PART B. LINEAR ALGEBRA, VECTOR CALCULUS

CHAPTER 6. Linear Algebra: Matrices, Vectors, Determinants. Linear Systems of Equations

Sec. 6.1 Basic Concepts. Matrix Addition, Scalar Multiplication

Problem Set 6.1. Page 309

5. **Matrix addition, scalar multiplication.** First multiply D by 5, then C by 3. This gives

$$5\mathbf{D} = 5\begin{bmatrix} 4 & 0 & -4 \\ -3 & 4 & 9 \end{bmatrix} = \begin{bmatrix} 20 & 0 & -20 \\ -15 & 20 & 45 \end{bmatrix}$$

and

$$3\mathbf{C} = 3\begin{bmatrix} 6 & 0 & 3 \\ 1 & 0 & -5 \end{bmatrix} = \begin{bmatrix} 18 & 0 & 9 \\ 3 & 0 & -15 \end{bmatrix}.$$

The resulting matrices have the same size as the given ones because scalar multiplication does not alter the size of a matrix. Hence these matrices both have the size 2×3, so that the operations of addition and subtraction are defined for these matrices. Subtraction gives

$$5\mathbf{D} - 3\mathbf{C} = \begin{bmatrix} 20-18 & 0-0 & -20-9 \\ -15-3 & 20-0 & 45-(-15) \end{bmatrix} = \begin{bmatrix} 2 & 0 & -29 \\ -18 & 20 & 60 \end{bmatrix}.$$

The second task is very simple. Use (6) and conclude that

$$5\mathbf{D}^T - 3\mathbf{C}^T = (5\mathbf{D} - 3\mathbf{C})^T.$$

That is, you obtain the answer simply by taking the transpose of the matrix just obtained,

$$\begin{bmatrix} 2 & -18 \\ 0 & 20 \\ -29 & 60 \end{bmatrix}.$$

The first row (the second row) of that matrix becomes the first colum (the second column) of the transpose, which has size 3×2.

11. **Vectors** are special matrices (having a single column or a single row), and operations with them are the same as those for general matrices (and somewhat simpler). **c** and **d** are column vectors and they have the same number of components. These two properties are preserved under scalar multiplication. Hence $3(\mathbf{c} - 4\mathbf{d})$ is defined. You obtain

$$\mathbf{c} = \begin{bmatrix} 9 \\ 5 \\ 7 \end{bmatrix}, \quad 4\mathbf{d} = 4\begin{bmatrix} 2 \\ -2 \\ 6 \end{bmatrix} = \begin{bmatrix} 8 \\ -8 \\ 24 \end{bmatrix}, \quad \text{hence} \quad \mathbf{c} - 4\mathbf{d} = \begin{bmatrix} 1 \\ 13 \\ -17 \end{bmatrix}, \quad 3(\mathbf{c} - 4\mathbf{d}) = \begin{bmatrix} 3 \\ 39 \\ -51 \end{bmatrix}.$$

17. Proof of (4a). A and B are assumed to be general 2 × 3 matrices, that is,

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}.$$

By the definition of matrix addition,

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & a_{13} + b_{13} \\ a_{21} + b_{21} & a_{22} + b_{22} & a_{23} + b_{23} \end{bmatrix}$$

and

$$\mathbf{B} + \mathbf{A} = \begin{bmatrix} b_{11} + a_{11} & b_{12} + a_{12} & b_{13} + a_{13} \\ b_{21} + a_{21} & b_{22} + a_{22} & b_{23} + a_{23} \end{bmatrix}.$$

Now comes the idea of the proof. Use the *commutativity of the addition of numbers* to show that the two sums of A and B are equal. Indeed, $a_{11} + b_{11} = b_{11} + a_{11}$ and similarly for the other five entries of A + B and B + A. Hence corresponding entries of the matrix A + B and of the matrix B + A are equal. By the definition of the equality of matrices this proves (4a) for any 2 × 3 matrices A and B, that is,

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}.$$

Sec. 6.2 Matrix Multiplication

Problem Set 6.2. Page 319

3. Matrix multiplication. $\mathbf{C}^2$ is obtained from C by a straightforward application of the definition of matrix multiplication,

$$\mathbf{C}^2 = \begin{bmatrix} 4 & 6 & 2 \\ 6 & 0 & 3 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} 4 & 6 & 2 \\ 6 & 0 & 3 \\ 2 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 56 & 30 & 24 \\ 30 & 45 & 9 \\ 24 & 9 & 14 \end{bmatrix}.$$

Sample calculation: Denoting the entries of $\mathbf{C}^2$ by d_{jk}, you obtain

$$d_{11} = c_{11}^2 + c_{12}c_{21} + c_{13}c_{31} = 4^2 + 6^2 + 2^2 = 56$$

$$d_{12} = c_{11}c_{12} + c_{12}c_{22} + c_{13}c_{32} = 4 \cdot 6 + 6 \cdot 0 + 2 \cdot 3 = 30.$$

Furthermore, $\mathbf{C}^T\mathbf{C} = \mathbf{C}\mathbf{C}^T = \mathbf{C}^2$ because C is symmetric (definition in Sec. 6.1).

