# 3

# $\mathfrak{X}$-local formations

In 1985 P. Förster [För85b] presented a common extension of the Gaschütz-Lubeseder-Schmid and Baer theorems (see Section 2.2). He introduced the concept of $\mathfrak{X}$-local formation, where $\mathfrak{X}$ is a class of simple groups with a completeness property. If $\mathfrak{X} = \mathfrak{J}$, the class of all simple groups, $\mathfrak{X}$-local formations are exactly the local formations and when $\mathfrak{X} = \mathbb{P}$, the class of all abelian simple groups, the notion of $\mathfrak{X}$-local formation coincides with the concept of Baer-local formation. P. Förster also defined a Frattini-like subgroup $\Phi^*_{\mathfrak{X}}(G)$ in each group $G$, which enables him to introduce the concept of $\mathfrak{X}$-saturation. Förster's definition of $\mathfrak{X}$-saturation is not the natural one if our aim is to generalise the concepts of saturation and soluble saturation. Since $\mathrm{O}_{\mathfrak{J}}(G) = G$ and $\mathrm{O}_{\mathbb{P}}(G) = G_{\mathfrak{S}}$, we would expect the $\mathfrak{X}$-Frattini subgroup of a group $G$ to be defined as $\Phi\bigl(\mathrm{O}_{\mathfrak{X}}(G)\bigr)$, where $\mathrm{O}_{\mathfrak{X}}(G)$ is the largest normal subgroup of $G$ whose composition factors belong to $\mathfrak{X}$. We have that $\Phi\bigl(\mathrm{O}_{\mathfrak{X}}(G)\bigr)$ is contained in $\Phi^*_{\mathfrak{X}}(G)$, but the equality does not hold in many cases. Nevertheless, Förster proved that $\mathfrak{X}$-saturated formations are exactly the $\mathfrak{X}$-local ones. If $\mathfrak{X} = \mathfrak{J}$, then we obtain as a special case the Gaschütz-Lubeseder-Schmid theorem. When $\mathfrak{X} = \mathbb{P}$, Baer's theorem appears as a corollary of Förster's result. Since $\Phi\bigl(\mathrm{O}_{\mathfrak{X}}(G)\bigr)$ is contained in $\Phi^*_{\mathfrak{X}}(G)$ for every group $G$, we can deduce from Förster's theorem that every $\mathfrak{X}$-local formation fulfils the following property:

$$\text{A group } G \text{ belongs to } \mathfrak{F} \text{ if and only if } G/\Phi\bigl(\mathrm{O}_{\mathfrak{X}}(G)\bigr) \text{ belongs to } \mathfrak{F}. \tag{3.1}$$

Therefore from the very beginning the following question naturally arises:

**Open question 3.0.1.** *Let $\mathfrak{F}$ be a formation with the property* (3.1). *Is $\mathfrak{F}$ $\mathfrak{X}$-local?*

After studying general properties of $\mathfrak{X}$-local formations in Section 3.1, we draw near the solution of Question 3.0.1 in Section 3.2. Products of $\mathfrak{X}$-local formations are the theme of Section 3.3, whereas some partially saturated formations are studied in Section 3.4.

*Throughout this chapter, $\mathfrak{X}$ denotes a fixed class of simple groups satisfying $\pi(\mathfrak{X}) = \operatorname{char}\mathfrak{X}$.*

## 3.1 $\mathfrak{X}$-local formations

This section is devoted to study some basic facts on $\mathfrak{X}$-local formations. We investigate the behaviour of $\mathfrak{X}$-local formations as classes of groups, focussing our attention on some distinguished $\mathfrak{X}$-local formation functions defining them.

We begin with the concept of $\mathfrak{X}$-local formation due to Förster [För85b].

Denote by $\mathfrak{J}$ the class of all simple groups. For any subclass $\mathfrak{Y}$ of $\mathfrak{J}$, we write $\mathfrak{Y}' = \mathfrak{J} \setminus \mathfrak{Y}$. Let $\mathrm{E}\,\mathfrak{Y}$ be the class of groups whose composition factors belong to $\mathfrak{Y}$. It is clear that $\mathrm{E}\,\mathfrak{Y}$ is a Fitting class, and so each group $G$ has a largest normal $\mathrm{E}\,\mathfrak{Y}$-subgroup, the $\mathrm{E}\,\mathfrak{Y}$-radical $\mathrm{O}_{\mathfrak{Y}}(G)$. A chief factor of $G$ which belongs to $\mathrm{E}\,\mathfrak{Y}$ is called a $\mathfrak{Y}$-*chief factor*, and if, moreover, $p$ divides the order of a $\mathfrak{Y}$-chief factor $H/K$ of $G$, we shall say that $H/K$ is a $\mathfrak{Y}_p$-*chief factor* of $G$.

Sometimes it will be convenient to identify the prime $p$ with the cyclic group $C_p$ of order $p$.

**Definition 3.1.1 (P. Förster).** *An* $\mathfrak{X}$-formation function $f$ *associates with each* $X \in (\operatorname{char}\mathfrak{X}) \cup \mathfrak{X}'$ *a formation* $f(X)$ *(possibly empty). If* $f$ *is an* $\mathfrak{X}$-*formation function, then the* $\mathfrak{X}$-local formation $\mathrm{LF}_{\mathfrak{X}}(f)$ *defined by* $f$ *is the class of all groups* $G$ *satisfying the following two conditions:*

1. *if* $H/K$ *is an* $\mathfrak{X}_p$-*chief factor of* $G$*, then* $G/\mathrm{C}_G(H/K) \in f(p)$*, and*
2. $G/K \in f(E)$*, whenever* $G/K$ *is a monolithic quotient of* $G$ *such that the composition factor of its socle* $\mathrm{Soc}(G/K)$ *is isomorphic to* $E$*, if* $E \in \mathfrak{X}'$.

*Remarks 3.1.2.* 1. It is obvious from the definition that $\mathrm{LF}_{\mathfrak{X}}(f)$ is $\mathrm{Q}$-closed.

2. Applying Theorem 1.2.34, it is only necessary to consider the $\mathfrak{X}_p$-chief factors of a given chief series of a group $G$ in order to check whether or not $G$ satisfies Condition 1.

3. If, for some prime $p \in \operatorname{char}\mathfrak{X}$, $f(p) = \emptyset$, then every $\mathfrak{X}$-chief factor of a group $G \in \mathrm{LF}_{\mathfrak{X}}(f)$ is a $p'$-group.

4. If, for some $S \in \mathfrak{X}'$, $f(S) = \emptyset$, then a group $G \in \mathrm{LF}_{\mathfrak{X}}(f)$ cannot have a monolithic quotient whose socle is in $\mathrm{E}(S)$. Consequently $\mathrm{LF}_{\mathfrak{X}}(f) \subseteq \mathrm{E}\big((S)'\big)$.

5. If $f(S) \neq \emptyset$ for some $S \in \mathfrak{X}'$, then $\mathrm{LF}_{\mathfrak{X}}(f) \subseteq \mathrm{E}\big((S)'\big) \circ f(S)$.

Remark 3.1.2 (5) is a consequence of the following lemma:

**Lemma 3.1.3.** *Let* $G$ *be a group and let* $\{M_i : 1 \leq i \leq s\}$ *be the set of all minimal normal subgroups of* $G$*. Then, for each* $1 \leq i \leq s$*,* $G$ *has a normal subgroup* $N_i$ *such that* $G/N_i$ *is monolithic and* $\mathrm{Soc}(G/N_i)$ *is* $G$-*isomorphic to* $M_i$*. Moreover* $G \in \mathrm{R}_0(\{G/N_i : 1 \leq i \leq s\})$*.*

*Proof.* For each $1 \leq i \leq s$, we consider an element $N_i$ of maximal order of the family $\{T_i \trianglelefteq G : T_i \cap M_i = 1\}$. Then $G/N_i$ is monolithic, $\mathrm{Soc}(G/N_i)$ is $G$-isomorphic to $M_i$ and $G \in \mathrm{R}_0(\{G/N_i : 1 \leq i \leq s\})$. □

*Proof (of Remark 3.1.2 (5)).* Assume that $G \in \mathrm{LF}_{\mathfrak{X}}(f)$ and $f(S) \neq \emptyset$ for some $S \in \mathfrak{X}'$. Then every minimal normal subgroup of $G/N$, for $N = \mathrm{O}_{(S)'}(G)$, is in $\mathrm{E}(S)$. Therefore $G/N \in f(S)$ by the above lemma. In particular, $G \in \mathrm{E}\big((S)'\big) \circ f(S)$. Remark 3.1.2 (5) is proved. □

We can now deduce the following result.

**Theorem 3.1.4.** *Let $f$ be an $\mathfrak{X}$-formation function. Then $\mathrm{LF}_{\mathfrak{X}}(f)$ is a formation.*

*Proof.* We prove that $\mathrm{LF}_{\mathfrak{X}}(f)$ is $\mathrm{R}_0$-closed. Let $N_1$ and $N_2$ be two different minimal normal subgroups of a group $G$ such that $G/N_i \in \mathrm{LF}_{\mathfrak{X}}(f)$ $(i = 1, 2)$. We see that $G$ satisfies Condition 1 of Definition 3.1.1. If $N_1 \in \mathrm{E}(\mathfrak{X}')$, then clearly $G \in \mathrm{LF}_{\mathfrak{X}}(f)$. Hence we may assume that $N_1 \in \mathrm{E}\,\mathfrak{X}$. Let $p$ be a prime dividing $|N_1|$. Then $N_1N_2/N_1$ is an $\mathfrak{X}_p$-chief factor of $G/N_2$ and $\mathrm{Aut}_G(N_1) \cong \mathrm{Aut}_{G/N_2}(N_1N_2/N_2)$ and $G/N_2 \in \mathrm{LF}_{\mathfrak{X}}(f)$. Therefore $G/\mathrm{C}_G(N_1) \in f(p)$. Since $G/N_1 \in \mathrm{LF}_{\mathfrak{X}}(f)$, by appealing to the generalised Jordan-Hölder theorem (1.2.34), we infer that $G$ satisfies Condition 1.

Consider now a monolithic quotient $G/K$ of $G$ such that $\mathrm{Soc}(G/K) \in \mathrm{E}(S)$ for some simple group $S \in \mathfrak{X}'$. If $f(S) = \emptyset$, then $\mathrm{LF}_{\mathfrak{X}}(f) \subseteq \mathrm{E}\big((S)'\big)$ by Remark 3.1.2 (4). Therefore $G/N_i \in \mathrm{E}\big((S)'\big)$ for $i \in \{1, 2\}$. This implies $G \in \mathrm{E}\big((S)'\big)$, contrary to supposition. Hence $f(S) \neq \emptyset$ and so $G/N_i \in \mathrm{E}\big((S)'\big) \circ f(S)$ by Remark 3.1.2 (5). In particular, $G/K \in \mathrm{E}\big((S)'\big) \circ f(S)$ because the latter class is a formation. Since $\mathrm{O}_{(S)'}(G/K) = 1$, it follows that $G/K \in f(S)$. Hence $G$ satisfies Condition 2 of Definition 3.1.1.

Consequently $G \in \mathrm{LF}_{\mathfrak{X}}(f)$. Applying Remark 3.1.2 (1) and [DH92, II, 2.6], $\mathrm{LF}_{\mathfrak{X}}(f)$ is a formation. □

**Definition 3.1.5.** *A formation $\mathfrak{F}$ is said to be* $\mathfrak{X}$-local *if $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(f)$ for some $\mathfrak{X}$-formation function $f$. In this case we say that $f$ is an* $\mathfrak{X}$-local definition *of $\mathfrak{F}$ or $f$* defines *$\mathfrak{F}$.*

*Examples 3.1.6.* 1. Each formation $\mathfrak{F}$ is $\mathfrak{X}$-local for $\mathfrak{X} = \emptyset$ because $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(f)$, where $f(S) = \mathfrak{F}$ for all $S \in \mathfrak{J}$.

2. If $\mathfrak{X} = \mathfrak{J}$, then an $\mathfrak{X}$-formation function is simply a formation function and the $\mathfrak{X}$-local formations are exactly the local formations.

3. If $\mathfrak{X} = \mathbb{P}$, then an $\mathfrak{X}$-formation function is a Baer function and the $\mathfrak{X}$-local formations are exactly the Baer-local ones.

*Remarks 3.1.7.* Let $f$ and $f_i$ be $\mathfrak{X}$-formation functions for all $i \in \mathcal{I}$.

1. $\bigcap_{i\in\mathcal{I}} \mathrm{LF}_{\mathfrak{X}}(f_i) = \mathrm{LF}_{\mathfrak{X}}(g)$, where $g(S) = \bigcap_{i\in\mathcal{I}} f_i(S)$ for all $S \in (\mathrm{char}\,\mathfrak{X}) \cup \mathfrak{X}'$.

2. Let $N \trianglelefteq G$ and $G/N \in \mathrm{LF}_{\mathfrak{X}}(f)$. If $N \in \mathrm{E}\,\mathfrak{X}$ and $G/\mathrm{C}_G(N) \in f(p)$ for all $p \mid |N|$, then $G \in \mathrm{LF}_{\mathfrak{X}}(f)$.

*Proof.* 1. This follows immediately from the definition of $\mathfrak{X}$-local formation.

2. If $H/K$ is an $\mathfrak{X}_p$-chief factor of $G$ above $N$, then $G/\mathrm{C}_G(H/K) \in f(p)$ because $G/N \in \mathrm{LF}_{\mathfrak{X}}(f)$. Let $H/K$ be an $\mathfrak{X}_p$-chief factor of $G$ below $N$. Then $\mathrm{C}_G(N) \leq \mathrm{C}_G(H/K)$ and so $G/\mathrm{C}_G(H/K) \in \mathrm{Q}\, f(p) = f(p)$. By the generalised Jordan-Hölder theorem (1.2.34), we have that $G$ satisfies Condition 1 of Definition 3.1.1.

Let $K$ be a normal subgroup of $G$ such that $G/K$ is a monolithic group with $\mathrm{Soc}(G/K) \in \mathrm{E}(S)$, $S \in \mathfrak{X}'$. Then, since $N \in \mathrm{E}\,\mathfrak{X}$, we have that $N \leq K$. Therefore $G/K \in f(S)$ because $G/N \in \mathrm{LF}_{\mathfrak{X}}(f)$.

Consequently $G \in \mathrm{LF}_{\mathfrak{X}}(f)$. □

**Definition 3.1.8.** *Let $p \in \operatorname{char}\mathfrak{X}$. Then the subgroup $\mathrm{C}^{\mathfrak{X}_p}(G)$ is defined to be the intersection of the centralisers of all $\mathfrak{X}_p$-chief factors of $G$, with $\mathrm{C}^{\mathfrak{X}_p}(G) = G$ if $G$ has no $\mathfrak{X}_p$-chief factors.*

*Remark 3.1.9.* Let $\mathrm{LF}_{\mathfrak{X}}(f)$ be an $\mathfrak{X}$-local formation. Then $G$ satisfies Condition 1 of Definition 3.1.1 if and only if $G/\mathrm{C}^{\mathfrak{X}_p}(G) \in f(p)$ for all $p \in \operatorname{char}\mathfrak{X}$ such that $f(p) \neq \emptyset$.

Note that $\big(\mathrm{C}^{\mathfrak{X}_p}(G)\big)^{\theta}$ is contained in $\mathrm{C}^{\mathfrak{X}_p}(G^{\theta})$ for every epimorphism $\theta$ of $G$. Therefore, by [DH92, IV, 1.10], the class $\mathrm{Q}\big(G/\mathrm{C}^{\mathfrak{X}_p}(G) : G \in \mathfrak{F}\big)$ is a formation, for each formation $\mathfrak{F}$.

Let $N$ be a normal subgroup of $G$ and let $H/K$ be a chief factor of $G$ below $N$. Then, by [DH92, A, 4.13 (c)], $H/K$ is a direct product of chief factors of $N$. Therefore we have

**Proposition 3.1.10.** $\mathrm{C}^{\mathfrak{X}_p}(G) \cap N = \mathrm{C}^{\mathfrak{X}_p}(N)$ *for all normal subgroups $N$ of $G$.*

Let $f_1$ and $f_2$ be two $\mathfrak{X}$-formation functions. We write $f_1 \leq f_2$ if $f_1(S) \subseteq f_2(S)$ for all $S \in (\operatorname{char}\mathfrak{X}) \cup \mathfrak{X}'$. Note that in this case $\mathrm{LF}_{\mathfrak{X}}(f_1) \subseteq \mathrm{LF}_{\mathfrak{X}}(f_2)$. By Remark 3.1.7 (1), each $\mathfrak{X}$-local formation $\mathfrak{F}$ has a unique $\mathfrak{X}$-formation function $\underline{f}$ defining $\mathfrak{F}$ such that $\underline{f} \leq f$ for each $\mathfrak{X}$-formation function $f$ such that $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(f)$. We say that $\underline{f}$ is the *minimal $\mathfrak{X}$-local definition* of $\mathfrak{F}$. This $\mathfrak{X}$-local formation function will always be denoted by the use of a "lower bar."

Moreover if $\mathfrak{Y}$ is a class of groups, the intersection of all $\mathfrak{X}$-local formations containing $\mathfrak{Y}$ is the smallest $\mathfrak{X}$-local formation containing $\mathfrak{Y}$. Such $\mathfrak{X}$-local formation is denoted by $\mathrm{form}_{\mathfrak{X}}(\mathfrak{Y})$. If $\mathfrak{X} = \mathfrak{J}$, we also write $\mathrm{lform}(\mathfrak{Y}) = \mathrm{form}_{\mathfrak{J}}(\mathfrak{Y})$, and if $\mathfrak{X} = \mathbb{P}$, $\mathrm{form}_{\mathbb{P}}(\mathfrak{Y})$ is usually denoted by $\mathrm{bform}(\mathfrak{Y})$.

**Theorem 3.1.11.** *Let $\mathfrak{Y}$ be a class of groups. Then $\mathfrak{F} = \mathrm{form}_{\mathfrak{X}}(\mathfrak{Y}) = \mathrm{LF}_{\mathfrak{X}}(\underline{f})$, where*

$$\underline{f}(p) = \mathrm{Q}\,\mathrm{R}_0\big(G/\mathrm{C}_G(H/K) : G \in \mathfrak{Y} \text{ and } H/K \text{ is an } \mathfrak{X}_p\text{-chief factor of } G\big),$$

*if $p \in \operatorname{char}\mathfrak{X}$, and*

$$\underline{f}(S) = \mathrm{Q}\,\mathrm{R}_0\big(G/L : G \in \mathfrak{Y},\ G/L \text{ is monolithic, and } \mathrm{Soc}(G/L) \in \mathrm{E}(S)\big),$$

*if $S \in \mathfrak{X}'$. Moreover $\underline{f}(p) = \mathrm{Q}\big(G/\mathrm{C}^{\mathfrak{X}_p}(G) : G \in \mathfrak{F}\big)$ for all $p \in \operatorname{char}\mathfrak{X}$ such that $\underline{f}(p) \neq \emptyset$.*

*Proof.* Let $g$ be an $\mathfrak{X}$-formation function such that $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(g)$. Since $\mathrm{LF}_{\mathfrak{X}}(\underline{f})$ is an $\mathfrak{X}$-local formation containing $\mathfrak{Y}$, we have $\mathfrak{F} \subseteq \mathrm{LF}_{\mathfrak{X}}(\underline{f})$. Assume that $\mathrm{LF}_{\mathfrak{X}}(\underline{f}) \neq \mathfrak{F}$. Then $\mathrm{LF}_{\mathfrak{X}}(\underline{f}) \setminus \mathfrak{F}$ contains a group $G$ of minimal order. Such a $G$ has a unique minimal normal subgroup $N$ by [DH92, II, 2.5] and $G/N \in \mathfrak{F}$. If $N$ is an $\mathfrak{X}'$-chief factor of $G$, then $G \in \underline{f}(S)$ for some $S \in \mathfrak{X}'$. This implies that $G \in \mathrm{Q}\,\mathrm{R}_0\,\mathfrak{Y} \subseteq \mathfrak{F}$, a contradiction. Therefore $N \in \mathrm{E}\,\mathfrak{X}$. Let $p$ be a prime divisor of $|N|$. Then $G/\mathrm{C}_G(N) \in \underline{f}(p)$. Now if $X$ is a group in $\mathfrak{Y}$ and $H/K$ is an $\mathfrak{X}_p$-chief factor of $X$, then $X/\mathrm{C}_X(H/K) \in g(p)$ because $\mathfrak{Y} \subseteq \mathfrak{F}$. Therefore $\underline{f}(p) \subseteq g(p)$, and so $G/\mathrm{C}_G(N) \in g(p)$. Applying Remark 3.1.7 (2), $G \in \mathfrak{F}$, contrary to hypothesis. Consequently $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(\underline{f})$.

Let $p \in \operatorname{char}\mathfrak{X}$ and $t(p) = \mathrm{Q}\big(G/\mathrm{C}^{\mathfrak{X}_p}(G) : G \in \mathfrak{F}\big)$. We know that $t(p)$ is a formation. Moreover, if $G \in \mathfrak{F}$ and $\underline{f}(p) \neq \emptyset$, then $G/\mathrm{C}^{\mathfrak{X}_p}(G) \in \underline{f}(p)$. Therefore $t(p) \subseteq \underline{f}(p)$. On the other hand, if $X \in \mathfrak{Y}$, then $X/\mathrm{C}^{\mathfrak{X}_p}(X) \in t(p)$. Hence $X/\mathrm{C}_X(H/K) \in t(p)$ for every $\mathfrak{X}_p$-chief factor $H/K$ of $X$. This means that $\underline{f}(p) \subseteq t(p)$ and the equality holds. This completes the proof of the theorem. □

*Remark 3.1.12.* If $\mathfrak{F}$ is a local formation and $\underline{f}$ is the smallest local definition of $\mathfrak{F}$, then $\underline{f}(p) = \mathrm{Q}\big(G/\mathrm{O}_{p',p}(G) : G \in \mathfrak{F}\big)$ for each $p \in \operatorname{char}\mathfrak{F}$ (cf. [DH92, IV, 3.10]). The equality $\underline{f}(p) = \mathrm{Q}\big(G/\mathrm{O}_{p',p}(G) : G \in \mathfrak{F}\big)$ does not hold for $\mathfrak{X}$-local formations in general: Let $\mathfrak{X} = (C_2)$ and consider $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(f)$, where $f(2) = \mathfrak{S}$ and $f(S) = \mathfrak{E}$ for all $S \in \mathfrak{X}'$. Then $\mathrm{Alt}(5) \in \mathfrak{F}$ and so $\mathrm{Alt}(5) \in \mathrm{Q}\big(G/\mathrm{O}_{2',2}(G) : G \in \mathfrak{F}\big)$. Since $\underline{f}(2) \subseteq \mathfrak{S}$, it follows that $\mathrm{Alt}(5) \notin \underline{f}(2)$. Consequently $\underline{f}(2) \neq \mathrm{Q}\big(G/\mathrm{O}_{2',2}(G) : G \in \mathfrak{F}\big)$.

**Corollary 3.1.13.** *Let $\mathfrak{X}$ and $\bar{\mathfrak{X}}$ be classes of simple groups such that $\bar{\mathfrak{X}} \subseteq \mathfrak{X}$. Then every $\mathfrak{X}$-local formation is $\bar{\mathfrak{X}}$-local.*

*Proof.* Let $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(\underline{f})$ be an $\mathfrak{X}$-local formation. Since $\operatorname{char}\bar{\mathfrak{X}} \subseteq \operatorname{char}\mathfrak{X}$, we can consider the $\bar{\mathfrak{X}}$-formation function $g$ defined by

$$\begin{aligned} g(p) &= \underline{f}(p) && \text{if } p \in \operatorname{char}\bar{\mathfrak{X}},\\ g(E) &= \mathfrak{F} && \text{if } E \in \bar{\mathfrak{X}}'. \end{aligned}$$

It is clear that $\mathfrak{F} \subseteq \mathrm{LF}_{\bar{\mathfrak{X}}}(g)$. Suppose that $\mathfrak{F} \neq \mathrm{LF}_{\bar{\mathfrak{X}}}(g)$, and choose a group $Y$ of minimal order in $\mathrm{LF}_{\bar{\mathfrak{X}}}(g) \setminus \mathfrak{F}$. Then $Y$ has a unique minimal normal subgroup $N$, and $G/N \in \mathfrak{F}$. If $N \in \mathrm{E}(\bar{\mathfrak{X}}')$, then $G \in \mathfrak{F}$, which contradicts the choice of $G$. Therefore $N \in \mathrm{E}\,\bar{\mathfrak{X}}$ and $G/\mathrm{C}_G(N) \in \underline{f}(p)$ for each prime $p$ dividing $|N|$. Applying Remark 3.1.7 (2), we conclude that $G \in \mathfrak{F}$, contrary to supposition. Hence $\mathfrak{F} = \mathrm{LF}_{\bar{\mathfrak{X}}}(g)$ and $\mathfrak{F}$ is $\bar{\mathfrak{X}}$-local. □

By [DH92, IV, 3.8], each local formation $\mathfrak{F} = \mathrm{LF}(f)$ can be defined by a formation function $g$ given by $g(p) = \mathfrak{F} \cap \mathfrak{S}_p f(p)$ for all primes $p$. The corresponding result for $\mathfrak{X}$-local formations is the following:

**Theorem 3.1.14.** *Let $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(f)$ be an $\mathfrak{X}$-local formation defined by the $\mathfrak{X}$-formation function $f$. Set*

$$f^*(p) = \mathfrak{F} \cap \mathfrak{S}_p f(p) \qquad \text{for all } p \in \operatorname{char}\mathfrak{X},$$
$$f^*(S) = \mathfrak{F} \cap f(S) \qquad \text{for all } S \in \mathfrak{X}'.$$

*Then:*

1. *$f^*$ is an $\mathfrak{X}$-formation function such that $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(f^*)$.*
2. *$\mathfrak{S}_p f^*(p) = f^*(p)$ for all $p \in \operatorname{char}\mathfrak{X}$.*

*Proof.* 1. It is clear that $f^*$ is an $\mathfrak{X}$-formation function. Let $\mathfrak{F}^* = \mathrm{LF}_{\mathfrak{X}}(f^*)$ and let $G \in \mathfrak{F}^*$. If $H/K$ is an $\mathfrak{X}_p$-chief factor of $G$, then $G/\mathrm{C}_G(H/K) \in \mathfrak{F} \cap \mathfrak{S}_p f(p)$. Since, by [DH92, A, 13.6], $\mathrm{O}_p\big(G/\mathrm{C}_G(H/K)\big) = 1$, it follows that $G/\mathrm{C}_G(H/K) \in f(p)$. Now if $G/L$ is a monolithic quotient of $G$ with $\mathrm{Soc}(G/L) \in \mathrm{E}(S)$ for some $S \in \mathfrak{X}'$, it follows that $G/L \in f(S)$. Therefore $G \in \mathfrak{F}$.

Now if $H/K$ is an $\mathfrak{X}_p$-chief factor of a group $A \in \mathfrak{F}$, then $A/\mathrm{C}_A(H/K) \in \mathrm{Q}\,\mathfrak{F} \cap f(p) \subseteq f^*(p)$. If $A/L$ is a monolithic quotient of $A$ with $\mathrm{Soc}(A/L) \in \mathrm{E}(S)$, $S \in \mathfrak{X}'$, then $A/L \in \mathrm{Q}\,\mathfrak{F} \cap f(S) \subseteq f^*(S)$. This implies that $A \in \mathfrak{F}^*$ and therefore $\mathfrak{F} = \mathfrak{F}^*$.

2. Let $G \in \mathfrak{S}_p f^*(p)$, $p \in \operatorname{char}\mathfrak{X}$. Then $G/\mathrm{O}_p(G) \in f^*(p)$ and so $G \in \mathfrak{S}_p f(p)$ because $\mathrm{O}_p\big(G/\mathrm{O}_p(G)\big) = 1$. Moreover $G/\mathrm{O}_p(G) \in \mathfrak{F}$. If $H/K$ is an $\mathfrak{X}_p$-chief factor of $G$ below $\mathrm{O}_p(G)$, then $\mathrm{O}_p(G) \leq \mathrm{C}_G(H/K)$ by [DH92, B, 3.12 (b)] and so $G/\mathrm{C}_G(H/K) \in \mathrm{Q}\, f(p) = f(p)$. If $G/L$ is a monolithic quotient of $G$ such that $\mathrm{Soc}(G/L) \in \mathrm{E}(S)$, $S \in \mathfrak{X}'$, it follows that $\mathrm{O}_p(G) \leq L$. Therefore $G/L \in \mathrm{Q}\, f^*(p) = f^*(p) \subseteq \mathfrak{F}$ and so $G/L \in f(S)$. This proves that $G \in \mathfrak{F}$. Consequently $G \in f^*(p)$ and $\mathfrak{S}_p f^*(p) = f^*(p)$. ☐

**Definition 3.1.15.** *Let $f$ be an $\mathfrak{X}$-formation function defining an $\mathfrak{X}$-local formation $\mathfrak{F}$. Then $f$ is called:*

1. integrated *if $f(S) \subseteq \mathfrak{F}$ for all $S \in (\operatorname{char}\mathfrak{X}) \cup \mathfrak{X}'$,*
2. full *if $\mathfrak{S}_p f(p) = f(p)$ for all $p \in \operatorname{char}\mathfrak{X}$.*

Let $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(f)$ be an $\mathfrak{X}$-local formation. Then the $\mathfrak{X}$-formation function $g$ given by $g(S) = \mathfrak{F} \cap f(S)$ for all $S \in (\operatorname{char}\mathfrak{X}) \cup \mathfrak{X}'$ is an integrated $\mathfrak{X}$-local definition of $\mathfrak{F}$. Moreover $f^*$ is, according to the above theorem, an integrated and full $\mathfrak{X}$-local definition of $\mathfrak{F}$.

