6

$\mathfrak{F}$-subnormality

How a subgroup can be embedded in a group is always a question of particular interest for clearing up the structure of finite groups.

One of the most important subgroup embedding properties is the subnormality, transitive closure of the relation of normality. This property was extensively studied by H. Wielandt (see [Wie94a]). For an excellent survey of the theory of subnormal subgroups, we refer the reader to J. C. Lennox and S. E. Stonehewer [LS87].

In finite groups the significance of the subnormal subgroups is apparent since they are precisely those subgroups which occur as terms of composition series, the factors of which are of great importance in describing the group structure.

Let $\mathfrak{F}$ be a saturated formation of full characteristic. If G is a soluble group, the $\mathfrak{F}$-normaliser D of G associated with a Hall system Σ of G is contained in the $\mathfrak{F}$-projector E of G in which Σ reduces (see [DH92, V, 4.11] and Theorem 4.2.9). In 1969, T. O. Hawkes [Haw69] analysed how D is embedded in E. It turns out that D can be joined to E by means of a maximal chain of $\mathfrak{F}$-normal subgroups, that is, D is $\mathfrak{F}$-subnormal in E ([DH92, V, 4.12]). The $\mathfrak{F}$-subnormality could be regarded, in the soluble universe, as the natural extension of the subnormality to formation theory. In fact, most of the results concerning subnormal subgroups can be read off by specialising to the case where $\mathfrak{F}$ is the formation of all nilpotent groups.

Our objective in this chapter is to present the main results of the $\mathfrak{F}$-subnormal subgroups. They are primarily connected with the study of subnormal subgroups properties by the methods of formation theory.

6.1 Basic properties

In the sequel, $\mathfrak{F}$ *will denote a non-empty formation.*

A subgroup U of a group G is called $\mathfrak{F}$-*normal* in G if $G/\operatorname{Core}_G(U) \in \mathfrak{F}$;

otherwise U is said to be $\mathfrak{F}$-*abnormal* in G. This definition was introduced in Definition 2.3.22 (1) for maximal subgroups.

Illustrations 6.1.1. 1. A subgroup U is $\mathfrak{F}$-normal in G if and only if $G^{\mathfrak{F}}$ is contained in U.

2. A maximal subgroup is normal in G if and only if it is $\mathfrak{N}$-normal in G. In general, a subgroup U is subnormal in G provided that U is $\mathfrak{N}$-subnormal in G.
3. If $\mathfrak{F} = \mathrm{LF}(F)$ is a saturated formation, a maximal subgroup M of G is $\mathfrak{F}$-normal in G if and only if $G/\operatorname{Core}_G(M) \in F(p)$ for every prime p dividing $\left|\operatorname{Soc}\bigl(G/\operatorname{Core}_G(M)\bigr)\right|$.

Definition 6.1.2. *A subgroup H of a group G is said to be* $\mathfrak{F}$-subnormal *in G if either $H = G$ or there exists a chain of subgroups*

$$H = H_0 < \cdots < H_n = G$$

such that H_{i-1} is an $\mathfrak{F}$-normal maximal subgroup of H_i for $i = 1, \ldots, n$. We shall write $H\,\mathfrak{F}\text{-sn}\,G$; $\mathrm{S}_{n\mathfrak{F}}(G)$ will denote the set of all $\mathfrak{F}$-subnormal subgroups of a group G. It is clear that $\mathrm{S}_{n\mathfrak{F}}$ is a subgroup functor.

Remark 6.1.3. Assume that $\mathfrak{F} = \mathfrak{N}$, the formation of all nilpotent groups. Then $\mathrm{S}_{n\mathfrak{N}}(G) \subseteq \mathrm{S}_n(G)$ for all groups G by Illustration 6.1.1 (2). However the equality does not hold in general because if $G = \mathrm{Alt}(5)$, then $1 \in \mathrm{S}_n(G) \setminus \mathrm{S}_{n\mathfrak{N}}(G)$. Nevertheless, if G is soluble, then $\mathrm{S}_n(G) = \mathrm{S}_{n\mathfrak{N}}(G)$.

To avoid the above situation, O. H. Kegel [Keg78] introduced a little bit different notion of $\mathfrak{F}$-subnormality. It unites the notions of subnormal and $\mathfrak{F}$-subnormal subgroup.

Definition 6.1.4. *A subgroup U of a group G is called* K-$\mathfrak{F}$-subnormal *subgroup of G if either $U = G$ or there is a chain of subgroups*

$$U = U_0 \leq U_1 \leq \cdots \leq U_n = G$$

such that U_{i-1} is either normal in U_i or U_{i-1} is $\mathfrak{F}$-normal in U_i, for $i = 1, \ldots, n$. We shall write $U\,\mathrm{K}\text{-}\mathfrak{F}\text{-sn}\,G$ and denote $\mathrm{S}_{n\,K\text{-}\mathfrak{F}}(G)$ the set of all K-$\mathfrak{F}$-subnormal subgroups of a group G. Clearly $\mathrm{S}_{n\,K\text{-}\mathfrak{F}}$ is a subgroup functor.

Remark 6.1.5. $\mathrm{S}_{n\mathrm{K}\text{-}\mathfrak{N}}(G) = \mathrm{S}_n(G)$ for every group G.

Let e be one of the functors $\mathrm{S}_{n\mathfrak{F}}$ or $\mathrm{S}_{n\mathrm{K}\text{-}\mathfrak{F}}$.

Lemma 6.1.6. e *is inherited, that is, if G is a group, we have*

1. *If $H \in \mathrm{e}(K)$ and $K \in \mathrm{e}(G)$, then $H \in \mathrm{e}(G)$.*
2. *If $N \trianglelefteq G$ and $U/N \in \mathrm{e}(G/N)$, then $U \in \mathrm{e}(G)$.*
3. *If $H \in \mathrm{e}(G)$ and $N \trianglelefteq G$, then $HN/N \in \mathrm{e}(G/N)$.*

Proof. It is obvious from the definitions that Statements 1 and 2 are fulfilled in both cases. We show that Statement 3 is satisfied when $\mathrm{e} = \mathrm{s}_{n\mathfrak{F}}$. Let H be an $\mathfrak{F}$-subnormal subgroup of G and let N be a normal subgroup of G. Proceeding by induction on $|G|$, we may clearly suppose that $H \neq G$. Let X be an $\mathfrak{F}$-normal maximal subgroup of G such that H is contained in X and H is $\mathfrak{F}$-subnormal in X. If $N \leq X$, then HN/N is $\mathfrak{F}$-subnormal in X/N by induction. Since X/N is $\mathfrak{F}$-normal in G/N, it follows that HN/N is $\mathfrak{F}$-subnormal in G/N by Assertion 1. Therefore we may assume that N is not contained in X and so $G = NX$. By induction, $H(X \cap N)/(X \cap N)$ is $\mathfrak{F}$-subnormal in $X/(X \cap N) \cong G/N$. Hence HN/N is $\mathfrak{F}$-subnormal in G/N. □

Lemma 6.1.7. *Assume that $\mathfrak{F}$ is subgroup-closed.*

1. *If H is a subgroup of a group G and $G^{\mathfrak{F}} \leq H$, then $H \in \mathrm{e}(G)$.*
2. *If $H \in \mathrm{e}(G)$ and $K \leq G$, then $H \cap K \in \mathrm{e}(K)$, that is, e is w-inherited.*
3. *If $\{H_i : 1 \leq i \leq n\} \subseteq \mathrm{e}(G)$, then $\bigcap_{i=1}^{n} H_i \in \mathrm{e}(G)$.*

Proof.
1. It follows at once from the fact that $X^{\mathfrak{F}} \leq G^{\mathfrak{F}}$ for all subgroups X of G.
2. Let $\mathrm{e} = \mathrm{s}_{n\mathfrak{F}}$. Proceeding by induction on $|G|$, we may clearly assume that $H \neq G$. Then there exists an $\mathfrak{F}$-normal maximal subgroup M of G such that $H \leq M$ and H is $\mathfrak{F}$-subnormal in M. Since $K^{\mathfrak{F}} \leq G^{\mathfrak{F}} \leq M$, it follows that $M \cap K$ is $\mathfrak{F}$-subnormal in K by Assertion 1. On the other hand, $H \cap K$ is $\mathfrak{F}$-subnormal in $M \cap K$ by induction. Therefore $H \cap K$ is $\mathfrak{F}$-subnormal in K.
3. It follows at once applying Lemma 5.1.5, as e is w-inherited, and using induction on n. □

Example 6.1.8. Lemma 6.1.7 (2) does not remain true if $\mathfrak{F}$ is not subgroup-closed. Let $\mathfrak{F} = \mathrm{LF}(f)$, where $f(2) = \mathfrak{S}_2\,\mathrm{Q}\,\mathrm{R}_0\big(\mathrm{Sym}(3)\big)$, $f(3) = \mathfrak{S}_3\mathfrak{S}_2$ and $\mathrm{f}(p) = \emptyset$ for all $p > 3$. If $G = \mathrm{Sym}(4)$ and H is a Sylow 3-subgroup of G, then $H \in \mathrm{s}_{n\mathfrak{F}}(G)$ ($H \leq \mathrm{Sym}(3) \leq G$). However $H \notin \mathrm{s}_{n\mathfrak{F}}\big(\mathrm{Alt}(4)\big)$.

The theory of $\mathfrak{F}$-subnormal subgroups is relevant only in the case of persistence in intermediate subgroups. Therefore

> *Unless otherwise stated, we stipulate that for the rest of the chapter the formation $\mathfrak{F}$ is closed under the operation of taking subgroups.*

Lemma 6.1.9. *Let G be a group.*

1. *If A is a K-$\mathfrak{F}$-subnormal subgroup of G, then $A^{\mathfrak{F}}$ is subnormal in G.*
2. *Let $\mathfrak{H} = \mathrm{E}\,\mathrm{K}(\mathfrak{F})$. Then $A^{\mathfrak{H}} = G^{\mathfrak{H}}$ for every $\mathfrak{F}$-subnormal subgroup of G.*
3. *If $1 \in \mathrm{s}_{n\mathfrak{F}}(G)$, then $\mathrm{s}_n(G) \subseteq \mathrm{s}_{n\mathfrak{F}}(G)$.*
4. *If G is a p-group for some prime p and $1 \in \mathrm{s}_{n\mathfrak{F}}(C_p)$, then $\mathrm{s}_{n\mathfrak{F}}(G) = \mathrm{s}_n(G) = \mathrm{s}(G)$.*

Proof. 1. We argue by induction on $|G|$. If $A = G$, then $G^{\mathfrak{F}}$ is normal in G, and the statement is true. Suppose $A < G$ and let X be an $\mathfrak{F}$-normal maximal subgroup of G containing A such that A is K-$\mathfrak{F}$-subnormal in X. Then $A^{\mathfrak{F}}$ is subnormal in X by induction. Since $A^{\mathfrak{F}} \leq X^{\mathfrak{F}}$, it follows that $A^{\mathfrak{F}}$ is subnormal in $X^{\mathfrak{F}}$. Moreover $G^{\mathfrak{F}}$ is contained in X. Hence $X^{\mathfrak{F}}$ is subnormal in $G^{\mathfrak{F}}$. This implies that $X^{\mathfrak{F}}$ is subnormal in G, hence so is $A^{\mathfrak{F}}$. A similar argument could be applied if X is a normal subgroup of G.

2. Proceeding by induction on $|G|$, we may assume that $A < G$. We argue as in Assertion 1 and use the same notation. It follows that $A^{\mathfrak{H}} = X^{\mathfrak{H}}$ for an $\mathfrak{F}$-normal maximal subgroup X of G such that A is $\mathfrak{F}$-subnormal in X. Moreover $G^{\mathfrak{H}}$ is contained in X as $G/\operatorname{Core}_G(X) \in \mathfrak{F} \subseteq \mathfrak{H}$. Now $X^{\mathfrak{F}}G^{\mathfrak{H}}/G^{\mathfrak{H}}$ belongs to $\mathfrak{H}$ because it is subnormal in $G/G^{\mathfrak{H}}$, by Statement 1 and Lemma 6.1.6 (3), and $\mathfrak{H}$ is closed under taking subnormal subgroups. Hence $X/X^{\mathfrak{F}} \cap G^{\mathfrak{H}}$ belongs to $\mathfrak{H}$. It implies that $X^{\mathfrak{H}}$ is contained in $G^{\mathfrak{H}}$. Note that every composition factor of $G^{\mathfrak{H}}/X^{\mathfrak{H}}$ belongs to $\mathrm{K}(\mathfrak{F})$. Therefore $G^{\mathfrak{H}} = (G^{\mathfrak{H}})^{\mathfrak{H}}$ is contained in $X^{\mathfrak{H}}$ and so $A^{\mathfrak{H}} = X^{\mathfrak{H}} = G^{\mathfrak{H}}$.

3. Since $\mathrm{S}_{n\mathfrak{F}}$ is a w-inherited functor, the result follows from Lemma 5.1.4.

4. It is enough to show that $1 \in \mathrm{S}_{n\mathfrak{F}}(G)$. Assume that it is not true and let G be a counterexample of minimal order. Let M be a maximal subgroup of G. The minimal choice of G implies that $1 \in \mathrm{S}_{n\mathfrak{F}}(M)$. Since $|G/M| = p$, it follows that $M/M \in \mathrm{S}_{n\mathfrak{F}}(G/M)$. Hence $M \in \mathrm{S}_{n\mathfrak{F}}(G)$ by Lemma 6.1.6 (2). Therefore $1 \in \mathrm{S}_{n\mathfrak{F}}(G)$. This contradiction shows that no counterexample exists. □

Proposition 6.1.10. *If $G \in \mathrm{E}\,\mathrm{K}(\mathfrak{F})$, then $\mathrm{S}_{n\mathfrak{F}}(G) = \mathrm{S}_{n\,K\text{-}\mathfrak{F}}(G)$.*

Proof. The inclusion $\mathrm{S}_{n\mathfrak{F}}(G) \subseteq \mathrm{S}_{n\mathrm{K}\text{-}\mathfrak{F}}(G)$ follows from the definitions.

Let $H \in \mathrm{S}_{n\mathrm{K}\text{-}\mathfrak{F}}(G)$. We prove that $H \in \mathrm{S}_{n\mathfrak{F}}(G)$ by induction on $|G|$. We may assume that $H \neq G$. Let N be a minimal normal subgroup of G. Then $G/N \in \mathrm{E}\,\mathrm{K}(\mathfrak{F})$ and $HN/N \in \mathrm{S}_{n\mathrm{K}\text{-}\mathfrak{F}}(G/N)$ by Lemma 6.1.6 (3). Consequently $HN/N \in \mathrm{S}_{n\mathfrak{F}}(G/N)$ by induction. This implies that HN is $\mathfrak{F}$-subnormal in G by Lemma 6.1.6 (2). Moreover $HN \in \mathrm{E}\,\mathrm{K}(\mathfrak{F})$ by Lemma 6.1.9 (1). Assume that HN is a proper subgroup of G. Since H is K-$\mathfrak{F}$-subnormal in HN by Lemma 6.1.7 (2), it follows that H is $\mathfrak{F}$-subnormal in HN by induction. Hence $H \in \mathrm{S}_{n\mathfrak{F}}(G)$, as required. Hence we may suppose that $G = HN$ for every minimal normal subgroup N of G. In particular, $\operatorname{Core}_G(H) = 1$. On the other hand, $H^{\mathfrak{F}}$ is subnormal in G by Lemma 6.1.9 (1) and so N normalises $H^{\mathfrak{F}}$ by [DH92, A, 14.3]. Thus $H^{\mathfrak{F}}$ is normal in G. This implies that $H^{\mathfrak{F}} \subseteq \operatorname{Core}_G(H) = 1$. Consequently $G/N \in \mathfrak{F}$ for each minimal normal subgroup N of G. If $G \in \mathfrak{F}$, then H is clearly $\mathfrak{F}$-subnormal in G. Hence we may assume that $G \notin \mathfrak{F}$ and therefore $G \in \mathrm{b}(\mathfrak{F})$. This means that G is a monolithic group, and $G^{\mathfrak{F}} = \operatorname{Soc}(G)$ is the unique minimal normal subgroup of G. Let M be a proper subgroup of G such that $H \in \mathrm{S}_{n\mathrm{K}\text{-}\mathfrak{F}}(M)$ and either $M \trianglelefteq G$ or $G^{\mathfrak{F}}$ is contained in M. If the second condition holds, then $NH = G$ is contained in M, contrary to supposition. Therefore $M \trianglelefteq G$. Since $G^{\mathfrak{F}}$ is not contained in M, it follows that $M = 1 = H$ and $G = \operatorname{Soc}(G)$ is a simple group. Therefore

$G \in \mathfrak{F} \cap \mathrm{b}(\mathfrak{F})$. This contradiction leads to $G \in \mathfrak{F}$ and so H is $\mathfrak{F}$-subnormal in G. □

Proposition 6.1.11. *Let $\mathfrak{F}$ be a saturated formation and let G be a group with an $\mathfrak{F}$-subnormal subgroup H such that $G = H\,\mathrm{F}^*(G)$. If $H \in \mathfrak{F}$, then $G \in \mathfrak{F}$.*

Proof. We argue by induction on $|G|$. Suppose that H is a proper subgroup of G and let M be an $\mathfrak{F}$-normal maximal subgroup of G such that $H \leq M$ and H is $\mathfrak{F}$-subnormal in M. Then $M = H\,\mathrm{F}^*(M)$. By induction, $M \in \mathfrak{F}$. Assume $G \notin \mathfrak{F}$. By Proposition 2.3.16, M is an $\mathfrak{F}$-projector of G. This is impossible because $G = G^{\mathfrak{F}}M$ and $G^{\mathfrak{F}}$ is contained in M. Consequently $G \in \mathfrak{F}$. □

6.2 $\mathfrak{F}$-subnormal closure

Let $\mathfrak{F}$ be a formation. By Lemma 6.1.7 (3), intersections of $\mathfrak{F}$-subnormal subgroups are $\mathfrak{F}$-subnormal. Therefore for any subset X of a group G, there exists a unique smallest $\mathfrak{F}$-subnormal subgroup of G containing X, the *$\mathfrak{F}$-subnormal closure* of X in G. We write $\mathrm{S}_G(X;\mathfrak{F})$ to denote this subgroup. It is clear that the same argument can be applied to K-$\mathfrak{F}$-subnormal subgroups. Consequently there exists a unique K-$\mathfrak{F}$-subnormal subgroup of G containing X, the *K-$\mathfrak{F}$-closure* of X in G. It is denoted by $\mathrm{S}_G(X;\text{K-}\mathfrak{F})$.

When $\mathfrak{F} = \mathfrak{N}$, the formation of all nilpotent groups, the subgroup $\mathrm{S}_G(X) = \mathrm{S}_G(X;\text{K-}\mathfrak{F})$ is the subnormal closure of X in G, that is, the smallest subnormal subgroup of G containing X.

The normal closure of X in G is generated by all of the conjugates of X in G and we might wonder whether or not the subnormal closure is generated by some natural subset of the set of these conjugates. Let us say that two subsets $X, Y \subseteq G$ are *strongly conjugate* if they are conjugate in $\langle X, Y\rangle$. It is rather clear that $\mathrm{S}_G(X)$ must contain all strong conjugates of X. In fact, the following powerful result, due to D. Bartels, is true.

Theorem 6.2.1 ([Bar77]). *Let X be a subset of a group G. Then $\mathrm{S}_G(X) = \langle Y \subseteq G : Y$ is strongly conjugate to X in $G\rangle$.*

The first part of this section is devoted to prove this theorem. First of all, we introduce some notation.

Notation 6.2.2. Let X and Y be subsets of a group G. We write:

- $X\,\sigma\,Y$ if X and Y are strongly conjugate in G.
- $X\,\sigma^{\infty}\,Y$ if there are subsets $X = X_0, X_1, \dots, X_n = Y$ such that $X_i\,\sigma\,X_{i+1}$ for all i, $0 \leq i < n$ (n natural number).
- $X =_U Y$ if X and Y are conjugate in the subgroup U of G.
- $X =_{\cdot\cdot G} Y$ if $\mathrm{S}_G(X) = \mathrm{S}_G(Y) = S$ and $X =_S Y$.
- $\mathrm{K}_G(X) = \langle Y \subseteq G : X\,\sigma\,Y\rangle$.

It is clear that σ^∞ and $=_{\cdot\cdot G}$ are equivalence relations on the set of all subsets of G.

Lemma 6.2.3. *Let X and Y be subsets of a group G such that $X\,\sigma\, Y$. Then $X =_{\cdot\cdot G} Y$.*

Proof. Denote $J := \langle X, Y\rangle$. Since $X\,\sigma\,Y$, there exists an element $g \in J$ such that $Y = X^g$. In particular, $\langle X^J\rangle$, the normal closure of X in J, is equal to J. Applying [DH92, A, 14.1], $\mathrm{S}_G(X)\cap J$ is subnormal in J and contains X. Since $J = \langle X^J\rangle$, it follows that $\mathrm{S}_G(X)\cap J = J$ and so $\mathrm{S}_G(J) = \mathrm{S}_G(X)$. Analogously $\mathrm{S}_G(J) = \mathrm{S}_G(Y)$. Therefore $X =_{\cdot\cdot G} Y$. □

Lemma 6.2.4. *Let X be a subset of a group G. Then*

$$\mathrm{S}_G(X) = \langle Y \subseteq G : X =_{\cdot\cdot G} Y\rangle.$$

Proof. Denote $A = \langle Y \subseteq G : X =_{\cdot\cdot G} Y\rangle$. Then $A = \langle X^g : g \in \mathrm{S}_G(X)\rangle$ by Lemma 6.2.3. It is clear that A is normal in $\mathrm{S}_G(X)$. Hence A is subnormal in G. Since A contains X, it follows that $A = \mathrm{S}_G(X)$. □

By Lemma 6.2.3, $X\,\sigma\,Y$ implies $X =_{\cdot\cdot G} Y$. Hence $\mathrm{K}_G(X) \subseteq \mathrm{S}_G(X)$ for every subgroup X of G.

Lemma 6.2.5. *Let X be a subset of a group G. Then*

1. $\mathrm{K}_G(X) = \langle Y \subseteq G : X\,\sigma^\infty\, Y\rangle$.
2. $X\,\sigma^\infty\, X^g$ *for all* $g \in \mathrm{K}_G(X)$.

Proof. 1. It is clear that $\mathrm{K}_G(X) \leq \langle Y \subseteq G : X\,\sigma^\infty\,Y\rangle$. Let $Y \subseteq G$ such that $X\,\sigma^\infty\,Y$. We have to show that $Y \subseteq \mathrm{K}_G(X)$. There is a natural number n and there are subsets $X = X_0, X_1, \dots, X_n = Y$ such that $X_i\,\sigma\,X_{i+1}$ for all i, $0 \leq i < n$. Suppose inductively that we have already shown that $X_0, X_1, \dots, X_{n-1}$ are contained in $\mathrm{K}_G(X)$. Since $\mathrm{K}_G(X) = \langle Z : X\,\sigma\,Z\rangle$, we may assume that $n > 1$. There exists an element $g \in \langle X_0, X_1, \dots, X_{n-1}\rangle \subseteq \mathrm{K}_G(X)$ such that $X^g = X_0^g = X_{n-1}$. Then $Y \leq \mathrm{K}_G(X^g)$, and since σ is G-invariant, it follows that $\mathrm{K}_G(X^g) = \mathrm{K}_G(X)^g = \mathrm{K}_G(X)$, and the induction step is complete.

2. Let Y be a subset of G. Let y be an element of $Y \cup Y^{-1}$ and assume that $X\,\sigma\,Y$. Then $X^y\,\sigma\,Y^y$ and $Y^y\,\sigma\,Y$, whence $X^y\,\sigma^\infty\,X$.

If $g \in \mathrm{K}_G(X)$, then $g = g_1 \cdots g_t$, where $g_i \in Y_i \cup Y_i^{-1}$, $X\,\sigma\,Y_i$, for all i, $1 \leq i \leq t$. If $t = 1$, then $X^{g_1}\,\sigma^\infty\,X$ by the above argument. Suppose inductively that $X^{(g_1\cdots g_{t-1})}\,\sigma^\infty\,X$. Then $X^{g_t^{-1}}\,\sigma^\infty\,X^{(g_1\cdots g_{t-1})}$ because $X^{g_t^{-1}}\,\sigma^\infty\,X$. Hence $X\,\sigma^\infty\,X^g$. □

Proposition 6.2.6. *For any subset X of a group G, the following statements are equivalent:*

1. $\mathrm{K}_G(X) = \mathrm{S}_G(X)$.
2. The equivalence relations σ^∞ *and* $=_{\cdot\cdot G}$ *coincide when restricted to the conjugacy class of* X *in* G.

Proof. Assume that $\mathrm{K}_G(X) = \mathrm{S}_G(X)$. Then $X =_{\cdot\cdot G} Y$ implies that $Y = X^g$ for some $g \in \mathrm{S}_G(X)$. By Lemma 6.2.5 (2), $X \,\sigma^\infty\, Y$. Since $=_{\cdot\cdot G}$ is a transitive relation, $X \,\sigma^\infty\, Y$ implies $X =_{\cdot\cdot G} Y$ by Lemma 6.2.3. Thus Statement 2 holds.

Conversely, assume Statement 2. Since $\mathrm{K}_G(X) = \langle Y \subseteq G : X \,\sigma^\infty\, Y\rangle$ by Lemma 6.2.5 (1), it follows that $\mathrm{K}_G(X) = \langle Y \subseteq G : X =_{\cdot\cdot G} Y\rangle$, which is equal to $\mathrm{S}_G(X)$ by Lemma 6.2.4. □

Lemma 6.2.7. *Let* X_0 *and* X_1 *be subsets of a group* G *such that* $X_0 \subseteq \langle X_1\rangle$. *Then* $\mathrm{K}_G(X_0) \leq \mathrm{K}_G(X_1)$.

Proof. Let t be an element of G such that $t \in \langle X_0, X_0^t\rangle$. Then obviously $t \in \langle X_1, X_1^t\rangle$. Hence $X_0 \,\sigma\, Y$ for some $Y \subseteq G$ implies that there is a subset W of G such that $Y \subseteq W$ and $X_1 \,\sigma\, W$. The lemma follows by definition of $\mathrm{K}_G(X_1)$. □

Lemma 6.2.8. *Let* G *be a group and let* N *be a normal subgroup of* G. *Let* $X \subseteq G$ *and let* Y_1/N *be a subset of* G/N *such that* $XN/N \,\sigma\, Y_1/N$. *Then there exists a subset* Y *of* G *such that* $X \,\sigma\, Y$ *and* $Y_1 = YN$.

Proof. Let

$$\mathcal{A} := \{V \subseteq G : VN/N = Y_1/N \text{ and } X =_{\cdot\cdot G} V\}.$$

Since $XN/N \,\sigma\, Y_1/N$, it follows that XN/N and Y_1/N are conjugate in $\mathrm{S}_{G/N}(XN/N) = \mathrm{S}_G(X)N/N$. Hence $Y_1 = X^z N$ for some $z \in \mathrm{S}_G(X)$. It is clear that $X =_{\cdot\cdot G} X^z$ and so $X^z = V \in \mathcal{A}$. This shows that $\mathcal{A}$ is non-empty. Let W be an element of $\mathcal{A}$ such that $\langle X, W\rangle$ has minimal order. Since $XN/N \,\sigma\, WN/N$, there exists an element $t \in \langle X, W\rangle$ such that $WN/N = X^t N/N = Y_1/N$. It is clear that $X =_{\cdot\cdot G} X^t$. Hence X^t belongs to $\mathcal{A}$. The minimal choice of $\langle X, W\rangle$ implies that $\langle X, X^t\rangle = \langle X, W\rangle$ and so $X \,\sigma\, X^t$ $(= Y)$. □

Corollary 6.2.9. *For any subset* X *of a group* G *and for any* $N \trianglelefteq G$, $\mathrm{K}_G(X)N/N = \mathrm{K}_{G/N}(XN/N)$.

Proposition 6.2.10. *For any subset* X *of a group* G, *the relations* σ^∞ *and* $=_{\cdot\cdot G}$ *coincide on the conjugacy class of* X *in* G.

Proof. Assume that the result is false, and let (G, X) be a counterexample with $|G| + |\langle X\rangle|$ as small as possible. Clearly $X \neq \emptyset$ and the conjugacy class of X in G splits into σ^∞-equivalence classes; we denote the set of these equivalence classes by Ω. Since $X \,\sigma^\infty\, Y$ implies $X =_{\cdot\cdot G} Y$ for all $Y \subseteq G$ by Lemma 6.2.3, it follows from our choice of (G, X) that Ω contains at least two elements. It is clear that G acts transitively by conjugation on Ω in the obvious way.

Let $K = \mathrm{K}_G(X)$. By Proposition 6.2.6, K is a proper subgroup of G. For any non-trivial normal subgroup N of G, the relations σ^∞ and $=\cdot\cdot_G$ coincide on the conjugacy class of XN/N in G/N by minimality of G. Hence $\mathrm{K}_{G/N}(XN/N) = \mathrm{S}_{G/N}(XN/N) = \mathrm{S}_G(X)N/N$ by Proposition 6.2.6, and so $KN/N = \mathrm{K}_{G/N}(XN/N)$ is subnormal in G/N. In particular, KN is subnormal in G. Suppose that $Z = KN$ is a proper subgroup of G. Then $K = \mathrm{K}_Z(X) = \mathrm{S}_Z(X)$ by the choice of G. Hence K is subnormal in Z and so is in G. Proposition 6.2.6 implies that the relations σ^∞ and $=\cdot\cdot_G$ coincide on the conjugacy class of X in G. This is a contradiction against the choice of (G, X). Consequently, $G = KN$ for any non-trivial normal subgroup N of G. From this we conclude that $\mathrm{Core}_G(K) = 1$ and $\langle X^G\rangle$, the normal closure of X in G, is equal to G.

Let p be a prime dividing $|\langle X\rangle|$ and let Q be a Sylow p-subgroup of $\langle X\rangle$. By Lemma 6.2.7, $\mathrm{K}_G(Q)$ is contained in K. Suppose that Q is a proper subgroup of $\langle X\rangle$. The minimal choice of (G, X) implies that $\mathrm{K}_G(Q)$ is subnormal in G. Let N be a minimal normal subgroup of G. By [DH92, A, 14.3], N normalises $\mathrm{K}_G(Q)$. Since $G = KN$, it follows that $\langle \mathrm{K}_G(Q)^G\rangle = \langle \mathrm{K}_G(Q)^K\rangle$ is a subgroup of K. Hence $\langle \mathrm{K}_G(Q)^G\rangle$ is contained in $\mathrm{Core}_G(K) = 1$. This contradiction shows that $Q = \langle X\rangle$ and $\langle X\rangle$ is a p-group.

For any subgroup U of G, let $[U]$ denote the set

$$[U] = \{\omega \in \Omega : \text{there exists } X^g \in \omega \text{ such that } X^g \subseteq U\}.$$

The following statements hold:

1. *For any proper subgroup U of G and for every Sylow p-subgroup P of U, $[U] = [P]$.*

It is clear that $[P] \subseteq [U]$. Conversely, let $\omega \in [U]$ and let $Y \in \omega$ be a subset of U. Let $L = \mathrm{S}_U(Y)$. Since L is subnormal in U, it follows that $L \cap P$ is a Sylow p-subgroup of L. Hence Y^z is contained in P for some $z \in L$. It is clear that $Y =\cdot\cdot_G Y^z$. Since $=\cdot\cdot_G$ and σ^∞ coincide on the conjugacy class of Y in U by induction, we have that $Y\,\sigma^\infty\,Y^z$. Hence $\omega \in [P]$.

2. *$[U]$ is a proper subset of Ω for any proper subgroup U of G.*

Assume that $[U] = \Omega$. Then $\Omega = [P]$ for some Sylow p-subgroup P of U. Since $\Omega \neq \emptyset$, it follows $P \neq 1$ and so $\mathrm{Z}(P) \neq 1$. Note that if $x \in \mathrm{Z}(P)$ and $\omega \in \Omega$, then $\omega^x = \omega$ because x centralises an element of ω. Hence $\mathrm{Z}(P)$ acts trivially on Ω. Since $\Omega = [P] = [P^g]$ for all $g \in G$, it follows that $\mathrm{Z}(P^g)$ acts trivially on Ω. This implies that $N = \langle \mathrm{Z}(P)^G\rangle$ acts trivially on Ω. Let ω_0 be the element of Ω such that $X \in \omega_0$. If $z \in K$, then $X^z \in \omega_0$ by Lemma 6.2.5 (2). Hence $\omega_0^z = \omega_0$. Let g be an element of G. There exist $z \in K$ and $n \in N$ such that $g = zn$. It follows that $\omega_0^g = \omega_0$ and so $X^g\,\sigma^\infty\,X$ for all $g \in G$. Therefore $\mathrm{K}_G(X) = \langle X^G\rangle = G$. This contradiction shows that $[U] \neq \Omega$.

3. *Any maximal subgroup M of G such that $[M] \neq \emptyset$ contains a Sylow p-subgroup of G.*

Let P be a Sylow p-subgroup of M. By Statement 1 and Statement 2, $[M] = [P] \neq \Omega$. Note that if $\omega \in [P]$ and $g \in M \cup \mathrm{N}_G(P)$, then $\omega^g \in [P]$.

Hence $\langle M, \mathrm{N}_G(P)\rangle$ is not transitive on Ω. Therefore $\langle M, \mathrm{N}_G(P)\rangle$ is a proper subgroup of G. In particular, $\mathrm{N}_G(P) \leq M$ and so P is a Sylow p-subgroup of G.

4. *Any Sylow p-subgroup P of G is contained in a unique maximal subgroup of G.*

Obviously G is not a p-group. Let P be contained in $L \cap M$, where L and M are maximal subgroups of G. Then $[L] = [M] = [P]$ by Statement 1 and $[P] \neq \Omega$ by Statement 2. This implies that $\langle L, M\rangle$ is not transitive on Ω. Hence $G \neq \langle L, M\rangle$ and $L = M$.

5. *X is contained in a unique maximal subgroup of G.*

Suppose that X is contained in at least two maximal subgroups L and M of G. Choose L and M such that the Sylow p-subgroups of $L \cap M$ have maximal order. There exist Sylow p-subgroups R and S of L and M respectively such that $R \cap S$ is a Sylow p-subgroup of $L \cap M$ containing X. By Statement 4, R and S are Sylow p-subgroups of G. Moreover $R \neq S$ by Statement 4. From this we conclude that $R \cap S$ is a proper subgroup of $R_1 = \mathrm{N}_R(R \cap S)$. Since $N = \mathrm{N}_G(R \cap S)$ is a proper subgroup of G, this implies N is contained in M by our choice of M and L. The same argument with L and S replacing M and R yields $N \leq L$. But then $R \cap S < R_1 \leq M \cap L$ and $R \cap S$ is a Sylow p-subgroup of $M \cap L$. This contradiction proves Statement 5.

Now from Statement 5 we deduce the final contradiction, thus proving the lemma. We know that K is a proper subgroup of G. Let M be the unique maximal subgroup of G containing X. Since $\langle X^G\rangle = G$, it follows that $M = \mathrm{N}_G(M)$. Let $g \in G \setminus M$. Then $G = \langle X, X^g\rangle$. This implies $X \,\sigma\, X^g$ and therefore we have $G = K$. □

Combining Proposition 6.2.6 and Proposition 6.2.10, we have:

Theorem 6.2.11. $\mathrm{S}_G(X) = \mathrm{K}_G(X)$ *for any subset X of G.*

Let X be a subset of G and $g \in G$ such that $g \in \langle X, X^g\rangle$. Then $g \in \langle \mathrm{S}_G(X), \mathrm{S}_G(X)^g\rangle \leq \mathrm{S}_G\big(\mathrm{S}_G(X)\big) = \mathrm{S}_G(X)$. Hence the following result is true.

Corollary 6.2.12. $\mathrm{S}_G(X) = \big\langle g \in G : g \in \langle X, X^g\rangle\big\rangle$.

Let H be a subgroup of a group G. If A is a subgroup of G, containing H, then $HA^{\mathfrak{N}}$ is a subnormal subgroup of A containing H. Now if $g \in G$ and $g \in \langle H, H^g\rangle = J$, then the normal closure of H in J is equal to J. The subnormality of $HJ^{\mathfrak{N}}$ in J implies that $J = HJ^{\mathfrak{N}}$ and $g \in H\langle H, H^g\rangle^{\mathfrak{N}}$. Moreover there exists $z \in \langle H, H^g\rangle^{\mathfrak{N}}$ such that $J = \langle H, H^z\rangle$. Thus we have shown the following:

Theorem 6.2.13. *Let H be a subgroup of a group G. Then*

$$\mathrm{S}_G(H) = \big\langle H^g : g \in \langle H, H^g\rangle^{\mathfrak{N}}\big\rangle = \big\langle g \in G : g \in H\langle H, H^g\rangle^{\mathfrak{N}}\big\rangle.$$

The descriptions of the subnormal closure provide a proof of the following subnormality criterion due to Wielandt.

Theorem 6.2.14 ([Wie74]). *Let H be a subgroup of a group G. The following statements are pairwise equivalent:*

1. *H is subnormal in G.*
2. *H is subnormal in $\langle H, g\rangle$ for all $g \in G$.*
3. *H is subnormal in $\langle H, H^g\rangle$ for all $g \in G$.*
4. *If $g \in G$ and $g \in \langle H, H^g\rangle$, then $g \in H$.*

Moreover, they are equivalent to:

5. *If $g \in G$ and $g \in \langle H, H^g\rangle^{\mathfrak{N}}$, then $g \in H$.*

Remark 6.2.15. Theorem 6.2.11 does not provide a description of the $\mathfrak{N}$-subnormal closure. Let $G = \mathrm{Alt}(5)$ and $H = \{1\}$. Then $\mathrm{S}_G(H) = H$ and $\mathrm{S}_G(H; \mathfrak{N}) = G$.

If G is a soluble group, then $\mathrm{S}_G(H) = \mathrm{S}_G(H; \mathrm{K}\text{-}\mathfrak{N}) = \mathrm{S}_G(H; \mathfrak{N})$ by Proposition 6.1.10. In this context, the following conjecture arises.

Conjecture 6.2.16 (K. Doerk). Let $\mathfrak{F}$ be a saturated formation and $\pi = \operatorname{char} \mathfrak{F}$. Given a subgroup H of a soluble group $G \in \mathfrak{S}_\pi$, the $\mathfrak{F}$-subnormal closure of H in G is the subgroup $\mathrm{S}_G(H; \mathfrak{F}) = \big\langle g \in G : g \in H\langle H, H^g\rangle^{\mathfrak{F}}\big\rangle$.

A. Ballester-Bolinches and M. D. Pérez-Ramos [BBPR91] confirmed Conjecture 6.2.16. In fact, they showed that the conjecture is valid for groups with soluble $\mathfrak{F}$-residual, that is, groups in the class $\mathfrak{S}\mathfrak{F}$.

Henceforth in the rest of the section

$\mathfrak{F} = \mathrm{LF}(F)$ *will denote a subgroup-closed saturated formation of characteristic π.*

The proof of Doerk's conjecture depends heavily on the following extension of Theorem 6.2.14 to subgroup-closed saturated formations.

Theorem 6.2.17 ([BBPR91]). *For a subgroup H of a π-group $G \in \mathfrak{S}\mathfrak{F}$, the following statements are pairwise equivalent:*

1. *H is $\mathfrak{F}$-subnormal in G*
2. *H is $\mathfrak{F}$-subnormal in $\langle H, x\rangle$ for every $x \in G$.*
3. *H is $\mathfrak{F}$-subnormal in $\langle H, H^x\rangle$ for every $x \in G$.*
4. *If T is a subgroup of G such that T is contained in $\langle H, T\rangle^{\mathfrak{F}}$, then T is contained in H.*
5. *If $x \in G$ and $x \in \langle H, x\rangle^{\mathfrak{F}}$, it follows that $x \in H$.*
6. *If $x \in G$ and $x \in \langle H, H^x\rangle^{\mathfrak{F}}$, it follows that $x \in H$.*

Proof. 3 implies 1. We argue by induction on $|G|$. We can assume that $G^{\mathfrak{F}} \neq 1$ by Lemma 6.1.7 (1). Let N be a minimal normal subgroup of G such that N is contained in $G^{\mathfrak{F}}$. By induction, HN/N is $\mathfrak{F}$-subnormal in G/N and so HN is $\mathfrak{F}$-subnormal in G by Lemma 6.1.6 (2). If HN were a proper subgroup of G, then H would be $\mathfrak{F}$-subnormal in $HN \in \mathfrak{S}\mathfrak{F}$ by induction. Applying

Lemma 6.1.6 (1), H is $\mathfrak{F}$-subnormal in G and the implication is true. Hence we can suppose $G = HN$ and $G \neq H$. Since N is soluble, H is a maximal subgroup of G. If H is a normal subgroup of G, then H is $\mathfrak{F}$-subnormal in G because $G \in \mathrm{E}\,\mathrm{K}(\mathfrak{F})$. If H is not normal in G, there exists an element $x \in G$ such that $H \neq H^x$. Then $G = \langle H, H^x \rangle$ and H is $\mathfrak{F}$-subnormal in G by Statement 3.

By Lemma 6.1.7 (2), 1 implies 2 and 2 implies 3. Consequently, 1, 2, and 3 are pairwise equivalent.

It is clear that 4 implies 5 and 5 implies 6 because $X^{\mathfrak{F}} \leq Y^{\mathfrak{F}} \leq G^{\mathfrak{F}}$ if $X \leq Y \leq G$.

1 implies 4. Suppose that H is $\mathfrak{F}$-subnormal in G and T is a subgroup of G such that T is contained in $\langle H, T \rangle^{\mathfrak{F}}$. Then $\langle H, T \rangle = H\langle H, T \rangle^{\mathfrak{F}}$. If H were a proper subgroup of $\langle H, T \rangle$, there would exist an $\mathfrak{F}$-normal maximal subgroup M of $\langle H, T \rangle$ containing H. Since $\langle H, T \rangle^{\mathfrak{F}} \leq M$, we would have $M = \langle H, T \rangle$. This contradiction yields $H = \langle H, T \rangle$ and T is contained in H.