5. Vectors. The product $\mathbf{a}^T\mathbf{d}$ is not defined because a is a column vector, so that $\mathbf{a}^T$ is a row vector, and so is d, but the product of two row vectors (with more than one component) is not defined. (Neither is the product of two column vectors.) $\mathbf{a}^T\mathbf{d}^T$ is defined because a is a column vector, so that $\mathbf{a}^T$ is a row vector, and $\mathbf{d}^T$ is a column vector, and in this product you multiply row times column, as usual in matrix multiplication,

$$\mathbf{a}^T\mathbf{d}^T = \begin{bmatrix} 1 & 4 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} = 1 \cdot 4 + 4 \cdot 3 + 3 \cdot 0 = 16.$$

d a is defined and equals 16. This follows by calculation or by taking the transpose of the product just computed and noting that transposition of a 1 × 1 matrix (a number) has no effect. Products of vectors as just discussed are of great practical importance. Products, such as **a d**, which will give a 3 × 3 matrix, occur once in a while; they are of minor practical importance, but may add to your understanding of matrix multiplication. In this you multiply row times column, but here rows and columns have just a single entry.

This looks as follows

$$\mathbf{ad} = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} \begin{bmatrix} 4 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 3 & 0 \\ 16 & 12 & 0 \\ 12 & 9 & 0 \end{bmatrix}.$$

The first entry of the product is $1 \cdot 4 = 4$, the next $1 \cdot 3 = 3$, the third $1 \cdot 0 = 0$, and so on. Note that the first row of the product matrix is **d**, the second is 4**d**, and the third is 3**d**.

13. **Markov process.** The matrix of the Markov process is given as

$$\begin{array}{cc} & \text{To I} \quad \text{To II} \\ \mathbf{A} = & \begin{bmatrix} 0.5 & 0.5 \\ 0.2 & 0.8 \end{bmatrix} \begin{array}{l} \text{From I} \\ \text{From II} \end{array} \end{array}$$

For good understanding, give the process an interpretation. For instance, I and II are two kinds of soap that initially sell equally well, as expressed by the starting vector [0.7 0.7], measured in millions of bars sold per month. Then the first entry 0.5 shows that somebody who is using brand I will continue to use it with probability 0.5, hence he or she will switch to brand II with probability 0.5. A person who is using II will again use II with the high probability 0.8 and will try out I with the much smaller probability of 0.2. From this interpretation you can already see what will happen in the long run, namely, the sale of II will constantly increase with time. The calculation is as follows. (The notation is that of Example 14 in the book.)

$$\mathbf{y}^T = \mathbf{x}^T\mathbf{A} = [0.7 \quad 0.7] \begin{bmatrix} 0.5 & 0.5 \\ 0.2 & 0.8 \end{bmatrix} = [0.49 \quad 0.91].$$

In the next step calculate $\mathbf{y}^T\mathbf{A} = [0.427 \quad 0.973]$. Note that by substituting $\mathbf{y}^T = \mathbf{x}^T\mathbf{A}$ into $\mathbf{y}^T\mathbf{A}$ you get

$$\mathbf{y}^T\mathbf{A} = \mathbf{x}^T\mathbf{A}^2,$$

but this would not be a practical way of calculating the vectors in this iteration. And so on for the further steps.

Sec. 6.3 Linear Systems of Equations. Gauss Elimination

Problem Set 6.3. Page 329

13. **Gauss elimination.** It is, of course, perfectly acceptable to do the Gauss elimination in terms of equations rather than in terms of the augmented matrix. However, as soon as you feel sufficiently acquainted with matrices, you may wish to save work of writing by operating on matrices. The unknowns are eliminated in the order in which they occur in each equation. Hence start with w. Since w does not occur in the first equation, you must **pivot**. Take the second equation as your pivot equation in this first step. This gives the new augmented matrix

$$\left[\begin{array}{cccc|c} 2 & -3 & -3 & 6 & 2 \\ 0 & 5 & 5 & -10 & 0 \\ 4 & 1 & 1 & -2 & 4 \end{array}\right].$$