It is known (cf. [DH92, IV, 3.7]) that if $\mathfrak{X} = \mathfrak{J}$, then every $\mathfrak{X}$-local formation has a unique integrated and full $\mathfrak{X}$-local definition, the canonical one. This is not true in general. In fact, if $\emptyset \neq \mathfrak{X} \neq \mathfrak{J}$, we can find an $\mathfrak{X}$-local formation with several integrated and full $\mathfrak{X}$-local definitions.

*Example 3.1.16.* Let $\emptyset \neq \mathfrak{X} \neq \mathfrak{J}$. Then we can consider $X \in \mathfrak{J} \setminus \mathfrak{X}$ and a prime $p \in \operatorname{char}\mathfrak{X}$. The formation $\mathfrak{F} = \mathfrak{S}_p$ is an $\mathfrak{X}$-local formation which can be $\mathfrak{X}$-locally defined by the following integrated and full $\mathfrak{X}$-formation functions:

$$f_1(S) = \begin{cases} \mathfrak{S}_p & \text{if } S \cong C_p, \\ \emptyset & \text{if } S \ncong C_p, \end{cases}$$

and

$$f_2(S) = \begin{cases} \mathfrak{S}_p & \text{if } S \cong C_p, \\ \mathfrak{S}_p & \text{if } S \cong X, \\ \emptyset & \text{otherwise} \end{cases}$$

for all $S \in (\operatorname{char}\mathfrak{X}) \cup \mathfrak{X}'$.

We say that an $\mathfrak{X}$-formation function $f$ defining an $\mathfrak{X}$-local formation $\mathfrak{F}$ is a *maximal integrated* $\mathfrak{X}$-formation function if $g \leq f$ for each integrated $\mathfrak{X}$-formation function $g$ such that $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(g)$.

The next result shows that every $\mathfrak{X}$-local formation can be $\mathfrak{X}$-locally defined by a maximal integrated $\mathfrak{X}$-formation function $F$. Moreover $F$ is full.

**Theorem 3.1.17.** *Let $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(\underline{f})$ be an $\mathfrak{X}$-local formation. Then:*

1. *$\mathfrak{F}$ is $\mathfrak{X}$-locally defined by the integrated and full $\mathfrak{X}$-formation function $F$ given by $F(p) = \mathfrak{S}_p\underline{f}(p)$ for all $p \in \operatorname{char}\mathfrak{X}$ and $F(S) = \mathfrak{F}$ for all $S \in \mathfrak{X}'$.*
2. *For each $p \in \operatorname{char}\mathfrak{X}$, $F(p) = (G : C_p \wr G \in \mathfrak{F})$.*
3. *If $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(g)$, then $F(p) = \mathfrak{F} \cap \mathfrak{S}_p g(p)$ for all $p \in \operatorname{char}\mathfrak{X}$.*

*Proof.* 1. Since $\underline{f} \leq F$, it follows that $\mathfrak{F} \subseteq \mathrm{LF}_{\mathfrak{X}}(F)$. Suppose, by way of contradiction, that the equality does not hold and let $G$ be a group of minimal order in $\mathrm{LF}_{\mathfrak{X}}(F)\setminus\mathfrak{F}$. Then the group $G$ has a unique minimal normal subgroup, $N$ say, and $G/N \in \mathfrak{F}$. Furthermore $N \in \mathrm{E}\,\mathfrak{X}$ because otherwise $G \in F(S)$ for some $S \in \mathfrak{X}'$ and then $G \in \mathfrak{F}$, contrary to supposition. Let $p$ be a prime dividing $|N|$. Then $G/\mathrm{C}_G(N) \in \mathfrak{S}_p\underline{f}(p)$ and so $G/\mathrm{C}_G(N) \in \underline{f}(p)$ because $\mathrm{O}_p\big(G/\mathrm{C}_G(N)\big) = 1$ by [DH92, A, 13.6 (b)]. Then Remark 3.1.7 (2) implies that $G \in \mathfrak{F}$. This contradiction yields $\mathrm{LF}_{\mathfrak{X}}(F) \subseteq \mathfrak{F}$ and then $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(F)$. It is clear that $F$ is full. Let $p \in \operatorname{char}\mathfrak{X}$. If possible, choose a group $G$ of minimal order in $F(p) \setminus \mathfrak{F}$. We know that $G$ has a unique minimal normal subgroup $N$ and, since $\underline{f}(p) \subseteq \mathfrak{F}$, $\mathrm{O}_p(G) \neq 1$. Hence $N$ is a $p$-group. Moreover $G/N \in \mathfrak{F}$ and $G/\mathrm{C}_G(N) \in \underline{f}(p)$ because $\mathrm{O}_p(G)$ centralises $N$. But then $G \in \mathfrak{F}$. This contradicts the choice of $G$, and so we conclude that $F(p) \subseteq \mathfrak{F}$.

2. Let $p \in \operatorname{char}\mathfrak{X}$ and let $\bar{F}(p)$ denote the class $(G : C_p \wr G \in \mathfrak{F})$. If $G \in F(p)$, then $C_p \wr G \in \mathfrak{S}_p F(p) = F(p) \subseteq \mathfrak{F}$ by Statement 1. Hence $G \in \bar{F}(p)$ and so $F(p) \subseteq \bar{F}(p)$. Now consider a group $G \in \bar{F}(p)$ and set $W = C_p \wr G$. Denote $B = C_p^\natural$ the base group of $W$ and $A = \bigcap\{\mathrm{C}_W(H/K) : H \leq B \text{ and } H/K \text{ is a chief factor of } W\}$. Since $W \in \mathfrak{F}$, it follows that $W/A \in F(p)$. On the other hand, $A$ acts as a group of operators for $B$ by conjugation and $A$ stabilises a chain of subgroups of $B$. Applying [DH92, A, 12.4], we have that $A/\mathrm{C}_A(B)$ is a $p$-group. Then $A$ is itself a $p$-group because $\mathrm{C}_A(B) = B$ by [DH92, A, 18.8]. Consequently $W \in F(p)$ and $G \in \mathrm{Q}\,F(p) = F(p)$. This proves that $\bar{F}(p) = F(p)$.

3. Let $g$ be an $\mathfrak{X}$-formation function such that $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(g)$. Since $\underline{f} \leq g$, it follows that $F(p) = \mathfrak{S}_p\underline{f}(p) \subseteq \mathfrak{F} \cap \mathfrak{S}_p g(p) = g^*(p)$ for all $p \in \operatorname{char}\mathfrak{X}$. Let $X$ be a group in $g^*(p)$ and set $W = C_p \wr X$. As above, denote by $B = C_p^{\natural}$ the base group of $W$. Then $W/B \in g^*(p)$. Moreover $W/B \in \mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(g^*)$ by Theorem 3.1.14. Applying Remark 3.1.7 (2), we conclude that $W \in \mathfrak{F}$. Hence $X \in F(p)$ and $F(p) = g^*(p)$. □

Let $g$ be an integrated $\mathfrak{X}$-formation function defining an $\mathfrak{X}$-local formation $\mathfrak{F}$. Then $g(p) \subseteq \mathfrak{F} \cap \mathfrak{S}_p g(p) = F(p)$ for all $p \in \operatorname{char}\mathfrak{X}$ by Theorem 3.1.17 (3). Therefore $g \leq F$. We shall say that $F$ is the *canonical* $\mathfrak{X}$-local definition of $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(F)$. As in the case of local formations, the canonical $\mathfrak{X}$-local definition will be identified by the use of an uppercase Roman letter. Hence if we write $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(F)$, we are assuming that $F$ is the canonical $\mathfrak{X}$-local definition of $\mathfrak{F}$.

**Corollary 3.1.18.** *Let $\mathfrak{F}$ be an $\mathfrak{X}$-local formation and $\mathfrak{Y} \subseteq \mathfrak{X}$. Let $F_1$ and $F_2$ be the canonical $\mathfrak{Y}$-local and $\mathfrak{X}$-local definitions of $\mathfrak{F}$, respectively. Then $F_1(p) = F_2(p)$ for all $p \in \operatorname{char}\mathfrak{Y}$.*

*Proof.* Applying Corollary 3.1.13, we know that $\mathfrak{F}$ is $\mathfrak{Y}$-local. Let $p$ be a prime in $\operatorname{char}\mathfrak{Y}$. Then $p \in \operatorname{char}\mathfrak{X}$ and by Theorem 3.1.17 (2) we have that $F_1(p) = (G : C_p \wr G \in \mathfrak{F}) = F_2(p)$. □

Taking $\mathfrak{Y} = (C_p)$, $p \in \operatorname{char}\mathfrak{X}$ in Corollary 3.1.18 and, applying Theorem 3.1.11 and Theorem 3.1.17, we have:

**Corollary 3.1.19.** *Let $\mathfrak{F}$ be an $\mathfrak{X}$-local formation. If $p \in \operatorname{char}\mathfrak{X}$, then*

$$F(p) = \mathfrak{S}_p \,\mathrm{Q}\,\mathrm{R}_0\big(G/\mathrm{C}_G(H/K) : G \in \mathfrak{F},\ H/K \text{ is an abelian } p\text{-chief factor of } G\big).$$

**Corollary 3.1.20.** *Let $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(\underline{f}) = \mathrm{LF}_{\mathfrak{X}}(F)$ and $\mathfrak{G} = \mathrm{LF}_{\mathfrak{X}}(\underline{g}) = \mathrm{LF}_{\mathfrak{X}}(G)$ be $\mathfrak{X}$-local formations. Then any two of the following statements are equivalent:*

1. $\mathfrak{F} \subseteq \mathfrak{G}$
2. $F \leq G$
3. $\underline{f} \leq \underline{g}$

**Corollary 3.1.21 ([BBCER05, Lemma 4.5]).** *Let $\mathfrak{F}$ be a formation and let $\{\mathfrak{X}_i : i \in \mathcal{I}\}$ be a family of classes of simple groups such that $\pi(\mathfrak{X}_i) = \operatorname{char}\mathfrak{X}_i$ for all $i \in \mathcal{I}$. Put $\mathfrak{X} = \bigcup_{i \in \mathcal{I}} \mathfrak{X}_i$. If $\mathfrak{F}$ is $\mathfrak{X}_i$-local for all $i \in \mathcal{I}$, then $\mathfrak{F}$ is $\mathfrak{X}$-local.*

*Proof.* First of all, note that $\pi(\mathfrak{X}) = \operatorname{char}\mathfrak{X}$.

Applying Theorem 3.1.17, $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}_i}(F_i)$, where

$$F_i(S) = \begin{cases} (G : C_p \wr G \in \mathfrak{F}) & \text{if } S \cong C_p,\ p \in \operatorname{char}\mathfrak{X}_i, \\ \mathfrak{F} & \text{if } S \in \mathfrak{X}_i', \end{cases}$$

for all $i \in \mathcal{I}$.

Let $f$ be the $\mathfrak{X}$-formation function defined by

$$f(S) = \begin{cases} (G : C_p \wr G \in \mathfrak{F}) & \text{if } S \cong C_p,\ p \in \operatorname{char} \mathfrak{X}, \\ \mathfrak{F} & \text{if } S \in \mathfrak{X}'. \end{cases}$$

It is clear that $\mathfrak{F} \subseteq \mathrm{LF}_{\mathfrak{X}}(f)$. Assume that the inclusion is proper and derive a contradiction. Let $G \in \mathrm{LF}_{\mathfrak{X}}(f) \setminus \mathfrak{F}$ of minimal order. Then $G$ has a unique minimal normal subgroup $N$ such that $G/N \in \mathfrak{F}$. It is clear that $N \in \mathrm{E}\,\mathfrak{X}$ because otherwise $G \in \mathfrak{F}$. Hence $N \in \mathrm{E}\,\mathfrak{X}_i$ for some $i \in \mathcal{I}$ and $G/\mathrm{C}_G(N) \in f(p) = F_i(p)$ for all $p \in \pi(N)$. Therefore $G \in \mathrm{LF}_{\mathfrak{X}_i}(F_i) = \mathfrak{F}$. This is a contradiction. Consequently $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(f)$ and $\mathfrak{F}$ is an $\mathfrak{X}$-local formation. □

When $\mathfrak{X}$ is the class of all abelian simple groups, we have $\mathfrak{X} = \bigcup_{p \in \mathbb{P}} (C_p)$. Therefore

**Corollary 3.1.22 ([BBCER05, Corollary 4.6]).** *A formation $\mathfrak{F}$ is Baer-local if and only if $\mathfrak{F}$ is $(C_p)$-local for every prime $p$.*

A natural question arising from the above discussion is whether an $\mathfrak{X}$-local formation has a unique upper bound for all its $\mathfrak{X}$-local definitions, that is, if $\mathfrak{F}$ can be $\mathfrak{X}$-locally defined by an $\mathfrak{X}$-formation function $F^0$ such that $f \leq F^0$ for each $\mathfrak{X}$-local definition $f$ of $\mathfrak{F}$. If such $F^0$ exists, we will refer to it as the *maximal $\mathfrak{X}$-local definition* of $\mathfrak{F}$.

In [Doe73], K. Doerk presented a beautiful result showing that in the soluble universe each local formation has a maximal local definition (see also [DH92, V, 3.18]). The same author, P. Schmid [Sch74], and L. A. Shemetkov [She78] posed the problem of whether every local formation of finite groups has a maximal local definition. The answer is "no" as the following example shows:

*Example 3.1.23 ([Sal85]).* Let $\mathfrak{F} = \mathfrak{S}$ be the local formation of all soluble groups. Then $\mathfrak{F} = \mathrm{LF}(f_1) = \mathrm{LF}(f_2)$, where $f_1$ and $f_2$ are the formation functions defined by

$$\begin{aligned} &f_1(2) = \mathrm{D}_0\bigl(\mathfrak{S}, \mathrm{Alt}(5)\bigr), \\ &f_1(p) = \mathfrak{S} && \text{for each prime } p \neq 2, \\ &f_2(3) = f_2(5) = \mathrm{D}_0\bigl(\mathfrak{S}, \mathrm{Alt}(5)\bigr), \\ &f_2(p) = \mathfrak{S} && \text{for each prime } p \neq 3,\ 5. \end{aligned}$$

Assume that $\mathfrak{F}$ has a maximal local definition, $F^0$ say. Then $f_i \leq F^0$ for $i = 1, 2$. This implies that $\mathrm{Alt}(5) \in \mathrm{LF}(F^0) = \mathfrak{F}$, a contradiction. Therefore $\mathfrak{F}$ does not have a maximal local definition.

Perhaps the most simple example of a local formation with a maximal local ($\mathfrak{J}$-local) definition is given by the class $\mathfrak{E}_\pi$ of all $\pi$-groups for a set of primes $\pi$. It is rather clear that

$$\widehat{F}(p) = \begin{cases} \mathfrak{E} & \text{if } p \in \pi, \\ \emptyset & \text{if } p \notin \pi, \end{cases}$$

defines the maximal local definition of $\mathfrak{E}_\pi$.

In the following we shall give a description of $\mathfrak{X}$-local formations with a maximal $\mathfrak{X}$-local definition. The main source for this description is P. Förster and E. Salomon [FS85].

The following concept, introduced for local formations in [Sal85], turns out to be crucial.

**Definition 3.1.24 ([FS85]).** *Let $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(F)$ be an $\mathfrak{X}$-local formation. Denote by $\mathrm{b}_{\mathfrak{X}}(\mathfrak{F})$ the class of all groups $G \in \mathrm{b}(\mathfrak{F})$ such that $\mathrm{Soc}(G) \in \mathrm{E}\,\mathfrak{X}$. A group $G \in \mathrm{b}_{\mathfrak{X}}(\mathfrak{F})$ is called* $\mathfrak{X}$-dense *with respect to $\mathfrak{F}$ if $G \in \mathrm{b}\big(F(p)\big)$ for each prime $p$ dividing $|\mathrm{Soc}(G)|$. Further, $\mathrm{b}(\mathfrak{F})$ is said to be* $\mathfrak{X}$-wide *if there does not exist an $\mathfrak{X}$-dense group $G \in \mathrm{b}_{\mathfrak{X}}(\mathfrak{F})$.*

Note that a group $G \in \mathrm{b}_{\mathfrak{X}}(\mathfrak{F})$ with abelian socle cannot be $\mathfrak{X}$-dense because $F$ is full.

*Remark 3.1.25.* Let $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(F)$ and $G \in \mathrm{b}_{\mathfrak{X}}(\mathfrak{F})$. $G$ is $\mathfrak{X}$-dense with respect to $\mathfrak{F}$ if and only if there exists an $\mathfrak{X}$-formation function $f$ such that $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(f)$ and $G \in \mathrm{b}\big(f(p)\big)$ for all primes $p$ dividing $|\mathrm{Soc}(G)|$.

*Proof.* If $G$ is $\mathfrak{X}$-dense with respect to $\mathfrak{F}$, then we take $f = F$. Conversely, assume that $G \in \mathrm{b}\big(f(p)\big)$ for all $p \in \pi\big(\mathrm{Soc}(G)\big)$ for some $\mathfrak{X}$-formation function $f$ such that $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(f)$. Then $G/\mathrm{Soc}(G) \in \mathfrak{F} \cap \mathfrak{S}_p f(p) = F(p)$ for all $p \in \pi\big(\mathrm{Soc}(G)\big)$ by Theorem 3.1.17 (3). Since $G \notin \mathfrak{F}$, it follows that $G \in \mathrm{b}\big(F(p)\big)$ for every prime $p$ dividing $|\mathrm{Soc}(G)|$. This is to say that $G$ is $\mathfrak{X}$-dense with respect to $\mathfrak{F}$. □

*Examples 3.1.26.* 1. Suppose that $\mathfrak{X}$ contains a non-abelian group $S$. Then $S$ is $\mathfrak{X}$-dense with respect to any $\mathfrak{X}$-local formation $\mathfrak{F}$ satisfying $S \notin \mathfrak{F}$ and $C_p \in \mathfrak{F}$ for all $p \in \pi(S)$. For example, $\mathfrak{F} = \mathfrak{N}$ or $\mathfrak{S}$.

2. Let $\mathfrak{F} = \mathfrak{N}\mathfrak{F}_0$ for some formation $\mathfrak{F}_0$. Let $\mathfrak{R}_{\mathfrak{X}}$ denote the class of all $\mathfrak{X}$-groups without abelian chief factors; it is clear that $\mathfrak{R}_{\mathfrak{X}} = \mathfrak{R}_{\mathfrak{X}}^2$ is a Fitting formation. It follows that $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(F)$ where $F(p) = \mathfrak{S}_p\mathfrak{F}_0$ for all $p \in \mathrm{char}\,\mathfrak{X}$, and $F(S) = \mathfrak{F}$ for all $S \in \mathfrak{X}'$. Then $\mathrm{b}(\mathfrak{F})$ is $\mathfrak{X}$-wide if and only if $\mathfrak{R}_{\mathfrak{X}}\mathfrak{F}_0 = \mathfrak{F}_0$.

*Proof.* 1. It is obvious.

2. It is rather clear that $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(F)$. Suppose that $\mathrm{b}(\mathfrak{F})$ is $\mathfrak{X}$-wide and $\mathfrak{R}_{\mathfrak{X}}\mathfrak{F}_0 \neq \mathfrak{F}_0$. Let $G \in \mathfrak{R}_{\mathfrak{X}}\mathfrak{F}_0 \setminus \mathfrak{F}_0$ be a group of minimal order. Then $G$ has a unique minimal normal subgroup $N$ such that $G/N \in \mathfrak{F}_0$. Since $G \notin \mathfrak{F}_0$, then $N$ is a non-abelian $\mathfrak{X}$-group. If $G \in \mathfrak{F}$, then $G \in \mathfrak{F}_0$ because $\mathrm{F}(G) = 1$, contrary to supposition. Hence $G \in \mathrm{b}(\mathfrak{F})$. Moreover $G \notin \mathfrak{S}_p\mathfrak{F}_0$ for all $p \in \pi(N)$. This means that $G \in \mathrm{b}\big(F(p)\big)$ for all $p \in \pi(N)$ and so $G$ is $\mathfrak{X}$-dense with respect to $\mathfrak{F}$. This is a contradiction. Hence $\mathfrak{R}_{\mathfrak{X}}\mathfrak{F}_0 \subseteq \mathfrak{F}_0$ and the equality holds.

Conversely, assume that $\mathfrak{R}_{\mathfrak{X}}\mathfrak{F}_0 = \mathfrak{F}_0$ and suppose that there exists $G \in \mathrm{b}_{\mathfrak{X}}(\mathfrak{F})$ such that $G \in \mathrm{b}\big(F(p)\big) = \mathrm{b}(\mathfrak{S}_p\mathfrak{F}_0)$ for each $p \in \pi\big(\mathrm{Soc}(G)\big)$. Let $p$ and $q$ be two different primes dividing $|\mathrm{Soc}(G)|$. Then $G/N \in \mathfrak{S}_p\mathfrak{F}_0 \cap \mathfrak{S}_q\mathfrak{F}_0$. Therefore $G \in \mathfrak{R}_{\mathfrak{X}}\mathfrak{F}_0 = \mathfrak{F}_0$. This contradicts the fact that $G \in \mathrm{b}(\mathfrak{F})$. Consequently $\mathrm{b}(\mathfrak{F})$ is $\mathfrak{X}$-wide. □

For each prime $p$, denote $\mathfrak{E}(p)$ the class of all groups such that $p$ divides the order of every chief factor of $G$. Then it is clear that $\mathfrak{E}(p) = \big(\mathfrak{E}(p)\big)^2$ is a Fitting formation and $\mathfrak{E}(p) \cap \mathfrak{S} = \mathfrak{S}_p$.

Note that if $p \in \operatorname{char}\mathfrak{X}$, then $\mathfrak{E}(p) \cap \mathrm{E}\,\mathfrak{X} = \mathrm{E}(\mathfrak{X}_p)$.

*Remark 3.1.27.* Let $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(f) = \mathrm{LF}_{\mathfrak{X}}(F)$ be an $\mathfrak{X}$-local formation. Then $F(p) = \mathfrak{F} \cap \mathfrak{E}(p)f(p)$ for each $p \in \operatorname{char}\mathfrak{X}$.

*Proof.* Let $p \in \operatorname{char}\mathfrak{X}$. By Theorem 3.1.17 (3), $F(p) = \mathfrak{F} \cap \mathfrak{S}_p f(p)$. Therefore $F(p) \subseteq \mathfrak{F} \cap \mathfrak{E}(p)f(p)$. Assume that the equality does not hold and let $G \in \big(\mathfrak{F} \cap \mathfrak{E}(p)f(p)\big) \setminus F(p)$ of minimal order. Then $G$ has a unique minimal normal subgroup $N$ such that $N \in \mathfrak{E}(p)$ and $G/N \in F(p)$. Since $F$ is full, we have that $N$ is not a $p$-group. Hence $\mathrm{C}_G(N) = 1$ and so $G \in F(p)$ because $G \in \mathfrak{F}$. This contradiction yields $F(p) = \mathfrak{F} \cap \mathfrak{E}(p)f(p)$. □

Let $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(f)$ be an $\mathfrak{X}$-local formation. Denote $\bar{f}$ the following $\mathfrak{X}$-formation function:

$$\bar{f}(p) = \begin{cases} \mathfrak{E}(p)f(p) & \text{if } p \in \operatorname{char}\mathfrak{X}, \\ f(S) & \text{if } S \in \mathfrak{X}'. \end{cases}$$

In general, $\mathfrak{F} \neq \mathrm{LF}_{\mathfrak{X}}(\bar{f})$; take $\mathfrak{F} = \mathfrak{N}$, $\mathfrak{X} = \mathfrak{J}$, and $f(p) = (1)$ for all $p \in \mathbb{P}$. However:

**Theorem 3.1.28.** *Let $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(f) = \mathrm{LF}_{\mathfrak{X}}(F)$ be an $\mathfrak{X}$-local formation with $f$ integrated. The following statements are pairwise equivalent:*

1. $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(\bar{f})$;
2. $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(\bar{F})$;
3. $\mathrm{b}(\mathfrak{F})$ *is $\mathfrak{X}$-wide.*

*Proof.* 1 implies 2. Let $p \in \operatorname{char}\mathfrak{X}$. Then, by Theorem 3.1.17 (3) $F(p) = \mathfrak{F} \cap \mathfrak{S}_p f(p) \subseteq \mathfrak{E}(p)f(p)$. Consequently $\mathfrak{E}(p)F(p) \subseteq \mathfrak{E}(p)f(p)$. It is then clear that $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(\bar{F})$.

2 implies 3. Let $G \in \mathrm{b}_{\mathfrak{X}}(\mathfrak{F})$ be an $\mathfrak{X}$-dense group with respect to $\mathfrak{F}$. Then $\mathrm{Soc}(G) \in \mathrm{E}\,\mathfrak{X}$ and so $\mathrm{Soc}(G) \in \mathfrak{E}(p)$ for all primes $p$ dividing $|\mathrm{Soc}(G)|$. Therefore $G \in \mathfrak{E}(p)F(p)$. Applying Remark 3.1.7 (2), we have that $G \in \mathrm{LF}_{\mathfrak{X}}(\bar{F}) = \mathfrak{F}$, contrary to the choice of $G$. Hence $\mathrm{b}_{\mathfrak{X}}(\mathfrak{F})$ is wide.

3 implies 1. Suppose that $\mathrm{b}_{\mathfrak{X}}(\mathfrak{F})$ is $\mathfrak{X}$-wide. Since $f \leq \bar{f}$, it follows that $\mathfrak{F} \subseteq \mathrm{LF}_{\mathfrak{X}}(\bar{f})$. Hence the burden of the proof is to show that $\mathrm{LF}_{\mathfrak{X}}(\bar{f}) \subseteq \mathfrak{F}$. Assume that this is not true, and let $G$ be a group of minimal order in $\mathrm{LF}_{\mathfrak{X}}(\bar{f}) \setminus \mathfrak{F}$.

It follows easily that $G$ has a unique minimal normal subgroup, $N$ say, and $G/N \in \mathfrak{F}$. If $N \in \mathrm{E}(\mathfrak{X}')$, then $G \in \bar{f}(S) = \mathfrak{F}$ for some simple group $S \in \mathfrak{X}'$, contrary to supposition. Hence $N \in \mathrm{E}\,\mathfrak{X}$ and so $G/\mathrm{C}_G(N) \in \mathfrak{E}(p)f(p)$ for each prime $p$ dividing $|N|$. If $N$ is abelian, then $G/\mathrm{C}_G(N) \in \mathfrak{F} \cap \mathfrak{E}(p)f(p) = F(p)$ by Remark 3.1.27. Now applying Remark 3.1.7 (2), $G \in \mathfrak{F}$, which is not the case. Hence $N$ is non-abelian and then $\mathrm{C}_G(N) = 1$. Then $G/N \in \mathfrak{F} \cap \mathfrak{E}(p)f(p) = F(p)$ for all primes $p$ dividing $|N|$. Since $G \notin F(p)$, we have that $G$ is $\mathfrak{X}$-dense with respect to $\mathfrak{F}$, and we have reached a final contradiction. Therefore $\mathrm{LF}_{\mathfrak{X}}(\bar{f}) \subseteq \mathfrak{F}$ and the equality holds. □

The next result shows that the $\mathfrak{X}$-local formations of $\mathfrak{X}$-wide boundary are precisely those for which a partial converse of Theorem 3.1.17 (3) holds.

**Theorem 3.1.29.** *Let $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(F)$ be an $\mathfrak{X}$-local formation. Then the following statements are equivalent:*

1. $\mathrm{b}(\mathfrak{F})$ *is $\mathfrak{X}$-wide.*
2. *If $f$ is an $\mathfrak{X}$-formation function such that $\mathfrak{F} \cap \mathfrak{S}_p f(p) = F(p)$ for all $p \in \operatorname{char}\mathfrak{X}$, and $f(S) = \mathfrak{F}$ for all $S \in \mathfrak{X}'$, then $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(f)$.*

*Proof.* 1 implies 2. Let $f$ be an $\mathfrak{X}$-formation function such that $\mathfrak{F} \cap \mathfrak{S}_p f(p) = F(p)$ for all $p \in \operatorname{char}\mathfrak{X}$ and $f(S) = \mathfrak{F}$ for all $S \in \mathfrak{X}'$. Denote $\mathfrak{F}_1 = \mathrm{LF}_{\mathfrak{X}}(f)$. It is clear that $\mathfrak{F} \subseteq \mathfrak{F}_1$ because $F(p) \subseteq \mathfrak{S}_p f(p)$ for all $p \in \operatorname{char}\mathfrak{X}$. Suppose that $\mathfrak{F}_1$ is not contained in $\mathfrak{F}$ and let $G \in \mathfrak{F}_1 \setminus \mathfrak{F}$ of minimal order. As usual, $G$ has a unique minimal normal subgroup $N$ such that $G/N \in \mathfrak{F}$. Moreover $N \in \mathrm{E}\,\mathfrak{X}$ and $G/\mathrm{C}_G(N) \in f(p)$ for all $p \in \pi(N)$. If $N$ were abelian, then $G/\mathrm{C}_G(N) \in \mathfrak{F} \cap f(p) \subseteq F(p)$ and we would have $G \in \mathfrak{F}$ by Remark 3.1.7 (2). This would contradict the choice of $G$. Hence $N$ should be non-abelian and so $G \in f(p)$ for all $p \in \pi(N)$. This implies that $G/N \in \mathfrak{F} \cap f(p) \subseteq F(p)$. Since $G \in \mathrm{b}(\mathfrak{F})$, we have that $G \notin F(p)$. Hence $G$ is $\mathfrak{X}$-dense with respect to $\mathfrak{F}$ and $\mathrm{b}(\mathfrak{F})$ is not $\mathfrak{X}$-wide. This is a contradiction. Consequently $\mathfrak{F}_1 \subseteq \mathfrak{F}$ and the equality holds.