To complete the proof we now show that 6 implies 1. We proceed by induction on $|G|$. Let $x \in G$ and $T = \langle H, H^x \rangle$. If T is a proper subgroup of G, then by induction H is $\mathfrak{F}$-subnormal in T. Since 3 is equivalent to 1, we may assume that $T = G$ for some $x \in G$. By Lemma 6.1.7 (1), $HG^{\mathfrak{F}}$ is $\mathfrak{F}$-subnormal in G. Hence, if $HG^{\mathfrak{F}}$ were a proper subgroup of G, then H would be $\mathfrak{F}$-subnormal in $HG^{\mathfrak{F}}$ by induction. Therefore H would be $\mathfrak{F}$-subnormal in G by Lemma 6.1.6 (1). Therefore we may suppose $G = \langle H, H^x \rangle = HG^{\mathfrak{F}} = H\langle H, H^x \rangle^{\mathfrak{F}}$. In particular, $x = ht$ for some $h \in H$ and $t \in \langle H, H^x \rangle^{\mathfrak{F}} = \langle H, H^t \rangle^{\mathfrak{F}}$. Applying Statement 6, it follows that $t \in H$. Hence $x \in H$ and $H = G$ is $\mathfrak{F}$-subnormal in G. The circle of implications is now complete. □

If H is a subgroup of a group G, denote

$$\mathrm{T}_G(H; \mathfrak{F}) = \big\langle x \in G : x \in H\langle H, H^x \rangle^{\mathfrak{F}} \big\rangle.$$

Lemma 6.2.18. *If N is a normal subgroup of a group G and H is a subgroup of G, then*

$$\mathrm{T}_{G/N}(HN/N; \mathfrak{F}) = T_G(H; \mathfrak{F})N/N.$$

Proof. Denote with bars the images in $\bar{G} = G/N$. It is clear that $\overline{T_G(H; \mathfrak{F})} = \mathrm{T}_G(H; \mathfrak{F})N/N$ is contained in $\mathrm{T}_{G/N}(HN/N; \mathfrak{F})$. Consider now $\bar{g} \in \langle \bar{H}, \bar{H}^{\bar{g}} \rangle^{\mathfrak{F}}$. Then there exists an element $z \in \langle H, H^g \rangle^{\mathfrak{F}}$ such that $\bar{z} = \bar{g}$. Hence the set $\mathcal{L} = \{z \in \langle H, H^g \rangle^{\mathfrak{F}} : \bar{z} = \bar{g}\}$ is non-empty. Let $t \in \mathcal{L}$ such that $\langle H, H^t \rangle^{\mathfrak{F}}$ has minimal order. Then $\langle H, H^g \rangle^{\mathfrak{F}} N = \langle H, H^t \rangle^{\mathfrak{F}} N$ and $t = xn$ for some $x \in \langle H, H^t \rangle^{\mathfrak{F}}$ and $n \in N$. It is clear that $x \in \langle H, H^g \rangle^{\mathfrak{F}}$ and $\bar{x} = \bar{t}$. Hence $x \in \mathcal{L}$. The minimal choice of t implies that $\langle H, H^x \rangle^{\mathfrak{F}} = \langle H, H^t \rangle^{\mathfrak{F}}$. Therefore $\bar{g} \in \overline{\mathrm{T}_G(H; \mathfrak{F})}$ and the equality holds. □

Theorem 6.2.19 ([BBPR91]). *Let G be a π-group with soluble $\mathfrak{F}$-residual. Let H be a subgroup of G. Then $\mathrm{S}_G(H; \mathfrak{F}) = \mathrm{T}_G(H; \mathfrak{F}) = \big\langle T \leq G : T \leq H\langle H, T \rangle^{\mathfrak{F}} \big\rangle$.*

Proof. Write $S = \mathrm{T}_G(H; \mathfrak{F})$. If L is an $\mathfrak{F}$-subnormal subgroup of G containing H, then S is contained in L by Theorem 6.2.17. Thus the first equality holds if we prove that S is $\mathfrak{F}$-subnormal in G. We argue by induction on $|G|$. Since $G^{\mathfrak{F}}$ is soluble and G is a π-group, it follows that $G^{\mathfrak{F}}$ is a proper subgroup of G. Of course, it may be assumed that $G^{\mathfrak{F}} \neq 1$. Let N be a minimal normal subgroup of G contained in $G^{\mathfrak{F}}$. Then, by Lemma 6.2.18, $SN/N = \mathrm{T}_{G/N}(HN/N; \mathfrak{F})$. Hence SN/N is $\mathfrak{F}$-subnormal in G/N. By Lemma 6.1.6 (2), SN is $\mathfrak{F}$-subnormal in G. Suppose that $SN = X$ is a proper subgroup of G. Then $S = \mathrm{T}_X(H; \mathfrak{F})$ is $\mathfrak{F}$-subnormal in X by induction. By Lemma 6.1.6 (1), S is $\mathfrak{F}$-subnormal in G. Therefore we must have $G = SN$ and thus S is a maximal subgroup of G because N is abelian and S is a proper subgroup of G. This argument also yields $\mathrm{Core}_G(S) = 1$. Therefore G is a primitive group of type 1 and $N = \mathrm{Soc}(G) = \mathrm{C}_G(N)$.

Suppose, by way of contradiction, that S is not $\mathfrak{F}$-subnormal in G and let us choose H of minimal order among those subgroups of G such that $\mathrm{T}_G(H; \mathfrak{F})$ is not $\mathfrak{F}$-subnormal in G. If M is a maximal subgroup of H satisfying $H = \mathrm{T}_H(M; \mathfrak{F})$, then $H \leq \mathrm{T}_G(M; \mathfrak{F})$ and $\mathrm{T}_G(M; \mathfrak{F})$ is $\mathfrak{F}$-subnormal in G. Consequently S is contained in $\mathrm{T}_G(M; \mathfrak{F})$ and S is $\mathfrak{F}$-subnormal in G, contrary to the choice of H. Therefore, each maximal subgroup of H is $\mathfrak{F}$-subnormal in H. This implies that every primitive epimorphic image of H belongs to $\mathfrak{F}$. Hence $H \in \mathfrak{F}$ because $\mathfrak{F}$ is saturated. Let N_0 be a minimal H-invariant subgroup of N. Put $A = HN_0$. If $A = G$, then $H = S$. By Theorem 6.2.17, S is $\mathfrak{F}$-subnormal in G, contrary to supposition. Hence A is a proper subgroup of G. Suppose that A is not an $\mathfrak{F}$-group. Then $N_0 = A^{\mathfrak{F}}$ and $A = \mathrm{S}_A(H; \mathfrak{F}) = \mathrm{T}_A(H; \mathfrak{F}) \leq S$. This is a contradiction. Therefore $A \in \mathfrak{F}$. Let $\mathrm{Soc}_H(N)$ be the product of all minimal H-invariant subgroups of N. Since $\mathfrak{F}$ is a formation, it follows that $H \,\mathrm{Soc}_H(N) \in \mathfrak{F}$. Suppose that $HN \notin \mathfrak{F}$ and let L be an $\mathfrak{F}$-maximal subgroup of HN containing $H \,\mathrm{Soc}_H(N)$. Then $HN = L(HN)^{\mathfrak{F}}$ and $L \cap (HN)^{\mathfrak{F}} = 1$ by Theorem 4.2.17. But then, since $(HN)^{\mathfrak{F}} \neq 1$, we have that $1 \neq (HN)^{\mathfrak{F}} \cap \mathrm{Soc}_H(N) \leq (HN)^{\mathfrak{F}} \cap L$, which is a contradiction. Therefore $HN \in \mathfrak{F}$. Let $1 = N_0 \trianglelefteq N_1 \trianglelefteq N_2 \trianglelefteq \cdots \trianglelefteq N_r = N$ be an H-composition series of N. Then $H \big/ \mathrm{C}_H(N_i/N_{i-1}) \in F(p)$, for $i = 1, \ldots, r$, and p the prime dividing $|N|$. Hence $H^{F(p)} \leq \bigcap\{\mathrm{C}_H(N_i/N_{i-1}) : i = 1, \ldots, r\}$ and so that $H^{F(p)} \big/ \mathrm{C}_{H^{F(p)}}(N) = H^{F(p)}$ is a p-group by [DH92, A, 12.4]. Therefore $H \in \mathfrak{S}_p F(p) = F(p)$.

Consider now $g \in \langle H, H^g \rangle^{\mathfrak{F}} \setminus H$. It is clear that H is a proper subgroup of $T = \langle H, H^g \rangle \notin \mathfrak{F}$. Obviously $T = HT^{\mathfrak{F}}$ is contained in S. Denote $T^{\mathfrak{F}} = R$. Let $1 = K_0 \trianglelefteq K_1 \trianglelefteq \cdots \trianglelefteq K_s = N$ be a T-composition series of N. If every T-chief factor K_j/K_{j-1}, $j \in \{1, \ldots, s\}$, is centralised by R, it follows, arguing as above, that R is a p-group and since $H \in F(p)$, it follows that $T \in F(p) \subseteq \mathfrak{F}$, contrary to the choice of T. Consequently, there exists a T-chief factor K_i/K_{i-1}, $i \in \{1, \ldots, s\}$, such that R is not contained in $\mathrm{C}_T(K_i/K_{i-1})$. Write $L = K_iT = K_i(RH)$, and denote with bars the images in $\bar{L} = L/K_{i-1}$. We have that $\bar{K}_i \leq \bar{L}^{\mathfrak{F}}$, because otherwise $\bar{K}_i \cap \bar{L}^{\mathfrak{F}} = 1$, and then $\bar{R} \leq \mathrm{C}_{\bar{L}}(\bar{K}_i)$, contradicting our choice of $\bar{K}_i$. Therefore $\bar{L}^{\mathfrak{F}} = \bar{K}_i\bar{R}$ by

Proposition 2.2.8. Assume that $|\bar{L}| < |G|$. Then $\mathrm{S}_{\bar{L}}(\bar{H}, \mathfrak{F}) = \mathrm{T}_{\bar{L}}(\bar{H}, \mathfrak{F}) = \bar{L}$ by induction. Applying Lemma 6.2.18, $\mathrm{T}_{\bar{L}}(\bar{H}; \mathfrak{F}) = \mathrm{T}_L(H; \mathfrak{F})K_{i-1}/K_{i-1}$. Hence $\mathrm{T}_L(H; \mathfrak{F})K_{i-1} = K_i RH$. If $\mathrm{T}_L(H; \mathfrak{F}) \cap K_i = 1$, then $\mathrm{T}_L(H; \mathfrak{F}) = RH$ and $K_i = K_{i-1}$. This is a contradiction. Thus $1 \neq K_i \cap \mathrm{T}_L(H; \mathfrak{F}) \leq S \cap N$, which is also impossible. Therefore $|\bar{L}| = |G|$, that is, $G = NT$ and $S = T = \langle H, g\rangle$. Let $n \in N$ such that $[H, n] \neq 1$ and consider $M = \langle H, H^{ng}\rangle$. Since $G = HG^{\mathfrak{F}}$, it follows that $M < G$, because otherwise $ng \in S$, and so $n \in S$, contradicting our supposition. Let L be a maximal subgroup of G containing M. If $L = S$, then $H^n \leq S$. Hence $1 \neq [h, n] = h^{-1}h^n \in S \cap N$, for some $h \in H$. This is impossible. If N were contained in L, then L would contain H^g, and so $S = \langle H, H^g\rangle \leq L$. This would be a contradiction. Hence $\mathrm{Core}_G(L) = 1$ and $L = H(L \cap G^{\mathfrak{F}}) = HL^{\mathfrak{F}}$. Our choice of G implies that $L = \mathrm{T}_L(H; \mathfrak{F}) \leq S$, and we have reached the desired contradiction. Therefore $\mathrm{T}_G(H; \mathfrak{F})$ is $\mathfrak{F}$-subnormal in G and $\mathrm{S}_G(H; \mathfrak{F}) = \mathrm{T}_G(H; \mathfrak{F})$.

On the other hand, it is clear that $\mathrm{S}_G(H; \mathfrak{F})$ is contained in $\mathrm{L}_G(H; \mathfrak{F}) = \big\langle T \leq G : T \leq H\langle H, T\rangle^{\mathfrak{F}}\big\rangle$. Now, if K is an $\mathfrak{F}$-subnormal subgroup of G containing H and T is a generator of $\mathrm{L}_G(H; \mathfrak{F})$, it follows that $T \leq K\langle K, T\rangle^{\mathfrak{F}}$. Thus, if $t \in T$, then $t = k_t x_t$ with $k_t \in K$, $x_t \in \langle K, T\rangle^{\mathfrak{F}}$. Denote by $R = \langle x_t : t \in T\rangle$. Then $\langle K, T\rangle = \langle K, R\rangle$ and $R \leq \langle K, R\rangle^{\mathfrak{F}}$. Since K is $\mathfrak{F}$-subnormal in G, it follows that $R \leq K$ by Theorem 6.2.17. Consequently T is contained in K and $\mathrm{L}_G(H; \mathfrak{F}) \leq K$. Since $\mathrm{S}_G(H; \mathfrak{F})$ is $\mathfrak{F}$-subnormal in G, $\mathrm{S}_G(H; \mathfrak{F})$ contains $\mathrm{L}_G(H; \mathfrak{F})$ and the proof of the theorem is complete. □

Open question 6.2.20. *Let $\mathfrak{F}$ be a saturated formation of characteristic π. Is it possible to find a useful description for the $\mathfrak{F}$-subnormal closure of a subgroup H of a π-group G?.*

6.3 Lattice formations

One of the most striking results in the theory of subnormal subgroups is the celebrated "join" theorem, proved by H. Wielandt in 1939: the subgroup generated by two subnormal subgroups of a finite group is itself subnormal. As a result, the set of all subnormal subgroups of a group is a sublattice of the subgroup lattice.

Let $\mathfrak{F}$ be a formation. One might wonder whether the set of $\mathfrak{F}$-subnormal subgroups of a group forms a sublattice of the subgroup lattice. The answer is in general negative.

Example 6.3.1 ([BBPR91]). Let $\mathfrak{F}$ be the formation of all 2-nilpotent groups and $G = \mathrm{Sym}(4)$. By [DH92, A, 10.9], G has an irreducible and faithful module V over $\mathrm{GF}(3)$. Let $R = [V]G$ be the corresponding semidirect product. If P is a Sylow 2-subgroup of G, then VP is an $\mathfrak{F}$-normal maximal subgroup of R. Since $VP \in \mathfrak{F}$, it follows that P is $\mathfrak{F}$-subnormal in R. However, if $x \in G \setminus \mathrm{N}_R(P)$, then $G = \langle P, P^x\rangle$ is not $\mathfrak{F}$-subnormal in R.

Therefore the following question naturally arises:

Which are the formations $\mathfrak{F}$ for which the set $\mathrm{s}_{n\mathfrak{F}}(G)$ is a sublattice of the subgroup lattice of G for every group G?

This question was first proposed by L. A. Shemetkov in his monograph [She78] in 1978 and it appeared in the *Kourovka Notebook* in 1984 as Problem 9.75 [MK84].

In 1992, A. Ballester-Bolinches, K. Doerk, and M. D. Pérez-Ramos gave in [BBDPR92] the answer to that question in the soluble universe for saturated formations.

On the other hand, O. H. Kegel [Keg78] showed that if $\mathfrak{F}$ is a subgroup-closed formation such that $\mathfrak{F}\mathfrak{F} = \mathfrak{F}$, then the set of all K-$\mathfrak{F}$-subnormal subgroups of a group G is a sublattice of the subgroup lattice of G for every group G. He also asks for other formations enjoying the lattice property for K-$\mathfrak{F}$-subnormal subgroups.

In 1993, A. F. Vasil'ev, S. F. Kamornikov, and V. N. Semenchuk [VKS93] published the extension of the lattice results of [BBDPR92] to the general finite universe. They also proved that the problems of O. H. Kegel and L. A. Shemetkov are equivalent for saturated formations.

Our objective in this section is to give a full account of the above results.

In the sequel, $\mathfrak{F}$ will be a (subgroup-closed) formation.

Definition 6.3.2. *We say that $\mathfrak{F}$ is a* lattice *(respectively,* K-lattice*)* formation *if the set of all $\mathfrak{F}$-subnormal (respectively, K-$\mathfrak{F}$-subnormal) subgroups is a sublattice of the lattice of all subgroups in every group.*

The next result provides a criterion for a saturated formation to be a lattice formation.

Theorem 6.3.3. *Any two of the following assertions about a saturated formation $\mathfrak{F}$ are equivalent:*

1. *$\mathfrak{F}$ is a lattice formation.*
2. *If A and B are $\mathfrak{F}$-subnormal $\mathfrak{F}$-subgroups of a group G, then $\langle A, B\rangle$ is an $\mathfrak{F}$-subgroup of G.*
3. *$\mathfrak{F}$ is a Fitting class and the $\mathfrak{F}$-radical $G_{\mathfrak{F}}$ of a group G contains every $\mathfrak{F}$-subnormal $\mathfrak{F}$-subgroup of G.*

Proof. Assume, arguing by contradiction, that $\mathfrak{F}$ is a lattice formation such that $\mathfrak{F}$ does not satisfy Statement 2. Let G be a group of minimal order among the groups X having two $\mathfrak{F}$-subnormal $\mathfrak{F}$-subgroups H and K such that $\langle H, K\rangle$ is not an $\mathfrak{F}$-group. Among the pairs (H, K) of $\mathfrak{F}$-subnormal $\mathfrak{F}$-subgroups of G such that $\langle H, K\rangle \notin \mathfrak{F}$, we choose a pair (A, B) with $|A|+|B|$ maximal. Because of Lemma 6.1.7 (2) and the choice of G, it must be $G = \langle A, B\rangle$. Moreover if N is a minimal normal subgroup of G, it follows that $G/N \in \mathfrak{F}$ because G/N is generated by the $\mathfrak{F}$-subnormal $\mathfrak{F}$-subgroups AN/N and BN/N. Therefore G is in the boundary of $\mathfrak{F}$. In particular, G is a monolithic primitive group.

Put $N = \mathrm{Soc}(G) = G^{\mathfrak{F}}$. By Lemma 6.1.7 (2) and Proposition 6.1.11, $AN = AF^*(AN)$ and $BN = BF^*(BN)$ are $\mathfrak{F}$-groups. Applying Lemma 6.1.7 (1), we have that AN and BN are $\mathfrak{F}$-subnormal subgroups of G. The choice of the pair (A, B) yields $N \leq A \cap B$.

Let H be a minimal supplement to N in G. By [DH92, A, 9.2(c)], we have $H \cap N \leq \Phi(H)$; since $H/(H \cap N) \cong HN/N = G/N \in \mathfrak{F}$, it follows that $H \in \mathrm{E}_{\Phi}\,\mathfrak{F} = \mathfrak{F}$. On the other hand, $A = N(A \cap H)$ and $B = N(B \cap H)$. By Lemma 6.1.6 (1) and Lemma 6.1.7 (1), $A \cap H$ is $\mathfrak{F}$-subnormal in G. Hence the normal closure $(A \cap H)^H$ of $A \cap H$ in H is $\mathfrak{F}$-subnormal in G. Note that $N(A \cap H)^H$ is normal in G and A is contained in $N(A \cap H)^H$. Therefore $G = N\big((A \cap H)^H(B \cap H)\big)$. Since $(A \cap H)^H$ and $B \cap H$ are $\mathfrak{F}$-subnormal in G, it follows that $(A \cap H)^H(B \cap H)$ is an $\mathfrak{F}$-subnormal $\mathfrak{F}$-subgroup of G. Applying Proposition 6.1.11, $G \in \mathfrak{F}$ and we have reached the desired contradiction. Therefore $G \in \mathfrak{F}$. We have proved that 1 implies 2.

2 implies 3. Suppose that G is a group such that $G = N_1N_2$ with $N_i \trianglelefteq G$ and $N_i \in \mathfrak{F}$ for $i = 1, 2$. Then $N_i \in \mathrm{E\,K}(\mathfrak{F})$, $i = 1, 2$, and so $G \in \mathrm{E\,K}(\mathfrak{F})$. Applying Proposition 6.1.10, N_i are $\mathfrak{F}$-subnormal in G for $i = 1, 2$. By Statement 2, $G \in \mathfrak{F}$ and we have shown that $\mathfrak{F}$ is N_0-closed. Therefore $\mathfrak{F}$ is a Fitting class because $\mathfrak{F}$ is subgroup-closed.

Let G be a group and $A = \langle X \in \mathfrak{F} : X \text{ is } \mathfrak{F}\text{-subnormal in } G\rangle$. Then A is normal in G and $A \in \mathfrak{F}$ by Statement 2. Hence A is contained in the $\mathfrak{F}$-radical $G_{\mathfrak{F}}$ of G.

3 implies 1. Suppose that $\mathfrak{F}$ is not a lattice formation and derive a contradiction. Let G be a counterexample with least possible order. Then G has two $\mathfrak{F}$-subnormal subgroups U and V such that $\langle U, V\rangle$ is not $\mathfrak{F}$-subnormal. If N is a minimal normal subgroup of G, then $\langle U, V\rangle N/N$ is $\mathfrak{F}$-subnormal in G/N by Lemma 6.1.6 (3). Hence $\langle U, V\rangle N$ is $\mathfrak{F}$-subnormal in G by Lemma 6.1.6 (2). Assume that $\langle U, V\rangle N$ is a proper subgroup of G. Then U and V are $\mathfrak{F}$-subnormal in $\langle U, V\rangle N$ by Lemma 6.1.7 (2). Hence $\langle U, V\rangle$ is $\mathfrak{F}$-subnormal in $\langle U, V\rangle N$ by the minimal choice of G. Therefore $\langle U, V\rangle$ is $\mathfrak{F}$-subnormal in G, contrary to supposition. Hence $G = \langle U, V\rangle N$ for every minimal normal subgroup N of G.

On taking N contained in $\mathrm{Core}_G(\langle U, V\rangle)$, if this is non-trivial, we can conclude $G = \langle U, V\rangle$. This is not possible. Thus $\mathrm{Core}_G(\langle U, V\rangle) = 1$. On the other hand, $U^{\mathfrak{F}}$ and $V^{\mathfrak{F}}$ are subnormal in G by Lemma 6.1.9 (1) and so N normalises $\langle U^{\mathfrak{F}}, V^{\mathfrak{F}}\rangle$ by [DH92, A, 14.3 and 14.4]. Hence $\big\langle \langle U^{\mathfrak{F}}, V^{\mathfrak{F}}\rangle^G \big\rangle = \big\langle \langle U^{\mathfrak{F}}, V^{\mathfrak{F}}\rangle^{\langle U,V\rangle} \big\rangle \leq \mathrm{Core}_G(\langle U, V\rangle) = 1$. This yields $U \in \mathfrak{F}$ and $V \in \mathfrak{F}$. By Statement 3, U and V are contained in $G_{\mathfrak{F}}$ and so $G = G_{\mathfrak{F}}N$. On taking $N \leq G_{\mathfrak{F}}$, we conclude that $G = G_{\mathfrak{F}} \in \mathfrak{F}$. In particular, $\langle U, V\rangle$ is $\mathfrak{F}$-subnormal in G. This is the final contradiction. □

Corollary 6.3.4. *Let $\mathfrak{F}$ be a saturated lattice formation. If $G \in \mathrm{E\,K}(\mathfrak{F})$, then $G_{\mathfrak{F}} = \langle X \in \mathfrak{F} : X \text{ is } \mathfrak{F}\text{-subnormal in } G\rangle$.*

Proof. Applying Proposition 6.1.10, every subnormal subgroup of G is $\mathfrak{F}$-subnormal. Hence $G_{\mathfrak{F}} \leq \langle X \in \mathfrak{F} : X \text{ is } \mathfrak{F}\text{-subnormal in } G\rangle$ and the equality holds by Theorem 6.3.3 (3). □

Remark 6.3.5. If $\mathfrak{F} = \mathfrak{S}_p$ for some prime p, there exist groups G such that $1 = \langle X \in \mathfrak{F} : X \text{ is } \mathfrak{F}\text{-subnormal in } G\rangle < G_{\mathfrak{F}} = \mathrm{O}_p(G) < G^{\mathfrak{F}} = \mathrm{O}^p(G)$.

A well-known result of Baer asserts that if p is a prime, then a p-element x of a group G lies in $\mathrm{O}_p(G)$ if, and only if, any two conjugates of x generate a p-subgroup of G. As a consequence a subgroup H of a group G is contained in the Hall π-subgroup of $\mathrm{F}(G)$, π a set of primes, if, and only if, $\langle H, H^g\rangle$ is a nilpotent π-group for every $g \in G$ ([DH92, A, 14.11]). This result does not hold for saturated Fitting formations. For instance, if $\mathfrak{F}$ is the class of all groups with nilpotent length at most 2 and $G = \mathrm{Sym}(4)$, then $\langle H, H^g\rangle \in \mathfrak{F}$ for every subgroup H generated by a transposition and every $g \in G$. However H is not contained in $\mathrm{Alt}(4) = G_{\mathfrak{N}^2}$.

Our next theorem shows that lattice formations $\mathfrak{F}$ do enjoy the above property in groups with soluble residual. This result was proved in the soluble universe in [BBDPR92].

Theorem 6.3.6. *Let $\mathfrak{F}$ be a lattice formation of characteristic π. For a subgroup H of a π-group $G \in \mathfrak{S}\mathfrak{F}$, the following statements are equivalent:*

1. *H is contained in the $\mathfrak{F}$-radical $G_{\mathfrak{F}}$ of G;*
2. *$\langle H, H^g\rangle$ is an $\mathfrak{F}$-group for every $g \in G$.*

Proof. 1 implies 2. If H is contained in $G_{\mathfrak{F}}$, then $\langle H, H^g\rangle \le G_{\mathfrak{F}}$ for all $g \in G$. Hence $\langle H, H^g\rangle$ is an $\mathfrak{F}$-group for all $g \in G$.

2 implies 1. By Lemma 6.1.7 (1), the subgroup H is $\mathfrak{F}$-subnormal in $\langle H, H^g\rangle$ for all $g \in G$. By Theorem 6.2.17, H is $\mathfrak{F}$-subnormal in G. Since $H \in \mathfrak{F}$, it follows that $H \le G_{\mathfrak{F}}$ by Theorem 6.3.3 (3). □

Lemma 6.3.7. *Let $\mathfrak{F}$ be a K-lattice formation. Then $\mathfrak{F}$ is a lattice formation.*

Proof. Assume the result is false and let G be a group of minimal order among the groups X for which $\mathrm{s}_{n\mathfrak{F}}(X)$ is not a sublattice of the subgroup lattice of X. Then G has two $\mathfrak{F}$-subnormal subgroups U and V such that $\langle U, V\rangle$ is not $\mathfrak{F}$-subnormal in G. Let N be a minimal normal subgroup of G. Then $\langle U, V\rangle N$ is $\mathfrak{F}$-subnormal in G by Lemma 6.1.6 (3) and Lemma 6.1.6 (2). Put $\mathfrak{H} = \mathrm{E\,K}(\mathfrak{F})$. Applying Lemma 6.1.9 (2), $U^{\mathfrak{H}} = V^{\mathfrak{H}} = G^{\mathfrak{H}}$. If $G \notin \mathfrak{H}$, then $N \le G^{\mathfrak{H}}$ and so $\langle U, V\rangle = \langle U, V\rangle N$. This contradiction yields $G \in \mathfrak{H}$ and so $\mathrm{s}_{n\mathfrak{F}}(G) = \mathrm{s}_{n\mathrm{K}\text{-}\mathfrak{F}}(G)$ by Proposition 6.1.10. We have reached a contradiction. Therefore $\mathfrak{F}$ is a lattice formation. □

Lemma 6.3.8. *Let $\mathfrak{F}$ be a saturated K-lattice formation. Then every K-$\mathfrak{F}$-subnormal $\mathfrak{F}$-subgroup of a group G is contained in the $\mathfrak{F}$-radical of G.*

Proof. We proceed by induction on $|G|$; we may clearly suppose that $G \notin \mathfrak{F}$. Let $1 \ne H$ be a K-$\mathfrak{F}$-subnormal subgroup of G such that $H \in \mathfrak{F}$. Let N be a minimal normal subgroup of G. By Lemma 6.1.6 (3), HN/N is K-$\mathfrak{F}$-subnormal in G/N. Applying induction $HN/N \le (G/N)_{\mathfrak{F}} = A/N$. If A is a proper subgroup of G, then H is contained in $A_{\mathfrak{F}}$ because H is K-$\mathfrak{F}$-subnormal in A by Lemma 6.1.7 (2). Since $A_{\mathfrak{F}}$ is a normal $\mathfrak{F}$-subgroup of G, it follows that $A_{\mathfrak{F}} \leq G_{\mathfrak{F}}$. Therefore H is contained in $G_{\mathfrak{F}}$. There remains the possibility that $G/N \in \mathfrak{F}$ for all minimal normal subgroups N of G. Then G is in the boundary of $\mathfrak{F}$ and so G is a primitive group and $N = G^{\mathfrak{F}}$ is the unique minimal normal subgroup of G. Assume that $N \notin \mathfrak{F}$. Then $(HN)_{\mathfrak{F}} \cap N = 1$. This implies that $(HN)_{\mathfrak{F}} \leq \mathrm{C}_G(N) \leq N$. On the other hand, H is a proper subgroup of G and therefore H is contained in a proper subgroup M of G such that either $M \trianglelefteq G$ or $G^{\mathfrak{F}} \leq M$. In both cases $N \leq M$ and so HN is a proper subgroup of G. By induction, $H \leq (HN)_{\mathfrak{F}} \leq (HN)_{\mathfrak{F}} \cap N = 1$. This contradiction implies that $N \in \mathfrak{F}$. Therefore $G \in \mathrm{E\,K}(\mathfrak{F})$ and so H is $\mathfrak{F}$-subnormal in G by Proposition 6.1.10. In this case, H is contained in $G_{\mathfrak{F}}$ by Lemma 6.3.7 and Theorem 6.3.3 (3). This is the final contradiction. □

Theorem 6.3.9. *Let $\mathfrak{F}$ be a saturated formation. Then $\mathfrak{F}$ is a lattice formation if and only if $\mathfrak{F}$ is a K-lattice formation.*

Proof. Only the necessity of the condition is in doubt. Assume, arguing by contradiction, that $\mathfrak{F}$ is a lattice formation and there exists a group G for which $\mathrm{S}_{n\mathrm{K}\text{-}\mathfrak{F}}(G)$ is not a sublattice of the subgroup lattice of G. Furthermore let G be a group of smallest order with this property. Then G has two K-$\mathfrak{F}$-subnormal subgroups U and V such that $\langle U, V\rangle$ is not K-$\mathfrak{F}$-subnormal in G. Let N be a minimal normal subgroup of G. Since UN/N and VN/N are K-$\mathfrak{F}$-subnormal in G by Lemma 6.1.6 (3), it follows that $\langle U, V\rangle N/N$ is K-$\mathfrak{F}$-subnormal in G/N by the minimal choice of G. Hence $\langle U, V\rangle N$ is K-$\mathfrak{F}$-subnormal in G by Lemma 6.1.6 (2). If $\langle U, V\rangle N$ were a proper subgroup of G, then $\langle U, V\rangle$ would be K-$\mathfrak{F}$-subnormal in $\langle U, V\rangle N$ by minimality of G (note that U and V are K-$\mathfrak{F}$-subnormal in $\langle U, V\rangle$ by Lemma 6.1.7 (2)). Applying Lemma 6.1.6 (1), $\langle U, V\rangle$ is K-$\mathfrak{F}$-subnormal in G. This contradiction yields $G = \langle U, V\rangle N$ for every minimal normal subgroup N of G. In particular, $\mathrm{Core}_G(\langle U, V\rangle) = 1$. By Lemma 6.1.9 (1), $U^{\mathfrak{F}}$ and $V^{\mathfrak{F}}$ are subnormal subgroups of G. Therefore $\langle U^{\mathfrak{F}}, V^{\mathfrak{F}}\rangle = D$ is subnormal in G and $\mathrm{Soc}(G) \leq \mathrm{N}_G(D)$ by [DH92, A, 14.3 and 14.4]. Hence $D^G = D^{\langle U,V\rangle N} = D^{\langle U,V\rangle} \leq \langle U, V\rangle$. This means that $D^G \leq \mathrm{Core}_G(\langle U, V\rangle) = 1$. Hence U and V belong to $\mathfrak{F}$. Applying Lemma 6.3.8, $\langle U, V\rangle$ is contained in $G_{\mathfrak{F}}$. Hence $G = G_{\mathfrak{F}}N$ for every minimal normal subgroup N of G. In particular, $G = G_{\mathfrak{F}}$ and $\langle U, V\rangle$ is K-$\mathfrak{F}$-subnormal in G by Lemma 6.1.7 (1). This is the desired contradiction. □

Lemma 6.3.10. *Let $\{\mathfrak{F}_i : i \in \mathcal{I}\}$ be a family of saturated lattice formations. Then $\mathfrak{F} = \bigcap_{i\in\mathcal{I}} \mathfrak{F}_i$ is a saturated lattice formation.*

Proof. It is sufficient to see that $\mathfrak{F}$ satisfies Statement 3 of Theorem 6.3.3. It is clear that $\mathfrak{F}$ is a saturated Fitting formation. Moreover $X^{\mathfrak{F}_i}$ is contained in $X^{\mathfrak{F}}$ for every group X, $i \in \mathcal{I}$. Hence every $\mathfrak{F}$-subnormal subgroup is $\mathfrak{F}_i$-subnormal for all $i \in \mathcal{I}$ by Lemma 6.1.7 (1).

Let G be a group and let H be an $\mathfrak{F}$-subnormal $\mathfrak{F}$-subgroup of G. Then H is an $\mathfrak{F}_i$-subnormal $\mathfrak{F}_i$-subgroup of G for every $i \in \mathcal{I}$. By Theorem 6.3.3 (3),

H is contained in $\bigcap_{i\in\mathcal{I}} G_{\mathfrak{F}_i}$, which is a normal $\mathfrak{F}$-subgroup of G because $\mathfrak{F}_i$ is subgroup-closed for every $i \in \mathcal{I}$. Therefore H is contained in $G_{\mathfrak{F}}$ and $\mathfrak{F}$ is a lattice formation. □

Lemma 6.3.11. *Let $\mathcal{I}$ be a non-empty set. For each $i \in \mathcal{I}$, let $\mathfrak{F}_i$ be a subgroup-closed saturated lattice formation. Assume that $\pi(\mathfrak{F}_i) \cap \pi(\mathfrak{F}_j) = \emptyset$ for all i, $j \in \mathcal{I}$, $i \neq j$. Then $\mathfrak{F} = \times_{i\in\mathcal{I}} \mathfrak{F}_i$ is a subgroup-closed saturated lattice formation.*

Proof. By Remark 2.2.13, $\mathfrak{F}$ is a subgroup-closed saturated formation.

Assume that $\mathfrak{F}$ does not satisfy Statement 2 of Theorem 6.3.3 and derive a contradiction. Let G be a counterexample of minimal order. Then G has two $\mathfrak{F}$-subnormal $\mathfrak{F}$-subgroups A and B such that $\langle A, B\rangle$ is not an $\mathfrak{F}$-group. Then obviously $A \neq 1$ and $B \neq 1$. Observe that $\langle A, B\rangle$ and any epimorphic image of G inherits the conditions of G. Therefore $G = \langle A, B\rangle$ and $G/N \in \mathfrak{F}$ for every minimal normal subgroup N of G. Since $G \notin \mathfrak{F}$, it follows that $N = G^{\mathfrak{F}}$ is the unique minimal normal subgroup of G and $\mathrm{C}_G(N) \leq N$. Since A is $\mathfrak{F}$-subnormal in AN by Lemma 6.1.7 (2) and N is a quasinilpotent normal subgroup of G, it follows that AN belongs to $\mathfrak{F}$ by Proposition 6.1.11. Hence there exists $i \in \mathcal{I}$ such that $N \in \mathfrak{F}_i$. Moreover, $\mathrm{C}_G(N) \leq N$ forces $AN \in \mathfrak{F}_i$. The same arguments can be applied to B. We then conclude that AN, $BN \in \mathfrak{F}_i$. Since $G/N \in \mathfrak{F}$ and $\mathfrak{F} = \times_{i\in\mathcal{I}} \mathfrak{F}_i$, it follows that G/N has a normal $\pi(\mathfrak{F}_i)$-Hall subgroup. Since AN/N and BN/N are $\pi(\mathfrak{F}_i)$-groups, we have that G/N is a $\pi(\mathfrak{F}_i)$-group. In particular, G is a $\pi(\mathfrak{F}_i)$-group and so A and B are $\mathfrak{F}_i$-subnormal $\mathfrak{F}_i$-subgroups of G. Therefore $G = \langle A, B\rangle \in \mathfrak{F}_i \subseteq \mathfrak{F}$ by Theorem 6.3.3 (2). This contradiction confirms that $\mathfrak{F}$ is a lattice formation. □

Let $\mathfrak{Z}$ be a class of groups. A group G is called S-*critical for* $\mathfrak{Z}$, or simply $\mathfrak{Z}$-*critical*, if G is not in $\mathfrak{Z}$ but all proper subgroups of G are in $\mathfrak{Z}$. Critical groups associated with some classes of groups will play a central role in Section 6.4.

Lemma 6.3.12. *Let $\mathfrak{F}$ be a saturated Fitting formation. Assume that each of the following conditions holds:*

1. $\mathfrak{F} = \mathfrak{S}_p\mathfrak{F}$ *for all* $p \in \operatorname{char}\mathfrak{F}$.
2. $\mathfrak{F}$ *is an* $\mathfrak{F}^2$*-normal Fitting class.*
3. *Every $\mathfrak{F}$-critical group G with $\Phi(G) = 1$ is either cyclic or G is monolithic such that* $\operatorname{Soc}(G)$ *is non-abelian and* $G/\operatorname{Soc}(G)$ *is a cyclic group of prime power order.*

Then $\mathfrak{F}$ is a lattice formation.

Proof. It will be established that every $\mathfrak{F}$-subnormal $\mathfrak{F}$-subgroup H of a group G is contained in the $\mathfrak{F}$-radical of G. This will be accomplished by induction on $|G|$, which we suppose greater than 1. Obviously we may suppose $G \notin \mathfrak{F}$ and $1 \neq H < G$. Let N be a minimal normal subgroup of G. Then HN/N is an

$\mathfrak{F}$-subnormal $\mathfrak{F}$-subgroup of G/N by Lemma 6.1.6 (3). By induction, HN/N is contained in $(G/N)_{\mathfrak{F}} = T/N$. If T is a proper subgroup of G, then H is $\mathfrak{F}$-subnormal in T by Lemma 6.1.7 (2) and H is contained in $T_{\mathfrak{F}}$ by induction. Since T is normal in G, it follows that $T_{\mathfrak{F}} \leq G_{\mathfrak{F}}$ and H is contained in $G_{\mathfrak{F}}$. Hence we may assume that $G/N \in \mathfrak{F}$ for every minimal normal subgroup N of G. This implies that G is a monolithic primitive group and $N = G^{\mathfrak{F}}$ is the unique minimal normal subgroup of G and $\mathrm{C}_G(N) \leq N$. By Lemma 6.1.7 (2) and Proposition 6.1.11, $HN = HF^*(HN)$ is an $\mathfrak{F}$-group. Hence $N \in \mathfrak{F}$ and $G \in \mathfrak{F}^2$. By Statement 2, $G_{\mathfrak{F}}$ is the $\mathfrak{F}$-injector of G.

If N were abelian, then N would be a p-group for some prime $p \in \operatorname{char} \mathfrak{F}$. Then $G \in \mathfrak{S}_p\mathfrak{F} = \mathfrak{F}$, contrary to supposition. Hence N is non-abelian. If $HG_{\mathfrak{F}}$ were a proper subgroup of G, then H would be contained in $(HG_{\mathfrak{F}})_{\mathfrak{F}}$. Thus $HG_{\mathfrak{F}} \in \mathfrak{F}$ and $HG_{\mathfrak{F}} = G_{\mathfrak{F}}$ by the $\mathfrak{F}$-maximality of $G_{\mathfrak{F}}$ in G. Consequently we may assume that $G = HG_{\mathfrak{F}}$. Let M be a maximal subgroup of G containing $G_{\mathfrak{F}}$. Then $M = (H \cap M)G_{\mathfrak{F}}$ and $H \cap M$ is $\mathfrak{F}$-subnormal in M by Lemma 6.1.7 (2). Since $H \cap M \in \mathfrak{F}$, it follows that H is contained in $M_{\mathfrak{F}}$ by induction. This forces $M \in \mathfrak{F}$ and so $M = G_{\mathfrak{F}}$ by the $\mathfrak{F}$-maximality of $G_{\mathfrak{F}}$ in G. Hence $G/G_{\mathfrak{F}}$ is a cyclic group of order p, for a prime number $p \in \operatorname{char} \mathfrak{F}$. Let H_p and J be Sylow p-subgroups of H and $G_{\mathfrak{F}}$, respectively, such that $P = H_pJ$ is a Sylow p-subgroup of G ([Hup67, VI, 4.7]). Then $G = PG_{\mathfrak{F}}$. Consider the subgroup PN of G. Since $N = G^{\mathfrak{F}}$, it follows that PN is $\mathfrak{F}$-subnormal in G by Lemma 6.1.9 (1). Moreover, PN is the product of its $\mathfrak{F}$-subnormal $\mathfrak{F}$-subgroups H_pN and JN. If $G = PN$, then H_pN is subnormal in G. Since $H_pN \in \mathfrak{F}$, it follows that $H_pN \leq G_{\mathfrak{F}}$. Consequently $G = G_{\mathfrak{F}}$, contrary to the choice of G. Hence we may assume that PN is a proper subgroup of G. By induction $PN \in \mathfrak{F}$. This implies that P is $\mathfrak{F}$-subnormal in G.