It is practical to indicate the operations after the corresponding row, as shown in the book. You obtain the next matrix row by row. Copy Row 1. This is the pivot row in the first step. w does not occur in Eq. 2, so you need not operate on Row 2 and simply copy it. To eliminate w from Eq. 3, subtract twice Row 1 from Row 3. Mark this after Row 3 of the next matrix, which, accordingly, takes the form

$$\left[\begin{array}{cccc|c} 2 & -3 & -3 & 6 & 2 \\ 0 & 5 & 5 & -10 & 0 \\ 0 & 7 & 7 & -14 & 0 \end{array}\right] \quad \text{Row 3} - 2\text{Row 1.}$$

Note well that rows may be left unlabeled if you do not operate on them. And the row numbers occurring in labels always refer to the *previous* matrix just as in the book. w has now been eliminated. Turn to the next unknown, x. Copy the first two rows of the present matrix and operate on Row 3 by subtracting from it 1.4 times Row 2 because this will eliminate x from Eq. 3. The result is surprising–not really if you have already noticed that you have produced two proportional rows. The calculation gives

$$\left[\begin{array}{cccc|c} 2 & -3 & -3 & 6 & 2 \\ 0 & 5 & 5 & -10 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \text{Row 3} - 1.4\text{Row 2.}$$

Note that if the last entry were not 0, the system would have no solution. Now solve the second equation,

$$5x + 5y - 10z = 0.$$

Solving for x, you obtain

$$x = \frac{1}{5}(10z - 5y) = 2z - y$$

where y and z remain arbitrary. Had you solved for y, you would have obtained the answer given in Appendix 2, namely, $y = 2z - x$ with arbitrary x and z. As a third possibility, you could solve for z and leave x and y arbitrary, $z = (x + y)/2$. Each of these three possibilities is acceptable. Finally, find w from the first equation of the system after pivoting, which is the second equation of the system in the form given, namely, $2w - 3x - 3y + 6z = 2$. This gives

$$w = \frac{1}{2}(2 + 3(2z - y) + 3y - 6z) = 1.$$

17. **Electrical network.** The elements of the circuits (batteries and Ohm's resistors in the present case) are given. The first step is the introduction of letters and directions for the unknown currents, which you want to determine. This has already been done in the figure of the network as shown. You do not know the directions of the currents. This does not matter. You make a choice, and if an unknown current comes out negative, this means that you have chosen the wrong direction and the current actually flows in the opposite direction. There are three currents I_1, I_2, I_3; hence you need three equations. An obvious choice is the right node, at which I_3 flows in and I_1 and I_2 flow out; thus, by KCL (Kirchhoff's current law, Sec. 1.7),

$$I_3 = I_1 + I_2.$$

The left node would do equally well. Can you see that you would get the same equation (except for a minus sign by which all three currents are now multiplied)? Two further equations are obtained from KVL (Kirchhoff's voltage law, Sec. 1.7), one for the upper circuit and one for the lower. In the upper circuit you have a voltage drop of $2I_1$ across the left resistor (which has resistance $R = 2$), a voltage drop of $1 \cdot I_3$ across the lower resistor (whose resistance is 1 and through which the current I_3 is flowing), and a voltage drop $2I_1$ across the right resistor. Hence the sum of the voltage drops is $2I_1 + I_3 + 2I_1 = 4I_1 + I_3$. By KVL this sum must equal the electromotive force 16 on the upper circuit; here resistance is measured in ohms and voltage in volts. Thus, the second equation for determining the currents is

$$4I_1 + I_3 = 16.$$

A third equation is obtained by KVL from the lower circuit. The voltage drop across the left resistor is $4I_2$ because this resistor has resistance 4 ohms and the current I_2 is flowing through it, causing the drop. A second voltage drop occurs across the upper (horizontal) resistor in the circuit, namely $1 \cdot I_3$, as before. The sum of these two voltage drops must equal the electromotive force of 32 volts in this circuit, again by KVL. This gives

$$4I_2 + I_3 = 32.$$

Hence the system of the three equations for the three unknowns, properly ordered, is

$$\begin{aligned} I_1 + I_2 - I_3 &= 0 \\ 4I_1 \qquad + I_3 &= 16 \\ 4I_2 + I_3 &= 32. \end{aligned}$$

In your further work use the corresponding augmented matrix (write a vertical bar before the entries on the right if you still need it)

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 4 & 0 & 1 & 16 \\ 0 & 4 & 1 & 32 \end{bmatrix}.$$

Subtract 4 times Row 1 (your pivot row) from Row 2, obtaining

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & -4 & 5 & 16 \\ 0 & 4 & 1 & 32 \end{bmatrix} \quad \text{Row 2} - 4\,\text{Row 1}.$$