2 implies 1. Let $f$ be the $\mathfrak{X}$-formation function given by $f(p) = \mathfrak{E}(p)F(p)$ for all $p \in \operatorname{char}\mathfrak{X}$ and $f(S) = F(S) = \mathfrak{F}$ for all $S \in \mathfrak{X}'$. Then, by Remark 3.1.27, we have $\mathfrak{F} \cap \mathfrak{S}_p f(p) = \mathfrak{F} \cap \mathfrak{E}(p)F(p) = F(p)$ for all $p \in \operatorname{char}\mathfrak{X}$. Consequently $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(f)$ by Statement 2. Applying Theorem 3.1.28, we conclude that $\mathrm{b}(\mathfrak{F})$ is $\mathfrak{X}$-wide. □

**Theorem 3.1.30.** *Let $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(F)$ be an $\mathfrak{X}$-local formation with a maximal $\mathfrak{X}$-local definition. Then* $\mathrm{b}(\mathfrak{F})$ *is $\mathfrak{X}$-wide.*

*Proof.* Let $p \in \operatorname{char}\mathfrak{X}$ and define the following $\mathfrak{X}$-formation function: $F_p(p) = \mathfrak{E}(p)F(p)$ and $F_p(S) = F(S)$ for every $S \in (\operatorname{char}\mathfrak{X}) \cup \mathfrak{X}'$ such that $S \not\cong C_p$. Then $F \leq F_p$. Hence $\mathfrak{F} \subseteq \mathrm{LF}_{\mathfrak{X}}(F_p)$. We suppose that $\mathfrak{F} \neq \mathrm{LF}_{\mathfrak{X}}(F_p)$ and derive a contradiction. Let $G \in \mathrm{LF}_{\mathfrak{X}}(F_p) \setminus \mathfrak{F}$ be a group of minimal order. Then $G$ has a unique minimal normal subgroup $N$ and $G/N \in \mathfrak{F}$. If $N \in \mathrm{E}(\mathfrak{X}')$, then $G \in F(S)$ for some $S \in \mathfrak{X}'$ and so $G \in \mathfrak{F}$, which is a contradiction. Hence

$N \in \mathrm{E}\,\mathfrak{X}$. Suppose that $N$ is abelian. Since $G \notin \mathfrak{F}$, we conclude that $N$ is a $p$-group. But in this case $G/\mathrm{C}_G(N) \in \mathfrak{E}(p)F(p) \cap \mathfrak{F} = F(p)$ by Remark 3.1.27. Hence $G \in \mathfrak{F}$ by Remark 3.1.7 (2). Consequently $N$ should be non-abelian. Let $q$ be a prime different from $p$ such that $q$ divides the order of $N$. Then $G \in F_p(q) = F(q) \subseteq \mathfrak{F}$. This is the desired contradiction.

Therefore $\mathfrak{F}_p = \mathrm{LF}_{\mathfrak{X}}(F_p) = \mathfrak{F}$ for all $p \in \operatorname{char}\mathfrak{X}$. Let $g$ be the maximal $\mathfrak{X}$-local definition of $\mathfrak{F}$. Then $\mathfrak{E}(p)F(p) \subseteq g(p)$ for all $p \in \operatorname{char}\mathfrak{X}$. Consequently $\mathfrak{F} = \mathrm{LF}(\bar{F})$. Applying Theorem 3.1.28, $\mathrm{b}(\mathfrak{F})$ is $\mathfrak{X}$-wide. □

Let $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(F)$ be an $\mathfrak{X}$-local formation. Define

$$\underline{F}(S) = \begin{cases} \mathrm{h}\Big(\mathrm{b}\big(F(p)\big) \cap \mathfrak{F}\Big) & \text{if } S = p \in \operatorname{char}\mathfrak{X}, \\ \mathrm{h}\big(\mathrm{b}_S(\mathfrak{F})\big) & \text{if } S \in \mathfrak{X}' \end{cases}$$

$$\widehat{F}(S) = \begin{cases} \Big(G : \mathrm{Q}\,\mathrm{R}_0\big(F(p) \cup \{G\}\big) \subseteq \underline{F}(p)\Big) & \text{if } S = p \in \operatorname{char}\mathfrak{X}, \\ \mathrm{h}\big(\mathrm{b}_S(\mathfrak{F})\big) & \text{if } S \in \mathfrak{X}' \setminus \mathbb{P}, \\ \Big(G : \mathrm{Q}\,\mathrm{R}_0\big(F(q) \cup \{G\}\big) \subseteq \underline{F}(q)\Big) & \text{if } S \in \mathfrak{X}' \cap \mathbb{P}. \end{cases}$$

Note that $\mathrm{h}\big(\mathrm{b}_S(\mathfrak{F})\big)$ is a saturated formation for all $S \in \mathfrak{X}' \setminus \mathbb{P}$ by Example 2.3.21. Moreover $\mathrm{Q}\,\widehat{F}(p) = \widehat{F}(p)$ for each prime $p$.

**Lemma 3.1.31.** $\underline{F}(p) \cap \mathfrak{F} = \widehat{F}(p) \cap \mathfrak{F} = F(p)$ *for each prime* $p$.

*Proof.* $F(p) \subseteq \widehat{F}(p) \cap \mathfrak{F} \subseteq \underline{F}(p) \cap \mathfrak{F}$. Now if $p \in \operatorname{char}\mathfrak{X}$, then $\underline{F}(p) \cap \mathfrak{F} \subseteq F(p)$ by using familiar arguments. If $p \in \mathfrak{X}'$, then $F(p) = \mathfrak{F}$. Therefore in both cases $\underline{F}(p) \cap \mathfrak{F} \subseteq F(p)$ and $F(p) = \underline{F}(p) \cap \mathfrak{F}$. □

**Lemma 3.1.32.** *Let* $p$ *be a prime. If* $\mathfrak{L}$ *is a formation contained in* $\underline{F}(p)$*, then* $\mathrm{Q}\,\mathrm{R}_0\big(F(p) \cup \mathfrak{L}\big)$ *is contained in* $\underline{F}(p)$*.*

*Proof.* It is enough to prove $\mathrm{R}_0\big(F(p) \cup \mathfrak{L}\big) \subseteq \underline{F}(p)$ since $\underline{F}(p)$ is a homomorph. Suppose that $\mathrm{R}_0\big(F(p) \cup \mathfrak{L}\big)$ is not contained in $\underline{F}(p)$ and take $G \in \mathrm{R}_0\big(F(p) \cup \mathfrak{L}\big) \setminus \underline{F}(p)$ of minimal order. Then $G^{F(p)} \neq 1 \neq G^{\mathfrak{L}}$ and $G \notin \underline{F}(p)$. Furthermore, there exists a normal subgroup $K$ of $G$ such that $G/K \in \mathrm{b}\big(F(p)\big) \cap \mathfrak{F}$ or $G/K \in \mathrm{b}_p(\mathfrak{F})$ according whether $p \in \operatorname{char}\mathfrak{X}$ or $p \in \mathfrak{X}'$. Suppose that $K \cap G^{F(p)} \neq 1$ and let $N$ be a minimal normal subgroup of $G$ such that $N$ is contained in $K \cap G^{F(p)}$. By the choice of $G$, we have $G/N \in \underline{F}(p)$. Hence $G/K \in \underline{F}(p)$. This is impossible. Consequently $K \cap G^{F(p)} = 1$ and, analogously, $K \cap G^{\mathfrak{L}} = 1$. Assume that $p \in \operatorname{char}\mathfrak{X}$. Then $G/K \in \mathfrak{F}$. Thus $G \in \mathrm{R}_0\,\mathfrak{F} = \mathfrak{F}$. This implies that $G/G^{\mathfrak{L}} \in \mathfrak{L} \cap \mathfrak{F} \subseteq \underline{F}(p) \cap \mathfrak{F} = F(p)$ by Lemma 3.1.31 and so $G^{F(p)} \leq G^{\mathfrak{L}}$. Since $G^{F(p)} \cap G^{\mathfrak{L}} = 1$, it follows that $G \in F(p)$. This contradicts the choice of $G$. Now suppose that $p \in \mathfrak{X}'$. In this case $G/K \in \mathrm{b}_p(\mathfrak{F})$. Let $L/K = \mathrm{Soc}(G/K)$. Then $L = G^{\mathfrak{F}}K = G^{\mathfrak{F}} \times K$ and so $G^{\mathfrak{F}}$ is a minimal normal subgroup of $G$. Let $B$ be a minimal normal subgroup contained in $G^{\mathfrak{L}}$. Then $G/B \in \mathrm{h}\big(\mathrm{b}_p(\mathfrak{F})\big)$ by the choice of $G$. Suppose that $G/B \notin \mathfrak{F}$. Then $G$ has a factor group, $G/T$ say, such that $B \le T$ and $G/T \in \mathrm{b}(\mathfrak{F})$. Set $M/T = \mathrm{Soc}(G/T)$. Then $M = G^{\mathfrak{F}}T$ because $G^{\mathfrak{F}}$ is a minimal normal subgroup of $G$. Therefore $M/T$ is a $p$-group and $G/T \in \mathrm{b}_p(\mathfrak{F})$. This is a contradiction. Consequently $\mathrm{Q}\,\mathrm{R}_0\big(F(p) \cup \mathfrak{L}\big)$ is contained in $\underline{F}(p)$. □

**Theorem 3.1.33 ([FS85]).** *Let $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(F)$ be an $\mathfrak{X}$-local formation. Then $\mathfrak{F}$ possesses a maximal $\mathfrak{X}$-local definition if and only if $\mathrm{b}(\mathfrak{F})$ is $\mathfrak{X}$-wide and, for each prime $p$, there exists a unique maximal formation contained in $\underline{F}(p)$. In this case, $\widehat{F}$ is an $\mathfrak{X}$-formation function and $\widehat{F}$ is the maximal $\mathfrak{X}$-local definition of $\mathfrak{F}$.*

*Proof.* First, suppose that $\mathfrak{F}$ possesses a maximal $\mathfrak{X}$-local definition, $g$ say. Then $\mathrm{b}(\mathfrak{F})$ is $\mathfrak{X}$-wide by Theorem 3.1.30. Let $p$ be a prime in $\mathrm{char}\,\mathfrak{X}$. Then $g(p) \cap \mathfrak{F}$ is contained in $F(p)$ by Theorem 3.1.17 (3). Hence $g(p)$ is contained in $\mathrm{h}\Big(\mathrm{b}(F(p)) \cap \mathfrak{F}\Big) = \underline{F}(p)$.

Assume now that $p \in \mathfrak{X}' \cap \mathbb{P}$ and $g(p)$ is not contained in $\underline{F}(p)$. Let $G$ be a group of least order in $g(p) \setminus \underline{F}(p)$. Then $G \in \mathrm{b}_p(\mathfrak{F})$, and from $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(g)$ we readily get that $G \in \mathfrak{F}$, the desired contradiction. Consequently $g(p) \subseteq \underline{F}(p)$. Let $\mathfrak{L}$ be a formation contained in $\underline{F}(p)$. By Lemma 3.1.32, $\mathrm{Q}\,\mathrm{R}_0\big(F(p) \cup \mathfrak{L}\big) \subseteq \underline{F}(p)$. Consider the $\mathfrak{X}$-formation function defined by setting

$$g_1(q) = \begin{cases} \mathrm{Q}\,\mathrm{R}_0\big(F(p) \cup \mathfrak{L}\big) & \text{if } p = q, \\ F(q) & \text{if } p \neq q \end{cases}$$

and $g_1(S) = g(S)$ for every $S \in \mathfrak{X}' \setminus \mathbb{P}$. Since $g_1(p) \cap \mathfrak{F} \subseteq F(p)$ by Lemma 3.1.31 and Lemma 3.1.32, we immediately have that $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(g_1)$. The maximal character of $g$ implies that $g_1(p) \subseteq g(p)$. Thus $\mathfrak{L} \subseteq g(p)$. Consequently, $g(p)$ is the unique maximal formation contained in $\underline{F}(p)$.

Conversely, suppose that $\mathrm{b}(\mathfrak{F})$ is $\mathfrak{X}$-wide and for each prime $p$, there exists a unique maximal formation, $g(p)$, contained in $\underline{F}(p)$. Consider the $\mathfrak{X}$-formation function $g_1$ defined by $g_1(p) = g(p)$ for every prime $p$ and $g_1(S) = \mathrm{h}\big(\mathrm{b}_S(\mathfrak{F})\big)$ for every $S \in \mathfrak{X}' \setminus \mathbb{P}$. Clearly $\mathfrak{F} \subseteq \mathrm{LF}_{\mathfrak{X}}(g_1)$ because $F(S) \subseteq g(p)$ for all $p$ and $\mathfrak{F} \subseteq g_1(S)$ for all $S \in \mathfrak{X}' \setminus \mathbb{P}$. If $\mathfrak{F} \neq \mathrm{LF}_{\mathfrak{X}}(g_1)$, then a group $G \in \mathrm{LF}_{\mathfrak{X}}(g_1) \setminus \mathfrak{F}$ of minimal order would be an $\mathfrak{X}$-dense group. Since $\mathrm{b}(\mathfrak{F})$ is $\mathfrak{X}$-wide, we conclude that $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(g_1)$. On the other hand, let $j$ be an $\mathfrak{X}$-formation function such that $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(j)$. Then, for all $p$, we have $j(p) \cap \mathfrak{F} \subseteq F(p)$. Consequently, $j(p) \subseteq \underline{F}(p)$ and then $j(p) \subseteq g(p)$. Furthermore, it is clear that $j(S) \subseteq g_1(S)$ for every $S \in \mathfrak{X}' \setminus \mathbb{P}$. Consequently, $g_1$ is the maximal $\mathfrak{X}$-local definition of $\mathfrak{F}$.

Note that in this case $g(p) = \widehat{F}(p)$ and $g(S) = \widehat{F}(S)$ for all $S \in \mathfrak{X}' \setminus \mathbb{P}$. Therefore $\widehat{F}$ is an $\mathfrak{X}$-formation function and it is actually the maximal $\mathfrak{X}$-local definition of $\mathfrak{F}$. □

**Proposition 3.1.34.** *Let $\mathfrak{Y} \subseteq \mathfrak{X}$ be classes of simple groups. If $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(F)$ has a maximal $\mathfrak{X}$-local definition, then $\mathfrak{F}$ has a unique maximal $\mathfrak{Y}$-local definition. If, in addition, $\mathrm{char}\,\mathfrak{X} = \mathrm{char}\,\mathfrak{Y}$, then the converse is valid if, and only if, $\mathrm{b}(\mathfrak{F})$ is $\mathfrak{X}$-wide.*

*Proof.* Note that $\mathfrak{F} = \mathrm{LF}_{\mathfrak{Y}}(F_1)$, where $F_1(p) = F(p)$ for all $p \in \operatorname{char}\mathfrak{Y}$ and $F_1(S) = \mathfrak{F}$ for all $S \in \mathfrak{Y}'$ (see Corollary 3.1.13). Therefore if $\mathfrak{F}$ has a maximal $\mathfrak{X}$-local definition, then $\mathrm{b}(\mathfrak{F})$ is $\mathfrak{X}$-wide (and so $\mathrm{b}(\mathfrak{F})$ is $\mathfrak{Y}$-wide) and $\widehat{F}(p) = \widehat{F_1}(p)$ for all $p \in \operatorname{char}\mathfrak{Y}$ is a formation. We are left to show that $\widehat{F_1}(p)$ is a formation for all $p \in (\operatorname{char}\mathfrak{X}) \cap \mathfrak{Y}'$. To see this, we prove that $\underline{F}_1(p) = \mathfrak{G} = \mathrm{h}\big(\mathrm{b}_q(\mathfrak{F})\big)$ contains a unique maximal formation. Set $\mathfrak{H} = f(\mathfrak{G}) = (G : H/K$ is $\mathfrak{G}$-central in $G$ for every chief factor of $G)$. Applying Theorem 2.3.20, $\mathfrak{H}$ is a formation. Suppose that $\mathfrak{H}$ is not contained in $\mathfrak{G}$ and let $G \in \mathfrak{H} \setminus \mathfrak{G}$ be a group of minimal order. Then $G \in \mathrm{b}(\mathfrak{G}) = \mathrm{b}_q(\mathfrak{F})$ and so $G$ is a monolithic group. Moreover $X = [N]\big(G/\mathrm{C}_G(N)\big) \in \mathfrak{G}$. If $X \notin \mathfrak{F}$, then $X \in \mathrm{b}_q(\mathfrak{F})$, because $G/\mathrm{C}_G(N) \in \mathfrak{F}$. Hence $X \in \mathfrak{G} \cap \mathrm{b}_q(\mathfrak{F}) = \emptyset$. This is a contradiction. Therefore $X \in \mathfrak{F}$ and $G/\mathrm{C}_G(N) \in F(p)$. Applying Remark 3.1.7 (2), we conclude that $G \in \mathrm{LF}_{\mathfrak{X}}(F) = \mathfrak{F}$. We have obtained a contradiction. Consequently $\mathfrak{H} \subseteq \mathfrak{G}$. Let now $\mathfrak{L}$ be a formation contained in $\mathfrak{G}$. Then by Theorem 2.3.20 (2), $\mathfrak{L} \subseteq \mathfrak{H}$. This means that $\widehat{F_1}(p)$ is a formation. By Theorem 3.1.33, it follows that $\mathfrak{F}$ has a maximal $\mathfrak{Y}$-local definition.

Now if $\operatorname{char}\mathfrak{X} = \operatorname{char}\mathfrak{Y}$, then $F(p) = F_1(p)$ for all $p \in \operatorname{char}\mathfrak{X}$. Consequently if $\mathfrak{F}$ has a maximal $\mathfrak{Y}$-local definition, then $\widehat{F}(p)$ is a formation for all $p \in \operatorname{char}\mathfrak{X}$. By Theorem 3.1.30, $\mathfrak{F}$ has a maximal $\mathfrak{X}$-local definition if, and only if, $\mathrm{b}(\mathfrak{F})$ is $\mathfrak{X}$-wide. □

*Examples 3.1.35.* 1. Let $\mathfrak{F} = \mathfrak{S}$ be the $\mathfrak{J}$-local (local) formation of all soluble groups. Then $\mathfrak{F} = \mathrm{LF}_{\mathfrak{J}}(F)$ where $F(p) = \mathfrak{F}$ for all $p \in \mathbb{P}$. Hence $\widehat{F}(p) = \mathfrak{E}$ and so $\widehat{F}$ is a $\mathfrak{J}$-formation function. However, $\mathfrak{F}$ does not have a maximal $\mathfrak{J}$-local definition (see Example 3.1.23).

This example shows that the requirement that $\mathrm{b}(\mathfrak{F})$ be $\mathfrak{X}$-wide cannot be removed from Theorem 3.1.33.

2. Let $\mathfrak{F}_0$ be the class of all groups whose Frattini chief factors have odd order. Then $\mathfrak{F}_0$ is a formation and $\mathfrak{R}_{\mathfrak{J}}\mathfrak{F}_0 = \mathfrak{F}_0$. Let $\mathfrak{F} = \mathfrak{N}\mathfrak{F}_0$. Applying Example 3.1.26 (2), we have that $\mathfrak{F}$ is a $\mathfrak{J}$-local formation with $\mathfrak{J}$-wide boundary. Assume that $\mathfrak{F} \neq \mathfrak{S}_2\mathfrak{F}_0$ and let $G \in \mathfrak{F}\setminus\mathfrak{S}_2\mathfrak{F}_0$ be a group of minimal order. Then $G$ has a unique minimal normal subgroup $N$. Moreover $G/N \in \mathfrak{S}_2\mathfrak{F}_0$. Since $G \notin \mathfrak{F}_0$, we conclude that $N$ is a $p$-group for some odd prime $p$. Hence $\mathrm{F}(G)$ is a $p$-group. This implies that $G \in \mathfrak{F}_0$ because $G/\mathrm{F}(G)$ has no Frattini 2-chief factors. This is a contradiction. Consequently $\mathfrak{F} = \mathfrak{S}_2\mathfrak{F}_0$ and $\mathfrak{F} = \mathrm{LF}_{\mathfrak{J}}(F)$, where

$$F(p) = \begin{cases} \mathfrak{F} & \text{if } p = 2, \\ \mathfrak{F}_0 & \text{if } p \neq 2. \end{cases}$$

Then $\underline{F}(q) = \mathrm{h}\Big(\mathrm{b}\big(F(q)\big) \cap \mathfrak{F}\Big) = \mathrm{h}\big(\mathrm{b}(\mathfrak{F}_0) \cap \mathfrak{F}\big) = \mathrm{h}\big(\mathrm{b}(\mathfrak{F}_0)\big) = \mathfrak{F}_0$ for each odd prime $q$ (note $\mathrm{b}(\mathfrak{F}_0) \subseteq \mathfrak{F}$). Consequently

$$\underline{F}(p) = \widehat{F}(p) = \begin{cases} \mathfrak{E} & \text{if } p = 2, \\ \mathfrak{F}_0 & \text{if } p \neq 2 \end{cases}$$

and $\underline{F} = \widehat{F}$ is a $\mathfrak{J}$-formation function. Applying Theorem 3.1.33, we have that $\underline{F}$ is the maximal $\mathfrak{J}$-local definition of $\mathfrak{F}$.

Let $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(F)$ be an $\mathfrak{X}$-local formation. In contrast to the condition that $\mathrm{b}(\mathfrak{F})$ is $\mathfrak{X}$-wide, the other condition from Theorem 3.1.33 — namely, that $\widehat{F}(X)$ be a formation for all $X \in \mathbb{P}$ — is not always easy to check when a concrete formation $\mathfrak{F}$ is given. We give an example of a local formation for which $\widehat{F}$ is not a formation function.

*Example 3.1.36 ([FS85]).* Let $\mathfrak{R} = \mathfrak{R}_{\mathfrak{J}}$ be the formation composed of all groups whose chief factors are non-abelian. Consider the local formation $\mathfrak{F} = \mathfrak{N}\mathfrak{R}\mathfrak{N}$. Then the canonical definition $F$ of $\mathfrak{F}$ is given by $F(p) = \mathfrak{S}_p\mathfrak{R}\mathfrak{N}$ for all $p$. Applying Examples 3.1.26 (2), we have that $\mathrm{b}(\mathfrak{F})$ is $\mathfrak{J}$-wide because $\mathfrak{R}_{\mathfrak{J}}(\mathfrak{R}\mathfrak{N}) = \mathfrak{R}\mathfrak{N}$.

Let $S = \mathrm{SL}(2,5)$. By [DH92, B, 10.9], $S$ has an irreducible module $V$ over $\mathrm{GF}(p)$ such that $\mathrm{Ker}(S \,\mathrm{on}\, V) = \mathrm{C}_S(V) = \mathrm{Z}(S)$. Let $X = [V]S$ be the corresponding semidirect product, and let $Y = S \wr_{\mathrm{Z}(X)} X$ be the wreath product of $S$ with respect to the permutation representation of $S$ with $X$ with respect to the permutation representation of $X$ on the set of cosets of $\mathrm{Z}(X) = \mathrm{Z}(S)$ in $X$. As usual, for any subgroup $U$ of $S$, $U^\natural = U \times \cdots \times U$ ($|X/\mathrm{Z}(X)|$ copies of $U$) shall denote the canonical subgroup of $S^\natural$, the base group of $Y$, isomorphic to a direct product of $|X/\mathrm{Z}(X)|$ copies of $S$. Note that $\mathrm{Z}(X) \leq \mathrm{Z}(Y)$ and $\mathrm{Z}(S)^\natural X/\mathrm{Z}(X)$ is $X$-isomorphic to the regular wreath product $C_2 \wr_{\mathrm{reg}} X/\mathrm{Z}(X)$ and this is isomorphic to the semidirect product of the regular $X/\mathrm{Z}(X)$-module over $\mathrm{GF}(2)$ with $X/\mathrm{Z}(X)$. Therefore there exists a normal subgroup $Z$ of $Y$ such that $Z \leq \mathrm{Z}(S)^\natural$ and $Z(S)^\natural/Z$ is a cyclic group of order 2.

We consider now $G = Y/Z$. It is clear that $S$ is isomorphic to a quotient of $G$. Let $A = Z(X)Z/Z$ and $B = Z(S)^\natural/Z$. It is clear that $A$ and $B$ are subgroups of order 2 contained in $Z(G)$ such that $A \cap B = 1$. Hence there exists $D \leq \mathrm{Z}(G)$ of order 2 such that $D \cap A = D \cap B = 1$. In particular $G \in \mathrm{R}_0(G/A, G/D)$.

Assume that $p$ is a prime and $p > 5$. Then Förster and Salomon [FS85, Example 4.1] proved that $G/A$, $G/D \in \widehat{F}(p)$.

However since $\widehat{F}(p)$ is Q-closed, $S$ is isomorphic to a quotient of $G$ and $S \in \mathrm{b}\big(F(p)\big) \cap \mathfrak{F}$, it follows that $G \notin \widehat{F}(p)$. This shows that $\widehat{F}(p)$ is not a formation and hence $\mathfrak{F} = \mathfrak{N}\mathfrak{R}\mathfrak{N}$ does not have a maximal $\mathfrak{J}$-local definition as $\mathfrak{J}$-local formation.

The above example can be modified to show that $\mathrm{h}\big(\mathrm{b}_q(\mathfrak{F})\big)$, $\mathfrak{F}$ an $\mathfrak{X}$-local formation and $q \in \mathfrak{X}' \cap \mathbb{P}$, does not always contain a unique largest formation.

*Example 3.1.37.* Let $\mathfrak{F} = \mathfrak{S}_p\mathfrak{R}\mathfrak{N}$ as in the above example. Suppose that $\mathfrak{X} = \emptyset$. Put $q = 2$ and take $G$, $A$, $D$ as in Example 3.1.36. Then $\mathrm{Q}\,\mathrm{R}_0(G/A) \cup \mathrm{Q}\,\mathrm{R}_0(G/D) \subseteq \mathrm{h}\big(\mathrm{b}_2(\mathfrak{F})\big)$, but $G \in \mathrm{R}_0(G/A, G/D)$ does not belong to $\mathrm{h}\big(\mathrm{b}_2(\mathfrak{F})\big)$. Consequently $\mathfrak{F}$ is an $\emptyset$-local formation without a maximal $\emptyset$-local definition.

*Proof.* First of all, we know that $S = \mathrm{SL}(2,5)$ is a quotient of $G$ and $S \in \mathrm{b}_2(\mathfrak{F})$. Therefore $G \notin \mathrm{h}\bigl(\mathrm{b}_2(\mathfrak{F})\bigr)$. Moreover, $G \in \mathrm{R}_0(G/A, G/D)$. Now let $\mathfrak{B}_1 = \mathrm{b}_2(\mathfrak{F}) \cap \mathfrak{N}\mathfrak{R}\mathfrak{N}$ and $\mathfrak{B}_2 = \mathrm{b}_2(\mathfrak{F}) \setminus \mathfrak{N}\mathfrak{R}\mathfrak{N}$. Thus $\mathrm{b}_2(\mathfrak{F}) = \mathfrak{B}_1 \cup \mathfrak{B}_2$ and $\mathrm{h}\bigl(\mathrm{b}_2(\mathfrak{F})\bigr) = \mathrm{h}(\mathfrak{B}_1) \cap \mathrm{h}(\mathfrak{B}_2)$. Förster and Salomon [FS85, Example 4.1] proved that $\mathrm{Q}\,\mathrm{R}_0(G/A) \cup \mathrm{Q}\,\mathrm{R}_0(G/D) \subseteq \mathrm{h}(\mathfrak{B}_1)$. Moreover $\mathfrak{B}_2$ is a class composed by primitive groups. Hence $\mathrm{h}(\mathfrak{B}_2)$ is a Schunck class by Corollary 2.3.11. Note that $[H/K]*(G/A) \in \mathrm{h}(\mathfrak{B}_2)$ for each chief factor $H/K$ of $G/A$ (and the same applies to $G/D$). This implies that $G/A$ and $G/B$ belong to $f\bigl(\mathrm{h}(\mathfrak{B}_2)\bigr)$, which is the largest formation contained in $\mathrm{h}(\mathfrak{B}_2)$ by Theorem 2.3.20 (3). Hence $\mathrm{Q}\,\mathrm{R}_0(G/A) \cup \mathrm{Q}\,\mathrm{R}_0(G/D) \subseteq \mathrm{h}(\mathfrak{B}_2)$. □

In [DH92, pages 364 and 365], the authors study the effect of some closure operations on a local formation. More precisely, they prove:

> Let $\mathfrak{F} = \mathrm{LF}(f)$ be a local formation and let $\mathrm{C}$ be one of the closure operations $\mathrm{S}$, $\mathrm{S}_n$, or $\mathrm{N}_0$.
> 1. If $f(p) = \mathrm{C}\,f(p)$ for all $p \in \mathbb{P}$, then $\mathfrak{F} = \mathrm{C}\,\mathfrak{F}$, and
> 2. if $\mathfrak{F} = \mathrm{C}\,\mathfrak{F}$, and $F$ is the canonical local definition of $\mathfrak{F}$, then $F(p) = \mathrm{C}\,F(p)$ for all $p \in \mathbb{P}$.

