1

Maximal subgroups and chief factors

1.1 Primitive groups

This book, devoted to classes of finite groups, begins with the study of a class, the class of primitive groups, with no hereditary properties, the usual requirement for a class of groups, but whose importance is overwhelming to understand the remainder. We shall present the classification of primitive groups made by R. Baer and the refinement of this classification known as the O'Nan-Scott Theorem. The book of H. Kurzweil and B. Stellmacher [KS04], recently appeared, presents an elegant proof of this theorem. Our approach includes the results of F. Gross and L. G. Kovács on induced extensions ([GK84]) which are essential in some parts of this book.

We will assume our reader to be familiar with the basic concepts of permutation representations: G-sets, orbits, faithful representation, stabilisers, transitivity, the Orbit-Stabiliser Theorem, ... (see [DH92, A, 5]). In particular we recall that the stabilisers of the elements of a transitive G-set are conjugate subgroups of G and any transitive G-set Ω is isomorphic to the G-set of right cosets of the stabiliser of an element of Ω in G.

Definition 1.1.1. *Let G be a group and Ω a transitive G-set. A subset $\Phi \subseteq \Omega$ is said to be a* block *if, for every $g \in G$, we have that $\Phi^g = \Phi$ or $\Phi^g \cap \Phi = \emptyset$.*

Given a G-set Ω, trivial examples of blocks are $\emptyset$, Ω and any subset with a single element $\{\omega\}$, for any $\omega \in \Omega$. In fact, these are called *trivial blocks*.

Proposition 1.1.2. *Let G be a group which acts transitively on a set Ω and $\omega \in \Omega$. There exists a bijection*

$$\{\text{block } \Phi \text{ of } \Omega : \omega \in \Phi\} \longrightarrow \{H \leq G : G_\omega \leq H\}$$

which preserves the containments.

Proof. Given a block Φ in Ω such that $\omega \in \Phi$, then $G_\Phi = \{g \in G : \Phi^g = \Phi\}$ is a subgroup of G and the stabiliser G_ω is a subgroup of G_Φ. Conversely, if H is a subgroup of G containing G_ω, then the set $\Phi = \{\omega^h : h \in H\}$ is a block and $\omega \in \Phi$. These are the mutually inverse bijections required. □

The following result is well-known and its proof appears, for instance, in Huppert's book [Hup67, II, 1.2].

Theorem 1.1.3. *Let G be a group which acts transitively on a set Ω and assume that Φ is a non-trivial block of the action of G on Ω. Set $H = \{g \in G : \Phi^g = \Phi\}$. Then H is a subgroup of G.*

Let $\mathcal{T}$ be a right transversal of H in G. Then

1. *$\{\Phi^t : t \in \mathcal{T}\}$ is a partition of Ω.*
2. *We have that $|\Omega| = |\mathcal{T}||\Phi|$. In particular $|\Phi|$ divides $|\Omega|$.*
3. *The subgroup H acts transitively on Φ.*

Notation 1.1.4. If H is a subgroup of a group G, the *core of H in G* is the subgroup

$$\mathrm{Core}_G(H) = \bigcap_{g \in G} H^g.$$

Along this chapter, in order to make the notation more compact, the core of a subgroup H in a group G will often be denoted by H_G instead of $\mathrm{Core}_G(H)$.

Theorem 1.1.5. *Let G be a group. The following conditions are equivalent:*

1. *G possesses a faithful transitive permutation representation with no non-trivial blocks;*
2. *there exists a core-free maximal subgroup of G.*

Proof. 1 implies 2. Suppose that there exists a transitive G-set Ω with no non-trivial blocks and consider any $\omega \in \Omega$. The action of G on Ω is equivalent to the action of G on the set of right cosets of G_ω in G. The kernel of this action is $\mathrm{Core}_G(G_\omega)$ and, by hypothesis, is trivial. By Proposition 1.1.2, if H is a subgroup containing G_ω, there exists a block $\Phi = \{\omega^h : h \in H\}$ of Ω such that $\omega \in \Phi$ and $H = G_\Phi = \{g \in G : \Phi^g = \Phi\}$. Since G has no non-trivial blocks, either $\Phi = \{\omega\}$ or $\Phi = \Omega$. If $\Phi = \{\omega\}$, then $G_\omega = H$ and if $\Phi = \Omega$, then $H = G_\Omega = G$. Hence the stabiliser G_ω is a core-free maximal subgroup of G.

2 implies 1. If U is a core-free maximal subgroup of G, then the action of G on the set of right cosets of U in G is faithful and transitive. By maximality of U, this action has no non-trivial blocks by Proposition 1.1.2. □

Definitions 1.1.6. *A a faithful transitive permutation representation of a group is said to be* primitive *if it does not have non-trivial blocks.*

A primitive group *is a group which possesses a primitive permutation representation. Equivalently, a group is primitive if it possesses a core-free maximal subgroup.*

A primitive pair *is a pair* (G,U)*, where* G *is a primitive group and* U *a core-free maximal subgroup of* G*,*

Each conjugacy class of core-free maximal subgroups affords a faithful transitive and primitive permutation representation of the group. Thus, in general, it is more precise to speak of primitive pairs. Consider, for instance, the alternating group of degree 5, $G = \mathrm{Alt}(5)$. There exist three conjugacy classes of maximal subgroups, namely the normalisers of each type of Sylow subgroup. Obviously all of them are core-free. This gives three non-equivalent primitive representations of degrees 5 (for the normalisers of the Sylow 2-subgroups), 10 (for the normalisers of the Sylow 3-subgroups) and 6 (for the normalisers of the Sylow 5-subgroups).

The remarkable result that follows, due to R. Baer, classifies all primitive groups (a property defined in terms of maximal subgroups) according to the structure of the *socle*, i.e. the product of all minimal normal subgroups.

Theorem 1.1.7 ([Bae57]).

1. *A group* G *is primitive if and only if there exists a subgroup* M *of* G *such that* $G = MN$ *for all minimal normal subgroups* N *of* G*.*
2. *Let* G *be a primitive group. Assume that* U *is a core-free maximal subgroup of* G *and that* N *is a non-trivial normal subgroup of* G*. Write* $C = \mathrm{C}_G(N)$*. Then* $C \cap U = 1$*. Moreover, either* $C = 1$ *or* C *is a minimal normal subgroup of* G*.*
3. *If* G *is a primitive group and* U *is a core-free maximal subgroup of* G*, then exactly one of the following statements holds:*
   a) $\mathrm{Soc}(G) = S$ *is a self-centralising abelian minimal normal subgroup of* G *which is complemented by* U*:* $G = US$ *and* $U \cap S = 1$*.*
   b) $\mathrm{Soc}(G) = S$ *is a non-abelian minimal normal subgroup of* G *which is supplemented by* U*:* $G = US$*. In this case* $\mathrm{C}_G(S) = 1$*.*
   c) $\mathrm{Soc}(G) = A \times B$*, where* A *and* B *are the two unique minimal normal subgroups of* G *and both are complemented by* U*:* $G = AU = BU$ *and* $A \cap U = B \cap U = A \cap B = 1$*. In this case* $A = \mathrm{C}_G(B)$*,* $B = \mathrm{C}_G(A)$*, and* A*,* B *and* $AB \cap U$ *are non-abelian isomorphic groups.*

Proof. 1. If G is a primitive group, and U is a core-free maximal subgroup of G, then it is clear that $G = UN$ for every minimal normal subgroup N of G. Conversely, if there exists a subgroup M of G, such that $G = MN$ for every minimal normal subgroup N of G and U is a maximal subgroup of G such that $M \leq U$, then U cannot contain any minimal normal subgroup of G, and therefore U is a core-free maximal subgroup of G.

2. Since U is core-free in G, we have that $G = UN$. Since N is normal, then C is normal in G and then $C \cap U$ is normal in U. Since $C \cap U$ centralises N, then $C \cap U$ is in fact normal in G. Therefore $C \cap U = 1$.

If $C \neq 1$, consider a minimal normal subgroup X of G such that $X \leq C$. Since X is not contained in U, then $G = XU$. Then $C = C \cap XU = X(C \cap U) = X$.

3. Let us assume that N_1, N_2, and N_3 are three pairwise distinct minimal normal subgroups. Since $N_1 \cap N_2 = N_1 \cap N_3 = N_2 \cap N_3 = 1$, we have that $N_2 \times N_3 \leq \mathrm{C}_G(N_1)$. But then $\mathrm{C}_G(N_1)$ is not a minimal normal subgroup of G, and this contradicts 2. Hence, in a primitive group there exist at most two distinct minimal normal subgroups.

Suppose that N is a non-trivial abelian normal subgroup of G. Then $N \leq \mathrm{C}_G(N)$. Since by 2, $\mathrm{C}_G(N)$ is a minimal normal subgroup of G, we have that N is self-centralising. Thus, in a primitive group G there exists at most one abelian minimal normal subgroup N of G. Moreover, $G = NU$ and N is self-centralising. Then $N \cap U = \mathrm{C}_G(N) \cap U = 1$.

If there exists a unique minimal non-abelian normal subgroup N, then $G = NU$ and $\mathrm{C}_G(N) = 1$.

If there exist two minimal normal subgroups A and B, then $A \cap B = 1$ and then $B \leq \mathrm{C}_G(A)$ and $A \leq \mathrm{C}_G(B)$. Since $\mathrm{C}_G(A)$ and $\mathrm{C}_G(B)$ are minimal normal subgroups, we have that $B = \mathrm{C}_G(A)$ and $A = \mathrm{C}_G(B)$. Now $A \cap U = \mathrm{C}_G(B) \cap U = 1$ and $B \cap U = \mathrm{C}_G(A) \cap U = 1$. Hence $G = AU = BU$.

Since $A = \mathrm{C}_G(B)$, it follows that B is non-abelian. Analogously we have that A is non-abelian.

By the Dedekind law [DH92, I, 1.3], we have $A(AB \cap U) = AB = B(AB \cap U)$. Hence $A \cong A/(A \cap B) \cong AB/B \cong B(AB \cap U)/B = AB \cap U$. Analogously $B \cong AB \cap U$. □

Baer's theorem enables us to classify the primitive groups as three different types.

Definition 1.1.8. *A primitive group G is said to be*

1. a primitive group of type 1 *if G has an abelian minimal normal subgroup,*
2. a primitive group of type 2 *if G has a unique non-abelian minimal normal subgroup,*
3. a primitive group of type 3 *if G has two distinct non-abelian minimal normal subgroups.*

We say that G is a monolithic primitive group *if G is a primitive group of type 1 or 2.*

Definition 1.1.9. *Let U be a maximal subgroup of a group G. Then U/U_G is a core-free maximal subgroup of the quotient group G/U_G. Then U is said to be*

1. a maximal subgroup of type 1 *if G/U_G is a primitive group of type 1,*
2. a maximal subgroup of type 2 *if G/U_G is a primitive group of type 2,*
3. a maximal subgroup of type 3 *if G/U_G is a primitive group of type 3.*

We say that U is a monolithic maximal subgroup *if G/U_G is a monolithic primitive group.*

Obviously all primitive soluble groups are of type 1. For these groups, there exists a well-known description called Galois' theorem. The proof appears in Huppert's book [Hup67, II, 3.2 and 3.3].

Theorem 1.1.10. *1. (Galois) If G is a soluble primitive group, then all core-free maximal subgroups are conjugate.*

2. If N is a self-centralising minimal normal subgroup of a soluble group G, then G is primitive, N is complemented in G, and all complements are conjugate.

Remarks 1.1.11. 1. The statement of Theorem 1.1.10 (1) is also valid if G is p-soluble for all primes dividing the order of $\mathrm{Soc}(G)$.

2. If G is a primitive group of type 1, then its minimal normal subgroup N is an elementary abelian p-subgroup for some prime p. Hence, N is a vector space over the field $\mathrm{GF}(p)$. Put $\dim N = n$, i.e. $|N| = p^n$. If M is a core-free subgroup of G, then M is isomorphic to a subgroup of $\mathrm{Aut}(N) = \mathrm{GL}(n, p)$. Therefore G can be embedded in the affine group $\mathrm{AGL}(n, p) = [C_p^n]\,\mathrm{GL}(n, p)$ in such a way that N is the translation group and $G \cap \mathrm{GL}(n, p)$ acts irreducibly on N. Thus, clearly, primitive groups of type 1 are not always soluble.

3. In his book B. Huppert shows that the affine group $\mathrm{AGL}(3, 2) = [C_2 \times C_2 \times C_2]\,\mathrm{GL}(3, 2)$ is an example of a primitive group of type 1 with non-conjugate core-free maximal subgroups (see [Hup67, page 161]).

4. Let G be a primitive group of type 2. If N is the minimal normal subgroup of G, then N is a direct product of copies of some non-abelian simple group and, in particular, the order of N has more than two prime divisors. If p is a prime dividing the order of N and $P \in \mathrm{Syl}_p(N)$, then $G = \mathrm{N}_G(P)N$ by the Frattini argument. Since P is a proper subgroup of N, then $\mathrm{N}_G(P)$ is a proper subgroup of G. If U is a maximal subgroup of G such that $\mathrm{N}_G(P) \leq U$, then necessarily U is core-free. Observe that if $P_0 \in \mathrm{Syl}_p(G)$ such that $P \leq P_0$, then $P = P_0 \cap N$ is normal in P_0 and so $P_0 \leq U$. In other words, U has p'-index in G. This argument can be done for each prime dividing $|N|$. Hence, the set of all core-free maximal subgroups of a primitive group of type 2 is not a conjugacy class.

5. In non-soluble groups, part 2 of Theorem 1.1.10 does not hold in general. Let G be a non-abelian simple group, p a prime dividing $|G|$ and $P \in \mathrm{Syl}_p(G)$. Suppose that P is cyclic. Let $G_{\Phi,p}$ be the maximal Frattini extension of G with p-elementary abelian kernel $A = \mathrm{A}_p(G)$ (see [DH 92; Appendix β] for details of this construction). Write $J = \mathrm{J}(KG)$ for the Jacobson radical of the group algebra KG of G, over the field $K = \mathrm{GF}(p)$. Then the section $N = A/AJ$ is irreducible and $\mathrm{C}_G(N) = \mathrm{O}_{p',p}(G) = 1$. Consequently $G_{\Phi,p}/AJ$ is a group with a unique minimal normal subgroup, isomorphic to N, self-centralising and non-supplemented.

In primitive groups of type 1 or 3, the core-free maximal subgroups complement each minimal subgroup. This characterises these types of primitive groups. In case of primitive groups of type 2 we will see later that the minimal normal subgroup could be complemented by some core-free maximal subgroup in some cases; but even then, there are always core-free maximal subgroups supplementing and not complementing the socle.

Proposition 1.1.12 ([Laf84a]). *For a group G, the following are pairwise equivalent:*

1. *G is a primitive group of type 1 or 3;*
2. *there exists a minimal normal subgroup N of G complemented by a subgroup M which also complements $\mathrm{C}_G(N)$;*
3. *there exists a minimal normal subgroup N of G such that G is isomorphic to the semidirect product $X = [N]\big(G/\mathrm{C}_G(N)\big)$.*

Proof. Clearly 1 implies 2. For 2 implies 1 observe that, since $N \cap M_G = 1$, then $M_G \leq \mathrm{C}_G(N)$. But, since also $M_G \cap \mathrm{C}_G(N) = 1$, we have that $M_G = 1$. Suppose that S is a proper subgroup of G such that $M \leq S$. Then the subgroup $S \cap N$ is normal in S and is centralised by $\mathrm{C}_G(N)$. Hence $S \cap N$ is normal in $S\,\mathrm{C}_G(N) = G$. By minimality of N, we have that $S \cap N = 1$ and then $S = M$. Then M is a core-free maximal subgroup of G and the group G is primitive. Observe that the minimal normal subgroup of a primitive group of type 2 has trivial centraliser.

2 implies 3. Observe that $G = NM$, with $N \cap M = 1$, and $M \cong G/\mathrm{C}_G(N)$. The map $\alpha\colon G \longrightarrow [N]\big(G/\mathrm{C}_G(N)\big)$ given by $(nm)^\alpha = \big(n, m\,\mathrm{C}_G(N)\big)$ is the desired isomorphism.

3 implies 2. Write $C = \mathrm{C}_G(N)$. Assume that there exists an isomorphism

$$\alpha\colon [N](G/C) \longrightarrow G$$

and consider the following subgroups $N^* = \big(\{(n, C) : n \in N\}\big)^\alpha$, $M^* = \big(\{(1, gC) : g \in G\}\big)^\alpha$, and $C^* = \big(\{(n, gC) : ng \in C\}\big)^\alpha$. For each $n \in N$, the element $(n^{-1}, nC)^\alpha$ is a non-trivial element of C^*. Hence $C^* \neq 1$. It is an easy calculation to show that N^* is a minimal normal subgroup of G, $C^* = \mathrm{C}_G(N^*)$ and M^* complements N^* and C^*. ☐

Corollary 1.1.13. *The following conditions for a group G are equivalent:*

1. *G is a primitive group of type 3.*
2. *The group G possesses two distinct minimal normal subgroups N_1, N_2, such that*
   a) *N_1 and N_2 have a common complement in G;*
   b) *the quotient groups G/N_i, for $i = 1$, 2, are primitive groups of type 2.*

Proof. 1 implies 2. By Theorem 1, if G is a primitive group of type 3, then G possesses two distinct minimal normal subgroups N_1, N_2 which have a common complement M in G. Observe that $M \cong G/N_1$ and N_2N_1/N_1 is a minimal normal subgroup of G/N_1. If $gN_1 \in \mathrm{C}_{G/N_1}(N_2N_1/N_1)$, then $[n, g] \in N_1$, for all $n \in N_2$. But then $[n, g] \in N_1 \cap N_2 = 1$, and therefore $g \in \mathrm{C}_G(N_2) = N_1$. Hence

$C_{G/N_1}(N_2N_1/N_1) = 1$. Consequently G/N_1 is a primitive group of type 2 and therefore so are M and G/N_2.

2 implies 1. Let M be a common complement of N_1 and N_2. Then $G/N_i \cong M$ is a primitive group of type 2 such that $\mathrm{Soc}(G/N_i) = N_1N_2/N_i$ and $C_G(N_1N_2/N_i) = N_i$. Therefore $C_G(N_2) = N_1$ and $C_G(N_1) = N_2$. By Proposition 1.1.12, this means that G is a primitive group of type 3. □

Proposition 1.1.14 ([Laf84a]). *For a group G, the following statements are pairwise equivalent.*

1. *G is a primitive group of type 2.*
2. *G possesses a minimal normal subgroup N such that $C_G(N) = 1$.*
3. *There exists a primitive group X of type 3 such that $G \cong X/A$ for a minimal normal subgroup A of X.*

Proof. 3 implies 2 is Corollary 1.1.13 and 2 implies 1 is the characterisation of primitive groups of type 2 in Theorem 1. Thus it only remains to prove that 1 implies 3. If G is a primitive group of type 2 and N is the unique minimal normal subgroup of G, then N is non-abelian and $C_G(N) = 1$. By Proposition 1.1.12, the semidirect product $X = [N]G$ is a primitive group of type 3. Clearly if $A = \{(n, 1) : n \in N\}$, then $X/A \cong G$. □

Consequently, if M is a core-free maximal subgroup of a primitive group G of type 3, then M is a primitive group of type 2 and $\mathrm{Soc}(M)$ is isomorphic to a minimal normal subgroup of G.

According to Baer's Theorem, the socle of a primitive group of type 2 is a non-abelian minimal normal subgroup and therefore is a direct product of copies of a non-abelian simple group (see [Hup67, I, 9.12]). Obviously, the simplest examples of primitive groups of type 2 are the non-abelian simple groups. Observe that if S is a non-abelian simple group, then $Z(S) = 1$ and we can identify S and the group of inner automorphisms $\mathrm{Inn}(S)$ and write $S \le \mathrm{Aut}(S)$. Since $C_{\mathrm{Aut}(S)}(S) = 1$, any group G such that $S \le G \le \mathrm{Aut}(S)$ is a primitive group of type 2 such that $\mathrm{Soc}(G)$ is a non-abelian simple group. Conversely, if G is a primitive group of type 2 and $S = \mathrm{Soc}(G)$ is a simple group, then, since $C_G(S) = 1$, we can embed G in $\mathrm{Aut}(S)$.

Definition 1.1.15. *An* almost simple *group G is a subgroup of* $\mathrm{Aut}(S)$ *for some simple group S, such that $S \le G$.*

If G is an almost simple group and $S \le G \le \mathrm{Aut}(S)$, for a non-abelian simple group S, then $C_G(S) = 1$. Hence G possesses a unique minimal normal subgroup S and every maximal subgroup U of G such that $S \not\le U$ is core-free in G.

Proposition 1.1.16. *Suppose that S is a non-abelian simple group and let G be an almost simple group such that $S \le G \le \mathrm{Aut}(S)$. If U is a core-free maximal subgroup of G, then $U \cap S \ne 1$.*

Proof. Recall Schreier's conjecture ([KS04, page 151]) which states that the group of outer automorphisms $\mathrm{Out}(S) = \mathrm{Aut}(S)/\mathrm{Inn}(S)$ of a non-abelian simple group S is always soluble. The classification of simple groups has allowed us to check that this conjecture is true.

Suppose that $U \cap S = 1$. We know that $U \cong US/S \leq \mathrm{Aut}(S)/\mathrm{Inn}(S)$ and, by Schreier's conjecture ([KS04, page 151]) we deduce that U is soluble. Let Q be a minimal normal subgroup of U. Then Q is an elementary abelian q-group for some prime q. Observe that $\mathrm{C}_G(Q)$ is normalised by U. Therefore $\mathrm{C}_S(Q)$ is normalised by U and then $U\,\mathrm{C}_S(Q)$ is a subgroup of G. Since U is maximal in G and $\mathrm{C}_G(S) = 1$, then $\mathrm{C}_S(Q) = 1$. The q-group Q acts fixed-point-freely on S and then S is a q'-group. By the Odd Order Theorem ([FT63]), we have that $q \neq 2$. Now Q acts by conjugation on the elements of the set $\mathrm{Syl}_2(S)$ and by the Orbit-Stabiliser Theorem ([DH92, A, 5.2]) we deduce that Q normalises some $P \in \mathrm{Syl}_2(S)$. If P and $P^{x^{-1}}$, for $x \in S$, are two Sylow 2-subgroups of S which are normalised by Q, then Q, $Q^x \in \mathrm{Syl}_q\big(\mathrm{N}_{QS}(P)\big)$ and there exists an element $g \in \mathrm{N}_{QS}(P)$, such that $Q^g = Q^x$. Write $g = yz$, with $y \in Q$ and $z \in S$. Then $Q^x = Q^z$ with $z \in \mathrm{N}_S(P)$. Hence $[Q, xz^{-1}] \leq Q \cap S = 1$ and $xz^{-1} \in \mathrm{C}_S(Q) = 1$. Therefore $x = z \in \mathrm{N}_S(P)$ and we conclude that Q normalises exactly one Sylow 2-subgroup P of S. Hence $\mathrm{N}_G(Q) \leq \mathrm{N}_G(P)$. But $U = \mathrm{N}_G(Q)$, by maximality of U. The subgroup UP is a proper subgroup of G which contains properly the maximal subgroup U. This is a contradiction. Hence $U \cap S \neq 1$. □

For our purposes, it will be necessary to embed the primitive group G in a larger group. Suppose that $\mathrm{Soc}(G) = S_1 \times \cdots \times S_n$, where the S_i are copies of a non-abelian simple group S, i.e. $\mathrm{Soc}(G) \cong S^n$, the direct product of n copies of S. Since $\mathrm{C}_G\big(\mathrm{Soc}(G)\big) = 1$, the group G can be embedded in $\mathrm{Aut}(S^n)$. The automorphism group of a direct product of copies of a non-abelian simple group has a well-known structure: it is a *wreath product*.

Thus, the study of some relevant types of subgroups of groups which are wreath products and the analysis of some special types of subgroups of a direct product of isomorphic non-abelian simple groups will be essential.

Definition 1.1.17. *Let X and H be two groups and suppose that H has a permutation representation φ on a finite set $\mathcal{I} = \{1, \dots, n\}$ of n elements. The* wreath product *$X \wr_\varphi H$ (or simply $X \wr H$ if the action is well-known) is the semidirect product $[X^\natural]H$, where $X^\natural$ is the direct product of n copies of X: $X^\natural = X_1 \times \cdots \times X_n$, with $X_i = X$ for all $i \in \mathcal{I}$, and the action is*

$$(x_1, \dots, x_n)^h = (x_{1^{(h^{-1})\varphi}}, \dots, x_{n^{(h^{-1})\varphi}}) \tag{1.1}$$

for $h \in H$ and $x_i \in X$, for all $i \in \mathcal{I}$.

The subgroup $X^\natural$ is called the base group *of $X \wr H$.*

Remarks 1.1.18. Consider a wreath product $G = X \wr_\varphi H$.

1. If φ is faithful, then $\mathrm{C}_G(X^\natural) \leq X^\natural$.

2. For any $g \in G$, then $g = xh$, with $x \in X^\natural$ and $h \in H$. For each $i = 1, \dots, n$, we have that $X_i^g = X_i^h = X_{i^{h^\varphi}}$.

3. Thus, the group G acts on $\mathcal{I}$ by the following rule: if $i \in \mathcal{I}$, for any $g = xh \in G$, with $x \in X^\natural$ and $h \in H$, then $i^g = i^{h^\varphi}$. In particular $i^h = i^{h^\varphi}$, if $h \in H$.

4. If $\mathcal{S} \subseteq \mathcal{I}$, then write

$$\pi_{\mathcal{S}} : X^\natural \longrightarrow \prod_{j \in \mathcal{S}} X_j$$

for the projection of $X^\natural$ onto $\prod_{j \in \mathcal{S}} X_j$. Then for any $y \in X^\natural$ and any $g \in G$, we have that

$$(y^g)^{\pi_{\mathcal{S}^g}} = (y^{\pi_{\mathcal{S}}})^g.$$

Proposition 1.1.19. *Let S be a non-abelian simple group and write $S^n = S_1 \times \cdots \times S_n$ for the direct product of n copies $S_1, \dots, S_n$ of S, for some positive integer n. Then the minimal normal subgroups of S^n are exactly the S_i, for any $i = 1, \dots, n$,*

Proof. Let N be a minimal normal subgroup of S^n. Suppose that $N \cap S_i = 1$ for all $i = 1, \dots, n$. Then N centralises all S_i and hence $N \leq \mathrm{Z}(S^n) = 1$. This is a contradiction. Therefore $N \cap S_i = N$ for some index i. Then $N = S_i$. □

Proposition 1.1.20. *Let S be a non-abelian simple group and write $S^n = S_1 \times \cdots \times S_n$ for the direct product of n copies $S_1, \dots, S_n$ of S, for some positive integer n. Then $\mathrm{Aut}(S^n) \cong \mathrm{Aut}(S) \wr \mathrm{Sym}(n)$, where $\mathrm{Sym}(n)$ is the symmetric group of degree n.*

Proof. If σ is a permutation in $\mathrm{Sym}(n)$, the map α_σ defined by

$$(x_1, \dots, x_n)^{\alpha_\sigma} = (x_{1^{\sigma^{-1}}}, \dots, x_{n^{\sigma^{-1}}})$$

is an element of $\mathrm{Aut}\,(S^n)$ associated with σ. Now $H = \{\alpha_\sigma \in \mathrm{Aut}\,(S^n)\colon \sigma \in \mathrm{Sym}(n)\}$ is a subgroup of $\mathrm{Aut}(S^n)$ and $\sigma \longmapsto \alpha_\sigma$ defines an isomorphism between $\mathrm{Sym}(n)$ and H. By Proposition 1.1.19, the minimal normal subgroups of the direct product $S_1 \times \cdots \times S_n$ are exactly the $S_1, \dots, S_n$. Therefore, if $\gamma \in \mathrm{Aut}(S^n)$, then there exists a $\sigma \in \mathrm{Sym}(n)$ such that $S_i^\gamma = S_{i^\sigma} = S_i^{\alpha_\sigma}$, for all $i = 1, \dots, n$.

Let D be the subgroup of all elements β in $\mathrm{Aut}(S^n)$ such that $S_i^\beta = S_i$ for all i. The maps $\beta_1, \dots, \beta_n$ defined by $(x_1, \dots, x_n)^\beta = (x_1^{\beta_1}, \dots, x_n^{\beta_n})$ are automorphisms of S and the map $\beta \mapsto (\beta_1, \dots, \beta_n)$ defines an isomorphism between D and $\mathrm{Aut}(S)^n$. Moreover, by Proposition 1.1.19 again, if $\beta \in D$ and $\gamma \in \mathrm{Aut}(S^n)$, then $(S_i^\gamma)^\beta = S_i^\gamma$. This means that D is a normal subgroup of $\mathrm{Aut}(S^n)$.

Observe that $\alpha_\sigma \in D$ if and only if $\sigma = 1$, or, in other words, $D \cap H = 1$. Moreover for all $\gamma \in \mathrm{Aut}(S^n)$, we have that $\gamma\alpha_\sigma^{-1} \in D$. Therefore $\mathrm{Aut}(S^n) = [D]H$. This allows us to define a bijective map between $\mathrm{Aut}(S^n)$ and $\mathrm{Aut}(S) \wr \mathrm{Sym}(n)$ which is an isomorphism. □

F. Gross and L. G. Kovács published in [GK84] a construction of groups, the so-called *induced extensions*, which is crucial to understand the structure of a, non-necessarily finite, group that possesses a normal subgroup which is a direct product of copies of a group. It is clear that primitive groups of type 2 are examples of this situation. We present in the sequel an adaptation of this construction to finite groups.

Proposition 1.1.21. *Consider the following diagram of groups and group homomorphisms:*

$$\begin{array}{ccc} & & Z \\ & & \big\downarrow g \\ X & \xrightarrow{\ f\ } & Y \end{array} \tag{1.2}$$

where g is a monomorphism. Let G be the following subset of X:

$$G = \{x \in X : x^f = z^g \text{ for some } z \in Z\},$$

and the following mapping

$$h\colon G \longrightarrow Z \quad x^h = x^{fg^{-1}} \text{ for every } x \in G.$$

Then G is a subgroup of X and h is a well-defined group homomorphism such that the following diagram of groups and group homomorphisms is commutative:

$$\begin{array}{ccc} G & \xrightarrow{\ h\ } & Z \\ \big\downarrow \iota & & \big\downarrow g \\ X & \xrightarrow{\ f\ } & Y \end{array}$$

(where ι is the canonical inclusion of G in X). Moreover $\mathrm{Ker}(h^\iota) = \mathrm{Ker}(f)$.

Further, if (G_0, ι_0, h_0) is a triple, with G_0 a group, $\iota_0\colon G_0 \longrightarrow X$ a monomorphism and $h_0\colon G_0 \longrightarrow Z$ is a group homomorphism, such that the diagram

$$\begin{array}{ccc} G_0 & \xrightarrow{\ h_0\ } & Z \\ \big\downarrow \iota_0 & & \big\downarrow g \\ X & \xrightarrow{\ f\ } & Y \end{array}$$

is commutative, then there exists a monomorphism $\Phi\colon G_0 \longrightarrow G$, such that $\Phi h = h_0$, $\Phi\iota = \iota_0$ and $\big(\mathrm{Ker}(h_0)\big)^\Phi \leq \big(\mathrm{Ker}(h)\big)^\iota = \mathrm{Ker}(f)$.

Proof. It is an easy exercise to prove that G is a subgroup of X and, since g is a monomorphism, the mapping h is a well-defined group homomorphism. It is not difficult to see that $\mathrm{Ker}(h)^\iota = \mathrm{Ker}(f)$.

For the second statement, let $x \in G_0$ and observe that x^{h_0} is an element of Z such that $(x^{h_0})^g = (x^{\iota_0})^f$, and then $x^{\iota_0} \in G$ and $(x^{\iota_0})^h = x^{h_0}$. Write $\Phi\colon G_0 \longrightarrow G$ such that $x^\Phi = x^{\iota_0}$. □

Definition 1.1.22. *The triple* (G, ι, h) *introduced in Proposition 1.1.21 is said to be the* pull-back *of the diagram* (1.2).

Proposition 1.1.23. *Consider the following extension of groups:*

$$1 \longrightarrow K \longrightarrow X \xrightarrow{f} Y \longrightarrow 1$$

and a monomorphism $g\colon Z \longrightarrow Y$. *Consider the triple* (G, ι, h), *the pull-back of the diagram* (1.2).

1. *There exists an extension*

$$Eg\colon 1 \longrightarrow K \longrightarrow G \xrightarrow{h} Z \longrightarrow 1$$

such that the following diagram of groups and group homomorphisms is commutative:

$$\begin{array}{ccccccccc} Eg\colon 1 & \longrightarrow & K & \longrightarrow & G & \xrightarrow{h} & Z & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle g} & & \\ E\colon 1 & \longrightarrow & K & \longrightarrow & X & \xrightarrow{f} & Y & \longrightarrow & 1 \end{array}$$

2. *Moreover, if*

$$E_0\colon 1 \longrightarrow K \longrightarrow G_0 \xrightarrow{h_0} Z \longrightarrow 1$$

is another extension such that the diagram

$$\begin{array}{ccccccccc} E_0\colon 1 & \longrightarrow & K & \longrightarrow & G_0 & \xrightarrow{h_0} & Z & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \iota_0} & & \downarrow{\scriptstyle g} & & \\ E\colon 1 & \longrightarrow & K & \longrightarrow & X & \xrightarrow{f} & Y & \longrightarrow & 1 \end{array}$$

is commutative, there exists a group isomorphism $\Phi\colon G_0 \longrightarrow G$ *such that* $\Phi h = h_0$, $\Phi\iota = \iota_0$ *and* $\Phi|_K = \mathrm{id}_K$.

Proof. The proof of 1 is a direct exercise. To see 2, first notice that, by the Short Five Lemma ([Hun80, IV, 1.17]), the homomorphism ι_0 is a monomorphism. By Proposition 1.1.21, there exists a group monomorphism $\Phi\colon G_0 \longrightarrow G$ such that $\Phi h = h_0$, $\Phi\iota = \iota_0$ and $\Phi|_K = \mathrm{id}_K$. Furthermore, since $|G| = |Z|/|K| = |G_0|$, we have that Φ is an isomorphism. □

Definition 1.1.24. *The extension* Eg *is said to be the* pull-back extension of the extension E and the monomorphism g.

Hypotheses 1.1.25. *Let* B *be a group. Assume that* C *a subgroup of a group* B *such that* $|B : C| = n$ *and let* $\mathcal{T} = \{t_1 = 1, \dots, t_n\}$ *be a right transversal* *of C in B. Then B, acting by right multiplication on the set of right cosets of C in B, induces a transitive action $\rho\colon B \longrightarrow \mathrm{Sym}(n)$ on the set of indices $\mathcal{I} = \{1, \dots, n\}$ in the following way. For each $i \in \mathcal{I}$ and each $h \in B$, the element $t_i h$ belongs to some coset Ct_j, i.e. $t_i h = c_{i,h} t_j$, for some $c_{i,h} \in C$. Then $i^{h^\rho} = j$. Write $P = B^\rho \leq \mathrm{Sym}(n)$.*

Let $\alpha\colon A \longrightarrow B$ be a group homomorphism and write $C = A^\alpha$ and $S =$ $\mathrm{Ker}(\alpha)$. Write $W = A \wr_\rho P$. There exists an induced epimorphism $\bar{\alpha}\colon A \wr_\rho P \longrightarrow C \wr_\rho P$ defined by $\big((a_1, \dots, a_n)x\big)^{\bar{\alpha}} = (a_1^\alpha, \dots, a_n^\alpha)x$, for $a_1, \dots, a_n \in A$ and $x \in P$. Write $M = \mathrm{Ker}(\bar{\alpha})$. Observe that $(a_1, \dots, a_n)x \in M$ if and only if $a_j^\alpha = 1$, for all $j \in \mathcal{I}$ and $x = 1$. This is to say that $M = \mathrm{Ker}(\bar{\alpha}) =$ $\mathrm{Ker}(\alpha) \times \dots \times \mathrm{Ker}(\alpha) = S_1 \times \dots \times S_n$. We have the exact sequence:

$$E\colon 1 \longrightarrow M \longrightarrow A \wr_\rho P \xrightarrow{\ \bar{\alpha}\ } C \wr_\rho P \longrightarrow 1$$

Lemma 1.1.26. *Assume the hypotheses and notation of Hypotheses 1.1.25.*

1. *The mapping $\lambda = \lambda_{\mathcal{T}}\colon B \longrightarrow C \wr_\rho P$ such that $h^\lambda = (c_{1,h}, \dots, c_{n,h})h^\rho$, for any $h \in B$, is a group monomorphism.*
2. *Consider the pull-back exact sequence $E\lambda$:*

$$\begin{array}{ccccccccc}
E\lambda\colon 1 & \longrightarrow & M & \longrightarrow & G & \xrightarrow{\ \sigma\ } & B & \longrightarrow & 1 \\
 & & \big\downarrow \mathrm{id} & & \big\downarrow & & \big\downarrow \lambda & & \\
E\colon 1 & \longrightarrow & M & \longrightarrow & A \wr_\rho P & \xrightarrow{\ \bar{\alpha}\ } & C \wr_\rho P & \longrightarrow & 1
\end{array}$$

Then, the isomorphism class of the group G is independent from the choice of transversal of C in B.

Proof. 1. Let h, $h' \in B$. Observe that

$$c_{i,hh'} t_{i^{(hh')^\rho}} = t_i hh' = c_{i,h} t_{i^{h^\rho}} h' = c_{i,h} c_{i^{h^\rho},h'} t_{i^{(hh')^\rho}}.$$

Hence, by (1.1) in Definition 1.1.17, we have that

$$\begin{aligned}
h^\lambda h'^\lambda &= (c_{1,h}, \dots, c_{n,h})h^\rho (c_{1,h'}, \dots, c_{n,h'})h'^\rho \\
&= (c_{1,h}, \dots, c_{n,h})(c_{1,h'}, \dots, c_{n,h'})^{(h^\rho)^{-1}} (hh')^\rho \\
&= (c_{1,h}, \dots, c_{n,h})(c_{1^{h^\rho},h'}, \dots, c_{n^{h^\rho},h'})(hh')^\rho \\
&= (c_{1,hh'}, \dots, c_{n,hh'})(hh')^\rho = (hh')^\lambda
\end{aligned}$$

and λ is a group homomorphism.

Suppose that $h^\lambda = h'^\lambda$. Then $(c_{1,h}, \dots, c_{n,h})h^\rho = (c_{1,h'}, \dots, c_{n,h'})h'^\rho$ and therefore, since $C \wr_\rho P_n = [C^n]P_n$ is a semidirect product, we have that

$$c_{j,h} = c_{j,h'} = c_j, \quad j \in \mathcal{I}; \qquad h^\rho = h'^\rho = \tau.$$

Therefore, for any index $j \in \mathcal{I}$, we have that $t_j h = c_j t_{j^\tau} = t_j h'$ and then $h = t_j^{-1} c_j t_{j^\tau} = h'$. Hence λ is a group monomorphism.