Let A be a maximal subgroup of G such that $A \neq G_{\mathfrak{F}}$. Then $G = AG_{\mathfrak{F}}$ and $A = A_p(A \cap G_{\mathfrak{F}})$ for some Sylow p-subgroup A_p of G. Without loss of generality we may assume that A_p is contained in P. Then, by Lemma 6.1.6 (1), A_p is $\mathfrak{F}$-subnormal in G because A_p is $\mathfrak{F}$-subnormal in P by Lemma 6.1.9 (4). Since A_p and $A \cap G_{\mathfrak{F}}$ are two $\mathfrak{F}$-subnormal $\mathfrak{F}$-subgroups of A, it follows that $A \in \mathfrak{F}$ by induction. Therefore G is an $\mathfrak{F}$-critical group. By Statement 3, G/N is a cyclic group of order p^α for some $\alpha \geq 1$. But then $G = PN$. This contradicts our supposition. Therefore G satisfies Statement 3 of Theorem 6.3.3 and $\mathfrak{F}$ is a lattice formation. □

Example 6.3.13. Let $\mathfrak{F} = \mathfrak{S}_\pi$ be the class of all soluble π-groups for a set of primes π. Then $\mathfrak{F}$ is a lattice formation as $\mathfrak{F}$ satisfies Statements 1–3 of Lemma 6.3.12.

There exist non-soluble saturated lattice formations as the next example due to A. F. Vasil'ev, S. F. Kamornikov, and V. N. Semenchuk [VKS93] shows:

Example 6.3.14. Let S be a non-abelian simple group with the property that if $T < S$, then T is soluble (e. g., $G = \mathrm{Alt}(5)$). Let $\mathfrak{F} = \mathfrak{S}_\pi \, \mathrm{D}_0(1, S)$, for $\pi = \pi(S)$. By Proposition 2.2.11, $\mathfrak{F}$ is a formation. Moreover, by [DH92, II, 1.9], $\mathfrak{F}$ is S_n-closed. It is not difficult to prove that $\mathfrak{F}$ is also N_0-closed and saturated. Hence $\mathfrak{F}$ is a saturated Fitting formation contained in $\mathfrak{E}_\pi$.

Suppose, by way of contradiction, that $\mathfrak{F}$ is not subgroup-closed, and choose a group G of minimal order such that $G \in \mathfrak{F}$ and G has a subgroup $H \notin \mathfrak{F}$. Let N be a normal soluble π-subgroup of G such that $G/N \in \mathrm{D}_0(1, S) \subseteq \mathfrak{F}$. If $N \neq 1$, then $HN/N \in \mathfrak{F}$ by the minimal choice of G. Hence $H \in \mathfrak{S}_\pi\mathfrak{F} = \mathfrak{F}$, contrary to supposition. Therefore $N = 1$ and $G = S_1 \times \cdots \times S_n$ is a direct product of copies of S. For $i = 1, \dots, n$, let π_i denote the projection of G onto the ith component of the direct product. Let $\mathcal{A}$ denote the subset of $\{1, \dots, n\}$ defined by

$$i \in \mathcal{A} \quad \text{if and only if} \quad \pi_i(H) = S_i.$$

Set $K = \bigcap_{i\in\mathcal{A}} \mathrm{Ker}\big((\pi_i)_H\big)$ and $K^* = \bigcap_{i\notin\mathcal{A}} \mathrm{Ker}\big((\pi_i)_H\big)$. Then $H/K \in \mathrm{D}_0(1, S)$ and H/K^* is soluble. Since $H/KK^* \in \mathrm{D}_0(1, S)$ and H/KK^* is soluble, it follows that $H = KK^* = K \times K^*$ as $K \cap K^* = 1$. Hence $H \in \mathfrak{F}$. This contradiction shows that $\mathfrak{F}$ is subgroup-closed.

Assume that $\mathfrak{F}$ is not a lattice formation and choose a group G of minimal order having an $\mathfrak{F}$-subnormal $\mathfrak{F}$-subgroup H which is not contained in $G_\mathfrak{F}$. Clearly $H \neq 1$. By familiar arguments, $G \in \mathrm{b}(\mathfrak{F})$ and so G is a monolithic primitive group. Let N be the unique minimal normal subgroup of G. If N is abelian, then $G \in \mathfrak{S}_\pi\mathfrak{F} = \mathfrak{F}$, which contradicts our assumption. Hence N is non-abelian and $\mathrm{C}_G(N) = 1$. By Lemma 6.1.7 (2) and Proposition 6.1.11, $HN = HF^*(HN)$ is an $\mathfrak{F}$-group. Since $\mathrm{C}_G(N) = 1$, it follows that HN has no normal soluble π-subgroups. Thus $HN \in \mathrm{D}_0(1, S)$ and $HN = N \leq G_\mathfrak{F}$. This is the final contradiction. Applying Theorem 6.3.3 (3), $\mathfrak{F}$ is a lattice formation. □

We have now arrived at our first main objective, namely the classification of the subgroup-closed saturated lattice formations.

Theorem 6.3.15. *Let $\mathfrak{F} = \mathrm{LF}(F)$ be a saturated formation. Then $\mathfrak{F}$ is a lattice formation if and only if $\mathfrak{F} = \mathfrak{M} \times \mathfrak{G}$ for some subgroup-closed saturated formations $\mathfrak{M}$ and $\mathfrak{G}$ satisfying the following conditions:*

1. *$\pi(\mathfrak{M}) \cap \pi(\mathfrak{G}) = \emptyset$.*
2. *There exists a set of prime numbers π^* and a partition $\{\pi_i : i \in \mathcal{I}\}$ of π^* such that $\mathfrak{G} = \bigtimes_{i\in\mathcal{I}} \mathfrak{S}_{\pi_i}$.*
3. *$\mathfrak{M} = \mathfrak{S}_p\mathfrak{M}$ for all $p \in \pi(\mathfrak{M})$ and $\mathfrak{M}$ is an $\mathfrak{M}^2$-normal Fitting class.*
4. *Every non-cyclic $\mathfrak{M}$-critical group G with $\Phi(G) = 1$ is a primitive group of type 2 such that $G/\mathrm{Soc}(G)$ is a cyclic group of prime power order.*

Proof. First of all, applying Proposition 3.1.40, $F(p)$ is a subgroup-closed formation for every prime $p \in \pi = \mathrm{char}\,\mathfrak{F}$.

Assume that $\mathfrak{F}$ is a lattice formation. For the ease of reading we break the argument into separately-stated steps.

1. *For each $p \in \pi$, every primitive group G of type 1 in $\mathfrak{F} \cap \mathrm{b}\big(F(p)\big)$ is cyclic.*

It is clear that $N = \mathrm{Soc}(G)$, the unique minimal normal subgroup of G, is a q-group for some prime $q \neq p$. By [DH92, B, 10.9], G has an irreducible and faithful G-module V over $\mathrm{GF}(p)$. We claim that G has a unique core-free maximal subgroup, which provides the result.

Suppose that M_1 and M_2 are maximal subgroups of G such that $M_1 \neq M_2$ and $\mathrm{Core}_G(M_i) = 1$ for $i = 1, 2$ and derive a contradiction. Then $M_i \in F(p)$, $i = 1, 2$. Consider the semidirect product $H = [V]G$, with respect to the action of G on V. Clearly $H \notin \mathfrak{F}$ because $G \notin F(p)$. Hence $H^{\mathfrak{F}} = V$ and G is not $\mathfrak{F}$-subnormal in H. But for $i = 1, 2$, VM_i is an $\mathfrak{F}$-normal maximal subgroup of H, and M_i is $\mathfrak{F}$-subnormal in VM_i because $VM_i \in \mathfrak{S}_p F(p) = F(p) \subseteq \mathfrak{F}$, that is, M_i is $\mathfrak{F}$-subnormal in H (Lemma 6.1.6 (1) and Lemma 6.1.7 (1)). Since $\mathfrak{F}$ is a lattice formation, it follows that $G = \langle M_1, M_2 \rangle$ is $\mathfrak{F}$-subnormal in H, contrary to supposition.

2. *If p and q belong to π, and $q \in \mathrm{char}\, F(p)$, then $p \in \mathrm{char}\, F(q)$.*

Assume that $C_p \notin F(q)$ and consider an irreducible and faithful C_q-module V over $\mathrm{GF}(p)$ ([DH92, B, 10.9]). Then the semidirect product $[V]C_q$, with respect to the action of C_q on V, is a non-cyclic primitive group of type 1 in $\mathfrak{F} \cap \mathrm{b}\big(F(p)\big)$. This contradicts Step 1. Therefore $C_p \in F(q)$ and $p \in \mathrm{char}\, F(q)$.

3. *If p, $q \in \pi$ and $p \in \mathrm{char}\, F(q)$, then $\mathrm{char}\, F(p) = \mathrm{char}\, F(q)$.*

If $r \in \mathrm{char}\, F(q) \setminus \mathrm{char}\, F(p)$, then $r \neq q$ and $C_q \in F(r)$, because of Step 2. Consider an irreducible and faithful C_q-module V over $\mathrm{GF}(r)$. Then $[V]C_q \in \mathfrak{F} \cap \mathrm{b}\big(F(p)\big)$ and $[V]C_q$ is non-cyclic primitive group of type 1. This contradicts Step 1. Therefore $\mathrm{char}\, F(q) \subseteq \mathrm{char}\, F(p)$ and analogously $\mathrm{char}\, F(p) \subseteq \mathrm{char}\, F(q)$.

4. *If p, $q \in \pi$ and $p \in \mathrm{char}\, F(q)$, then $\mathfrak{S}_p \subseteq F(q)$.*

Since $F(q)$ is subgroup-closed, and a p-group of order p^n is isomorphic with a subgroup of the n-fold iterated wreath product $H_n = \big(\dots(C_p \wr C_p)\dots\big) \wr C_p$, it is enough to prove that $H_n \in F(q)$ for all $n \in \mathbb{N}$. Denote inductively $H_1 = C_p$ and $H_n = H_{n-1} \wr C_p$ for $n \geq 2$. We can assume that $p \neq q$. Since $\mathrm{Z}(H_n)$ is cyclic, H_n has a unique minimal normal subgroup, and consequently there exists an irreducible and faithful H_n-module V over $\mathrm{GF}(q)$ by [DH92, B, 10.9]. Consider the semidirect product $G = [V]H_n$, with respect to the action of H_n on V. If $(H_{n-1})^{\natural}$ denotes the base group of H_n, then $H_n = (H_{n-1})^{\natural} C_p$. Since $(H_{n-1})^{\natural}$ and C_p are $F(q)$-groups, it follows that $V(H_{n-1})^{\natural}$ and VC_p belong to $F(q)$. Moreover they are $\mathfrak{F}$-subnormal in G. Hence $G \in \mathfrak{F}$ by Theorem 6.3.3 (2) and so $H_n \in F(q)$.

5. *If p, $q \in \pi$ and $q \in \mathrm{char}\, F(q)$, then $\mathfrak{S}_p F(q) = F(q)$.*

Assume that $F(q) \neq \mathfrak{S}_p F(q)$ and derive a contradiction. Let G be a group of minimal order in the supposed non-empty class $\mathfrak{S}_p F(q) \setminus F(q)$. Then, since $F(q)$ is a subgroup-closed formation, G has a unique minimal normal subgroup, M say, and $G/M \in F(q)$. Moreover M is a p-group and every maximal subgroup of G belongs to $F(q)$. If $M \leq \Phi(G)$, then $G \in \mathfrak{F}$ and we may argue as in Step 1 to obtain that G is cyclic. Consequently $G \in F(q)$ by Step 4.

This contradiction implies that M is not contained in $\Phi(G)$. Let R be a maximal subgroup of G such that $G = MR$. Then $M \cap R = 1$, $R \in F(q)$ and $M = G^{\mathfrak{F}}$. Clearly we may assume that $p \neq q$. Hence considering a faithful and irreducible G-module over GF(q), it is rather clear that R must be a cyclic r-group for some prime r, $r \in \pi$. From Step 3 and Step 4, it follows that $G \in \mathfrak{S}_p\mathfrak{S}_r \subseteq \mathfrak{S}_pF(p) = F(p) \subseteq \mathfrak{F}$. This is the desired contradiction. Therefore $F(q) = \mathfrak{S}_pF(q)$.

Calling two elements p, $q \in \pi$ *equivalent* if and only if $\operatorname{char} F(p) = \operatorname{char} F(q)$, we obtain an equivalence relation on π whose equivalence classes $\{\pi_i : i \in \mathcal{I}\}$ form a partition of π.

Let $p \in \pi_i$, $i \in \mathcal{I}$. Since $F(p)$ is a subgroup-closed formation, it follows that $F(p) \subseteq \mathfrak{E}_{\pi_i}$. If $2 \notin \pi_i$, then every group in $\mathfrak{E}_{\pi_i}$ is soluble by the Odd Order Theorem [FT63]. Therefore $F(p) \subseteq \mathfrak{S}_{\pi_i}$. In fact, we have:

6. *If $p \in \pi_i$, $i \in \mathcal{I}$, and $2 \notin \pi_i$, then $F(p) = \mathfrak{S}_{\pi_i}$.*

Assume that $F(p) \neq \mathfrak{S}_{\pi_i}$ and choose a group $G \in \mathfrak{S}_{\pi_i} \setminus F(p)$ of minimal order. Then G has a unique minimal normal subgroup N, N is a q-group for some prime $q \in \pi_i$, and $G/N \in F(p)$. By Step 5, $G \in \mathfrak{S}_qF(p) = F(p)$. This contradiction forces $F(p) = \mathfrak{S}_{\pi_i}$.

Put $\mathfrak{M} = (1)$ if $2 \notin \pi$ and $\mathfrak{M} = \mathfrak{F} \cap \mathfrak{E}_{\pi_{i_0}}$ if $2 \in \pi_{i_0}$ for some $i_0 \in I$.

Assume that $2 \in \pi$. Then $\{\pi_i : i \neq i_0\}$ is a partition of $\pi^* = \pi \setminus \pi_{i_0}$. Let $\mathfrak{G} = \times_{i \in \mathcal{I}\setminus\{i_0\}} \mathfrak{S}_{\pi_i}$. Then $\mathfrak{G}$ is a subgroup-closed lattice-formation by Lemma 6.3.11 and Example 6.3.13.

7. $\mathfrak{F} = \mathfrak{M} \times \mathfrak{G}$, $\pi(\mathfrak{M}) \cap \pi(\mathfrak{G}) = \emptyset$.

It is clear that $\mathfrak{M} \times \mathfrak{G}$ is contained in $\mathfrak{F}$. Suppose, for a contradiction, that this inclusion is proper, and choose a group G of minimal order in $\mathfrak{F} \setminus (\mathfrak{M} \times \mathfrak{G})$. Then G is a monolithic primitive group because $\mathfrak{M} \times \mathfrak{G}$ is a saturated formation. Let N be the unique minimal normal subgroup of G. Suppose that N is non-abelian. Then 2 divides $|N|$ and $G \in F(2) \subseteq \mathfrak{F} \cap \mathfrak{E}_{\pi_{i_0}} = \mathfrak{M}$, contrary to our choice of G. Therefore N is abelian. Let p be the prime dividing $|N|$. Then $G \in F(p)$. If $p \in \pi_{i_0}$, it follows that $G \in \mathfrak{M}$. If $p \in \pi_i$ for some $i \in \mathcal{I}\setminus\{i_0\}$, we have $G \in \mathfrak{G}$. In both cases, $G \in \mathfrak{M} \times \mathfrak{G}$, another contradiction.

Evidently, $\pi_{i_0} = \pi(\mathfrak{M})$ and $\pi^* = \pi \setminus \pi_{i_0} = \pi(\mathfrak{G})$. Hence $\pi(\mathfrak{M}) \cap \pi(\mathfrak{G}) = \emptyset$.

8. $\mathfrak{M} = \mathfrak{S}_p\mathfrak{M}$ *for all $p \in \pi(\mathfrak{M})$.*

Let $p \in \pi(\mathfrak{M})$. Assume that $\mathfrak{S}_p\mathfrak{M}$ is not contained in $\mathfrak{M}$ and derive a contradiction. Let $G \in \mathfrak{S}_p\mathfrak{M} \setminus \mathfrak{M}$ be a group of minimal order. By familiar reasoning, G is a primitive group of type 1 and $N = \operatorname{Soc}(G)$ is an abelian p-group. Our eventual goal is to show that every core-free maximal subgroup M of G is cyclic. Suppose that M_1 and M_2 are maximal subgroups of M such that $M_1 \neq M_2$. Since $\mathfrak{S}_p\mathfrak{M}$ is subgroup-closed, it follows that $NM_i \in \mathfrak{M} \subseteq \mathfrak{F}$ for $i = 1, 2$. Moreover, NM_i are $\mathfrak{F}$-subnormal in G because $N = G^{\mathfrak{F}}$ (Lemma 6.1.7 (1)). By Theorem 6.3.3 (2), $G = N\langle M_1, M_2\rangle \in \mathfrak{F}$, which contradicts the assumption that $G \notin \mathfrak{M}$. Therefore M has a unique maximal subgroup and so M is a cyclic group of prime power order. By Step 4, $M \in F(q)$, where $q \in \pi(M)$. Therefore $G \in \mathfrak{S}_pF(q) = F(q)$ by Step 5. We conclude then that $G \in \mathfrak{F} \cap \mathfrak{E}_{\pi_{i_0}}$. This final contradiction completes the proof.

9. $\mathfrak{M}$ *is an* $\mathfrak{M}^2$*-normal Fitting class.*

It is clear that $\mathfrak{M}$ is a Fitting class. Let $G \in \mathfrak{M}^2$ be a group and let J be an $\mathfrak{M}$-maximal subgroup of G containing $G_{\mathfrak{M}}$. Since $G/G_{\mathfrak{M}} \in \mathfrak{M} \subseteq \mathfrak{F}$, it follows that $G^{\mathfrak{F}} \leq G_{\mathfrak{M}}$. Consequently J is $\mathfrak{F}$-subnormal in G by Lemma 6.1.7 (1). Applying Theorem 6.3.3 (3), $J \leq G_{\mathfrak{F}} = G_{\mathfrak{M}}$ because $G \in \mathfrak{E}_{\pi_{i_0}}$. Therefore $G_{\mathfrak{F}}$ is $\mathfrak{F}$-maximal in G. Let H be a subnormal subgroup of G. Then H is actually $\mathfrak{F}$-subnormal because $G \in \mathrm{E\,K}(\mathfrak{F})$ (Proposition 6.1.10). In addition, $H \cap G_{\mathfrak{F}}$ is contained in $H_{\mathfrak{F}}$ as $H \cap G_{\mathfrak{F}}$ is an $\mathfrak{F}$-subnormal $\mathfrak{F}$-subgroup of H (Lemma 6.1.7 (2) and Theorem 6.3.3 (3)). Consequently $H \cap G_{\mathfrak{F}} = H_{\mathfrak{F}} = H_{\mathfrak{M}}$ and $H_{\mathfrak{M}} = H \cap G_{\mathfrak{M}}$ is $\mathfrak{M}$-maximal in H. This means that $G_{\mathfrak{M}}$ is an $\mathfrak{M}$-injector of G.

10. *If G is a non-cyclic $\mathfrak{M}$-critical group and $\Phi(G) = 1$, then G is a primitive group of type 2 such that* $\mathrm{Soc}(G)$ *is non-abelian and* $G/\mathrm{Soc}(G)$ *is a cyclic group of prime power order.*

Let G be a non-cyclic $\mathfrak{M}$-critical group such that $\Phi(G) = 1$. Then G is a monolithic primitive group because $\mathfrak{M}$ is saturated. Assume that $N = \mathrm{Soc}(G)$ is abelian. Then $N < G$ because G is non-cyclic, and so N is a p-group for some prime $p \in \pi(\mathfrak{M})$. Hence $G \in \mathfrak{S}_p\mathfrak{M} = \mathfrak{M}$ by Step 8. This contradiction implies that N is non-abelian. Suppose that $N < G$. Let M_1 and M_2 be two different maximal subgroups of G containing N. Then $M_i \in \mathfrak{M}$ and M_i is $\mathfrak{F}$-subnormal in G, $i = 1, 2$, as $N = G^{\mathfrak{F}}$ (Lemma 6.1.7 (1)). Applying Theorem 6.3.3 (2), $G \in \mathfrak{F}$. Since $G \in \mathfrak{E}_{\pi_{i_0}}$, it follows that $G \in \mathfrak{M}$, contrary to the assumption that G is $\mathfrak{M}$-critical. This contradiction proves that G/N has a unique maximal subgroup and so G/N is cyclic of prime power order.

Applying Lemma 6.3.12, $\mathfrak{M}$ is a lattice formation.

Conversely, assume that $\mathfrak{F} = \mathfrak{M} \times \mathfrak{G}$ for subgroup-closed saturated formations $\mathfrak{M}$ and $\mathfrak{G}$ satisfying Statements 1 to 4. By Example 6.3.13, Lemma 6.3.11, and Lemma 6.3.12, $\mathfrak{F}$ is a lattice formation. □

Corollary 6.3.16. *Let $\mathfrak{F}$ be a saturated formation of soluble groups of characteristic π. Then $\mathfrak{F}$ is a lattice formation if and only if there exists a partition $\{\pi_i : i \in I\}$ of π such that $\mathfrak{F} = \bigtimes_{i\in I} \mathfrak{S}_{\pi_i}$.*

Corollary 6.3.16 holds not only for subgroup-closed saturated formations but also for S_n-closed saturated ones. This was proved in [VKS93].

Lockett [Loc71] described the $\mathfrak{F}$-injectors of soluble π-groups, here $\mathfrak{F}$ is a lattice formation of soluble groups of characteristic π. It turns out that if G is a soluble π-group, the $\mathfrak{F}$-injectors of G are exactly the $\mathfrak{F}$-maximal subgroups of G containing $G_{\mathfrak{F}}$, that is, $\mathfrak{F}$ is a dominant Fitting class in $\mathfrak{S}_\pi$.

Theorem 6.3.17 ([Loc71]). *Let π be a non-empy set of primes and let G be a soluble π-group. Assume that $\{\pi_i : i \in \mathcal{I}\}$ is a partition of π and $\mathfrak{F} = \bigtimes_{i\in\mathcal{I}} \mathfrak{S}_{\pi_i}$. For each $i \in \mathcal{I}$, let V_i be a Hall π_i-subgroup of $C_i = \mathrm{C}_G\big(\bigtimes_{j\neq i} \mathrm{O}_{\pi_j}(G)\big)$. Then:*

1. $[V_i, V_j] = 1$ *if* $i \neq j$,
2. *the subgroup* $\langle V_i : i \in \mathcal{I}\rangle = \mathsf{X}_{i\in\mathcal{I}} V_i$ *is an* $\mathfrak{F}$*-subgroup of* G *containing* $G_{\mathfrak{F}} = \mathsf{X}_{i\in\mathcal{I}} \mathrm{O}_{\pi_i}(G)$.

Let $\mathrm{I}(G)$ *be the set of all such subgroups* $\langle V_i : i \in \mathcal{I}\rangle$ *obtained from the various choices of* $V_i \in \mathrm{Hall}_{\pi_i}(C_i)$. *Then*

3. *if* W *is an* $\mathfrak{F}$*-subgroup of* G *containing* $G_{\mathfrak{F}}$, *then* $W \leq V$ *for some* $V \in \mathrm{I}(G)$.
4. $\mathrm{I}(G) = \mathrm{Inj}_{\mathfrak{F}}(G)$.

Proof. It is clear that $\mathrm{F}(G)$ is contained in $G_{\mathfrak{F}} = \mathsf{X}_{i\in\mathcal{I}} \mathrm{O}_{\pi_i}(G)$ and $\mathrm{C}_G\big(\mathrm{F}(G)\big)$ is contained in $\mathrm{F}(G)$ because G is soluble.

1. Take $i, j \in I$, $i \neq j$. Then $[V_i, C_j] \leq C_i \cap C_j \leq \mathrm{C}_G(G_{\mathfrak{F}}) \leq \mathrm{C}_G\big(\mathrm{F}(G)\big) \leq \mathrm{F}(G) \leq G_{\mathfrak{F}}$. Therefore C_j normalises $V_i G_{\mathfrak{F}} = V_i \times \mathsf{X}_{i\neq j} \mathrm{O}_{\pi_j}(G)$. Since $V_i = \mathrm{O}_{\pi_i}(V_i G_{\mathfrak{F}})$, it follows that C_j normalises V_i. In particular, V_j normalises V_i. By a similar argument V_i normalises V_j. Hence $[V_i, V_j] \leq V_i \cap V_j = 1$.

2. We deduce at once from Statement 1 that $\langle V_i : i \in \mathcal{I}\rangle$ is the direct product of its Hall π_i-subgroups and also that $G_{\mathfrak{F}} \leq \langle V_i : i \in \mathcal{I}\rangle \in \mathfrak{F}$.

3. Let $i \in \mathcal{I}$. Since $W \in \mathfrak{F}$, the Hall π_i-subgroup W_i of W centralises $\mathrm{O}_{\pi_j}(W)$, which contains $\mathrm{O}_{\pi_j}(G)$ by assumption, $i \neq j$. Therefore W_i is contained in a Hall π_i-subgroup, V_i say, of $\mathrm{C}_G\big(\mathsf{X}_{j\neq i} \mathrm{O}_{\pi_i}(G)\big)$. Hence $W = \mathsf{X}_{i\in\mathcal{I}} W_i \leq \mathsf{X}_{i\in\mathcal{I}} V_i \in \mathrm{I}(G)$.

4. It is enough to prove that $\mathrm{I}(G)$ is a conjugacy class of G.

Let $V = \mathsf{X}_{i\in\mathcal{I}} V_i$ and $\bar{V} = \mathsf{X}_{i\in\mathcal{I}} \bar{V}_i$ be two typical elements of $\mathrm{I}(G)$. For each $i \in \mathcal{I}$, there exists an element $x_i \in \mathrm{C}_G\big(\mathsf{X}_{j\neq i} \mathrm{O}_{\pi_j}(G)\big)$ such that $\bar{V}_i = V_i^{x_i}$. Let $x = \prod_{i\in\mathcal{I}} x_i$, where the product may be taken in any order. If $i \neq j$, the element x_j normalises each conjugate of V_i, and therefore $V^x = \mathsf{X}_{i\in\mathcal{I}} V_i^x = \mathsf{X}_{i\in\mathcal{I}} V_i^{x_i} = \bar{V}$. □

The next result, due to A. Ballester-Bolinches, K. Doerk, and M. D. Pérez-Ramos [BBDPR92], shows that these injectors have a good behaviour with respect to $\mathfrak{F}$-subnormal subgroups.

Theorem 6.3.18. *Let* $\mathfrak{F}$ *be a lattice formation of soluble groups of characteristic* π. *If* G *is a soluble* π*-group and* V *is an* $\mathfrak{F}$*-injector of* G *and* H *is an* $\mathfrak{F}$*-subnormal subgroup of* G, *then* $V \cap H$ *is an* $\mathfrak{F}$*-injector of* H.

Proof. Assume that the result is not true and let G be a counterexample of minimal order. Clearly we may suppose that H is an $\mathfrak{F}$-normal maximal subgroup of G. Hence $G/\mathrm{Core}_G(H)$ is a π_i-group for some member π_i of $\{\pi_i : i \in \mathcal{I}\}$, where $\{\pi_i : i \in \mathcal{I}\}$ is the partition of π such that $\mathfrak{F} = \mathsf{X}_{i\in\mathcal{I}} \mathfrak{S}_{\pi_i}$. Write $\pi = \pi_i$. Then $\mathrm{Core}_G(H)$ contains every Hall π'-subgroup of G.

Note that $H_{\mathfrak{F}}$ is contained in $G_{\mathfrak{F}}$ because $H_{\mathfrak{F}}$ is an $\mathfrak{F}$-subnormal $\mathfrak{F}$-subgroup of G (Lemma 6.1.6 (1) and Theorem 6.3.3 (3)). Let V be an $\mathfrak{F}$-injector of G such that $V \cap H$ is not an $\mathfrak{F}$-injector of H. Since $H_{\mathfrak{F}}$ is contained in $V \cap H$, it follows that $V \cap H$ is not $\mathfrak{F}$-maximal in H. Let R be an $\mathfrak{F}$-maximal subgroup of H containing $V \cap H$. It is clear that R is an $\mathfrak{F}$-injector of H. Since the

Hall π'-subgroup $V_{\pi'}$ of V is contained in the Hall π'-subgroup $R_{\pi'}$ of R and $R_{\pi'}$ is contained in $\mathrm{Core}_G(H)$, it follows that $V_{\pi'} = R_{\pi'}$ as $V \cap \mathrm{Core}_G(H)$ is an $\mathfrak{F}$-injector of $\mathrm{Core}_G(H)$. On the other hand, according to Lockett's result, the Hall π-subgroup V_π of V is a Hall π-subgroup of $C = \mathrm{C}_G\big(\mathsf{X}_{j\neq i} \mathrm{O}_{\pi_j}(G)\big)$ and the Hall π-subgroup R_π of R is a Hall π-subgroup of $\mathrm{C}_H\big(\mathsf{X}_{j\neq i} \mathrm{O}_{\pi_j}(H)\big)$. Moreover $V_\pi \cap H \leq R_\pi$. Since $G/\mathrm{Core}_G(H)$ is a π-group, it follows that $\mathsf{X}_{j\neq i} \mathrm{O}_{\pi_j}(G) = \mathsf{X}_{j\neq i} \mathrm{O}_{\pi_j}(H)$ and so there exists an element $g \in C$ such that $V_\pi^g \cap H = R_\pi$. If $C = G$, then V_π is a Hall π-subgroup of G. Thus $G = \mathrm{Core}_G(H)V_\pi$ and $V_\pi^g \cap H = V_\pi^h \cap H$ for some $h \in \mathrm{Core}_G(H)$. This implies that $|V_\pi \cap H| = |R_\pi|$ and $R_\pi = V_\pi \cap H$, contrary to our supposition. Consequently C is a proper subgroup of G. Since C is normal in G, it follows that $V \cap C$ is an $\mathfrak{F}$-injector of C. Moreover $H \cap C$ is $\mathfrak{F}$-subnormal in C by Lemma 6.1.7 (2). The minimality of G yields $V \cap H \cap C$ is an $\mathfrak{F}$-injector of $H \cap C$. In particular $V \cap H \cap C = R \cap C$. Since R_π is a Hall π-subgroup of $R \cap C$, we have that $V_\pi \cap H = R_\pi$. This contradiction proves the result. □

The following result is a characterisation of saturated lattice formations of soluble groups by means of properties of Fitting type. Most of the work is already contained in the above theorem.

Theorem 6.3.19. *Let $\mathfrak{F}$ be a saturated formation of soluble groups of characteristic π. The following statements are pairwise equivalent:*

1. *$\mathfrak{F}$ is a lattice formation.*
2. *$\mathfrak{F}$ is a Fitting class satisfying that if G is a soluble π-group, V is an $\mathfrak{F}$-injector of G and H is an $\mathfrak{F}$-subnormal subgroup of G, then $V \cap H$ is an $\mathfrak{F}$-injector of H.*
3. *$\mathfrak{F}$ is a Fitting class and if H is an $\mathfrak{F}$-subnormal $\mathfrak{F}$-subgroup of a soluble π-group G, then $\langle H, H^g\rangle \in \mathfrak{F}$ for every $g \in G$.*

Proof. Applying Theorem 6.3.3 (3) and Theorem 6.3.18, we have that 1 implies 2.

Assume that Statement 2 holds. Let H be an $\mathfrak{F}$-subnormal $\mathfrak{F}$-subgroup of a soluble π-group G. If $g \in G$, then H^g is contained in every $\mathfrak{F}$-injector of G. Therefore $\langle H, H^g\rangle \in \mathfrak{F}$.

Suppose, arguing by contradiction, that the statement 3 is true but $\mathfrak{F}$ is not a lattice formation. On this supposition, by Theorem 6.3.3 (3), there exists a group G of minimal order having an $\mathfrak{F}$-subnormal $\mathfrak{F}$-subgroup $1 \neq H$ which is not contained in the $\mathfrak{F}$-radical of G. If N is a minimal normal subgroup of G, then HN/N is contained in the $\mathfrak{F}$-radical K/N of G/N by minimality of G. Since H is $\mathfrak{F}$-subnormal in K by Lemma 6.1.7 (2), and $K_{\mathfrak{F}}$ is contained in $G_{\mathfrak{F}}$, it follows that $K = G$. Hence $G/N \in \mathfrak{F}$ for every minimal normal subgroup of G. Thus G is a monolithic primitive group. Let $A = G^{\mathfrak{F}}$ be the unique minimal normal subgroup of G. By Lemma 6.1.7 (2) and Proposition 6.1.11, $T = HA = HF^*(T)$ is an $\mathfrak{F}$-group. Note that $\langle T, T^g\rangle$ is $\mathfrak{F}$-subnormal in G for all $g \in G$ because A is contained in T (Lemma 6.1.7 (1)). Applying

Statement 3, it follows that $\langle T^G \rangle \in \mathfrak{F}$. Since $\langle T^G \rangle$ is normal in G and $\mathfrak{F}$ is a Fitting class, T^G is contained in the $\mathfrak{F}$-radical of G. In particular, H is contained in $G_{\mathfrak{F}}$, which contradicts our assumption. Hence $\mathfrak{F}$ is a lattice formation. □

Remark 6.3.20. In [BBMPPR00], it is proved that an s_n-closed saturated formation of soluble groups of full characteristic satisfying Statement 3 of Theorem 6.3.19 is actually subgroup-closed. Therefore Theorem 6.3.19 hold not only for subgroup-closed saturated formations of soluble groups, but also for s_n-closed ones.

We round this section off with a characterisation of lattice formations of soluble groups.

It is not always true in general that a lattice formation of soluble groups is saturated. It is enough to consider the formation of all abelian groups. In the sequel we shall take a closer look at this family of formations, following ideas of A. F. Vasil'ev and S. F. Kamornikov [VK02].

Therefore until further notice we make the following general assumption.

Hypothesis 6.3.21. *$\mathfrak{F}$ is not only a subgroup-closed formation but also soluble.*

Let $\mathfrak{Z}_{\mathfrak{F}}$ be the class of all groups G such that every subgroup of G is $\mathfrak{F}$-subnormal in G. The basic properties of $\mathfrak{F}$-subnormal subgroups imply that $\mathfrak{Z}_{\mathfrak{F}}$ is an homomorph containing $\mathfrak{F}$.

The formation of all abelian groups shows that the inclusion could be proper. Moreover it is rather easy to see that $\mathfrak{Z}_{\mathfrak{F}} = \mathfrak{F}$ if $\mathfrak{F}$ is saturated.

We gather together in a convenient "portmanteau" lemma some relevant properties of $\mathfrak{Z}_{\mathfrak{F}}$, when $\mathfrak{F}$ is a lattice formation.

Lemma 6.3.22. *Let $\mathfrak{F}$ be a lattice formation. Then:*

1. *$\mathfrak{Z}_{\mathfrak{F}}$ is a subgroup-closed formation of soluble groups.*
2. *$\pi(\mathfrak{F}) = \pi(\mathfrak{Z}_{\mathfrak{F}})$ and $\mathfrak{Z}_{\mathfrak{F}}$ contains all nilpotent $\pi(\mathfrak{F})$-groups.*
3. *$\mathfrak{Z}_{\mathfrak{F}}$ is a Fitting class.*

Proof. 1. First we prove that $\mathfrak{Z}_{\mathfrak{F}}$ is a soluble class. Suppose, by way of contradiction, that $\mathfrak{Z}_{\mathfrak{F}}$ is not contained in $\mathfrak{S}$. Then $\mathfrak{Z}_{\mathfrak{F}} \setminus \mathfrak{S}$ is not empty. Let G be a group of minimal order in $\mathfrak{Z}_{\mathfrak{F}} \setminus \mathfrak{S}$. By familiar reasoning, G is a non-abelian simple group such that every subgroup of G is $\mathfrak{F}$-subnormal in G. Let M be a maximal subgroup of G. Then $1 \neq M$ and $G^{\mathfrak{F}}$ is contained in M because M is $\mathfrak{F}$-subnormal in G. But then $G^{\mathfrak{F}} = 1$ because G is simple. This means that $G \in \mathfrak{F}$ and so G is soluble, contrary to supposition. Hence $\mathfrak{Z}_{\mathfrak{F}}$ is composed of soluble groups.

It is clear that $\mathfrak{Z}_{\mathfrak{F}}$ is a homomorph. Let N_1 and N_2 be minimal normal subgroups of a group G such that $N_1 \cap N_2 = 1$ and $G/N_i \in \mathfrak{Z}_{\mathfrak{F}}$ for $i = 1, 2$. Let P be a Sylow subgroup of G. Then our assumption implies that PN_i/N_i is

$\mathfrak{F}$-subnormal in G/N_i for $i = 1, 2$. By Lemma 6.1.7 (3), $PN_1 \cap PN_2 = P$ is $\mathfrak{F}$-subnormal in G. Moreover since N_i/N_i is $\mathfrak{F}$-subnormal in G/N_i, it follows that N_i is $\mathfrak{F}$-subnormal in G for $i = 1, 2$ by Lemma 6.1.6 (2). Hence $1 = N_1 \cap N_2$ is $\mathfrak{F}$-subnormal in G. By Lemma 6.1.7 (2), 1 is $\mathfrak{F}$-subnormal in P. Therefore every subgroup of P is $\mathfrak{F}$-subnormal in G by Lemma 6.1.9 (3) and Lemma 6.1.6 (2). Since every subgroup of G is generated by its subgroups of prime power order, and $\mathfrak{F}$ is a lattice formation, it follows that $G \in \mathfrak{Z}_{\mathfrak{F}}$. Applying [DH92, II, 2.6], $\mathfrak{Z}_{\mathfrak{F}}$ is R_0-closed and so $\mathfrak{Z}_{\mathfrak{F}}$ is a formation.

Let $G \in \mathfrak{Z}_{\mathfrak{F}}$ and let H be a subgroup of G. Since every subgroup of H is $\mathfrak{F}$-subnormal in G, it follows by Lemma 6.1.7 (2) that $H \in \mathfrak{Z}_{\mathfrak{F}}$. Hence $\mathfrak{Z}_{\mathfrak{F}}$ is subgroup-closed.

2. It is clear that $\pi(\mathfrak{F}) \subseteq \pi(\mathfrak{Z}_{\mathfrak{F}})$ because $\mathfrak{F} \subseteq \mathfrak{Z}_{\mathfrak{F}}$. Let $p \in \pi(\mathfrak{Z}_{\mathfrak{F}})$ and let G be a group in $\mathfrak{Z}_{\mathfrak{F}}$ such that p divides $|G|$. Then $C_p \in \mathfrak{Z}_{\mathfrak{F}}$ because $\mathfrak{Z}_{\mathfrak{F}}$ is subgroup-closed. Hence 1 is $\mathfrak{F}$-subnormal in C_p and so $C_p \in \mathfrak{F}$. This shows that $p \in \pi(\mathfrak{F})$.

If P is a p-group for some prime $p \in \pi(\mathfrak{F}) = \pi(\mathfrak{Z}_{\mathfrak{F}})$, then $C_p \in \mathfrak{Z}_{\mathfrak{F}}$ because $\mathfrak{Z}_{\mathfrak{F}}$ is subgroup-closed. By Lemma 6.1.9 (4), every subgroup of P is $\mathfrak{F}$-subnormal in P. Hence $P \in \mathfrak{Z}_{\mathfrak{F}}$. This implies that every nilpotent $\pi(\mathfrak{F})$-group is a $\mathfrak{Z}_{\mathfrak{F}}$-group.

3. It is clear that only the N_0-closure of $\mathfrak{Z}_{\mathfrak{F}}$ needs checking. Let A and B be normal subgroups of a group G such that $G = AB$ and A and B belong to $\mathfrak{Z}_{\mathfrak{F}}$. We prove that $G \in \mathfrak{Z}_{\mathfrak{F}}$ by induction on the order of G. If G is nilpotent, then $G \in \mathfrak{Z}_{\mathfrak{F}}$ because $\pi(G) \subseteq \pi(\mathfrak{Z}_{\mathfrak{F}})$ and $\mathfrak{Z}_{\mathfrak{F}}$ contains all nilpotent $\pi(\mathfrak{Z}_{\mathfrak{F}})$-groups. Hence we may suppose that G is not nilpotent. Let P be a Sylow subgroup of G. Then P is the product of $P \cap A$ and $P \cap B$, which are obviously normal subgroups of P. Moreover, $P \cap A$ and $P \cap B$ are $\mathfrak{Z}_{\mathfrak{F}}$-subgroups of P because $\mathfrak{Z}_{\mathfrak{F}}$ is subgroup-closed. Since P is a proper subgroup of G, the induction hypothesis leads to the conclusion that $P \in \mathfrak{Z}_{\mathfrak{F}}$. Furthermore, G is soluble because $\mathfrak{Z}_{\mathfrak{F}}$ is composed of soluble groups. Therefore $G \in \mathrm{K\,E}(\mathfrak{F})$ as $\pi(\mathfrak{F}) = \pi(\mathfrak{Z}_{\mathfrak{F}}) = \operatorname{char} \mathfrak{Z}_{\mathfrak{F}}$. This implies that A and B are $\mathfrak{F}$-subnormal in G by Proposition 6.1.10. Since A and B belong to $\mathfrak{Z}_{\mathfrak{F}}$, it follows that $P \cap A$ and $P \cap B$ are $\mathfrak{F}$-subnormal subgroups of G by Lemma 6.1.6 (1). Therefore $P = (P \cap A)(P \cap B)$ is $\mathfrak{F}$-subnormal in G because $\mathfrak{F}$ is a lattice formation. Hence every subgroup of P is $\mathfrak{F}$-subnormal in G Lemma 6.1.6 (1). Since every subgroup of G is generated by its subgroups of prime power order, it follows that $G \in \mathfrak{Z}_{\mathfrak{F}}$. Consequently $\mathfrak{Z}_{\mathfrak{F}}$ is N_0-closed and so $\mathfrak{Z}_{\mathfrak{F}}$ is a Fitting class. □

Combining Theorem 2.5.2 and Lemma 6.3.22, we have:

Proposition 6.3.23. *Let $\mathfrak{F}$ be a lattice formation. Then $\mathfrak{Z}_{\mathfrak{F}}$ is a saturated formation.*

Theorem 6.3.24. *Let $\mathfrak{F}$ be a lattice formation. Then $\mathfrak{Z}_{\mathfrak{F}}$ is a lattice formation.*

Proof. By Theorem 6.3.3 (3), it is sufficient to prove that every $\mathfrak{Z}_{\mathfrak{F}}$-subnormal $\mathfrak{Z}_{\mathfrak{F}}$-subgroup H of a group G is contained in the $\mathfrak{Z}_{\mathfrak{F}}$-radical $G_{\mathfrak{Z}_{\mathfrak{F}}}$ of G.