The natural question is: can the above results be extended to $\mathfrak{X}$-local formations? If $\mathrm{C} = \mathrm{S}$, 1 is not always true (compare with [För85b, Lemma 3.13]).

*Example 3.1.38.* Let $\mathfrak{X} = (C_2)$ and $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(f)$, where $f(2) = (1)$ and $f(S) = \mathfrak{E}$ if $S \not\cong C_2$. It is clear that $\mathrm{S}\,f(S) = f(S)$ for all $S \in (\operatorname{char}\mathfrak{X}) \cup \mathfrak{X}'$, but $\mathfrak{F}$ is not $\mathrm{S}$-closed because $\mathrm{Alt}(5) \in \mathfrak{F}$ but $\mathrm{Alt}(4) \notin \mathfrak{F}$.

Our next result shows that 1 is true for $\mathrm{C} = \mathrm{S}_n$ or $\mathrm{N}_0$.

**Proposition 3.1.39.** *Let $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(f)$ be an $\mathfrak{X}$-local formation and let $\mathrm{C}$ be one of the closure operations $\mathrm{S}_n$ or $\mathrm{N}_0$. If $f(S) = \mathrm{C}\,f(S)$ for all $S \in (\operatorname{char}\mathfrak{X}) \cup \mathfrak{X}'$, then $\mathfrak{F} = \mathrm{C}\,\mathfrak{F}$.*

*Proof.* Let $\mathrm{C} = \mathrm{S}_n$. Let $G \in \mathfrak{F}$, and let $N$ be a normal subgroup of $G$. We prove that $N \in \mathfrak{F}$ by induction on $|G|$. Let $A$ be a minimal normal subgroup of $G$. Then $NA/A \in \mathfrak{F}$. If $B$ were another minimal normal subgroup of $G$, then $NB/B \in \mathfrak{F}$. This would imply $N \in \mathfrak{F}$. Consequently we may assume that $A = \mathrm{Soc}(G)$ is the unique minimal normal subgroup of $G$. Let $p \in \operatorname{char}\mathfrak{X}$. Then $N/\bigl(N \cap \mathrm{C}^{\mathfrak{X}_p}(G)\bigr) \cong N\,\mathrm{C}^{\mathfrak{X}_p}(G)/\mathrm{C}^{\mathfrak{X}_p}(G)$ and $N\,\mathrm{C}^{\mathfrak{X}_p}(G)/\mathrm{C}^{\mathfrak{X}_p}(G)$ is a normal subgroup of $G/\mathrm{C}^{\mathfrak{X}_p}(G) \in f(p)$. Since $N \cap \mathrm{C}^{\mathfrak{X}_p}(G) = \mathrm{C}^{\mathfrak{X}_p}(N)$ by Proposition 3.1.10, it follows that $N/\mathrm{C}^{\mathfrak{X}_p}(N) \in f(p)$.

Assume now that $N/L$ is a monolithic quotient of $N$ such that $T/L = \mathrm{Soc}(N/L) \in \mathrm{E}(S)$ for some simple group $S \in \mathfrak{X}'$. If $A$ is not contained in $L$, then $T/L$ is contained in $AL/L \neq 1$ and so $A \in \mathrm{E}(S)$. Since $G$ is a monolithic $\mathfrak{F}$-group, it follows that $G \in f(S)$. Hence $N \in \mathrm{S}_n\,f(S) = f(S)$ and $N/L \in \mathrm{Q}\,f(S) = f(S)$. Suppose that $A$ is contained in $L$. We have that $N/A \in \mathfrak{F}$ by induction. Therefore $N/L \in f(S)$ because $N/L$ is isomorphic to a monolithic quotient of $N/A$ whose socle belongs to $\mathrm{E}(S)$. Therefore $N \in \mathfrak{F}$ and $\mathfrak{F} = \mathrm{S}_n\,\mathfrak{F}$.

Now suppose $\text{C} = \text{N}_0$. Applying [DH92, II, 2.11], it is enough to show that $G \in \mathfrak{F}$ provided that $G = N_1N_2$, where $N_i$ is a normal subgroup of $G$ and $N_i \in \mathfrak{F}_i$, $i \in \{1,2\}$. We argue by induction on $|G|$. It is rather clear that we may assume that $G$ has a unique minimal normal subgroup, $A$ say, and $G/A \in \mathfrak{F}$. Let $p \in \operatorname{char}\mathfrak{X}$. Then $G/\operatorname{C}^{\mathfrak{X}_p}(G) = \big(N_1 \operatorname{C}^{\mathfrak{X}_p}(G)/\operatorname{C}^{\mathfrak{X}_p}(G)\big)\big(N_2 \operatorname{C}^{\mathfrak{X}_p}(G)/\operatorname{C}^{\mathfrak{X}_p}(G)\big)$. Moreover $N_i \operatorname{C}^{\mathfrak{X}_p}(G)/\operatorname{C}^{\mathfrak{X}_p}(G) \cong N_i/\big(N_i \cap \operatorname{C}^{\mathfrak{X}_p}(G)\big) = N_i/\operatorname{C}^{\mathfrak{X}_p}(N_i) \in f(p)$. Hence $G/\operatorname{C}^{\mathfrak{X}_p}(G) \in \text{N}_0\, f(p) = f(p)$.

Suppose that $G/L$ is a monolithic quotient of $G$ such that $\operatorname{Soc}(G/L) \in \text{E}(S)$ for some simple group $S \in \mathfrak{X}'$. If $L \neq 1$, then $G/L \in \mathfrak{F}$ by induction. This implies $G/L \in f(S)$. Thus we may assume that $L = 1$. In this case $A \in \text{E}(S)$. It is clear that $\operatorname{Soc}(N_i) \in \text{E}(S)$ for $i \in \{1,2\}$. Therefore, applying Remark 3.1.2 (5), $N_i \in f(S)$ because $N_i \in \mathfrak{F}$, $i \in \{1,2\}$. Consequently $G \in \text{N}_0\, f(S) = f(S)$ and $G \in \mathfrak{F}$. We conclude that $\mathfrak{F}$ is $\text{N}_0$-closed. □

The next proposition shows that Statement 2 holds for $\mathfrak{X}$-local formations.

**Proposition 3.1.40.** *Let $\mathfrak{F} = \operatorname{LF}_{\mathfrak{X}}(F)$ be an $\mathfrak{X}$-local formation. If* $\text{C}$ *is one of the closure operations* $\text{S}$, $\text{S}_n$, *or* $\text{N}_0$ *and $\mathfrak{F} = \text{C}\,\mathfrak{F}$, then $F(S) = \text{C}\, F(S)$ for all $S \in (\operatorname{char}\mathfrak{X}) \cup \mathfrak{X}'$.*

*Proof.* If $S \in \mathfrak{X}'$, then $F(S) = \mathfrak{F}$. Hence we have to prove that $F(p) = \text{C}\, F(p)$ for all $p \in \operatorname{char}\mathfrak{X}$.

Assume $\text{C} = \text{S}$ and $p \in \operatorname{char}\mathfrak{X}$. Let $G \in F(p)$, and let $H$ be a subgroup of $G$. Then if $W = C_p \wr G$, we know that $W \in \mathfrak{F}$. Hence $BH \in \mathfrak{F}$, where $B$ is the base group of $W$. Therefore $BH/\operatorname{C}^{\mathfrak{X}_p}(BH) \in F(p)$. Now $\operatorname{C}^{\mathfrak{X}_p}(BH)$ centralises every chief factor of $BH$ below $B$. Since $B \leq \operatorname{C}^{\mathfrak{X}_p}(BH)$ and $\operatorname{C}_W(B) = B$, we have that $\operatorname{C}^{\mathfrak{X}_p}(BH)/B$ is a $p$-group by [DH92, A, 12.4]. Thus $H \in F(p)$ and $F(p)$ is subgroup-closed.

The case $\text{C} = \text{S}_n$ is analogous.

Now assume that $\text{C} = \text{N}_0$. By [DH92, II, 2.11], it will suffice to show that if $G = N_1N_2$ with $N_i$ a normal subgroup of $G$ and $N_i \in F(p)$, $i = 1, 2$, then $G \in F(p)$. Let $W = C_p \wr G$ with $B$ as the base group of $W$. Note that $W = (BN_1)(BN_2)$, $BN_i \trianglelefteq W$, and $BN_i \in \mathfrak{S}_pF(p) = F(p) \subseteq \mathfrak{F}$ for $i = 1, 2$. Therefore $W \in \text{N}_0\,\mathfrak{F} = \mathfrak{F}$. By Theorem 3.1.17 (3), $G \in F(p)$. □

Given a group $G$, denote by $\text{S}_{\mathfrak{X}}(G)$ the set of all subgroups $H$ of $G$ such that $H \in \text{E}\,\mathfrak{X}$. If $\mathfrak{L}$ is a class of groups, write $\mathfrak{L}(\mathfrak{X}) = \big(G : \text{S}_{\mathfrak{X}}(G) \subseteq \mathfrak{L}\big)$. It is clear that $\mathfrak{L}(\mathfrak{X})$ is the unique largest subgroup-closed class such that $\mathfrak{L}(\mathfrak{X}) \cap \text{E}\,\mathfrak{X} \subseteq \mathfrak{L}$.

If $\mathfrak{F}$ is a formation, then $\mathfrak{F}(\mathfrak{X})$ is clearly a formation, but if $\mathfrak{F}$ is an $\mathfrak{X}$-local formation, then $\mathfrak{F}(\mathfrak{X})$ is not an $\mathfrak{X}$-local formation in general as the next example shows.

*Example 3.1.41.* Consider $\mathfrak{X} = \mathfrak{J}$, the class of all simple groups, let $G = \operatorname{Alt}(5)$, and let $\mathfrak{F} = \mathfrak{N}^2\,\text{D}_0(1, G)$. In this case, $\mathfrak{F}(\mathfrak{X})$ is the class of all groups $U$ such that every subgroup of $U$ belongs to $\mathfrak{F}$. Hence $G$ belongs to $\mathfrak{F}(\mathfrak{X})$. If $\mathfrak{F}(\mathfrak{X})$ were an $\mathfrak{X}$-local formation, then $[V]G$ would be an $\mathfrak{F}(\mathfrak{X})$-group for every irreducible and faithful $\mathrm{GF}(2)G$-module $V$. In particular, if $D$ is the dihedral group of order 10, then $VD \in \mathfrak{F}$. This would be a contradiction. Hence $\mathfrak{F}(\mathfrak{X})$ is not an $\mathfrak{X}$-local formation.

The next result provides precise conditions to ensure that $\mathfrak{F}(\mathfrak{X})$ is again an $\mathfrak{X}$-local formation.

**Theorem 3.1.42 ([BB91]).** *Let $\mathfrak{F}$ be an $\mathfrak{X}$-local formation. The following statements are pairwise equivalent:*

1. *For each primitive group $G$ of type 2 in $\mathfrak{F}(\mathfrak{X})$ such that $\mathrm{Soc}(G) \in \mathrm{E}\,\mathfrak{X}$, and for every irreducible and faithful $G$-module $V$ over $\mathrm{GF}(p)$, $p \in \pi\big(\mathrm{Soc}(G)\big)$, the corresponding semidirect product $[V]G$ is an $\mathfrak{F}(\mathfrak{X})$-group.*
2. *For each primitive group $G$ of type 2 in $\mathfrak{F}(\mathfrak{X})$ such that $\mathrm{Soc}(G) \in \mathrm{E}\,\mathfrak{X}$ and for every irreducible and faithful $G$-module $V$ over $\mathrm{GF}(p)$, $p \in \pi\big(\mathrm{Soc}(G)\big)$, and for every $X \in \mathrm{S}_{\mathfrak{X}}(G)$ such that $G = X\,\mathrm{Soc}(G)$, the semidirect product $[V]X$ is an $\mathfrak{F}$-group.*
3. *$\mathfrak{F}(\mathfrak{X})$ is an $\mathfrak{X}$-local formation.*

*Proof.* 2 implies 3. Suppose $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(F)$. Define $F^*(p) = F(p)(\mathfrak{X})$, for each prime $p \in \mathrm{char}\,\mathfrak{X}$ and $F^*(E) = F(E)(\mathfrak{X})$, for every $E \in \mathfrak{X}'$. Then $F^*$ is an $\mathfrak{X}$-formation function. We see that $\mathfrak{F}(\mathfrak{X}) = \mathrm{LF}_{\mathfrak{X}}(F^*)$. Assume that $\mathfrak{F}(\mathfrak{X})$ is not contained in $\mathrm{LF}_{\mathfrak{X}}(F^*)$ and derive a contradiction. We choose a group $G \in \mathfrak{F}(\mathfrak{X}) \setminus \mathrm{LF}_{\mathfrak{X}}(F^*)$ of minimal order. Using familiar arguments, we have that $G$ is a monolithic group. Denote $N = \mathrm{Soc}(G)$. If $N \in \mathrm{E}(\mathfrak{X}')$, then $G \in F(E)(\mathfrak{X})$ for some $E \in \mathfrak{X}'$ and so $G \in \mathrm{LF}_{\mathfrak{X}}(F^*)$, which is a contradiction. Hence $N \in \mathrm{E}\,\mathfrak{X}$. Suppose that $N$ is abelian. Then $N$ is a $p$-group for some prime $p \in \mathrm{char}\,\mathfrak{X}$. Let $X$ be a subgroup of $G$ such that $X \in \mathrm{E}\,\mathfrak{X}$. Without loss of generality, we may assume that $N$ is contained in $X$. Certainly $X \in \mathfrak{F}(\mathfrak{X})$ as $\mathfrak{F}(\mathfrak{X})$ is subgroup-closed. If $X$ is a proper subgroup of $G$, then $X \in \mathrm{LF}_{\mathfrak{X}}(F^*)$ by the choice of $G$. This implies that $X/\mathrm{C}^h_X(N) \in F^*(p)$, where $\mathrm{C}^h_X(N)$ is the intersection of the centralisers in $X$ of all chief factors of $X$ below $N$. Applying [DH92, A, 2.11], $\mathrm{C}^h_X(N)/\mathrm{C}_X(N)$ is a $p$-group. Hence $X/\mathrm{C}_X(N) \in F^*(p)$ and so $X/\mathrm{C}_X(N) \in F(p)$. If $X = G$, then $G/\mathrm{C}_G(N) \in F(p)$ because $\mathfrak{F}$ is $\mathfrak{X}$-local. Consequently $G/\mathrm{C}_G(N) \in F^*(p)$. Applying Remark 3.1.7 (2), we have that $G \in \mathrm{LF}_{\mathfrak{X}}(F^*)$ and we have the desired contradiction. Therefore $N$ is a non-abelian group. Let $p$ be a prime dividing the order of $N$ and let $X \in \mathrm{E}\,\mathfrak{X}$. Assume that $T = XN$ is a proper subgroup of $G$. Arguing as above, $T = XN \in \mathrm{LF}_{\mathfrak{X}}(F^*)$ and $\mathrm{C}^h_T(N) \cong \mathrm{C}^h_T(N)/\mathrm{C}_T(N)$ is a $p$-group (note that $\mathrm{C}_T(N) = 1$). Hence $T/\mathrm{C}^h_T(N) \in F^*(p)$. Since $X\,\mathrm{C}^h_T(N)/\mathrm{C}^h_T(N) \in \mathrm{S}_{\mathfrak{X}}\big(T/\mathrm{C}^h_T(N)\big)$, it follows that $X\,\mathrm{C}^h_T(N)/\mathrm{C}^h_T(N)$ is in $F(p)$ and so $X \in F(p)$. Suppose that $T = G$ and consider an irreducible and faithful $G$-module $V$ over $\mathrm{GF}(p)$ (such $V$ exists by [DH92, B, 10.9]). By Statement 2, the semidirect product $[V]X$ is an $\mathfrak{F}$-group. It implies that $X \in F(p)$. Therefore $G \in F^*(p)$ and $G \in \mathrm{LF}_{\mathfrak{X}}(F^*)$ and we have the desired contradiction.

On the other hand, taking into account that $\mathrm{LF}_{\mathfrak{X}}(F^*)$ is subgroup-closed, it is easy to see that $\mathrm{LF}_{\mathfrak{X}}(F^*)$ is contained in $\mathfrak{F}(\mathfrak{X})$. Consequently $\mathfrak{F}(\mathfrak{X})$ is an $\mathfrak{X}$-local formation.

3 implies 1. Taking into account that the $\mathfrak{F}(\mathfrak{X})$ can be locally defined by an $\mathfrak{X}$-formation function, it is clear that if $G$ is a primitive group of type 2 in $\mathfrak{F}(\mathfrak{X})$ and $\mathrm{Soc}(G) \in \mathrm{E}\,\mathfrak{X}$, then the semidirect product $[V]G$ is an $\mathfrak{F}(\mathfrak{X})$-group for every irreducible and faithful $G$-module $V$ over $\mathrm{GF}(p)$, $p \in \pi\big(\mathrm{Soc}(G)\big)$. Hence Statement 1 holds.

Finally, it is clear that 1 implies 2. The circle of implications is now complete. ☐

*Example 3.1.43.* Assume that $\mathfrak{X}$ is the class of all simple groups and consider the class $\mathfrak{F} = \big(G : \mathrm{Alt}(5) \notin \mathrm{Q}(G)\big)$. Then $\mathrm{b}(\mathfrak{F}) = \big(\mathrm{Alt}(5)\big)$. Hence $\mathfrak{F}$ is a saturated formation by Example 2.3.21. If $G$ is a primitive group of type 2 in $\mathfrak{F}(\mathfrak{X})$, then every subgroup of $[V]X$ is an $\mathfrak{F}$-group, for every subgroup $X$ of $G$ such that $G = X\,\mathrm{Soc}(G)$ and for every irreducible and faithful $G$-module $V$ over $\mathrm{GF}(p)$, $p \in \pi\big(\mathrm{Soc}(G)\big)$. Consequently $\mathfrak{F}(\mathfrak{X})$ is a saturated formation. It is clear that $\mathfrak{F}(\mathfrak{X})$ is the largest subgroup-closed formation contained in $\mathfrak{F}$.

## 3.2 A generalisation of Gaschütz-Lubeseder-Schmid-Baer theorem

In this section we study two different Frattini-like subgroups associated with a class of simple groups which lead to the corresponding notion of saturation. We then present an extension of Gaschütz-Lubeseder-Schmid and Baer theorems.

We begin with the following definition due to P. Förster.

**Definition 3.2.1 ([För85b]).** *Let $G$ be a group. For a prime $p$, we define $\Phi^p_{\mathfrak{X}}(G)$ as follows:*

- *If $\mathrm{O}_{p'}(G) = 1$,*

$$\Phi^p_{\mathfrak{X}}(G) = \begin{cases} \Phi(G) & \text{if } \mathrm{Soc}\big(G/\Phi(G)\big) \text{ and } \Phi(G) \text{ belong to } \mathrm{E}\,\mathfrak{X}, \\ \Phi\big(\mathrm{O}_{\mathfrak{X}}(G)\big) & \text{otherwise.} \end{cases}$$

- *In general, $\Phi^p_{\mathfrak{X}}(G)$ is the subgroup of $G$ such that $\Phi^p_{\mathfrak{X}}(G)/\mathrm{O}_{p'}(G) = \Phi^p_{\mathfrak{X}}\big(G/\mathrm{O}_{p'}(G)\big)$.*
- *Finally put $\Phi^*_{\mathfrak{X}}(G) = \mathrm{O}_{\mathfrak{X}}(G) \cap \bigcap_{p \in \mathrm{char}\,\mathfrak{X}} \Phi^p_{\mathfrak{X}}(G)$.*

If $q$ is a prime such that $q \notin \mathrm{char}\,\mathfrak{X}$, then $\Phi^*_{\mathfrak{X}}(G)$ is a $q'$-group because $\pi(\mathfrak{X}) = \mathrm{char}\,\mathfrak{X}$. Hence $\Phi^*_{\mathfrak{X}}(G) \leq \mathrm{O}_{q'}(G) \leq \Phi^q_{\mathfrak{X}}(G)$. Consequently $\Phi^*_{\mathfrak{X}}(G) = \mathrm{O}_{\mathfrak{X}}(G) \cap \bigcap_{p \in \mathbb{P}} \Phi^p_{\mathfrak{X}}(G)$.

The basic properties of $\Phi^*_{\mathfrak{X}}(G)$ are displayed in the next result.

**Proposition 3.2.2.** *Let $G$ be a group.*
1. *$\Phi^*_{\mathfrak{X}}(G)$ and $\Phi^p_{\mathfrak{X}}(G)$, $p$ a prime, are characteristic subgroups of $G$.*
2. *$\Phi\big(\mathrm{O}_{\mathfrak{X}}(G)\big) \leq \Phi^*_{\mathfrak{X}}(G) \leq \mathrm{O}_{\mathfrak{X}}(G) \cap \Phi(G)$.*
3. *Let $p$ be a prime. If $\mathrm{O}_{p'}(G) = 1$, then $\Phi^*_{\mathfrak{X}}(G) = \Phi^p_{\mathfrak{X}}(G)$.*
4. *Let $p$ be a prime. If $N$ is a normal subgroup of $G$ contained in $\Phi^p_{\mathfrak{X}}(G)$, then $\mathrm{O}_{p'}(G/N) = \mathrm{O}_{p'}(G)N/N$.*
5. *If $N$ is a normal subgroup of $G$ contained in $\Phi^*_{\mathfrak{X}}(G)$, then $\Phi^*_{\mathfrak{X}}(G/N) = \Phi^*_{\mathfrak{X}}(G)/N$.*

*Proof.* 1. It is clear.

2. Let $p$ be a prime. Then $\Phi^*_{\mathfrak{X}}(G)\,\mathrm{O}_{p'}(G)/\,\mathrm{O}_{p'}(G)$ is isomorphic to a subgroup of $\Phi\big(G/\,\mathrm{O}_{p'}(G)\big)$, which is a $p$-group. Hence $\Phi^*_{\mathfrak{X}}(G) \cap \mathrm{O}_{p'}(G)$ is a normal Hall $p'$-subgroup of $\Phi^*_{\mathfrak{X}}(G)$ and so $\Phi^*_{\mathfrak{X}}(G)$ is $p$-nilpotent. Therefore $\Phi^*_{\mathfrak{X}}(G)$ is nilpotent.

Assume, arguing by contradiction, that $\Phi^*_{\mathfrak{X}}(G)$ is not contained in $\Phi(G)$. Then there exists a maximal subgroup $M$ of $G$ such that $G = M\Phi^*_{\mathfrak{X}}(G)$. Since $\Phi^*_{\mathfrak{X}}(G)$ is nilpotent, we can find a prime $p$ and a Sylow $p$-subgroup $P$ of $\Phi^*_{\mathfrak{X}}(G)$ such that $G = MP$. In particular, $\mathrm{O}_{p'}(G)$ is contained in $M$. Hence $\Phi^p_{\mathfrak{X}}(G)/\,\mathrm{O}_{p'}(G)$ is a subgroup of $M/\,\mathrm{O}_{p'}(G)$ and so $\Phi^*_{\mathfrak{X}}(G) \leq M$. This contradiction leads to $\Phi^*_{\mathfrak{X}}(G) \leq \Phi(G)$. Now

$$\begin{aligned}\Phi\big(\mathrm{O}_{\mathfrak{X}}(G)\big)\,\mathrm{O}_{p'}(G)/\,\mathrm{O}_{p'}(G) &\leq \Phi\big(\mathrm{O}_{\mathfrak{X}}(G)\,\mathrm{O}_{p'}(G)\big)\,\mathrm{O}_{p'}(G)/\,\mathrm{O}_{p'}(G)\\ &\leq \Phi\big(\mathrm{O}_{\mathfrak{X}}(G)\,\mathrm{O}_{p'}(G)/\,\mathrm{O}_{p'}(G)\big) \leq \Phi\left(\mathrm{O}_{\mathfrak{X}}\big(G/\,\mathrm{O}_{p'}(G)\big)\right) \leq \Phi^p_{\mathfrak{X}}(G)/\,\mathrm{O}_{p'}(G)\end{aligned}$$

for each prime $p$. Consequently $\Phi\big(\mathrm{O}_{\mathfrak{X}}(G)\big) \leq \Phi^*_{\mathfrak{X}}(G)$.

3. Suppose that $\mathrm{O}_{p'}(G) = 1$ for some prime $p$. Since $\Phi^p_{\mathfrak{X}}(G)$ is contained in $\Phi(G)$, it follows that $\Phi^p_{\mathfrak{X}}(G)$ is a $p$-group. Hence if $q$ is a prime, $q \neq p$, we have that $\Phi^p_{\mathfrak{X}}(G) \leq \mathrm{O}_{q'}(G) \leq \Phi^q_{\mathfrak{X}}(G)$. Therefore $\Phi^p_{\mathfrak{X}}(G) \leq \Phi^*_{\mathfrak{X}}(G)$ and so $\Phi^*_{\mathfrak{X}}(G) = \Phi^p_{\mathfrak{X}}(G)$.

4. Let $N$ be a normal subgroup of $G$ such that $N \leq \Phi^p_{\mathfrak{X}}(G)$ for some prime $p$. Put $Q/N = \mathrm{O}_{p'}(G/N)$ and $M = N \cap \mathrm{O}_{p'}(G)$. Then $N\,\mathrm{O}_{p'}(G)/\,\mathrm{O}_{p'}(G) \leq \Phi^p_{\mathfrak{X}}(G)/\,\mathrm{O}_{p'}(G) \leq \Phi\big(G/\,\mathrm{O}_{p'}(G)\big)$, which is a $p$-group. Therefore $N/M$ is a $p$-group. Since $(Q/M)\big/(N/M)$ is a $p'$-group, it follows that $Q/M = (N/M)(H/M)$ for some Hall $p'$-subgroup $H/M$ of $Q/M$. It is clear that $H$ is a Hall $p'$-subgroup of $Q \trianglelefteq G$. Moreover the Hall $p'$-subgroups of $Q$ are conjugate. Therefore $G = \mathrm{N}_G(H)N$ by the Frattini argument. Since $N\,\mathrm{O}_{p'}(G)/\,\mathrm{O}_{p'}(G)$ is contained in $\Phi\big(G/\,\mathrm{O}_{p'}(G)\big)$, it follows that $G = \mathrm{N}_G(H)$ and $H \leq \mathrm{O}_{p'}(G)$. Consequently $Q/N = \mathrm{O}_{p'}(G)N/N$.

5. Let $N$ be a normal subgroup of $G$ contained in $\Phi^*_{\mathfrak{X}}(G)$. Let $p$ be a prime. Suppose that $\mathrm{O}_{p'}(G/N) = 1$. Then $\mathrm{O}_{p'}(G)$ is contained in $N$ by Statement 4. Moreover $\Phi^p_{\mathfrak{X}}(G/N)$ is $\Phi(G/N) = \Phi(G)/N$ or $\Phi\big(\mathrm{O}_{\mathfrak{X}}(G/N)\big) = \Phi\big(\mathrm{O}_{\mathfrak{X}}(G)/N\big)$. Suppose that $\Phi^p_{\mathfrak{X}}(G/N) \neq \Phi\big(\mathrm{O}_{\mathfrak{X}}(G)/N\big)$. Then $\mathrm{Soc}\,\big((G/N)/\big(\Phi(G)/N\big)\big)$ and $\Phi(G)/N$ belongs to $\mathrm{E}\,\mathfrak{X}$ and for $\mathrm{Soc}\,\big(\big(G/\,\mathrm{O}_{p'}(G)\big)/\big(\Phi(G)/\,\mathrm{O}_{p'}(G)\big)\big)$ and $\Phi\big(G/\,\mathrm{O}_{p'}(G)\big)$ the same is true. Hence we have that $\Phi^p_{\mathfrak{X}}\big(G/\,\mathrm{O}_{p'}(G)\big) = \Phi(G)/\,\mathrm{O}_{p'}(G)$ and $\Phi^p_{\mathfrak{X}}(G/N) = \Phi^p_{\mathfrak{X}}(G)/N$.