2. Let $\mathcal{T}' = \{t'_1, \dots, t'_n\}$ be some other right transversal of C in B such that $Ct'_i = Ct_i$, for each $i \in \mathcal{I}$: there exist elements $b_1, \dots, b_n \in C$ such that $t'_i = b_i t_i$, for $i = 1, \dots, n$. For each $i \in \mathcal{I}$ and each $h \in B$, the element $t'_i h$ belongs to the coset $Ct'_j = Ct_j$, for $i^{h^\rho} = j$, and $t'_i h = c'_{i,h} t'_j$, for some $c'_{i,h} \in C$. Then

$$t'_i h = b_i t_i h = b_i c_{i,h} t_j = c'_{i,h} t'_j = c'_{i,h} b_j t_j \quad \text{and} \quad c_{i,h} = b_i^{-1} c'_{i,h} b_j$$

and it appears the element $(b_1, \dots, b_n) \in C^\natural$ associated with $\mathcal{T}'$. Then, for $\lambda' = \lambda_{\mathcal{T}'}$, we have that

$$\begin{aligned} h^\lambda &= (c_{1,h}, \dots, c_{n,h}) h^\rho = \big((b_1, \dots, b_n)^{-1} (c'_{1,h}, \dots, c'_{n,h})(b_{1^{h^\rho}}, \dots, b_{n^{h^\rho}})\big) h^\rho \\ &= \big((b_1, \dots, b_n)^{-1} (c'_{1,h}, \dots, c'_{n,h})(b_1, \dots, b_n)^{(h^{-1})^\rho}\big) h^\rho \\ &= (b_1, \dots, b_n)^{-1} (c'_{1,h}, \dots, c'_{n,h}) h^\rho (b_1, \dots, b_n) \\ &= \big((c'_{1,h}, \dots, c'_{n,h}) h^\rho\big)^{(b_1, \dots, b_n)} = (h^{\lambda'})^{(b_1, \dots, b_n)}, \end{aligned}$$

for any $h \in B$, and then $\big(\mathrm{Im}(\lambda')\big)^{(b_1, \dots, b_n)} = \mathrm{Im}(\lambda)$. For each $i \in \mathcal{I}$, let a_i be an element of A such that $a_i^\alpha = b_i$. This is to say that $(a_1, \dots, a_n)^{\bar{\alpha}} = (b_1, \dots, b_n)$. If $x \in G$, then

$$(x^{(a_1, \dots, a_n)})^{\bar{\alpha}} = (x^{\bar{\alpha}})^{(b_1, \dots, b_n)} = (h^\lambda)^{(b_1, \dots, b_n)} = h^{\lambda'}$$

and then $x^{(a_1, \dots, a_n)} \in G^* = \{w \in W : \ w^{\bar{\alpha}} = h^{\lambda'} \text{ for some } h \in B\}$, which is the pull-back defined with the monomorphism λ':

$$\begin{array}{rccccccccc} E\lambda'\colon 1 & \longrightarrow & M & \longrightarrow & G^* & \xrightarrow{\ \sigma'\ } & B & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow{\scriptstyle \lambda'} & & \\ E\colon 1 & \longrightarrow & M & \longrightarrow & A \wr_\rho P & \xrightarrow{\ \bar{\alpha}\ } & C \wr_\rho P & \longrightarrow & 1 \end{array}$$

Thus, $G^* = G^a$ for some $a \in A^\natural$ associated with the transversals $\mathcal{T}$ and $\mathcal{T}'$, i.e. the pull-back groups constructed from two different transversals are conjugate in W. In other words, the isomorphism class of the group G is independent from the choice of transversal. □

Definition 1.1.27 ([GK84]). *In the above situation and with that notation, we will say that $E\lambda$ is the* induced extension defined by *$\alpha\colon A \longrightarrow B$.*

Recall that G is a subgroup of $W = A \wr_\rho P$ defined by:

$$G = \{x \in W : x^{\bar{\alpha}} = h^\lambda, \text{ for some } h \in B\}$$

and σ is defined by $\sigma = \bar{\alpha}|_G \lambda^{-1}$.

Proposition 1.1.28. *With the notation introduced above, we have the following.*

1. $\mathrm{N}_G(A_1) = \mathrm{N}_G(S_1) = \mathrm{N}_G(S_2 \times \cdots \times S_n) = N = \{x \in W : x^{\bar{\alpha}} = h^\lambda$, *for some* $h \in C\}$.
2. $N/(S_2 \times \cdots \times S_n) \cong A$. *Moreover, the image of* $M/(S_2 \times \cdots \times S_n)$ *under this isomorphism is* $S = \mathrm{Ker}(\alpha)$.
3. *In particular* $N^\sigma = C$ *and* $|G : N| = |B : C| = n$. *Thus, if* $\rho' \colon G \longrightarrow \mathrm{Sym}(n)$ *is the action of* G *on the right cosets of* N *in* G *by multiplication, then* $\rho' = \sigma\rho$.
4. *The set* $\{S_1, \dots, S_n\}$ *is the conjugacy class of the subgroup* S_1 *in* G.

Proof. 1. We can consider the subgroup

$$N = \{w \in W : w^{\bar{\alpha}} = h^\lambda, \text{ for some } h \in C\}.$$

Observe that if $(a_1, \dots, a_n)x \in N$, for $a_i \in A$ and $x \in P$, then there exists $h \in C$, such that

$$h^\lambda = (c_{1,h}, \dots, c_{n,h})h^\rho = (a_1^\alpha, \dots, a_n^\alpha)x.$$

Since $h \in C$, it is clear that $c_{1,h} = h$ and h^ρ belongs to the stabiliser P_1 of 1. In other words

$$N \le A_1 \times (A_2 \times \cdots \times A_n)P_1 = \mathrm{N}_W(A_1) = \mathrm{N}_W(S_1) = \mathrm{N}_W(S_2 \times \cdots \times S_n)$$

and hence $N \le \mathrm{N}_G(A_1)$. Conversely, if $(a_1, \dots, a_n)x \in \mathrm{N}_G(A_1)$, then $x \in P_1$ and there exists $h \in B$ such that $a_i^\alpha = c_{i,h}$ and $x = h^\rho \in P_1$, i.e. $1^{h^\rho} = 1$. Hence $h = t_1 h = c_{1,h} t_1 = c_{1,h} = a_1^\alpha \in C$. Then $\mathrm{N}_G(A_1) \le N$. Hence $N = \mathrm{N}_G(A_1) = \mathrm{N}_G(S_1)$.

2. Consider the projection $e_1 \colon A_1 \times (A_2 \times \cdots \times A_n)P_1 = \mathrm{N}_W(A_1) \longrightarrow A$ on the first component. Obviously, $\mathrm{Ker}(e_1) = (A_2 \times \cdots \times A_n)P_1$.

Let e be the restriction to N of the projection e_1:

$$e = e_1|_N \colon N \longrightarrow A.$$

Observe that if $x \in N$, then $x^{\bar{\alpha}} = c^\lambda$ for some $c \in C$. We can characterise this $c = x^\sigma$ in the following way. Assume that $x = (a_1, \dots, a_n)y$. Then $x^{\bar{\alpha}} = (a_1^\alpha, \dots, a_n^\alpha)y = c^\lambda = (c, c_{2,c}, \dots, c_{n,c})c^\rho$. Hence $c = a_1^\alpha = x^{e\alpha}$.

We have that $\mathrm{Ker}(e) = \mathrm{Ker}(e_1) \cap N$. If $x \in \mathrm{Ker}(e)$, then $x^{\bar{\alpha}} = (x^{e\alpha})^\lambda = 1$. Thus $x \in \mathrm{Ker}(\bar{\alpha}) = M$ and then $\mathrm{Ker}(e) \le M$. Therefore $\mathrm{Ker}(e) = \mathrm{Ker}(e_1) \cap M = (A_2 \times \cdots \times A_n)P_1 \cap M = S_2 \times \cdots \times S_n$.

For any $a \in A$, consider the element $c = a^\alpha \in C$. Then $c^\rho \in P_1$ and $c_{i,c} = t_i c t_j^{-1} \in C$, where $j = i^{c^\rho}$, for $i = 2, \dots, n$. Since $C = A^\alpha$, there exist elements $a_2, \dots, a_n$ in A such that $a_j^\alpha = c_{j,c}$, for $j = 2, \dots, n$. The element $x = (a, a_2, \dots, a_n)c^\rho \in N$, since $x^{\bar{\alpha}} = (a^\alpha, a_2^\alpha, \dots, a_n^\alpha)c^\rho = (c, c_{2,c}, \dots, c_{n,c})c^\rho = c^\lambda$. Now $x^e = a$, and then e is an epimorphism.

Hence

$$N/\mathrm{Ker}(e) = N/(S_2 \times \cdots \times S_n) \cong A.$$

Finally observe that $M^e \cong M/\mathrm{Ker}(e|_M) = M/(S_2 \times \cdots \times S_n) \cong S$. Since $M^e \le S = \mathrm{Ker}(\alpha)$ and these two subgroups have the same order, equality holds.

3. Choose a right transversal of N in G, $\{g_1 = 1, \ldots, g_n\}$ such that $g_i^\sigma = t_i$. Then for each $g \in G$, we have that $g_i g = x_{i,g} g_{i^{g^{\rho'}}}$, for some $x_{i,g} \in N$. Then

$$c_{i,g^\sigma} t_{i^{g^{\sigma\rho}}} = t_i g^\sigma = g_i^\sigma g^\sigma = x_{i,g}^\sigma g_{i^{g^{\rho'}}}^\sigma = x_{i,g}^\sigma t_{i^{g^{\rho'}}}$$

and then $i^{g^{\sigma\rho}} = i^{g^{\rho'}}$, for every $i \in \mathcal{I}$. Therefore $g^{\sigma\rho} = g^{\rho'}$ for each $g \in G$, and then $\sigma\rho = \rho'$.

4. Observe that for each $i \in \mathcal{I}$, the permutation t_i^ρ moves 1 to i. Therefore, having in mind (1.1) of Definition 1.1.17, we see that $S_1^{g_i} = S_i$, and then $\{S_1, \ldots, S_n\}$ is the conjugacy class of the subgroup S_1 in G. □

We prove next that in fact the structure of the group G analysed in Proposition 1.1.28 characterises the induced extensions.

Theorem 1.1.29. *Let G be a group. Suppose that we have in G the following situation: there exist a normal subgroup M of G and a normal subgroup S of M such that $\{S_1, \ldots, S_n\}$ is the set of all conjugate subgroups of S in G and $M = S_1 \times \cdots \times S_n$. Write $N = \mathrm{N}_G(S_1)$ and $K = S_2 \times \cdots \times S_n$.*

Let $\alpha\colon N/K \longrightarrow G/M$ be defined by $(Kx)^\alpha = Mx$. Then G is the induced extension defined by α.

Proof. Let $\sigma\colon G \longrightarrow G/M$ and $e\colon N \longrightarrow N/K$ be the natural epimorphisms. If $\mathcal{T} = \{t_1 = 1, \ldots, t_n\}$ is a right transversal of N in G, then $\mathcal{T}^\sigma$ is a right transversal of N/M in G/M. Consider $\rho\colon G/M \longrightarrow \mathrm{Sym}(n)$ the permutation representation of G/M on the right cosets of N/M in G/M. Then $\bar\rho = \sigma\rho$ is the permutation representation of G on the right cosets of N in G. Write $P = G^{\bar\rho} = (G/M)^\rho$. Let

$$\bar\lambda = \lambda_{\mathcal{T}}\colon G \longrightarrow N \wr_{\bar\rho} P$$

be the embedding of G into $N \wr_{\bar\rho} P$ defined in Lemma 1.1.26 and

$$\lambda = \lambda_{\mathcal{T}^\sigma}\colon G/M \longrightarrow (N/M) \wr_\rho P$$

be the embedding of G/M into $(N/M) \wr_\rho P$. As usual, for each $x \in G$, write $t_i x = c_{i,x} t_j$, for some $c_{i,x} \in N$, and $i^{x^{\bar\rho}} = j$. Observe that $c_{i,g^\sigma} = (c_{i,g})^\sigma$. Write $S_i = S^{t_i}$. For each $i \in \mathcal{I} = \{1, \ldots, n\}$, write also $K_i = \prod_{j \in \mathcal{I}\setminus\{i\}} S_j$. Then $K = K_1$ and $K_i = K^{t_i}$.

If we write $\bar\sigma\colon N \wr_{\bar\rho} P \longrightarrow (N/M) \wr_\rho P$ for the epimorphism induced by σ, then $\sigma\lambda = \bar\lambda\bar\sigma$. Consider

$$\bar e\colon N \wr P \longrightarrow (N/K) \wr P, \quad \text{induced by } e$$

and

$$\bar\alpha\colon (N/K) \wr P \longrightarrow (N/M) \wr P, \quad \text{induced by } \alpha.$$

Since $e\alpha = \sigma|_N$, we find that $\bar e\bar\alpha = \bar\sigma$. Therefore $\bar\lambda\bar e\bar\alpha = \bar\lambda\bar\sigma = \sigma\lambda$ and the following diagram is commutative:

$$\begin{array}{ccc} G & \xrightarrow{\ \sigma\ } & G/M \\ \big\downarrow{\scriptstyle \bar{\lambda}\bar{e}} & & \big\downarrow{\scriptstyle \lambda} \\ (N/K) \wr P & \xrightarrow{\ \bar{\alpha}\ } & (N/M) \wr P \end{array}$$

The commutativity of the diagram shows that $M^{\bar{\lambda}\bar{e}\bar{\alpha}} = M^{\sigma\lambda} = 1$ and then $M^{\bar{\lambda}\bar{e}} \leq \mathrm{Ker}(\bar{\alpha})$.

Consider an element $x \in G$ such that $x^{\bar{\lambda}} = (c_{1,x}, \dots, c_{n,x})x^{\bar{\rho}} \in G^{\bar{\lambda}} \cap \mathrm{Ker}(\bar{e})$. Then we have $1 = (Kc_{1,x}, \dots, Kc_{n,x})x^{\rho}$. This means that $x^{\rho} = \mathrm{id}$ and $c_{i,x} \in K$, for $i \in \mathcal{I}$. Therefore, $c_{i,x} = t_i x t_i^{-1}$, for $i \in \mathcal{I}$. Hence, $x \in \bigcap_{i=1}^{n} K^{t_i} = \bigcap_{i=1}^{n} K_i = 1$. Therefore $G^{\bar{\lambda}} \cap \mathrm{Ker}(\bar{e}) = 1$ and then $\bar{\lambda}\bar{e}$ is a monomorphism. Observe that $\mathrm{Ker}(\bar{\alpha}) = (M/K)^{\natural} = (M/K)_1 \times \cdots \times (M/K)_n$ and then $|\mathrm{Ker}(\bar{\alpha})| = |M|$. Thus, the restriction $\bar{\lambda}\bar{e}|_M : M \longrightarrow \mathrm{Ker}(\bar{\alpha})$ is an isomorphism.

Therefore, the following diagram is commutative:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & M & \longrightarrow & G & \xrightarrow{\ \sigma\ } & G/M & \longrightarrow & 1 \\ & & \big\downarrow & & \big\downarrow{\scriptstyle \bar{\lambda}\bar{e}} & & \big\downarrow{\scriptstyle \lambda} & & \\ 1 & \longrightarrow & \mathrm{Ker}(\bar{\alpha}) & \longrightarrow & (N/K) \wr P & \xrightarrow{\ \bar{\alpha}\ } & (N/M) \wr P & \longrightarrow & 1 \end{array}$$

Therefore G is the induced extension defined by α. □

Remark 1.1.30. We are interested in the action of the group G on the normal subgroup $M = S_1 \times \cdots \times S_n$, when G is an induced extension. We keep the notation of Theorem 1.1.29. The action of the group N on S, $\psi\colon N \longrightarrow \mathrm{Aut}(S)$, is defined by conjugation: if $x \in N$, then x^{ψ} is the automorphism of S given by the conjugation in N by the element x: for every $s \in S$, we have $s^{x^{\psi}} = s^x$.

The induced extension G can be considered as a subgroup of the wreath product $W = N \wr_{\rho} P$, via the embedding

$$\bar{\lambda} = \lambda_{\mathcal{T}} : G \longrightarrow N \wr_{\bar{\rho}} P \quad \text{given by } x^{\bar{\lambda}} = (c_{1,x}, \dots, c_{n,x})g^{\bar{\rho}}, \text{ for all } x \in G.$$

If $(x_1, \dots, x_n) \in M = S_1 \times \cdots \times S_n$ and $x \in G$, then, by Definition 1.1.17,

$$(x_1, \dots, x_n)^x = \left(x_1^{c_{1,x}^{\psi}}, \dots, x_n^{c_{n,x}^{\psi}}\right)^{x^{\bar{\rho}}} = (y_1, \dots, y_n),$$

where $x_i^{c_{i,x}^{\psi}} = y_{i^{x^{\bar{\rho}}}}$, for $i \in \{1, \dots, n\}$.

Proposition 1.1.31. *In the hypotheses 1.1.25, assume that S is a group and C acts on S by a group homomorphism $\psi\colon C \longrightarrow \mathrm{Aut}(S)$. Then the group B acts on the direct product $S^n = S_1 \times \cdots \times S_n$ by a group homomorphism*

$$\psi^B : B \longrightarrow C^{\psi} \wr_{\rho} P \leq \mathrm{Aut}(S^n)$$

such that for $(x_1,\dots,x_n)\in S^n$ *and* $h\in B$*, then*

$$(x_1,\dots,x_n)^{h^{\psi^B}}=(y_1,\dots,y_n),\ \text{where } x_i^{c_{i,h}^{\psi}}=y_{i^{h^{\bar\rho}}},\ \text{for } i\in\{1,\dots,n\}. \tag{1.3}$$

Moreover, $\operatorname{Ker}(\psi^B)=\operatorname{Core}_B\big(\operatorname{Ker}(\psi)\big)$.

Proof. If $\bar\psi\colon C\wr_\rho P\longrightarrow C^\psi\wr_\rho P$ is induced by ψ and λ is the monomorphism of Lemma 1.1.26, then $\psi^B=\lambda\bar\psi$. Clearly ψ^B is a group homomorphism. Observe that $h\in\operatorname{Ker}(\psi^B)$ if and only if h^ρ is the identity permutation and $c_{i,h}\in\operatorname{Ker}(\psi)$, for all $i\in\mathcal{I}$. This means that $t_iht_i^{-1}=c_{i,h}\in\operatorname{Ker}(\psi)$, for all $i\in\mathcal{I}$. And this is equivalent to saying that $h\in\operatorname{Core}_B\big(\operatorname{Ker}(\psi)\big)$. In other words, $\operatorname{Ker}(\psi^B)=\operatorname{Core}_B\big(\operatorname{Ker}(\psi)\big)$. □

These observations motivate the following definition.

Definition 1.1.32. *With the notation of Proposition 1.1.31, the action* ψ^B *is called* the induced B-action from ψ*, and the* B*-group* (S^n,ψ^B) *is the* induced B-group*.*

The semidirect product $[S^n]_{\psi^B}B=[S_1\times\cdots\times S_n]B$ *is called the* twisted wreath product *of* S *by* B*; it is denoted by* $S\wr_{(C,\psi)}B$*.*

Thus, if G is the induced extension defined by the map $\alpha\colon N/K\longrightarrow G/M$ as in Theorem 1.1.29, then the conjugacy action of G on the normal subgroup $M=S_1\times\cdots\times S_n$ is the induced G-action from the conjugacy action of $N=\mathrm{N}_G(S_1)$ on S_1.

Remarks 1.1.33. 1. The structure of induced B-group does not depend, up to equivalence of B-groups, on the chosen transversal of C in B.

2. The construction of induced actions is motivated by the classical construction of induced modules. If S is a C-module, the induced B-action gives to S^n the well-known structure of induced B-module: $S^n\cong S^B$. This explains the name and the notation.

Proposition 1.1.34. *Let* S *and* B *be groups and* C *a subgroup of* B*. Suppose that* (S,ψ) *is a* C*-group and consider the twisted wreath product* $G=S\wr_{(C,\psi)}B$*. Then*

1. $\mathrm{N}_B(S_1)=C$ *and* $\mathrm{C}_B(S_1)=\operatorname{Ker}(\psi)$*.*
2. $\mathrm{C}_B(S^\natural)=\operatorname{Core}_B\big(\operatorname{Ker}(\psi)\big)$*. Moreover if* $\operatorname{Core}_B(C)=1$*, then* $\mathrm{C}_G(S^\natural)=\mathrm{Z}(S^\natural)$*.*

Proof. 1. If $h\in\mathrm{N}_B(S_1)$, then, by (1.3), $1^{h^\rho}=1$ and $h=c_{1,h}\in C$. Conversely, if $c\in C$, then $c=c_{1,c}$ and $1^{c^\rho}=1$; moreover $(x,1,\dots,1)^{c^{\rho}}=(x^{c^\psi},1,\dots,1)$. Hence $C\le\mathrm{N}_B(S_1)$.

Observe that the elements of $\mathrm{C}_B(S_1)$ are elements $c\in C$ such that $c^\psi=\mathrm{id}_{S_1}$. Hence $\mathrm{C}_B(S_1)=\operatorname{Ker}(\psi)$.

2. Observe that $S_1^{t_i} = S_i$, for all $i \in \mathcal{I}$. Therefore $\mathrm{C}_B(S^\natural) = \bigcap_{i=1}^n \mathrm{C}_B(S_i) = \bigcap_{i=1}^n \mathrm{C}_B(S_1^{t_i}) = \mathrm{Core}_B\big(\mathrm{C}_B(S_1)\big) = \mathrm{Core}_B\big(\mathrm{Ker}(\psi)\big)$.

If $(x_1, \dots, x_n)h \in \mathrm{C}_G(S^\natural)$, then $h \in \bigcap_{i=1}^n \mathrm{N}_B(S_i) = \mathrm{Core}_B(C) = 1$. Therefore $(x_1, \dots, x_n) \in \mathrm{Z}(S^\natural)$. □

If $1 \longrightarrow M \longrightarrow G \longrightarrow B \longrightarrow 1$ is the induced extension defined by a group homomorphism $\alpha\colon A \longrightarrow B$, then G splits over M if and only if G is isomorphic to the twisted wreath product $S \wr_{(C,\psi)} B$. F. Gross and L. G. Kovács characterise when the induced extension splits. This characterisation, which will be crucial in Chapter 7, is just a consequence of a deep analysis of the supplements of M in G.

Theorem 1.1.35 (([GK84])). *Let G be a group in which there exists a normal subgroup M of G such that $M = S_1 \times \cdots \times S_n$, where $\{S_1, \dots, S_n\}$ is the set of all conjugate subgroups of a normal subgroup S_1 of M. Write $N = \mathrm{N}_G(S_1)$ and $K = S_2 \times \cdots \times S_n$.*

1. *Let L/K be a supplement of M/K in N/K. Then, there exists a supplement H of M in G satisfying the following:*
   a) *$L = (H \cap N)K$ and $H \cap M = (H \cap S_1) \times \cdots \times (H \cap S_n)$. Further, $\{H \cap S_1, \dots, H \cap S_n\}$ is a conjugacy class in H, and $H \cap S_1 = L \cap S_1$.*
   b) *Suppose that H_0 is a supplement of M in G such that $H_0 \cap N \leq L$. Then there is an element $k \in K$ such that $H_0^k \leq H$. Moreover, $H_0^k = H$ if and only if $L = (H_0 \cap N)K$ and $H_0 \cap M = (H_0 \cap S_1) \times \cdots \times (H_0 \cap S_n)$.*
   c) *In particular, H is unique up to conjugacy under K.*
2. *Suppose that H is a supplement M in G such that $H \cap M = (H \cap S_1) \times \cdots \times (H \cap S_n)$. Write $L = (H \cap N)K$. Assume further that R is a subgroup of G such that $G = RM$. Then the following are true:*
   a) *R is conjugate in G to a subgroup of H if and only if $R \cap N$ is conjugate in N to a subgroup of L.*
   b) *R is conjugate to H in G if and only if $(R \cap N)K$ is conjugate to L in N and also $R \cap M = (R \cap S_1) \times \cdots \times (R \cap S_n)$.*
3. *There is a bijection between, on the one hand, the conjugacy classes in G of supplements H of M in G such that $H \cap M = (H \cap S_1) \times \cdots \times (H \cap S_n)$, and, on the other hand, the conjugacy classes in N/K of supplements L/K of M/K in N/K, Moreover, under this bijection, we have the following:*
   a) *the conjugacy classes in G of supplements U of M which are maximal subgroups of G such that $U \cap M = (U \cap S_1) \times \cdots \times (U \cap S_n)$ are in one-to-one correspondence with the conjugacy classes in N/K of supplements of M/K which are maximal subgroups of N/K.*
   b) *the conjugacy classes in G of complements of M, if any, are in one-to-one correspondence with the conjugacy classes in N/K of complements of M/K.*

Proof. By Theorem 1.1.29, the group G is the induced extension defined by $\alpha\colon N/K \longrightarrow G/M$ given by $(Kx)^\alpha = Mx$, for all $x \in G$. Let $\mathcal{T} = \{t_1 =$

$1, \dots, t_n\}$ be a right transversal of N in G and write $\rho\colon G \longrightarrow \mathrm{Sym}(n)$ the permutation representation of G on the right cosets of N in G. As usual, for each $x \in G$, write $t_i x = c_{i,x} t_j$, for some $c_{i,x} \in N$, and $i^{x^\rho} = j$. Write $S_i = S^{t_i}$. For each $i \in \mathcal{I} = \{1, \dots, n\}$, write also $K_i = \prod_{j \in \mathcal{I} \setminus \{i\}} S_j$. Then $K = K_1$ and $K_i = K^{t_i}$. For $P = G^\rho$, let λ be the embedding of G into $(N/K) \wr_\rho P$ defined by $\lambda\colon G \longrightarrow (N/K) \wr_\rho P$ such that $x^\lambda = (Kc_{1,x}, \dots, Kc_{n,x})x^\rho$, for any $x \in G$.

1a. Define

$$H = \big((L/K) \wr_\rho P\big)^{\lambda^{-1}} = \{x \in G : c_{i,x} \in L, \text{ for all } i \in \mathcal{I}I\}.$$

This subgroup H satisfies the required properties.

Fix an element $g \in G$. Then, for each $i \in \mathcal{I}$, we have that $c_{i,g} \in N = ML$ and there exists $m_{i,g} \in M$ such that $m_{i,g}^{-1} c_{i,g} \in L$.

Observe that, if $m \in M$, then $c_{i,m} = m^{t_i^{-1}}$. Then

$$m^\lambda = (Km^{t_1^{-1}}, \dots, Km^{t_n^{-1}}) = (Km, Km^{t_2^{-1}}, \dots, Km^{t_n^{-1}}).$$

Write $m = (s_1, \dots, s_n)$. Then, for any $i \in \mathcal{I}$, using (1.1) in Definition 1.1.17, $(m^{t_i^{-1}})^{\pi_1} = s_i$, since $1^{t_i^\rho} = i$. Therefore $(s_1, \dots, s_n)^\lambda = (Ks_1, \dots, Ks_n)$.

Since the restriction of λ to M is an isomorphism onto $(M/K)^\natural$, i.e. $M^\lambda = (M/K)^\natural$, there exists a unique $m_g \in M$ such that $m_g^\lambda = (Km_{1,g}, \dots, Km_{n,g})$. Hence

$$\begin{aligned}(m_g^{-1} g)^\lambda = (m_g^\lambda)^{-1} g^\lambda &= (Km_{1,g}^{-1}, \dots, Km_{n,g}^{-1})(Kc_{1,g}, \dots, Kc_{n,g})g^\rho = \\ &= (Km_{1,g}^{-1} c_{1,g}, \dots, Km_{n,g}^{-1} c_{n,g})g^\rho \in (L/K) \wr_\rho P,\end{aligned}$$

and then $m_g^{-1} g \in H$. Hence, $G = HM$.

Observe that $Km_{i,g} = Kc_{i,m_g} = Km_g^{t_i^{-1}}$. If $g \in L$, then we can choose $m_{1,g} = 1$, and then $m_g \in K$. Thus $m_g^{-1} g \in H \cap N$. Then $L \le K(H \cap N)$. On the other hand, if $h \in H \cap N$, then $h = c_{1,h} \in L$. Hence $L = K(H \cap N)$.

If $m = (s_1, \dots, s_n) \in M \cap H$, then $Ks_i \in L/K$, for all $i \in \mathcal{I}$. Observe that, for any $i \in \mathcal{I}$, we have that $(1, \dots, s_i, \dots, 1)^\lambda = (K, \dots, Ks_i, \dots, K) \in \big((L \cap M)/K\big)^\natural$ and then $(1, \dots, s_i, \dots, 1) \in H \cap S_i$. Hence, $H \cap M = (H \cap S_1) \times \dots \times (H \cap S_n)$.

Since $G = HM$, we can choose the transversal $\mathcal{T} \subseteq H$. Hence, for all $i \in \mathcal{I}$, we have that $H \cap S_i = (H \cap S_1)^{t_i}$. Therefore $\{H \cap S_1, \dots, H \cap S_n\}$ is a conjugacy class in H. Moreover $(L \cap M)/K = \big((H \cap N)K \cap M\big)/K = (H \cap M)K/K = (H \cap S_1)K/K \cong H \cap S_1$ and also $(L \cap M)/K = (L \cap S_1)K/K \cong L \cap S_1$. Hence $|H \cap S_1| = |L \cap S_1|$. Since $H \cap S_1 = H \cap N \cap S_1 \le L \cap S_1$, we have the equality $H \cap S_1 = L \cap S_1$.

1b. Assume now that H_0 is a subgroup of G such that $G = MH_0$ and $H_0 \cap N \le L$. For each $i \in \mathcal{I}$, there must be an element $m_i \in M$ such that $t_i \in m_i^{-1} H_0$, i.e. $m_i t_i \in H_0$. We may choose $m_1 = 1$. Now, there exists a unique $k \in M$ such that

$$(K, Km_2, \ldots, Km_n) = k^\lambda = (Kk^{t_1^{-1}}, \ldots, Kk^{t_n^{-1}}).$$

This implies that $k \in K$ and $t_i k t_i^{-1} m_i^{-1} \in K$, for all $i \in \mathcal{I}$. We show that $H_0^{k^{-1}} \leq H$.

Let $x \in H_0$ and consider $y = x^{k^{-1}}$. Observe that, for all $i \in \mathcal{I}$, $Mt_i x = Mt_i x^{k^{-1}} = Mt_i y$ and then $i^{x^\rho} = i^{y^\rho}$. Now

$$\begin{aligned} c_{i,y} &= t_i y t_{i^{y^\rho}}^{-1} = t_i y t_{i^{x^\rho}}^{-1} = t_i k x k^{-1} t_{i^{x^\rho}}^{-1} = \\ &= t_i k (t_i^{-1} m_i^{-1} m_i t_i) x (t_{i^{x^\rho}}^{-1} m_{i^{x^\rho}}^{-1} m_{i^{x^\rho}} t_{i^{x^\rho}}) k^{-1} t_{i^{x^\rho}}^{-1} = \\ &= (t_i k t_i^{-1} m_i^{-1})(m_i t_i x t_{i^{x^\rho}}^{-1} m_{i^{x^\rho}}^{-1})(m_{i^{x^\rho}} t_{i^{x^\rho}} k^{-1} t_{i^{x^\rho}}^{-1}). \end{aligned}$$

Now observe that $m_i t_i$ and $t_{i^{x^\rho}}^{-1} m_{i^{x^\rho}}^{-1}$ are in H_0 and then, $m_i t_i x t_{i^{x^\rho}}^{-1} m_{i^{x^\rho}}^{-1} \in H_0$. On the other hand, $t_i x t_{i^{x^\rho}}^{-1} = c_{i,x} \in N$, and then $m_i t_i x t_{i^{x^\rho}}^{-1} m_{i^{x^\rho}}^{-1} \in N$. Since $t_i k t_i^{-1} m_i^{-1} \in K$ and also $m_{i^{x^\rho}} t_{i^{x^\rho}} k^{-1} t_{i^{x^\rho}}^{-1} \in K$, we have that

$$c_{i,y} = (t_i k t_i^{-1} m_i^{-1})(m_i t_i x t_{i^{x^\rho}}^{-1} m_{i^{x^\rho}}^{-1})(m_{i^{x^\rho}} t_{i^{x^\rho}} k^{-1} t_{i^{x^\rho}}^{-1}) \in K(H_0 \cap N)K \leq L$$

for all $i \in \mathcal{I}$. This means that $y \in H$.

Assume that $H_0^k \leq H$, for $k \in K$. Clearly, if $H_0^k = H$, then $L = (H_0 \cap N)K$ and $H_0 \cap M = (H_0 \cap S_1) \times \cdots \times (H_0 \cap S_n)$. Conversely, suppose that $L = (H_0 \cap N)K$ and $H_0 \cap M = (H_0 \cap S_1) \times \cdots \times (H_0 \cap S_n)$. Observe that H_0^k satisfies the same properties. Thus, we can assume that $H_0 \leq H$.

As in 1a, since $G = H_0 M$, we have that $\{H_0 \cap S_1, \ldots, H_0 \cap S_n\}$ is a conjugacy class in H_0, and $H_0 \cap S_1 = L \cap S_1$. Hence, $|H \cap S_1| = |H_0 \cap S_1|$, and then $H \cap S_1 = H_0 \cap S_1$. Therefore, $H \cap M = H_0 \cap M$. Then, from $G = H_0 M = HM$, we deduce that $|G : H_0| = |M : M \cap H_0| = |M : M \cap H| = |G : H|$. Hence, $|H| = |H_0|$ and then, $H = H_0$.

Part 1c is a direct consequence of 1b.

2a. Clearly L/K is a supplement of M/K in N/K. By 1c, the subgroup H is determined, up to conjugacy in K, by L.

Suppose that $G = RM$ and $R \cap N$ is conjugate to a subgroup of L in N. Since $N = RM \cap N = (R \cap N)M$, there is an element $m \in M$ such that $(R \cap N)^m \leq L$. Write $H_0 = R^m$. Then $G = H_0 M$ and $H_0 \cap N \leq L$. It follows, from 1b, that H_0 is conjugate to a subgroup of H. Hence R is conjugate to a subgroup of H. Conversely, if R is conjugate to a subgroup of H, then, since $G = RM$, we have that $R^m \leq H$, for some $m \in M$. Then $(R \cap N)^m = R^m \cap N \leq H \cap N \leq L$.

2b. If $G = RM$ and L is conjugate to $(R \cap N)K$ in N, there is an element $m \in M$ such that $L = \big((R \cap N)K\big)^m = (R \cap N)^m K = (R^m \cap N)K$. If $R \cap M = (R \cap S_1) \times \cdots \times (R \cap S_n)$, by 1b, we deduce that $H_0 = R^m$ is conjugate to H. The rest of 2b follows easily.

3. The bijection follows easily from 1 and 2.

3a. Let L be a maximal subgroup of N such that $K \leq L$ and $N = LM$ and consider one of the supplements U of M in G determined by the conjugacy class of L in N under the bijection. Suppose that $U \leq H < G$. Then $N = (H \cap N)M$. Set $L_0 = (H \cap N)K$. Then L_0/K is a supplement of M/K in N/K. Clearly $L = (U \cap N)K \leq L_0$. By maximality of L, we have that $L = L_0$. But then $H \cap N \leq L$ and, by 1b, $H^k \leq U$, for some $k \in K$. Clearly, this implies that $U = H$. Hence U is maximal in G.

Conversely, let U be a maximal subgroup of G which supplements M in G such that $U \cap M = (U \cap S_1) \times \cdots \times (U \cap S_n)$. Write $L = (U \cap N)K$. Suppose that $L \leq L_0 < N$. Consider a supplement R of M in G determined by L_0 under the bijection. Then $L_0 = (R \cap N)K$. Since $U \cap N \leq L_0$, then $U^k \leq R$, for some $k \in K$. By maximality of U, we have that $R = U^k$. This implies that L and L_0 are conjugate in N and, since $L \leq L_0$, equality holds.

3b. Observe that if L/K is a complement of M/K in N/K, then $L \cap S_1 = 1$. Hence $H \cap S_1 = 1$ and therefore $H \cap M = 1$. This is to say that H is a complement of M in G. Conversely, if H is a complement of M in G, then $(H \cap N)K \cap M = (H \cap N \cap M)K = K$. □

The following result, due also to F. Gross and L. G. Kovács, is an application of the induced extension procedure to the construction of groups which are not semidirect products. We will use it in Chapter 5.

Theorem 1.1.36 ([GK84]). *Let B be any finite simple group. Then there exists a finite group G with a minimal normal subgroup M such that M is a direct product of copies of* Alt(6)*, the alternating group of degree* 6*, the quotient group G/M is isomorphic to B and G does not split over M.*

Proof. Consider the group $A = \mathrm{Aut}\big(\mathrm{Alt}(6)\big)$. Let D denote the normal subgroup of inner automorphisms, $D \cong \mathrm{Alt}(6)$, of A. It is well-known that the quotient group A/D is isomorphic to an elementary abelian 2-group of order 4 and A does not split over D, i.e. there is no complement of D in A (see [Suz82]).

By the Odd Order Theorem ([FT63]), the Sylow 2-subgroups of B are nontrivial. By the Burnside Transfer Theorem (see [Suz86, 5.2.10, Corollary 2]), a Sylow 2-subgroup of B cannot by cyclic. By a theorem of R. Brauer and M. Suzuki (see [Suz86, page 306]), the Sylow 2-subgroups of G cannot by isomorphic to a quaternion group. Hence a Sylow 2-subgroup of B has two transpositions generating a dihedral 2-group (see [KS04, 5.3.7 and 1.6.9]). Therefore B must contain a subgroup G which is elementary abelian of order 2. Then there is a homomorphism α of A into B such that $A^\alpha = C$ and Ker $(\alpha) = D$

Now let G be the induced extension defined by $\alpha \colon A \longrightarrow B$. Since A does not split over D, the group G has the required properties. □

Let G be a group which is an induced extension of a normal subgroup $M = S_1 \times \cdots \times S_n$. We have presented above a complete description of those supplements of M in G whose intersection with M is a direct product of the projections in each component $H \cap M = (H \cap S_1) \times \cdots \times (H \cap S_n)$. But nothing is said about those supplements H whose projections $\pi_i\colon H \cap M \longrightarrow S_i$ are surjective. Subgroups D of a direct product M such that all projections $\pi_i\colon D \longrightarrow S_i$ are surjective are fully described by M. Aschbacher and L. Scott in [AS85]. In the sequel we present here an adaptation of their results suitable for our purposes.

Definition 1.1.37. *Let $G = \prod_{i=1}^n S_i$ be a direct product of groups. A subgroup H of G is said to be* diagonal *if each projection $\pi_i : H \longrightarrow S_i$, $i = 1, \dots, n$, is injective.*

If each projection $\pi_i\colon H \longrightarrow S_i$ is an isomorphism, then the subgroup H is said to be a full diagonal subgroup.

Obviously if H is a full diagonal subgroup of $G = \prod_{i=1}^n S_i$, then all the S_i are isomorphic. Observe that if $x = (x_1, \dots, x_n) \in H$, then $x_i = x^{\pi_i}$, for all $i = 1, \dots, n$, and then $x = (x_1, x_1^{\pi_1^{-1}\pi_2}, \dots, x_1^{\pi_1^{-1}\pi_n})$. All $\varphi_i = \pi_1^{-1}\pi_i$ are isomorphisms of S_1 and then $\varphi = (\varphi_1 = 1, \varphi_2, \dots, \varphi_n) \in \mathrm{Aut}(S_1)^n$. Conversely, given a group S and $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_n) \in \mathrm{Aut}(S)^n$, it is clear that $\{x^\varphi = (x^{\varphi_1}, x^{\varphi_2}, \dots, x^{\varphi_n}) : x \in S\}$ is a full diagonal subgroup of S^n.

More generally, given a direct product of groups $G = \prod_{i=1}^n S_i$ such that all S_i are isomorphic copies of a group S, to each pair $(\mathit{\Delta}, \varphi)$, where $\mathit{\Delta} = \{\mathcal{I}_1, \dots, \mathcal{I}_l\}$ is a partition of the set $\mathcal{I} = \{1, \dots, n\}$ and $\varphi = (\varphi_1, \dots, \varphi_n) \in \mathrm{Aut}(S)^n$, we associate a direct product $D_{(\mathit{\Delta},\varphi)} = D_1 \times \dots \times D_l$, where each D_j is a full diagonal subgroup of the direct product $\prod_{i \in \mathcal{I}_j} S_i$ defined by the automorphisms $\{\varphi_i : i \in \mathcal{I}_j\}$. It is easy to see that if $\mathit{\Gamma}$ is a partition of $\mathcal{I}$ refining $\mathit{\Delta}$, then $D_{(\mathit{\Delta},\varphi)} \le D_{(\mathit{\Gamma},\varphi)}$. In particular, the trivial partition $\mathit{\Omega} = \bigl\{\{1\}, \dots \{n\}\bigr\}$ of $\mathcal{I}$ gives $D_{(\mathit{\Omega},\varphi)} = G$, for any $\varphi \in \mathrm{Aut}(S)^n$.

For groups S with trivial centre, the group G can be embedded in the wreath product $W = \mathrm{Aut}(S) \wr \mathrm{Sym}(n)$. In particular, if S is a non-abelian simple group, then $G \le \mathrm{Aut}(S^n)$. In the group W the conjugacy by the element $\varphi \in W$ makes sense and $D_{(\mathit{\Delta},\varphi)} = D_{(\mathit{\Delta},\mathrm{id})}^{\varphi}$, where id denotes the n-tuple composed by all identity isomorphisms.

