x), (∀x)(∀y)x = g(y)}
*j. {(∃x)(∃y)f(x,y) = f(y,x), (∀x) [f(x,a) = f(a,x) ⊃ ~ a = x]}
k. {(∀x)[Hx ⊃ (∀y)Txy], (∃x)Hf(x), ~ (∃x)Txx}
*l. {Hf(a,b), (∀x)(Hx ⊃ ~ Gx), (∃y)Gy}
m. {(∃x)Fx ⊃ (∃x)(∃y)f(y) = x, (∃x)Fx}
*n. {(∃x)[x = f(s) & (∀y)(y = f(s) ⊃ y = x)]}

2. Determine, for each of the following sentences, whether it is quantificationally true, quantificationally false, or quantificationally indeterminate.

a. a = b ≡ b = a
*b. (~ a = b & ~ b = c) ⊃ ~ a = c
c. (Gab & ~ Gba) ⊃ ~ a = b
*d. (∀x)(∃y)x = y
e. Fa ≡ (∃x)(Fx & x = a)
*f. ~ (∃x)x = a
g. (∀x)x = a ⊃ [(∃x)Fx ⊃ (∀x)Fx]
*h. (∀x)(∀y)x = y
i. (∀x)(∀y) ~ x = y
*j. (∃x)(∃y)x = y
k. (∃x)(∃y) ~ x = y
*l. (∀x)(∀y)[x = y ⊃ (Fx ≡ Fy)]
m. (∀x)(∀y)[(Fx ≡ Fy) ⊃ x = y]
*n. (∀x)(∀y)[x = y ⊃ (∀z)(Fxz ≡ Fyz)]
o. [(∃x)Gax & ~ (∃x)Gxa] ⊃ (∀x)(Gxa ⊃ ~ x = a)

3. Determine which of the following sentences are quantificationally true.

a. (∃x)x = f(a)
*b. (∀x)(∃y)y = f(x)
c. (∃x)(∃y)x = y
*d. (∃x)(∃y)x = f(y)
e. (∀x)[Gx ⊃ (∃y)f(x) = y]

*f. $(\forall x)(\forall y)[x = y \supset f(x) = f(y)]$
g. $(\forall y) \sim [(\forall x)x = y \vee (\forall x)f(x) = y]$
*h. $(\forall x)(\exists y)[y = f(x) \;\&\; (\forall z)(z = f(x) \supset z = y)]$

4. Determine which of the following pairs of sentences are quantificationally equivalent.

a.	$\sim a = b$	$\sim b = a$
*b.	$(\exists x) \sim x = a$	$(\exists x) \sim x = b$
c.	$(\forall x)x = a$	$(\forall x)x = b$
*d.	$a = b \;\&\; b = c$	$a = c \;\&\; b = c$
e.	$(\forall x)(\forall y)x = y$	$(\forall x)x = a$
*f.	$(\forall x)(\exists y)x = y$	$(\forall y)(\exists x)x = y$
g.	$(\forall x)(Fx \supset x = a)$	$(\forall x)(Fa \supset x = a)$
*h.	$(\forall x)(x = a \vee x = b)$	$(\forall x)x = a \vee (\forall x)x = b$
i.	$(\forall x)Fx \vee (\forall x) \sim Fx$	$(\forall y)(Fy \supset y = b)$
*j.	$a = b$	$(\forall y)(y = a \supset y = b)$
k.	$(\exists x)(x = a \;\&\; x = b)$	$a = b$

5. Determine which of the following arguments are quantificationally valid.

a. $a = b \;\&\; \sim Bab$
$\sim (\forall x)Bxx$

*b. $Ge \supset d = e$
$Ge \supset He$
$Ge \supset Hd$

c. $(\forall z)(Gz \supset (\forall y)(Ky \supset Hzy))$
$(Ki \;\&\; Gj) \;\&\; i = j$
Hii

*d. $(\exists x)(Hx \;\&\; Mx)$
$Ms \;\&\; \sim Hs$
$(\exists x)((Hx \;\&\; Mx) \;\&\; \sim x = s)$

e. $a = b$
$Ka \vee \sim Kb$

*f. $(\exists x) \sim Pxx \supset \sim a = a$
$a = c$
Pac

g. $(\forall x)(x = a \vee x = b)$
$(\exists x)(Fxa \;\&\; Fbx)$
$(\exists x)Fxx$

*h. $(\exists x)Fxa$
$(\forall y)(y = a \supset y = b)$
$(\exists y)Fyy$

i. $(\forall x)(\forall y)(Fxy \vee Fyx)$
$a = b$
$(\forall x)(Fxa \vee Fbx)$

*j. $(\exists x)Fxa \;\&\; (\exists x)Fxb$
$\sim a = b$
$(\forall x)(\forall y)((Fxa \;\&\; Fyb) \supset \sim x = y)$

k. $(\forall x)(Fx \equiv \sim Gx)$
Fa
Gb
$\sim a = b$

*l. $\sim (\exists x)Fxx$
$(\forall x)(\forall y)(Fxy \supset \sim x = y)$

m. $\dfrac{(\forall x)(\forall y)x = y}{\sim(\exists x)(\exists y)(Fx \,\&\, \sim Fy)}$

*n. $\dfrac{(\forall x)(\sim x = a \equiv (\exists y)Gyx) \quad Gbc}{\sim c = a}$

o. $\dfrac{(\forall x)(Hx \supset Hf(x)) \quad (\exists z) \sim Hf(z)}{\sim (\forall x)Hx}$

*p. $\dfrac{(\forall y)(Hy \supset g(y) = y) \quad (\exists x) \sim g(x) = x}{(\exists x) \sim Hx}$

q. $\dfrac{(\forall x)(\forall y)(Hxy \equiv \sim Hyx) \quad (\exists x)[Hxf(x) \,\&\, \sim Hf(x)x]}{\sim (\forall x)f(x) = x}$

*r. $\dfrac{(\exists x)h(x) = x \quad (\forall x)(Fx \supset \sim Fh(x)) \quad (\exists x) \sim Fx}{}$

s. $\dfrac{(\forall x)[Px \supset (Ox \vee \sim x = f(b))] \quad (\exists x)[(Px \,\&\, \sim Ox) \,\&\, x = f(b)]}{Ob}$

*t. $\dfrac{(\forall x)(\forall y)(Hxy \supset f(x) = y) \quad (\exists x)Hxx}{(\exists x)f(x) = x}$

6. Determine which of the following claims are true.

a. $\{(\forall x)(Fx \supset (\exists y)(Gyx \,\&\, \sim y = x)), (\exists x)Fx)\} \models (\exists x)(\exists y) \sim x = y$
*b. $\{\sim (\exists x)(Fxa \vee Fxb), (\forall x)(\forall y)(Fxy \supset \sim x = y)\} \models \sim a = b$
c. $\{(\forall x)(Fx \supset \sim x = a), (\exists x)Fx\} \models (\exists x)(\exists y) \sim x = y$
*d. $\{(\forall x)(\exists y)(Fxy \,\&\, \sim x = y), a = b, Fab\} \models (\exists y)(Fay \,\&\, y = b)$
e. $\{(\exists w)(\exists z) \sim w = z, (\exists w)Hw\} \models (\exists w) \sim Hw$
*f. $\{(\exists w)(\forall y)Gwy, (\exists w)(\forall y)(\sim w = y \supset \sim Gwy)\} \models (\exists z) \sim Gzz$
g. $\{(\forall x)(\forall y)((Fx \equiv Fy) \equiv x = y), (\exists z)Fz\} \models (\exists x)(\exists y)(\sim x = y \,\&\, (Fx \,\&\, \sim Fy))$
*h. $\{(\forall x)(\exists y)y = f(x)\} \models (\exists z)z = f(a)$
i. $\{(\forall x)(\forall y)[\sim x = g(y) \supset Gxy], \sim (\exists x)Gax\} \models (\exists x)a = g(x)$

9.6 FINE-TUNING THE TREE METHOD FOR *PLE*

The last tree that we presented in Section 9.5 contained an unending branch in the making, due to a sentence that contained an existential quantifier within the scope of a universal quantifier. We introduced a new rule in Section 9.4, Existential Decomposition-2, as well as a systematic method of constructing trees for $PL_=$ to ensure that such branches would neither prevent discovering completed open branches (where they exist) nor prevent closing trees for inconsistent sets. Because the tree method for *PLE* includes all of the rules for $PL_=$ we will clearly need to address infinite branches arising from the interplay between existential and universal quantifiers here as well. But the inclusion of functors in *PLE* creates an additional source of nonterminating branches in trees for finite sets of sentences. Consider a tree for the set $\{(\forall x)Hf(x)\}$:

1.	(∀x)H*f*(x)	SM
2.	H*f*(a)	1 ∀D
3.	H*f*(*f*(a))	1 ∀D
4.	H*f*(*f*(*f*(a)))	1 ∀D

•
•
•

To qualify as a completed open branch, every universally quantified sentence on that branch must be decomposed at least once and must be decomposed to every closed term occurring on the branch. To satisfy the first requirement, we first decompose the universally quantified sentence occurring on line 1 using the constant 'a'. This introduces *two* new closed terms to the one branch of the tree: 'a' and '*f*(a)' (both occurring within the formula 'H*f*(a)'). The universal quantification has just been decomposed using 'a', but now we must also decompose it using the closed term '*f*(a)'. This produces line 3, and a new closed term, '*f*(*f*(a))', which triggers another decomposition of the universally quantified sentence, producing a new closed term, '*f*(*f*(*f*(a)))', at line 4, and so on. Clearly this branch will never close and will never become a completed open branch.

As for *PL*, we would like to have a tree system for *PLE* such that every finite inconsistent set has a closed tree *and* every finite set with a finite model has a finite tree with a completed open branch. We just saw that we cannot produce a finite tree for the set {(∀x)H*f*(x)}, given the methods presented for *PLE* in Section 9.5. Yet this set has a finite model, for example, any interpretation that includes the following assignments:

UD: The set {2}
Hx: x is even
f(x): x

Here *f*(x) is just x, and the only value of x in this UD is 2, and 2 is even. So the single member of {(∀x)H*f*(x)} is true on any interpretation that includes these assignments. In this section we will modify our definition of completed open branches, add a new decomposition rule for constructing *PLE* trees, and then present a systematic method for constructing trees such that our desiderata are satisfied.

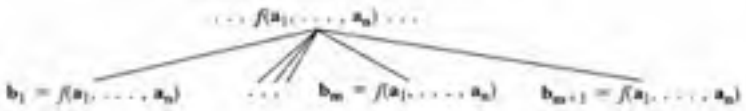
We will modify the definition of completed open branches in several ways. The first of these will be to drop the requirement that universally quantified sentences be decomposed using *every* closed term occurring on a branch, and adopt the weaker requirement that universally quantified sentences must be decomposed using every *constant* occurring on a branch (and that at least one constant must be so used). This will cut short the infinite branch that we just saw in progress for the set {(∀x)H*f*(x)}, but without other changes it will also count the single branch of the following tree as a completed open branch:

1.	(∀x)Bx	SM
2.	~ Bg(a)	SM
3.	Ba	1 ∀D

We clearly don't want to count this as a completed open branch, for the set {(∀x)Bx, ~ Bg(a)} is quantificationally inconsistent. Because we are required to decompose universally quantified sentences only with constants, we will add a new rule that identifies the individuals denoted by closed complex terms with individuals identified by constants, a rule that will lead to a closed tree for this set once we make a final modification of the definition of open branches. More generally, we will then have the results we desire for all cases: closed trees for inconsistent sets and completed open branches for finite consistent sets with finite models.

The new rule is called *Complex Term Decomposition*:[8]

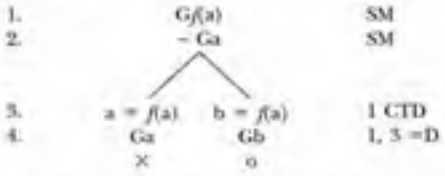
Complex Term Decomposition (CTD)



where $f(\mathbf{a}_1, \ldots, \mathbf{a}_n)$ is a closed complex term occurring within a literal on some branch, whose arguments $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are individual constants; $\mathbf{b}_1, \ldots, \mathbf{b}_m$ are the constants that already occur on that branch, and $\mathbf{b}_{m+1}$ is a constant that is foreign to that branch.

The expression '. . . $f(\mathbf{a}_1, \ldots, \mathbf{a}_n)$ . . .' stands for any literal that contains the complex term '$f(\mathbf{a}_1, \ldots, \mathbf{a}_n)$'. This rule bears an obvious affinity to Existential Decomposition-2: it branches out based on the constants that occur on the branch containing the complex term being decomposed, and generates one additional branch with a constant that was foreign to that branch. At the end of each of the new branches is an identity sentence, with one of the constants on the left-hand side and the complex term being decomposed on the right-hand side.

The following tree for the set {G*f*(a), ~ Ga} illustrates the use of Complex Term Decomposition:



The tree begins with the set members on the first two lines. The sentence on line 1 is a literal containing the closed complex term '*f*(a)', so Complex Term Decomposition must be applied to this term. Prior to applying the rule there is one constant, 'a', that occurs on the single branch. So one of the new branches must end with the identity sentence 'a = *f*(a)', while the other ends with an

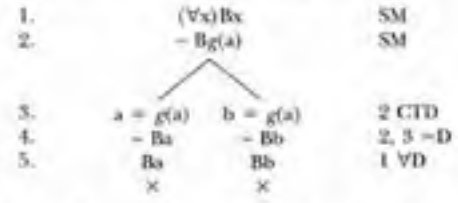
[8] This is a slight variation of the rule introduced in Merrie Bergmann, "Finite Tree Property for First-Order Logic with Identity and Functions," *Notre Dame Journal of Formal Logic*, 46 (2005), pp. 175–180.

identity sentence that has a new constant on the left-hand side. (Note that we do not check off the sentence containing the closed term that is being decomposed. Checks will continue to indicate completed *sentence* decomposition only.) The tree is then extended to line 4, because the identity sentences on line 3 must be decomposed by substituting the constants for 'f(a)' in the literal 'Gf(a)'. The left branch closes, as it should, because if 'a' and 'f(a)' denote the same individual, then the set members state that this individual both does and does not have the property G. The right branch is a completed open branch, confirming that the set {Gf(a), ~ Ga} is quantificationally consistent. (Strictly speaking, the formula 'b = b' should also appear on the right branch by virtue of applying Identity Decomposition to the single formula on line 3 of that branch. But here, as in Section 9.5, we omit identity formulas in which the same term appears on both sides of the identity predicate because such formulas will never cause a branch to close.) Because the open branch contains two individual constants, it indicates that we can construct a model for the set using a UD with at least two members, for example, any interpretation that includes the following assignments:

UD: {1, 2}
Gx: x is even
a: 1
b: 2
f(x): 3 − x

(In addition, the fact that the left branch, the only branch that contains exactly one individual constant, closes tells us that any model for this set *must* have at least two members in its UD.)

Using Complex Term Decomposition we can produce a closed tree for the quantificationally inconsistent set {(∀x)Bx, ~ Bg(a)}:



Although we dropped the requirement that universally quantified sentences must be decomposed using closed complex terms as well as constants, this tree nevertheless closes because of the identity sentences that were generated on line 3 by Complex Term Decomposition. Branching to those sentences says that either the constant 'a' or the constant 'b' denotes the individual that the complex term 'g(a)' denotes. Further, the identity sentences must themselves be decomposed with Identity Decomposition, and respectively substituting the constants 'a' and 'b' for 'g(a)' in the sentence '~ Bg(a)' produces '~ Ba' on the left branch and '~ Bb' on the right branch. Finally, because universally quantified sentences *must* be decomposed using all the *constants* occurring on a

branch, we add the substitution instances on line 5 that respectively contradict the sentences on line 4 of the two branches. So the tree closes.

On the other hand, Complex Term Decomposition and our modified requirement for decomposing universally quantified sentences produce a completed open branch on a tree for the set $\{(\forall x)Hf(x)\}$:

1.	$(\forall x)Hf(x)$			SM
2.	$Hf(a)$			1 ∀D
3.	$a = f(a)$		$b = f(a)$	2 CTD
4.	Ha		Hb	2, 3 =D
5.	o		$Hf(b)$	1 ∀D

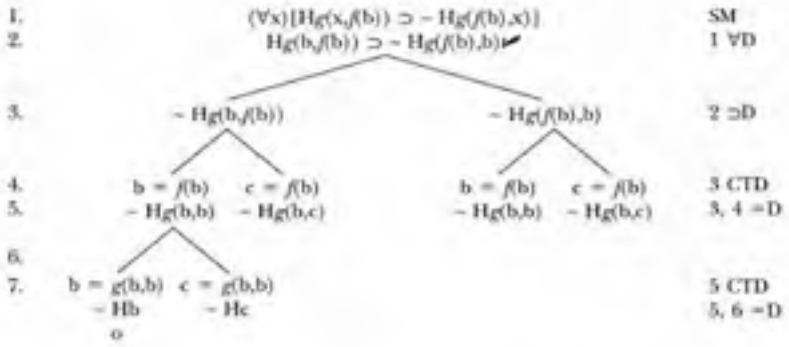
After adding the sentences on line 3 with Complex Term Decomposition, we substitute the constants for the complex term 'f(a)' in the literal on line 2 to generate the literals on line 4. The left branch is now completed: We have decomposed the universal quantification on line 1 with the only constant on the branch (this occurs at line 2), we have decomposed the complex term on line 2 (although we haven't stated so, there will be a requirement that all complex terms must be decomposed), and the identity on line 3 has also been decomposed as many times as it can be (it produces only the sentence on line 4). It is also open, indicating that the set being tested is quantificationally consistent. The right branch is also open, but it is not completed. On line 5 we have added another substitution instance of the universal quantification, because the right branch now contains the constant 'b' along with the constant 'a'. This generates a new closed complex term, to which Complex Term Decomposition must be applied. An infinite branch is in the making here, for Complex Term Decomposition will produce a new branch with the constant 'c', which must be used to form a substitution instance of the universal quantification on line 1, and so on. The important point is, however, that we have managed to produce a completed open branch. That branch contains exactly one constant, confirming (as we already saw) that there is a model for the set $\{(\forall x)Hf(x)\}$ in which the UD has exactly one member.

Having seen Complex Term Decomposition in action, we will now examine a tree that illustrates some of the finer points of this rule:

1.	$(\exists x)Hg(x,f(x))$ ✔						SM
2.	$Hg(a,f(a))$						1 ∃D2
3.	$a = f(a)$				$b = f(a)$		2 CTD
4.	$Hg(a,a)$				$Hg(a,b)$		2, 3 =D
5.	$a = g(a,a)$	$b = g(a,a)$		$a = g(a,b)$	$b = g(a,b)$	$c = g(a,b)$	4 CTD
6.	Ha	Hb		Ha	Hb	Hc	4, 5 =D
	o	o		o	o	o	

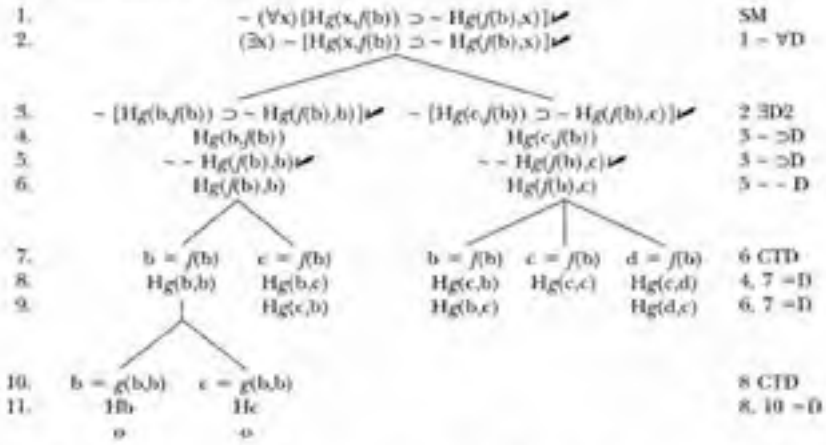
(In this section, as in Section 9.4, we will always use Existential Decomposition-2 to decompose existentially quantified sentences.) The sentence on line 1 contains the complex term '$g(x,f(x))$' but that term is not decomposed because it is not a *closed* term. The sentence on line 2 contains two complex terms: '$f(a)$' and '$g(a,f(a))$'. Both are closed, but only '$f(a)$' gets decomposed (on line 3). The term '$g(a,f(a))$' does not get decomposed because Complex Term Decomposition only applies if all of the arguments to the main functor are constants, but the argument '$f(a)$' is not a constant. After Identity Decomposition adds identity sentences at line 4, the tree contains two new complex terms: '$g(a,a)$' and '$g(a,b)$'. These terms are closed, and the arguments in both are exclusively constants, so they must be decomposed—this is done on line 5. Identity Decomposition then produces the sentences on line 6, and all five branches become completed open branches.

Now we will use Complex Term Decomposition to produce trees that show that the sentence '$(\forall x)[Hg(x,f(b)) \supset \sim Hg(f(b),x)]$' is quantificationally indeterminate. The first tree shows that this sentence is not quantificationally false:



The leftmost branch is a completed open branch. The remaining branches are incomplete, but since the tree does have at least one completed open branch there is no need to complete them. In constructing the tree we reached the sentences '~ Hb' and '~ Hc' through repeated applications of Complex Term Decomposition and Identity Decomposition, applying CTD first to line 3 to produce the identities on line 4, then to '$g(b,b)$' as that closed term occurs in a literal on line 5. The following tree establishes that '$(\forall x)[Hg(x,f(b)) \supset \sim Hg(f(b),x)]$' also is not quantificationally true, and

hence that it is quantificationally indeterminate:



The left two branches of this tree are completed open branches, so there is no point in continuing to work on the other branches. Note that the closed term '*f*(b)' occurs in literals on lines 4 and 6. Nonetheless we applied CTD to this closed term only once—at line 7, citing line 6. We could equally well have cited line 4. (There is no point to applying CTD twice to the same closed term, as the results will always be the same.) Also note that we applied Identity Decomposition at line 8 to lines 4 and 7 on all five branches. At line 9 we applied it only to the branches where a new literal is yielded.

Before we give our official definition of open branches for trees of *PLE*, we pause to consider the restriction, in the statement of Complex Term Decomposition, that the complex term being decomposed must be a closed term *whose arguments are individual constants*. In each of the preceding three trees there are closed complex terms whose arguments include complex terms. Why are we not required to decompose these complex terms? Keep in mind that the point of Complex Term Decomposition was to ensure that for every closed complex term occurring on a branch, there is a constant on that branch that denotes the same individual (so that when decomposing universally quantified sentences, we only need to generate substitution instances that are formed from individual constants that occur on the branch). It turns out that this requirement, that for every closed complex term occurring on a branch there be a constant that denotes the same individual, is met as long as we are careful to apply CTD to all complex terms in which the arguments are all constants, and to decompose identity sentences wherever they occur. To see this, take as an example the term '*g*(b,*f*(b))' that occurs at line 4 in the literal on the left branch of the preceding tree. The *required* decompositions on the two completed open branches below line 4 guarantee that on each of these branches there is an individual constant that denotes the same member of the UD as

'g(b,f(b))'. More specifically, consider the completed open branch on the left. The identity sentence on line 7 states that 'b' and 'f(b)' denote the same individual, and from this it follows that 'g(b,b)' must denote the same individual as the more complex term 'g(b,f(b))'. Moreover, the identity sentence on line 10 states that 'b' and 'g(b,b)' denote the same individual—from which it follows that on any interpretation represented by this branch, 'b' denotes the same individual as the complex term 'g(b,f(b))'. Similarly, it follows from the identity sentences on the right completed open branch that on any interpretation represented by that branch, 'c' denotes the same individual as 'g(b,f(b))'.

The availability of CTD justifies our relaxing the requirement in clause 3 of our definition of a completed open branch so as *not* to require that universally quantified sentences be decomposed to substitution instances formed from closed complex terms. Such decompositions are still allowed and will sometimes produce closed trees sooner than will using CTD, but they are not required for a branch to be a completed open branch. It turns out that we can similarly loosen the requirement concerning the use of Identity Decomposition. So our revised definition of a completed open branch for trees for *PLE* is:

> A **completed open branch** is a finite open branch on which each sentence is one of the following:
>
> 1. A literal that is not an identity sentence
> 2. A compound sentence that is not a universally quantified sentence and is decomposed
> 3. A universally quantified sentence $(\forall \mathbf{x})\mathbf{P}$ such that $\mathbf{P}(\mathbf{a}/\mathbf{x})$ also occurs on that branch *for each constant* **a** occurring on the branch and $\mathbf{P}(\mathbf{a}/\mathbf{x})$ occurs on the branch for at least one constant **a**
> 4. A sentence of the form $\mathbf{a} = \mathbf{t}$, where **a** is an individual constant and **t** is a closed term such that the branch also contains, for every literal **P** on that branch containing **t**, every sentence $\mathbf{P}(\mathbf{a}//\mathbf{t})$ that can be obtained from **P** by Identity Decomposition
>
> and on which Complex Term Decomposition has been applied to every closed complex term occurring in a literal on the branch whose arguments are all individual constants.

All branches that we have marked as completed open branches on trees in this section meet this definition (except that we do not bother to show the required identity sentences in which the same term occurs on both sides of the identity predicate since these will never lead to closed branches). Note that Identity Decomposition can still be used on sentences of the form $\mathbf{t}_1 = \mathbf{t}_2$ when $\mathbf{t}_1$ is a complex term, but not using it is not necessary for producing closed branches. So, where $\mathbf{t}_1 = \mathbf{t}_2$ occurs on a branch and $\mathbf{t}_1$ is a complex term, Identity Decomposition should be used if we suspect that doing so will quickly produce a closed branch, but not otherwise.

Before turning to the second task of this section, that of developing a tree construction procedure that yields a finite tree wherever one exists, we shall construct several more trees using Complex Term Decomposition. First we shall use the tree method to determine the quantificational status of the sentence '$(\forall x)(\exists y)Pf(x,y)$'. Here is a tree for the negation of the sentence:

1.	$-(\forall x)(\exists y)Pf(x,y)$ ✓	SM
2.	$(\exists x)-(\exists y)Pf(x,y)$ ✓	1 – ∀D
3.	$-(\exists y)Pf(a,y)$ ✓	2 ∃D2
4.	$(\forall y)-Pf(a,y)$	3 – ∃D
5.	$-Pf(a,a)$	4 ∀D
6.	$a = f(a,a)$ | $b = f(a,a)$	5 CTD
7.	$-Pa$ | $-Pb$	5, 6 =D
	o	

The left-hand branch is a completed open branch. The sentences on lines 1–3 have been checked off. The sentences on lines 5 and 7 are literals that are not identity sentences. The universally quantified sentence on line 3 has been decomposed to a substitution instance containing 'a', the only constant on the left-hand branch. Complex Term Decomposition has been applied to the one relevant closed term, '$f(a,a)$', and Identity Decomposition has been applied as required to the identity sentence on line 6 and the literal on line 5 containing the constant 'a'. So '$(\forall x)(\exists y)Pf(x,y)$' is not quantificationally false. (Note that the right-hand branch of the preceding tree is not a completed open branch. It contains a constant, 'b', for which the universally quantified sentence on line 4 has not been decomposed.)

We now construct a tree for the sentence itself, to determine whether it is quantificationally false or quantificationally indeterminate:

1.	$(\forall x)(\exists y)Pf(x,y)$	SM
2.	$(\exists y)Pf(a,y)$ ✓	1 ∀D
3.	$Pf(a,a)$ | $Pf(a,b)$	2 ∃D2
4.	$a = f(a,a)$ | $b = f(a,a)$	3 CTD
5.	Pa | Pb	3, 4 =D
	o	

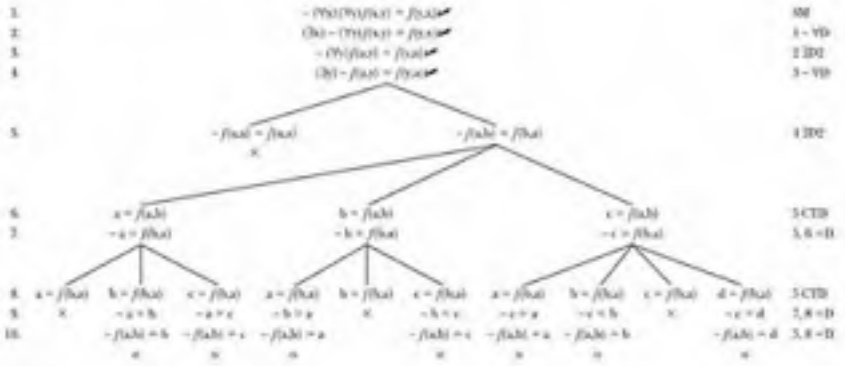
At this point the tree has one completed open branch, the leftmost branch. The other branches are not completed open branches because each contains the constant 'b', for which the universally quantified sentence on line 1 has not been decomposed. Because the tree has at least one completed open branch, the sentence we are testing is true on at least one interpretation and is therefore not quantificationally false. This, together with the tree we constructed for

the negation of the sentence, establishes that the sentence '$(\forall x)(\exists y)Pf(x,y)$' is quantificationally indeterminate.

We shall next show that '$(\forall x)(\forall y)f(x,y) = f(y,x)$' is quantificationally indeterminate. We start by constructing a tree for the unit set of the sentence:

1.	$(\forall x)(\forall y)f(x,y) = f(y,x)$		SM
2.	$(\forall y)f(a,y) = f(y,a)$		1 VD
3.	$f(a,a) = f(a,a)$		2 VD
4.	$a = f(a,a)$	$b = f(a,a)$	3 CTD
5.	$f(a,a) = a$	$f(a,a) = b$	3, 4 =D
	O		

The left-hand branch is, as of line 5, a completed open branch. So the sentence is true on at least one interpretation and is therefore not quantificationally false. The right-hand branch is not completed, as it does not contain substitution instances of the sentences on lines 1 and 2 that can be formed using the constant 'b' that occurs on that branch. Here is a tree for the unit set of the negation of our sentence:



This tree has seven completed open branches, so the sentence at line 1 is not quantificationally false, and the sentence '$(\forall x)(\forall y)f(x,y) = f(y,x)$' is not quantificationally true. This establishes that '$(\forall x)(\forall y)f(x,y) = f(y,x)$' is quantificationally indeterminate.

As we did for *PL*, we will present a procedure for constructing trees for *PLE* in a systematic fashion that will always, in a finite number of steps, find a completed open branch if one exists or close the tree if it can be closed. The System for *PLE* is somewhat more complicated than that for *PL*, owing to the

presence of identity sentences and complex terms:

The System for PLE

List the members of the set to be tested.

Exit Conditions: Stop if

a. The tree closes.
b. An open branch becomes a completed open branch.

Construction Procedures:

Stage 1: Decompose all truth-functionally compound and existentially quantified sentences and each resulting sentence that is itself either a truth-functional compound or an existentially quantified sentence.

Stage 2: For each universally quantified sentence $(\forall \mathbf{x})\mathbf{P}$ on the tree:

i) Enter $\mathbf{P}(\mathbf{a}/\mathbf{x})$ on each open branch passing through $(\forall \mathbf{x})\mathbf{P}$ for each individual constant **a** already occurring on that branch.

ii) On each open branch passing through $(\forall \mathbf{x})\mathbf{P}$ on which no constant occurs, enter $\mathbf{P}(\mathrm{a}/\mathbf{x})$.

iii) Enter $\mathbf{P}(\mathrm{t}/\mathbf{x})$ on an open branch passing through $(\forall \mathbf{x})\mathbf{P}$ for a closed complex term **t** if and only if doing so closes the branch.

Repeat this process until every universally quantified sentence on the tree, including those added as a result of this process, has been so decomposed.

Stage 3: Apply Complex Term Decomposition to every complex term on an open branch whose arguments are all constants and to which Complex Term Decomposition has not already been applied.

Stage 4: For every sentence of the form $\mathbf{t}_1 = \mathbf{t}_2$ occurring on an open branch, apply Identity Decomposition as follows:

i) Where $\mathbf{t}_1$ is an individual constant, apply Identity Decomposition until every open branch passing through $\mathbf{t}_1 = \mathbf{t}_2$ also contains, for every literal **P** containing $\mathbf{t}_2$ on that branch, every sentence $\mathbf{P}(\mathbf{t}_1//\mathbf{t}_2)$ that can be obtained from **P** by Identity Decomposition.

ii) Where $\mathbf{t}_1$ is a closed complex term, apply Identity Decomposition to $\mathbf{t}_1 = \mathbf{t}_2$ and a literal **P** containing $\mathbf{t}_2$ that occurs on a branch passing through $\mathbf{t}_1 = \mathbf{t}_2$ if and only if only doing so closes the branch.

Return to Stage 1.

Note that Stage 3 does not require us to apply Complex Term Decomposition to the same complex term on a branch more than once, even though the term may occur in more than one literal on that branch.

Stage 4 ensures that after passing through that stage every sentence of the form **a** = **t** on every open branch meets the requirements of clause 4 of the definition of a completed open branch. That is, if the branch is not completed at this point, it will *not* be because we have failed to apply Identity Decomposition the required number of times.[9] Stages 2 and 4 each contain instructions to apply a decomposition rule in certain cases only if doing so closes a branch. The decompositions in question do not need to be done to meet the requirements for having a completed open branch, and doing such decompositions when the result is not a closed branch can produce infinite branches in cases where completed open branches are possible.

Nor do such decompositions need to be done to produce closed branches—the branches in question will eventually close even without these decompositions—but they can result in less complex trees than would otherwise be produced by The System. To illustrate this consider the following tree that we constructed in Section 9.5 for the set {(∀x) ~ Bx, B*f*(c)}:

1.	(∀x) ~ Bx	SM
2.	B*f*(c)	SM
3.	~ B*f*(c)	1 ∀D
	×	

This tree, which serves to establish the inconsistency of the set being tested, is a systematic tree and it illustrates the reason for decomposing universally quantified sentences with complex terms when the result is a closed branch. If we never used complex terms to decompose universally quantified sentences, The System would still produce a closed tree but it would be more complex:

1.	(∀x) ~ Bx		SM
2.	B*f*(c)		SM
3.	~ Bc		1 ∀D
4.	c = *f*(c)	d = *f*(c)	2 CTD
5.	Bc	|	4, 2 =D
6.	×	Bd	4, 2 =D
7.		~ Bd	1 ∀D
		×	

Here we only used constants when decomposing the universally quantified sentence, but the use of Complex Term Decomposition ensured that we nevertheless ended up with a closed tree.

We conclude with some examples that use The System for *PLE*. The sentence '(∀x)(∃y)y = *f*(*f*(x))' is quantificationally true. So the truth-tree for the negation of that sentence should close, and it does:

[9] In Chapter 11, various results about the tree system for *PLE* presuppose that Identity Composition is thoroughly applied as required by clause 4 of the definition of a completed open branch of a *PLE* tree. This is why we have previously allowed informally, *but not formally*, omitting applications of Identity Decomposition that yield sentences of the form t = t. Our examples of systematic trees for *PLE* will include all such identity sentences on completed open branches.

1.	$\sim (\forall x)(\exists y)y = f(f(x))$ ✓	SM
2.	$(\exists x) \sim (\exists y)y = f(f(x))$ ✓	1 ~ ∀D
3.	$\sim (\exists y)y = f(f(a))$ ✓	2 ∃D2
4.	$(\forall y) \sim y = f(f(a))$	3 ~ ∃D
5.	$\sim a = f(f(a))$	4 ∀D
6.	$\sim f(f(a)) = f(f(a))$	4 ∀D
	×	

We closed this systematic tree by taking advantage of the instruction that a universally quantified sentence be decomposed to a substitution instance formed from a complex term if and only if doing so closes the branch.

On the other hand, the sentence '$(\forall x)(\forall y)y = f(f(x))$' is quantificationally indeterminate. The following trees show that the sentence is not quantificationally true:

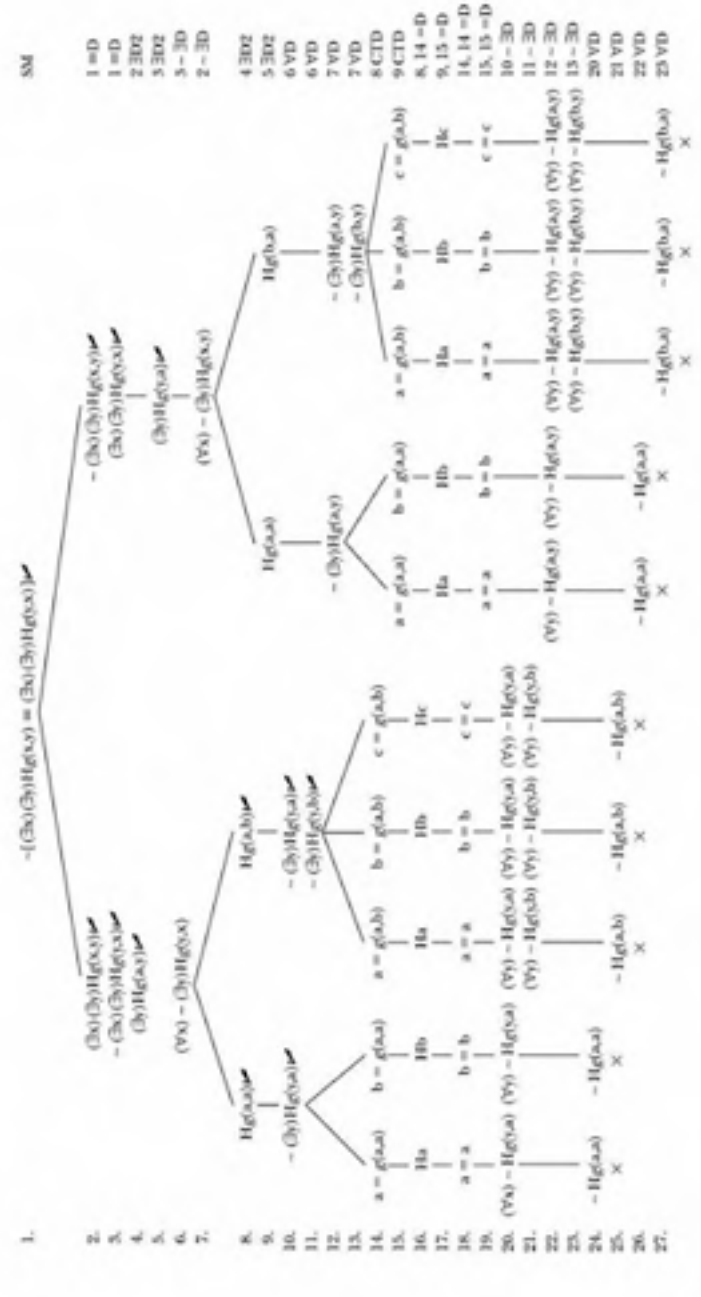
1.			$\sim (\forall x)(\forall y)y = f(f(x))$ ✓			SM
2.			$(\exists x) \sim (\forall y)y = f(f(x))$ ✓			1 ~ ∀D
3.			$\sim (\forall y)y = f(f(a))$ ✓			2 ∃D2
4.			$(\exists y) \sim y = f(f(a))$ ✓			3 ~ ∀D
5.	$\sim a = f(f(a))$				$\sim b = f(f(a))$	4 ∃D2
6.	$a = f(a)$	$b = f(a)$	$a = f(a)$	$b = f(a)$	$c = f(a)$	3 CTD
7.	$\sim a = f(a)$	$\sim a = f(b)$	$\sim b = f(a)$	$\sim b = f(b)$	$\sim b = f(c)$	5, 6 =D
8.	$a = a$	$b = b$	$a = a$	$b = b$	$c = c$	6, 6 =D
9.	$\sim a = a$		$\sim b = a$			6, 7 =D
	×		O			

Recall that the identity sentences on line 8 are required on completed open branches. At line 9 the leftmost branch closes owing to the sentence just entered on that branch, while the third branch becomes a completed open branch. The following tree shows that the sentence '$(\forall x)(\forall y)y = f(f(x))$' is also not quantificationally false:

1.	$(\forall x)(\forall y)y = f(f(x))$		SM
2.	$(\forall y)y = f(f(a))$		1 ∀D
3.	$a = f(f(a))$		2 ∀D
4.	$a = f(a)$	$b = f(a)$	3 CTD
5.	|	$a = f(b)$	3, 4 =D
6.	$a = a$	$b = b$	4, 4 =D
	O		

Here, when we applied Identity Decomposition at line 5 using the sentences from lines 3 and 4 we did not add '$a = f(a)$' to the left branch since it already occurred on that branch. The left branch (but not the right one) became a completed open branch after adding the final sentence at line 6.

As a final example we shall construct a substantially more complicated systematic tree. This tree establishes that the sentence '$(\exists x)(\exists y)Hg(x,y) = (\exists x)(\exists y)Hg(y,x)$' is quantificationally true:



We begin work on the tree by decomposing all the truth-functionally compound sentences and existentially quantified sentences on the tree, and all those that are added by doing those decompositions. This Stage 1 work takes us through line 9 of the tree. At that point we move to Stage 2 of The System and, on each branch, decompose all universally quantified sentences on that branch using constants already on the branch. This takes us through line 13. We next move to Stage 3 and apply Complex Term Decomposition on each branch to every complex term occurring in a literal on that branch. We complete this stage at line 15. We move to Stage 4 and apply Identity Decomposition as directed at lines 16 through 19. At this point we have moved through The System once, but the tree has not closed and does not have a completed open branch. So we return, as instructed, to Stage 1. We apply Negated Existential Decomposition yielding lines 20–23. We then move to Stage 2 and apply Universal Decomposition, yielding lines 24–27. After these four applications of Universal Decomposition, every branch is closed and we have a closed tree.

Note that The System specifies, at Stage 3, that we apply Complex Term Decomposition. We did so in this tree, even though the results (the introduction of the literals 'Ha', 'Hb', and 'Hc' on appropriate branches) play no role in closing any branch. It is common, with systematic trees, for a tree to contain entries that are not germane to the final result. This is the price we pay for being sure we explore all possibilities.

9.6E EXERCISES

1. Construct systematic trees to determine, for each of the following sets, whether that set is quantificationally consistent. State your result. If you abandon a tree, explain why.

a. $\{(\forall x)(\forall y)[\sim x = g(y) \supset Gxy], \sim (\exists x)Gax\}$

*b. $\{(\forall x)(Gx \supset Gh(x)), (\exists x)(Gx \,\&\, \sim Gh(x))\}$

c. $\{(\exists x)(\exists y)Hf(x,y), \sim (\exists x)Hx\}$

*d. $\{(\exists x)(\forall y)\ x = f(y), (\exists x)(\forall y) \sim x = f(y)\}$

e. $(\forall x)\ Lxf(x), (\exists y) \sim Lf(y)y\}$

*f. $\{(\forall x) \sim x = f(x), (\exists x)\ x = f(f(x))\}$

g. $\{(\forall x)(Gx \supset \sim Gh(x)), (\exists x)(\sim Gx \,\&\, \sim Gh(x))\}$

*h. $\{(\forall x)\ x = f(x), (\exists x) \sim x = f(f(x))\}$

2. Show that the following sentences are not quantificationally true by constructing an appropriate systematic truth-tree.

a. $(\forall x)(Pf(x) \supset Px)$

*b. $(\forall x)(\forall y)(x = g(y) \vee y = g(x))$

c. $(\exists x)(\forall y)x = g(y)$

*d. $(\forall x) \sim x = f(x)$

e. $(\forall x)(\forall y)(Dh(x,y) \supset Dh(y,x))$

*f. $(\exists x)(\exists y) \sim (x = f(y) \vee y = f(x))$

3. Show that the following sentences are quantificationally true by constructing an appropriate systematic tree.

a. $(\forall x)(\exists y)y = f(f(x))$
*b. $(\forall x)(\forall y)(\forall z)((y = f(x) \ \& \ z = f(x)) \supset y = z)$
c. $(\forall x)Lf(x) \supset (\forall x)Lf(f(x))$
*d. $(\exists y)y = g(f(a))$

4. Construct systematic trees to determine, for each of the following sentences, whether that sentence is quantificationally true, quantificationally false, or quantificationally indeterminate. In each case state your result. If you abandon a tree, explain why.

a. $(\exists x)f(x) = x$
*b. $(\forall x)Gf(x)x$
c. $(\forall x)(\exists y)y = f(f(x))$
*d. $(\forall x)(Fx \vee \sim Fg(x))$

5. Construct systematic trees to determine which of the following arguments are quantificationally valid. In each case state your result. If you abandon a tree, explain why.

a. $(\exists x)(Fg(x) \ \& \sim Hg(x))$
$(\forall x)(Fx \supset Hx)$
$\sim Ra$

c. $a = f(b) \ \& \ b = f(a)$
$(\exists x)(\sim x = a \ \& \sim x = b)$

*b. $(\forall x) \ Pf(f(x))$
$Pf(a)$

*d. $(\forall x)(Fx \vee Fg(x))$
$(\forall x)(Fx \vee Fg(g(x)))$

6. Construct systematic trees to determine which of the following pairs of sentences are quantificationally equivalent. In each case state your result. If you abandon a tree, explain why.

a. $(\forall x)(\exists y)y = f(x)$ $(\forall x)(\exists y)x = f(y)$
*b. $Labf(b)$ $Laf(b)b$
c. $(\exists x)x = x$ $(\exists x)x = f(x)$
*d. $(\forall x)Bh(x)x$ $(\forall x)Bxh(x)$

7. Construct systematic trees to determine which of the following alleged entailments hold. In each case state your result. If you abandon a tree, explain why.

a. $[(\forall x)(\forall y)x = g(x,y)] \vDash (\forall x)x = g(x,x)$
*b. $[(\exists x)(\forall y) \ x = g(y)] \vDash h(a) = g(a)$
c. $[(\forall x)x = f(f(x))] \vDash (\forall x)x = f(x)$
*d. $[(\forall x)x = f(x)] \vDash (\forall x)x = f(f(x))$

SUMMARY

Key Semantic Properties

QUANTIFICATIONAL INCONSISTENCY: A finite set Γ of sentences of *PL/PLE* is *quantificationally inconsistent* if and only if Γ has a closed truth-tree.

QUANTIFICATIONAL CONSISTENCY: A finite set Γ of sentences of *PL/PLE* is *quantificationally consistent* if and only if Γ is not quantificationally inconsistent, that is, if and only if Γ does not have a closed truth-tree.

QUANTIFICATIONAL TRUTH: A sentence **P** of *PL/PLE* is *quantificationally true* if and only if the set $\{\sim \mathbf{P}\}$ has a closed truth-tree.

QUANTIFICATIONAL FALSITY: A sentence **P** of *PL/PLE* is *quantificationally false* if and only if the set $\{\mathbf{P}\}$ has a closed truth-tree.

QUANTIFICATIONAL INDETERMINACY: A sentence **P** of *PL/PLE* is *quantificationally indeterminate* if and only if neither the set $\{\mathbf{P}\}$ nor the set $\{\sim \mathbf{P}\}$ has a closed truth-tree.

QUANTIFICATIONAL EQUIVALENCE: Sentences **P** and **Q** of *PL/PLE* are *quantificationally equivalent* if and only if the set $\{\sim (\mathbf{P} \equiv \mathbf{Q})\}$ has a closed truth-tree.

QUANTIFICATIONAL ENTAILMENT: A finite set Γ of sentences of *PL/PLE quantificationally entails* a sentence **P** of *PL/PLE* if and only if $\Gamma \cup \{\sim \mathbf{P}\}$ has a closed truth-tree.

QUANTIFICATIONAL VALIDITY: An argument of *PL/PLE* with a finite number of premises is *quantificationally valid* if and only if the set consisting of the premises and the negation of the conclusion has a closed truth-tree.

Key Truth-Tree Concepts

CLOSED BRANCH OF A TRUTH-TREE FOR A SET OF SENTENCES OF *PL*: A branch containing both an atomic sentence and the negation of that sentence.

CLOSED BRANCH OF A TRUTH-TREE FOR A SET OF SENTENCES OF *PLE*: A branch containing both an atomic sentence and the negation of that sentence or a sentence of the form $\sim \mathbf{t} = \mathbf{t}$.

CLOSED TRUTH-TREE: A truth-tree each of whose branches is closed.

OPEN BRANCH: A branch that is not closed.

COMPLETED OPEN BRANCH OF A TRUTH-TREE FOR A SET OF SENTENCES OF *PL*: A finite open branch on which each sentence is one of the following:

1. A literal (an atomic sentence or the negation of an atomic sentence)
2. A compound sentence that is not a universally quantified sentence and is decomposed
3. A universally quantified sentence $(\forall \mathbf{x})\mathbf{P}$ such that $\mathbf{P}(\mathbf{a}/\mathbf{x})$ also occurs on that branch *for each constant* **a** occurring on the branch and at least one substitution instance $\mathbf{P}(\mathbf{a}/\mathbf{x})$ occurs on the branch

(SECTION 9.5 ACCOUNT) COMPLETED OPEN BRANCH OF A TRUTH-TREE FOR A SET OF SENTENCES OF *PLE*: A finite open branch on which each sentence is one of the following:

1. A literal that is not an identity sentence
2. A compound sentence that is not a universally quantified sentence and is decomposed
3. A universally quantified sentence $(\forall \mathbf{x})\mathbf{P}$ such that $\mathbf{P}(\mathbf{t}/\mathbf{x})$ also occurs on that branch *for each closed individual term* **t** occurring on the branch and at least one substitution instance $\mathbf{P}(\mathbf{t}/\mathbf{x})$ occurs on the branch
4. A sentence of the form $\mathbf{t}_1 = \mathbf{t}_2$, where $\mathbf{t}_1$ and $\mathbf{t}_2$ are closed terms such that the branch also contains, for every literal **P** on that branch containing $\mathbf{t}_2$, every sentence $\mathbf{P}(\mathbf{t}_1//\mathbf{t}_2)$ that can be obtained from **P** by Identity Decomposition

(SECTION 9.6 ACCOUNT) COMPLETED OPEN BRANCH OF A TRUTH-TREE FOR A SET OF SENTENCES OF *PLE*: A finite open branch on which each sentence is one of the following:

1. A literal that is not an identity sentence
2. A compound sentence that is not a universally quantified sentence and is decomposed
3. A universally quantified sentence $(\forall \mathbf{x})\mathbf{P}$ such that $\mathbf{P}(\mathbf{a}/\mathbf{x})$ also occurs on that branch *for each constant* **a** occurring on the branch and $\mathbf{P}(\mathbf{a}/\mathbf{x})$ occurs on the branch for at least one constant **a**
4. A sentence of the form $\mathbf{a} = \mathbf{t}$, where a is an individual constant and **t** is a closed term such that the branch also contains, for every literal **P** on that branch containing **t**, every sentence $\mathbf{P}(\mathbf{a}//\mathbf{t})$ that can be obtained from **P** by Identity Decomposition

and on which Complex Term Decomposition has been applied to every closed complex term occurring in a literal on the branch whose arguments are all individual constants.

NONTERMINATING BRANCH: An open branch that never closes and will never, in a finite number of steps, become a completed open branch.

COMPLETED TRUTH-TREE: A truth-tree each of whose branches is either closed or is a completed open branch.

OPEN TRUTH-TREE: A truth-tree that is not closed.

Chapter 10
PREDICATE LOGIC: DERIVATIONS

10.1 THE DERIVATION SYSTEM *PD*

In this chapter we develop natural deduction systems for predicate logic. The first system, *PD* (for *p*redicate *d*erivations), contains exactly two rules for each logical operator, just as *SD* contains exactly two rules for each sentential connective. It provides syntactic methods for evaluating sentences and sets of sentences of *PL*, just as the natural deduction system *SD* provides methods for evaluating sentences and sets of sentences of *SL*. *PD* is both complete and sound: for any set Γ of sentences of *PL* and any sentence **P** of *PL*,

$\Gamma \models \mathbf{P}$ if and only if $\Gamma \vdash \mathbf{P}$ in *PD*.

That is, a sentence **P** of *PL* is quantificationally entailed by a set Γ of sentences of *PL* if and only if **P** is derivable from Γ in *PD*. We prove this in Chapter 11.

The derivation rules of *PD* include all the derivation rules of *SD*, with the understanding that they apply to sentences of *PL*. So the following is a derivation in *PD*:

Derive: ~ (∀x)Hx

1	(∀x)Hx ⊃ ~ (∃y)Py	Assumption
2	(∃y)Py	Assumption
3	| (∀x)Hx	A / ~ I
4	| ~ (∃y)Py	1, 3 ⊃E
5	| (∃y)Py	2 R
6	~ (∀x)Hx	3–5 ~ I

The strategies we used with *SD* are also useful when working in *PD*. Those strategies are based on careful analyses of the goal or goals of a derivation—the structure of the sentence or sentences to be derived—and the structure of accessible sentences. They can be summarized thus:

- If the sentence that is the current goal can be derived by applying an elimination rule or some sequence of elimination rules to accessible sentences, then that is the strategy to follow.
- If the current goal can be obtained by an introduction rule, that is the strategy to follow.
- In most cases the successful strategy will make use of several of these approaches, working from the "bottom up" and from the "top down" as the occasion indicates.
- When using a negation rule try to use a negation that is readily available as the $-\mathbf{Q}$ that the rule requires within the negation subderivation.
- If a sentence is derivable from a set of sentences, then it is derivable using a negation rule as the primary strategy. So if no other strategy suggests itself it is useful to consider a negation strategy. But like all strategies, just because a negation strategy is available doesn't mean it is always the best choice.
- There will often be more than one plausible strategy, and often more than one will lead to success.

The new rules of *PD* call for some new strategies. We will introduce these as we introduce the new derivation rules of *PD*. *PD* contains four new rules, Universal Elimination, Universal Introduction, Existential Elimination, and Existential Introduction. Each of the new rules involves a quantified sentence and a substitution instance of that sentence. The elimination rule for the universal quantifier is Universal Elimination:

Universal Elimination ($\forall$E)

$(\forall \mathbf{x})\mathbf{P}$
$\mathbf{P}(\mathbf{a}/\mathbf{x})$

Here we use the expression '$\mathbf{P}(\mathbf{a}/\mathbf{x})$' to stand for a substitution instance of the quantified sentence $(\forall \mathbf{x})\mathbf{P}$. $\mathbf{P}(\mathbf{a}/\mathbf{x})$ is obtained from the quantified sentence by dropping the initial quantifier and replacing every occurrence of $\mathbf{x}$ with $\mathbf{a}$. We will refer to the constant $\mathbf{a}$ that is substituted for the variable $\mathbf{x}$ as the **instantiating constant** for the rule $\forall$E (and similarly for the other rules introduced on the following pages).

The rule Universal Elimination allows us to infer, from a universally quantified sentence, any substitution instance of that sentence. To understand the rule consider the simple argument

All philosophers are somewhat strange.
Socrates is a philosopher.

Socrates is somewhat strange.

The first premise makes a universal claim: it says that each thing is such that if it is a philosopher then it is somewhat strange. We can symbolize this claim as '(∀y)(Py ⊃ Sy)'. The second premise can be symbolized as 'Ps' and the conclusion as 'Ss'. Here is a derivation of the conclusion from the premises.

Derive: Ss

1	(∀y)(Py ⊃ Sy)	Assumption
2	Ps	Assumption
3	Ps ⊃ Ss	1 ∀E
4	Ss	2, 3 ⊃E

The sentence on line 3 is a substitution instance of the quantified sentence on line 1. When we remove the initial (and only) quantifier from '(∀y)(Py ⊃ Sy)' we get the open sentence 'Py ⊃ Sy', which contains two free occurrences of 'y'. Replacing both occurrences with the constant 's' yields the substitution instance 'Ps ⊃ Ss' on line 3, justified by ∀E. We then use Conditional Elimination to obtain 'Ss'.

This simple derivation illustrates the first new strategy for constructing derivations in *PD*:

- When using Universal Elimination use goal sentences as guides to which constant to use in forming the substitution instance of the universally quantified sentence.

At line 3 in the above derivation we could have entered 'Pa ⊃ Sa', or any other substitution instance of '(∀y)(Py ⊃ Sy)'. But obviously only the substitution instance using 's' is of any use in completing the derivation.

That Universal Elimination is truth-preserving should be apparent. 'Ps ⊃ Ss' says of one thing (whatever thing 's' designates) exactly what '(∀y)(Py ⊃ Sy)' says of each thing in the UD. If 'All philosophers are somewhat strange' is true, then it is true of David Hume that if he is a philosopher he is somewhat strange, and true of Isaac Newton that if he is a philosopher he is somewhat strange, and true of Marie Curie that if she is a philosopher she is somewhat strange, and true of the Milky Way that if it is a philosopher then it is somewhat strange.

The instantiating constant employed in Universal Elimination may or may not already occur in the quantified sentence. The following is a correct use of Universal Elimination:

1	(∀x)Lxa	Assumption
2	Laa	1 ∀E

If we take our one assumption to symbolize 'Everyone loves Alice', with 'a' designating Alice, then clearly it follows that Tom, or whomever t designates, loves Alice. The following is also a correct use of Universal Elimination:

1	$(\forall x)Lxa$	Assumption
2	Laa	1 ∀E

If everyone loves Alice, then it follows that Alice loves Alice, that is, that Alice loves herself.

The introduction rule for existential quantifiers is Existential Introduction:

Existential Introduction (∃I)

$$\begin{array}{|l} \mathbf{P(a/x)} \\ \\ \triangleright\ (\exists \mathbf{x})\mathbf{P} \end{array}$$

This rule allows us to infer an existentially quantified sentence from any one of its substitution instances. Here is an example:

1	Fa	Assumption
2	$(\exists y)Fy$	1 ∃I

That Existential Introduction is truth-preserving should also be obvious. If the thing designated by the constant 'a' is F, then at least one thing is F. For example, if Alfred is a father, then it follows that someone is a father.

The following derivation uses Existential Introduction three times:

1	Faa	Assumption
2	$(\exists y)Fya$	1 ∃I
3	$(\exists y)Fyy$	1 ∃I
4	$(\exists y)Fay$	1 ∃I

These uses are all correct because the sentence on line 1 is a substitution instance of the sentence on line 2, and of the sentence on line 3, and of the sentence on line 4. If Alice is fond of herself, then it follows that someone is fond of Alice, that someone is fond of her/himself, and that Alice is fond of someone.

The strategy for using Existential Introduction is straightforward:

- When the goal to be derived is an existentially quantified sentence establish a substitution instance of that sentence as a subgoal, with the intent of applying Existential Introduction to that subgoal to obtain the goal.