Assume now that $\Phi^p_{\mathfrak{X}}(G/N) = \Phi\big(\mathrm{O}_{\mathfrak{X}}(G)/N\big)$. Then $\Phi^p_{\mathfrak{X}}(G)/\mathrm{O}_{p'}(G) = \Phi\big(\mathrm{O}_{\mathfrak{X}}(G)/\mathrm{O}_{p'}(G)\big)$. By [DH92, A, 9.3 (c)], it follows that $\Phi^p_{\mathfrak{X}}(G)$ is nilpotent. Hence $\Phi^p_{\mathfrak{X}}(G)$ is contained in $\Phi\big(\mathrm{O}_{\mathfrak{X}}(G)\big)\mathrm{O}_{p'}(G)$ by [DH92, A, 9.11]. Therefore $\Phi^p_{\mathfrak{X}}(G)/N$ is contained in $\Phi\big(\mathrm{O}_{\mathfrak{X}}(G)\big)N/N \le \Phi\big(\mathrm{O}_{\mathfrak{X}}(G/N)\big)$. Since $N/\mathrm{O}_{p'}(G)$ is contained in $\Phi\big(\mathrm{O}_{\mathfrak{X}}(G)/\mathrm{O}_{p'}(G)\big)$, it follows that $\Phi^p_{\mathfrak{X}}(G/N)$ is isomorphic to $\Phi^p_{\mathfrak{X}}(G)/N$. It leads to $\Phi^p_{\mathfrak{X}}(G)/N = \Phi^p_{\mathfrak{X}}(G/N)$.

Assume now that $\mathrm{O}_{p'}(G/N) = \mathrm{O}_{p'}(G)N/N \neq 1$. Denote with bars the images in $\bar{G} = G/\mathrm{O}_{p'}(G)$. Since $\mathrm{O}_{p'}(\bar{G}/\bar{N}) = 1$ and $\bar{N} \le \Phi^p_{\mathfrak{X}}(\bar{G})$, it follows that $\Phi^p_{\mathfrak{X}}(\bar{G}/\bar{N}) = \Phi^p_{\mathfrak{X}}(\bar{G})/\bar{N}$. By definition of $\Phi^p_{\mathfrak{X}}(G)$, we have that $\Phi^p_{\mathfrak{X}}(\bar{G}) = \overline{\Phi^p_{\mathfrak{X}}(G)}$. Therefore the image of $\Phi^p_{\mathfrak{X}}(\bar{G}/\bar{N})$ under the natural isomorphism between $\bar{G}/\bar{N}$ and $G/N\,\mathrm{O}_{p'}(G)$ is $\Phi^p_{\mathfrak{X}}(G)/N\,\mathrm{O}_{p'}(G)$. This implies that $\Phi^p_{\mathfrak{X}}\big(G/N\,\mathrm{O}_{p'}(G)\big) = \Phi^p_{\mathfrak{X}}(G)/N\,\mathrm{O}_{p'}(G)$. On the other hand, by definition we have $\Phi^p_{\mathfrak{X}}(G/N)/\mathrm{O}_{p'}(G/N) = \Phi^p_{\mathfrak{X}}(G/N)/N\,\mathrm{O}_{p'}(G)/N = \Phi^p_{\mathfrak{X}}\big((G/N)/\big(N\,\mathrm{O}_{p'}(G)/N\big)\big)$. Now the image of $\Phi^p_{\mathfrak{X}}\big((G/N)/\big(N\,\mathrm{O}_{p'}(G)/N\big)\big)$ under the natural isomorphism between the groups $(G/N)/\big(N\,\mathrm{O}_{p'}(G)/N\big)$ and $G/N\,\mathrm{O}_{p'}(G)$ is the subgroup $\Phi^p_{\mathfrak{X}}\big(G/N\,\mathrm{O}_{p'}(G)\big)$. Therefore we have that $\Phi^p_{\mathfrak{X}}(G)/N = \Phi^p_{\mathfrak{X}}(G/N)$.

Consequently $\Phi^p_{\mathfrak{X}}(G)/N = \Phi^p_{\mathfrak{X}}(G/N)$ for all primes $p$ and so $\Phi^*_{\mathfrak{X}}(G)/N = \Phi^*_{\mathfrak{X}}(G/N)$. □

*Remark 3.2.3.* If $N$ is a normal subgroup of a group $G$, then $\Phi(G)N/N \le \Phi(G/N)$ and $\Phi(N) \le \Phi(G)$ ([DH92, A, 9.2]). This is not true for $\Phi^*_{\mathfrak{X}}(G)$ in general, as the next examples show.

*Examples 3.2.4.* 1. Let $H = \mathrm{SL}(2,5)$. Then $H$ has an irreducible module $V$ over $\mathrm{GF}(2)$ such that $\mathrm{Ker}(H \,\mathrm{on}\, V) = \mathrm{Z}(H)$ (cf. [DH92, B, 10.9]). Let $G = [V]H$ be the corresponding semidirect product. Put $\mathfrak{X} = (C_2)$. Then $\Phi^*_{\mathfrak{X}}(G) = \Phi(G) = \Phi(H)$ and $\Phi^*_{\mathfrak{X}}(G/V) = 1$.

2. If $G_1 = G \times \mathrm{Alt}(5)$, where $G$ and $\mathfrak{X}$ are as in 1, it follows that $\Phi(H) = \Phi^*_{\mathfrak{X}}(G) \not\le \Phi^*_{\mathfrak{X}}(G_1) = 1$.

If $\mathfrak{X} = \mathfrak{J}$, then $\Phi^*_{\mathfrak{X}}(G) = \Phi(G)$ for every group $G$ by Proposition 3.2.2 (2). However, if $\emptyset \neq \mathfrak{X} \neq \mathfrak{J}$, then we can find a group $G$ such that $\Phi\big(\mathrm{O}_{\mathfrak{X}}(G)\big)$ is a proper subgroup of $\Phi^*_{\mathfrak{X}}(G)$ as the next example shows.

*Example 3.2.5 ([BBCER05]).* Assume that $\emptyset \neq \mathfrak{X} \neq \mathfrak{J}$. Then there exist a non-abelian simple group $S \in \mathfrak{X}'$ and a prime $p \in \pi(S)$ such that $p \in \mathrm{char}\,\mathfrak{X}$. It is certainly true that $\mathrm{char}\,\mathfrak{X}$ is the set of all prime numbers. Suppose that $\mathrm{char}\,\mathfrak{X} \neq \mathbb{P}$ and take $p \in \mathrm{char}\,\mathfrak{X}$ and $q \notin \mathrm{char}\,\mathfrak{X}$. If $S$ is the alternating group of degree $p+q$, then $S \in \mathfrak{X}'$ and $p \in \mathrm{char}\,\mathfrak{X} \cap \pi(S)$. Let $T$ be the group algebra $\mathrm{GF}(p)S$ and consider $G = [T]S$, the corresponding semidirect product. It is rather clear that $\Phi(G) = \mathrm{Rad}\,T$. Since $\mathrm{O}_{p'}(G) = 1$ and $\Phi(G)$ and $\mathrm{Soc}\big(G/\Phi(G)\big)$ belong to $\mathrm{E}\,\mathfrak{X}$, we have that $\Phi^*_{\mathfrak{X}}(G) = \Phi^p_{\mathfrak{X}}(G) = \Phi(G)$ by Proposition 3.2.2 (3). It is certainly true that $\Phi(G) \neq 1$ because $\mathrm{Rad}\,T \neq 1$. However, $\mathrm{O}_{\mathfrak{X}}(G) = T$ and $\Phi(T) = 1$.

This example shows, in particular, that $\Phi^*_{\mathfrak{X}}(G)$ is not always the Frattini subgroup of the soluble radical when $\mathfrak{X}$ is the class of all abelian simple groups.

In [BBCER05] another Frattini-like subgroup associated with a class of simple groups is introduced and analysed. It is smaller than Förster's one and coincides with the Frattini subgroup of the $\mathrm{E}\,\mathfrak{X}$-radical except in a very few number of cases. We present here a slight variation of this subgroup as it appears in [BBCER05].

**Definition 3.2.6.** *Let $p$ be a prime. A group $G$ belongs to $\mathrm{A}_{\mathfrak{X}_p}(\mathfrak{P}_2)$ provided that $G$ is monolithic and there exists an elementary abelian normal $p$-subgroup $N$ of $G$ such that*

1. *$N \le \Phi(G)$ and $G/N$ is a primitive group of type 2,*
2. *$\mathrm{Soc}(G/N) \in \mathrm{E}\,\mathfrak{X} \setminus \mathfrak{E}_{p'}$, and*
3. *$\mathrm{C}^h_G(N) \le N$, where*

$$\mathrm{C}^h_G(N) := \bigcap \{\mathrm{C}_G(H/K) : H/K \text{ is a chief factor of } G \text{ below } N\}.$$

The next result shows that $\mathrm{A}_{\mathfrak{X}_p}(\mathfrak{P}_2) \neq \emptyset$ if $\mathfrak{X}$ contains non-abelian simple groups.

**Proposition 3.2.7.** *Let $G$ be a primitive group of type 2 such that $\mathrm{Soc}(G) \in \mathrm{E}\,\mathfrak{X}$. Then, for each prime $p \in \pi\bigl(\mathrm{Soc}(G)\bigr)$, there exists a group $E_p \in \mathrm{A}_{\mathfrak{X}_p}(\mathfrak{P}_2)$ and a minimal normal $p$-subgroup $T_p$ of $E_p$ contained in $\Phi(E_p)$ such that $E_p/\mathrm{C}_{E_p}(T_p)$ is isomorphic to $G$.*

*Proof.* Note that $p \in \mathrm{char}\,\mathfrak{X}$ because $\pi(\mathfrak{X}) = \mathrm{char}\,\mathfrak{X}$. Let $E_p$ be the maximal Frattini extension of $G$ with $p$-elementary abelian kernel $\mathrm{A}_p(G)$.Then $E_p/\mathrm{A}_p(G) \cong G$ and $\mathrm{A}_p(G) = \Phi(E_p)$ (see [GS78]). Moreover, by [GS78, Theorem 1], we have that $\mathrm{Ker}\bigl(G \,\mathrm{on}\, \mathrm{Soc}\bigl(\mathrm{A}_p(G)\bigr)\bigr) = \mathrm{O}_{p',p}(G) = 1$. Hence there exists a minimal normal subgroup $T_p$ of $E_p$ such that $T_p \le \mathrm{A}_p(G)$ and $\mathrm{C}_{E_p}(T_p) = \mathrm{A}_p(G)$. If $E_p$ is monolithic, then clearly $E_p \in \mathrm{A}_{\mathfrak{X}_p}(\mathfrak{P}_2)$ and the proposition is proved. Suppose that $E_p$ is not monolithic. By Lemma 3.1.3, there exists a normal subgroup $N$ of $E_p$ such that $N \cap T_p = 1$, $E_p/N$ is monolithic, and $\mathrm{Soc}(E_p/N) = T_pN/N$. Now $N \le \mathrm{C}_{E_p}(T_p) = \mathrm{A}_p(G) = \Phi(E_p)$ and $\mathrm{C}_{E_p/N}(T_pN/N) = \mathrm{C}_{E_p}(T_p)/N = \Phi(E_p)/N = \Phi(E_p/N)$. Therefore $E_p/N \in \mathrm{A}_{\mathfrak{X}_p}(\mathfrak{P}_2)$ and $T_pN/N$ is a minimal normal subgroup of $E_p/N$ such that $(E_p/N)\big/\mathrm{C}_{E_p/N}(T_p/N) \cong E_p/\mathrm{C}_{E_p}(T) \cong G$. □

**Definition 3.2.8.** *The* $\mathfrak{X}$-Frattini subgroup *of a group $G$ is the subgroup $\Phi_{\mathfrak{X}}(G)$ defined as follows:*

$$\Phi_{\mathfrak{X}}(G) := \begin{cases} \Phi\bigl(\mathrm{O}_{\mathfrak{X}}(G)\bigr) & \text{if } G \notin \mathrm{A}_{\mathfrak{X}_p}(\mathfrak{P}_2) \text{ for all } p \in \mathrm{char}\,\mathfrak{X}, \\ \Phi(G) & \text{otherwise.} \end{cases}$$

It is clear that $\Phi_{\mathfrak{X}}(G)$ is a characteristic subgroup of $G$. Moreover if $\mathfrak{X} = \mathfrak{J}$, then obviously $\Phi_{\mathfrak{X}}(G) = \Phi(G)$ and if $\mathfrak{X} = \mathbb{P}$, then $\mathrm{A}_{\mathfrak{X}_p}(\mathfrak{P}_2) = \emptyset$ for all $p \in \operatorname{char}\mathfrak{X}$. Hence $\Phi_{\mathfrak{X}}(G) = \Phi(G_{\mathfrak{S}})$ for every group $G$. Moreover,

**Proposition 3.2.9.** *Let $G$ be a group. Then $\Phi_{\mathfrak{X}}(G)$ is contained in $\Phi^*_{\mathfrak{X}}(G)$.*

*Proof.* We know, by Proposition 3.2.2 (2), that $\Phi\big(\mathrm{O}_{\mathfrak{X}}(G)\big)$ is contained in $\Phi^*_{\mathfrak{X}}(G)$. Suppose now that $G \in \mathrm{A}_{\mathfrak{X}_p}(\mathfrak{P}_2)$ for some prime $p \in \operatorname{char}\mathfrak{X}$. Then $\mathrm{O}_{p'}(G) = 1$ and $\Phi(G)$ is a $p$-group. Since $\Phi(G)$ and $\mathrm{Soc}\big(G/\Phi(G)\big)$ belong to $\mathrm{E}\,\mathfrak{X}$, $\Phi^p_{\mathfrak{X}}(G) = \Phi(G)$. In addition, $\Phi^p_{\mathfrak{X}}(G) = \Phi^*_{\mathfrak{X}}(G)$ by Proposition 3.2.2 (3). Therefore $\Phi(G) = \Phi_{\mathfrak{X}}(G) = \Phi^*_{\mathfrak{X}}(G)$. □

*Remarks 3.2.10.* 1. Example 3.2.5 shows that the equality $\Phi_{\mathfrak{X}}(G) = \Phi^*_{\mathfrak{X}}(G)$ does not hold in general.

2. If $\mathfrak{X}_1 \subseteq \mathfrak{X}_2$, then $\Phi_{\mathfrak{X}_1}(G) \leq \Phi_{\mathfrak{X}_2}(G)$ for all groups $G$.

By definition, if $G \notin \mathrm{A}_{\mathfrak{X}_p}(\mathfrak{P}_2)$ for $p \in \operatorname{char}\mathfrak{X}$, then $\Phi_{\mathfrak{X}}(G) = \Phi\big(\mathrm{O}_{\mathfrak{X}}(G)\big)$. We do not know whether in groups belonging to $\mathrm{A}_{\mathfrak{X}_p}(\mathfrak{P}_2)$ for some $p \in \operatorname{char}\mathfrak{X}$ the above equality holds. This raises the following question:

**Open question 3.2.11.** *Let $\mathfrak{X}$ be a class of simple groups such that* $\operatorname{char}\mathfrak{X} = \pi(\mathfrak{X})$ *and let $p \in \operatorname{char}\mathfrak{X}$. If $G \in \mathrm{A}_{\mathfrak{X}_p}(\mathfrak{P}_2)$, is it true that $\Phi(G) = \Phi\big(\mathrm{O}_{\mathfrak{X}}(G)\big)$?*

Moreover, the compatibility of $\Phi_{\mathfrak{X}}(G)$ with quotients of $G$ is not visible and doubtful. In fact, we do not know whether $\Phi_{\mathfrak{X}}(G/N) = \Phi_{\mathfrak{X}}(G)/N$ for $N \trianglelefteq G$ such that $N \leq \Phi_{\mathfrak{X}}(G)$.

In the sequel, using the ideas contained in the paper [BBCER05], we shall prove that the $\mathfrak{X}$-local formations are exactly those formations which are closed under extensions by the Frattini-like subgroups studied above. It leads to extensions of the Gaschütz-Lubeseder-Schmid and Baer theorems.

We begin with the following definitions.

**Definitions 3.2.12.** *Let $\mathfrak{F}$ be a formation. We say that:*

1. *$\mathfrak{F}$ is* $\mathfrak{X}$-saturated (N) *if $\mathfrak{F}$ contains a group $G$ whenever it contains $G/\Phi\big(\mathrm{O}_{\mathfrak{X}}(G)\big)$.*
2. *$\mathfrak{F}$ is* $\mathfrak{X}$-saturated (F) *if $G \in \mathfrak{F}$ provided that $G/\Phi^*_{\mathfrak{X}}(G) \in \mathfrak{F}$.*
3. *$\mathfrak{F}$ is* $\mathfrak{X}$-saturated *if $G \in \mathfrak{F}$ provided that $G/\Phi_{\mathfrak{X}}(G) \in \mathfrak{F}$.*
4. *$G$ has property $\mathfrak{X}_*$ if $\mathfrak{F}$ contains every group $G \in \mathrm{A}_{\mathfrak{X}_p}(\mathfrak{P}_2)$, $p \in \operatorname{char}\mathfrak{X}$, whenever it contains $G/\Phi(G)$.*

*Remarks 3.2.13.* Let $\mathfrak{F}$ be a formation.

1. $\mathfrak{F}$ is $\mathfrak{X}$-saturated if and only if $\mathfrak{F}$ is $\mathfrak{X}$-saturated (N) and $\mathfrak{F}$ has property $\mathfrak{X}_*$.
2. If $\mathfrak{F}$ is $\mathfrak{X}$-saturated (F), then $\mathfrak{F}$ is $\mathfrak{X}$-saturated.
3. If $\mathfrak{X} = \mathfrak{J}$, then $\mathfrak{F}$ is $\mathfrak{X}$-saturated if and only if $\mathfrak{F}$ is saturated.
4. If $\mathfrak{X} \subseteq \mathbb{P}$, then $\mathfrak{F}$ is $\mathfrak{X}$-saturated if and only if $\mathfrak{F}$ is $\mathfrak{X}$-saturated (N).

The main result in this section is the following.

**Theorem 3.2.14.** *Let $\mathfrak{F}$ be a formation. The following statements are pairwise equivalent:*

1. *$\mathfrak{F}$ is $\mathfrak{X}$-local.*
2. *$\mathfrak{F}$ is $\mathfrak{X}$-saturated (F).*
3. *$\mathfrak{F}$ is $\mathfrak{X}$-saturated.*
4. *$\mathfrak{F}$ is $\mathfrak{X}$-saturated (N) and $\mathfrak{F}$ has property $\mathfrak{X}_*$.*

We begin with some preliminary results.

**Lemma 3.2.15.** *Let $p$ be a prime in $\operatorname{char}\mathfrak{X}$, let $G$ be a group, and let $N$ be a normal subgroup of $G$ such that $N \leq \mathrm{O}_{\mathfrak{X}}(G)$. Then $\mathrm{C}^{\mathfrak{X}_p}\big(G/\Phi(N)\big) = \mathrm{C}^{\mathfrak{X}_p}(G)/\Phi(N)$.*

*Proof.* Put $A/\Phi(N) = \mathrm{C}^{\mathfrak{X}_p}\big(G/\Phi(N)\big)$. It is clear that $A$ is a normal subgroup of $G$ such that $\Phi(N) \leq \mathrm{O}_{p',p}(G) \leq \mathrm{C}^{\mathfrak{X}_p}(G) \leq A$. We prove that $A \leq \mathrm{C}^{\mathfrak{X}_p}(G)$; we consider $A$ acting on $G$ and $N$ by conjugation, and define the following formation function:

$$f(q) = \begin{cases} (1) & \text{for } q = p, \\ \mathfrak{E} & \text{for } q \neq p. \end{cases}$$

Next we see that $A$ acts $f$-hypercentrally on $N$ (cf. [DH92, IV, 6.2]). Let $H/K$ be an $A$-composition factor of $G$ between $A \cap N$ and $N$. Since $[A, N] \leq A \cap N$, it is true that $\mathrm{C}_A(H/K) = A$. Let $H/K$ be a chief factor of $G$ between $\Phi(N)$ and $A \cap N$ such that $p$ divides $|H/K|$. Then $H/K$ is an $\mathfrak{X}_p$-chief factor of $G$ because $N \leq \mathrm{O}_{\mathfrak{X}}(G)$. Hence $\mathrm{C}_A(H/K) = A$ and so $A$ centralises every $A$-composition factor of $N$ between $K$ and $H$. It yields that $A$ acts $f$-hypercentrally on $N/\Phi(N)$. By [DH92, IV, 6.7], $A$ acts $f$-hypercentrally on $N$.

Let $H/K$ be an $\mathfrak{X}_p$-chief factor of $G$ below $\Phi(N)$. Since $H/K$ is a minimal normal subgroup of $G/K$ and $H/K \leq A/K$, we can apply [DH92, A, 4.13] to conclude that $H/K = L_1/K \times \cdots \times L_r/K$, where $L_i/K$ is a minimal normal subgroup of $A/K$ for all $1 \leq i \leq r$. Since $L_i/K$ is an $A$-composition factor of $N$ and $p$ divides $|L_i/K|$, it follows that $A \leq \mathrm{C}_G(L_i/K)$. Hence $\mathrm{C}_A(H/K) = A$. Consequently $A$ centralises all $\mathfrak{X}_p$-chief factors of $G$ below $\Phi(N)$ and so $A \leq \mathrm{C}^{\mathfrak{X}_p}(G)$. □

**Theorem 3.2.16.** *If $\mathfrak{F}$ is an $\mathfrak{X}$-local formation, then $\mathfrak{F}$ is $\mathfrak{X}$-saturated (F).*

*Proof.* Let $G$ be a group such that $G/\Phi^*_{\mathfrak{X}}(G) \in \mathfrak{F}$. We prove that $G \in \mathfrak{F}$ by induction on $|G|$. Let $p$ be a prime in $\operatorname{char}\mathfrak{X}$. Then $G/\Phi^p_{\mathfrak{X}}(G) \in \mathfrak{F}$ and $\Phi^p_{\mathfrak{X}}(G)/\mathrm{O}_{p'}(G) = \Phi^*_{\mathfrak{X}}\big(G/\mathrm{O}_{p'}(G)\big)$ by Proposition 3.2.2 (3). Consequently, if $\mathrm{O}_{p'}(G) \neq 1$, we have $G/\mathrm{O}_{p'}(G) \in \mathfrak{F}$. This implies that every $\mathfrak{X}_p$-chief factor $H/K$ of $G$ is $G$-isomorphic to an $\mathfrak{X}_p$-chief factor of $G/\mathrm{O}_{p'}(G)$. Hence $G/\mathrm{C}_G(H/K) \in F(p)$, where $F$ is the canonical $\mathfrak{X}$-local definition of $\mathfrak{F}$.

We may assume that $\mathrm{O}_{p'}(G) = 1$ for some prime $p \in \operatorname{char}\mathfrak{X}$. In this case $\Phi^p_{\mathfrak{X}}(G) = \Phi^*_{\mathfrak{X}}(G)$ is a $p$-group. Suppose that $\Phi^*_{\mathfrak{X}}(G) = \Phi\big(\mathrm{O}_{\mathfrak{X}}(G)\big)$. Then $p$ divides $\big|\mathrm{O}_{\mathfrak{X}}(G)/\Phi\big(\mathrm{O}_{\mathfrak{X}}(G)\big)\big|$ and so $G$ has an $\mathfrak{X}_p$-chief factor above

$\Phi\bigl(\mathrm{O}_{\mathfrak{X}}(G)\bigr)$. In particular, $F(p) \neq \emptyset$. Since $G/\Phi\bigl(\mathrm{O}_{\mathfrak{X}}(G)\bigr) \in \mathfrak{F}$, we have that $\Bigl(G/\Phi\bigl(\mathrm{O}_{\mathfrak{X}}(G)\bigr)\Bigr)\Big/\,\mathrm{C}^{\mathfrak{X}_p}\Bigl(G/\Phi\bigl(\mathrm{O}_{\mathfrak{X}}(G)\bigr)\Bigr) \in F(p)$. By Lemma 3.2.15, it follows $G/\,\mathrm{C}^{\mathfrak{X}_p}(G) \in F(p)$. We conclude then that $G$ satisfies Condition 1 in Definition 3.1.1.

Assume now that $\Phi^*_{\mathfrak{X}}(G) \neq \Phi\bigl(\mathrm{O}_{\mathfrak{X}}(G)\bigr)$, then $\Phi(G)$ and $\mathrm{Soc}\bigl(G/\Phi(G)\bigr) = \mathrm{F}'(G)/\Phi(G)$ belong to $\mathrm{E}\,\mathfrak{X}$ and $\Phi^*_{\mathfrak{X}}(G) = \Phi(G)$. Note that in this case $p$ divides the order of every $\mathfrak{X}$-chief factor of $G$ below $\mathrm{F}'(G)$. Let $T$ be the intersection of the centralisers in $G$ of the $\mathfrak{X}_p$-chief factors of $G$ between $\Phi(G)$ and $\mathrm{F}'(G)$. Then $G/T \in F(p)$ because $G/\Phi(G) \in \mathfrak{F}$. Moreover, $T/\Phi(G)$ centralises $\mathrm{F}'(G)/\Phi(G)$ because $\mathrm{F}'(G)/\Phi(G)$ is a direct product of $\mathfrak{X}_p$-chief factors of $G$. By [För85b, Satz 1.2], $T/\Phi(G)$ is a $p$-group. This yields $T$ is a $p$-group and so $G \in F(p)$.

Consequently, in both cases, $G$ satisfies Condition 1 in Definition 3.1.1.

Let $L$ be a normal subgroup of $G$ such that $G/L$ is monolithic and $\mathrm{Soc}(G/L)$ belongs to $\mathrm{E}(S)$ for some $S \in \mathfrak{X}'$. Then $\Phi^*_{\mathfrak{X}}(G) \leq \mathrm{O}_{\mathfrak{X}}(G) \leq L$ and so $G/L \in \mathfrak{F} = F(S)$. Hence $G$ satisfies Condition 2 in Definition 3.1.1 and therefore $G \in \mathfrak{F}$. This is to say that $\mathfrak{F}$ is $\mathfrak{X}$-saturated (F). ◻

**Lemma 3.2.17.** *Let $p$ be a prime and let $\mathfrak{F}$ be a $(C_p)$-saturated formation.*

1. *Let $X$ be a group, and let $M$, $N$ be $\mathrm{GF}(p)X$-modules with $N$ irreducible and $X$ acting faithfully on $M$. If $[M]X \in \mathfrak{F}$, then $[N]X \in \mathfrak{F}$.*
2. *Let $N$ be an elementary abelian normal $p$-subgroup of a group $G$. Assume that $[N](G/N) \in \mathfrak{F}$ and that $C_p \in \mathfrak{F}$. Then $G \in \mathfrak{F}$.*

*Proof.* 1 and 2 follow from the proofs of [DH92, IV, 4.1] and [DH92, IV, 4.15], respectively, taking into account that the Hartley group used there plays the role of the normal $p$-subgroup. ◻

**Lemma 3.2.18.** *Let $\mathfrak{F}$ be a $(C_p)$-saturated formation, $p$ a prime. If $X \in \mathrm{R}_0\bigl(G/\,\mathrm{C}_G(H/K) : G \in \mathfrak{F}$ and $H/K$ is an abelian $p$-chief factor of $G\bigr)$, then $[N]X \in \mathfrak{F}$ for every irreducible $\mathrm{GF}(p)X$-module.*

*Proof.* The group $X$ has a set $\{N_1, \dots, N_n\}$ of normal subgroups satisfying:

1. $X/N_i$ is isomorphic to $G_i/\,\mathrm{C}_{G_i}(H_i/K_i)$, where $G_i \in \mathfrak{F}$ and $H_i/K_i$ is an abelian $p$-chief factor of $G_i$,
2. $\bigcap_{i=1}^n N_i = 1$.

By Corollary 2.2.5, $[H_i/K_i](X/N_i) \in \mathfrak{F}$, $1 \leq i \leq n$. Note that $H_i/K_i$ can be regarded as $X$-modules over $\mathrm{GF}(p)$ and $\mathrm{Ker}(X \,\mathrm{on}\, H_i/K_i) = N_i$, $1 \leq i \leq n$. Moreover, the semidirect product $[H_i/K_i]X$ has normal subgroups $H_i/K_i$ and $N_i$ satisfying $[H_i/K_i]X\big/(H_i/K_i)$, $[H_i/K_i]X/N_i \in \mathfrak{F}$. Therefore $[H_i/K_i]X \in \mathrm{R}_0\,\mathfrak{F} = \mathfrak{F}$, $1 \leq i \leq n$. Put $M = H_1/K_1 \times \cdots \times H_n/K_n$. Then $M$ is an $X$-module and $\mathrm{Ker}(X \,\mathrm{on}\, M) = \bigcap_{i=1}^n N_i = 1$. Hence $X$ acts faithfully on $M$. Consider the set $\{M_1, \dots, M_n\}$ of normal subgroups of $[M]X$: $M_1 = H_2/K_2 \times \cdots \times H_n/K_n, \dots, M_n = H_1/K_1 \times \cdots \times H_{n-1}/K_{n-1}$ and $M_i =$

$H_1/K_1 \times \cdots \times H_{i-1}/K_{i-1} \times H_{i+1}/K_{i+1} \times \cdots \times H_n/K_n$, $2 \leq i \leq n-1$. Then $\bigcap_{j=1}^n M_j = 1$ and $[M]X/M_j \cong [H_j/K_j]X \in \mathfrak{F}$. Therefore $[M]X \in \mathrm{R}_0\,\mathfrak{F} = \mathfrak{F}$. By Lemma 3.2.17, $[N]X \in \mathfrak{F}$ for every irreducible $\mathrm{GF}(p)X$-module. □

**Theorem 3.2.19.** *If $\mathfrak{F}$ is an $\mathfrak{X}$-saturated formation, then $\mathfrak{F}$ is $\mathfrak{X}$-local.*

*Proof.* By Remark 3.2.10 (3.2), $\mathfrak{F}$ is a $(C_p)$-saturated formation for all $p \in \operatorname{char}\mathfrak{X}$.