Lemma 1.1.38. *Let H be a full diagonal subgroup of the direct product $G = \prod_{i=1}^n S_i$, where the S_i are copies of a non-abelian simple group S. Then H is self-normalising in G.*

Proof. Since H is a full diagonal subgroup of G, all π_i are isomorphisms of H onto S_i. Observe that $(x_1, \dots, x_n) \in H$ if and only if $x_j = x_1^{\pi_1^{-1}\pi_j}$, for $j = 2, \dots, n$ and for all $x_1 \in S$. Write $\varphi_j = \pi_1^{-1}\pi_j$, for $j = 2, \dots, n$.

If $g = (g_1, \dots, g_n) \in \mathrm{N}_G(H)$, then for all $x \in S$ we have that

$$(x, x^{\varphi_2}, \dots, x^{\varphi_n})^g = \bigl(x^{g_1}, (x^{\varphi_2})^{g_2}, \dots, (x^{\varphi_n})^{g_n}\bigr) \in H.$$

Hence, for $j = 2, \dots, n$, $(x^{\varphi_j})^{g_j} = (x^{g_1})^{\varphi_j} = (x^{\varphi_j})^{g_1^{\varphi_j}}$ and the automorphism $g_1^{\varphi_j} g_j^{-1}$ is the trivial automorphism of S_j. Hence $g_1^{\varphi_j} = g_j$ and $g \in H$. This is to say that H is self-normalising in G. ☐

Proposition 1.1.39. *Suppose that H is a subgroup of the direct product $G = \prod_{i=1}^{n} S_i$, where the S_i are non-abelian simple groups for all $i \in \mathcal{I} = \{1, \ldots, n\}$. Assume that all projections $\pi_i : H \longrightarrow S_i$, $i \in \mathcal{I}$, are surjective.*

1. *There exists a partition Δ of $\mathcal{I}$ such that the subgroup H is the direct product*
$$H = \prod_{\mathcal{D} \in \Delta} H^{\pi_{\mathcal{D}}},$$
where
   a) *each $H^{\pi_{\mathcal{D}}}$ is a full diagonal subgroup of $\prod_{i\in\mathcal{D}} S_i$,*
   b) *the partition Δ is uniquely determined by H in the sense that if $H = \prod_{\mathcal{D}\in\Delta} H^{\pi_{\mathcal{D}}} = \prod_{\mathcal{G}\in\Gamma} H^{\pi_{\mathcal{G}}}$, for Δ and Γ partitions of $\mathcal{I}$, then $\Delta = \Gamma$, and*
   c) *if $H \leq K \leq G$, then $K = \prod_{\mathcal{G}\in\Gamma} H^{\pi_{\mathcal{G}}}$, where Γ is a partition of $\mathcal{I}$ which refines Δ.*
2. *Suppose that the S_i are isomorphic copies of a non-abelian simple group S, for all $i \in \mathcal{I}$, i.e. $G \cong S^n$. Let U be a subgroup of $\mathrm{Aut}(G)$. Then U, acting by conjugation on the simple components S_i of $\mathrm{Soc}\big(\mathrm{Aut}(G)\big)$, is a permutation group on the set $\{S_1, \ldots, S_n\}$ (and therefore on $\mathcal{I}$).*
   *Observe that the action of U on $\mathcal{I}$ induces an action on the set of all partitions of $\mathcal{I}$. We can say that a partition Δ of $\mathcal{I}$ is U-*invariant *if $\Delta^x = \Delta$ for all $x \in U$.*
   If H is U-invariant, i.e. $U \leq \mathrm{N}_{\mathrm{Aut}(G)}(H)$, then the partition Δ is a U-invariant set of blocks of the action of U on $\mathcal{I}$.
3. *In the situation of 2, if Γ is a U-invariant partition of $\mathcal{I}$ which refines Δ and every member of Γ is again a block for the action of U on $\mathcal{I}$, then the subgroup $K = \prod_{\mathcal{G}\in\Gamma} H^{\pi_{\mathcal{G}}}$ is also U-invariant.*

Proof. 1a. Let $\mathcal{D}$ be a subset of $\mathcal{I}$ minimal such that the subgroup $D = H \cap \big(\prod_{i\in\mathcal{D}} S_i\big)$ is non-trivial. It is clear that D is a normal subgroup of H and then every projection of D is a normal subgroup of the corresponding projection of H. Since, by minimality of $\mathcal{D}$, D^{π_j} is non-trivial, for each $j \in \mathcal{D}$, we have that $D^{\pi_j} = S_j$. Moreover, for each $j \in \mathcal{D}$, we have that $\mathrm{Ker}(\pi_j) \cap D = H \cap \big(\prod_{i\in\mathcal{D}, i\neq j} S_i\big) = 1$, by minimality of $\mathcal{D}$. Therefore D is a full diagonal subgroup of $\prod_{i\in\mathcal{D}} S_i$. Let $E = H^{\pi_{\mathcal{D}}}$ be the image of the projection of H in $\prod_{i\in\mathcal{D}} S_i$. Then $D = D^{\pi_{\mathcal{D}}}$ is normal in E. By Lemma 1.1.38, $D = E$. Write $F = H \cap \prod_{i\notin\mathcal{D}} S_i$. Clearly $D \times F \leq H$. For each $x \in H$, we can write $x = x_1 x_2$, where x_1 is the projection of x onto $\prod_{i\in\mathcal{D}} S_i$ and x_2 is the projection of x onto $\prod_{i\notin\mathcal{D}} S_i$. Observe that $x_1 \in D \leq H$ and then $x_2 \in F$. This is to say that $H = D \times F$. Now the result follows by induction on the cardinality of $\mathcal{I}$.

To prove 1b suppose that $H = \prod_{\mathcal{D}\in\Delta} H^{\pi_{\mathcal{D}}} = \prod_{\mathcal{G}\in\Gamma} H^{\pi_{\mathcal{G}}}$, for Δ and Γ partitions of $\mathcal{I}$. Observe that for each $\mathcal{D} \in \Delta$, since $H^{\pi_{\mathcal{D}}}$ is a full diagonal subgroup of $\prod_{i\in\mathcal{D}} S_i$, we have that the following statements are equivalent for a non-trivial element $h \in H$:

1. $h \in H^{\pi_{\mathcal{D}}}$;
2. $h^{\pi_i} \neq 1$ if and only if $i \in \mathcal{D}$;
3. there exists an $i \in \mathcal{D}$ such that $h^{\pi_i} \neq 1$ and for each $\mathcal{D}' \in \Delta$, with $\mathcal{D}' \neq \mathcal{D}$, there exists a $j \in \mathcal{D}'$ such that $h^{\pi_j} = 1$.

Suppose that $h \in H^{\pi_{\mathcal{D}}}$. Then $h^{\pi_i} \neq 1$, for all $i \in \mathcal{D}$, and $h^{\pi_j} = 1$, for all $j \notin \mathcal{D}$. If $i \in \mathcal{D}$, there exists $\mathcal{G} \in \Gamma$ such that $i \in \mathcal{G}$. Thus $h \in H^{\pi_{\mathcal{G}}}$ and in fact $\mathcal{D} = \mathcal{G}$. Hence $\Delta = \Gamma$.

1c. Suppose finally that K is a subgroup of G containing H. Obviously, the projections $\pi_i \colon K \longrightarrow S_i$ are surjective. Then, by the above arguments, we have that $K = \prod_{\mathcal{G} \in \Gamma} K^{\pi_{\mathcal{G}}}$, where Γ is a partition of $\mathcal{I}$, and, for each $\mathcal{G} \in \Gamma$, $K^{\pi_{\mathcal{G}}}$ is a full diagonal subgroup of $\prod_{j \in \mathcal{G}} S_j$. In particular, for all $i \in \mathcal{G}$, the S_i are isomorphic to a non-abelian simple group $S_{\mathcal{G}}$. Since $H = \prod_{\mathcal{D} \in \Delta} H^{\pi_{\mathcal{D}}}$, we have $H^{\pi_{\mathcal{G}}} = \prod_{\mathcal{D} \cap \mathcal{G}, \mathcal{D} \in \Delta} H^{\pi_{\mathcal{D} \cap \mathcal{G}}}$. If $\mathcal{G} \cap \mathcal{D} \neq \emptyset$, then $H^{\pi_{\mathcal{D} \cap \mathcal{G}}} \cong S_{\mathcal{G}}$. Observe that $H^{\pi_{\mathcal{G}}}$ is a direct product contained in $K^{\pi_{\mathcal{G}}} \cong S_{\mathcal{G}}$. This implies that the direct product has a unique component which is equal to $K^{\pi_{\mathcal{G}}}$. Hence, for each $\mathcal{G} \in \Gamma$, $H^{\pi_{\mathcal{G}}} = K^{\pi_{\mathcal{G}}}$, and $\mathcal{G} \subseteq \mathcal{D}$, for some $\mathcal{D} \in \Delta$, i.e. Γ is a partition of $\mathcal{I}$ which refines Δ.

2. By Proposition 1.1.20, we can consider that U is a subgroup of the wreath product $A \wr \operatorname{Sym}(n)$, for $A = \operatorname{Aut}(S)$ and S a non-abelian simple group such that $S \cong S_i$, for all $i \in \mathcal{I}$. We see in Remark 1.1.18 (2) of that U acts by conjugation on the set $\{A_1, \dots, A_n\}$ of factors of the base group. Since S is the unique minimal normal subgroup of A, the group U acts by conjugation on $\{S_1, \dots, S_n\}$.

Suppose that H is U-invariant. Then, for any $x \in U$, by Remark 1.1.18 (4), we have

$$H = H^x = \prod_{\mathcal{D} \in \Delta} (H^{\pi_{\mathcal{D}}})^x = \prod_{\mathcal{D} \in \Delta} (H^x)^{\pi_{\mathcal{D}^x}} = \prod_{\mathcal{D}^x \in \Delta^x} H^{\pi_{\mathcal{D}^x}}$$

and then $\Delta = \Delta^x$, by 1b. Hence Δ is U-invariant. Moreover $\mathcal{D}^x$ is an element of the partition Δ. Therefore either $\mathcal{D} = \mathcal{D}^x$ or $\mathcal{D} \cap \mathcal{D}^x = \emptyset$. Hence the elements of Δ are blocks for the action of U on $\mathcal{I}$.

3. This follows immediately from Remark 1.1.18 (4): for any $x \in U$, we have

$$K^x = \prod_{\mathcal{G} \in \Gamma} (H^{\pi_{\mathcal{G}}})^x = \prod_{\mathcal{G} \in \Gamma} (H^x)^{\pi_{\mathcal{G}^x}} = \prod_{\mathcal{G} \in \Gamma} H^{\pi_{\mathcal{G}}} = K,$$

and therefore K is U-invariant. □

The purpose of the following is to present a proof of the Theorem of O'Nan and Scott classifying all primitive groups of type 2. The first version of this theorem, stated by Michael O'Nan and Leonard Scott at the symposium on Finite Simple Groups at Santa Cruz in 1979, appeared in the proceedings in [Sco80] but one of the cases, the primitive groups whose socle is complemented by a maximal subgroup, is omitted. In [Cam81], P. J. Cameron presented an outline of the proof of the O'Nan-Scott Theorem again with the same omission. Finally, in [AS85] a corrected and expanded version of the theorem appears. Independently, L. G. Kovács presented in [Kov86] a completely different approach to the same result.

We are indebted to P. Jiménez-Seral for her kind contributions in [JS96]. These personal notes, written for a doctoral course at the Universidad de Zaragoza and adapted for her students, are motivated mainly by the self-contained version of the O'Nan-Scott Theorem which appears in [LPS88].

To study the structure of a primitive group G of type 2 whose socle $\mathrm{Soc}(G)$ is non-simple, we will follow the following strategy. We observe that in general, for any supplement M of $\mathrm{Soc}(G)$ in G, we have that M is a maximal subgroup of G if and only if $M \cap \mathrm{Soc}(G)$ is a maximal M-invariant subgroup of $\mathrm{Soc}(G)$. We will focus our attention in the structure of the intersection $U \cap \mathrm{Soc}(G)$ of a core-free maximal subgroup U of G with the socle.

General remarks and notation 1.1.40. We fix here the main notation which is used in our study of primitive groups of type 2 in the sequel. We also review some previously known facts and make some remarks. All these observations give rise to the first steps of the classification theorem of O'Nan and Scott.

Let G be a primitive group of type 2.

1. Write $\mathrm{Soc}(G) = S_1 \times \cdots \times S_n$ where the S_i are copies of a non-abelian simple group S, for $i \in \mathcal{I} = \{1, \dots, n\}$. Write also $K_j = \prod_{i \in \mathcal{I}, i \neq j} S_i$, for each $j \in \mathcal{I}$.

2. Write $N = \mathrm{N}_G(S_1)$ and $C = \mathrm{C}_G(S_1)$. Let $\psi\colon N \longrightarrow \mathrm{Aut}(S_1)$ denote the conjugacy action of $N = \mathrm{N}_G(S_1)$ on S_1. Sometimes we will make the identification $S_1^{\psi} = \mathrm{Inn}(S_1) = S_1$.

3. The quotient group $X = N/C$ is an almost simple group with $\mathrm{Soc}(X) = S_1C/C$.

4. Suppose that U is a core-free maximal subgroup of G.

5. The subgroup $U \cap \mathrm{Soc}(G)$ is maximal with respect to being a proper U-invariant subgroup of $\mathrm{Soc}(G)$.

6. By Proposition 1.1.19, the group G, acting by conjugation on the elements of the set $\{S_1, \dots, S_n\}$, induces the structure of a G-set on $\mathcal{I}$. Write $\rho\colon G \longrightarrow \mathrm{Sym}(n)$ for this action. The kernel of this action is $\mathrm{Ker}(\rho) = \bigcap_{i=1}^{n} \mathrm{N}_G(S_i) = Y$. Therefore G/Y is isomorphic to a subgroup $G^{\rho} = P_n$ of $\mathrm{Sym}(n)$. For any $g \in G$, we write g^{ρ} for the image of g in P_n.

Moreover, since $\mathrm{Soc}(G)$ is a minimal normal subgroup, the conjugacy action of G on $\{S_1, \dots, S_n\}$, and on $\mathcal{I}$, is transitive. Observe that $S_{i^{x^{\rho}}} = S_i^x$ and $K_{i^{x^{\rho}}} = K_i^x$, for $x \in G$ and $i \in \mathcal{I}$.

It is worth remarking here that the action of $\mathrm{Soc}(G)$ on $\mathcal{I}$ is trivial. Therefore if H is a supplement of $\mathrm{Soc}(G)$ in G and Δ is a partition of $\mathcal{I}$, then Δ is H-invariant if and only if Δ is G-invariant. Also, a subset $\mathcal{D} \subseteq \mathcal{I}$ is block for the action of H if and only if $\mathcal{D}$ is a block for the action of G.

Since the set $\{S_1, \dots, S_n\}$ is a conjugacy class of subgroups of G, we have that $Y = \mathrm{Core}_G(N)$. In particular $\mathrm{Soc}(G) \leq Y$.

Now U is core-free and maximal in G and therefore $G = UY$. This means that if τ is a permutation of $\mathcal{I}$ in P_n, there exists an element $x \in U$ such that the conjugation by x permutes the S_i in the same way τ does: $S_{i^\tau} = S_i^x$, for all $i \in \mathcal{I}$. In other words, $x^\rho = \tau$. This is to say that the projection of U onto P_n is surjective.

7. The stabiliser of the element 1 for the action of G on $\mathcal{I}$ is $N = \mathrm{N}_G(S_1)$. Therefore $|G : N| = n$. Observe that $N = \mathrm{N}_G(S_1) = \mathrm{N}_G(K_1)$. Let $\mathcal{T} = \{1 = t_1, t_2, \dots, t_n\}$ be a right transversal of N in G such that $S_1^{t_i} = S_i$, for $i \in \mathcal{I}$.

8. Observe that $\mathrm{Soc}(G) \leq N$ and then $G = UN$. For this reason the transversal $\mathcal{T}$ can be chosen such that $\mathcal{T} \subseteq U$.

9. Write $V = U \cap N = \mathrm{N}_U(S_1)$. Then $\mathcal{T}$ is a right transversal of V in U. Observe that $N = N \cap U \,\mathrm{Soc}(G) = (N \cap U)\,\mathrm{Soc}(G) = V\,\mathrm{Soc}(G) = VCS_1$.

10. The conjugation in S_1 by the elements of V induces a group homomorphism $\varphi \colon V \longrightarrow \mathrm{Aut}(S_1)$. It is clear that $\mathrm{Ker}(\varphi) = \mathrm{C}_U(S_1)$.

11. For any $i \in \mathcal{I}$, we have

$$t_i g = a_{i,g} t_j, \text{ with } a_{i,g} \in N \text{ and } i^{g^\rho} = j.$$

Moreover, since $\mathcal{T} \subseteq U$, if $g \in U$, then $a_{i,g} \in V$.

12. Denote with a star $(*)$ the projection of N in X: if $a \in N$, then $a^* = aC \in X$.

13. The group G is the induced extension defined by $\alpha \colon N/K_1 \longrightarrow G/\mathrm{Soc}(G)$. Hence, the action of G on $\mathrm{Soc}(G)$ is the induced G-action from ψ:

$$\psi^G \colon G \longrightarrow X \wr_\rho P_n \leq \mathrm{Aut}(S^n),$$

given by $g^{\psi^G} = (a_{1,g}^*, \dots, a_{n,g}^*) g^\rho$, for any $g \in G$. Observe that $\mathrm{Ker}(\psi^G) = \mathrm{Core}_G\big(\mathrm{Ker}(\psi)\big) = \mathrm{Core}_G\big(\mathrm{C}_G(S_1)\big) = 1$. Hence ψ^G is injective. In other words, ψ^G is an embedding of G into the wreath product $X \wr_\rho P_n$, and then into $\mathrm{Aut}(S^n)$. We identify G and G^{ψ^G}.

With this identification, $\mathrm{N}_G(S_1) = G \cap (X_1 \times [X_2 \times \dots \times X_n]P_{n-1})$, where P_{n-1} is the stabiliser of 1. If $g \in \mathrm{N}_G(S_1)$, then g^ρ fixes 1, i.e. $g^\rho \in P_{n-1}$. Moreover $a_{1,g} = g$. Hence $g^{\psi^G} = (g^*, a_{2,g}^*, \dots, a_{n,g}^*) g^\rho \in (X_1 \times [X_2 \times \dots \times X_n]P_{n-1})$. Hence the projection of $\mathrm{N}_G(S_1)$ on X_1 is surjective.

14. Observe that, for each $i \in \mathcal{I}$, any element x_i of S_i can be written as $x_i = e_i^{t_i}$, for certain $e_i \in S_1$. For any $j \neq i$, we have that $x_i^{t_j^{-1}} \in S_k$, for some $k \neq 1$ and therefore $x_i^{t_j^{-1}} \in \mathrm{C}_G(S_1)$. This implies that $a_{j,x_i}^* = 1$ for any $j \neq i$. Moreover $a_{i,x_i} = e_i$. Also it is clear that x_i normalises all the S_j, for $j = 1, \dots, n$ and then ${x_i}^\rho = 1$. Hence $x_i^{\psi^G} = e_i^*$. This is to say that, with the identification of 2, $S_i^{\psi^G} = S_i$, for all $i \in \mathcal{I}$, and then $\mathrm{Soc}(G)^{\psi^G} = S^\natural$.

15. For each $i \in \mathcal{I}$, the quotient group $Y\,\mathrm{C}_G(S_i)/\,\mathrm{C}_G(S_i)$ is isomorphic to a subgroup of $\mathrm{Aut}(S_i)$ and then $Y/\bigcap_{i=1}^n \mathrm{C}_Y(S_i) \cong Y$ is embedded in $\mathrm{Aut}(S_1) \times \dots \times \mathrm{Aut}(S_n)$. Observe that the kernel of the homomorphism which assigns to each n-tuple of $\mathrm{Aut}(S_1) \times \dots \times \mathrm{Aut}(S_n)$ the n-tuple of the corresponding projections of $\mathrm{Out}(S_1) \times \dots \times \mathrm{Out}(S_n)$ is $\mathrm{Soc}(G)$. Hence the quotient group $Y/\mathrm{Soc}(G)$ is isomorphic to a subgroup of $\mathrm{Out}(S_1) \times \dots \times \mathrm{Out}(S_n)$. Hence,

by the Schreier's conjecture ([KS04, page 151]), the group $Y/\operatorname{Soc}(G) = \big(Y \cap U\operatorname{Soc}(G)\big)/\operatorname{Soc}(G) = (Y \cap U)\operatorname{Soc}(G)/\operatorname{Soc}(G) \cong (U \cap Y)/\big(U \cap \operatorname{Soc}(G)\big)$ is soluble.

16. As in Remarks 1.1.18, if $\mathcal{S} \subseteq \mathcal{I}$, then we write

$$\pi_{\mathcal{S}}\colon \operatorname{Soc}(G) \longrightarrow \prod_{j\in\mathcal{S}} S_j$$

for the projection of $\operatorname{Soc}(G)$ onto $\prod_{j\in\mathcal{S}} S_j$. If $\mathcal{S} = \{j\}$, then the projection onto S_j is denoted simply π_j.

17. Write $R_j = \big(U \cap \operatorname{Soc}(G)\big)^{\pi_j}$. Since the action of G on $\mathcal{I}$ is transitive and $G = U\operatorname{Soc}(G)$, then all projections R_j, $j = 1, \dots, n$ are conjugate by elements of U. Hence $U \cap \operatorname{Soc}(G) \le R_1 \times \cdots \times R_n = R_1 \times R_1^{t_2} \times \cdots \times R_1^{t_n}$.

18. By Remark 1.1.18 (4), if $y \in U \cap \operatorname{Soc}(G)$ and $g \in V$, then $(y^g)^{\pi_1} = (y^{\pi_1})^g$. This is to say that R_1 is a V-invariant subgroup of S_1.

Therefore $R_1 \times \cdots \times R_n = R_1 \times R_1^{t_2} \times \cdots \times R_1^{t_n}$ is a V-invariant subgroup of $\operatorname{Soc}(G)$.

19. By 5 and 18, we have two possibilities for each R_i:

a) either R_i is a proper subgroup of S_i; in this case,

$$U \cap \operatorname{Soc}(G) = R_1 \times \cdots \times R_n = (U \cap S_1) \times \cdots \times (U \cap S_n),$$

b) or $R_i = S_i$, i.e. the projections of $U \cap \operatorname{Soc}(G)$ on each S_i are surjective.

20. Let us deal first with the Case 19a: suppose that R_1 is a proper subgroup of S_1. Suppose that $R_1 \le T_1 < S_1$ and T_1 is a V-invariant subgroup of S_1. Then $T_1 \times T_1^{t_2} \cdots \times T_1^{t_n}$ is U-invariant in $\operatorname{Soc}(G)$ and, by 5, we have that $T_1 \times T_1^{t_2} \cdots \times T_1^{t_n} = U \cap \operatorname{Soc}(G) = R_1 \times \cdots \times R_n$. Hence $R_1 = T_1$.

This means that if R_1 is a proper subgroup of S_1, then R_1 is maximal with respect to being a proper V-invariant subgroup of S_1.

21. If the projection π_1 of $U \cap \operatorname{Soc}(G)$ on S_1 is not surjective, then two possibilities arise:

a) either $R_1 = 1$, i.e. $U \cap \operatorname{Soc}(G) = 1$: the core-free maximal subgroup U complements $\operatorname{Soc}(G)$;
b) or $1 \ne R_1 < S_1$.

22. Suppose that $1 \ne R_1 < S_1$. Then, by 19a, $R_1 = U \cap S_1$ and then $R_1 \le V$. Hence $R_1 = V \cap S_1$.

Moreover, if we suppose that there exists a proper subgroup M of N such that $VC \le M < N$, then $M \cap S_1$ is a V-invariant subgroup of S_1 and $R_1 \le M \cap S_1$. Observe that if $S_1 \le M$, then $N = VCS_1 \le M$ and $N = M$, against our assumption. Hence, $R_1 \le M \cap S_1 \ne S_1$. By maximality of R_1, we have that $R_1 = M \cap S_1$ and then $M = M \cap CVS_1 = CV(M \cap S_1) = CVR_1 = CV$. Therefore VC is a maximal subgroup of N.

23. Now we consider the Case 19b. Assume that H is a supplement of $\operatorname{Soc}(G)$ in G and we suppose that the projection of $H \cap \operatorname{Soc}(G)$ on each component S_i of $\operatorname{Soc}(G)$ is surjective. Then, by Proposition 1.1.39, there exists an H-invariant partition Δ of $\mathcal{I}$ into blocks for the action of H on $\mathcal{I}$ such that

$$H \cap \mathrm{Soc}(G) = \prod_{\mathcal{D} \in \Delta} \bigl(H \cap \mathrm{Soc}(G)\bigr)^{\pi_{\mathcal{D}}},$$

and, for each $\mathcal{D} \in \Delta$, the projection $\bigl(H \cap \mathrm{Soc}(G)\bigr)^{\pi_{\mathcal{D}}}$ is a full diagonal subgroup of the direct product $\prod_{i \in \mathcal{D}} S_i$.

Now we prove that H is maximal in G if and only if Δ is a minimal non-trivial G-invariant partition of $\mathcal{I}$ in blocks for the action of G on $\mathcal{I}$.

Suppose $1 < \Gamma < \Delta$, where all are H-invariant partitions of $\mathcal{I}$ into blocks for the action of H on $\mathcal{I}$. Then by Proposition 1.1.39 (3), the product of projections of $H \cap \mathrm{Soc}(G)$ obtained from Γ is an H-invariant subgroup J of $\mathrm{Soc}(G)$. By Proposition 1.1.39 (1b), $H \cap \mathrm{Soc}(G) < J < \mathrm{Soc}(G)$. But if H is maximal in G, then $H \cap \mathrm{Soc}(G)$ is maximal as an H-invariant subgroup of $\mathrm{Soc}(G)$ as in 5. Hence H is not maximal in G.

Now suppose $H < L < G$. Then $H \,\mathrm{Soc}(G) = L \,\mathrm{Soc}(G) = G$ implies $H \cap \mathrm{Soc}(G) < L \cap \mathrm{Soc}(G) < \mathrm{Soc}(G)$. Then, by Proposition 1.1.39 (1c), $L \cap \mathrm{Soc}(G)$ is the product of projections of $H \cap \mathrm{Soc}(G)$ (which are the same as the projections of $L \cap \mathrm{Soc}(G)$) obtained from a non-trivial proper refinement Γ of Δ. Then by Proposition 1.1.39 (2), Γ is L-invariant so, like Δ, it is an H-invariant set of blocks for the action of H on $\mathcal{I}$. Thus if Δ is a minimal such partition of $\mathcal{I}$, then H is maximal in G.

Finally, any H-invariant block is G-invariant, by 6.

24. If the projection of $U \cap \mathrm{Soc}(G)$ on each component S_i of $\mathrm{Soc}(G)$ is surjective, then $U \cap \mathrm{Soc}(G) = D_1 \times \cdots \times D_l$, with $1 \le l < n$, and each D_i is isomorphic to S. Hence $\mathrm{Soc}(G) = \bigl(U \cap \mathrm{Soc}(G)\bigr)K_1$ and then $G = UK_1$.

25. In this study we have observed three different types of core-free maximal subgroups U of a primitive group G of type 2 according to the image of the projection $\pi_1 \colon U \cap \mathrm{Soc}(G) \longrightarrow S_1$.

a) $\bigl(U \cap \mathrm{Soc}(G)\bigr)^{\pi_1} = S_1$, i.e. the projection π_1 of $U \cap \mathrm{Soc}(G)$ on S_1 is surjective.
b) $1 \ne R_1 = \bigl(U \cap \mathrm{Soc}(G)\bigr)^{\pi_1} < S_1$, i.e. the image of the projection π_1 of $U \cap \mathrm{Soc}(G)$ on S_1 is a non-trivial proper subgroup of S_1. In this case

$$1 \ne U \cap \mathrm{Soc}(G) = R_1 \times \cdots \times R_n = (U \cap S_1) \times \cdots \times (U \cap S_n).$$

c) $\bigl(U \cap \mathrm{Soc}(G)\bigr)^{\pi_1} = 1$, i.e. U is a complement of $\mathrm{Soc}(G)$ in G.

26. With all the above remarks, we have a first approach to the O'Nan-Scott classification of primitive groups of type 2. We have the following five situations:

a) $\mathrm{Soc}(G)$ is a simple group, i.e. $n = 1$: the group G is almost simple;
b) $n > 1$ and $U \cap \mathrm{Soc}(G) = D$ is a full diagonal subgroup of $\mathrm{Soc}(G)$;
c) $n > 1$ and $U \cap \mathrm{Soc}(G) = D_1 \times \cdots \times D_l$, a direct product of l subgroups, with $1 < l < n$, such that, for each $j = 1, \dots, l$, the subgroup D_j is a full diagonal subgroup of a direct product $\prod_{i \in \mathcal{I}_j} S_i$, and $\{\mathcal{I}_1, \dots, \mathcal{I}_l\}$ is a minimal non-trivial G-invariant partition of $\mathcal{I}$ in blocks for the action of U on $\mathcal{I}$;

d) $n > 1$ and the projection $R_1 = \big(U \cap \mathrm{Soc}(G)\big)^{\pi_1}$ is a non-trivial proper subgroup of S_1; here, $R_1 = VC \cap S_1$ and VC/C is a maximal subgroup of X.
e) $U \cap \mathrm{Soc}(G) = 1$.

This enables us to describe all configurations of primitive groups of type 2.

Proposition 1.1.41. *Let S be a non-abelian simple group and consider an almost simple group X such that $S \leq X \leq \mathrm{Aut}(S)$. Let P_n be a primitive group of permutations of degree n. Construct the wreath product $W = X \wr P_n$ and consider the subgroups $D_X = \{(x, \dots, x) : x \in X\} \leq X^\natural$ and $D_S = \{(s, \dots, s) : s \in S\} \leq S^\natural$. Clearly $P_n \leq \mathrm{C}_W(D_X)$. Suppose that U is a subgroup of W such that $D_S \leq U \leq D_X \times P_n$, and the projection of U on P_n is surjective.*

Then the group $G = S^\natural U$ is a primitive group of type 2 and U is a core-free maximal subgroup of G.

Proof. It is clear that $S^\natural$ is a minimal normal subgroup of G and $\mathrm{C}_G(S^\natural) = 1$. Hence G is a primitive group of type 2 and $\mathrm{Soc}(G) = S^\natural$.

Observe that $D_S = \mathrm{Soc}(G) \cap D_X = \mathrm{Soc}(G) \cap U$. Since P_n is a primitive group, the action of U on the elements of the set $\{S_1, \dots, S_n\}$ is primitive and there are no non-trivial blocks. By 1.1.40 (23), U is a maximal subgroup of G. ☐

Definition 1.1.42. *A primitive pair (G, U) constructed as in Proposition 1.1.41 is called a* primitive pair with simple diagonal action.

A detailed and complete study of these primitive groups of simple diagonal type appears in [Kov88].

Remarks 1.1.43. In a primitive pair (G, U) with simple diagonal action, we have the following.

1. $U \cap \mathrm{Soc}(G) = D_S \neq 1$: this is the case 26b in 1.1.40
2. $D_S \cap (S_2 \times \cdots \times S_n) = 1$ and $\mathrm{Soc}(G) = D_S(S_2 \times \cdots \times S_n)$. Hence $\mathrm{N}_G(S_1) = \mathrm{N}_U(S_1)\,\mathrm{Soc}(G) = \mathrm{N}_U(S_1)(S_2 \times \cdots \times S_n)$, and analogously for the centraliser. Hence

$$\mathrm{N}_G(S_1)/\,\mathrm{C}_G(S_1) \cong \mathrm{N}_U(S_1)/\,\mathrm{C}_U(S_1).$$

Proposition 1.1.44. *Let (Z, H) be a primitive pair such that either Z is an almost simple group or (Z, H) is a primitive pair with simple diagonal action. Write $T = \mathrm{Soc}(Z)$. Given a positive integer $k > 1$, let P_k be a transitive group of degree k and construct the wreath product $W = Z \wr P_k$. Write P_{k-1} for the stabiliser of 1.*

Consider a subgroup $G \leq W$ such that

1. $\mathrm{Soc}(W) = T^\natural = T_1 \times \cdots \times T_k \leq G$,
2. the projection of G onto P_k is surjective,

3. the projection of $\mathrm{N}_G(T_1) = \mathrm{N}_W(T_1) \cap G = (Z_1 \times [Z_2 \times \cdots \times Z_k]P_{k-1}) \cap G$ *onto* Z_1 *is surjective.*

Put $U = G \cap (H \wr P_k)$*. Then* G *is a primitive group of type 2 and* U *is a core-free maximal subgroup of* G*.*

Proof. Set $M = H \cap T$; clearly $\mathrm{N}_Z(M) = H$. With the obvious notation, write $M^\natural = M_1 \times \cdots \times M_k$. Then clearly $H \wr P_k \leq \mathrm{N}_W(M^\natural)$. Moreover if $(z_1, \ldots, z_k)x \in \mathrm{N}_W(M^\natural)$, then $z_i \in \mathrm{N}_{Z_i}(M_i) = H_i$ for any $i = 1, \ldots, k$. Hence $H \wr P_k = \mathrm{N}_W(M^\natural)$ and therefore $U = \mathrm{N}_G(M^\natural)$.

Notice that $T_1 \times \cdots \times T_k$ is a minimal normal subgroup of G and $\mathrm{C}_G(T_1 \times \cdots \times T_k) = 1$. Hence G is a primitive group of type 2 and $\mathrm{Soc}(G) = T_1 \times \cdots \times T_k$.

Clearly $G = U \mathrm{Soc}(G)$. Since W is a semidirect product, every element of W can be written uniquely as a product of an element of $Z^\natural$ and an element of P_k. Hence, if $(h_1, \ldots, h_k)x \in T^\natural$, for $x \in P_k$ and $h_i \in H_i$, $i = 1, \ldots, k$, then $x = 1$ and $h_i \in T_i \cap H_i = M_i$. Hence $U \cap \mathrm{Soc}(G) = M^\natural$. In particular, U is core-free in G. Let us see that U is a maximal subgroup of G.

Observe that $\mathrm{N}_G(T_1) = \mathrm{N}_W(T_1) \cap U \mathrm{Soc}(G) = \mathrm{N}_U(T_1) \mathrm{Soc}(G)$. Let V_1 be the projection of $\mathrm{N}_U(T_1)$ on Z_1. It is clear that V_1 is contained in the projection of U on Z_1, i.e. $V_1 \leq H_1$. Since the projection of $\mathrm{N}_G(T_1)$ onto Z_1 is surjective and the projection of $\mathrm{Soc}(G)$ on Z_1 is T_1, then $Z_1 = V_1T_1$. Since clearly $M_1 \leq \mathrm{N}_G(T_1)$, then $M_1 \leq V_1 \leq H_1$, so $V_1 \cap T_1 = M_1$ and by easy order calculations, $V_1 = H_1$.

Let L be an intermediate subgroup $U \leq L < G$. By the above arguments, the projection of $\mathrm{N}_L(T_1)$ on Z_1 is an intermediate subgroup between H_1 and Z_1. By maximality of H in Z, we have that this projection is either H_1 or Z_1.

Write Q_i for the projection of $L \cap \mathrm{Soc}(G)$ on T_i, for $i = 1, \ldots, k$. Since L acts transitively by conjugation on the elements of the set $\{T_1, \ldots, T_k\}$, we have that all Q_i are isomorphic to a subgroup Q such that $M \leq Q \leq T$ and $L \cap \mathrm{Soc}(G) \leq Q_1 \times \cdots \times Q_k$. The subgroup $L \cap \mathrm{Soc}(G)$ is normal in L and then in $\mathrm{N}_L(T_1)$. Hence Q_1 is normal in the projection of $\mathrm{N}_L(T_1)$ on Z_1. If this projection is H_1, then Q is normal in H and then $M \leq Q \leq H \cap T = M$, i.e. $Q = M$. In this case $L = L \cap U \mathrm{Soc}(G) = U\bigl(L \cap \mathrm{Soc}(G)\bigr) = U$.

Suppose that the projection of $\mathrm{N}_L(T_1)$ on Z_1 is the whole of Z_1. Then Q is a normal subgroup of Z and therefore $Q = T$. If for each $i = 1, \ldots, k$ we write $T_i = S_{i1} \times \cdots \times S_{ir}$, where all the S_{ij} are isomorphic copies of a non-abelian simple group S, then we can put

$$\mathrm{Soc}(G) = (S_{11} \times \cdots \times S_{1r}) \times \cdots \times (S_{k1} \times \cdots \times S_{kr}).$$

The projection of $L \cap \mathrm{Soc}(G)$ on each simple component is surjective. By Remark 1.1.40 (23), $L \cap \mathrm{Soc}(G) = \prod_{\mathcal{D} \in \mathit{\Delta}} \bigl(L \cap \mathrm{Soc}(G)\bigr)^{\pi_{\mathcal{D}}}$ is a direct product of full diagonal subgroups and the partition $\mathit{\Delta}$ of the set $\{11, \ldots, 1r, \ldots, k1, \ldots, kr\}$ associated with $L \cap \mathrm{Soc}(G)$ is a set of blocks for the action of L. Observe that $M_1 \times 1 \times \cdots \times 1 \leq L \cap \mathrm{Soc}(G)$. If Z is an almost simple group, then $r = 1$ and $\mathcal{D} = \{1\}$ is a block of $\mathit{\Delta}$. Hence, in this case, $\mathit{\Delta}$ is the trivial partition of

$\{1,\ldots,k\}$. If (Z,H) is a primitive pair of simple diagonal action, then M is a full diagonal subgroup of T. Hence the set $\{11,\ldots,1r\}$ is the union set of some members $\mathcal{D}_1,\ldots,\mathcal{D}_l$ of the partition Δ. Since the projection of $L\cap\mathrm{Soc}(G)$ on T_1 is surjective, then $T_1=\prod_{i=1}^{l}\big(L\cap\mathrm{Soc}(G)\big)^{\pi_{\mathcal{D}_i}}\cong S_1\times\cdots\times S_l$ (here the S_i's are simply the names of the projections). Hence $l=r$. Since L is transitive on the T_i's, so that because the blocks corresponding to T_1 have one element, all the blocks do. In other words, $L\cap\mathrm{Soc}(G)=T_1\times\cdots\times T_k$. Hence $L=G$. □

Definition 1.1.45. *A primitive pair* (G,U) *constructed as in Proposition 1.1.44 is called a* primitive pair with product action.

A detailed and complete study of these primitive groups in product action appears in [Kov89].

Remarks 1.1.46. 1. If (Z,H) is a primitive pair, then Z is a permutation group on the set of right cosets of H in Z and the cardinality of Ω is $|Z:H|$ (the degree of the permutation group Z). Now, if (G,U) is a primitive pair with product action, as in Proposition 1.1.44, then the degree of the permutation group G is

$$|G:U|=|G:G\cap(H\wr P_k)|=|W:H\wr P_k|=|Z:H|^k.$$

2. Observe that we have two different types of primitive pairs with product action:

a) If Z is an almost simple group, $T=\mathrm{Soc}(Z)$, and $R=H\cap T$, then $1\neq R<T$ and the projection $R_1=\big(U\cap\mathrm{Soc}(G)\big)^{\pi_1}$ is a non-trivial proper subgroup of T_1, by Proposition 1.1.16; this is Case 26d in 1.1.40.

b) If (Z,H) is a primitive pair with simple diagonal action, then $U\cap\mathrm{Soc}(G)=D_1\times\cdots\times D_k$ a direct product of k full diagonal subgroups, with $1<k<n$; here we are in Case 26c of 1.1.40.

Examples 1.1.47. 1. Let S be a non-abelian simple group and H a maximal subgroup of S. If C is a cyclic group of order 2, construct the wreath product $G=S\wr C$ with respect to the regular action. The group G is a primitive group of type 2 and $\mathrm{Soc}(G)=S^{\natural}=S_1\times S_2$.

Consider the diagonal subgroup $D=\{(x,x):x\in S\}$. Then $U=D\times C$ is a core-free maximal subgroup of G and (G,U) is a primitive pair with diagonal action.

Consider now the subgroup $U^*=H\wr C=[H_1\times H_2]C$. Then U^* is also a core-free maximal subgroup of G and the pair (G,U^*) is a primitive pair with product action.