Universal Introduction and Existential Elimination are somewhat more complicated than the two rules just considered. We begin with Universal Introduction:

Universal Introduction (∀I)

$$\begin{array}{|l} \mathbf{P(a/x)} \\ \\ \hline (\forall \mathbf{x})\mathbf{P} \end{array}$$

provided that

(i) **a** does not occur in an open assumption.
(ii) **a** does not occur in $(\forall \mathbf{x})\mathbf{P}$.

Here, again, we will call the constant **a** in $\mathbf{P(a/x)}$ the *instantiating constant*. This rule specifies that under certain conditions we can infer a universally quantified sentence from one of its substitution instances. At first glance this might seem implausible, for how can we infer, from a claim that a particular thing is of a certain sort, that *everything* is of that sort? The answer, of course, lies in the restrictions specified in the "provided that" clause.

Here is a very simple example. The sentences '(∀x)Fx' and '(∀y)Fy' are what we might call notational variants of each other. They both say that everything is F. So we should be able to derive each from the other. Below we derive the second from the first:

Derive: (∀y)Fy

1	(∀x)Fx	Assumption
2	Fb	1 ∀E
3	(∀y)Fy	2 ∀I

As required by the first restriction on Universal Introduction, 'b' does not occur in any open assumption, so the rule is correctly applied at line 3. This derivation contains only one assumption, at line 1, and 'b' does not occur in that assumption. This means that we have not assumed any particular information about whatever the constant 'b' might designate. No matter how 'b' is interpreted, the sentence on line 2 must be true if the assumption on line 1 is true. But if 'Fb' is true no matter how 'b' is interpreted, then the universally quantified sentence on line 3 clearly will be true as well.

The kind of reasoning that Universal Introduction is based on is common in mathematics. Suppose we want to establish that no even positive integer greater than 2 is prime. [A prime is a positive integer that is evenly divisible only by itself and 1, and is not 1.] We might reason thus:

> Consider any even positive integer *i* greater than 2. Because *i* is even, *i* must be evenly divisible by 2. But since *i* is not 2 (it is greater than 2), it follows that *i* is evenly divisible by at least three positive integers: 1, 2, and

i itself. So it is not the case that *i* is evenly divisible only by itself and 1, and *i* cannot be prime. Therefore no even positive integer greater than 2 is prime.

It would exhibit a misunderstanding of this reasoning to reply "but the positive integer *i* you considered might have been 4, and while the reasoning does hold of 4—it is not prime—that fact alone doesn't show that the reasoning holds of every even positive integer greater than 2. You haven't considered 6 and 8 and 10 and. . . ." It would be a misunderstanding because in saying "Consider any even positive integer *i* greater than 2" we don't mean "Pick one". We say "Consider any even positive integer *i* . . ." because it is easier to construct the argument when we are speaking, grammatically, in the singular ("*i* is . . .", "*i* is not . . ."). But what we are really saying is "Consider what we know about all positive integers that are even and greater than two . . ." So the proof is a proof about all such integers. Similarly, in derivations we often use an individual constant to reason about all cases of a certain sort.

Suppose we want to establish that '$(\forall x)[Fx \supset (Fx \vee Gx)]$' can be derived from no assumptions. (This will, of course, establish that this sentence of *PL* is a theorem in *PD*.) Here is one such derivation:

Derive: $(\forall x)[Fx \supset (Fx \vee Gx)]$

1	| Fc	A / ⊃I
2	| Fc ∨ Gc	1 ∨I
3	Fc ⊃ (Fc ∨ Gc)	1–2 ⊃I
4	$(\forall x)[Fx \supset (Fx \vee Gx)]$	3 ∀I

The sentence on line 3 follows from the subderivation on lines 1–2, no matter what the constant 'c' designates. The subderivation establishes that no matter what c is, if it is F then it is F or G. Hence we are justified in deriving the universal quantification on line 4. Note that although 'c' occurs in the assumption on line 1, that assumption is not open at line 4, so we have not run afoul of the first restriction on the rule Universal Introduction.

On the other hand, Universal Introduction is misused in the following attempted derivation:

Derive: $(\forall y)Fy$

1	Fb & ~ Fc	Assumption	
2	Fb	1 &E	
3	$(\forall y)Fy$	2 ∀I	**MISTAKE!**

'Fb' does follow from line 1. But from the truth of 'Fb & ~ Fc', where 'b' designates some one member of the UD, it does not follow that 'Fb' is true no matter what member of the UD we might take 'b' to designate. It only follows that 'Fb' is true as long as 'b' in 'Fb' designates the same thing as it does in 'Fb & ~ Fc'. So it is incorrect to infer that *everything* is F, and we want to block the move to line 3. (From the fact that Beth is a faculty member and Carl is

not it does not follow that everyone is a faculty member.) The rule Universal Introduction prevents us from citing line 2 as a justification for line 3 by stipulating that the instantiating constant (the **a** of the substitution instance $\mathbf{P}(\mathbf{a}/\mathbf{x})$ of $(\forall\mathbf{x})\mathbf{P}$) not occur in an open assumption. In our current example 'b' occurs in the open assumption on line 1.

The rule Universal Introduction contains a second restriction, namely that the instantiating constant not occur in the derived universally quantified sentence. The following attempt at a derivation illustrates why this restriction is needed:

Derive: (∀x)Lxh

1	(∀x)Lxx	Assumption	
2	Lhh	1 ∀E	
3	(∀x)Lxh	2 ∀I	**MISTAKE!**

Suppose we interpret 'Lhh' as 'Henry loves Henry'. While line 2 does follow from line 1, it does not follow from line 1 that everyone loves Henry. The second restriction on Universal Introduction has been violated: the instantiating constant 'h' in the substitution instance on line 2 occurs in the sentence we tried to derive by Universal Introduction on line 3.

The strategy associated with Universal Introduction is

- When the current goal is a universally quantified sentence make a substitution instance of that quantified sentence a subgoal, with the intent of applying Universal Introduction to derive the goal from the subgoal. Make sure that the two restrictions on Universal Introduction will be met: use an instantiating constant in the substitution instance that does not occur in the universally quantified goal sentence and that does not occur in any assumption that is open at the line where the substitution instance is entered.

Here is the elimination rule for existential quantifiers:

Existential Elimination (∃E)

$(\exists\mathbf{x})\mathbf{P}$

$\mathbf{P}(\mathbf{a}/\mathbf{x})$
$\mathbf{Q}$

▷ $\mathbf{Q}$

provided that

(i) **a** does not occur in an open assumption.
(ii) **a** does not occur in $(\exists\mathbf{x})\mathbf{P}$.
(iii) **a** does not occur in **Q**.

The idea behind this rule is that if we have an existentially quantified sentence $(\exists \mathbf{x})\mathbf{P}$ then we know that something is of the sort specified by **P**, though not which thing. If, by assuming an arbitrary substitution instance $\mathbf{P}(\mathbf{a}/\mathbf{x})$ of $(\exists \mathbf{x})\mathbf{P}$, we can derive a sentence **Q** that makes no mention of the instantiating constant **a** in $\mathbf{P}(\mathbf{a}/\mathbf{x})$, then we can discharge the assumed substitution instance and simply infer **Q**.

We illustrate a simple use of Existential Elimination by deriving '$(\exists x)(Gx \vee Fx)$' from $\{(\exists z)Fz \,\&\, (\forall y)Hy\}$.

Derive: $(\exists x)(Gx \vee Fx)$

1	$(\exists z)Fz \,\&\, (\forall y)Hy$	Assumption
2	$(\exists z)Fz$	1 &E
3	$\quad Fb$	A / $\exists$E
4	$\quad Gb \vee Fb$	3 $\vee$I
5	$\quad (\exists x)(Gx \vee Fx)$	4 $\exists$I
6	$(\exists x)(Gx \vee Fx)$	2, 3–5 $\exists$E

'Existential Elimination' may seem like an odd name for the rule we used at line 6 of the above derivation, because the sentence entered at line 6 is itself an existentially quantified sentence. But remember that what is common to all elimination rules is that they are rules that *start* with a sentence with a specified main logical operator and produce a sentence that may or may not have that operator as a main logical operator. Here Existential Elimination cites the existentially quantified sentence at line 2, along with the subderivation beginning with a substitution instance of that sentence. Note that we have met all the restrictions on Existential Elimination. The instantiating constant 'b' does not occur in an assumption that is open as of line 6. Nor does 'b' occur in '$(\exists z)Fz$'. Finally, 'b' does not occur in the sentence that is derived, at line 6, by Existential Elimination. All three of these restrictions are necessary, as we will illustrate next.

Two specific strategies are associated with the rule Existential Elimination. The first is this:

- When one or more of the currently accessible sentences in a derivation is an existentially quantified sentence, consider using Existential Elimination to obtain the current goal. Assume a substitution instance that contains a constant that does not occur in the existential quantification, in an open assumption, or in the current goal. Work within the Existential Elimination subderivation to derive the current goal.

In other words, whenever an existentially quantified sentence is accessible consider making Existential Elimination the primary strategy for obtaining

the current goal, doing the work required to obtain the current goal within the scope of the Existential Elimination subderivation. This is often necessary to avoid violating the restrictions on Existential Elimination. For example, in the previous derivation we had to use Existential Introduction within the scope of the assumption on line 3—because trying to derive '$Gb \vee Fb$' by Existential Elimination at line 5, prior to applying Existential Generalization, would violate the third restriction on Existential Elimination:

Derive: $(\exists x)(Gx \vee Fx)$

1	$(\exists z)Fz \ \& \ (\forall y)Hy$	Assumption	
2	$(\exists z)Fz$	1 &E	
3	$\quad Fb$	A / $\exists$E	
4	$\quad Gb \vee Fb$	3 $\vee$I	
5	$Gb \vee Fb$	2, 3–4 $\exists$E	**MISTAKE!**
6	$(\exists x)(Gx \vee Fx)$	5 $\exists$I	

Line 5 is a mistake because the instantiating constant 'b' occurs in the sentence we are trying to obtain by Existential Elimination, in violation of the third restriction on Existential Elimination. From the truth of '$(\exists z)Fz$' it does not follow that the individual designated by 'b' is either G or F—although it does follow, as in the previous derivation, that *something* is either G or F. This is why, in the correctly done derivation, we used Existential Introduction inside of the Existential Elimination subderivation. Doing so results in a sentence that does not contain the instantiating constant 'b' and that therefore can correctly be moved out of the subderivation by Existential Elimination.

Here is another example in which the third restriction on Existential Elimination is violated:

Derive: $(\exists z)Fbz$

1	$(\exists z)Fzz$	Assumption	
2	$\quad Fbb$	A / $\exists$E	
3	$\quad (\exists z)Fbz$	3 $\exists$I	
4	$(\exists z)Fbz$	1, 2–3 $\exists$E	**MISTAKE!**

The instantiating constant 'b' occurs in the sentence on line 4, in violation of restriction (iii) on Existential Elimination. It is clear that we don't want the above to count as a derivation. Given the assumption on line 1 we know that something bears F to itself. At line 2 we assume that thing is b (knowing that this may not be the case). Line 3 certainly follows from line 2. *If* b bears F to itself then b does bear F to something. But line 4, where we have given up the assumption that it is b that bears F to itself, does not follow from the sentence at line 1, which is the single open assumption as of line 4. From the fact that

something bears F to itself it does not follow that it is b that does so. Contrast the preceding derivation with the following:

Derive: $(\exists y)Fyy$

1	$(\exists z)Fzz$		Assumption
2	| Fbb		A / $\exists E$
3	| $(\exists y)Fyy$		2 $\exists I$
4	$(\exists y)Fyy$		1, 2–3 $\exists E$

Here 'b' does not occur in the sentence at line 4, so the third restriction on Existential Elimination is not violated. We have used Existential Elimination to show that '$(\exists y)Fyy$' follows from '$(\exists z)Fzz$', which should be no surprise since these sentences are clearly equivalent.

We will now examine some misuses of Existential Elimination that illustrate why the two other restrictions on Existential Elimination are also necessary.

Derive: $(\forall x)Fx$

1	$Gb \supset (\forall x)Fx$	Assumption	
2	$(\exists z)Gz$	Assumption	
3	| Gb	A / $\exists E$	
4	| $(\forall x)Fx$	1, 3 $\supset E$	
5	$(\forall x)Fx$	2, 3–4 $\exists E$	**MISTAKE!**

From line 1 we know that if a particular thing, namely b, is G, then everything is F. And from line 2 we know that something is G. But we do not know that it is b that is G. So we should not be able to infer, as we have here tried to do at line 5, that everything is F. Line 5 is a mistaken application of Existential Elimination because restriction (i) has not been met. The assumption at line 1, which contains the instantiating constant 'b', is still open as of line 5. The rationale for restriction (i) should now be clear. Existential Elimination uses a substitution instance of an existentially quantified claim to show what follows from the existentially quantified claim. But the constant used in the substitution instance, the instantiating constant, should be arbitrary, in the sense that no assumptions have been made concerning the thing designated by that constant. If the instantiating constant occurs in an open assumption then it is not arbitrary, because the open assumption gives us specific information about whatever the constant designates. If it is true that if Bob graduates then everyone is happy and it is true that someone graduates, it doesn't follow that everyone is happy—because even if *someone* graduates Bob might not.

We now turn to the rationale for the second restriction. Consider the following mistaken derivation:

Derive: (∃w)Lww

1	(∀y)(∃z)Lzy	Assumption	
2	(∃z)Lza	1 ∀E	
3	| Laa	A / ∃E	
4	| (∃w)Lww	3 ∃I	
5	(∃w)Lww	2, 3–4 ∃E	**MISTAKE!**

Suppose we take positive integers as our UD and interpret 'Lxy' as 'x is greater than y' On this interpretation the sentence on line 1 says that every positive integer is such that there is a positive integer greater than it, which is of course true. But the sentence on line 5 says there is a positive integer that is greater than itself, which is obviously false. The problem is that the instantiating constant 'a' used at line 3 to form a substitution instance of the sentence '(∃z)Lza' occurs in '(∃x)Lza', violating the second restriction. If we only know that *something* stands in the relation L to a, we should not assume that that something is in fact a itself.

Universal Elimination produces a substitution of the universally quantified sentence to which it is applied. Existential Instantiation does not, in general, produce a substitution instance of the existentially quantified sentence to which it is applied. Indeed the sentence it produces may bear no resemblance, by any normal standard of resemblance, to the existentially quantified sentence to which it is applied. Here is a case in point:

Derive: (∃x)Hx

1	(∃z)Gz	Assumption
2	(∀y)(Gy ⊃ Hc)	Assumption
3	| Gb	A / ∃E
4	| Gb ⊃ Hc	2 ∀E
5	| Hc	3, 4 ⊃E
6	Hc	1, 3–5 ∃E
7	(∃x)Hx	6 ∃I

Here the sentence derived at line 6 bears no resemblance to the existentially quantified sentence at line 1. The existential quantified sentence tells us that something is G. At line 3 we assume that that thing is b. The constant 'b' is not used earlier in the derivation, so we are committed to nothing about b other than its being G. At line 4 we use Universal Elimination to obtain 'Gb ⊃ Hc', and then we use Conditional Elimination at line 5 to obtain 'Hc'. At the point we apply Existential Elimination (line 6) there is here no open assumption that contains 'b'—the only open assumptions are those on line 1 and line 2—so the first restriction on Existential Elimination is met. The second and third restrictions are also met since 'b' occurs in neither '(∃z)Gz' nor 'Hc'. So it is correct to derive 'Hc' by Existential Elimination at line 6, although, as noted, it bears no resemblance to the existentially quantified sentence at line 1.

Note that in this case we were able to move 'Hc' out of the Existential Elimination subderivation prior to using Existential Introduction. We could do this because 'c' was not the instantiating constant for our use of Existential Elimination. However, we could also have applied Existential Introduction within the subderivation;

Derive: $(\exists x)Hx$

1	$(\exists z)Gz$	Assumption
2	$(\forall y)(Gy \supset Hc)$	Assumption
3	$\quad Gb$	A / $\exists$E
4	$\quad Gb \supset Hc$	2 $\forall$E
5	$\quad Hc$	3, 4 $\supset$E
6	$\quad (\exists x)Hx$	5 $\exists$I
7	$(\exists x)Hx$	1, 3–6 $\exists$E

Existential Elimination is an elimination rule not in the sense that it produces a substitution instance of an existentially quantified sentence—in general it does not—but rather that it provides a strategy for working from a substitution instance of an existentially quantified sentence to some desired sentence that does not contain the instantiating constant and hence that does not assume that it is the thing designated by that particular constant that accounts for the truth of the existentially quantified sentence.

There is a second important strategy associated with Existential Elimination. We will use it to show that the set $\{(\exists x) \sim Fx, (\forall x)Fx\}$ is inconsistent in *PD*. The foregoing set obviously is inconsistent, but demonstrating this is not as easy as it might seem. We might start as follows:

Derive: $Fa, \sim Fa$

1	$(\exists x) \sim Fx$	Assumption	
2	$(\forall x)Fx$	Assumption	
3	$\quad Fa$	2 $\forall$E	
4	$\quad \sim Fa$	1 $\exists$E	**MISTAKE!**

Line 4 is an obvious misuse of Existential Elimination. A more promising approach might be as follows:

Derive: $Fa, \sim Fa$

1	$(\exists x) \sim Fx$	Assumption
2	$(\forall x)Fx$	Assumption
3	$\quad \sim Fa$	A / $\exists$E
4	$\quad Fa$	2 $\forall$E
5	$\quad \sim Fa$	3 R

We have derived our goal sentences, but only within the scope of our Existential Elimination subderivation. And since 'a' is the instantiating constant of the assumption at line 3, we cannot hope to move either 'Fa' or '~ Fa' out from the scope of the assumption at line 3 by Existential Elimination. The situation we

are in is not an uncommon one. We need to use Existential Elimination, and we can derive a contradiction within the Existential Elimination subderivation, but the contradictory sentences we derive cannot be moved outside that subderivation because they contain the instantiating constant of the assumption.

The strategy we will use in situations such as this makes use of the fact that we can derive contradictory sentences within the Existential Elimination subderivation. Since we can do this, we can also derive any sentence we want by use of the appropriate negation rule. In our present case we want to derive a sentence and its negation, to show that the set we are working from is inconsistent in *PD*. There are no negations among our primary assumptions. We know taking 'Fa' and '~ Fa' as our ultimate goals will not work (so long as '~ Fa' remains as our Existential Elimination assumption at line 3). So we will take a sentence that is accessable, '(∀x)Fx', and its negation as our ultimate goals, and we will derive '~ (∀x)Fx' by Negation Introduction within our Existential Elimination subderivation, and then move it out of that subderivation by Existential Elimination:

Derive: (∀x)Fx, ~ (∀x)Fx

1	(∃x) ~ Fx	Assumption
2	(∀x)Fx	Assumption
3	~ Fa	A / ∃E
4	(∀x)Fx	A / ~ I
5	Fa	4 ∀E
6	~ Fa	3 R
7	~ (∀x)Fx	4–6 ~ I
8	~ (∀x)Fx	1, 3–7 ∃E
9	(∀x)Fx	2 R

What may strike one as odd about this derivation is that we are assuming, at line 4, a sentence that is already accessible (as the assumption on line 2). But the point of making this assumption of a sentence we already have is to derive its negation, which we do at line 7. Negation Introduction requires us to assume this sentence, even though it also occurs at line 2, before we can apply that rule. At line 4 we could, of course, have equally well assumed '(∃x) ~ Fx', in which case our ultimate goals would have been '(∃x) ~ Fx' and '~ (∃x) ~ Fx'.

The strategy we are illustrating can be put thus:

- When contradictory sentences are available within an Existential Elimination subderivation but cannot be moved out of that subderivation without violating the restrictions on Existential Elimination, derive another sentence—one that is contradictory to a sentence accessible outside the Existential Elimination subderivation and one that can be moved out. That sentence will be derivable by the appropriate negation strategy (because contradictory sentences are available within the Existential Elimination subderivation).

Using this strategy will frequently involve assuming, as the assumption of a negation strategy, a sentence that is already accessible outside the Existential Elimination subderivation.

Here is another derivation problem in which this strategy is useful:

Derive: ~ (∃x)Fx

1	(∀x) ~ Fx	Assumption	
2	| (∃x)Fx	A / ~ I	
3	| | Fa	A / ∃E	
4	| | ~ Fa	1 ∀E	
5	| | Fa	3 R	
6	| ~ (∃x)Fx	3–5 ~ I	**MISTAKE!**
7	~ (∃x)Fx	2, 3–6 ∃E	**MISTAKE!**

We are trying to derive a negation, '~ (∃x)Fx', and so assume '(∃x)Fx' at line 2. Clearly an Existential Elimination strategy is now called for, and accordingly we assume 'Fa' at line 3. It is now easy to derive the contradictory sentences 'Fa' and '~ Fa', and we do so at lines 4 and 5. But line 6 is a mistake. Our primary strategy is Negation Introduction and we have derived a sentence and its negation; but we have done so only within the scope of an additional assumption, the one at line 3 that begins our Existential Elimination strategy. Line 6 is a mistake because 'Fa' and '~ Fa' have been derived, not from just the assumptions on lines 1 and 2, but also using the assumption on line 3. We need to complete our Existential Elimination strategy before using Negation Introduction. And what we want our Existential Elimination strategy to yield is a sentence that can serve as one of the contradictory sentences we need to complete the Negation Introduction subderivation we began at line 2.

Two sentences are accessible outside our Existential Elimination subderivation—those on lines 1 and 2 ('(∀x) ~ Fx' and '(∃x)Fx') and obtaining the negation of either one of these by Existental Elimination will allow us to complete the derivation. Here is a successful derivation in which we derive '~ (∀x) ~ Fx' by Existential Elimination.

Derive: ~ (∃x)Fx

1	(∀x) ~ Fx	Assumption
2	| (∃x)Fx	A / ~ I
3	| | Fa	A / ∃E
4	| | | (∀x) ~ Fx	A / ~ I
5	| | | ~ Fa	1 ∀E
6	| | | Fa	3 R
7	| | ~ (∀x) ~ Fx	4–6 ~ I
8	| ~ (∀x) ~ Fx	2, 3–7 ∃E
9	| (∀x) ~ Fx	1 R
10	~ (∃x)Fx	2–9 ~ I

After making the assumption at line 3 we realize we can derive the contradictory sentences 'Fa' and '~ Fa'. Because we want to obtain '~ (∀x) ~ Fx' by Existential Elimination, we assume '(∀x) ~ Fx' at line 4 and derive '~ Fa' and 'Fa' within the scope of that assumption, allowing us to then derive '~ (∀x) ~ Fx' by Negation Introduction.

Alternatively, we could have used '(∃x)Fx' as an assumption at line 4, derived 'Fa' and '~ Fa', obtained '~ (∃x)Fx' by Negation Elimination, moved that sentence out of the scope of the assumption made at line 3 by Existential Elimination, and then reiterated '(∃x)Fx' within the scope of the assumption '(∃x)Fx' so as to have the contradictory sentences we need to finish the derivation with Negation Introduction. Note also that the assumption at line 4 is necessary to obtain its negation even though the sentence we assume is already available as an earlier assumption (on line 1). This process of making an assumption of a sentence that is already available outside the scope of an Existential Elimination strategy within that strategy *in order to obtain its negation* is extremely useful and frequently called for, as we will see in examples and exercises later in this chapter.

Here is a further example illustrating this new strategy:

Derive: ~ (∃x) (Fx & ~ Gx)

1	(∀x) (~ Gx ⊃ ~ Fx)	Assumption
	~ (∃x) (Fx & ~ Gx)	

Since our primary goal is a negation we plan to use Negation Introduction, and since the assumption of that strategy will be an existentially quantified sentence, we will use Existential Elimination within the Negation Introduction subderivation:

Derive: ~ (∃x) (Fx & ~ Gx)

1	(∀x) (~ Gx ⊃ ~ Fx)	Assumption
2	(∃x) (Fx & ~ Gx)	A / ~ I
3	Fa & ~ Ga	A / ∃E
G	~ (∃x) (Fx & ~ Gx)	

Following our new strategy we will begin a Negation Introduction subderivation inside of the Existential Elimination subderivation, assuming one of the sentences that is accessible from outside of that subderivation. In this example there are again two such sentences, '$(\forall x)(\sim Gx \supset \sim Fx)$' and '$(\exists x)(Fx \,\&\, \sim Gx)$'. We arbitrarily select the latter as the assumption of the inner Negation Introduction subderivation and complete the derivation as follows:

Derive: $\sim (\exists x)(Fx \,\&\, \sim Gx)$

1	$(\forall x)(\sim Gx \supset \sim Fx)$	Assumption
2	$(\exists x)(Fx \,\&\, \sim Gx)$	A / $\sim$I
3	$Fa \,\&\, \sim Ga$	A / $\exists$E
4	$(\exists x)(Fx \,\&\, \sim Gx)$	A / $\sim$I
5	$\sim Ga \supset \sim Fa$	1 $\forall$E
6	$\sim Ga$	3 &E
7	$\sim Fa$	5, 6 $\supset$E
8	Fa	3 &E
9	$\sim (\exists x)(Fx \,\&\, \sim Gx)$	4–8 $\sim$I
10	$\sim (\exists x)(Fx \,\&\, \sim Gx)$	2, 3–9 $\exists$E
11	$(\exists x)(Fx \,\&\, \sim Gx)$	2 R
12	$\sim (\exists x)(Fx \,\&\, \sim Gx)$	2–11 $\sim$I

Although the assumption at line 4 is an existentially quantified sentence, there is no need to use Existential Elimination. We can derive the contradictory pair of sentences 'Fa' and '~ Fa' without making any additional assumptions.

We have specified strategies for using each of the four new quantifier rules. Now that we have introduced all the rules of *PD* a note about applying those rules is in order. The quantifier introduction and elimination rules, like all the rules of *PD*, are *rules of inference*. That is, they apply only to whole sentences, *not* to subsentential components of sentences that may or may not themselves be sentences. The only sentences that quantifier elimination rules can be applied to are sentences whose main logical operators are quantifiers. Moreover, the quantifier *introduction* rules generate only sentences whose main logical operators are quantifiers. The following examples illustrate some common types of mistakes that ignore these points about the quantifier rules of *PD*.

Derive: $Fa \supset Ha$

1	$(\forall x)Fx \supset Ha$	Assumption	
2	$Fa \supset Ha$	1 $\forall$E	**MISTAKE!**

The sentence on line 1 is not a universally quantified sentence. Rather, it is a material conditional, so Universal Elimination cannot be applied to it. Obviously, the sentence on line 2 does not follow from the sentence on line 1. From that fact that if everything is F then a is H it does not follow that if a (which is only one thing) is F then a is H. If it is the case that if everyone is funny

then Albert is happy, it does not follow that if Albert (and perhaps no one else) is funny then Albert is happy.

Here is another example illustrating a similar mistake:

Derive: Ga

1	Fa	Assumption	
2	$(\forall x)(Fx \supset (\forall y)Gy)$	Assumption	
3	$(\forall x)(Fx \supset Ga)$	2 ∀E	**MISTAKE!**
4	$Fa \supset Ga$	3 ∀E	
5	Ga	1, 4 ⊃E	

Line 3 is a mistake even though the sentence it cites, '$(\forall x)(Fx \supset (\forall y)Gy)$', is a universally quantified sentence. It is a mistake because it attempts to apply Universal Elimination to '$(\forall y)Gy$', which occurs only as a component of the sentence on line 2. Rules of inference can only be applied to sentences that are not components of larger sentences. Universal Elimination can only produce a substitution instance, for example '$Fa \supset (\forall y)Gy$', of the entire sentence on line 2.

We hasten to add that it *is* possible to derive 'Ga' from the sentences on lines 1 and 2 but a different strategy is required:

Derive: Ga

1	Fa	Assumption	
2	$(\forall x)(Fx \supset (\forall y)Gy)$	Assumption	
3	$Fa \supset (\forall y)Gy$	2 ∀E	
4	$(\forall y)Gy$	1, 3 ⊃E	
5	Ga	4 ∀E	

Here Universal Elimination has only been applied to entire sentences occurring on earlier lines.

The following also illustrates a misuse of a quantifier rule:

Derive: $(\exists z)Fz \supset Gb$

1	$Fa \supset Gb$	Assumption	
2	$(\exists z)Fz \supset Gb$	1 ∃I	**MISTAKE!**

Existential Introduction produces existentially quantified sentences, and the sentence on line 2 is a material conditional, not an existentially quantified sentence. Nor do we want to be able to derive the sentence on line 2. If it is true that if Alice flirts then Bob grins it does not follow that if anyone flirts, Bob grins. A correct use of the rule would be

Derive: $(\exists z)(Fz \supset Gb)$

1	$Fa \supset Gb$	Assumption	
2	$(\exists z)(Fz \supset Gb)$	1 ∃I	

If it is true that if Alice flirts then Bob grins, it is true that there is someone (Alice) such that if that person flirts, Bob grins.

In the following attempted derivation, the use of Universal Elimination is incorrect because the sentence on line 1 is not a universally quantified sentence. Rather, it is the *negation* of a universally quantified sentence:

Derive: ~ Fb

1	~ (∀y)Fy	Assumption	
2	~ Fb	1 ∀E	**MISTAKE!**

From the fact that not everyone flirts it does not follow that Bob doesn't.

Having introduced all the rules of *PD* we can now define the basic syntactic concepts of *PD*, which parallel those of *SD*:

Derivability in PD: A sentence **P** of *PL* is *derivable in PD* from a set Γ of sentences of *PL* if and only if there is a derivation in *PD* in which all the primary assumptions are members of Γ and **P** occurs within the scope of only the primary assumptions.

Validity in PD: An argument of *PL* is *valid in PD* if and only if the conclusion of the argument is derivable in *PD* from the set consisting of the premises. An argument of *PL* is *invalid in PD* if and only if it is not valid in *PD*.

Theorem in PD: A sentence **P** of *PL* is a *theorem in PD* if and only if **P** is derivable in *PD* from the empty set.

Equivalence in PD: Sentences **P** and **Q** of *PL* are *equivalent in PD* if and only if **Q** is derivable in *PD* from {**P**} and **P** is derivable in *PD* from {**Q**}.

Inconsistency in PD: A set Γ of sentences of *PL* is *inconsistent in PD* if and only if there is a sentence **P** such that both **P** and ~ **P** are derivable in *PD* from Γ. A set Γ is *consistent in PD* if and only if it is not inconsistent in *PD*.

10.1E EXERCISES

1. Construct derivations that establish the following claims:

a. {(∀x)Fx} ⊢ (∀y)Fy
*b. {Fb, Gb} ⊢ (∃x)(Fx & Gx)
c. {(∀x)(∀y)Hxy} ⊢ (∃x)(∃y)Hxy
*d. {(∃x)(Fx & Gx)} ⊢ (∃y)Fy & (∃w)Gw
e. {(∀x)(∀y)Hxy, Hab ⊃ Kg} ⊢ Kg
*f. {(∀x)(Fx ≡ Gx), (∀y)(Gy ≡ Hy)} ⊢ (∀x)(Fx ≡ Hx)
g. {(∀x)Sx, (∃y)Sy ⊃ (∀w)Ww} ⊢ (∃y)Wy
*h. {(∀y)Hyy, (∃z)Bz} ⊢ (∃x)(Bx & Hxx)
i. {(∀x)(∀y)Lxy, (∃w)Hww} ⊢ (∃x)(Lxx & Hxx)
*j. {(∀x)(Fx ⊃ Lx), (∃y)Fy} ⊢ (∃x)Lx

2. Identify the mistake in each of the following attempted derivations, and explain why it is a mistake.

a. Derive: Na

1	(∀x)Hx ⊃ ~ (∃y)Ky	Assumption
2	Ha ⊃ Na	Assumption
3	Ha	1 ∀E
4	Na	2, 3 ⊃E

*b. Derive: (∀x)(Bx & Mx)

1	Bk	Assumption
2	(∀x)Mx	Assumption
3	Mk	2 ∀E
4	Bk & Mk	1, 3 &I
5	(∀x)(Bx & Mx)	4 ∀I

c. Derive: (∃x)Cx

1	(∃y)Fy	Assumption
2	(∀w)(Fw ≡ Cw)	Assumption
3	Fa	1 ∃E
4	Fa ≡ Ca	2 ∀∃E
5	Ca	3, 4 ≡E
6	(∃x)Cx	5 ∃I

*d. Derive: (∃z)Gz

1	(∀x)(Fx ⊃ Gx)	Assumption
2	(∃y)Fy	Assumption
3	Fa	A / ∃E
4	Fa ⊃ Ga	1 ∀E
5	Ga	3, 4 ⊃E
6	Ga	2, 3–5 ∃E
7	(∃z)Gz	6 ∃I

e. Derive: (∃y)(∀x)Ayx

1	(∀x)(∃y)Ayx	Assumption
2	(∀x)Aax	1 ∀E
3	(∃y)(∀x)Ayx	2 ∃I

*f. Derive: ~ Rba

1	(∃x)Rxx	Assumption
2	(∀x)(∀y)(Rxy ⊃ ~ Ryx)	Assumption
3	Raa	A / ∃E
4	(∀y)(Ray ⊃ ~ Rya)	2 ∀E
5	Raa ⊃ ~ Raa	2 ∀E
6	~ Raa	3, 5 ⊃E
7	(∀x) ~ Rxx	6 ∀I
8	(∀x) ~ Rxx	1, 3–7 ∃E

10.2 USING DERIVATIONS TO ESTABLISH SYNTACTIC PROPERTIES OF *PD*

In this section we will work through a series of derivations, illustrating both strategies that are useful in constructing derivations in *PD* and how derivations are used to establish that various syntactic properties of *PD* hold of sentences and sets of sentences of PL.

We begin by repeating the strategies we have enumerated as useful in constructing derivations:

- If the sentence that is the current goal can be derived by applying an elimination rule or some sequence of elimination rules to accessible sentences, then that is the strategy to follow.
- If the current goal can be obtained by an introduction rule, that is the strategy to follow.
- In most cases the successful strategy will make use of several of these approaches, working from the "bottom up" and from the "top down" as the occasion indicates.
- When using a negation rule try to use a negation that is readily available as the **– Q** that the rule requires within the negation subderivation.
- If a sentence is derivable from a set of sentences, then it is derivable using a negation rule as the primary strategy. So if no other strategy suggests itself it is useful to consider a negation strategy. But like all strategies, just because a negation strategy is available doesn't mean it is always the best choice.
- When using Universal Elimination use goal sentences as guides when choosing the instantiating constant.
- When the goal to be derived is an existentially quantified sentence make a substitution instance of that sentence a subgoal, with the intent of applying Existential Introduction to that subgoal to obtain the goal.
- When the current goal is a universally quantified sentence make a substitution instance of that quantified sentence a subgoal, with the intent of applying Universal Introduction to that subgoal. Make sure the two restrictions on the instantiating constant for the use of Universal Introduction are met. Be sure to choose an instantiating constant that does not occur in the universally quantified sentence that is the goal and that does not occur in any assumption that will be open when Universal Introduction is applied to derive that goal.
- When one of the accessible assumptions is an existentially quantified sentence, consider using Existential Elimination to obtain the current goal. Set up an Existential Elimination subderivation, and

continue working within that subderivation until a sentence that does not contain the constant used to form the substitution instance that is the assumption of that subderivation is derived.

- When contradictory sentences are available within an Existential Elimination subderivation but cannot be moved out of that subderivation without violating the restrictions on Existential Elimination, derive another sentence—one that is contradictory to a sentence accessible outside the Existential Elimination subderivation and that does not contain the instantiating constant for this use of Existential Elimination. That sentence will be derivable by the appropriate negation strategy (using the contradictory sentences that are available within the Existential Elimination subderivation).
- There will often be more than one plausible strategy, and often more than one will lead to success. Rather than trying to figure out which of these is the most promising it is often wise to just pick one and pursue it.

ARGUMENTS

An argument of *PL* is valid in *PD* if and only if the conclusion can be derived from the set consisting of the argument's premises. The following argument is valid in *PD*, as we will now show.

$$\frac{(\exists x)(Fx \,\&\, Gx)}{(\exists y)Fy \,\&\, (\exists z)Gz}$$

The single premise is an existentially quantified sentence—which suggests we will need to use Existential Elimination. The conclusion is a conjunction, suggesting Conjunction Introduction as a strategy. We will have to use both strategies, and since it is in general wise to do as much work as possible within an Existential Elimination strategy (so as to avoid violating the third restriction on Existential Elimination), we will make that strategy our primary strategy. We begin as follows:

Derive: $(\exists y)Fy \,\&\, (\exists z)Gz$

1	$(\exists x)(Fx \,\&\, Gx)$	Assumption
2	$\quad Fb \,\&\, Gb$	A / $\exists$E
	$\quad\quad\vdots$	
G	$\quad (\exists y)Fy \,\&\, (\exists z)Gz$	—, — &I
G	$(\exists y)Fy \,\&\, (\exists z)Gz$	1, 2–— $\exists$E

We will try to derive the conclusion of the argument *within* the scope of the Existential Elimination subderivation because doing so will avoid violating the third restriction on Existential Elimination, that the instantiating constant not occur in the derived sentence. In our derivation 'b' is the instantiating constant and it does not occur in the conclusion of the argument.

Our current goal is a conjunction and can be obtained by Conjunction Introduction. The completed derivation is

Derive: $(\exists y)Fy \,\&\, (\exists z)Gz$

1	$(\exists x)(Fx \,\&\, Gx)$	Assumption
2	$\quad Fa \,\&\, Ga$	A / $\exists$E
3	$\quad Fa$	2 &E
4	$\quad (\exists y)Fy$	3 $\exists$I
5	$\quad Ga$	2 &E
6	$\quad (\exists z)Gz$	5 $\exists$I
7	$\quad (\exists y)Fy \,\&\, (\exists z)Gz$	4, 6 &I
8	$(\exists y)Fy \,\&\, (\exists z)Gz$	1, 2–7 $\exists$E

The following argument is also valid in *PD*:

$$\begin{array}{l}(\forall x)(Nx \supset Ox)\\ \sim (\exists y)Oy\\ \hline \sim (\exists x)Nx\end{array}$$

If each thing is such that if it is N then it is also O, and nothing is O, then clearly nothing is N. Since the conclusion of this argument is a negation we will use Negation Introduction as our primary strategy:

Derive: $\sim (\exists x)Nx$

1	$(\forall x)(Nx \supset Ox)$	Assumption
2	$\sim (\exists y)Oy$	Assumption
3	$\quad (\exists x)Nx$	A / $\sim$E
G	$\sim (\exists x)Nx$	3–__ $\sim$I

Only one negation, '~ (∃y)Oy' is readily available, so we will use it as ~ **Q** and try to also derive '(∃y)Oy'.

Derive: ~ (∃x)Nx

1	(∀x)(Nx ⊃ Ox)	Assumption
2	~ (∃y)Oy	Assumption
3	(∃x)Nx	A / ~ E
G	(∃y)Oy	
	~ (∃y)Oy	2 R
G	~ (∃x)Nx	3–__ ~ I

Since one of the accessible sentences, '(∃x)Nx' is an existentially quantified sentence, we will try to obtain our current goal, '(∃y)Oy', by Existential Elimination:

Derive: ~ (∃x)Nx

1	(∀x)(Nx ⊃ Ox)	Assumption
2	~ (∃y)Oy	Assumption
3	(∃x)Nx	A / ~ E
4	Na	A / ∃E
G	(∃y)Oy	
G	(∃y)Oy	3, 4–__ ∃E
G	~ (∃y)Oy	2 R
G	~ (∃x)Nx	3–__ ~ I

Looking at the sentences on lines 1 and 4, we see that we will be able to derive 'Oa' by Conditional Elimination after applying Universal Elimination to the sentence on line 1, with 'a' as the instantiating constant. And from

'Oa' we can obtain '(∃y)Oy' by Existential Introduction. So the completed derivation is

Derive: ~ (∃x)Nx

1	(∀x)(Nx ⊃ Ox)	Assumption
2	~ (∃y)Oy	Assumption
3	(∃x)Nx	A / ~ E
4	Na	A / ∃E
5	Na ⊃ Oa	1 ∀E
6	Oa	4, 5 ⊃E
7	(∃y)Oy	6 ∃I
8	(∃y)Oy	3, 4–7 ∃E
9	~ (∃y)Oy	2 R
10	~ (∃x)Nx	3–9 ~ I

We will next consider two arguments, both of which involve relational predicates and quantifiers with overlapping scope. The first is

(∀x)(∀y)(Hxy ⊃ ~ Hyx)
(∀x)(∃y)Hxy

(∀x)(∃y) ~ Hxy

We set up our derivation as follows:

Derive: (∀x)(∃y) ~ Hxy

1	(∀x)(∀y)(Hxy ⊃ ~ Hyx)	Assumption
2	(∀x)(∃y)Hxy	Assumption
G	(∀x)(∃y) ~ Hyx	

Here our assumptions and our goal sentence are all universally quantified sentences. So we will clearly be using Universal Elimination and Universal Introduction. We should also note that the sentence on line 2 will yield an existentially quantified sentence when we apply Universal Elimination to it. This makes

it likely we will be using an Existential Elimination strategy. These considerations suggest the following structure:

Derive: (∀x)(∃y) ~ Hxy

1	(∀x)(∀y)(Hxy ⊃ ~ Hyx)	Assumption
2	(∀x)(∃y)Hxy	Assumption
3	(∃y)Hay	2 ∀E
4	Hab	A / ∃E
G	(∀x)(∃y) ~ Hyx	3, 4–__ ∃E

On line 4 we chose an instantiating constant that does not appear earlier in the derivation, so that the restrictions on the instantiating constant can be met. Clearly at some point we will obtain '(∀x)(∃y) ~ Hyx' by Universal Introduction. The question is whether we will use Universal Introduction before or after ending our Existential Elimination subderivation. We have stressed in earlier examples that it is generally wise to do as much work as possible within Existential Elimination subderivations. This might suggest that we try to obtain '(∀x)(∃y) ~ Hyx' within our Existential Elimination subderivation. But this is in fact a bad idea for this derivation. The substitution instance of '(∀x)(∃y) ~ Hyx' we will be able to obtain is '(∃y) ~ Hya', in which 'a' is the instantiating constant. The first restriction on Universal Introduction requires that the instantiating constant not occur in any open assumption. But 'a' does occur in 'Hab', the assumption on line 4. So we cannot apply Universal Introduction within the scope of that assumption.

A strategy that will work is to obtain '(∃y) ~ Hya' by Existential Elimination and then, after the assumption 'Hab' is discharged, to apply Universal Introduction. Note that our advice—to do as much work within Existential Elimination subderivations as possible, still holds. The current case is simply a reminder that doing as much work as possible within an Existential Elimination subderivation means, in part, doing as much work as can be done without violating the restrictions on the rules we use.

We have now settled on the following strategy:

Derive: (∀x)(∃y) ~ Hxy

1	(∀x)(∀y)(Hxy ⊃ ~ Hyx)	Assumption
2	(∀x)(∃y)Hxy	Assumption
3	(∃y)Hay	2 ∀E
4	Hab	A / ∃E
G	(∃y) ~ Hya	
G	(∃y) ~ Hya	3, 4–__ ∃E
G	(∀x)(∃y) ~ Hyx	__ ∀I

Our current goal is '(∃y) ~ Hya'. We would like to use Existential Introduction to derive this sentence, which means we first have to derive a substitution instance of this sentence. Looking at our first assumption, '(∀x)(∀y)(Hxy ⊃ ~ Hyx)', we see that with two applications of Universal Elimination we can obtain 'Hab ⊃ ~ Hba', then we can use Conditional Elimination to derive '~ Hba', a substitution instance of our goal, '(∃y) ~ Hya', and from 'Hab ⊃ ~ Hba' and the assumption 'Hab' on line 4. Our completed derivation is

Derive: (∀x)(∃y) ~ Hyx

1	(∀x)(∀y)(Hxy ⊃ ~ Hyx)	Assumption
2	(∀x)(∃y)Hxy	Assumption
3	(∃y)Hay	2 ∀E
4	Hab	A / ∃E
5	(∀y)(Hay ⊃ ~ Hya)	1 ∀E
6	Hab ⊃ ~ Hba	5 ∀E
7	~ Hba	4, 6 ⊃E
8	(∃y) ~ Hya	7 ∃I
9	(∃y) ~ Hya	3, 4–8 ∃E
10	(∀x)(∃y) ~ Hyx	9 ∀I

We have met all the restrictions for using each of the two rules Existential Elimination and Universal Introduction. The constant we had to worry about in using Existential Elimination is 'b', for it is the instantiating constant used to form a substitution instance of '(∃y)Hay' at line 4. By choosing 'b' as the instantiating constant we were able to meet all the restrictions on Existential Elimination: 'b' does not occur in any assumption that is open at line 9, does not occur in the existentially quantified sentence '(∃y)Hay' at line 3, and does not occur in the sentence '(∃y) ~ Hya' derived by Existential Elimination at line 9.

Our next argument is somewhat more complex, having one premise that contains three quantifiers:

(∀x)[(∃z)Fxz ⊃ (∀y)Fxy]
(∃x)(∃y)Fxy

(∃x)(∀w)Fxw

The first premise says that each thing x is such that if x bears F to something, then x bears F to everything. The second premise says that there is a thing x that does bear F to something. And the conclusion says there is something x that bears F to everything. The argument is valid in *PD*, and the derivation is not as difficult as may be feared. We will take our first clue from the second

assumption, which begins with two existential quantifiers. This suggests we will be using Existential Elimination twice, as follows:

Derive: $(\exists x)(\forall y)Fxy$

1	$(\forall x)[(\exists z)Fxz \supset (Ay)Fxy]$	Assumption
2	$(\exists x)(\exists y)Fxy$	Assumption
3	$(\exists y)Fay$	A / $\exists$E
4	Fab	A / $\exists$E
G	$(\exists x)(\forall w)Fxw$	$\exists$I
G	$(\exists x)(\forall w)Fxw$	3, 4–__ $\exists$E
G	$(\exists x)(\forall w)Fxw$	2, 3–__ $\exists$E

We will now use Universal Elimination to produce a conditional to which we can apply Conditional Elimination after applying Existential Introduction to the assumption on line 4, being careful to choose an instantiating constant that will produce a match between the conditional and the existentially quantified sentence we generate. Here the instantiating constant 'a' does the trick:

Derive: $(\exists x)(\forall y)Fxy$

1	$(\forall x)[(\exists z)Fxz \supset (\forall y)Fxy]$	Assumption
2	$(\exists x)(\exists y)Fxy$	Assumption
3	$(\exists y)Fay$	A / $\exists$E
4	Fab	A / $\exists$E
5	$(\exists z)Faz \supset (\forall y)Fay$	1 $\forall$E
6	$(\exists z)Faz$	4 $\exists$I
7	$(\forall y)Fay$	5, 6 $\supset$E
G	$(\exists x)(\forall w)Fxw$	$\exists$I
G	$(\exists x)(\forall w)Fxw$	3, 4–__ $\exists$E
G	$(\exists x)(\forall w)Fxw$	2, 3–__ $\exists$E

Our current goal is '$(\exists x)(\forall w)Fxw$'. To obtain it, by Existential Introduction, we need to first derive a substitution instance of that sentence, say '$(\forall w)Faw$'. We have already derived '$(\forall y)Fay$'. This is not the sentence we need, because it contains the variable 'y' where we want 'w'. But we can easily obtain the substitution instance we want by using Universal Elimination (with a new

instantiating constant) followed by Universal Introduction using the variable 'y' instead of the variable 'w'. We do this at lines 8 and 9, completing the derivation:

Derive: $(\exists x)(\forall y)Fxy$

1	$(\forall x)[(\exists z)Fxz \supset (\forall y)Fxy]$	Assumption
2	$(\exists x)(\exists y)Fxy$	Assumption
3	$(\exists y)Fay$	A / ∃E
4	Fab	A / ∃E
5	$(\exists z)Faz \supset (\forall y)Fay$	1 ∀E
6	$(\exists z)Faz$	4 ∃I
7	$(\forall y)Fay$	5, 6 ⊃E
8	Fac	7 ∀E
9	$(\forall w)Faw$	8 ∀I
10	$(\exists x)(\forall w)Fxw$	9 ∃I
11	$(\exists x)(\forall w)Fxw$	3, 4–10 ∃E
12	$(\exists x)(\forall w)Fxw$	2, 3–11 ∃E

As a final example of contructing a derivation to establish the validity of an argument in *PD* we will tackle a considerably harder problem. In Chapter 1 we considered the following argument:

Everyone loves a lover.
Tom loves Alice.

Everyone loves everyone.

There we argued that despite its initial implausibility, this argument is valid if the predicate 'loves' is being used unambiguously. Our reasoning went like this: Because Tom loves Alice, Tom is a lover. And since everyone loves a lover, everyone loves Tom. But then everyone is a lover, and since everyone loves a lover, everyone loves everyone. Here is a symbolization of the argument in *PL*:

$(\forall x)[(\exists y)Lxy \supset (\forall z)Lzx]$
Lta

$(\forall x)(\forall y)Lxy$

If we take the UD to be the set of all people, interpreting 'lxy' as 'x loves y' and assigning Tom to 't' and Alice to 'a' then line 1 is a correct symbolization of the first premise of our argument, which can be parsed as 'Each person x is such that if there is someone that x loves (if x is a lover), then

everyone loves x'. To show that this argument is valid in *PD*, we begin a derivation as follows:

Derive: (∀x)(∀y)Lxy

1	(∀x)[(∃y)Lxy ⊃ (∀z)Lzx]	Assumption
2	Lta	Assumption
G	(∀x)(∀y)Lxy	

As in the last example, it appears that our ultimate goal will be obtained by Universal Introduction, and indeed that our penultimate goal will also be obtained by this rule. Our work would be over if we could proceed as follows:

Derive: (∀x)(∀y)Lxy

1	(∀x)[(∃y)Lxy ⊃ (∀z)Lzx]	Assumption	
2	Lta	Assumption	
3	(∀y)Lty	2 ∀I	**MISTAKE!**
4	(∀x)(∀y)Lxy	3 ∀I	**MISTAKE!**

But of course we cannot do this. Both line 3 and line 4 are in violation of the restrictions on Universal Introduction. In each case the constant we are replacing, first 'a' and then 't', occurs in an open assumption (at line 2). To use Universal Introduction we need to obtain a sentence like 'Lta' but formed from other constants, any other constants. We select 'c' and 'd':

Derive: (∀x)(∀y)Lxy

1	(∀x)[(∃y)Lxy ⊃ (∀z)Lzx]	Assumption
2	Lta	Assumption
G	Lcd	
G	(∀y)Lcy	— ∀I
G	(∀x)(∀y)Lxy	— ∀I

How might we obtain our current goal, 'Lcd'? Recall the reasoning we did in English: from Lta we can infer that Tom is a lover—and we mirror this inference in *PD* by obtaining '(∃y)Lty' by Existential Introduction. In English we reasoned that if Tom is a lover, then everyone loves Tom. We can mirror this in *PD* by applying Universal Elimination to line 1. And since we have

established that Tom is a lover, we can infer that everyone loves him. So we have:

Derive: $(\forall x)(\forall y)Lxy$

1	$(\forall x)[(\exists y)Lxy \supset (\forall z)Lzx]$	Assumption
2	Lta	Assumption
3	$(\exists y)Lty$	2 $\exists$I
4	$(\exists y)Ly \supset (\forall z)Lzt$	1 $\forall$E
5	$(\forall z)Lzt$	3, 4 $\supset$E
G	Lcd	
G	$(\forall y)Lcy$	— $\forall$I
G	$(\forall x)(\forall y)Lxy$	— $\forall$I

It is because neither 'c' nor 'd' occur in an open assumption that we will be able to derive our final goal by two uses of Universal Introduction. In other words, 'Lcd' says that whoever 'c' might be, and whoever 'd' might be, c loves d, which is tantamount to everyone loves everyone. But how do we get, in *PD*, from line 5 to 'Lcd'? From line 5 we can bet 'Ldt' by Universal Elimination. But how does this help us get 'Lcd'? One difference between these two sentences is that 'd' occurs in the first position after 'L' in the first, and in the second position in the second. We also note that line 1, which is our symbolization of 'Everyone loves a lover' contains two occurrences of the two-place predicate 'L', with 'x' occurring in the first position after L in the first occurrence, and in the second position in the second occurrence. So perhaps we can use this sentence to move 'd' from the first position after L to the second position. (Remember that 'Everyone loves a lover' does say that if someone loves then that person gets loved.) Following this clue we proceed as follows:

Derive: $(\forall x)(\forall y)Lxy$

1	$(\forall x)[(\exists y)Lxy \supset (\forall z)Lzx]$	Assumption
2	Lta	Assumption
3	$(\exists y)Lty$	2 $\exists$I
4	$(\exists y)Ly \supset (\forall z)Lzt$	1 $\forall$E
5	$(\forall z)Lzt$	3, 4 $\supset$E
6	Ldt	5 $\forall$E
7	$(\exists y)Ldy$	6 $\exists$I
8	$(\exists y)Ldy \supset (\forall z)Lzd$	1 $\forall$E
9	$(\forall z)Lzd$	7, 8 $\supset$E
10	Lcd	9 $\forall$E
11	$(\forall y)Lcy$	10 $\forall$I
12	$(\forall x)(\forall y)Lxy$	11 $\forall$I

Our derivation is now complete. The corresponding English reasoning, from line 5 on, goes thus. Line 5 tells us everyone loves Tom. That means d, whoever that might be, loves Tom. And that makes d a lover, that is, there is someone d loves—as line 7 asserts. And because everyone loves a lover, if d loves someone then everyone loves d. And since d does love someone, everyone loves

d. And if everyone loves d, then c loves d. Since c and d may designate any members of the UD, this amounts to everyone loves everyone.

There are opportunities to take wrong turns in this derivation. For example, if we had made our goal, at line 6, 'Lct' (because it "resembles" 'Lcd' more than does 'Ldt') while continuing to have 'Lcd' a goal at line 10, we would have ended up, at line 9, with '$(\forall z)$Lzc' from which 'Lcd' cannot be obtained. Of course, the easy solution to this misstep would have been to change our goal at line 10 to 'Ldc'.

THEOREMS

'$(\forall z)[Fz \supset (Fz \vee Gz)]$' is a theorem in *PD*. To prove that it is such we need to derive it from the empty set, which means we will need a derivation that has no primary assumptions. The most plausible strategy for obtaining this sentence is Universal Introduction.

Derive: $(\forall z)[Fz \supset (Fz \vee Gz)]$

G	$Fb \supset (Fb \vee Gb)$	
G	$(\forall z)[Fz \supset (Fz \vee Gz)]$	__ ∀I

Our current goal is a material conditional and can be obtained by Conditional Introduction, using Disjunction Introduction to derive '$Fb \vee Gb$' within the Conditional Introduction subderivation.:

Derive: $(\forall z)[Fz \supset (Fz \vee Gz)]$

1	Fb	A / ⊃I
2	$Fb \vee Gb$	2 ∨I
3	$Fb \supset (Fb \vee Gb)$	1–2 ⊃I
4	$(\forall z)[Fz \supset (Fz \vee Gz)]$	4 ∀I

We have met both of the restrictions on Universal Introduction. The instantiating constant 'b' does not occur in any assumption that is open at line 4 and does not occur in the sentence derived on line 4 by Universal Introduction.

To prove the theorem '$(\exists x)Fx \supset (\exists x)(Fx \vee Gx)$' we will use Conditional Introduction, Existential Elimination, and Existential Introduction as well as Disjunction Introduction. The proof is straightforward:

Derive: $(\exists x)Fx \supset (\exists x)(Fx \vee Gx)$

1	$(\exists x)Fx$	A / ⊃I
2	Fa	A / ∃E
3	$Fa \vee Ga$	2 ∨I
4	$(\exists x)(Fx \vee Gx)$	3 ∃I
5	$(\exists x)(Fx \vee Gx)$	1, 2–4 ∃E
6	$(\exists x)Fx \supset (\exists x)(Fx \vee Gx)$	1–5 ⊃I

We used Conditional Introduction as our primary strategy because our ultimate goal is a material conditional. We used Existential Elimination within that strategy because the assumption that begins the Conditional Introduction subderivation is an existentially quantified sentence. And we used Existential Introduction at line 4, within our Existential Elimination subderivation, to generate the consequent of the goal conditional. The consequent does not contain the instantiating constant 'a' and can therefore be pulled out of the Existential Elimination subderivation.

The third theorem we will prove is '$(\exists x)(\forall y)Fxy \supset (\exists x)(\exists y)Fxy$'. This is also a material conditional, and our primary strategy will again be Conditional Introduction:

Derive: $(\exists x)(\forall y)Fxy \supset (\exists x)(\exists y)Fxy$

1		$(\exists x)(\forall y)Fxy$	Assumption
G		$(\exists x)(\exists y)Fxy$	
G		$(\exists x)(\forall y)Fxy \supset (\exists x)(\exists y)Fxy$	1–__ ⊃I

Our current goal is an existentially quantified sentence, '$(\exists x)(\exists y)Fxy$'. The most obvious way to obtain it is by two uses of Existential Introduction. We know that when we assume a substitution instance of '$(\exists x)(\forall y)Fxy$'—with the intention of eventually using Existential Elimination—we will have to continue working within that subderivation until we obtain a sentence that does not contain the instantiating constant. This suggests that the goal '$(\exists x)(\forall y)Fxy$', which contains no constants, should also be the goal of the Existential Elimination subderivation:

Derive: $(\exists x)(\forall y)Fxy \supset (\exists x)(\exists y)Fxy$

1		$(\exists x)(\forall y)Fxy$	Assumption
2		$(\forall y)Fay$	A / ∃E
G		$(\exists x)(\exists y)Fxy$	
G		$(\exists x)(\exists y)Fxy$	1, 2–__ ∃E
G		$(\exists x)(\forall y)Fxy \supset (\exists x)(\exists y)Fxy$	1–__ ⊃I

Completing this derivation is now straightforward. We apply Universal Elimination to the sentence on line 2 to produce 'Fab' and then use Existential Introduction twice to derive '(∃x)(∃y)Fxy'.

Derive: (∃x)(∀y)Fxy ⊃ (∃x)(∃y)Fxy

1	(∃x)(∀y)Fxy	Assumption
2	(∀y)Fay	A / ∃E
3	Fab	2 ∀E
4	(∃y)Fay	3 ∃I
5	(∃x)(∃y)Fxy	4 ∃I
6	(∃x)(∃y)Fxy	1, 2–5 ∃E
7	(∃x)(∀y)Fxy ⊃ (∃x)(∃y)Fxy	1–6 ⊃I

We have met all the restrictions on Existential Elimination. The instantiating constant 'a' does not occur in any assumption that is open as of line 6. The constant 'a' also does not occur in the existentially quantified sentence to which we are applying Existential Elimination, and it does not occur in the sentence derived at line 6 by Existential Elimination.