Suppose, by way of contradiction, that there exists a group G of minimal order having a $\mathfrak{Z}_{\mathfrak{F}}$-subnormal $\mathfrak{Z}_{\mathfrak{F}}$-subgroup H such that H is not contained in $G_{\mathfrak{Z}_{\mathfrak{F}}}$. Among the $\mathfrak{Z}_{\mathfrak{F}}$-subnormal $\mathfrak{Z}_{\mathfrak{F}}$-subgroups of G that are not contained in $G_{\mathfrak{Z}_{\mathfrak{F}}}$, let H be one of maximal order. Let N be a minimal normal subgroup of G. Then HN/N is a $\mathfrak{Z}_{\mathfrak{F}}$-subnormal $\mathfrak{Z}_{\mathfrak{F}}$-subgroup of G/N by Lemma 6.1.6 (3). The choice of G implies that $HN/N \leq (G/N)_{\mathfrak{Z}_{\mathfrak{F}}} = L/N$. Assume that L is a proper subgroup of G. The minimality of G forces the conclusion that H is contained in $L_{\mathfrak{Z}_{\mathfrak{F}}}$ as H is $\mathfrak{Z}_{\mathfrak{F}}$-subnormal $\mathfrak{Z}_{\mathfrak{F}}$-subgroup of L. Since L is normal in G, it follows that $L_{\mathfrak{Z}_{\mathfrak{F}}}$ is contained in $G_{\mathfrak{Z}_{\mathfrak{F}}}$. This contradiction shows that $L = G$ and so $G/N \in \mathfrak{Z}_{\mathfrak{F}}$ for every minimal normal subgroup of G. Consequently G is a monolithic primitive group and $N = G^{\mathfrak{Z}_{\mathfrak{F}}}$ is the unique minimal normal subgroup of G. Moreover $\mathrm{C}_G(N) \leq N$. By Lemma 6.1.7 (2) and Proposition 6.1.11, $HN = HF^*(HN)$ is an $\mathfrak{Z}_{\mathfrak{F}}$-group. Since HN is $\mathfrak{Z}_{\mathfrak{F}}$-subnormal in G by Lemma 6.1.7 (1), it follows that $N \leq H$ by the choice of the pair (G, H). Therefore $H = N(H \cap M)$, where M is a core-free maximal subgroup of G complementing N in G. On the other hand, H/N is $\mathfrak{F}$-subnormal in G/N. Hence H is $\mathfrak{F}$-subnormal in G by Lemma 6.1.6 (2). Since $H \in \mathfrak{Z}_{\mathfrak{F}}$, it follows that $H \cap M = X$ is also $\mathfrak{F}$-subnormal in G Lemma 6.1.6 (1). Consequently X^M, the normal closure of X in M, is $\mathfrak{F}$-subnormal in G because $\mathfrak{F}$ is a lattice formation. Furthermore $X^M \in \mathfrak{Z}_{\mathfrak{F}}$ because $M \in \mathfrak{Z}_{\mathfrak{F}}$ and $\mathfrak{Z}_{\mathfrak{F}}$ is subgroup-closed. Since X^M is $\mathfrak{Z}_{\mathfrak{F}}$-subnormal in G, it follows that X^M is $\mathfrak{Z}_{\mathfrak{F}}$-subnormal in NX^M by Lemma 6.1.7 (2), and NX^M belongs to $\mathfrak{Z}_{\mathfrak{F}}$ by Proposition 6.1.11. Since NX^M is a normal subgroup of G and $\mathfrak{Z}_{\mathfrak{F}}$ is a Fitting class, it follows that $H \leq NX^M \leq G_{\mathfrak{Z}_{\mathfrak{F}}}$. This contradiction proves the theorem. □

We are now in a position to state and prove Vasil'ev and Kamornikov's characterisation of lattice formations of soluble groups.

Theorem 6.3.25 ([VK02]). *Let $\mathfrak{F}$ be a formation of soluble groups. The following statements are pairwise equivalent:*

1. *The set of all K-$\mathfrak{F}$-subnormal subgroups is a sublattice of the subgroup lattice of every group.*
2. *The set of all $\mathfrak{F}$-subnormal subgroups is a sublattice of the subgroup lattice of every group.*
3. *There exists a partition $\{\pi_i : i \in \mathcal{I}\}$ of the set $\pi(\mathfrak{F})$ such that $\mathfrak{F} = \times_{i \in \mathcal{I}} \mathfrak{F}_{\pi_i}$, where $\mathfrak{F}_{\pi_i} = \mathfrak{F} \cap \mathfrak{S}_{\pi_i}$. Moreover, $\mathfrak{F}_{\pi_i} = \mathfrak{S}_{\pi_i}$ for all $i \in \mathcal{I}$ such that $|\pi_i| > 1$.*

Proof. Of the three statements in the theorem, it follows that 1 implies 2.

Assume that Statement 2 holds. Then the preceding results show that $\mathfrak{Z}_{\mathfrak{F}}$ is a saturated lattice formation of soluble groups. Hence, by Theorem 6.3.15, there exists a partition $\{\pi_i : i \in \mathcal{I}\}$ of $\pi := \operatorname{char} \mathfrak{Z}_{\mathfrak{F}} = \pi(\mathfrak{Z}_{\mathfrak{F}}) = \pi(\mathfrak{F})$ such that $\mathfrak{Z}_{\mathfrak{F}} = \times_{i \in \mathcal{I}} \mathfrak{S}_{\pi_i}$. Hence $\mathfrak{F} = \times_{i \in \mathcal{I}} \mathfrak{F}_{\pi_i}$, where $\mathfrak{F}_{\pi_i} = \mathfrak{F} \cap \mathfrak{S}_{\pi_i}$ for all $i \in \mathcal{I}$.

Let G be a soluble primitive π_i-group for some $i \in \mathcal{I}$. Then $G \in \mathfrak{Z}_{\mathfrak{F}}$ and so every subgroup of G is $\mathfrak{F}$-subnormal in G. In particular, $G \in \mathfrak{F}$ and thus $G \in \mathfrak{F}_{\pi_i}$. Suppose, in addition, that $|\pi_i| \geq 2$. If A is a π_i-group such that $|\pi(A)| \geq 2$, then by a theorem of Hawkes [Haw75, Theorem 1], A is isomorphic to a subgroup of a multiprimitive $\pi(A)$-group G, that is, every epimorphic image of G is primitive. Since G is primitive and G is a π_i-group, it follows that $G \in \mathfrak{F}_{\pi_i}$. Hence $A \in \mathfrak{F}_{\pi_i}$ because $\mathfrak{F}_{\pi_i}$ is subgroup-closed. If A is a π_i-group, then A is isomorphic to a subgroup of a π_i-group B such that $|\pi(B)| \geq 2$ as $|\pi_i| \geq 2$. Hence $A \in \mathfrak{F}_{\pi_i}$. Consequently $\mathfrak{F}_{\pi_i} = \mathfrak{S}_{\pi_i}$ for all $i \in \mathcal{I}$ such that $|\pi_i| \geq 2$ and Statement 3 is true.

To complete the proof we now show that 3 implies 1. Suppose that $\mathfrak{F}$ is a formation such that $\mathfrak{F} = \times_{i \in \mathcal{I}} \mathfrak{F}_{\pi_i}$ for a partition $\{\pi_i : i \in \mathcal{I}\}$ of $\pi = \pi(\mathfrak{F})$. Assume, in addition, that $\mathfrak{F}_{\pi_i} = \mathfrak{S}_{\pi_i}$ if $|\pi_i| \geq 2$. Consider the subgroup-closed formation $\mathfrak{H} = \times_{i \in \mathcal{I}} \mathfrak{S}_{\pi_i}$. By Lemma 6.3.11, $\mathfrak{H}$ is a saturated lattice formation and $\operatorname{char} \mathfrak{H} = \pi$. We aim to show that $\mathrm{s}_{n\mathrm{K}\text{-}\mathfrak{H}}(G) = \mathrm{s}_{n\mathrm{K}\text{-}\mathfrak{F}}(G)$ for every group G. Assume, arguing by contradiction, there exists a group G of minimal order such that $\mathrm{s}_{n\mathrm{K}\text{-}\mathfrak{H}}(G) \neq \mathrm{s}_{n\mathrm{K}\text{-}\mathfrak{F}}(G)$. Clearly $\mathrm{s}_{n\mathrm{K}\text{-}\mathfrak{F}}(G) \subseteq \mathrm{s}_{n\mathrm{K}\text{-}\mathfrak{H}}(G)$ because $\mathfrak{F} \subseteq \mathfrak{H}$. Hence there exists a subgroup $H \in \mathrm{s}_{n\mathrm{K}\text{-}\mathfrak{H}}(G) \setminus \mathrm{s}_{n\mathrm{K}\text{-}\mathfrak{F}}(G)$. Then H is a proper subgroup of G and thus there exists a subgroup M of G such that either M is normal in G or M is an $\mathfrak{H}$-normal maximal subgroup of G. Since H is K-$\mathfrak{H}$-subnormal in M, it follows that H is K-$\mathfrak{F}$-subnormal in M by minimality of G. If M were normal in G, we would have that H would be K-$\mathfrak{F}$-subnormal in G. This would contradict our choice of H. Hence M is an $\mathfrak{H}$-normal maximal subgroup of G and so $G^{\mathfrak{H}}$ is contained in M. Then $G/\operatorname{Core}_G(M)$ is a π_i-group for some $\pi_i \subseteq \pi$ as $G/\operatorname{Core}_G(M)$ is a soluble primitive $\mathfrak{H}$-group. Note that $|\pi_i| > 1$ because M is not normal in G. Therefore $G/\operatorname{Core}_G(M) \in \mathfrak{F}_{\pi_i}$. This means that M is $\mathfrak{F}$-normal in G and H is K-$\mathfrak{F}$-normal in G, contrary to our initial supposition. Therefore $\mathrm{s}_{n\mathrm{K}\text{-}\mathfrak{F}}(X) = \mathrm{s}_{n\mathrm{K}\text{-}\mathfrak{H}}(X)$ for all groups X. Applying Corollary 6.3.16 and Theorem 6.3.9, the set $\mathrm{s}_{n\mathrm{K}\text{-}\mathfrak{F}}(X)$ is a sublattice of the subgroup lattice of X for all groups X. ◻

Example 6.3.26. Let $\mathfrak{F}$ be the formation of all abelian groups. Then $\mathfrak{F}$ is a lattice formation of soluble groups such that $\mathfrak{Z}_{\mathfrak{F}} = \mathfrak{N}$, the class of all nilpotent groups. It is clear that $\mathfrak{F}_p \neq \mathfrak{S}_p$ for all $p \in \pi(\mathfrak{F}) = \mathbb{P}$.

In [VK01], A. F. Vasil'ev and S. F. Kamornikov consider w-inherited subgroup functors f enjoying the following property:

> If G is a group and $H, K \in \mathrm{f}(G)$, then $H \cap K \in \mathrm{f}(G)$ and $\langle H, K\rangle \in \mathrm{f}(G)$.

They called them *subgroup NTL-functors*. The techniques employed in this section allow them to prove the following nice result in the universe of all soluble groups:

Theorem 6.3.27. *Let* f *be a subgroup NTL-functor. Then:*

1. *The class* $\chi_{\mathrm{f}} = \{G \mid \mathrm{f}(G) = \mathrm{s}(G)\}$ *is a subgroup-closed saturated formation,*

2. *there exists a partition* $\{\pi_i \mid i \in \mathcal{I}\}$ *of* $\pi(\chi_{\mathrm{f}})$ *such that* $\chi_{\mathrm{f}} = \times_{i\in\mathcal{I}} \mathfrak{S}_{\pi_i}$,
3. *For every group* G, $\mathrm{f}(G) = \mathrm{s}_{n\chi_{\mathrm{f}}}(G)$.

Consequently the subgroup NTL-functors in the soluble universe are exactly $\mathrm{f} = \mathrm{s}_{n\mathfrak{F}}$, for some subgroup-closed saturated lattice formation $\mathfrak{F}$.

The authors also consider the problem in the general finite universe. The best they were able to prove is the following:

Proposition 6.3.28. *Let* f *be a subgroup NTL-functor. Then:*

1. *The class* $\chi_{\mathrm{f}} = \{G \mid \mathrm{f}(G) = \mathrm{s}(G)\}$ *is a subgroup-closed solubly saturated formation,*
2. *For every group* G, $\mathrm{s}_{n\chi_{\mathrm{f}}}(G)$ *is contained in* $\mathrm{f}(G)$.

Consequently the following question remains open.

Open question 6.3.29 ([VK01]). *Let* f *be a subgroup NTL-functor. Is there a solubly saturated formation* $\mathfrak{F}$ *such that* $\mathrm{f} = \mathrm{s}_{n\mathfrak{F}}$*?.*

The reader is referred to [KS03] for more information about subgroup functors and classes of groups.

Postscript

Lattice formations have been also involved in the study of $\mathfrak{F}$-normality associated with subgroup-closed saturated formations $\mathfrak{F}$ in the soluble universe. As it is known, this notion was primarily associated with maximal subgroups.

A first attempt to give a definition valid for arbitrary subgroups was made by A. Ballester-Bolinches, K. Doerk, and M. D. Pérez-Ramos in [BBDPR95]. In the case $\mathfrak{F} = \mathfrak{N}$, the class of all nilpotent groups, the $\mathfrak{F}$-normality coincides with the classical normality and, for a general subgroup-closed saturated formation $\mathfrak{F}$, the $\mathfrak{F}$-subnormality turns out to be associated naturally with the $\mathfrak{F}$-normality in the obvious way. However, the results concerning lattice properties of $\mathfrak{F}$-normal subgroups differ from the corresponding ones for $\mathfrak{F}$-subnormal subgroups.

More recently, M. Arroyo-Jordá and M. D. Pérez-Ramos [AJPR01] study an alternative definition of $\mathfrak{F}$-normality, the $\mathfrak{F}$-Dnormality. It was suggested by K. Doerk. This new definition satisfies all the desired properties. Moreover, in this case, lattice formations turn out to be the subgroup-closed saturated formations for which the set of all $\mathfrak{F}$-Dnormal subgroups is a sublattice of the subgroup lattice in every soluble group.

The same authors [AJPR04a], [AJPR04b], studied Fitting classes with stronger closure properties involving $\mathfrak{F}$-subnormal subgroups, for a lattice formation $\mathfrak{F}$ of full characteristic.

Definition 6.3.30. *1. Let $\mathfrak{F}$ be a lattice formation containing the class $\mathfrak{N}$ of nilpotent groups. A class $\mathfrak{X}$ of groups is said to be an $\mathfrak{F}$-Fitting class if: a) for every $G \in \mathfrak{X}$ and every $\mathfrak{F}$-subnormal subgroup H of G we have $H \in \mathfrak{X}$; and b) for $G = \langle H, K\rangle$ with H, K $\mathfrak{F}$-subnormal in G, if $H, K \in \mathfrak{X}$, then $G \in \mathfrak{X}$.*

2. A subgroup of a group G is said to be an $(\mathfrak{X}, \mathfrak{F})$-injector if, for every $\mathfrak{F}$-subnormal subgroup K of G, $V \cap K$ is $\mathfrak{X}$-maximal in K.

Every $\mathfrak{F}$-Fitting class is also a Fitting class. They proved in [AJPR04b] the following nice result (see Theorem 2.4.26):

Theorem 6.3.31. *Let $\mathfrak{F}$ be a lattice formation containing $\mathfrak{N}$, and $\mathfrak{X}$ an $\mathfrak{F}$-Fitting class. Then for every group G, a subgroup V of G is an $(\mathfrak{X}, \mathfrak{F})$-injector if and only if it is an $\mathfrak{X}$-injector.*

6.4 $\mathfrak{F}$-subnormal subgroups and $\mathfrak{F}$-critical groups

We saw in Section 6.3 that if $\mathfrak{F}$ is a saturated formation, then $\mathfrak{F}$ is a lattice formation if and only if $\mathfrak{F}$ contains all groups generated by two $\mathfrak{F}$-subnormal $\mathfrak{F}$-subgroups (Theorem 6.3.3 (2)). As a consequence, a saturated lattice formation $\mathfrak{F}$ enjoys the following property:

If A and B are $\mathfrak{F}$-subnormal $\mathfrak{F}$-subgroups of a group G and $G = AB$, then $G \in \mathfrak{F}$. (6.1)

It turns out that Condition (6.1) is not sufficient for a subgroup-closed saturated formation to be a lattice formation: the formation of all p-nilpotent groups, p a prime, satisfies Condition (6.1), but it is not a lattice formation (see Example 6.3.1). Moreover, the formation of all groups with nilpotent length at most two does not satisfy Condition (6.1). Consequently the question of determining the subgroup-closed saturated formations which are closed under taking products of $\mathfrak{F}$-subnormal subgroups arises (see [MK99, Problem 14.99]). This problem has already been settled and solved in the soluble universe by A. Ballester-Bolinches in [BB92] for subgroup-closed saturated formations of full characteristic (see also [Sem92]).

The first result of this section puts a rich source of subgroup-closed saturated formations satisfying Condition 6.1 at our disposal.

Proposition 6.4.1. *Let $\mathfrak{F}$ be a saturated formation. Suppose that, for every $p \in \pi = \operatorname{char} \mathfrak{F}$, there exists a set of primes $\pi(p)$ with $p \in \pi(p)$ such that $\mathfrak{F}$ is locally defined by the formation function f given by $f(p) = \mathfrak{E}_{\pi(p)}$ if $p \in \pi$ and $f(q) = \emptyset$ if $q \notin \pi$. Then $\mathfrak{F}$ is closed under taking products of $\mathfrak{F}$-subnormal subgroups.*

Proof. Assume that the result is false and derive a contradiction. Then there exists a group G of minimal order with two $\mathfrak{F}$-subnormal $\mathfrak{F}$-subgroups A and B

such that $G = AB$ and $G \notin \mathfrak{F}$. If N is a minimal normal subgroup of G, then it is clear that G/N is the product of the $\mathfrak{F}$-subnormal $\mathfrak{F}$-subgroups AN/N and BN/N by Lemma 6.1.6 (3). The choice of G implies that $G/N \in \mathfrak{F}$. Therefore G is in the boundary of $\mathfrak{F}$ and so G is a monolithic primitive group. Then $N = G^{\mathfrak{F}}$ is the unique minimal normal subgroup of G and $\mathrm{C}_G(N) \leq N$. By Lemma 6.1.7 (2) and Proposition 6.1.11, $AN = AF^*(AN)$ is an $\mathfrak{F}$-group. Analogously $BN \in \mathfrak{F}$. Since $\mathrm{C}_G(N) \subseteq N$, it follows that AN and BN belong to $\mathfrak{E}_{\pi(p)}$ for all $p \in \pi(N)$. Therefore $G \in \mathfrak{E}_{\pi(p)}$ for each prime p dividing $|N|$. This implies that $G \in \mathfrak{F}$, contrary to our supposition. Consequently $\mathfrak{F}$ is closed under taking products of $\mathfrak{F}$-subnormal products. □

Note that the above result also holds if we replace $\mathfrak{E}_{\pi(p)}$ by $\mathfrak{S}_{\pi(p)}$, for all $p \in \operatorname{char} \mathfrak{F}$.

Unfortunately, the converse of Proposition 6.4.1 is not true in general, as the following example shows.

Example 6.4.2. Let S be a non-abelian simple group, and consider the saturated formation $\mathfrak{H} = \big(G : S \notin \mathrm{Q}(G)\big)$. Let $\mathfrak{F}$ be the largest subgroup-closed formation contained in $\mathfrak{H}$. By Theorem 3.1.42, $\mathfrak{F} = \big(G : \mathrm{S}(G) \subseteq \mathfrak{H}\big)$ is saturated. In addition, $\mathfrak{F}$ cannot be locally defined by a formation function as in Proposition 6.4.1.

We assert that $\mathfrak{F}$ is closed under taking products of $\mathfrak{F}$-subnormal subgroups. Suppose, for a contradiction, that this is not true and let G be a counterexample of least order. Then G has two proper $\mathfrak{F}$-subnormal $\mathfrak{F}$-subgroups A and B such that $G = AB$ and $G \notin \mathfrak{F}$. Let N be a minimal normal subgroup of G. Since G/N is a product of the $\mathfrak{F}$-subnormal $\mathfrak{F}$-subgroups AN/N and BN/N, the choice of G implies that $G/N \in \mathfrak{F}$. Therefore G is in the boundary of $\mathfrak{F}$ and so G is a monolithic primitive group. In particular, $N = G^{\mathfrak{F}}$ is the unique minimal normal subgroup of G. Assume $G \notin \mathfrak{H}$. Then $G \in \mathrm{b}(\mathfrak{H}) = (S)$. Hence G is non-abelian and simple. This implies that $N = G$ and therefore $G = A = B$, contrary to supposition. Consequently $G \in \mathfrak{H}$. Since $G \notin \mathfrak{F}$, it follows that $\mathrm{S}(G)$ is not contained in $\mathfrak{H}$. Among the proper subgroups X of G not belonging to $\mathfrak{H}$, we choose H of minimal order. Then every proper subgroup of H belongs to $\mathfrak{H}$. Applying [DH92, III, 2.2(c)], there exists a normal subgroup K of H such that $H/K \in \mathrm{b}(\mathfrak{H})$. Hence H/K is a non-abelian simple group. Since $H/H \cap N$ belongs to $\mathfrak{F}$, it follows that $H = (H \cap N)K$. It $H \cap N$ were a proper subgroup of H, we would have $H \cap N \in \mathfrak{F}$ and so $H/K \in \mathfrak{F} \subseteq \mathfrak{H}$, contrary to supposition. Hence $H \cap N = H$ and H is a subgroup of N. By Lemma 6.1.7 (2) and Proposition 6.1.11, $AN = HF^*(AN)$ is an $\mathfrak{F}$-group. Consequently $N \in \mathfrak{F}$ and so $H \in \mathfrak{F}$. This final contradiction proves that $\mathfrak{F}$ is closed under taking $\mathfrak{F}$-subnormal subgroups.

At the time of writing no useful characterisation of subgroup-closed saturated formations satisfying (6.1) is known. The picture improves, however, if attention is confined just to subgroup-closed saturated formations of soluble groups. The following result supports this view.

Theorem 6.4.3. *Let $\mathfrak{F}$ be a saturated formation of soluble groups of characteristic π. The following statements are equivalent:*

1. *For each prime $p \in \pi$, there exists a set of primes $\pi(p)$, with $p \in \pi(p)$, such that $\mathfrak{F}$ is locally defined by the formation function f given by $f(p) = \mathfrak{S}_{\pi(p)}$ and $f(q) = \emptyset$ if $q \notin \pi$.*
2. *$\mathfrak{F}$ satisfies Condition* (6.1).

Proof. It follows at once from Proposition 6.4.1 that 1 implies 2.

2 implies 1. We are assuming in this chapter that $\mathfrak{F}$ is subgroup-closed. Hence, for every $p \in \pi$, $F(p)$ is a subgroup-closed formation by Proposition 3.1.40. Therefore $F(p)$ is contained in $\mathfrak{F} \cap \mathfrak{E}_{\pi(p)} \subseteq \mathfrak{S}_{\pi(p)}$, where $\pi(p) = \operatorname{char} F(p)$. Suppose, for a contradiction, that the inclusion is proper and choose a group G of minimal order in $(\mathfrak{F} \cap \mathfrak{S}_{\pi(p)}) \setminus F(p)$. Then every proper subgroup of G belongs to $F(p)$ and G is a soluble monolithic group. Assume that G contains two inconjugate maximal subgroups, L and M say. Then $G = ML$ by [DH92, A, 16.2]. Moreover M and L belong to $F(p)$. Let W be a faithful G-module over $\mathrm{GF}(p)$ and denote by $Z = [W]G$ the corresponding semidirect product. Then $Z^{\mathfrak{F}}$ is contained in W and therefore WM and WL are two $\mathfrak{F}$-subnormal subgroups of Z by Lemma 6.1.7 (1). Moreover WM an WL belong to $\mathfrak{S}_p F(p) = F(p)$. Hence WM and WL are $\mathfrak{F}$-groups. Since $Z = (WM)(WL)$ and $\mathfrak{F}$ satisfies (6.1), it follows that $Z \in \mathfrak{F}$. This implies that $G \in F(p)$, which is clearly not the case. Hence G has a single conjugacy class of maximal subgroups. This implies that G is a cyclic group whose order is a power of a prime, q say. Moreover, $q \in \pi(p)$. On the other hand, it is rather easy to see that $\mathfrak{F}$ is clearly a Fitting class as $\mathfrak{F}$ is closed under taking products of $\mathfrak{F}$-subnormal subgroups. Hence $F(p)$ is also a Fitting class by Proposition 3.1.40. Since $q \in \pi(p)$, it follows that $F(p)$ contains $\mathfrak{S}_q$ by [DH92, IX, 1.9]. Hence $G \in F(p)$ and we have reached a contradiction. Consequently $F(p) = \mathfrak{F} \cap \mathfrak{S}_{\pi(p)}$ for all $p \in \pi$. It remains to prove that $\mathfrak{F} = \mathrm{LF}(f)$, where f is the formation function defined by $f(p) = \mathfrak{S}_{\pi(p)}$, $p \in \pi$, and $f(q) = \emptyset$ if $q \notin \pi$. To this end assume, by way of contradiction, that $\mathfrak{M} = \mathrm{LF}(f)$ is not contained in $\mathfrak{F}$ and let G be a group of minimal order in $\mathfrak{M} \setminus \mathfrak{F}$. Then G is a soluble primitive group and $N = \mathrm{Soc}(G) = G^{\mathfrak{F}}$ is the unique minimal normal subgroup of G. Let q be the prime dividing $|N|$. Then $q \in \pi$ and $G/N \in f(q) = \mathfrak{S}_{\pi(q)}$. Hence $G/N \in \mathfrak{F} \cap \mathfrak{S}_{\pi(q)} = F(q)$. By Remark 3.1.7 (2), $G \in \mathfrak{F}$. It follows that our supposition is wrong and hence $\mathfrak{M}$ is contained in $\mathfrak{F}$. Since $\mathfrak{F}$ is obviously contained in $\mathfrak{M}$, we have $\mathfrak{F} = \mathfrak{M}$ and the proof of the theorem is complete. □

From now on we focus our attention on formations whose associated critical groups have special properties. In order to carry out our task we shall need some definitions.

Recall that if $\mathfrak{Z}$ be a class of groups, a group G is called S-*critical for* $\mathfrak{Z}$, or simply $\mathfrak{Z}$-*critical*, if G is not in $\mathfrak{Z}$ but all proper subgroups of G are in $\mathfrak{Z}$. Following [DH92, VII, 6.1], we denote $\mathrm{Crit}_{\mathrm{S}}(\mathfrak{Z})$ the class of all $\mathfrak{Z}$-critical groups. The motivation for investigating such minimal classes is that detailed knowledge of groups that just fail to have a group theoretic property is likely to give some insight into just what makes a group have the property. The minimal classes have been investigated for a number of classes of groups. For instance, O. J. Schmidt (see [Hup67, III, 5.2]) studied the $\mathfrak{N}$-critical groups. These groups are also called *Schmidt groups*. They have a very restricted structure and they are useful in proving a known result of H. Wielandt about groups with nilpotent Hall π-subgroups (see [Hup67, III, 5.8]). K. Doerk [Doe66] studied the critical groups with respect to the class of all supersoluble groups and R. W. Carter, B. Fischer, and T. O. Hawkes (see [DH92, VII, Section 6]) used a method of extreme classes to study the soluble $\mathfrak{F}$-critical groups in the case when $\mathfrak{F}$ is the formation of all soluble groups with nilpotent length less than or equal to r. K. Doerk and T. O. Hawkes [DH92, VII, 6.18] gave a complete description of a soluble group G which is not in $\mathfrak{F}$ but all maximal subgroups are in $\mathfrak{F}$, where $\mathfrak{F}$ is an arbitrary (not necessarily subgroup-closed) saturated formation of soluble groups (note that such a group G is $\mathfrak{F}$-critical if $\mathfrak{F}$ is a subgroup-closed formation). This result was extended by A. Ballester-Bolinches and M. C. Pedraza-Aguilera to the general universe of all finite groups in [BBPA96].

The reader is referred to [Rob02], [BBERR05], and [BBERss] for further information about critical groups associated with some interesting classes of groups.

A useful property for a formation $\mathfrak{F}$ in this connection is that of having $\mathfrak{F}$-critical groups with a well-known structure. For instance, if $\mathfrak{F}$ is either a soluble saturated lattice formation or the formation of all p-nilpotent groups for a prime p, then every $\mathfrak{F}$-critical group is either a Schmidt group or a cyclic group of prime order. Therefore a subgroup-closed class $\mathfrak{Z}$ is contained in $\mathfrak{F}$ if and only if $\mathfrak{F}$ contains every Schmidt group and every cyclic group of prime order in $\mathfrak{Z}$.

This raises the following question.

Which are the saturated formations $\mathfrak{F}$ such that every $\mathfrak{F}$-critical group is either a Schmidt group or a cyclic group of prime order?

This question was proposed by L. A. Shemetkov in [MK92, Problem 9.74]. Hence we shall say that a formation $\mathfrak{F}$ has the *Shemetkov property* or $\mathfrak{F}$ is a Š-*formation* if every $\mathfrak{F}$-critical group is a Schmidt group or a cyclic group of prime order.

The first investigation of Š-formations was taken up by V. N. Semenchuk and A. F. Vasil'ev [SV84] in the soluble realm. A. Ballester-Bolinches and M. D. Pérez-Ramos [BBPR95] determined necessary and sufficient conditions for a subgroup-closed saturated formation to be a Š-formation. This result can be used to give examples of subgroup-closed saturated Š-formations of different nature.

On the other hand, L. A. Shemetkov [She92, Problem 10.22] proposes the following question:

Let $\mathfrak{F}$ be a subgroup-closed Š-formation. Is $\mathfrak{F}$ saturated?

A. N. Skiba [Ski90] answered this question affirmatively in the soluble universe. However his result does not remain true in the general case as A. Ballester-Bolinches and M. D. Pérez-Ramos showed in [BBPR96b]. In this paper, they gave a criterion for a subgroup-closed Š-formation to be saturated from which Skiba's result emerges. An alternative approach to Shemetkov's question is due to S. F. Kamornikov [Kam94]. There he proved that a subgroup-closed Š-formation is a Baer-local formation.

We shall begin our treatment of this material with a general result concerning formations $\mathfrak{F}$ whose $\mathfrak{F}$-critical groups have the composition factors of their $\mathfrak{F}$-residual in a class of simple groups $\mathfrak{X}$. We shall then specialise $\mathfrak{X}$ to $\mathfrak{J}$ and in this class aim to give a detailed account of the present state of knowledge.

Theorem 6.4.4 ([BB05]). *Let $\emptyset \neq \mathfrak{X}$ be a class of simple groups satisfying $\pi(\mathfrak{X}) = \operatorname{char}\mathfrak{X}$. Denote $\mathfrak{Y} = \mathfrak{X} \cap \mathbb{P}$, the abelian groups in $\mathfrak{X}$.*

For a formation $\mathfrak{F}$, the following statements are equivalent:

1. *Every $\mathfrak{F}$-critical group G such that $G^{\mathfrak{F}}$ is contained in $\mathrm{O}_{\mathfrak{X}}(G)$ is either a Schmidt group or a cyclic group of prime order.*
2. *Every $\mathfrak{F}$-critical group G whose $\mathfrak{F}$-residual is contained in $\mathrm{O}_{\mathfrak{X}}(G)$ is soluble, $\mathfrak{F}$ is a $\mathfrak{Y}$-local formation, and for each prime $p \in \mathfrak{Y} \cap \operatorname{char}\mathfrak{F}$ there exists a set of primes $\pi(p)$ with $p \in \pi(p)$ such that $\mathfrak{F}$ is $\mathfrak{Y}$-locally defined by the $\mathfrak{Y}$-formation function f given by*

$$f(S) = \begin{cases} \mathfrak{E}_{\pi(p)} & \text{if } S \cong C_p,\ p \in \mathfrak{Y} \cap \operatorname{char}\mathfrak{F}, \\ \emptyset & \text{if } S \cong C_p,\ p \in \mathfrak{Y} \setminus \operatorname{char}\mathfrak{F}, \\ \mathfrak{F} & \text{if } S \in \mathfrak{X}' \cup (\mathfrak{X} \setminus \mathfrak{Y}). \end{cases}$$

Proof. 1 implies 2. It is clear that every $\mathfrak{F}$-critical group G with $G^{\mathfrak{F}} \leq \mathrm{O}_{\mathfrak{X}}(G)$ is soluble by Statement 1. The next stage of the proof is to show that $\mathfrak{F}$ is a $\mathfrak{Y}$-local formation. Applying Lemma 3.1.21, it is enough to prove that $\mathfrak{F}$ is (C_p)-local for all primes $p \in \mathfrak{Y}$.

Let p be a prime in $\mathfrak{Y}$. By Theorem 3.1.11, the smallest (C_p)-local formation $\mathfrak{F}_1$ containing $\mathfrak{F}$ is (C_p)-locally defined by the (C_p)-local formation function f given by

$$\begin{aligned} \underline{f}(p) &= \mathrm{Q}\,\mathrm{R}_0\big(A/\mathrm{C}_A(H/K) : A \in \mathfrak{F} \\ &\qquad \text{and } H/K \text{ is an abelian } p\text{-chief factor of } A\big), \qquad \text{and} \\ \underline{f}(S) &= \mathrm{Q}\,\mathrm{R}_0\big(A/L : A/L \text{ is monolithic and if } \mathrm{Soc}(A/L) \in \mathrm{E}(S)\big) \quad \text{if } S \in (C_p)'. \end{aligned}$$

Denote $\pi(p) = \pi\big(\mathfrak{S}_p\underline{f}(p)\big)$, and consider the (C_p)-local formation $\mathfrak{M} = \mathrm{LF}_{(C_p)}(g)$, where

$$\begin{aligned} g(p) &= \mathfrak{E}_{\pi(p)}, \\ g(S) &= \mathfrak{F} \qquad\qquad \text{if } S \in (C_p)'. \end{aligned}$$

It is clear that $\mathfrak{F}$ is contained in $\mathfrak{M}$. Assume that $\mathfrak{M}$ is not contained in $\mathfrak{F}$ and derive a contradiction. Let G be a group of minimal order in the non-empty class $\mathfrak{M} \setminus \mathfrak{F}$. Since G is monolithic and $G \notin \mathfrak{F}$, it follows that $N := \mathrm{Soc}(G)$ is a p-group. Then $p \in \operatorname{char} \mathfrak{F}$. Moreover there exists a subgroup H of G such that $H \in \mathrm{Crit}_{\mathrm{S}}(\mathfrak{F})$. By Statement 1, H is a Schmidt group as $\pi(G) \subseteq \operatorname{char} \mathfrak{F}$.

Suppose that H is a proper subgroup of G. Since $HN/N \in \mathfrak{F}$ and $H \notin \mathfrak{F}$, it follows that $H \cap N \neq 1$. In particular, $|H| = p^a q^b$ for some prime $q \neq p$ and one of the non-trivial Sylow subgroups of H is normal in H by [Hup67, III, 5.2]. Assume that a Sylow q-subgroup Q of H is normal in H. Then $H/Q \in \mathfrak{F}$ because $p \in \operatorname{char} \mathfrak{F}$ and so $\mathfrak{M}$ contains $\mathfrak{S}_p$. Since $H/(H \cap N)$ belongs to $\mathfrak{F}$, it follows that $H \in \mathrm{R}_0 \, \mathfrak{F} = \mathfrak{F}$. This would contradict the choice of G. Therefore H has a normal Sylow p-subgroup and a Sylow q-subgroup of H is cyclic by [Hup67, III, 5.2].

Suppose that q does not belong to $\pi(p)$. Then $H \cap N$ is a Sylow p-subgroup of H. Assume not, and let P be a Sylow p-subgroup of H containing $H \cap N$. Since $[P, Q]{=}P$ (see the proof of [Hup67, III, 5.2(c)]), it follows that $Q(H \cap N)/(H \cap N)$ is not contained in $\mathrm{O}_{p',p}(H/H \cap N)$. This implies that $q \in \pi(p)$, contrary to our supposition. Hence $H \cap N$ is a Sylow p-subgroup of G. Let $C = \mathrm{C}_G(N)$. If H is a subgroup of C, then H is nilpotent. This is not possible. Hence H is not contained in C. Since $H \cap N$ is contained in C, it follows that q divides $|G/C|$. Denote $A = [N](G/C)$. By Corollary 2.2.5, $A \in \mathfrak{M}$. Hence $q \in \pi(p)$. This contradiction proves that $q \in \pi(p)$.

The definition of $\pi(p)$ implies the existence of a group $B \in \mathfrak{F}$ and an abelian p-chief factor L/M of B satisfying that q divides $\bigl|B\big/\mathrm{C}_B(L/M)\bigr|$. By Corollary 2.2.5, $C = [L/M]\bigl(B\big/\mathrm{C}_B(L/M)\bigr) \in \mathfrak{F}$. Denote $V = L/M$ and $B^* = B\big/\mathrm{C}_B(L/M)$, and $E = \langle g\,\mathrm{C}_B(L/M)\rangle$ for some element $g \in B$ such that $\mathrm{o}\bigl(g\,\mathrm{C}_B(L/M)\bigr) = q$. It is clear that V is a faithful and irreducible B^*-module over $\mathrm{GF}(p)$. Moreover V, regarded as E-module, is completely reducible by Maschke's theorem [DH92, B, 4.5]. Since V is faithful for B^* and E is a cyclic group of order q, we can find an irreducible E-submodule W of V such that W is a faithful E-module. Let $F = WE$ be the corresponding semidirect product. Then F is isomorphic to $\mathrm{E}(q|p)$, the unique Schmidt primitive group defined in [DH92, B, 12.5]. Then $F \in \mathfrak{F}$ because $\mathfrak{F}$ is subgroup-closed. If $\Phi\bigl(\mathrm{O}_p(H)\bigr) \neq 1$, then $H/\Phi\bigl(\mathrm{O}_p(H)\bigr)$ is isomorphic to $\mathrm{E}(q|p)$ constructed above. This implies that $H/\Phi\bigl(\mathrm{O}_p(H)\bigr) \in \mathfrak{F} \subseteq \mathfrak{M}$ and so $H \in \mathfrak{M}$ because $\mathfrak{M}$ is (C_p)-saturated by Theorem 3.2.14. The minimality of G implies that $H \in \mathfrak{F}$, and this contradicts the fact that $H \in \mathrm{Crit}_{\mathrm{S}}(\mathfrak{F})$. Thus our supposition is false and $\Phi\bigl(\mathrm{O}_p(H)\bigr) = 1$. But then H is isomorphic to $\mathrm{E}(q|p) \in \mathfrak{F}$ and we have reached the contradiction that $H \in \mathfrak{F}$.

Consequently $H = G$ and G is a Schmidt group with a normal Sylow p-subgroup, P say. Let Q be a Sylow q-subgroup of G. Since N is the unique minimal normal subgroup of G and $\Phi(Q)$ is normal in G, we have that $\Phi(Q) = 1$ and Q is a cyclic group of order q. Note that $\Phi(P)$ is elementary abelian by [DH92, VII, 6.18] and $G/\Phi(P) \cong \mathrm{E}(q|p)$ and $\Phi(P) = \Phi(G)$. Hence G is an epimorphic image of the maximal Frattini extension E of $\mathrm{E}(q|p)$ with p-elementary abelian kernel (see [DH92, Appendix β]). Denote by A the kernel of the above extension. Then E/A is isomorphic to $\mathrm{E}(q|p)$ and $A \leq \Phi(E)$. Moreover $\mathrm{C}_E\bigl(\mathrm{Soc}(A)\bigr)$ is a p-group by [GS78, Theorem 1]. Let Q^* be a Sylow q-subgroup of E. If AQ^* belongs to $\mathfrak{F}$, then E is $\mathfrak{F}$-critical because $G \notin \mathfrak{F}$ and A is contained in each maximal subgroup of E (note that every Sylow subgroup of E belongs to $\mathfrak{F}$). This implies that E is a Schmidt group. In particular, AQ^* is nilpotent and then $1 \neq Q^* \leq \mathrm{C}_E\bigl(\mathrm{Soc}(G)\bigr)$. This contradiction yields $AQ^* \notin \mathfrak{F}$. In this case, we can find a subgroup J of AQ^* such that $J \in \mathrm{Crit}_{\mathrm{S}}(\mathfrak{F})$. By Statement 1, J should be a Schmidt group with an elementary abelian Sylow p-subgroup. This implies that J is isomorphic to $\mathrm{E}(q|p) \in \mathfrak{F}$, and we have a contradiction. Therefore $\mathfrak{F} = \mathfrak{M}$ is a $\mathfrak{Y}$-local formation, and we have completed the proof of the implication.

2 implies 1. Suppose that $\mathfrak{F}$ is a $\mathfrak{Y}$-local formation and there exists a set of primes $\pi(p)$ with $p \in \pi(p)$, for each $p \in \pi = \mathrm{char}\,\mathfrak{F}$, such that $\mathfrak{F}$ is $\mathfrak{Y}$-locally defined by the $\mathfrak{Y}$-formation function f given by $f(p) = \mathfrak{E}_{\pi(p)}$ if $p \in \pi \cap \mathfrak{Y}$, $f(q) = \emptyset$, if $p \in \mathfrak{Y} \setminus \pi$, and $f(E) = \mathfrak{F}$ for every simple group $E \in \mathfrak{X}' \cup (\mathfrak{X} \setminus \mathfrak{Y})$. We shall prove that every group in $\mathrm{Crit}_{\mathrm{S}}(\mathfrak{F})$ whose $\mathfrak{F}$-residual is an $\mathfrak{X}$-group is a Schmidt group or a cyclic group of prime order.

Let G be a group in $\mathrm{Crit}_{\mathrm{S}}(\mathfrak{F})$ such that $G^{\mathfrak{F}} \leq \mathrm{O}_{\mathfrak{X}}(G)$. By Condition 2, G is soluble. We prove by induction on $|G|$ that G is a Schmidt group or a cyclic group of prime order.