Note well that the pivot row (or pivot equation) remains untouched as it is. The new pivot row is Row 2. Use it to eliminate I_2 (its coefficient 4) from Row 3. For this you must add it to Row 3, obtaining

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & -4 & 5 & 16 \\ 0 & 0 & 6 & 48 \end{bmatrix} \quad \text{Row 3} + \text{Row 2}.$$

The system has now reached triangular form. Back substitution begins. You may perhaps first write the transformed system in terms of equations,

$$\begin{aligned} I_1 + I_2 - I_3 &= 0 \\ -4I_2 + 5I_3 &= 16 \\ 6I_3 &= 48. \end{aligned}$$

From Eq. 3 you obtain $I_3 = 48/6 = 8$. With this and $5I_3 = 40$, Eq. 2 gives

$$I_2 = \frac{1}{-4}(16 - 40) = 6.$$

Finally, Eq. 1 gives

$$I_1 = -I_2 + I_3 = 2.$$

Sec. 6.4 Rank of a Matrix. Linear Independence. Vector Space

Problem Set 6.4. Page 336

1. **Linear independence and dependence** are concepts of general importance, for instance, in dropping unnecessary vectors from a set of vectors, keeping only the essential ones that make up a possibly much smaller and therefore simpler set to work with. To show linear independence or dependence in the problem, use (1), with a simpler notation for the scalars, say, a, b, c. Then (1) takes the form

$$a\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} + b\begin{bmatrix} 1 & 1 & 0 \end{bmatrix} + c\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = \mathbf{0}.$$

This vector equation is equivalent to three equations for the three components. You can first do the three scalar multiplications and then add the three resulting vectors. This gives

$$\begin{aligned} &\begin{bmatrix} a & 0 & 0 \end{bmatrix} + \begin{bmatrix} b & b & 0 \end{bmatrix} + \begin{bmatrix} c & c & c \end{bmatrix} \\ &= \begin{bmatrix} a+b+c & b+c & c \end{bmatrix} = \mathbf{0}. \end{aligned}$$

The equation for the third component gives $c = 0$. This reduces the equation for the second component to $b = 0$. Finally, the equation for the first component is now reduced to $a = 0$. Hence the only solution of (1) is $a = 0, b = 0, c = 0$, which means linear independence.

15. Rank of a matrix. To determine the rank of a matrix, you can use the definition and work with rows, or Theorem 1 and work with columns, and reduce that matrix as shown in Example 6 in the text. The determination of a rank by inspection will hardly be possible in practice. The given matrix is 4×3. Hence its rank can be 3 at most, but may be less. Whether using rows or using columns will be better, this is hard to say; so start with rows, that is, consider the matrix as given and reduce it as in the Gauss elimination, first taking Row 1 as pivot row and then Row 2. This looks as follows. The given matrix is

$$\begin{bmatrix} 3 & 1 & 4 \\ 0 & 5 & 8 \\ -3 & 4 & 4 \\ 1 & 2 & 4 \end{bmatrix}$$

Generate zeros in the first column:

$$\begin{bmatrix} 3 & 1 & 4 \\ 0 & 5 & 8 \\ 0 & 5 & 8 \\ 0 & 5/3 & 8/3 \end{bmatrix} \begin{matrix} \\ \\ \text{Row 3 + Row 1} \\ \text{Row 4 − 1/3 Row 1} \end{matrix}$$

(Can you see already here that rank A = 2? But go on, to see how the algorithm works.) Now take the second row as pivot row and generate zeros in the second column:

$$\begin{bmatrix} 3 & 1 & 4 \\ 0 & 5 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \\ \text{Row 3 − Row 2} \\ \text{Row 4 − 1/3 Row 2} \end{matrix}$$

Your task is finished; you see that the number of linearly independent rows of this matrix, hence of the given matrix, is 2. Hence it has rank 2.

You may be curious to see whether you could have saved work by taking columns instead of rows. This question seems natural since the matrix has only 3 columns (and its rank cannot exceed 3), whereas it has 4 rows. Write the matrix in transposed form, for convenient work. Use the first row of the transpose (the first column of the given matrix) as pivot row and then the second row in the second step.

$$\begin{bmatrix} 3 & 0 & -3 & 1 \\ 1 & 5 & 4 & 2 \\ 4 & 8 & 4 & 4 \end{bmatrix}$$

Generate zeros in the first column of this transposed matrix.

$$\begin{bmatrix} 3 & 0 & -3 & 1 \\ 0 & 5 & 5 & 5/3 \\ 0 & 8 & 8 & 8/3 \end{bmatrix} \begin{matrix} \\ \text{Row 2 − 1/3 Row 1} \\ \text{Row 3 − 4/3 Row 1} \end{matrix}$$

Now generate a zero in the second column of this matrix, and this will automatically reduce the last row to zero.

$$\begin{bmatrix} 3 & 0 & -3 & 1 \\ 0 & 5 & 5 & 5/3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{Row 3} - 8/5\,\text{Row 2}$$

The matrix has a maximum number of 2 linearly independent rows. Theorem 1 now implies that the given matrix has rank 2. This confirms your previous result.