It is worth noting that since there are no restrictions on Existential Introduction, we could have entered 'Faa' rather than 'Fab' at line 3 (there are also no restrictions on Universal Elimination), and then applied Existential Introduction twice.

The last theorem we will consider is the quantified sentence '(∃x)(Fx ⊃ (∀y)Fy)'. At first glance it appears that we should use Existential Introduction to derive this sentence from some substitution instance, for example, 'Fa ⊃ (∀y)Fy' and so the latter sentence should be a subgoal. However, this will not work! 'Fa ⊃ (∀y)Fy' is not quantificationally true and therefore cannot be derived in *PD* from no assumptions. So we must choose another strategy. Our primary strategy will be Negation Elimination and the proof will be quite complicated:

Derive: (∃x)(Fx ⊃ (∀y)Fy)

1	~ (∃x)(Fx ⊃(∀y)Fy)	A / ~ E
G	(∃x)(Fx ⊃ (∀y)Fy)	
	~ (∃x)(Fx ⊃ (∀y)Fy)	1 R
G	(∃x)(Fx ⊃ (∀y)Fy)	1–~ ~ E

We have selected Negation Elimination as our primary strategy because there is no plausible alternative to that strategy. We have selected '(∃x)(Fx ⊃ (∀y)Fy)'

and '~ (∃x)(Fx ⊃ (∀y)Fy)' as the contradictory sentences we will derive within that strategy because the latter sentence is our assumption on line 1 and therefore available for use. The question now is how to derive '(∃x)(Fx ⊃ (∀y)Fy)'. Since this is an existentially quantified sentence we will attempt to derive it by Existential Introduction: first deriving the substitution instance 'Fa ⊃ (∀y)Fy' of that sentence (any other instantiating constant could be used). The substitution instance should be derivable using Conditional Introduction:

Derive: (∃x)(Fx ⊃ (∀y)Fy)

1		~ (∃x)(Fx ⊃ (∀y)Fy)	A / ~ E
2		Fa	A / ⊃I
G		(∀y)Fy	
G		Fa ⊃ (∀y)Fy	2–G ⊃I
G		(∃x)(Fx ⊃ (∀y)Fy)	G ∃I
		~ (∃x)(Fx ⊃ (∀y)Fy)	1 R
G		(∃x)(Fx ⊃ (∀y)Fy)	1–G ~ E

Our new goal is '(∀y)Fy', a universally quantified sentence. We cannot obtain it by applying Universal Introduction to the sentence on line 2, because 'a' here occurs in an open assumption. So we will try to obtain a different substitution instance of '(∀y)Fy', 'Fb', and we will try to derive this substitution instance using Negation Elimination:

1		~ (∃x)(Fx ⊃ (∀y)Fy)	A / ~ E
2		Fa	A / ⊃I
3		~ Fb	A / ~ E
G		Fb	
G		(∀y)Fy	G ∀I
G		Fa ⊃ (∀y)Fy	2–G ⊃I
G		(∃x)(Fx ⊃ (∀y)Fy)	
G		~ (∃x)(Fx ⊃ (∀y)Fy)	1 R
		(∃x)(Fx ⊃ (∀y)Fy)	1–G ~ E

We now have to decide on the sentence and its negation to be derived within the Negation Elimination subderivation. Two negations are accessible at this point: '~ Fb' and '~ (∃x)(Fx ⊃ (∀y)Fy)'. We will make the latter sentence and '(∃x)(Fx ⊃ (∀y)Fy)' our goals as picking 'Fb' and '~ Fb' as goals appears to be unpromising (there is no obvious way to derive 'Fb' from the assumptions

on lines 1–3). We plan to derive '$(\exists x)(Fx \supset (\forall y)Fy)$' using Existential Introduction:

1	$\sim (\exists x)(Fx \supset (\forall y)Fy)$	A / ~ E
2	Fa	A / ⊃I
3	~ Fb	A / ~ E
G	Fb ⊃ (∀y)Fy	
G	(∃x)(Fx ⊃ (∀y)Fy)	__ ∃I
G	~ (∃x)(Fx ⊃ (∀y)Fy)	1 R
G	Fb	
G	(∀y)Fy	__ ∀I
G	Fa ⊃ (∀y)Fy	2–__ ⊃I
G	(∃x)(Fx ⊃ (∀y)Fy)	
G	~ (∃x)(Fx ⊃ (∀y)Fy)	1 R
	(∃x)(Fx ⊃ (∀y)Fy)	1–__ ~ E

We have selected 'b' as the instantiating constant in our new goal because we anticipate using Conditional Introduction to derive 'Fb ⊃ (∀y)Fy', and this use of 'b' will give us 'Fb' as an assumption, something that is likely to be useful as we already have '~ Fb' at line 3.

1	$\sim (\exists x)(Fx \supset (\forall y)Fy)$	A / ~ E
2	Fa	A / ⊃I
3	~ Fb	A / ~ E
4	Fb	A / ⊃I
G	(∀y)Fy	
G	Fb ⊃ (∀y)Fy	
G	(∃x)(Fx ⊃ (∀y)Fy)	__ ∃I
G	~ (∃x)(Fx ⊃ (∀y)Fy)	1 R
G	Fb	
G	(∀y)Fy	__ ∀I
G	Fa ⊃ (∀y)Fy	2–__ ⊃I
G	(∃x)(Fx ⊃ (∀y)Fy)	
G	~ (∃x)(Fx ⊃ (∀y)Fy)	1 R
G	(∃x)(Fx ⊃ (∀y)Fy)	1–__ ~ E

Our new goal is '(∀y)Fy' and since 'Fb' and '~ Fb' are both accessible, we can easily derive it using Negation Elimination, completing the derivation:

Derive: (∃x)(Fx ⊃ (∀y)Fy)

1	~ (∃x)(Fx ⊃ (∀y)Fy)	A / ~ E
2	Fa	A / ⊃I
3	~ Fb	A / ~ I
4	Fb	A / ⊃I
5	~ (∀y)Fy	A / ~ E
6	Fb	4 R
7	~ Fb	3 R
8	(∀y)Fy	5–7 ~ E
9	Fb ⊃ (∀y)Fy	4–8 ⊃I
10	(∃x)(Fx ⊃ (∀y)Fy)	9 ∃I
11	~ (∃x)(Fx ⊃ (∀y)Fy)	1 R
12	Fb	3–11 ~ E
13	(∀y)Fy	12 ∀I
14	Fa ⊃ (∀y)Fy	2–13 ⊃I
15	(∃x)(Fx ⊃ (∀y)Fy)	14 ∃I
16	~ (∃x)(Fx ⊃ (∀y)Fy)	1 R
17	(∃x)(Fx ⊃ (∀y)Fy)	1–16 ~ E

This is a complex derivation, as we warned it would be. In the end we used the same pair of contradictory sentences in two Negation Elimination subderivations. This sometimes happens.

EQUIVALENCE

To show that sentences **P** and **Q** of *PL* are equivalent in *PD* we must derive each from the unit set of the other. As our first example we take the sentences '(∀x)(Fa ⊃ Fx)' and 'Fa ⊃ (∀x)Fx'. We begin by deriving the second of these sentences from the first, and since our goal sentence in this derivation is a material conditional, we will use Conditional Introduction:

Derive: Fa ⊃ (∀x)Fx

1	(∀x)(Fa ⊃ Fx)	Assumption
2	Fa	A / ⊃I
G	(∀x)Fx	
G	Fa ⊃ (∀x)Fx	2–G ⊃I

We cannot derive our present goal, '(∀x)Fx', by simply applying Universal Introduction to 'Fa' at line 2, for the sentence on line 2 is an open assumption and 'a' occurs in that sentence. We can rather try to derive a different substitution instance of '(∀x)Fx', say 'Fb', and then apply Universal Introduction. And this is easy to do by applying Universal Elimination to the sentence on line 1 (being careful to use an instantiating constant other than 'a'), and then using Conditional Introduction:

Derive: Fa ⊃ (∀x)Fx

1	(∀x)(Fa ⊃ Fx)	Assumption
2	| Fa	A / ⊃I
3	| Fa ⊃ Fb	1 ∀E
4	| Fb	2, 3 ⊃E
5	| (∀x)Fx	4 ∀I
6	Fa ⊃ (∀x)Fx	2–5 ⊃I

We have met both restrictions on Universal Introduction at line 5: the instantiating constant 'b' does not occur in any open assumption; nor does it occur in the derived sentence '(∀x)Fx'.

We must now derive '(∀x)(Fa ⊃ Fx)' from 'Fa ⊃ (∀x)Fx'. A plausible start is

Derive: (∀x)(Fa ⊃ Fx)

1	Fa ⊃ (∀x)Fx	Assumption
2	| Fa	A / ⊃I
	|	
	|	
G	| Fb	
G	Fa ⊃ Fb	2–__ ⊃I
G	(∀x)(Fa ⊃ Fx)	__ ∀I

We plan to derive the last sentence by Universal Introduction, and the substitution instance on the prior line by Conditional Introduction. And we can now see how to complete the derivation. We can apply Conditional Elimination to the sentences on lines 1 and 2 to derive '(∀x)Fx', from which we can then derive 'Fb':

Derive: (∀x)(Fa ⊃ Fx)

1	Fa ⊃ (∀x)Fx	Assumption
2	| Fa	A / ⊃I
3	| (∀x)Fx	1, 2 ⊃E
4	| Fb	3 ∀E
5	Fa ⊃ Fb	2–4 ⊃I
6	(∀x)(Fa ⊃ Fx)	5 ∀I

Having derived each member of our pair of sentences from the other, we have demonstrated that the sentences '$(\forall x)(Fa \supset Fx)$' and '$Fa \supset (\forall x)Fx$' are equivalent in *PD*.

We will next show that '$(\forall x)Fx \supset Ga$' and '$(\exists x)(Fx \supset Ga)$' are equivalent in *PD*. It is reasonably straightforward to derive '$(\forall x)Fx \supset Ga$' from '$(\exists x)(Fx \supset Ga)$'. We begin with

Derive: $(\forall x)Fx \supset Ga$

1	$(\exists x)(Fx \supset Ga)$	Assumption
2	$(\forall x)Fx$	A / $\supset$I
G	Ga	
G	$(\forall x)Fx \supset Ga$	2–G $\supset$I

We will complete the derivation by using Existential Elimination—being careful to use an instantiating constant other than 'a' (because 'a' occurs in 'Ga', the sentence we plan to derive with Existential Elimination):

Derive: $(\forall x)Fx \supset Ga$

1	$(\exists x)(Fx \supset Ga)$	Assumption
2	$(\forall x)Fx$	A / $\supset$I
3	$Fb \supset Ga$	A / $\exists$E
4	Fb	2 $\forall$E
5	Ga	3–4 $\supset$E
6	Ga	1, 3–5 $\exists$E
7	$(\forall x)Fx \supset Ga$	2–6 $\supset$I

Our use of Existential Elimination at line 6 meets all three restrictions on that rule: the instantiating constant 'b' does not occur in '$(\exists x)(Fx \supset Ga)$', does not occur in any assumption that is open at line 6, and does not occur in the sentence 'Ga' that we derived with Existential Elimination.

Deriving '$(\exists x)(Fx \supset Ga)$' from '$(\forall x)Fx \supset Ga$' is a somewhat more challenging exercise. Since our primary goal is an existentially quantified sentence, both Existential Introduction and Negation Elimination suggest themselves as primary strategies. We have opted to use Negation Elimination, and since the assumption that begins that strategy is a negation, we will make it and the

sentence of which it is a negation our goals within the Negation Elimination subderivation:

Derive: $(\exists x)(Fx \supset (\forall y)Fy)$

1	$(\forall x)Fx \supset Ga$	Assumption
2	$\sim (\exists x)(Fx \supset Ga)$	A / ~ E
G	$(\exists x)(Fx \supset Ga)$	
	$\sim (\exists x)(Fx \supset Ga)$	2 R
G	$(\exists x)(Fx \supset Ga)$	1–2 ~ E

When two primary strategies suggest themselves, it is frequently useful to use one as a secondary strategy within the other, primary, strategy. Here we will use Existential Introduction as a secondary strategy: We will try to obtain the goal '$(\exists x)(Fx \supset Ga)$' by Existential Introduction, first using Conditional Introduction to derive an appropriate substitution instance of the goal sentence:

Derive: $(\exists x)(Fx \supset (\forall y)Fy)$

1	$(\forall x)Fx \supset Ga$	Assumption
2	$\sim (\exists x)(Fx \supset Ga)$	A / ~ E
3	Fa	A / $\supset$I
G	Ga	
G	$Fa \supset Ga$	3–4 $\supset$I
G	$(\exists x)(Fx \supset Ga)$	5 $\exists$I
	$\sim (\exists x)(Fx \supset Ga)$	1 R
G	$(\exists x)(Fx \supset Ga)$	1–7 ~ E

The current goal, 'Ga', can be derived by Conditional Elimination using the sentence on line 1 *if* we can first derive the antecedent '$(\forall x)Fx$' of that sentence. It is not easy to see how the antecedent might be derived, but one strategy is to try to first derive a substitution instance in which the instantiating constant does not occur in an open assumption. This rules out 'Fa'. So we will try

to derive 'Fb', and since no more direct strategy suggests itself at this point, we'll try to derive 'Fb' by Negation Elimination:

Derive: $(\exists x)(Fx \supset (\forall y)Fy)$

1	$(\forall x)Fx \supset Ga$	Assumption
2	$\sim (\exists x)(Fx \supset Ga)$	A / ~ E
3	Fa	A / ⊃I
4	$\sim Fb$	A / ~ E
G	Fb	4–__ ~ E
G	$(\forall x)Fx$	__ ∀I
G	Ga	1, __ ⊃E
G	$Fa \supset Ga$	3–__ ⊃I
G	$(\exists x)(Fx \supset Ga)$	
	$\sim (\exists x)(Fx \supset Ga)$	1 R
G	$(\exists x)(Fx \supset Ga)$	1–__ ~ E

Given '~ Fb' at line 4 we can obtain 'Fb ⊃ Ga'. We know we can do this because we know that given the negation of the antecedent of any conditional we can derive the conditional—as the following schema demonstrates:

n	$\sim \mathbf{P}$	
n+1	$\mathbf{P}$	A / ⊃I
n+2	$\sim \mathbf{Q}$	A / ~ E
n+3	$\mathbf{P}$	n+1 R
n+4	$\sim \mathbf{P}$	n R
n+5	$\mathbf{Q}$	n+2–n+4 ~ E
n+6	$\mathbf{P} \supset \mathbf{Q}$	n+1–n+5 ⊃I

Once we derive 'Fb ⊃ Ga' we can obtain '$(\exists x)(Fx \supset Ga)$' by Existential Introduction. Because we already have the negation of that sentence at line 3 we can see our way clear to deriving a sentence and its negation as follows:

Derive: $(\exists x)(Fx \supset (\forall y)Fy)$

1	$(\forall x)Fx \supset Ga$	Assumption
2	$\sim (\exists x)(Fx \supset Ga)$	A / ~ E
3	Fa	A / ⊃I
4	$\sim Fb$	A / ~ E
5	Fb	A / ⊃I
6	$\sim Ga$	A / ~ E
7	Fb	5 R
8	$\sim Fb$	4 R
9	Ga	6–8 ~ E
10	$Fb \supset Ga$	5–9 ⊃I
11	$(\exists x)(Fx \supset Ga)$	10 ∃I
12	$\sim (\exists x)(Fx \supset Ga)$	2 R
13	Fb	4–12 ~ E
14	$(\forall x)Fx$	13 ∀I
15	Ga	1, 14 ⊃E
16	$Fa \supset Ga$	3–15 ⊃I
17	$(\exists x)(Fx \supset Ga)$	16 ∃I
18	$\sim (\exists x)(Fx \supset Ga)$	1 R
19	$(\exists x)(Fx \supset Ga)$	1–18 ~ E

We will conclude our discussion of Equivalence in *PD* by deriving each of the following sentences from the unit set of the other:

$(\forall x)[Fx \supset (\exists y)Gxy]$ $\qquad$ $(\forall x)(\exists y)(Fx \supset Gxy)$

Establishing that these sentences are equivalent in *PD* is substantially more difficult than was establishing equivalence in our last example, in large part because in these sentences the existentially quantified formulas occur *within the scope of* universal quantifiers. We begin by deriving '$(\forall x)(\exists y)(Fx \supset Gxy)$' from '$[(\forall x)[Fx \supset (\exists y)Gxy]]$'. Since our one primary assumption will be a universally quantified sentence, as will our goal, it is plausible to expect that we will use both Universal Elimination and Universal Introduction:

Derive: $(\forall x)(\exists y)(Fx \supset Gxy)$

1	$(\forall x)[Fx \supset (\exists y)Gxy]$	Assumption
2	$Fa \supset (\exists y)Gay$	1 ∀E
G	$(\exists y)(Fa \supset Gay)$	
G	$(\forall x)(\exists y)(Fx \supset Gxy)$	___ ∀I

It is now tempting to make 'Fa ⊃ Gab' our next subgoal, to be derived using Conditional Introduction, from which '(∃y)(Fa ⊃ Gay)' can be derived by Existential Introduction:

Derive: (∀x)(∃y)(Fx ⊃ Gxy)

1	(∀x)[Fx ⊃ (∃y)Gxy]	Assumption
2	Fa ⊃ (∃y)Gay	1 ∀E
3	Fa	A / ⊃I
G	Gab	
G	Fa ⊃ Gab	3–G ⊃I
G	(∃y)(Fa ⊃ Gay)	G ∃I
G	(∀x)(∃y)(Fx ⊃ Gxy)	G ∀I

'(∃y)Gay' can be derived from lines 2 and 3 by Conditional Elimination. We might then plan to use Existential Elimination to get from '(∃y)Gay' to the current goal sentence 'Gab'. But we have to be careful here. If we want to derive 'Gab' by Existential Elimination then the instantiating constant for Existential Elimination has to be a constant other than 'b'.

Derive: (∀x)(∃y)(Fx ⊃ Gxy)

1	(∀x)[Fx ⊃ (∃y)Gxy]	Assumption
2	Fa ⊃ (∃y)Gay	1 ∀E
3	Fa	A / ⊃I
4	(∃y)Gay	2, 3 ⊃E
5	Gac	A / ∃E
G	Gab	
G	Gab	4, 5–G ∃E
G	Fa ⊃ Gab	3–G ⊃I
G	(∃y)(Fa ⊃ Gay)	G ∃I
G	(∀x)(∃y)(Fx ⊃ Gxy)	G ∀I

But how do we get from 'Gac' to 'Gab'? A negation strategy might work, but it would be complicated as there are no negations among the accessible sentences.

It is time to consider an alternative strategy. We will try to obtain our penultimate goal, '(∃y)(Fa ⊃ Gay)', by Negation Elimination rather than by Existential Introduction:

Derive: (∀x)(∃y)(Fx ⊃ Gxy)

1	(∀x)[Fx ⊃ (∃y)Gxy]	Assumption
2	~ (∃y)(Fa ⊃ Gay)	A / ~ E
G	(∃y)(Fa ⊃ Gay)	15 ∃I
G	~ (∃y)(Fa ⊃ Gay)	2 R
G	(∃y)(Fa ⊃ Gay)	2–17 ~ E
G	(∀x)(∃y)(Fx ⊃ Gxy)	18 ∀I

It may appear that because '(∃y)(Fa ⊃ Gay)' is still our goal we are making no progress. But this is not so, for we now have an additional assumption to work from. We will now proceed much as we did previously in our first attempt at this derivation:

Derive: (∀x)(∃y)(Fx ⊃ Gxy)

1	(∀x)[Fx ⊃ (∃y)Gxy]	Assumption
2	~ (∃y)(Fa ⊃ Gay)	A / ~ E
3	Fa	A / ⊃I
4	Fa ⊃ (∃y)Gay	1 ∀E
5	(∃y)Gay	3, 4 ⊃E
6	Gac	A / ∃E
G	Gab	
G	Gab	5, 6–__ ∃E
G	Fa ⊃ Gab	3–__ ⊃I
G	(∃y)(Fa ⊃ Gay)	15 ∃I
G	~ (∃y)(Fa ⊃ Gay)	2 R
G	(∃y)(Fa ⊃ Gay)	2–17 ~ E
G	(∀x)(∃y)(Fx ⊃ Gxy)	18 ∀I

Once again we want to get from 'Gac' to 'Gab'. But this time we do have an accessible negation, '~ (∃y)(Fa ⊃ Gay)'. So we will use a negation strategy, assuming '~ Gab' and seeking to derive '(∃y)(Fa ⊃ Gay)' along with reiterating its negation:

Derive: $(\forall x)(\exists y)(Fx \supset Gxy)$

1	$(\forall x)[Fx \supset (\exists y)Gxy]$	Assumption
2	$\sim(\exists y)(Fa \supset Gay)$	A / ~ E
3	Fa	A ⊃I
4	$Fa \supset (\exists y)Gay$	1 ∀E
5	$(\exists y)Gay$	3, 4 ⊃E
6	Gac	A / ∃E
7	~ Gab	A ~ E
G	$(\exists y)(Fa \supset Gay)$	
G	$\sim(\exists y)(Fa \supset Gay)$	2 R
G	Gab	7–__ ~ E
G	Gab	5, 6–__ ∃E
G	$Fa \supset Gab$	3–__ ⊃I
G	$(\exists y)(Fa \supset Gay$	__ ∃I
	$\sim(\exists y)(Fa \supset Gay)$	2 R
G	$(\exists y)(Fa \supset Gay)$	2–__ ~ E
G	$(\forall x)(\exists y)(Fx \supset Gxy)$	__ ∀I

What remains is to derive '$(\exists y)(Fa \supset Gay)$'. This is easily done. We assume 'Fa', derive 'Gac' by Reiteration, derive '$Fa \supset Gac$' by Conditional Introduction, and then '$(\exists y)(Fa \supset Gay)$' by Existential Introduction. The derivation is then complete:

Derive: $(\forall x)(\exists y)(Fx \supset Gxy)$

1	$(\forall x)[Fx \supset (\exists y)Gxy]$	Assumption
2	$\sim(\exists y)(Fa \supset Gay)$	A / ~ E
3	Fa	A ⊃I
4	$Fa \supset (\exists y)Gay$	1 ∀E
5	$(\exists y)Gay$	3, 4 ⊃E
6	Gac	A / ∃E
7	~ Gab	A ~ E
8	Fa	A / ⊃I
9	Gac	6 R
10	$Fa \supset Gac$	8–9 ⊃I
11	$(\exists y)(Fa \supset Gay)$	10 ∃I
12	$\sim(\exists y)(Fa \supset Gay)$	2 R
13	Gab	7–12 ~ E
14	Gab	5, 6–13 ∃E
15	$Fa \supset Gab$	3–14 ⊃I
16	$(\exists y)(Fa \supset Gay)$	15 ∃I
17	$\sim(\exists y)(Fa \supset Gay)$	2 R
18	$(\exists y)(Fa \supset Gay)$	2–17 ~ E
19	$(\forall x)(\exists y)(Fx \supset Gxy)$	18 ∀I

We must now derive '$(\forall x)[Fx \supset (\exists y)Gxy]$' from '$(\forall x)(\exists y)(Fx \supset Gxy)$'. This will be an easier task since we can derive '$(\exists y)(Fa \supset Gay)$' by Universal Elimination and then do the bulk of the derivation within an Existential Elimination subderivation:

Derive: $(\forall x)[Fx \supset (\exists y)Gxy]$

1	$(\forall x)(\exists y)(Fx \supset Gxy)$	Assumption
2	$(\exists y)(Fa \supset Gay)$	1 ∀E
3	$Fa \supset Gab$	A / ∃E
4	Fa	A ⊃I
5	Gab	3, 4 ⊃E
6	$(\exists y)Gay$	5 ∃I
7	$Fa \supset (\exists y)Gay$	4–6 ⊃I
8	$Fa \supset (\exists y)Gay$	3, 4–7 ∃E
9	$(\forall x)[Fx \supset (\exists y)Gxy]$	8 ∀I

The instantiating constant 'b' for our use of Existential Elimination does not occur in the existentially quantified sentence '$(\exists y)(Fa \supset Gay)$' in any assumption that is open at line 8, or in the sentence '$Fa \supset (\exists y)Gay$' obtained by Existential Elimination. (In this case we could also have applied Universal Introduction *within* the Existential Elimination subderivation and then moved '$(\forall x)[Fx \supset (\exists y)Gxy]$' out of that subderivation.) This completes our demonstration that '$(\forall x)[Fx \supset (\exists y)Gxy]$' and '$(\forall x)(\exists y)(Fx \supset Gxy)$' are equivalent in *PD*.

INCONSISTENCY

We next turn our attention to demonstrating that sets of sentences of *PL* are inconsistent in *PD*. Recall that a set of sentences is inconsistent in *PD* if we can derive both a sentence **Q** and its negation ~ **Q** from the set. As our first example we will show that the set $\{(\forall x)(Fx \equiv Gx), (\exists y)(Fy \cdot \sim Gy)\}$ is inconsistent in *PD*. It is quite apparent that this set is inconsistent. If each thing is F if and only if it is G then contrary to what the second sentence says there cannot be something that is F and is not G. Because this set does not contain a negation, it is not obvious what our **Q** and ~ **Q** should be. We will use the set member '$(\forall x)(Fx \equiv Gx)$' as **Q**, making ~ **Q** '$\sim (\forall x)(Fx \equiv Gx)$':

Derive: $(\forall x)(Fx = Gx), \sim (\forall x)(Fx = Gx)$

1	$(\forall x)(Fx = Gx)$	Assumption
2	$(\exists y)(Fy \mathbin{\&} \sim Gy)$	Assumption
G	$\sim (\forall x)(Fx = Gx)$	
	$(\forall x)(Fx = Gx)$	1 R

The second assumption suggests using Existential Elimination, and we know it is wise to do as much of the work of the derivation as possible within the Existential Elimination subderivation:

Derive: $(\forall x)(Fx = Gx), \sim (\forall x)(Fx = Gx)$

1	$(\forall x)(Fx = Gx)$	Assumption
2	$(\exists y)(Fy \mathbin{\&} \sim Gy)$	Assumption
3	$Fa \mathbin{\&} \sim Ga$	A / $\exists$E
G	$\sim (\forall x)(Fx = Gx)$	
G	$\sim (\forall x)(Fx = Gx)$	2, 3–__ $\exists$E
	$(\forall x)(Fx = Gx)$	1 R

Our current goal is a negation, which we will try to derive using Negation Introduction. We assume '$(\forall x)(Fx = Gx)$' *even though that sentence is one of our primary assumptions and hence already accessible.* We assume it because Negation Introduction requires that we assume the sentence whose negation we wish to derive:

Derive: $(\forall x)(Fx = Gx), \sim (\forall x)(Fx = Gx)$

1	$(\forall x)(Fx = Gx)$	Assumption
2	$(\exists y)(Fy \mathbin{\&} \sim Gy)$	Assumption
3	$Fa \mathbin{\&} \sim Ga$	A / $\exists$E
4	$(\forall x)(Fx = Gx)$	A / ~ I
G	$\sim (\forall x)(Fx = Gx)$	4–__ ~ I
G	$\sim (\forall x)(Fx = Gx)$	2, 3–__ $\exists$E
G	$(\forall x)(Fx = Gx)$	1 R

We are now finally in a position where we can work profitably from the "top down". From line 4 we can derive 'Fa ≡ Ga' by Biconditional Elimination; from line 3 we can derive 'Fa'; and then it is easy to derive both 'Ga' and '~ Ga':

Derive: (∀x)(Fx ≡ Gx), ~ (∀x)(Fx ≡ Gx)

1	(∀x)(Fx ≡ Gx)	Assumption
2	(∃y)(Fy & ~ Gy)	Assumption
3	Fa & ~ Ga	A / ∃E
4	(∀x)(Fx ≡ Gx)	A / ~ I
5	Fa ≡ Ga	4 ∀E
6	Fa	3 E
7	Ga	5, 6 ≡E
8	~ Ga	3 &E
9	~ (∀x)(Fx ≡ Gx)	4–8 ~ I
10	~ (∀x)(Fx ≡ Gx)	2, 3–9 ∃E
11	(∀x)(Fx ≡ Gx)	1 R

Had we taken '(∃y)(Fy & ~ Gy)' and '~ (∃y)(Fy & ~ Gy)' as our **Q** and ~ **Q** we would have produced the following very similar derivation:

Derive: (∃y)(Fy & ~ Gy), ~ (∃y)(Fy & ~ Gy)

1	(∀x)(Fx ≡ Gx)	Assumption
2	(∃y)(Fy & ~ Gy)	Assumption
3	Fa & ~ Ga	A / ∃E
4	(∃y)(Fy & ~ Gy)	A / ~ I
5	Fa ≡ Ga	1 ∀E
6	Fa	3 &E
7	Ga	5, 6 ≡E
8	~ Ga	3 &E
9	~ (∃y)(Fy & ~ Gy)	4–8 ~ I
10	~ (∃y)(Fy & ~ Gy)	2, 3–9 ∃E
11	(∃y)(Fy & ~ Gy)	2 R

We will next demonstrate that {(∀z)(Hz ⊃ (∃y)Gzy), (∃w)Hw, (∀x) ~ (∃y)Gxy} is inconsistent in *PD*. Though the set includes no negations, we can immediately derive one, say '~ (∃y)Gay', by applying Universal Elimination to '(∀x) ~ (∃y)Gxy'. So we will take '(∃y)Gay' and '~ (∃y)Gay' as our goals:

Derive: (∃y)Gay, ~ (∃y)Gay

1	(∀z)(Hz ⊃ (∃y)Gzy)	Assumption
2	(∃w)Hw	Assumption
3	(∀x) ~ (∃y)Gxy	Assumption
G	(∃y)Gay	
	~ (∃y)Gay	3 ∀E

Our assumptions include the existentially quantified sentence '(∃w)Hw', so we will try to derive '(∃y)Gay' by Existential Elimination—which means we will have to be careful to pick a constant other than 'a' as the instantiating constant in our Existential Elimination assumption:

Derive: (∃y)Gay, ~ (∃y)Gay

1	(∀z)(Hz ⊃ (∃y)Gzy)	Assumption
2	(∃w)Hw	Assumption
3	(∀x) ~ (∃y)Gxy	Assumption
4	Hb	A / ∃E
G	(∃y)Gay	
G	(∃y)Gay	2, 4–__ ∃E
	~ (∃y)Gay	3 ∀E

There is a problem in the offing here. We used 'b' as the instantiating constant at line 4 because 'a' occurs in the sentence we hope to obtain by Existential Elimination, '(∃y)Gay'. This means that we will be able to obtain '(∃y)Gby', but not '(∃y)Gay' by applying Universal Elimination to line 1 (obtaining 'Hb ⊃ (∃y)Gby' and then doing Conditional Elimination. So we need an alternative strategy for obtaining our current goal, '(∃y)Gay'. We will use Negation Elimination:

Derive: (∃y)Gay, ~ (∃y)Gay

1	(∀z)(Hz ⊃ (∃y)Gzy)	Assumption
2	(∃w)Hw	Assumption
3	(∀x) ~ (∃y)Gxy	Assumption
4	Hb	A / ∃E
5	~ (∃y)Gay	A / ~ E
G	(∃y)Gay	5–__ ~ E
G	(∃y)Gay	2, 4–6 ∃E
	~ (∃y)Gay	3 ∀E

We can now complete the derivation by deriving both '(∃y)Gby' and '~ (∃y)Gby' within the scope of the assumption on line 5, the first by the steps mentioned previously, the second by applying Universal Elimination to the sentence on line 3.

Derive: (∃y)Gay, ~ (∃y)Gay

1	(∀z) (Hz ⊃ (∃y)Gzy)	Assumption
2	(∃w)Hw	Assumption
3	(∀x) ~ (∃y)Gxy	Assumption
4	Hb	A / ∃E
5	~ (∃y)Gay	A / ~ E
6	Hb ⊃ (∃y)Gby	1 ∀E
7	(∃y)Gby	4, 6 ⊃E
8	~ (∃y)Gby	3 ∀E
9	(∃y)Gay	5–8 ~ E
10	(∃y)Gay	2, 4–9 ∃E
11	~ (∃y)Gay	3 ∀E

The technique of using a negation strategy within an Existential Elimination subderivation, as we have just done, is useful as a way of generating a sentence that does not violate any of the restrictions on Existential Elimination. It is useful whenever we can see that some sentence and its negation are derivable within the Existential Elimination subderivation, but those sentences contain a constant that keeps us from moving either out from the Existential Elimination subderivation by Existential Elimination. In such a case we can always derive a sentence that does not contain the Existential Elimination subderivation's instantiating constant. We can do this by assuming the negation of the desired sentence and deriving the contradictory sentences within the negation elimination subderivation.

10.2E EXERCISES

Note: Here, as always, the *Student Solutions Manual* contains answers to all unstarred exercises. In addition, when an exercise is preceded by a number sign (#) the *Solutions Manual* contains a detailed account of how the derivation given in the *Solutions Manual* is constructed.

1. Construct derivations that establish the validity of the following arguments:

a. (∀y)[Fy ⊃ (Gy & Hy)]
(∀x)(Fx ⊃ Hx)

*b. (∀x)(Fx ≡ Gx)
(∃x)Fx
(∃x)(Fx & Gx)

#c. (∀y)[Gy ⊃ (Hy & Fy)]
(∃x)Gx
(∃z)Fz

*d. (∀x)[Fx ⊃ (Gx & Hx)]
(∃y)(Fy & Dy)
(∃z)Gz

e. $(\exists x)Fx \supset (\forall x)Gx$
Fa
$(\forall x)(Gx \supset Hx)$
$\therefore (\forall x)Hx$

*f. $(\forall y)[(Hy \ \& \ Fy) \supset Gy]$
$(\forall z)Fz$
$\therefore (\forall x)(Hx \supset Gx)$

g. $(\forall x)Fx \vee (\forall x)Gx$
$\therefore (\forall x)(Fx \vee Gx)$

*h. $(\forall x)(Dx \equiv \sim Gx)$
$(\forall y)(Gy \supset Hy)$
$(\exists z) \sim Hz$
$\therefore (\exists z)Dz$

#i. $(\forall x)(Fx \supset Hx)$
$(\forall y)(Gy \supset Hy)$
$\therefore (\forall y)[(Fy \vee Gy) \supset Hy]$

*j. $(\exists y)(Fy \vee Gy)$
$(\forall x)(Fx \supset Hx)$
$(\forall x)(Gx \supset Hx)$
$\therefore (\exists z)Hz$

k. $(\exists x)Hx$
$(\forall x)(Hx \supset Rx)$
$(\exists x)Rx \supset (\forall x)Gx$
$\therefore (\forall x)(Fx \supset Gx)$

*l. $\sim (\exists x)Fx \equiv (\forall y)Gy$
$(\forall y) \sim Fy$
$\therefore (\exists y)Gy$

m. $(\forall x)Fx \vee (\forall y) \sim Gy$
$Fa \supset Hb$
$\sim Gb \supset Jb$
$\therefore (\exists y)(Hy \vee Jy)$

*n. $Fa \vee (\forall x) \sim Fx$
$(\exists y)Fy$
$\therefore Fa$

2. Prove that the following sentences of *PL* are theorems of *PD*:

a. $Fa \supset (\exists y)Fy$
*b. $(\forall x)Fx \supset (\exists y)Fy$
c. $(\forall x)[Fx \supset (Gx \supset Fx)]$
*d. $\sim Fa \supset \sim (\forall x)Fx$
e. $\sim (\exists x)Fx \supset (\forall x) \sim Fx$
*f. $(\exists x)(\exists y)Fxy \supset (\exists y)(\exists x)Fyx$
g. $Fa \vee (\exists y) \sim Fy$
*h. $(\forall x)(Hx \supset Ix) \supset [(\exists x)Hx \supset (\exists x)Ix]$
#i. $[(\forall x)Fx \vee (\forall x)Gx] \supset (\forall x)(Fx \vee Gx)$
*j. $[(\forall x)Fx \ \& \ (\exists y)Gy] \supset (\exists x)(Fx \ \& \ Gx)$
k. $(\exists x)(Fx \ \& \ Gx) \supset [(\exists x)Fx \ \& \ (\exists x)Gx]$
*l. $[(\exists x)Fx \vee (\exists x)Gx] \supset (\exists x)(Fx \vee Gx)$
m. $(\forall x)Hx \equiv \sim (\exists x) \sim Hx$

3. Construct derivations that establish that the following pairs of sentences are equivalent in *PD*:

a. $(\forall x)(Fx \ \& \ Gx)$ — $(\forall x)Fx \ \& \ (\forall x)Gx$
*b. $(\forall x)(Fx \supset Ga)$ — $(\exists x)Fx \supset Ga$
c. $(\forall x)Fx$ — $\sim (\exists x) \sim Fx$
*d. $(\exists y)(Fy \ \& \ (\forall x)Gx)$ — $(\exists y)(\forall x)(Fy \ \& \ Gx)$
#e. $(\exists x)Fx$ — $\sim (\forall x) \sim Fx$
*f. $(\exists x)(Fx \ \& \sim Gx)$ — $\sim (\forall x)(Fx \supset Gx)$

g. $(\forall z)(Hz \supset \sim Iz)$ $\quad \sim (\exists y)(Hy \ \& \ Iy)$
*h. $(\exists x)(Fa \supset Gx)$ $\quad Fa \supset (\exists x)Gx$
i. $(\forall x)(\exists y)(Fx \supset Gy)$ $\quad (\forall x)(Fx \supset (\exists y)Gy)$

4. Construct derivations that establish that the following sets are inconsistent in *PD*:

a. $\{(\forall x)(Fx \equiv \sim Fx)\}$
*b. $\{(\forall x)Hx, (\forall y) \sim (Hy \vee Gyy)\}$
#c. $\{\sim (\forall x)Fx, \sim (\exists x) \sim Fx\}$
*d. $\{\sim (\forall x) \sim Fx, \sim (\exists x)Fx\}$
e. $\{(\forall x)(Fx \supset Gx), (\exists x)Fx, \sim (\exists x)Gx\}$
*f. $\{(\forall z) \sim Fz, (\exists z)Fz\}$
g. $\{(\forall x)Fx, (\exists y) \sim Fy\}$
*h. $\{(\exists y)(Hy \ \& \ Jy), (\forall x) \sim Jx\}$
i. $\{(\forall x)(Hx \equiv \sim Gx), (\exists x)Hx, (\forall x)Gx\}$
*j. $\{(\forall z)(Hz \supset Iz), (\exists y)(Hy \ \& \sim Iy)\}$
k. $\{(\forall z)[Rz \supset (Tz \ \& \sim Mz)], (\exists y)(Ry \ \& \ My)\}$
*l. $\{(\forall x)(Fx \supset Gx), (\forall x)(Fx \supset \sim Gx), (\exists x)Fx\}$

5. Construct derivations that establish the following:

a. $\{(\exists y)(\forall x)Fxy\} \vdash (\forall x)(\exists y)Fxy$
*b. $\{(\forall z)(Gz \supset (\exists x)Fxz), (\forall x)Gx\} \vdash (\forall z)(\exists x)Fxz$
c. $\{(\exists x)Fxxx\} \vdash (\exists x)(\exists y)(\exists z)Fxyz$
*d. $\{(\forall x)(\forall y)(Bx \supset Txy\} \vdash (\forall x)(\forall y)[(Bx \ \& \ Ny) \supset Txy]$
e. $\{(\forall x)(Fx \supset (\exists y)Gxy), (\exists x)Fx\} \vdash (\exists x)(\exists y)Gyx$
*f. $\{(\forall x)(\exists y)Gxy, (\forall x)(\forall y)(Hxy \supset \sim Gxy)\} \vdash (\forall x)(\exists z) \sim Hxz$
g. $\{(\forall x)(\forall y)(Hxy \supset \sim Hyx), (\exists x)(\exists y)Hxy\} \vdash (\exists x)(\exists y) \sim Hyx$
*h. $\{(\forall x)(\forall y)Fxy \vee (\forall x)(\forall y)Gxy\} \vdash (\forall x)(\forall y)(Fxy \vee Gxy)$
i. $\{\sim (\exists x)(\exists y)Rxy, (\forall x)(\forall y)(\sim Hxy \equiv Rxy)\} \vdash (\forall x)(\forall y)Hxy$
*j. $\{(\forall x)(\forall y)(Fxy \equiv \sim Gyx), (\exists z)(\exists w)Gzw\} \vdash (\exists x)(\exists y) \sim Fxy$

6. Construct derivations that establish the validity of the following arguments:

a. $(\forall x)(Fx \supset Gba)$
$(\exists x)Fx$
$\therefore (\exists y)Gya$

*b. $(\forall x)(Hx \supset (\forall y)Rxyb)$
$(\forall x)(\forall z)(Razx \supset Szzz)$
$\therefore Ha \supset (\exists x)Sxcc$

c. $(\exists x)(\exists y)(Fxy \vee Fyx)$
$\therefore (\exists x)(\exists y)Fxy$

*d. $(\forall x)(Fxa \supset Fax)$
$(\exists x)(Hx \ \& \sim Fax)$
$\therefore \sim (\forall y)(Hy \supset Fya)$

e. $(\forall x)(\forall y)[(\exists z)(Fyz \ \& \sim Fzx) \supset Gxy]$
$\sim (\exists x)Gxx$
$\therefore (\forall z x)(Faz \supset Fza)$

*f. $(\forall x)(\forall y)(Dxy \supset Cxy)$
$(\forall x)(\exists y)Dxy$
$(\forall x)(\forall y)(Cxy \supset Cyx)$
$\therefore (\exists x)(\exists y)(Cxy \ \& \ Cyx)$

g. $(\forall x)(Fx \supset (\exists y)Gxy$
$(\forall x)(\forall y) \sim Gxy$
$\therefore (\forall x) \sim Fx$

*h. $(\forall x)(Fx \supset (\exists y)Gxy)$
$(\forall x)(\forall y)(Gxy \supset Hxy)$
$\sim (\exists x)(\exists y)Hxy$
$\therefore \sim (\exists x)Fx$

7. Prove that the following sentences of *PL* are theorems of *PD*:

a. $(\forall x)(\exists z)(Fxz \supset Fzx)$
*b. $(\forall x)Fxx \supset (\forall x)(\exists y)Fxy$
c. $(\forall x)(\forall y)Gxy \supset (\forall z)Gzz$
*d. $(\exists x)Fxx \supset (\exists x)(\exists y)Fxy$
e. $(\forall x)Lxx \supset (\exists x)(\exists y)(Lxy \mathbin{\&} Lyx)$
*f. $(\exists x)(\forall y)Lxy \supset (\exists x)Lxx$
#g. $(\exists x)(\forall y)Fxy \supset (\exists x)(\exists y)Fxy$
*h. $(\forall x)(Fx \supset (\exists y)Gya) \supset (Fb \supset (\exists y)Gya)$
i. $(\exists x)(\exists y)(Lxy \equiv Lyx)$
*j. $(\exists x)(\forall y)Hxy \supset (\forall y)(\exists x)Hxy$
k. $(\forall x)(\forall y)(\forall z)Gxyz \supset (\forall x)(\forall y)(\forall z)(Gxyz \supset Gzyx)$
*l. $(\forall x)(Fx \supset (\exists y)Gyx) \supset ((\exists x)Fx \supset (\exists x)(\exists y)Gxy)$
m. $(\forall x)(\forall y)(Fxy \equiv Fyx) \supset \sim (\exists x)(\exists y)(Fxy \mathbin{\&} \sim Fyx)$
*n. $(\exists x)(Fx \supset (\forall y)Fy)$

8. Construct derivations that establish that the following pairs of sentences are equivalent in *PD*:

a.	$(\forall x)(Fx \supset (\exists y)Gya)$	$(\exists x)Fx \supset (\exists y)Gya$
*b.	$(\forall x)(Fx \supset (\forall y)Gy)$	$(\forall x)(\forall y)(Fx \supset Gy)$
#c.	$(\exists x)[Fx \supset (\forall y)Hxy]$	$(\exists x)(\forall y)(Fx \supset Hxy)$
*d.	$(\forall x)(\forall y)(Fxy \supset Gy)$	$(\forall y)[(\exists x)Fxy \supset Gy]$
e.	$(\forall x)(\forall y)(Fxy \equiv \sim Gyx)$	$(\forall x)(\forall y) \sim (Fxy \equiv Gyx)$

9. Construct derivations that establish that the following sets are inconsistent in *PD*:

a. $\{(\forall x)(\forall y)[(Ex \mathbin{\&} Ey) \supset Txy], (Ea \mathbin{\&} Eb) \mathbin{\&} \sim Tab\}$
*b. $\{(\forall x)(\exists y)Lyx, \sim (\exists x)Lxb\}$
c. $\{\sim (\exists x)Fxx, (\exists x)(\forall y)Fxy\}$
*d. $\{(\forall x)(\forall y)(Fxy \supset Fyx), Fab, \sim (\exists z)Fza\}$
e. $\{(\forall x)(\exists y)Lxy, (\forall y) \sim Lay\}$
*f. $\{(\exists x)(\forall y)Gxy, \sim (\forall y)(\exists x)Gxy\}$
g. $\{(\forall x)[Hx \supset (\exists y)Lyx], (\exists x) \sim (\exists y)Lyx, (\forall x)Hx\}$
*h. $\{\sim (\exists x)Fxx, (\forall x)[(\exists y)Fxy \supset Fxx], (\exists x)(\exists y)Fxy\}$
#i. $\{(\forall x)(\exists y)Fxy, (\exists z) \sim (\exists w)Fzw\}$
*j. $\{(\forall x)(\forall y)(Gxy \equiv Gyx), (\exists x)(\exists y)(Gxy \mathbin{\&} \sim Gyx)\}$
k. $\{(\forall x)(\forall y)(Fxy \vee Gxy), (\exists x)(\exists y)(\sim Fxy \mathbin{\&} \sim Gxy)\}$
*l. $\{(\forall x)(Fx \supset [(\exists y)Gy \supset (\forall y)Gy]), (\exists x)(Fx \mathbin{\&} Gx), (\exists y) \sim Gy\}$

10.3 THE DERIVATION SYSTEM *PD+*

PD+ is a derivation system that includes all the rules of *PD*, the rules of replacement that distinguish *SD+* from *SD*, and one additional rule of replacement that are unique to *PD+*. *PD+* is no stronger than *PD*; however, derivations in *PD+* are often shorter than the corresponding derivations in *PD*. The rules of replacement in *PD+* apply to subformulas of sentences as well as to complete

sentences. In the following example each of the replacement rules has been applied to a subformula of the sentence on the previous line:

1	$(\forall x)[(Fx \ \& \ Hx) \supset (\exists y)Nxy]$	Assumption
2	$(\forall x)[\sim (Fx \ \& \ Hx) \vee (\exists y)Nxy]$	1 Impl
3	$(\forall x)[\sim (Fx \ \& \ Hx) \vee \sim\sim (\exists y)Nxy]$	2 DN
4	$(\forall x) \sim [(Fx \ \& \ Hx) \ \& \sim (\exists y)Nxy]$	3 DeM
5	$(\forall x) \sim [(Hx \ \& \ Fx) \ \& \sim (\exists y)Nxy]$	4 Com

Here Implication was applied to the subformula '(Fx & Hx) ⊃ (∃y)Nxy' of the sentence on line 1 to produce the subformula '~ (Fx & Hx) ∨ (∃y)Nxy' of the sentence on line 2. Double Negation was applied to the subformula '(∃y)Nxy' of the sentence on line 2, to produce the subformula '~ ~ (∃y)Nxy' of the sentence on line 3. De Morgan was applied to the subformula '~ (Fx & Hx) ∨ ~ ~ (∃y)Nxy' of the sentence on line 3 to produce the subformula '~ [(Fx & Hx) & ~ (∃y)Nxy]' of the sentence on line 4. Finally, Commutation was applied to the subformula 'Fx & Hx' of the sentence on line 4 to produce the 'Hx & Fx' of the sentence on line 5.

In applying rules of replacement in *PD+* it is important to correctly identify subformulas of sentences. Consider the following:

1	$(\forall x)[Lx \vee (\exists y)(Bxy \vee Jxy)]$	Assumption	
2	$(\forall x)[(Lx \vee (\exists y)Bxy) \vee Jxy]$	1 Assoc	**MISTAKE!**

Line 2 is a mistake because the immediate subformula of the sentence on line 1 is not of the form $\mathbf{P} \vee (\mathbf{Q} \vee \mathbf{R})$. Rather, it is of the form $\mathbf{P} \vee (\exists \mathbf{x})(\mathbf{Q} \vee \mathbf{R})$.

In addition to the rules of replacement of *SD+*, *PD+* contains **Quantifier Negation.** Where **P** is an open sentence of *PL* in which **x** occurs free, the rule is

Quantifier Negation (QN)

$\sim(\forall \mathbf{x})\mathbf{P} \Leftrightarrow (\exists \mathbf{x}) \sim \mathbf{P}$
$\sim(\exists \mathbf{x})\mathbf{P} \Leftrightarrow (\forall \mathbf{x}) \sim \mathbf{P}$

As with all rules of replacement, Quantifier Negation can be applied to subformulas within a sentence, as well as to an entire sentence. All these are proper uses of Quantifier Negation:

1	$\sim (\exists y) \sim (\forall x)(Fx \supset (\exists z) \sim Gxy)$	Assumption
2	$(\forall y) \sim\sim (\forall x)(Fx \supset (\exists z) \sim Gxy)$	1 QN
3	$(\forall y) \sim (\exists x) \sim (Fx \supset (\exists z) \sim Gxy)$	2 QN
4	$(\forall y) \sim (\exists x) \sim (Fx \supset \sim (\forall z)Gxy)$	3 QN

The definitions of the basic concepts of *PD+* strictly parallel the definitions of the basic concepts of *PD*, in all cases replacing '*PD*' with '*PD+*'. Consequently the tests for the various syntactic properties are carried out in

the same way. The important difference between *PD* and *PD+* is that *PD*, with fewer rules, provides theoretical elegance and *PD+*, with more rules, provides practical ease.

In Section 10.2 we proved that '$(\exists x)(Fx \supset (\forall y)Fy)$' is a theorem in *PD*. Our derivation was 17 lines long. We repeat it here.

Derive: $(\exists x)(Fx \supset (\forall y)Fy)$

1	$\sim (\exists x)(Fx \supset (\forall y)Fy)$	A / ~ E
2	Fa	A / ⊃I
3	~ Fb	A / ~ I
4	Fb	A / ⊃I
5	$\sim (\forall y)Fy$	A / ~ E
6	Fb	4 R
7	~ Fb	3 R
8	$(\forall y)Fy$	5–7 ~ E
9	$Fb \supset (\forall y)Fy$	4–8 ⊃I
10	$(\exists x)(Fx \supset (\forall y)Fy)$	9 ∃I
11	$\sim (\exists x)(Fx \supset (\forall y)Fy)$	1 R
12	Fb	3–11 ~ E
13	$(\forall y)Fy$	12 ∀I
14	$Fa \supset (\forall y)Fy$	2–13 ⊃I
15	$(\exists x)(Fx \supset (\forall y)Fy)$	14 ∃I
16	$\sim (\exists x)(Fx \supset (\forall y)Fy)$	1 R
17	$(\exists x)(Fx \supset (\forall y)Fy)$	1–16 ~ E

We can show that this sentence is a theorem in *PD+* in just 10 lines:

Derive: $(\exists x)(Fx \supset (\forall y)Fy)$

1	$\sim (\exists x)(Fx \supset (\forall y)Fy)$	A / ~ E
2	$(\forall x) \sim (Fx \supset (\forall y)Fy)$	1 QN
3	$\sim (Fa \supset (\forall y)Fy)$	2 ∀E
4	$\sim (\sim Fa \vee (\forall y)Fy)$	3 Impl
5	$\sim \sim Fa \,\&\, \sim (\forall y)Fy$	4 DeM
6	$\sim \sim Fa$	5 &E
7	Fa	6 DN
8	$\sim (\forall y)Fy$	5 &E
9	$(\forall y)Fy$	7 ∀I
10	$(\exists x)(Fx \supset (\forall y)Fy)$	1–9 ~ E

In Section 10.2 it took us 19 lines to derive '$(\exists x)(Fx \supset Ga)$' from $\{(\forall x)Fx \supset Ga\}$. We repeat our derivation here:

Derive: $(\exists x)(Fx \supset (\forall y)Fy)$

1	$(\forall x)Fx \supset Ga$	Assumption
2	$\sim (\exists x)(Fx \supset Ga)$	A / ~ E
3	Fa	A / ⊃I
4	$\sim Fb$	A / ~ E
5	Fb	A / ⊃I
6	$\sim Ga$	A / ~ E
7	Fb	5 R
8	$\sim Fb$	4 R
9	Ga	6–8 ~ E
10	$Fb \supset Ga$	5–9 ⊃I
11	$(\exists x)(Fx \supset Ga)$	10 ∃I
12	$\sim (\exists x)(Fx \supset Ga)$	2 R
13	Fb	4–12 ~ E
14	$(\forall x)Fx$	13 ∀I
15	Ga	1, 14 ⊃E
16	$Fa \supset Ga$	3–15 ⊃I
17	$(\exists x)(Fx \supset Ga)$	16 ∃I
18	$\sim (\exists x)(Fx \supset Ga)$	1 R
19	$(\exists x)(Fx \supset Ga)$	1–18 ~ E

We can derive '$(\exists x)(Fx \supset Ga)$' from $\{(\forall x)Fx \supset Ga\}$ in just 12 lines in *PD+*:

Derive: $(\exists x)(Fx \supset (\forall y)Fy)$

1	$(\forall x)Fx \supset Ga$	Assumption
2	$\sim (\exists x)(Fx \supset Ga)$	A / ~ E
3	$(\forall x) \sim (Fx \supset Ga)$	2 QN
4	$\sim (Fb \supset Ga)$	3 ∀E
5	$\sim (\sim Fb \vee Ga)$	4 Impl
6	$\sim \sim Fb \,\&\, \sim Ga$	5 DeM
7	$\sim \sim Fb$	6 &E
8	Fb	7 DN
9	$(\forall x)Fx$	8 ∀I
10	Ga	1, 9 ⊃E
11	$\sim Ga$	6 &E
12	$(\exists x)(Fx \supset Ga)$	2–11 ~ E

10.5E EXERCISES

1. Show that each of the following derivability claims holds in *PD+*.

a. $\{\sim (\forall y)(Fy \,\&\, Gy)\} \vdash (\exists y)(\sim Fy \vee \sim Gy)$

*b. $\{(\forall w)(Lw \supset Mw),\ (\forall y)(My \supset Ny)\} \vdash (\forall w)(Lw \supset Nw)$

c. $\{(\exists z)(Gz \,\&\, Az),\ (\forall y)(Cy \supset \sim Gy)\} \vdash (\exists z)(Az \,\&\, \sim Cz)$

*d. [~ (∃x)(~ Rx & Sxx), Sjj] ⊢ Rj
e. [(∀x)[(~ Cxb ∨ Hx) ⊃ Lxx], (∃y) ~ Lyy] ⊢ (∃x)Cxb
*f. [(∀x)Fx, (∀z)Hz] ⊢ ~ (∃y)(~ Fy ∨ ~ Hy)

2. Show that each of the following arguments is valid in *PD+*.

a. (∀x) ~ Jx
(∃y)(Hby ∨ Ryy) ⊃ (∃x)Jx
(∀y) ~ (Hby ∨ Ryy)

*b. ~ (∃x)(∀y)(Pxy & ~ Qxy)
(∀x)(∃y)(Pxy ⊃ Qxy)

c. (∀x) ~ ((∀y)Hyx ∨ Tx)
~ (∃y)(Ty ∨ (∃x) ~ Hxy)
(∀x)(∀y)Hxy & (∀x) ~ Tx

*d. (∀z)(Lz ≡ Hz)
(∀x) ~ (Hx ∨ ~ Bx)
~ Lb

e. (∀z)[Kzz ⊃ (Mz & Nz)]
(∃z) ~ Nz
(∃x) ~ Kxx

*f. (∃x)[~ Bxm & (∀y)(Cy ⊃ ~ Gxy)]
(∀z)[~ (∀y)(Wy ⊃ Gzy) ⊃ Bzm]
(∀x)(Cx ⊃ ~ Wx)

g. (∃z)Qz ⊃ (∀w)(Lww ⊃ ~ Hw)
(∃x)Bx ⊃ (∀y)(Ay ⊃ Hy)
(∃w)(Qw & Bw) ⊃ (∀y)(Lyy ⊃ ~ Ay)

*h. (∀y)(Kby ⊃ ~ Hy)
(∀x)[(∃y)(Kby & Qxy) ⊃ (∃z)(~ Hz & Qxz)]

i. ~ (∀x)(~ Px ∨ ~ Hx) ⊃ (∀x)[Cx & (∀y)(Ly ⊃ Axy)]
(∃x)[Hx & (∀y)(Ly ⊃ Axy)] ⊃ (∀x)(Rx & (∀y)Bxy)
~ (∀x)(∀y)Bxy ⊃ (∀x)(~ Px ∨ ~ Hx)

3. Show that each of the following sentences is a theorem in *PD+*.
a. (∀x)(Ax ⊃ Bx) ⊃ (∀x)(Bx ∨ ~ Ax)
*b. (∀x)(Ax ⊃ (Ax ⊃ Bx)) ⊃ (∀x)(Ax ⊃ Bx)
c. ~ (∃x)(Ax ∨ Bx) ⊃ (∀x) ~ Ax

*d. $(\forall x)(Ax \supset Bx) \vee (\exists x)Ax$
e. $(\exists x)Ax \supset (\exists x)Bx) \supset (\exists x)(Ax \supset Bx)$
*f. $(\forall x)(\exists y)(Ax \vee By) \equiv (\exists y)(\forall x)(Ax \vee By)$

4. Show that the members of each of the following pairs of sentences are equivalent in *PD+*.

a. $\sim (\forall x)(Ax \supset Bx)$ — $(\exists x)(Ax \,\&\, \sim Bx)$
*b. $(\exists x)(\exists y)Axy \supset Aab$ — $(\exists x)(\exists y)(Axy \equiv Aab)$
c. $\sim (\forall x) \sim [(Ax \,\&\, Bx) \supset Cx]$ — $(\exists x)[\sim Ax \vee (\sim Cx \supset \sim Bx)]$
*d. $\sim (\forall x)(\exists y)[(Ax \,\&\, Bx) \vee Cy]$ — $(\exists x)(\forall y)[\sim (Cy \vee Ax) \vee \sim (Cy \vee Bx)]$
e. $(\forall x)(Ax \equiv Bx)$ — $\sim (\exists x)[(\sim Ax \vee \sim Bx) \,\&\, (Ax \vee Bx)]$
*f. $(\forall x)(Ax \,\&\, (\exists y) \sim Bxy)$ — $\sim (\exists x)[\sim Ax \vee (\forall y)(Bxy \,\&\, Bxy)]$

5. Show that each of the following sets of sentences is inconsistent in *PD+*.

a. $\{[(\forall x)(Mx \equiv Jx) \,\&\, \sim Mc] \,\&\, (\forall x)Jx\}$
*b. $\{\sim Fa, \sim (\exists x)(\sim Fx \vee \sim Fx)\}$
c. $\{(\forall x)(\forall y)Lxy \supset \sim (\exists x)Tz, (\forall x)(\forall y)Lxy \supset ((\exists w)Cww \vee (\exists z)Tz), (\sim (\forall x)(\forall y)Lxy \vee (\forall z)Bzzk) \,\&\, (\sim (\forall z)Bzzk \vee \sim (\exists w)Cww), (\forall x)(\forall y)Lxy\}$
*d. $\{(\exists x)(\forall y)(Hxy \supset (\forall w)Jww), (\exists x) \sim Jxx \,\&\, \sim (\exists x) \sim Hxm\}$
e. $\{(\forall x)(\forall y)(Gxy \supset Hc), (\exists x)Gix \,\&\, (\forall x)(\forall y)(\forall z)Lxyz, \sim Lcib \vee \sim (Hc \vee Hc)\}$
*f. $\{(\forall x)[(Sx \,\&\, Bxx) \supset Kax], (\forall x)(Hx \supset Bxx), (\exists x)(Sx \,\&\, Hx), (\forall x) \sim (Kax \,\&\, Hx)\}$

6. a. Show that Universal Introduction and Universal Elimination are eliminable in *PD+* by developing routines that can be used in place of these rules to obtain the same results. (*Hint:* Consider using Quantifier Negation, Existential Introduction, and Existential Elimination.)
*b. Show that Existential Introduction and Existential Elimination are eliminable in *PD+* by developing routines that can be used in place of these rules to obtain the same results. (*Hint:* Consider using Quantifier Negation, Universal Introduction, and Universal Elimination.)