If G is a p-group for some prime p and G has not order p, then $p \in \pi \cap \mathfrak{Y}$ and so G is an $\mathfrak{E}_{\pi(p)}$-group. In particular, G is an $\mathfrak{F}$-group. This contradicts our choice of G. Hence G is cyclic group of prime order.

Assume that G has not prime power order and there exists a minimal normal subgroup B of G such that G/B is not an $\mathfrak{F}$-group. Then B has to be contained in $\Phi(G)$ because G is $\mathfrak{F}$-critical. Therefore $G/B \in \mathrm{Crit}_{\mathrm{S}}(\mathfrak{F})$. By induction, G/B is either a Schmidt group or a cyclic group of prime order. If G/B is a cyclic group of prime order, then so is G. This contradiction shows that G/B is a Schmidt group. Let p be the prime dividing $|B|$. Then G is a $\{p,q\}$-group, for some prime $q \neq p$ and either G has a normal Sylow p-subgroup or G has a normal Sylow q-subgroup. Suppose that G has a normal Sylow p-subgroup, P say. Then $G^{\mathfrak{F}}$ is a p- group and so $p \in \pi \cap \mathfrak{Y}$. Since G is not nilpotent because it is $\mathfrak{F}$-critical, then there exists a q-element $g \in G$ such that g does not centralise P. Let us choose g of minimal order. Then every proper subgroup of $N = \langle g \rangle$ centralises P. Consequently PN is a Schmidt group. Suppose that PN is a proper subgroup of G. Then $PN \in \mathfrak{F}$. Hence $PN/\mathrm{O}_{p',p}(PN)$ belongs to $\mathfrak{E}_{\pi(p)}$. It follows that $q \in \mathfrak{E}_{\pi(p)}$, $G \in f(p) = \mathfrak{E}_{\pi(p)}$ and G is an $\mathfrak{F}$-group. This contradiction yields $G = PN$ and G is a Schmidt group. If G has a normal Sylow q-subgroup, a similar argument can be used to conclude that G is a Schmidt group.

Consequently we may assume that $G/B \in \mathfrak{F}$ for every minimal normal subgroup B of G. Then $N = G^{\mathfrak{F}} = \mathrm{Soc}(G)$ is a minimal normal subgroup of G and it is a p-group for some prime $p \in \pi \cap \mathfrak{Y}$. If N is contained in

$\Phi(G)$, then $\mathrm{O}_{p',p}(G/N) = \mathrm{O}_{p',p}(G)/N = \mathrm{O}_p(G)/N$ and so $G/\mathrm{O}_p(G) \in \mathfrak{E}_{\pi(p)}$. Hence $G \in f(p) = \mathfrak{E}_{\pi(p)}$ and G is an $\mathfrak{F}$-group. This contradiction implies that $\Phi(G) = 1$ and G is a monolithic primitive group. If $N = G$, then G is a cyclic group of prime order. Thus we may assume that N is a proper subgroup of G. Since $G \notin \mathfrak{F}$, it follows that $G \notin f(p) = \mathfrak{E}_{\pi(p)}$. Hence there exists an element $g \in G$ whose order is a prime $q \notin \pi(p)$. Denote $A = \langle g \rangle$. If NA were a proper subgroup of G, then $NA \in \mathfrak{F}$. Hence $NA/\mathrm{O}_{p',p}(NA) \cong A$ belongs to $\mathfrak{E}_{\pi(p)}$. This contradiction yields $G = AN$ and then every maximal subgroup of G is nilpotent. Consequently G is a Schmidt group and the Statement 1 of the theorem is now clear. □

If $\mathfrak{X} = \mathfrak{J}$, the class of all simple groups, then $\mathfrak{Y}$ is the class of all abelian simple groups. Therefore we have:

Corollary 6.4.5. *Let $\mathfrak{F}$ be a formation. The following statements are equivalent:*

1. *Every $\mathfrak{F}$-critical group is either a Schmidt group or a cyclic group of prime order.*
2. *Every $\mathfrak{F}$-critical group is soluble, $\mathfrak{F}$ is solubly saturated and, for each prime $p \in \operatorname{char}\mathfrak{F}$, there exists a set of primes $\pi(p)$ with $p \in \pi(p)$ such that $\mathfrak{F}$ is $\mathbb{P}$-locally defined by the $\mathbb{P}$-formation function f given by*

$$f(S) := \begin{cases} \mathfrak{E}_{\pi(p)} & \text{if } S \cong C_p,\ p \in \operatorname{char}\mathfrak{F}, \\ \emptyset & \text{if } S \cong C_p,\ p \notin \operatorname{char}\mathfrak{F}, \\ \mathfrak{F} & \text{if } S \in \mathfrak{J} \setminus \mathbb{P}. \end{cases}$$

In particular, every subgroup closed Š-formation is solubly saturated (see [Kam94]). It is clear that every soluble formation is saturated if and only if it is solubly saturated. Hence combining Theorem 6.4.4 and Theorem 6.4.3 we have

Corollary 6.4.6. *Let $\mathfrak{F}$ be a soluble Š-formation. Then $\mathfrak{F}$ is saturated and it is closed under taking products of $\mathfrak{F}$-subnormal subgroups.*

The saturated formation $\mathfrak{S}$ of all soluble groups shows that the converse of the above result does not hold.

There also exist non-saturated Š-formations.

Example 6.4.7 ([BBPR96b]). Let

$$\mathfrak{F} = (G : \text{every } \{3,5\}\text{-subgroup of } G \text{ is nilpotent}).$$

By [DH92, VII, 6.5], $\mathfrak{F}$ is a subgroup-closed formation. Let G be an $\mathfrak{F}$-critical group. Then G has a $\{3,5\}$-subgroup H such that H is not nilpotent. The choice of G implies that $H = G$. Especially, G is a $\{3,5\}$-group which is not nilpotent but all its subgroups are nilpotent. Therefore G is a Schmidt group.

Take $G = \mathrm{Alt}(5)$, the alternating group of degree 5. Then $G \in \mathfrak{F}$. Let E be the maximal Frattini extension of G with 5-elementary abelian kernel. Then $E/\Phi(E)$ is isomorphic to G and $\mathrm{C}_G\big(\Phi(E)\big) = \mathrm{O}_{5'}(G) = 1$ by [GS78, Proposition 5]. If $\mathfrak{F}$ were saturated, it would be true that $E \in \mathfrak{F}$. But this is not true because $\Phi(E)P$, for a Sylow 3-subgroup P of E, is not nilpotent inasmuch as P does not centralise $\Phi(E)$.

The following result provides a criterion for a Š-formation to be saturated.

Theorem 6.4.8 ([BBPR96b]). *Let $\mathfrak{F}$ be a Š-formation. The following statements are equivalent:*

1. *$\mathfrak{F}$ is a saturated formation.*
2. *Let G be a primitive group of type 2 such that $G \in \mathfrak{F}$. If p is a prime dividing $|\mathrm{Soc}(G)|$ and V is an irreducible and faithful G-module over* $\mathrm{GF}(p)$*, then every Schmidt subgroup isomorphic to* $\mathrm{E}(q|p)$ *of $[V]G$ belongs to $\mathfrak{F}$.*

Proof. If $\mathfrak{F}$ is a saturated formation, then the statement 2 is always true: Let $G \in \mathfrak{F}$ be a primitive group of type 2; then $G \in F(p)$ for every prime $p \in \pi\big(\mathrm{Soc}(G)\big)$, where F is the canonical local definition of $\mathfrak{F}$. The semidirect product $[V]G \in \mathfrak{S}_p F(p) = F(p) \subseteq \mathfrak{F}$, for each irreducible and faithful G-module V over $\mathrm{GF}(p)$ and p dividing the order of $\mathrm{Soc}(G)$. Now the result is clear because $\mathfrak{F}$ is subgroup-closed.

To complete the proof we now show that 2 implies 1. By Corollary 6.4.5, $\mathfrak{F}$ is solubly saturated and, for each prime $p \in \mathrm{char}\,\mathfrak{F}$, there exists a set of primes $\pi(p)$ with $p \in \pi(p)$ such that $\mathfrak{F} = \mathrm{LF}_{\mathbb{P}}(f)$, where f is the $\mathbb{P}$-formation function given by

$$f(S) := \begin{cases} \mathfrak{E}_{\pi(p)} & \text{if } S \cong C_p,\ p \in \mathrm{char}\,\mathfrak{F}, \\ \emptyset & \text{if } S \cong C_p,\ p \notin \mathrm{char}\,\mathfrak{F}, \\ \mathfrak{F} & \text{if } S \in \mathfrak{J} \setminus \mathbb{P}. \end{cases}$$

Applying Theorem 3.4.5, the formation $\mathfrak{H} = \mathrm{LF}(f)$ is the largest saturated formation contained in $\mathfrak{F}$. Suppose, by way of contradiction, that the class $\mathfrak{F} \setminus \mathfrak{H}$ is non empty, and let G be a group of minimal order in this class. Then G is a primitive group of type 2 and, since $G \notin \mathfrak{H}$, there exists a prime $p \in \pi\big(\mathrm{Soc}(G)\big) \subseteq \mathrm{char}\,\mathfrak{F}$ such that $G \notin f(p) = \mathfrak{E}_{\pi(p)}$. Consequently there exists an element $g \in G$ of order q, for some prime $q \notin \pi(p)$. Furthermore, by [DH92, B, 10.9], G has an irreducible and faithful module V over $\mathrm{GF}(p)$. Let $X = [V]G$ be the corresponding semidirect product. Denote $A = \langle g \rangle$ and consider the subgroup VA of X. V, regarded as an A-module, is semisimple by [DH92, B, 4.5]. Moreover, since V is faithful and A is a cyclic group of order q, we can find an irreducible A-submodule W of V such that W is a faithful A-module. Let $B = WA$ be the corresponding semidirect product. It is clear that B is a Schmidt group which is isomorphic to $\mathrm{E}(q|p)$. By Condition 2, $B \in \mathfrak{F}$. It yields $B/\mathrm{C}_B(W) \in f(p)$. Hence $q \in \pi(p)$. This contradiction shows that $\mathfrak{H} = \mathfrak{F}$ and $\mathfrak{F}$ is saturated. □

Remark 6.4.9. Let $\mathfrak{F}$ be a saturated Š-formation. According to Corollary 6.4.5, $\mathfrak{F} = \mathrm{LF}_{\mathbb{P}}(f)$, where f is the $\mathbb{P}$-formation function given by $f(p) = \mathfrak{E}_{\pi(p)}$, $p \in \pi(p)$, if $p \in \operatorname{char}\mathfrak{F}$, $f(p) = \emptyset$ if $p \notin \operatorname{char}\mathfrak{F}$ and $f(S) = \mathfrak{F}$ if $S \in \mathfrak{J} \setminus \mathbb{P}$. By Theorem 3.1.17, the canonical $\mathbb{P}$-local definition of $\mathfrak{F}$, F say, is given by

$$F(S) = \begin{cases} \mathfrak{F} \cap \mathfrak{E}_{\pi(p)} & \text{if } p \in \operatorname{char}\mathfrak{F}, \\ \emptyset & \text{if } p \notin \operatorname{char}\mathfrak{F}, \\ \mathfrak{F} & \text{if } S \in \mathfrak{J} \setminus \mathbb{P}. \end{cases}$$

Furthermore, by Corollary 3.1.18, the canonical local definition of $\mathfrak{F}$ is $F(p) = \mathfrak{F} \cap \mathfrak{E}_{\pi(p)}$ if $p \in \operatorname{char}\mathfrak{F}$ and $F(p) = \emptyset$ otherwise. Using familiar arguments it can be proved that $\mathfrak{F} = \mathrm{LF}(f)$.

Unfortunately, not every saturated formation which is locally defined as above is a Š-formation.

Example 6.4.10 ([BBPR95]). Consider $\mathfrak{F} = \mathrm{LF}(f)$ which is locally defined by the formation function given by $f(2) = f(3) = \mathfrak{E}_{\{2,3\}}$, $f(5) = \mathfrak{E}_{\{2,5\}}$, and $f(q) = \emptyset$ if $q \neq 2$, 3, 5. Then $\mathfrak{F}$ is subgroup-closed and $\operatorname{char}\mathfrak{F} = \{2, 3, 5\}$; $\mathrm{Alt}(5)$ is $\mathfrak{F}$-critical but it is neither a Schmidt group nor a cyclic group of prime order.

For saturated formations of soluble groups, the following characterisation holds.

Theorem 6.4.11. *Let $\mathfrak{F}$ be a saturated formation of soluble groups. Then every soluble $\mathfrak{F}$-critical group is either a Schmidt group or a cyclic group of prime order if and only if $\mathfrak{F}$ satisfies the following condition: there exists a formation function f, defined by $f(p) = \mathfrak{S}_{\pi(p)}$ for a set of primes $\pi(p)$ such that $p \in \pi(p)$ if $p \in \operatorname{char}\mathfrak{F}$ and $f(p) = \emptyset$ otherwise, such that $\mathfrak{F} = \mathrm{LF}(f)$.*

Proof. Assume that every soluble $\mathfrak{F}$-critical group is either a Schmidt group or a cyclic group of prime order. Let $\mathfrak{F}$ be the canonical local definition of $\mathfrak{F} = \mathrm{LF}(F)$. Since $\mathfrak{F}$ is subgroup-closed, it follows that $F(p)$ is subgroup-closed for each $p \in \pi = \operatorname{char}\mathfrak{F}$ by Proposition 3.1.40. Hence $F(p)$ is contained in $\mathfrak{F} \cap \mathfrak{S}_{\pi(p)}$, where $\pi(p) = \operatorname{char} F(p)$ for every $p \in \pi$. Assume that there exists a prime $p \in \pi$ such that $F(p) \neq \mathfrak{S}_{\pi(p)} \cap \mathfrak{F}$ and let G be a group of minimal order in the non-empty class $(\mathfrak{F} \cap \mathfrak{S}_{\pi(p)}) \setminus F(p)$. Then $1 \neq \mathrm{Soc}(G)$ is the unique minimal normal subgroup of G which is not a p-group. By [DH92, B, 10.9] there exists an irreducible and faithful G-module V over $\mathrm{GF}(p)$. Let $X = [V]G$ be the corresponding semidirect product. It is clear that X is a primitive group and $V = \mathrm{Soc}(X)$ is the unique minimal normal subgroup of X. Since $G \notin F(p)$, we have that X is not an $\mathfrak{F}$-group. Let M be a maximal subgroup of X. If $\mathrm{Core}_X(M) = 1$, then M is isomorphic to G. Hence $M \in \mathfrak{F}$. Assume that $\mathrm{Core}_X(M) \neq 1$. Then $V \leq M$ and $M \cap G$ is a maximal subgroup of G. In this case $M \cap G \in F(p)$ and so $M \in \mathfrak{S}_p F(p) = F(p) \subseteq \mathfrak{F}$. Hence X is an $\mathfrak{F}$-critical soluble group. By hypothesis, X is a Schmidt group (clearly X cannot be a cyclic group of prime order). In particular G is a nilpotent group. Assume that $\mathrm{Soc}(G)$ is a proper subgroup of G. Let A be a maximal subgroup of G containing $\mathrm{Soc}(G)$. Then VA is a maximal subgroup of X and so VA is nilpotent. Let q be the prime dividing $|\mathrm{Soc}(G)|$. Then the Sylow q-subgroup A_q of A is non-trivial and $A_q \leq \mathrm{C}_{VA}(V) = V$. This contradiction yields $\mathrm{Soc}(G) = G$ and hence G is a cyclic group of order $q \in \pi(p) = \mathrm{char}\, F(p)$. This means that $G \in F(p)$, contrary to our supposition. Consequently, for each prime $p \in \pi$, we have that $F(p) = \mathfrak{F} \cap \mathfrak{S}_{\pi(p)}$, where $\pi(p) = \mathrm{char}\, F(p)$.

We are now close to completing the proof of the implication. Let f be the formation function given by $f(p) = \mathfrak{S}_{\pi(p)}$ if $p \in \pi$ and $f(q) = \emptyset$, if $q \neq \pi$ and $F(q) = \emptyset$. It is clear that $\mathfrak{F}$ is contained in $\mathrm{LF}(f)$. Assume that the equality is not true and take a group $G \in \mathrm{LF}(f) \setminus \mathfrak{F}$ of minimal order. Since $\mathrm{LF}(f)$ is composed of soluble groups, it follows that G is a soluble primitive group. Let p be the prime dividing $|Soc(G)|$. Then $G/N \in \mathfrak{S}_{\pi(p)} \cap \mathfrak{F} = F(p)$. Consequently $G \in \mathfrak{S}_p F(p) \subseteq \mathfrak{F}$, and we have reached a contradiction. Hence $\mathfrak{F} = \mathrm{LF}(f)$.

Suppose now that there exists a formation function f, defined by $f(p) = \mathfrak{S}_{\pi(p)}$ for a set of primes $\pi(p)$ such that $p \in \pi(p)$ if $p \in \mathrm{char}\, \mathfrak{F}$ and $f(p) = \emptyset$ otherwise, such that $\mathfrak{F} = \mathrm{LF}(f)$. Let G a soluble $\mathfrak{F}$-critical group. Assume that $\Phi(G) = 1$. Then G is a primitive group. Let p be the prime dividing $\mathrm{Soc}(G)$. If $q \neq p$ is a prime dividing the order of G, and $g \in G$ is an element of G of order q, then g does not centralise N. Denote $A = \langle g \rangle$. If NA were a proper subgroup of G, then $NA \in \mathfrak{F}$. Hence $NA/\mathrm{O}_{p',p}(NA) \cong A$ belongs to $\mathfrak{S}_{\pi(p)}$ and $q \in \pi(p)$. Since G does not belong to $\mathfrak{S}_{\pi(p)}$, it follows that $G = AN$ for some subgroup A of G of prime order. This means that G is a Schmidt group. Hence, in this case, G is either a Schmidt group or a cyclic group of prime order.

Assume that $\Phi(G) \neq 1$. The group $G^* = G/\Phi(G)$ is an $\mathfrak{F}$-critical group and $\Phi(G^*) \neq 1$. The above argument implies that G^* is either a Schmidt group or a cyclic group of prime order. Consequently, G is a Schmidt group and the other implication of the theorem is now clear. □

We now present a set of necessary and sufficient conditions for a saturated formation to be a Š-formation.

Theorem 6.4.12 ([BBPR95]). *Let $\mathfrak{F}$ be a saturated formation. Then $\mathfrak{F}$ is a Š-formation if and only if $\mathfrak{F}$ satisfies the following two conditions:*

1. *There exists a formation function f, defined by $f(p) = \mathfrak{E}_{\pi(p)}$ for a set of primes $\pi(p)$ such that $p \in \pi(p)$ if $p \in \mathrm{char}\, \mathfrak{F}$ and $f(p) = \emptyset$ otherwise, such that $\mathfrak{F} = \mathrm{LF}(f)$; this formation function f satisfies the following property:*
   If $G \in \mathrm{Crit_S}(\mathfrak{F}) \cap \mathrm{b}(\mathfrak{F})$ and G is an almost simple group such that $G \notin f(p)$ for some prime $p \in \pi\big(\mathrm{Soc}(G)\big)$, then $G \notin f(q)$ for each prime $q \in \pi\big(\mathrm{Soc}(G)\big)$. (6.2)
2. *$\mathrm{Crit_S}(\mathfrak{F}) \cap \mathrm{b}(\mathfrak{F})$ does not contain non-abelian simple groups.*

Proof. Denote $\pi := \operatorname{char} \mathfrak{F}$. If $\mathfrak{F}$ is a Š-formation, every group $G \in \operatorname{Crit}_{\mathrm{S}}(\mathfrak{F}) \cap \mathrm{b}(\mathfrak{F})$ has abelian socle. Bearing in mind Remark 6.4.9, only the sufficiency of the conditions is in doubt.

Assume that there exists a set of primes $\pi(p)$ with $p \in \pi(p)$, for each $p \in \pi$, such that $\mathfrak{F}$ is locally defined by the formation function f given by $f(p) = \mathfrak{E}_{\pi(p)}$ if $p \in \pi$, and $f(q) = \emptyset$ if $q \notin \pi$. Then $\mathfrak{F} = \mathrm{LF}_{\mathbb{P}}(\hat{f})$, where $\hat{f}$ is the $\mathbb{P}$-formation function defined by $\hat{f}(p) = f(p)$ for all $p \in \mathbb{P}$ and $\hat{f}(S) = \mathfrak{F}$ for all $S \in \mathfrak{J} \setminus \mathbb{P}$ (see Corollary 3.1.13 and Corollary 3.1.18). By Corollary 6.4.5, it will be sufficient to show that every $\mathfrak{F}$-critical group is soluble to conclude that $\mathfrak{F}$ is a Š-formation. Suppose that $\operatorname{Crit}_{\mathrm{S}}(\mathfrak{F}) \setminus \mathfrak{S}$ is not empty and derive a contradiction. Let G be a group of minimal order in $\operatorname{Crit}_{\mathrm{S}}(\mathfrak{F}) \setminus \mathfrak{S}$. Then $\Phi(G) = 1$ and G is a monolithic primitive group in $\mathrm{b}(\mathfrak{F})$. Let $N = \operatorname{Soc}(G)$ be the unique minimal normal subgroup of G. If $N = G$, then G is simple. Since this contradicts 2, we must have $N < G$, so that $N \in \mathfrak{F}$. Assume that N is non-abelian. Then $\mathrm{C}_G(N) = 1$ and $\pi(N) \subseteq \pi(p)$, for every $p \in \pi(N)$. Now since $G \notin \mathfrak{F}$, there exists a prime $q \in \pi(G)$ such that $q \notin \pi(p)$ for some prime $p \in \pi(N)$; in particular, $q \notin \pi(N)$. Let g be an element of G of order q. Denote $A = \langle g \rangle$. The group A operates by conjugation on N and $(|N|, |A|) = 1$. By [DH92, I, 1.3], there exists an A-invariant Sylow p-subgroup N_p of N. Since G is not soluble, it follows that $N_p A$ is a proper subgroup of G. Hence $N_p A \in \mathfrak{F}$. Since N_p is normal in $N_p A$ and $q \notin \pi(p)$, it follows that $A \le \mathrm{C}_G(N_p)$. On the other hand, $N = N_1 \times \cdots \times N_r$ is a direct product of non-abelian simple groups N_i, $1 \le i \le r$, which are pairwise isomorphic. Since $N_p = (N_1)_p \times \cdots \times (N_r)_p$ for some Sylow p-subgroup $(N_i)_p$ of N_i, $1 \le i \le r$, and $(N_i)_p \le N_i \cap N_i^g$, it follows that $N_i = N_i^g$, $1 \le i \le r$. Hence A normalises N_i for all $i \in \{1, \dots, r\}$. Suppose that $N_i A \ne G$ for every $i \in \{1, \dots, r\}$. Then $N_i A \in \mathfrak{F}$. Consequently $A \le \mathrm{C}_G(N_i)$ for every $i \in \{1, \dots, r\}$ because $q \notin \pi(p)$. This implies that $A \le \mathrm{C}_G(N) = 1$, which is impossible. We conclude for this contradiction that $G = N_i A$ for some $i \in \{1, \dots, r\}$ and $N = N_i$ is a non-abelian simple group. In particular, G is an almost simple group. We may apply now Condition 1 and deduce that $G \notin f(r)$ for each $r \in \pi(N)$. But the above argument shows that A centralises a Sylow r-subgroup of N for each $r \in \pi(N)$. Hence $A \le \mathrm{C}_G(N) = 1$, and again we have a contradiction. Therefore N must be abelian. Let p be the prime dividing $|N|$. Then $p \in \pi$ and $G \notin f(p)$ because $G \notin \mathfrak{F}$. Let g be an element of G whose order is a prime $q \notin \pi(p)$. Denote again $A = \langle g \rangle$. If $NA = G$, then G is soluble. This contradiction implies that NA is a proper subgroup of G. But in this case $q \in \pi(p)$ because N is self-centralising in G, contrary to our initial supposition that $q \notin \pi(p)$.

Thus we are forced to the conclusion that every $\mathfrak{F}$-critical group is soluble and $\mathfrak{F}$ is a Š-formation. □

Remark 6.4.13. None of the conditions 1 and 2 can be dispensed with in Theorem 6.4.12 (see [BBPR95, Examples]).

With the help of the preceding theorem we can now give examples of subgroup-closed saturated Š-formations of different nature. The simplest example is the formation $\mathfrak{F}$ of the p-nilpotent groups, p a prime number. It is clear that $\mathfrak{F} = \mathrm{LF}(f)$, where $f(p) = \mathfrak{S}_p$ and $f(q) = \mathfrak{E}$ for every prime $q \neq p$. Hence $\mathfrak{F}$ belongs to the family of saturated formations described in Theorem 6.4.12. Let $G \in \mathrm{Crit_S}(\mathfrak{F}) \cap \mathrm{b}(\mathfrak{F})$. Then G is not a p'-group. Thus $p \in \pi(G)$. Since G is not p-nilpotent, we can apply the p-nilpotence criterion of Frobenius [Hup67, IV, 5.8] to conclude that $G = \mathrm{N}_G(P)$ for some p-subgroup $1 \neq P$ of G. Hence $\mathrm{Soc}(G)$ is abelian and $\mathfrak{F}$ satisfies Conditions 1 and 2 of Theorem 6.4.12. Consequently $\mathfrak{F}$ is a Š-formation. This is a classical result due to Itô ([Hup67, IV, 5.4]).

Less trivial is the following result.

Theorem 6.4.14 ([BBPR95]). *Let $\{\pi_i : i \in \mathcal{I}\}$ be a family of pairwise disjoint sets of primes and put $\pi = \bigcup\{\pi_i : i \in \mathcal{I}\}$. Let $\mathfrak{F}$ be the saturated formation locally defined by the formation function f given by $f(p) = \mathfrak{E}_{\pi_i}$ if $p \in \pi_i$, $i \in \mathcal{I}$, and $f(q) = \emptyset$ if $q \notin \pi$. Then $\mathfrak{F}$ is a subgroup-closed saturated Š-formation.*

Proof. By Proposition [DH92, IV, 3.14], $\mathfrak{F}$ is a subgroup-closed saturated formation. It is clear that $\pi = \mathrm{char}\,\mathfrak{F}$. Note that a group G belongs to $\mathfrak{F}$ if and only if G has a normal Hall π_i-subgroup for every $i \in \mathcal{I}$.

We claim that $\mathfrak{F}$ satisfies Conditions 1 and 2 of Theorem 6.4.12. On one hand, the formation function defined above satisfies Condition 1. On the other hand, assume that G is a non-abelian simple group in $\mathrm{Crit_S}(\mathfrak{F}) \cap \mathrm{b}(\mathfrak{F})$ and derive a contradiction. Then $2 \in \mathrm{char}\,\mathfrak{F}$, by the Odd Order Theorem [FT63], and so there exists an element $i \in \mathcal{I}$ such that $2 \in \pi_i$. Denote $\pi_1 = \pi \setminus \pi_i$ and $\pi_2 = \pi_i$. If X is a group in $\mathfrak{F}$, we denote by X_1 the normal Hall π_1-subgroup of X. The normal Hall π_2-subgroup of X is denoted by X_2. We reach a contradiction after the following steps:

Step 1. Let M be a maximal subgroup of G such that $M_1 \neq 1$ and $M_2 \neq 1$. Then $\mathrm{Syl}_p(M) \subseteq \mathrm{Syl}_p(G)$, for every prime p dividing the order of M.

Let p be a prime dividing $|M|$ and let $M_p \in \mathrm{Syl}_p(M)$. There exists a Sylow p-subgroup G_p of G such that $M_p \subseteq G_p$. Assume, arguing by contradiction, that M_p is a proper subgroup of G_p. Then M_p is a proper subgroup of $T_p = \mathrm{N}_{G_p}(M_p)$. Suppose that $p \in \pi_2$ (similar arguments can be used if $p \in \pi_1$). In this case, we have that $M_1 \leq \mathrm{N}_G(M_p)$ and so $\langle M_1, T_p\rangle \leq \mathrm{N}_G(M_p)$, which is a proper subgroup of G because G is a non-abelian simple group. Let L be a maximal subgroup of G such that $\mathrm{N}_G(M_p) \leq L$. Then $L = L_1 \times L_2$ because $L \in \mathfrak{F}$. Furthermore, $L_1 \neq 1$ and $L_2 \neq 1$ as $p \in \pi(L)$ and $M_1 \leq L_1$. Hence $\langle M_2, L_2\rangle \leq \mathrm{N}_G(M_1) = M$ and so M_p is a Sylow p-subgroup of L_2. But then T_p is a Sylow p-subgroup of L_2 containing properly a Sylow p-subgroup of L_2. This contradiction yields $M_p = G_p$ and M_p is a Sylow p-subgroup of G.

Step 2. Let p be a prime in π_1 and let $1 \neq P$ be a p-subgroup of G. Then $\mathrm{N}_G(P)$ is of Glauberman type with respect to the prime p (cf. [Gor80, 4.1, page 281]).

It is clear that $N_G(P)$ is a proper subgroup of G. Hence $N_G(P) \in \mathfrak{F}$ and $N_G(P) = N_G(P)_1 \times N_G(P)_2$. In particular, $N_G(P)$ is p-soluble. Suppose that $p = 3$, then $SL(2,3)$ is not involved in G because the Sylow 2-subgroup of $SL(2,3)$ is not centralised by a Sylow 3-subgroup of $SL(2,3)$. Hence $N_G(P)$ is strongly p-soluble in the sense of [Gor80, page 234]. Moreover, $O_p\big(N_G(P)\big) \neq 1$ and p is an odd prime. Therefore we can apply [Gor80, pages 268–269] to conclude that $N_G(P)$ is p-constrained and p-stable. By [Gor80, Theorem 8.2.11, page 279] we have that $N_G(P)$ is of Glauberman type with respect to the prime p.

Step 3. Let p be a prime in π_1 and let $1 \neq P$ be a Sylow p-subgroup of G. Then $P \leq N'$, where $N = N_G\big(ZJ(P)\big)$ and $ZJ(P)$ is the centre of the Thompson subgroup of P.

By Step 2, the normaliser of every nonidentity p-subgroup of G is of Glauberman type with respect to the prime p. Applying [Gor80, Theorem 8.4.3, page 282], we conclude that $P \cap G' = P \cap G = P \cap N'$.

Step 4. Let M be a maximal subgroup of G. Then M is either a π_1-group or a π_2-group.

Since G is $\mathfrak{F}$-critical, we have that every maximal subgroup of G belongs to $\mathfrak{F}$. Assume that the above statement is not true. Then the set

$$\Sigma := \{M : M \text{ is a maximal subgroup of } G,\ M_1 \neq 1, \text{ and } M_2 \neq 1\}$$

is non-empty. We define a binary relation $\mathcal{R}$ in Σ by $M \mathcal{R} L$ if and only if $M_2 \leq L_2$. Clearly $\mathcal{R}$ is reflexive and transitive. Moreover, if $M \mathcal{R} L$ and $L \mathcal{R} M$, then $M_2 = L_2$ and so $M = N_G(M_2) = N_G(L_2) = L$. Hence $(\Sigma, \mathcal{R})$ is a partially ordered set. Let M be a maximal element of $(\Sigma, \mathcal{R})$. Since $M_1 \neq 1$ and M_1 is soluble, by the Feit-Thompson theorem, we have that $(M_1)'$ is a proper subgroup of M_1. Let p be a prime dividing $|M_1 : (M_1)'|$ and let P be a Sylow p-subgroup of M. Then, by Step 1, P is a Sylow p-subgroup of G and moreover $P \leq N'$, where $N = N_G\big(ZJ(P)\big)$, by Step 3. Clearly N is a proper subgroup of G and $M_2 \leq N$. Let L be a maximal subgroup of G containing N. Then $M_2 \leq L_2$. Moreover $L_1 \neq 1$ because $p \in \pi_1$. Therefore $L \in \Sigma$ and $M \mathcal{R} L$. By the maximality of M, we have that $L = M$. In particular, P is contained in $(M_1)'$ because $N' \leq (M_1)' \times (M_2)'$. Hence $|M_1 : (M_1)'|$ is a p'-number, contrary to our supposition.

Step 5. G has a maximal subgroup of odd order.

If every maximal subgroup of G were of even order, then G would be an $\mathfrak{E}_{\pi_2}$-group by Step 4. This would imply that $G \in \mathfrak{F}$, and we would have a contradiction. Hence we conclude that G has a maximal subgroup of odd order.

Applying [LS91, Theorem 2], we have that G is one of the following groups: $\mathrm{Alt}(p)$, p a prime number, $p \equiv 3 \pmod 4$ and $p \neq 7, 11, 23$; $L_2(q)$, $q \equiv 3 \pmod 4$, $L_p^{\varepsilon}(q)$, $\varepsilon = \pm 1$, p odd prime, and $G \neq U_3(3)$ or $U_5(2)$; M_{23}, Th, F_2, or F_1.

In the remaining steps we rule out the above possibilities for the non-abelian simple group G.

Step 6. G is not of the type $\mathrm{Alt}(p)$, *p a prime number, $p \equiv 3 \pmod 4$ and $p \neq 7, 11, 23$.*

Suppose that $G = \mathrm{Alt}(p)$ for some prime p, $p \equiv 3 \pmod 4$. It is clear that $\mathrm{Alt}(p-1)$ is a maximal subgroup of G, $\mathrm{Alt}(p-1) \in \mathfrak{F}$ and $\mathrm{Alt}(p-1) \in \mathfrak{E}_{\pi_2}$. Let P be a Sylow p-subgroup of G and let M be a maximal subgroup of G such that $\mathrm{N}_G(P) \leq M$. By Step 4, M is either a π_1-group or a π_2-group. If $M \in \mathfrak{E}_{\pi_2}$, then $G \in \mathfrak{E}_{\pi_2}$ and so $G \in \mathfrak{F}$. This contradiction yields $M \in \mathfrak{E}_{\pi_1}$. Hence $M = P = \mathrm{N}_G(P)$ and we have a contradiction.

Step 7. $G \neq \mathrm{L}_2(q)$, $q \equiv 3 \pmod 4$.

Assume that $G = \mathrm{L}_2(q)$, for some $q \equiv 3 \pmod 4$. Then by [LS91, Theorem 2], if M is a maximal subgroup of odd order, then M is isomorphic to a semidirect product of an elementary abelian group of order q and a cyclic group of order $(q-1)/2$. On the other hand, by the theorem of Dickson [Hup67, II, 8.27], G has a subgroup H which is isomorphic to the dihedral group of order $2\big((q-1)/2\big)$. Then $H \in \mathfrak{E}_{\pi_2}$ by Step 4, and therefore $M \in \mathfrak{E}_{\pi_2}$. It means that $G \in \mathfrak{E}_{\pi_2}$. This contradiction confirms Step 7.

Step 8. $G \neq \mathrm{L}_p^{\varepsilon}(q)$, $\varepsilon \in \{\pm 1\}$, p odd prime.

Assume that $G = \mathrm{L}_p^{\varepsilon}(q)$ for some odd prime p, $\varepsilon \in \{\pm 1\}$ and $G \neq \mathrm{U}_3(3)$ or $\mathrm{U}_5(2)$. Again, by [LS91, Theorem 2], if M is a maximal subgroup of G of odd order, then the order of M is $p\big((q^p - \varepsilon)/(q-\varepsilon)(q-\varepsilon, p)\big)$. From Tables 3.5A and 3.5B and the corresponding results of Chapter 4 of [KL90], it follows that there exists a proper subgroup M of G of even order such that $p \in \pi(M)$. Hence $M \in \mathfrak{E}_{\pi_2}$ and then $G \in \mathfrak{E}_{\pi_2}$. This contradiction proves Step 8.

Step 9. G is not of type M_{23}, Th, F_2, *or* F_1.

Using the *Atlas* [CCN+85] as reference for the list of maximal subgroups of G, we see that in this case G should be a π_2-group. This final contradiction proves the theorem. □

We now turn our attention to an application of Theorem 6.4.12 leading to a characterisation of the subgroup-closed saturated Š-formations.

Theorem 6.4.15 ([BBPR96b]). *Let $\mathfrak{F} = \mathrm{LF}(F)$ be a subgroup-closed saturated formation. Denote $\pi = \operatorname{char} \mathfrak{F}$ and $\pi(p) = \operatorname{char} F(p)$, for every $p \in \pi$. Any two of the following statements are equivalent:*

1. *$\mathfrak{F}$ is a Š-formation.*
2. *A π-group G belongs to $\mathfrak{F}$ if and only if $\mathrm{N}_G(Q)/\mathrm{C}_G(Q)$ belongs to $\mathfrak{E}_{\pi(p)}$ for each p-subgroup Q of G and each prime $p \in \pi$.*
3. *A π-group G belongs to $\mathfrak{F}$ if and only if $\mathrm{N}_G(Q) \in \mathfrak{E}_{\pi\setminus\{p\}}\mathfrak{E}_{\pi(p)}$ for each non-trivial p-subgroup Q of G and each prime $p \in \pi$.*

Proof. 1 implies 2. Assume that $\mathfrak{F}$ is a Š-formation. According to Theorem 6.4.12, we have that $F(p) = \mathfrak{E}_{\pi(p)} \cap \mathfrak{F}$, for every $p \in \pi$. Let G be a π-group in $\mathfrak{F}$. Suppose that a prime $p \in \pi$ is fixed and let Q be a p-subgroup of G. Then $\mathrm{N}_G(Q) \in \mathfrak{F}$ because $\mathfrak{F}$ is subgroup-closed. In particular $\mathrm{N}_G(Q)/\mathrm{O}_{p'}\big(\mathrm{N}_G(Q)\big) \in F(p) \subseteq \mathfrak{E}_{\pi(p)}$. Since Q is a normal p-subgroup of $\mathrm{N}_G(Q)$, it follows that $\mathrm{O}_{p'}\big(\mathrm{N}_G(Q)\big) \leq \mathrm{C}_G(Q)$. This means that

$\mathrm{N}_G(Q)/\mathrm{C}_G(Q) \in \mathfrak{E}_{\pi(p)}$. Conversely, assume that G is a π-group such that $\mathrm{N}_G(Q)/\mathrm{C}_G(Q)$ belongs to $\mathfrak{E}_{\pi(p)}$ for each p-subgroup Q of G and each $p \in \pi$, but G is not an $\mathfrak{F}$-group. If we choose G of minimal order among the groups $X \notin \mathfrak{F}$ satisfying the above property, we have that G is an $\mathfrak{F}$-critical group because this property holds in every subgroup of G. Since $\mathfrak{F}$ is a Š-formation, it follows that G is a Schmidt group. In particular $\pi(G) = \{p, q\}$ for two distinct primes p and q in π and G has a normal Sylow p-subgroup, P say. By hypothesis, we have that $G/\mathrm{C}_G(P) \in \mathfrak{E}_{\pi(p)}$. If q were not in $\pi(p)$, it would be true that $Q \leq \mathrm{C}_G(P)$. This is not possible. Hence $q \in \pi(p)$ and then $Q \in \mathfrak{E}_{\pi(p)} \cap \mathfrak{F} = F(p)$. Therefore $G \in \mathfrak{S}_p F(p) = F(p) \subseteq \mathfrak{F}$. This contradiction yields $G \in \mathfrak{F}$.

2 implies 1. We see that $\mathfrak{F}$ satisfies the Statements 1 and 2 of Theorem 6.4.12.

(a) *For each prime $p \in \pi$, we have that $F(p) = \mathfrak{E}_{\pi(p)} \cap \mathfrak{F}$.*

Let p be a prime in π. Since $F(p)$ is subgroup-closed by Proposition 3.1.40, it follows that $F(p)$ is contained in $\mathfrak{E}_{\pi(p)} \cap \mathfrak{F}$. Assume, by way of contradiction, that $F(p) \neq \mathfrak{E}_{\pi(p)} \cap \mathfrak{F}$ and let G be a group of minimal order in the non-empty class $(\mathfrak{E}_{\pi(p)} \cap \mathfrak{F}) \setminus F(p)$. Then $1 \neq \mathrm{Soc}(G)$ is the unique minimal normal subgroup of G and it is not a p-group. By [DH92, B, 10.9], there exists an irreducible and faithful G-module over $\mathrm{GF}(p)$. Let $X = [V]G$ be the corresponding semidirect product. X is a primitive group and $X \notin \mathfrak{F}$ because $G \notin F(p)$. Let q be a prime in π and let Q be a non-trivial q-subgroup of G. Suppose that $p \neq q$. Then $\mathrm{N}_X(Q)$ is a proper subgroup of G because V is the unique minimal normal subgroup of X. Let L be a maximal subgroup of G containing $\mathrm{N}_X(Q)$. If $\mathrm{Core}_X(L) = 1$, then L is isomorphic to G and if $\mathrm{Core}_G(L) \neq 1$, then $V \leq L$ and $L = V(G \cap L)$. Since $G \cap L \in F(p)$, it follows that $L \in \mathfrak{S}_p F(p) = F(p) \subseteq \mathfrak{F}$. In both cases, $L \in \mathfrak{F}$. Therefore $\mathrm{N}_X(Q) \in \mathfrak{F}$ and so $\mathrm{N}_X(Q)/\mathrm{C}_X(Q) \in \mathfrak{E}_{\pi(q)}$. Now, if $p = q$ and $\mathrm{N}_X(Q)$ is a proper subgroup of G, we can argue as above to conclude that $\mathrm{N}_X(Q)/\mathrm{C}_X(Q) \in \mathfrak{E}_{\pi(p)}$. If Q is a normal subgroup of X, then $V = Q$ and $X/\mathrm{C}_X(Q) \in \mathfrak{E}_{\pi(p)}$ because it is isomorphic to G. Since X is a π-group, we can apply Statement 2 to conclude that $X \in \mathfrak{F}$. This contradiction shows that $F(p) = \mathfrak{E}_{\pi(p)} \cap \mathfrak{F}$.