Bearing in mind Theorem 3.1.17, the natural candidate $f$ for an $\mathfrak{X}$-local definition of $\mathfrak{F}$ is given by

$$\begin{aligned} f(p) &= \mathfrak{S}_p\,\mathrm{Q}\,\mathrm{R}_0\big(G/\,\mathrm{C}_G(H/K) : G \in \mathfrak{F} \text{ and} \\ &\qquad H/K \text{ is an abelian } p\text{-chief factor of } G\big) && \text{for } p \in \operatorname{char}\mathfrak{X}, \\ f(S) &= \mathfrak{F} && \text{for } S \in \mathfrak{X}'. \end{aligned}$$

It is clear that $f$ is an $\mathfrak{X}$-formation function.

Put $\mathfrak{H} = \mathrm{LF}_{\mathfrak{X}}(f)$. Suppose that $\mathfrak{F}$ is not contained in $\mathfrak{H}$ and let $G \in \mathfrak{F} \setminus \mathfrak{H}$ of minimal order. We shall show that this supposition leads to a contradiction. Since $\mathfrak{H}$ is a formation, it follows that $G$ has a unique minimal normal subgroup, $N$ say, and that $G/N \in \mathfrak{H}$. If $N$ has composition type $S \in \mathfrak{X}'$, then $G \in f(S) = \mathfrak{F}$. This is impossible. Therefore $N$ is an $\mathfrak{X}$-chief factor of $G$. If $N$ is non-abelian, then $G$ is a primitive group of type 2. Let $p$ be a prime divisor of $|N|$. Then $p \in \operatorname{char}\mathfrak{X}$ and, by Proposition 3.2.7, there exists $E \in \mathrm{A}_{\mathfrak{X}_p}(\mathfrak{P}_2)$ such that $E/\,\mathrm{C}_E(T) \cong G$ for some minimal normal subgroup $T$ of $G$. Moreover $T$ is a $p$-group. Since $\Phi(E) = \mathrm{C}_E(T) = \Phi_{\mathfrak{X}}(E)$ and $\mathfrak{F}$ is $\mathfrak{X}$-saturated, it follows that $E \in \mathfrak{F}$. This means that $G \in f(p)$. Then we conclude that $G \in \mathfrak{F}$ because $\mathrm{O}_p(G) = 1$. But $G \notin \mathfrak{F}$ by supposition, and so we must have that $N$ is a $p$-group for some prime $p \in \operatorname{char}\mathfrak{X}$. In this case, $G/\,\mathrm{C}_G(N) \in f(p)$ and so $G \in \mathfrak{H}$ by Remark 3.1.7 (2), and we reach a contradiction. Therefore $\mathfrak{F} \subseteq \mathfrak{H}$.

Suppose that $\mathfrak{H}$ is not contained in $\mathfrak{F}$, and let $G$ be a group of minimal order in $\mathfrak{H} \setminus \mathfrak{F}$. Then, as usual, $G$ has a unique minimal normal subgroup $N$ and $G/N \in \mathfrak{F}$. Moreover neither $N \in \mathrm{E}(\mathfrak{X}')$ nor $N$ is a non-abelian $\mathrm{E}\,\mathfrak{X}$-group because $G \notin \mathfrak{F}$. Consequently, $N$ is an abelian $p$-group for some prime $p \in \operatorname{char}\mathfrak{X}$. In particular, $f(p) \neq \emptyset$ and therefore $\mathfrak{H}$ contains the cyclic group of order $p$. By Corollary 2.2.5, $A = [N](G/N) \in \mathfrak{H}$. Assume that $N < \mathrm{C}_G(N)$. Then $M = (G/N) \cap \mathrm{C}_A(N)$ is a non-trivial normal subgroup of $A$. Since $|A/M| < |G|$, we have that $A/M \in \mathfrak{F}$ by minimality of $G$. Hence $A \cong A/(N \cap M) \in \mathrm{R}_0\,\mathfrak{F} = \mathfrak{F}$. We can apply Lemma 3.2.17 (2) and deduce that $G \in \mathfrak{F}$. This is a contradiction. Hence we must have $\mathrm{C}_G(N) = N$ and so $G/N \in f(p)$. Since $\mathrm{O}_p(G/N) = 1$ by [DH92, B, 3.12 (b)], it follows that $G/N \in \mathrm{Q}\,\mathrm{R}_0\big(B/\,\mathrm{C}_B(H/K) : B \in \mathfrak{F}$ and $H/K$ is an abelian $p$-chief factor of $B\big)$. This yields that $G/N \cong X/T$ for some normal subgroup $T$ of

$$X \in \mathrm{R}_0\big(B/\,\mathrm{C}_B(H/K) : B \in \mathfrak{F} \text{ and } H/K \text{ is an abelian } p\text{-chief factor of } B\big).$$

Now $N$ can be regarded as an irreducible $X$-module over $\mathrm{GF}(p)$ such that $T = \mathrm{Ker}(X \text{ on } N)$. By Lemma 3.2.18, we have $[N]X \in \mathfrak{F}$. Consequently $G \cong$

$[N](G/N) \cong [N](X/T)$ belongs to $\mathfrak{F}$. We have reached a contradiction. Hence $\mathfrak{H} \subseteq \mathfrak{F}$ and the equality holds. □

Return for the moment to Theorem 3.2.14. It can be deduced at once from Theorem 3.2.16, Theorem 3.2.19, and Remarks 3.2.13.

Note that the Gaschütz-Lubeseder-Schmid theorem is a special case of Theorem 3.2.14 when $\mathfrak{X} = \mathfrak{J}$, the class of all simple groups.

Another generalisation of Gaschütz's concept of local formation in the general finite universe is due to L. A. Shemetkov, who introduced in 1973 the notion of composition formation. The most general version of these kind of formations was presented in [She01]. Let us describe Shemetkov's approach. Let $\mathfrak{Y} \neq \emptyset$ be a class of simple groups. A function which associates with every group $A \in \mathfrak{Y}$ a formation $f(A)$ and with every group $B \in \mathfrak{Y}'$ a formation $\emptyset \neq f(\mathfrak{Y}')$ is called a $\mathrm{C}_{\mathfrak{Y}}$*-satellite*. If $f$ is a $\mathrm{C}_{\mathfrak{Y}}$-satellite, then the class $\mathrm{CF}_{\mathfrak{Y}}(f)$ of all groups $G$ satisfying:

1. if $H/K$ is a $\mathfrak{Y}$-chief factor of $G$ and $S$ is the composition factor of $H/K$, then $G/\mathrm{C}_G(H/K) \in f(S)$, and
2. $G/\mathrm{O}_{\mathfrak{Y}}(G) \in f(\mathfrak{Y}')$

is a formation.

We say that a formation $\mathfrak{F}$ is a $\mathfrak{Y}$*-composition formation* if $\mathfrak{F} = \mathrm{CF}_{\mathfrak{Y}}(f)$ for some $\mathrm{C}_{\mathfrak{Y}}$-satellite $f$.

*Remark 3.2.20.* Let $\emptyset \neq \mathfrak{Y}$ be a class of simple groups. Denote $\mathfrak{X} = \operatorname{char}\mathfrak{Y} = \{C_p : p \in \operatorname{char}\mathfrak{Y}\}$. Then the $\mathfrak{Y}$-composition formations are exactly the $\mathfrak{X}$-saturated ones.

*Proof.* Let $\mathfrak{F} = \mathrm{CF}_{\mathfrak{Y}}(f)$ be a $\mathfrak{Y}$-composition formation. Then it is clear that $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(f_0)$, where $f_0$ is the $\mathfrak{X}$-formation function defined by

$$f_0(S) = \begin{cases} f(p) & \text{if } S \cong C_p \in \mathfrak{X}, \\ \mathfrak{F} & \text{if } S \in \mathfrak{X}'. \end{cases}$$

By Theorem 3.2.14, $\mathfrak{F}$ is $\mathfrak{X}$-saturated.

Conversely, suppose that $\mathfrak{F}$ is an $\mathfrak{X}$-saturated formation. Then, by Theorem 3.2.14, $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(F)$, where $F$ is the canonical $\mathfrak{X}$-local definition of $\mathfrak{F}$. We define a $\mathrm{C}_{\mathfrak{Y}}$-satellite $f$ by the following formula:

$$f(S) = \begin{cases} F(p) & \text{if } S \cong C_p \in \mathfrak{X}, \\ \mathfrak{F} & \text{if } S \in \mathfrak{X}'. \end{cases}$$

Then $\mathfrak{F} = \mathrm{CF}_{\mathfrak{Y}}(f)$. □

Assume that $\mathfrak{X} \subseteq \mathbb{P}$, then $\mathfrak{F}$ is $\mathfrak{X}$-saturated if and only if $\mathfrak{F}$ is $(C_p)$-saturated for all $p \in \operatorname{char}\mathfrak{X}$ by Theorem 3.2.14, Corollary 3.1.13 and Corollary 3.1.21. Therefore we have:

**Corollary 3.2.21 ([She97, Theorem 3.2], [She01, Lemma 7]).** *Let $\mathfrak{F}$ be a formation, $\emptyset \neq \mathfrak{Y}$ a non-empty class of simple groups and $\pi = \operatorname{char} \mathfrak{Y}$. The following statements are pairwise equivalent:*

1. *$\mathfrak{F}$ is closed under extensions by the Frattini subgroup of a normal soluble $\pi$-subgroup.*
2. *$\mathfrak{F}$ contains each group $G$ provided that $\mathfrak{F}$ contains $G/\Phi\big(\mathrm{F}(G)_\pi\big)$, where $\mathrm{F}(G)_\pi$ is the Hall $\pi$-subgroup of the Fitting subgroup of $G$.*
3. *A group $G$ belongs to $\mathfrak{F}$ if and only if $G/\Phi\big(\mathrm{O}_p(G)\big)$ belongs to $\mathfrak{F}$ for all $p \in \pi$.*
4. *$\mathfrak{F}$ is a $\mathfrak{Y}$-composition formation.*

When $\mathfrak{Y} = \mathbb{P}$, the class of all abelian simple groups, we have:

**Corollary 3.2.22 ([För84a, Korollar 3.11]).** *Let $\mathfrak{F}$ be a formation. The following statements are pairwise equivalent:*

1. *$\mathfrak{F}$ is solubly saturated.*
2. *A group $G$ belongs to $\mathfrak{F}$ if and only if $G/\Phi\big(\mathrm{F}(G)\big) \in \mathfrak{F}$.*
3. *$\mathfrak{F}$ contains a group $G$ provided that $\mathfrak{F}$ contains $G/\Phi\big(\mathrm{O}_p(G)\big)$ for every prime $p$.*

*Final remark 3.2.23.* In the sequel we make use of the fact that the concepts of "$\mathfrak{X}$-saturated formation" and "$\mathfrak{X}$-local formation" are equivalent without appealing to Theorem 3.2.14.

## 3.3 Products of $\mathfrak{X}$-local formations

As a point of departure, consider the following observations: if $\mathfrak{F}$ and $\mathfrak{G}$ are saturated formations, then the formation product $\mathfrak{F} \circ \mathfrak{G}$ is again saturated ([DH92, IV, 3.13 and 4.8]). However, the formation product of two solubly saturated formations is not solubly saturated in general as the following example shows.

*Example 3.3.1 ([Sal85]).* Let $\mathfrak{F} = \mathrm{D}_0\big(1, \mathrm{Alt}(5)\big)$ and $\mathfrak{G} = \mathfrak{S}_2$. Then it is clear that $\mathfrak{F}$ and $\mathfrak{G}$ are solubly saturated. Assume that $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$ is solubly saturated. Then $\mathfrak{H} = \mathrm{LF}_{\mathbb{P}}(H)$, where $H$ is the canonical $\mathbb{P}$-local definition of $\mathfrak{H}$. Since $\mathfrak{G} \subseteq \mathfrak{H}$, it follows that $H(2) \neq \emptyset$. Consider $G = \mathrm{SL}(2,5)$. Then $G/\mathrm{Z}(G) \in \mathfrak{H}$ and $G/\mathrm{C}_G\big(\mathrm{Z}(G)\big) \in H(2)$. Applying Remark 3.1.7 (2), we have that $G \in \mathfrak{H}$. This is not true. Hence $\mathfrak{H}$ is not solubly saturated.

Taking the above example into account, the following question arises:

> *Which are the precise conditions on two $\mathfrak{X}$-local formations $\mathfrak{F}$ and $\mathfrak{G}$ to ensure that $\mathfrak{F} \circ \mathfrak{G}$ is an $\mathfrak{X}$-local formation?*

The problem of the existence of solubly saturated factorisations of solubly saturated formations was taken up by Salomon [Sal85]. A complete answer to the general question was obtained in [BBCER06].

In the first part of the section we are concerned with the above question. We stay close to the treatment presented in [BBCER06].

*In the following* $\mathfrak{F}$ *and* $\mathfrak{G}$ *are formations and* $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$

If $p \in \operatorname{char} \mathfrak{X}$, denote

$$G_{\mathfrak{X}}(p) = \mathfrak{S}_p \operatorname{Q} \operatorname{R}_0\bigl(G/\operatorname{C}_G(H/K) : G \in \mathfrak{G} \text{ and } H/K \text{ is an } \mathfrak{X}_p\text{-chief factor of } G\bigr).$$

By Theorem 3.1.11, the smallest $\mathfrak{X}$-local formation $\operatorname{form}_{\mathfrak{X}}(\mathfrak{G})$ containing $\mathfrak{G}$ is $\mathfrak{X}$-locally defined by the $\mathfrak{X}$-formation function $G$ given by $G(p) = G_{\mathfrak{X}}(p)$, $p \in \operatorname{char} \mathfrak{X}$, and $G(S) = \mathfrak{F}$ for every $S \in \mathfrak{X}'$.

The next theorem provides an $\mathfrak{X}$-local definition of $\operatorname{form}_{\mathfrak{X}}(\mathfrak{H})$.

**Theorem 3.3.2.** *Assume that* $\mathfrak{F}$ *is an* $\mathfrak{X}$*-local formation defined by an integrated* $\mathfrak{X}$*-formation function* $f$*. Then the smallest* $\mathfrak{X}$*-local formation* $\operatorname{form}_{\mathfrak{X}}(\mathfrak{H})$ *containing* $\mathfrak{H}$ *is* $\mathfrak{X}$*-locally defined by the* $\mathfrak{X}$*-formation function* $h$ *given by*

$$h(p) = \begin{cases} f(p) \circ \mathfrak{G} & \text{if } \mathfrak{S}_p \subseteq \mathfrak{F} \\ G_{\mathfrak{X}}(p) & \text{if } \mathfrak{S}_p \not\subseteq \mathfrak{F} \end{cases} \qquad p \in \operatorname{char} \mathfrak{X}$$

$$h(S) = \mathfrak{H} \qquad S \in \mathfrak{X}'$$

*Proof.* It is clear that $h$ is an $\mathfrak{X}$-formation function. We set $\bar{\mathfrak{H}} = \operatorname{LF}_{\mathfrak{X}}(h)$ and first prove that $\mathfrak{H} \subseteq \bar{\mathfrak{H}}$. Assume that $\mathfrak{H} \setminus \bar{\mathfrak{H}}$ contains a group $G$ of minimal order. Then $G$ has a unique minimal normal subgroup $N$ and $G/N \in \bar{\mathfrak{H}}$. Let $A = G^{\mathfrak{G}} \trianglelefteq G$. If $A = 1$, then $G \in \mathfrak{G} \subseteq \bar{\mathfrak{H}}$, contrary to supposition. Therefore $N$ is contained in $A$. If $N$ were an $\mathfrak{X}'$-chief factor of $G$, since $G/N \in \bar{\mathfrak{H}}$, $G$ would satisfy the first condition to belong to $\bar{\mathfrak{H}}$. Since $G \in \mathfrak{H}$, the second condition would also be satisfied, bearing in mind that $h(S) = \mathfrak{H}$ for every simple group $S \in \mathfrak{X}'$. This would imply that $G \in \bar{\mathfrak{H}}$. Hence $N \in \operatorname{E}\mathfrak{X}$. Applying [DH92, A, 4.13], $N = N_1 \times \cdots \times N_n$, where $N_i$ is a minimal normal subgroup of $A$, $1 \leq i \leq n$. Since $A \in \mathfrak{F}$, it follows that $f(p) \neq \emptyset$ for each prime $p$ dividing $|N|$. Moreover $A/\operatorname{C}_N(N_i) \in f(p)$, for all $i \in \{1, \dots, n\}$, and $p \mid |N|$. Consequently $\bigl(G/\operatorname{C}_G(N)\bigr)^{\mathfrak{G}} \cong A/\operatorname{C}_A(N) \in \operatorname{R}_0 f(p) = f(p)$ and so $G/\operatorname{C}_G(N) \in f(p) \circ \mathfrak{G} = h(p)$ for all $p \mid |N|$. Hence, applying Remark 3.1.7 (2), we have that $G \in \bar{\mathfrak{H}}$. This contradiction proves that $\mathfrak{H} \subseteq \bar{\mathfrak{H}}$. Since $\bar{\mathfrak{H}}$ is $\mathfrak{X}$-local, it follows that $\operatorname{form}_{\mathfrak{X}}(\mathfrak{H}) \subseteq \bar{\mathfrak{H}}$.

On the other hand, we know by Theorem 3.1.17 that $\operatorname{form}_{\mathfrak{X}}(\mathfrak{H}) = \operatorname{LF}_{\mathfrak{X}}(H)$, where $H$ is the $\mathfrak{X}$-formation function defined by

$$\begin{cases} H(p) = H_{\mathfrak{X}}(p) & \text{if } p \in \operatorname{char}(\mathfrak{X}) \\ H(E) = \mathfrak{H} & \text{if } E \in \mathfrak{X}' \end{cases}$$

Suppose that $\bar{\mathfrak{H}}$ is not contained in $\operatorname{form}_{\mathfrak{X}}(\mathfrak{H})$ and choose a group $Z \in \bar{\mathfrak{H}} \setminus \operatorname{form}_{\mathfrak{X}}(\mathfrak{H})$ of minimal order. Then $Z$ has a unique minimal normal subgroup $N$ and $Z/N \in \operatorname{form}_{\mathfrak{X}}(\mathfrak{H})$. Moreover it is clear that $N \in \mathrm{E}\,\mathfrak{X}$. Let $p$ be a prime dividing $|N|$. If $\mathfrak{S}_p \not\subseteq \mathfrak{F}$, then $h(p) = G_{\mathfrak{X}}(p)$. Since $Z \in \bar{\mathfrak{H}}$, we have that $Z/\mathrm{C}_Z(N) \in G_{\mathfrak{X}}(p) \subseteq H(p)$. Assume we are in the case $\mathfrak{S}_p \subseteq \mathfrak{F}$. Then $Z/\mathrm{C}_Z(N) \in h(p) = f(p) \circ \mathfrak{G}$ and $C_p \wr \big(Z/\mathrm{C}_Z(N)\big) \in \mathfrak{S}_p\big(f(p) \circ \mathfrak{G}\big) \subseteq \mathfrak{S}_p f(p) \circ \mathfrak{G}$. By Theorem 3.1.17, we know that $\mathfrak{S}_p f(p) \subseteq \mathfrak{F}$ and, hence, $C_p \wr \big(Z/\mathrm{C}_Z(N)\big) \in \mathfrak{F} \circ \mathfrak{G} \subseteq \operatorname{form}_{\mathfrak{X}}(\mathfrak{H})$. This implies that $Z/\mathrm{C}_Z(N) \in H_{\mathfrak{X}}(p) = H(p)$ by Theorem 3.1.17. Applying Remark 3.1.7 (2), we can conclude that $Z \in \operatorname{form}_{\mathfrak{X}}(\mathfrak{H})$. This contradiction shows that $\bar{\mathfrak{H}} \subseteq \operatorname{form}_{\mathfrak{X}}(\mathfrak{H})$ and, hence, $\bar{\mathfrak{H}} = \operatorname{form}_{\mathfrak{X}}(\mathfrak{H})$. □

The following definition was introduced in [Sal85] for Baer-local formations.

**Definition 3.3.3.** *We say that the boundary* $\mathrm{b}(\mathfrak{H})$ *is* $\mathfrak{X}\mathfrak{G}$-free *if every group* $G \in \mathrm{b}(\mathfrak{H})$ *such that* $\operatorname{Soc}(G)$ *is a* $p$*-group for some prime* $p \in \operatorname{char}\mathfrak{X}$ *satisfies that* $G/\mathrm{C}_G\big(\operatorname{Soc}(G)\big) \notin G_{\mathfrak{X}}(p)$.

*Remark 3.3.4.* Note that in Example 3.3.1, $\mathrm{b}(\mathfrak{H})$ is not $\mathbb{P}\mathfrak{G}$-free.

The next result provides a test for $\mathfrak{X}$-locality of $\mathfrak{H}$ in terms of its boundary.

**Theorem 3.3.5.** *Assume that* $\mathfrak{F}$ *is* $\mathfrak{X}$*-local. Then* $\mathfrak{H}$ *is an* $\mathfrak{X}$*-local formation if and only if* $\mathrm{b}(\mathfrak{H})$ *is* $\mathfrak{X}\mathfrak{G}$*-free.*

*Proof.* Suppose that $\mathfrak{H}$ is $\mathfrak{X}$-local. Then $\mathfrak{H} = \mathrm{LF}_{\mathfrak{X}}(H)$, where $H$ is the canonical $\mathfrak{X}$-local definition of $\mathfrak{H}$. Let $G$ be a group in $\mathrm{b}(\mathfrak{H})$ such that $\operatorname{Soc}(G)$ is a $p$-group for some $p \in \operatorname{char}\mathfrak{X}$. If $G/\mathrm{C}_G\big(\operatorname{Soc}(G)\big)$ were in $G_{\mathfrak{X}}(p)$, then we would have that $G/\mathrm{C}_G\big(\operatorname{Soc}(G)\big) \in H_{\mathfrak{X}}(p) = H(p)$, since $\mathfrak{G} \subseteq \mathfrak{H}$. By Remark 3.1.7 (2), it would imply that $G \in \mathfrak{H}$. This would be a contradiction. Therefore $G/\mathrm{C}_G\big(\operatorname{Soc}(G)\big) \notin G_{\mathfrak{X}}(p)$ and $\mathrm{b}(\mathfrak{H})$ is $\mathfrak{X}\mathfrak{G}$-free.

Conversely, suppose that $\mathrm{b}(\mathfrak{H})$ is $\mathfrak{X}\mathfrak{G}$-free. Consider an integrated $\mathfrak{X}$-local definition $f$ of $\mathfrak{F}$. By Theorem 3.3.2, $\operatorname{form}_{\mathfrak{X}}(\mathfrak{H}) = \mathrm{LF}_{\mathfrak{X}}(h)$, where

$$h(p) = \begin{cases} f(p) \circ \mathfrak{G} & \text{if } \mathfrak{S}_p \subseteq \mathfrak{F} \\ G_{\mathfrak{X}}(p) & \text{if } \mathfrak{S}_p \not\subseteq \mathfrak{F} \end{cases} \qquad p \in \operatorname{char}\mathfrak{X}$$

$$h(S) = \mathfrak{H} \qquad S \in \mathfrak{X}'$$

We shall prove that $\mathfrak{H} = \operatorname{form}_{\mathfrak{X}}(\mathfrak{H})$. Assume that this is not the case and choose a group $G$ of minimal order in $\operatorname{form}_{\mathfrak{X}}(\mathfrak{H}) \setminus \mathfrak{H}$. Then $G \in \mathrm{b}(\mathfrak{H})$ and so $G$ has a unique minimal normal subgroup, $N$ say, and $G/N \in \mathfrak{H}$. If $N$ were an $\mathfrak{X}'$-group, we would have that $G \in h(S)$ for some $S \in \mathfrak{X}'$. This would imply that $G \in \mathfrak{H}$, contrary to supposition. Hence $N$ is an $\mathfrak{X}$-chief factor of $G$. Let $p$ be a prime dividing $|N|$. Since $p \in \operatorname{char}\mathfrak{X}$, it follows that $G/\mathrm{C}_G(N) \in h(p)$. Since $h(p) \subseteq \mathfrak{S}_p\mathfrak{H}$ and $\mathrm{O}_p\big(G/\mathrm{C}_G(N)\big) = 1$, we have that $G/\mathrm{C}_G(N) \in \mathfrak{H}$. Therefore $\mathrm{C}_G(N) \neq 1$ and so $N$ is an abelian $p$-group.

Assume that $\mathfrak{S}_p$ is not contained in $\mathfrak{F}$. Then $h(p) = G_{\mathfrak{X}}(p)$. We conclude that $\mathrm{b}(\mathfrak{H})$ is not $\mathfrak{X}\mathfrak{G}$-free. This contradiction shows that $\mathfrak{S}_p$ is contained in $\mathfrak{F}$. Then $G/\mathrm{C}_G(N) \in f(p) \circ \mathfrak{G}$. It follows that $G^{\mathfrak{G}}/\mathrm{C}_{G^{\mathfrak{G}}}(N) \in f(p)$. Since $G^{\mathfrak{G}}/N \in \mathfrak{F}$, we can apply Remark 3.1.7 (2) to conclude that $G^{\mathfrak{G}} \in \mathfrak{F}$, that is, $G \in \mathfrak{H}$. This contradiction shows that $\mathrm{form}_{\mathfrak{X}}(\mathfrak{H})$ is contained in $\mathfrak{H}$ and, therefore, $\mathfrak{H}$ is $\mathfrak{X}$-local. □

*Example 3.3.6.* Let $S$ be a non-abelian simple group with trivial Schur multiplier. Consider $\mathfrak{F} = \mathrm{D}_0(1, S)$, the formation of all groups which are a direct product of copies of $S$ together with the trivial group. Let $\mathfrak{X}$ be a class of simple groups such that $S \notin \mathfrak{X}$. Notice that $\mathfrak{F}$ is $\mathfrak{X}$-local. Let $\mathfrak{G}$ be any formation. Suppose that $G \in \mathrm{b}(\mathfrak{H})$, $N = \mathrm{Soc}(G)$ is the minimal normal subgroup of $G$, and $N$ is a $p$-group for some $p \in \mathrm{char}\,\mathfrak{X}$. If $G/\mathrm{C}_G(N) \in G_{\mathfrak{X}}(p)$, then $N \leq \mathrm{Z}(G^{\mathfrak{G}})$ because $1 \neq G^{\mathfrak{G}} \leq \mathrm{C}_G(N)$. Since $G/N \in \mathfrak{H}$, it follows that $G^{\mathfrak{G}}/N \in \mathfrak{F}$. Assume that $G^{\mathfrak{G}}/N \neq 1$. This implies that $G^{\mathfrak{G}}/N$, a direct product of copies of $S$, has non-trivial Schur multiplier, contrary to [Suz82, Exercise 4 (c), page 265]. Thus $G^{\mathfrak{G}} = N$ and then $G \in \mathrm{form}_{\mathfrak{X}}(\mathfrak{H})$ by Remark 3.1.7 (2). Therefore if $\mathrm{form}_{\mathfrak{X}}(\mathfrak{G}) \subseteq \mathfrak{N}_{p'}\mathfrak{G}$ for all primes $p \in \mathrm{char}(\mathfrak{X})$, it follows that $G \in \mathfrak{G}$, and this contradicts our choice if $G$. Hence $\mathrm{b}(\mathfrak{H})$ is $\mathfrak{X}\mathfrak{G}$-free and $\mathfrak{H}$ is $\mathfrak{X}$-local by Theorem 3.3.5. Consequently, $\mathfrak{H}$ is $\mathfrak{X}$-local for all formations $\mathfrak{G}$ satisfying $\mathrm{form}_{\mathfrak{X}}(\mathfrak{G}) \subseteq \mathfrak{N}_{p'}\mathfrak{G}$ for all primes $p \in \mathrm{char}(\mathfrak{X})$.