2. Let G be the primitive group of Example 1 and construct the wreath product $W=G\wr Z$ with respect to the regular action of the cyclic group Z of order 2. Then, the socle of W is isomorphic to the direct product of four copies of S: $\mathrm{Soc}(W)=S_1\times S_2\times S_3\times S_4$. Moreover $\mathrm{Soc}(W)$ is complemented by a 2-subgroup P isomorphic to the wreath product $C_2\wr C_2$, that is, isomorphic to the dihedral group of order 8. The group W is a primitive group of type 2.

If we consider the maximal subgroup U of G and construct $M = U \wr Z$, we obtain a core-free maximal subgroup of index $|W : M| = |S|^2$ such that $M \cap \mathrm{Soc}(W) = D_1 \times D_2$. Taking now the maximal subgroup U^* of G, then the subgroup $M^* = U^* \wr Z$ is another core-free maximal subgroup of W of index $|S : H|^4$ such that $M^* \cap \mathrm{Soc}(G) = H_1 \times H_2 \times H_3 \times H_4$.

Therefore the pairs (W, M) and (W, M^*) are non-equivalent primitive pairs of type 2 with product action.

Write $D_S = \{(s, s, s, s) : s \in S\}$, the full diagonal subgroup of $\mathrm{Soc}(W)$. Observe that M contains properly the subgroup $M_0 = D_S \times P$ and therefore M_0 is non-maximal in W.

According to Remark 1.1.40 (26), there still remains another structure of primitive group of type 2 to describe: those primitive groups of type 2 with the special property that the core-free maximal subgroup is a complement of the socle. This new configuration is in fact a twisted wreath product.

Theorem 1.1.48. *1. If (G, U) is a primitive pair of type 2 and $U \cap \mathrm{Soc}(G) = 1$, then, with the notation of Definition 1.1.32, $G \cong S \wr_{(V,\varphi)} U$.*

2. Conversely, let S be a non-abelian simple group and a group U with a subgroup V such that there exists a group homomorphism $\varphi\colon V \longrightarrow \mathrm{Aut}(S)$. Construct the twisted wreath product $G = S \wr_{(V,\varphi)} U$. If $\mathrm{Core}_U(V) = 1$ then G is a primitive group of type 2. Moreover, if U is maximal in G, then (G, U) is a primitive pair of type 2. By construction, $U \cap \mathrm{Soc}(G) = 1$.

Proof. 1. Recall that G is the induced extension defined by $\alpha\colon N/K_1 \longrightarrow G/\mathrm{Soc}(G)$. Hence $\mathrm{Soc}(G)$ is the induced U-group from the action φ of V on S (see Remark 1.1.40 (10)). Since G splits on $\mathrm{Soc}(G)$, then G is isomorphic to the twisted wreath product $G \cong S \wr_{(V,\varphi)} U$.

2. To prove the converse, it is enough to recall that in the twisted wreath product $G = S \wr_{(V,\varphi)} U$, we have that $\mathrm{C}_G(Z^\natural) = \mathrm{Z}(S^\natural) = 1$, by Proposition 1.1.34, and the conclusion follows. □

Definition 1.1.49. *A primitive pair (G, U) constructed as in Theorem 1.1.48 is called* a primitive pair with twisted wreath product action.

Maximal subgroups of a primitive group G of type 2 complementing $\mathrm{Soc}(G)$ are called by some authors *small maximal subgroups*.

Obviously one can wonder about the existence of primitive groups of type 2 with small maximal subgroups. P. Förster, in [För84a], gives sufficient conditions for U, V, and S to obtain a primitive group with small maximal subgroups.

Theorem 1.1.50 ([För84a]). *Let U be a group with a non-abelian simple non-normal subgroup S such that whenever A is a non-trivial subgroup of U such that $S \leq \mathrm{N}_U(A)$, then $S \leq A$. Write $V = \mathrm{N}_U(S)$ and $\varphi\colon V \longrightarrow \mathrm{Aut}(S)$ for the obvious group homomorphism induced by the conjugation. Construct the twisted wreath product $G = S \wr_{(V,\varphi)} U$.*

Then G is a primitive group of type 2 such that $\mathrm{Soc}(G) = S^\natural$, *the base group, is complemented by a maximal subgroup of G isomorphic to U.*

Proof. First we see that if $\mathrm{C}_U(S) \neq 1$, then, by hypothesis, we have that $S \leq \mathrm{C}_U(S)$ and this contradicts the fact that S is a non-abelian simple group. Hence $\mathrm{C}_U(S) = 1$ and φ is in fact a monomorphism of V into $\mathrm{Aut}(S)$ and V is an almost simple group such that $\mathrm{Soc}(V) = S$.

Write $n = |U : V|$ and $S^\natural = S_1 \times \cdots \times S_n$. Since U acts a transitive permutation group by right multiplication on the set of right cosets of V in U, and then on the set $\mathcal{I} = \{1, \dots, n\}$, $S^\natural$ is a minimal non-abelian subgroup of G. Moreover, if $C = \mathrm{Core}_U(V) \neq 1$, then $S \leq \mathrm{N}_U(C) = U$. Now C is an almost simple group with $\mathrm{Soc}(C) = S$. Hence S is normal in U, giving a contradiction. Hence $C = \mathrm{Core}_U(V) = 1$. Therefore, to prove that (G, U) is a primitive pair of type 2 with twisted wreath product action by Theorem 1.1.48, it only remains to prove U is a maximal subgroup of G. To do this, let M be a maximal subgroup of G such that $U \leq M$. Observe that $M = M \cap G = M \cap U \,\mathrm{Soc}(G) = U\big(M \cap \mathrm{Soc}(G)\big)$. All projections $R_j = \big(M \cap \mathrm{Soc}(G)\big)^{\pi_j}$, for $j \in \mathcal{I}$, are conjugate by elements of M, that is, all R_i are isomorphic to the subgroup R_1 and $S_1 \cap U \leq R_1 \leq S_1$ and $M \cap \mathrm{Soc}(G) \leq R_1 \times \cdots \times R_k$. Observe that $V \leq \mathrm{N}_G(S_1)$ by (1.3) in Proposition 1.1.31, since $v = v_{1,v}$, for all $v \in V$, and $1^v = 1$. By 1.1.18 (4), $(y^v)^{\pi_1} = (y^{\pi_1})^v$, for all $y \in M \cap \mathrm{Soc}(G)$. Since the subgroup S normalises $M \cap \mathrm{Soc}(G)$, then S normalises $R_1 = \big(M \cap \mathrm{Soc}(G)\big)^{\pi_1}$. The automorphisms induced in S_1 by S are the inner automorphisms. Hence R_1 is a normal subgroup of S_1, and, since S_1 is a simple group, we have that $R_1 = 1$ or $R_1 = S_1$. In the first case, we have that $M \cap \mathrm{Soc}(G) = 1$ and then $M = U$. Thus, assume that the projections π_j are surjective, for all $j \in \mathcal{I}$.

By 1.1.40 (23), there exists a minimal non-trivial M-invariant partition Δ of $\mathcal{I}$ in blocks for the action of M on $\mathcal{I}$ such that

$$M \cap \mathrm{Soc}(G) = \prod_{\mathcal{D} \in \Delta} \big(M \cap \mathrm{Soc}(G)\big)^{\pi_{\mathcal{D}}},$$

and, for each $\mathcal{D} \in \Delta$, the projection $\big(M \cap \mathrm{Soc}(G)\big)^{\pi_{\mathcal{D}}}$ is a full diagonal subgroup of the direct product $\prod_{i \in \mathcal{D}} S_i$.

For each $y \in M \cap \mathrm{Soc}(G)$ ad $x \in M$, we have that $(y^x)^{\pi_{\mathcal{D}^x}} = (y^{\pi_{\mathcal{D}}})^x$ for any $\mathcal{D} \in \Delta$. Suppose that Δ_0 is an orbit of the action of M on Δ. Then the subgroup

$$T = \prod_{\mathcal{D} \in \Delta_0} \big(M \cap \mathrm{Soc}(G)\big)^{\pi_{\mathcal{D}}}$$

is normal in M. If Δ_0 is a proper subset of Δ, then there exists some j which is not in a member of Δ_0. Then S_j centralises T and then T is normal in $\langle M, S_j \rangle$. Since T is a proper subgroup of $\mathrm{Soc}(G)$, we have that $S_j \leq M$, by maximality of M. But this implies that $\mathrm{Soc}(G) \leq M$, and this is not true. Hence, M acts transitively on Δ. And so does U, since $M = U\big(M \cap \mathrm{Soc}(G)\big)$.

Assume that each member $\mathcal{D}$ of Δ has m elements of $\mathcal{I}$ and $|\Delta| = l$, i.e. $n = lm$. Since Δ is a non-trivial partition, then $m > 1$.

Suppose that $l = 1$. This means that $M \cap \mathrm{Soc}(G)$ is a full diagonal subgroup of $\mathrm{Soc}(G)$. Hence $M = [M \cap \mathrm{Soc}(G)]U$ and $M \cap \mathrm{Soc}(G)$ is a normal subgroup of M which is isomorphic to S (π_1 is an isomorphism between $M \cap \mathrm{Soc}(G)$ and S_1). This gives a homomorphism $\psi\colon U \longrightarrow \mathrm{Aut}(S)$ whose restriction to V is the monomorphism φ. Notice that $\mathrm{Ker}(\psi)$ is a normal subgroup of U and, by hypothesis, if $\mathrm{Ker}(\psi) \neq 1$, then $S \leq \mathrm{Ker}(\psi)$. This contradicts the fact that φ is a monomorphism. Therefore $\mathrm{Ker}(\psi) = 1$ and ψ is a monomorphism. Since $S^\psi = \mathrm{Inn}(S)$ is normal in $U^\psi \leq \mathrm{Aut}(S)$, then S is normal in U. But this contradicts the fact that $\mathrm{Core}_U(S) = 1$. Hence $l > 1$.

The partition Δ has l members which are blocks for the action of M (or U) on $\mathcal{I}$. Write $\Delta = \{\mathcal{D}_1, \dots, \mathcal{D}_l\}$. The subgroup U acts transitively on Δ. We can assume without loss of generality that $1 \in \mathcal{D}_1$. Let U_1 denote the stabiliser of $\mathcal{D}_1$ by the action of U on Δ. Clearly $|U : U_1| = l$.

For any $x \in V$, since $V \leq \mathrm{N}_G(S_1)$, then $1^x = 1$ and $1 \in \mathcal{D}_1 \cap \mathcal{D}_1^x$. Hence $\mathcal{D}_1 = \mathcal{D}_1^x$ and $x \in U_1$. Therefore $V \leq U_1$. Since $D_1 = \big(M \cap \mathrm{Soc}(G)\big)^{\pi_{\mathcal{D}_1}} \cong S$, there exists a group homomorphism $\psi\colon U_1 \longrightarrow \mathrm{Aut}(D_1) \cong \mathrm{Aut}(S)$ whose restriction to V is the monomorphism φ. Repeating the arguments of the above paragraph, we obtain that S^ψ is normal in U_1^ψ and then $U_1 \leq \mathrm{N}_U(S) = V$. Therefore $V = U_1$.

But now we have that $l = |U : U_1| = |U : V| = n$, and then $m = 1$. This is the final contradiction. Thus we deduce that U is a maximal subgroup of G. ◻

Remarks 1.1.51. 1. Examples of pairs U, S satisfying the conditions of the hypothesis of Theorem 1.1.48 are $S = \mathrm{Alt}(n)$ and $U = \mathrm{Alt}(n+1)$, for $n \geq 5$. In this case S is maximal in U. Also $S = \mathrm{PSL}(2, p^n)$ and $U = \mathrm{PSL}(2, p^{2n})$, for $p^n \geq 3$ satisfies the hypothesis. Here $\mathrm{N}_U(S) \cong \mathrm{PGL}(2, p^n)$ is maximal in U.

2. In [Laf84b], J. Lafuente proved that if G is a primitive group of type 2 and U is a small maximal subgroup of G, then U is also a primitive group of type 2 and each simple component of $\mathrm{Soc}(U)$ is isomorphic to a section of the simple component of $\mathrm{Soc}(G)$.

The O'Nan-Scott Theorem proves that these are all possible configurations of primitive groups of type 2.

Theorem 1.1.52 (M. O'Nan and L. Scott). *Let G be a primitive group of type 2 and U a core-free maximal subgroup of G. Then one of the following holds:*

1. *G is an almost simple group;*
2. *(G, U) is equivalent to a primitive pair with simple diagonal action; in this case $U \cap \mathrm{Soc}(G)$ is a full diagonal subgroup of $\mathrm{Soc}(G)$;*
3. *(G, U) is equivalent to a primitive pair with product action such that $U \cap \mathrm{Soc}(G) = D_1 \times \cdots \times D_l$, a direct product of $l > 1$ subgroups such that, for each $j = 1, \dots, l$, the subgroup D_j is a full diagonal subgroup of a direct product $\prod_{i \in \mathcal{I}_j} S_i$, and $\{\mathcal{I}_1, \dots, \mathcal{I}_l\}$ is a minimal non-trivial G-invariant partition of $\mathcal{I}$ in blocks for the action of U on $\mathcal{I}$.*

4. *(G,U) is equivalent to a primitive pair with product action such that the projection $R_1 = \big(U \cap \mathrm{Soc}(G)\big)^{\pi_1}$ is a non-trivial proper subgroup of S_1; in this case $R_1 = VC \cap S_1$ and VC/C is a maximal subgroup of X;*
5. *(G,U) is equivalent to a primitive pair with twisted wreath product action; in this case $U \cap \mathrm{Soc}(G) = 1$.*

Proof. Recall that by 1.1.40 we can distinguish five different cases.

Case 1. If $n = 1$, then G is an almost simple group. Thus we suppose that $n > 1$.

Case 2. Assume that $n > 1$ and $U \cap \mathrm{Soc}(G) = D$ is a full diagonal subgroup Then there exist automorphisms $\varphi_i \in \mathrm{Aut}(S)$, $i \in \mathcal{I}$, such that $D = U \cap \mathrm{Soc}(G) = \{(x^{\varphi_1}, x^{\varphi_2}, \dots, x^{\varphi_n}) : x \in S\}$. Since D is normal in U and U is maximal in G, we have that $U = \mathrm{N}_G(D)$. Let P_n be the permutation group induced by the conjugacy action of G on the simple components of $\mathrm{Soc}(G)$: $P_n = G/Y$ (see 1.1.40 (13)). By 1.1.40 (23), the group P_n is transitive and primitive. We embed G in $X \wr P_n$ as in 1.1.40 (13) and then in $\mathrm{Aut}(S) \wr P_n$. Consider $\varphi = (\varphi_1^{-1}, \dots, \varphi_n^{-1}) \in \mathrm{Aut}(S)^n \leq \mathrm{Aut}(S) \wr P_n$. By conjugation by φ in $\mathrm{Aut}(S) \wr P_n$, we have that $D^{\varphi} = D_S = \{(x, \dots, x) : x \in S\}$ and $U^{\varphi} = \mathrm{N}_{G^{\varphi}}(D_S) = G^{\varphi} \cap (D_X \times P_n)$, where $D_X = \{(x, \dots, x) : x \in X\}$. Then $G^{\varphi} = U^{\varphi} S^{\natural}$ and, since $S_i^{\varphi} = S_i$, for all $i \in \mathcal{I}$, the action of U^{φ} and of U on $\mathcal{I}$ are the same. Hence, the projection of U^{φ} onto P_n is surjective. By Proposition 1.1.41, we have that $(G^{\varphi}, U^{\varphi})$ is a primitive pair with simple diagonal action and is equivalent to (G,U).

Case 3. Assume that $n > 1$ and $U \cap \mathrm{Soc}(G) = D_1 \times \cdots \times D_l$, a direct product of $l > 1$ subgroups such that, for each $j = 1, \dots, l$, the subgroup D_j is a full diagonal subgroup of a direct product $\prod_{i \in \mathcal{I}j} S_i$, and $\{\mathcal{I}_1, \dots, \mathcal{I}_l\}$ is a minimal non-trivial U-invariant partition of $\mathcal{I}$ in blocks for the action of U on $\mathcal{I}$.

Suppose that the S_i are ordered in such a way that $\mathcal{I}_1 = \{1, \dots, m\}$. Write $K = S_1 \times \cdots \times S_m$, $N^* = \mathrm{N}_G(K)$, $C^* = \mathrm{C}_G(K)$. Observe that $\mathcal{I}_1$ is a minimal block for the action of G on $\mathcal{I}$. Then N^* acts transitively and primitively on $\mathcal{I}_1$. Hence, $X^* = N^*/C^*$ is a primitive group whose socle is $\mathrm{Soc}(X^*) = KC^*/C^*$. Put $V^* = U \cap N^*$. Since $\mathrm{Soc}(G) \leq N^*$, then $N^* = N^* \cap U \,\mathrm{Soc}(G) = V^* \,\mathrm{Soc}(G) = V^* C^* K$. Moreover $K \cap V^* = K \cap N^* \cap U = K \cap U = D_1$. Let $\{g_1 = 1, \dots, g_l\}$ be a right transversal of V^* in U (and of N^* in G). We can assume that this transversal is ordered in such a way that $D_1^{g_i} = D_i$, for $i = 1, \dots, l$, and put $K_i = K^{g_i}$, for $i = 1, \dots, l$. Then G acts transitively, by conjugation of the K_i's, on the set $\{K_1, \dots, K_l\}$.

Clearly D_1 is a V^*-invariant subgroup of K. Suppose that $D_1 \leq T_1 < K_1$ and T_1 is a V^*-invariant subgroup of K_1. Then $T_1 \times T_1^{g_2} \cdots \times T_1^{g_l}$ is U-invariant in $\mathrm{Soc}(G)$ and, by maximality of U, we have that $T_1 \times T_1^{g_2} \cdots \times T_1^{g_l} = U \cap \mathrm{Soc}(G) = D_1 \times \cdots \times D_l$. Hence $D_1 = T_1$. In other words, D_1 is maximal as V^*-invariant subgroup of K and then a maximal V^*C^*-invariant subgroup of K. Suppose that $s \in S_1 \cap V^* C^*$. There exist $v \in V^*$ and $c \in C^*$, such that $s = vc$. Now $v = sc^{-1} \in \mathrm{C}_G(S_i)$, for $i = 2, \dots, m$ and $v \in S_1 \,\mathrm{C}_G(S_1) \leq \mathrm{N}_G(S_1)$.

Consider the element $(t, t^{\varphi_2} \dots, t^{\varphi_m}) \in D_1$ associated with some $t \in S_1$; then $(t, t^{\varphi_2} \dots, t^{\varphi_m})^v = (t^v, t^{\varphi_2} \dots, t^{\varphi_m}) \in D_1$, since D_1 is normal in V^*. Hence $t^v = t$. This happens for any $t \in S_1$ and therefore $v \in \mathrm{C}_G(S_1)$. Hence $s \in \mathrm{C}_{S_1}(S_1) = 1$. Therefore $S_1 \cap V^*C^* = 1$ and then $K \neq K \cap V^*C^*$. Since $D_1 \leq V^*C^* \cap K \leq K$ and D_1 is maximal as V^*C^*-subgroup of K, we have that $D_1 = V^*C^* \cap K$. And, finally, if M is a maximal subgroup of N^* such that $V^*C^* \leq M$, then $M \cap K$ is a V^*C^*-invariant subgroup of K containing D_1. Hence $D_1 = V^*C^* \cap K = M \cap K$. Now $M = M \cap N^* = M \cap V^*C^*K = V^*C^*(M \cap K) = V^*C^*$. Therefore V^*C^*/C^* is a core-free maximal subgroup of X^*.

Observe that $(V^*C^*/C^*) \cap \mathrm{Soc}(X^*) = D_1C^*/C^*$ is a full diagonal subgroup of $\mathrm{Soc}(X^*)$. Thus X^* is a group of Case 2. Hence $(X^*, V^*C^*/C^*)$ is a primitive pair with simple diagonal action.

Write $P_l = G/\big(\bigcap_{i=1}^{l} \mathrm{N}_G(K_i)\big)$ for the permutation group induced by the action of G by conjugation of the K_i's. For any $g \in G$, we write g^ρ for the projection of g in P_l. On the other hand, for each $g \in G$ and each $i \in \{1, \dots, l\}$, let $a_{i,g}$ be the element of N^* such that $g_i g = a_{i,g} g_j$, for some j. For any $a \in N^*$, write $\bar{a} = aC^*$ for the projection of a on X^*. Consider the conjugacy action $\psi\colon N^* \longrightarrow \mathrm{Aut}(K)$ and the induced G-action on $(X^*)^\natural$:

$$\psi^G\colon G \longrightarrow X^* \wr P_l \quad \text{given by} \quad g^{\psi^G} = (\bar{a}_{1,g}, \dots, \bar{a}_{l,g})g^\rho, \quad \text{for any } g \in G.$$

Arguing as in 1.1.40 (13–14), we have that

1. the map ψ^G is a group homomorphism and is injective; the projection of G^{ψ^G} on P_l is surjective;
2. $\mathrm{N}_G(K_1)^{\psi^G} = G^{\psi^G} \cap (X_1^* \times [X_2^* \times \cdots \times X_l^*]P_{l-1})$, where P_{l-1} is the stabiliser of 1. The image of $\mathrm{N}_G(K_1)$ by the projection on the first component of $(X^*)^\natural$ is the whole of X_1^*;
3. the elements of $\mathrm{Soc}(G)$ can be written as $(e_1, e_2^{g_2}, \dots, e_l^{g_l})$, for certain $e_1, \dots, e_l \in K_1$. The image by ψ^G of the elements of the socle is
$$(e_1, e_2^{g_2}, \dots, e_l^{g_l})^{\psi^G} = (\bar{e}_1, \bar{e}_2, \dots, \bar{e}_l),$$
and then $(KC^*/C^*)^\natural = \mathrm{Soc}(X^* \wr P_l) \leq G^{\psi^G}$.

Now, for any $g \in U$, since the $g_i \in U$, we have that $a_{i,g} \in N^* \cap U = V^*$. Hence $U^{\psi^G} \leq G^{\psi^G} \cap \big((V^*C^*/C^*) \wr P_l\big)$. Since V^*C^*/C^* is maximal in X^* and U^{ψ^G} is maximal in G^{ψ^G}, we have that $U^{\psi^G} = G^{\psi^G} \cap \big((V^*C^*/C^* \wr P_1\big)$. By Proposition 1.1.44, this means that (G, U) is equivalent to (G^{ψ^G}, U^{ψ^G}) which is a primitive pair with product action.

Case 4. Suppose now $n > 1$ and the projection $R_1 = \big(U \cap \mathrm{Soc}(G)\big)^{\pi_1}$ is a non-trivial proper subgroup of S_1.

Moreover, $R_1 = VC \cap S_1$ and VC is a maximal subgroup of N.

Consider the embedding $\psi^G\colon G \longrightarrow X \wr P_n$ of 1.1.40 (13). Then X is almost simple and G is isomorphic to a subgroup G^{ψ^G} of $X \wr P_n$ satisfying all conditions of Proposition 1.1.44. Hence $U^{\psi^G} \leq G^{\psi^G} \cap \bigl((VC/C) \wr P_n\bigr)$. Since VC/C is maximal in X and U^{ψ^G} is maximal in G^{ψ^G}, we have that $U^{\psi} = G^{\psi} \cap \bigl((VC/C) \wr P_n\bigr)$. Therefore (G,U) is equivalent to a primitive pair with product action.

Case 5. Assume finally that $U \cap \operatorname{Soc}(G) = 1$. Then, by Theorem 1.1.48, $G \cong S \wr_{(V,\varphi)} U$ and the pair (G,U) is equivalent to a primitive pair with twisted wreath product action. □

If U is a core-free maximal subgroup of a primitive group G of type 2, then there are exactly three different possibilities as we saw in 1.1.40 (25):

1. $\bigl(U \cap \operatorname{Soc}(G)\bigr)^{\pi_1} = S_1$, i.e. the projection π_1 of $U \cap \operatorname{Soc}(G)$ on S_1 is surjective.
2. $1 \neq R_1 = \bigl(U \cap \operatorname{Soc}(G)\bigr)^{\pi_1} < S_1$, i.e. the image of the projection π_1 of $U \cap \operatorname{Soc}(G)$ on S_1 is a non-trivial proper subgroup of S_1.
$$1 \neq U \cap \operatorname{Soc}(G) = R_1 \times \cdots \times R_n = (U \cap S_1) \times \cdots \times (U \cap S_n).$$
3. $\bigl(U \cap \operatorname{Soc}(G)\bigr)^{\pi_1} = 1$, i.e. U is a complement of $\operatorname{Soc}(G)$ in G.

As we saw in 1.1.35, in a primitive group G of type 2, there exists a bijection between

1. the set of all conjugacy classes of maximal subgroups U of G such that the projection $\bigl(U \cap \operatorname{Soc}(G)\bigr)^{\pi_1}$ is a proper subgroup of S_1,
2. the set of all conjugacy classes of maximal subgroups of $N/(S_2 \times \cdots \times S_n)$ supplementing $\operatorname{Soc}(G)/(S_2 \times \cdots \times S_n)$.

Under this bijection, the complements, if any, of $\operatorname{Soc}(G)$ in G are in correspondence with the complements of $\operatorname{Soc}(G)/K_1$ in N/K_1. Thus, this bijection works in Cases 2 and 3. Since core-free maximal subgroups of Case 2 occur in every primitive group of type 2, these are called *frequent maximal subgroups* by some authors. We complete this study in the following way.

Proposition 1.1.53. *Let G be a primitive group of type 2. There exist bijections between the following sets:*

1. *the set of all conjugacy classes of maximal subgroups U of G such that the projection $\bigl(U \cap \operatorname{Soc}(G)\bigr)^{\pi_1}$ is a non-trivial proper subgroup of S_1,*
2. *the set of all conjugacy classes of maximal subgroups of $N/(S_2 \times \cdots \times S_n)$ supplementing but not complementing $\operatorname{Soc}(G)/(S_2 \times \cdots \times S_n)$, and*
3. *the set of all conjugacy classes of core-free maximal subgroups of X.*

Proof. We only have to see the bijection between the sets in 2 and 3. Write $K = S_2 \times \cdots \times S_n$ and observe that if L/C is core-free maximal subgroup of X, then obviously L/K is a maximal subgroup of N/K and $N = L\operatorname{Soc}(G)$. If L/K complements $\operatorname{Soc}(G)/K$ in N/K, then $K = L \cap \operatorname{Soc}(G)$; in particular $L \cap S_1 = 1$. But $L \cap S_1C = C(L \cap S_1) = C$ and this contradicts the fact that $(L/C) \cap (S_1C/C)$ is non-trivial by Proposition 1.1.16. Thus L does not complement $\mathrm{Soc}(G)/K$ in N.

Conversely, let L/K be a maximal subgroup of N/K such that $N = L\,\mathrm{Soc}(G)$ and $K < \mathrm{Soc}(G) \cap L$. Let us see that $C \leq L$. Consider $L_0/K = \mathrm{Core}_{N/K}(L/K)$. Since $\mathrm{Soc}(G)/K$ is a minimal normal subgroup of N/K, then $L_0/K \leq \mathrm{C}_{N/K}\big(\mathrm{Soc}(G)/K\big) = C/K$ and $L_0 \leq C$. If $L_0 = C$, then $C \leq L$ and we are done. Suppose that C/L_0 is nontrivial. Since L/L_0 is a core-free maximal subgroup of N/L_0, it is clear that N/L_0 is a primitive group. Observe that $\mathrm{Soc}(G)L_0/L_0$ is a minimal normal subgroup of N/L_0 and $\mathrm{C}_{N/L_0}\big(\mathrm{Soc}(G)L_0/L_0\big) = C/L_0$. Since we are assuming that C/L_0 is nontrivial, the primitive group N/L_0 is of type 3. Hence L/L_0 complements $\mathrm{Soc}(G)L_0/L_0$. This is to say that $L \cap \mathrm{Soc}(G) \leq L_0$, i.e. $L \cap \mathrm{Soc}(G) = L_0 \cap \mathrm{Soc}(G)$. Therefore $L \cap \mathrm{Soc}(G)$ is a normal subgroup of N between K and $\mathrm{Soc}(G)$. Since $\mathrm{Soc}(G)/K \cong S$, a non-abelian simple group, and L supplements $\mathrm{Soc}(G)$ in N, we have that $K = \mathrm{Soc}(G) \cap L$. This is not possible. □

As we saw in 1.1.35, the existence of complements of the socle in a primitive group G of type 2 is characterised by the existence of complements of $\mathrm{Soc}(G)/(S_2 \times \cdots \times S_n)$ in $\mathrm{N}_G(S_1)/(S_2 \times \cdots \times S_n)$. We wonder whether it is possible to obtain a characterisation of the existence of complements of $\mathrm{Soc}(G)$ in G in terms of complements of $\mathrm{Soc}(X)$ in X as we saw in 1.1.53 for supplements. The answer is partially affirmative.

Corollary 1.1.54. *With the notation of 1.1.40, let G be a primitive group of type 2 such that* $\mathrm{Soc}(X)$ *is complemented in* X. *Then* $\mathrm{Soc}(G)$ *is complemented in* G.

The converse does not hold in general.

Proof. Suppose that there exists a subgroup $Y \leq N$ such that $C \leq Y$ and $N = YS_1$ and $Y \cap S_1C = C$. Then it is clear that

$$S_2 \times \cdots \times S_n \leq Y \cap \mathrm{Soc}(G) \leq Y \cap S_1C \cap \mathrm{Soc}(G) = C \cap \mathrm{Soc}(G) = S_2 \times \cdots \times S_n$$

and therefore Y is a complement of $\mathrm{Soc}(G)/(S_2 \times \cdots \times S_n)$ in $N/(S_2 \times \cdots \times S_n)$. The conclusion follows by Theorem 1.1.35.

It is well-known that if $S = \mathrm{Alt}(6)$, the alternating group of degree 6, the automorphism group $A = \mathrm{Aut}(S)$ is an almost simple group whose socle is non-complemented. With the cyclic group $C \cong C_2$ we consider the regular wreath product $H = A \wr C$. In H we consider the diagonal subgroups $D_S = \{(x,x) : x \in S)\}$ and $D_A = \{(x,x) : x \in A\}$. Then $\mathrm{N}_H(D_S) = D_AC$. Since $D_S \cong S$, the conjugacy action of $\mathrm{N}_H(D_S)$ on D_S gives a group homomorphism $\varphi\colon \mathrm{N}_H(D_S) \longrightarrow \mathrm{Aut}(S)$. We construct the twisted wreath product $G = S \wr_{(\mathrm{N}_H(D_S),\varphi)} H$. Then $\mathrm{Soc}(G) = S_1 \times \cdots \times S_n$ is a minimal normal subgroup of G and it is the direct product of $n = |H : \mathrm{N}_H(D)|$ copies of S. Moreover since $\mathrm{Core}_H\big(\mathrm{N}_H(D_S)\big) = 1$, then $\mathrm{C}_G\big(\mathrm{Soc}(G)\big) = 1$ by Proposition 1.1.34 (2). Hence G is a primitive group of type 2. Clearly $\mathrm{Soc}(G)$ is complemented in G. $\mathrm{N}_H(S_1) = \mathrm{N}_H(D_S) = D_A C$ and $\mathrm{C}_H(S_1) = \mathrm{Ker}(\varphi) = \mathrm{C}_H(D_S) = C$. Hence, $X \cong D_A \cong A$ and $\mathrm{Soc}(X)$ is not complemented in X. □

Primitive pairs (G, U) of diagonal type, i.e. core-free maximal subgroups U of primitive groups G of type 2 such that the projection π_1 of $U \cap \mathrm{Soc}(G)$ on S_1 is surjective, appear in Cases (2) and (3) of the O'Nan-Scott Theorem. In this case $U \cap \mathrm{Soc}(G)$ is a direct product of l full diagonal subgroups, with $1 \le l < n$, and $U = \mathrm{N}_G(D)$.

Proposition 1.1.55. *Let G be a primitive group of type 2. Given a minimal non-trivial partition $\Delta = \{\mathcal{I}_1, \dots, \mathcal{I}_l\}$ of $\mathcal{I}$ in blocks for the action of G on $\mathcal{I}$ and a subgroup $D = D_1 \times \cdots \times D_l$, where D_j is a full diagonal subgroup of $\prod_{i \in \mathcal{I}_j} S_i$, for each $j = 1, \dots, l$, associated with Δ. The following statements are pairwise equivalent:*

1. *there exists a maximal subgroup U of G such that $U \cap \mathrm{Soc}(G) = D$;*
2. *$\mathrm{N}_G(D)$ is a maximal subgroup of G;*
3. *$G = \mathrm{N}_G(D)\,\mathrm{Soc}(G)$.*

Proof. 1 implies 2. Suppose that there exists a maximal subgroup U of G such that $U \cap \mathrm{Soc}(G) = D$. Then $U \le \mathrm{N}_G(D)$ and, by maximality of U in G, we have that $U = \mathrm{N}_G(D)$.

2 implies 3. Observe that $\mathrm{N}_G(D) \cap \mathrm{Soc}(G) = \mathrm{N}_{\mathrm{Soc}(G)}(D) = D$, by Lemma 1.1.38, and then $\mathrm{Soc}(G) \not\le \mathrm{N}_G(D)$. Therefore $G = \mathrm{N}_G(D)\,\mathrm{Soc}(G)$.

3 implies 1. Let H be a maximal subgroup of G such that $\mathrm{N}_G(D) \le H$. Then $D = \mathrm{N}_{\mathrm{Soc}(G)}(D) = \mathrm{Soc}(G) \cap \mathrm{N}_G(D) \le \mathrm{Soc}(G) \cap H$. Then $H \cap \mathrm{Soc}(G)$ is a direct product of full diagonal subgroups associated with a partition of $\mathcal{I}$ which refines $\{\mathcal{I}_1, \dots, \mathcal{I}_l\}$, by Proposition 1.1.39. By minimality of the blocks, we have that $H \cap \mathrm{Soc}(G) = D$ and therefore $H = \mathrm{N}_G(D)$. □

Example 1.1.56. We construct a primitive group G of type 2 with no maximal subgroup of diagonal type. Consider the symmetric group of degree 5, $H \cong \mathrm{Sym}(5)$ and denote with S the alternating group of degree 5. If C is a cyclic group of order 2, let G be the regular wreath product $G = H \wr C$. Then $\mathrm{Soc}(G) = S_1 \times S_2 \cong \mathrm{Alt}(5) \times \mathrm{Alt}(5)$. Any full diagonal subgroup of $\mathrm{Soc}(G)$ is isomorphic to $\mathrm{Alt}(5)$ and its normaliser N is isomorphic to $\mathrm{Sym}(5) \times C_2$. Observe that $|G/\mathrm{Soc}(G)| = 8 > 4 = |N\,\mathrm{Soc}(G)/\mathrm{Soc}(G)|$. Hence N does not satisfy 3. Clearly $N\,\mathrm{Soc}(G)$ is a normal maximal subgroup of G containing N.

Proposition 1.1.57. *Let G be a primitive group of type 2. Two maximal subgroups U, U^* of G, such that $U \cap \mathrm{Soc}(G)$ and $U^* \cap \mathrm{Soc}(G)$ are direct products of full diagonal subgroups, are conjugate in G if and only if $U \cap \mathrm{Soc}(G)$ and $U^* \cap \mathrm{Soc}(G)$ are conjugate in $\mathrm{Soc}(G)$.*

Proof. Suppose that $U^g = U^*$ for some $g \in G$. Then $g = xh$, with $x \in \mathrm{N}_G\big(U \cap \mathrm{Soc}(G)\big)$ and $h \in \mathrm{Soc}(G)$. Hence $U^* \cap \mathrm{Soc}(G) = \big(U \cap \mathrm{Soc}(G)\big)^g = \big(U \cap$

$\mathrm{Soc}(G)\big)^h$. Conversely, if $U^* \cap \mathrm{Soc}(G) = \big(U \cap \mathrm{Soc}(G)\big)^h$ for some $h \in \mathrm{Soc}(G)$, then $U^* = \mathrm{N}_G\big(U^* \cap \mathrm{Soc}(G)\big) = \mathrm{N}_G\Big(\big(U \cap \mathrm{Soc}(G)\big)^h\Big) = \mathrm{N}_G\big(U \cap \mathrm{Soc}(G)\big)^h = U^h$. □

1.2 A generalisation of the Jordan-Hölder theorem

In the first book dedicated to Group Theory, the celebrated *Traité des substitutions et des équations algébriques* ([Jor70]), published in Paris in 1870, the author, C. Jordan, presents the first version of a theorem known as the Jordan-Hölder Theorem: *The length of all composition series of a finite group is an invariant of the group and the orders of the composition factors are uniquely determined by the group.* Nineteen years later, in 1889, O. Hölder ([Höl89]) completed his contribution to the theorem proving that not only the orders but even the composition factors are uniquely determined by the group.

In recent years a number of generalisations of the classic Jordan-Hölder Theorem have been done. For example it has been proved that *given two chief series of a finite group G, there is a one-to-one correspondence between the chief factors of the series, corresponding factors being G-isomorphic, such that the Frattini chief factors of one series correspond to the Frattini chief factors of the other* (see [DH92, A, 9.13]). This result was first published by R. W. Carter, B. Fischer, and T. O. Hawkes (see [CFH68]) for soluble groups, and for finite groups in general by J. Lafuente (see [Laf78]). A further contribution is given by D. W. Barnes (see [Bar72]), for soluble groups, and again by J. Lafuente [Laf89] for finite groups in general, describing the bijection in terms of common supplements.

But if we restrict our arguments to a proper subset of the set of all maximal subgroups, we find that this is no longer true. For instance, in the elementary abelian group G of order 4, there are three maximal subgroups, say A, B, and C. If we consider the set $\mathbf{X} = \{A, B\}$, the maximal subgroup B is a common complement in $\mathbf{X}$ for the chief factors A and C. Also G/A is complemented by $A \in \mathbf{X}$. However G/C has no complement in $\mathbf{X}$.

In general, the key of the proof of these Jordan-Hölder-type theorems is to prove the result in the particular case of two pieces of chief series of a group G of the form

$$1 < N_1 < N_1 \times N_2 \qquad 1 < N_2 < N_1 \times N_2$$

where N_1 and N_2 are minimal normal subgroups of G. It is not difficult to prove that if N_1N_2/N_1 is supplemented by a maximal subgroup M, then M also supplements N_2 (see Lemma 1.2.16), but the converse is not true. The particular case in which N_1 and N_2 are supplemented and either N_1N_2/N_1 or N_1N_2/N_2 is a Frattini chief factor is the hardest one (see [DH92, A, 9.12]) and, in fact, proving the generalised Jordan-Hölder Theorem is reduced to proving that, in the above situation, N_1N_2/N_1 and N_1N_2/N_2 are simultaneously Frattini chief factors of G.

For this reason J. Lafuente, in [Laf89], wonders about the precise condition on a set $\mathbf{X}$ of maximal subgroups of a group G which allows a proof that, in the above situation, if N_1 and N_2 have supplements in $\mathbf{X}$, then N_1N_2/N_1 and N_1N_2/N_1 possess simultaneously supplements in $\mathbf{X}$, or, in other words, which is the precise condition on $\mathbf{X}$ to prove a Jordan-Hölder-type Theorem. In this section we present, among other related results, an answer to this question.

Definition 1.2.1. *Given a group G and two normal subgroups K, H of G such that $K \leq H$, we say that the section H/K is a* chief factor *of G if there is no normal subgroup of G between K and H, i.e. if N is a normal subgroup of G and $K \leq N \leq H$, then either $H = N$ or $K = N$.*

Equivalently, H/K is a chief factor of G if H/K is a minimal normal subgroup of G/K.

Hence H/K is a direct product of copies of a simple group and we have two possibilities:

1. either H/K is abelian, and there exists a prime p such that H/K is an elementary abelian p-group, or
2. H/K is non-abelian, and there exists a non-abelian simple group S such that $H/K \cong S_1 \times \cdots \times S_n$, where $S_i \cong S$ for all $i = 1, \dots, n$.

Given a group G and two normal subgroups K, H of G such that $K \leq H$, the group G acts by conjugation on the cosets of the section H/K: for $h \in H$ and $g \in G$, then $(hK)^g = h^gK$. This action of G on H/K defines a group homomorphism $\varphi\colon G \longrightarrow \mathrm{Aut}(H/K)$ such that

$$\mathrm{Ker}(\varphi) = \mathrm{C}_G(H/K) = \{g \in G : h^gK = hK \text{ for all } h \in H\}.$$

We say that $\mathrm{C}_G(H/K)$ is the *centraliser* of H/K in G. We write $\mathrm{Aut}_G(H/K) = \mathrm{Im}(\varphi) \cong G/\mathrm{C}_G(H/K)$ for the group of automorphisms of H/K induced by the conjugation of the elements of G. The set of G composed of all elements which induce inner automorphisms on H/K is the subset $\mathrm{C}^*_G(H/K) = H\,\mathrm{C}_G(H/K)$.