10.4 THE DERIVATION SYSTEM *PDE*

The symbolic language *PLE* extends *PL* to include sentences that contain functors and the identity predicate. Accordingly we need to extend the derivation system *PD* developed earlier in this chapter to allow for derivations that include these new sentences of *PLE*. We shall do so by adding an introduction rule and an elimination rule for the identity predicate, and then modifying the quantifier rules so as to allow for sentences containing functors. The resulting *extended* predicate derivation system is called *PDE*.

The introduction rule for '=' is

Identity Introduction (=I)

$\triangleright$ | $(\forall x)x = x$

Identity Introduction is unlike other introduction rules in that it appeals to no previous line or lines of the derivation. Rather, it allows sentences of the specified form to be entered on any line of any derivation, no matter what

sentences, if any, occur earlier in the derivation.[1] Identity Introduction is truth-preserving because every sentence that can be introduced by it, that is every sentence of the form $(\forall \mathbf{x})\mathbf{x} = \mathbf{x}$, is quantificationally true. These sentences simply say of each thing that it is identical to itself. Here is a very simple derivation of a theorem using the rule Identity Introduction:

Derive: a = a

1	(∀y)y = y	=I
2	a = a	1 ∀E

Notice that the sentence on line 1 is *not an assumption*.

The elimination rule for "=" is

Identity Elimination (=E)

$$\triangleright \begin{array}{|l} \mathbf{t_1 = t_2} \\ \mathbf{P} \\ \mathbf{P(t_1//t_2)} \end{array} \quad \text{or} \quad \triangleright \begin{array}{|l} \mathbf{t_1 = t_2} \\ \mathbf{P} \\ \mathbf{P(t_2//t_1)} \end{array}$$

where $\mathbf{t_1}$ and $\mathbf{t_2}$ are closed terms.

The notation

$$\mathbf{P(t_1//t_2)}$$

is read '**P** with one or more occurrences of $\mathbf{t_2}$ replaced by $\mathbf{t_1}$'. Similarly $\mathbf{P(t_2//t_1)}$ is read '**P** with one or more occurrences of $\mathbf{t_1}$ replaced by $\mathbf{t_2}$'. Recall that the closed terms of *PLE* are the individual constants together with complex terms such as '*f*(a,b)' and '*f*(*g*(a,b),c)' that contain no variables. Identity Elimination is a truth-preserving rule because it permits the replacement of one closed term with another in a sentence only if those closed terms designate the same thing ($\mathbf{t_1 = t_2}$ says that $\mathbf{t_1}$ and $\mathbf{t_2}$ do designate the same thing). The following simple examples illustrate the use of this rule:

Derive: Hda

1	c = d	Assumption
2	Hca	Assumption
3	Hda	1, 2 =E

The following three derivations are very similar but not identical:

Derive: (∀x)(Fxh ⊃ Ghx)

1	h = e	Assumption
2	(∀y)(Fye ⊃ Gey)	Assumption
3	(∀y)(Fyh ⊃ Ghy)	1, 2 =E
4	Fah ⊃ Gha	3 ∀E
5	(∀x)(Fxh ⊃ Ghx)	4 ∀I

[1] Metaformulas (such as '(∀x)x=x') that specify sentences that can be introduced without reference to previous sentences occurring in a derivation are usually called **axiom schemas.** An axiom schema is a metaformula such that every formula having its form may be entered in a derivation. Some derivation systems rely primarily on axiom schemas; these are called **axiomatic systems.**

Derive: (∀x)(Fxe ⊃ Ghx)

1	h = e	Assumption
2	(∀y)(Fye ⊃ Gey)	Assumption
3	(∀y)(Fye ⊃ Ghy)	1, 2 =E
4	Fae ⊃ Gha	3 ∀E
5	(∀x)(Fxe ⊃ Ghx)	4 ∀I

Derive: (∀x)(Fxh ⊃ Gex)

1	h = e	Assumption
2	(∀y)(Fye ⊃ Gey)	Assumption
3	(∀y)(Fyh ⊃ Gey)	1, 2 =E
4	Fah ⊃ Gea	3 ∀E
5	(∀x)(Fxh ⊃ Gex)	4 ∀I

In the first derivation we replaced, at line 3, both occurrences of 'e' in line 2 with 'h'. In the second derivation we replaced, at line 3, only the second occurrence of 'e' in line 2 with 'h'. And in the third derivation we replaced, at line 3, only the first occurrence of 'e' in line 2 with 'h'. All of these are appropriate uses of Identity Decomposition.

Derive: Hc

1	(∀y)H*f*(a,x)	Assumption
2	c = *f*(a,b)	Assumption
3	H*f*(a,b)	1 ∀E
4	Hc	2, 3 =E

Derive: Wab

1	Haa ⊃ Waa	Assumption
2	Hab	Assumption
3	a = b	Assumption
4	Hab ⊃ Wab	1, 3 =E
5	Wab	2, 4 =E

Note that from lines 1 through 3 we can obtain, by Identity Elimination, not just 'Hab ⊃ Wab' but a host of additional sentences, including those on lines 5 through 8 below:

1	Haa ⊃ Waa	Assumption
2	Hab	Assumption
3	a = b	Assumption
4	Hab ⊃ Wab	1, 3 =E
5	Hbb ⊃ Wbb	1, 3 =E
6	Haa ⊃ Wbb	1, 3 =E
7	Hbb ⊃ Waa	1, 3 =E
8	Hba ⊃ Wba	1, 3 =E

But these additional sentences do not advance us toward our goal of 'Wab'. There are alternative ways of deriving 'Wab'. Here is one:

Derive: Wab

1	Haa ⊃ Waa	Assumption
2	Hab	Assumption
3	a = b	Assumption
4	Haa	2, 3 =E
5	Waa	1, 4 ⊃E
6	Wab	3, 5 =E

Consider next these derivations:

Derive: Had

1	c = d	Assumption
2	Hac	Assumption
3	Had	1, 2 =E

Derive: (∀y)(Fyh ⊃ Ghy)

1	h = e	Assumption
2	(∀y)(Fye ⊃ Gey)	Assumption
3	(∀y)(Fyh ⊃ Ghy)	1, 2 =E
4	Fah ⊃ Gha	3 ∀E
5	(∀x)(Fxh ⊃ Ghx)	4 ∀I

Derive: Hc

1	(∀y)H*f*(a,x)	Assumption
2	c = *f*(a,b)	Assumption
3	H*f*(a,b)	1 ∀E
4	Hc	2, 3 =E

The sentence '(a = b & b = c) ⊃ a = c' says that if a is identical to b, and b is identical to c, then a is identical to c. As expected, it is a theorem of *PDE*. Here is a proof:

Derive: (a = b & b = c) ⊃ a = c

1	a = b & b = c	A / ⊃I
2	a = b	1 &E
3	b = c	1 &E
4	a = c	2, 3 =E
5	(a = b & b = c) ⊃ a = c	1–4 ⊃I

In considering this example one might well ask whether the justification for line 4 indicates that we have replaced 'b' in line 3 with 'a', based on the identity at

line 2, or that we replaced 'b' in line 2 with 'c' based on the identity at line 3. Fortunately, both replacements are allowed so the justification can be understood either way.

As we have already seen, sentences of the form $\mathbf{t_1 = t_1}$ are normally obtained by Identity Introduction, as in

1	b = b ⊃ Fb	Assumption
2	(∀x)x = x	=I
3	b = b	2 ∀E
4	Fb	1, 3 ⊃E

In special circumstances we can obtain a sentence of the form $\mathbf{a = a}$ by Identity Elimination. This happens when the **a** of **a** = **a** already occurs in an accessible identity sentence. Here is an example:

1	b = b ⊃ Fb	Assumption
2	a = b	Assumption
3	b = b	2, 2 IE
4	Fb	1, 3 ⊃E

Identity Elimination allows us, given a sentence of the form $\mathbf{t_1 = t_2}$, to replace any occurrence of $\mathbf{t_1}$ with $\mathbf{t_2}$ in any sentence that contains $\mathbf{t_1}$, and vice versa. In our example we have the identity sentence 'a = b' and that very sentence contains 'a', so we can replace the 'a' in 'a = b' with 'b', and we do so at line 3.

As we saw in Chapter 7, the identity predicate is useful in symbolizing sentences containing definite descriptions. Consider the argument:

> The Roman general who defeated Pompey conquered Gaul.
> Julius Caesar is a Roman general, and he defeated Pompey.
> ---
> Julius Caesar conquered Gaul.

This argument can be symbolized in *PLE* as:

> (∃x)[((Rx & Dxp) & (∀y)[(Ry & Dyp) ⊃ y = x]) & Cxg]
> Rj & Djp
> ---
> Cjg

This argument is valid, for if there is one and only one thing that is a Roman general and defeated Pompey, and if Julius Caesar is a Roman general who defeated Pompey, then Caesar is *the* Roman general who defeated Pompey, and

is therefore someone who conquered Gaul. We can show this argument is valid in *PDE*:

Derive: Cjg

1		(∃x) [((Rx & Dxp) & (∀y)[(Ry & Dyp) ⊃ y = x]) & Cxg]	Assumption
2		Rj & Djp	Assumption
3	|	((Ra & Dap) & (∀y)[(Ry & Dyp) ⊃ y = a]) & Cag	A / ∃E
4	|	(Ra & Dap) & (∀y)[(Ry & Dyp) ⊃ y = a]	3 &E
5	|	(∀y)[(Ry & Dyp) ⊃ y = a]	4 &E
6	|	(Rj & Djp) ⊃ j = a	5 ∀E
7	|	j = a	2, 6 ⊃E
8	|	Cag	3 &E
9	|	Cjg	7, 8 =E
10		Cjg	1, 3–9 ∃E

Here is another argument that involves a definite description.

The primary author of the Declaration of Independence was a slave owner.
Thomas Jefferson was the primary author of the Declaration of Independence.

Thomas Jefferson was a slave owner.

The conclusion of this argument can be symbolized as 'Ot' where 'Ox' is interpreted as 'x owns at least one slave' and 't' designates Thomas Jefferson. To symbolize the premises we need a way of saying there was one and only one primary author of the Declaration of Independence. We can do so as follows:

$$(\exists x)[Px \ \& \ (\forall z)(Pz \supset z = x)]$$

We are here using 'Px' for 'x is a primary author of the Declaration of Independence'. This sentence of *PL* can be read as 'There is at least one thing x that is a primary author of the Declaration of Independence and each thing z that is a primary author of the Declaration of Independence is identical to x.' The full argument can now be symbolized as:

$$(\exists x)([Px \ \& \ (\forall z)(Pz \supset z = x)] \ \& \ Ox)$$
$$Pt \ \& \ (\forall z)(Pz \supset z = t)$$

$$Ot$$

We can construct a derivation that establishes that the above argument is valid in *PDE*. Here is a start:

Derive: Ot

1	$(\exists x)([Px \ \& \ (\forall z)(Pz \supset z = x)] \ \& \ Ox)$	Assumption
2	$Pt \ \& \ (\forall z)(Pz \supset z = t)$	Assumption
3	$[Pa \ \& \ (\forall z)(Pz \supset z = a)] \ \& \ Oa$	A / ∃E
G	Ot	
G	Ot	1, 2–__ ∃E

Our intent is to derive the final goal using Existential Elimination. If we can derive 'Ot' within the Existential Elimination subderivation we will be able to move it out of that subderivation because 't' is not the instantiating constant in our assumption at line 3 (it is for this reason that we picked a constant other than 't' as our instantiating constant at line 3). 'Oa' can be derived immediately from line 3 by Conjunction Elimination. What remains is to get to a point where we can use Identity Elimination to infer 'Ot' from 'Oa' and an appropriate identity sentence, either 'a = t' or 't = a'.

Derive: Ot

1	$(\exists x)([Px \ \& \ (\forall z)(Pz \supset z = x)] \ \& \ Ox$	Assumption
2	$Pt \ \& \ (\forall z)(Pz \supset z = t)$	Assumption
3	$[Pa \ \& \ (\forall z)(Pz \supset z = a)] \ \& \ Oa$	A / ∃E
4	Oa	3 &E
G	a = t	
G	Ot	4, __ =E
G	Ot	1, 2–__ ∃E

Identity sentences are obtainable both from line 2 and from line 3. This suggests two strategies, and both will work. First we will try to obtain 'a = t'. We start by obtaining '$(\forall z)(Pz \supset z = t)$' from line 2 by Conjuction Elimination and then '$Pa \supset a = t$' by Universal Elimination. And 'Pa' is available from line 3

by two uses of Conjunction Elimination. This will allow us to complete the derivation:

Derive: Ot

1		$(\exists x)([Px \ \& \ (\forall z)(Pz \supset z = x) \ \& \ Ox)$	Assumption
2		$Pt \ \& \ (\forall z)(Pz \supset z = t)$	Assumption
3		$[Pa \ \& \ (\forall z)(Pz \supset z = a)] \ \& \ Oa$	A / ∃E
4		Oa	3 &E
5		$(\forall z)(Pz \supset z = t)$	2 &E
6		$Pa \supset a = t$	5 ∀E
7		$Pa \ \& \ (\forall z)(Pz \supset z = a)$	3 &E
8		Pa	7 &E
9		$a = t$	6, 8 ⊃E
10		Ot	4, 9 =E
11		Ot	1, 2–10 ∃E

We could also have completed our derivation by deriving the identity sentence 't = a' as follows:

Derive: Ot

1		$(\exists x)([Px \ \& \ (\forall z)(Pz \supset z = x)] \ \& \ Ox$	Assumption
2		$Pt \ \& \ (\forall z)(Pz \supset z = t)$	Assumption
3		$[Pa \ \& \ (\forall z)(Pz \supset z = a)] \ \& \ Oa$	A / ∃E
4		Oa	3 &E
5		$Pa \ \& \ (\forall z)(Pz \supset z = a)$	3 &E
6		$(\forall z)(Pz \supset z = a)$	5 &E
7		$Pt \supset t = a$	6 ∀E
8		Pt	2 &E
9		$t = a$	7, 8 ⊃E
10		Ot	4, 9 =E
11		Ot	1, 2–10 ∃E

When we formulated Identity Elimination we did so in a way that allows for the presence of complex terms in *PLE*. Two of our quantifier rules, Existential Introduction and Universal Elimination, need to be modified so that they too allow for the presence of complex terms. The other rules of *PD* function without modification as part of *PDE*. We recast Existential Introduction and Universal Elimination as follows:

Existential Introduction (∃I)

		$\mathbf{P(t/x)}$
▷		$(\exists \mathbf{x})\mathbf{P}$

where **t** is any closed term

Universal Elimination (∀E)

(∀x)P
▷ P(t/x)

where **t** is any closed term

Consider the following simple derivations:

Derive: (∃z)Fz

1	(∀y)Fy	Assumption
2	Fa	1 ∀E
3	(∃z)Fz	2 ∃I

Derive: (∃z)Fg(z)

1	(∀y)Fy	Assumption
2	Fg(a)	1 ∀E
3	(∃z)Fg(z)	2 ∃I

In the first derivation 'Fa' is the substitution instance associated with both the use of Universal Elimination *and* the use of Existential Introduction. In the terminology of previous sections, 'a' is the instantiating constant for these uses of the two rules. In the second derivation 'Fg(a)' is the substitution instance associated with both the use of Universal Elimination and the use of Existential Introduction. However, the instantiating term in the use of Universal Elimination is 'g(a)' (we have replaced 'y' with 'g(a)') whereas the instantiating term in the use of Existential Introduction is 'a', *not* 'g(a)' (we replaced the constant 'a' with the variable 'z'). Since the individual term used to form substitution instances associated with the quantifier rules is sometimes an individual constant and sometimes a closed complex term, we will hereafter speak, with reference to substitution instances and uses of Existential Introduction and Universal Elimination, of the **instantiating term** rather than the instantiating constant.

But we will not modify Existential Elimination and Universal Introduction so as to allow substitution instances used in these rules to be formed from complex terms and so we will continue to talk, with reference to these latter rules, only of the **instantiating constant.** To understand why we will not modify Universal Introduction to allow for complex instantiating terms, consider the following attempt at a derivation:

Derive: (∀x)Ex

1	(∀x)Ed(x)	Assumption	
2	Ed(a)	1 ∀E	
3	(∀x)Ex	2 ∀I	**MISTAKE!**

If this were a legitimate derivation in PDE then the following argument would be valid in *PDE*:

$$\frac{(\forall x)\mathrm{E}d(x)}{(\forall x)\mathrm{E}x}$$

We do not want this argument to be valid in *PDE*. If our UD is the set of positive integers and we interpret 'Ex' as 'x is even' and 'd(x)' as 'x times 2', the premise says that each positive integer is such that 2 times that integer is even, which is true. The conclusion says that each positive integer is even, which is false. The problem is in the attempted inference of line 3 from line 2. The expression 'd(a)' cannot designate an arbitrarily selected member of the UD; rather it can refer only to a member of the UD that is the value of the function d for some member a of the UD. On the interpretation given previously, for example, 'd(a)' can only refer to even numbers.

For similar reasons, we continue to require that in using Existential Elimination the instantiating term must be an individual constant, not a closed complex term. Here is a failed derivation that would be allowed if we dropped this requirement:

Derive: $(\exists x)\mathrm{O}d(x)$

1	$(\exists x)\mathrm{O}x$	Assumption	
2	$\mathrm{O}d(a)$	A / $\exists$E	
3	$(\exists x)\mathrm{O}d(x)$	2 $\exists$I	
4	$(\exists x)\mathrm{O}d(x)$	1, 2–3 $\exists$E	**MISTAKE!**

To see why we do not want this derivation to go through suppose we again use the set of positive integers as our UD and interpret 'Ox' as 'x is odd' and 'd(x)' as 'x times 2'. Then the primary assumption says that there is a positive integer that is odd, which is true. The sentence on line 4 says there is an integer that is 2 times some positive integer and that is odd, and this is false. The problem is that the assumption on line 2 contains information about the individual that is assumed to have property O—namely that it is the value of the function d for some member of the UD, while the existentially quantified sentence on line 1 does not contain this information. The requirement that the assumption for an Existential Elimination subderivation be a substitution instance formed from a constant guarantees that the assumption does not contain information that is absent from the existentially quantified sentence. Hence we continue to require that in using Existential Elimination the assumed substitution instance must be formed using an individual constant.

Having said that, it is important to note that while for Universal Introduction and Existential Elimination the instantiating term must be a constant,

the substitution instances associated with these rules may contain complex terms. For example, the following is a correctly done derivation:

Derive: $(\forall y)Ed(y)$

1	$(\forall x)Ex$	Assumption
2	$Ed(a)$	1 ∀∃
3	$(\forall y)Ed(y)$	2 ∀I

Here 'a' is the instantiating constant for the use of Universal Introduction: In moving from line 2 to line 3 we replaced 'a' with 'y'. But 'd(a)' is the instantiating term associated with Universal Elimination. In moving from line 1 to line 2 we replace 'x' with 'd(a)'. So 'Ed(a)' is a substitution instance of '$(\forall x)Ex$' because it is the result of replacing every occurrence of 'x' in 'Ex' with 'd(a)' and 'Ed(a)' is a substitution instance of '$(\forall y)Ed(y)$' because it is the result of replacing every occurrence of 'y' in '$Ed(y)$' with 'a'.

And the following is an allowed use of Existential Elimination:

Derive:

1	$(\exists x)Fg(x)$	Assumption
2	$Fg(b)$	A / ∃E
3	$(\exists z)Fz$	2 ∃I
4	$(\exists z)Fz$	1, 2–3 ∃E

Here 'Fg(b)' is a substitution instance of '$(\exists x)Fg(x)$' and also a substitution instance of '$(\exists z)Fz$'. In its role as a substitution instance of '$(\exists x)Fg(x)$', the instantiating term is 'b'; in its role as a substitution instance of '$(\exists z)Fz$', 'g(b)' is the instantiating term.

Here are the quantifier rules, modified as appropriate for the system *PDE*.

Universal Elimination (∀E)

$$\begin{array}{l} \quad (\forall \mathbf{x})\mathbf{P} \\ \triangleright \quad \mathbf{P}(\mathbf{t}/\mathbf{x}) \end{array}$$

Existential Introduction (∃I)

$$\begin{array}{l} \quad \mathbf{P}(\mathbf{t}/\mathbf{x}) \\ \triangleright \quad (\exists \mathbf{x})\mathbf{P} \end{array}$$

where **t** is a closed term

Universal Introduction (∀I)

$$\begin{array}{l} \quad \mathbf{P}(\mathbf{a}/\mathbf{x}) \\ \triangleright \quad (\forall \mathbf{x})\mathbf{P} \end{array}$$

provided that:
(i) **a** does not occur in an open assumption.
(ii) **a** does not occur in $(\forall \mathbf{x})\mathbf{P}$.

Existential Elimination (∃E)

$$\begin{array}{l} \quad (\exists \mathbf{x})\mathbf{P} \\ \quad \quad \mathbf{P}(\mathbf{a}/\mathbf{x}) \\ \quad \quad \mathbf{Q} \\ \triangleright \quad \mathbf{Q} \end{array}$$

provided that:
(i) **a** does not occur in an open assumption.
(ii) **a** does not occur in $(\exists \mathbf{x})\mathbf{P}$.
(iii) **a** does not occur in **Q**.

where **a** is an individual constant.

The definitions of the syntactic properties of sentences and sets of sentences in *PDE* (equivalence, validity, etc.) are all carried over from *PD*, substituting '*PDE*' for '*PD*' in each of the definitions.

In the rest of this section we will illustrate the use of the quantifier rules, as modified for *PDE*, by doing a series of derivations that establish various syntactic properties of sentences and sets of sentences of *PLE*.

ARGUMENTS

We begin by showing that the following argument is valid in *PDE*.

$$\begin{array}{l}(\forall x)(\forall y)(Fx \supset Gxy)\\ (\exists x)F\mathit{f}(x)\\ \hline (\exists x)(\exists y)Gxy\end{array}$$

Since the second premise is an existentially quantified sentence we will use Existential Elimination as our primary strategy:

Derive: $(\exists x)(\exists y)Gxy$

1	$(\forall x)(\forall y)(Fx \supset Gxy)$	Assumption
2	$(\exists x)F\mathit{f}(x)$	Assumption
3	$F\mathit{f}(a)$	A / ∃E
G	$(\exists x)(\exists y)Gxy$	
G	$(\exists x)(\exists y)Gxy$	2, 3–G ∃E

Two applications of Universal Elimination produce a material conditional that has '$F\mathit{f}(a)$' as its antecedent:

Derive: $(\exists x)(\exists y)Gxy$

1	$(\forall x)(\forall y)(Fx \supset Gxy)$	Assumption
2	$(\exists x)F\mathit{f}(x)$	Assumption
3	$F\mathit{f}(a)$	A / ∃E
4	$(\forall y)(F\mathit{f}(a) \supset G\mathit{f}(a)y)$	1 ∀E
5	$F\mathit{f}(a) \supset G\mathit{f}(a)b$	4 ∀E
G	$(\exists x)(\exists y)Gxy$	
G	$(\exists x)(\exists y)Gxy$	2, 3–G ∃E

We can derive 'G*f*(a)b' from lines 3 and 5 by Conditional Elimination, and then we can derive our current goal with two applications of Existential Introduction:

Derive: $(\exists x)(\exists y)Gxy$

1	$(\forall x)(\forall y)(Fx \supset Gxy)$	Assumption
2	$(\exists x)F\mathit{f}(x)$	Assumption
3	$F\mathit{f}(a)$	A / $\exists$E
4	$(\forall y)(F\mathit{f}(a) \supset G\mathit{f}(a)y)$	1 $\forall$E
5	$F\mathit{f}(a) \supset G\mathit{f}(a)b$	4 $\forall$E
6	$G\mathit{f}(a)b$	3, 6 $\supset$E
7	$(\exists y)G\mathit{f}(a)y$	6 $\exists$I
8	$(\exists x)(\exists y)Gxy$	7 $\exists$I
9	$(\exists x)(\exists y)Gxy$	2, 3–8 $\exists$E

Both Universal Elimination and Existential Introduction allow the associated substitution instance to be formed from a closed complex term, as we have done here (the substitution instance on line 4 of the universally quantified sentence on line 1 is formed using the complex term '*f*(a)', as is the substitution instance on line 7 of the existentially quantified sentence on line 8).

We next show that the following argument is valid in *PDE*:

$a = g(b)$
$(\forall x)(Fxa \supset (\forall y)Gyx)$
$(\exists y)Fyg(b)$

$(\exists x)(\forall y)Gyx$

We will proceed much as in the previous example, using Existential Elimination as our primary strategy. But this example also requires the use of Identity Elimination:

Derive: $(\exists x)(\forall y)Gyx$

1	$a = g(b)$	Assumption
2	$(\forall x)(Fxa \supset (\forall y)Gyx)$	Assumption
3	$(\exists y)Fyg(b)$	Assumption
4	$Fcg(b)$	A / $\exists$E
5	$Fca \supset (\forall y)Gyc$	2 $\forall$E
6	Fca	1, 4 =E
7	$(\forall y)Gya$	5, 6 $\supset$E
8	$(\exists x)(\forall y)Gyx$	7 $\exists$I
9	$(\exists x)(\forall y)Gyx$	3, 4–8 $\exists$E

At line 6 we replaced '$g(b)$' in '$Fcg(b)$' with 'a'.

THEOREMS

The sentence '$(\forall z)(\forall y)(z = y \supset y = z)$' says of each pair of things that if the first is identical to the second, then the second is identical to the first. Our derivation will end with two uses of Universal Introduction:

Derive: $(\forall z)(\forall y)(z = y \supset y = z)$

G	$b = c \supset c = b$	
G	$(\forall y)(b = y \supset y = b)$	__ ∀I
G	$(\forall z)(\forall y)(z = y \supset y = z)$	__ ∀I

It is important that we use two different constants to form the goal at the third line from the bottom. If we had picked '$b = b \supset b = b$' as our goal we would not be able to derive '$(\forall y)(b = y \supset y = b)$' by Universal Introduction, as the second restriction on that rule prohibits the instantiating term from occurring in the sentence that is derived by the rule. We will use Conditional Introduction to derive the goal '$b = c \supset c = b$':

Derive: $(\forall z)(\forall y)(z = y \supset y = z)$

1	$b = c$	A / ⊃I
G	$c = b$	
G	$b = c \supset c = b$	1–__ ⊃I
G	$(\forall y)(b = y \supset y = b)$	__ ∀I
G	$(\forall z)(\forall y)(z = y \supset y = z)$	__ ∀I

We can finish the derivation by using Identity Introduction to derive '$(\forall y)y = y$' (or any other sentence of this form), then deriving either '$b = b$' or '$c = c$'—it doesn't matter which—by Universal Elimination and then using Identity Elimination to derive '$c = b$':

Derive: $(\forall z)(\forall y)(z = y \supset y = z)$

1	$b = c$	A / ⊃I
2	$(\forall x)x = x$	=I
3	$c = c$	2 ∀E
4	$c = b$	1, 3 =E
5	$b = c \supset c = b$	1–4 ⊃I
6	$(\forall y)(b = y \supset y = b)$	5 ∀I
7	$(\forall z)(\forall y)(z = y \supset y = z)$	6 ∀I

Once we have '$c = c$' at line 3 we can use Identity Elimination, replacing the second occurrence of 'c' in '$c = c$' with 'b', based on the identity at line 1.

The sentence '$(\forall x)(\forall y)(\forall z)[(x = f(z) \ \& \ y = f(z)) \supset x = y]$' is also a theorem of *PDE*. We will work from the bottom up, anticipating three applications of Universal Introduction:

Derive: $(\forall x)(\forall y)(\forall z)[(x = f(z) \ \& \ y = f(z)) \supset x = y]$

1	|	
	|	
	|	
G	$[(a = f(c) \ \& \ b = f(c)) \supset a = b]$	
G	$(\forall z)[(a = f(z) \ \& \ b = f(z)) \supset a = b]$	__ ∀I
G	$(\forall y)(\forall z)[(a = f(z) \ \& \ y = f(z)) \supset a = y]$	__ ∀I
G	$(\forall x)(\forall y)(\forall z)[(x = f(z) \ \& \ y = f(z)) \supset x = y]$	__ ∀I

Our current goal is a material conditional, so we will try to obtain it by Conditional Introduction, assuming '$(a = f(c) \ \& \ b = f(c))$' and deriving '$a = b$'. The latter can be derived using Conjunction Elimination and Identity Elimination:

1	$(a = f(c) \ \& \ b = f(c))$	A/ ⊃ I
2	$a = f(c)$	1 &E
3	$b = f(c)$	1 &E
4	$a = b$	2, 3 =E
5	$[(a = f(c) \ \& \ b = f(c)) \supset a = b]$	1–4 ⊃E
6	$(\forall z)[(a = f(z) \ \& \ b = f(z)) \supset a = b]$	5 ∀I
7	$(\forall y)(\forall z)[(a = f(z) \ \& \ y = f(z)) \supset a = y]$	6 ∀I
8	$(\forall x)(\forall y)(\forall z)[(x = f(z) \ \& \ y = f(z)) \supset x = y]$	7 ∀I

INCONSISTENCY

The set $\{(\forall x)(Fx \vee (\exists y)Gxy), \sim Fg(a,b), g(a,b) = c, \sim (\exists y)Gcy\}$ is inconsistent in *PDE*. To show this we need to derive a sentence **Q** and its negation ~ **Q**. We will use '~ Fg(a,b)' as ~ **Q** and we will use Disjunction Elimination as our primary strategy:

Derive: $Fg(a,b), \sim Fg(a,b)$

1	$(\forall x)(Fx \vee (\exists y)Gxy)$	Assumption
2	$\sim Fg(a,b)$	Assumption
3	$g(a,b) = c$	Assumption
4	$\sim (\exists y)Gcy$	Assumption
5	$Fc \vee (\exists y)Gcy$	1 ∀E
6	| Fc	A / ∨E
7	| $Fg(a,b)$	3, 6 =E
8	| $(\exists y)Gcy$	A / ∨E
	|	
G	| $Fg(a,b)$	
G	$Fg(a,b)$	5, 6–7, 8–__ ∨E
	$\sim Fg(a,b)$	2 R

Our remaining task is to derive '$Fg(a,b)$'. Doing so is not difficult because both '$\sim (\exists y)Gcy$' and '$(\exists y)Gcy$' are available to us, at lines 4 and 8, respectively. So we will use Negation Elimination to complete the derivation:

Derive: $Fg(a,b)$, $\sim Fg(a,b)$

1	$(\forall y)(Fx \vee (\exists y)Gxy)$	Assumption
2	$\sim Fg(a,b)$	Assumption
3	$g(a,b) = c$	Assumption
4	$\sim (\exists y)Gcy$	Assumption
5	$Fc \vee (\exists y)Gcy$	1 ∀E
6	$\quad Fc$	A / ∨E
7	$\quad Fg(a,b)$	3, 6 =E
8	$\quad (\exists y)Gcy$	A / ∨E
9	$\quad\quad \sim Fg(a,b)$	A / ~ E
10	$\quad\quad (\exists y)Gcy$	8 R
11	$\quad\quad \sim (\exists y)Gcy$	4 R
12	$\quad Fg(a,b)$	9–11 ~ E
13	$Fg(a,b)$	5, 6–7, 8–12 ∨E
14	$\sim Fg(a,b)$	2 R

There is an important difference between *PD+* and our latest system, *PDE*. Although both are extensions of *PD* in the sense that each adds new rules to *PD*, *PD+* is not stronger than *PD*. Everything derivable in *PD+* is derivable in *PD*. However, *PDE*, with two new identity rules and modifications of two of *PD*'s quantifier rules, allows us to derive results in *PDE* that are not derivable in *PD*. The previous examples in this section involving the identity predicate and complex terms illustrate this.

However, it should be clear that we can augment the rules of *PDE* with the additional rules of *PD+* to form a derivation system *PDE+* that is equivalent to *PDE*. Here is a short derivation in *PDE+*:

Derive: $\sim (\exists x)f(x) = x$

1	$(\forall x)(\forall y)(f(x) = y \supset \sim f(y) = x)$	Assumption
2	$\quad f(a) = a$	A / ~ I
3	$\quad (\forall y)(f(a) = y \supset \sim f(y) = a)$	1 ∀E
4	$\quad f(a) = a \supset \sim f(a) = a)$	3 ∀E
5	$\quad \sim f(a) = a$	2, 4 ⊃E
6	$\quad f(a) = a$	2 R
7	$\sim f(a) = a$	2–6 ~ I
8	$(\forall x) \sim f(x) = x$	7 ∀I
9	$\sim (\exists x)f(x) = x$	8 QN

10.4E EXERCISES

1. Show that each of the following is a theorem in *PLE*.

a. $a = b \supset b = a$

*b. $(a = b \ \& \ b = c) \supset a = c$

c. $(\sim a = b \ \& \ b = c) \supset \sim a = c$

*d. $\sim a = b \equiv \sim b = a$

e. $\sim a = c \supset (\sim a = b \vee \sim b = c)$

2. Show that each of the following is valid in *PDE*.

a. $a = b \ \& \sim Bab$

$\sim (\forall x)Bcx$

*b. $Ge \supset d = e$

$Ge \supset He$

$Ge \supset Hd$

c. $(\forall z)[Gz \supset (\forall y)(Ky \supset Hzy)]$

$(Ki \ \& \ Gj) \ \& \ i = j$

Hii

*d. $(\exists x)(Hx \ \& \ Mx)$

$Ms \ \& \sim Hs$

$(\exists x)[(Hx \ \& \ Mx) \ \& \sim x = s]$

e. $a = b$

$Ka \vee \sim Kb$

3. Show that each of the following is a theorem in *PDE*.

a. $(\forall x)(x = x \vee \sim x = x)$

*b. $(\forall x)(\forall y)(x = x \ \& \ y = y)$

c. $(\forall x)(\forall y)(x = y \equiv y = x)$

*d. $(\forall x)(\forall y)(\forall z)[(x = y \ \& \ y = z) \supset x = z]$

e. $\sim (\exists x) \sim x = x$

4. Symbolize each of the following arguments in *PLE* and show that each argument is valid in *PDE*.

a. The number 2 is not identical to 4. The numbers 2 and 4 are both even numbers. Therefore there are at least two different even numbers.

*b. Hyde killed some innocent person. But Jekyll is Hyde. Jekyll is a doctor. Hence some doctor killed some innocent person.

c. Shakespeare didn't admire himself, but the queen admired Bacon. Thus Shakespeare isn't Bacon since Bacon admired everybody who was admired by somebody.

*d. Rebecca loves those and only those who love her. The brother of Charlie loves Rebecca. Sam is Charlie's brother. So Sam and Rebecca love each other.

e. Somebody robbed Peter and paid Paul. Peter didn't rob himself. Paul didn't pay himself. Therefore the person who robbed Peter and paid Paul was neither Peter nor Paul.

5. Which of the following illustrate mistakes in *PDE*? Explain what each mistake is.

a.
1		$(\exists x)Sx$	Assumption
2		$Sg(f)$	Assumption
3		$(\exists x)Sg(x)$	2 ∃I
4		$(\exists x)Sg(x)$	1, 2–3 ∃E

*b.
1		$(\exists x)Sg(x,x)$	Assumption
2		$Sg(i,i)$	Assumption
3		$(\exists x)Sg(i,x)$	2 ∃I
4		$(\exists x)Sg(i,x)$	1, 2–3 ∃E

c.
1		$(\exists x)Hxg(x)$	Assumption
2		$Heg(e)$	A / ∃E
3		$(\exists y)Hyg(y)$	2 ∃I
4		$(\exists y)Hyg(y)$	1, 2–3 ∃E

*d.
1		$(\forall x)Rf(x)$	Assumption
2		$Rf(a)$	1 ∀E
3		$(\forall z)Rf(z)$	2 ∀I

e.
1		$(\forall x)Lxxx$	Assumption
2		$L_f(a,a)a$	1 ∀E
3		$(\forall x)L_f(x,x)x$	2 ∀I

*f.
1		$(\forall x)Mx$	Assumption
2		$Mf(f(a))$	1 ∀E
3		$(\exists x)Mf(x)$	2 ∃I

g.
1		$(\forall x)Rf(x,x)$	Assumption
2		$Rf(c,c)$	1 ∀E
3		$(\forall y)Ry$	2 ∀I

*h.
1		$(\forall x)Jx$	Assumption
2		$Jf(f(a))$	1 ∀E
3		$(\exists y)Jf(f(y))$	2 ∃I

i. 1 | $(\forall x)Jx$ — Assumption
2 | $Jf(g(a,b))$ — 1 ∀E
3 | $(\exists x)Jf(g(x,b))$ — 2 ∃I

*j. 1 | $(\forall x)Lx$ — Assumption
2 | $Lf(a,a)$ — 1 ∀E
3 | $(\forall x)Lf(a,x)$ — 2 ∀I

6. Show that each of the following is a theorem in *PDE*.

a. $(\forall x)(\exists y)f(x) = y$
*b. $(\forall x)(\forall y)(\forall z)[(f(x) = g(x,y) \ \& \ g(x,y) = h(x,y,z)) \supset f(x) = h(x,y,z)]$
c. $(\forall x)Ff(x) \supset (\forall x)Ff(g(x))$
*d. $(\forall x)[\sim f(x) = x \supset (\forall y)(f(x) = y \supset \sim x = y)]$
e. $(\forall x)(f(f(x)) = x \supset f(f(f(f(x)))) = x)$
*f. $(\forall x)(\forall y)(\forall z)[(f(g(x)) = y \ \& \ f(y) = z) \supset f(f(g(x))) = z]$
g. $(\forall x)(\forall y)[(f(x) = y \ \& \ f(y) = x) \supset x = f(f(x))]$

7. Show that each of the following is valid in *PDE*.

a. $(\forall x)(Bx \supset Gxf(x))$
$(\forall x)Bf(x)$
$(\forall x)Gf(x)f(f(x))$

*b. $(\forall x)(Kx \vee Hg(x))$
$(\forall x)(Kg(x) \vee Hg(g(x)))$

c. $(\forall x)(\forall y)(f(x) = y \supset Myxc)$
$\sim Mbac \ \& \sim Mabc$
$\sim f(a) = b$

*d. $\sim (\exists x)Rx$
$(\forall x) \sim Rf(x,g(x))$

e. $(\exists x)(\forall y)(\forall z)Lxyz$
$(\exists x)Lxf(x)g(x)$

*f. $(\forall x)[\sim Lxf(x) \vee (\exists y)Ng(y)]$
$(\exists x)Lf(x)f(f(x)) \supset (\exists x)Ng(y)$

g. $(\forall x)[Zx \supset (\forall y)(\sim Dxy \equiv Hf(f(y)))]$
$(\forall x)(Zx \ \& \sim Hx)$
$(\forall x)Df(x)f(x)$

*h. $(\forall x)(\forall y)(\exists z)Sf(x)yz$
$(\forall x)(\forall y)(\forall z)(Sxyz \supset \sim (Cxyz \vee Mzyx))$
$(\exists x)(\exists y) \sim (\forall z)Mzg(y)f(g(x))$

GLOSSARY[2]

DERIVABILITY IN *PD*: A sentence **P** of *PL* is *derivable in PD* from a set Γ of sentences of *PL* if and only if there is a derivation in *PD* in which all the primary assumptions are members of Γ and **P** occurs within the scope of only the primary assumptions.

VALIDITY IN *PD*: An argument of *PL* is *valid in PD* if and only if the conclusion of the argument is derivable in *PD* from the set consisting of the premises. An argument of *PL* is *invalid in PD* if and only if it is not valid in *PD*.

THEOREM IN *PD*: A sentence **P** of *PL* is a *theorem in PD* if and only if **P** is derivable in *PD* from the empty set.

EQUIVALENCE IN *PD*: Sentences **P** and **Q** of *PL* are *equivalent in PD* if and only if **Q** is derivable in *PD* from {**P**} and **P** is derivable in *PD* from {**Q**}.

INCONSISTENCY IN *PD*: A set Γ of sentences of *PL* is *inconsistent in PD* if and only if there is a sentence **P** of *PL* such that both **P** and $\sim$ **P** are derivable in *PD* from Γ. A set Γ of sentences of *PL* is *consistent in PD* if and only if it is not inconsistent in *PD*.

[2]Similar definitions hold for the derivation systems *PD+*, *PDE*, and *PDE+*.

Chapter 11

PREDICATE LOGIC: METATHEORY

11.1 SEMANTIC PRELIMINARIES FOR *PL*

We have been tacitly assuming that our semantic and syntactic concepts of predicate logic coincide. For example, we have assumed that a sentence **P** of *PL* is quantificationally true if and only if **P** is a theorem in *PD*, and also that **P** is quantificationally true if and only if {~ **P**} has a closed truth-tree. In this chapter we shall show that our semantic and syntactic concepts do coincide. We shall establish four major results: the soundness and completeness of the natural deduction systems *PD*, *PD+*, and *PDE*, and the soundness and completeness of the truth-tree method developed in Chapter 9. The results we establish are part of the **metatheory** of predicate logic.

In our proofs of the adequacy of the natural deduction systems and the tree method, we shall use some fundamental semantic results that may seem obvious but that nevertheless must be proved. The purpose of this section is to establish these results. One may skim over this section on the first reading without working through all the proofs but should keep in mind that later metatheoretic proofs depend on the results presented here.

Given any formula **P**, variable **x**, and constant **a**, let **P**(**a**/**x**) be the formula that results from replacing every free occurrence of **x** in **P** with **a**. Our first result establishes that any variable assignment **d** will treat **P**(**a**/**x**) exactly

as $\mathbf{d}[\mathbf{I}(\mathbf{a})/\mathbf{x}]$ treats $\mathbf{P}$. If $\mathbf{d}$ satisfies $\mathbf{P}(\mathbf{a}/\mathbf{x})$, then the variable assignment that is just like $\mathbf{d}$ except that it assigns the denotation of $\mathbf{a}$ to $\mathbf{x}$ will satisfy $\mathbf{P}$, and vice versa. This should not be surprising, for if $\mathbf{x}$ is used to refer to exactly the same thing as $\mathbf{a}$, we would expect $\mathbf{P}$ and $\mathbf{P}(\mathbf{a}/\mathbf{x})$ to behave the same way.

> **11.1.1:** Let $\mathbf{P}$ be a formula of *PL*, let $\mathbf{P}(\mathbf{a}/\mathbf{x})$ be the formula that results from replacing every free occurrence of $\mathbf{x}$ in $\mathbf{P}$ with an individual constant $\mathbf{a}$, let $\mathbf{I}$ be an interpretation, and let $\mathbf{d}$ be a variable assignment for $\mathbf{I}$. Then $\mathbf{d}$ satisfies $\mathbf{P}(\mathbf{a}/\mathbf{x})$ on $\mathbf{I}$ if and only if $\mathbf{d}[\mathbf{I}(\mathbf{a})/\mathbf{x}]$ satisfies $\mathbf{P}$ on $\mathbf{I}$.

To prove the result, we shall use mathematical induction on the number of occurrences of logical operators—truth-functional connectives and quantifiers—that occur in $\mathbf{P}$.

> *Basis clause:* If $\mathbf{P}$ is a formula that contains zero occurrences of logical operators, then $\mathbf{d}$ satisfies $\mathbf{P}(\mathbf{a}/\mathbf{x})$ if and only if $\mathbf{d}[\mathbf{I}(\mathbf{a})/\mathbf{x}]$ satisfies $\mathbf{P}$.
>
> **Proof of basis clause:** If $\mathbf{P}$ contains zero occurrences of logical operators, then $\mathbf{P}$ is either a sentence letter or a formula of the form $\mathbf{At}_1 \ldots \mathbf{t}_n$, where $\mathbf{A}$ is a predicate and $\mathbf{t}_1, \ldots, \mathbf{t}_n$ are individual constants or variables. If $\mathbf{P}$ is a sentence letter, then $\mathbf{P}(\mathbf{a}/\mathbf{x})$ is simply $\mathbf{P}$—a sentence letter alone does not contain any variables to be replaced. $\mathbf{d}$ satisfies $\mathbf{P}(\mathbf{a}/\mathbf{x})$, then, if and only if $\mathbf{I}(\mathbf{P}) = \mathbf{T}$. And $\mathbf{d}[\mathbf{I}(\mathbf{a})/\mathbf{x}]$ satisfies $\mathbf{P}$ if and only if $\mathbf{I}(\mathbf{P}) = \mathbf{T}$. So $\mathbf{d}$ satisfies $\mathbf{P}(\mathbf{a}/\mathbf{x})$ if and only if $\mathbf{d}[\mathbf{I}(\mathbf{a})/\mathbf{x}]$ satisfies $\mathbf{P}$.
>
> If $\mathbf{P}$ has the form $\mathbf{At}_1 \ldots \mathbf{t}_n$, then $\mathbf{P}(\mathbf{a}/\mathbf{x})$ is $\mathbf{At}'_1 \ldots \mathbf{t}'_n$, where $\mathbf{t}'_i$ is $\mathbf{a}$ if $\mathbf{t}_i$ is $\mathbf{x}$ and $\mathbf{t}'_i$ is just $\mathbf{t}_i$ otherwise. By the definition of satisfaction,
>
> a. $\mathbf{d}$ satisfies $\mathbf{At}'_1 \ldots \mathbf{t}'_n$ if and only if $<\mathrm{den}_{\mathbf{I},\mathbf{d}}(\mathbf{t}'_1), \mathrm{den}_{\mathbf{I},\mathbf{d}}(\mathbf{t}'_2), \ldots, \mathrm{den}_{\mathbf{I},\mathbf{d}}(\mathbf{t}'_n)>$ is a member of $\mathbf{I}(\mathbf{A})$.
>
> b. $\mathbf{d}[\mathbf{I}(\mathbf{a})/\mathbf{x}]$ satisfies $\mathbf{At}_1 \ldots \mathbf{t}_n$ if and only if $<\mathrm{den}_{\mathbf{I},\mathbf{d}[\mathbf{I}(\mathbf{a})/\mathbf{x}]}(\mathbf{t}_1), \mathrm{den}_{\mathbf{I},\mathbf{d}[\mathbf{I}(\mathbf{a})/\mathbf{x}]}(\mathbf{t}_2), \ldots, \mathrm{den}_{\mathbf{I},\mathbf{d}[\mathbf{I}(\mathbf{a})/\mathbf{x}]}(\mathbf{t}_n)>$ is a member of $\mathbf{I}(\mathbf{A})$.

But now we note that

> c. $<\mathrm{den}_{\mathbf{I},\mathbf{d}}(\mathbf{t}'_1), \mathrm{den}_{\mathbf{I},\mathbf{d}}(\mathbf{t}'_2), \ldots, \mathrm{den}_{\mathbf{I},\mathbf{d}}(\mathbf{t}'_n)> =$
> $<\mathrm{den}_{\mathbf{I},\mathbf{d}[\mathbf{I}(\mathbf{a})/\mathbf{x}]}(\mathbf{t}_1), \mathrm{den}_{\mathbf{I},\mathbf{d}[\mathbf{I}(\mathbf{a})/\mathbf{x}]}(\mathbf{t}_2), \ldots, \mathrm{den}_{\mathbf{I},\mathbf{d}[\mathbf{I}(\mathbf{a})/\mathbf{x}]}(\mathbf{t}_n)>$.

Consider: If $\mathbf{t}_i$ is a constant, then $\mathbf{t}'_i$ is $\mathbf{t}_i$ and so $\mathrm{den}_{\mathbf{I},\mathbf{d}}(\mathbf{t}'_i) = \mathbf{I}(\mathbf{t}_i)$ and $\mathrm{den}_{\mathbf{I},\mathbf{d}[\mathbf{I}(\mathbf{a})/\mathbf{x}]}(\mathbf{t}_i) = \mathbf{I}(\mathbf{t}_i)$. If $\mathbf{t}_i$ is any variable other than $\mathbf{x}$, then $\mathbf{t}'_i$ is $\mathbf{t}_i$ and so $\mathrm{den}_{\mathbf{I},\mathbf{d}}(\mathbf{t}'_i) = \mathbf{d}(\mathbf{t}_i) = \mathbf{d}_{[\mathbf{I}(\mathbf{a})/\mathbf{x}]}(\mathbf{t}_i) = \mathrm{den}_{\mathbf{I},\mathbf{d}[\mathbf{I}(\mathbf{a})/\mathbf{x}]}(\mathbf{t}_i)$—the assignment of $\mathbf{I}(\mathbf{a})$ to $\mathbf{x}$ in the variable assignment does not affect the value assigned to $\mathbf{t}_i$ in this case. If $\mathbf{t}_i$ is the variable $\mathbf{x}$, then $\mathbf{t}'_i$ is $\mathbf{a}$ and $\mathrm{den}_{\mathbf{I},\mathbf{d}}(\mathbf{a}) = \mathbf{I}(\mathbf{a}) = \mathbf{d}[\mathbf{I}(\mathbf{a})/\mathbf{x}](\mathbf{x}) = \mathrm{den}_{\mathbf{I},\mathbf{d}[\mathbf{I}(\mathbf{a})/\mathbf{x}]}(\mathbf{x})$. (The variant ensures that the denotations of $\mathbf{x}$ and of $\mathbf{a}$ coincide.)

Because the **n**-tuples are the same **n**-tuple, we conclude from (a) and (b) that **d** satisfies $\mathbf{At}'_1 \ldots \mathbf{t}'_n$ if and only if $\mathbf{d}[\mathbf{I}(\mathbf{a})/\mathbf{x}]$ satisfies $\mathbf{At}_1 \ldots \mathbf{t}_n$.

The basis clause—in particular, the case where an atomic formula has the form $\mathbf{At}_1 \ldots \mathbf{t}_n$—is the crux of our proof. Having shown that at the atomic level the thesis we are proving holds, it is straightforward to show that the addition of connectives and quantifiers to build larger formulas does not change matters. The inductive step in the proof of 11.1.1 is

> *Inductive step:* If every formula **P** that contains **k** or fewer occurrences of logical operators is such that **d** satisfies $\mathbf{P}(\mathbf{a}/\mathbf{x})$ if and only if $\mathbf{d}[\mathbf{I}(\mathbf{a})/\mathbf{x}]$ satisfies **P**, then every formula **P** that contains **k** + 1 occurrences of logical operators is such that **d** satisfies $\mathbf{P}(\mathbf{a}/\mathbf{x})$ if and only if $\mathbf{d}[\mathbf{I}(\mathbf{a})/\mathbf{x}]$ satisfies **P**.
>
> **Proof of inductive step:** Letting **k** be an arbitrary positive integer, we assume that the inductive hypothesis holds—that our claim is true of every formula with **k** or fewer occurrences of logical operators. We must show that it follows that the claim is also true of every formula **P** with **k** + 1 occurrences of logical operators. We consider each form that **P** may have.
>
> **Case 1:** **P** has the form $\sim \mathbf{Q}$. Then $\mathbf{P}(\mathbf{a}/\mathbf{x})$ is $\sim \mathbf{Q}(\mathbf{a}/\mathbf{x})$, the negation of $\mathbf{Q}(\mathbf{a}/\mathbf{x})$ (that is, any replacements of **x** that were made had to be made within **Q**). By the definition of satisfaction,
>
> a. **d** satisfies $\sim \mathbf{Q}(\mathbf{a}/\mathbf{x})$ if and only if it does not satisfy $\mathbf{Q}(\mathbf{a}/\mathbf{x})$.
>
> Because $\mathbf{Q}(\mathbf{a}/\mathbf{x})$ contains fewer than **k** + 1 occurrences of logical operators, it follows from the inductive hypothesis that
>
> b. **d** does not satisfy $\mathbf{Q}(\mathbf{a}/\mathbf{x})$ if and only if $\mathbf{d}[\mathbf{I}(\mathbf{a})/\mathbf{x}]$ does not satisfy **Q**.
>
> And, by the definition of satisfaction,
>
> c. $\mathbf{d}[\mathbf{I}(\mathbf{a})/\mathbf{x}]$ does not satisfy **Q** if and only if $\mathbf{d}[\mathbf{I}(\mathbf{a})/\mathbf{x}]$ does satisfy $\sim \mathbf{Q}$.
>
> So, by (a)–(c), **d** satisfies $\sim \mathbf{Q}(\mathbf{a}/\mathbf{x})$ if and only if $\mathbf{d}[\mathbf{I}(\mathbf{a})/\mathbf{x}]$ satisfies $\sim \mathbf{Q}$.
>
> **Case 2:** **P** has the form $\mathbf{Q} \,\&\, \mathbf{R}$. Then $\mathbf{P}(\mathbf{a}/\mathbf{x})$ is
>
> $\mathbf{Q}(\mathbf{a}/\mathbf{x}) \,\&\, \mathbf{R}(\mathbf{a}/\mathbf{x})$
>
> —all replacements of **x** occurred within **Q** and **R**. By the definition of satisfaction,
>
> a. **d** satisfies $\mathbf{Q}(\mathbf{a}/\mathbf{x}) \,\&\, \mathbf{R}(\mathbf{a}/\mathbf{x})$ if and only if **d** satisfies $\mathbf{Q}(\mathbf{a}/\mathbf{x})$ and **d** satisfies $\mathbf{R}(\mathbf{a}/\mathbf{x})$.

Both conjuncts contain fewer than **k** + 1 occurrences of logical operators so, by the inductive hypothesis,

b. **d** satisfies **Q**(**a**/**x**) if and only if **d**[**I**(**a**)/**x**] satisfies **Q**.
c. **d** satisfies **R**(**a**/**x**) if and only if **d**[**I**(**a**)/**x**] satisfies **R**.

By the definition of satisfaction,

d. **d**[**I**(**a**)/**x**] satisfies both **Q** and **R** if and only if **d**[**I**(**a**)/**x**] satisfies **Q** & **R**.

By (a)–(d), then, **d** satisfies **Q**(**a**/**x**) & **R**(**a**/**x**) if and only if **d**[**I**(**a**)/**x**] satisfies **Q** & **R**.

Cases 3–5: The proofs for the case in which **P** has one of the forms **Q** ∨ **R**, **Q** ⊃ **R**, and **Q** ≡ **R** are similar to that of Case 2 and are left as exercises.

Case 6: **P** has the form (∀**y**)**Q**. We must consider two possibilities. If **y** is not the variable **x** that **a** is replacing in (∀**y**)**Q**, then **P**(**a**/**x**) is (∀**y**)**Q**(**a**/**x**)—all replacements of **x** are made within **Q**. By the definition of satisfaction,

a. **d** satisfies (∀**y**)**Q**(**a**/**x**) if and only if, for every member **u** of the UD, **d**[**u**/**y**] satisfies **Q**(**a**/**x**).

Because **Q** contains fewer than **k** + 1 occurrences of logical operators, it follows from the inductive hypothesis that for every member **u** of the UD,

b. **d**[**u**/**y**] satisfies **Q**(**a**/**x**) if and only if **d**[**u**/**y**, **I**(**a**)/**x**] satisfies **Q**.

Each variant **d**[**u**/**y**, **I**(**a**)/**x**] is identical to **d**[**I**(**a**)/**x**, **u**/**y**] because **x** and **y** are not the same variable, and hence neither of the assignments within the brackets can override the other. So every member **u** of the UD is such that

c. **d**[**u**/**y**, **I**(**a**)/**x**] satisfies **Q** if and only if **d**[**I**(**a**)/**x**, **u**/**y**] satisfies **Q**.

And, by the definition of satisfaction again,

d. Every member **u** of the UD is such that **d**[**I**(**a**)/**x**, **u**/**y**] satisfies **Q** if and only if **d**[**I**(**a**)/**x**] satisfies (∀**y**)**Q**.

So, by (a)–(d), in the case where **y** is not the variable **x** that **a** is replacing, **d** satisfies (∀**y**)**Q**(**a**/**x**) if and only if **d**[**I**(**a**)/**x**] satisfies (∀**y**)**Q**.

If **P** is $(\forall \mathbf{x})\mathbf{Q}$, where **x** is the variable that **a** is replacing, then $\mathbf{P}(\mathbf{a}/\mathbf{x})$ is also $(\forall \mathbf{x})\mathbf{Q}$. Because **a** replaces only *free* occurrences of **x** in **P** and **x** does not occur free in **P**, no replacements are made within **Q**. By the definition of satisfaction,

a. **d** satisfies $(\forall \mathbf{x})\mathbf{Q}$ (which is our $\mathbf{P}(\mathbf{a}/\mathbf{x})$) if and only if, for every member **u** of the UD, $\mathbf{d}[\mathbf{u}/\mathbf{x}]$ satisfies **Q**.

b. $\mathbf{d}[\mathbf{I}(\mathbf{a})/\mathbf{x}]$ satisfies $(\forall \mathbf{x})\mathbf{Q}$ (which is our **P**) if and only if, for every member **u** of the UD, $\mathbf{d}[\mathbf{I}(\mathbf{a})/\mathbf{x}, \mathbf{u}/\mathbf{x}]$ satisfies **Q**.