(b) $\mathrm{Crit_S}(\mathfrak{F}) \cap \mathrm{b}(\mathfrak{F})$ *does not contain primitive groups of type 2.*

Assume that G is a primitive group of type 2 in $\mathrm{Crit_S}(\mathfrak{F}) \cap \mathrm{b}(\mathfrak{F})$. Since G is $\mathfrak{F}$-critical, it follows that G is a π-group. On the other hand, applying Statement 2, we can determine a prime $p \in \pi$ and a p-subgroup Q of G such that $\mathrm{N}_G(Q)/\mathrm{C}_G(Q) \notin \mathfrak{E}_{\pi(p)}$. Then Q is non-trivial. Suppose that $\mathrm{N}_G(Q)$ is a proper subgroup of G. Then $\mathrm{N}_G(Q) \in \mathfrak{F} \subseteq \mathfrak{E}_{\pi\setminus\{p\}}\mathfrak{E}_{\pi(p)}$. This means that $\mathrm{N}_G(Q)/\mathrm{C}_G(Q) \in \mathfrak{E}_{\pi(p)}$, contrary to our supposition. Hence Q is a normal subgroup of G. But then $\mathrm{Soc}(G) \leq Q$ and $\mathrm{Soc}(G)$ is abelian. This contradiction confirms Statement b.

From Statements a and b we deduce that $\mathfrak{F}$ enjoys the properties given in Theorem 6.4.12. This means that $\mathfrak{F}$ is a Š-formation.

Assume now that G is a π-group. Let p be a prime in π and let Q be a non-trivial p-subgroup of G. If $\mathrm{N}_G(Q) \in \mathfrak{E}_{\pi\setminus\{p\}}\mathfrak{E}_{\pi(p)}$, then $\mathrm{N}_G(Q)/\mathrm{C}_G(Q) \in \mathfrak{E}_{\pi(p)}$.

This elementary remark proves that 2 implies 3. Now, if Statement 3 holds, we can repeat the arguments used in the proof of 2 implies 1 to conclude that $\mathfrak{F}$ is a Š-formation. Consequently 3 implies 1 and the circle of implications is complete. □

Illustration 6.4.16. Let $\mathfrak{F}$ be the saturated formation of p-nilpotent groups, p a prime number. It is clear that $\mathfrak{F} = \mathrm{LF}(F)$, where $F(p) = \mathfrak{S}_p$ and $F(q) = \mathfrak{F}$ for every prime $q \neq p$. We have seen above that $\mathfrak{F}$ is a Š-formation. Therefore $\mathfrak{F}$ satisfies Condition 2 of Theorem 6.4.15. Hence a group G is p-nilpotent if and only if $\mathrm{N}_G(Q)/\mathrm{C}_G(Q)$ is a p-group for every p-subgroup Q of G. The statement 3 of this theorem says that a group G is p-nilpotent if and only if $\mathrm{N}_G(Q)$ is p-nilpotent for every p-subgroup Q of G. These statements are two equivalent forms of the well known p-nilpotence criterion due to Frobenius.

The next topic we broach concerns the relation between the $\mathfrak{F}$-residual of a group and the subgroup generated by the $\mathfrak{F}$-residuals of some of its $\mathfrak{F}$-critical subgroups. The springboard for these results was a theorem of Berkovich [Ber99] stating that the nilpotent residual of a group G is the subgroup generated by the nilpotent residuals of the subgroups A of G such that $A/\Phi(A)$ is a Schmidt group.

Berkovich's result is a particular case of a more general theorem as we shall see below.

Let $\mathfrak{F}$ be a formation. Denote by $\mathfrak{B}_{\mathfrak{F}}$ the class of all groups G such that $G/\Phi(G)$ is an $\mathfrak{F}$-critical group. Note that if $\mathfrak{F} = \mathfrak{N}$, the class of all nilpotent groups, $\mathfrak{B}_{\mathfrak{N}}$ is the class of all groups such that $G/\Phi(G)$ is a Schmidt group (see [Ber99]).

Let G be a group and let $\mathrm{T}(G) = \langle A^{\mathfrak{F}} : A \leq G; A \in \mathfrak{B}_{\mathfrak{F}} \rangle$ if $\mathfrak{B}_{\mathfrak{F}} \cap \mathrm{s}(G) \neq \emptyset$; otherwise, we let $\mathrm{T}(G) = 1$.

Theorem 6.4.17 ([ABB02]). *Let $\mathfrak{F}$ be a saturated formation, and let G be a group. Then $\mathrm{T}(G) = G^{\mathfrak{F}}$.*

Proof. Clearly $X^{\mathfrak{F}} \leq G^{\mathfrak{F}}$ for every subgroup X of G because $\mathfrak{F}$ is subgroup-closed. Hence $T = \mathrm{T}(G)$ is contained in $G^{\mathfrak{F}}$.

Assume, arguing by contradiction, that $G/T \notin \mathfrak{F}$. Then G/T has an $\mathfrak{F}$-critical subgroup, A/T say. Choose now a minimal supplement A_0 of T in A. Then $A_0 \cap T$ is contained in $\Phi(A_0)$. Since A/T is isomorphic to $A_0/A_0 \cap T$, it follows that $A_0/A_0 \cap T$ is $\mathfrak{F}$-critical. Therefore the factor group $(A_0/A_0 \cap T)/\Phi(A_0/A_0 \cap T)$ is also $\mathfrak{F}$-critical because $\mathfrak{F}$ is saturated. It means that $A_0 \in \mathfrak{B}_{\mathfrak{F}}$ and so $A_0^{\mathfrak{F}}$ is contained in T. Hence $A_0^{\mathfrak{F}} \leq A_0 \cap T \leq \Phi(A_0)$. It follows that $A_0/\Phi(A_0) \in \mathfrak{F}$. Now since $\mathfrak{F}$ is saturated, we conclude that $A_0 \in \mathfrak{F}$. This contradiction completes the proof. □

We continue the section with an application of Theorem 6.4.17 leading to a characterisation of the Š-formations in the soluble universe among the subgroup-closed saturated formations. It rests on the following result.

Theorem 6.4.18 ([ABB02]). *Let $\mathfrak{F}$ be a saturated formation of soluble groups of full characteristic such that every soluble group in* $\mathrm{Crit_S}(\mathfrak{F})$ *is a Schmidt group. If A is a group in $\mathfrak{B}_{\mathfrak{F}}$, then $A^{\mathfrak{N}} = A^{\mathfrak{F}}$.*

Proof. We shall argue by induction on $|A|$. Firstly, if $\Phi(A) = 1$, then A is an $\mathfrak{F}$-critical group. Assume that A is not soluble. Then A is a non-abelian simple group. In this case $A^{\mathfrak{F}} = A = A^{\mathfrak{N}}$. Thus we may suppose that A is soluble and then A is a Schmidt group. Hence there exists a normal abelian Sylow p-subgroup of A, P say, for some prime p. It is rather clear that P coincides with both the nilpotent residual and the $\mathfrak{F}$-residual of A. Hence we can assume that $\Phi(A) \neq 1$. Let N be a minimal normal subgroup of A contained in $\Phi(A)$. Then $A/N \in \mathfrak{B}_{\mathfrak{F}}$. Hence the induction hypothesis implies that $(A/N)^{\mathfrak{F}} = (A/N)^{\mathfrak{N}}$. This yields $A^{\mathfrak{F}}N = A^{\mathfrak{N}}N$.

If $A^{\mathfrak{N}} \cap N = 1$, then $A^{\mathfrak{N}} = A^{\mathfrak{N}} \cap A^{\mathfrak{F}}N = A^{\mathfrak{F}}(A^{\mathfrak{N}} \cap N) = A^{\mathfrak{F}}$. Thus we can suppose that N is contained in $A^{\mathfrak{N}}$ and $A^{\mathfrak{N}} = A^{\mathfrak{F}}N$. If N is contained in $A^{\mathfrak{F}}$, then $A^{\mathfrak{N}} = A^{\mathfrak{F}}$ and the theorem is true. Consequently we shall assume that $N \cap A^{\mathfrak{F}} = 1$, and hence $\Phi(A) \cap A^{\mathfrak{F}} = 1$. Note that we can suppose that $A/\Phi(A)$ is an extension of a p-group by a q-group for some primes p and q. Since this class is a saturated formation, we have that A is also an extension of a p-group by a q-group. Therefore $A^{\mathfrak{N}}$ is a p-group, and consequently N is a p-group, too.

Let us have a look now at the structure of the $\mathfrak{F}$-group $A/A^{\mathfrak{F}}$. Given a subgroup H of A, denote by $\bar{H}$ the corresponding subgroup $HA^{\mathfrak{F}}/A^{\mathfrak{F}}$ of $A/A^{\mathfrak{F}} = \bar{A}$. By Theorem 6.4.11, we have that the class $\mathfrak{F}$ is locally defined by a formation function f given by $f(r) = \mathfrak{S}_{\pi(r)}$, where $\pi(r)$ is a set of primes such that $r \in \pi(r)$, for all primes r. Now note that $\bar{N}$ is a minimal normal subgroup of $\bar{A}$. Therefore, $\bar{A}/\mathrm{C}_{\bar{A}}(\bar{N}) \in \mathfrak{S}_{\pi(p)}$. We can conclude that $A/\mathrm{C}_A(N) \cong \bar{A}/\mathrm{C}_{\bar{A}}(\bar{N}) \in \mathfrak{S}_{\pi(p)}$. If $q \in \pi(p)$, then $A \in \mathfrak{S}_{\pi(p)} = f(p)$ and so $A \in \mathfrak{F}$, against the supposition that $A/\Phi(A)$ is $\mathfrak{F}$-critical. Therefore $q \notin \pi(p)$ and we have that $A/\mathrm{C}_A(N)$ is a p-group. Since the normal Sylow p-subgroup of A centralises N, it follows that N is central in G. On the other hand, $A/\Phi(A)$ is an $\mathfrak{F}$-critical group with trivial Frattini subgroup. Since $A^{\mathfrak{F}} \cap \Phi(A) = 1$ and $A^{\mathfrak{F}}\Phi(A)/\Phi(A) = \big(A/\Phi(A)\big)^{\mathfrak{F}}$, it follows that $A^{\mathfrak{F}}$ is abelian. But the equality $A^{\mathfrak{N}} = A^{\mathfrak{F}} \times N$ yields that $A^{\mathfrak{N}}$ is complemented in A by a Carter subgroup of A by Theorem 4.2.17. We can conclude that there exists a Carter subgroup C of A such that $A = A^{\mathfrak{N}}C$ and $A^{\mathfrak{N}} \cap C = 1$. Now N is central in A. Hence $N \leq \mathrm{N}_G(C) = C$. Consequently $N \leq A^{\mathfrak{N}} \cap C = 1$, contrary to supposition. This final contradiction proves the result. □

Theorem 6.4.19 ([ABB02]). *Let $\mathfrak{F}$ be a saturated Fitting formation of soluble groups of full characteristic. The following statements are equivalent:*

1. *Every soluble group in* $\mathrm{Crit_S}(\mathfrak{F})$ *is a Schmidt group.*
2. *$G^{\mathfrak{F}} = \langle A^{\mathfrak{N}} : A \leq G; A \in \mathfrak{B}_{\mathfrak{F}}\rangle$ for every group G.*

Proof. By Theorem 6.4.17, we have that $G^{\mathfrak{F}} = \langle A^{\mathfrak{F}} : A \leq G, A \in \mathfrak{B}_{\mathfrak{F}}\rangle$. Hence if every soluble group in $\mathrm{Crit_S}(\mathfrak{F})$ is a Schmidt group, we apply Theorem 6.4.18

to conclude that $G^{\mathfrak{F}} = \langle A^{\mathfrak{N}} : A \leq G; A \in \mathfrak{B}_{\mathfrak{F}} \rangle$ for every group G. Therefore 1 implies 2. Therefore, only the sufficiency of the Condition 2 is in doubt. To prove that every soluble group in $\mathrm{Crit}_{\mathrm{S}}(\mathfrak{F})$ is a Schmidt group, we shall use Theorem 6.4.11. Write $\mathfrak{F} = \mathrm{LF}(F)$, where F denotes the canonical local definition of $\mathfrak{F}$. Consider any prime p. We prove that $F(p) = \mathfrak{S}_{\pi(p)} \cap \mathfrak{F}$, where $\pi(p) = \operatorname{char} F(p)$. Since $\mathfrak{F}$ is a subgroup-closed Fitting formation, we have that $F(p)$ is subgroup-closed Fitting formation by Proposition 3.1.40. Since F is integrated, we have that $F(p) \subseteq \mathfrak{S}_{\pi(p)} \cap \mathfrak{F}$. Assume that $\mathfrak{S}_{\pi(p)} \cap \mathfrak{F} \neq F(p)$ and take a group G in $(\mathfrak{S}_{\pi(p)} \cap \mathfrak{F}) \setminus F(p)$ of minimal order. By familiar reasoning, $1 \neq \mathrm{Soc}(G)$ is the unique minimal normal subgroup of G. Moreover, $\mathrm{Soc}(G)$ cannot be a p-group since, being F full, it holds that $F(p) = \mathfrak{S}_p F(p)$. Note that, in fact, $\mathrm{O}_p(G) = 1$. By [DH92, B, 10.9], there exists an irreducible and faithful G-module V over $\mathrm{GF}(p)$. Consider now the corresponding semidirect product $X = [V]G$. Note that if $X \in \mathfrak{F}$, then $X/\mathrm{C}_X(V) \in F(p)$ and thus $X/V \in F(p)$. This is impossible because $G \cong X/V$. Therefore $X \notin \mathfrak{F}$ and X is in fact an $\mathfrak{F}$-critical group.

We are ready at this point to reach our final contradiction. Since $X^{\mathfrak{F}} = \langle A^{\mathfrak{N}} : A \leq X; A \in \mathfrak{B}_{\mathfrak{F}} \rangle$, and $X \in \mathfrak{B}_{\mathfrak{F}}$, we have that $X^{\mathfrak{F}} = X^{\mathfrak{N}} = V$. Therefore $G \cong X/V = X/X^{\mathfrak{N}}$ is nilpotent. Then G is a q-group for some prime $q \in \operatorname{char} F(p)$. Since $F(p)$ is a Fitting class of soluble groups, it follows that $\mathfrak{S}_q$ is contained in $F(p)$ by [DH92, IX, 1.9] and then $G \in F(p)$. This contradiction yields $F(p) = \mathfrak{S}_{\pi(p)} \cap \mathfrak{F}$. It follows then that $\mathfrak{F} = \mathrm{LF}(f)$, where f is the formation function defined by $f(p) = \mathfrak{S}_{\pi(p)}$ if $p \in \operatorname{char} \mathfrak{F}$ and $f(p) = \emptyset$ otherwise. Applying Theorem 6.4.11, every soluble group in $\mathrm{Crit}_{\mathrm{S}}(\mathfrak{F})$ is a Schmidt group. □

Remark 6.4.20. The formation in the above theorem is not a Š-formation in general (see Example 6.4.10).

We close our extended treatment of Š-formations with a survey describing another context where this family of saturated formations appears.

In [Keg65] O. H. Kegel introduced the notion of a triple factorisation. This is a factorisation of a group G involving three subgroups A, B, and C of the type $G = AB = AC = BC$. The evidence is that the existence of a triple factorisation can have greater consequences for the group structure than does a single factorisation. For example, Kegel shows that a group which has a triple factorisation by nilpotent groups is nilpotent. Consequently, it seems natural to wonder which are the saturated formations $\mathfrak{F}$ which are closed under taking triple factorisations. The first contribution to the solution of this problem was made by Vasil'ev [Vas87, Vas92] in the soluble universe. The following three results are proved in that universe.

Theorem 6.4.21 (Vasil'ev). *Let $\mathfrak{F}$ be an s_n-closed saturated formation. Then the following statements are equivalent:*

1. *$\mathfrak{F}$ is a Š-formation.*
2. *(Kegel's property) $\mathfrak{F}$ contains each group $G = AB = AC = BC$ where A, B, $C \in \mathfrak{F}$.*
3. *$\mathfrak{F}$ contains each group having three pairwise non-conjugate maximal subgroups belonging to $\mathfrak{F}$.*

The above result has been improved in [BBPAMP00].

Theorem 6.4.22. *Let $\mathfrak{F}$ be an s_n-closed saturated formation of full characteristic. The following statements are equivalent.*

1. *$\mathfrak{F}$ is a Š-formation.*
2. *$\mathfrak{F}$ satisfies the property:*
   If G is a group of the form $G = AB = AC = BC$, where A and B are $\mathfrak{F}$-subgroups of G and C is an $\mathfrak{F}$-subnormal $\mathfrak{F}$-subgroup of G, then G is an $\mathfrak{F}$-group.

In the study of factorised groups, the case of a triply factorised group $G = AB = AC = BC$ where C is a normal subgroup of G is of particular interest. For instance, the factoriser of a normal subgroup of a factorised group always has this form. Hence the following characterisation of the above formations is also of interest ([BBPAMP00]).

Theorem 6.4.23. *Let $\mathfrak{F}$ be an s_n-closed saturated formation of full characteristic. The following statements are equivalent:*

1. *$\mathfrak{F}$ is a Š-formation.*
2. *$\mathfrak{F}$ satisfies the property:*
   If G is a group of the form $G = AB = AC = BC$, where A and B are $\mathfrak{F}$-subgroups of G and C is a normal subgroup of G, then $G^{\mathfrak{F}} = C^{\mathfrak{F}}$.

Bearing in mind the above results, a natural question arises:

Let $\mathfrak{F}$ be a Fitting formation, non-necessarily subgroup-closed, with the Kegel property. Is $\mathfrak{F}$ saturated?

This question, proposed by Vasil'ev in the *Kourovka Notebook* [MK99] for formations of soluble groups, was partially answered in [BBE05].

Theorem 6.4.24. *Let $\mathfrak{F}$ be a Fitting formation with the following property:*

for every prime $p \in \operatorname{char}\mathfrak{F}$, whenever G is a primitive $\mathfrak{F}$-group whose socle is a p-group, all groups $\mathrm{E}(q|p)$ are in $\mathfrak{F}$ for all primes $q \neq p$ such that q divides $|G/\operatorname{Soc}(G)|$. (6.3)

Then $\mathfrak{F}$ satisfies the Kegel property if and only if $\mathfrak{F}$ is a subgroup-closed Š-formation.

Note that if $\mathfrak{F}$ is saturated, then $\mathfrak{F}$ satisfies (6.3).

Let $\mathfrak{F}$ be a Fitting formation. If for some primes p, q, the group $\mathrm{E}(q|p) \in \mathfrak{F}$, then $\mathfrak{S}_p(C_q) \subseteq \mathfrak{F}$ by [DH92, XI, 2.5]. Since $\mathfrak{S}_p(C_q) \subseteq \mathfrak{F} \cap \mathfrak{N}^2$ and $\mathfrak{F} \cap \mathfrak{N}^2$ is a Fitting formation of metanilpotent groups, it follows that $\mathfrak{S}_p\mathfrak{S}_q \subseteq \mathfrak{F} \cap \mathfrak{N}^2 \subseteq \mathfrak{F}$ by [DH92, XI, 2.4]. Hence $\mathrm{E}(q|p) \in \mathfrak{F}$ if and only if $\mathfrak{S}_p\mathfrak{S}_q \subseteq \mathfrak{F}$.

6.5 Wielandt operators

One of the significant properties of subnormal subgroups is that the nilpotent residual of the subgroup generated by two subnormal subgroups of a group is the subgroup generated by the nilpotent residuals of the subgroups. This is a consequence of an elegant theory of operators created by H. Wielandt for proving results on permutability of subnormal subgroups.

For a group G and the lattice $\mathrm{s}_n(G)$ of all subnormal subgroups of G, a map $\omega\colon \mathrm{s}_n(G) \longrightarrow \mathrm{s}_n(G)$ is called a *Wielandt operator* in G if, for any H, $K \in \mathrm{s}_n(G)$, the following conditions are satisfied:

$$\langle H, K\rangle^\omega = \langle H^\omega, K^\omega\rangle, \tag{6.4}$$

$$\text{if } H \trianglelefteq K, \text{ then } H^\omega \trianglelefteq K. \tag{6.5}$$

Here, of course, H^ω denotes the image of H under the map ω. Note that Condition 6.5 implies that H^ω is a normal subgroup of H.

The importance of the theory of operators is suggested by the following result of H. Wielandt.

Theorem 6.5.1 ([Wie57]). *Let φ and ψ be two Wielandt operators in a group G. Assume that two subnormal subgroups H and K of G are permutable if $H = H^\varphi = H^\psi$. Then $A^\varphi B^\psi = B^\psi A^\varphi$ for any pair (A, B) of subnormal subgroups of G.*

It is a consequence of the above result that each new operator leads to the discovering of a new case of permutability of subnormal subgroups and gives new insights on the construction of subnormal subgroup generation. Wielandt's theory of operators is clearly of interest in relation to the theory of classes of groups and may repay further study.

Suppose that a Wielandt operator ω is defined in all groups G. If ω satisfies $(X^\omega)^\alpha = (X^\alpha)^\omega$ for any homomorphism α of a group X, then the class $\mathfrak{F} := (X : X^\omega = 1)$ is a Fitting formation and G^ω is the $\mathfrak{F}$-residual of G for every group G. Conversely if $\mathfrak{F}$ is a Fitting formation, then the map $\delta\colon \mathrm{s}_n(G) \longrightarrow \mathrm{s}_n(G)$, $H^\delta = H^{\mathfrak{F}}$ for all $H \in \mathrm{s}_n(G)$, defines a Wielandt operator in every group G, permuting with all homomorphisms provided that δ satisfies Condition 6.4.

Consequently, the problem of finding Wielandt operators which are permutable with homomorphisms is reduced to the description of Fitting formations $\mathfrak{F}$ satisfying the following property:

If U and V are subnormal subgroups of a group G, then $\langle U, V\rangle^{\mathfrak{F}} = \langle U^{\mathfrak{F}}, V^{\mathfrak{F}}\rangle$. (6.6)

Let us state this property in a formal definition.

Definition 6.5.2. *Let $\mathfrak{F}$ be a formation. We say that $\mathfrak{F}$* satisfies the Wielandt property for residuals *if whenever U and V are subnormal subgroups of $\langle U, V\rangle$ in a group G, then $\langle U, V\rangle^{\mathfrak{F}} = \langle U^{\mathfrak{F}}, V^{\mathfrak{F}}\rangle$.*

The formations appearing in this section are not subgroup-closed in general.

Not all formations have the Wielandt property for residuals. For instance, let $\mathfrak{F}$ be the saturated formation composed of all groups with no epimorphic image isomorphic to Alt(5). Then if G is the symmetric group of degree 5, it follows that $G^{\mathfrak{F}} = 1 \neq \langle \mathrm{Alt}(5), 1\rangle^{\mathfrak{F}} = \mathrm{Alt}(5)$. In fact, we have:

Proposition 6.5.3. *Let $\mathfrak{F}$ be a formation. If $\mathfrak{F}$ satisfies the Wielandt property for residuals, then $\mathfrak{F}$ is a Fitting formation.*

Proof. Let G be a group in $\mathfrak{F}$, and N a subnormal subgroup of G. Then $N^{\mathfrak{F}} \leq \langle N^{\mathfrak{F}}, G^{\mathfrak{F}}\rangle = \langle N, G\rangle^{\mathfrak{F}} = G^{\mathfrak{F}} = 1$. Hence $N \in \mathfrak{F}$ and $\mathfrak{F}$ is s_n-closed.

Suppose that $G = N_1N_2$ for normal subgroups N_1 and N_2 such that $N_i \in \mathfrak{F}$, $i = 1, 2$. Then $G^{\mathfrak{F}} = N_1^{\mathfrak{F}}N_2^{\mathfrak{F}} = 1$. This means that $G \in \mathfrak{F}$ and $\mathfrak{F}$ is N_0-closed.

Consequently, $\mathfrak{F}$ is a Fitting class. □

The validity of the converse is not known at the time of writing and seems to be quite difficult.

Our aim in the first part of this section is to show that many of the known Fitting formations have the Wielandt property for residuals.

The procedure we describe here is based on the papers [KS95] and [BBCE01].

The basic strategy is the following: first we prove a reduction theorem for a minimal counterexample. This allows us to reduce the problem in many cases to considering a restricted class of groups in the boundary of the formation. As an application, we deduce that many known Fitting formations have the Wielandt property for residuals.

The main obstacle in giving the complete answer for the problem is in understanding the restriction of an irreducible module to a subnormal subgroup. Although a certain amount of information can be derived from repeated application of the Clifford theorems, the closed relation between the components of the restriction is lost. In particular, for a subnormal subgroup, it is difficult to find the relationship between the kernels of the action of the subnormal subgroup on each component of the restriction.

We begin by describing two ways to obtain new formations with the Wielandt property from some old ones.

Proposition 6.5.4. *If $\mathfrak{F}_1$, $\mathfrak{F}_2$, and $\mathfrak{F}_i$, $i \in \mathcal{I}$, are formations satisfying the Wielandt property for residuals, then*

1. *the formation $\mathfrak{F}_1 \circ \mathfrak{F}_2$ satisfies the Wielandt property, and*
2. *the formation $\bigcap_{i\in\mathcal{I}} \mathfrak{F}_i$ satisfies the Wielandt property.*

Proof. 1. We have that $X^{\mathfrak{F}_1\circ\mathfrak{F}_2} = (X^{\mathfrak{F}_2})^{\mathfrak{F}_1}$ by Proposition 2.2.11 (4) for any group X. Let G be a group and U and V subgroups of G such that U and V are subnormal subgroups of $H = \langle U, V\rangle$. Then $H^{\mathfrak{F}_1\circ\mathfrak{F}_2} = (H^{\mathfrak{F}_2})^{\mathfrak{F}_1} = \langle U^{\mathfrak{F}_2}, V^{\mathfrak{F}_2}\rangle^{\mathfrak{F}_1} = \langle U^{\mathfrak{F}_1\circ\mathfrak{F}_2}, V^{\mathfrak{F}_1\circ\mathfrak{F}_2}\rangle$.

2. We have that $X^{\bigcap_{i\in\mathcal{I}}\mathfrak{F}_i} = \prod_{i\in\mathcal{I}} X^{\mathfrak{F}_i}$ for any group X, where in the product only a finite set of residuals appear since X is finite. Consider a group G and U and V subgroups of G such that U and V are subnormal subgroups of $H = \langle U, V\rangle$. Then $H^{\bigcap_{i\in\mathcal{I}}\mathfrak{F}_i} = \prod_{i\in\mathcal{I}} H^{\mathfrak{F}_i} = \prod_{i\in\mathcal{I}}\langle U^{\mathfrak{F}_i}, V^{\mathfrak{F}_i}\rangle = \langle\prod_{i\in\mathcal{I}} U^{\mathfrak{F}_i}, \prod_{i\in\mathcal{I}} V^{\mathfrak{F}_i}\rangle = \langle U^{\bigcap_{i\in\mathcal{I}}\mathfrak{F}_i}, V^{\bigcap_{i\in\mathcal{I}}\mathfrak{F}_i}\rangle$. □

Note that if $\mathfrak{F}$ is a Fitting formation, then $U^{\mathfrak{F}}$ is contained in $G^{\mathfrak{F}}$ for every subnormal subgroup U of G. Therefore it is always true that $\langle U^{\mathfrak{F}}, V^{\mathfrak{F}}\rangle$ is contained in $\langle U, V\rangle^{\mathfrak{F}}$ provided that U and V are subnormal in $\langle U, V\rangle$. If U and V permute, the equality holds as the next result shows.

Proposition 6.5.5. *Let $\mathfrak{F}$ be a Fitting formation. If U and V are subgroups of G such that $UV = VU$ and U and V are subnormal in UV, then $(UV)^{\mathfrak{F}} = U^{\mathfrak{F}}V^{\mathfrak{F}}$.*

Proof. Assume that the result is false and let G be a counterexample of least order. Let U and V be subnormal subgroups of $UV = VU$ such that $|U| + |V|$ is maximal doing false the result. Clearly U and V are proper subgroups of G and $G = UV$. Let N be a proper normal subgroup of G such that $U \leq N$. Then $N = U(V \cap N)$. The minimality of G yields $N^{\mathfrak{F}} = U^{\mathfrak{F}}(V \cap N)^{\mathfrak{F}}$. If U is a proper subgroup of N, then $G^{\mathfrak{F}} = N^{\mathfrak{F}}V^{\mathfrak{F}}$ by the maximality of the pair (U, V). Hence $G^{\mathfrak{F}} = U^{\mathfrak{F}}V^{\mathfrak{F}}$. This contradiction shows that U and V are maximal normal subgroups of G. Thus $U^{\mathfrak{F}}$ and $V^{\mathfrak{F}}$ are normal in G. Assume that one of them, $U^{\mathfrak{F}}$ say, is not trivial, and let N be a minimal normal subgroup of G such that $N \leq U^{\mathfrak{F}}$. It follows that G/N is a group generated by the subnormal subgroups UN/N and VN/N of G/N. Then, by minimality of G, we have that $G^{\mathfrak{F}} = U^{\mathfrak{F}}(V^{\mathfrak{F}}N) = U^{\mathfrak{F}}V^{\mathfrak{F}}$, contrary to our initial supposition. Hence $U^{\mathfrak{F}} = 1 = V^{\mathfrak{F}}$ or, equivalently, U and V are in $\mathfrak{F}$. Since $\mathfrak{F}$ is a Fitting class, we deduce that $G \in \mathfrak{F}$, i.e. $G^{\mathfrak{F}} = 1$. This final contradiction proves the proposition. □

Corollary 6.5.6. *Let $\mathfrak{F}$ be a Fitting formation. If U and V are subgroups of a group G such that U and V are subnormal in $\langle U, V\rangle$ and $U \in \mathfrak{F}$, then $\langle U, V\rangle^{\mathfrak{F}} = V^{\mathfrak{F}}$.*

Proof. Since U is a subnormal subgroup of $\langle U, V\rangle$ and $U \in \mathfrak{F}$, we have that U is contained in the $\mathfrak{F}$-radical $\langle U, V\rangle_{\mathfrak{F}}$ of $\langle U, V\rangle$. Hence $\langle U, V\rangle = \langle U, V\rangle_{\mathfrak{F}}V$. By Proposition 6.5.5, we deduce that $\langle U, V\rangle^{\mathfrak{F}} = (\langle U, V\rangle_{\mathfrak{F}})^{\mathfrak{F}}V^{\mathfrak{F}} = V^{\mathfrak{F}}$. □

A well-known result of H. Wielandt (see [Wie94b]) asserts that the Fitting subgroup of a group G normalises the nilpotent residual of each subnormal subgroup of G. The next corollary extends this result to an arbitrary Fitting formation.

Corollary 6.5.7. *Let $\mathfrak{F}$ be a Fitting formation. If U and V are subgroups of a group G such that U and V are subnormal in $\langle U, V\rangle$, it follows that $U_{\mathfrak{F}}$ normalises $V^{\mathfrak{F}}$. In particular, $G_{\mathfrak{F}}$ normalises the $\mathfrak{F}$-residual $H^{\mathfrak{F}}$ of each subnormal subgroup H of G.*

Proof. Consider the subgroup $K = \langle U_{\mathfrak{F}}, V\rangle$ generated by its subnormal subgroups $U_{\mathfrak{F}}$ and V. Then $K^{\mathfrak{F}} = V^{\mathfrak{F}}$ by Corollary 6.5.6 and $K^{\mathfrak{F}}$ is normal in K. Hence $U_{\mathfrak{F}}$ normalises $V^{\mathfrak{F}}$. □

Let $\mathfrak{F}$ be a Fitting formation. Given a group X, we denote by $\mathcal{W}(X, \mathfrak{F})$ the set of all pairs (A, B) such that A and B are subnormal subgroups of $\langle A, B\rangle \leq X$ and $\langle A^{\mathfrak{F}}, B^{\mathfrak{F}}\rangle < \langle A, B\rangle^{\mathfrak{F}}$. Let $\mathcal{B}(\mathfrak{F})$ denote the class of all groups X such that $\mathcal{W}(X, \mathfrak{F}) \neq \emptyset$. If $\mathfrak{F}$ does not satisfy the Wielandt property for residuals, then the class $\mathcal{B}(\mathfrak{F})$ is non-empty. In the following we analyse the structure of a group G of minimal order in $\mathcal{B}(\mathfrak{F})$. We consider a pair (U, V) in $\mathcal{W}(G, \mathfrak{F})$ such that $|U|+|V|$ is maximal. Denote $H = \langle U^{\mathfrak{F}}, V^{\mathfrak{F}}\rangle$ and $A = U \cap V$. By Proposition 6.5.5, U and V do not permute. In particular, neither U nor V is normal in G. Moreover, $U^{\mathfrak{F}} \neq 1 \neq V^{\mathfrak{F}}$ by Corollary 6.5.6.

Statement 6.5.8. $G = \langle U, V\rangle$. *Moreover,* $U^{\mathfrak{F}} \neq V^{\mathfrak{F}}$.

Proof. By minimality of G, it is clear that $G = \langle U, V\rangle$ and $1 \neq G^{\mathfrak{F}}$. If $N = U^{\mathfrak{F}} = V^{\mathfrak{F}}$, then N is normal in G. The minimal choice of G implies that $G^{\mathfrak{F}}/N = \langle U^{\mathfrak{F}}/N, V^{\mathfrak{F}}/N\rangle = 1$. Then $N = G^{\mathfrak{F}}$. This contradiction yields $U^{\mathfrak{F}} \neq V^{\mathfrak{F}}$. □

Statement 6.5.9. $G^{\mathfrak{F}} = HN$ *for every minimal normal subgroup* N *of* G. *In particular,* H *is core-free in* G. *Moreover,* H *is normal in* $G^{\mathfrak{F}}$.

Proof. Let N be a minimal normal subgroup of G. We consider $G/N = \langle UN/N, VN/N\rangle$. By minimality of G, we deduce that $G^{\mathfrak{F}}N = HN$. If N is not contained in $G^{\mathfrak{F}}$, then $N \cap G^{\mathfrak{F}} = 1$. This means that $G^{\mathfrak{F}}N = G^{\mathfrak{F}} \times N$. Since $H \leq G^{\mathfrak{F}}$, it follows that $H \cap N = 1$. But $G^{\mathfrak{F}}N = HN$ implies that $|G^{\mathfrak{F}}| = |H|$ and then $G^{\mathfrak{F}} = H$, contrary to our supposition. Hence $\mathrm{Soc}(G) \leq G^{\mathfrak{F}}$ and $G^{\mathfrak{F}} = HN$ for any minimal normal subgroup N of G.

By [DH92, A, 14.3], $\mathrm{Soc}(G)$ normalises H because H is subnormal in G. Hence H is normal in $G^{\mathfrak{F}}$.

Assume that H is not core-free in G. Then H contains a minimal normal subgroup of G, N say. Hence $G^{\mathfrak{F}} = G^{\mathfrak{F}}N = HN = H$, against to our choice of G. Therefore H is core-free in G. □

Statement 6.5.10. *If* $\mathrm{Soc}(G)$ *is non-abelian, then* $\mathrm{Soc}(G)$ *is a minimal normal subgroup of* G *and* G *is in the boundary of* $\mathfrak{F}$. *In this case,* $G^{\mathfrak{F}}$ *is the unique minimal normal subgroup of* G.

Proof. First, note that for every minimal normal subgroup N of G, since $H \cap N$ is normal in N, we have that $N = (H \cap N) \times N^*$ and $G^{\mathfrak{F}} = H \times N^*$ with $N^* \neq 1$. This implies that H centralises N^*. If there exist two minimal normal subgroups N_1 and N_2 of G, then $G^{\mathfrak{F}} = H \times N_i^* \leq \mathrm{C}_G(N_{3-i}^*)$, for $i = 1, 2$. Therefore $N_i^* \leq \mathrm{Z}(G^{\mathfrak{F}})$ and both N_1 and N_2 are abelian. In other words, if $\mathrm{Soc}(G)$ is not a minimal normal subgroup of G, then $\mathrm{Soc}(G)$ is abelian.

Assume that $N = \mathrm{Soc}(G)$ is non-abelian. Then N is a minimal normal subgroup of G and $\mathrm{C}_G(N) = 1$. It is clear that N is a direct product of copies of a non-abelian simple group, E say. This means that $N \in \mathrm{D}_0(1, E) = \mathfrak{X}$. Then $G^{\mathfrak{F}}/H \in \mathfrak{X}$ and $(G^{\mathfrak{F}})^{\mathfrak{X}} \leq H$. Since $(G^{\mathfrak{F}})^{\mathfrak{X}}$ is normal in G, it follows that $(G^{\mathfrak{F}})^{\mathfrak{X}} = 1$ by Statement 6.5.9. Hence $G^{\mathfrak{F}} \in \mathfrak{X}$. Assume that N is a proper subgroup of $G^{\mathfrak{F}}$. Then there exists a copy of E centralising N. This is a contradiction. Hence $N = G^{\mathfrak{F}}$. In particular, G is in the boundary of $\mathfrak{F}$. □

Statement 6.5.11. *If* $\mathrm{Soc}(G)$ *is abelian, then* $G^{\mathfrak{F}}$ *is an elementary abelian* p*-group for some prime* p*.*

Proof. Let N be a minimal normal subgroup of G. By Statement 6.5.9, $G^{\mathfrak{F}} = HN$. Since $\mathrm{Soc}(G)$ is abelian, N is an elementary abelian p-group for some prime p. In particular, $\mathrm{O}^p(G^{\mathfrak{F}})$ and $(G^{\mathfrak{F}})'$ are normal subgroups of G contained in H. Since H is core-free in G, $\mathrm{O}^p(G^{\mathfrak{F}}) = (G^{\mathfrak{F}})' = 1$, and $G^{\mathfrak{F}}$ is an abelian p-group.

If, on the other hand, $\Phi(G^{\mathfrak{F}}) = 1$, then we can take N to be contained in $\Phi(G^{\mathfrak{F}})$. In this case, $G^{\mathfrak{F}} = HN = H$. This contradiction leads to $\Phi(G^{\mathfrak{F}}) = 1$, and $G^{\mathfrak{F}}$ is an elementary abelian p-group. □

Statement 6.5.12. $H = U^{\mathfrak{F}}V^{\mathfrak{F}}$*. Furthermore,* $U^{\mathfrak{F}}$ *and* $V^{\mathfrak{F}}$ *are proper subgroups of* U *and* V*, respectively.*

Proof. Whether or not $\mathrm{Soc}(G)$ is abelian, every subnormal subgroup of $G^{\mathfrak{F}}$ is a normal subgroup of $G^{\mathfrak{F}}$. In particular, $U^{\mathfrak{F}}$ and $V^{\mathfrak{F}}$ are normal in $G^{\mathfrak{F}}$. Therefore $H = U^{\mathfrak{F}}V^{\mathfrak{F}}$. Assume that $U^{\mathfrak{F}} = U$. Then U normalises $V^{\mathfrak{F}}$. This would imply that $V^{\mathfrak{F}}$ is normal in G, contrary to Statement 6.5.9. Therefore $U^{\mathfrak{F}} < U$ and $V^{\mathfrak{F}} < V$. □

Statement 6.5.13. $A = G_{\mathfrak{F}}$ *and* $G^{\mathfrak{F}}$ *is contained in* A*. Moreover,*

1. *A is a maximal normal subgroup of U and V, and G/A is a q-group for some prime $q \in \mathrm{char}\,\mathfrak{F}$;*
2. *if $\mathrm{Soc}(G)$ is a p-group, then $p \in \mathrm{char}\,\mathfrak{F}$; and*
3. *$G^{\mathfrak{N}} = \mathrm{O}^q(G)$ is contained in A.*

Proof. Let M be a proper subnormal subgroup of G such that $U \leq M$ and consider the subgroup $Y = \langle U, M \cap V\rangle$. The lattice properties of the subnormal subgroups imply that Y is subnormal in G. Furthermore Y is contained in M. Assume that U is a proper subgroup of Y. Then, by maximality of the pair (U, V), we have that $G^{\mathfrak{F}} = \langle Y^{\mathfrak{F}}, V^{\mathfrak{F}}\rangle$. By minimality of G, it follows that $Y^{\mathfrak{F}} = \langle U^{\mathfrak{F}}, (M \cap V)^{\mathfrak{F}}\rangle$. Therefore $G^{\mathfrak{F}} = \langle U^{\mathfrak{F}}, (M \cap V)^{\mathfrak{F}}, V^{\mathfrak{F}}\rangle = \langle U^{\mathfrak{F}}, V^{\mathfrak{F}}\rangle = H$ and we have reached a contradiction. Hence $U = Y$ and so $M \cap V \leq U$. In particular $M \cap V = U \cap V = A$. The arguments for a proper subnormal subgroup of G containing V are analogous.

Let M be a maximal normal subgroup of G such that $U \leq M$. By the foregoing arguments, we have that $M \cap V = A$. Therefore, A is a normal subgroup of V. Moreover, $V/A = V/(V \cap U) = V/(M \cap V) \cong VM/M = G/M$ is a simple group (note that V is not contained in M). Analogously, we deduce that A is normal in U and that U/A is a simple group. This implies that A is a normal subgroup of G and that A is a maximal normal subgroup of U and V. Since $A^{\mathfrak{F}}$ is a normal subgroup of G contained in H, it follows that $A^{\mathfrak{F}} = 1$ by Statement 6.5.9. This means that $A \in \mathfrak{F}$ and A is contained in $G_{\mathfrak{F}}$.

Since $G_{\mathfrak{F}}$ is normal in G, we have that $(UG_{\mathfrak{F}})^{\mathfrak{F}} = U^{\mathfrak{F}}$ by Corollary 6.5.6. Therefore $UG_{\mathfrak{F}}$ is a proper subnormal subgroup of G. Assume that $U < UG_{\mathfrak{F}}$. Since $G = \langle UG_{\mathfrak{F}}, V\rangle$, we deduce that $G^{\mathfrak{F}} = \langle (UG^{\mathfrak{F}})^{\mathfrak{F}}, V^{\mathfrak{F}}\rangle = \langle U^{\mathfrak{F}}, V^{\mathfrak{F}}\rangle$ by maximality of the pair (U, V), contrary to supposition. Hence $G_{\mathfrak{F}}$ is a subgroup of U. Analogously, $G_{\mathfrak{F}}$ is contained in V, and we have the equality.