Definition 1.2.2. *Given a chief factor H/K of a group G, the* inneriser *of H/K in G is the subgroup*

$$\mathrm{C}^*_G(H/K) = H\,\mathrm{C}_G(H/K).$$

It is clear that if H/K is abelian, then $\mathrm{C}^*_G(H/K) = \mathrm{C}_G(H/K)$

Definition 1.2.3. *Let G be a group and let F_1 and F_2 two chief factors of G. A map $\gamma\colon F_1 \longrightarrow F_2$ is a G-*isomorphism *if γ is a group isomorphism and $(x^g)^\gamma = (x^\gamma)^g$, for any $x \in F_1$ and any $g \in G$.*

*Two chief factors F_1, F_2 of G are G-*isomorphic *if there exists a G-isomorphism $\gamma\colon F_1 \longrightarrow F_2$.*

If two chief factors F_1, F_2 of G are G-isomorphic, then write $F_1 \cong_G F_2$.

Proposition 1.2.4. *Let G be a group and let H_1/K_1 and H_2/K_2 be two chief factors of G.*

1. *If H_1/K_1 and H_2/K_2 are G-isomorphic, then $\mathrm{C}_G(H_1/K_1) = \mathrm{C}_G(H_2/K_2)$.*
2. *In general, the converse of 1 is not true.*
3. *Suppose that H_1/K_1 and H_2/K_2 are non-abelian. Then H_1/K_1 and H_2/K_2 are G-isomorphic if and only if $\mathrm{C}_G(H_1/K_1) = \mathrm{C}_G(H_2/K_2)$.*

Proof. Since clearly 1 is true, we prove 3 and give a counterexample to prove 2.

Suppose that H_1/K_1 and H_2/K_2 are non-abelian chief factors of G such that $C = \mathrm{C}_G(H_1/K_1) = \mathrm{C}_G(H_2/K_2)$. We have that $K_i \leq C \cap H_i \leq H_i$, for $i = 1, 2$. Since the chief factors are non-abelian, H_i is not contained in C. Therefore $K_i = C \cap H_i$, for $i = 1, 2$. Hence, $H_i/K_i \cong_G H_iC/C$, for $i = 1, 2$. Observe that H_1C/C is a minimal normal subgroup of the group G/C with trivial centraliser. This means that G/C is a primitive group of type 2, by Proposition 1.1.14. Since H_2C/C is also a minimal normal subgroup of G/C, then $H_1C = H_2C$. Hence H_1/K_1 and H_2/K_2 are G-isomorphic.

To see that this does not hold when the chief factors are abelian, let P be an extraspecial p-group, p an odd prime, of order p^3. Let F be a field of characteristic q, with $q \neq p$, such that F contains a primitive p-th root of unity. Then there exist $p-1$ non-equivalent irreducible and faithful P-modules over F of dimension p (see [DH92, B, 9.16]). Since $p - 1 > 1$, we can consider two non-isomorphic such P-modules, V_1, V_2. If V is the direct sum $V = V_1 \oplus V_2$, construct the semidirect product $G = [V]P$. The group G has two isomorphic minimal normal subgroups V_1, V_2 such that $\mathrm{C}_G(V_i) = V$, for $i = 1, 2$. But V_1 and V_2 are not G-isomorphic. □

Observe that in a primitive group G of type 3, the two minimal normal subgroups are not G-isomorphic. In other words, G-isomorphism is an equivalence relation in the set of all chief factors of G which is too "narrow" to include the case of the relation between the two minimal normal subgroups of a primitive group of type 3. J. Lafuente and P. Förster [För83] propose two equivalent "enlargements" of G-isomorphism. Here we follow Lafuente's definition.

Definition 1.2.5. *Let G be a group. We say that two given chief factors of G are G-*connected *if either they are G-isomorphic or there exists a normal subgroup N of G such that G/N is a primitive group of type 3 whose minimal normal subgroups are G-isomorphic to the given chief factors.*

Obviously, in a group G, two abelian chief factors are G-connected if and only if they are G-isomorphic.

Proposition 1.2.6 ([Laf84a]). *In a group G, the relation of being G-connected is an equivalence relation on the set of all chief factors of G.*

Proof. The only non-obvious property to prove is transitivity. Let F_1, F_2, F_3 be chief factors of G such that F_1 is G-connected to F_2 and F_2 is G-connected to F_3. We may suppose that no two are G-isomorphic. Therefore

1. there exists a normal subgroup N of G such that G/N is a primitive group of type 3 whose minimal normal subgroups are $A/N \cong_G F_1$ and $B/N \cong_G F_2$, and
2. there exists a normal subgroup M of G such that G/M is a primitive group of type 3 whose minimal normal subgroups are $C/M \cong_G F_2$ and $D/M \cong_G F_3$.

Observe that $\mathrm{C}_G(F_2) = \mathrm{C}_G(B/N) = A$ and also $\mathrm{C}_G(F_2) = \mathrm{C}_G(C/M) = D$. Hence $A = D$. Moreover $N \leq NM \leq A$ and A/N is a chief factor. If $N = NM$, then $M \leq N \leq A$ and $M = N$. This implies that $F_1 \cong_G F_3$ and, in particular, F_1 and F_3 are G-connected.

Now suppose that $A = MN$. Then the group G/A is isomorphic to $(G/N)\big/(A/N)$, which is the quotient group of a primitive group of type 3 over one of its minimal normal subgroups. Therefore G/A is a primitive group of type 2 by Corollary 1.1.13. On the other hand $BA/A \cong_G B/(B \cap A) = B/N \cong_G F_2$ and, since $M = A \cap C$, we have that $CA/A \cong_G C/(C \cap A) = C/M \cong_G F_2$, so BA/A and CA/A are minimal normal subgroups of G/A. Hence $AC = AB$. Analogously, working with G/B, we obtain that $AB = BC$.

Note that if C is contained in B, then $AB = B$ and then $A = B$, giving a contradiction. If B is contained in C, then $AB = C$. Since $M < A \leq AB = C$ and C/M is a chief factor of G, we have that $A = C$ and then $A = B$, which gives again a contradiction. Hence the subgroup $E = B \cap C$ is a proper subgroup of B and of C. Consider the group G/E. We have that

$$B/E \cong_G BC/C = AC/C \cong_G A/(A \cap C) = A/M \cong_G F_3$$

and then

$$\mathrm{C}_G(B/E) = \mathrm{C}_G(F_3) = \mathrm{C}_G(A/M) = C.$$

Also

$$C/E \cong_G BC/B = AB/B \cong_G A/(A \cap B) = A/N \cong_G F_1$$

and then

$$\mathrm{C}_G(C/E) = \mathrm{C}_G(F_1) = \mathrm{C}_G(A/N) = B.$$

On the other hand, let U, V be maximal subgroups of G such that $N \leq U$ and U is a common complement of A/N and B/N and $M \leq V$ and V is a common complement of A/M and C/M. Consider the subgroup $X = (U \cap V)E$.

If $X = G$, then $U = U \cap X = (U \cap V)(U \cap E) = (U \cap V)(N \cap C) = (U \cap V)(N \cap M) = U \cap V$. This contradicts the fact that $U \neq V$. Hence X is a proper subgroup of G. Now we have:

$$XB = (U \cap V)B = (U \cap VN)B = UB = G$$

and

$$XC = (U \cap V)C = (UM \cap V)C = VC = G.$$

Moreover $B \cap X$ is a normal subgroup of X and $(B \cap X)/E$ is centralised by $\mathrm{C}_G(B/E) = C$. Hence $B \cap X$ is a normal subgroup of $XC = G$. Since B/E is a chief factor of G and X is a proper subgroup of G, then $B \cap X = E$. Analogously $C \cap X = E$. In other words, the subgroup X is a common complement of B/E and C/E. By Corollary 1.1.13, the group $G/(B \cap C)$ is a primitive group of type 3. Consequently, F_1 is G-connected to F_3. □

Definition 1.2.7. *Let H/K be a chief factor of a group G.*

1. *We say that H/K is a* Frattini chief factor *of G if $H/K \leq \Phi(G/K)$.*
2. *If there exists a proper subgroup M of G such that $G = MH$ and $K \leq H \cap M$, we say that H/K is a* supplemented chief factor *of G and M is a* supplement *of H/K in G. If H/K in a non-Frattini chief factor of G, then H/K is supplemented in G by a maximal subgroup of G.*
3. *If H/K is a chief factor of G supplemented by a subgroup M of G and $K = H \cap M$, then we say that H/K is a* complemented chief factor *of G and M is a* complement *of H/K in G.*

Remarks 1.2.8. Let G be a group and H/K a supplemented chief factor of G. Consider a maximal subgroup M of G supplementing H/K in G. Clearly, in the quotient group G/M_G, the maximal subgroup M/M_G is core-free. Therefore G/M_G is a primitive group. We get $K = H \cap M_G$ and then note that if $M_G < X < HM_G$ and X is normal in G, then $X = M_G(X \cap H)$, where $K \leq X \cap H \leq H$. Hence $X \cap H = K$ or H. In both cases we have a contradiction. Thus HM_G/M_G is a minimal normal subgroup of the primitive group G/M_G.

1. Note that if M is a maximal subgroup of type 1 or 3 of a group G, then each chief factor of G supplemented by M is in fact complemented by M. In these cases, HM_G/M_G is a minimal normal subgroup of the primitive group G/M_G, which is of type 1 or 3, and then $M \cap HM_G = M_G$. Therefore $M \cap H = M_G \cap H = K$, as claimed.

2. Observe that $HM_G/M_G \cong_G H/K$. Write

$$C = \mathrm{C}_G(H/K) = \mathrm{C}_G(HM_G/M_G).$$

a) If H/K is abelian, then the primitive group G/M_G is of type 1; in this case $C = HM_G$ and $M/M_G \cong G/C$; therefore G/M_G is isomorphic to the semidirect product $[H/K](G/C)$.
b) if H/K is non-abelian, then two cases arise:
   i. If $C = M_G$, then G/M_G is a primitive group of type 2; clearly $\mathrm{Soc}(G/C) = HC/C \cong_G H/K$.
   ii. If M_G is contained in C, then G/M_G is a primitive group of type 3 whose minimal normal subgroups are HM_G/M_G and C/M_G; in this case G/C is a primitive group of type 2 and $\mathrm{Soc}(G/C) = HC/C \cong_G$

H/K. If S is a maximal subgroup supplementing HC/C in G, then $G = HS$ and $K = H \cap C = H \cap S_G$. Hence S is also a supplement of H/K in G and $S_G = C$ as in 2(b)i.

Hence for any supplemented chief factor H/K of G, there exists a maximal subgroup M of G supplementing H/K in G such that G/M_G is a monolithic primitive group. We say then that M is a *monolithic supplement* of H/K in G. This observation leads us to two definitions.

Definition 1.2.9. *For any chief factor H/K of a group G, we define the* primitive group associated *with H/K in G to be*

1. *the semidirect product $[H/K]\big(G/\operatorname{C}_G(H/K)\big)$, if H/K is abelian, or*
2. *the quotient group $G/\operatorname{C}_G(H/K)$, if H/K is non-abelian.*

Notation 1.2.10. The primitive group associated with H/K is denoted by $[H/K] * G$.

It is easy to see that if H/K is a supplemented chief factor of a group G, and M is a monolithic supplement of H/K in G, then $[H/K] * G \cong G/M_G$.

Definition 1.2.11. *Let H/K be a supplemented chief factor of the group G. Assume that M is a maximal subgroup G supplementing H/K in G such that G/M_G is a monolithic primitive group. We say that the chief factor* $\operatorname{Soc}(G/M_G) = HM_G/M_G$ *is the* precrown *of G associated with M and H/K, or simply, a* precrown *of G associated with H/K.*

Remarks 1.2.12. 1. If H/K is a non-abelian chief factor of the group G, then for each maximal subgroup M of G supplementing H/K in G such that G/M_G is a monolithic primitive group, we have that $M_G = \operatorname{C}_G(H/K)$. Therefore the unique precrown of G associated with H/K is

$$\begin{aligned}\operatorname{Soc}(G/M_G) &= HM_G/M_G \\ &= H\operatorname{C}_G(H/K)/\operatorname{C}_G(H/K) = \operatorname{C}^*_G(H/K)/\operatorname{C}_G(H/K).\end{aligned}$$

2. If H/K is a complemented abelian chief factor of G and M is a complement of H/K in G, then the precrown of G associated with M and H/K is

$$\operatorname{Soc}(G/M_G) = HM_G/M_G = \operatorname{C}_{G/M_G}(HM_G/M_G) = \operatorname{C}_G(H/K)/M_G.$$

For this reason it is interesting to know how many different precrowns are associated with a particular abelian chief factor. The answer, in a soluble group, is particularly elegant.

Proposition 1.2.13. *Let H/K be a complemented chief factor of a soluble group G. Then the function which assigns to each conjugacy class of complements of H/K in G, $\{M^g : g \in G\}$ say, the common core M_G of its elements* *induces a bijection between the set of all conjugacy classes of complements of H/K in G and the set of all normal subgroups of G which complement H/K in $\mathrm{C}_G(H/K)$.*

Therefore there exists a bijection between the set of all precrowns of G associated with H/K and the set of all conjugacy classes of complements of H/K in G.

Proof. Write $C = \mathrm{C}_G(H/K)$. Let N be a normal subgroup of G such that $C = HN$ and $H \cap N = K$. Then $HN/N \cong_G H/K$ and HN/N is a self-centralising minimal normal subgroup of the group G/N. By Theorem 1.1.10, HN/N is complemented in G/N and all complements are conjugate. If M/N is one of these complements, then $N = M_G$. Hence the correspondence is surjective.

Let M and S be two complements of H/K in G such that $N = M_G = S_G$. Then G/N is a soluble primitive group such that and S/N, M/N are complements of $\mathrm{Soc}(G/N) = HN/N$. By Theorem 1.1.10, there exists an element $g \in G$ such that $S^g = M$. Hence the correspondence is injective.

Finally observe that, since H/K is abelian, the precrowns of G associated with H/K have a common numerator $\mathrm{C}_G(H/K)$ and different denominators M_G, one for each conjugacy class of complements of H/K in G. □

Our next goal is to give a characterisation of the property of being G-connected. Observe that in a primitive group G of type 3, if A and B are the minimal normal subgroups, then $\mathrm{C}^*_G(A) = \mathrm{C}^*_G(B) = AB = \mathrm{Soc}(G)$. This means that two G-connected chief factors have the same inneriser. But this cannot be a characterisation as we can see from the example in Proposition 1.2.4 (2). To characterise the property of being G-connected in terms of the inneriser we have to be more precise.

But before that we have to include here a technical lemma, which will be crucial in our presentation.

Lemma 1.2.14 (see [För88]).

1. *Let $N_1, \dots, N_n$ be normal subgroups of a group G ($n \geq 2$), and consider $N = \prod_{i=1}^n N_i$. Suppose that $\bigcap_{i=1}^n N_i = 1$ and that $|N| = \prod_{i=1}^n |N/N_i|$. For $i = 1, \dots, n$, write $p_i \colon G/N_i \longrightarrow G/N$ for the natural projection: $(gN_i)^{p_i} = gN$, for all $g \in G$. Then the following statements are equivalent:*
   a) *There exists a subgroup U of G which complements all the N_i's in G.*
   b) *There exist group isomorphisms $\varphi_i \colon G/N_1 \longrightarrow G/N_i$, for $i = 2, \dots, n$, such that $\varphi_i p_i = p_1$, for all $i = 2, \dots, n$.*
2. *Let N_1 and N_2 be two normal subgroups of a group G such that $N_1 \cap N_2 = 1$. Write $N = N_1N_2$. Suppose that, for $i = 1, 2$, there exist group isomorphisms γ_i between G/N_i and a semidirect product $X = [Z]Y$, where Z is a normal subgroup of X, such that $(N/N_i)^{\gamma_i} = Z$.*
   Then there exists a subgroup H of G such that $G = HN$ and $H \cap N = 1$. For such H the following statements are equivalent:

a) there exists a subgroup U of G such that $H \le U$ and U is a common complement of N_1 and N_2 in G, and

b) $N_1 \cong_H N_2$.

If, moreover, the N_i, $i = 1, 2$, are abelian, then each of the previous statements is equivalent to

c) $N_1 \cong_G N_2$.

Proof. 1. Define $\varphi\colon N \longrightarrow N/N_1 \times \cdots \times N/N_n$, by $x^\varphi = (xN_1, \dots, xN_n)$, for every $x \in N$. It is clear that φ is a group homomorphism. If $x \in \operatorname{Ker}(\varphi)$, then $x \in \bigcap_{i=1}^n N_i = 1$. Moreover, since $|N| = \prod_{i=1}^n |N/N_i|$, we have that φ is an isomorphism.

Suppose that there exist group isomorphisms $\varphi_i\colon G/N_1 \longrightarrow G/N_i$, for $i = 2, \dots, n$, such that $\varphi_i p_i = p_1$, for all $i = 2, \dots, n$. Given $g_1N_1 \in G/N_1$, we consider $g_iN_i = (g_1N_1)^{\varphi_i}$, for $i = 2, \dots, n$. Then $(g_1N_1)^{\varphi_i p_i} = g_iN$ and $(g_1N_1)^{p_1} = g_1N$. Hence $g_1^{-1}g_i \in N$, for all $i = 1, \dots, n$.

Since φ is an isomorphism, there exists a unique element $x_0 \in N$ such that

$$(N_1, g_1^{-1}g_2N_2, \dots, g_1^{-1}g_nN_n) = (x_0N_1)^\varphi = (x_0N_1, \dots, x_0N_n)$$

and then $x_0 \in N_1$ and $x_0^{-1}g_1^{-1}g_i \in N_i$, for $i = 2, \dots, n$. Therefore $g_iN_i = g_1x_0N_i$, for all $i = 2, \dots, n$. Then, $(g_1x_0N_1)^{\varphi_i} = (g_1N_1)^{\varphi_i} = g_iN_i = g_1x_0N_i$. For the element $g = g_1x_0 \in g_1N_1 \cap g_2N_2 \cap \cdots \cap g_nN_n$, we have that $(gN_1)^{\varphi_i} = gN_i$, for $i = 2, \dots, n$.

For each $i = 1, \dots, n$, we choose a system of coset representatives $\mathcal{U}_i = \{x_{1i}, \dots, x_{ri}\}$ of N_i in G, such that $(x_{k1}N_1)^{\varphi_i} = x_{ki}N_i$ for all $i = 2, \dots, n$ and all $k = 1, \dots, r$. The above arguments show that there exist $z_k \in x_{k1}N_1 \cap x_{k2}N_2 \cap \cdots \cap x_{kn}N_n$ such that $(z_kN_1)^{\varphi_i} = z_kN_i$, for all $i = 2, \dots, n$. Thus we obtain a common system of coset representatives $U = \{z_1, \dots, z_k\}$ of all the N_i's in G.

Let us prove that U is a subgroup of G. If we suppose that $x_{11}N_1 = N_1$, which forces $z_1N_1 = N_1$, we obtain $N_i = N_1^{\varphi_i} = (z_1N_1)^{\varphi_i} = z_1N_i$, for all $i = 2, \dots, n$. Hence $z_1 \in \bigcap_{i=1}^n N_i = 1$ and $1 \in U$.

Suppose that $(z_kN_1)^{-1} = z_tN_1$ for some t. Then $z_kz_t \in N_1$. Hence $z_kz_t \in \bigcap_{i=1}^n N_i = 1$. Therefore $z_k^{-1} = z_t \in U$.

For $z_k, z_j \in U$, we have that $z_kz_jN_1 = z_tN_1$ for some t. Then $z_t^{-1}z_kz_j \in N_1$. As above this implies that $z_t^{-1}z_kz_j \in \bigcap_{i=1}^n N_i = 1$ and $z_kz_j = z^t \in U$.

Therefore U is a subgroup of G and is the required common complement of all the N_i's in G.

To prove the converse, let U be a common complement of the N_i's in G and define $\varphi_i\colon G/N_1 \longrightarrow G/N_i$ by $(gN_1)^{\varphi_i} = uN_i$, where $g = un$, $u \in U$, and $n \in N_i$. This is a well-defined homomorphism and it is injective. Since all the N_i have a common complement, they have, in particular, the same order and $|G/N_1| = |G/N_i|$, for all $i = 2, \dots, n$. Then the φ_i are group isomorphisms. Finally note that, for all $i = 2, \dots, n$ and all $g \in G$, $(gN_1)^{\varphi_i p_i} = uN = gN = (gN_1)^{p_1}$, i.e. $\varphi_i p_i = p_1$.

2. Since Y is a complement of Z in X and γ_i is a group isomorphism, then $H_i/N_i = Y^{\gamma_i^{-1}}$ is a complement of $Z^{\gamma_i^{-1}} = N/N_i$ in G/N_i, for each $i = 1, 2$. Consider the subgroup $H = H_1 \cap H_2$. Observe that $G = H_1N = H_1N_2 = H_2N_1$. Then $HN = (H_1 \cap H_2)N = (H_1 \cap H_2N_1)N_2 = H_1N_2 = G$, and $H \cap N = H_1 \cap H_2 \cap N = N_1 \cap N_2 = 1$.

Suppose that there exists a subgroup U of G such that $H \leq U$ and U is a common complement of N_1 and N_2 in G. Consider the isomorphisms φ_i between $G/N_i = UN_i/N_i$ and U defined by $(uN_i)^{\varphi_i} = u$. Then $(N/N_i)^{\varphi_i} = U \cap N$. Write τ_i for the restriction of φ_i to N/N_i.

Consider also the isomorphisms $\rho_i \colon N/N_i \longrightarrow N_{3-i}$, $i = 1, 2$, given by $(nN_i)^{\rho_i} = n_{3-i}$, for all $n \in N$, where $n = n_1n_2$, $n_1 \in N_1$ and $n_2 \in N_2$.

Consider the isomorphism $\psi = \rho_1^{-1}\tau_1\tau_2^{-1}\rho_2$ between N_2 and N_1. It is not difficult to see that if $n_2 \in N_2$, then $n_2^{\psi} = n_1^{-1}$, where $n_2 = un_1$ for $u \in U$ and $n_1 \in N_1$. The fact that ψ is H-invariant is an easy consequence of the fact that U is H-invariant. Therefore 2a implies 2b.

Conversely, if φ is an H-isomorphism between N_1 and N_2, then $T = \{aa^{\varphi} : a \in N_1\}$ is a subgroup of $N = N_1N_2$, and $H \leq \mathrm{N}_G(T)$. Consider $U = HT$. Since $N = TN_i$, then $G = UN_i$, for $i = 1, 2$. Moreover, $U \cap N_i \leq HT \cap N = T(H \cap N) = T$, and then $U \cap N_i \leq T \cap N_i = 1$, for $i = 1, 2$. Hence U is a common complement of N_1 and N_2 in G. Therefore 2b implies 2a.

If, moreover, the N_i, $i = 1, 2$, are abelian and 2a is true, then it is easy to see that any H-isomorphism between N_1 and N_2 is a G-isomorphism. □

Proposition 1.2.15. *Let G be a group and H_i/K_i, $i = 1, 2$, two supplemented chief factors of G. Then the following are equivalent.*

1. *H_1/K_1 and H_2/K_2 are G-connected;*
2. *for each $i = 1, 2$, there exists a precrown C_i/R_i associated with H_i/K_i, such that*
   a) *$C_1 = C_2$, and*
   b) *there exists a common complement U of the factors $R_i/(R_1 \cap R_2)$ in G, $i = 1, 2$.*

Proof. 1 implies 2. If the H_i/K_i, $i = 1, 2$, are abelian, then $H_1/K_1 \cong_G H_2/K_2$. In this case $C_1 = \mathrm{C}_G(H_1/K_1) = C_2 = \mathrm{C}_G(H_2/K_2) = C$. Hence the numerators of the precrowns coincide. For each $i = 1, 2$, let M_i be a complement of H_i/K_i in G. Then $C = H_1(M_1)_G = H_2(M_2)_G$. If $R = (M_1)_G = (M_2)_G$, then both chief factors have the same precrown C/R and we can take $U = G$. Otherwise $R_1 = (M_1)_G \neq (M_2)_G = R_2$. We can assume without loss of generality that $R_1 \cap R_2 = 1$. In particular, $C = R_1 \times R_2$ and $R_1 \cong_G R_2 \cong_G H_1/K_1$.

Note that $G/R_1 \cong G/R_2 \cong [H_1/K_1](G/C)$ and the isomorphisms map the C/R_i onto H_1/K_1. By the previous lemma, there exists a common complement to R_1 and R_2 in G.

Suppose now that the H_i/K_i, $i = 1, 2$, are non-abelian and $H_1/K_1 \cong_G H_2/K_2$. Then they have the same precrown and we can take G as complement of the trivial factor.

Assume finally that H_i/K_i, $i = 1, 2$, are non-abelian and there exists a normal subgroup N of G such that G/N is a primitive group of type 3 with minimal normal subgroups A_1/N and A_2/N such that $A_1/N \cong_G H_1/K_1$ and $A_2/N \cong_G H_2/K_2$. Clearly $\mathrm{C}_G(A_1/N) = A_2$ and $\mathrm{C}_G(A_2/N) = A_1$. Hence the precrown of G associated with H_1/K_1 and with A_1/N is A_1A_2/A_2 and the precrown of G associated with H_2/K_2 and with A_2/N is A_1A_2/A_1. Since $A_1 \cap A_2 = N$ and G/N is a primitive group of type 3, the conclusion follows easily from Theorem 1 (3c).

2 implies 1. Suppose that there exist normal subgroups C, R_1, R_2 of G such that C/R_i is a precrown associated with H_i/K_i and there exists a common complement U of the factors $R_i/(R_1 \cap R_2)$ in G, $i = 1, 2$.

If H_1/K_1 and H_2/K_2 are non-abelian, then $R_i = \mathrm{C}_G(H_i/K_i)$ and G/R_i is a primitive group of type 2, $i = 1, 2$. If $R_1 = R_2$, then H_1/K_1 and H_2/K_2 are G-isomorphic and then G-connected. If $R_1 \neq R_2$, we apply Corollary 1.1.13 to conclude that $G/(R_1 \cap R_2)$ is a primitive group of type 3 whose minimal normal subgroups are $R_i/(R_1 \cap R_2) \cong_G H_i/K_i$, $i = 1, 2$. Therefore H_1/K_1 and H_2/K_2 are G-connected.

Assume that H_1/K_1 and H_2/K_2 are abelian. If $R_1 = R_2$, then H_1/K_1 and H_2/K_2 are G-isomorphic and if $R_1 \neq R_2$, then both factors are G-isomorphic to $\mathrm{Soc}(G/U_G)$. In both cases, they are G-connected. □

Lemma 1.2.16 ([Bra88]). *Let G be a group and suppose that Z, Y, X, W are normal subgroups of G such that $Z = XY$ and $X \cap Y = W$.*

1. *If Z/X is complemented in G by M, then Y/W is complemented in G by M.*
2. *Moreover, if M complements Z/X and S complements X/W, then $(M \cap S)Y$ complements Z/Y; in this case $M \cap S$ complements Z/W in G.*
3. *Parts 1 and 2 hold in terms of supplements.*

When Y/W is a non-abelian chief factor of G, we can say even more:

4. *the set of monolithic supplements of Y/W in G coincides with the set of monolithic supplements of Z/X in G;*
5. *moreover, if X/W is an abelian chief factor of G then the (possibly empty) set of complements of X/W in G coincides with the set of complements of Z/Y in G.*

Proof. 1, 3. If $G = MZ$ and $X \leq Z \cap M$, then $G = MY$. Moreover $W = X \cap Y \leq M \cap Z \cap Y = M \cap Y$. Then M is a supplement of Y/W in G.

2, 3. If $G = MZ$ with $X \leq Z \cap M$ and $G = SX$ with $W \leq S \cap X$, then $\big((M \cap S)Y\big)Z = (M \cap S)Z = (M \cap S)XY = (M \cap SX)Y = MY = M(XY) = MZ = G$. Moreover $\big((M \cap S)Y\big) \cap Z = (M \cap S \cap Z)Y$ contains $(X \cap S)Y$ and $Y = WY \leq (X \cap S)Y$. Hence $(M \cap S)Y$ is a supplement of

Z/Y in G. Moreover, in this case, $G = (M \cap S)Z$ and W is contained in $S \cap X = S \cap X \cap Z \leq M \cap S \cap Z$. This is to say that $M \cap S$ supplements Z/W in G.

A substitution of the above inequalities by equalities gives the result in terms of complements.

For the remainder of the proof we can suppose without loss of generality that $W = 1$ and then Y is a non-abelian minimal normal subgroup of G centralising X.

4. If M is a monolithic supplement of Y in G then $G = MY$ and the group G/M_G is a monolithic primitive group of non-abelian socle. Then we have $\mathrm{Soc}(G/M_G) = M_GY/M_G$ and $\mathrm{C}_G(Y) = \mathrm{C}_G(M_GY/M_G) = M_G$. Hence $X \leq \mathrm{C}_G(Y) = M_G \leq M$. Then $G = MZ$ with $X \leq Z \cap M$ and M is a monolithic supplement of Z/X in G. Conversely, if M is a monolithic supplement of Z/X in G, then, by Statement 3, M supplements Y in G.

5. Suppose that X is an abelian minimal normal subgroup of G complemented by M. Then $\mathrm{C}_G(X) = XM_G$ and then $Z = X \times (Z \cap M_G)$. Since Y is non-abelian, this implies that $Y = Z'$ is contained in $Z \cap M_G$. Then Y is contained in M and M complements Z/Y. Note that the roles of X and Y in the original hypothesis can be interchanged without loss. Hence, by Statement 1, the (possibly empty) set of complements in G of X coincides with the set of complements of Z/Y in G. □

Lemma 1.2.17 (see [Haw67]). *Let U and S be two maximal subgroups of a group G such that $U_G \neq S_G$. Suppose that U and S supplement the same chief factor H/K of G. Then $M = (U \cap S)H$ is a maximal subgroup of G such that $M_G = H(U_G \cap S_G)$.*

1. *Assume that H/K is abelian. Then M is a maximal subgroup of type 1 and complements the chief factors $U_G/(U_G \cap S_G)$ and $S_G/(U_G \cap S_G)$. Moreover $M \cap U = M \cap S = U \cap S$.*
2. *Assume that H/K is non-abelian. Then either U or S is of type 3. Suppose that U is of type 3 and S is monolithic. Then $U_G < S_G = \mathrm{C}_G(H/K)$. Moreover M is a maximal subgroup of type 2 of G such that M supplements the chief factor S_G/U_G.*
3. *Assume that U and S are of type 3. Then M is a maximal subgroup of type 3 of G such that M complements the chief factors HS_G/M_G and HU_G/M_G. Moreover $M \cap U = M \cap S = U \cap S$.*

Proof. 1. Assume that H/K is abelian and denote $C = \mathrm{C}_G(H/K)$. First observe that $M \cap U = H(U \cap S) \cap U = (H \cap U)(U \cap S) = K(U \cap S) = U \cap S$, since $H \cap U = K$, by the abelian nature of H/K. Analogously $M \cap S = U \cap S$. Hence M is a proper subgroup of G. Note also that $C = U_GH = S_GH = U_GS_G$ and $U_G/(U_G \cap S_G)$ is a G-chief factor which is G-isomorphic to the precrown C/S_G. Hence $U_G/(U_G \cap S_G)$ is G-isomorphic to H/K. Now, $MU_G = (U \cap S)HU_G = (U \cap SU_G)H = UH = G$ and $U_G \cap S_G \leq M \cap U_G$ and then we deduce that M is a maximal subgroup of G which complements $U_G/(U_G \cap S_G)$. The same argument holds for the chief factor $S_G/(U_G \cap S_G)$. Since M also complements the chief factor $C/(U_G \cap S_G)H$, we have that $M_G = H(U_G \cap S_G)$.

2. Assume that H/K is non-abelian. If U and S were both monolithic, of type 2, then $U_G = S_G = \mathrm{C}_G(H/K)$. This is not true by hypothesis and then either U or S is of type 3.

Assume that U is of type 3 and S is monolithic. It is clear that $S_G = \mathrm{C}_G(H/K)$. Observe that HU_G/U_G is a chief factor of G which is G-isomorphic to H/K. Then HU_G/U_G and S_G/U_G are the two minimal normal subgroups of the primitive group G/U_G of type 3. Both are complemented by U; in particular, $G = US_G$.

Observe that $MS_G = H(U \cap S)S_G = H(US_G \cap S) = HS = G$ and $M \cap S_G = (U \cap S)H \cap S_G$ contains $U_GH \cap S_G = U_G(H \cap S_G) = U_GK = U_G$ and then M supplements the chief factor S_G/U_G.

Now the group $G/U_GH = (M/U_GH)(S_GH/U_GH)$ is primitive of type 2. If the normal subgroup M_G/U_GH were non-trivial, then S_GH would be contained in M_G and so $S_G \leq M$. This is not possible. Hence $M_G = U_GH$.

Consider a subgroup T such that $U \cap S \leq T \leq U$. Then $S = (U \cap S)S_G \leq TS_G \leq US_G = G$. By maximality of S in G we have that either $S = TS_G$ or $G = TS_G$. Observe that $T \cap S_G = U \cap S_G = U_G$, and then, $U \cap TS_G = T(U \cap S_G) = T(T \cap S_G) = T$, so $U \cap S = T$ or $U = U \cap G = T$. This means that $U \cap S$ is a maximal subgroup of U. In the isomorphism $U/(U \cap H) \cong G/H$, the image of $(U \cap S)/(U \cap H)$ is M/H. Hence M is a maximal subgroup of G of type 2.

3. Assume now that U and S are maximal subgroups of type 3: the quotient groups G/U_G and G/S_G are primitive groups of type 3.

If $C = \mathrm{C}_G(H/K)$, then U complements the chief factors HU_G/U_G and C/U_G. Analogously, S complements the chief factors HS_G/S_G and C/S_G. In particular, $U_G \not\leq S_G$ and $S_G \not\leq U_G$. Therefore $G = US_G = SU_G$. Now, by an analogous argument to that presented at the end of 2, we have that $M = (U \cap S)H$ is a maximal subgroup of G.

On the other hand, since C/S_G and C/U_G are chief factors of G and $U_G \neq S_G$, then $C = U_GS_G$. Write $L = U_G \cap S_G$. Observe that $HU_G/HL \cong_G U_G/(U_G \cap HL) = U_G/L \cong_G C/S_G$ and then HU_G/HL is a chief factor of G and $\mathrm{C}_G(HU_G/HL) = \mathrm{C}_G(C/S_G) = HS_G$. Similarly HS_G/HL is a chief factor of G and $\mathrm{C}_G(HS_G/HL) = HU_G$. Hence the quotient group $G^* = G/HL$ has two minimal normal subgroups, namely $N = HS_G/HL$ and $\mathrm{C}_{G^*}(N) = HU_G/HL$. Observe that $M(S_GH) = (U \cap S)S_GH = (US_G \cap S)H = SH = G$. Because U complements C/U_G, we have that $U \cap U_GS_G = U_G$, so $U \cap S_G = U_G \cap S_G = L$ and $M \cap HS_G = (U \cap S \cap HS_G)H = (U \cap S_G)H = HL$. Analogously $G = M(U_GH)$ and $M \cap HU_G = HL$. Therefore, the maximal subgroup $M^* = M/HL$ of G^* complements N and $\mathrm{C}_{G^*}(N)$. By Proposition 1.1.12, the group G^* is a primitive group of type 3. Hence $M_G = HL$.

Finally observe that $M \cap U = H(U \cap S) \cap U = (H \cap U)(U \cap S) = K(U \cap S) = U \cap S$. Analogously $M \cap S = U \cap S$. □

Definitions 1.2.18. *Let* $\mathbf{X}$ *be a set of maximal subgroups of a group* G.

1. *If* $\mathbf{X}$ *is non-empty, then the* $\mathbf{X}$-Frattini *subgroup of* G *is defined to be the intersection of the cores of all members of* $\mathbf{X}$. *It is denoted by* $\Phi_{\mathbf{X}}(G)$. *If* $\mathbf{X} = \emptyset$, *we define* $\Phi_{\mathbf{X}}(G) = G$.
2. *Let* H/K *be a chief factor of* G. *We say that* H/K *is an* $\mathbf{X}$-supplemented *(respectively,* $\mathbf{X}$-complemented*) chief factor if it has a supplement (respectively, complement) in* $\mathbf{X}$*; otherwise* H/K *is said to be an* $\mathbf{X}$-Frattini *chief factor.*
3. *If* C^*/N *is a precrown of* G *associated with an* $\mathbf{X}$*-supplemented chief factor* H/K *of* G, *we shall say that* C^*/N *is an* $\mathbf{X}$-precrown *of* G *associated with* H/K.

Notation 1.2.19. Let N be a normal subgroup of a group G and let $\mathbf{X}$ be a set of maximal subgroups of G. We write

$$\mathbf{X}/N = \{Z/N : Z \in \mathbf{X} \text{ and } N \text{ is contained in } Z\}$$

and if $\varphi\colon G \longrightarrow H$ is a group homomorphism, we write

$$\mathbf{X}^{\varphi} = \{S^{\varphi}\colon S \in \mathbf{X}\}.$$

The following lemma will be used frequently in the sequel.

Lemma 1.2.20. *Let* $\mathbf{X}$ *be a set of maximal subgroups of a group* G. *Let* H/K *be a chief factor of a group* G.

1. H/K *is an* $\mathbf{X}$*-Frattini chief factor of* G *if and only if* $H/K \leq \Phi_{\mathbf{X}/K}(G/K)$.
2. *If* A *is a normal subgroup of* G *contained in* K, *then* H/K *is* $\mathbf{X}$*-Frattini in* G/K *if and only if* $(H/A)/(K/A)$ *is* $\mathbf{X}/A$*-Frattini in* G/A. *Furthermore, if* H/K *is* $\mathbf{X}$*-supplemented in* G, *then a maximal subgroup* $U \in \mathbf{X}$ *is a supplement of* H/K *in* G *if and only if* U/A *is an* $\mathbf{X}/A$*-supplement of* $(H/A)/(K/A)$ *in* G/A.

Definition 1.2.21. *A set* $\mathbf{X}$ *of maximal subgroups of a group* G *is said to be* solid for the Jordan-Hölder theorem, *or simply* JH-solid, *if it satisfies the following condition:*

(JH) If U, $S \in \mathbf{X}$ *with* $U_G \neq S_G$ *and both supplement a chief factor* H/K *of* G, *then there exists* $M \in \mathbf{X}$ *such that* $M_G = (U_G \cap S_G)H$.

Applying Lemma 1.2.17, the set of all maximal subgroups of a group G that supplement a single chief factor, the set $\operatorname{Max}(G)$ of all maximal subgroups of a group G, and the set $\operatorname{Max}^*(G)$ of all monolithic maximal subgroups of a group G are JH-solid.

Note that

$$\Phi(G) = \bigcap\{M \in \operatorname{Max}(G)\} = \bigcap\{M \in \operatorname{Max}^*(G)\}.$$

We will use the following results in inductive arguments.

Theorem 1.2.22. *Let G be a group factorised as $G = MN$, where M is a subgroup of G and N is a normal subgroup of G. Then $G/N \cong M/(N \cap M)$, and we have the following.*

1. *If*
$$N = H_n < \cdots < H_0 = G \tag{1.4}$$
is a piece of chief series of G, then
$$M \cap N = M \cap H_n < \cdots < M \cap H_0 = M \tag{1.5}$$
is a piece of chief series of M. If S is a maximal subgroup of G which supplements a chief factor H_i/H_{i+1} in (1.4), then $M \cap S$ is a maximal subgroup of M which supplements the chief factor $(H_i \cap M)/(H_{i+1} \cap M)$ in (1.5). Moreover, the core of $M \cap S$ in M is $(M \cap S)_M = M \cap S_G$.
2. *Conversely, if*
$$M \cap N = M_n < \cdots < M_0 = M \tag{1.6}$$
is a piece of chief series of M, then
$$N = M_nN < \cdots < M_0N = MN = G \tag{1.7}$$
is a piece of chief series of G. If U is a maximal subgroup of M which supplements a chief factor M_i/M_{i+1} in (1.6), then UN is a maximal subgroup of G which supplements the chief factor $M_iN/M_{i+1}N$ in (1.7). Moreover, the core of UN in G is $(UN)_G = U_MN$.