What is $\mathbf{d}[\mathbf{I}(\mathbf{a})/\mathbf{x}, \mathbf{u}/\mathbf{x}]$? This variable assignment is just $\mathbf{d}[\mathbf{u}/\mathbf{x}]$—the first assignment made to **x** within the brackets is overridden by the second. So

c. Every member **u** of the UD is such that $\mathbf{d}[\mathbf{I}(\mathbf{a})/\mathbf{x}, \mathbf{u}/\mathbf{x}]$ satisfies **Q** if and only if every member **u** of the UD is such that $\mathbf{d}[\mathbf{u}/\mathbf{x}]$ satisfies **Q**.

Therefore, by (a)–(c), $\mathbf{d}[\mathbf{I}(\mathbf{a})/\mathbf{x}]$ satisfies $(\forall \mathbf{x})\mathbf{Q}$ if and only if **d** satisfies $(\forall \mathbf{x})\mathbf{Q}$.

Case 7: **P** has the form $(\exists \mathbf{y})\mathbf{Q}$. Again we consider two possibilities. If **y** is not the variable **x** that **a** is replacing, then $\mathbf{P}(\mathbf{a}/\mathbf{x})$ is $(\exists \mathbf{y})\mathbf{Q}(\mathbf{a}/\mathbf{x})$. By the definition of satisfaction,

a. **d** satisfies $(\exists \mathbf{y})\mathbf{Q}(\mathbf{a}/\mathbf{x})$ if and only if, for some member **u** of the UD, $\mathbf{d}[\mathbf{u}/\mathbf{y}]$ satisfies $\mathbf{Q}(\mathbf{a}/\mathbf{x})$.

By the inductive hypothesis, because $\mathbf{Q}(\mathbf{a}/\mathbf{x})$ contains fewer than $\mathbf{k} + 1$ occurrences of logical operators,

b. $\mathbf{d}[\mathbf{u}/\mathbf{y}]$ satisfies $\mathbf{Q}(\mathbf{a}/\mathbf{x})$ if and only if $\mathbf{d}[\mathbf{u}/\mathbf{y}, \mathbf{I}(\mathbf{a})/\mathbf{x}]$ satisfies **Q**.

Because **y** and **x** are different variables, $\mathbf{d}[\mathbf{u}/\mathbf{y}, \mathbf{I}(\mathbf{a})/\mathbf{x}]$ is the same variable assignment as $\mathbf{d}[\mathbf{I}(\mathbf{a})/\mathbf{x}, \mathbf{u}/\mathbf{y}]$. So

c. $\mathbf{d}[\mathbf{u}/\mathbf{y}, \mathbf{I}(\mathbf{a})/\mathbf{x}]$ satisfies $\mathbf{Q}(\mathbf{a}/\mathbf{x})$ if and only if $\mathbf{d}[\mathbf{I}(\mathbf{a})/\mathbf{x}, \mathbf{u}/\mathbf{y}]$ satisfies **Q**.

and by the definition of satisfaction,

d. $\mathbf{d}[\mathbf{I}(\mathbf{a})/\mathbf{x}, \mathbf{u}/\mathbf{y}]$ satisfies **Q** if and only if $\mathbf{d}[\mathbf{I}(\mathbf{a})/\mathbf{x}]$ satisfies $(\exists \mathbf{y})\mathbf{Q}$.

It follows from (a)–(d) that in the case where **y** and **x** are different variables, **d** satisfies $(\exists \mathbf{y})\mathbf{Q}(\mathbf{a}/\mathbf{x})$ if and only if $\mathbf{d}[\mathbf{I}(\mathbf{a})/\mathbf{x}]$ satisfies $(\exists \mathbf{y})\mathbf{Q}$.

If **P** is $(\exists \mathbf{x})\mathbf{Q}$, where **x** is the variable that **a** is replacing, then $\mathbf{P}(\mathbf{a}/\mathbf{x})$ is $(\exists \mathbf{x})\mathbf{Q}$—no replacements are made within **Q** because **x** is not free in $(\exists \mathbf{x})\mathbf{Q}$. So we must show that $\mathbf{d}[\mathbf{I}(\mathbf{a})/\mathbf{x}]$ satisfies $(\exists \mathbf{x})\mathbf{Q}$ if and only if **d** satisfies $(\exists \mathbf{x})\mathbf{Q}$. By the definition of satisfaction,

a. $\mathbf{d}[\mathbf{I}(\mathbf{a})/\mathbf{x}]$ satisfies $(\exists \mathbf{x})\mathbf{Q}$ if and only if, for some member **u** of the UD, $\mathbf{d}[\mathbf{I}(\mathbf{a})/\mathbf{x}, \mathbf{u}/\mathbf{x}]$ satisfies **Q**.

$\mathbf{d}[\mathbf{I}(\mathbf{a})/\mathbf{x}, \mathbf{u}/\mathbf{x}]$ is just $\mathbf{d}[\mathbf{u}/\mathbf{x}]$—the second assignment to **x** overrides the first—and so

b. $\mathbf{d}[\mathbf{I}(\mathbf{a})/\mathbf{x}, \mathbf{u}/\mathbf{x}]$ satisfies **Q** if and only if $\mathbf{d}[\mathbf{u}/\mathbf{x}]$ satisfies **Q**.

By the definition of satisfaction,

c. $\mathbf{d}[\mathbf{u}/\mathbf{x}]$ satisfies **Q** if and only if **d** satisfies $(\exists \mathbf{x})\mathbf{Q}$.

Therefore, by (a)–(c), $\mathbf{d}[\mathbf{I}(\mathbf{a})/\mathbf{x}]$ satisfies $(\exists \mathbf{x})\mathbf{Q}$ if and only if **d** satisfies $(\exists \mathbf{x})\mathbf{Q}$.

With the basis clause and the inductive step established, the conclusion of the argument is also established—every formula **P** is such that **d** satisfies $\mathbf{P}(\mathbf{a}/\mathbf{x})$ on **I** if and only if $\mathbf{d}[\mathbf{I}(\mathbf{a})/\mathbf{x}]$ satisfies **P** on **I**. And that completes the proof of result 11.1.1.

The second result will enable us to prove a claim that was made in Chapter 8: that for any interpretation and any sentence of *PL*, either all variable assignments satisfy the sentence or none do. We used this claim in defining truth and falsehood for sentences: A sentence is true on an interpretation if it is satisfied by all variable assignments and false if it is satisfied by none. The reason this claim turns out to be true is that there are no free variables in sentences. Result 11.1.2 assures us that only the values that a variable assignment assigns to the variables that are free in a formula play a role in determining whether the formula is satisfied:

11.1.2: Let **I** be an interpretation, **d** a variable assignment for **I**, and **P** a formula of *PL*. Then **d** satisfies **P** on **I** if and only if **P** is satisfied on **I** by every variable assignment that assigns the same values to the free variables in **P** as does **d**.

Proof: Let **I** be an interpretation, **d** a variable assignment for **I**, and **P** a formula of *PL*. We shall prove 11.1.2 by mathematical induction on the number of occurrences of logical operators in **P**.

Basis clause: If **P** is a formula that contains zero occurrences of logical operators, then **d** satisfies **P** if and only if **P** is satisfied by every variable assignment that assigns the same values to the free variables in **P** as does **d**.

Proof of basis clause: If **P** contains zero occurrences of logical operators, then **P** is either a sentence letter or a formula of the form $\mathbf{At}_1 \ldots \mathbf{t}_n$. If **P** is a sentence letter, then any variable assignment satisfies **P** on **I** if and only if $\mathbf{I}(\mathbf{P}) = \mathbf{T}$. Therefore **d** satisfies **P** if and only if every variable assignment that assigns the same values to the free variables in **P** as **d** satisfies **P**.

If **P** has the form $\mathbf{At}_1 \ldots \mathbf{t}_n$, then by the definition of satisfaction,

a. **d** satisfies **P** if and only if $<\mathrm{den}_{\mathbf{I},\mathbf{d}}(\mathbf{t}_1), \mathrm{den}_{\mathbf{I},\mathbf{d}}(\mathbf{t}_2), \ldots, \mathrm{den}_{\mathbf{I},\mathbf{d}}(\mathbf{t}_n)>$ is a member of $\mathbf{I}(\mathbf{A})$.

And where $\mathbf{d}'$ is a variable assignment that assigns the same values to the free variables in **P** as does **d**,

b. $\mathbf{d}'$ satisfies **P** if and only if $<\mathrm{den}_{\mathbf{I},\mathbf{d}'}(\mathbf{t}_1), \mathrm{den}_{\mathbf{I},\mathbf{d}'}(\mathbf{t}_2), \ldots, \mathrm{den}_{\mathbf{I},\mathbf{d}'}(\mathbf{t}_n)>$ is a member of $\mathbf{I}(\mathbf{A})$.

But now we note that

c. $<\mathrm{den}_{\mathbf{I},\mathbf{d}}(\mathbf{t}_1), \mathrm{den}_{\mathbf{I},\mathbf{d}}(\mathbf{t}_2), \ldots, \mathrm{den}_{\mathbf{I},\mathbf{d}}(\mathbf{t}_n)> = <\mathrm{den}_{\mathbf{I},\mathbf{d}'}(\mathbf{t}_1), \mathrm{den}_{\mathbf{I},\mathbf{d}'}(\mathbf{t}_2), \ldots, \mathrm{den}_{\mathbf{I},\mathbf{d}'}(\mathbf{t}_n)>$.

For if $\mathbf{t}_i$ is a constant, then $\mathrm{den}_{\mathbf{I},\mathbf{d}}(\mathbf{t}_i) = \mathbf{I}(\mathbf{t}_i)$ and $\mathrm{den}_{\mathbf{I},\mathbf{d}'}(\mathbf{t}_i) = \mathbf{I}(\mathbf{t}_i)$. If $\mathbf{t}_i$ is a variable, then $\mathbf{t}_i$ is free in $\mathbf{At}_1 \ldots \mathbf{t}_n$ and is therefore by stipulation assigned the same value by $\mathbf{d}'$ as it is assigned by **d**. So $\mathrm{den}_{\mathbf{I},\mathbf{d}}(\mathbf{t}_i) = \mathbf{d}(\mathbf{t}_i) = \mathbf{d}'(\mathbf{t}_i) = \mathrm{den}_{\mathbf{I},\mathbf{d}'}(\mathbf{t}_i)$. Hence, by (a)–(c), we conclude that **d** satisfies $\mathbf{At}_1 \ldots \mathbf{t}_n$ if and only if every variable assignment $\mathbf{d}'$ that assigns the same values to the free variables in $\mathbf{At}_1 \ldots \mathbf{t}_n$ as **d** satisfies $\mathbf{At}_1 \ldots \mathbf{t}_n$.

Inductive step: If every sentence **P** that contains **k** or fewer occurrences of logical operators is such that **d** satisfies **P** on **I** if and only if **P** is satisfied by every variable assignment that assigns the same values to the free variables in **P** as **d**, then every sentence **P** that contains $\mathbf{k} + 1$ occurrences of logical operators is such that **d** satisfies **P** on **I** if and only if **P** is satisfied by every variable assignment that assigns the same values to the free variables in **P** as **d**.

Proof of inductive step: Assume that, for an arbitrary positive integer **k**, the inductive hypothesis is true. We shall show that on this assumption our claim must also be true of every sentence **P** that contains $\mathbf{k} + 1$ occurrences of logical operators. Let **I** be an interpretation and **d** a variable assignment for **I**. We consider each form that **P** may have.

Case 1: **P** has the form $-\mathbf{Q}$. By the definition of satisfaction,

a. **d** satisfies $-\mathbf{Q}$ if and only if **d** does not satisfy **Q**.

Because **Q** contains fewer than **k** + 1 occurrences of logical operators, it follows from the inductive hypothesis that

b. **d** does not satisfy **Q** if and only if every variable assignment that assigns to the free variables in **Q** the same values as **d** assigns to those variables does not satisfy **Q**.

And, by the definition of satisfaction,

c. Every variable assignment that assigns to the free variables in **Q** the same values as **d** assigns to those variables fails to satisfy **Q** if and only if every such assignment does satisfy ~ **Q**.

The variable assignments that assign the same values to the free variables of **Q** as does **d** are the variable assignments that assign the same values to the free variables of ~ **Q** as does **d** because **Q** and ~ **Q** contain the same free variables. So, by (a)–(c), **d** satisfies ~ **Q** if and only if every variable assignment that assigns the same values to the free variables in ~ **Q** as does **d** satisfies ~ **Q**.

Case 2: **P** has the form **Q** ∨ **R**. By the definition of satisfaction,

a. **d** satisfies **Q** ∨ **R** if and only if either **d** satisfies **Q** or **d** satisfies **R**.

Because **Q** and **R** each contain fewer than **k** + 1 occurrences of logical operators, it follows by the inductive hypothesis that

b. **d** satisfies **Q** if and only if every variable assignment that assigns to the free variables in **Q** the same values as **d** satisfies **Q**.

c. **d** satisfies **R** if and only if every variable assignment that assigns to the free variables in **R** the same values as **d** satisfies **R**.

By (a)–(c),

d. **d** satisfies **Q** ∨ **R** if and only if either every variable assignment that assigns to the free variables in **Q** the same values as **d** satisfies **Q** or every variable assignment that assigns to the free variables in **R** the same values as **d** satisfies **R**.

Because every variable that is free in **Q** is also free in **Q** ∨ **R**,

e. Every variable assignment that assigns the same values as **d** to the free variables in **Q** ∨ **R** is a variable assignment that assigns the same values as **d** to the free variables in **Q**.

(The converse does not hold.) And, for a similar reason,

f. Every variable assignment that assigns the same values as **d** to the free variables in **Q** ∨ **R** is a variable assignment that assigns the same values as **d** to the free variables in **R**.

Assume now that **d** satisfies **Q** ∨ **R**. By (d), either (i) every variable assignment that assigns to the free variables in **Q** the same values as **d** satisfies **Q** or (ii) every variable assignment that assigns to the free variables in **R** the same values as **d** satisfies **R**. If (i) is the case, then, by (e), we may conclude that every variable assignment that assigns the same values as **d** to the free variables in **Q** ∨ **R** satisfies **Q** and hence **Q** ∨ **R**. If (ii) is the case, then, by (f), we may conclude that every variable assignment that assigns the same values as **d** to the free variables in **Q** ∨ **R** satisfies **R** and hence **Q** ∨ **R**. Either way, then, we conclude that if **d** satisfies **Q** ∨ **R**, then every variable assignment that assigns the same values to the free variables in **Q** ∨ **R** as **d** satisfies **Q** ∨ **R**.

Conversely, if every variable assignment that assigns the same values to the free variables in **Q** ∨ **R** as does **d** satisfies **Q** ∨ **R**, then, trivially, **d** satisfies **Q** ∨ **R**.

Cases 3–5: **P** has one of the forms **Q** & **R**, **Q** ⊃ **R**, or **Q** ≡ **R**. These cases are left as an exercise.

Case 6: **P** has the form (∀**x**)**Q**. By the definition of satisfaction,

a. **d** satisfies (∀**x**)**Q** if and only if every member **u** of the UD is such that **d**[**u**/**x**] satisfies **Q**.

Because **Q** contains fewer than **k** + 1 occurrences of connectives, it follows from the inductive hypothesis that

b. Each member **u** of the UD is such that **d**[**u**/**x**] satisfies **Q** if and only if every variable assignment that assigns the same values to the free variables in **Q** as **d**[**u**/**x**] satisfies **Q**.

It follows from (a) and (b) that

c. **d** satisfies (∀**x**)**Q** if and only if for each member **u** of the UD every variable assignment that assigns the same values to the free variables in **Q** as **d**[**u**/**x**] satisfies **Q**.

Because the variables other than **x** that are free in **Q** are also free in (∀**x**)**Q**, every variable assignment that assigns the same values to the

free variables in **Q** as **d**[**u**/**x**] is a variant **d**'[**u**/**x**] of a variable assignment **d**' that assigns the same values to the free variables in (∀**x**)**Q** as **d**, and vice versa. So (c) is equivalent to

d. **d** satisfies (∀**x**)**Q** if and only if for each member **u** of the UD every variable assignment **d**' that assigns the same values to the free variables in (∀**x**)**Q** as **d** is such that **d**'[**u**/**x**] satisfies **Q**.

If follows by the definition of satisfaction that

e. **d** satisfies (∀**x**)**Q** if and only if every variable assignment that assigns the same values to the free variables in (∀**x**)**Q** as **d** satisfies (∀**x**)**Q**.

Case 7: **P** has the form (∃**x**)**Q**. This case is left as an exercise.

It follows immediately from 11.1.2 that

11.1.3: For any interpretation **I** and sentence **P** of *PL*, either every variable assignment for **I** satisfies **P** or no variable assignment for **I** satisfies **P**.

Proof: Let **d** be any variable assignment. Because **P** is a sentence and hence contains no free variables, every variable assignment for **I** assigns the same values to the free variables in **P** as does **d**. By result 11.1.2, then, **d** satisfies **P** if and only if every variable assignment satisfies **P**. Therefore either every variable assignment satisfies **P** or none does.

Each of the following results, which can be established using results 11.1.1–11.1.3, states something that we would hope to be true of quantified sentences of *PL*.

11.1.4: For any universally quantified sentence (∀**x**)**P** of *PL*, {(∀**x**)**P**} quantificationally entails every substitution instance of (∀**x**)**P**.

Proof: Let (∀**x**)**P** be any universally quantified sentence, let **P**(**a**/**x**) be a substitution instance of (∀**x**)**P**, and let **I** be an interpretation on which (∀**x**)**P** is true. Then, by 11.1.3, every variable assignment satisfies (∀**x**)**P**, and so, for every variable assignment **d** and every member **u** of the UD, **d**[**u**/**x**] satisfies **P**. In particular, for every variable assignment **d** the variant **d**[**I**(**a**)/**x**] must satisfy **P**. By 11.1.1, then, every variable assignment **d** satisfies **P**(**a**/**x**), so **P**(**a**/**x**) is also true on **I**.

11.1.5: Every substitution instance **P**(**a**/**x**) of an existentially quantified sentence (∃**x**)**P** is such that {**P**(**a**/**x**)} ⊨ (∃**x**)**P**.

Proof: See Exercise 3.

11.1.4 and 11.1.5 are results that were used to motivate informally two of the quantifier rules in Chapter 10, Universal Elimination and Existential Introduction, and they will play a role in our proof of the soundness of *PD*. We also want to ensure that the motivations for Universal Introduction and Existential Elimination were correct. Prior to showing this, we establish two further results that we shall need:

> **11.1.6:** Let **I** and **I'** be interpretations that have the same UD and that agree on the assignments made to each individual constant, predicate, and sentence letter in a formula **P** (that is, **I** and **I'** assign the same values to those symbols). Then each variable assignment **d** satisfies **P** on interpretation **I** if and only if **d** satisfies **P** on interpretation **I'**.

In stating result 11.1.6, we have made use of the fact that if two interpretations have the same UD then every variable assignment for one interpretation is a variable assignment for the other—because the collection of objects that can be assigned to the variables of *PL* is the same. The result should sound obvious: If two interpretations with identical universes of discourse treat the nonlogical symbols of **P** in the same way, and if the free variables are interpreted the same way on the two interpretations, then **P** says the same thing on both interpretations. The values that **I** and **I'** assign to other symbols of *PL* have no bearing on what **P** says; **P** must either be satisfied by the variable assignment on both interpretations or be satisfied on neither.

> **Proof of 11.1.6:** Let **P** be a formula of *PL* and let **I** and **I'** be interpretations that have the same UD and that agree on the values assigned to each nonlogical symbol in **P**. We shall now prove, by mathematical induction on the number of occurrences of logical operators in **P**, that a variable assignment satisfies **P** on interpretation **I** if and only if it satisfies **P** on interpretation **I'**.
>
> *Basis clause:* If **P** contains zero occurrences of logical operators, then a variable assignment satisfies **P** on **I** if and only if it satisfies **P** on **I'**.
>
> **Proof of basis clause:** Let **d** be a variable assignment. If **P** is a sentence letter, then **d** satisfies **P** on **I** if and only if $\mathbf{I}(\mathbf{P}) = \mathbf{T}$, and **d** satisfies **P** on **I'** if and only if $\mathbf{I}'(\mathbf{P}) = \mathbf{T}$. $\mathbf{I}(\mathbf{P}) = \mathbf{I}'(\mathbf{P})$ because we have stipulated that **I** and **I'** assign the same values to the nonlogical symbols in **P**. So **d** satisfies **P** on **I** if and only if **d** satisfies **P** on **I'**.
>
> If **P** is an atomic formula $\mathbf{At_1 \ldots t_n}$ then by the definition of satisfaction,
>
> a. **d** satisfies **P** on **I** if and only if $<\text{den}_{\mathbf{I},\mathbf{d}}(\mathbf{t_1}), \text{den}_{\mathbf{I},\mathbf{d}}(\mathbf{t_2}), \ldots, \text{den}_{\mathbf{I},\mathbf{d}}(\mathbf{t_n})>$ is a member of $\mathbf{I}(\mathbf{A})$.

b. **d** satisfies **P** on **I′** if and only if $<den_{I',d}(t_1), den_{I',d}(t_2), \ldots, den_{I',d}(t_n)>$ is a member of $I'(A)$.

We note that

c. $<den_{I,d}(t_1), den_{I,d}(t_2), \ldots, den_{I,d}(t_n)> = <den_{I',d}(t_1), den_{I',d}(t_2), \ldots, den_{I',d}(t_n)>$.

If t_i is a constant, then $den_{I,d}(t_i) = I(t_i)$, $den_{I',d}(t_i) = I'(t_i)$, and $I'(t_i) = I(t_i)$ since **I** and **I′** assign the same values to the nonlogical symbols in **P**; and if t_i is a variable, then $den_{I,d}(t_i) = d(t_i) = den_{I',d}(t_i)$. Moreover,

d. $I(A) = I'(A)$, by our assumption about **I** and **I′**.

So, by (c) and (d),

e. $<den_{I,d}(t_1), den_{I,d}(t_2), \ldots, den_{I,d}(t_n)>$ is a member of $I(A)$ if and only if $<den_{I',d}(t_1), den_{I',d}(t_2), \ldots, den_{I',d}(t_n)>$ is a member of $I'(A)$.

And, by (a), (b), and (e), it follows that **d** satisfies $At_1 \ldots t_n$ on **I** if and only if it does so on **I′**.

Inductive step: If every formula **P** that contains **k** or fewer occurrences of logical operators is such that a variable assignment satisfies **P** on **I** if and only if it satisfies **P** on **I′**, then every formula **P** that contains **k** + 1 occurrences of logical operators is such that a variable assignment satisfies **P** on **I** if and only if it satisfies **P** on **I′**.

Proof of inductive step: We shall consider the forms that **P** may have.

Case 1: **P** has the form $-Q$. By the definition of satisfaction,

a. A variable assignment **d** satisfies $-Q$ on **I** if and only if it does not satisfy **Q** on **I**.

Because **Q** contains fewer than **k** + 1 occurrences of logical operators, it follows from the inductive hypothesis that

b. A variable assignment fails to satisfy **Q** on **I** if and only if it fails to satisfy **Q** on **I′**.

By the definition of satisfaction again,

c. A variable assignment fails to satisfy **Q** on **I′** if and only if it does satisfy $-Q$ on **I′**.

So, by (a)–(c), a variable assignment satisfies $-Q$ on **I** if and only if it satisfies $-Q$ on **I′**.

Case 2: **P** has the form **Q** & **R**. By the definition of satisfaction,

a. A variable assignment satisfies **Q** & **R** on **I** if and only if it satisfies both **Q** and **R** on **I**.

Q and **R** each contain **k** or fewer occurrences of logical operators, and so by the inductive hypothesis,

b. A variable assignment satisfies both **Q** and **R** on **I** if and only if it satisfies both **Q** and **R** on **I′**.

By the definition of satisfaction again,

c. A variable assignment satisfies both **Q** and **R** on **I′** if and only if it satisfies **Q** & **R** on **I′**.

By (a)–(c), a variable assignment satisfies **Q** & **R** on **I** if and only if it satisfies **Q** & **R** on **I′**.

Cases 3–5: **P** has the form **Q** ∨ **R**, **Q** ⊃ **R**, or **Q** ≡ **R**. We omit proofs for these cases as they are strictly analogous to Case 2.

Case 6: **P** has the form (∀**x**)**Q**. By the definition of satisfaction,

a. A variable assignment **d** satisfies (∀**x**)**Q** on **I** if and only if for every member **u** of **I**'s UD, **d**[**u**/**x**] satisfies **Q** on **I**.

b. **d** satisfies (∀**x**)**Q** on **I′** if and only if, for every member **u** of **I′**'s UD (which is the same as **I**'s UD), **d**[**u**/**x**] satisfies **Q** on **I′**.

Because **Q** contains fewer than **k** + 1 occurrences of logical operators, it follows from the inductive hypothesis that

c. For each member **u** of the common UD, **d**[**u**/**x**] satisfies **Q** on **I** if and only if **d**[**u**/**x**] satisfies **Q** on **I′**.

By (a)–(c), it follows that a variable assignment satisfies (∀**x**)**Q** on **I** if and only if it satisfies (∀**x**)**Q** on **I′**.

Case 7: **P** has the form (∃**x**)**Q**. This case is similar to Case 6.

That completes the proof of the inductive step, and we may now conclude that 11.1.6 is true of every formula **P** and pair of interpretations that agree on their assignments to nonlogical symbols in **P**.

Result 11.1.7 follows as an immediate consequence of 11.1.6.

11.1.7: Let **I** and **I′** be interpretations that have the same UD and that agree on the assignments made to each individual constant, predicate, and sentence letter in a sentence **P**. Then **P** is true on **I** if and only if **P** is true on **I′**.

Proof: Let **I** and **I'** be as specified for a sentence **P**. If **P** is true on **I**, then, by 11.1.2, **P** is satisfied by every variable assignment on **I**. By 11.1.6, this is the case if and only if **P** is satisfied by every variable assignment on **I'**, that is, if and only if **P** is true on **I'**.

With results 11.1.6 and 11.1.7 at hand, we may now show that our motivations for the rules Universal Introduction and Existential Elimination are correct.

11.1.8: Let **a** be a constant that does not occur in $(\forall \mathbf{x})\mathbf{P}$ or in any member of the set Γ. Then if $\Gamma \models \mathbf{P}(\mathbf{a}/\mathbf{x})$, $\Gamma \models (\forall \mathbf{x})\mathbf{P}$.

11.1.9: Let **a** be a constant that does not occur in the sentences $(\exists \mathbf{x})\mathbf{P}$ and **Q** and that does not occur in any member of the set Γ. Then if $\Gamma \models (\exists \mathbf{x})\mathbf{P}$ and $\Gamma \cup \{\mathbf{P}(\mathbf{a}/\mathbf{x})\} \models \mathbf{Q}$, $\Gamma \models \mathbf{Q}$ as well.

Result 11.1.8 says, in effect, that if a sentence containing an individual constant is quantificationally entailed by a set of sentences, and if no sentence in the set contains that constant (no specific assumptions were made about the individual designated by that constant), then what **P** says with that constant may be said about everything. And result 11.1.9 says that if a set of sentences quantificationally entails an existentially quantified sentence, and if we can take a substitution instance of that sentence involving a new constant, add it to the set, and find that a sentence making no mention of the individual designated by the constant is quantificationally entailed, then **Q** must have been entailed by the set without the substitution instance thrown in. Intuitively this is so because **Q** does not draw any conclusion about the individual designated by the constant in the substitution instance, and so all that was really needed to entail **Q** was the existential claim that the set entails, and not a specific claim about the individual in question. We shall prove 11.1.8 here; 11.1.9 is left as an exercise.

Proof of 11.1.8: Assume that $\Gamma \models \mathbf{P}(\mathbf{a}/\mathbf{x})$, where **a** does not occur in $(\forall \mathbf{x})\mathbf{P}$ or in any member of Γ. We shall assume, contrary to what we want to show, that Γ does not quantificationally entail $(\forall \mathbf{x})\mathbf{P}$—that there is at least one interpretation, call it **I**, on which every member of Γ is true and $(\forall \mathbf{x})\mathbf{P}$ is false. We shall use **I** as the basis for constructing an interpretation **I'** on which every member of Γ is true and the substitution instance $\mathbf{P}(\mathbf{a}/\mathbf{x})$ is false, contradicting our original assumption. Having done so, we may conclude that if Γ does quantificationally entail $\mathbf{P}(\mathbf{a}/\mathbf{x})$ it must also quantificationally entail $(\forall \mathbf{x})\mathbf{P}$.

So assume that **I** is an interpretation on which every member of Γ is true and on which $(\forall \mathbf{x})\mathbf{P}$ is false. Because $(\forall \mathbf{x})\mathbf{P}$ is false, there is no variable assignment for **I** that satisfies $(\forall \mathbf{x})\mathbf{P}$. That is, for every variable assignment **d**, there is at least one member **u** of the UD such that $\mathbf{d}[\mathbf{u}/\mathbf{x}]$ does not satisfy **P**. Choose one of these members, calling it **u**, and let **I'** be the interpretation that is just like **I** except that it assigns **u** to **a** (all other assignments made by **I** remain the same). It is

now straightforward to show that every member of Γ is true on $\mathbf{I}'$ and $\mathbf{P}(\mathbf{a}/\mathbf{x})$ is false. That every member of Γ' is true on $\mathbf{I}'$ follows from 11.1.7 because $\mathbf{I}$ and $\mathbf{I}'$ assign the same values to all the nonlogical symbols of *PL* other than **a**, and, by stipulation, **a** does not occur in any member of Γ. $\mathbf{I}$ and $\mathbf{I}'$ therefore agree on the values assigned to the nonlogical symbols of each sentence in Γ, and each of these sentences must be true on $\mathbf{I}'$ because it is true on $\mathbf{I}$.

On our assumption that $\mathbf{d}[\mathbf{u}/\mathbf{x}]$ does not satisfy $\mathbf{P}$ on $\mathbf{I}$ it follows from 11.1.6 that $\mathbf{d}[\mathbf{u}/\mathbf{x}]$ does not satisfy $\mathbf{P}$ on $\mathbf{I}'$ (again $\mathbf{I}$ and $\mathbf{I}'$ assign the same values to all the nonlogical constants in $\mathbf{P}$; **a** does not occur in $\mathbf{P}$). By the way we have constructed $\mathbf{I}'$, $\mathbf{u}$ is $\mathbf{I}'(\mathbf{a})$ and so $\mathbf{d}[\mathbf{u}/\mathbf{x}]$ is $\mathbf{d}[\mathbf{I}'(\mathbf{a})/\mathbf{x}]$. Result 11.1.1 tells us that $\mathbf{d}[\mathbf{I}'(\mathbf{a})/\mathbf{x}]$ satisfies $\mathbf{P}$ on $\mathbf{I}'$ if and only if $\mathbf{d}$ satisfies $\mathbf{P}(\mathbf{a}/\mathbf{x})$ on Γ'. So, because $\mathbf{d}[\mathbf{I}'(\mathbf{a})/\mathbf{x}]$ does not satisfy $\mathbf{P}$ on $\mathbf{I}'$, $\mathbf{d}$ does not satisfy $\mathbf{P}(\mathbf{a}/\mathbf{x})$ on Γ'. By 11.1.3, then, no variable assignment satisfies $\mathbf{P}(\mathbf{a}/\mathbf{x})$ on $\mathbf{I}'$, and it is therefore false on this interpretation. But this contradicts our first assumption, that $\Gamma \vDash \mathbf{P}(\mathbf{a}/\mathbf{x})$, and so we conclude that if $\Gamma' \vDash \mathbf{P}(\mathbf{a}/\mathbf{x})$, then $\Gamma \vDash (\forall \mathbf{x})\mathbf{P}$ as well.

As a consequence of 11.1.8, we know that the rule Universal Introduction is indeed truth-preserving.

We shall state four more semantic results that will be needed in the sections that follow and that the reader should now be able to prove. The proofs are left as exercises. The first result relies on 11.1.6 and 11.1.7, much as the proofs of 11.1.8 and 11.1.9 do.

11.1.10: If **a** does not occur in any member of the set $\Gamma \cup \{(\exists \mathbf{x})\mathbf{P}\}$ and if the set is quantificationally consistent, then the set $\Gamma \cup \{(\exists \mathbf{x})\mathbf{P}, \mathbf{P}(\mathbf{a}/\mathbf{x})\}$ is also quantificationally consistent.

Results 11.1.11 and 11.1.12 concern interpretations of a special sort: interpretations on which every member of the UD has a name.

11.1.11: Let $\mathbf{I}$ be an interpretation on which each member of the UD is assigned to at least one individual constant. Then, if every substitution instance of $(\forall \mathbf{x})\mathbf{P}$ is true on $\mathbf{I}$, so is $(\forall \mathbf{x})\mathbf{P}$.

11.1.12: Let $\mathbf{I}$ be an interpretation on which each member of the UD is assigned to at least one individual constant. Then, if every substitution instance of $(\exists \mathbf{x})\mathbf{P}$ is false on $\mathbf{I}$, so is $(\exists \mathbf{x})\mathbf{P}$.

Result 11.1.13 says that, if we rename the individual designated by some individual constant in a sentence $\mathbf{P}$ with a constant that does not already occur in $\mathbf{P}$, then, for any interpretation on which $\mathbf{P}$ is true, there is a closely related interpretation (one that reflects the renaming) on which the new sentence is true.

11.1.13: Let $\mathbf{P}$ be a sentence of *PL*, let **b** be an individual constant that does not occur in $\mathbf{P}$, and let $\mathbf{P}(\mathbf{b}/\mathbf{a})$ be the sentence that results

from replacing every occurrence of the individual constant **a** in **P** with **b**. Then if **P** is true on an interpretation **I**, **P**(**b**/**a**) is true on the interpretation **I'** that is just like **I** except that it assigns **I**(**a**) to **b** (**I'**(**b**) = **I**(**a**)).

11.1E EXERCISES

***1.** Prove Cases 3–5 in the proof of result 11.1.1.

***2.** Prove Cases 3–5 and 7 in the proof of result 11.1.2.

***3.** Prove result 11.1.5.

***4.** Prove result 11.1.9.

5. Prove result 11.1.10.

6. Prove result 11.1.11.

***7.** Prove result 11.1.12.

***8.** Prove result 11.1.13.

11.2 SEMANTIC PRELIMINARIES FOR *PLE*

When we turn to the metatheory for *PLE*, we shall need versions of Section 11.1's semantic results that apply to sentences containing the identity operator and complex terms. In this section we discuss the changes that must be made in the statement of the semantic results and in their proofs.

Starting with 11.1.1, we note that we must generalize the result to read:

> Let **P** be a formula of *PLE*, let **P**(**t**/**x**) be the formula that results from replacing every free occurrence of **x** in **P** with a closed term **t**, let **I** be an interpretation, let **d** be a variable assignment for **I**, and let $\mathbf{d}' = \mathbf{d}[\mathrm{den}_{\mathbf{I},\mathbf{d}}(\mathbf{t})/\mathbf{x}]$ (that is, **d'** is just like **d** except that it assigns to **x** whatever **d** and **I** assign to **t**). Then **d** satisfies **P**(**t**/**x**) on **I** if and only if **d'** satisfies **P** on **I**.

To modify the proof of 11.1.1, we first establish a result concerning complex terms:

> **11.2.1:** Let **t** be a complex term of *PLE*, let **t**($\mathbf{t}_c$/**x**) be the term that results from replacing every occurrence of the variable **x** in **t** with a closed term $\mathbf{t}_c$, let **I** be an interpretation, let **d** be a variable assignment for **I**, and let $\mathbf{d}' = \mathbf{d}[\mathrm{den}_{\mathbf{I},\mathbf{d}}(\mathbf{t}_c)/\mathbf{x}]$. Then $\mathrm{den}_{\mathbf{I},\mathbf{d}'}(\mathbf{t}) = \mathrm{den}_{\mathbf{I},\mathbf{d}}(\mathbf{t}(\mathbf{t}_c/\mathbf{x}))$.

The result states that, if the sole difference between two complex terms $\mathbf{t}_1$ and $\mathbf{t}_2$ is that one contains the closed term $\mathbf{t}_c$ where the other contains the variable $\mathbf{x}$, then if $\mathbf{x}$ and $\mathbf{t}_c$ denote the same individual so do the complex terms $\mathbf{t}_1$ and $\mathbf{t}_2$. We shall prove 11.2.1 by mathematical induction on the number of occurrences of functors in the term.

Basis clause: If a complex term $\mathbf{t}$ contains one functor, then $\text{den}_{\mathcal{I},\mathbf{d}'}(\mathbf{t}) = \text{den}_{\mathcal{I},\mathbf{d}}(\mathbf{t}(\mathbf{t}_c/\mathbf{x}))$.

Proof of basis clause: If a complex term $\mathbf{t}$ contains one functor, then $\mathbf{t}$ is $f(\mathbf{t}_1, \ldots, \mathbf{t}_n)$ where f is a functor, each $\mathbf{t}_i$ is either a variable or a constant, and $\mathbf{t}(\mathbf{t}_c/\mathbf{x})$ is $f(\mathbf{t}_1', \ldots, \mathbf{t}_n')$ where $\mathbf{t}_i'$ is $\mathbf{t}_i$ if $\mathbf{t}_i$ is not $\mathbf{x}$, and $\mathbf{t}_i'$ is $\mathbf{t}_c$ if $\mathbf{t}_i$ is $\mathbf{x}$.

As in the proof of the basis clause of 11.1.1, we note that if $\mathbf{t}_i$ is a constant or variable other than $\mathbf{x}$ then $\text{den}_{\mathcal{I},\mathbf{d}'}(\mathbf{t}_i) = \text{den}_{\mathcal{I},\mathbf{d}}(\mathbf{t}_i)$—since $\mathbf{d}'$ does not differ from $\mathbf{d}$ in a way that affects the denotation of these terms. If $\mathbf{t}_i$ is $\mathbf{x}$, then $\mathbf{t}_i'$ is $\mathbf{t}_c$, and by the definition of how $\mathbf{d}'$ was constructed, $\text{den}_{\mathcal{I},\mathbf{d}'}(\mathbf{x}) = \text{den}_{\mathcal{I},\mathbf{d}}(\mathbf{t}_c)$. So we know that $\langle \text{den}_{\mathcal{I},\mathbf{d}'}(\mathbf{t}_1), \text{den}_{\mathcal{I},\mathbf{d}'}(\mathbf{t}_2), \ldots, \text{den}_{\mathcal{I},\mathbf{d}'}(\mathbf{t}_n)\rangle = \langle \text{den}_{\mathcal{I},\mathbf{d}}(\mathbf{t}_1'), \text{den}_{\mathcal{I},\mathbf{d}}(\mathbf{t}_2'), \ldots, \text{den}_{\mathcal{I},\mathbf{d}}(\mathbf{t}_n')\rangle$. Therefore the $\mathbf{n}+1$-tuple $\langle \text{den}_{\mathcal{I},\mathbf{d}'}(\mathbf{t}_1), \text{den}_{\mathcal{I},\mathbf{d}'}(\mathbf{t}_2), \ldots, \text{den}_{\mathcal{I},\mathbf{d}'}(\mathbf{t}_n), \mathbf{u}\rangle$ is a member of $\mathbf{I}(f)$ if and only if $\langle \text{den}_{\mathcal{I},\mathbf{d}}(\mathbf{t}_1'), \text{den}_{\mathcal{I},\mathbf{d}}(\mathbf{t}_2'), \ldots, \text{den}_{\mathcal{I},\mathbf{d}}(\mathbf{t}_n'), \mathbf{u}\rangle$ is a member of $\mathbf{I}(f)$ since these are the same $\mathbf{n}$-tuple, so $\text{den}_{\mathcal{I},\mathbf{d}'}(f(\mathbf{t}_1, \ldots, \mathbf{t}_n)) = \text{den}_{\mathcal{I},\mathbf{d}}(f(\mathbf{t}_1', \ldots, \mathbf{t}_n'))$.

Inductive step: If every complex term $\mathbf{t}$ that contains $\mathbf{k}$ or fewer functors is such that $\text{den}_{\mathcal{I},\mathbf{d}}(\mathbf{t}) = \text{den}_{\mathcal{I},\mathbf{d}'}(\mathbf{t}(\mathbf{t}_c/\mathbf{x}))$, then every complex term that contains $\mathbf{k}+1$ functors is such that $\text{den}_{\mathcal{I},\mathbf{d}}(\mathbf{t}) = \text{den}_{\mathcal{I},\mathbf{d}'}(\mathbf{t}(\mathbf{t}_c/\mathbf{x}))$.

Proof of inductive step: Letting $\mathbf{k}$ be an arbitrary positive integer, we assume that the inductive hypothesis holds—that our claim is true of every complex term that contains $\mathbf{k}$ or fewer functors. We must show that the claim is also true of every complex term that contains $\mathbf{k}+1$ functors. If $\mathbf{t}$ contains $\mathbf{k}+1$ functors, then $\mathbf{t}$ is $f(\mathbf{t}_1, \ldots, \mathbf{t}_n)$, where each $\mathbf{t}_i$ has $\mathbf{k}$ or fewer functors, and $\mathbf{t}(\mathbf{t}_c/\mathbf{x})$ is $f(\mathbf{t}_1', \ldots, \mathbf{t}_n')$, where each $\mathbf{t}_i'$ is $\mathbf{t}_i(\mathbf{t}_c/\mathbf{x})$. So each $\mathbf{t}_i$ falls under the inductive hypothesis; that is, $\text{den}_{\mathcal{I},\mathbf{d}'}(\mathbf{t}_i) = \text{den}_{\mathcal{I},\mathbf{d}}(\mathbf{t}_i(\mathbf{t}_c/\mathbf{x}))$, and so $\langle \text{den}_{\mathcal{I},\mathbf{d}'}(\mathbf{t}_1), \text{den}_{\mathcal{I},\mathbf{d}'}(\mathbf{t}_2), \ldots, \text{den}_{\mathcal{I},\mathbf{d}'}(\mathbf{t}_n)\rangle = \langle \text{den}_{\mathcal{I},\mathbf{d}}(\mathbf{t}_1'), \text{den}_{\mathcal{I},\mathbf{d}}(\mathbf{t}_2'), \ldots, \text{den}_{\mathcal{I},\mathbf{d}}(\mathbf{t}_n')\rangle$. Therefore the $\mathbf{n}+1$-tuple $\langle \text{den}_{\mathcal{I},\mathbf{d}'}(\mathbf{t}_1), \text{den}_{\mathcal{I},\mathbf{d}'}(\mathbf{t}_2), \ldots, \text{den}_{\mathcal{I},\mathbf{d}'}(\mathbf{t}_n), \mathbf{u}\rangle$ is a member of $\mathbf{I}(f)$ if and only if $\langle \text{den}_{\mathcal{I},\mathbf{d}}(\mathbf{t}_1'), \text{den}_{\mathcal{I},\mathbf{d}}(\mathbf{t}_2'), \ldots, \text{den}_{\mathcal{I},\mathbf{d}}(\mathbf{t}_n'), \mathbf{u}\rangle$ is a member of $\mathbf{I}(f)$, so $\text{den}_{\mathcal{I},\mathbf{d}'}(f(\mathbf{t}_1, \ldots, \mathbf{t}_n)) = \text{den}_{\mathcal{I},\mathbf{d}}(f(\mathbf{t}_1', \ldots, \mathbf{t}_n'))$.

With result 11.2.1 in hand, we can modify the basis clause of result 11.1.1 as follows:

Basis clause: If $\mathbf{P}$ is a formula that contains zero occurrences of logical operators, then $\mathbf{d}$ satisfies $\mathbf{P}(\mathbf{t}/\mathbf{x})$ if and only if $\mathbf{d}'$ satisfies $\mathbf{P}$.

Proof of basis clause: If $\mathbf{P}$ is a formula that contains zero occurrences of logical operators, then $\mathbf{P}$ is either a sentence letter or a formula of

the form $\mathbf{At}_1 \ldots \mathbf{t}_n$, where $\mathbf{A}$ is a predicate and $\mathbf{t}_1, \ldots, \mathbf{t}_n$ are terms. If $\mathbf{P}$ is a sentence letter, then $\mathbf{P}(\mathbf{t}/\mathbf{x})$ is simply $\mathbf{P}$—a sentence letter alone does not contain any variables to be replaced. $\mathbf{d}$ satisfies $\mathbf{P}(\mathbf{t}/\mathbf{x})$, then, if and only if $\mathbf{I}(\mathbf{P}) = \mathbf{T}$. And $\mathbf{d}'$ satisfies $\mathbf{P}$ if and only if $\mathbf{I}(\mathbf{P}) = \mathbf{T}$. So $\mathbf{d}$ satisfies $\mathbf{P}(\mathbf{t}/\mathbf{x})$ if and only if $\mathbf{d}'$ satisfies $\mathbf{P}$.

If $\mathbf{P}$ has the form $\mathbf{At}_1 \ldots \mathbf{t}_n$, then $\mathbf{P}(\mathbf{t}/\mathbf{x})$ is $\mathbf{At}'_1 \ldots \mathbf{t}'_n$, where $\mathbf{t}'_i$ is $\mathbf{t}$ if $\mathbf{t}_i$ is $\mathbf{x}$ and $\mathbf{t}'_i$ is just $\mathbf{t}_i$ otherwise. By the definition of satisfaction,

a. $\mathbf{d}$ satisfies $\mathbf{At}'_1 \ldots \mathbf{t}'_n$ if and only if $<\mathrm{den}_{\mathbf{I},\mathbf{d}}(\mathbf{t}'_1), \mathrm{den}_{\mathbf{I},\mathbf{d}}(\mathbf{t}'_2), \ldots, \mathrm{den}_{\mathbf{I},\mathbf{d}}(\mathbf{t}'_n)>$ is a member of $\mathbf{I}(\mathbf{A})$.

b. $\mathbf{d}'$ satisfies $\mathbf{At}_1 \ldots \mathbf{t}_n$ if and only if $<\mathrm{den}_{\mathbf{I},\mathbf{d}'}(\mathbf{t}_1), \mathrm{den}_{\mathbf{I},\mathbf{d}'}(\mathbf{t}_2), \ldots, \mathrm{den}_{\mathbf{I},\mathbf{d}'}(\mathbf{t}_n)>$ is a member of $\mathbf{I}(\mathbf{A})$.

But now we note that

c. $<\mathrm{den}_{\mathbf{I},\mathbf{d}}(\mathbf{t}'_1), \mathrm{den}_{\mathbf{I},\mathbf{d}}(\mathbf{t}'_2), \ldots, \mathrm{den}_{\mathbf{I},\mathbf{d}}(\mathbf{t}'_n)> = <\mathrm{den}_{\mathbf{I},\mathbf{d}'}(\mathbf{t}_1), \mathrm{den}_{\mathbf{I},\mathbf{d}'}(\mathbf{t}_2), \ldots, \mathrm{den}_{\mathbf{I},\mathbf{d}'}(\mathbf{t}_n)>$.

Consider: If $\mathbf{t}_i$ is a constant, then $\mathbf{t}'_i$ is $\mathbf{t}_i$ and so $\mathrm{den}_{\mathbf{I},\mathbf{d}}(\mathbf{t}'_i) = \mathbf{I}(\mathbf{t}_i)$ and $\mathrm{den}_{\mathbf{I},\mathbf{d}'}(\mathbf{t}_i) = \mathbf{I}(\mathbf{t}_i)$. If $\mathbf{t}_i$ is a variable other than $\mathbf{x}$, then $\mathbf{t}'_i$ is $\mathbf{t}_i$ and so $\mathrm{den}_{\mathbf{I},\mathbf{d}}(\mathbf{t}'_i) = \mathbf{d}(\mathbf{t}_i) = \mathbf{d}'(\mathbf{t}_i) = \mathrm{den}_{\mathbf{I},\mathbf{d}'}(\mathbf{t}_i)$—the variation in the variable assignment does not affect the value assigned to $\mathbf{t}_i$ in this case. If $\mathbf{t}_i$ is the variable $\mathbf{x}$, then $\mathbf{t}'_i$ is $\mathbf{t}$ and $\mathrm{den}_{\mathbf{I},\mathbf{d}}(\mathbf{t}) = \mathrm{den}_{\mathbf{I},\mathbf{d}'}(\mathbf{x})$. (The variant $\mathbf{d}'$ was defined in a way that ensures that the denotations of $\mathbf{x}$ and of $\mathbf{t}$ coincide.) And if $\mathbf{t}_i$ is a complex term, it follows from 11.2.1 that $\mathrm{den}_{\mathbf{I},\mathbf{d}}(\mathbf{t}'_i) = \mathrm{den}_{\mathbf{I},\mathbf{d}'}(\mathbf{t}_i)$.

Because the $\mathbf{n}$-tuples are the same $\mathbf{n}$-tuple, we conclude from (a) and (b) that $\mathbf{d}$ satisfies $\mathbf{At}'_1 \ldots \mathbf{t}'_n$ if and only if $\mathbf{d}'$ satisfies $\mathbf{At}_1 \ldots \mathbf{t}_n$.

We must also add a new case to the proof of the basis clause, to cover formulas of the form $\mathbf{t}_1 = \mathbf{t}_2$. We leave this as an exercise. The rest of the proof of 11.1.1 remains the same, except that we replace $\mathbf{d}[\mathbf{I}(\mathbf{a})/\mathbf{x}]$ with $\mathbf{d}'$ (which is shorthand for $\mathbf{d}[\mathrm{den}_{\mathbf{I},\mathbf{d}}(\mathbf{t}_c)/\mathbf{x}]$) throughout.

The proof of 11.1.2 is modified in a similar way. First, we need to prove

11.2.2: Let $\mathbf{I}$ be an interpretation, $\mathbf{d}$ a variable assignment for $\mathbf{I}$, and $\mathbf{t}$ a complex term of *PLE*. Then, for any variable assignment $\mathbf{d}'$ that assigns the same values to the variables in $\mathbf{t}$ as does $\mathbf{d}$, $\mathrm{den}_{\mathbf{I},\mathbf{d}'}(\mathbf{t}) = \mathrm{den}_{\mathbf{I},\mathbf{d}}(\mathbf{t})$.

Proof: See Exercise 2.

With this result the basis clause in the proof of 11.1.2 must be modified to include atomic formulas containing complex terms along the same lines that we modified the basis clause in the proof of 11.1.1, and also to include formulas of the form $\mathbf{t}_1 = \mathbf{t}_2$. Both modifications are left as exercises.

The proof of result 11.1.6 can be similarly modified, once we have established

11.2.3: Let **t** be a complex term of *PLE* and let **I** and **I'** be interpretations that have the same UD and that agree on the values assigned to each individual constant and functor in **t**. Then, for any variable assignment **d**, $\text{den}_{\mathbf{I},\mathbf{d}}(\mathbf{t}) = \text{den}_{\mathbf{I}',\mathbf{d}}(\mathbf{t})$.

Proof: See Exercise 3.

Result 11.1.6 must itself be changed to say:

> Let **I** and **I'** be interpretations that have the same UD and that agree on the assignments made to each individual constant, functor, predicate, and sentence letter in a formula **P**. Then each variable assignment **d** satisfies **P** on interpretation **I** if and only if **d** satisfies **P** on interpretation **I'**.

The basis clause must be modified to cover formulas containing complex terms, as well as formulas of the form $\mathbf{t}_1 = \mathbf{t}_2$. This is left as an exercise.

The proofs of results 11.1.3–11.1.5 and 11.1.7–11.1.13 for *PLE* are the same as for *PL*, except for the following changes:

1. The proofs must use the modified versions of 11.1.1, 11.1.2, and 11.1.6 in order to apply to *PLE*.
2. Where '**a**' and '**P**(**a**/**x**)' are used in results 11.1.4 and 11.1.5 to refer to substitution instances of **P**, we need to use '**t**' and '**P**(**t**/**x**)' instead to allow for instantiation with arbitrary terms.
3. In 11.1.7, **I** and **I'** must also agree on the assignments made to each functor.
4. Results 1.1.11 and 1.1.12 are true for *PLE* in two senses: We can change 'every substitution instance' to 'every substitution instance in which the instantiating individual term is a constant', or we can leave the phrase as it is, to include substitution instances with instantiation by all closed terms, complex ones as well as constants.

Finally we shall need two additional semantic results for *PLE*:

11.2.4: For any closed terms $\mathbf{t}_1$ and $\mathbf{t}_2$, if **P** is a sentence that contains $\mathbf{t}_1$, then $\{\mathbf{t}_1 = \mathbf{t}_2, \mathbf{P}\} \vdash \mathbf{P}(\mathbf{t}_2//\mathbf{t}_1)$, and if **P** is a sentence that contains $\mathbf{t}_2$, then $\{\mathbf{t}_1 = \mathbf{t}_2, \mathbf{P}\} \vdash \mathbf{P}(\mathbf{t}_1//\mathbf{t}_2)$.

Proof: See Exercise 4.

11.2.5: If a quantificationally consistent set Γ contains a sentence with a complex term $f(\mathbf{a}_1, \ldots, \mathbf{a}_n)$, where $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are constants, and the constant $\mathbf{b}$ does not occur in Γ, then the set $\Gamma \cup \{\mathbf{b} = f(\mathbf{a}_1, \ldots, \mathbf{a}_n)\}$ is also quantificationally consistent.

Proof: See Exercise 5.

11.2E EXERCISES

***1.** Show the changes that must be made in the basis clauses of the proofs of the following results so that they cover formulas containing complex terms and formulas of the form $\mathbf{t}_1 = \mathbf{t}_2$:
a. Result 11.1.1
b. Result 11.1.2
c. Result 11.1.6

***2.** Prove result 11.2.2.

***3.** Prove result 11.2.3.

4. Prove result 11.2.4.

***5.** Prove result 11.2.5.

11.3 THE SOUNDNESS OF *PD*, *PD*+, AND *PDE*

We shall now establish the soundness of our natural deduction systems. A natural deduction system is said to be **sound** for predicate logic if every rule in that system is truth-preserving—that is, if no derivation that uses the rules of that system can lead from true assumptions to a false conclusion. The *Soundness Metatheorem* for *PD* is

Metatheorem 11.3.1: If $\Gamma \vdash \mathbf{P}$ in *PD*, then $\Gamma \models \mathbf{P}$.

(If $\mathbf{P}$ is derivable from Γ in *PD*, then $\mathbf{P}$ is quantificationally entailed by Γ.) To establish Metatheorem 11.3.1, we shall prove that each sentence in a derivation is quantificationally entailed by the set of open assumptions within whose scope the sentence lies. It will then follow that the last sentence of any derivation is quantificationally entailed by the set of open assumptions of that derivation—and hence that $\Gamma \models \mathbf{P}$ if $\Gamma \vdash \mathbf{P}$. (As in Chapter 6, we drop 'in *PD*' when we use the single turnstile here, and we use the double turnstile to signify *quantificational* entailment.) The proof is by mathematical

induction and is, in outline, like the proof that we presented in Chapter 6 establishing the soundness of *SD* for sentential logic. In fact, we shall see that much of the proof in Chapter 6 can be used here—for in Chapter 6 we showed that the rules for the truth-functional connectives are all sound for sentential logic, and with a change from talk of truth-value assignments to talk of interpretations, those rules are established to be sound for predicate logic in the same way. The bulk of the proof will therefore concentrate on the rules for quantifier introduction and elimination.

In our proof we shall use several semantic results that were presented in Section 11.1 along with the following result:

11.3.2: If $\Gamma \models \mathbf{P}$ and Γ^* is a superset of Γ, then $\Gamma^* \models \mathbf{P}$.

Proof: If every member of Γ^* is true on an interpretation $\mathbf{I}$, then every member of its subset Γ is true on $\mathbf{I}$. And if $\Gamma \models \mathbf{P}$, then $\mathbf{P}$ is also true on $\mathbf{I}$. Hence $\Gamma^* \models \mathbf{P}$.

Letting $\mathbf{P_i}$ be the sentence at position $\mathbf{i}$ in a derivation and letting $\Gamma_\mathbf{i}$ be the set of assumptions that are open at position $\mathbf{i}$ (and hence within whose scope $\mathbf{P_i}$ lies), the proof of Metatheorem 11.3.1 by mathematical induction is

Basis clause: $\Gamma_\mathbf{1} \models \mathbf{P_1}$.

Inductive step: If $\Gamma_\mathbf{i} \models \mathbf{P_i}$ for every position $\mathbf{i}$ in a derivation such that $\mathbf{i} \leq \mathbf{k}$, then $\Gamma_{\mathbf{k}+\mathbf{1}} \models \mathbf{P_{k+1}}$.

Conclusion: Every sentence in a derivation is quantificationally entailed by the set of open assumptions in whose scope it lies (for every position $\mathbf{k}$ in a derivation, $\Gamma_\mathbf{k} \models \mathbf{P_k}$).

Proof of basis clause: The first sentence in any derivation in *PD* is an assumption, and it lies in its own scope. $\Gamma_\mathbf{1}$ is just $\{\mathbf{P_1}\}$, and it is trivial that $\{\mathbf{P_1}\} \models \mathbf{P_1}$.

Proof of inductive step: We choose an arbitrary position $\mathbf{k}$ and assume the inductive hypothesis: For every position $\mathbf{i}$ such that $\mathbf{i} \leq \mathbf{k}$, $\Gamma_\mathbf{i} \models \mathbf{P_i}$. We must now show that the same holds for position $\mathbf{k} + 1$. We shall show this by considering the justifications that might be used for the sentence at position $\mathbf{k} + 1$, establishing that the entailment claim holds no matter which justification is used.

Cases 1–12: $\mathbf{P_{k+1}}$ is justified by one of the rules of *SD*. For each of these cases, use the corresponding case from the proof of the soundness of *SD* in Section 6.3, changing talk of truth-value assignments to talk of interpretations, and talk of truth-functional concepts (inconsistency and so on) to talk of quantificational concepts.