Assume that U/A is a non-abelian simple group. Then U/A normalises V/A by Theorem 2.2.19, and V is normal in G. This contradiction yields that U/A is a cyclic group of prime order, q say. The same argument for V proves that V/A is a cyclic group of prime order, r say. If $r \neq q$, then $[U/A, V/A] \leq [\mathrm{O}_q(G/A), \mathrm{O}_r(G/A)] = 1$. Then $G/A = U/A \times V/A$ is abelian, and U and V are normal subgroups of G. This possibility cannot happen. Therefore, $r = q$ and G/A is a group generated by two subnormal q-subgroups, U/A and V/A, and so G/A is a q-group. Suppose that $A = 1$. Then G is a q-group. If $q \notin \operatorname{char}\mathfrak{F}$, then $U^{\mathfrak{F}} = U$ and $V^{\mathfrak{F}} = V$, contrary to Statement 6.5.12. Therefore we must have $q \in \operatorname{char}\mathfrak{F}$ and so $G \in \mathfrak{F}$ by [DH92, IX, 1.9]. This contradiction yields $A \neq 1$ and then A contains a minimal normal subgroup of G. On the other hand, we can assume that either $\mathrm{Soc}(G)$ is an elementary abelian p-group for some prime p, or $\mathrm{Soc}(G)$ is a non-abelian minimal normal subgroup of G by Statements 6.5.10 and 6.5.11. In both cases, we have that $\mathrm{Soc}(G)$ and $G^{\mathfrak{F}}$ are subgroups of A and Statement 2 holds. Since $G^{\mathfrak{F}} \leq A$, we have that the q-group G/A belongs to $\mathfrak{F}$. Therefore $q \in \operatorname{char}\mathfrak{F}$ and Statement 1 holds.

Now we prove that if M is a maximal normal subgroup of G, then A is contained in M. Assume that there exists a maximal normal subgroup M of G such that A is not contained in M. Then $G = AM$, $U = A(U \cap M)$, and $V = A(V \cap M)$. The subnormal subgroup $T = \langle U \cap M, V \cap M\rangle$ is a supplement of A in G and $A \cap M \leq T \leq M$. Hence $M = G \cap M = TA \cap M = T(A \cap M) = T$. By minimality of G, $M^{\mathfrak{F}} = \langle (U \cap M)^{\mathfrak{F}}, (V \cap M)^{\mathfrak{F}}\rangle$. On the other hand, since $G = MU$, by Proposition 6.5.5, it follows that $G^{\mathfrak{F}} = M^{\mathfrak{F}}U^{\mathfrak{F}} = \langle (U \cap M)^{\mathfrak{F}}, (V \cap M)^{\mathfrak{F}}, U^{\mathfrak{F}}\rangle = \langle (V \cap M)^{\mathfrak{F}}, U^{\mathfrak{F}}\rangle = \langle U^{\mathfrak{F}}, V^{\mathfrak{F}}\rangle$, contrary to our initial supposition. Therefore, every maximal normal subgroup of G contains A.

Clearly, $G^{\mathfrak{N}}$ is contained in A. Since every maximal subgroup of $G/G^{\mathfrak{N}}$ is normal in $G/G^{\mathfrak{N}}$, it follows that $A/G^{\mathfrak{N}} \leq \Phi(G/G^{\mathfrak{N}})$. Since G/A is a q-group, we deduce that $G/G^{\mathfrak{N}}$ is a q-group. Then $G^{\mathfrak{N}} = \mathrm{O}^q(G)$ and 3 holds. □

Next, we assume that $\mathrm{Soc}(G)$ is abelian. This implies that $G^{\mathfrak{F}}$ is an elementary abelian p-group for some prime $p \in \operatorname{char}\mathfrak{F}$ by Statements 6.5.11 and 6.5.13. Denote $B = G^{\mathfrak{F}}$. Then B is a G-module over the field $\mathrm{GF}(p)$.

Statement 6.5.14. *B is a completely reducible A-module over* GF(p).

Proof. We denote by $\mathrm{J}(B_A)$ the intersection of all maximal A-submodules of B_A. Since A is normal in G, the action of G permutes these maximal submodules, and thus $\mathrm{J}(B_A)$ is a normal subgroup of G.

Suppose that $\mathrm{J}(B_A) \neq 0$, and let N be a minimal normal subgroup of G such that $N \leq \mathrm{J}(B_A)$. By Statement 6.5.9, we have, in additive notation, that $B = H + N$. Since B, H, and N are A-submodules and $N \leq \mathrm{J}(B_A)$, we have that $B = H$ by Nakayama's lemma ([HB82a, VII, 6.4]). This is a contradiction. Therefore, $\mathrm{J}(B_A) = 0$, and B is a completely reducible A-module over GF(p) by [HB82a, VII, 1.6]. □

It is clear that H is an A-submodule of B.

Statement 6.5.15. *Let Z be an arbitrary irreducible A-submodule of H. Then if Z_1 is an irreducible A-submodule of B, then there exists $g \in G$ such that Z_1 is A-isomorphic to Z^g.*

Proof. Let Z be an irreducible A-submodule of H and consider the normal closure $\langle Z^G \rangle = \sum_{g \in G} Z^g$. Then $\langle Z^G \rangle_A$ is a completely reducible A-module and is a direct sum of its irreducible submodules which are isomorphic to some conjugate of Z. Let N be a minimal normal subgroup of G such that $N \leq \langle Z^G \rangle$. Hence N_A is again a completely reducible A-module and is a direct sum of its irreducible submodules, which are isomorphic to some conjugate of Z. On the other hand, $B = H + N$ by Statement 6.5.9. Therefore, every A-composition factor of B/H is isomorphic to a conjugate of Z.

Let Z_1 be an irreducible A-submodule of B. The normal closure $N_1 = \langle Z_1^G \rangle$ is not contained in H, and every A-composition factor of N_1 is isomorphic to a conjugate of Z_1. Again by Statement 6.5.9, $B = H + N_1$ and so every A-composition factor of B/H is isomorphic to a conjugate of Z_1. This implies that Z_1 is A-isomorphic to a conjugate of Z. □

The following lemma is needed in the proof of our next statement.

Lemma 6.5.16. *Let K be a field of characteristic p, and let G be a group with a normal subgroup N such that G/N is a p-group. If W is an irreducible KN-module, then the induced KG-module W^G has all of its composition factors isomorphic.*

Proof. Let T be the inertia subgroup of W in G. First note that $(W^T)_N = \bigoplus_g Wg$, where g runs over a transversal of N in T. This is a particular case of Mackey's theorem ([DH92, B, 6.21]). Since T is the inertia subgroup of W in G, we have that $Wg \cong W$ for all $g \in T$. Therefore $(W^T)_N$ is homogeneous, and all of its composition factors are isomorphic to W. In particular, if U/V is a composition factor of W^T, then $(U/V)_N$ is homogeneous, and all its composition factors are isomorphic to W.

If U/V is a composition factor of W^T, then the G-module $(U/V)^G \cong U^G/V^G$ is irreducible by [DH92, B, 7.4]. It is thus sufficient to prove that all composition factors of W^T are isomorphic. Let U/V be a composition factor of W^T. Then, by [DH92, B, 8.3], $(U/V)_N$ is an irreducible N-module. Hence $(U/V)_N$ is isomorphic to W. By [DH92, B, 5.17], all composition factors of W^T are isomorphic. □

Statement 6.5.17. *If $p = q$, then all composition factors of B are isomorphic.*

Proof. Suppose that $p = q$. Let Z be an irreducible A-submodule of B. By Lemma 6.5.16, the induced module Z^G has all its composition factors isomorphic. Let M be a composition factor of B, and let Z_1 be an irreducible A-submodule of M_A. By Statement 6.5.15, Z_1 is A-isomorphic to Z^g for some $g \in G$. Then $Z_1^{g^{-1}}$ is an irreducible A-submodule of M which is isomorphic to Z. In other words, M_A has an irreducible submodule isomorphic to Z, that is, $0 \neq \mathrm{Hom}_{KA}(Z, M_A)$. By Nakayama's reciprocity theorem [DH92, B, 6.5], it follows that $0 \neq \mathrm{Hom}_{KG}(Z^G, M)$. Therefore a composition factor of Z^G is isomorphic to M, and then all composition factors of Z^G are isomorphic to M. □

Statement 6.5.18. *If $p \neq q$, then $B = \mathrm{Soc}(G)$.*

Proof. Let Z be an irreducible A-submodule of B_A. Since $p \neq q$, it follows that Z^G is a completely reducible G-module by [HB82a, VII, 9.4].

Denote by α the inclusion of Z in B_A. Applying [HB82a, VII, 4.4], there exists a KG-homomorphism $\alpha' \colon Z^G \longrightarrow B$ such that $(z \otimes g)\alpha' = z^g$ for all $g \in G$ and all $z \in Z$. Hence $\mathrm{Im}(\alpha') = \langle Z^G \rangle$, the normal closure of Z in G. Therefore $\langle Z^G \rangle$ is a completely reducible G-module and $\langle Z^G \rangle \leq \mathrm{Soc}(G)$. In particular, Z is contained in $\mathrm{Soc}(G)$. Since, by Statement 6.5.14, B_A is a completely reducible A-module, it follows that B is contained in $\mathrm{Soc}(G)$ and the equality holds by Statement 6.5.9. □

The most important examples of Fitting formations are as follows:

1. The solubly saturated Fitting formations (see Chapter 3).
2. The Fitting formations constructed by Fitting families of modules ([DH92, Chapter IX, 2, Construction F]). Fix a prime r. Let K be an extension field of $\mathrm{GF}(r)$. For any r-soluble group G, denote $\mathfrak{T}_K(G)$ the class of all irreducible KG-modules V such that V is a composition factor of the module $W^K = W \otimes K$, where W is an r-chief factor of G.

Suppose that, for every group G, a class of irreducible KG-modules $\mathfrak{M}(G)$ is defined. Then the class $\mathfrak{M} := \bigcup_G \mathfrak{M}(G)$ is called a *Fitting family* if it satisfies the four properties listed in Definition 2.5.5. Applying Theorem 2.5.6, the class

$$\mathfrak{T}(1, \mathfrak{M}) = \bigl(G : G \text{ is } r\text{-soluble and } \mathfrak{T}_K(G) \subseteq \mathfrak{M}(G)\bigr)$$

is a Fitting formation provided that $\mathfrak{M}$ is a Fitting family.

In both cases, we have a way to distinguish between the abelian r-chief factors of any group X in the following sense:

1. If $\mathfrak{F}$ is a solubly saturated formation defined by the canonical $\mathbb{P}$-local formation function F, then an abelian r-chief factor M of X can be $\mathfrak{F}$-central if $X/\mathrm{C}_X(M) \in F(r)$ or $\mathfrak{F}$-eccentric otherwise.
2. If $\mathfrak{F} = \mathfrak{T}(1, \mathfrak{M})$ is a Fitting formation constructed by a Fitting family of modules $\mathfrak{M}$, then an abelian r-chief factor M of X can be such that all composition factors of M^K are in $\mathfrak{M}(X)$ or not.

Let X be an arbitrary group, and let M be an X-module over $\mathrm{GF}(r)$. Denote by $\mathrm{Irr}(M)$ the class of all irreducible X-modules occurring as composition factors of M.

Suppose that $\mathfrak{F}$ is either a solubly saturated formation or a Fitting formation defined by a Fitting family of modules, and let $\mathrm{Mod}_{\mathfrak{F}}(U)$ denote the class of all irreducible U-modules occurring as

1. $\mathfrak{F}$-central chief factors of U below B, if $\mathfrak{F}$ is a solubly saturated Fitting formation, or

2. abelian chief factors M of U below B such that every composition factor of M^K is in $\mathfrak{M}(U)$, if $\mathfrak{F} = \mathfrak{T}(1, \mathfrak{M})$ is a Fitting formation constructed by a Fitting family of modules $\mathfrak{M}$.

Analogously, let $\mathrm{Mod}_{\mathfrak{F}}(V)$ denote the corresponding set for V.

Statement 6.5.19. *If $\mathfrak{F}$ is either a solubly saturated Fitting formation or a Fitting formation defined by a Fitting family of modules, then G is in the boundary of $\mathfrak{F}$.*

Proof. Assume first that $p = q$. In this case, all composition factors of B are isomorphic G-modules by Statement 6.5.17. We consider a G-composition series of B, $0 = B_0 \leq B_1 \leq \cdots \leq B_r = B$ say.

The composition factor B_1 is a minimal normal subgroup of G and so $B = HB_1$ by Statement 6.5.9. Since H is core-free in G, it follows that $B_1 \neq U^{\mathfrak{F}}$. Moreover, B_1 is a completely reducible U-module by Clifford's theorems [DH92, B, 7.3]. It then decomposes as $B_1 = B_1' \oplus B_1^*$, where $B_1' = B_1 \cap U^{\mathfrak{F}}$. Since B_1 is not contained in $U^{\mathfrak{F}}$, we have that $B_1^* \neq 0$.

Let M be a U-composition factor of B_1^*. Then M is isomorphic to a U-composition factor of B_1/B_1', which is a section of $U/U^{\mathfrak{F}} \in \mathfrak{F}$. This implies that $M \in \mathrm{Mod}_{\mathfrak{F}}(U)$ and $\mathrm{Irr}(B_1^*)$ is contained in $\mathrm{Mod}_{\mathfrak{F}}(U)$.

Assume now that $B_1' \neq 0$ and let M be an irreducible U-submodule of B_1'. Then $U^{\mathfrak{F}} = M \oplus M_1$, for some U-submodule M_1 of $U^{\mathfrak{F}}$. Since $U/M_1 \notin \mathfrak{F}$ and $U/U^{\mathfrak{F}} \in \mathfrak{F}$, it follows that M is not in $\mathrm{Mod}_{\mathfrak{F}}(U)$. Consequently $\mathrm{Irr}(B_1') \cap \mathrm{Mod}_{\mathfrak{F}}(U) = \emptyset$ if $B_1' \neq 0$.

The arguments for V are completely analogous. Hence B_1, considered as V-module, decomposes as $B_1 = B_1'' \oplus B_1^{**}$ where $B_1'' \leq B_1 \cap V^{\mathfrak{F}}$ and B_1^{**} is a non-trivial V-submodule of B_1 such that $\mathrm{Irr}(B_1^{**})$ is contained in $\mathrm{Mod}_{\mathfrak{F}}(V)$. Moreover, $\mathrm{Irr}(B_1'') \cap \mathrm{Mod}_{\mathfrak{F}}(V) = \emptyset$ provided that $B_1'' \neq 0$.

Let $B' = U^{\mathfrak{F}} + B_{r-1}$. Assume that $B'/B_{r-1} \neq 0$ and let M/B_{r-1} be an irreducible U-submodule of B'/B_{r-1}. Since B'/B_{r-1} is completely reducible as U-module, it follows that $B'/B_{r-1} = M/B_{r-1} \oplus M_1/B_{r-1}$. Note that

$M_1 = M_1 \cap (U^{\mathfrak{F}} + B_{r-1}) = (M_1 \cap U^{\mathfrak{F}}) + B_{r-1}$ and $U^{\mathfrak{F}} + M_1 = U^{\mathfrak{F}} + B_{r-1} = B'$. Therefore $U^{\mathfrak{F}}/(M_1 \cap U^{\mathfrak{F}})$ is U-isomorphic to $(U^{\mathfrak{F}} + M_1)/M_1 \cong B'/M_1$ and B'/M_1 is U-isomorphic to M/B_{r-1}. Since $M/B_{r-1} \neq 0$, we have that $U^{\mathfrak{F}}$ is not contained in M_1 and so $U/(M_1 \cap U^{\mathfrak{F}})$ is not in $\mathfrak{F}$. Hence $M/B_{r-1} \notin \mathrm{Mod}_{\mathfrak{F}}(U)$. Consequently $\mathrm{Irr}(B'/B_{r-1}) \cap \mathrm{Mod}_{\mathfrak{F}}(U) = \emptyset$.

The same argument holds for V, that is, if $B'' = V^{\mathfrak{F}} + B_{r-1}$, then $\mathrm{Irr}(B''/B_{r-1}) \cap \mathrm{Mod}_{\mathfrak{F}}(V) = \emptyset$ provided that $B''/B_{r-1} \neq 0$.

Suppose that $r \geq 2$. Then the composition factors B_1 and B/B_{r-1} are different. Furthermore $B = B' + B''$.

Assume that $B'/B_{r-1} = 0$. Then $B = B''$ and $\mathrm{Irr}(B/B_{r-1}) \cap \mathrm{Mod}_{\mathfrak{F}}(V) = \emptyset$. Let $\varphi\colon B/B_{r-1} \longrightarrow B_1$ be a G-isomorphism. Since φ is a V-isomorphism, it follows that $\mathrm{Irr}(B_1) \cap \mathrm{Mod}_{\mathfrak{F}}(V) = \emptyset$. This is a contradiction because B_1^{**} is a non-trivial V-submodule of B_1 such that $\mathrm{Irr}(B_1^{**}) \subseteq \mathrm{Mod}_{\mathfrak{F}}(V)$. Consequently $B'/B_{r-1} \neq 0$ and $B''/B_{r-1} \neq 0$. Moreover $\varphi(B'/B_{r-1})$ is contained in B_1' and $\varphi(B''/B_{r-1})$ is contained in B_1''. Hence $B_1 = \varphi(B/B_{r-1}) = \varphi(B'/B_{r-1} + B''/B_{r-1}) = \varphi(B'/B_{r-1}) + \varphi(B''/B_{r-1}) = B_1' + B_1'' \leq (B_1 \cap U^{\mathfrak{F}}) + (B_1 \cap V^{\mathfrak{F}}) \leq H$. This is a contradiction. Therefore $r = 1$ and B is an irreducible G-module.

Consider now the case where $p \neq q$. Then, by Statement 6.5.18, B is a completely reducible G-module. By Clifford's theorem, $U^{\mathfrak{F}}$ is a completely reducible U-module. If M is an irreducible U-submodule of $U^{\mathfrak{F}}$, then there exists a U-submodule M_0 of $U^{\mathfrak{F}}$ such that $U^{\mathfrak{F}} = M \oplus M_0$. Since U/M_0 is not in $\mathfrak{F}$, we have that $M \notin \mathrm{Mod}_{\mathfrak{F}}(U)$. That is, $\mathrm{Irr}(U^{\mathfrak{F}}) \cap \mathrm{Mod}_{\mathfrak{F}}(U) = \emptyset$. On the other hand, since $U/U^{\mathfrak{F}} \in \mathfrak{F}$, it follows that $M \in \mathrm{Mod}_{\mathfrak{F}}(U)$, for every chief factor M of U between $U^{\mathfrak{F}}$ and U. That is, $\mathrm{Irr}(B/U^{\mathfrak{F}})$ is contained in $\mathrm{Mod}_{\mathfrak{F}}(U)$. With a similar argument, we deduce the corresponding result for V. Hence

$$\mathrm{Irr}(B_U/U^{\mathfrak{F}}) \cap \mathrm{Irr}(U^{\mathfrak{F}}) = \emptyset = \mathrm{Irr}(V^{\mathfrak{F}}) \cap \mathrm{Irr}(B_V/V^{\mathfrak{F}}).$$

Now suppose that $B = N_1 \times \cdots \times N_r$, where N_i is a minimal normal subgroup of G, $i = 1, \dots, r$, and $r \geq 2$. Each N_i can be decomposed as $N_i = N_i^* \oplus (N_i \cap U^{\mathfrak{F}})$, where N_i^* is a complement of $N_i \cap U^{\mathfrak{F}}$ in N_i as U-modules. Then $B_U = (N_1^* \oplus \cdots \oplus N_r^*) \oplus \big((N_1 \cap U^{\mathfrak{F}}) \oplus \cdots \oplus (N_r \cap U^{\mathfrak{F}})\big)$. Denote $B^* = N_1^* \oplus \cdots \oplus N_r^*$. Then $U^{\mathfrak{F}} \cap B^* = 0$ because $\mathrm{Irr}(U^{\mathfrak{F}}) \cap \mathrm{Irr}(B_U/U^{\mathfrak{F}}) = \emptyset$. Hence $U^{\mathfrak{F}} = (N_1 \cap U^{\mathfrak{F}}) \oplus \cdots \oplus (N_r \cap U^{\mathfrak{F}})$. The same arguments hold for V: $V^{\mathfrak{F}} = (N_1 \cap V^{\mathfrak{F}}) \oplus \cdots \oplus (N_r \cap V^{\mathfrak{F}})$. Comparing the two decomposition as vector spaces, $B = N_1 + H = N_1 + U^{\mathfrak{F}} + V^{\mathfrak{F}} = N_1 + \big((N_1 \cap U^{\mathfrak{F}}) \oplus \cdots \oplus (N_r \cap U^{\mathfrak{F}})\big) + \big((N_1 \cap V^{\mathfrak{F}}) \oplus \cdots \oplus (N_r \cap V^{\mathfrak{F}})\big) = N_1 \oplus \big((N_2 \cap U^{\mathfrak{F}}) + (N_2 \cap V^{\mathfrak{F}})\big) \oplus \cdots \oplus \big((N_r \cap U^{\mathfrak{F}}) + (N_r \cap V^{\mathfrak{F}})\big) = N_1 \oplus \cdots \oplus N_r$, we deduce that $N_i = (N_i \cap U^{\mathfrak{F}}) + (N_i \cap V^{\mathfrak{F}})$ for each $i \geq 2$. Therefore $N_i \leq U^{\mathfrak{F}} + V^{\mathfrak{F}} = H$ for each $i \geq 2$. This contradiction leads to $r = 1$ and B is a minimal normal subgroup of G.

Consequently, in both cases, we have that G is a monolithic group in the boundary of $\mathfrak{F}$. □

For convenience, we incorporate the class of all groups satisfying the above statements in a formal definition.

Definition 6.5.20. *Let $\mathfrak{F}$ be a Fitting formation. Define* $\mathrm{b}_3(\mathfrak{F})$ *as the class of all triples* (G, U, V) *such that*

1. $G \in \mathrm{b}(\mathfrak{F})$ *and* U *and* V *are subnormal subgroups of* G*;*
2. $G = \langle U, V\rangle$*;*
3. $U \cap V = G_{\mathfrak{F}} \neq 1$*;*
4. $U/G_{\mathfrak{F}}$ *and* $V/G_{\mathfrak{F}}$ *are cyclic groups of order* q*, a prime.*

Note that if $(G, U, V) \in \mathrm{b}_3(\mathfrak{F})$, then $G^{\mathfrak{F}}$ is contained in $G_{\mathfrak{F}}$ and $G/G_{\mathfrak{F}}$ is a q-group, $q \in \operatorname{char}\mathfrak{F}$.

The above statements lead to the following result.

Theorem 6.5.21. *Let $\mathfrak{F}$ be a Fitting formation. Suppose that either*

1. $\mathfrak{F}$ *is a solubly saturated Fitting formation, or*
2. $\mathfrak{F} = \mathfrak{T}(1, \mathfrak{M})$ *is a Fitting formation defined by a Fitting family of modules* $\mathfrak{M}$ *constructed over an extension field* K *of* $\mathrm{GF}(r)$*.*

Then the following statements are equivalent:

1. $\mathfrak{F}$ *satisfies the Wielandt property for residuals.*
2. *For every triple* $(G, U, V) \in \mathrm{b}_3(\mathfrak{F})$*, we have that* $G^{\mathfrak{F}} = \langle U^{\mathfrak{F}}, V^{\mathfrak{F}}\rangle$*.*

Applying Theorem 6.5.21, a large number of Fitting formations satisfying the Wielandt property for residuals appear.

Corollary 6.5.22. *Let $\mathfrak{F}$ be a Fitting formation. Then $\mathfrak{F}$ satisfies the Fitting property for residuals provided that one of the following conditions hold:*

1. $\mathfrak{S}_p\mathfrak{F} = \mathfrak{F}$*, for all primes* $p \in \operatorname{char}\mathfrak{F}$*.*
2. $\mathfrak{F}\mathfrak{S}_p = \mathfrak{F}$*, for all primes* $p \in \operatorname{char}\mathfrak{F}$*.*
3. $\mathfrak{F}$ *is solubly saturated, and its boundary is composed of non-abelian simple groups.*
4. $\operatorname{char}\mathfrak{F} = \emptyset$*.*
5. $\mathfrak{F} = \mathrm{E}\,\mathfrak{X}$ *for some class* $\mathfrak{X}$ *of simple groups.*
6. $\mathfrak{F} = \mathrm{D}_0(1, \mathfrak{X}_1)$*, where* $\mathfrak{X}_1$ *is a class of non-abelian simple groups.*

Let p be a prime, and let $\mathfrak{M}_p$ be the class of all groups whose abelian p-chief factors are central. It is rather clear that $\mathfrak{M}_p$ is a Fitting formation. Moreover, $\mathfrak{M}_p$ is solubly saturated by Lemma 3.2.15 and $\mathfrak{M}_p \cap \mathfrak{S}$ is the class of all soluble p-nilpotent groups. The $\mathfrak{M}_p$-radical of a group G is the intersection of the centralisers of the abelian p-chief factors of G. This subgroup also appears when a $\mathbb{P}$-local definition of a solubly saturated formation is considered (see Section 3.2).

Corollary 6.5.23. *Let p be a prime. Then $\mathfrak{M}_p$ satisfies the Wielandt property for residuals.*

Proof. Applying Theorem 6.5.21, we need only consider triples in $\mathrm{b}_3(\mathfrak{M}_p)$.

Suppose that (G, U, V) is a triple in $\mathrm{b}_3(\mathfrak{M}_p)$. Then $G = \langle U, V\rangle$ is a monolithic group in $\mathrm{b}(\mathfrak{M}_p)$, U and V are subnormal subgroups of G, $U \cap V$ is the $\mathfrak{M}_p$-radical of G and $G/(U\cap V)$ is a q-group for some prime $q \in \operatorname{char}\mathfrak{M}_p = \mathbb{P}$. Denote by N the $\mathfrak{M}_p$-residual of G. Then N is an abelian p-group contained in $U \cap V = A$. Since N is a completely reducible A-module, it follows that $A \leq \mathrm{C}_G(N)$. Consequently $G = QA = Q\,\mathrm{C}_G(N)$ for every Sylow q-subgroup Q of G. Let $B = NQ$. We have that B is soluble and N is a minimal normal subgroup of Q. It is clear that Q does not centralise N because $G \notin \mathfrak{M}_p$. This implies that $B^{\mathfrak{M}_p} = N$. On the other hand, $U = A(Q \cap U)$ and $V = A(Q\cap V)$. Hence $G = A\langle Q\cap U, Q\cap V\rangle$ and $Q = \langle Q\cap U, Q\cap V\rangle$. It means that $B = N\langle Q\cap U, Q\cap V\rangle = \langle N(Q\cap U), N(Q\cap V)\rangle = \langle U\cap B, V\cap B\rangle$. Note that $B^{\mathfrak{M}_p} = B^{\mathfrak{F}}$, where $\mathfrak{F}$ is the saturated formation of all p'-nilpotent groups. Combining Proposition 6.5.4 (1) and Corollary 6.5.22 (1), it follows that $\mathfrak{F}$ satisfies the Wielandt property for residuals. Therefore $B^{\mathfrak{M}_p} = \langle (U\cap B)^{\mathfrak{M}_p}, (V\cap B)^{\mathfrak{M}_p}\rangle \leq \langle U^{\mathfrak{M}_p}, V^{\mathfrak{M}_p}\rangle$ and so $N = \langle U^{\mathfrak{M}_p}, V^{\mathfrak{M}_p}\rangle$. ◻

Let $\mathfrak{F}$ be a solubly saturated formation. Then, applying Theorem 3.2.14, there exists a Baer function f such that $\mathfrak{F} = \mathrm{LF}_{\mathbb{P}}(f)$. Denote $\operatorname{Supp}(f) = \{p \in \mathbb{P} : f(p) \neq \emptyset\} \cup \{S \in \mathfrak{J}\setminus\mathbb{P} : f(S) \neq \emptyset\}$. Then it rather clear that $\mathfrak{F} = \bigcap_{p\in\operatorname{Supp}(f)} \mathfrak{M}_p \circ f(p) \cap \bigcap_{S\in\operatorname{Supp}(f)\setminus\mathbb{P}} \mathrm{E}\big((S)'\big)\circ f(S)$ by Remarks 3.1.2 and Remark 3.1.9.

Therefore, applying Proposition 6.5.4 and Corollary 6.5.23, we have:

Theorem 6.5.24 ([KS95]). *Let $\mathfrak{F}$ be a solubly saturated formation and let f be a Baer function $\mathbb{P}$-locally defining $\mathfrak{F}$. If for all $S \in \operatorname{Supp}(f)$, $f(S)$ satisfies the Wielandt property for residuals, then $\mathfrak{F}$ satisfies the Wielandt property for residuals.*

Corollary 6.5.25. *Let $\mathfrak{F}$ be a saturated formation locally defined by a formation function f. If for all primes p, the formations $f(p)$ satisfy the Wielandt property for residuals, then $\mathfrak{F}$ satisfies the Wielandt property for residuals.*

Proof. Set
$$g(J) = \begin{cases} f(p) & \text{when } J \cong C_p,\ p \in \mathbb{P} \text{ and} \\ \bigcap_{p \mid |J|} f(p) & \text{when } J \in \mathfrak{J}\setminus\mathbb{P}, \end{cases}$$
then it is clear that $\mathfrak{F} = \mathrm{LF}_{\mathbb{P}}(g)$. Applying Proposition 6.5.4 (2), $g(J)$ satisfies the Wielandt property for residuals if $J \in \mathfrak{J}\setminus\mathbb{P}$. By Theorem 6.5.24, $\mathfrak{F}$ satisfies the Wielandt property for residuals. ◻

Corollary 6.5.26. *Any soluble subgroup-closed Fitting formation satisfies the Wielandt property for residuals.*

Proof. Any soluble subgroup-closed Fitting formation $\mathfrak{F}$ is a primitive saturated formation. Therefore, $\mathfrak{F}$ has a local definition f such that $f(p)$ satisfies the Wielandt property for residuals for all prime numbers p (see [DH92, page 497]). ◻

Example 6.5.27. (see Example 2.2.17) Let $\mathfrak{Q}$ be the Fitting formation of all quasinilpotent groups. Then $\mathfrak{Q}$ is a solubly saturated formation $\mathbb{P}$-locally defined by the $\mathbb{P}$-local formation function f given by

$$f(S) = \begin{cases} (1) & \text{when } S \cong C_p, \text{ and} \\ \mathrm{D}_0(1, S) & \text{when } S \in \mathfrak{J} \setminus \mathbb{P}. \end{cases}$$

Since $f(S)$ satisfies the Wielandt property for residuals for all S, it follows that $\mathfrak{Q}$ satisfies the Wielandt property for residuals by Theorem 6.5.24.

In the next examples, we work in the universe of all soluble groups.

Let $\mathfrak{X}_i$ be Fitting formations, $i = 1, 2$. For every group G, denote by $\mathfrak{M}(G)$ the class of all irreducible KG-modules V such that $V = U \otimes W$ with U π-special, W π'-special, and $G/\operatorname{Ker}(G \operatorname{on} U) \in \mathfrak{X}_1$ and $G/\operatorname{Ker}(G \operatorname{on} W) \in \mathfrak{X}_2$. Applying Theorem 2.5.10, $\mathfrak{M} = \mathfrak{M}(K, \mathcal{P}, \mathfrak{X}_1, \mathfrak{X}_2) = \bigcup_G \mathfrak{M}(G)$ is a Fitting family. Let $\mathfrak{T}(1, \mathfrak{M}) = \mathfrak{T}(1, r, \mathcal{P}, \mathfrak{X}_1, \mathfrak{X}_2)$be the Fitting formation defined by $\mathfrak{M}$.

Theorem 6.5.28. *Let π be a set of primes and consider the partition $\mathcal{P} = \{\pi, \pi'\}$ of the set of all prime numbers. The Fitting formation $\mathfrak{F} = \mathfrak{T}(1, \mathfrak{M}) = \mathfrak{T}(1, r, \mathcal{P}, \mathfrak{X}_1, \mathfrak{X}_2)$ satisfies the Wielandt property for residuals in the following case: $\mathfrak{X}_1 = \mathfrak{S}_\rho$ and $\mathfrak{X}_2 = \mathfrak{S}_\sigma$ for some sets of primes ρ and σ (not both empty).*

The following result is used in the proof of Theorem 6.5.28. It can be proved by using similar arguments to those used in the proof of [HB82a, VII, 9.13].

Lemma 6.5.29. *Let N be a normal subgroup of G, and let V_1 and V_2 be two KG-modules such that*

1. *$(V_1)_N$ is absolutely irreducible, and*
2. *V_2 is absolutely irreducible and $(V_2)_N$ is homogeneous, and all of its constituents are isomorphic to $(V_1)_N$. Write $(V_2)_N \cong s(V_1)_N$.*

Then there exists an irreducible $K(G/N)$-module W with $\dim W = s$ such that $V_2 \cong V_1 \otimes W$.

Proof (of Theorem 6.5.28). We use only the restriction on the $\mathfrak{X}_i$ at one point, and so have written the proof as far as possible to be independent of that hypothesis. Applying Theorem 6.5.21, we need only consider groups in $\mathrm{b}_3(\mathfrak{F})$. Hence we suppose that G is in the boundary of $\mathfrak{F}$ and, moreover, that U and V are subnormal subgroups of G satisfying $G = \langle U, V \rangle$, $A = U \cap V = G_{\mathfrak{F}} \neq 1$, and U/A and V/A are of prime order q, $q \in \operatorname{char} \mathfrak{F}$. Note that G/A is a q-group and $\mathrm{O}^q(G) = G^{\mathfrak{N}} = \mathrm{O}^q(U) = \mathrm{O}^q(V) = \mathrm{O}^q(A)$. Furthermore, G has a unique minimal normal subgroup $B = G^{\mathfrak{F}}$ which is a p-group for some prime p. First, we observe that $p = r$ (the characteristic of K), since otherwise all r-chief factors would come from $G/B \in \mathfrak{F}$, and so G would be in $\mathfrak{F}$. We are working with a field K which is algebraically closed. However, when dealing with dimensions of KX-modules for a subgroup X of G, we can assume that K is a splitting field for G and all its subgroups. In fact, by Brauer's theorem [HB82a, VII, 2.6], we can assume that K is a finite Galois extension of $k = \mathrm{GF}(p)$. We are interested in the behaviour of the irreducible components of B^K. By [HB82a, VII, 1.15], the KG-module B^K is completely reducible. Let N be an irreducible component of B^K. Applying [HB82a, VII, 1.18 (b)], every irreducible KG-submodule of B^K is G-isomorphic to N^η for some $\eta \in \mathrm{G}(K/k)$.

We collect some properties we need. First, if L is a normal subgroup of G, then a KL-module Q is π-special if and only if all of its G-conjugates are π-special, and $L/\operatorname{Ker}(L \operatorname{on} Q) \in \mathfrak{X}_i$ if and only if the same is true for all of the G-conjugates of Q. Further, Q is π-special if and only if all of its Galois conjugates are special and $L/\operatorname{Ker}(L \operatorname{on} Q) \in \mathfrak{X}_i$ if and only if the same is true for all of the Galois conjugates of Q.

Clearly we may assume that $q \in \pi$. Suppose, by way of contradiction, that $B \neq \langle U^{\mathfrak{F}}, V^{\mathfrak{F}} \rangle$. Then $\operatorname{Core}_G(\langle U^{\mathfrak{F}}, V^{\mathfrak{F}} \rangle) = 1$. We have that B_U is completely reducible as U-module and so $B = U^{\mathfrak{F}} \oplus B_0$, with $B \neq B_0 \neq 0$. It follows that $(U^{\mathfrak{F}})^K$ can contain no components in $\mathfrak{M}(U)$ and $(B_0)^K$ must have all its components in $\mathfrak{M}(U)$. Let N be an irreducible component of B^K. If no component of $(B_U)^K$ is in $\mathfrak{M}(U)$, then no component of $(B_U)^K$ is in $\mathfrak{M}(U)$ and thus $B_0 = 0$. This is a contradiction. If every component of N_U is in $\mathfrak{M}(U)$, then every component of $(B_U)^K$ is in $\mathfrak{M}(U)$. This implies that $U^{\mathfrak{F}} = 1$ (or, equivalently, $B = B_0$). It is also a contradiction. Hence, if we denote by $D(N_U)$ the sum of all irreducible KU-submodules of N_U which do not lie in $\mathfrak{M}(U)$, then $0 \neq D(N_U) \neq N$. In particular, N_U is not a homogeneous module. Similar remarks apply to V.

Furthermore, $N_A = N_1 \oplus \cdots \oplus N_t$, where the N_i are irreducible KA-modules, all conjugate by elements of G. Since $A \in \mathfrak{F}$, for each i we have that $N_i = Z_i \otimes X_i$, where Z_i is a π-special irreducible KA-module with $A/\operatorname{Ker}(A \operatorname{on} Z_i) \in \mathfrak{X}_1$ and X_i is a π'-special irreducible KA-module with $A/\operatorname{Ker}(A \operatorname{on} X_i) \in \mathfrak{X}_2$. Note that since all N_i are G-conjugates, so are the Z_i and the X_i, because if $N_i^g \cong N_1$ for some $G \in G$, then $Z_i^g \otimes X_i^g \cong (Z_i \otimes X_i)^g = N_i^g \cong N_1 = Z_1 \otimes X_1$, and thus $Z_i^g \cong Z_1$ and $X_i^g \cong X_1$, by [CK87, 2.4].

We break the proof into a number of cases.

Case 1. Suppose that all of the Z_i, as well as all of the X_i, are isomorphic. This is equivalent to saying that N_A is homogeneous. If $p = q$, then N_A is irreducible by [DH92, B, 8.3]. This implies that N_U is irreducible and either $N \in \mathfrak{M}(U)$ or $N \notin \mathfrak{M}(U)$. This contradiction yields $p \neq q$. Since N_U is a completely irreducible U-module, we can write $N_U = L_1 \oplus \cdots \oplus L_u$, where L_i are irreducible KU-modules. Analogously, $N_V = P_1 \oplus \cdots \oplus P_v$, where P_i are irreducible KV-modules.

If L_j is an irreducible component of N_U such that N_i is a component of $(L_j)_A$, then $(L_j)_A \cong t_j N_i$ for some t_j. Since q divides $|K|-1$ by [HB82a, VII, 2.6], we have that t_j is either 1 or q by [DH92, B, 8.5].

Analogously, if P_k is an irreducible component of N_V such that N_i is a component of $(P_k)_A$, then either $(P_k)_A = N_i$ or $(P_k)_A \cong qN_i$. We have that

$N_A = N_1 \oplus \cdots \oplus N_r$, and each irreducible component N_i is $N_i = Z \otimes X$, with Z a π-special KA-module and X a π'-special KA-module. Applying [CK87, 2.3], there is a unique π'-special KU-module Y contained in X^U such that $X = Y_A$. Moreover, Z^U is completely reducible by [HB82a, VII, 9.4]. Let W be an irreducible component of Z^U. By the Nakayama's reciprocity theorem, $0 \neq \mathrm{Hom}_U(Z^U, W) \cong \mathrm{Hom}_A(Z, W_A)$ ([DH92, B, 6.5]). Therefore Z is an irreducible component of W_A. Since Z is π-special, then so is W by [CK87, 2.3]. It is clear that the inertia subgroup of Z in U is the whole U. Then W_A is homogeneous, i.e. $W_A \cong tZ$. Again, by [DH92, B, 8.5], either $t = 1$ or $t = q$. Assume that $t = q$. Therefore, we have that $\dim W = \dim Z^U = q \dim Z$. This implies that $W \cong Z^U$ and Z^U is a π-special KU-module. Let L be any irreducible KU-module such that $Z \otimes X$ is a component of L_A. It follows that $Z^U \otimes Y$ is irreducible by [CK87, 2.4]. By [HB82a, VII, 4.5 (a)], we have that $(Z \otimes X)^U = (Z \otimes Y_A)^U \cong Z^U \otimes Y$. Applying Nakayama's reciprocity theorem ([DH92, B, 6.5]), it follows that $0 \neq \mathrm{Hom}_A(Z \otimes X, L_A) \cong \mathrm{Hom}_U\big((Z \otimes X)^U, L\big)$. Consequently $Z^U \otimes Y \cong L$. This implies that $L_i \cong Z^U \otimes Y$ for all $i \in \{1, \ldots, u\}$ and N_U is homogeneous, contrary to $0 \neq D(N_U) \neq N$. Hence $t = 1$, and W has the same dimension as Z. Consequently, $W \otimes Y$ is an irreducible KU-module with $(W \otimes Y)_A = Z \otimes X$.

For any irreducible component L_j of N_U, it follows from Lemma 6.5.29 that $L_j = (W \otimes Y) \otimes J_j$, where J_j is an irreducible $K(U/A)$-module (regarded as KU-module) and $\dim J_j = 1$ or q. Since U/A is cyclic, it follows that $\dim J_j = 1$ by [DH92, B, 9.2]. Hence $(L_j)_A = N_i$.

Arguing with V, we have that if $N_V = P_1 \oplus \cdots \oplus P_v$, with the P_i irreducible V-modules, and P_k is an irreducible component of N_V such that N_i is a component of $(P_k)_A$, then $(P_k)_A = N_i$ is irreducible.

It implies that N_i is in fact U-module and V-module. Therefore $N_i = N$ is an irreducible G-module. This is a contradiction.

Case 2. Suppose that not all of the X_i are isomorphic. We let T denote the inertia subgroup of X_1 and note that $A \leq T \neq G$. Since G/A is a q-group generated by U/A and V/A, we have that there is a maximal normal subgroup M of G satisfying $T \leq M$ and so either U or V is not contained in M. We may suppose that U is not contained in M. Recall that all X_i are isomorphic to G-conjugates of X_1, and so the inertia subgroups are conjugate in G. It then follows that U is not contained in the inertia subgroup of any X_i. Now let L be a component of N_U and suppose that N_1 is a component of L_A. If L is π-factorable, then $L = D \otimes E$ with D π-special and E π'-special. Note that $L_A = D_A \otimes E_A$; if $D_A = D_1 \oplus \cdots \oplus D_m$ with all D_i irreducible A-modules, then D_i is π-special for all $i \in \{1, \ldots, m\}$ by [CK87, 2.2]. Suppose that E_A is irreducible. Then $L_A = (D_1 \otimes E_A) \oplus \cdots \oplus (D_m \otimes E_A)$. Therefore E_A is isomorphic to X_1 by [CK87, 2.4], and then U is contained in the inertia subgroup of X_1, contrary to supposition. Hence we cannot have E_A irreducible. By Clifford's theorem, since the inertia subgroup of X_1 in U is A, we have that E is the direct sum of $q = |U/A|$ irreducible modules conjugate to X_1. But then the dimension of E is not a π'-number. This contradiction yields that L cannot be a π-factorable module. It follows that no component of N_U can be π-factorable, and so no component of N_U can be in $\mathfrak{M}(U)$, i.e. $N_U = D(N_U)$, and we have reached a contradiction.