Lemma 1.2.23. *Let $\mathbf{X}$ be a JH-solid set of maximal subgroups of a group G and N a normal subgroup of G.*

1. *The set $\mathbf{X}/N$ is a JH-solid set of maximal subgroups of G/N.*
2. *Suppose that the subgroup M supplements N in G: $G = MN$. Then the set*
$$(\mathbf{X} \cap M)/(N \cap M) = \{(S \cap M)/(N \cap M) : N \leq S \in \mathbf{X}\}$$
is a JH-solid set of maximal subgroups of $M/(N \cap M)$.
Moreover, if φ is the isomorphism between G/N and $M/(N \cap M)$ then we have that $(\mathbf{X}/N)^{\varphi} = (\mathbf{X} \cap M)/(M \cap N)$.

Now we can prove the announced strengthened form of the Jordan-Hölder theorem for chief series of finite groups and give an answer to Lafuente's question. To do this we proceed following Lafuente's arguments in [Laf89]. It must be observed that these arguments deal with the modular lattice of all normal subgroups of a group in which we can use the Duality Principle (see [Bir69, Chapter 1, Theorem 2]).

Notation 1.2.24. If A/B and C/D are sections of a group G, then we write $A/B \ll C/D$ (or $C/D \gg A/B$) if $C = AD$ and $B = A \cap D$.

Observe that if $A/B \ll C/D$, then $A/B \cong_G C/D$. In particular, A/B is a chief factor of G if and only if C/D is a chief factor of G.

Lemma 1.2.25. *Let K and H be normal subgroups of a group G and let*

$$K = Y_0 < Y_1 < \cdots < Y_{m-1} < Y_m = H$$

be a piece of chief series between K and H. Suppose that X^*/X is a chief factor of G between H and K.

1. *If $X^*Y_j = XY_j$, then $X^*Y_k = XY_k$ for $j \leq k \leq m$.*
2. *If $X^* \cap Y_{j-1} = X \cap Y_{j-1}$, then $X^* \cap Y_{k-1} = X \cap Y_{k-1}$, for $1 \leq k \leq j$.*
3. *If $X^*Y_{j-1} > XY_{j-1}$, then $X^*Y_{k-1} > XY_{k-1}$, for $1 \leq k \leq j$ and $X^* \cap Y_{j-1} = X \cap Y_{j-1}$. In this case,*
$$X^*Y_{j-1}/XY_{j-1} \gg X^*Y_{k-1}/XY_{k-1} \gg X^*/X.$$
4. *If $X^* \cap Y_j > X \cap Y_j$, then $X^* \cap Y_k > X \cap Y_k$, for $j \leq k \leq m$ and $X^*Y_j = XY_j$. Moreover*
$$X^*/X \gg (X^* \cap Y_k)/(X \cap Y_k) \gg (X^* \cap Y_j)/(X \cap Y_j).$$

Proof. Note that Statement 1 and its dual, which is Statement 2, are obvious.

3. By Statement 1, if $X^*Y_{j-1} > XY_{j-1}$, then $X^*Y_{k-1} > XY_{k-1}$, for $1 \leq k \leq j$. On the other hand, we have

$$(X^*Y_{k-1})(XY_{j-1}) = X^*Y_{j-1} \qquad X^*Y_{k-1} = X^*(XY_{k-1}).$$

Moreover $X \leq X(X^* \cap Y_{j-1}) = X^* \cap XY_{j-1} \leq X^*$. Since X^*/X is a chief factor of G, then either $X = X(X^* \cap Y_{j-1}) = X^* \cap XY_{j-1}$ or $X^* \cap XY_{j-1} = X^*$. In the last case $X^* \leq XY_{j-1}$ and then $X^*Y_{j-1} = XY_{j-1}$, contrary to our supposition. Hence $X^* \cap Y_{j-1} \leq X$ and then $X^* \cap Y_{j-1} = X \cap Y_{j-1}$. By Statement 2, $X^* \cap Y_{k-1} = X \cap Y_{k-1}$. Hence

$$X^* \cap XY_{k-1} = X(X^* \cap Y_{k-1}) = X(X \cap Y_{k-1}) = X$$

and

$$\begin{aligned} XY_{j-1} \cap X^*Y_{k-1} &= (XY_{j-1} \cap X^*)Y_{k-1} \\ &= X(Y_{j-1} \cap X^*)Y_{k-1} = X(X \cap Y_{j-1})Y_{k-1} = XY_{k-1}. \end{aligned}$$

Statement 4 is dual of Statement 3. □

Definition 1.2.26. *Let A/B, A/C and C/D be chief factors of a group G such that $A/B \gg C/D$. If $\mathbf{X}$ is a set of maximal subgroups of G, such that A/B is $\mathbf{X}$-Frattini and C/D is $\mathbf{X}$-supplemented, we will say that the situation $A/B \gg C/D$ is an* $\mathbf{X}$-crossing. *We write $[A/B \gg C/D]$ to denote an* $\mathbf{X}$-c*rossing.*

Remark 1.2.27. 1. If $A/B \gg C/D$ and A/B is $\mathbf{X}$-supplemented, then C/D is $\mathbf{X}$-supplemented, by Lemma 1.2.16.

2. If $[A/B \gg C/D]$ is an $\mathbf{X}$-crossing, then C/D is abelian. If C/D is a non-abelian $\mathbf{X}$-supplemented chief factor, then A/B is also $\mathbf{X}$-supplemented, by Lemma 1.2.16 (4), against our supposition.

Next we see a characterisation of JH-solid sets of monolithic maximal subgroups in terms of $\mathbf{X}$-crossing situations.

Theorem 1.2.28. *Let* $\mathbf{X}$ *be a set of maximal subgroups of a group* G*.*

1. *Assume that* $\mathbf{X}$ *is JH-solid. Let* Z/Y*,* Y/W *and* X/W *be chief factors of* G*. If* $[Z/X \gg Y/W]$ *is an* $\mathbf{X}$*-crossing, then* $[Z/Y \gg X/W]$ *is an* $\mathbf{X}$*-crossing. Moreover, in this case, a maximal subgroup* $U \in \mathbf{X}$ *supplements* Y/W *if and only if* U *supplements* X/W*.*
2. *Conversely, assume that* $\mathbf{X}$ *is a monolithic set of maximal subgroups of* G *such that whenever we have chief factors* Z/Y*,* Y/W *and* X/W *of* G *such that* $[Z/X \gg Y/W]$ *is an* $\mathbf{X}$*-crossing, then* $[Z/Y \gg X/W]$ *is an* $\mathbf{X}$*-crossing. Then* $\mathbf{X}$ *is JH-solid.*

Proof. 1. We can assume that $W = 1$. We have to prove that if X and Y are minimal normal subgroups of G, Z/X is $\mathbf{X}$-Frattini chief factor and Y is $\mathbf{X}$-suplemented, then Z/Y is $\mathbf{X}$-Frattini and X is $\mathbf{X}$-supplemented.

Assume that U is an $\mathbf{X}$-supplement of Y. If $X \leq U$, then $G = UZ$ and $X \leq U \cap Z$, so U supplements Z/X. This contradiction yields that X is not contained in U and then U supplements X. Suppose that, in this case, there exists $S \in \mathbf{X}$ supplementing Z/Y. Then S also supplements X. Since $Y \not\leq U_G$ and $Y \leq S_G$, by the property (JH), there exists $M \in \mathbf{X}$ such that $M_G = (U_G \cap S_G)X$. If $Z \leq M$, then $Z = Z \cap M_G = X(U_G \cap S_G \cap Z) = X(U_G \cap Y) = X$, which is a contradiction. Hence M supplements Z/X, which we have supposed to be $\mathbf{X}$-Frattini. We deduce that Z/Y must be an $\mathbf{X}$-Frattini chief factor of G.

2. Suppose that we have U, $S \in \mathbf{X}$, both supplementing the same chief factor H/K of G and $U_G \neq S_G$. Since U and S are monolithic, the chief factor H/K must be abelian, by Lemma 1.2.17 (2). Therefore $K = U \cap H = U_G \cap H = S_G \cap H = S \cap H$.

Observe that $C = \mathrm{C}_G(H) = HS_G = HU_G = U_G S_G$. Write $A = U_G \cap S_G$. Then

$$C/HA = HU_G/HA \cong_G U_G/(U_G \cap HA) = U_G/A \cong_G C/S_G$$

and then C/HA is a chief factor of G and $C/HA \gg U_G/A$. Observe that U_G/A is $\mathbf{X}$-complemented by S. Suppose that C/HA is $\mathbf{X}$-Frattini. Then $[C/HA \gg U_G/A]$ is an $\mathbf{X}$-crossing. By hypothesis, $[C/U_G \gg HA/A]$ is an $\mathbf{X}$-crossing. But C/U_G is obviously $\mathbf{X}$-complemented by U. This contradiction yields that C/HA is $\mathbf{X}$-complemented in G, i. e. there exists $M \in \mathbf{X}$ such that $G = MC$ and $HA = M_G$. Therefore $\mathbf{X}$ is JH-solid. □

Proposition 1.2.29. *With the hypotheses of Lemma 1.2.25, assume that* X^*/X *is an* $\mathbf{X}$*-supplemented chief factor of* G*. Let*

$$j' = \max\{j : X^*Y_{j-1}/XY_{j-1} \text{ is } \mathbf{X}\text{-supplemented chief factor of } G\}$$

and set $Y^* = Y_{j'}$ *and* $Y = Y_{j'-1}$*. Then* Y^*/Y *is* $\mathbf{X}$*-supplemented.*

Furthermore the following conditions are satisfied:

1. *If* $X^*Y^* = XY^*$*, then* $X^*Y^* = XY^* = X^*Y$*. Write* $R^* = X^*Y^*$ *and* $R = XY$*. Then* $X^*/X \ll R^*/R \gg Y^*/Y$*. Moreover* $X^* \cap Y = X \cap Y = X \cap Y^*$*. Write* $S = X \cap Y$ *and* $S^* = X^* \cap Y^*$*, then* $X^*/X \gg S^*/S \ll Y^*/Y$*.*
2. *If* $X^*Y^* \neq XY^*$*, then* $[X^*Y^*/XY^* \gg X^*Y/XY]$ *is an* $\mathbf{X}$*-crossing and* $X^*/X \ll X^*Y/XY$ *and* $XY^*/XY \gg Y^*/Y$*.*

In particular, in both cases X^*Y/XY *and* XY^*/XY *are* $\mathbf{X}$*-supplemented chief factors of* G*.*

Proof. Observe that $X^*Y_0/XY_0 = X^*/X$ is $\mathbf{X}$-supplemented. Hence j' is well-defined.

Assume that $XY^* = XY$. Then $X^*Y^* = X^*Y$. So $X^*Y^*/XY^* = X^*Y/XY$ is $\mathbf{X}$-supplemented, giving a contradiction to the election of j'. Therefore $XY^*/XY \gg Y^*/Y$ and XY^*/XY is a chief factor.

1. Assume that $X^*Y^* = XY^*$. Then $XY \leq X^*Y \leq X^*Y^* = XY^*$. Therefore $X^*Y^* = X^*Y$ because $X^*Y > XY$ by hypothesis. From part 3 of Lemma 1.2.25, it follows that $X^*/X \ll R^*/R \gg Y^*/Y$. On the other hand, $X^* = XY^* \cap X^* = (X^* \cap Y^*)X$. Hence $X^*/X \gg (X^* \cap Y^*)/(X \cap Y^*)$. Now, from part 3 of Lemma 1.2.25, $X^* \cap Y = X \cap Y = X \cap Y^*$. Thus, $X^*/X \gg S^*/S \ll Y^*/Y$.

In this case $R^*/R = X^*Y^*/XY = XY^*/XY = X^*Y/XY$ is $\mathbf{X}$-supplemented, by definition of j'.

2. Now consider $X^*Y^* \neq XY^*$. From the choice of j', it follows that X^*Y^*/XY^* is an $\mathbf{X}$-Frattini chief factor of G. Then $XY \leq XY^* \cap X^*Y \leq X^*Y$. If $XY^* \cap X^*Y = X^*Y$, it follows that $X^*Y^* = XY^*$ contrary to our assumption. Hence $XY = XY^* \cap X^*Y$ and $[X^*Y^*/XY^* \gg X^*Y/XY]$ is an $\mathbf{X}$-crossing. Moreover $X^*/X \ll X^*Y/XY$ and $XY^*/XY \gg Y^*/Y$.

Since $[X^*Y^*/XY^* \gg X^*Y/XY]$ is an $\mathbf{X}$-crossing, we have that X^*Y/XY and XY^*/XY are $\mathbf{X}$-supplemented chief factors of G. □

Proposition 1.2.30. *With the hypotheses of Lemma 1.2.25, assume that* X^*/X *is an* $\mathbf{X}$*-Frattini chief factor of* G*. Let*

$$j' = \min\{j : (X^* \cap Y_j)/(X \cap Y_j) \text{ is an } \mathbf{X}\text{-Frattini chief factor of } G\}$$

and set $Y^* = Y_{j'}$ *and* $Y = Y_{j'-1}$*. Then* Y^*/Y *is* $\mathbf{X}$*-Frattini.*

Furthermore the following conditions are satisfied:

1. *If* $X^* \cap Y = X \cap Y$*, then* $X \cap Y = X \cap Y^* = X^* \cap Y$*. Write* $S^* = X^* \cap Y^*$ *and* $S = X \cap Y$*. Then* $X^*/X \gg S^*/S \ll Y^*/Y$*. Moreover* $X^*Y = X^*Y^* = XY^*$*. Write* $R = XY$ *and* $R^* = X^*Y^*$*, then* $X^*/X \ll R^*/R \gg Y^*/Y$*.*

2. *If* $X^* \cap Y \neq X \cap Y$, *then* $[(X^* \cap Y^*)/(X \cap Y^*) \gg (X^* \cap Y)/(X \cap Y)]$ *is an* **X**-*crossing and* $X^*/X \gg (X^* \cap Y^*)/(X \cap Y^*)$ *and* $(X^* \cap Y^*)/(X^* \cap Y) \ll Y^*/Y$.

In particular, in both cases $(X^* \cap Y^*)/(X^* \cap Y)$ *and* $(X^* \cap Y^*)/(X \cap Y^*)$ *are* **X**-*Frattini chief factors of* G.

Proof. This is the dual statement of Proposition 1.2.29. □

Definition 1.2.31. *Given a set* **X** *of maximal subgroups of a group* G, *we say that two chief factors of* G, *say* X^*/X *and* Y^*/Y, *are* **X**-related *if one of these properties is satisfied:*

1. *There exists an* **X**-*supplemented chief factor* R^*/R *such that* $X^*/X \ll R^*/R \gg Y^*/Y$,
2. *There exists an* **X**-*crossing* $[A/Z \gg T/B]$ *such that* $X^*/X \ll Z/B$ *and* $T/B \gg Y^*/Y$.
3. *There exists an* **X**-*Frattini chief factor* S^*/S *such that* $X^*/X \gg S^*/S \ll Y^*/Y$,
4. *There exists an* **X**-*crossing* $[A/Z \gg T/B]$ *such that* $X^*/X \gg A/Z$ *and* $A/T \ll Y^*/Y$.

The importance of the **X**-relation becomes clear in the following theorem.

Theorem 1.2.32. *Let* **X** *be a JH-solid set of maximal subgroups of a group* G. *If the chief factors* X^*/X *and* Y^*/Y *are* **X**-*related, then*

1. X^*/X *and* Y^*/Y *are* G-*connected, and*
2. X^*/X *is* **X**-*Frattini if and only if* Y^*/Y *is* **X**-*Frattini.*
3. *If* X^*/X *and* Y^*/Y *are* **X**-*supplemented, there exists a common* **X**-*supplement to both.*

Furthermore, if **X** *is composed of monolithic maximal subgroups of* G *then any two* **X**-*related chief factors are* G-*isomorphic.*

Proof. 1. Observe that in Cases 1 and 3 of the definition of **X**-relation, we have that X^*/X is G-isomorphic to Y^*/Y. Suppose that there exists an **X**-crossing $[A/Z \gg T/B]$ such that $X^*/X \ll Z/B$ and $T/B \gg Y^*/Y$. Since **X** is JH-solid, there exists a common **X**-supplement U of Z/B and T/B, by Theorem 1.2.28. Then TU_G/U_G and ZU_G/U_G are minimal normal subgroups of the primitive group G/U_G. If $ZU_G = TU_G$, then $Z/B \cong_G T/B$; in this case $X^*/X \cong_G Y^*/Y$. Otherwise G/U_G is a primitive group of type 3 whose minimal normal subgroups are TU_G/U_G and ZU_G/U_G. Since $X^*/X \cong_G Z/B$ and $Y^*/Y \cong_G T/B$, then X^*/X and Y^*/Y are G-connected. The analysis of Case 4 is analogous.

Observe that if all elements of **X** are monolithic maximal subgroups of G, then necessarily $ZU_G = TU_G$ in the above analysis. Therefore $X^*/X \cong_G Y^*/Y$.

2. If X^*/X is $\mathbf{X}$-Frattini, then we are not in Case 1 of the definition of $\mathbf{X}$-relation. Suppose that there exists an $\mathbf{X}$-crossing $[A/Z \gg T/B]$ such that $X^*/X \ll Z/B$ and $T/B \gg Y^*/Y$. Then $[A/T \gg Z/B]$ is an $\mathbf{X}$-crossing by Theorem 1.2.28. Then Z/B is $\mathbf{X}$-complemented. This implies that X^*/X is $\mathbf{X}$-supplemented by Lemma 1.2.16. Therefore we are not in Case 2 of Definition 1.2.31 either. If we are in Case 3, then Y^*/Y is $\mathbf{X}$-Frattini by Lemma 1.2.16. In Case 4, $[A/T \gg Z/B]$ is an $\mathbf{X}$-crossing by Theorem 1.2.28 and again Y^*/Y is $\mathbf{X}$-Frattini by Lemma 1.2.16.

3. If X^*/X and Y^*/Y are $\mathbf{X}$-supplemented, we are either in Case 1 or in Case 2 of Definition 1.2.31. In Case 1, if U is an $\mathbf{X}$-supplement of R^*/R, then U supplements X^*/X and Y^*/Y, In Case 2, there exists an $\mathbf{X}$-crossing $[A/Z \gg T/B]$ such that $X^*/X \ll Z/B$ and $T/B \gg Y^*/Y$. By Theorem 1.2.28, we know that there exists a common $\mathbf{X}$-supplement U to Z/B and T/B. By Lemma 1.2.16, U also $\mathbf{X}$-supplements X^*/X and Y^*/Y, □

Lemma 1.2.33. *Under the hypotheses of Lemma 1.2.25, assume that X^*/X and Y_j/Y_{j-1} are $\mathbf{X}$-related.*

1. *X^*/X and Y_j/Y_{j-1} are $\mathbf{X}$-supplemented in G if and only if X^*Y_{j-1}/XY_{j-1} is $\mathbf{X}$-supplemented in G.*
2. *X^*/X and Y_j/Y_{j-1} are $\mathbf{X}$-Frattini if and only if $(X^* \cap Y_j)/(X \cap Y_j)$ are $\mathbf{X}$-Frattini.*

Proof. 1. Set $Y^* = Y_j$, $Y = Y_{j-1}$ and assume that there exists an $\mathbf{X}$-supplemented chief factor R^*/R such that $X^*/X \ll R^*/R \gg Y^*/Y$. Since $(X^*Y)R = R^*$ and $XY \leq R$, then $XY < X^*Y$. By part 3 of Lemma 1.2.25, $X^*Y/XY \gg X^*/X$ and in particular, X^*Y/XY is a chief factor. On the other hand, $XY \leq X^*Y \cap R \leq X^*Y$. As X^*Y is not contained in R, then $R^*/R \gg X^*Y/XY$. Therefore X^*Y_{j-1}/XY_{j-1} is $\mathbf{X}$-supplemented in G. Now suppose that there exists an $\mathbf{X}$-crossing $[A/Z \gg T/B]$ such that $Z/B \gg X^*/X$y $T/B \gg Y^*/Y$. Since $XY \leq B$ and $(X^*Y)B = Z$, we have that $X^*Y > XY$ and, as above, $X^*/X \ll X^*Y/XY$. Now $Z = (X^*Y)B$ and $X^*Y \cap B = Y(X^* \cap B) = XY$. Hence $Z/B \gg X^*Y/XY$. Therefore X^*Y_{j-1}/XY_{j-1} is $\mathbf{X}$-supplemented in G.

The converse follows from part 3 of Lemma 1.2.25.

2. This is the dual statement of 1. □

Theorem 1.2.34. *Let G be a group and $\mathbf{X}$ a JH-solid set of maximal subgroups of G. For any pair K, H of normal subgroups of G such that $K < H$ and two pieces of chief series of G between K and H*

$$K = X_0 \leq X_1 \leq \cdots \leq X_n = H$$

and

$$K = Y_0 \leq Y_1 \leq \cdots \leq Y_m = H,$$

then $n = m$ and there exists a unique permutation $\sigma \in \mathrm{Sym}(n)$ such that X_i/X_{i-1} and $Y_{i^\sigma}/Y_{i^\sigma-1}$ are $\mathbf{X}$-related, for $1 \leq i \leq n$. Furthermore

$$i^\sigma = \max\{j : X_iY_{j-1}/X_{i-1}Y_{j-1} \textit{ is } \mathbf{X}\textit{-supplemented}\}$$

if X_i/X_{i-1} *is* **X**-*supplemented, and*

$$i^\sigma = \min\{j : (X_i \cap Y_j)/(X_{i-1} \cap Y_j) \textit{ is } \mathbf{X}\textit{-Frattini}\}$$

if X_i/X_{i-1} *is* **X**-*Frattini.*

Proof. We can assume without loss of generality that $m \leq n$. Write $X^* = X_i$, $X = X_{i-1}$, $Y^* = Y_{i^\sigma}$ and $Y = Y_{i^\sigma-1}$.

By Proposition 1.2.29, if X^*/X is **X**-supplemented, then so is Y^*/Y. Furthermore, if $X^*Y^* = XY^*$, then $X^*/X \ll R^*/R \gg Y^*/Y$, where $R^* = X^*Y^* = X^*Y$ and $R = XY$, by part 1 of Proposition 1.2.29. Hence R^*/R is **X**-supplemented by definition of i^σ. So, this is Case 1 of the definition of **X**-relation. And if $X^*Y^* \neq XY^*$, then we are in Case 2 of Definition 1.2.31 by part 2 of Proposition 1.2.29.

Dually, by Proposition 1.2.30, if X^*/X is **X**-Frattini, then so is Y^*/Y. Furthermore, if $X^* \cap Y^* = X \cap Y^*$, then $X^*/X \gg S^*/S \ll Y^*/Y$, where $S^* = X^* \cap Y^*$ and $S = X \cap Y$, by part 1 of Proposition 1.2.30. Hence S^*/S is **X**-Fratttini by definition of i^σ. So, this is Case 3 of the definition of **X**-relation. and if $X^* \cap Y^* \neq X \cap Y$, then we are in Case 4 of Definition 1.2.31.

Therefore, in any case, X_i/X_{i-1} and $Y_{i^\sigma}/Y_{i^\sigma-1}$ are **X**-related, for $1 \leq i \leq n$.

Now we prove that the map $\sigma : \{1, \ldots, n\} \to \{1, \ldots, m\}$ defined above is injective. Write $Z^* = X_k$ and $Z = X_{k-1}$, where $i < k$ and $i^\sigma = k^\sigma$.

Suppose that X^*/X is **X**-supplemented; then so are Y^*/Y and Z^*/Z. Assume that $X^*Y^* = XY^*$. From $X^* \leq Z$ we get that $ZY^* = ZY$. Since Z^*/Z is **X**-supplemented and $k^\sigma = j$, Z^*Y/ZY is a chief factor of G and then $ZY = ZY^* < Z^*Y = Z^*Y^*$. By part 2 of Proposition 1.2.29, $ZY^*/ZY \gg Y^*/Y$. In particular $ZY^* > ZY$ and yields a contradiction. Hence $X^*Y^* > XY^*$. Then $[X^*Y^*/XY^* \gg X^*Y/XY]$ is an **X**-crossing by part 2 of Proposition 1.2.29. The chief factor X^*Y^*/X^*Y is **X**-Frattini. Since $k^\sigma = j$, then Z^*Y/ZY and ZY^*/ZY are **X**-supplemented chief factors of G. As $X^* \leq Z$ gives $X^*Y \leq ZY$ and $X^*Y^* \leq ZY^*$. Observe that $ZY^* = (ZY)(X^*Y^*)$. Moreover $ZY \cap X^*Y^* = X^*(Z \cap Y^*)Y$ In the situation $Y \leq (Z \cap Y^*)Y \leq Y^*$ and Y^*/Y chief factor of G, we cannot have $ZY \cap Y^* = Y^*$, since this would imply $Y^* \leq ZY$ and then $ZY^* = ZY$ and this contradicts the fact that ZY^*/ZY is a chief factor. Hence $ZY \cap X^*Y^* = X^*Y$. In other words, $ZY^*/ZY \gg X^*Y^*/X^*Y$ and we deduce that ZY^*/ZY is **X**-Frattini by Lemma 1.2.16. This is a contradiction.

We have shown that the restriction to σ to the subset $\mathcal{I}$ of $\{1, \ldots, n\}$ composed of all indices i corresponding to **X**-supplemented chief factors X_i/X_{i-1}, is injective. Applying dual arguments we show that the restriction of σ to the subset of $\{1, \ldots, n\} \setminus \mathcal{I}$ composed of all indices i corresponding to **X**-Frattini chief factors X_i/X_{i-1}, is injective. By the arguments at the beginning of the proof, σ is injective. Therefore $n = m$ and σ is a permutation of the set $\{1, \ldots, n\}$.

Finally if τ is any permutation with the above properties, then the definition of σ requires that $i^\tau \leq i^\sigma$ for all $i \in \mathcal{I}$ and $i^\tau \geq i^\sigma$ for all $i \in \{1, \dots, n\} \setminus \mathcal{I}$ by Lemma 1.2.33. Consequently $\sigma = \tau$. □

Remark 1.2.35. By Theorem 1.2.32, the bijection constructed in Theorem 1.2.34 satisfies that if X_i/X_{i-1} and $Y_{i^\sigma}/Y_{i^\sigma-1}$ are **X**-supplemented, there exists a common **X**-supplement to both. Clearly when X_i/X_{i-1}, and $Y_{i^\sigma}/Y_{i^\sigma-1}$, is abelian we can change the **X**-supplementation by **X**-complementation. But we can go further and say the same even for non-abelian **X**-complemented chief factors. We know, by Theorem 1.1.48, the existence of non-abelian chief factors complemented by maximal subgroups. Observe that if X_i/X_{i-1} and $Y_{i^\sigma}/Y_{i^\sigma-1}$ are **X**-complemented non-abelian chief factors, then we are in Case 1 of Definition 1.2.31, since Case 2 is not possible by Remark 2 of 1.2.27. If U is an **X**-complement of the non-abelian chief factor X_i/X_{i-1} and $Y_{i^\sigma}/Y_{i^\sigma-1} \ll R^*/R \gg X_i/X_{i-1}$, then U also supplements R^*/R, by of Lemma 1.2.16 (4), and the same for $Y_{i^\sigma}/Y_{i^\sigma-1}$. Observe that U/U_G is a small maximal subgroup of the primitive group G/U_G of type 2. Then, $Soc(G/U_G) = X_iU_G/U_G = R^*U_G/U_G = Y_{i^\sigma}U_G/U_G$ and $U_G = U \cap Y_{i^\sigma}U_G$. Thus, $U \cap Y_{i^\sigma} = U_G \cap Y_{i^\sigma} = Y_{i^\sigma-1}$ and U complements $Y_{i^\sigma}/Y_{i^\sigma-1}$.

Theorem 1.2.36. *Let G be a group and* **X** *a set of monolithic maximal subgroups of G. Then the following conditions are equivalent:*

1. **X** *is a JH-solid set.*
2. *For any pair K, H of normal subgroups of G such that $K < H$ and two pieces of chief series of G between K and H*

$$K = X_0 \leq X_1 \leq \cdots \leq X_n = H \quad \text{and} \quad K = Y_0 \leq Y_1 \leq \cdots \leq Y_m = H,$$

   then $n = m$ and there exists $\sigma \in \mathrm{Sym}(n)$ such that
   a) $X_i/X_{i-1} \cong_G Y_{i^\sigma}/Y_{i^\sigma-1}$;
   b) X_i/X_{i-1} is **X***-Frattini if and only if $Y_{i^\sigma}/Y^{i^\sigma-1}$ is* **X***-Frattini;*
   c) if X_i/X_{i-1} is **X***-supplemented (respectively, complemented) in G, there exists a maximal subgroup $U \in$* **X** *of G such that G supplements (respectively, complements) both X_i/X_{i-1} and $Y_{i^\sigma}/Y_{i^\sigma-1}$.*

Proof. After Theorem 1.2.34 we have only to see that 2 implies 1.

Suppose that we have $U, S \in$ **X**, both supplementing the same chief factor H/K of G and $U_G \neq S_G$. Since U and S are monolithic, the chief factor H/K must be abelian, by Lemma 1.2.17 (2). Therefore $K = U \cap H = U_G \cap H = S_G \cap H = S \cap H$.

Observe that $C = \mathrm{C}_G(H) = HS_G = HU_G = U_GS_G$. Write $A = U_G \cap S_G$. Then

$$C/HA = HU_G/HA \cong_G U_G/(U_G \cap HA) = U_G/A \cong_G C/S_G$$

and then C/HA is a chief factor of G and $C/HA \gg U_G/A$. Observe that U_G/A is **X**-complemented by S and C/U_G is obviously **X**-complemented by U. By

Statement 2, all chief factors of G between C and A are $\mathbf{X}$-complemented. In particular C/HA is $\mathbf{X}$-complemented in G, i. e. there exists a maximal subgroup $M \in \mathbf{X}$ such that $G = MC$ and $HA = M \cap C$. This implies that $M_G = HA$. Therefore $\mathbf{X}$ is JH-solid. □

Corollary 1.2.37. *If $\mathbf{X}$ is a JH-solid set of maximal subgroups of a group G and H is a normal subgroup of G such that all chief factors H/K_i, $i = 1, \dots, n$, of G are $\mathbf{X}$-supplemented, and $\bigcap_{i=1}^{n} K_i = K$, then every chief factor between K and H is $\mathbf{X}$-supplemented.*

Proof. Denote $K^j = \bigcap_{i=1}^{j} K_i$ and $K^0 = H$. Then

$$K = K^n \leq K^{n-1} \leq \cdots \leq K^0 = H$$

is a piece of a chief series of G. Assume that $K^i \neq K^{i+1}$. Then $H = K^i K_{i+1}$, K^i/K^{i+1} is a chief factor of G and $K^i/K^{i+1} \cong_G H/K_{i+1}$. If M is an $\mathbf{X}$-supplement of H/K_{i+1} in G, then M is an $\mathbf{X}$-supplement of K^i/K^{i+1} in G by Lemma 1.2.16 (1). We deduce that all chief factors in the above series are $\mathbf{X}$-supplemented. Now apply Theorem 1.2.36 to conclude the proof. □

Corollary 1.2.38. *Let $\mathbf{X}$ be a JH-solid set of monolithic maximal subgroups of a group G and write $R = \Phi_{\mathbf{X}}(G)$. Suppose that N is a normal subgroup of G such that $N = N_1 \times \cdots \times N_n$, where N_i is a minimal normal subgroups of G, $1 \leq i \leq n$. If $R \cap N = 1$, then every chief factor of G below N is X-supplemented in G.*

Proof. We use induction on n. If $n = 1$, the result is obvious. Thus we assume that $n \geq 2$.

If N_1 is $\mathbf{X}$-Frattini, then $N_1 \leq R \cap N = 1$, giving a contradiction. Hence there exists $M \in \mathbf{X}$ such that $G = MN_1$. The quotient group G/M_G is a monolithic primitive group and then $NM_G/M_G = N_1M_G/M_G = \mathrm{Soc}(G/M_G)$. Then $N = N_1 \times (N \cap M_G)$. By Theorem 1.2.36, every piece of chief series of G between N_1 and N has exactly $n-1$ chief factors and so every piece of chief series of G below $N_0 = N \cap M_G$ has exactly $n-1$ chief factors. Since the normal subgroup N_0 is contained in $\mathrm{Soc}(G)$, we have that N_0 can be written as a direct product of $n-1$ minimal normal subgroups of G. Since $R \cap N_0 = 1$, it follows that every chief factor of G below N_0 is $\mathbf{X}$-supplemented by induction. Since clearly M supplements N/N_0, we have that all chief factors of G below N are $\mathbf{X}$-supplemented, by Theorem 1.2.36. □

Observe that in a primitive group G of type 3 with minimal normal subgroups N_1 and N_2, if M is a core-free maximal subgroup, then $\mathbf{X} = \{M\}$ is a JH-solid set of maximal subgroups of G, $R = M_G = 1$, and $N = \mathrm{Soc}(G)$ satisfies that $R \cap N = 1$. However neither N/N_1 nor N/N_2 are $\mathbf{X}$-supplemented.

Remarks 1.2.39. 1. Given a modular lattice $\mathcal{L}$, J. Lafuente in [Laf89] introduced the concept of *M-set in* $\mathcal{L}$ and he proved a general Jordan-Hölder theorem in modular lattices with an M-set.

In fact, Theorem 1.2.28 shows that, for a set of maixmal subgroups $\mathbf{X}$ of a group G, the set $\mathcal{M}_{\mathbf{X}}$ of all $\mathbf{X}$-supplemented chief factors of G is an M-set in the modular lattice $\mathcal{N}$ of all normal subgroups of G if and only if $\mathbf{X}$ is JH-solid.

2. For JH-solid sets containing some maximal subgroups of type 3, a converse of Theorem 1.2.34, giving an equivalence analogous to Theorem 1.2.36, does not hold.

Let T be a non-abelian simple group and consider the group G which is the direct product of three copies of T: $G = T_1 \times T_2 \times T_3$. Suppose that $\mathbf{X}$ is the set whose elements are three monolithic maximal subgroups M_1, M_2, and M_3, such that $(M_i)_G = T_j \times T_k$, where $\{i, j, k\} = \{1, 2, 3\}$. Consider the subgroups $U_1 = \Delta_{23} \times T_1 = \{(x, y, y) : x, y \in T\}$, which is a maximal subgroup of type 3 of G such that $(U_1)_G = T_1$, and $U_2 = \Delta_{13} \times T_2 = \{(x, y, x) : x, y \in T\}$, a maximal subgroup of type 3 of G such that $(U_2)_G = T_2$. The set $\mathbf{X} \cup \{U_1, U_2\}$ is not a JH-solid set of maximal subgroups: the minimal normal subgroup T_3 is supplemented by U_1 and U_2 but no maximal subgroup of $\mathbf{X} \cup \{U_1, U_2\}$ has core $\big((U_1)_G \cap (U_2)_G\big)T_3 = T_3$.

On the other hand, it is easy to see that no chief factor of G is $\mathbf{X}$-Frattini, and that any two G-isomorphic chief factors are supplemented by exactly one element of $\mathbf{X}$, so the conditions of Theorem 1.2.36 (2) hold. In other words, $\mathbf{X}$ is a JH-solid set of maximal subgroups of G.

1.3 Crowns

The concept of crown of a soluble group was introduced in [Gas62]. In this seminal paper, W. Gaschütz analyses the structure of the chief factors of a soluble group G as G-modules. Associated with a G-module $\mathfrak{a}$, there exists a section of the group, called $\mathfrak{a}$-*Kopf*, or crown in English, such that, viewed as a G-module, is completely reducible and homogeneous with a composition series of length the number of complemented chief factors G-isomorphic to $\mathfrak{a}$ in any chief series of G. These crowns are complemented sections of G.

The study of non-soluble chief factors made by J. Lafuente in [Laf84a], and, in particular, the introduction of the concept of G-connected chief factors, allowed him to discover that some sections associated with non-abelian chief factors can be constructed enjoying similar properties to Gaschütz's crowns. This originated the concept of crown of a non-abelian chief factor.

Given a group G, fixing a JH-solid set of maximal subgroups $\mathbf{X}$ of G and restricting ourselves to $\mathbf{X}$-supplemented chief factors, we can presume, after the results of Section 1.2, that most of the known results on crowns hold for the so-called $\mathbf{X}$-crowns. The aim of this section is to present results in this direction.

Let us start with the following observations. Let G be a group and H/K a non-abelian chief factor of G. If there exists a maximal subgroup M of G of type 3 complementing H/K, then the primitive group G/M_G has two minimal normal subgroups, namely HM_G/M_G and C/M_G, where $C = \mathrm{C}_G(H/K)$ and $HM_G \cap C = M_G$. In this case $\mathrm{C}^*_G(H/K) = HC$. By Remark 1.2.8 (2b), there exists a monolithic maximal supplement S of HM_G/M_G such that $S_G = C$. Analogously, since $\mathrm{C}_G(C/M_G) = HM_G$, there exists a monolithic maximal supplement T of C/M_G such that $T_G = HM_G$. This means that, although the sets

$$\begin{aligned}\mathcal{E}_1 = \{N : C^*/N \text{ is a precrown associated with a chief factor}\\ G\text{-connected to } H/K\}\\ \mathcal{E}_2 = \{M_G : M \text{ is a maximal subgroup of } G \text{ supplementing}\\ \text{a chief factor } G\text{-connected to } H/K\}, \text{and}\\ \mathcal{E}_3 = \{M_G : M \text{ is a maximal subgroup of } G \text{ supplementing}\\ \text{a chief factor } G\text{-isomorphic to } H/K\}\end{aligned}$$

in general are different, in fact

$$\bigcap\{N : N \in \mathcal{E}_1\} = \bigcap\{N : N \in \mathcal{E}_2\} = \bigcap\{N : N \in \mathcal{E}_3\}.$$

If we replace the set of all maximal subgroups for a proper JH-solid subset, the above equalities are not longer true.

Let G be a primitive group of type 3 with minimal normal subgroups A and B. If M and S are monolithic maximal subgroups with $M_G = A$ and $S_G = B$, and $\mathbf{X} = \{M, S\}$, then $\mathbf{X}$ is JH-solid and

$$\begin{aligned}\mathcal{E}_4 &= \{M_G : M \text{ is a maximal subgroup in } \mathbf{X}\\ &\qquad \text{supplementing a chief factor } G\text{-connected to } A\}\\ &= \{A, B\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{E}_5 &= \{M_G : M \text{ is a maximal subgroup in } \mathbf{X}\\ &\qquad \text{supplementing a chief factor } G\text{-isomorphic to } A\}\\ &= \{B\}.\end{aligned}$$

Then

$$\bigcap\{N : N \in \mathcal{E}_4\} = 1 < B = \bigcap\{N : N \in \mathcal{E}_5\}.$$

These observations motivate the following definitions.

Definitions 1.3.1. *1. Let H/K be a supplemented chief factor of a group G and consider the set $\mathcal{E}$ composed of all cores of the monolithic maximal subgroups of G which supplement chief factors G-connected to H/K. Write $R = \bigcap\{N : N \in \mathcal{E}\}$ and $C^* = \mathrm{C}^*_G(H/K)$. Then we say that the factor C^*/R is the* crown of G associated with *H/K.*

2. *Let* $\mathbf{X}$ *be a JH-solid set of monolithic maximal subgroups of a group* G *and* H/K *an* $\mathbf{X}$*-supplemented chief factor of* G*. Write* $C^* = \mathrm{C}^*_G(H/K)$ *and consider the normal subgroup*

$$R_{\mathbf{X}} = \bigcap\{M_G : M \in \mathbf{X} \text{ and } M \text{ supplements a chief factor } G\text{-connected to } H/K\}.$$

Then $C^*/R_{\mathbf{X}}$ *is the* $\mathbf{X}$-crown of G associated with H/K*.*

Obviously a crown of G associated with a supplemented chief factor of G is just an $\mathbf{X}$-crown of G for the set $\mathbf{X} = \mathrm{Max}^*(G)$ of all monolithic maximal subgroups of G.