Case 13: $\mathbf{P}_{k+1}$ is justified by Universal Elimination. Then $\mathbf{P}_{k+1}$ is a sentence $\mathbf{Q(a/x)}$ derived as follows:

h	$(\forall \mathbf{x})\mathbf{Q}$	
k + 1	$\mathbf{Q(a/x)}$	**h** ∀E

where every assumption that is open at position **h** is also open at position **k** + 1 (because $(\forall \mathbf{x})\mathbf{Q}$ at position **h** is accessible at position **k** + 1)—so $\Gamma_{\mathbf{h}}$ is a subset of $\Gamma_{\mathbf{k+1}}$. By the inductive hypothesis, $\Gamma_{\mathbf{h}} \vDash (\forall \mathbf{x})\mathbf{Q}$. It follows, by 11.3.2, that the superset $\Gamma_{\mathbf{k+1}} \vDash (\forall \mathbf{x})\mathbf{Q}$. By 11.1.4, which says that a universally quantified sentence quantificationally entails every one of its substitution instances, $\mathbf{Q(a/x)}$ is true on every interpretation on which $(\forall \mathbf{x})\mathbf{Q}$ is true. So $\Gamma_{\mathbf{k+1}} \vDash \mathbf{Q(a/x)}$ as well.

Case 14: $\mathbf{P}_{k+1}$ is justified by Existential Introduction. Then $\mathbf{P}_{k+1}$ is a sentence $(\exists \mathbf{x})\mathbf{Q}$ derived as follows:

h	$\mathbf{Q(a/x)}$	
k + 1	$(\exists \mathbf{x})\mathbf{Q}$	**h** ∃I

where every assumption that is open at position **h** is also open at position **k** + 1. So $\Gamma_{\mathbf{h}}$ is a subset of $\Gamma_{\mathbf{k+1}}$. By the inductive hypothesis, $\Gamma_{\mathbf{h}} \vDash \mathbf{Q(a/x)}$ and so, by 11.3.2, $\Gamma_{\mathbf{k+1}} \vDash \mathbf{Q(a/x)}$. By 11.1.5, we know that $(\exists \mathbf{x})\mathbf{Q}$ is true on every interpretation on which $\mathbf{Q(a/x)}$ is true, so $\Gamma_{\mathbf{k+1}} \vDash (\exists \mathbf{x})\mathbf{Q}$ as well.

Case 15: $\mathbf{P}_{k+1}$ is justified by Universal Introduction. Then $\mathbf{P}_{k+1}$ is a sentence $(\forall \mathbf{x})\mathbf{Q}$ derived as follows:

h	$\mathbf{Q(a/x)}$	
k + 1	$(\forall \mathbf{x})\mathbf{Q}$	**h** ∀I

where every assumption that is open at position **h** is also open at position **k** + 1—so $\Gamma_{\mathbf{h}}$ is a subset of $\Gamma_{\mathbf{k+1}}$—and in addition **a** does not occur in $(\forall \mathbf{x})\mathbf{Q}$ or in any member of $\Gamma_{\mathbf{k+1}}$ because the rule ∀I stipulates this. By the inductive hypothesis, $\Gamma_{\mathbf{h}} \vDash \mathbf{Q(a/x)}$. Because $\Gamma_{\mathbf{h}}$ is a subset of $\Gamma_{\mathbf{k+1}}$, it follows from 11.3.2 that $\Gamma_{\mathbf{k+1}} \vDash \mathbf{Q(a/x)}$. And because **a** does not occur in $(\forall \mathbf{x})\mathbf{Q}$ or in any member of $\Gamma_{\mathbf{k+1}}$, it follows from 11.1.8, which we repeat here, that $\Gamma_{\mathbf{k+1}} \vDash (\forall \mathbf{x})\mathbf{Q}$ as well.

11.1.8: Let **a** be a constant that does not occur in $(\forall \mathbf{x})\mathbf{P}$ or in any member of the set Γ. Then if $\Gamma \vDash \mathbf{P(a/x)}$, $\Gamma \vDash (\forall \mathbf{x})\mathbf{P}$.

Case 16: $\mathbf{P_{k+1}}$ is justified by Existential Elimination. Then $\mathbf{P_{k+1}}$ is derived as follows:

h	$(\exists \mathbf{x})\mathbf{Q}$	
j	$\mathbf{Q(a/x)}$	
m	$\mathbf{P_{k+1}}$	
k + 1	$\mathbf{P_{k+1}}$	**h, j-m** $\exists$E

where every member of $\Gamma_{\mathbf{h}}$ is a member of $\Gamma_{\mathbf{k+1}}$ and every member of $\Gamma_{\mathbf{m}}$ except $\mathbf{Q(a/x)}$ is a member of $\Gamma_{\mathbf{k+1}}$ (if any other assumptions in $\Gamma_{\mathbf{m}}$ were closed prior to position $\mathbf{k + 1}$, then the subderivation **j-m** would not be accessible at position $\mathbf{k + 1}$). Because every member of $\Gamma_{\mathbf{m}}$ except $\mathbf{Q(a/x)}$ is a member of $\Gamma_{\mathbf{k+1}}$, $\Gamma_{\mathbf{m}}$ is a subset of $\Gamma_{\mathbf{k+1}} \cup \{\mathbf{Q(a/x)}\}$. Moreover **a** does not occur in $(\exists \mathbf{x})\mathbf{Q}$, $\mathbf{P_{k+1}}$, or any member of $\Gamma_{\mathbf{k+1}}$ because the rule $\exists$E stipulates this. By the inductive hypothesis, $\Gamma_{\mathbf{h}} \vDash (\exists \mathbf{x})\mathbf{Q}$, and so because $\Gamma_{\mathbf{h}}$ is a subset of $\Gamma_{\mathbf{k+1}}$, it follows from 11.3.2 that $\Gamma_{\mathbf{k+1}} \vDash (\exists \mathbf{x})\mathbf{Q}$. Also by the inductive hypothesis, $\Gamma_{\mathbf{m}} \vDash \mathbf{P_{k+1}}$, and so, because $\Gamma_{\mathbf{m}}$ is a subset of $\Gamma_{\mathbf{k+1}} \cup \{\mathbf{Q(a/x)}\}$, it follows from 11.3.2 that $\Gamma_{\mathbf{k+1}} \cup \{\mathbf{Q(a/x)}\} \vDash \mathbf{P_{k+1}}$. Because **a** does not occur in $(\exists \mathbf{x})\mathbf{Q}$, $\mathbf{P_{k+1}}$, or any member of $\Gamma_{\mathbf{k+1}}$, it follows from the last two entailments noted that $\Gamma_{\mathbf{k+1}} \vDash \mathbf{P_{k+1}}$, by 11.1.9, which we repeat here.

11.1.9: Let **a** be a constant that does not occur in the sentences $(\exists \mathbf{x})\mathbf{P}$ and **Q** and that does not occur in any member of the set Γ. Then, if $\Gamma \vDash (\exists \mathbf{x})\mathbf{P}$ and $\Gamma \cup \{\mathbf{P(a/x)}\} \vDash \mathbf{Q}$, $\Gamma \vDash \mathbf{Q}$ as well.

This completes the proof of the inductive step; all of the derivation rules of *PD* are truth-preserving. Note that, in establishing that the two quantifier rules $\forall$I and $\exists$E are truth-preserving, we made essential use of the restrictions that those rules place on the instantiating constant **a**—the restrictions were included in those rules to ensure that they would be truth-preserving. Having established that the inductive step is true, we may conclude that every sentence in a derivation of *PD* is quantificationally entailed by the set of open assumptions in whose scope it lies. Therefore, if $\Gamma \vdash \mathbf{P}$ in *PD*, then $\Gamma \vDash \mathbf{P}$. This establishes Metatheorem 11.3.1, the Soundness Metatheorem for *PD*.

The proof that *PD+* is sound for predicate logic involves the additional steps of showing that the rules of replacement of *SD+*, the three derived rules of *SD+*, and the rule Quantifier Negation are all truth-preserving. The steps in the soundness proof for *SD+*, which show that its rules are truth-preserving for sentential logic, can with appropriate adjustments to quantificational talk be converted into steps showing that the rules are truth-preserving for quantificational logic. We leave the proof that the additional rule of replacement in *PD+*, Quantifier Negation, is truth-preserving as an exercise.

Finally we can prove that *PDE* is sound for predicate logic with identity and functions by extending the inductive step of the proof for *PD* to cover the two identity rules, Identity Introduction and Identity Elimination, and by making one change in the basis clause of the soundness proof. We note that, since we have shown in Section 11.2 that all of the semantic results in Section 11.1 can be extended to predicate logic with identity and functions, a soundness proof for *PDE* can refer to all of those results. In particular, even though the rules ∀E and ∃I have been changed for *PDE*, the proof for Cases 13 and 14 in the inductive step of the soundness proof for *PD* will remain the same except that in place of the substitution instance $\mathbf{Q(a/x)}$ we now have a substitution instance $\mathbf{Q(t/x)}$, where $\mathbf{t}$ is any closed term.

We first discuss the change in the basis clause of the soundness proof. In the basis clause for *PD*, we said that the first sentence in a derivation is an assumption. This is not always the case in *PDE*; the first sentence *can* be an assumption, but it can also be a sentence of the form $(\forall \mathbf{x})\mathbf{x} = \mathbf{x}$, introduced by Identity Introduction. So the proof of the basis clause will look like this:

> The first sentence in a derivation in *PDE* is either an assumption or a sentence introduced by Identity Introduction. If the first sentence is an assumption, then it lies in its own scope. In this case Γ_1 is just $\{\mathbf{P}_1\}$, and it is trivial that $\{\mathbf{P}_1\} \vdash \mathbf{P}_1$.
>
> If the first sentence is introduced by Identity Introduction, then Γ_1 is empty—there are no assumptions, and hence no open assumptions, at that point. So it remains to be shown that $\varnothing$ truth-functionally entails every sentence of the form $(\forall \mathbf{x})\mathbf{x} = \mathbf{x}$—that is, that every such sentence is quantificationally true. This was proved in Exercise 8.7.10a.

We add the following two cases to the proof of the inductive step for *PD*:

> **Case 17:** $\mathbf{P}_{k+1}$ is introduced by Identity Introduction. Then $\mathbf{P}_{k+1}$ is a sentence of the form $(\forall \mathbf{x})\mathbf{x} = \mathbf{x}$ derived as follows:
>
> $$\begin{array}{l|ll} \mathbf{k+1} & (\forall \mathbf{x})\mathbf{x} = \mathbf{x} & =\mathrm{I} \end{array}$$
>
> We have already noted that the empty set quantificationally entails every sentence of the form $(\forall \mathbf{x})\mathbf{x} = \mathbf{x}$. And the empty set is a subset of Γ_{k+1}, so, by 11.3.2, $\Gamma_{k+1} \vdash (\forall \mathbf{x})\mathbf{x} = \mathbf{x}$.
>
> **Case 18:** $\mathbf{P}_{k+1}$ is introduced by Identity Elimination. Then $\mathbf{P}_{k+1}$ is derived as follows:
>
> $$\begin{array}{rl|ll} \mathbf{h} & \mathbf{t_1 = t_2} & & \\ \mathbf{j} & \mathbf{P} & & \\ \mathbf{k+1} & \mathbf{P(t_1//t_2)} & \mathbf{h, j} & =\mathrm{E} \end{array} \quad \text{or} \quad \begin{array}{rl|ll} \mathbf{h} & \mathbf{t_1 = t_2} & & \\ \mathbf{j} & \mathbf{P} & & \\ \mathbf{k+1} & \mathbf{P(t_2//t_1)} & \mathbf{h, j} & =\mathrm{E} \end{array}$$

where both Γ_h and Γ_j are subsets of Γ_{k+1} (because the sentences at positions **h** and **j** are accessible at position **k** + 1). By the inductive hypothesis, $\Gamma_h \vdash t_1 = t_2$ and $\Gamma_j \vdash P$. Because these are both subsets of Γ_{k+1}, it follows, by 11.3.2, that $\Gamma_{k+1} \vdash t_1 = t_2$ and $\Gamma_{k+1} \vdash P$.

Let **I** be any interpretation on which all the members of Γ_{k+1} are true. Then $t_1 = t_2$ and **P** are both true because they are quantificationally entailed by Γ_{k+1}. It follows from 11.2.4, which we repeat here:

11.2.4: For any closed terms t_1 and t_2, if **P** is a sentence that contains t_1, then $\{t_1 = t_2, P\} \vdash P(t_2//t_1)$, and if **P** is a sentence that contains t_2, then $\{t_1 = t_2, P\} \vdash P(t_1//t_2)$.

that the sentence at position **k** + 1 is true as well. So $\Gamma_{k+1} \vdash P_{k+1}$.

These changes establish that *PDE* is sound for predicate logic with identity and functions; like *PD* and *PD+*, *PDE* never leads from true premises to a false conclusion.

11.3E EXERCISES

1. Using Metatheorem 11.3.1, prove the following:
   a. Every argument of *PL* that is valid in *PD* is quantificationally valid.
   b. Every sentence of *PL* that is a theorem in *PD* is quantificationally true.
   *c. Every pair of sentences **P** and **Q** of *PD* that are equivalent in *PD* are quantificationally equivalent.

2. Prove the following (to be used in Exercise 3) by mathematical induction:

   11.3.4. Let **P** be a formula of *PL* and **Q** a subformula of **P**. Let $[P](Q_1//Q)$ be a sentence that is the result of replacing one or more occurrences of **Q** in **P** with a formula Q_1. If **Q** and Q_1 contain the same nonlogical symbols and variables, and if on any interpretation **Q** and Q_1 are satisfied by exactly the same variable assignments, then on any interpretation **P** and $[P](Q_1//Q)$ are satisfied by exactly the same variable assignments.

3. Using 11.3.4, show how we can establish, as a step in an inductive proof of the soundness of *PD+*, that Quantifier Negation is truth-preserving for predicate logic.

*4.a. Suppose that the rule ∀I did not have the restriction that the instantiating constant **a** in the sentence $P(a/x)$ to which ∀I applies must not occur in any open assumption. Explain why *PD* would *not* be sound for predicate logic in this case.
   b. Suppose that the rule ∃E did not have the restriction that the instantiating constant **a** in the assumption $P(a/x)$ must not occur in the sentence **Q** that is derived. Explain why *PD* would *not* be sound for predicate logic in this case.

11.4 THE COMPLETENESS OF *PD*, *PD*+, AND *PDE*

In this section we shall prove that our natural deduction systems are **complete** for predicate logic. A natural deduction system is complete for predicate logic if, whenever a sentence is quantificationally entailed by a set of sentences, there is at least one derivation of the sentence from members of that set in the natural deduction system. Metatheorem 11.4.1 is the *Completeness Metatheorem* for *PD*.

Metatheorem 11.4.1: If $\Gamma \models \mathbf{P}$, then $\Gamma \vdash \mathbf{P}$ in *PD*.

We shall prove the Completeness Metatheorem for *PD* in a manner analogous to that in which the completeness of *SD* for sentential logic was shown in Chapter 6. We note that the Completeness Metatheorem follows almost immediately from the Inconsistency Lemma for predicate logic:

> **11.4.2** (the *Inconsistency Lemma*): If a set Γ of sentences of *PL* is quantificationally inconsistent, then Γ is also inconsistent in *PD*.

To see how Metatheorem 11.4.1 follows, assume that, for some set Γ and sentence $\mathbf{P}$, $\Gamma \models \mathbf{P}$ (this is the antecedent of the metatheorem). Then the set $\Gamma \cup \{\sim \mathbf{P}\}$ is quantificationally inconsistent (see Exercise 1). It follows, from Lemma 11.4.2, that $\Gamma \cup \{\sim \mathbf{P}\}$ is also inconsistent in *PD*. And from this it follows that $\Gamma \vdash \mathbf{P}$ in *PD* (see Exercise 2).

So the bulk of this section is devoted to proving Lemma 11.4.2. We shall do so by proving its contrapositive:

> If a set Γ of sentences of *PL* is consistent in *PD*, then Γ is quantificationally consistent.

If the contrapositive is true, then we may conclude of any set that is quantificationally *in*consistent that it is also inconsistent in *PD*. The proof of the contrapositive is in four steps:

> *Step 1 in proof of Lemma 11.4.2:* We shall prove in result 11.4.3 that, for any set Γ that is consistent in *PD*, if we double the subscript of every individual constant in Γ (so that every resulting subscript will be even), then the resulting set Γ_e is also consistent in *PD*. We call such a set an **evenly subscripted set.**
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> *Step 2 in proof of Lemma 11.4.2:* We shall then show that, because there are infinitely many individual constants (namely, all the oddly subscripted constants) that do not occur in the sentences of any evenly subscripted set, every evenly subscripted set Γ that is

consistent in *PD* is a subset of a set that is *maximally consistent in PD* and that is ∃-*complete*. This will be established as result 11.4.4, the Maximal Consistency Lemma for predicate logic. Maximal consistency is defined as it was for *SD*:

> A set Γ of sentences of *PL* is **maximally consistent in *PD*** if and only if Γ is consistent in *PD*, and for every sentence **P** of *PD* that is not a member of Γ, Γ ∪ {**P**} is inconsistent in *PD*.

If a set is maximally consistent in *PD*, then adding even one new sentence to the set makes it inconsistent in *PD*. ∃-completeness (read aloud as *existential completeness*) is a new concept, but a simple one:

> A set Γ of sentences of *PL* is **∃-complete** if and only if, for each sentence in Γ that has the form (∃**x**)**P**, at least one substitution instance of (∃**x**)**P** is also a member of Γ.

If, for example, '(∃**x**)G**x**' is a member of a set that is ∃-complete, then at least one of 'G**a**', 'G**b**', 'G**c**', . . . is also a member of the set.

Step 3 in proof of Lemma 11.4.2: We shall next show that there is a straightforward way to construct a model for every set that is maximally consistent in *PD* and that is ∃-complete, from which it follows that any such set is quantificationally consistent and hence that all of its subsets are also quantificationally consistent. This will be established as result 11.4.8, the Consistency Lemma for predicate logic. It follows from the Consistency Lemma that the evenly subscripted set from which we built the maximally consistent set in Step 2 must be quantificationally consistent.

Step 4 in proof of Lemma 11.4.2: Finally we shall show, in result 11.4.9, that the set that we began with, whose subscripts were doubled in Step 1, must be quantificationally consistent as well.

It follows from Steps 1–4 that every set of sentences that is consistent in *PD* is also quantificationally consistent and therefore that the contrapositive of this statement, the Consistency Lemma 11.4.2, is true.

So let us turn to result 11.4.3, which will establish *Step 1*:

> **11.4.3:** Let Γ be a set of sentences of *PL* and let Γ_e be the set that results from doubling the subscript of every individual constant that occurs in any member of Γ. Then if Γ is consistent in *PD*, Γ_e is also consistent in *PD*.

Proof: Assume that Γ is consistent in *PD* and that, contrary to what we wish to prove, Γ_* is *in*consistent in *PD*. Then there is a derivation of the sort

1	$\mathbf{P}_1$
2	$\mathbf{P}_2$
.	.
n	$\mathbf{P}_n$
.	.
k	$\mathbf{Q}$
.	.
p	$\sim\mathbf{Q}$

where $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$ are members of Γ_*. We shall convert this derivation into a derivation that shows that Γ is inconsistent in *PD*, contradicting our first assumption. Our strategy, not surprisingly, will be to halve the subscript of every evenly subscripted individual constant occurring in the derivation, thus converting each of $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$ back to a member of the original Γ. There is a complication, though—in so doing we may end up with a sequence in which either an $\exists$E restriction or an $\forall$I restriction is violated. If, for example, $\mathbf{P}_1$ in the derivation above is 'Ba_2', and later in the derivation the sentence '$(\forall x)\mathrm{L}x$' is legally derived from the sentence 'La_1' by $\forall$I (note that an odd subscript *may* occur in the above derivation *after* the primary assumptions), then in changing $\mathbf{P}_1$ to 'Ba_1' we shall have introduced an individual constant into a primary assumption that *prevents* the later use of $\forall$I. Our first step is thus to ensure that this will not happen.

When the rule $\exists$E is used to justify a sentence, let the constant $\mathbf{a}$ that is the instantiating constant in the substitution instance $\mathbf{P}(\mathbf{a}/\mathbf{x})$ that begins the cited subderivation be called the *instantiating constant for that use of* $\exists E$. Similarly the *instantiating constant for a use of* $\forall I$ is the instantiating constant $\mathbf{a}$ in the sentence $\mathbf{P}(\mathbf{a}/\mathbf{x})$ cited. Let $\mathbf{a}_1, \ldots, \mathbf{a}_m$ be the distinct constants that are instantiating constants for uses of $\exists$E and $\forall$I in the above derivation. Let $\mathbf{b}_1, \ldots, \mathbf{b}_m$ be distinct constants that have odd subscripts that are larger than the subscript of any constant occurring in the derivation. (Because every derivation is a finite sequence, we know that of the constants occurring in our derivation there is one that has the largest subscript—and, whatever this largest subscript may be, there are infinitely many odd numbers that are larger.) We replace each sentence $\mathbf{R}$ in the original derivation with a sentence $\mathbf{R}^*$ that is the result of first replacing each occurrence of $\mathbf{a}_i$ in $\mathbf{R}$, $1 \leq i \leq m$, with $\mathbf{b}_i$, and then halving every even subscript in a constant in the resulting sentence.

We claim that the resulting sequence is a derivation in *PD* of $\mathbf{Q}^*$ and $\sim\mathbf{Q}^*$ from members of the set Γ. First note that, for every new primary assumption $\mathbf{P}_i^*$, $\mathbf{P}_i^*$ is a member of Γ. This is because none of

$\mathbf{a}_1, \ldots, \mathbf{a}_m$ can occur in a primary assumption of the original derivation (lest an instantiating constant restriction be violated—for these are the instantiating constants for uses of ∃E and ∀I in that derivation). So $\mathbf{P}^*$ is just $\mathbf{P}_i$ with all its individual constant subscripts halved—that is, a member of the set Γ from which Γ_n was constructed by doubling subscripts. It remains to be shown that the resulting sequence counts as a derivation in *PD*—that every sentence in that sequence can be justified. This is left as an exercise.

Step 2 in our proof of Lemma 11.4.2 is contained in the Maximal Consistency Lemma for predicate logic:

> **11.4.4** (the *Maximal Consistency Lemma*): If Γ is an evenly subscripted set of sentences that is consistent in *PD*, then Γ is a subset of at least one set of sentences that is both maximally consistent in *PD* and ∃-complete.

We shall establish this lemma by showing how, beginning with Γ, to construct a superset that has the two properties. We assume that the sentences of *PL* have been enumerated, that is, that they have been placed in a one-to-one correspondence with the positive integers so that there is a first sentence, a second sentence, a third sentence, and so on. The enumeration can be done analogously to the enumeration of the sentences of *SL* in Section 6.4; we leave proof of this as an exercise (Exercise 4). We shall now build a sequence of sets Γ_1, Γ_2, Γ_3, . . . starting with an evenly subscripted set Γ that is consistent in *PD* and considering each sentence in the enumeration, adding the sentence if it can consistently be added, and, if the added sentence is existentially quantified, adding one of its substitution instances as well. The sequence is constructed as follows:

1. Γ_1 is Γ.
2. Γ_{i+1} is
   - (i) $\Gamma_i \cup \{\mathbf{P}_i\}$, if $\Gamma_i \cup \{\mathbf{P}_i\}$ is consistent in *PD* and $\mathbf{P}_i$ does not have the form $(\exists\mathbf{x})\mathbf{P}$, or
   - (ii) $\Gamma_i \cup \{\mathbf{P}_i, \mathbf{P}_i^*\}$, if $\Gamma \cup \{\mathbf{P}_i\}$ is consistent in *PD* and $\mathbf{P}_i$ has the form $(\exists\mathbf{x})\mathbf{Q}$, where $\mathbf{P}_i^*$ is a substitution instance $\mathbf{Q}(\mathbf{a}/\mathbf{x})$ of $(\exists\mathbf{x})\mathbf{Q}$ and $\mathbf{a}$ is the alphabetically earliest constant that does not occur in $\mathbf{P}_i$ or in any sentence in Γ_i, or
   - (iii) Γ_i, if $\Gamma_i \cup \{\mathbf{P}_i\}$ is inconsistent in *PD*.

As an example of (ii), if Γ_i is the set $[(\forall x)(Fxa \supset Gx), \sim Hc \vee (\exists y)]yy]$ and $\mathbf{P}_i$ is '$(\exists z)(Kz \,\&\, (\forall y)Fzy)$', then $\Gamma_i \cup \{\mathbf{P}_i\}$ is quantificationally consistent, and so $\mathbf{P}_i$ will be added to the set—but we must add a substitution instance of $\mathbf{P}_i$ as well. The alphabetically earliest constant that does not occur in $\mathbf{P}_i$ or in any member of Γ_i is 'b', and so this will be the instantiating constant. Γ_{i+1} is

therefore

$$[(\forall x)(Fxa \supset Gx),\ \sim Hc \vee (\exists y)Jyy,\ (\exists z)(Kz\ \&\ (\forall y)Fzy),\ Kh\ \&\ (\forall y)Fby]$$

The reason for using an instantiating constant that does not already occur in Γ_i will become clear shortly when we prove that each set in the sequence is consistent in *PD*. Here it is important to note that, for any set in the sequence, there is always at least one individual constant that does not already occur in that set. This is because the set that we started with is evenly subscripted, and so we know that infinitely many oddly subscripted individual constants do not occur in Γ; and each subsequent set adds at most one new individual constant, still leaving infinitely many individual constants yet to be used. Thus the requirement in condition (ii)—that **a** be a *new* constant—is a requirement that can always be satisfied.

Because the sequence $\Gamma_1, \Gamma_2, \Gamma_3, \ldots$ is infinitely long, there is no last member in the set. We want a set that contains all the sentences in these sets, so we let Γ^* be the set that contains every sentence that occurs in some set in the infinite sequence $\Gamma_1, \Gamma_2, \Gamma_3, \ldots$. We shall show that Γ^* is both maximally consistent in *PD* and $\exists$-complete. To show that Γ^* is maximally consistent in *PD*, we first prove that each set Γ_i in the sequence is consistent in *PD*, using mathematical induction.

Basis clause: Γ_1 is consistent in *PD*.

Proof of basis clause: By definition, Γ_1 is Γ, a set that is consistent in *PD*.

Inductive step: If for every $\mathbf{i} \leq \mathbf{k}$, Γ_i is consistent in *PD*, then $\Gamma_{\mathbf{k}+1}$ is consistent in *PD*.

Proof of inductive step: If $\Gamma_{\mathbf{k}+1}$ is formed in accordance with condition (i), then $\Gamma_{\mathbf{k}+1}$ is obviously consistent in *PD*. If $\Gamma_{\mathbf{k}+1}$ is formed in accordance with condition (ii), then $\Gamma_k \cup \{\mathbf{P}_k\}$ is consistent—as stipulated by condition (ii). We need to show that it follows that $\Gamma_k \cup \{\mathbf{P}_k, \mathbf{P}_k^*\}$, which is what $\Gamma_{\mathbf{k}+1}$ was defined to be in this case, is also consistent in *PD*. Because the instantiating constant in $\mathbf{P}_k^*$ does not occur in any member of $\Gamma_k \cup \{\mathbf{P}_k\}$, the consistency of $\Gamma_{\mathbf{k}+1}$ follows immediately from result 11.4.5, the proof of which is left as an exercise.

11.4.5. If **a** does not occur in any member of the set $\Gamma \cup \{(\exists \mathbf{x})\mathbf{P}\}$, and if the set is consistent in *PD*, then the set $\Gamma \cup \{(\exists \mathbf{x})\mathbf{P}, \mathbf{P}(\mathbf{a}/\mathbf{x})\}$ is also consistent in *PD*.

(This is why condition (ii) stipulated that **a** be a new constant—the fact that **a** does not occur in the set $\Gamma \cup \{(\exists \mathbf{x})\mathbf{P}\}$ is crucial to the proof of the inductive step.) Finally, if $\Gamma_{\mathbf{k}+1}$ is formed in accordance with condition (iii), then $\Gamma_{\mathbf{k}+1}$ is Γ_k, and Γ_k is, by the inductive hypothesis, consistent in *PD*. So, no matter which condition was applied in its construction, $\Gamma_{\mathbf{k}+1}$ is consistent in *PD*.

Having established both premises, we conclude that every set in the sequence $\Gamma_1, \Gamma_2, \Gamma_3, \ldots$ is consistent in *PD*.

We now need to show that the set Γ^*, which contains all the sentences that occur in any set in the sequence, is itself consistent in *PD*. We shall show this by assuming that it is not consistent in *PD* and deriving a contradiction. So assume that Γ^* is not consistent in *PD*. Then there is a finite nonempty subset Γ' of Γ^* that is inconsistent in *PD* (the proof is analogous to that in the proof of 6.4.6). Because Γ' is finite, some sentence in Γ', say, $\mathbf{P}_j$, occurs later in our enumeration of the sentences of *PL* than any other sentence in Γ'. Every member of Γ' is thus a member of Γ_{j+1}, by the way we constructed the sets in the sequence. (If the ith sentence is added to one of the sets, it is added by the time that Γ_{i+1} is constructed.) It follows that Γ_{j+1} is also inconsistent in *PD* (the proof is analogous to the proof of 6.4.7). But we have just proved that every set in the sequence is consistent in *PD*, so we conclude that, contrary to our assumption, Γ^* is also consistent in *PD*.

That Γ^* is *maximally* consistent in *PD* is proved in exactly the manner that the parallel result in Section 6.4 was proved—for any sentence $\mathbf{P}_k$, if $\Gamma^* \cup \{\mathbf{P}_k\}$ is consistent in *PD*, then the subset Γ_k of Γ^* is such that $\Gamma_k \cup \{\mathbf{P}_k\}$ is consistent in *PD*, and so by the construction of the sequence of sets, $\mathbf{P}_k$ is a member of Γ_{k+1} and hence of Γ^*.

Finally the proof that Γ^* is $\exists$-complete is left as an exercise. This completes the proof of the Maximal Consistency Lemma 11.4.4—because every member of the original set Γ is also a member of Γ^*, it follows that every evenly subscripted set Γ of sentences of *PL* that is consistent in *PD* is a subset of at least one set of sentences that is both maximally consistent in *PD* and $\exists$-complete.

We now turn to *Step 3* in our proof of the Inconsistency Lemma 11.4.2. We must prove that every set Γ that is both maximally consistent in *PD* and $\exists$-complete is consistent in *PD*. To do this we need the following preliminary results:

11.4.6: If $\Gamma \vdash \mathbf{P}$ and Γ is a subset of a set Γ^* that is maximally consistent in *PD*, then $\mathbf{P} \in \Gamma^*$.

Proof: See Exercise 9.

11.4.7: Every set Γ^* of sentences that is both maximally consistent in *PD* and $\exists$-complete has the following properties:

a. $\mathbf{P} \in \Gamma^*$ if and only if $-\mathbf{P} \notin \Gamma^*$.
b. $\mathbf{P} \,\&\, \mathbf{Q} \in \Gamma^*$ if and only if $\mathbf{P} \in \Gamma^*$ and $\mathbf{Q} \in \Gamma^*$.
c. $\mathbf{P} \vee \mathbf{Q} \in \Gamma^*$ if and only if either $\mathbf{P} \in \Gamma^*$ or $\mathbf{Q} \in \Gamma^*$.
d. $\mathbf{P} \supset \mathbf{Q} \in \Gamma^*$ if and only if either $\mathbf{P} \notin \Gamma^*$ or $\mathbf{Q} \in \Gamma^*$.
e. $\mathbf{P} \equiv \mathbf{Q} \in \Gamma^*$ if and only if either $\mathbf{P} \in \Gamma^*$ and $\mathbf{Q} \in \Gamma^*$ or $\mathbf{P} \notin \Gamma^*$ and $\mathbf{Q} \notin \Gamma^*$.
f. $(\forall \mathbf{x})\mathbf{P} \in \Gamma^*$ if and only if, for every individual constant $\mathbf{a}$, $\mathbf{P}(\mathbf{a}/\mathbf{x}) \in \Gamma^*$.
g. $(\exists \mathbf{x})\mathbf{P} \in \Gamma^*$ if and only if, for at least one individual constant $\mathbf{a}$, $\mathbf{P}(\mathbf{a}/\mathbf{x}) \in \Gamma^*$.

Proof: The proofs that (a)–(e) hold for sets of sentences that are maximally consistent in *PD* and ∃-complete parallel exactly the corresponding proofs in Section 6.4, using result 11.4.6 instead of 6.4.5. (In those proofs we did not appeal to a property like ∃-completeness, and we do not need to appeal to it here in establishing (a)–(e).)

Proof of (f): Assume that $(\forall \mathbf{x})\mathbf{P} \in \Gamma^*$. For any substitution instance $\mathbf{P}(\mathbf{a}/\mathbf{x})$ of $(\forall \mathbf{x})\mathbf{P}$, $\{(\forall \mathbf{x})\mathbf{P}\} \vdash \mathbf{P}(\mathbf{a}/\mathbf{x})$ (by ∀E); so, by 11.4.6, every substitution instance is a member of Γ^* as well. Now assume that $(\forall \mathbf{x})\mathbf{P} \notin \Gamma^*$. Then $\sim (\forall \mathbf{x})\mathbf{P} \in \Gamma^*$, by (a). The following derivation shows that $\{\sim (\forall \mathbf{x})\mathbf{P}\} \vdash (\exists \mathbf{x}) \sim \mathbf{P}$:

1	$\sim (\forall \mathbf{x})\mathbf{P}$	Assumption
2	$\quad \sim (\exists \mathbf{x}) \sim \mathbf{P}$	A /∼ E
3	$\quad\quad \sim \mathbf{P}(\mathbf{a}/\mathbf{x})$	A /∼ E
4	$\quad\quad (\exists \mathbf{x}) \sim \mathbf{P}$	3 ∃I
5	$\quad\quad \sim (\exists \mathbf{x}) \sim \mathbf{P}$	2 R
6	$\quad \mathbf{P}(\mathbf{a}/\mathbf{x})$	3–5 ∼ E
7	$\quad (\forall \mathbf{x})\mathbf{P}$	6 ∀I
8	$\quad \sim (\forall \mathbf{x})\mathbf{P}$	1 R
9	$(\exists \mathbf{x}) \sim \mathbf{P}$	2–8 ∼ E

(We assume that the constant **a** does not occur in **P**.) Therefore, by 11.4.6, $(\exists \mathbf{x}) \sim \mathbf{P}$ is also a member of Γ^*. Because Γ^* is ∃-complete, some substitution instance $\sim \mathbf{P}(\mathbf{a}/\mathbf{x})$ of $(\exists \mathbf{x}) \sim \mathbf{P}$ is a member of Γ^* as well, and it therefore follows from (a) that $\mathbf{P}(\mathbf{a}/\mathbf{x}) \notin \Gamma^*$. So, if $(\forall \mathbf{x})\mathbf{P} \notin \Gamma^*$, then there is at least one substitution instance of $(\forall \mathbf{x})\mathbf{P}$ that is not a member of Γ^*.

Proof of (g): Assume that $(\exists \mathbf{x})\mathbf{P} \in \Gamma^*$. Then, because Γ^* is ∃-complete, at least one substitution instance of $(\exists \mathbf{x})\mathbf{P}$ is also a member of Γ^*. Now assume that $(\exists \mathbf{x})\mathbf{P} \notin \Gamma^*$. If some substitution instance $\mathbf{P}(\mathbf{a}/\mathbf{x})$ of $(\exists \mathbf{x})\mathbf{P}$ is a member of Γ^*, then because $\{\mathbf{P}(\mathbf{a}/\mathbf{x})\} \vdash (\exists \mathbf{x})\mathbf{P}$ (by ∃I), it follows from 11.4.6 that, contrary to our assumption, $(\exists \mathbf{x})\mathbf{P}$ is also a member of Γ^*. So, if $(\exists \mathbf{x})\mathbf{P} \notin \Gamma^*$, then none of its substitution instances is a member of Γ^*.

We can now complete the proof of *Step 3* by establishing the Consistency Lemma for predicate logic:

11.4.8 (the *Consistency Lemma*): Every set of sentences of *PL* that is both maximally consistent in *PD* and ∃-complete is quantificationally consistent.

We shall prove the Consistency Lemma by showing how to construct a model for any set Γ^* that is both maximally consistent in *PD* and ∃-complete, that is, an interpretation $\mathbf{I}^*$ on which every member of Γ^* is true. We begin by associating with each individual constant a distinct positive integer—the positive

integer **i** will be associated with the alphabetically ith constant. The number 1 will be associated with 'a', 2 with 'b', . . . , 22 with 'v', 23 with 'a_1', and so on. $\mathbf{I}^*$ is then defined as follows:

1. The UD is the set of positive integers.
2. For each sentence letter **P**, $\mathbf{I}^*(\mathbf{P}) = \mathbf{T}$ if and only if $\mathbf{P} \in \Gamma^*$.
3. For each individual constant **a**, $\mathbf{I}^*(\mathbf{a})$ is the positive integer associated with **a**.
4. For each **n**-place predicate **A**, $\mathbf{I}^*(\mathbf{A})$ includes all and only those **n**-tuples $\langle \mathbf{I}^*(\mathbf{a}_1), \ldots, \mathbf{I}^*(\mathbf{a}_n) \rangle$ such that $\mathbf{A}\mathbf{a}_1 \ldots \mathbf{a}_n \in \Gamma^*$.

The major feature of this interpretation is that, for each atomic sentence **P** of *PL*, **P** will be true on $\mathbf{I}^*$ if and only if $\mathbf{P} \in \Gamma^*$. That is why we defined condition 4 (as well as condition 2) as we did. And to be sure that condition 4 can be met, we must have condition 3, which ensures that each individual constant designates a different member of the UD. This is necessary because, for example, if 'Fa' is a member of Γ^* and 'Fb' is not a member, then if 'a' and 'b' designated the same integer—say, 1—condition 4 would require that the 1-tuple $\langle 1 \rangle$ both be and not be in the extension of 'F'. (In addition, condition 3 ensures that every member of the UD is named by a constant, which we shall shortly see is also important when we look at the truth-values that quantified sentences receive on $\mathbf{I}^*$.) With all the atomic sentences in Γ^* true and all other atomic sentences false, it follows that truth-functionally compound and quantified sentences are true on $\mathbf{I}^*$ if and only if they are members of Γ^*.

We complete the proof of the Consistency Lemma by establishing, by mathematical induction on the number of occurrences of logical operators in sentences of *PL*, that each sentence **P** of *PL* is true on $\mathbf{I}^*$ if and only if $\mathbf{P} \in \Gamma^*$.

Basis clause: Each sentence **P** that contains zero occurrences of logical operators is true on $\mathbf{I}^*$ if and only if $\mathbf{P} \in \Gamma^*$.

Proof of basis clause: Either **P** is a sentence letter or **P** has the form $\mathbf{A}\mathbf{a}_1 \ldots \mathbf{a}_m$. If **P** is a sentence letter, then, by part 2 of the definition of $\mathbf{I}^*$, it follows that **P** is true on $\mathbf{I}^*$ if and only if $\mathbf{P} \in \Gamma^*$.

If **P** has the form $\mathbf{A}\mathbf{a}_1 \ldots \mathbf{a}_m$, then **P** is true on $\mathbf{I}^*$ if and only if $\langle \mathbf{I}^*(\mathbf{a}_1), \ldots, \mathbf{I}^*(\mathbf{a}_m) \rangle \in \mathbf{I}^*(\mathbf{A})$. Part 4 of the definition of $\mathbf{I}^*$ stipulates that $\langle \mathbf{I}^*(\mathbf{a}_1), \ldots, \mathbf{I}^*(\mathbf{a}_m) \rangle \in \mathbf{I}^*(\mathbf{A})$ if and only if $\mathbf{A}\mathbf{a}_1 \ldots \mathbf{a}_m \in \Gamma^*$. So in this case as well, **P** is true on $\mathbf{I}^*$ if and only if $\mathbf{P} \in \Gamma^*$.

Inductive step: If each sentence **P** that contains **k** or fewer occurrences of logical operators is such that **P** is true on $\mathbf{I}^*$ if and only if $\mathbf{P} \in \Gamma^*$, then each sentence **P** that contains $\mathbf{k} + 1$ occurrences of logical operators is such that **P** is true on $\mathbf{I}^*$ if and only if $\mathbf{P} \in \Gamma^*$.

Proof of inductive step: We assume that, for an arbitrary positive integer **k**, the inductive hypothesis is true. We must show that on this assumption it follows that any sentence **P** that contains $\mathbf{k} + 1$ occurrences

of logical operators is such that **P** is true on $\mathbf{I}^*$ if and only if $\mathbf{P} \in \Gamma^*$. We consider the forms that the sentence **P** may have.

Cases 1–5: **P** has one of the forms $\sim\mathbf{Q}$, $\mathbf{Q} \,\&\, \mathbf{R}$, $\mathbf{Q} \vee \mathbf{R}$, $\mathbf{Q} \supset \mathbf{R}$, or $\mathbf{Q} \equiv \mathbf{R}$. The proofs for these five cases are analogous to the proofs for the parallel cases for *SI*. in Section 6.4, so we omit them here.

Case 6: **P** has the form $(\forall\mathbf{x})\mathbf{Q}$. Assume that $(\forall\mathbf{x})\mathbf{Q}$ is true on $\mathbf{I}^*$. Then every substitution instance $\mathbf{Q}(\mathbf{a}/\mathbf{x})$ of $(\forall\mathbf{x})\mathbf{Q}$ is true on $\mathbf{I}^*$ because, by 11.1.4, $\{(\forall\mathbf{x})\mathbf{Q}\}$ quantificationally entails every one of its substitution instances. Each substitution instance contains fewer than $\mathbf{k} + 1$ occurrences of connectives, and so, by the inductive hypothesis, each substitution instance is a member of Γ^* since it is true on $\mathbf{I}^*$. It follows from part (f) of 11.4.7 that $(\forall\mathbf{x})\mathbf{Q}$ is also a member of Γ^*.

Now assume that $(\forall\mathbf{x})\mathbf{Q}$ is false on $\mathbf{I}^*$. In this case we shall make use of result 11.1.11, which we repeat here:

11.1.11: Let **I** be an interpretation on which each member of the UD is assigned to at least one individual constant. Then, if every substitution instance of $(\forall\mathbf{x})\mathbf{P}$ is true on **I**, so is $(\forall\mathbf{x})\mathbf{P}$.

$\mathbf{I}^*$ is an interpretation of the type specified in 11.1.11: Every positive integer in the UD is designated by the individual constant with which we have associated that integer. It follows, then, that if every substitution instance of $(\forall\mathbf{x})\mathbf{Q}$ is true on $\mathbf{I}^*$, then so is $(\forall\mathbf{x})\mathbf{Q}$. Therefore, if $(\forall\mathbf{x})\mathbf{Q}$ is false on $\mathbf{I}^*$, at least one of its substitution instances $\mathbf{Q}(\mathbf{a}/\mathbf{x})$ must also be false on $\mathbf{I}^*$. Because $\mathbf{Q}(\mathbf{a}/\mathbf{x})$ contains fewer than $\mathbf{k} + 1$ occurrences of logical operators, it follows from the inductive hypothesis that $\mathbf{Q}(\mathbf{a}/\mathbf{x}) \notin \Gamma^*$. And so, by part (f) of 11.4.7, $(\forall\mathbf{x})\mathbf{Q} \notin \Gamma^*$.

Case 7: **P** has the form $(\exists\mathbf{x})\mathbf{Q}$. Assume that $(\exists\mathbf{x})\mathbf{Q}$ is true on $\mathbf{I}^*$. Then it follows from 11.1.12, which we repeat here, that at least one substitution instance $\mathbf{Q}(\mathbf{a}/\mathbf{x})$ of $(\exists\mathbf{x})\mathbf{Q}$ is true on $\mathbf{I}^*$, for every member of the UD is designated by an individual constant.

11.1.12: Let **I** be an interpretation on which each member of the UD is assigned to at least one individual constant. Then, if every substitution instance of $(\exists\mathbf{x})\mathbf{P}$ is false on **I**, so is $(\exists\mathbf{x})\mathbf{P}$.

Because the substitution instance $\mathbf{Q}(\mathbf{a}/\mathbf{x})$ contains fewer than $\mathbf{k} + 1$ occurrences of logical operators, it follows from the inductive hypothesis that $\mathbf{Q}(\mathbf{a}/\mathbf{x}) \in \Gamma^*$. So, by part (g) of 11.4.7, $(\exists\mathbf{x})\mathbf{Q}$ is also a member of Γ^*.

Now assume that $(\exists\mathbf{x})\mathbf{Q}$ is false on $\mathbf{I}^*$. Because each substitution instance $\mathbf{Q}(\mathbf{a}/\mathbf{x})$ is such that $\{\mathbf{Q}(\mathbf{a}/\mathbf{x})\} \models (\exists\mathbf{x})\mathbf{Q}$ (this is result 11.1.5), it follows that every substitution instance $\mathbf{Q}(\mathbf{a}/\mathbf{x})$ is also false on $\mathbf{I}^*$. Each of these substitution instances contains fewer than $\mathbf{k} + 1$ occurrences of logical operators, and so it follows from the inductive hypothesis that

no substitution instance of $(\exists \mathbf{x})\mathbf{Q}$ is a member of Γ^*. Finally, by part (g) of 11.4.7, it follows that $(\exists \mathbf{x})\mathbf{Q}$ is not a member of Γ^* either.

That completes the proof of the inductive step, and we may now conclude that every sentence $\mathbf{P}$ of *PL* is such that it is true on $\mathbf{I}^*$ if and only if it is a member of Γ^*. So $\mathbf{I}^*$ is a model of Γ^*, and we conclude that Γ^* is quantificationally consistent. Lemma 11.4.8 is therefore true: Every set that is both maximally consistent in *PD* and $\exists$-complete is quantificationally consistent. Lemmas 11.4.4 and 11.4.8 together establish that every evenly subscripted set of sentences of *PL* that is consistent in *PD* is also quantificationally consistent.

Step 4 of the proof of the Inconsistency Lemma 11.4.2 is contained in result 11.4.9:

> **11.4.9:** Let Γ be a set of sentences of *PL* and let Γ_e be the set that results from doubling the subscript of every individual constant that occurs in any member of Γ. Then, if Γ_e is quantificationally consistent, Γ is quantificationally consistent as well.
>
> **Proof:** See Exercise 8.

We have now completed the four steps in the proof of the Inconsistency Lemma 11.4.2, so we may conclude that, if a set Γ of sentences of *PL* is quantificationally inconsistent, then Γ is also inconsistent in *PD*. And this establishes the completeness of *PD* for predicate logic. If $\Gamma \vDash \mathbf{P}$, then $\Gamma \cup \{\sim \mathbf{P}\}$ is quantificationally inconsistent. By the Inconsistency Lemma, $\Gamma \cup \{\sim \mathbf{P}\}$ is also inconsistent in *PD*, and hence $\Gamma \vdash \mathbf{P}$ in *PD*. And that is what Metatheorem 11.4.1, the Completeness Theorem for *PD*, states.

Because *PD* is complete for predicate logic, so is *PD*+. Every rule of *PD* is a rule of *PD*+, and so every derivation in *PD* is a derivation in *PD*+. So, if $\Gamma \vDash \mathbf{P}$, then $\Gamma \vdash \mathbf{P}$ in *PD*+ because we know, by Metatheorem 11.4.1, that $\Gamma \vdash \mathbf{P}$ in *PD*.

We also want to be sure that *PDE* is complete for predicate logic with identity and functions. The completeness proof for *PDE* is similar to the completeness proof for *PD*, but there are some important changes. Results 11.4.3 and 11.4.9 must now take into account sentences containing the identity predicate and complex terms; the necessary changes are left as an exercise. Maximal consistency is defined for *PDE* as it was for *PD*, while the definition of $\exists$-completeness must be modified slightly:

> A set Γ of sentences of *PLE* is $\exists$-complete if and only if, for each sentence in Γ that has the form $(\exists \mathbf{x})\mathbf{P}$, at least one substitution instance of $(\exists \mathbf{x})\mathbf{P}$ in which the instantiating individual term is a constant is also a member of Γ.

The proof of the Maximal Consistency Lemma 11.4.4 for *PDE*—that every evenly subscripted set of sentences that is consistent in *PDE* is a subset of a set

of sentences that is both maximally consistent in *PDE* and ∃-complete—is just like the proof for *PD* except that *PLE* and *PDE*, rather than *PL* and *PD*, are spoken of. However, the proof of the Consistency Lemma 11.4.8 is different because the model $\mathbf{I}^*$ that is constructed for a maximally consistent and ∃-complete set of sentences must be defined differently.

The interpretation $\mathbf{I}^*$ of the maximally consistent and ∃-complete set Γ^* that we constructed in the proof of the Consistency Lemma 11.4.8 stipulated that a distinct positive integer be associated with each individual constant and that

> 3. For each individual constant **a**, $\mathbf{I}^*(\mathbf{a})$ is the positive integer associated with **a**.

This will not do in the case of *PDE*, for suppose that Γ, and consequently its superset Γ^*, contains a sentence $\mathbf{a} = \mathbf{b}$, where **a** and **b** are different constants. If we interpret the constants of *PLE* in accordance with condition 3, **a** and **b** will denote *different* members of the UD, and hence $\mathbf{a} = \mathbf{b}$ will be false. But the interpretation is supposed to make all members of Γ^*, including $\mathbf{a} = \mathbf{b}$, true. So we shall have to change condition 3 to take care of the case where a sentence like $\mathbf{a} = \mathbf{b}$ is a member of the set Γ^*. We shall also have to interpret the functors in the language, and to do so in a way that makes sentences containing complex terms true if and only if those sentences are members of Γ.

Before turning to the construction of an interpretation for Γ^*, however, we first establish some facts about sets of sentences that are maximally consistent in *PDE* and ∃-complete. As the reader may easily verify, the properties listed in result 11.4.7 remain true for maximally consistent, ∃-complete sets of sentences of *PDE*. We add three additional properties to the list in result 11.4.7:

> h. For every closed term **t**, $\mathbf{t} = \mathbf{t} \in \Gamma^*$.
>
> **Proof:** Let **t** be any closed term. $\varnothing \vdash \mathbf{t} = \mathbf{t}$, by =I and ∀E. Because the empty set is a subset of Γ^*, it follows from 11.4.6 that $\mathbf{t} = \mathbf{t} \in \Gamma^*$.
>
> i. If a sentence $\mathbf{t}_1 = \mathbf{t}_2$, where $\mathbf{t}_1$ and $\mathbf{t}_2$ are closed terms, is a member of Γ^*, then
>
> a. If **Q** is a sentence in which $\mathbf{t}_1$ occurs, $\mathbf{Q} \in \Gamma^*$ if and only if every sentence $\mathbf{Q}(\mathbf{t}_2//\mathbf{t}_1)$ (every sentence obtained by replacing one or more occurrences of $\mathbf{t}_1$ in **Q** with $\mathbf{t}_2$) is a member of Γ^*.
>
> b. If **Q** is a sentence in which $\mathbf{t}_2$ occurs, $\mathbf{Q} \in \Gamma^*$ if and only if every sentence $\mathbf{Q}(\mathbf{t}_1//\mathbf{t}_2)$ is a member of Γ^*.
>
> **Proof:** Let $\mathbf{t}_1 = \mathbf{t}_2$ be a sentence that is a member of Γ^* and let **Q** be a sentence in which $\mathbf{t}_1$ occurs. Assume that $\mathbf{Q} \in \Gamma^*$. Every sentence $\mathbf{Q}(\mathbf{t}_2//\mathbf{t}_1)$ is derivable from the set $\{\mathbf{t}_1 = \mathbf{t}_2, \mathbf{Q}\}$ by =E. Therefore, by 11.4.6, every sentence $\mathbf{Q}(\mathbf{t}_2//\mathbf{t}_1)$ is a member of Γ^*. Now assume that $\mathbf{Q} \notin \Gamma^*$. Every sentence $\mathbf{Q}(\mathbf{t}_2//\mathbf{t}_1)$ is such that $\{\mathbf{t}_1 = \mathbf{t}_2, \mathbf{Q}(\mathbf{t}_2//\mathbf{t}_1)\} \vdash \mathbf{Q}$,

by =E—use $\mathbf{t}_1$ to replace every occurrence of $\mathbf{t}_2$ that replaced $\mathbf{t}_1$ in $\mathbf{Q}(\mathbf{t}_2//\mathbf{t}_1)$, and the result is $\mathbf{Q}$ once again. So, if any sentence $\mathbf{Q}(\mathbf{t}_2//\mathbf{t}_1)$ is in Γ^*, then, by 11.4.6, $\mathbf{Q}$ must be as well. Therefore, if $\mathbf{Q} \notin \Gamma^*$, then no sentence $\mathbf{Q}(\mathbf{t}_2//\mathbf{t}_1)$ is a member of Γ^*.

Similar reasoning shows that, if $\mathbf{t}_1 = \mathbf{t}_2 \in \Gamma^*$ and $\mathbf{Q}$ is a sentence in which $\mathbf{t}_2$ occurs, then $\mathbf{Q} \in \Gamma^*$ if and only if every sentence $\mathbf{Q}(\mathbf{t}_1//\mathbf{t}_2)$ is a member of Γ^*.

j. For each $\mathbf{n}$-place functor f and $\mathbf{n}$ terms $\mathbf{t}_1, \ldots, \mathbf{t}_n$, there is at least one constant $\mathbf{b}$ such that the formula $f(\mathbf{t}_1, \ldots, \mathbf{t}_n) = \mathbf{b}$ is a member of Γ^*.

Proof: By property (h), $f(\mathbf{t}_1, \ldots, \mathbf{t}_n) = f(\mathbf{t}_1, \ldots, \mathbf{t}_n) \in \Gamma^*$. Since $f(\mathbf{t}_1, \ldots, \mathbf{t}_n) = f(\mathbf{t}_1, \ldots, \mathbf{t}_n) \vdash (\exists \mathbf{x}) f(\mathbf{t}_1, \ldots, \mathbf{t}_n) = \mathbf{x}$, the sentence $(\exists \mathbf{x}) f(\mathbf{t}_1, \ldots, \mathbf{t}_n) = \mathbf{x}$ must also be a member of Γ^*, by 11.4.6. And because Γ^* is $\exists$-complete, it follows (from our revised definition of $\exists$-completeness) that there is at least one constant $\mathbf{b}$ such that the formula $f(\mathbf{t}_1, \ldots, \mathbf{t}_n) = \mathbf{b}$ is also a member of Γ^*.

We now turn to the proof of the Consistency Lemma 11.4.8 for *PDE*—that every set of sentences of *PLE* that is both maximally consistent in *PDE* and $\exists$-complete is also quantificationally consistent. Let Γ^* be a set of sentences that is both maximally consistent in *PLE* and $\exists$-complete. We associate positive integers with the individual constants of *PLE* as follows:

First associate the positive integer $\mathbf{i}$ with the alphabetically $\mathbf{i}$th individual constant of *PLE*. Let $\mathbf{p}$ designate this association and let $\mathbf{p}(\mathbf{a})$ stand for the integer that has been associated with the constant $\mathbf{a}$. Thus $\mathbf{p}$('a') is 1, $\mathbf{p}$('b') is 2, and so on.

Now we define a second association, which we shall designate with $\mathbf{q}$. For each constant $\mathbf{a}$, $\mathbf{q}(\mathbf{a}) = \mathbf{p}(\mathbf{a}')$, where $\mathbf{a}'$ is the alphabetically earliest constant such that $\mathbf{a} = \mathbf{a}'$ is a member of Γ^*.

Note that for each constant $\mathbf{a}$ property (h) of maximally consistent, $\exists$-complete sets assures us that $\mathbf{a} = \mathbf{a} \in \Gamma^*$, and so we can be certain that $\mathbf{q}$ assigns a value to $\mathbf{a}$ because there is always at least one $\mathbf{a}'$ such that $\mathbf{a} = \mathbf{a}' \in \Gamma^*$. According to the definition, $\mathbf{q}$('a') is always 1 since property (h) assures us that 'a = a' is a member of Γ^*, and because 'a' is the alphabetically earliest constant of *PLE*, there can be no earlier constant that stands to the right of the identity predicate in a sentence containing 'a' to the left. But for any other constant, the value that it receives from $\mathbf{q}$ depends on the identity sentences that the particular set Γ^* contains. Suppose that 'b = a', 'b = b', 'b = e', and 'b = m_{22}' are the only identity sentences in Γ^* that contain 'b' to the left of the identity predicate. In this case there is an alphabetically earlier constant to the right, namely, 'a', and this is the alphabetically earliest constant so occurring. So $\mathbf{q}$('b') = $\mathbf{p}$('a') = 1. If 'c = c', 'c = f', and 'c = g_3' are the only identity

sentences in Γ^* that contain 'c' to the left of the identity predicate, then 'c' is the alphabetically earliest constant occurring to the right, and so $\mathbf{q}(\text{'c'}) = \mathbf{p}(\text{'c'}) = 3$. The definition of $\mathbf{q}$ plays a role in ensuring that identity sentences come out true on the interpretation that we shall construct if and only if they are members of Γ^*, as a consequence of

11.4.10: For any constants $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{q}(\mathbf{a}) = \mathbf{q}(\mathbf{b})$ if and only if $\mathbf{a} = \mathbf{b} \in \Gamma^*$.

Proof: Let $\mathbf{a}'$ be the alphabetically earliest constant such that $\mathbf{a} = \mathbf{a}' \in \Gamma^*$. (Remember that property (h) guarantees that there is at least one such constant.) Then

a. $\mathbf{q}(\mathbf{a}) = \mathbf{p}(\mathbf{a}')$.

Let $\mathbf{b}'$ be the alphabetically earliest constant such that $\mathbf{b} = \mathbf{b}' \in \Gamma^*$. Then

b. $\mathbf{q}(\mathbf{b}) = \mathbf{q}(\mathbf{b}')$.

It follows from (a) and (b) that

c. $\mathbf{q}(\mathbf{a}) = \mathbf{q}(\mathbf{b})$ if and only if $\mathbf{q}(\mathbf{a}') = \mathbf{p}(\mathbf{b}')$.

And because $\mathbf{p}$ associates different values with different constants,

d. $\mathbf{p}(\mathbf{a}') = \mathbf{p}(\mathbf{b}')$ if and only if $\mathbf{a}'$ and $\mathbf{b}'$ are the same constant.

From (c) and (d) we conclude that

e. $\mathbf{q}(\mathbf{a}) = \mathbf{q}(\mathbf{b})$ if and only if $\mathbf{a}'$ and $\mathbf{b}'$ are the same constant.

Assume that $\mathbf{q}(\mathbf{a}) = \mathbf{q}(\mathbf{b})$. It follows from (e) that $\mathbf{a}'$ and $\mathbf{b}'$ are the same constant. Therefore, because $\mathbf{b} = \mathbf{b}' \in \Gamma^*$, it follows trivially that $\mathbf{b} = \mathbf{a}'$, which is the same sentence, is a member of Γ^*. And because $\mathbf{a} = \mathbf{a}' \in \Gamma^*$, it follows from property (i) of maximally consistent, $\exists$-complete sets that $\mathbf{a} = \mathbf{b} \in \Gamma^*$ ($\mathbf{a} = \mathbf{b}$ is a sentence $\mathbf{a} = \mathbf{a}'$ ($\mathbf{b}//\mathbf{a}'$)).