As an application of Theorem 3.3.5 we have:

**Theorem 3.3.7.** *Assume that $\mathfrak{F}$ is $\mathfrak{X}$-local and $\mathfrak{G}$ is a formation satisfying one of the following conditions:*

1. *$\mathfrak{G}$ is $\mathfrak{X}$-local, or*
2. *$\mathfrak{S}_p\mathfrak{G} = \mathfrak{G}$ for all $p \in \mathrm{char}\,\mathfrak{X} \setminus \mathrm{char}\,\mathfrak{F}$.*

*Then $\mathfrak{H}$ is $\mathfrak{X}$-local if $\mathfrak{F}$ and $\mathfrak{G}$ satisfy the following condition:*

$$\text{If } p \in \mathrm{char}\,\mathfrak{X} \cap \pi(\mathfrak{F}) \text{ and } \mathfrak{S}_p \subseteq \mathfrak{G}, \text{ then } \mathfrak{S}_p \subseteq \mathfrak{F}. \tag{3.2}$$

*Proof.* Consider the canonical $\mathfrak{X}$-local definition $F$ of $\mathfrak{F}$. We will obtain a contradiction by assuming that $\mathfrak{H}$ is not $\mathfrak{X}$-local. Then, by Theorem 3.3.5, there exists a group $G \in \mathrm{b}(\mathfrak{H})$ such that $N = \mathrm{Soc}(G)$ is the unique minimal normal subgroup of $G$, $N$ is a $p$-group for some prime $p \in \mathrm{char}\,\mathfrak{X}$ and $G/\mathrm{C}_G(N) \in G_{\mathfrak{X}}(p)$. Since $G_{\mathfrak{X}}(p) \subseteq \mathfrak{S}_p\mathfrak{G}$ and $\mathrm{O}_p\big(G/\mathrm{C}_G(N)\big) = 1$, it follows that $G/\mathrm{C}_G(N) \in \mathfrak{G}$. Then $G^{\mathfrak{G}} \leq \mathrm{C}_G(N)$. Since $G^{\mathfrak{G}} \neq 1$, it follows that $N \leq G^{\mathfrak{G}}$. Hence $N \leq \mathrm{Z}(G^{\mathfrak{G}})$. Moreover $G^{\mathfrak{G}}/N \in \mathfrak{F}$ because $G/N \in \mathfrak{H}$. Suppose that $N$ is not contained in $\Phi(G^{\mathfrak{G}})$. Then there exists a maximal subgroup $M$ of $G^{\mathfrak{G}}$ such that $G^{\mathfrak{G}} = MN$. Notice that $M$ is normal in $G^{\mathfrak{G}}$. Then $\mathrm{O}^p(G^{\mathfrak{G}})$ is contained in $M$ and is a normal subgroup of $G$. If $\mathrm{O}^p(G^{\mathfrak{G}}) \neq 1$, it follows that $N \leq \mathrm{O}^p(G^{\mathfrak{G}}) \leq M$. This contradiction proves that $G^{\mathfrak{G}}$ is a $p$-group. Assume that $p \notin \mathrm{char}\,\mathfrak{F}$. In this case, since $G^{\mathfrak{G}}/N \in \mathfrak{F}$, it follows that $N = G^{\mathfrak{G}}$. This means that $G/N \in \mathfrak{G}$. If $\mathfrak{G}$ is $\mathfrak{X}$-local, we conclude that $G \in \mathfrak{G}$ by Remark 3.1.7 (2). If $\mathfrak{G}$ is not $\mathfrak{X}$-local, we have $G \in \mathfrak{S}_p\mathfrak{G} = \mathfrak{G}$ because $p \notin \mathrm{char}\,\mathfrak{F}$.

In both cases, we reach a contradiction. Hence we have that $p \in \operatorname{char}\mathfrak{F}$. In this case $F(p) \neq \emptyset$. In particular, $\mathfrak{S}_p \subseteq \mathfrak{F}$ as $\mathfrak{F}$ is $\mathfrak{X}$-local. Therefore $G^{\mathfrak{G}} \in \mathfrak{F}$. This contradiction proves that $N$ is contained in $\Phi(G^{\mathfrak{G}})$. This implies that $p$ divides $|G^{\mathfrak{G}}/N|$ and so $p \in \pi(\mathfrak{F})$. If $p \in \operatorname{char}\mathfrak{F}$, then $F(p) \neq \emptyset$ and $G^{\mathfrak{G}} \in \mathfrak{F}$ as $\mathfrak{F}$ is $\mathfrak{X}$-local and Remark 3.1.7 (2) can be applied. Suppose that $p \notin \operatorname{char}\mathfrak{F}$. If $\mathfrak{G}$ is $\mathfrak{X}$-local, we have that $\mathfrak{S}_p \subseteq \mathfrak{G}$ because $G_{\mathfrak{X}}(p) \neq \emptyset$. The same holds if $\mathfrak{S}_p\mathfrak{G} = \mathfrak{G}$. Hence if $p \notin \operatorname{char}\mathfrak{F}$, we have that $\mathfrak{S}_p$ is contained in $\mathfrak{G}$. By Condition (3.2), $\mathfrak{S}_p \subseteq \mathfrak{F}$. This contradiction completes the proof. □

Since local formations are $\mathfrak{X}$-local for every class of simple groups $\mathfrak{X}$ (see Corollary 3.1.13), we obtain as a special case of Theorem 3.3.7 the following results:

**Corollary 3.3.8.** *Suppose that either of the following conditions is fulfilled:*

1. *$\mathfrak{F}$ is local and $\mathfrak{G}$ is $\mathfrak{X}$-local.*
2. *$\mathfrak{F}$ is local and $\mathfrak{S}_p\mathfrak{G} = \mathfrak{G}$ for all $p \in \operatorname{char}\mathfrak{X} \setminus \operatorname{char}\mathfrak{F}$.*

*Then $\mathfrak{H}$ is an $\mathfrak{X}$-local formation.*

*Proof.* If $\mathfrak{F}$ is local, then condition (3.2) in Theorem 3.3.7 is satisfied, since $\mathfrak{S}_p \subseteq \mathfrak{F}$ for every $p \in \pi(\mathfrak{F})$. □

**Corollary 3.3.9 ([DH92, IV, 3.13 and 4.8]).** *$\mathfrak{H}$ is a local formation if either of the following conditions is satisfied:*

1. *$\mathfrak{F}$ and $\mathfrak{G}$ are both local.*
2. *$\mathfrak{F}$ is local and $\mathfrak{S}_p\mathfrak{G} = \mathfrak{G}$ for all $p \notin \operatorname{char}\mathfrak{F}$.*

Example 3.3.6 shows that there are cases in which a product of an $\mathfrak{X}$-local formation and a non $\mathfrak{X}$-local formation is $\mathfrak{X}$-local. This observation leads to the following question:

*Are there $\mathfrak{X}$-local products of non $\mathfrak{X}$-local formations?*

The local version of the above question is the one appearing in *The Kourovka Notebook* ([MK90]) as Question 9.58. It was posed by L. A. Shemetkov and A. N. Skiba and answered affirmatively in several papers (see [BBPR98, Ved88, Vor93]).

The next example gives a positive answer to the above question when $|\operatorname{char}\mathfrak{X}| \geq 2$.

*Example 3.3.10 ([BBPR98]).* Assume that $p$ and $q$ are different primes in $\operatorname{char}\mathfrak{X}$. Consider the formations $\mathfrak{F} = \mathfrak{S}_p\mathfrak{A}_q \cap \mathfrak{A}_q\mathfrak{S}_p$ and $\mathfrak{G} = \mathfrak{S}_q\mathfrak{A}_p$, where $\mathfrak{A}_r$ denotes the formation of all abelian $r$-groups for a prime $r$. $\mathfrak{F}$ is not $(C_q)$-local and $\mathfrak{G}$ is not $(C_p)$-local. Therefore, by Corollary 3.1.13, $\mathfrak{F}$ and $\mathfrak{G}$ are not $\mathfrak{X}$-local. However $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$ is local and so it is $\mathfrak{X}$-local.

Note that if the formation of all $p$-groups, $p$ a prime, were a product of two proper subformations, Question 9.58 in [MK90] would be solved automatically. Perhaps it was the reason to put forward the following question in *The Kourovka Notebook* [MK90]:

> Question 10.72 (Shemetkov). *To prove indecomposability of $\mathfrak{S}_p$, $p$ a prime, into a product of two non-trivial subformations.*

This question was solved positively by L. A. Shemetkov and A. N. Skiba in [SS89].

We present a general version of this conjecture as a corollary of a more general result at the end of the section.

On the other hand, bearing in mind Example 3.3.10, the following question naturally arises:

> Which are the precise conditions on two formations $\mathfrak{F}$ and $\mathfrak{G}$ to ensure that $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$ is $\mathfrak{X}$-local?

Our next results answer this question.

*Notation 3.3.11.* If $\mathfrak{Y}$ is a class of groups, denote $\mathfrak{Y}^{\mathfrak{G}} = (Y^{\mathfrak{G}} : Y \in \mathfrak{Y})$.

**Lemma 3.3.12.** *If $T$ is a group such that $T \notin \mathfrak{G}$ and $\mathfrak{S}_p(T) \subseteq \mathfrak{H}$ for some prime $p$, then $\mathfrak{S}_p(T^{\mathfrak{G}}) \subseteq \mathfrak{F}$.*

*Proof.* Let $Z$ be a group in $\mathfrak{S}_p(T^{\mathfrak{G}})$. Then $Z$ has a normal subgroup $P$ such that $P$ is a $p$-group and $Z/P$ is isomorphic to $T^{\mathfrak{G}} \neq 1$. Assume that $p^s$ is the exponent of the abelian $p$-group $P/P'$. Consider $Q = P \wr_{\text{nat}} H$, where $H = \langle (1, 2, \dots, p^s) \rangle$ is a cyclic group of order $p^s$ regarded as a subgroup of the symmetric group of degree $p^s$. Here the wreath product is taken with respect to the natural permutation representation of $H$ of degree $p^s$. Set $D = \{(a, \dots, a) : a \in P\}$ the diagonal subgroup of $P^\natural$, the base group of $Q$. Since $a^{p^s} \in P'$, we have that $D$ is contained in $[P^\natural, H]$ by [DH92, A, 18.4]. In particular $D$ is contained in $Q'$. Let $Y = Q \wr T$ be the regular wreath product of $Q$ with $T$. Since $Q \in \mathfrak{S}_p(T) \subseteq \mathfrak{H}$, it follows that $Q \in \mathfrak{H}$. Therefore $Y^{\mathfrak{G}} \in \mathfrak{F}$. Now, by Proposition 2.2.8, we know that $Y^{\mathfrak{G}} = (B \cap Y^{\mathfrak{G}})T^{\mathfrak{G}}$, where $B = Q^\natural$ is the base group of $Y$. Now, by [DH92, A, 18.8], $BT^{\mathfrak{G}}$ is isomorphic to $(Q^n) \wr T^{\mathfrak{G}}$, where $n = |T : T^{\mathfrak{G}}|$ and $C' \leq [C, T^{\mathfrak{G}}]$, for $C = (Q^n)^\natural$, by virtue of [DH92, A, 18.4]. This implies that $B' = [B, T^{\mathfrak{G}}] \leq [B, Y^{\mathfrak{G}}] \leq B \cap Y^{\mathfrak{G}}$. Hence $B'T^{\mathfrak{G}}$ is contained in $Y^{\mathfrak{G}}$. Applying Theorem 2.2.6, $B'T^{\mathfrak{G}} \in \mathfrak{F}$. Therefore $\big((Q')^n\big) \wr T^{\mathfrak{G}} \in \mathfrak{F}$. Since $P$ is isomorphic to a subgroup of $Q'$, it follows that $P^n \wr T^{\mathfrak{G}} \in \mathfrak{F}$ by Theorem 2.2.6. Since $P$ can be regarded as a subgroup of $P^n$, we have that $P \wr T^{\mathfrak{G}}$ is a subgroup of $P^n \wr T^{\mathfrak{G}}$ supplementing the Fitting subgroup of $P^n \wr T^{\mathfrak{G}}$. Applying again Theorem 2.2.6, we have that $P \wr T^{\mathfrak{G}} \in \mathfrak{F}$. By [DH92, A, 18.9], $Z$ is isomorphic to a subgroup of $P \wr T^{\mathfrak{G}}$ supplementing the Fitting subgroup of $P \wr T^{\mathfrak{G}}$. Therefore $Z \in \mathfrak{F}$ by virtue of Theorem 2.2.6. This completes the proof of the lemma. □

**Theorem 3.3.13.** *$\mathfrak{H}$ is an $\mathfrak{X}$-local formation if and only if the following two conditions hold:*

1. *If $p \in (\operatorname{char} \mathfrak{X}) \cap \big(\operatorname{char} \operatorname{form}_{\mathfrak{X}}(\mathfrak{H})\big)$ and $H_{\mathfrak{X}}(p)$ is not contained in $\mathfrak{G}$, then $\mathfrak{S}_p H_{\mathfrak{X}}(p)^{\mathfrak{G}} \subseteq \mathfrak{F}$.*
2. *If $p \in (\operatorname{char} \mathfrak{X}) \cap \big(\operatorname{char} \operatorname{form}_{\mathfrak{X}}(\mathfrak{H})\big)$, $G \in \mathrm{b}(\mathfrak{H})$, and $N = \operatorname{Soc}(G)$ is a $p$-group, then $[N](G/N) \notin \mathfrak{H}$.*

*Proof.* Assume that $\mathfrak{H}$ is an $\mathfrak{X}$-local formation, that is, $\mathfrak{H} = \operatorname{form}_{\mathfrak{X}}(\mathfrak{H})$. We know that $\mathfrak{H} = \mathrm{LF}_{\mathfrak{X}}(H)$, where $H$ is the $\mathfrak{X}$-formation function defined in Theorem 3.1.17. Consider a prime $p \in \operatorname{char}(\mathfrak{X})$ and assume there exists a group $T \in H_{\mathfrak{X}}(p) \setminus \mathfrak{G}$. We have that $\mathfrak{S}_p(T) \subseteq \mathfrak{S}_p H_{\mathfrak{X}}(p) = H_{\mathfrak{X}}(p) \subseteq \mathfrak{H}$. Hence, by Lemma 3.3.12, we have that $\mathfrak{S}_p(T^{\mathfrak{G}}) \subseteq \mathfrak{F}$. Now consider a group $G$ in $\mathfrak{S}_p H_{\mathfrak{X}}(p)^{\mathfrak{G}}$. Then $G$ has a normal $p$-subgroup $N$ such that $G/N \cong \bar{T}^{\mathfrak{G}}$, where $\bar{T} \in H_{\mathfrak{X}}(p)$. If $\bar{T}^{\mathfrak{G}} \neq 1$, we have just proved that $\mathfrak{S}_p(\bar{T}^{\mathfrak{G}}) \subseteq \mathfrak{F}$ and, therefore, $G \in \mathfrak{F}$. If $\bar{T}^{\mathfrak{G}} = 1$, then $G \in \mathfrak{S}_p$. Consider the group $A := G \times T^{\mathfrak{G}}$. We have that $A \in \mathfrak{S}_p(T^{\mathfrak{G}}) \subseteq \mathfrak{F}$ and, therefore, $G \in \mathrm{Q}(\mathfrak{F}) = \mathfrak{F}$. We conclude that $\mathfrak{S}_p H_{\mathfrak{X}}(p)^{\mathfrak{G}} \subseteq \mathfrak{F}$ and Statement 1 holds.

Let $G$ be a group in $\mathrm{b}(\mathfrak{H})$ such that $N = \operatorname{Soc}(G)$ is a $p$-group for a prime $p \in (\operatorname{char} \mathfrak{X}) \cap \big(\operatorname{char} \operatorname{form}_{\mathfrak{X}}(\mathfrak{H})\big)$. Note that $N$ is a minimal normal subgroup of $G$. If $H := [N](G/N) \in \mathfrak{H}$, we would have that $H/\operatorname{C}_H(N) \in H_{\mathfrak{X}}(p)$ and, therefore, $G/\operatorname{C}_G(N) \in H_{\mathfrak{X}}(p)$. Since $G/N \in \mathfrak{H}$, this would imply by Remark 3.1.7 (2) that $G \in \mathrm{LF}_{\mathfrak{X}}(H) = \mathfrak{H}$. This contradiction proves Condition 2.

To prove the sufficiency, assume that $\mathfrak{H}$ is the product of $\mathfrak{F}$ and $\mathfrak{G}$ and $\mathfrak{H}$ satisfies Conditions 1 and 2. We will obtain a contradiction by supposing that $\operatorname{form}_{\mathfrak{X}}(\mathfrak{H}) \setminus \mathfrak{H}$ contains a group $G$ of minimal order. Such a $G$ has a unique minimal normal subgroup, $N$, and $G/N \in \mathfrak{H}$. This is to say that $G \in \mathrm{b}(\mathfrak{H})$. If $N \in \mathrm{E}(\mathfrak{X}')$, then there exists $S \in \mathfrak{X}'$ such that $G \in H(S) = \mathfrak{H}$, contrary to supposition. Therefore $N \in \mathrm{E}\,\mathfrak{X}$. Let $p$ be a prime dividing $|N|$. Then $G/\operatorname{C}_G(N) \in H_{\mathfrak{X}}(p)$. In particular $p \in (\operatorname{char} \mathfrak{X}) \cap \big(\operatorname{char} \operatorname{form}_{\mathfrak{X}}(\mathfrak{H})\big)$. If $N$ were non-abelian, then $\operatorname{C}_G(N) = 1$ and $G \in H_{\mathfrak{X}}(p)$. This would imply that $G \in \mathfrak{H}$ because $\operatorname{O}_p(G) = 1$. It would contradict the choice of $G$. Therefore $N$ is an abelian $p$-group. Applying Corollary 2.2.5, $A = [N](G/N) \in \operatorname{form}_{\mathfrak{X}}(\mathfrak{H})$. Suppose that the intersection $B$ of $\operatorname{C}_A(N)$ with $G/N$ is non-trivial. Then $B \trianglelefteq A$ and $A/B \in \mathfrak{H}$ by the choice of $G$. Since $G/N \in \mathfrak{H}$, we have that $A \in \mathrm{R}_0\,\mathfrak{H} = \mathfrak{H}$. This contradicts Statement 2. Hence $B = 1$ and $N = \operatorname{C}_G(N)$. In particular $G \in H_{\mathfrak{X}}(p) \setminus \mathfrak{G}$. Applying Statement 1, we have that $\mathfrak{S}_p H_{\mathfrak{X}}(p)^{\mathfrak{G}} \subseteq \mathfrak{F}$. We deduce then that $G^{\mathfrak{G}} \in \mathfrak{F}$ and so $G \in \mathfrak{H}$. We have reached a final contradiction. Therefore $\operatorname{form}_{\mathfrak{X}}(\mathfrak{H}) \subseteq \mathfrak{H}$ and $\mathfrak{H}$ is $\mathfrak{X}$-local. □

*Remark 3.3.14.* If $\mathfrak{X} = \mathfrak{J}$, then Condition 1 implies Condition 2 in the above theorem.

*Proof.* Assume that $\mathfrak{H}$ satisfies Condition 1. Let $G \in \mathrm{b}(\mathfrak{H})$ such that $N = \operatorname{Soc}(G)$ is the unique minimal normal subgroup of $G$. Suppose that $N$ is a $p$-group for some $p \in (\operatorname{char} \mathfrak{X}) \cap \big(\operatorname{char} \operatorname{form}_{\mathfrak{X}}(\mathfrak{H})\big)$.

Suppose that $\Phi(G) = 1$. Then $G$ is a primitive group, $\mathrm{C}_G(N) = N$ and $G$ is isomorphic to $[N](G/N)$. Therefore, $[N](G/N) \notin \mathfrak{H}$ and the remark follows. Now assume that $\Phi(G) \neq 1$. Consider $T/N := \mathrm{O}_{p'}(G/N)$. Since $T/N$ is $p$-nilpotent and $N \leq \Phi(G)$, we have by [Hup67, VI, 6.3] that $T$ is $p$-nilpotent. This implies that $T = N$ because otherwise we would find a non-trivial normal $p'$-subgroup of $G$. Hence, $\mathrm{O}_{p'}(G/N) = 1$. Consequently, $G \in H_{\mathfrak{X}}(p)$ by [DH92, IV, 3.7]. By Condition 1, $\mathfrak{S}_p(G^{\mathfrak{G}}) \subseteq \mathfrak{F}$. In particular, $G^{\mathfrak{G}} \in \mathfrak{F}$. We conclude that $G \in \mathfrak{H}$, which contradicts our supposition. □

**Corollary 3.3.15 ([BBPR98, Theorem A]).** *A formation product $\mathfrak{H}$ of two formations $\mathfrak{F}$ and $\mathfrak{G}$ is local if and only if $\mathfrak{H}$ satisfies the following condition:*

*If $p \in \operatorname{char} \operatorname{lform}(\mathfrak{H})$ and $H_{\mathfrak{J}}(p)$ is not contained in $\mathfrak{G}$, then*

$$\mathfrak{S}_p H_{\mathfrak{J}}(p)^{\mathfrak{G}} \subseteq \mathfrak{F}.$$

When a product is $\mathfrak{X}$-local, the formation $\mathfrak{G}$ has a very nice property.

**Corollary 3.3.16.** *If $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$ is $\mathfrak{X}$-local, then $\operatorname{form}_{\mathfrak{X}}(\mathfrak{G}) \subseteq \mathfrak{N}_{p'}\mathfrak{G}$ for all primes $p \in \operatorname{char}(\mathfrak{X}) \setminus \pi(\mathfrak{F})$.*

*Proof.* Let $p \in \operatorname{char}(\mathfrak{X}) \setminus \pi(\mathfrak{F})$. By Theorem 3.3.13, we have that $H_{\mathfrak{X}}(p) \subseteq \mathfrak{G}$. Consider the canonical $\mathfrak{X}$-formation function $G$ defining $\operatorname{form}_{\mathfrak{X}}(\mathfrak{G})$. Suppose that $\operatorname{form}_{\mathfrak{X}}(\mathfrak{G})$ is not contained in $\mathfrak{N}_{p'}\mathfrak{G}$, and let $G \in \operatorname{form}_{\mathfrak{X}}(\mathfrak{G}) \setminus \mathfrak{N}_{p'}\mathfrak{G}$ be a group of minimal order. Then $G \in \mathfrak{H}$ and $G$ has a unique minimal normal subgroup, $N$ say. In addition, $N \leq G^{\mathfrak{G}}$ and $G/N \in \mathfrak{N}_{p'}\mathfrak{G}$. If $N \in \mathrm{E}\,\mathfrak{X}'$, it follows that $G \in G(S)$ for some $S \in \mathfrak{X}'$. But, in this case, $G \in \mathfrak{G}$. This is a contradiction. Hence $N$ is an $\mathrm{E}\,\mathfrak{X}$-group. Consider $q \in \pi(N)$. If $N$ were non-abelian, then $G$ would belong to $G(q) \subseteq \mathfrak{S}_q\mathfrak{G}$. Hence $G \in \mathfrak{G}$ because $\mathrm{O}_q(G) = 1$. This would contradict our assumption. Therefore $N$ is an elementary abelian $q$-group for some prime $q \in \operatorname{char} \mathfrak{X}$. Assume that $\Phi(G) = 1$. Then $G$ is a primitive group and $N = \mathrm{C}_G(N)$. Therefore $G \in G(q)$. If $p \neq q$, then $G \in \mathfrak{N}_{p'}\mathfrak{G}$ because $G(q) \subseteq \mathfrak{S}_q\mathfrak{G}$ and if $p = q$, then $G \in \mathfrak{S}_p H_{\mathfrak{X}}(p) = H_{\mathfrak{X}}(p) \subseteq \mathfrak{G}$. In both cases, we reach a contradiction. Hence $N$ is contained in $\Phi(G)$. If $p \neq q$, then $\mathrm{F}(G)$ is a $p'$-group and $G/\mathrm{F}(G) \cong (G/N)\big/\mathrm{F}(G/N) \in \mathfrak{G}$. Hence, $G \in \mathfrak{N}_{p'}\mathfrak{G}$, contrary to supposition. Assume that $p = q$. Then, since $G/N \in \mathfrak{N}_{p'}\mathfrak{G}$, it follows that $(G/N)^{\mathfrak{G}} = G^{\mathfrak{G}}/N$ is a $p'$-group. Thus $G^{\mathfrak{G}}/N$ is contained in $\mathrm{O}_{p'}(G/N)$ which is trivial by [Hup67, VI, 6.3]. Hence $N = G^{\mathfrak{G}}$. Since $G \in \mathfrak{H}$, we have that $G^{\mathfrak{G}} = N \in \mathfrak{F}$ and $p \in \pi(\mathfrak{F})$. This final contradiction proves that $\operatorname{form}_{\mathfrak{X}}(\mathfrak{G}) \subseteq \mathfrak{N}_{p'}\mathfrak{G}$. □

If $\mathfrak{X} = \mathfrak{J}$, we have:

**Corollary 3.3.17 ([She84]).** *If $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$ is local, then $\operatorname{lform}(\mathfrak{G})$ is contained in $\mathfrak{N}_{p'}\mathfrak{G}$ for all primes $p \notin \pi(\mathfrak{F})$.*

This result motivates the following definition.

**Definition 3.3.18.** *Let $\omega$ be a non-empty set of primes, and let $\mathfrak{F}$ be a formation.*

1. *(see [She84]) $\mathfrak{F}$ is said to be* $\omega$-local *if* $\text{lform}(\mathfrak{F})$ *is contained in* $\mathfrak{N}_{\omega'}\mathfrak{F}$.
2. *(see [SS00a]) $\mathfrak{F}$ is called* $\omega$-saturated *if the condition* $G/\big(\Phi(G) \cap \mathrm{O}_\omega(G)\big) \in \mathfrak{F}$ *always implies* $G \in \mathfrak{F}$.

*When $\omega = \{p\}$, we shall say* $p$-local *(respectively,* $p$-saturated*) instead of $\{p\}$-local (respectively, $\{p\}$-saturated).*

*Remarks 3.3.19.* Let $\emptyset \neq \omega$ be a set of primes and let $\mathfrak{F}$ be a formation.

1. $\mathfrak{F}$ is $\omega$-local if and only if $\mathfrak{F}$ is $p$-local for all $p \in \omega$. Hence $\mathfrak{F}$ is local if and only if $\mathfrak{F}$ is $p$-local for all primes $p$.
2. If $\mathfrak{F}$ is an $\omega$-local formation, then $\mathfrak{F}$ is $\omega$-saturated.
3. If $\mathfrak{F}$ is $\omega$-saturated, then $\mathfrak{N}_{\omega'}\mathfrak{F}$ is local. Therefore every $\omega$-saturated formation is $\omega$-local (see [SS95]).
4. Every formation composed of $\omega'$-groups is $\omega$-saturated.
5. Every $\omega$-saturated formation is $\mathfrak{X}_\omega$-saturated, where $\mathfrak{X}_\omega$ is the class of all simple $\omega$-groups.

*Proof.* 1, 2, and 4 are clear. To prove Statement 3, suppose that $\mathfrak{F}$ is $\omega$-saturated. If $q$ is a prime such that $q \in \omega'$, then $\mathfrak{H} = \mathfrak{N}_{\omega'}\mathfrak{F}$ is $q$-saturated. Assume that $p$ is a prime in $\omega$ such that $\mathfrak{H}$ is not $p$-saturated. Then there exists a group $G$ such that $G/\big(\Phi(G) \cap \mathrm{O}_p(G)\big) \in \mathfrak{H}$ but $G \notin \mathfrak{H}$. Let us choose $G$ of least order. Then $G$ has a unique minimal normal subgroup $N$, $N$ is contained in $\Phi(G) \cap \mathrm{O}_p(G)$, and $G/N \in \mathfrak{H}$. Since $\mathfrak{F}$ is contained in $\mathfrak{H}$, it follows that $G^{\mathfrak{F}} \neq 1$ and so $N$ is also contained in $G^{\mathfrak{F}}$. Now $\mathrm{O}_{p'}(G/N) = 1$ and $G^{\mathfrak{F}}/N$ is a $p'$-group because $G/N \in \mathfrak{H}$. This implies that $G^{\mathfrak{F}} = N$. But then $G/N \in \mathfrak{F}$ and so $G \in \mathfrak{F}$ because $\mathfrak{F}$ is $p$-saturated. This contradiction shows that $\mathfrak{H}$ is $p$-saturated for all $p \in \omega$. Therefore $\mathfrak{H}$ is saturated. In particular, $\text{lform}(\mathfrak{F}) \subseteq \mathfrak{N}_{\omega'}\mathfrak{F}$ and $\mathfrak{F}$ is $\omega$-local.

5 follows directly from the fact that $\Phi_{\mathfrak{X}_\omega}(G) \subseteq \Phi(G) \cap \mathrm{O}_\omega(G)$ for every group $G$. □

The family of $\mathfrak{X}_\omega$-saturated formations does not coincide with the one of $\omega$-saturated formations in general. This follows from the fact that there exist Baer-local formations which are not $\omega$-saturated for any non-empty set of primes $\omega$.

*Example 3.3.20 ([BBCER03]).* Let $\mathfrak{Y} = \{\mathrm{Alt}(n) : n \geq 5\}$ and $\mathfrak{F} = \mathrm{E}\,\mathfrak{Y}$. It is clear that $\mathfrak{F}$ is a Baer-local formation. In particular, $\mathfrak{F}$ is $\mathfrak{X}$-saturated for every $\mathfrak{X} \subseteq \mathbb{P}$ by Corollary 3.1.13.

Assume that $\mathfrak{F}$ is $p$-saturated for a prime $p$. If $p \geq 5$, set $k := p$; otherwise, set $k := 5$. As $p$ divides $|\mathrm{Alt}(k)|$, there exists a group $E$ with a normal elementary abelian $p$-subgroup $A \neq 1$ such that $A \leq \Phi(E)$ and $E/A \cong \mathrm{Alt}(k)$ ([DH92, B, 11.8]). Then $E/\big(\Phi(E) \cap \mathrm{O}_p(E)\big) \cong E/A \in \mathfrak{F}$. Therefore $E \in \mathfrak{F}$, and we have a contradiction.

This implies that $\mathfrak{F}$ is not $\omega$-saturated for any non-empty set of primes $\omega$. In particular, $\mathfrak{F}$ is $(C_2)$-saturated but not 2-saturated.

Suppose that $\mathfrak{G}$ is a $p$-saturated formation, $p$ a prime. Then $\mathrm{lform}(G) \subseteq \mathfrak{N}_{p'}\mathfrak{G}$. Therefore $G(p) \subseteq \mathfrak{N}_{p'}\mathfrak{G}$ and so $G(p) = G_{\mathfrak{J}}(p) \subseteq \mathfrak{G}$. The converse is also true as the following lemma shows.