Theorem 1.3.2. *Let* $\mathbf{X}$ *be a JH-solid set of monolithic maximal subgroups of a group* G *and* H/K *an* $\mathbf{X}$*-supplemented chief factor of* G*. Write* $C^*/R_{\mathbf{X}}$ *for the* $\mathbf{X}$*-crown of* G *associated with* H/K*. Then*

$$C^*/R_{\mathbf{X}} = \mathrm{Soc}(G/R_{\mathbf{X}}).$$

Furthermore

1. *every minimal normal subgroup of* $G/R_{\mathbf{X}}$ *is an* $\mathbf{X}$*-supplemented chief factor of* G *which is* G*-connected to* H/K*, and*
2. *no* $\mathbf{X}$*-supplemented chief factor of* G *over* C^* *or below* $R_{\mathbf{X}}$ *is* G*-connected to* H/K*.*

In other words, there exist m *normal subgroups* $A_1, \dots, A_m$ *of* G *such that*

$$C^*/R_{\mathbf{X}} = A_1/R_{\mathbf{X}} \times \cdots \times A_m/R_{\mathbf{X}}$$

where $A_i/R_{\mathbf{X}}$ *is an* $\mathbf{X}$*-supplemented chief factor* G*-connected to* H/K*, for* $i = 1, \dots, m$*, and* m *is the number of* $\mathbf{X}$*-supplemented chief factors* G*-connected to* H/K *in each chief series of* G*. Moreover,* $\Phi(G/R_{\mathbf{X}}) = \mathrm{O}_{q'}(G/R_{\mathbf{X}}) = 1$*, for each prime* q *dividing the order of* $|H/K|$*.*

Proof. We can write $R_{\mathbf{X}} = R = N_1 \cap \cdots \cap N_r$, such that C^*/N_i are $\mathbf{X}$-precrowns associated with chief factors G-connected to H/K and r is minimal with this property. Consider the group monomorphism

$$\begin{aligned} \psi\colon C^*/R = C^*/(N_1 \cap \cdots \cap N_r) &\longrightarrow C^*/N_1 \times \cdots \times C^*/N_r \\ c(N_1 \cap \cdots \cap N_r) &\longmapsto (cN_1, \dots, cN_r) \end{aligned}$$

for any $c \in C^*$. Observe that ψ is compatible with the action of G:

$$\begin{aligned} \big(c(N_1 \cap \cdots \cap N_r)^{\psi}\big)^g &= (cN_1, \dots, cN_r)^g = (c^gN_1, \dots, c^gN_r) \\ &= \big(c^g(N_1 \cap \cdots \cap N_r)\big)^{\psi}. \end{aligned}$$

From minimality of r, we have that $C^* = N_i(N_1 \cap \cdots \cap N_{i-1})$, for $i \le r$, and then

$$(N_1 \cap \cdots \cap N_{i-1})/(N_1 \cap \cdots \cap N_i) \cong_G C^*/N_i.$$

Therefore the chain

$$R = (N_1 \cap \cdots \cap N_r) \le (N_1 \cap \cdots \cap N_{r-1}) \le \cdots \le N_1 \le C^*$$

is a piece of chief series of G and each chief factor is G-connected to H/K.

Hence the order $|C^*/R| = |H/K|^r$ and ψ is an isomorphism. By Corollary 1.2.37, every chief factor of G between R and C^* is $\mathbf{X}$-supplemented in G. Therefore, there exist r normal subgroups $A_1, \ldots, A_r$ of G such that

$$C^*/R = A_1/R \times \cdots \times A_r/R,$$

where A_i/R is a $\mathbf{X}$-supplemented chief factor G-connected to H/K, $i = 1, \ldots, r$.

Suppose that H_0/K_0 is a $\mathbf{X}$-supplemented chief factor of G which is G-connected to H/K and let $M \in \mathbf{X}$ be a supplement of H_0/K_0 in G. Then $H_0 \le C^*$. Observe that since $R \le M_G$, then $H_0 \not\le R$. Therefore no $\mathbf{X}$-supplemented chief factor of G over C^* or below R is G-connected to H/K.

By Theorem 1.2.36, the number of $\mathbf{X}$-supplemented chief factors G-connected to H/K in each chief series of G is an invariant of the group and coincides with the length of any piece of chief series of G between R and C^*.

If B/R is a minimal normal subgroup of G/R and $B \cap C^* = R$, then $B \le \mathrm{C}_G(A_1/R)$ which is contained in C^* by Proposition 1.2.15. This contradiction implies that $C^*/R = \mathrm{Soc}(G/R)$. Since every minimal normal subgroup of G/R is supplemented in G/R, we have that $\Phi(G/R) = 1 = \mathrm{O}_{q'}(G/R)$, for each prime q dividing the order of $|H/K|$. □

Corollary 1.3.3 ([Laf84a]). *Two supplemented chief factors of a group G define the same crown of G if and only if they are G-connected.*

Let C^*/R be the $\mathbf{X}$-crown of G associated with an $\mathbf{X}$-supplemented chief factor H/K. Applying Theorem 1.3.2, we have that $C^*/R = (R_{\mathbf{X}}/R) \times (C_0/R)$, and the $\mathbf{X}$-crown of G associated to H/K is isomorphic to C_0/R which is a direct product of $\mathbf{X}$-supplemented components of C^*/R.

Corollary 1.3.4. *Let $\mathbf{X}$ be a JH-solid set of monolithic maximal subgroups of a group G. Let H/K be an $\mathbf{X}$-supplemented chief factor of a group G and write C^*/R for the $\mathbf{X}$-crown of G associated with H/K. Then*

1. *if H/K is abelian and p is the prime dividing $|H/K|$, then $C^* = \mathrm{C}_G(H/K) = C$ and*

$$C/R = \mathrm{Soc}(G/R) = \mathrm{F}(G/R) = \mathrm{O}_p(G/R)$$

is a completely reducible and homogeneous G-module over $\mathrm{GF}(p)$ whose composition factors are G-isomorphic to H/K and the length of a composition series of C/R, as G-module, is the number of $\mathbf{X}$-complemented G-chief factors G-isomorphic to H/K in each chief series of G;

2. *if H/K is non-abelian, then $\{A_j/R : j = 1, \dots, m\}$ is the set of all minimal normal subgroups of G/R; in particular, if C^*/R is a chief factor of G, then $R = \mathrm{C}_G(H/K)$ and $G/R \cong [H/K] * G$ is a primitive group of type 2.*

Proof. Applying Theorem 1.3.2, $C^*/R = Soc(G/R)$.

1. If H/K is abelian, then H/K is a p-group for some prime p and $C^* = C = \mathrm{C}_G(H/K)$ is the common centraliser of the chief factors of G between C and R. Then $\mathrm{C}_{G/R}(C/R) = C/R = F(G/R)$ and Statement 1 follows from Theorem 1.3.2.

2. Suppose now that H/K is non-abelian. Then $\{A_j/R : j = 1, \dots, m\}$, as in Theorem 1.3.2, are the minimal normal subgroups of G/R. Finally observe that if C^*/R is a chief factor, then C^*/R is the **X**-precrown of G associated with H/K and $R = \mathrm{C}_G(H/K)$. □

Our main goal is now to prove that in every group G, we can order in some sense the **X**-crowns of G to obtain a chief series of G in which some G-isomorphic images of the **X**-crowns are placed one after the other, possibly separated by **X**-Frattini chief factors, and all **X**-supplemented chief factors which are G-connected are consecutive.

We need a technical proposition to explore how the crowns of the quotient group are related to the crowns of the original group. We will use it in inductive arguments.

Proposition 1.3.5. *Let* **X** *be a JH-solid set of monolithic maximal subgroups of a group G and let N be a normal subgroup of G contained in some maximal subgroup of G in* **X**.

1. *For any* **X***-crown C^*/R of G, either*
   a) *$C^* \leq RN$ or*
   b) *$RN < C^*$ and $(C^*/N)\big/(RN/N)$ is an* **X**/*N-crown of G/N.*
2. *For any* **X**/*N-crown $(C_0^*/N)\big/(R_0/N)$ of G/N, there is an* **X***-crown C^*/R of G such that $C_0^* = C^*$ and $R_0 = RN$.*

Proof. 1. Assume that C^* is not contained in RN. Then, applying Corollary 1.3.4, there exists a minimal normal subgroup A/R of G/R such that A is not contained in RN. Therefore AN/RN is a chief factor of G which is G-isomorphic to A/R. Hence $RN < AN \leq \mathrm{C}_G^*(AN/RN) = C^*$. Applying Theorem 1.3.2, AN/RN is **X**-supplemented and clearly $(C^*/N)\big/(NR/N)$ is the **X**/N-crown of G/N associated with the chief factor $AN/N/RN/N$ of G/N.

2. Let $(C_0^*/N)\big/(R_0/N)$ be the **X**/N-crown of G/N associated with an **X**/N-supplemented chief factor $(H/N)\big/(K/N)$ of G/N. Then $(H/N)\big/(K/N)$ is G-isomorphic to the chief factor H/K of G and H/K is **X**-supplemented in G. Consider the **X**-crown C^*/R of G associated with H/K. It follows that $C_0^*/N = \mathrm{C}_{G/N}^*\big((H/N)\big/(K/N)\big) = \mathrm{C}_G^*(H/K)/N$ and then $C_0 = C^*$.

On the other hand, it is clear that $RN \leq R_0$. In addition, every chief factor of a given chief series of G between RN and R_0 is $\mathbf{X}$-supplemented in G and G-connected to H/K. Since, by Theorem 1.3.2, the number of $\mathbf{X}/N$-supplemented chief factors of each chief series of G/N which are G/N-connected to $(H/N)\big/(K/N)$ is exactly the number of chief factors of G/N between R_0/N and C^*/N, we have that $RN = R_0$. □

Lemma 1.3.6. *Let G be a group with $\Phi(G) = 1$. There exists a crown C^*/R and a non-trivial normal subgroup D of G such that $C^* = R \times D$.*

Proof. We argue by induction on the order of G. Let M be a minimal normal subgroup of G. Since $\Phi(G) = 1$, it follows that M is supplemented in G and we can consider the crown C_0^*/R_0 and a precrown C_0^*/N_0 associated with M in G. We know that $C_0^* = N_0 \times M$.

If $N_0 = R_0$, then the normal subgroup $D = M$ and the crown $C^*/R = C_0^*/R_0$ fulfils our requirements.

Assume that $R_0 < N_0$. This means that $R_0 \times M < C_0^*$. Write $F/M = \Phi(G/M)$. By Proposition 1.3.5, $(C_0^*/M)\big/(R_0M/M)$ is a crown of G/M associated with the chief factors of G/M, i.e. the chief factors of G over M, which are G-connected to M. Since, by Theorem 1.3.2, $\Phi\big((G/M)\big/(R_0M/M)\big) = 1$, we have that $F \leq R_0M$ and then $F = M \times (F \cap R_0)$. Put $N = F \cap R_0$. Suppose that $N \neq 1$, and let A be a minimal normal subgroup of G contained in N. Recall that all monolithic maximal subgroups of G form a JH-solid set and their intersection is $\Phi(G)$. Since obviously $MA \cap \Phi(G) = 1$, we can apply Corollary 1.2.38 and deduce that the chief factor MA/M is supplemented in G. But this contradicts the fact that $MA/M \leq F/M = \Phi(G/M)$. Therefore $F = M$ and $\Phi(G/M) = 1$. By induction, there exists a crown $(C_1^*/M)\big/(R_1/M)$ and a non-trivial normal subgroup D_1/M of G/M, such that $C_1^*/M = (R_1/M) \times (D_1/M)$.

Suppose first that $(C_1^*/M)\big/(R_1/M)$ is the crown associated with the chief factors G-connected to M. Then $C_1^* = C_0^*$ and $R_1 = R_0 \times M$. In this case, we take $D = D_1$ and $C^*/R = C_0^*/R_0$. Note that $M = D_1 \cap R_1 = D_1 \cap (R_0 \times M) = (D_1 \cap R_0) \times M$. Hence $D_1 \cap R_0 = 1$.

Suppose now that the chief factors of G between M and D_1 are not G-connected to M. If $C_0^*/M \leq (R_0M/M)(D_1/M)$, then $C_0^* = R_0(C_0^* \cap D_1)$. Then $C_0^*/R_0 \cong_G (C_0^* \cap D_1)/(R_0 \cap D_1)$ and $M \leq C_0^* \cap D_1$. Hence all chief factors of G between $(R_0 \cap D_1) \times M$ and $C_0^* \cap D_1$ are G-connected to M by Theorem 1.3.2. Since no chief factor of G between M and D_1 is G-connected to M, we deduce that $C_0^* \cap D_1 = (R_0 \cap D_1) \times M$. Then $C_0^* = R_0M$, against our assumption. Hence, by Proposition 1.3.5, we have that $(R_0M/M)(D_1/M) < C_0^*/M$ and then $R_0 \leq R_0M \leq R_0D_1 \leq C_0^*$. Applying Theorem 1.3.2, every chief factor of G between R_0M and R_0D_1 is G-connected to M. Since $D_1R_0/MR_0 \cong_G D_1/M(D_1 \cap R_0)$ and we are assuming that all chief factors of G between M and D_1 are not G-connected to M, we have that $D_1 = M(D_1 \cap R_0)$. In this case, take $D = D_1 \cap R_0 \neq 1$ and $C^* = C_1^*$. This completes the proof. □

We prove now the corresponding result for a JH-solid set $\mathbf{X}$ of monolithic maximal subgroups of a group G.

Proposition 1.3.7. *Let G be a group and $\mathbf{X}$ a JH-solid set of monolithic maximal subgroups of G such that $\Phi_{\mathbf{X}}(G) = 1$. There exists an $\mathbf{X}$-crown $C^*/R_{\mathbf{X}}$ of G and a non-trivial normal subgroup D of G such that $C^* = R_{\mathbf{X}} \times D$.*

Proof. Observe first that $\Phi(G) \leq \Phi_{\mathbf{X}}(G) = 1$. By Lemma 1.3.6, there exists a crown C^*/R and a non-trivial normal subgroup D of G such that $C^* = R \times D$. Consider the G-isomorphism $\varphi\colon C^*/R \longrightarrow D$. If $C^*/R = (A_1/R) \times \cdots \times (A_r/R)$, then all the images $(A_i/R)^{\varphi} = N_i$ are minimal normal subgroups of G below D, the N_i are G-connected, C^*/R is the crown of G associated with them and $D = N_1 \times \cdots \times N_r$. Moreover, by Theorem 1.2.38, every chief factor of G below D is $\mathbf{X}$-supplemented in G. Hence $R = R_{\mathbf{X}}$ and $C^*/R = C^*/R_{\mathbf{X}}$ is the $\mathbf{X}$-crown of G associated with the N_i. □

Theorem 1.3.8 (see [För88]). *Let $\mathbf{X}$ be a non-empty JH-solid set of monolithic maximal subgroups of a group G and.*

1. *Let $C_1^*/R_1, \dots, C_n^*/R_n$ denote the $\mathbf{X}$-crowns of G. Then there exists a permutation $\sigma \in \operatorname{Sym}(n)$ and a chain of normal subgroups of G*

   $$1 = C_{(0)} \leq R_{(1)} < C_{(1)} \leq R_{(2)} < C_{(2)} \leq \cdots < C_{(n-1)} \leq R_{(n)} < C_{(n)} \leq G$$

   such that $G/C_{(n)} = \Phi_{\mathbf{X}/C_{(n)}}(G/C_{(n)})$ (including the case $G = C_{(n)}$) and for $i = 1, \dots, n$, we have

   $$R_{(i)}/C_{(i-1)} = \Phi_{\mathbf{X}/C_{(i-1)}}(G/C_{(i-1)}), \quad C^*_{i^\sigma} = R_{i^\sigma} C_{(i)}, \quad R_{i^\sigma} \cap C_{(i)} = R_{(i)}.$$

2. *Moreover, if N is a normal subgroup of G and $C_{(k-1)} \leq N \leq R_{(k)}$, for some $k \in \{1, \dots, n\}$,*

   $$1 = N/N = C_{(k-1)}N/N \leq R_{(k)}/N < C_{(k)}/N \leq R_{(k+1)}/N < \cdots < C_{(n)}/N \leq G/N$$

   is a chain of G/N enjoying the corresponding property.

Proof. 1. We use induction on $|G|$. Clearly $\Phi_{\mathbf{X}}(G)$ is contained in each R_i. Moreover, every $\mathbf{X}$-supplemented chief factor of G is G-isomorphic to an $\mathbf{X}/\Phi_{\mathbf{X}}(G)$-supplemented chief factor of $G/\Phi_{\mathbf{X}}(G)$. Hence, by Proposition 1.3.5 (2), we can assume without loss of generality that $\Phi_{\mathbf{X}}(G) = R_{(1)} = 1$.

By Proposition 1.3.7, there exists an $\mathbf{X}$-crown C_k^*/R_k of G and a normal subgroup $C_{(1)}$ of G such that $C_k^* = R_k \times C_{(1)}$. If $G = C_{(1)}$, the result is trivial. If $C_{(1)}$ is a proper subgroup of G and $\mathbf{X}/C_{(1)} = \emptyset$, then $\Phi_{\mathbf{X}/C_{(1)}}(G/C_{(1)}) = G/C_{(1)}$ or, in other words, no maximal subgroup of G in $\mathbf{X}$ contains $C_{(1)}$. Hence no chief factor of G over $C_{(1)}$ is $\mathbf{X}$-supplemented.

In this case there exists exactly one $\mathbf{X}$-crown of G and the theorem holds trivially. Assume that $\bar{\mathbf{X}} = \mathbf{X}/C_{(1)}$ is non-empty, i.e. $C_{(1)}$ is contained in some maximal subgroup of G in $\mathbf{X}$. Then we can apply the inductive hypothesis to the quotient group $\bar{G} = G/C_{(1)}$. Observe that if $C_j^* \leq R_j C_{(1)}$, for some $j \neq k$, then $C_j^*/R_j \cong_G (C_j^* \cap C_{(1)})/(R_j \cap C_{(1)})$ and the chief factors of G between C_j^* and R_j are G-connected to some chief factors of G below $C_{(1)}$ and therefore to the chief factors in C_k^*/R_k, which is not possible by Theorem 1.3.2. Hence, by Proposition 1.3.5, $R_j C_{(1)} < C_j^*$, for all $j \in \{1, \ldots, n\} \setminus \{k\}$. Therefore $\{\bar{C}_j^*/\bar{R}_j : j \neq k\}$ are the $\mathbf{X}$-crowns of $\bar{G}$ and, by induction, there exists a bijection $\tau \colon \{2, \ldots, n\} \longrightarrow \{1, \ldots, n\} \setminus \{k\}$, and a chain of normal subgroups of $\bar{G}$

$$1 = \bar{C}_{(1)} \leq \bar{R}_{(2)} < \bar{C}_{(2)} \leq \bar{R}_{(3)} < \bar{C}_{(3)} \leq \cdots < \bar{C}_{(n-1)} \leq \bar{R}_{(n)} < \bar{C}_{(n)} \leq \bar{G}$$

such that $\bar{G}/\bar{C}_{(n)} = \Phi_{\bar{\mathbf{X}}/\bar{C}_{(n)}}(\bar{G}/\bar{C}_{(n)})$, and for $i = 2, \ldots, n+1$, we have

$$\bar{R}_{(i)}/\bar{C}_{(i-1)} = \Phi_{\mathbf{X}/\bar{C}_{(i-1)}}(\bar{G}/\bar{C}_{(i-1)}), \qquad \bar{C}^*_{i^\tau} = \bar{R}_{i^\tau}\bar{C}_{(i)}, \qquad \bar{R}_{i^\tau} \cap \bar{C}_{(i)} = \bar{R}_{(i)}.$$

Now, just take the inverse images $R_{(j)}/C_{(1)} = \bar{R}_{(j)}$ and $C_{(j)}/C_{(1)} = \bar{C}_{(j)}$, for $j = 2, \ldots, n$. The required permutation is σ such that $1^\sigma = k$ and $i^\sigma = i^\tau$, for $i = 2, \ldots, n$.

2. Assume that N is a normal subgroup of G such that $C_{(k-1)} \leq N \leq R_{(k)}$. Every $\mathbf{X}$-supplemented chief factor H/K of G such that $N \leq K$ is G-isomorphic to an $\mathbf{X}/N$-supplemented chief factor of G/N and therefore is G-connected to some chief factor between $R_{(j)}/N$ and $C_{(j)}/N$, for some $j \geq k$. The $\mathbf{X}/N$-crown of G/N associated with $(H/N)/(K/N)$ is $(C^*_{j^\sigma}/N)/(R_{j^\sigma}/N)$ and clearly we have that $C^*_{j^\sigma}/N = (R_{j^\sigma}/N)(C_{(j)}/N)$ and $(R_{j^\sigma}/N) \cap (C_{(j)}/N) = R_{(j)}/N$. In addition, $R_{(i)}/C_{(i-1)}$ is equal to $\Phi_{\mathbf{X}/C_{(i-1)}}(G/C_{(i-1)})$. Hence

$$(R_{(i)}/N)/(C_{(i-1)}/N) = \Phi_{(\mathbf{X}/N)/(C_{(i-1)}/N)}\big((G/N)/(C_{(i-1)}/N)\big)$$

for all $i = k+1, \ldots, n$. Now $R_{(k)}/C_{(k-1)} = \Phi_{\mathbf{X}/C_{(k-1)}}(G/C_{(k-1)})$ implies that $\Phi_{\mathbf{X}/N}(G/N) = R_{(k)}/N$. □

Now, the result we were looking for becomes clear.

Corollary 1.3.9 (see [Gas62] and [För88]). *Let $\mathbf{X}$ be a JH-solid set of monolithic maximal subgroups of a group G. If $C_1^*/R_1, \ldots, C_n^*/R_n$ are the $\mathbf{X}$-crowns of G, there exists a permutation $\sigma \in \mathrm{Sym}(n)$ and a chief series of G*

$$\begin{aligned}1 &= F_{1,0} < F_{1,1} < \cdots < F_{1,m_1} = N_{1,0} < N_{1,1} < \cdots < N_{1,k_1}\\ &= F_{2,0} < F_{2,1} < \cdots < F_{2,m_2} = N_{2,0} < N_{2,1} < \cdots < N_{2,k_2}\\ &\ldots\\ &= F_{n,0} < F_{n,1} < \cdots < F_{n,m_n} = N_{n,0} < N_{n,1} < \cdots < N_{n,k_n}\\ &= F_{n+1,0} < F_{n+1,1} < \cdots < F_{n+1,m_{n+1}} = G\end{aligned}$$

such that

1. *the* $F_{i,j}/F_{i,j-1}$ *are* **X**-*Frattini chief factors of* G,
2. *the* $N_{i,j}/N_{i,j-1}$ *are* **X**-*supplemented chief factors of* G *satisfying that* $N_{i,j}/N_{i,j-1}$ *is* G-*connected to* $N_{i',j'}/N_{i',j'-1}$ *if and only if* $i = i'$; *moreover* $C_{i^\sigma}/R_{i^\sigma}$ *is the* **X**-*crown associated with* $N_{i,j}/N_{i,j-1}$;
3. $F_{i,m_i} / F_{i,j} = \Phi_{\mathbf{X}/F_{i,j}}(G/F_{i,j})$, *for each* $i = 1, \dots, n{+}1$ *and* $j = 1, \dots, m_i{-}1$.

Let **X** be a JH-solid set of monolithic maximal subgroups of a group G. Then if $C^*/R_{\mathbf{X}}$ is the **X**-crown of G associated with a chief factor H/K, and $R_{\mathbf{X}} = G_0 < G_1 < \cdots < G_n = C^*$ is a piece of chief series of G, then the subgroup

$$V = \bigcap_{i=1}^{n} \{M_i : M_i \text{ is an } \mathbf{X}\text{-supplement of } G_i/G_{i-1}\}.$$

is a supplement (if H/K is abelian, then V is a complement) of $C^*/R_{\mathbf{X}}$ in G, by repeated applications of Lemma 1.2.16 (2). However, this supplement depends on the choice of the chief series and on the choice of the maximal subgroups and it is not preserved by epimorphic images. The following example is illustrative of these problems.

Example 1.3.10. Denote by N the elementary abelian group of order 3^2. The cyclic group Z of order 2 acts on N by inversion. Form the semidirect product $G = [N]Z$ and write $A = \langle a \rangle$, $N = \langle a, b \rangle$, and $Z = \langle z \rangle$. Consider the JH-solid set of maximal subgroups $\mathbf{X} = \{M_1 = \langle a, z \rangle, M_2 = \langle b, az \rangle, M_3 = \langle ab, z \rangle, M_4 = \langle a^2 b, z \rangle\}$. The **X**-crown of G associated with any of the chief factors below N is $N = \mathrm{C}_G(A)$. All subgroups of the form $V_{ij} = M_i \cap M_j$, $i \neq j$, are complements of N in G. Note that $\bigcap_{i=1}^{4} M_i = 1$.

Consider now the group G/A. Observe that $\mathbf{X}/A = \{M_1/A\}$ and the $\mathbf{X}/A$-crown of G/A associated with N/A is N/A itself. Notice that the subgroup $V_{23}A/A = \langle a, bz \rangle / A$ is a complementchief factor!complemented of N/A in G/A which does not belong to $\mathbf{X}/A$.

Proposition 1.3.11. *Let* G *be a group and* **X** *a JH-solid set of monolithic maximal subgroups of* G. *Assume that if* U *and* S *are two distinct elements of* **X**, *then* $U_G \neq S_G$. *Let* $C^*/R_{\mathbf{X}}$ *be the* **X**-*crown of* G *associated with the* **X**-*supplemented chief factor* F. *Consider the set*

$$\mathbf{X}_F = \{M \in \mathbf{X} : M \text{ supplements a chief factor } G\text{-connected to } F\}.$$

We define the subgroup $T = \mathrm{T}(G, \mathbf{X}, F) = \bigcap \{M : M \in \mathbf{X}_F\}$. *Clearly* $T_G = R_{\mathbf{X}}$.

1. *Assume that if* U *and* S *are two distinct elements of* **X** *and both supplement a chief factor* H/K *of* G, *then* $M = (U \cap S)H \in \mathbf{X}$. *Then the subgroup* T *satisfies the following properties.*

a) *For any piece of chief series of G, $R_{\mathbf{X}} = G_0 < G_1 < \cdots < G_n = C^*$ and any family $\{M_i \in \mathbf{X} : i = 1, \ldots, n\}$ such that M_i is a supplement of G_i/G_{i-1}, for each $i = 1, \ldots, n$, we have*

$$\mathrm{T}(G, \mathbf{X}, F) = \bigcap_{i=1}^{n} M_i$$

and $\mathrm{T}(G, \mathbf{X}, F)$ is a supplement (a complement, if F is abelian) of $C^*/R_{\mathbf{X}}$ in G.

b) *For any normal subgroup N of G such that F is G-connected with an $\mathbf{X}/N$-supplemented chief factor F_1 of G/N, then $\mathrm{T}(G/N, \mathbf{X}/N, F_1) = TN/N$.*

2. *Conversely, assume that the subgroup T satisfies the above Conditions 1a and 1b. Then, if U and S are elements of $\mathbf{X}_F$ such that $U_G \neq S_G$, and both supplement a chief factor H/K of G, then $M = (U \cap S)H \in \mathbf{X}_F$.*

Proof. 1. a) Fix a piece of chief series of G, $R_{\mathbf{X}} = G_0 < G_1 < \cdots < G_n = C^*$ and a family $\{M_i \in \mathbf{X} : i = 1, \ldots, n\}$ such that M_i is a supplement of G_i/G_{i-1}, for each $i = 1, \ldots, n$ and write $D = \bigcap_{i=1}^{n} M_i$. If $\mathbf{X}_F = \{M_i \in \mathbf{X} : i = 1, \ldots, n\}$, then there is nothing to prove.

Assume that there exists $U \in \mathbf{X}_F \setminus \{M_i \in \mathbf{X} : i = 1, \ldots, n\}$. Then U supplements G_j/G_{j-1}, for some $j = 1, \ldots, n$. Since U and M_j are distinct monolithic $\mathbf{X}_F$-supplements of the same chief factor G_j/G_{j-1} and $U_G \neq M_{j_G}$, we have that G_j/G_{j-1} is abelian by Lemma 1.2.17 (2), and so is $C^*/R_{\mathbf{X}}$. Therefore U and M_j complement G_j/G_{j-1}. By hypothesis, $M = (U \cap M_j)G_j \in \mathbf{X}_F$. Now we have that $M_j \cap M = (M_j \cap U)(M_j \cap G_j) = (M_j \cap U)G_{j-1} = M_j \cap U$ and analogously $U \cap M = M_j \cap U$. Then $D \cap U = D \cap M$. Observe that M complements a chief factor G_k/G_{k-1}, for some $k > j$. If $M = M_k$, then $D \cap U = D$. If $M \neq M_k$, repeat the previous argument replacing U by M and M_j by M_k. Observe also that G_n/G_{n-1} is self-centralising in G/G_{n-1} and so the latter group is primitive. Hence $(M_n)_G = G_{n-1}$. Therefore M_n is the unique maximal subgroup of G in $\in \mathbf{X}_F$ complementing the last chief factor. Since the other possible maximal subgroups in $\mathbf{X}_F$ do not change the intersection, it follows that $\mathrm{T}(G, \mathbf{X}, F) = \bigcap_{i=1}^{n} M_i$.

Moreover, if we apply repeatedly Lemma 1.2.16 (2), we deduce that the subgroup $T = \bigcap_{i=1}^{n} M_i$ is a supplement (complement if the crown is abelian) of $C^*/R_{\mathbf{X}}$ in G.

b) Let N be a minimal normal subgroup of G such that G/N has an $\mathbf{X}/N$-supplemented chief factor F_1 which is G-connected to F. By Proposition 1.3.5, we have that $R_{\mathbf{X}}N < C^*$ and $(C^*/N)/(R_{\mathbf{X}}N/N)$ is the $\mathbf{X}/N$-crown of G/N associated with F_1. If $N \leq R_{\mathbf{X}}$, it is clear that $T/N = \mathrm{T}(G/N, \mathbf{X}/N, F_1)$. Assume that N is G-connected to F, i.e. $R_{\mathbf{X}} < R_{\mathbf{X}}N$. We consider a piece of chief series of G

$$R_{\mathbf{X}} = G_0 < G_1 = R_{\mathbf{X}}N < G_2 \cdots < G_n = C^*.$$

By Statement 1a we have that

$$T = \mathrm{T}(G, \mathbf{X}, F) = \bigcap_{i=1}^{n} \{M_i : M_i \text{ is an } \mathbf{X}\text{-supplement of } G_i/G_{i-1}\}.$$

Since $N \leq \bigcap_{i=2}^{n} M_i$ and $G = M_1 N$, we have that

$$TN = \left(\bigcap_{i=1}^{n} M_i\right) N = M_1 N \cap \left(\bigcap_{i=2}^{n} M_i\right) = \bigcap_{i=2}^{n} M_i,$$

and

$$TN/N = \bigcap_{i=2}^{n} (M_i/N) = \mathrm{T}(G/N, \mathbf{X}/N, F).$$

An inductive argument proves the validity of the Statement 1b for any normal subgroup N of G such that F is G-connected with a chief factor of G/N.

2. Assume that the subgroup T satisfies Statement 1a and Statement 1b and suppose that U and S are elements of $\mathbf{X}_F$ such that $U_G \neq S_G$, and both supplement the same chief factor H/K of G. Since U and S are monolithic and $U_G \neq S_G$, H/K is abelian by Lemma 1.2.17 (2).

Observe that $C^* = C = \mathrm{C}_G(H/K) = HU_G = HS_G$ and $K = U_G \cap H = S_G \cap H$. Suppose that $R_{\mathbf{X}} < U_G \cap S_G$. Let $N/R_{\mathbf{X}}$ be a chief factor of G such that $N \leq U_G \cap S_G$. It is clear that $F_2 = (HN/N)\big/(KN/N)$ is a chief factor of G/N and the $\mathbf{X}/N$-crown of G/N associated with F_2 is C/N. We see that in the group G/N all hypotheses hold for $\mathbf{X}/N$ and $\mathrm{T}(G/N, \mathbf{X}/N, F_2) = TN/N$. To see that TN/N satisfies Statement 1a, let $1 = N/N = G_1/N < \cdots < G_n/N = C/N$ be a piece of chief series of G/N and M_i/N an $\mathbf{X}/N$-complement of $(G_i/N)\big/(G_{i-1}/N)$ for $i = 2, \ldots, n$. Let M_1 be an $\mathbf{X}$-complement of $N/R_{\mathbf{X}}$. Then $R_{\mathbf{X}} < N = G_1 < \cdots < G_n = C$ is a piece of chief series of G and M_i is an $\mathbf{X}$-complement of G_i/G_{i-1}, for $i = 1, \ldots, n$. Since G satisfies Statement 1a, we have that $T = \bigcap_{i=1}^{n} M_i$ and then $TN = M_1 N \cap (\bigcap_{i=2}^{n} M_i) = \bigcap_{i=2}^{n} M_i$. Since T satisfies Statement 1b, we have $TN/N = \bigcap_{i=2}^{n}(M_i/N)$ and TN/N satisfies Statement 1a. Clearly, TN/N satisfies Statement 1b.

Arguing by induction, we have that the maximal subgroup

$$M/N = \big((U/N) \cap (S/N)\big)(HN/N) = \big((U \cap S)H\big)/N \in \mathbf{X}_F/N$$

and then $M = (U \cap S)H \in \mathbf{X}_F$. Hence, we can assume that $U_G \cap S_G = R_{\mathbf{X}} = 1$. This implies that $K = 1$ and H is a minimal normal subgroup of G. Observe that also U_G and S_G are minimal normal subgroups of G. We can consider these three different pieces of chief series of G below C:

$$1 < H < C \qquad 1 < U_G < C \qquad 1 < S_G < C.$$

By Statement 1a, applied to the second or the third piece of chief series, we have that $T = U \cap S = \bigcap\{M : M \in \mathbf{X}_F\}$. In other words, for all $M \in \mathbf{X}_F$, we have that $U \cap S \leq M$. Since $\mathbf{X}$ is JH-solid, the number of $\mathbf{X}$-complemented chief factors of G which are G-isomorphic to H is the same in any chief series by Theorem 1.2.36. Hence, there exists an $\mathbf{X}$-complement of C/H in G. If M is such a complement, then $H \leq M$. Therefore $(U \cap S)H \leq M$. But $(U \cap S)H$ is a maximal subgroup of G, by Lemma 1.2.17. Therefore $M = (U \cap S)H \in \mathbf{X}_F$. □

1.4 Systems of maximal subgroups

JH-solid sets of monolithic maximal subgroups are characterised by their excellent adequacy to the Jordan-Hölder correspondence, as we saw in Theorem 1.2.36, but are not strong enough to fulfil some expected properties when working with supplements of $\mathbf{X}$-crowns. A supplement of a particular $\mathbf{X}$-crown C^*/R of a group G is obtained by the intersection of an $\mathbf{X}$-supplement of each chief factor in a piece of chief series of G passing through R and C^*, applying repeatedly Lemma 1.2.16. If we want these supplements of $\mathbf{X}$-crowns to be preserved by epimorphic images and to be independent of the choice of the chief series and of the choice of maximal subgroups, the JH-solid set of monolithic maximal subgroups $\mathbf{X}$ have to satisfy some rather stronger conditions characterised in Proposition 1.3.11. A *subsystem of maximal subgroups* of a group G is in fact a JH-solid set of monolithic maximal subgroups of G, with different cores, and satisfying the properties stated in Proposition 1.3.11.

Why are we interested in supplements of $\mathbf{X}$-crowns? The answer will be clear in Section 4.3 where the *subgroups of prefrattini type* are introduced. W. Gaschütz constructed his celebrated prefrattini subgroups, in [Gas62], by intersecting complements of (abelian) crowns. Several generalisations of prefrattini subgroups are constructed by intersecting some cleverly chosen maximal subgroups. The key is these "clever" choice of supplements. Within the limits of the soluble groups, maximal subgroups into which a fixed Hall system reduces are used. But the extension of these ideas to a general non necessarily soluble group required of a new arithmetical-free method of choice of maximal subgroups. Subsystems of maximal subgroups are the answer and, supporting this idea, we will show that in a soluble group G, given a system of maximal subgroups $\mathbf{X}$ of G, there exists a Hall system Σ of G such that $\mathbf{X}$ is the set of all maximal subgroups of G into which Σ reduces. Thus, the original method for soluble groups due to Gaschütz is included in our theory.

In this way from soluble to finite, we lose the arithmetical properties. This is no surprising since they characterise solubility. But we find deep relations between maximal subgroups hidden behind the luxuriant Hall theory.

Definition 1.4.1. *Let G be a group. We say that two maximal subgroups U, S of G are* core-related *in G if $U_G = S_G$.*

It is clear that the core-relation is an equivalence relation in the set $\mathrm{Max}(G)$ of all maximal subgroups of a group G.

By Theorem 1.1.10, the core-relation coincides with conjugacy in soluble groups. Moreover, by Lemma 1.2.17 (2), two monolithic maximal subgroups supplementing the same non-abelian chief factor are core-related.

Definitions 1.4.2. *1. Let* $\mathbf{X}$ *be a, possibly empty, set of monolithic maximal subgroups of* G*. We will say that* $\mathbf{X}$ *is a* subsystem of maximal subgroups *of* G *provided the following two properties are satisfied:*

a) if $U,\ S \in \mathbf{X}$ *and* $U \neq S$*, then* $U_G \neq S_G$*, and*

b) if $U,\ S \in \mathbf{X}$*,* $U \neq S$ *and both complement the same abelian chief factor* H/K *of* G*, then* $M = (U \cap S)H \in \mathbf{X}$*.*

2. If a subsystem of maximal subgroups $\mathbf{X}$ *is a complete set of representatives of the core-relation in the set* $\mathrm{Max}^*(G)$ *of all monolithic maximal subgroups of* G*, then we will say that* $\mathbf{X}$ *is a* system of maximal subgroups *of* G*.*

Since Condition 1b of the above definition only has an effect on maximal subgroups of type 1, we have that every subset of representatives of the core-relation in the set of maximal subgroups of type 2 is a subsystem of maximal subgroups.

If $\mathbf{X}$ is a subsystem of maximal subgroups of a group G, then $\mathbf{X}$ can be written as the disjoint union set $\mathbf{X} = \mathbf{X}_1 \cup \mathbf{X}_2$, where $\mathbf{X}_k = \{U \in \mathbf{X} : U \text{ is a maximal subgroup of type } k\}$ for $k = 1,\ 2$. On the other hand, if $F_1, \dots,\ F_n$ are representatives of the G-isomorphism classes of abelian chief factors of G, then $\mathbf{X}_1$ is a disjoint union set $\mathbf{X}_1 = \bigcup_{i=1}^{n} \mathbf{X}_{F_i}$, for $\mathbf{X}_{F_i} = \{U \in \mathbf{X} : U \text{ complements a chief factor } G\text{-isomorphic to } F_i\}$.

Clearly a subsystem of maximal subgroups is, in particular, a JH-solid set of monolithic maximal subgroups by Lemma 1.2.17.

Let $\mathbf{X}$ be a subsystem of maximal subgroups of a group G. If $g \in G$, denote $\mathbf{X}^g = \{S^g : S \in \mathbf{X}\}$. It is clear that $\mathbf{X}^g$ is again a subsystem of maximal subgroups of G.

We say that two subsystems of maximal subgroups $\mathbf{X_1}$ and $\mathbf{X_2}$ of a group G are *conjugate* in G, if there exists an element $g \in G$ such that $\mathbf{X_1}^g = \mathbf{X_2}$.

Proposition 1.4.3. *Let* G *be a group and* φ *an epimorphism of* G*. If* $\mathbf{X}$ *is a subsystem of maximal subgroups of* G*, then the set* $\mathbf{X}^\varphi = \{M^\varphi : \mathrm{Ker}(\varphi) \leq M \in \mathbf{X}\}$ *is a subsystem of maximal subgroups of* G^φ*.*

Conversely, if $\mathbf{Y}$ *is a subsystem of maximal subgroups of* G^φ*, then the set* $\mathbf{Y}^{\varphi^{-1}} = \{M \leq G : \mathrm{Ker}(\varphi) \leq M, M/\mathrm{Ker}(\varphi) \in \mathbf{Y}\}$ *is a subsystem of maximal subgroups of* G*.*

Proof. Let M^φ, S^φ be two distinct maximal subgroups of G^φ in $\mathbf{X}^\varphi$. Then M, S are two distinct maximal subgroups of G in $\mathbf{X}$ and then $M_G \neq S_G$. Moreover $\mathrm{Ker}(\varphi) \leq M \cap S$. It is clear that this implies that $(M^\varphi)_{G^\varphi} \neq (S^\varphi)_{G^\varphi}$.