Now assume that $\mathbf{a} = \mathbf{b} \in \Gamma^*$. Then, because $\mathbf{a} = \mathbf{a}' \in \Gamma^*$, it follows from property (i) that $\mathbf{b} = \mathbf{a}' \in \Gamma^*$, and because $\mathbf{b} = \mathbf{b}' \in \Gamma^*$ as well, it also follows from property (i) that $\mathbf{a} = \mathbf{b}' \in \Gamma^*$. $\mathbf{a}'$ was defined to be the alphabetically earliest constant that appears to the right of the identity predicate in an identity statement containing $\mathbf{a}$, and so from the fact that $\mathbf{a} = \mathbf{b}' \in \Gamma^*$ we conclude that $\mathbf{b}'$ is not alphabetically earlier than $\mathbf{a}'$. $\mathbf{b}'$ was defined to be the alphabetically earliest constant that appears to the right of the identity predicate in an identity statement containing $\mathbf{b}$, and so from the fact that $\mathbf{b} = \mathbf{a}' \in \Gamma^*$ we conclude that $\mathbf{a}'$ is not alphabetically earlier than $\mathbf{b}'$. These two observations establish

that $\mathbf{a}'$ and $\mathbf{b}'$ must be the same constant. So, from (e), we may conclude that $\mathbf{q}(\mathbf{a}) = \mathbf{q}(\mathbf{b})$.

Result 11.4.10 guarantees that if there is an identity statement in Γ^* that contains the individual constants $\mathbf{a}$ and $\mathbf{b}$ then $\mathbf{q}(\mathbf{a}) = \mathbf{q}(\mathbf{b})$, and if there is no identity statement in Γ^* that contains $\mathbf{a}$ and $\mathbf{b}$ then $\mathbf{q}(\mathbf{a}) \neq \mathbf{q}(\mathbf{b})$. And this fact will be crucial in our construction of an interpretation on which every member of a set that is both maximally consistent in *PDE* and $\exists$-complete is true. We turn now to the construction.

Let Γ^* be a set that is both maximally consistent in *PDE* and $\exists$-complete, and define the interpretation $\mathbf{I}^*$ as follows:

1. The UD is the set of positive integers that $\mathbf{q}$ associates with at least one individual constant of *PLE*.
2. For each sentence letter $\mathbf{P}$, $\mathbf{I}^*(\mathbf{P}) = \mathbf{T}$ if and only if $\mathbf{P} \in \Gamma^*$.
3. For each individual constant $\mathbf{a}$, $\mathbf{I}^*(\mathbf{a}) = \mathbf{q}(\mathbf{a})$.
4. For each $\mathbf{n}$-place functor f, $\mathbf{I}^*(f)$ is the set that includes all and only those $\mathbf{n} + 1$-tuples $\langle \mathbf{I}^*(\mathbf{a}_1), \ldots, \mathbf{I}^*(\mathbf{a}_n), \mathbf{I}^*(\mathbf{b})\rangle$, where $\mathbf{a}_1, \ldots, \mathbf{a}_n$ and $\mathbf{b}$ are individual constants such that $f(\mathbf{a}_1, \ldots, \mathbf{a}_n) = \mathbf{b} \in \Gamma^*$.
5. For each $\mathbf{n}$-place predicate $\mathbf{A}$ other than the identity predicate, $\mathbf{I}^*(\mathbf{A})$ is the set that includes all and only those $\mathbf{n}$-tuples $\langle \mathbf{I}^*(\mathbf{a}_1), \ldots, \mathbf{I}^*(\mathbf{a}_n)\rangle$ such that $\mathbf{A}\mathbf{a}_1 \ldots \mathbf{a}_n \in \Gamma^*$.

We must ensure that conditions 4 and 5 can be met.

For condition 4 we must ensure that the interpretation of f is indeed a function on the UD: that for each $\mathbf{n}$ members $\mathbf{u}_1, \ldots, \mathbf{u}_n$ of the UD there is exactly one member $\mathbf{u}_{n+1}$ of the UD such that $\langle \mathbf{u}_1, \ldots, \mathbf{u}_n, \mathbf{u}_{n+1}\rangle$ is a member of $\mathbf{I}^*(f)$. That there is *at least one* such member of the UD follows from the fact that every member of the UD is denoted by at least one individual constant (this is guaranteed by condition 1 of our definition of $\mathbf{I}^*$), and property (j) of sets that are maximally consistent in *PDE* and $\exists$-complete, which we repeat here:

j. For each $\mathbf{n}$-place functor f and $\mathbf{n}$ constants $\mathbf{a}_1, \ldots, \mathbf{a}_n$, there is at least one constant $\mathbf{b}$ such that the formula $f(\mathbf{a}_1, \ldots, \mathbf{a}_n) = \mathbf{b}$ is a member of Γ^*.

Given these two facts, condition 4 ensures that for each $\mathbf{n}$ members $\mathbf{u}_1, \ldots, \mathbf{u}_n$ of the UD there is at least one member $\mathbf{u}_{n+1}$ of the UD such that $\langle \mathbf{u}_1, \ldots, \mathbf{u}_n, \mathbf{u}_{n+1}\rangle$ is a member of $\mathbf{I}^*(f)$, for any functor f. To show that there is at most one such member $\mathbf{u}_{n+1}$, let us assume, to the contrary, that there is also a member of the UD $\mathbf{u}'_{n+1}$, where $\mathbf{u}'_{n+1} \neq \mathbf{u}_{n+1}$, such that $\langle \mathbf{u}_1, \ldots, \mathbf{u}_n, \mathbf{u}'_{n+1}\rangle$ is a member of $\mathbf{I}^*(f)$. This means that, in addition to the sentence $f(\mathbf{a}_1, \ldots, \mathbf{a}_n) = \mathbf{b}$, Γ includes a sentence $f(\mathbf{a}_1, \ldots, \mathbf{a}_n) = \mathbf{c}$, such that $\mathbf{I}^*(\mathbf{c}) = \mathbf{u}'_{n+1} \neq \mathbf{I}^*(\mathbf{b})$. Then $\mathbf{q}(\mathbf{a}) \neq \mathbf{q}(\mathbf{b})$ by virtue of clause 3 of the definition of $\mathbf{I}^*$. But this is impossible, since $\{f(\mathbf{a}_1, \ldots, \mathbf{a}_n) = \mathbf{b}, f(\mathbf{a}_1, \ldots, \mathbf{a}_n) = \mathbf{c}\} \vdash \mathbf{b} = \mathbf{c}$ by =E, so $\mathbf{b} = \mathbf{c} \in \Gamma^*$, by 11.4.6, and therefore $\mathbf{q}(\mathbf{c}) = \mathbf{q}(\mathbf{b})$, by 11.4.10. It follows that $\mathbf{I}^*(\mathbf{c}) = \mathbf{I}^*(\mathbf{b})$,

and so there is at most one member $\mathbf{u}_{n+1}$ of the UD such that $<\mathbf{u}_1, \ldots, \mathbf{u}_n, \mathbf{u}_{n+1}>$ is a member of $\mathbf{I}^*(f)$.

We must also ensure that condition 5 can be met, that is, that there are not two atomic sentences $\mathbf{A}\mathbf{a}_1 \ldots \mathbf{a}_n$ and $\mathbf{A}\mathbf{a}'_1 \ldots \mathbf{a}'_n$ such that one is in Γ^* and the other is not in Γ^*, yet $\langle \mathbf{I}^*(\mathbf{a}_1), \ldots, \mathbf{I}^*(\mathbf{a}_n)\rangle = \langle \mathbf{I}^*(\mathbf{a}'_1), \ldots, \mathbf{I}^*(\mathbf{a}'_n)\rangle$. In the case of *PD*, it was simple to show this, for distinct constants were interpreted to designate distinct individuals. However, $\mathbf{q}$ may assign the same positive integer to more than one constant, and as a consequence condition 5 may interpret several constants to designate the same value. Here our previous results will be useful. Suppose that the constants $\mathbf{a}_1, \ldots, \mathbf{a}_n$ and $\mathbf{a}'_1, \ldots, \mathbf{a}'_n$ are such that $\langle \mathbf{I}^*(\mathbf{a}_1), \ldots, \mathbf{I}^*(\mathbf{a}_n)\rangle = \langle \mathbf{I}^*(\mathbf{a}'_1), \ldots, \mathbf{I}^*(\mathbf{a}'_n)\rangle$. Then, by clause 3 of the definition of $\mathbf{I}^*$, we know that for each i, $\mathbf{q}(\mathbf{a}_i) = \mathbf{q}(\mathbf{a}'_i)$. It follows from 11.4.10 that for each i, $\mathbf{a}_i = \mathbf{a}'_i \in \Gamma^*$. Because $\mathbf{a}_1 = \mathbf{a}'_1$ is a member of Γ^*, property (i) assures us that $\mathbf{A}\mathbf{a}_1 \ldots \mathbf{a}_n$ is a member of Γ^* if and only if $\mathbf{A}\mathbf{a}'_1\mathbf{a}_2 \ldots \mathbf{a}_n$ is a member of Γ^*. And because $\mathbf{a}_2 = \mathbf{a}'_2$ is also a member of Γ^*, property (i) assures us that $\mathbf{A}\mathbf{a}'_1\mathbf{a}_2 \ldots \mathbf{a}_n$ is a member of Γ^* if and only if $\mathbf{A}\mathbf{a}'_1\mathbf{a}'_2 \ldots \mathbf{a}_n$ is a member of Γ^*, and so on until we note that because $\mathbf{a}_n = \mathbf{a}'_n$ is in Γ^*, $\mathbf{A}\mathbf{a}'_1\mathbf{a}'_2 \ldots \mathbf{a}_n$ is a member of Γ^* if and only if $\mathbf{A}\mathbf{a}'_1\mathbf{a}'_2 \ldots \mathbf{a}'_n$ is a member of Γ^*. We conclude that, if $\langle \mathbf{I}^*(\mathbf{a}_1), \ldots, \mathbf{I}^*(\mathbf{a}_n)\rangle = \langle \mathbf{I}^*(\mathbf{a}'_1), \ldots, \mathbf{I}^*(\mathbf{a}'_n)\rangle$, then $\mathbf{A}\mathbf{a}_1 \ldots \mathbf{a}_n \in \Gamma^*$ if and only if $\mathbf{A}\mathbf{a}'_1 \ldots \mathbf{a}'_n \in \Gamma^*$. So condition 5 can indeed be met.

To establish Lemma 11.4.8 for *PDE*—that every set Γ^* that is both maximally consistent in *PDE* and $\exists$-complete is also quantificationally consistent—we can prove by mathematical induction that a sentence $\mathbf{P}$ of *PDE* is true on $\mathbf{I}^*$ if and only if $\mathbf{P} \in \Gamma^*$. The proof is similar to that for *PD*, except that we must change the basis clause to consider closed complex terms as well as constants, and also to consider formulas containing the identity operator. We shall find the following result useful here:

> **11.4.11:** For any closed complex term $\mathbf{t}$ and variable assignment $\mathbf{d}$, $\text{den}_{\mathbf{I}^*,\mathbf{d}}(\mathbf{t}) = \mathbf{I}^*(\mathbf{a})$, where $\mathbf{a}$ is the alphabetically earliest individual constant such that $\mathbf{t} = \mathbf{a}$ is a member of Γ^*. (Property (j) of sets that are maximally consistent in *PDE* and $\exists$-complete guarantees that there *is* such a constant $\mathbf{a}$.)
>
> **Proof:** See Exercise 16.

Here is the revised proof.

> **Proof of basis clause:** Either $\mathbf{P}$ is a sentence letter or $\mathbf{P}$ has the form $\mathbf{A}\mathbf{t}_1 \ldots \mathbf{t}_n$ or $\mathbf{t}_1 = \mathbf{t}_2$. If $\mathbf{P}$ is a sentence letter, then, by clause 2 of the definition of $\mathbf{I}^*$, it follows that $\mathbf{P}$ is true on $\mathbf{I}^*$ if and only if $\mathbf{P} \in \Gamma^*$.
>
> If $\mathbf{P}$ has the form $\mathbf{A}\mathbf{t}_1 \ldots \mathbf{t}_n$ then $\mathbf{P}$ is true on $\mathbf{I}^*$ if and only if, for every $\mathbf{d}$, $<\text{den}_{\mathbf{I}^*,\mathbf{d}}(\mathbf{t}_1), \ldots, \text{den}_{\mathbf{I}^*,\mathbf{d}}(\mathbf{t}_n)>$ is a member of $\mathbf{I}^*(\mathbf{A})$. Now property (j) guarantees, for each complex term $\mathbf{t}_i$, that there is an alphabetically earliest constant $\mathbf{a}_i$ such that $\mathbf{t}_i = \mathbf{a}_i$ is a member of Γ^*. Moreover, by virtue of the rule =D, the set consisting of $\mathbf{P}$ and each of these identity sentences quantificationally entails $\mathbf{A}\mathbf{t}'_1 \ldots \mathbf{t}'_n$, where

$\mathbf{t}'_i$ is $\mathbf{t}_i$ if $\mathbf{t}_i$ is a constant and $\mathbf{t}'_i$ is $\mathbf{a}_i$ otherwise. So, by 11.4.6, $\mathbf{P} \in \Gamma^*$ if and only if the sentence $\mathbf{At}'_1 \ldots \mathbf{t}'_n$ is also a member of Γ^*. In addition, $\text{den}_{\mathbf{I}^*,\mathbf{d}}(\mathbf{t}_i) = \mathbf{I}^*(\mathbf{t}'_i)$, trivially if $\mathbf{t}_i$ is a constant and by 11.4.11 if $\mathbf{t}_i$ is a complex term. So $<\text{den}_{\mathbf{I}^*,\mathbf{d}}(\mathbf{t}_1), \ldots, \text{den}_{\mathbf{I}^*,\mathbf{d}}(\mathbf{t}_n)>$ is a member of $\mathbf{I}^*(\mathbf{A})$ if and only if $<\mathbf{I}^*(\mathbf{t}'_1), \ldots, \mathbf{I}^*(\mathbf{t}'_n)>$ is a member of $\mathbf{I}^*(\mathbf{A})$; and clause 5 in the definition of $\mathbf{I}^*$ guarantees that $\mathbf{At}'_1 \ldots \mathbf{t}'_n$ is a member of Γ^* if and only if $<\mathbf{I}^*(\mathbf{t}'_1), \ldots, \mathbf{I}^*(\mathbf{t}'_n)>$ is a member of $\mathbf{I}^*(\mathbf{A})$. We conclude that $\mathbf{At}_1 \ldots \mathbf{t}_n \in \Gamma^*$ if and only if $\mathbf{At}_1 \ldots \mathbf{t}_n$ is true on $\mathbf{I}^*$.

If $\mathbf{P}$ has the form $\mathbf{t}_1 = \mathbf{t}_2$, then $\mathbf{P}$ is true on $\mathbf{I}^*$ if and only if, for each variable assignment $\mathbf{d}$, $\text{den}_{\mathbf{I}^*,\mathbf{d}}(\mathbf{t}_1) = \text{den}_{\mathbf{I}^*,\mathbf{d}}(\mathbf{t}_2)$. Again, for each complex term $\mathbf{t}_i$, property (j) guarantees that there is an alphabetically earliest constant $\mathbf{a}_i$ such that $\mathbf{t}_i = \mathbf{a}_i$ is a member of Γ^* and so, by virtue of =D and result 11.4.6, $\mathbf{t}_1 = \mathbf{t}_2 \in \Gamma^*$ if and only if the sentence $\mathbf{t}'_1 = \mathbf{t}'_2$ is also a member of Γ^*, where $\mathbf{t}'_i$ is $\mathbf{t}_i$ if $\mathbf{t}_i$ is a constant and $\mathbf{t}'_i$ is $\mathbf{a}_i$ otherwise. Moreover $\text{den}_{\mathbf{I}^*,\mathbf{d}}(\mathbf{t}_i) = \mathbf{I}^*(\mathbf{t}'_i)$, trivially if $\mathbf{t}_i$ is a constant and by result 11.4.11 if $\mathbf{t}_i$ is a complex term. So $\text{den}_{\mathbf{I}^*,\mathbf{d}}(\mathbf{t}_1) = \text{den}_{\mathbf{I}^*,\mathbf{d}}(\mathbf{t}_2)$ if and only if $\mathbf{I}^*(\mathbf{t}'_1) = \mathbf{I}^*(\mathbf{t}'_2)$. By the way in which $\mathbf{I}^*$ was constructed, $\mathbf{I}^*(\mathbf{t}'_1) = \mathbf{I}^*(\mathbf{t}'_2)$ if and only if $\mathbf{q}(\mathbf{t}'_1) = \mathbf{q}(\mathbf{t}'_2)$. By result 11.4.10, $\mathbf{q}(\mathbf{t}'_1) = \mathbf{q}(\mathbf{t}'_2)$ if and only if $\mathbf{t}'_1 = \mathbf{t}'_2 \in \Gamma^*$. We may conclude that $\mathbf{t}_1 = \mathbf{t}_2 \in \Gamma^*$ if and only if $\mathbf{t}_1 = \mathbf{t}_2$ is true on $\mathbf{I}^*$.

Because every member of Γ^* is true on $\mathbf{I}^*$, Γ^* is quantificationally consistent. And, with Lemmas 11.4.4 and 11.4.8 established for *PDE*, along with the necessary modifications of 11.4.9 (see Exercise 11.4.15), we know that the Inconsistency Lemma 11.4.2 is also true for *PDE*. It follows that *PDE* is complete for predicate logic with identity and functions.

11.4E EXERCISES

***1.** Prove that if $\Gamma \models \mathbf{P}$ then $\Gamma \cup \{\sim \mathbf{P}\}$ is quantificationally inconsistent.

2. Prove that if $\Gamma \cup \{\sim \mathbf{P}\}$ is inconsistent in *PD* then $\Gamma \vdash \mathbf{P}$.

3. Using Metatheorem 11.4.1, prove the following:

a. Every argument of *PL* that is quantificationally valid is valid in *PD*.

b. Every sentence of *PL* that is quantificationally true is a theorem in *PD*.

*c. Every pair of sentences $\mathbf{P}$ and $\mathbf{Q}$ of *PL* that are quantificationally equivalent are equivalent in *PD*.

4. Prove that the sentences of *PL* can be enumerated. (*Hint:* See Section 6.4.)

5. Prove the following:

If $\Gamma \vdash \mathbf{P}$ and Γ is a subset of Γ', then $\Gamma' \vdash \mathbf{P}$.

6.a. Prove 11.4.5.

b. Prove that any set Γ^* constructed as in our proof of Lemma 11.4.4 is $\exists$-complete.

7. Prove that the sequence of sentences constructed in the proof of 11.4.3 is a derivation in *PD* by showing (by mathematical induction) that each sentence in the new sequence can be justified with the same rule as the corresponding sentence in the original derivation.

*8. Prove 11.4.9, using result 11.1.13.

*9. Prove 11.4.6.

10. Explain why, in Lemmas 11.4.4 and 11.4.8, we constructed a set that was both ∃-complete and maximally consistent in *PD*, rather than a set that was just maximally consistent in *PD*.

11. Let system *PD*° be just like *PD* except that the rule ∀E is replaced by the following rule:

*Universal Elimination** (∀E*)

| (∀x)**P**
| ~ (∃x) ~ **P**

Prove that the system *PD*° is complete for predicate logic.

*12. Let system *PD*° be just like *PD* except that the rules ∃E and ∃I are replaced by the following two rules:

*Existential Elimination** (∃E*)

| (∃x)**P**
| ~ (∀x) ~ **P**

*Existential Introduction** (∃I*)

| ~ (∀x) ~ **P**
| (∃x)**P**

Prove that system *PD*° is complete for predicate logic.

13. Using the results in the proof of Metatheorem 11.4.1, prove the following theorem (known as the *Löwenheim Theorem*):

If a sentence **P** of *PL* is not quantificationally false, then there is an interpretation with the set of positive integers as the UD on which **P** is true.

*14. Prove the following metatheorem (known as the *Löwenheim-Skolem Theorem*):

If a set Γ of sentences of *PL* is quantificationally consistent, then there is an interpretation with the set of positive integers as the UD on which every member of Γ is true.

*15. Show the changes that must be made in the proofs of 11.4.3 and 11.4.9 so that these results will hold for *PLE* and *PDE*. (*Hint*: Exercise 8 suggested that you use result 11.1.13 in proving 11.4.9; so you must check whether 11.1.13 needs to be changed.)

16. Prove result 11.4.11.

17. Show that the Löwenheim Theorem (and consequently the more general Löwenheim-Skolem Theorem) does *not* hold for *PLE*.

11.5 THE SOUNDNESS OF THE TREE METHOD

We have presented the tree method as a means of testing for semantic properties of sentences and sets of sentences in both sentential logic and predicate logic. In this section and the next we shall prove that the tree method in Chapter 9 fulfills a claim we have made: A finite set of sentences of *PL* is quantificationally inconsistent if and only if every systematic tree for that set closes. In this section we shall prove that the tree method is **sound** for predicate logic—that if a systematic tree for a set of sentences of *PL* closes, then the set is quantificationally inconsistent. We shall prove the same for predicate logic with identity and functions. In both cases we can then be assured that, if we pronounce a set of sentences inconsistent because a tree for that set closes, our pronouncement is correct. In the next section we shall prove that the tree method is **complete** for predicate logic—that if a finite set of sentences is quantificationally inconsistent, then every systematic tree for that set is bound to close. Knowing that the method is complete, we shall also know that open systematic trees do establish quantificational consistency. (With a simple adaptation of parts of our proofs, the soundness and completeness of the tree method for sentential logic can also be established. This will be addressed in the exercises.)

Our *Soundness Metatheorem* for the tree method is this:

Metatheorem 11.5.1: If a systematic tree for a set Γ of sentences of *PL* closes, then Γ is quantificationally inconsistent.

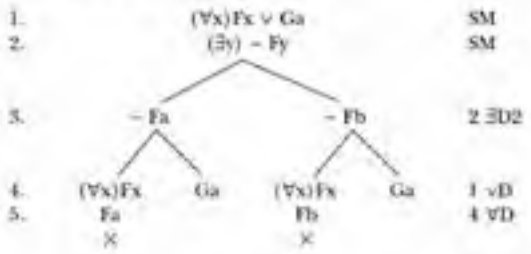
(As we shall see in the exercises for this section, soundness also holds for nonsystematic trees.) Our proof of Metatheorem 11.5.1 will rely heavily on the following observation about the decomposition rules used in constructing trees: Each rule is consistency-preserving in the sense that, if we have a consistent set of sentences and apply a decomposition rule to one of the sentences in that set, at least one of the sentences that results (there will be only one in the case of a nonbranching rule) can consistently be added to the set. As we build a tree for a set of sentences, we are, in effect, building supersets of the one we started with—the set of sentences occurring on a

branch is a superset of the original set. Given the sense in which the decomposition rules are consistency-preserving, at least one of the supersets formed on a branch by repeated application of decomposition rules will be quantificationally consistent. This will be important in establishing Metatheorem 11.5.1, for the supersets that comprise the branches of a closed tree are all quantificationally inconsistent, each such branch containing some literal and its negation. Because the decomposition rules are consistency-preserving in the sense described, it follows that the only way we can end up with every superset being quantificationally inconsistent (with a closed tree) is by starting with a set that is quantificationally inconsistent. And that is what Metatheorem 11.5.1 says.

Our observation that the decomposition rules are consistency-preserving must be proved. To facilitate our proof, we introduce the concept of a *level* of a tree. The first (occurrence of a) sentence on any tree is at level 1. For any other sentence **P**, **P** is at level **i** + 1, where **i** is the level of the sentence occurring immediately before **P** on the same branch of the tree. The line numbers used to annotate trees in Chapters 4 and 9 do not always correspond to levels because we adopted the convention in those chapters that only one decomposition rule can be cited on each line. Consider, for example, the tree on page 477. Lines 1–3 do correspond directly to levels 1–3 of that tree. Line 4, however, displays only one of the sentences occurring at level 4. The sentence in line 10 on the left-hand branch is *also* at level 4, for the sentence that occurs immediately before it on the same branch is at level 3. Similar observations hold for sentences further down the branches of the tree.

We shall establish that our decomposition rules are consistency-preserving by showing that each level **i** of a systematic tree for a quantificationally consistent set of sentences is such that either there is at least one branch that was completed prior to that level on which the sentences form a quantificationally consistent set (a quantificationally consistent superset has resulted from applying as many rules as could be applied) or there is at least one branch that extends at least as far as level **i** such that the sentences on that branch up to and including level **i** form a quantificationally consistent set (the rules thus far have preserved consistency).

As an example of what we want to prove, here is a completed open tree for the set $\{(\forall x)Fx \vee Ga, (\exists y) \sim Fy\}$:



Each of the levels 1–5 is such that our claim holds. At level 1 there is at least one branch such that the set of sentences occurring on that branch through and including level 1 form a quantificationally consistent set: {(∀x)Fx ∨ Ga}. At level 2 there is at least one branch such that the set of sentences occurring on that branch form a quantificationally consistent set: {(∀x)Fx ∨ Ga, (∃y) – Fy}. At level 3 there are two (and so at least one) such branches: {(∀x)Fx ∨ Ga, (∃y) – Fy, – Fa} and {(∀x)Fx ∨ Ga, (∃y) – Fy, – Fb}. At level 4 there are also two such branches: {(∀x)Fx ∨ Ga, (∃y) – Fy, – Fa, Ga} and {(∀x)Fx ∨ Ga, (∃y) – Fy, – Fb, Ga}. The branches that include '(∀x)Fx' are not candidates, but we have not claimed that quantificationally consistent sets will be found on *all* branches. At level 5 there is no branch that extends to that level that contains a quantificationally consistent set of sentences, but there is at least one branch that was completed at an earlier level, level 4, on which the sentences form a quantificationally consistent set. So the claim is true of all levels of this tree.

The fact that the claim holds for every level of a systematic tree for a quantificationally consistent set allows us to conclude that if a tree for a set of sentences closes then the set must be quantificationally inconsistent. Consider: If a tree closes, then the tree has only a finite number of levels, and the longest branch ends at a finite level. In the following closed tree the longest branch ends at level 4:

No branch that closes before the last level contains a quantificationally consistent set of sentences (each such branch containing a literal and its negation), and the branch that closes at the last level does not contain a quantificationally consistent set. So, at the last level of a closed tree, our claim about the levels of a tree for a quantificationally consistent set is not true; and we may conclude that the set for which the tree was constructed is therefore quantificationally *inconsistent*.

To prove our claim about the levels of any systematic tree for a quantificationally consistent set of sentences, we also introduce the concept of a path to a level of a tree. For any branch that extends to level i or further, we call that part of the branch that extends to level i a **path** to level i, and we say that the path **contains** the set of sentences that occur on it. In the last tree displayed, there is exactly one path to level 1, and it contains the set of sentences {(∀x)Fx ∨ Fa}. There is exactly one path to level 2, and it contains the set of sentences {(∀x)Fx ∨ Fa, – Fa}. There are two paths to level 3; one contains the set of sentences {(∀x)Fx ∨ Fa, – Fa, (∀x)Fx}, and the other contains the set {(∀x)Fx ∨ Fa, – Fa, Fa}. There is one path to level 4, and it contains the

set {(∀x)Fx ∨ Fa, ~ Fa, (∀x)Fx, Fa}. Finally a **completed** path to level **i** of a tree is a completed branch of that tree that ends at level **i**. We state our claim about the levels of a systematic tree for a quantificationally consistent set in terms of paths in the Consistent Branch Lemma:

> **11.5.2** (the *Consistent Branch Lemma*): Each level **i** of a tree for a quantificationally consistent set of sentences of *PL* is such that either (a) there is at least one completed path to a level earlier than **i** that contains a quantificationally consistent set of sentences or (b) there is at least one path to level **i** that contains a quantificationally consistent set of sentences.

We shall prove 11.5.2 by establishing a more specific claim (which will later be useful in proving that systematic trees for sets of sentences with finite models always have a completed open branch). Let Γ be a finite set of sentences of *PL* that is quantificationally consistent, and let **I** be an interpretation. We call interpretation **I′** a **path-variant** of **I** for path **p** of a tree for the set of sentences Γ if **I′** is just like **I** except that, for each constant **a** that occurs in some sentence on the path but not in any member of Γ, there is a member **u** of the UD such that $\mathbf{I}'(\mathbf{a}) = \mathbf{u}$ and such that, for every other constant **b** occurring on the path but not in Γ, $\mathbf{I}'(\mathbf{b}) \neq \mathbf{u}$. We shall show that

> **11.5.3:** If a finite set Γ of sentences of *PL* is true on an interpretation **I**, then each level **i** of a systematic tree for Γ is such that either (a) there is at least one completed path **p** to a level earlier than **i** that contains a set of sentences all of which are true on a path-variant of **I** for **p** or (b) there is at least one path **p** to level **i** that contains a set of sentences all of which are true on a path-variant of **I** for **p**.

We establish result 11.5.3 by mathematical induction on the levels of a systematic tree for a quantificationally consistent set of sentences of *PL*. Letting Γ be a finite set of sentences of *PL* and **I** an interpretation on which every member of Γ is true,

> *Basis clause:* Level 1 of a systematic tree for Γ is such that either (a) or (b) holds.
>
> *Inductive step:* If every level less than or equal to level **k** of a systematic tree for Γ is such that either (a) or (b) holds, then level **k** + 1 of a systematic tree for Γ is also such that either (a) or (b) holds.

> *Conclusion:* Every level of a systematic tree for Γ is such that either (a) or (b) holds.
>
> **Proof of basis clause:** There is exactly one path to level 1 of any tree, and that path contains the unit set {**P**}, where **P** is a member of the set Γ for which the tree is being constructed. **P** is true on **I** since every

member of Γ is, and $\mathbf{I}$ is in this case a path-variant of itself (since the path contains no constants that do not occur in Γ). So there is a path to level 1 that contains a set of sentences all of which are true on a path-variant of $\mathbf{I}$, and (b) holds for level 1.

Proof of inductive step: We assume the inductive hypothesis for an arbitrary positive integer $\mathbf{k}$: For each level $\mathbf{i}$ less than $\mathbf{k} + 1$ of a tree for Γ, either (a) or (b) holds. We must show that on this assumption either (a) or (b) holds for level $\mathbf{k} + 1$ of a tree for Γ as well. We first note that if (a) holds for an earlier level $\mathbf{i}$, then (a) holds for level $\mathbf{k} + 1$ as well. That is, if there is at least one completed path to a level earlier than $\mathbf{i}$ that contains a set of sentences all of which are true on a path-variant of $\mathbf{I}$ for that path, then that path is also a completed path to a level earlier than $\mathbf{k} + 1$.

Now we must consider the case where (a) does not hold for any level prior to $\mathbf{k} + 1$. In this case it follows from the inductive hypothesis that (b) holds for every level prior to $\mathbf{k} + 1$ and, in particular, that (b) is true of level $\mathbf{k}$. If, in addition, there is a *completed* path to level $\mathbf{k}$ that contains a set of sentences all of which are true on a path-variant of $\mathbf{I}$ for that path, then (a) is true of level $\mathbf{k} + 1$. If there is not such a path to level $\mathbf{k}$, we still know, because (b) is true of level $\mathbf{k}$, that at least one (noncompleted) path to level $\mathbf{k}$ contains a set of sentences all of which are true on a path-variant of $\mathbf{I}$ for that path. Call this path $\Gamma_{\mathbf{k}}$ and the variant $\mathbf{I}_{\Gamma_{\mathbf{k}}}$. Because the path is not complete at level $\mathbf{k}$, it is extended to level $\mathbf{k} + 1$ by application of some tree rule. We shall now consider the rules that might be used to extend the path to level $\mathbf{k} + 1$ and show that in each case application of the rule results in at least one path to level $\mathbf{k} + 1$ that contains a set of sentences all of which are true on a path-variant of $\mathbf{I}$ for that path—thereby establishing that (b) holds for level $\mathbf{k} + 1$ as well.

We divide the rules into six cases.

Case 1: The path $\Gamma_{\mathbf{k}}$ is extended to level $\mathbf{k} + 1$ by adding a set member at that level. Because decomposition rules apply *after* all set members have been entered, the only sentences on the path $\Gamma_{\mathbf{k}}$ are members of Γ, and the sentence entered at level $\mathbf{k} + 1$ is also a member of Γ. So there is a path to level $\mathbf{k} + 1$ that contains a subset of Γ, all members of which are true in $\mathbf{I}$, which in this case is a path-variant of itself.

Case 2: The path $\Gamma_{\mathbf{k}}$ is extended to level $\mathbf{k} + 1$ as a result of applying one of the nonbranching rules $-\,-\mathrm{D}$, $\&\mathrm{D}$, $-\vee\mathrm{D}$, $-\supset\mathrm{D}$, $-\forall\mathrm{D}$, or $-\exists\mathrm{D}$ to a sentence $\mathbf{P}$ on $\Gamma_{\mathbf{k}}$. In each case $[\mathbf{P}]$ quantificationally entails the sentence $\mathbf{Q}$ entered on level $\mathbf{k} + 1$ (see Exercise 11.5.3). Therefore all the sentences on $\Gamma_{\mathbf{k}}$ and the sentence $\mathbf{Q}$ are true on $\mathbf{I}_{\Gamma_{\mathbf{k}}}$, which is a path-variant of $\mathbf{I}$ for the extended path that we are considering (because none of the rules in this case add a new individual constant to the tree

and $\mathbf{I}_{\Gamma'_k}$ itself was a path-variant of $\mathbf{I}$ for Γ_k). Thus there is a path to level $\mathbf{k} + 1$—the path that extends Γ'_k to include $\mathbf{Q}$—that contains a set of sentences all of which are true on a path-variant of $\mathbf{I}$ for this path.

Case 3: The path is extended to form two paths to level $\mathbf{k} + 1$ as a result of applying one of the branching rules – &D, ∨D, or ⊃D to a sentence $\mathbf{P}$ on Γ'_k. Letting $\mathbf{Q}$ be one of the sentences that was entered on level $\mathbf{k} + 1$ and $\mathbf{R}$ the other, it can be shown that on any interpretation on which $\mathbf{P}$ is true either $\mathbf{Q}$ is true or $\mathbf{R}$ is true (see Exercise 4). Therefore either all the sentences on Γ'_k plus $\mathbf{Q}$ are true on $\mathbf{I}_{\Gamma'_k}$, which is a path-variant of $\mathbf{I}$ for the new path containing $\mathbf{Q}$, or all the sentences on Γ'_k plus $\mathbf{R}$ are true on $\mathbf{I}_{\Gamma'_k}$, which is a path-variant of $\mathbf{I}$ for the new path containing $\mathbf{R}$. Thus there is a path to level $\mathbf{k} + 1$—either the path that extends Γ'_k to include $\mathbf{Q}$ or the path that extends Γ'_k to include $\mathbf{R}$—that contains a set of sentences all of which are true on a path-variant of $\mathbf{I}$ for that path.

Case 4: The path is extended to level $\mathbf{k} + 1$ as a result of applying either =D or – =D (see Exercise 5).

Case 5: The path is extended to level $\mathbf{k} + 1$ as a result of applying ∀D. Then Γ'_k contains a sentence $(\forall \mathbf{x})\mathbf{P}$ such that $\mathbf{P}(\mathbf{a}/\mathbf{x})$ is entered at level $\mathbf{k} + 1$, where $\mathbf{a}$ is either 'a' if no constants occur on Γ'_k or the alphabetically earliest constant that does occur. Because $(\forall \mathbf{x})\mathbf{P}\mathbf{x} \vDash \mathbf{P}(\mathbf{a}/\mathbf{x})$ (result 11.1.4), $\mathbf{P}(\mathbf{a}/\mathbf{x})$ is true on $\mathbf{I}_{\Gamma'_k}$. If no constant occurred on Γ'_k, then $\mathbf{I}_{\Gamma'_k}$ is also a path-variant of $\mathbf{I}$ for the new path to level $\mathbf{k} + 1$ because $\mathbf{I}_{\Gamma'_k}(\mathbf{a}) \neq \mathbf{I}_{\Gamma'_k}(\mathbf{b})$ for any other constant $\mathbf{b}$ occurring on Γ'_k but not in Γ—there are no other constants $\mathbf{b}$ occurring on Γ'_k. If, on the other hand, $\mathbf{a}$ is a constant that already occurs on Γ'_k, then we have added no new constant to the path, and so $\mathbf{I}_{\Gamma'_k}$ is in this case also a path-variant of $\mathbf{I}$ for the new path to level $\mathbf{k} + 1$. Either way, there is a path to level $\mathbf{k} + 1$ that contains a set of sentences all of which are true on a path-variant of $\mathbf{I}$ for that path.

Case 6: The path is extended to level $\mathbf{k} + 1$ as a result of applying ∃D2. Then Γ'_k contains a sentence $(\exists \mathbf{x})\mathbf{P}$ such that $\mathbf{P}(\mathbf{a}_1/\mathbf{x})$, . . . , $\mathbf{P}(\mathbf{a}_m/\mathbf{x})$, $\mathbf{P}(\mathbf{a}_{m+1}/\mathbf{x})$ are entered on distinct paths to level $\mathbf{k} + 1$, where $\mathbf{a}_1, \ldots, \mathbf{a}_m$ are all the individual constants that occur in sentences on Γ'_k and $\mathbf{a}_{m+1}$ is the alphabetically earliest constant that does not occur on Γ'_k. We consider two possibilities:

a. If any one (or more) of $\mathbf{P}(\mathbf{a}_1/\mathbf{x}), \ldots, \mathbf{P}(\mathbf{a}_m/\mathbf{x})$ is true on $\mathbf{I}_{\Gamma'_k}$, then the path to level $\mathbf{k} + 1$ on which that substitution instance was entered is a path that contains a set of sentences all of which are true on a path-variant of $\mathbf{I}$ for that path ($\mathbf{I}_{\Gamma'_k}$ is a path-variant of the newly formed path because the substitution instance does not introduce a new constant).

b. Now consider the case where none of $\mathbf{P}(\mathbf{a}_1/\mathbf{x}), \ldots,$ $\mathbf{P}(\mathbf{a}_{m+1}/\mathbf{x})$ is true on $\mathbf{I}_{\Gamma_k}$. Because $(\exists\mathbf{x})\mathbf{P}$ is true on $\mathbf{I}_{\Gamma_k}$ and because $\mathbf{a}_{m+1}$ does not occur in any sentence on Γ_k, our proof of result 11.1.10 (Exercise 11.1.5) shows that $\mathbf{P}(\mathbf{a}_{m+1}/\mathbf{x})$ is true on an interpretation $\mathbf{I}'_{\Gamma_k}$ that is just like $\mathbf{I}_{\Gamma_k}$ except that $\mathbf{I}'_{\Gamma_k}(\mathbf{a}_{m+1}) = \mathbf{u}$, where $\mathbf{u}$ is a member of the UD such that $\mathbf{d}[\mathbf{u}/\mathbf{x}]$ satisfies $\mathbf{P}$ on $\mathbf{I}_{\Gamma_k}$. This member $\mathbf{u}$ is not assigned to any other individual constant $\mathbf{b}_i$ occurring on Γ_k but not in Γ (for, if it were, it would follow from result 11.1.13 that $\mathbf{P}(\mathbf{b}_i/\mathbf{x})$ is true on $\mathbf{I}_{\Gamma_k}$, which contradicts our assumption here). Thus $\mathbf{I}'_{\Gamma_k}$ is a path-variant of $\mathbf{I}$ for the path extended to level $\mathbf{k}+1$ by the addition of $\mathbf{P}(\mathbf{a}_{m+1}/\mathbf{x})$, and one on which every sentence in the new path is true.

Either way, then, there is a path to level $\mathbf{k}+1$ that contains a set of sentences all of which are true on a path-variant of $\mathbf{I}$ for that path.

We have considered each rule that might be used to extend the path Γ_k to level $\mathbf{k}+1$ and have shown that in each case there is at least one path to level $\mathbf{k}+1$ that contains a set of sentences all of which are true on a path-variant of $\mathbf{I}$ for that path. That completes the proof of the inductive step.

Therefore, result 11.5.3 holds for every level of a tree for a set of sentences all of which are true on interpretation $\mathbf{I}$.

The Consistent Branch Lemma 11.5.2 follows immediately from result 11.5.3, for in establishing the existence of paths containing sentences all of which are true on some path variant of $\mathbf{I}$, we have established that the set of sentences on each such path has a model and therefore forms a quantificationally consistent set.

Metatheorem 11.5.1 follows from the Consistent Branch Lemma and from the fact that the null tree, which is the single tree for the empty set of sentences of *PL*, is not closed (the null branch does not contain any sentences and therefore does not contain some atomic sentence and its negation). If a tree for a set Γ of sentences is closed, then every branch on that tree is closed and hence contains a literal and its negation. Any path to the last level of the tree is a closed branch, and hence the set of sentences on that branch is quantificationally inconsistent (because it contains some literal and its negation). So (b) does not hold for the last level of such a tree. Nor does (a); all completed paths to earlier levels are closed branches, and therefore the sets of sentences on those branches are also quantificationally inconsistent. Because the last level of any closed tree is such that neither (a) nor (b) holds, it follows, by the Consistent Branch Lemma, that the set for which the tree was constructed is quantificationally inconsistent. We conclude that the tree method is sound for predicate logic.

To establish that the tree method for predicate logic with identity and functions is also sound, we first note that the cases in the inductive proof of result 11.5.3 carry over to predicate logic with identity and functions. We must

also add two more cases to the inductive step, one to cover paths that are extended by an application of =D and one to cover paths that are extended by an application of CTD.

Case 7: The path Γ'_k is extended to level $k + 1$ as a result of applying =D. Then Γ'_k contains sentences $t_1 = t_2$ and P such that a sentence $P(t_1//t_2)$ was entered at level $k + 1$. It follows from 11.2.4, which we repeat here, that $P(t_1//t_2)$ is true on $\mathbf{I}_{\Gamma'_k}$.

11.2.4: For any closed terms t_1 and t_2, if P is a sentence that contains t_1, then $\{t_1 = t_2, P\} \vdash P(t_2//t_1)$, and if P is a sentence that contains t_2, then $\{t_1 = t_2, P\} \vdash P(t_1//t_2)$.

In addition, $\mathbf{I}_{\Gamma'_k}$ is a path-variant for the new path to level $k + 1$ because =D does not introduce new constants. Therefore there is a path to level $k + 1$ that contains a set of sentences all of which are true on some path-variant of $\mathbf{I}$ for that path.

Case 8: The path Γ'_k is extended to level $k + 1$ as a result of applying CTD. Then Γ'_k contains a literal sentence with a closed complex term $f(a_1, \ldots, a_n)$, where $a_1, \ldots, a_n$ are all constants, such that $b_1 = f(a_1, \ldots, a_n), \ldots, b_m = f(a_1, \ldots, a_n)$ $b_{m+1} = f(a_1, \ldots, a_n)$ are entered on distinct paths to level $k + 1$, where $b_1, \ldots, b_m$ are all the individual constants that occur in sentences on Γ'_k and b_{m+1} is the alphabetically earliest constant that does not occur on Γ'_k. We consider two possibilities:

a. If any one (or more) of $b_1 = f(a_1, \ldots, a_n), \ldots, b_m = f(a_1, \ldots, a_n)$ is true on $\mathbf{I}_{\Gamma'_k}$, then the path to level $k + 1$ on which that identity sentence was entered is a path that contains a set of sentences all of which are true on a path-variant of $\mathbf{I}$ for that path. ($\mathbf{I}_{\Gamma'_k}$ is a path-variant of the newly formed path because the identity sentence does not introduce a new constant.)

b. Now consider the case where none of $b_1 = f(a_1, \ldots, a_n), \ldots, b_m = f(a_1, \ldots, a_n)$ is true on $\mathbf{I}_{\Gamma'_k}$. Because b_{m+1} does not occur in any sentence on Γ'_k, our proof of result 11.2.5 (Exercise 11.2.7) shows that $\Gamma \cup \{b_{m+1} = f(a_1, \ldots, a_n)\}$ is true on an interpretation $\mathbf{I}'_{\Gamma'_k}$ that is just like $\mathbf{I}_{\Gamma'_k}$ except that $\mathbf{I}'_{\Gamma'_k}(b_{m+1}) = u$, where u is the member of the UD such that $\langle \mathbf{I}_{\Gamma'_k}(a_1), \ldots, \mathbf{I}_{\Gamma'_k}(a_n), u\rangle$ is a member of $\mathbf{I}_{\Gamma'_k}(f)$. This member u is not assigned to any other individual constant b_i occurring on Γ'_k but not in Γ (for, if it were, it would follow that $b_i = f(a_1, \ldots, a_n)$ is true on $\mathbf{I}_{\Gamma'_k}$, which contradicts our assumption here). Thus $\mathbf{I}'_{\Gamma'_k}$ is a path-variant of $\mathbf{I}_{\Gamma'_k}$ for the path extended to level $k + 1$ by the addition of $b_{m+1} = f(a_1, \ldots, a_n)$, and one on which every sentence in the new path is true.

Finally we must note that a branch of a tree for predicate logic with identity closes in one of *two* cases: Either the branch contains some literal and its negation or the branch contains a sentence of the form $\sim \mathbf{t} = \mathbf{t}$. In showing that Metatheorem 11.5.1 for predicate logic followed from the Consistent Branch Lemma 11.5.2, we made use of the fact that each closed branch contained a literal and its negation—arguing that the set of the sentences on that branch was therefore quantificationally inconsistent. In the present case we must also be sure that the set of sentences on a branch that closes because it contains a sentence $\sim \mathbf{t} = \mathbf{t}$ is quantificationally inconsistent. This is not hard to show: $\mathbf{t} = \mathbf{t}$ is quantificationally true, so $\sim \mathbf{t} = \mathbf{t}$ is quantificationally false, and therefore any set that contains $\sim \mathbf{t} = \mathbf{t}$ is quantificationally inconsistent. This and the addition of Cases 7 and 8 in the proof of result 11.5.3 suffice to show that the tree method is sound for predicate logic with identity and functions.

Result 11.5.3 also allows us to prove another claim made in Chapter 9, that trees constructed in accordance with The System have the **finite model property:**

Metatheorem 11.5.4: If a finite set Γ of sentences of *PL* has a finite model, that is, an interpretation with a finite UD on which every member of Γ is true, then every systematic tree for Γ will contain a *completed* open branch.[1]

In such a case, we shall be able to conclude in a *finite* number of steps that the set is quantificationally consistent.

Proof of Metatheorem 11.5.4: Let Γ be a finite set of sentences such that there is an interpretation $\mathbf{I}$ with a finite UD on which every member of Γ is true. By result 11.5.3, every level $\mathbf{i}$ of a systematic tree for Γ is such that either (a) there is at least one completed path to a level earlier than $\mathbf{i}$ that contains a set of sentences all of which are true on a path-variant of $\mathbf{I}$ for that path or (b) there is at least one path to level $\mathbf{i}$ that contains a set of sentences all of which are true on a path-variant of $\mathbf{I}$ for that path.

Consider, for any level $\mathbf{i}$, a path that satisfies either (a) or (b). There is a limit to the number of distinct individual constants not already occurring in Γ that can occur on this path (constants that were introduced by an application of $\forall$D or $\exists$D), namely, the *size* $\mathbf{n}$ of the finite UD for $\mathbf{I}$. For if a path contains more than $\mathbf{n}$ new individual constants, it cannot meet the condition in the definition of path-variants that each of these constants be assigned a member of the UD that is *different* from the members assigned to other new constants; there would not be enough members of the UD to go around. In addition, because Γ contains only a finite number of constants, a path that satisfies either (a) or (b) can contain only a finite number of constants.

[1] This metatheorem, along with result 11.5.3, is due to George Boolos, "Trees and Finite Satisfiability: Proof of a Conjecture of Burgess," *Notre Dame Journal of Formal Logic*, 25(3) (1984), 193–197.

We now show that a path of a systematic tree that contains only a finite number of individual constants must be finitely long. Each of the decomposition rules &D, ~&D, ∨D, ~∨D, ~⊃D, ~≡D, ~~D, ∀D, and ∃D2 produces sentences with fewer occurrences of logical operators than the sentence being decomposed. The rules ⊃D, ≡D, ~∃D, and ~∀D produce one or two sentences with the same number of occurrences of logical operators as the sentence being decomposed, but the sentences so produced have one of the forms ~**P** (in the case of ⊃D and ≡D), (∃**x**) ~**P** (in the case of ~∀D), or (∀**x**) ~**P** (in the case of ~∃D). Each of the latter sentences, if not a literal, will be decomposed by a rule that produces only sentences with fewer occurrences of logical operators. Because subsequent applications of decomposition rules produce sentences with fewer and fewer occurrences of logical operators, literals are eventually reached. The only way in which a branch of a systematic tree can continue indefinitely is through repeated instantiation of one or more universally quantified sentences by ∀D, each instantiation containing a different instantiating constant. But this cannot be the case with a branch that contains a finite number of individual constants. Therefore the paths that we are guaranteed by result 11.5.3 can be only finitely long.

In addition, The System was designed to guarantee that if a path *can* be completed (or closed) after a finite number of applications of rules, it *will* be completed. Stages 1 and 2 (and stage 3, in the case of *PLE*) each require that we decompose *all* sentences on the tree of the specified sort before going to the next stage, and at each stage there are only finitely many sentences. The System does not allow one branch to be developed indefinitely while others are ignored, and so a branch that can be completed after a finite number of steps will be completed.

We conclude that at some finite level **i** of a systematic tree for Γ, there is a path that meets condition (a) of result 11.5.3. In addition, because this path meets (a), it is a completed *open* path. This establishes Metatheorem 11.5.4.

Metatheorem 11.5.4 is also true of *PLE*; this proof is left as an exercise.

11.5E EXERCISES

***1.** Show that Metatheorem 11.5.1 holds for nonsystematic trees as well as for systematic ones. (Result 11.5.3 is not generally true of nonsystematic trees, so you should prove Lemma 11.5.2 directly by mathematical induction.)

2. Using Metatheorem 11.5.1, prove the following:

a. If a sentence **P** of *SL* is such that ⊨**P** has a closed truth-tree, then **P** is quantificationally false.

b. If a sentence **P** of *SL* is such that ⊭**P** has a closed truth-tree, then **P** is quantificationally true.

*c. If a set $\{\sim (\mathbf{P} \equiv \mathbf{Q})\}$ has a closed truth-tree, then **P** and **Q** are quantificationally equivalent.
d. If a set $\Gamma \cup \{\sim \mathbf{P}\}$ has a closed truth-tree, then $\Gamma \vDash \mathbf{P}$.
*e. If the set consisting of the premises and the negation of the conclusion of an argument has a closed truth-tree, then that argument is quantificationally valid.

3. Prove that, if a sentence **Q** is obtained from a sentence **P** by application of one of the following tree rules, then $\{\mathbf{P}\} \vDash \mathbf{Q}$.

a. $--$D
*b. &D
*c. $-\vee$D
d. $-\supset$D
e. $\forall$D
*f. $-\forall$D
*g. $-\exists$D

4. Prove that, if sentences **Q** and **R** are obtained from a sentence **P** by application of one of the following tree rules, then on any interpretation on which **P** is true, either **Q** is true or **R** is true.

a. $-$&D
*b. $\vee$D
*c. $\supset$D

5. Prove Case 4 in the inductive step of the proof of Lemma 11.5.2.

6. If we were to drop the rule $\forall$D from the tree method, would the method still be sound for predicate logic? Explain.

7. Explain how we can adapt the proof of Metatheorem 11.5.1 to establish that the tree method for *SL* is sound for sentential logic.

***8.** Prove that Metatheorem 11.5.4 is true of *PLE*.

11.6 THE COMPLETENESS OF THE TREE METHOD

In the last section we established that the tree method is sound for predicate logic—if a tree for a set of sentences of *PL* closes, then that set is quantificationally inconsistent. In this section we shall prove that the tree method is also **complete** for sentential logic. The *Completeness Metatheorem* for the tree method is as follows:

Metatheorem 11.6.1: If a finite set Γ sentences of *PL* is quantificationally inconsistent, then every systematic tree for Γ closes.

Whereas soundness ensures that we are correct in pronouncing a set inconsistent if we can construct a closed tree for that set, completeness ensures that we are correct in pronouncing a set consistent if a systematic tree for that set does not close. The requirement that the tree be systematic is important, as we shall see; and the reader should remember that a tree that is constructed in accordance with The System but is abandoned before every branch closes and

before at least one branch becomes a completed open branch does not count as a systematic tree.

We shall prove that the tree method is complete by establishing that the contrapositive of Metatheorem 11.6.1 is true—that if a systematic tree for a set of sentences of *PL* does not close, then the set is quantificationally consistent. There are three parts to the proof. First, we shall prove that, if a systematic tree fails to close and does not contain a completed open branch after a finite number of steps in its construction, then it has at least one branch with infinitely many sentences. Second, we shall prove that for any completed open branch or infinite branch of a systematic truth-tree, the set of sentences occurring on that branch is a special sort of set known as a *Hintikka set*.[2] Finally we shall present a method of constructing a model for any Hintikka set. This will establish that every Hintikka set is quantificationally consistent and consequently that the set of sentences occurring on either a completed open branch or an infinite branch of a systematic truth-tree is quantificationally consistent. Because each sentence in the set Γ for which a tree is constructed occurs on every branch of that tree, it follows that, if a systematic tree for Γ fails to close, Γ is a subset of a Hintikka set and is therefore also quantificationally consistent. Therefore, if a finite set Γ is quantificationally *in*consistent, then every systematic tree for Γ will close—and that is what Metatheorem 11.6.1 says.

Consider a systematic tree such that at no level **i** does the tree contain a completed open branch and at no level **i** is the tree closed. Our first task is to show that such a tree must contain an infinite branch. Because the tree fails to close or to contain a completed open branch at any level **i**, the tree must contain infinitely many sentences (strictly speaking, infinitely many *occurrences* of sentences—the sentences need not be distinct). The tree contains infinitely many sentences because it takes infinitely many steps to construct a systematic tree that neither is closed nor contains a completed open branch at any level, and each step in the construction involves adding at least one new sentence. It remains to be shown that a systematic truth-tree containing infinitely many sentences has at least one branch that is infinitely long—at least one branch that contains an infinite number of sentences. The reason that this needs to be *proved* is that a tree *could* contain infinitely many sentences and yet be such that each of its branches was only finitely long, *if* it contained infinitely many branches. So we need to establish the following lemma, the Infinite Branch Lemma:

> **11.6.2** (the *Infinite Branch Lemma*): Every systematic tree that contains an infinite number of occurrences of sentences has at least one branch that is infinitely long.[3]

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[2] These sets were first studied by J. Hintikka, in "Form and Content in Quantification Theory," *Acta Philosophica Fennica*, 8 (1955), 7–55; and "Notes on Quantification Theory," *Societas Scientiarum Fennica, Commentationes Physico-Mathematicae*, 17(12) (1955).

[3] This follows as a special case of a famous lemma known as *König's Lemma* (D. König, *Theorie der endlichen und unendlichen Graphen*, Leipzig, 1936).

Proof of Lemma 11.6.2: Some definitions will be useful for the proof. We shall say that a sentence **P** (throughout, read *occurrence of a sentence* whenever we speak of a sentence) in a tree is **above** sentence **Q** when **P** and **Q** lie on the same branch of the tree and **P** is at an earlier level of the tree. **Q** is an **immediate successor** of **P** if **P** and **Q** lie on the same branch and **P** is one level earlier than **Q**. Every sentence in a tree, except those that occur at the ends of branches, has a finite number of immediate successors—one if a nonbranching rule is applied, two if a branching rule other than ∃D2 is applied, and **m** + 1, where **m** is the number of individual constants already occurring on the sentence's branch, if ∃D2 is applied.

We shall now show that if a systematic tree contains infinitely many sentences then there is at least one infinite branch in the tree, by starting at level 1 and working down through the levels of the tree. The sentence at level 1 of such a tree—call it $\mathbf{P}_1$—is above every other sentence in the tree. Therefore this sentence is above infinitely many sentences (subtracting 1 from an infinite number leaves an infinite number). $\mathbf{P}_1$ has a finite number of successors at level 2. At least one of these immediate successors is above infinitely many sentences. Consider the possibility that each of the immediate successors—call them $\mathbf{Q}_1, \ldots, \mathbf{Q}_n$—is above only a finite number of sentences. Then **P** itself would be above only finitely many sentences: $\mathbf{Q}_1, \ldots, \mathbf{Q}_n$, $\mathbf{Q}_1$'s successors, ..., and $\mathbf{Q}_n$'s successors together would constitute only a finite number of sentences. $\mathbf{P}_1$ must therefore have at least one immediate successor $\mathbf{P}_2$ that is above an infinite number of sentences. The reasoning that we have just used can be generalized: If a sentence at any level is above infinitely many sentences, then at least one immediate successor of that sentence is above infinitely many sentences. So $\mathbf{P}_2$, being above infinitely many sentences, has an immediate successor $\mathbf{P}_3$ that is above infinitely many sentences, and $\mathbf{P}_3$ has an immediate successor $\mathbf{P}_4$ that is above infinitely many sentences, and so on, for each positive integer. The sentences $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$, $\mathbf{P}_4$, . . . constitute a branch with an infinite number of sentences. Therefore, if a systematic tree contains infinitely many sentences, then it has at least one branch that is infinitely long.

We may now conclude that a systematic tree that fails to close either has a completed open branch after a finite number of steps or has at least one infinite branch. This will be important in what follows.