Case 3. Suppose that all of the X_i are isomorphic. By Case 1, we may assume that not all the Z_i are isomorphic and let T be the inertia subgroup of Z_1. As before, it follows that we may suppose that U is not contained in the inertia subgroup of any Z_i.

Now let L be any irreducible KU-module such that $Z_1 \otimes X_1$ is a component of L_A. We then have that X_1 has a unique extension to a π'-special KU-module, $(X_1)^*$ say by [CK87, 2.3]. Also, since Z_1 is not U-invariant, we have that $(Z_1)^U$ is irreducible by [DH92, B, 7.8] and π-special by [CK87, 2.3]. It follows that $(Z_1)^U \otimes (X_1)^*$ is irreducible by [CK87, 2.4]. By [HB82a, VII, 4.5], we have that $(Z_1 \otimes X_1)^U = \big(Z_1 \otimes \big((X_1)^*\big)_A\big)^U \cong Z_1^U \otimes (X_1)^*$. Now $0 \neq \mathrm{Hom}_A(Z_1 \otimes X_1, L_A) \cong \mathrm{Hom}_U\big((Z_1 \otimes X_1)^U, L\big)$ by the Nakayama's reciprocity theorem ([DH92, B, 6.5]). Then L is isomorphic to $(Z_1)^U \otimes (X_1)^*$.

It follows that if $N_U = L_1 \oplus \cdots \oplus L_U$ with the L_i irreducible, then $L_i = (Z_i)^* \oplus (X_i)^*$ with $(Z_i)^*$ π-special and $(X_i)^*$ π'-special, $1 \leq i \leq u$. In each case the π'-special factor is isomorphic to $(X_1)^*$, and thus if $U/\operatorname{Ker}\big(U \operatorname{on}(X_1)^*\big)$ is not in $\mathfrak{X}_2$, then no component of N_U is in $\mathfrak{M}_U$, i.e. $N_U = D(N_U)$. This contradiction proves that $U/\operatorname{Ker}\big(U \operatorname{on}(X_1)^*\big) \in \mathfrak{X}_2$. Some of the L_j is in $\mathfrak{M}(U)$. Suppose $L_i \in \mathfrak{M}(U)$. Then the group $U/\operatorname{Ker}\big(U \operatorname{on}(Z_i)^*\big) \in \mathfrak{X}_1$. On the other hand, since $A \in \mathfrak{F}$, the group $A/\operatorname{Ker}(A \operatorname{on} Z_j)$ belongs to $\mathfrak{X}_1$ for all j. Recall that all Z_j are conjugate and then so are the $\operatorname{Ker}(A \operatorname{on} Z_j)$. Since $(Z_j)^* = (Z_j)^U$, we have that $\operatorname{Ker}\big(U \operatorname{on}(Z_j)^*\big) = \operatorname{Core}_U\big(\operatorname{Ker}(A \operatorname{on} Z_j)\big)$. Thus $A/\operatorname{Ker}\big(U \operatorname{on}(Z_j)^*\big) \in \mathfrak{X}_1$.

At this point, we must invoke the special form of $\mathfrak{X}_i$, $i = 1, 2$. Since $U/\operatorname{Ker}\big(U \operatorname{on}(Z_i)^*\big) \in \mathfrak{S}_\rho$ and $\operatorname{Ker}\big(U \operatorname{on}(Z_i)^*\big)$ is contained in A, we must have $Q \in \rho$. Then $U/\operatorname{Ker}\big(U \operatorname{on}(Z_j)^*\big)$ is a ρ-group and hence is in $\mathfrak{X}_1$ for all j. Thus $L_j \in \mathfrak{M}(U)$ for all j. In other words, $D(N_U) = 0$. This final contradiction proves $G^{\mathfrak{F}} = \langle U^{\mathfrak{F}}, V^{\mathfrak{F}} \rangle$ and then $\mathfrak{F}$ has the Wielandt property for residuals. □

Examples 6.5.30.
1. Set $\mathcal{P} = \big\{\pi = \{p\}, \pi' = \{p\}'\big\}$, p a prime, $\mathfrak{X}_1 = (1)$ and $\mathfrak{X}_2 = \mathfrak{S}$, then $\mathfrak{T}(1, p, \mathcal{P}, \mathfrak{X}_1, \mathfrak{X}_2) = \mathfrak{T}(1, \mathfrak{M}^p)$ are the Fitting classes introduced by T. O. Hawkes in [Haw70]. Applying Theorem 6.5.28, $\mathfrak{T}(1, \mathfrak{M}^p)$ satisfies the Wielandt property for residuals.
2. The Fitting formations studied by K. L. Haberl and H. Heineken in [HH84] can be seen as Fitting formations constructed by the Cossey-Kanes method with $\mathfrak{X}_1 = \mathfrak{S}$ and $\mathfrak{X}_2 = (1)$. Hence, by Theorem 6.5.28, these classes also satisfy the Wielandt property for residuals.

Let $\mathfrak{F}$ be a Fitting formation satisfying the Wielandt property for residuals. In general, the $\mathfrak{F}$-residual of a group generated by two $\mathfrak{F}$-subnormal subgroups is not the subgroup generated by their $\mathfrak{F}$-residuals, as the following example shows.

Example 6.5.31. Let $\mathfrak{F}$ be the saturated Fitting formation of all groups of nilpotent length at most 2. Then $\mathfrak{F}$ is a subgroup-closed formation of soluble groups. Applying Corollary 6.5.26, $\mathfrak{F}$ has the Wielandt property for residuals.

Let G be the symmetric group of degree 4. If A is the alternating group of degree 4 and B is a Sylow 2-subgroup of G, then A and B are both $\mathfrak{F}$-subnormal in $G = \langle A, B\rangle$, A and B belong to $\mathfrak{F}$, but $G \notin \mathfrak{F}$.

From this example the following problem arises:

> Find a precise description of those formations $\mathfrak{F}$ for which the $\mathfrak{F}$-residual of a group generated by two $\mathfrak{F}$-subnormal subgroups is the subgroup generated by their $\mathfrak{F}$-residuals.

We will be mainly concerned with this problem from now on. Our treatment of the question closely follows the approaches developed in the papers of S. F. Kamornikov [Kam96] and A. Ballester-Bolinches, M. C. Pedraza-Aguilera, and M. D. Pérez-Ramos [BBPAPR96], and A. Ballester-Bolinches [BB05].

For the purposes of this discussion, let $\mathfrak{F}$ be a fixed, but arbitrary, formation.

Definition 6.5.32. *1. We say that* $\mathfrak{F}$ has the generalised Wielandt property for residuals*,* $\mathfrak{F}$ *is a* GWP-formation *for short, if* $\mathfrak{F}$ *enjoys the following property:*

If G is a group generated by two $\mathfrak{F}$-subnormal subgroups A and B, then $G^{\mathfrak{F}} = \langle A^{\mathfrak{F}}, B^{\mathfrak{F}}\rangle$.

2. $\mathfrak{F}$ satisfies the Kegel-Wielandt property for residuals*,* $\mathfrak{F}$ *is a* KW-formation *for short, if* $\mathfrak{F}$ *has the following property:*

Let $G = \langle A, B\rangle$ be a group generated by two K-$\mathfrak{F}$-subnormal subgroups A and B. Then $G^{\mathfrak{F}} = \langle A^{\mathfrak{F}}, B^{\mathfrak{F}}\rangle$.

Obviously, every KW-formation is a GWP-formation. We show in the following that that the converse holds for saturated formations, and the soluble GWP-formations are exactly the soluble subgroup-closed saturated lattice formations. We need a couple of preliminary results.

To be a subgroup-closed Fitting formation is a necessary condition for a formation $\mathfrak{F}$ to have the generalised Wielandt property for residuals.

Lemma 6.5.33. *If $\mathfrak{F}$ is a GWP-formation, then $\mathfrak{F}$ is a subgroup-closed Fitting formation.*

Proof. First suppose, by way of contradiction, that $\mathfrak{F}$ is not subgroup-closed. Let G be an $\mathfrak{F}$-group of minimal order having a subgroup not in $\mathfrak{F}$ and, among subgroups of G not in $\mathfrak{F}$, let M be one of maximal order. Then M is a maximal subgroup of G. Since $G^{\mathfrak{F}} = 1$, it follows that M is $\mathfrak{F}$-subnormal in G. Since $\mathfrak{F}$ is a GWP-formation, we have that $M^{\mathfrak{F}} = \langle M^{\mathfrak{F}}, 1\rangle = \langle M^{\mathfrak{F}}, G^{\mathfrak{F}}\rangle = G^{\mathfrak{F}} = 1$. This contradicts the choice of G. Consequently $\mathfrak{F}$ is subgroup-closed. In particular, $\mathfrak{F}$ is S_n-closed. To complete the proof we now show that $\mathfrak{F}$ is N_0-closed. Suppose that this is not true and derive a contradiction. Let G be a group of minimal order having two normal subgroups N_1 and N_2 such that $G = N_1N_2$ and $N_i \in \mathfrak{F}$ for $i = 1, 2$. If N is a minimal normal subgroup of G, it follows that $G/N \in \mathfrak{F}$. Therefore G is in the boundary of $\mathfrak{F}$ and $N = G^{\mathfrak{F}}$ is the unique minimal normal subgroup of G. It is clear that $N_i \neq 1$, $i = 1, 2$. Hence N is contained in $N_1 \cap N_2$ and thus N_i is $\mathfrak{F}$-subnormal in G, $i = 1, 2$ by Lemma 6.1.7 (1). Since $\mathfrak{F}$ is a GWP-formation, it follows that $G^{\mathfrak{F}} = N_1^{\mathfrak{F}}N_2^{\mathfrak{F}} = 1$, contrary to supposition. Therefore $\mathfrak{F}$ is N_0-closed. The proof of the lemma is now complete. □

The following result is another step to attain our objectives.

Theorem 6.5.34. *Let $\mathfrak{F}$ be a GWP-formation. Then $\mathfrak{F}$ is a lattice formation.*

Proof. By Lemma 6.5.33, $\mathfrak{F}$ is subgroup-closed. Hence the intersection of $\mathfrak{F}$-subnormal subgroups of a group is $\mathfrak{F}$-subnormal by Lemma 6.1.7 (3).

Suppose that $\mathfrak{F}$ is not a lattice formation and derive a contradiction. By this supposition, there exists a group G of minimal order such that $\mathrm{S}_{n\mathfrak{F}}(G)$ is not a sublattice of the subgroup lattice of G. In particular, G has two $\mathfrak{F}$-subnormal subgroups A and B such that $\langle A, B\rangle$ is not $\mathfrak{F}$-subnormal in G.

Let N be a minimal normal subgroup of G. Then AN/N and BN/N are $\mathfrak{F}$-subnormal in G/N by Lemma 6.1.6 (3). Hence $\langle AN/N, BN/N\rangle = \langle A, B\rangle N/N$ is $\mathfrak{F}$-subnormal in G/N by minimality of G. Therefore $X = \langle A, B\rangle N$ is $\mathfrak{F}$-subnormal in G by Lemma 6.1.6 (2). Since A and B are $\mathfrak{F}$-subnormal in X by Lemma 6.1.7 (2), it follows that $\langle A, B\rangle$ is $\mathfrak{F}$-subnormal in X provided that X is a proper subgroup of G. This would imply the $\mathfrak{F}$-subnormality of $\langle A, B\rangle$ in G by Lemma 6.1.6 (1). Consequently $G = \langle A, B\rangle N$ for every minimal normal subgroup N of G. Hence either $G = \langle A, B\rangle$ or $\mathrm{Core}_G(\langle A, B\rangle) = 1$. If $G = \langle A, B\rangle$, then $\langle A, B\rangle$ is $\mathfrak{F}$-subnormal in G, contrary to supposition. Hence $\mathrm{Core}_G(\langle A, B\rangle) = 1$. On the other hand, $A^{\mathfrak{F}}$ and $B^{\mathfrak{F}}$ are subnormal subgroups of G by Lemma 6.1.9 (1). Hence $\langle A^{\mathfrak{F}}, B^{\mathfrak{F}}\rangle$ is subnormal in G and so N normalises $\langle A^{\mathfrak{F}}, B^{\mathfrak{F}}\rangle$ ([DH92, A, 14.3 and 14.4]). Since $\mathfrak{F}$ is a GWP-formation, we have that $\langle A^{\mathfrak{F}}, B^{\mathfrak{F}}\rangle = \langle A, B\rangle^{\mathfrak{F}}$. This implies that $\langle A, B\rangle^{\mathfrak{F}}$ is normal in G. Hence $\langle A, B\rangle^{\mathfrak{F}} \leq \mathrm{Core}_G(\langle A, B\rangle) = 1$ and $\langle A, B\rangle$ is an $\mathfrak{F}$-group. Let us consider the subgroup AN. Clearly N is not contained in A. If $N^{\mathfrak{F}} = N$, then no simple component of N belongs to $\mathfrak{F}$ and thus $(AN)^{\mathfrak{F}} = N$. This contradicts the fact that A is $\mathfrak{F}$-subnormal in AN (Lemma 6.1.6 (2)). Therefore $N \in \mathfrak{F}$. This implies that $G \in \mathrm{E\,K}(\mathfrak{F})$ and so N is $\mathfrak{F}$-subnormal in G by Proposition 6.1.10. Then N is an $\mathfrak{F}$-subnormal subgroup of AN by Lemma 6.1.6 (2). In particular, $(AN)^{\mathfrak{F}} = A^{\mathfrak{F}}N^{\mathfrak{F}} = 1 = (BN)^{\mathfrak{F}}$ because the property of $\mathfrak{F}$. Since $G = \langle AN, BN\rangle$ and $\mathfrak{F}$ is a GWP-formation, it follows that $G^{\mathfrak{F}} = \langle (AN)^{\mathfrak{F}}, (BN)^{\mathfrak{F}}\rangle = 1$. This final contradiction proves that $\mathfrak{F}$ is a lattice formation. □

A challenging unsolved problem in the theory of formations is whether the converse of Theorem 6.5.34 is true. The chance of finding the answer seems remote. With our present knowledge even the saturated case remains unanswered.

We shall prove a result that provides a test for a subgroup-closed saturated lattice formation to be a GWP-formation in terms of its boundary. This allows us to present the complete answer to the problem in the soluble universe and give interesting examples.

As in the case of groups generated by subnormal subgroups, we thought it would be desirable to collect the arguments common to our next results. Let $\mathfrak{F}$ be a subgroup-closed Fitting formation. Given a group Z, we denote by $\mathcal{R}(Z, \mathfrak{F})$ the set of all pairs (H, K) such that H and K are $\mathfrak{F}$-subnormal subgroups of $\langle H, K\rangle$ and $\langle H^{\mathfrak{F}}, K^{\mathfrak{F}}\rangle < \langle H, K\rangle^{\mathfrak{F}}$. Let $\mathcal{W}(\mathfrak{F})$ denote the class of all groups Z such that $\mathcal{R}(Z, \mathfrak{F}) \neq \emptyset$.

If $\mathfrak{F}$ is not a GWP-formation, then the class $\mathcal{W}(\mathfrak{F})$ is not empty.

In the following we analyse the structure of a group G of minimal order in $\mathcal{W}(\mathfrak{F})$. Then G has two $\mathfrak{F}$-subnormal subgroups A and B such that $\langle A, B\rangle^{\mathfrak{F}} \neq \langle A^{\mathfrak{F}}, B^{\mathfrak{F}}\rangle$. Choose A and B with $|A| + |B|$ maximal.

Arguing as in the subnormal case, we have:

Result 6.5.35. $G = \langle A, B\rangle$*, and*

Result 6.5.36. $\mathrm{Soc}(G) \leq G^{\mathfrak{F}}$ *and* $G^{\mathfrak{F}} = \langle A^{\mathfrak{F}}, B^{\mathfrak{F}}\rangle N$ *for any minimal normal subgroup of* G*. In particular,* $\mathrm{Core}_G(\langle A^{\mathfrak{F}}, B^{\mathfrak{F}}\rangle) = 1$.

Result 6.5.37. $\langle A^{\mathfrak{F}}, B^{\mathfrak{F}}\rangle$ *is normal in* $G^{\mathfrak{F}}$*.*

Proof. Applying Lemma 6.1.9 (1), $A^{\mathfrak{F}}$ and $B^{\mathfrak{F}}$ are subnormal subgroups of G. Hence $\mathrm{Soc}(G) \leq \mathrm{N}_G(\langle A^{\mathfrak{F}}, B^{\mathfrak{F}}\rangle)$ by [DH92, A, 14.3 and 14.4]. This implies that $\langle A^{\mathfrak{F}}, B^{\mathfrak{F}}\rangle$ is normal in $G^{\mathfrak{F}}$. □

Result 6.5.38. $G^{\mathfrak{F}} \in \mathrm{Q\,R}_0(N)$ *for any minimal normal subgroup* N *of* G*.*

Proof. Let N be a minimal normal subgroup of G. Then $G^{\mathfrak{F}} = \langle A^{\mathfrak{F}}, B^{\mathfrak{F}}\rangle N$ and $\langle A^{\mathfrak{F}}, B^{\mathfrak{F}}\rangle \trianglelefteq G^{\mathfrak{F}}$ by Results 6.5.36 and 6.5.37. Hence $(G^{\mathfrak{F}})^{\mathrm{Q\,R}_0(N)}$ is a normal subgroup of G contained in $\langle A^{\mathfrak{F}}, B^{\mathfrak{F}}\rangle$. Since $\mathrm{Core}_G(\langle A^{\mathfrak{F}}, B^{\mathfrak{F}}\rangle) = 1$ by Result 6.5.36, we have that $(G^{\mathfrak{F}})^{\mathrm{Q\,R}_0(N)} = 1$ and $G^{\mathfrak{F}} \in \mathrm{Q\,R}_0(N)$. □

Result 6.5.39. $N \in \mathfrak{F}$ *for any minimal normal subgroup* N *of* G*.*

Proof. Since $N^{\mathfrak{F}}$ is normal in G, we have that either $N^{\mathfrak{F}} = 1$ or $N^{\mathfrak{F}} = N$. Assume that $N^{\mathfrak{F}} = N$. By Lemma 6.1.6 (2), A is $\mathfrak{F}$-subnormal in AN. Hence $A^{\mathfrak{F}}$ is normal in AN by Lemma 6.1.9 (1) and [DH92, A, 14.3]. This implies that $AN/A^{\mathfrak{F}}N \in \mathfrak{F}$ and so $(AN)^{\mathfrak{F}} = A^{\mathfrak{F}}N$. Hence $AN = A(AN)^{\mathfrak{F}}$. The $\mathfrak{F}$-subnormality of A in AN yields $AN = A$. Since $\mathfrak{F}$ is subgroup-closed, it follows that $NA^{\mathfrak{F}}/A^{\mathfrak{F}} \in \mathfrak{F}$, whence $N/(N \cap A^{\mathfrak{F}}) \in \mathfrak{F}$. Therefore $N = N \cap A^{\mathfrak{F}}$ and $G^{\mathfrak{F}} = \langle A^{\mathfrak{F}}, B^{\mathfrak{F}}\rangle$, contrary to our initial supposition. Consequently $N \in \mathfrak{F}$. □

Result 6.5.40. $G^{\mathfrak{F}} \in \mathfrak{F}$.

Proof. Let N be a minimal normal subgroup of G contained in $G^{\mathfrak{F}}$. Then $N \in \mathfrak{F}$ by Result 6.5.39 and $G^{\mathfrak{F}} \in \mathrm{Q}\,\mathrm{R}_0(N)$ by Result 6.5.38. Hence $G^{\mathfrak{F}} \in \mathfrak{F}$. □

Result 6.5.41. *$G^{\mathfrak{F}}$ is contained in $A \cap B$. In particular, $A^{\mathfrak{F}}B^{\mathfrak{F}}$ is a subgroup of G.*

Proof. Clearly $AG^{\mathfrak{F}}$ is a proper subgroup of G. Hence $AG^{\mathfrak{F}}$ is contained in a maximal $\mathfrak{F}$-normal subgroup of G. The minimality of G yields $(AG^{\mathfrak{F}})^{\mathfrak{F}} = A^{\mathfrak{F}}$. Assume that A is a proper subgroup of $AG^{\mathfrak{F}}$. Since $AG^{\mathfrak{F}}$ is $\mathfrak{F}$-subnormal in G, it follows that $G^{\mathfrak{F}} = \langle (AG^{\mathfrak{F}})^{\mathfrak{F}}, B^{\mathfrak{F}} \rangle = \langle A^{\mathfrak{F}}, B^{\mathfrak{F}} \rangle$ by the choice of the pair (A, B), contrary to our initial supposition. Hence $A = AG^{\mathfrak{F}}$ and $G^{\mathfrak{F}}$ is contained in A. Analogously $G^{\mathfrak{F}}$ is contained in B. □

With the same arguments to those used in Statement 6.5.10, we have:

Result 6.5.42. *If $G^{\mathfrak{F}}$ is non-abelian, then G is in the boundary of $\mathfrak{F}$.*

Suppose now that there exists a family of subgroup-closed formations $\{\mathfrak{F}_i\}_{i \in \mathcal{I}}$ such that $\pi(\mathfrak{F}_i) \cap \pi(\mathfrak{F}_j) = \emptyset$, $i \neq j$, and $\mathfrak{F} = \times_{i \in \mathcal{I}} \mathfrak{F}_i$.

Result 6.5.43. *There exist $i, j \in I$ such that $G/G^{\mathfrak{F}} \in \mathfrak{F}_i$ and $G^{\mathfrak{F}} \in \mathfrak{F}_j$. Moreover if either $G \notin \mathrm{b}(\mathfrak{F})$ or $G^{\mathfrak{F}}$ is non-abelian, then $i = j$.*

Proof. By Result 6.5.40, we have that $G^{\mathfrak{F}} \in \mathfrak{F}$ and, by Result 6.5.38, $G^{\mathfrak{F}}$ is a direct product of copies of a simple group. Hence there exists $j \in I$ such that $G^{\mathfrak{F}} \in \mathfrak{F}_j$. On the other hand, $G/G^{\mathfrak{F}} = X_{i_1}/G^{\mathfrak{F}} \times \cdots \times X_{i_t}/G^{\mathfrak{F}}$, where $X_{i_k}/G^{\mathfrak{F}} \in \mathfrak{F}_{i_k}$ is a Hall $\pi(\mathfrak{F}_{i_k})$-subgroup of $G/G^{\mathfrak{F}}$, $1 \leq k \leq t$, for some set $\{i_1, \dots, i_t\} \subseteq I$. Let $k \in \{1, \dots, t\}$. Then $X_{i_k}/G^{\mathfrak{F}} = \langle (A \cap X_{i_k})/G^{\mathfrak{F}}, (B \cap X_{i_k})/G^{\mathfrak{F}} \rangle = \langle A \cap X_{i_k}, B \cap X_{i_k} \rangle / G^{\mathfrak{F}}$ and $X_{i_k} = \langle A \cap X_{i_k}, B \cap X_{i_k} \rangle$. Applying Lemma 6.1.7 (2), $A \cap X_{i_k}$ and $B \cap X_{i_k}$ are $\mathfrak{F}$-subnormal subgroups of X_{i_k}. Assume that X_{i_k} is a proper subgroup of G for all $k \in \{1, \dots, t\}$. Then $X_{i_k}^{\mathfrak{F}} = \langle (A \cap X_{i_k})^{\mathfrak{F}}, (B \cap X_{i_k})^{\mathfrak{F}} \rangle$ by the minimal choice of G, leading to $X_{i_k}^{\mathfrak{F}} = 1$. This is due to the fact that $X_{i_k}^{\mathfrak{F}}$ is a normal subgroup of G contained in $\langle A^{\mathfrak{F}}, B^{\mathfrak{F}} \rangle$ and $\mathrm{Core}_G(\langle A^{\mathfrak{F}}, B^{\mathfrak{F}} \rangle) = 1$ by Result 6.5.36. Hence $G \in \mathrm{N}_0\,\mathfrak{F} = \mathfrak{F}$, contrary to hypothesis. Therefore there exists an index $i = i_k \in \{i_1, \dots, i_t\}$ such that $X_i = G$. This means that $G/G^{\mathfrak{F}} \in \mathfrak{F}_i$.

Suppose that $i \neq j$. Then $G^{\mathfrak{F}}$ is a Hall $\pi(\mathfrak{F}_j)$-subgroup of G and there exists a Hall $\pi(\mathfrak{F}_i)$-subgroup C of G such that $G = G^{\mathfrak{F}}C$ and $G^{\mathfrak{F}} \cap C = 1$. It follows that $A/A^{\mathfrak{F}} = G^{\mathfrak{F}}/A^{\mathfrak{F}} \times (C \cap A)A^{\mathfrak{F}}/A^{\mathfrak{F}}$ and so A normalises $A^{\mathfrak{F}}B^{\mathfrak{F}}$. Analogously B normalises $A^{\mathfrak{F}}B^{\mathfrak{F}}$. It implies that $A^{\mathfrak{F}}B^{\mathfrak{F}} = 1$ by Result 6.5.36 and $G^{\mathfrak{F}}$ is a minimal normal subgroup of G. Hence $G \in \mathrm{b}(\mathfrak{F})$.

If $G^{\mathfrak{F}}$ is non-abelian, $\mathrm{C}_G(G^{\mathfrak{F}}) = 1$. Since $A = G^{\mathfrak{F}} \times (C \cap A)$ and $B = G^{\mathfrak{F}} \times (C \cap B)$, it follows that $A = B = G^{\mathfrak{F}}$. Then, by Results 6.5.35 and 6.5.41, $A = B = G$, and this contradicts our initial hypothesis.

Consequently if either $G \notin \mathrm{b}(\mathfrak{F})$ or $G^{\mathfrak{F}}$ is non-abelian, we have that $i = j$. □

Assume now that $\mathfrak{F}$ is a saturated GWP-formation. Then $\mathfrak{F}$ is a lattice formation by Theorem 6.5.34 and, since $\mathfrak{F}$ is saturated, it follows that $\mathfrak{F}$ is a K-lattice formation by Theorem 6.3.9.

Suppose that $\mathfrak{F}$ is not a KW-formation. Then there exists a group G and a pair (A, B) of K-$\mathfrak{F}$-subnormal subgroups of G such that $G = \langle A, B\rangle$ and $G^{\mathfrak{F}} \neq \langle A^{\mathfrak{F}}, B^{\mathfrak{F}}\rangle$. Let us take (A, B) satisfying $|A| + |B|$ maximal. Then, as in the above reductions, G enjoys the properties stated in Results 6.5.35, 6.5.36, 6.5.37, and 6.5.38. G also has the following property.

Result 6.5.44. *$G^{\mathfrak{F}} \in \mathfrak{F}$.*

Proof. Consider the subgroup $M = \langle A, B^{\mathfrak{F}}\rangle$. Suppose that $M = G$. Then $G = AG^{\mathfrak{F}}$. Since by Lemma 6.1.9 (1), $A^{\mathfrak{F}}$ is subnormal in G and $G^{\mathfrak{F}}$ is a direct product of isomorphic simple groups by Result 6.5.38, it follows that $G^{\mathfrak{F}}$ normalises $A^{\mathfrak{F}}$ and so $A^{\mathfrak{F}}$ is a normal subgroup of G. By Result 6.5.36, we have that $A \in \mathfrak{F}$. By virtue of Lemma 6.3.8, it follows that $A \leq G_{\mathfrak{F}}$. If $G_{\mathfrak{F}} \cap G^{\mathfrak{F}} \neq 1$, then $G^{\mathfrak{F}} \in \mathfrak{F}$ and the result follows. Hence $G_{\mathfrak{F}} \cap G^{\mathfrak{F}} = 1$ and $G = G_{\mathfrak{F}} \times G^{\mathfrak{F}}$. By Result 6.5.36, we have that $\mathrm{Soc}(G) \leq G^{\mathfrak{F}}$. It implies that $G_{\mathfrak{F}} = 1$ and so $A = 1$ and $G = B$, giving a contradiction. Therefore we may assume that M is a proper subgroup of G. The choice of G, Lemma 6.1.6 (1) and Lemma 6.1.7 (2) imply that $M^{\mathfrak{F}} = \langle A, B^{\mathfrak{F}}\rangle^{\mathfrak{F}} = \langle A^{\mathfrak{F}}, (B^{\mathfrak{F}})^{\mathfrak{F}}\rangle$.

Arguing in a similar way with B, we have $\langle A^{\mathfrak{F}}, B\rangle^{\mathfrak{F}} = \langle (A^{\mathfrak{F}})^{\mathfrak{F}}, B^{\mathfrak{F}}\rangle$. If either $A < \langle A, B^{\mathfrak{F}}\rangle$ or $B < \langle A^{\mathfrak{F}}, B\rangle$, it follows that $G^{\mathfrak{F}} = \big\langle \langle A, B^{\mathfrak{F}}\rangle^{\mathfrak{F}}, \langle A^{\mathfrak{F}}, B\rangle^{\mathfrak{F}}\big\rangle = \langle A^{\mathfrak{F}}, B^{\mathfrak{F}}\rangle$ by the choice of G (note that $\mathfrak{F}$ is subgroup-closed). This contradiction yields $A = \langle A, B^{\mathfrak{F}}\rangle$ and $B = \langle A^{\mathfrak{F}}, B\rangle$. Then $B^{\mathfrak{F}}$ is contained in A and $A^{\mathfrak{F}}$ is a normal subgroup of $G^{\mathfrak{F}}$. Hence $(A \cap G^{\mathfrak{F}})/A^{\mathfrak{F}}$ is a K-$\mathfrak{F}$-subnormal $\mathfrak{F}$-subgroup of $G^{\mathfrak{F}}/A^{\mathfrak{F}}$ by Lemma 6.1.7 (2) and Lemma 6.1.6 (3). Applying Lemma 6.3.8, $(A \cap G^{\mathfrak{F}})/A^{\mathfrak{F}}$ is contained in $(G^{\mathfrak{F}}/A^{\mathfrak{F}})_{\mathfrak{F}}$. If $A^{\mathfrak{F}} \neq A \cap G^{\mathfrak{F}}$, then $(G^{\mathfrak{F}}/A^{\mathfrak{F}})_{\mathfrak{F}} \neq 1$ and $G^{\mathfrak{F}} \in \mathfrak{F}$ because $G^{\mathfrak{F}}$ is a direct product of simple groups. Therefore we may assume $A^{\mathfrak{F}} = A \cap G^{\mathfrak{F}}$. In this case, $B^{\mathfrak{F}}$ is contained in $A^{\mathfrak{F}}$. Arguing in a similar way with B, we conclude that $A^{\mathfrak{F}}$ is contained in $B^{\mathfrak{F}}$. Consequently $A^{\mathfrak{F}} = B^{\mathfrak{F}}$ is a normal subgroup of G. By Result 6.5.36, A and B are $\mathfrak{F}$-groups. By Lemma 6.3.8, $G \in \mathfrak{F}$ and $G^{\mathfrak{F}} = 1$. □

This completes our preparations, and we can now deduce the main results.

Theorem 6.5.45. *Let $\mathfrak{F}$ be a saturated formation. Then:*
$\mathfrak{F}$ is a GWP-formation if and only if $\mathfrak{F}$ is a KW-formation.

Proof. Only the necessity of the condition is in doubt. Assume that $\mathfrak{F}$ is a GWP-formation which is not a KW-formation. Then there exists a group G and a pair (A, B) of K-$\mathfrak{F}$-subnormal subgroups of G such that $G = \langle A, B\rangle$ and $G^{\mathfrak{F}} \neq \langle A^{\mathfrak{F}}, B^{\mathfrak{F}}\rangle$. If $|A| + |B|$ maximal, then $G^{\mathfrak{F}} \in \mathfrak{F}$ by Result 6.5.44. Then $\mathrm{s}_{n\mathfrak{F}}(G) = \mathrm{s}_{n\mathrm{K}\text{-}\mathfrak{F}}(G)$ by Proposition 6.1.10. This contradiction yields that $\mathfrak{F}$ is a KW-formation. □

Our next main result provides a test for a subgroup-closed saturated lattice formation to have the generalised Wielandt property for residuals in terms of its boundary.

If $\mathfrak{F}$ is a subgroup-closed Fitting formation, let $\mathrm{b}_n(\mathfrak{F})$ denote the class of all groups $G \in \mathrm{b}(\mathfrak{F})$ such that $\mathrm{Soc}(G)$ is not abelian and G has the properties stated in Results 6.5.35–6.5.43.

Theorem 6.5.46. *Let $\mathfrak{F}$ be a subgroup-closed saturated lattice formation. Then $\mathfrak{F}$ is a GWP-formation if and only if the following condition is fulfilled by all groups $G \in \mathrm{b}_n(\mathfrak{F})$:*

If $G = \langle A, B\rangle$ with A and B $\mathfrak{F}$-subnormal subgroups of G, then $G^{\mathfrak{F}} = \langle A^{\mathfrak{F}}, B^{\mathfrak{F}}\rangle$. (6.7)

Proof. It is clear that only the sufficiency of the condition is in doubt.

Assume that Property (6.7) holds. We suppose that $\mathfrak{F}$ is not a GWP-formation and derive a contradiction. Since $\mathcal{W}(\mathfrak{F})$ is not empty, a group G of minimal order in $\mathcal{W}(\mathfrak{F})$ satisfies the properties stated in Results 6.5.35–6.5.43 for a pair of $\mathfrak{F}$-subnormal subgroups A and B of G with $|A| + |B|$ maximal.

Applying Theorem 6.3.15, $\mathfrak{F} = \mathfrak{M} \times \mathfrak{H}$, where $\mathfrak{M}$ is a subgroup-closed saturated Fitting formation such that $\mathfrak{S}_{\pi(\mathfrak{M})}\mathfrak{M} = \mathfrak{M}$ and $\mathfrak{H} = \mathsf{X}_{i\in\mathcal{I}}\, \mathfrak{S}_{\pi_i}$, with $\pi_l \cap \pi_k = \emptyset$ for all $k \neq l$ in I. Moreover $\pi(\mathfrak{M}) \cap \pi(\mathfrak{H}) = \emptyset$. Since $G \notin \mathrm{b}_n(\mathfrak{F})$, $G^{\mathfrak{F}}$ is an elementary abelian p-group for some prime p by Results 6.5.38 and 6.5.42. Therefore $G^{\mathfrak{F}} \in \mathfrak{M}$ or $G^{\mathfrak{F}} \in \mathfrak{S}_{\pi_i}$ for some $i \in \mathcal{I}$. In addition, by Result 6.5.43, $G/G^{\mathfrak{F}} \in \mathfrak{M}$ or $G/G^{\mathfrak{F}} \in \mathfrak{S}_{\pi_j}$ for some $j \in I$. If $G^{\mathfrak{F}} \in \mathfrak{M}$ and $G/G^{\mathfrak{F}} \in \mathfrak{M}$, then $p \in \pi(\mathfrak{M})$ and $G \in \mathfrak{S}_{\pi(\mathfrak{M})}\mathfrak{M} = \mathfrak{M} \subseteq \mathfrak{F}$, contradicting $G \in \mathcal{W}(\mathfrak{F})$. Assume now that $G/G^{\mathfrak{F}} \in \mathfrak{S}_{\pi_j}$ for some $j \in I$. Then $G^{\mathfrak{F}}$ is a Hall $\pi(\mathfrak{M})$-subgroup of G and there exists a Hall π_j-subgroup C of G such that $G = G^{\mathfrak{F}}C$ and $G^{\mathfrak{F}} \cap C = 1$. Then $A/A^{\mathfrak{F}} = G^{\mathfrak{F}}/A \times (C \cap A)A^{\mathfrak{F}}/A^{\mathfrak{F}}$. It follows that A normalises $A^{\mathfrak{F}}B^{\mathfrak{F}}$. Analogously B normalises $A^{\mathfrak{F}}B^{\mathfrak{F}}$. Consequently $A^{\mathfrak{F}}B^{\mathfrak{F}} = 1$ and A and B are $\mathfrak{F}$-groups. Since $\mathfrak{F}$ is a lattice formation and A and B are $\mathfrak{F}$-subnormal in G, we have that $G \in \mathfrak{F}$ by Theorem 6.3.3 (3). It contradicts our supposition. Suppose that $G^{\mathfrak{F}} \in \mathfrak{S}_{\pi_i}$. If $G/G^{\mathfrak{F}} \in \mathfrak{M}$ or $G/G^{\mathfrak{F}} \in \mathfrak{S}_{\pi_j}$ for some $j \in I$, $i \neq j$, we can argue as above and obtain a contradiction. Hence $G/G^{\mathfrak{F}} \in \mathfrak{S}_{\pi_i}$ and so $G \in \mathfrak{S}_{\pi_i} \subseteq \mathfrak{F}$, contradicting $G \in \mathcal{W}(\mathfrak{F})$. It follows that our supposition is wrong and hence $\mathfrak{F}$ is a GWP-formation. □

If $\mathfrak{F}$ is a soluble subgroup-closed saturated lattice formation, then $\mathrm{b}_n(\mathfrak{F}) = \emptyset$. Moreover, if $\mathfrak{F}$ is a soluble GWP-formation, then $\mathfrak{F}$ is a subgroup-closed Fitting formation by Lemma 6.5.33, and hence saturated by Theorem 2.5.2. Therefore we have:

Corollary 6.5.47 (see [Kam96, BBPAPR96]). *Let $\mathfrak{F}$ be a soluble formation. Then $\mathfrak{F}$ is a GWP-formation if and only if $\mathfrak{F}$ is a subgroup-closed saturated lattice formation.*

Another interesting examples of GWP-formations follow from the following result.

Corollary 6.5.48 ([Kam96]). *Let $\mathfrak{F}$ be a saturated formation representable as $\mathfrak{F} = \mathfrak{M} \times \mathfrak{H}$, where $\pi(\mathfrak{M}) \cap \pi(\mathfrak{H}) = \emptyset$, $\mathfrak{M} = \mathfrak{M}^2$ is a subgroup-closed saturated Fitting formation, $\mathfrak{H} = \times_{i \in \mathcal{I}} \mathfrak{S}_{\pi_i}$, and moreover $\pi_l \cap \pi_k = \emptyset$ for all $k \neq l$ in I. Then $\mathfrak{F}$ is a GWP-formation.*

Proof. Applying Theorem 6.3.15, $\mathfrak{F}$ is a subgroup-closed saturated lattice formation. Hence, by Theorem 6.5.46, it is enough to check the property in groups in $\mathrm{b}_n(\mathfrak{F})$ generated by two $\mathfrak{F}$-subnormal subgroups. Let G be one of them. Then $G \in \mathrm{b}(\mathfrak{F})$ and $\mathrm{Soc}(G) = G^{\mathfrak{F}}$ is non-abelian. Moreover, by Result 6.5.43, $G^{\mathfrak{F}} \in \mathfrak{M}$ and $G/G^{\mathfrak{F}} \in \mathfrak{M}$ (note that $\mathfrak{H}$ is a soluble formation). Hence $G \in \mathfrak{M}^2 = \mathfrak{M} \subseteq \mathfrak{F}$. This contradiction proves that $\mathfrak{F}$ is a GWP-formation. □

This completes our discussion about GWP-formations. We can turn this situation on its head and ask the following.

> Let $\mathfrak{F}$ be a subgroup-closed formation and let G be a group generated by two $\mathfrak{F}$-subnormal subgroups A and B of G. When do we have $G^{\mathfrak{F}} = \langle A^{\mathfrak{F}}, B^{\mathfrak{F}} \rangle$?

The question is answered in [BBEPA02] for subgroup-closed saturated formations. It is proved there that if G is a group whose derived subgroup is nilpotent, then $G^{\mathfrak{F}} = \langle A^{\mathfrak{F}}, B^{\mathfrak{F}} \rangle$ provided that A and B are $\mathfrak{F}$-subnormal in $G = \langle A, B \rangle$. Furthermore the class $\mathfrak{N}\mathfrak{A}$ of all groups whose derived subgroup is nilpotent is characterised as the largest subgroup-closed saturated formation enjoying that property.

Let $\mathfrak{F}$ be a GWP-formation. Then $\mathfrak{F}$ has the following property:

> If A and B are K-$\mathfrak{F}$-subnormal ($\mathfrak{F}$-subnormal) subgroups of a group G and $G = AB$, then $G^{\mathfrak{F}} = A^{\mathfrak{F}} B^{\mathfrak{F}}$. (6.8)

In general, Property 6.8 does not characterise the GWP-formations as the class of all 2-nilpotent groups shows. Hence the question of how one subgroup-closed formation satisfying Property 6.8 can be characterised arises. This question is closely related to the characterisation of the subgroup-closed formations satisfying Property 6.1.

The above question has a nice answer in the soluble universe for subgroup-closed saturated formations of full characteristic.

Theorem 6.5.49 ([BBPAPR96]). *Let $\mathfrak{F}$ be a subgroup-closed saturated formation of soluble groups of full characteristic. The following statements are pairwise equivalent:*

1. *$\mathfrak{F}$ satisfies the property:*
   If A and B are two $\mathfrak{F}$-subnormal subgroups of a soluble group G and $G = AB$, then $G^{\mathfrak{F}} = A^{\mathfrak{F}} B^{\mathfrak{F}}$.
2. *For each prime number p, there exists a set of primes $\pi(p)$ with $p \in \pi(p)$ such that $\mathfrak{F}$ is locally defined by the formation function f given by $f(p) = \mathfrak{S}_{\pi(p)}$.*
   These sets of primes satisfy the following property:

If $q \in \pi(p)$, then $\pi(q) \subseteq \pi(p)$ for every pair of primes p and q.

Let $\mathfrak{F}$ be a subgroup-closed saturated formation of full characteristic satisfying the conditions of the above theorem. Then a soluble group G is an $\mathfrak{F}$-group if and only if G has a normal $\pi(p)$-complement for every prime p, where $\pi(p)$ is the set of primes such that $p \in \pi(p)$.