If M^φ and S^φ are two maximal subgroups complementing an abelian chief factor H^φ/K^φ of G^φ, then H/K is an abelian chief factor of G which is complemented by M and S. Therefore $(M \cap S)H \in \mathbf{X}$. Hence $(M^\varphi \cap S^\varphi)H^\varphi \in \mathbf{X}^\varphi$.

For the converse, just notice that for any subgroup $H \leq G$ such that $\mathrm{Ker}(\varphi) \leq H$, we have $\big(H/\mathrm{Ker}(\varphi)\big)_{G^\varphi} = H_G/\mathrm{Ker}(\varphi)$. □

Notation 1.4.4. Bearing in mind Notation 1.2.19, if G is a group, N is a normal subgroup of G, and $\varphi\colon G \longrightarrow G/N$ is the canonical epimorphism, we write

$$\mathbf{X}^\varphi = \mathbf{X}/N = \{M/N : M \in \mathbf{X} \text{ and } N \leq M\}$$

for a subsystem of maximal subgroups $\mathbf{X}$ of G.

Corollary 1.4.5. *Let G be a group factorised as $G = MN$, where M is a subgroup of G and N is a normal subgroup of G. If $\mathbf{X}$ is subsystem of maximal subgroups of G and $\mathbf{Y}$ is a subsystem of maximal subgroups of M, then*

$$(\mathbf{X} \cap M)/(N \cap M) = \{(S \cap M)/(N \cap M) : S \in \mathbf{X}, N \leq S\}$$

is a subsystem of maximal subgroups of $M/(N \cap M)$ and

$$\mathbf{Y}N/N = \{SN/N : S \in \mathbf{Y}, N \cap M \leq S\}$$

is a subsystem of maximal subgroups of G/N.

Lemma 1.4.6. *Let C/R be the crown of a complemented abelian chief factor F of a group G.*

1. *Suppose that N is a normal subgroup of G such that $R \leq N < C$. If T is a complement of C/N in G, then the set*

   $$\mathbf{Y}(F, N, T) = \{TM : N \leq M < C \text{ and } C/M \text{ is a chief factor of } G\}$$

   is a subsystem of maximal subgroups of G.
   Moreover any chief factor of G between C and N is complemented by some maximal subgroup of $\mathbf{Y}(F, N, T)$ and $T = \bigcap\{U : U \in \mathbf{Y}(F, N, T)\}$.
2. *Let H/K be a chief factor of G such that $R \leq K < H < C$, T a complement of C/H in G, and U a complement of H/K in G. Then $S = T \cap U$ is a complement of C/K in G such that $T = SH$ and $\mathbf{Y}(F, H, T) \cup \{U\} \subseteq \mathbf{Y}(F, K, S)$.*
3. *If $\mathbf{X}$ is a subsystem of maximal subgroups of G such that F is $\mathbf{X}$-supplemented in G, and $T = \mathrm{T}(G, \mathbf{X}, F)$ is the complement of $C/R_{\mathbf{X}}$ defined in Proposition 1.3.11, then*

   $$\begin{aligned}\mathbf{Y}(F, R_{\mathbf{X}}, T) &= \mathbf{X}_F\\ &= \{U \in \mathbf{X} : U \text{ complements a chief factor } G\text{-isomorphic to } F\}.\end{aligned}$$

Proof. 1. Since F is abelian, $C = \mathrm{C}_G(F)$. Write $\mathbf{Y} = \mathbf{Y}(F, N, T)$ and consider $U = TM \in \mathbf{Y}$, for some normal subgroup M such that $N \leq M$ and C/M is a chief factor of G. It is clear that U complements C/M in G. Hence U is a maximal subgroup of G. Since $U_G < C$, it follows that $U_G = M$.

Let $U_1 = TM_1$ and $U_2 = TM_2$ be two elements of $\mathbf{Y}$, with M_1 and M_2 as in the definition of the elements of $\mathbf{Y}$. We have seen in the preceding paragraph that $(U_i)_G = M_i$, $i = 1, 2$. Clearly $U_1 \neq U_2$ implies that $(U_1)_G = M_1 \neq M_2 = (U_2)_G$. Suppose that U_1 and U_2 complement the same chief factor H/K of G. Observe that $C = HM_1 = HM_2 = M_1M_2$ and $M_1 \cap H = K = M_2 \cap H$. The subgroup $M_3 = (M_1 \cap M_2)H$ is a normal subgroup of G and $N \leq M_3 \leq C$. Moreover

$$\begin{aligned} C/M_3 &= HM_1/(M_1 \cap M_2)H \\ &\cong_G M_1/(M_1 \cap M_2)(M_1 \cap H) \\ &= M_1/(M_1 \cap M_2) \\ &\cong_G F \end{aligned}$$

and G/M_3 is a chief factor of G. By Lemma 1.2.17, the subgroup $(U_1 \cap U_2)H$ is maximal in G. Since $M_1 \cap TM_2 \leq C \cap TM_2 = M_2(C \cap T) = M_2$, it follows that $M_1 \cap TM_2 = M_1 \cap M_2$. Hence

$$(U_1 \cap U_2)H = (TM_1 \cap TM_2)H = T(M_1 \cap TM_2)H = T(M_1 \cap M_2)H = TM_3 \in \mathbf{Y}.$$

Consequently, $\mathbf{Y}$ is a subsystem of maximal subgroups of G.

Let H/K be a chief factor of G such that $N \leq K < H \leq C$. Let U be a complement of H/K in G and write $M = U_G$. Then $C = HM$ and $K = M \cap H$. Then $TM \in \mathbf{Y}(F, N, T)$. Now $(TM)H = TC = G$ and $TM \cap H \leq TM \cap C = M$. Hence $TM \cap H = M \cap H = K$ and TM complements H/K in G.

Clearly, $T \leq \bigcap\{U : U \in \mathbf{Y}(F, N, T)\}$. If $N = G_k \leq G_{k-1} \leq \cdots \leq G_0 = G$ is a piece of chief factor of G and, for $i = 1, \ldots, k$, U_i is a maximal subgroup in $\mathbf{Y}(F, N, T)$ complementing G_{i-1}/G_i, then $T = \bigcap_{i=1}^{k} U_i$, by Proposition 1.3.11. Hence, $T = \bigcap\{U : U \in \mathbf{Y}(F, N, T)\}$.

2. Applying Corollary 1.3.4, C/K is completely reducible G-module. Hence, by [DH92, A, 4.6], $C = HA$ for some normal subgroup A of G containing K such that $H \cap A = K$. By Lemma 1.2.16 (2), the subgroup $S = T \cap U$ is a complement of C/K in G and $SH = (T \cap U)H = T \cap UH = T$. If C/M is a chief factor of G such that $H \leq M$, then $SM = TM \in \mathbf{Y}(F, H, T)$. Hence $\mathbf{Y}(F, H, T) \subseteq \mathbf{Y}(F, K, S)$. Moreover C/U_G is a chief factor of G such that $K \leq U_G$. Since SU_G complements C/U_G in G, it follows that $U = SU_G$ is a maximal subgroup of G in $\mathbf{Y}(F, K, S)$.

3. Let TM be a maximal subgroup of G in $\mathbf{Y}(F, R_{\mathbf{X}}, T)$. The chief factor C/M is complemented by some maximal subgroup, U say, in $\mathbf{X}$. Since $U_G = M$, because C/M is self-centralising in G/M, it follows that $TM \leq U$. Hence $U = TM \in \mathbf{X}$. Therefore $\mathbf{Y}(F, R_{\mathbf{X}}, T) \subseteq \mathbf{X}_F$. If $U \in \mathbf{X}_F$, then $T \leq U$ and U complements $C/U_G \cong_G F$. Clearly $R_{\mathbf{X}} \leq U_G$ and $U = TU_G$. Hence $U \in \mathbf{Y}(F, R_{\mathbf{X}}, T)$. Therefore $\mathbf{X}_F \subseteq \mathbf{Y}(F, R_{\mathbf{X}}, T)$. □

Theorem 1.4.7. *Let G be a group. Every subsystem of maximal subgroups of G is contained in a system of maximal subgroups of G. In particular, every group possesses a system of maximal subgroups.*

Proof. Let $\mathbf{X}$ be a subsystem of maximal subgroups of G. Then $\mathbf{X} = \mathbf{X}_1 \cup \mathbf{X}_2$, where

$$\mathbf{X}_k = \{U \in \mathbf{X} : U \text{ is a maximal subgroup of type } k\}, \qquad \text{for } k = 1, 2.$$

Also, if $F_1, \ldots, F_n$ are representatives of the G-isomorphism classes of complemented abelian chief factors of G, we have that $\mathbf{X}_1 = \bigcup_{i=1}^{n} \mathbf{X}_{F_i}$, where $\mathbf{X}_{F_i} = \{U \in \mathbf{X} : U \text{ complements a chief factor } G\text{-isomorphic to } F_i\}$.

Fix a complemented abelian chief factor F which is $\mathbf{X}$-complemented in G. Consider its $\mathbf{X}$-crown $C/R_{\mathbf{X}}$ and the subgroup $T^0 = T(G, \mathbf{X}, F)$ as in the previous lemma. Then $\mathbf{Y}(F, R_{\mathbf{X}}, T^0) = \mathbf{X}_F$. If C/R is the crown of F and $R = G_r \leq G_{r-1} \leq \ldots \leq G_0 = R_{\mathbf{X}} \leq \ldots \leq C$ is a piece of chief series of G, applying Lemma 1.4.6 (2), we construct a series of subsystems of maximal subgroups

$$\mathbf{X}_F = \mathbf{Y}(F, G_0, T^0) \subseteq \mathbf{Y}(F, G_1, T^1) \subseteq \ldots \subseteq \mathbf{Y}(F, G_r, T^r) = \mathbf{Y}(F, R, T),$$

and T is a complement of the crown C/R such that $T^0 = TR_{\mathbf{X}}$.

Note that every complemented chief factor G-isomorphic to F lies between R and C and hence it is complemented by a maximal subgroup in $\mathbf{Y}(F, R, T)$ by Lemma 1.4.6 (1). Hence, $\mathbf{Y}(F, R, T)$ is a complete set of representatives of the core-relation in the set of all maximal subgroups of G which complement a chief factor G-isomorphic to F.

Now, it is rather clear that

$$\mathbf{Y}_1 = \bigcup_{i=1}^{n} \mathbf{Y}(F_i, R_i, T_i)$$

is a subsystem of maximal subgroups of G which is a complete set of representatives of the core-relation in the set of all maximal subgroups of G of type 1. Moreover $\mathbf{X}_1 \subseteq \mathbf{Y}_1$.

For the maximal subgroups of type 2, just note that we only have to complete $\mathbf{X}_2$ to a complete set of representatives $\mathbf{Y}_2$ of the core-relation in the set of all maximal subgroups of type 2 of G.

Consequently $\mathbf{Y} = \mathbf{Y}_1 \cup \mathbf{Y}_2$ is a system of maximal subgroups of G and $\mathbf{X} \subseteq \mathbf{Y}$. □

Corollary 1.4.8. *Let G be a group factorised as $G = MN$, where M is a subgroup of G and N is a normal subgroup of G. If $\mathbf{Y}$ is a subsystem of maximal subgroups of M, then there exists a system of maximal subgroups $\mathbf{X}$ of G such that*

$$\mathbf{Y}/(M \cap N) = (\mathbf{X} \cap M)/(N \cap M)$$

Proof. By Corollary 1.4.5, the set

$$\mathbf{Y}N/N = \{SN/N : S \in \mathbf{Y}, N \cap M \leq S\}$$

is a subsystem (a system, in fact) of maximal subgroups of G/N. By Proposition 1.4.3. the set

$$\mathbf{X}_0 = \{S \leq G : N \leq S, S/N \in \mathbf{Y}N/N\}$$

is a subsystem of maximal subgroups of G. By Theorem 1.4.7 there exists a system of maximal subgroups $\mathbf{X}$ of G such that $\mathbf{X}_0 \subseteq \mathbf{X}$.

Observe that if $S \in \mathbf{X}_0$, then $S = UN$ for some $U \in \mathbf{Y}$ such that $N \cap M \leq U$. Moreover $S \cap M = UN \cap M = U(N \cap M) = U$. Hence

$$\mathbf{Y}/(M \cap N) = (\mathbf{X}_0 \cap M)/(M \cap N) \subseteq (\mathbf{X} \cap M)/(M \cap N).$$

Observe that $(\mathbf{X} \cap M)/(M \cap N)$ is a system of maximal subgroups of $M/(N \cap M)$ and so is $\mathbf{Y}/(M \cap N)$. Hence equality holds. □

The following results analyse the behaviour of systems of maximal subgroups in some particular maximal subgroups called *critical subgroups*. These subgroups turn out to be crucial in the introduction of normalisers associated with some classes of groups in Chapter 4.

Definition 1.4.9. *Let G be a group. A monolithic maximal subgroup M of G is said to be a* critical subgroup of G *if M supplements the subgroup* $\mathrm{F}'(G) = \mathrm{Soc}\big(G \bmod \Phi(G)\big)$.

Since

$$\mathrm{F}'(G)/\Phi(G) = \mathrm{Soc}\big(G/\Phi(G)\big) = N_1/\Phi(G) \times \cdots \times N_n/\Phi(G)$$

for normal subgroups N_i of G such that each $N_i/\Phi(G)$ is a chief factor of G, we can say that a maximal subgroup M of G is critical if there exists a chief factor of G of the form $N/\Phi(G)$ supplemented by M.

If the group G is soluble, then $\mathrm{F}'(G) = \mathrm{F}(G)$, the Fitting subgroup of G. In this case, this definition coincides with that of [DH92, III, 6.4 (a)].

Proposition 1.4.10. *Let G be a group and N a normal subgroup of G. If M is a subgroup of G, then $\mathrm{F}'(M)N/N$ is contained in $\mathrm{F}'(MN/N)$. Consequently, if U is critical in M and $M \cap N$ is contained in U, then UN/N is critical in MN/N.*

Proof. Write $F/N = \Phi(MN/N)$ and recall that $\Phi(M) \leq F$. Let $K/\Phi(M)$ be a minimal normal subgroup of $M/\Phi(M)$. We have that $\Phi(M) \leq K \cap F \leq K$ and $K \cap F$ is normal in M. Hence either $\Phi(M) = K \cap F$ or $K \leq F$ by minimality of $K/\Phi(M)$. If $K \leq F$, then $KN/N \leq \mathrm{F}'(MN/N)$. Assume that $\Phi(M) = K \cap F$. It follows that KF/F is a minimal normal subgroup of MN/F. Hence $KN/N \leq \mathrm{F}'(MN/N)$ and $\mathrm{F}'(M)N/N \leq \mathrm{F}'(MN/N)$.

Assume that U is critical in M. Then $M = U\,\mathrm{F}'(M)$ and $MN/N = (UN/N)\big(\mathrm{F}'(M)N/N\big) = (UN/N)\,\mathrm{F}'(MN/N)$. If $M \cap N \leq U$, UN/N is maximal in MN/N. Hence, in this case, UN/N is critical in MN/N.

Proposition 1.4.11. *Let M be a critical subgroup of a group G. Suppose that H/K is a chief factor of G covered by M and avoided by $\Phi(G)$. Then we have the following.*

1. *The section $(H \cap M)/(K \cap M)$ is a chief factor of M such that $M \cap \mathrm{C}_G(H/K) = \mathrm{C}_M\big((H \cap M)/(K \cap M)\big)$.*
2. *$\mathrm{Aut}_G(H/K) \cong \mathrm{Aut}_M\big((H \cap M)/(K \cap M)\big)$.*
3. *$[H/K] * G \cong [(H \cap M)/(K \cap M)] * M$.*
4. *If U is a monolithic maximal subgroup of G which supplements H/K in G, then $U \cap M$ is a maximal subgroup of M which supplements $(H \cap M)/(K \cap M)$ in M.*

Proof. First of all, since $H = K(M \cap H)$, it follows that H/K is M-isomorphic to $(H \cap M)/(K \cap M)$. Therefore $M \cap \mathrm{C}_G(H/K) = \mathrm{C}_M\big((H \cap M)/(K \cap M)\big)$. We shall prove now that $G = M\,\mathrm{C}_G(H/K)$. Since M is critical in G, M is a supplement in G of a chief factor of G of the form $N/\Phi(G)$. Note that $H\Phi(G)/K\Phi(G)$ is G-isomorphic to H/K. Hence, by considering $H\Phi(G)/K\Phi(G)$ instead of H/K if necessary, we can assume that $\Phi(G) \leq K$.

If $G = MK$, then $G = M\,\mathrm{C}_G(H/K)$. Assume that $K \leq M$. Then $H \leq M$, since M covers H/K. Therefore $[H, N] \leq \Phi(G)$ and thus $N \leq \mathrm{C}_G(H/K)$. Consequently, in both cases, $G = M\,\mathrm{C}_G(H/K)$.

Now Statements 1 and 2 follow from [DH92, A, 13.9].

3. If H/K is non-abelian, then clearly $[H/K] * G \cong [(H \cap M)/(K \cap M)] * M$. If H/K is abelian, then the correspondence

$$\alpha\colon [H/K] * G \longrightarrow [(H \cap M)/(K \cap M)] * M,$$

given by

$$\big(xK, y\,\mathrm{C}_G(H/K)\big)^{\alpha} = \Big(x(K \cap M), y\,\mathrm{C}_M\big((H \cap M)/(K \cap M)\big)\Big)$$

for any $x \in H$, $y \in M$, is an isomorphism. Hence $[H/K] * G \cong [(H \cap M)/(K \cap M)] * M$.

4. Note that $H = K(M \cap H)$ because M covers H/K.

Let us prove first that if X is a monolithic maximal subgroup of G such that $X \cap M = U \cap M$ and $N \leq X$, then $X \cap M$ is a maximal subgroup of M which supplements $(H \cap M)/(K \cap M)$ in M.

Note that $X = X \cap MN = (X \cap M)N$. Let T be a subgroup such that $X \cap M \leq T \leq M$. Then $N \cap M \leq X \cap M \leq T$ and $X = (X \cap M)N \leq TN \leq MN = G$. By maximality of X in G, we have that either $X = TN$ or $TN = G$. If $X = TN$, then $X \cap M = TN \cap M = T(N \cap M) = T$. If $G = TN$, then $M = M \cap TN = T(M \cap N) = T$. Hence $X \cap M$ is a maximal subgroup of M.

Now consider the subgroup $(X \cap M)(H \cap M)$. Suppose that $(X \cap M)(H \cap M) = X \cap M$. This is to say that $M \cap H \leq M \cap X = U \cap M$ and then $H = K(M \cap H) \leq U$, which is a contradiction. Hence, by maximality of

$X \cap M$ in M, we have that $M = (X \cap M)(H \cap M) = (U \cap M)(H \cap M)$. Moreover $K \cap M$ is contained in $U \cap H \cap M$. Therefore $U \cap M$ supplements $(H \cap M)/(K \cap M)$ in M.

Clearly if $N \leq U$, we can apply the above arguments to $X = U$. Suppose that $G = UN$. If $U_G = M_G$, then $K = U_G) \cap H = M_G \cap H \leq M \cap H$ and then $H = K$. This contradiction yields $U_G \neq M_G$. Applying Lemma 1.2.17, the subgroup $X = (U \cap M)N$ is a maximal subgroup of G. Also by Lemma 1.2.17 (2), we have that $N/\Phi(G)$ is abelian. In particular $M \cap N = \Phi(G) \leq U \cap M$. Therefore $X \cap M = (U \cap M)(N \cap M) = U \cap M$ and $U \cap M$ supplements $(H \cap M)/(K \cap M)$ in M by the above arguments. □

Corollary 1.4.12. *Let M be a critical subgroup of a group G. Assume that U is a monolithic maximal subgroup of G such that $U_G \neq M_G$. Then $M \cap U$ is a monolithic maximal subgroup of M.*

Proof. Assume that U supplements a chief factor H/K of G. Suppose that H/K is supplemented by M. By Lemma 1.2.17 (2), the chief factor H/K is abelian. In this case U complements the chief factor C/U_G, for $C = \mathrm{C}_G(H/K)$, and this chief factor is covered by M, since $M_G \neq U_G$. Hence, we can assume that U supplements a chief factor covered by M. Since this chief factor is avoided by $\Phi(G)$, we have that $M \cap U$ is a maximal subgroup of M, by Proposition 1.4.11 (4). □

Theorem 1.4.13. *Let $\mathbf{X}$ be a subsystem of maximal subgroups of a group G and M a critical subgroup of G in $\mathbf{X}$. Consider the set*

$$\mathbf{X}_M = \{S \cap M : S \in \mathbf{X}, S \neq M\},$$

with no repetitions. Then

1. *if $G = MN$, for some chief factor $N/\Phi(G)$ of G, then $\mathbf{X}_M = \{S \cap M : N \leq S \in \mathbf{X}\}$;*
2. *$\mathbf{X}_M$ is a subsystem of maximal subgroups of M.*

Proof. 1. Let S be an element of $\mathbf{X}$ such that $S \cap M \in \mathbf{X}_M$ and $G = SN$. Then $N/\Phi(G)$ is abelian by Lemma 1.2.17 (2), $S^* = (S \cap M)N \in \mathbf{X}$ and $S^* \cap M = S \cap M$.

2. Assume that $G = MN$, for some chief factor $N/\Phi(G)$ of G. Applying Corollary 1.4.12, all elements of $\mathbf{X}_M$ are monolithic maximal subgroups of M. Consider two distinct maximal subgroups $S \cap M$ and $U \cap M$ in $\mathbf{X}_M$. By Statement 1, we can assume that $N \leq S \cap U$. By Theorem 1.2.22, we have that $S_G = N(S \cap M)_M \neq U_G = N(U \cap M)_M$. Hence $(S \cap M)_M \neq (U \cap M)_M$.

Suppose that $S \cap M$ and $U \cap M$ are distinct elements of $\mathbf{X}_M$, for S, $U \in \mathbf{X}$, and both complement the same abelian chief factor H/K of M. We can assume that $N \leq S \cap U$. Then $S = N(S \cap M)$ and $U = N(U \cap M)$. Since $M \cap N \leq S \cap M$, it follows that $H \cap M \cap N \leq H \cap M \cap S = K$. Therefore $H(M \cap N)/K(M \cap N) \cong_M H/K$. Clearly, $S \cap M$ and $U \cap M$ complement the chief factor $H(M \cap N)/K(M \cap N)$ of M. By Theorem 1.2.22, U and S complement the chief factor HN/KN of G. Thus, $(S \cap U)HN = (S \cap U)H$ is a maximal subgroup in $\mathbf{X}$, inasmuch as $\mathbf{X}$ is a subsystem of maximal subgroups of G. Therefore $(S \cap U)H \cap M = (S \cap U \cap M)H \in \mathbf{X}_M$.

Consequently, $\mathbf{X}_M$ is a subsystem of maximal subgroups of M. □

Theorem 1.4.14. *Let M be a critical subgroup of a group G. Assume that $\mathbf{Y}$ is a system of maximal subgroups of M. Then there exists a system of maximal subgroups $\mathbf{X}$ of G such that $M \in \mathbf{X}$ and $\mathbf{X}_M \subseteq \mathbf{Y}$.*

Proof. Without loss of generality we can assume that $\Phi(G) = 1$. Since M is critical in G, it follows that $G = NM$, for some minimal normal subgroup N of G.

Suppose that N is non-abelian and consider the following set of monolithic maximal subgroups of G

$$\mathbf{X} = \{SN : M \cap N \leq S \in \mathbf{Y}\} \cup \{M\}.$$

If U is a maximal subgroup of G and $N \cap U_G = 1$, then $G = UN$ and $U_G = \mathrm{C}_G(N) = M_G$, since N is non-abelian. If $N \leq U_G$, then $U \cap M$ is a maximal subgroup of M and there exists $S \in \mathbf{Y}$, such that $N \cap M \leq S_M = (U \cap M)_M$. Now observe that $SN_G = S_M N = U_G$. Therefore $\mathbf{X}$ is a complete set of representatives of the core-relation in G.

Suppose now that S_1 and S_2 are maximal subgroups of M in $\mathbf{Y}$ such that $M \cap N \leq S_1 \cap S_2$ and the maximal subgroups $U_1 = S_1N$ and $U_2 = S_2N$ of G complement the same abelian chief factor H/K of G. We see that $(U_1 \cap U_2)H \in \mathbf{X}$. Changing if necessary H/K by HN/KN, we can assume that $N \leq K$. Now S_1 and S_2 complement the abelian chief factor $(H \cap M)/(K \cap M)$ of M. Since $\mathbf{Y}$ is a system of maximal subgroups of M, the subgroup $(S_1 \cap S_2)(H \cap M)$ is in $\mathbf{Y}$. Since $N \cap M \leq H \cap M \leq (S_1 \cap S_2)(H \cap M)$, we have that $(S_1 \cap S_2)(H \cap M)N = (S_1 \cap S_2)H$ is a maximal subgroup of G in $\mathbf{X}$. Clearly $(S_1 \cap S_2)H = (U_1 \cap U_2)H$. This shows that $\mathbf{X}$ is a system of maximal subgroups of G and $M \in \mathbf{X}$.

Finally, if $S \in \mathbf{Y}$ and $M \cap N \leq S$, then $M \cap SN = S$. Hence $\mathbf{X}_M \subseteq \mathbf{Y}$.

Assume now that N is abelian. Hence $M \cap N = 1$. Write $\mathbf{Y} = \mathbf{Y}_1 \cup \mathbf{Y}_2$, where $\mathbf{Y}_i$ is the set of maximal subgroups of type i in $\mathbf{Y}$, for $i = 1, 2$. Let $\{F_1, \dots, F_n\}$ be a complete set of representatives of the M-isomorphism classes of abelian chief factors of M. Then $\mathbf{Y}_1 = \bigcup_{i=1}^{n} \mathbf{Y}_{F_i}$, where $\mathbf{Y}_{F_i} = \{S \in \mathbf{Y} : S$ complements a chief factor M-isomorphic to $F_i\}$.

Applying Theorem 1.2.22, $\mathbf{X}_2 = \{SN : S \in \mathbf{Y}_2\}$ is a complete set of representatives of the core-relation in the set of all maximal subgroups of type 2 of G. Note that $(\mathbf{X}_2)_M = \{SN \cap M : S \in \mathbf{Y}_2\} = \mathbf{Y}_2$.

Since $M \cong G/N$, we can find a complete set $\{L_1, \dots, L_n\}$ of representatives of the G/N-isomorphism (G-isomorphism) classes of abelian chief factors of G/N such that $L_i \cong F_i$, $1 \leq i \leq n$.

If N is not isomorphic to L_i for all $i = 1, \dots, n$, then all complements of N in G are core-related. In this case $\mathbf{X}_1 = \{SN : S \in \mathbf{Y}_1\} \cup \{M\}$ is a subsystem of maximal subgroups of G containing a representative of each equivalence class of the core-relation in the set of all maximal subgroups of G of type 1. Therefore $\mathbf{X} = \mathbf{X}_1 \cup \mathbf{X}_2$ is a system of maximal subgroups of G such that $\mathbf{X}_M = \mathbf{Y}$.

Suppose that N is G-isomorphic to some of L_i, $1 \leq i \leq n$. Let us assume that $N \cong_G L_n$. For each $i \in \{1, \dots, n-1\}$, denote $\mathbf{X}_{L_i} = \{SN : S \in \mathbf{Y}_{F_i}\}$. Then $(\mathbf{X}_{L_i})_M = \mathbf{Y}_{F_i}$ and $\mathbf{X}_{L_i}$ is a subsystem of maximal subgroups of G containing a representative of each equivalence class of the core-relation in the set of all complements of chief factors of G which are G-isomorphic to L_i.

If L_i is a Frattini chief factor of G/N, then all complements of N in G are core-related and $\mathbf{X}_1 = \{SN : S \in \mathbf{Y}_1\} \cup \{M\}$ is a system of maximal subgroups of G satisfying the condition of the theorem. Therefore we may assume that L_n is complemented, and so there exists a $\mathbf{Y}$-complemented chief factor A/B of M such that A/B is M-isomorphic to F_n.

Let C/R be the crown of G associated with N and AN/BN in G. By Proposition 1.3.5, RN is a proper subgroup of C and $(C/N)/(RN/N)$ is the crown of $(AN/N)/(BN/N)$ in G/N. Applying Proposition 1.3.11, $(C/N)/(RN/N)$ in G/N is complemented in G/N. Let T be a subgroup of M such TN/N is a complement of $(C/N)/(RN/N)$ in G/N. Since TN is a complement of C/RN in G and M is a complement of RN/R in G, it follows that $T = TN \cap M$ is a complement of C/R in G by Lemma 1.4.6 (2). In addition, applying Lemma 1.4.6 (1), the set $\mathbf{Y}(AN/BN, RN, TN)$, composed of all subgroups TK where K is a normal subgroup of G such that $RN \leq K$ and C/K is a chief factor of G, is a subsystem of maximal subgroups of G and

$$\begin{aligned}\mathbf{Y}(L_n, RN, TN) \cup \{M\} &\subseteq \mathbf{Y}(L_n, R, T)\\ &= \{TK : R \leq K \text{ and } C/K \text{ is chief factor of } G\}.\end{aligned}$$

Write $\mathbf{X}_{L_n} = \mathbf{Y}(L_n, R, T)$. Then $\mathbf{X}_{L_n}$ is a subsystem of maximal subgroups of G by Lemma 1.4.6 (1).

Consider a subgroup $U \in \mathbf{X}_{L_n}$, $U \neq M$. We see that $U \cap M \in \mathbf{Y}_{F_n}$. Suppose that $U = TK$ for some normal subgroup K of G such that $R \leq K$ and C/K is a chief factor of G. If K is contained in M_G, then $U = TK = M$ against our assumption. Hence we have that $G = MK$ and $C = M_G K$. In particular, $U_G \neq M_G$. Moreover, $(C \cap M)/(K \cap M)$ is a chief factor of M which is M-isomorphic to F_n and is complemented in M by the maximal subgroup $U \cap M$ of M by Proposition 1.4.11 (1) and (4). Note that $(C \cap M)/(K \cap M)$ is $\mathbf{Y}$-complemented in M because $\mathbf{Y}$ is a system of maximal subgroups of M. Consider a maximal subgroup $Y \in \mathbf{Y}_{F_n}$ which complements the chief factor $(C \cap M)/(K \cap M)$ in M. Applying Proposition 1.3.11, we have that T is contained in Y. Hence $U \cap M = T(K \cap M) \leq Y$. Maximality of $U \cap M$ in M forces $U \cap M = Y$. Therefore, $(\mathbf{X}_{L_n})_M \subseteq \mathbf{Y}_{F_n}$.

If U is a maximal subgroup of G which complements a chief factor isomorphic to L_n, then U complements the chief factor $C/U_G \cong_G L_n$. The maximal subgroup TU_G is in $\mathbf{X}_{L_n}$ and $\big(TU_G\big)_G = U_G$. Thus, $\mathbf{X}_{L_n}$ is a complete set of representatives for the core-relation in the set of all complements of chief factors G-isomorphic to L_n.

Consider the union set $\mathbf{X}_1 = \bigcup_{i=1}^{n} \mathbf{X}_{F_i}$ and $\mathbf{X} = \mathbf{X}_1 \cup \mathbf{X}_2$ is a system of maximal subgroups of G such that $\mathbf{X}_M \subseteq \mathbf{Y}$ and $M \in \mathbf{X}$. □

Theorem 1.4.15. *If N is a normal subgroup of a group G and $\mathbf{X}^*$ is system of maximal subgroups of G/N, then there exists a system of maximal subgroups $\mathbf{X}$ of G such that $\mathbf{X}/N = \mathbf{X}^*$.*

Proof. We argue by induction of the order of G. It is clear that $N \neq 1$. Assume that N is a minimal normal subgroup of G. It is clear that we can suppose that $N \cap \Phi(G) = 1$. Let M be a critical subgroup of G such that $G = MN$. If α is the isomorphism $G/N \cong M/(M \cap N)$, then $(\mathbf{X}^*)^\alpha = \{(U \cap M)/(N \cap M) : U/N \in \mathbf{X}^*\}$ is a system of maximal subgroups of $M/(N \cap M)$. By induction, there exists a system of maximal subgroups $\mathbf{X}(M)$ of M such that $\mathbf{X}(M)/(N \cap M) = (\mathbf{X}^*)^\alpha$. By Theorem 1.4.14 there exists a system of maximal subgroups $\mathbf{X}$ of G such that $\mathbf{X}_M \subseteq \mathbf{X}(M)$.

The set $(\mathbf{X}/N)^\alpha = \{(S \cap M)/(N \cap M) : S \in \mathbf{X}, N \leq S\}$ is a system of maximal subgroups of $M/(M \cap N)$ by Corollary 1.4.5. Notice that $(\mathbf{X}/N)^\alpha \subseteq \mathbf{X}(M)/(M \cap N) = (\mathbf{X}^*)^\alpha$ and then $(\mathbf{X}/N)^\alpha = (\mathbf{X}^*)^\alpha$. Consequently $\mathbf{X}/N = \mathbf{X}^*$ and the theorem is true.

Now assume that L is a minimal normal subgroup of G and L is a proper subgroup of N. By inductive hypothesis the theorem is true for the group G/L. Since $\mathbf{X}^{**} = \{(S/L)/(N/L) : S/N \in \mathbf{X}^*\}$ is a system of maximal subgroups of $(G/L)/(N/L)$, there exists a system of maximal subgroups $\mathbf{X}_0$ of G/L such that $\mathbf{X}_0/(N/L) = \mathbf{X}^{**}$. On the other hand, since for L the theorem is true, there exists a system of maximal subgroups $\mathbf{X}$ of G such that $\mathbf{X}/L = \mathbf{X}_0$. If $H \in \mathbf{X}$ and $N \leq H$, then $L \leq H$ and $H/L \in \mathbf{X}_0$, $(H/L)/(N/L) \in \mathbf{X}^{**}$, and then $H/N \in \mathbf{X}^*$. Consequently, $\mathbf{X}^* = \mathbf{X}/N$. □

Corollary 1.4.16. *Given a system of maximal subgroups $\mathbf{X}$ of a group G and a critical subgroup M of G such that $M \in \mathbf{X}$, there exists a system of maximal subgroups $\mathbf{Y}$ of M, such that $\mathbf{X}_M \subseteq \mathbf{Y}$.*

Proof. Assume that M supplements a chief factor $N/\Phi(G)$ of G. Denote by α the isomorphism $\alpha\colon G/N \longrightarrow M/(N \cap M)$. Then $(\mathbf{X} \cap N)/(M \cap N)$ is a system of maximal subgroups of $M/(N \cap M)$. By Theorem 1.4.15, there exists a system of maximal subgroups $\mathbf{Y}$ of M such that $\mathbf{Y}/(N \cap M) = (\mathbf{X} \cap M)/(M \cap N)$. Let $U \in \mathbf{X}$ with $U \neq M$. If $G = UN$, then $N/\Phi(G)$ is abelian by Lemma 1.2.17 (2), and $V = (U \cap M)N \in \mathbf{X}$. In this case, we have that $U \cap M = V \cap M$. Hence we can assume that $N \leq U$. Then $(U \cap M)/(N \cap M) \in (\mathbf{X} \cap N)/(M \cap N)$ and $U \cap M \in \mathbf{Y}$. Therefore $\mathbf{X}_M \subseteq \mathbf{Y}$. □

The soluble case is particularly interesting in this context. Given a Hall system Σ of a soluble group G, we consider the set

$$\mathbf{S}(\Sigma) = \{S \in \mathrm{Max}(G) : \Sigma \text{ reduces into } S\}.$$

Maximal subgroups are always pronormal (see [DH92, Section I, 6]) and therefore if M is a maximal subgroup of G, then Σ reduces into exactly one conjugate of M by a theorem due to Mann (see [DH92, I, 6.6]). Then $\mathbf{S}(\Sigma)$ is a complete set of representatives of the core-relation. By [DH92, I, 4.22], $\mathbf{S}(\Sigma)$ is indeed a system of maximal subgroups of G. the following result shows that all systems of maximal subgroups of the soluble group G arise in this manner.

Theorem 1.4.17. *Let $\mathbf{X}$ be a subsystem of maximal subgroups of a soluble group G. Then there exists a Hall system Σ of G such that Σ reduces into each maximal subgroup of G in $\mathbf{X}$.*

Proof. We argue by induction on the order of G. Let N be a minimal normal subgroup of G. Then $\mathbf{X}/N$ is a subsystem of maximal subgroups of G/N by Proposition 1.4.3. By induction there exists a Hall system Σ of G such that the Hall system $\Sigma N/N$ reduces into each maximal subgroup of G/N in $\mathbf{X}/N$. Hence Σ reduces into each maximal subgroup of G containing N and belonging to $\mathbf{X}$ by [DH92, I, 4.17 b]. In particular, we can assume that $\Phi(G) = 1$.

If no complement of N in G is in $\mathbf{X}$, then Σ reduces into each maximal subgroup of G in $\mathbf{X}$. Thus, we can assume that the set of complements $\{T_1, \dots, T_r\}$ of N in $\mathbf{X}$ is non-empty, i.e. $r \geq 1$. We can also assume that T_1 is not normal in G. By [DH92, I, 4.16] there exists an element $n \in N$ such that $\Sigma_0 = \Sigma^n$ reduces into T_1. Then $\Sigma_0 N/N = \Sigma N/N$. This means that we can assume without loss of generality that $\Sigma_0 = \Sigma$. If $r = 1$, then it is clear that Σ reduces into each maximal subgroup of G in $\mathbf{X}$. Suppose that $r > 1$. For $j \neq 1$, the subgroup $M = (T_1 \cap T_j)N$ is a maximal subgroup of G in $\mathbf{X}$. Since $\Sigma N/N$ reduces into M/N, it is clear that Σ reduces into M. Let p be the prime dividing the order of N and consider the Hall p'-subgroup Q of G in Σ. We know, by Lemma 1.2.17 (1), that M complements a p-chief factor of G. Hence $T_1 \cap T_j$ has p-index in G and so $Q \leq (T_1 \cap T_j)^a$ for some $a \in N$. This implies that Σ reduces into T_1^a and into T_j^a by [DH92, I, 4.20]. Since T_1 is pronormal in G, we have that $a \in T_1 \cap N = 1$ and then Σ reduces into T_j. Thus, in any case Σ reduces into T_i, for $i = 1, \dots, r$ and then Σ reduces into each maximal subgroup of G in $\mathbf{X}$. □

Corollary 1.4.18. *If G is a soluble group then:*

1. *the map*

$$\{\textit{Hall systems of } G\} \longrightarrow \{\textit{Systems of maximal subgroups of } G\}$$

such that the image of a Hall system Σ of G is the set $\mathbf{X}(\Sigma)$ given by

$$\mathbf{X}(\Sigma) = \{S \in \operatorname{Max}(G) : \Sigma \textit{ reduces into } S\},$$

is surjective.

2. *All systems of maximal subgroups of G are conjugate.*

3. *The number of systems of maximal subgroups of* G *is the index of the stabiliser* $\mathrm{N}_G\big(\mathbf{X}(\Sigma)\big) = \bigcap\{\mathrm{N}_G(S) : S \in \mathbf{X}(\Sigma)\}$.

Corollary 1.4.19. *A group* G *is soluble if and only if all systems of maximal subgroups of* G *are conjugate.*

Proof. Only the sufficiency of the condition is in doubt. Suppose that all systems of maximal subgroups are conjugate in G. If G is non-soluble, there exists a non-abelian chief factor H/K of G; then $G/\mathrm{C}_G(H/K)$ is a primitive group of type 2 by Proposition 1.1.14. Take S and U two maximal subgroups of G such that $S_G = U_G = \mathrm{C}_G(H/K)$, i.e. $S/\mathrm{C}_G(H/K)$ and $U/\mathrm{C}_G(H/K)$ are two core-free maximal subgroups of $G/\mathrm{C}_G(H/K)$. There exist two systems of maximal subgroups of G, $\mathbf{X}$ and $\mathbf{Y}$, such that $S \in \mathbf{X}$ and $U \in \mathbf{Y}$ by Theorem 1.4.14. Since $\mathbf{Y} = \mathbf{X}^g$ for some $g \in G$, then $U = S^g$ and all core-free maximal subgroups of $G/\mathrm{C}_G(H/K)$ are conjugate. But this contradicts the fact of being a primitive group of type 2 (see Remark 1.1.11 (4)). Therefore G is soluble. ◻