Turning to the second step of the proof of Metatheorem 11.6.1, we define a **Hintikka set** to be a set Γ of sentences of *PL* that has the following properties:

a. There is no atomic sentence **P** such that both **P** and ~ **P** are members of Γ.

b. If ~ ~ **P** ∈ Γ, then **P** ∈ Γ.

c. If **P** & **Q** ∈ Γ, then **P** ∈ Γ and **Q** ∈ Γ.

d. If ~ (**P** & **Q**) ∈ Γ, then either ~ **P** ∈ Γ or ~ **Q** ∈ Γ.

e. If $\mathbf{P} \vee \mathbf{Q} \in \Gamma$, then either $\mathbf{P} \in \Gamma$ or $\mathbf{Q} \in \Gamma$.
f. If $-(\mathbf{P} \vee \mathbf{Q}) \in \Gamma$, then $-\mathbf{P} \in \Gamma$ and $-\mathbf{Q} \in \Gamma$.
g. If $\mathbf{P} \supset \mathbf{Q} \in \Gamma$, then either $-\mathbf{P} \in \Gamma$ or $\mathbf{Q} \in \Gamma$.
h. If $-(\mathbf{P} \supset \mathbf{Q}) \in \Gamma$, then $\mathbf{P} \in \Gamma$ and $-\mathbf{Q} \in \Gamma$.
i. If $\mathbf{P} \equiv \mathbf{Q} \in \Gamma$, then either $\mathbf{P} \in \Gamma$ and $\mathbf{Q} \in \Gamma$ or $-\mathbf{P} \in \Gamma$ and $-\mathbf{Q} \in \Gamma$.
j. If $-(\mathbf{P} \equiv \mathbf{Q}) \in \Gamma$, then either $\mathbf{P} \in \Gamma$ and $-\mathbf{Q} \in \Gamma$ or $-\mathbf{P} \in \Gamma$ and $\mathbf{Q} \in \Gamma$.
k. If $(\forall \mathbf{x})\mathbf{P} \in \Gamma$, then at least one substitution instance of $(\forall \mathbf{x})\mathbf{P}$ is a member of Γ, and for every constant $\mathbf{a}$ that occurs in some sentence of Γ, $\mathbf{P}(\mathbf{a}/\mathbf{x}) \in \Gamma$.
l. If $-(\forall \mathbf{x})\mathbf{P} \in \Gamma$, then $(\exists \mathbf{x}) -\mathbf{P} \in \Gamma$.
m. If $(\exists \mathbf{x})\mathbf{P} \in \Gamma$, then for at least one constant $\mathbf{a}$, $\mathbf{P}(\mathbf{a}/\mathbf{x}) \in \Gamma$.
n. If $-(\exists \mathbf{x})\mathbf{P} \in \Gamma$, then $(\forall \mathbf{x}) -\mathbf{P} \in \Gamma$.

We call a branch of a tree a *Hintikka branch* if and only if the sentences on that branch constitute a Hintikka set. We now prove that every completed open branch and every infinite branch of a systematic tree is a Hintikka branch, which will establish the Hintikka Branch Lemma:

11.6.3 (the *Hintikka Branch Lemma*): Every systematic tree that is not closed has at least one Hintikka branch.

Afterward we shall show that every Hintikka set is quantificationally consistent.

Proof of Lemma 11.6.3: If a systematic tree fails to close, then either the tree has a completed open branch or, by Lemma 11.6.2, the tree has an infinite branch. We shall show that each of these two types of branches is a Hintikka branch—that is, that the set of sentences occurring on such a branch has properties (a)–(n).

First consider completed open branches. By definition a completed open branch is a finite branch that is open—there is no pair of literals $\mathbf{P}$ and $-\mathbf{P}$ such that both $\mathbf{P}$ and $-\mathbf{P}$ occur on the branch—and each sentence on that branch is one of the following:

1. A literal (an atomic sentence or the negation of an atomic sentence)
2. A sentence that is not universally quantified and that has been decomposed
3. A universally quantified sentence $(\forall \mathbf{x})\mathbf{P}$ such that at least one substitution instance of $(\forall \mathbf{x})\mathbf{P}$ occurs on the branch and, for each constant $\mathbf{a}$ occurring on the branch, $\mathbf{P}(\mathbf{a}/\mathbf{x})$ occurs on the branch

The set of sentences on a completed open branch has property (a) because there is no pair of literals **P** and $-$ **P** occurring on the branch. Every sentence that has one of the forms described in properties (b)–(j) and (l)–(n) has been decomposed (part 2 of the definition of a completed open branch), and so it is easily verified that the set of sentences on a completed open branch has those properties. (For example, if a sentence $--$ **P** occurs on a completed open branch and has been decomposed by an application of $--$ D, then **P** also occurs on that branch—which establishes property (b).) Finally the set of sentences on a completed open branch also has property (k), for part 3 of the definition of completed open branches stipulates that property (k) is satisfied. We conclude that the set of sentences occurring on a completed open branch has properties (a)–(n) and that the branch is therefore a Hintikka branch.

Now we turn to infinite branches. The System for tree construction was designed to guarantee that every infinite (nonterminating) branch is a Hintikka branch; we shall explain how it does so. First, a nonterminating branch is not closed (a branch that closes contains only finitely many sentences); so the set of sentences on such a branch must have property (a) of Hintikka sets. Second, the alternation of stages 1 and 2 of The System ensures that each nonliteral sentence that does not have the form $(\forall \mathbf{x})\mathbf{P}$ is decomposed a finite number of levels after the level on which it occurs, that for each universally quantified sentence $(\forall \mathbf{x})\mathbf{P}$ and constant **a** on a branch of the tree $\mathbf{P}(\mathbf{a}/\mathbf{x})$ is entered within a finite number of levels, and that at least one substitution instance $\mathbf{P}(\mathbf{a}/\mathbf{x})$ is entered. Each such addition yields only a finite number of levels, so every sentence on a branch must be decomposed if the branch is infinite. Therefore the set of sentences on a nonterminating branch satisfies properties (b)–(n) of Hintikka sets, as well as property (a). Every infinite branch of a systematic tree is therefore a Hintikka branch.

Finally we shall prove the Hintikka Set Lemma:

11.6.4 (the *Hintikka Set Lemma*): Every Hintikka set is quantificationally consistent.

From this it follows that if a systematic tree for a set Γ of sentences does not close then Γ is quantificationally consistent—for one of the branches that contains the sentences in Γ is a Hintikka branch.

Proof of Lemma 11.6.4: Let Γ be a Hintikka set of sentences of *PL*. We first associate with each individual constant of *PL* a distinct positive integer—**i** is associated with the alphabetically **i**th constant. We shall

prove that every sentence of Γ is true on the interpretation **I** defined as follows:

1. The UD is the set consisting of the positive integers that are associated with the individual constants occurring in members of Γ. If no member of Γ contains an individual constant, let the UD be the set {1}.
2. For each sentence letter **P**, $\mathbf{I}(\mathbf{P}) = \mathbf{T}$ if and only if $\mathbf{P} \in \Gamma$.
3. For each individual constant **a** that occurs in some sentence in Γ, $\mathbf{I}(\mathbf{a})$ is the positive integer associated with **a**. For each constant **a** that does not occur in any sentence of Γ, $\mathbf{I}(\mathbf{a})$ is the smallest positive integer in the UD (which, by specification 1, is nonempty).
4. For each **n**-place predicate **A**, $\mathbf{I}(\mathbf{A})$ includes all and only those **n**-tuples $\langle \mathbf{u}_1, \ldots, \mathbf{u}_n \rangle$ such that for some constants $\mathbf{a}_1, \ldots, \mathbf{a}_n$, $\mathbf{A}\mathbf{a}_1 \ldots \mathbf{a}_n \in \Gamma$ and $\langle \mathbf{I}(\mathbf{a}_1), \ldots, \mathbf{I}(\mathbf{a}_n) \rangle = \langle \mathbf{u}_1, \ldots, \mathbf{u}_n \rangle$.

We shall use mathematical induction to prove that every member of Γ is true on **I**. Our induction will not be on the number of occurrences of logical operators in a sentence since some of the clauses of the proof would not work in that case (see Exercise 11.6.5). Instead, we shall appeal to the *length* of a sentence. Where **P** is a formula of *PL*, let the *length* of **P** be the number of occurrences of sentence letters, predicates, and logical operators in **P**. No sentence of *PL* has length 0 since every sentence contains at least one sentence letter or predicate. So the basis clause begins with length 1.

Basis clause: Every sentence **P** of length 1 is such that if $\mathbf{P} \in \Gamma$ then $\mathbf{I}(\mathbf{P}) = \mathbf{T}$.

Inductive step: If every sentence **P** of length less than or equal to **k** is such that if $\mathbf{P} \in \Gamma$ then $\mathbf{I}(\mathbf{P}) = \mathbf{T}$, then every sentence **P** of length $\mathbf{k} + 1$ is such that if $\mathbf{P} \in \Gamma$ then $\mathbf{I}(\mathbf{P}) = \mathbf{T}$.

Conclusion: Every sentence **P** is such that if $\mathbf{P} \in \Gamma$, then $\mathbf{I}(\mathbf{P}) = \mathbf{T}$.

Proof of basis clause: A sentence of length 1 is an atomic sentence. (If a sentence contains any logical operators, then because it also must contain at least one sentence letter or predicate it has a length that is greater than 1.) If **P** is a sentence letter, then by part 2 of the definition of **I**, $\mathbf{I}(\mathbf{P}) = \mathbf{T}$ if $\mathbf{P} \in \Gamma$. If **P** is an atomic sentence of the form $\mathbf{A}\mathbf{a}_1 \ldots \mathbf{a}_n$, then, by part 4 of the definition of **I**, if $\mathbf{A}\mathbf{a}_1 \ldots \mathbf{a}_n \in \Gamma$ then $\langle \mathbf{I}(\mathbf{a}_1), \ldots, \mathbf{I}(\mathbf{a}_n) \rangle \in \mathbf{I}(\mathbf{A})$, and so $\mathbf{I}(\mathbf{P}) = \mathbf{T}$.

Proof of inductive step: We assume that the inductive hypothesis holds for some arbitrary positive integer **k**—that every sentence of length **k** or smaller that is a member of Γ is true on **I**. We must show that any

sentence **P** of length $\mathbf{k} + 1$ is also such that if it is a member of Γ then it is true on **I**. It is easy to verify that **P**, being nonatomic, must have one of the forms specified in properties (a)–(n) of Hintikka sets; and we shall consider each of these forms that **P** may have.

Case 1: **P** has the form $\sim \mathbf{Q}$, where **Q** is an atomic sentence. If $\sim \mathbf{Q} \in \Gamma$ then, by property (a) of Hintikka sets, $\mathbf{Q} \notin \Gamma$. If **Q** is a sentence letter then, by part 2 of the definition of **I**, $\mathbf{I}(\mathbf{Q}) = \mathbf{F}$ and so $\mathbf{I}(\sim \mathbf{Q}) = \mathbf{T}$.[8] If **Q** has the form $\mathbf{A}\mathbf{a}_1 \ldots \mathbf{a}_n$ then, by part 4 of the definition of **I**, $\langle \mathbf{I}(\mathbf{a}_1), \ldots, \mathbf{I}(\mathbf{a}_n)\rangle \notin \mathbf{I}(\mathbf{A})$. This is because each constant that occurs in some member of Γ designates a positive integer different from that designated by any other constant occurring in Γ (by part 3), and so there is no other set of constants occurring in Γ that also designate the members of the n-tuple $\langle \mathbf{I}(\mathbf{a}_1), \ldots, \mathbf{I}(\mathbf{a}_n)\rangle$, and consequently there can be no sentence $\mathbf{A}\mathbf{a}'_1 \ldots \mathbf{a}'_n$ in Γ such that $\langle \mathbf{I}(\mathbf{a}_1), \ldots, \mathbf{I}(\mathbf{a}_n)\rangle = \langle \mathbf{I}(\mathbf{a}'_1) \ldots \mathbf{I}(\mathbf{a}'_n)\rangle$. We may therefore conclude, from the fact that $\mathbf{A}\mathbf{a}_1 \ldots \mathbf{a}_n \notin \Gamma$, that the n-tuple $\langle \mathbf{I}(\mathbf{a}_1), \ldots, \mathbf{I}(\mathbf{a}_n)\rangle$ is not in the extension of **A**. So $\mathbf{I}(\mathbf{A}\mathbf{a}_1 \ldots \mathbf{a}_n) = \mathbf{F}$ and $\mathbf{I}(\sim \mathbf{A}\mathbf{a}_1 \ldots \mathbf{a}_n) = \mathbf{T}$.

Case 2: **P** has the form $\sim\sim \mathbf{Q}$. If $\sim\sim \mathbf{Q} \in \Gamma$ then, by property (b) of Hintikka sets, $\mathbf{Q} \in \Gamma$. The length of **Q** is less than $\mathbf{k} + 1$, so, by the inductive hypothesis, $\mathbf{I}(\mathbf{Q}) = \mathbf{T}$. Therefore $\mathbf{I}(\sim\sim \mathbf{Q}) = \mathbf{T}$ as well.

Case 3: **P** has the form $\mathbf{Q} \,\&\, \mathbf{R}$. If $\mathbf{Q} \,\&\, \mathbf{R} \in \Gamma$ then, by property (c) of Hintikka sets, $\mathbf{Q} \in \Gamma$ and $\mathbf{R} \in \Gamma$. By the inductive hypothesis (**Q** and **R** both having lengths less than $\mathbf{k} + 1$), $\mathbf{I}(\mathbf{Q}) = \mathbf{T}$ and $\mathbf{I}(\mathbf{R}) = \mathbf{T}$. So $\mathbf{I}(\mathbf{Q} \,\&\, \mathbf{R}) = \mathbf{T}$.

Case 4: **P** has the form $\sim (\mathbf{Q} \,\&\, \mathbf{R})$. If $\sim (\mathbf{Q} \,\&\, \mathbf{R}) \in \Gamma$ then, by property (d) of Hintikka sets, either $\sim \mathbf{Q} \in \Gamma$ or $\sim \mathbf{R} \in \Gamma$. The lengths of $\sim \mathbf{Q}$ and of $\sim \mathbf{R}$ are less than the length of $\sim (\mathbf{Q} \,\&\, \mathbf{R})$, so, by the inductive hypothesis, either $\mathbf{I}(\sim \mathbf{Q}) = \mathbf{T}$ or $\mathbf{I}(\sim \mathbf{R}) = \mathbf{T}$. If $\mathbf{I}(\sim \mathbf{Q}) = \mathbf{T}$, then $\mathbf{I}(\mathbf{Q}) = \mathbf{F}$, $\mathbf{I}(\mathbf{Q} \,\&\, \mathbf{R}) = \mathbf{F}$, and $\mathbf{I}(\sim (\mathbf{Q} \,\&\, \mathbf{R})) = \mathbf{T}$. If $\mathbf{I}(\sim \mathbf{R}) = \mathbf{T}$, then $\mathbf{I}(\sim (\mathbf{Q} \,\&\, \mathbf{R})) = \mathbf{T}$ as well. Either way, then, $\mathbf{I}(\sim (\mathbf{Q} \,\&\, \mathbf{R})) = \mathbf{T}$.

Cases 5–10: **P** has one of the forms $\mathbf{Q} \vee \mathbf{R}$, $\sim (\mathbf{Q} \vee \mathbf{R})$, $\mathbf{Q} \supset \mathbf{R}$, $\sim (\mathbf{Q} \supset \mathbf{R})$, $\mathbf{Q} \equiv \mathbf{R}$, or $\sim (\mathbf{Q} \equiv \mathbf{R})$ (see Exercise 3).

Case 11: **P** has the form $(\forall \mathbf{x})\mathbf{Q}$. If $\mathbf{P} \in \Gamma$ then, by property (k) of Hintikka sets, for every constant **a** that occurs in Γ, $\mathbf{Q}(\mathbf{a}/\mathbf{x}) \in \Gamma$ (and there is at least one such constant). Each substitution instance has a length smaller than $\mathbf{k} + 1$, so it follows from the inductive hypothesis that, for each of these sentences, $\mathbf{I}(\mathbf{Q}(\mathbf{a}/\mathbf{x})) = \mathbf{T}$. Moreover each member of the UD is designated by some constant occurring in Γ (by part 1 of the definition of the interpretation **I**—because at least one constant

[8] Note that we have here bypassed the intermediate step of observing that a sentence is true on an interpretation if and only if it is satisfied by every suitable assignment for that interpretation. We shall continue this practice until we reach Cases 11–14; the reader who so desires may fill in the intermediate steps for Cases 1–10 without much trouble.

occurs in Γ), so for each member of the UD there is a constant **a** such that $\mathbf{Q}(\mathbf{a}/\mathbf{x}) \in \Gamma$ and hence is true on **I**. It therefore follows from 11.6.5 that $\mathbf{I}((\forall \mathbf{x})\mathbf{Q}) = \mathbf{T}$.

11.6.5: Let **I** be an interpretation on which for each member **u** of the UD there is at least one constant **a** such that $\mathbf{I}(\mathbf{a}) = \mathbf{u}$ and $\mathbf{I}(\mathbf{P}(\mathbf{a}/\mathbf{x})) = \mathbf{T}$. Then $\mathbf{I}((\forall \mathbf{x})\mathbf{P}) = \mathbf{T}$.

Proof: See Exercise 4.

Case 12: **P** has the form $\sim(\forall \mathbf{x})\mathbf{Q}$. If $\sim(\forall \mathbf{x})\mathbf{Q} \in \Gamma$, then, by property (l) of Hintikka sets, $(\exists \mathbf{x}) \sim \mathbf{Q} \in \Gamma$, and by property (m) $\sim \mathbf{Q}(\mathbf{a}/\mathbf{x}) \in \Gamma$ for some constant **a**. $\sim \mathbf{Q}(\mathbf{a}/\mathbf{x})$ has a length less than $\mathbf{k} + 1$, so, by the inductive hypothesis, $\mathbf{I}(\sim \mathbf{Q}(\mathbf{a}/\mathbf{x})) = \mathbf{T}$ and therefore $\mathbf{I}(\mathbf{Q}(\mathbf{a}/\mathbf{x})) = \mathbf{F}$. Because $(\forall \mathbf{x})\mathbf{Q} \models \mathbf{Q}(\mathbf{a}/\mathbf{x})$ (result 11.1.4), it follows that $\mathbf{I}((\forall \mathbf{x})\mathbf{Q}) = \mathbf{F}$ and so $\mathbf{I}(\sim(\forall \mathbf{x})\mathbf{Q}) = \mathbf{T}$.

Cases 13 and 14: **P** has one of the forms $(\exists \mathbf{x})\mathbf{Q}$ or $\sim(\exists \mathbf{x})\mathbf{Q}$ (see Exercise 3).

That completes the proof of the inductive step. Therefore every sentence that is a member of the Hintikka set Γ is true on **I**, and this shows that Γ is quantificationally consistent.

The Hintikka Branch Lemma 11.6.3 and the Hintikka Set Lemma 11.6.4 can now be used to establish Metatheorem 11.6.1. If a systematic tree for a set Γ of sentences does not close, then the tree has at least one Hintikka branch (Lemma 11.6.3). The set of sentences on that Hintikka branch is quantificationally consistent (Lemma 11.6.4). Therefore, because every member of Γ lies on that branch (as well as on every other branch), Γ is quantificationally consistent. So if Γ is quantificationally *inconsistent*, then every systematic tree for Γ closes.

We note that the proof that we have just given is a *constructive* completeness proof. We have shown how, given a Hintikka branch of a systematic tree for a set of sentences Γ, to construct a model for Γ. This establishes a claim made in Chapter 9: An interpretation showing the quantificational consistency of Γ can always be constructed from a completed open branch of a tree for Γ.

Finally the tree method for predicate logic with identity and functions is also complete, and this can be shown by making appropriate changes in the proofs of Lemmas 11.6.3 and 11.6.4. We define a Hintikka set for *PLE* to be a set Γ that has the properties (a)–(n) of our earlier definition and that also has these properties:

o. No sentence of the form $\sim \mathbf{t} = \mathbf{t}$ is a member of Γ.

p. If $\mathbf{a} = \mathbf{t}$, where **a** is a constant, is a member of Γ, and a literal sentence $\mathbf{P} \in \Gamma$ contains **t**, then every sentence $\mathbf{P}(\mathbf{a}//\mathbf{t})$ is also a member of Γ.

q. If a complex term $f(\mathbf{a}_1, \ldots, \mathbf{a}_n)$ in which $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are individual constants occurs in any literal sentence in Γ, then, for at least one constant $\mathbf{b}$, $\mathbf{b} = f(\mathbf{a}_1, \ldots, \mathbf{a}_n) \in \Gamma$.

The proof of Lemma 11.6.3—that every systematic tree that does not close *has* a Hintikka branch—runs as before, except that we replace talk of the constant **a** occurring on a branch with talk of the closed term **t** occurring on a branch. We must also add the following:

> The set of sentences on a completed open branch of a systematic tree must have property (o), because by definition a branch that does not close does not contain a sentence of the form $\sim \mathbf{t} = \mathbf{t}$. Property (p) must hold by virtue of the requirement in clause 4 of the definition of a completed open branch. Property (q) must hold by virtue of the final component of the definition of a completed open branch.
>
> Similar remarks establish that infinite branches are also Hintikka branches: The cycle of stages 1–4 in The System, and the fact that each stage adds only a finite number of sentences, guarantees that every sentence on a branch will be decomposed if the branch is infinite. Note that the requirement in stage 2 of The System, that $\mathbf{P}(\mathbf{t}/\mathbf{x})$ be entered for a *complex* term **t** only if doing so will close the branch on which it is entered, plays a crucial role here. If The System allowed such substitutions for complex terms that did not close the branch, we could have a branch containing a sequence such as '$(\forall x)Gf(x)$', '$Gf(a)$', '$a = f(a)$', '$Gf(f(a))$', '$Gf(f(f(a)))$', '$Gf(f(f(f(a))))$', . . . in which '$f(a)$' has been continuously substituted for 'a' at the expense of decomposing other sentences on the branch. Similarly the restriction (ii) in stage 4 guarantees that we will not have an branch containing a sequence such as '$a = f(a)$', '$a = f(f(a))$', '$a = f(f(f(a)))$', . . . , or '$a = f(a)$', 'Ga', '$Gf(a)$', '$Gf(f(a))$', $Gf(f(f(a)))$, . . . in which '$f(a)$' has been continuously substituted for 'a' at the expense of decomposing other sentences on the branch.

To show that Lemma 11.6.4 holds for predicate logic with identity and functions, we must define the UD for the interpretation for a *PLE* Hintikka set Γ differently than it was defined for *PL*, as follows:

> Associate the positive integer **i** with the alphabetically **i**th individual constant of *PLE*. Let **p** designate this association and let $\mathbf{p}(\mathbf{a})$ stand for the integer that has been associated with the constant **a**. Next we define a second association, which we shall designate with **q**, as follows: $\mathbf{q}(\mathbf{a}) = \mathbf{p}(\mathbf{a}')$ if $\mathbf{a}'$ is the alphabetically earliest constant such that $\mathbf{a}' = \mathbf{a}$ is a member of Γ, and $\mathbf{q}(\mathbf{a}) = \mathbf{p}(\mathbf{a})$ otherwise.
>
> I. The UD is the set consisting of the positive integers that **q** assigns to the individual constants occurring in members of Γ. If no

member of Γ contains an individual constant, let the UD be the set {1}.

We change clause 3 in the interpretation constructed in Lemma 11.6.4 to this:

3. For each individual constant **a** that occurs in some sentence in Γ, $\mathbf{I}(\mathbf{a}) = \mathbf{q}(\mathbf{a})$. For each constant **a** that does not occur in any sentence of Γ, $\mathbf{I}(\mathbf{a})$ is the smallest positive integer in the UD.

We must also add a fifth clause to complete the definition of the interpretation:

5. For each **n**-place functor f, $\mathbf{I}(f)$ consists of all **n** + 1-tuples $\langle \mathbf{d}_1, \ldots, \mathbf{d}_n, \mathbf{d}_{n+1} \rangle$ of members of the UD such that either (i) there exist constants $\mathbf{a}_1, \ldots, \mathbf{a}_n, \mathbf{a}_{n+1}$ such that $\mathbf{a}_{n+1} = f(\mathbf{a}_1, \ldots, \mathbf{a}_n)$ is a member of Γ and $\mathbf{d}_i = \mathbf{I}(\mathbf{a}_i)$, $1 \leq i \leq n + 1$, or (ii) there are no such constants, and $\mathbf{d}_{n+1}$ is the smallest member of the UD.

We must ensure that clause 5 correctly defines the interpretation of an **n**-place functor as a function that assigns exactly one member of the UD to each **n**-tuple of members of the UD. It is clear that it assigns at least one member of the UD to each **n**-tuple, because if case (i) doesn't apply then case (ii) will. Moreover, if case (i) doesn't apply then case (ii) will assign at most one member. It remains to show that if case (i) applies it will also assign at most one member.

Now, if (i) assigned more than one member of the UD to some **n**-tuple of members of the UD, that would be because there existed constants $\mathbf{a}_1, \ldots, \mathbf{a}_n, \mathbf{a}_{n+1}$ such that $\mathbf{a}_{n+1} = f(\mathbf{a}_1, \ldots, \mathbf{a}_n)$ is a member of Γ and constants $\mathbf{b}_1, \ldots, \mathbf{b}_n, \mathbf{b}_{n+1}$ such that $\mathbf{b}_{n+1} = f(\mathbf{b}_1, \ldots, \mathbf{b}_n)$ is a member of Γ, where $\mathbf{I}(\mathbf{a}_i) = \mathbf{I}(\mathbf{b}_i)$ for $1 \leq i \leq n$ and $\mathbf{I}(\mathbf{a}_{n+1}) \neq \mathbf{I}(\mathbf{b}_{n+1})$. But note that if $\mathbf{I}(\mathbf{a}_i) = \mathbf{I}(\mathbf{b}_i)$ then, by the way we defined **I** via the association **q**, either (a) $\mathbf{a}_i = \mathbf{b}_i$ is a member of Γ, or (b) $\mathbf{b}_i = \mathbf{a}_i$ is a member of Γ, or (c) there is a constant $\mathbf{c}_i$ such that both $\mathbf{c}_i = \mathbf{a}_i$ and $\mathbf{c}_i = \mathbf{b}_i$ are members of Γ. We now perform the following substitutions in the sentences $\mathbf{a}_{n+1} = f(\mathbf{a}_1, \ldots, \mathbf{a}_n)$ and $\mathbf{b}_{n+1} = f(\mathbf{b}_1, \ldots, \mathbf{b}_n)$. For each $\mathbf{a}_i$ and $\mathbf{b}_i$ occurring on the right-hand side of these identities:

If (a) holds then replace $\mathbf{b}_i$ with $\mathbf{a}_i$ in $\mathbf{b}_{n+1} = f(\mathbf{b}_1, \ldots, \mathbf{b}_n)$.
Otherwise, if (b) holds then replace $\mathbf{a}_i$ with $\mathbf{b}_i$ in $\mathbf{a}_{n+1} = f(\mathbf{a}_1, \ldots, \mathbf{a}_n)$.
Otherwise, if (c) holds then replace $\mathbf{a}_i$ with $\mathbf{c}_i$ in $\mathbf{a}_{n+1} = f(\mathbf{a}_1, \ldots, \mathbf{a}_n)$ and replace $\mathbf{b}_i$ with $\mathbf{c}_i$ in $\mathbf{b}_{n+1} = f(\mathbf{b}_1, \ldots, \mathbf{b}_n)$.

Note that each replacement generates a sentence that is also a member of Γ by property (p) of Hintikka sets, and at the end of the replacements, the right-hand sides of the final identity statements will be identical. That is, we have shown that there are constants $\mathbf{d}_1, \ldots, \mathbf{d}_n$ such that both $\mathbf{a}_{n+1} = f(\mathbf{d}_1, \ldots, \mathbf{d}_n)$ and $\mathbf{b}_{n+1} = f(\mathbf{d}_1, \ldots, \mathbf{d}_n)$ are also members of the Hintikka set. By virtue of property (o), it follows that $\mathbf{a}_{n+1} = \mathbf{b}_{n+1}$, which is $\mathbf{a}_{n+1} = f(\mathbf{d}_1, \ldots, \mathbf{d}_n)$

$(\mathbf{b}_{n+1}//f(\mathbf{d}_1, \ldots, \mathbf{d}_n))$, is also in the Hintikka set, and so are $\mathbf{b}_{n+1} = \mathbf{a}_{n+1}$, $\mathbf{b}_{n+1} = \mathbf{b}_{n+1}$, and $\mathbf{a}_{n+1} = \mathbf{a}_{n+1}$, all by property (p). But then $\mathbf{I}(\mathbf{a}_{n+1}) = \mathbf{I}(\mathbf{b}_{n+1})$, contrary to our previous assumption. For it follows from the construction of $\mathbf{I}$ that, if no identity sentence with a constant that is alphabetically earlier than $\mathbf{a}_{n+1}$ or $\mathbf{b}_{n+1}$ occurring on the left-hand side is a member of Γ, then both constants will denote either $\mathbf{p}(\mathbf{a}_{n+1})$ or $\mathbf{p}(\mathbf{b}_{n+1})$, depending on which is alphabetically earlier. Or, if there is a constant $\mathbf{c}$ that is alphabetically earlier than both $\mathbf{a}_{n+1}$ and $\mathbf{b}_{n+1}$ such that either $\mathbf{c} = \mathbf{a}_{n+1}$ or $\mathbf{c} = \mathbf{b}_{n+1}$ is a member of Γ, then, because the identity sentences $\mathbf{a}_{n+1} = \mathbf{b}_{n+1}$ and $\mathbf{b}_{n+1} = \mathbf{a}_{n+1}$ are both members of Γ, it follows that both $\mathbf{c} = \mathbf{a}_{n+1}$ and $\mathbf{c} = \mathbf{b}_{n+1}$ must be members of Γ as well, and hence both $\mathbf{I}(\mathbf{a}_{n+1})$ and $\mathbf{I}(\mathbf{b}_{n+1})$ are defined to be $\mathbf{p}(\mathbf{c})$ for the alphabetically earliest such constant $\mathbf{c}$. We conclude that case (i) will not assign more than one member of the UD to any given $\mathbf{n}$-tuple of members of the UD.

We shall use the following result in the proof that every member of a Hintikka set is true on the interpretation $\mathbf{I}$ we have just defined:

> **11.6.6:** If a Hintikka set Γ contains a literal sentence $\mathbf{P}$ with a closed complex term $\mathbf{t}$, it also contains each sentence $\mathbf{P}(\mathbf{a}//\mathbf{t})$ for some constant $\mathbf{a}$ such that $\text{den}_{\mathbf{I}}(\mathbf{a}) = \text{den}_{\mathbf{I}}(\mathbf{t})$,[5] where $\mathbf{I}$ is the interpretation that has just been defined for the Hintikka set.
>
> **Proof:** We shall prove this using mathematical induction on the *complexity* of $\mathbf{t}$, which is defined recursively as follows:
>
> > If $\mathbf{t}$ is $f(\mathbf{a}_1, \ldots, \mathbf{a}_n)$, where each $\mathbf{a}_i$ is a constant, the complexity of $\mathbf{t}$ is 1.
> >
> > If $\mathbf{t}$ is $f(\mathbf{t}_1, \ldots, \mathbf{t}_n)$, where some $\mathbf{t}_i$ is not a constant, the complexity of $\mathbf{t}$ is 1 greater than the maximum complexity of the terms $\mathbf{t}_1, \ldots, \mathbf{t}_n$.
>
> So 'f(a, b)' has complexity 1, '$f(g(\text{a}), g(\text{b}))$' has complexity 2, and '$f(g(h(\text{a})), g(\text{b}))$' has complexity 3.
>
> *Basis clause:* 11.6.6 holds for every closed complex term $\mathbf{t}$ of complexity 1.
>
> **Proof of basis clause:** In this case the Hintikka set contains an identity sentence $\mathbf{a} = \mathbf{t}$, where $\mathbf{a}$ is a constant, by property (q) of Hintikka sets, and so, by property (p), it follows that each sentence $\mathbf{P}(\mathbf{a}//\mathbf{t})$ is also a member of Γ. In addition, $\text{den}_{\mathbf{I}}(\mathbf{t}) = \text{den}_{\mathbf{I}}(\mathbf{a})$ by the way that $\mathbf{I}(f)$ is defined in clause 5, since the Hintikka set contains the identity sentence $\mathbf{a} = \mathbf{t}$.
>
> *Inductive step:* If 11.6.6 holds for every closed complex term $\mathbf{t}$ of complexity $\mathbf{k}$ or less, then 11.6.6 holds for every closed complex term $\mathbf{t}$ of complexity $\mathbf{k} + 1$.

[5] Because all of the terms mentioned in this proof are closed terms, we omit reference to a variable assignment $\mathbf{d}$ when referring to the denotation of a term. We do so because the denotation is independent of any particular variable assignment.

Proof of inductive step: We assume that the inductive hypothesis holds—that is, that 11.6.6 is true of every closed complex term of complexity **k** or less. We must show that it follows that 11.6.6 also holds of every closed complex term of complexity **k** + 1. Let **t** be a closed complex term of complexity **k** + 1. Then **t** is $f(\mathbf{t}_1, \ldots, \mathbf{t}_n)$, where each $\mathbf{t}_i$ is of complexity **k** or less. It follows, by the inductive hypothesis, that for some constants $\mathbf{a}_1, \ldots, \mathbf{a}_n$ such that $\text{den}_\mathfrak{I}(\mathbf{a}_i) = \text{den}_\mathfrak{I}(\mathbf{t}_i)$, each formula that results from replacing one or more occurrences of $f(\mathbf{t}_1, \ldots, \mathbf{t}_n)$ in **P** with $f(\mathbf{a}_1, \ldots, \mathbf{a}_n)$ is a member of Γ. By property (q) of Hintikka sets, there is some constant **a** such that $\mathbf{a} = f(\mathbf{a}_1, \ldots, \mathbf{a}_n)$ is a member of Γ, and so, by property (p), each sentence $\mathbf{P}(\mathbf{a}//\mathbf{t})$ is also a member of Γ. Moreover, because the identity sentence $\mathbf{a} = f(\mathbf{a}_1, \ldots, \mathbf{a}_n)$ is a member of Γ, it follows from the definition of $\mathfrak{I}(f)$ in clause 5 that $\text{den}_\mathfrak{I}(\mathbf{a}) = \text{den}_\mathfrak{I}(f(\mathbf{a}_1, \ldots, \mathbf{a}_n))$, and because $\text{den}_\mathfrak{I}(\mathbf{a}_i) = \text{den}_\mathfrak{I}(\mathbf{t}_i)$ for each $\mathbf{a}_i$ and $\mathbf{t}_i$, it follows that $\text{den}_\mathfrak{I}(f(\mathbf{a}_1, \ldots, \mathbf{a}_n)) = \text{den}_\mathfrak{I}(f(\mathbf{t}_1, \ldots, \mathbf{t}_n))$; so $\text{den}_\mathfrak{I}(\mathbf{a}) = \text{den}_\mathfrak{I}(f(\mathbf{t}_1, \ldots, \mathbf{t}_n))$.

The proof of the basis clause of the inducive proof in 11.6.4 that every member of a Hintikka set Γ is true on the interpretation $\mathfrak{I}$ that we have just defined is changed as follows:

Proof of basis clause: A sentence of length 1 is an atomic sentence. If **P** is a sentence letter, then by part 2 of the definition of $\mathfrak{I}$, $\mathfrak{I}(\mathbf{P}) = \mathbf{T}$ if $\mathbf{P} \in \Gamma$.

If **P** is an atomic sentence of the form $\mathbf{At}_1 \ldots \mathbf{t}_n$, where **A** is not the identity predicate, then it follows from 11.6.6 that Γ also contains a sentence $\mathbf{Aa}_1 \ldots \mathbf{a}_n$ such that each $\mathbf{a}_i$ is a constant and $\text{den}_\mathfrak{I}(\mathbf{t}_i) = \text{den}_\mathfrak{I}(\mathbf{a}_i)$. By part 4 of the definition of $\mathfrak{I}$, if $\mathbf{Aa}_1 \ldots \mathbf{a}_n \in \Gamma$, then $\langle \mathfrak{I}(\mathbf{a}_1), \ldots, \mathfrak{I}(\mathbf{a}_n) \rangle \in \mathfrak{I}(\mathbf{A})$, and so $\mathfrak{I}(\mathbf{Aa}_1 \ldots \mathbf{a}_n) = \mathfrak{I}(\mathbf{At}_1 \ldots \mathbf{t}_n) = \mathbf{T}$.

If **P** is a sentence of the form $\mathbf{t}_1 = \mathbf{t}_2$, then it follows from 11.6.6 that Γ also contains a sentence $\mathbf{a}_1 = \mathbf{a}_2$ such that $\mathbf{a}_1$ and $\mathbf{a}_2$ are constants and $\text{den}_\mathfrak{I}(\mathbf{a}_1) = \mathbf{q}(\mathbf{a}_1) = \text{den}_\mathfrak{I}(\mathbf{t}_1)$ and $\text{den}_\mathfrak{I}(\mathbf{a}_2) = \mathbf{q}(\mathbf{a}_1) = \text{den}_\mathfrak{I}(\mathbf{t}_2)$. Since $\mathbf{a}_1 = \mathbf{a}_2$ is a member of Γ, it follows, by property (p) of *PLE*'s Hintikka sets, that $\mathbf{a}_1 = \mathbf{a}_1$ is also a member of Γ. Now, let $\mathbf{q}(\mathbf{a}_1)$ be $\mathbf{p}(\mathbf{b})$. Because $\mathbf{a}_1 = \mathbf{a}_1$ is a member of Γ, it follows by the way that **q** was defined that $\mathbf{b} = \mathbf{a}_1$ is a member of Γ and that **b** must be alphabetically earlier than or identical to $\mathbf{a}_1$. It also follows by property (p) that $\mathbf{b} = \mathbf{a}_2$ is a member of Γ. Let $\mathbf{q}(\mathbf{a}_2)$ be $\mathbf{p}(\mathbf{c})$. Since $\mathbf{a}_1 = \mathbf{a}_2$ is a member of Γ, it follows that **c** is the alphabetically earliest constant such that $\mathbf{c} = \mathbf{a}_2$ is a member of Γ. Now, **b** cannot be alphabetically earlier than **c**, since $\mathbf{b} = \mathbf{a}_2$ is a member of Γ. Further, by property (p), $\mathbf{c} = \mathbf{b}$ is also a member of Γ. Therefore $\mathbf{c} = \mathbf{a}_1$ is also a member of Γ, and so **c** cannot be alphabetically earlier than **b** since $\mathbf{q}(\mathbf{a}_1)$ is $\mathbf{p}(\mathbf{b})$. We conclude that **b** and **c** are the same constant, so $\mathbf{q}(\mathbf{a}_1) = \mathbf{q}(\mathbf{a}_2)$. Consequently $\text{den}_\mathfrak{I}(\mathbf{t}_1) = \text{den}_\mathfrak{I}(\mathbf{a}_1) = \mathbf{q}(\mathbf{a}_1) = \mathbf{q}(\mathbf{a}_2) = \text{den}_\mathfrak{I}(\mathbf{a}_2) = \text{den}_\mathfrak{I}(\mathbf{t}_2)$, so $\mathfrak{I}(\mathbf{t}_1 = \mathbf{t}_2) = \mathbf{T}$.

We must also change the proof of Case 1 in the inductive step:

Case 1: **P** has the form $\sim \mathbf{Q}$, where **Q** is an atomic sentence.

If **Q** is a sentence letter, then if $\sim \mathbf{Q} \in \Gamma$ it follows from part 2 of the definition of **I** that $\mathbf{I}(\mathbf{Q}) = \mathbf{F}$ since by property (a) of Hintikka sets $\mathbf{Q} \notin \Gamma$. Therefore $\mathbf{I}(\sim \mathbf{Q}) = \mathbf{T}$.

If **Q** has the form $\mathbf{At}_1 \ldots \mathbf{t}_n$, where **A** is not the identity predicate and $\sim \mathbf{Q} \in \Gamma$, then, by 11.6.6, there is a formula $\sim \mathbf{Aa}_1 \ldots \mathbf{a}_n$ in Γ in which every complex term $\mathbf{t}_i$ occurring in $\sim \mathbf{Q}$ has been replaced by a constant $\mathbf{a}_i$ such that $\mathrm{den}_{\mathbf{I}}(\mathbf{t}_i) = \mathrm{den}_{\mathbf{I}}(\mathbf{a}_i)$. By property (a) of Hintikka sets, $\mathbf{Aa}_1 \ldots \mathbf{a}_n \notin \Gamma$. It follows from part 4 of the definition of **I** that $\langle \mathbf{I}(\mathbf{a}_1), \ldots, \mathbf{I}(\mathbf{a}_n) \rangle \notin \mathbf{I}(\mathbf{A})$. For if it were, this would be because, for some $\mathbf{a}'_1, \ldots, \mathbf{a}'_n$, $\mathbf{Aa}'_1 \ldots \mathbf{a}'_n \in \Gamma$ and $\mathbf{I}(\mathbf{a}_i) = \mathbf{I}(\mathbf{a}'_i)$ for each $\mathbf{a}_i$. Because each of these constants occurs in a member of Γ, we would have $\mathbf{I}(\mathbf{a}_i) = \mathbf{q}(\mathbf{a}_i)$ and $\mathbf{I}(\mathbf{a}'_i) = \mathbf{q}(\mathbf{a}'_i)$ and so $\mathbf{q}(\mathbf{a}_i) = \mathbf{q}(\mathbf{a}'_i)$. From the last equation and the way that **q** was defined, it would follow for each **i** that either (a) $\mathbf{a}_i = \mathbf{a}'_i$ is a member of Γ, or (b) $\mathbf{a}'_i = \mathbf{a}_i$ is a member of Γ, or (c) there is a constant $\mathbf{c}_i$ such that both $\mathbf{c}_i = \mathbf{a}_i$ and $\mathbf{c}_i = \mathbf{a}'_i$ are members of Γ. We now perform the following substitutions in the sentences $\sim \mathbf{Aa}_1 \ldots \mathbf{a}_n$ and $\mathbf{Aa}'_1 \ldots \mathbf{a}'_n$. For each $\mathbf{a}_i$ and $\mathbf{a}'_i$:

> If (a) holds then replace $\mathbf{a}'_i$ with $\mathbf{a}_i$ in $\mathbf{Aa}'_1 \ldots \mathbf{a}'_n$.
> Otherwise, if (b) holds then replace $\mathbf{a}_i$ with $\mathbf{a}'_i$ in $\sim \mathbf{Aa}_1 \ldots \mathbf{a}_n$.
> Otherwise, if (c) holds then replace $\mathbf{a}_i$ with $\mathbf{c}_i$ in $\sim \mathbf{Aa}_1 \ldots \mathbf{a}_n$ and replace $\mathbf{a}'_i$ with $\mathbf{c}_i$ in $\mathbf{Aa}'_1 \ldots \mathbf{a}'_n$.

Note that each replacement generates a sentence that is also a member of Γ by property (p) of Hintikka sets, and at the end of the replacements, we shall have two literal sentences, one of which is the negation of the other and both of which are members of Γ. But this is impossible because of property (a) of Hintikka sets. So, since we have now established that $\langle \mathbf{I}(\mathbf{a}_1), \ldots, \mathbf{I}(\mathbf{a}_n) \rangle \notin \mathbf{I}(\mathbf{A})$, it follows that $\mathbf{I}(\sim \mathbf{Aa}_1 \ldots \mathbf{a}_n) = \mathbf{T}$. Consequently, because $\mathrm{den}_{\mathbf{I}}(\mathbf{t}_i) = \mathrm{den}_{\mathbf{I}}(\mathbf{a}_i)$ for each **I**, it also follows that $\mathbf{I}(\sim \mathbf{At}_1 \ldots \mathbf{t}_n)$ (that is, $\mathbf{I}(\sim \mathbf{Q})$) $= \mathbf{T}$.

If **Q** has the form $\mathbf{t}_1 = \mathbf{t}_2$ and $\sim \mathbf{Q} \in \Gamma$ then, by 11.6.6, there is a formula $\sim \mathbf{a}_1 = \mathbf{a}_2$ in Γ such that $\mathrm{den}_{\mathbf{I}}(\mathbf{t}_i) = \mathrm{den}_{\mathbf{I}}(\mathbf{a}_i)$. By property (a) of Hintikka sets, $\mathbf{a}_1 = \mathbf{a}_2 \notin \Gamma$. It follows from the definition of **I** that $\mathbf{q}(\mathbf{a}_1) \neq \mathbf{q}(\mathbf{a}_2)$. For if $\mathbf{q}(\mathbf{a}_1) = \mathbf{q}(\mathbf{a}_2)$, then either (a) $\mathbf{a}_1 = \mathbf{a}_2$ is a member of Γ, or (b) $\mathbf{a}_2 = \mathbf{a}_1$ is a member of Γ, or (c) there is a constant **b** such that $\mathbf{b} = \mathbf{a}_1$ and $\mathbf{b} = \mathbf{a}_2$ are both members of Γ. We have already shown that (a) does not hold. Case (b) does not hold, because if it did then, by property (p) of Hintikka sets, $\sim \mathbf{a}_2 = \mathbf{a}_2$ would be a member of Γ since $\sim \mathbf{a}_1 = \mathbf{a}_2$ is, but that is impossible by

property (o) of Hintikka sets. Case (c) does not hold, because if did then, by property (p) of Hintikka sets, $\sim \mathbf{b} = \mathbf{b}$ would be a member of Γ since $\sim \mathbf{a}_1 = \mathbf{a}_2$ is, but that is impossible by property (o) of Hintikka sets. Since $\mathbf{q}(\mathbf{a}_1) \neq \mathbf{q}(\mathbf{a}_2)$, $\mathbf{I}(\mathbf{a}_1 = \mathbf{a}_2) = \mathbf{F}$ and $\mathbf{I}(\sim \mathbf{a}_1 = \mathbf{a}_2) = \mathbf{T}$, so, since $\text{den}_{\mathbf{I}}(\mathbf{t}_i) = \text{den}_{\mathbf{I}}(\mathbf{a}_i)$ it follows that $\mathbf{I}(\sim \mathbf{t}_1 = \mathbf{t}_2) = \mathbf{T}$.

11.6E EXERCISES

1. Using Metatheorem 11.6.1, prove the following:

a. If **P** is quantificationally false, then every systematic tree for {**P**} closes.

b. If **P** is quantificationally true, then every systematic tree for {$\sim$ **P**} closes.

*c. If **P** and **Q** are quantificationally equivalent, then every systematic tree for {$\sim$ (**P** $\equiv$ **Q**)} closes.

d. If $\Gamma \models \mathbf{P}$, where Γ is finite, then every systematic tree for $\Gamma \cup \{\sim \mathbf{P}\}$ closes.

*e. If an argument of *PL* is quantificationally valid, then every systematic tree for the set consisting of the premises and the negation of the conclusion of that argument closes.

2.a. What is the length of each of the following sentences?

$(\forall y)\text{Wy} \supset \sim (\forall y)\text{Bya}$
$(\exists x)\text{Sxbc}$
$(\forall x)(\text{Mx} \equiv \sim (\exists y)\text{My})$

b. Show that the length of a sentence $\sim$ (**Q** & **R**) is greater than the length of $\sim$ **Q** and greater than the length of $\sim$ **R**.

*c. Show that the length of a sentence **Q** $\equiv$ **R** is greater than the length of $\sim$ **Q** and greater than the length of $\sim$ **R**.

d. Show that the length of a sentence $\sim (\forall \mathbf{x})\mathbf{Q}$ is greater than the length of $\sim \mathbf{Q}(\mathbf{a}/\mathbf{x})$.

3. Complete the following clauses in the inductive proof of the completeness of the tree method.

a. 5
*b. 6
c. 7
*d. 8
*e. 9
f. 10
g. 13
*h. 14

***4.** Prove result 11.6.5.

5. Which clauses in the inductive proof of the completeness of the tree method would have broken down if our induction had been on the number of occurrences of logical operators in the sentences of *PL*?

6. If the rule $\exists$D were not included in our tree rules, where would the proof of Metatheorem 11.6.1 break down?

7. Suppose that our rule ~ ∀D were replaced by the following rule:

   *Negated Universal Decomposition** (~ ∀D*)

   ~ (∀**x**)**P**

   ~ **P**(**a**/**x**)

   where **a** is a constant foreign to all preceding lines of the tree.

   Would the resulting system be complete for predicate logic? Explain.

8. Explain how we can adapt the proof of Metatheorem 11.6.1 to establish that the tree method for *SL* is complete for sentential logic.

9. Prove that every set of *PL* that is both maximally consistent and ∃-complete (as defined in Section 11.4) is a Hintikka set. Prove that every Hintikka set is ∃-complete. Prove that some Hintikka sets are not maximally consistent in *PD*.

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INDEX

INDEX OF SYMBOLS

DERIVATION RULES OF *SD*

Reiteration (R)

| P
▷ | P

'&' Rules

Conjunction Introduction (&I)

| P
| Q
▷ | P & Q

Conjunction Elimination (&E)

| P & Q
▷ | P

or

| P & Q
▷ | Q

'⊃' Rules

Conditional Introduction (⊃I)

| | P
| | Q
▷ | P ⊃ Q

Conditional Elimination (⊃E)

| P ⊃ Q
| P
▷ | Q

'~' Rules

Negation Introduction (~ I)

| | P
| | Q
| | ~ Q
▷ | ~ P

Negation Elimination (~ E)

| | ~ P
| | Q
| | ~ Q
▷ | P

'∨' Rules

Disjunction Introduction (∨I)

| P
▷ | P ∨ Q

or

| P
▷ | Q ∨ P

Disjunction Elimination (∨E)

| P ∨ Q
| | P
| | R
| | Q
| | R
▷ | R

'≡' Rules

Biconditional Introduction (≡I)

| | P
| | Q
| | Q
| | P
▷ | P ≡ Q

Biconditional Elimination (≡E)

| P ≡ Q
| P
▷ | Q

or

| P ≡ Q
| Q
▷ | P

DERIVATION RULES OF *SD+*

All the Derivation Rules of *SD* and Rules of Inference

Modus Tollens (MT)

$$\begin{array}{l|l} & \mathbf{P \supset Q} \\ & \mathbf{\sim Q} \\ \rhd & \mathbf{\sim P} \end{array}$$

Hypothetical Syllogism (HS)

$$\begin{array}{l|l} & \mathbf{P \supset Q} \\ & \mathbf{Q \supset R} \\ \rhd & \mathbf{P \supset R} \end{array}$$

Disjunctive Syllogism (DS)

$$\begin{array}{l|l} & \mathbf{P \vee Q} \\ & \mathbf{\sim P} \\ \rhd & \mathbf{Q} \end{array} \quad \text{or} \quad \begin{array}{l|l} & \mathbf{P \vee Q} \\ & \mathbf{\sim Q} \\ \rhd & \mathbf{P} \end{array}$$

Rules of Replacement

Commutation (Com)

$\mathbf{P \,\&\, Q \mathrel{\lhd\rhd} Q \,\&\, P}$
$\mathbf{P \vee Q \mathrel{\lhd\rhd} Q \vee P}$

Association (Assoc)

$\mathbf{P \,\&\, (Q \,\&\, R) \mathrel{\lhd\rhd} (P \,\&\, Q) \,\&\, R}$
$\mathbf{P \vee (Q \vee R) \mathrel{\lhd\rhd} (P \vee Q) \vee R}$

Implication (Impl)

$\mathbf{P \supset Q \mathrel{\lhd\rhd} \sim P \vee Q}$

Double Negation (DN)

$\mathbf{P \mathrel{\lhd\rhd} \sim\sim P}$

De Morgan (DeM)

$\mathbf{\sim (P \,\&\, Q) \mathrel{\lhd\rhd} \sim P \vee \sim Q}$
$\mathbf{\sim (P \vee Q) \mathrel{\lhd\rhd} \sim P \,\&\, \sim Q}$

Idempotence (Idem)

$\mathbf{P \mathrel{\lhd\rhd} P \,\&\, P}$
$\mathbf{P \mathrel{\lhd\rhd} P \vee P}$

Transposition (Trans)

$\mathbf{P \supset Q \mathrel{\lhd\rhd} \sim Q \supset \sim P}$

Exportation (Exp)

$\mathbf{P \supset (Q \supset R) \mathrel{\lhd\rhd} (P \,\&\, Q) \supset R}$

Distribution (Dist)

$\mathbf{P \,\&\, (Q \vee R) \mathrel{\lhd\rhd} (P \,\&\, Q) \vee (P \,\&\, R)}$
$\mathbf{P \vee (Q \,\&\, R) \mathrel{\lhd\rhd} (P \vee Q) \,\&\, (P \vee R)}$

Equivalence (Equiv)

$\mathbf{P \equiv Q \mathrel{\lhd\rhd} (P \supset Q) \,\&\, (Q \supset P)}$
$\mathbf{P \equiv Q \mathrel{\lhd\rhd} (P \,\&\, Q) \vee (\sim P \,\&\, \sim Q)}$

DERIVATION RULES OF *PD*

All the Derivation Rules of *SD* and

Universal Introduction (∀I)

▷ | **P**(**a**/**x**)
▷ | (∀**x**)**P**

provided that:
(i) **a** does not occur in an open assumption.
(ii) **a** does not occur in (∀**x**)**P**.

Universal Elimination (∀E)

▷ | (∀**x**)**P**
▷ | **P**(**a**/**x**)

Existential Introduction (∃I)

▷ | **P**(**a**/**x**)
▷ | (∃**x**)**P**

Existential Elimination (∃E)

▷ | (∃**x**)**P**
▷ | | **P**(**a**/**x**)
▷ | | **Q**
▷ | **Q**

provided that:
(i) **a** does not occur in an open assumption.
(ii) **a** does not occur in (∃**x**)**P**.
(iii) **a** does not occur in **Q**.

DERIVATION RULES OF *PD+*

All the Derivation Rules of *SD+* and of *PD* and

Quantifier Negation (QN)

– (∀**x**)**P** ◁ ▷ (∃**x**) – **P**
– (∃**x**)**P** ◁ ▷ (∀**x**) – **P**

DERIVATION RULES OF *PDE*

All of the Derivation Rules of *PD* and

Universal Elimination (∀E)

$$\begin{array}{l|l} \triangleright & (\forall \mathbf{x})\mathbf{P} \\ \triangleright & \mathbf{P}(\mathbf{t}/\mathbf{x}) \end{array}$$

where **t** is any closed term

Existential Introduction (∃I)

$$\begin{array}{l|l} \triangleright & \mathbf{P}(\mathbf{t}/\mathbf{x}) \\ \triangleright & (\exists \mathbf{x})\mathbf{P} \end{array}$$

where **t** is any closed term

Identity Introduction (=I)

$$\triangleright \mid (\forall \mathbf{x})\mathbf{x} = \mathbf{x}$$

Identity Elimination (=E)

$$\begin{array}{l|l} & \mathbf{t}_1 = \mathbf{t}_2 \\ & \mathbf{P} \\ \triangleright & \mathbf{P}(\mathbf{t}_1//\mathbf{t}_2) \end{array} \quad \text{or} \quad \begin{array}{l|l} & \mathbf{t}_1 = \mathbf{t}_2 \\ & \mathbf{P} \\ \triangleright & \mathbf{P}(\mathbf{t}_2//\mathbf{t}_1) \end{array}$$

where $\mathbf{t}_1$ and $\mathbf{t}_2$ are closed terms

TRUTH-TREE RULES FOR *SL*

Negated Negation Decomposition (~ ~ D)

~ ~ **P**✓
P

Conjunction Decomposition (&D)

P & **Q**✓
P
Q

Negated Conjunction Decomposition (~ &D)

~ (**P** & **Q**)✓
~ **P** | ~ **Q**

Disjunction Decomposition (∨D)

P ∨ **Q**✓
P | **Q**

Negated Disjunction Decomposition (~ ∨D)

~ (**P** ∨ **Q**)✓
~ **P**
~ **Q**

Conditional Decomposition (⊃D)

P ⊃ **Q**✓
~ **P** | **Q**

Negated Conditional Decomposition (~ ⊃D)

~ (**P** ⊃ **Q**)✓
P
~ **Q**

Biconditional Decomposition (≡D)

P ≡ **Q**✓
P | ~ **P**
Q | ~ **Q**

Negated Biconditional Decomposition (~ ≡D)

~ (**P** ≡ **Q**)✓
P | ~ **P**
~ **Q** | **Q**

TRUTH-TREE RULES FOR *PL*

Universal Decomposition (∀D)

(∀**x**)**P**
P(**a**/**x**)

Negated Universal Decomposition (~ ∀D)

~ (∀**x**)**P**✓
(∃**x**) ~ **P**

Existential Decomposition (∃D)

(∃**x**)**P**✓
P(**a**/**x**)

where **a** is a constant foreign to the branch on which **P**(**a**/**x**) is entered.

Existential Negated Decomposition (~ ∃D)

~ (∃**x**)**P**✓
(∀**x**) ~ **P**

ALTERNATE EXISTENTIAL DECOMPOSITION RULE

Existential Decomposition-2 (∃D2)

$$(\exists \mathbf{x})\mathbf{P}$$

$$\mathbf{P}(\mathbf{a}_1/\mathbf{x}) \quad \ldots \quad \mathbf{P}(\mathbf{a}_m/\mathbf{x}) \quad \mathbf{P}(\mathbf{a}_{m+1}/\mathbf{x})$$

where $\mathbf{a}_1, \ldots, \mathbf{a}_m$ are the constants that already occur on the branch on which Existential Decomposition-2 is being applied to decompose $(\exists \mathbf{x})\mathbf{P}$ and $\mathbf{a}_{m+1}$ is a constant that is foreign to that branch.[1]

TRUTH-TREE RULES FOR *PLE*

All of the Predicate Truth-Tree Rules and

Universal Decomposition (∀D)

$$(\forall \mathbf{x})\mathbf{P}$$

$$\mathbf{P}(\mathbf{t}/\mathbf{x})$$

where $\mathbf{t}$ is a closed term

Identity Decomposition (=D)

$$\mathbf{t}_1 = \mathbf{t}_2$$

$$\mathbf{P}$$

$$\mathbf{P}(\mathbf{t}_1//\mathbf{t}_2)$$

where $\mathbf{t}_1$ and $\mathbf{t}_2$ are closed individual terms and $\mathbf{P}$ is a literal containing $\mathbf{t}_2$

Complex Term Decomposition (CTD)

$$\ldots f(\mathbf{a}_1, \ldots, \mathbf{a}_n) \ldots$$

$$\mathbf{b}_1 = f(\mathbf{a}_1, \ldots, \mathbf{a}_n) \quad \ldots \quad \mathbf{b}_m = f(\mathbf{a}_1, \ldots, \mathbf{a}_n) \quad \mathbf{b}_{m+1} = f(\mathbf{a}_1, \ldots, \mathbf{a}_n)$$

where $f(\mathbf{a}_1, \ldots, \mathbf{a}_n)$ is a closed complex term occurring within a literal on some branch, whose arguments $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are individual constants; $\mathbf{b}_1, \ldots, \mathbf{b}_m$ are the constants that already occur on that branch, and $\mathbf{b}_{m+1}$ is a constant that is foreign to that branch.

[1] This Existential Decomposition rule is due to George Boolos, "Trees and Finite Satisfiability: Proof of a Conjecture of Burgess," *Notre Dame Journal of Formal Logic*, 25(3)(1984), 193–197.