**Lemma 3.3.21.** *$\mathfrak{G}$ is $p$-saturated if and only if $G(p) \subseteq \mathfrak{G}$.*

*Proof.* Only the sufficiency is in doubt. Suppose that $\mathfrak{G}$ is not $p$-saturated and $G_{\mathfrak{J}}(p) \subseteq \mathfrak{G}$. Let $G$ be a group of minimal order satisfying $G/\big(\Phi(G) \cap \mathrm{O}_p(G)\big) \in \mathfrak{G}$ and $G \notin \mathfrak{G}$. $G$ is a monolithic group and $N := \mathrm{Soc}(G) \leq \Phi(G) \cap \mathrm{O}_p(G)$. We have that $\mathrm{O}_{p',p}(G/N) = \mathrm{O}_{p',p}(G)/N$, since $N \leq \Phi(G)$. Moreover, $G/N \in \mathfrak{G}$ and, therefore, $G/\mathrm{O}_{p',p}(G) \in G_{\mathfrak{J}}(p)$, bearing in mind that $p \in \pi(G/N)$. Since $\mathrm{O}_{p',p}(G) = \mathrm{O}_p(G)$, $G \in G_{\mathfrak{J}}(p) \subseteq \mathfrak{G}$. This is not possible. □

**Theorem 3.3.22.** *Let $\mathfrak{F}$ and $\mathfrak{G}$ be formations. Let $\mathfrak{M}$ be a $p$-saturated formation contained in $\mathfrak{F} \circ \mathfrak{G}$, where $p$ is a prime. If $M_{\mathfrak{J}}(p)$ is not contained in $\mathfrak{G}$, then $\mathfrak{S}_p M_{\mathfrak{J}}(p)^{\mathfrak{G}} \subseteq \mathfrak{F}$.*

*Proof.* Assume that $\mathfrak{M}$ is $p$-saturated. Then $M_{\mathfrak{J}}(p)$ is contained in $\mathfrak{M}$ by Lemma 3.3.21. There exists a group $T \in M_{\mathfrak{J}}(p) \setminus \mathfrak{G}$. We have that $\mathfrak{S}_p(T) \subseteq M_{\mathfrak{J}}(p) \subseteq \mathfrak{M} \subseteq \mathfrak{F} \circ \mathfrak{G}$. Hence $\mathfrak{S}_p(T^{\mathfrak{G}}) \subseteq \mathfrak{F}$ by Lemma 3.3.12.

Now consider a group $G$ in $\mathfrak{S}_p M_{\mathfrak{J}}(p)^{\mathfrak{G}}$. Then $G$ has a normal $p$-subgroup $N$ such that $G/N \cong \bar{T}^{\mathfrak{G}}$, where $\bar{T} \in M_{\mathfrak{J}}(p)$. If $\bar{T}^{\mathfrak{G}} \neq 1$, we have just proved that $\mathfrak{S}_p(\bar{T}^{\mathfrak{G}}) \subseteq \mathfrak{F}$ and, therefore, $G \in \mathfrak{F}$. If $\bar{T}^{\mathfrak{G}} = 1$, then $G \in \mathfrak{S}_p$. Consider the group $A := G \times T^{\mathfrak{G}}$. We have that $A \in \mathfrak{S}_p(T^{\mathfrak{G}}) \subseteq \mathfrak{F}$ and, therefore, $G \in \mathrm{Q}(\mathfrak{F}) = \mathfrak{F}$. We conclude that $\mathfrak{S}_p M_{\mathfrak{J}}(p)^{\mathfrak{G}} \subseteq \mathfrak{F}$. □

**Corollary 3.3.23.** *Let $\mathfrak{F}$ and $\mathfrak{G}$ be formations and let $p$ be a prime. Then the following statements are equivalent:*

1. *$\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$ is a $p$-saturated formation.*
2. *If $H_{\mathfrak{J}}(p)$ is not contained in $\mathfrak{G}$, then $\mathfrak{S}_p H_{\mathfrak{J}}(p)^{\mathfrak{G}} \subseteq \mathfrak{F}$.*

*Proof.* 1 implies 2 by virtue of Theorem 3.3.22. Let us prove that 2 implies 1. We shall derive a contradiction by supposing that $H_{\mathfrak{J}}(p) \setminus \mathfrak{H}$ contains a group $G$ of minimal order. Then $G$ has a unique minimal normal subgroup $N$, and $G/N \in \mathfrak{H}$. Since $H_{\mathfrak{J}}(p)$ is contained in $\mathfrak{S}_p\mathfrak{H}$, it follows that $N$ is a $p$-group. It is clear that $H_{\mathfrak{J}}(p)$ is not contained in $\mathfrak{G}$. Hence $\mathfrak{S}_p H_{\mathfrak{J}}(p)^{\mathfrak{G}} \subseteq \mathfrak{F}$. Note that $N \leq G^{\mathfrak{G}}$ and $G^{\mathfrak{G}}/N \in H_{\mathfrak{J}}(p)^{\mathfrak{G}}$. Therefore $G^{\mathfrak{G}} \in \mathfrak{S}_p H_{\mathfrak{J}}(p)^{\mathfrak{G}} \subseteq \mathfrak{F}$. This contradiction shows that $H_{\mathfrak{J}}(p) \subseteq \mathfrak{H}$ and that $\mathfrak{H}$ is $p$-saturated by Lemma 3.3.21. □

Theorem 3.3.22 also confirms a more general version of the abovementioned conjecture of L. A. Shemetkov concerning the non-decomposability of the formation of all $p$-groups ($p$ a prime) as formation product of two non-trivial subformations.

**Corollary 3.3.24.** *Let $\mathfrak{F}$, $\mathfrak{G}$, and $\mathfrak{M}$ be formations such that $\mathfrak{M}$ is contained in $\mathfrak{F} \circ \mathfrak{G}$ and $\mathfrak{M}$ is p-saturated. If $\mathfrak{F} \subseteq \mathfrak{S}_p$ and $\mathfrak{F} \neq \mathfrak{S}_p$, then $\mathfrak{M} \subseteq \mathfrak{G}$.*

*Proof.* If $M_{\mathfrak{Z}}(p) = \emptyset$, it follows that $\mathfrak{M} \subseteq \mathfrak{E}_{p'}$. In this case, we have that $\mathfrak{M} \subseteq \mathfrak{E}_{p'} \cap (\mathfrak{F} \circ \mathfrak{G}) \subseteq \mathfrak{E}_{p'} \cap (\mathfrak{S}_p \circ \mathfrak{G})$. Therefore, $\mathfrak{M} \subseteq \mathfrak{G}$. If $M_{\mathfrak{Z}}(p) \neq \emptyset$, we have that $\mathfrak{M} \subseteq \mathfrak{E}_{p'} M_{\mathfrak{Z}}(p)$. If $M_{\mathfrak{Z}}(p)$ is contained in $\mathfrak{G}$, then $\mathfrak{M} \subseteq \big(\mathfrak{E}_{p'} M_{\mathfrak{Z}}(p)\big) \cap (\mathfrak{S}_p \mathfrak{G}) \subseteq (\mathfrak{E}_{p'} \mathfrak{G}) \cap (\mathfrak{S}_p \mathfrak{G}) = \mathfrak{G}$ and the result holds. Suppose that $M_{\mathfrak{Z}}(p)$ is not contained in $\mathfrak{G}$. Then $\mathfrak{S}_p M_{\mathfrak{Z}}(p)^{\mathfrak{G}}$ is contained in $\mathfrak{F}$ by Theorem 3.3.22. In particular, $\mathfrak{S}_p \subseteq \mathfrak{F}$, and we have a contradiction. □

## 3.4 $\omega$-local formations

The family of $\omega$-local formations, $\omega$ a set of primes, emerges naturally in the study of local formations that are products of two formations as it was observed in Section 3.3. There it is also proved that the $\omega$-local formations are exactly the ones which are closed under extensions by the Hall $\omega$-subgroup of the Frattini subgroup. In this section $\omega$-saturated formations are studied by using a functional approach. This method was initially proposed by L. A. Shemetkov in [She84] for $p$-local formations, and further developed in [SS00a, SS00b, BBS97].

The second part of the section is devoted to study the relation between $\omega$-saturated formations and $\mathfrak{X}$-local formations, where $\mathfrak{X}$ is a class of simple groups which is naturally associated with $\omega$.

**Definition 3.4.1 ([SS00a]).** *Let $\omega$ be a non-empty set of primes. Every function of the form*

$$f\colon \omega \cup \{\omega'\} \longrightarrow \{\textit{formations}\}$$

*is called an $\omega$-*local satellite*.*

*If $f$ is an $\omega$-local satellite, define the class*

$$\mathrm{LF}_\omega(f) = \big(G : G/G_{\omega d} \in f(\omega') \text{ and } G/\mathrm{O}_{p',p}(G) \in f(p) \text{ for all } p \in \omega \cap \pi(G)\big),$$

*where $G_{\omega d}$ is the product of all normal subgroups $N$ of $G$ such that every composition factor of $N$ is divisible by at least one prime in $\omega$ ($G_{\omega d} = 1$ if $\pi\big(\mathrm{Soc}(G)\big) \cap \omega = \emptyset$).*

If $f$ is an $\omega$-local satellite, we write $\mathrm{Supp}(f) = \big\{p \in \omega \cup \{\omega'\} : f(p) \neq \emptyset\big\}$. Denote $\pi_1 = \mathrm{Supp}(f) \cap \omega$, $\pi_2 = \omega \setminus \pi_1$. Then $\mathrm{LF}_\omega(f) = \big(\bigcap_{p\in\pi_2} \mathfrak{E}_{p'}\big) \cap \big(\bigcap_{p\in\pi_1} \mathfrak{E}_{p'}\mathfrak{S}_p \circ f(p)\big) \cap \mathfrak{E}_{\omega d} \circ f(w')$. Here $\mathfrak{E}_{\omega d}$ is the class of all groups $G$ such that every composition factor of $G$ is divisible by at least one prime in $\omega$. Since the intersection and the formation product of two formations are again formations, the above formula implies that $\mathrm{LF}_\omega(f)$ is a formation.

**Theorem 3.4.2 ([SS00a]).** *Let $\omega$ be a non-empty set of primes and let $\mathfrak{F}$ be a formation. The following statements are equivalent:*

*1. $\mathfrak{F}$ is $\omega$-saturated.*
*2. $\mathfrak{F} = \mathrm{LF}_\omega(f)$, where $f(p) = \mathfrak{F}_{\mathfrak{J}}(p)$, $p \in \omega$, and $f(\omega') = \mathfrak{F}$.*

*Proof.* 1 implies 2. It is clear that $\mathfrak{F} \subseteq \mathrm{LF}_\omega(f)$. Suppose that the equality does not hold and derive a contradiction. Choose a group $G \in \mathrm{LF}_\omega(f) \setminus \mathfrak{F}$ of minimal order. Then, as usual, $G$ has a unique minimal normal subgroup $N$ and $G/N \in \mathfrak{F}$. Since $G/G_{\omega d} \in f(\omega') = \mathfrak{F}$, it follows that $G_{\omega d} \neq 1$. This implies that $\pi(N) \cap \omega \neq \emptyset$. Let $p \in \omega$ be a prime dividing $|N|$. If $N$ were non-abelian, then $G \in \mathfrak{F}_{\mathfrak{J}}(p)$. Since, by Lemma 3.3.21, $\mathfrak{F}_{\mathfrak{J}}(p) \subseteq \mathfrak{F}$, we would have $G \in \mathfrak{F}$. This would be a contradiction. Therefore $N$ is an abelian $p$-group. Moreover $N \cap \Phi(G) = 1$ because $\mathfrak{F}$ is $\omega$-saturated. Hence $N = \mathrm{C}_G(N)$ and $G/N \in \mathfrak{F}_{\mathfrak{J}}(p)$. This implies that $G \in \mathfrak{S}_p\mathfrak{F}_{\mathfrak{J}}(p) = \mathfrak{F}_{\mathfrak{J}}(p)$, and we have a contradiction. Consequently $\mathfrak{F} = \mathrm{LF}_\omega(f)$.

2 implies 1. Suppose that $\mathfrak{F}$ is not $\omega$-saturated. Then there exists a prime $p \in \omega$ and a group $G$ such that $G/\big(\Phi(G) \cap \mathrm{O}_p(G)\big) \in \mathfrak{F}$ but $G \notin \mathfrak{F}$. Denote $L = \Phi(G) \cap \mathrm{O}_p(G)$. Then $(G/L)_{\omega d} = G_{\omega d}/L$ and $\mathrm{O}_{q',q}(G/L) = \mathrm{O}_{q',q}(G)/L$ for all primes $q$. Hence $G/G_{\omega d} \in f(\omega')$ and $G/\mathrm{O}_{q',q}(G) \in f(q)$ for all $q \in \omega \cap \pi(G)$ because $G/L \in \mathfrak{F}$. Consequently $G \in \mathfrak{F}$. This contradiction completes the proof of the theorem. ☐

*Remark 3.4.3.* An $\omega$-saturated formation can be $\omega$-locally defined by two distinguished $\omega$-local satellites: the minimal $\omega$-local satellite and the canonical one. Moreover, if $\mathfrak{Y}$ is a class of groups, the intersection of all $\omega$-local formations containing $\mathfrak{Y}$ is the smallest $\omega$-local formation containing $\mathfrak{Y}$. Such $\omega$-local formation is denoted by $\mathrm{lform}_\omega(\mathfrak{Y})$. It is clear that $\mathrm{lform}_\omega(\mathfrak{Y}) = \mathrm{LF}_\omega(f)$, where $f$ is given by:

$$\begin{aligned} f(p) &= \mathrm{Q}\,\mathrm{R}_0\big(G/\mathrm{O}_{p',p}(G) : G \in \mathfrak{Y}\big) && \text{if } p \in \pi(\mathfrak{Y}) \cap \omega, \\ f(p) &= \emptyset, && p \in \omega \setminus \pi(\mathfrak{Y}), \\ f(\omega') &= \mathrm{Q}\,\mathrm{R}_0(G/G_{\omega d} : G \in \mathfrak{Y}) && \end{aligned}$$

(see [SS00a] for details).

Let $\omega$ be a non-empty set of primes. One can ask the following question:

*Is it possible to ensure the existence of a class $\mathfrak{X}(\omega)$ of simple groups such that* $\operatorname{char}\mathfrak{X}(\omega) = \pi\big(\mathfrak{X}(\omega)\big)$ *satisfying that a formation is $\omega$-saturated if and only if it is $\mathfrak{X}(\omega)$-saturated?*

The following example shows that the answer is negative.

*Example 3.4.4 ([BBCER03]).* Consider the formation

$$\mathfrak{F} := (G : \text{all abelian composition factors of } G \text{ are isomorphic to } C_2).$$

Suppose that $\mathfrak{F}$ is $\mathfrak{X}$-saturated for a class $\mathfrak{X}$ containing a non-abelian simple group $E$ and $\pi(\mathfrak{X}) = \operatorname{char}\mathfrak{X}$. Let $p \neq 2$ be a prime dividing $|E|$. Then $p \in \operatorname{char}\mathfrak{X}$. Since $E \in \mathfrak{F}$, it follows that if $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(f)$, then $f(p) \neq \emptyset$. This means that $C_p \in \mathfrak{F}$. This is a contradiction. Hence $\mathfrak{X}$ should be composed of abelian simple groups. Since $\mathfrak{F}$ is solubly saturated, we have that $\mathfrak{F}$ is $\mathfrak{X}$-saturated exactly for the classes of simple groups $\mathfrak{X}$ contained in $\mathbb{P}$ by Corollary 3.1.13. Since $\mathfrak{F}$ is clearly 2-saturated, if we assume the existence of a class $\mathfrak{X}(2)$ fulfilling the property, it follows that $\mathfrak{X}(2) \subseteq \mathbb{P}$. This is not possible because the formation in Example 3.3.20 is $\mathfrak{X}(2)$-saturated but not 2-saturated.

The following theorem shows that an $\mathfrak{X}$-local formation always contains a largest $\omega$-local formation for $\omega = \operatorname{char}\mathfrak{X}$.

**Theorem 3.4.5 ([BBCS05]).** *Let $\mathfrak{X}$ be a class of simple groups such that $\omega = \operatorname{char}\mathfrak{X} = \pi(\mathfrak{X})$. Let $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(F)$ be an $\mathfrak{X}$-local formation. Then the $\omega$-local formation $\mathfrak{F}_\omega = \mathrm{LF}_\omega(f)$, where $f(p) = F(p)$ for every $p \in \omega$ and $f(\omega') = \mathfrak{F}$, is the largest $\omega$-local formation contained in $\mathfrak{F}$.*

*Proof.* Suppose, for a contradiction, that $\mathfrak{F}_\omega$ is not contained in $\mathfrak{F}$. Let $G$ be a group of minimal order in $\mathfrak{F}_\omega \setminus \mathfrak{F}$. Then, as usual, $G$ has a unique minimal normal subgroup $N$, and $G/N \in \mathfrak{F}$. If $G_{\omega d} = 1$, we would have that $G \in f(\omega') = \mathfrak{F}$, contradicting the choice of $G$. Assume that $G_{\omega d} \neq 1$. Then $N$ is contained in $G_{\omega d}$. This means that there exists a prime $p \in \omega$ dividing $|N|$. Hence $G/\mathrm{C}_G(N) \in f(p) = F(p)$. If $N$ is a $p$-group, it follows that $N$ is an $\mathfrak{X}$-chief factor of $G$. By Remark 3.1.7 (2), we conclude that $G \in \mathrm{LF}_{\mathfrak{X}}(F) = \mathfrak{F}$, against the choice of $G$. Hence $N$ is non-abelian and so $\mathrm{C}_G(N) = 1$ and $G \in F(p)$. Since $F(p) = \mathfrak{S}_p \underline{f}(p)$ and $\mathrm{O}_p(G) = 1$, it follows that $G \in \underline{f}(p) \subseteq \mathfrak{F}$. This contradiction proves that $\mathfrak{F}_\omega \subseteq \mathfrak{F}$.

Now let $\mathfrak{G} = \mathrm{LF}_\omega(g)$ be an $\omega$-local formation contained in $\mathfrak{F}$. Suppose, if possible, that $\mathfrak{G}$ is not contained in $\mathfrak{F}_\omega$ and let $A$ be a group of minimal order in the supposed non-empty class $\mathfrak{G} \setminus \mathfrak{F}_\omega$. Then $A$ has a unique minimal normal subgroup $B$, and $A/B \in \mathfrak{F}_\omega$. Since $A \in \mathfrak{G} \subseteq \mathfrak{F}$, we have that $A/A_{\omega d} \in \mathfrak{F} = f(\omega')$. Suppose that $p \in \omega \cap \pi(B)$. If $B$ is an $\mathfrak{X}$-chief factor of $A$, it follows that $A/\mathrm{C}_A(B) \in F(p) = f(p)$. If $B$ is an $\mathfrak{X}'$-chief factor of $A$, then $B$ is non-abelian and $A \cong A/\mathrm{C}_A(B) \in g(p)$. Then $\mathrm{O}_p(A) = 1$ and so, by [DH92, B, 10.9], $A$ has a faithful irreducible representation over $\mathrm{GF}(p)$. Let $M$ be the corresponding module and $G = [M]A$ the corresponding semidirect product. Let us see that $G \in \mathfrak{G}$. Since $M$ is contained in $G_{\omega d}$, it follows that $G/G_{\omega d} \in g(\omega')$ because $A/A_{\omega d} \in g(\omega')$. Moreover, we have that $G/\mathrm{C}_G(M) \cong A \in g(p)$. We can conclude that $G \in \mathfrak{G}$ and, consequently, $G = [M]A \in \mathfrak{F}$. This implies that $A \cong G/\mathrm{C}_G(M) \in f(p)$. Now we can state that $A \in \mathfrak{F}_\omega$, contradicting the choice of $A$. Therefore $\mathfrak{G}$ is contained in $\mathfrak{F}_\omega$. □

An immediate application of Theorem 3.4.5 is the following corollary:

**Corollary 3.4.6 ([BBCER03]).** *Let $\omega$ be a set of primes and let $\mathfrak{X}_\omega$ be the class of all simple $\omega$-groups. If $\mathfrak{F}$ is an $\mathfrak{X}_\omega$-local formation composed of $\omega$-separable groups, then $\mathfrak{F}$ is $\omega$-local.*

*Proof.* Suppose that $\mathfrak{F}$ is an $\mathfrak{X}_\omega$-local formation. According to Theorem 3.4.5, $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}_\omega}(F)$ contains a largest $\omega$-local formation $\mathfrak{F}_\omega$, where $f(p) = F(p)$ for every $p \in \omega$ and $f(\omega') = \mathfrak{F}$. Suppose that the inclusion is proper, and let $G$ be a group of minimal order in $\mathfrak{F} \setminus \mathfrak{F}_\omega$. Then $G$ has a unique minimal normal subgroup $N$, and $G/N \in \mathfrak{F}_\omega$. It is clear that $G/G_{\omega d} \in f(\omega') = \mathfrak{F}$. If $p \in \pi(N) \cap \omega$, it follows that $N$ is an $\omega$-group, since $G$ is $\omega$-separable. Hence, $N$ is an $\mathfrak{X}_\omega$-chief factor of $G$ and, therefore, $G/\mathrm{C}_G(N) \in F(p) = f(p)$. Taking into account that $G/N \in \mathfrak{F}_\omega$, we conclude that $G \in \mathfrak{F}_\omega$. This contradiction proves that $\mathfrak{F} = \mathfrak{F}_\omega$ is $\omega$-local. □

**Corollary 3.4.7 ([BBCER03]).** *Let $\mathfrak{F}$ be a formation composed of $\omega$-separable groups. Then $\mathfrak{F}$ is $\omega$-saturated if and only if $\mathfrak{F}$ is $\mathfrak{X}_\omega$-saturated, where $\mathfrak{X}_\omega$ is the class of all simple $\omega$-groups.*

The following consequence of Theorem 3.4.5 is of interest.

**Corollary 3.4.8 ([Sal85]).** *Every solubly saturated formation contains a maximal saturated formation with respect to inclusion.*

*Remarks 3.4.9.* 1. The converse of Corollary 3.4.8 does not hold. It is enough to consider $\mathfrak{F} = \mathrm{D}_0\big(\mathfrak{S}_2, \mathrm{Alt}(5)\big)$. By Lemma 2.2.3, $\mathfrak{F}$ is a formation. The group $\mathrm{SL}(2,5)$ shows that $\mathfrak{F}$ is not solubly saturated. However $\mathfrak{S}_2$ is the maximal saturated formation contained in $\mathfrak{F}$.

2. There exist formations not containing a maximal saturated formation as the Example 5.5 in [Sal85] shows: Let $\mathfrak{F}$ be the class of all soluble groups $G$ such that Sylow subgroups corresponding to different primes permute. By [Hup67, VI, 3.2], $\mathfrak{F}$ is a formation. Let $q$ be a prime and consider the formation function $f_q$ given by: $f_q(p) = \mathfrak{S}_{\{p,q\}}$ for all $p \in \mathbb{P}$. Then the saturated formation $\mathfrak{F}_q = \mathrm{LF}(f_q)$ is contained in $\mathfrak{F}$ by [Hup67, VI, 3.1]. Let $q_1$ and $q_2$ be two different primes and let $\mathfrak{F}_{q_1,q_2}$ be the smallest saturated formation containing $\mathfrak{F}_{q_1}$ and $\mathfrak{F}_{q_2}$. Then $C_{q_1} \times C_{q_2} \in F(p)$ for all $p \in \mathbb{P}$, where $F$ is the canonical local definition of $\mathfrak{F}_{q_1,q_2}$. This is due to the fact that $C_{q_1} \in F_{q_1}(p)$ and $C_{q_2} \in F_{q_2}(p)$, where $F_{q_1}$ and $F_{q_2}$ are the canonical local definitions of $\mathfrak{F}_{q_1}$ and $\mathfrak{F}_{q_2}$, respectively. Let $q_3$ be a prime, $q_3 \neq q_1, q_2$. By [DH92, B, 10.9], $C_{q_1} \times C_{q_2}$ has an irreducible and faithful module $M$ over $\mathrm{GF}(q_3)$. Let $G = [M](C_{q_1} \times C_{q_2})$ be the corresponding semidirect product. Then $G \in \mathfrak{F}_{q_1,q_2}$, but $G \notin \mathfrak{F}$. This shows that $\mathfrak{F}$ does not contain a maximal saturated formation with respect to the inclusion.

A natural question arising from the above results is the following:

*What are the precise conditions to ensure that an $\mathfrak{X}$-local formation is $\omega$-local for $\omega = \mathrm{char}\,\mathfrak{X}$?*

The next result gives the answer.

**Theorem 3.4.10.** *Let $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(\underline{f}) = \mathrm{LF}(F)$ be an $\mathfrak{X}$-local formation and $\omega = \mathrm{char}\,\mathfrak{X}$. The following conditions are pairwise equivalent:*

1. $\mathfrak{F}$ *is* $\omega$*-local.*
2. $\underline{f}(S) \subseteq \underline{f}(p)$ *for every* $S \in \mathfrak{X}'$ *and* $p \in \pi(S) \cap \omega$*.*
3. $\mathfrak{S}_p \underline{f}(S) \subseteq \mathfrak{F}$ *for every* $S \in \mathfrak{X}'$ *and* $p \in \pi(S) \cap \omega$*.*

*Proof.* 1 implies 2. Assume that $\mathfrak{F}$ is $\omega$-local. Then, by Theorem 3.4.5, $\mathfrak{F} = \mathrm{LF}_\omega(f)$, where

$$f(p) = F(p) = \mathfrak{S}_p \underline{f}(p) \qquad \text{if } p \in \omega,$$
$$f(\omega') = \mathfrak{F}.$$

Let $S \in \mathfrak{X}'$ and $p \in \pi(S) \cap \omega$. Then $S$ is non-abelian. By Theorem 3.1.11, $\underline{f}(S) = \mathrm{Q}\,\mathrm{R}_0\big(G/L : G \in \mathfrak{F},\ G/L \text{ is monolithic, and } \mathrm{Soc}(G/L) \in \mathrm{E}(S)\big)$.

Let $G$ be a group in $\mathfrak{F}$ and let $L$ be a normal subgroup of $G$ such that $G/L$ is monolithic and $\mathrm{Soc}(G/L) \in \mathrm{E}(S)$. Since $G/L$ is a primitive group of type 2, $L = \mathrm{C}_G\big(\mathrm{Soc}(G/L)\big)$. Moreover $G/L \in \mathfrak{F}$. This implies that $G/L \in F(p) = \mathfrak{S}_p \underline{f}(p)$. Hence $G/L \in \underline{f}(p)$ because $\mathrm{O}_p(G/L) = 1$. Consequently $\underline{f}(S) \subseteq \underline{f}(p)$ for all $p \in \pi(S) \cap \omega$.

2 implies 3. Let $S \in \mathfrak{X}'$ and $p \in \pi(S) \cap \omega$. Then $\mathfrak{S}_p \underline{f}(S) \subseteq \mathfrak{S}_p \underline{f}(p) = F(p) \subseteq \mathfrak{F}$.

3 implies 2. Applying Theorem 3.4.5, it is known that $\mathfrak{F}_\omega = \mathrm{LF}_\omega(f)$, where

$$f(p) = F(p) \qquad \text{if } p \in \omega, \text{ and}$$
$$f(\omega') = \mathfrak{F},$$

is the largest $\omega$-local formation contained in $\mathfrak{F}$. Suppose, by way of contradiction, that $\mathfrak{F}$ is not $\omega$-local. Then $\mathfrak{F}_\omega \neq \mathfrak{F}$. Let $G$ be a group of minimal order in $\mathfrak{F} \setminus \mathfrak{F}_\omega$. By a familiar argument, $G$ has a unique minimal normal subgroup $N$, and $G/N \in \mathfrak{F}_\omega$. If $\pi(N) \cap \omega = \emptyset$, then $G_{\omega d} = 1$ and so $G \in \mathfrak{F}_\omega$, which contradicts the fact that $G \notin \mathfrak{F}_\omega$. Therefore $\pi(N) \cap \omega \neq \emptyset$. Let $p$ be a prime in $\pi(N) \cap \omega$. If $N$ is an $\mathfrak{X}_p$-chief factor of $G$, $G/\mathrm{C}_G(N) \in F(p) = f(p)$. Assume that $N$ is an $\mathfrak{X}'$-chief factor of $G$ and $N \in \mathrm{E}(S)$. Then $S$ is non-abelian and so $\mathrm{O}_p(G) = 1$. By [DH92, B, 10.9], $G$ has an irreducible and faithful module $M$ over $\mathrm{GF}(p)$. Let $Z = [M]G$ be the corresponding semidirect product. Since $G \in \underline{f}(S)$, it follows that $Z \in \mathfrak{S}_p \underline{f}(S) \subseteq \mathfrak{F}$. This implies that $G \cong Z/\mathrm{C}_Z(M) \in F(p) = f(p)$. Consequently $G/\mathrm{C}_G(N) \in f(p)$ for all $p \in \pi(N) \cap \omega$ and $G \in \mathfrak{F}_\omega$. This contradicts our initial supposition. Therefore $\mathfrak{F} = \mathfrak{F}_\omega$ and $\mathfrak{F}$ is $\omega$-local. □