27. Row space. By definition, a matrix **B** is row-equivalent to a matrix **A** if **B** is obtained from **A** by elementary row operations (p. 326), that is, by

(a) Interchanging two rows of **A**,

(b) Adding a constant multiple of one row of **A** to another row of **A**,

(c) Multiplying a row by a nonzero constant .

By definition, the row space of **A** is the span of the row vectors of **A**; similarly for **B**, and you have to show that these two spans agree.

Clearly, operation (a) does not alter the span because the order in which one writes the vectors does not affect their span.

Consider the effect of (b). The row space of **A**, being the span of the row vectors of **A**, is also obtained as the span of the maximum number of *linearly independent* row vectors of **A**. But this maximum number (which equals rank **A**) is not altered by (b).

Finally, (c) has no influence on that span since multiplication by a constant that is not zero multiplies the terms of a linear combination by that constant, converting it to another linear combination already present because that span consists of *all* linear combinations of the row vectors of **A**. This completes the proof.

Sec. 6.6 Determinants. Cramer's Rule

Problem Set 6.6. Page 349

9. Determinant. The simplest way seems a development by any of the rows or columns, for instance, by the first row.

13. Determinant. In older books, determinants were evaluated "unsystematically" by looking for rows or columns that already had zeros and then increasing their number by suitable operations. In the light of programmability, this method has become obsolete and has been replaced by reduction to triangular form, as shown in Example 7 of the text. For special determinants this can be modified. If a square matrix is of the form

$$\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{bmatrix}$$

then you may perform the reduction only to the extent that you do not "destroy" the zeros in the two zero matrices. Accordingly, use Row 1 as pivot row, but reduce only Row 2, obtaining

$$\begin{vmatrix} 3 & 2 & 0 & 0 \\ 6 & 8 & 0 & 0 \\ 0 & 0 & 4 & 7 \\ 0 & 0 & 2 & 5 \end{vmatrix} = \begin{vmatrix} 3 & 2 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 7 \\ 0 & 0 & 2 & 5 \end{vmatrix} \quad \text{Row 2} - 2\,\text{Row 1}.$$

Then use Row 3 as pivot row for reducing Row 4. This gives

$$\begin{vmatrix} 3 & 2 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 7 \\ 0 & 0 & 0 & 1.5 \end{vmatrix} \quad \text{Row 4} - (1/2)\,\text{Row 3}.$$

This determinant can now be developed by columns, first by the first column, the resulting third order determinant by *its* first column, and so on. The result is $3 \cdot 4 \cdot 4 \cdot 1.5 = 72$.

15. Rank by determinants. Theorem 3 shows immediately that the rank must be at least 2 because the 2×2 submatrix in the left upper corner has determinant -4. To find out by determinants whether the rank is 3, you must compute the determinant of the matrix. The development by the first row is

$$\begin{vmatrix} 0 & 2 & -3 \\ 2 & 0 & 5 \\ -3 & 5 & 0 \end{vmatrix} = -2\begin{vmatrix} 2 & 5 \\ -3 & 0 \end{vmatrix} - 3\begin{vmatrix} 2 & 0 \\ -3 & 5 \end{vmatrix} = -2 \cdot 15 - 3 \cdot 10 = -60.$$

Since this determinant is not zero, Theorem 3 implies that the matrix has rank 3. To check this result, apply row reduction to the matrix obtained by pivoting (by intechanging Rows 1 and 2).

19. Cramer's rule. The determinant of the system is

$$D = \begin{vmatrix} 3 & 7 & 8 \\ 2 & 0 & 9 \\ -4 & 1 & -26 \end{vmatrix} = 101.$$

The determinants D_1, D_2, D_3 needed in (13) are

$$D_1 = \begin{vmatrix} -13 & 7 & 8 \\ -5 & 0 & 9 \\ 2 & 1 & -26 \end{vmatrix} = -707$$

$$D_2 = \begin{vmatrix} 3 & -13 & 8 \\ 2 & -5 & 9 \\ -4 & 2 & -26 \end{vmatrix} = 0$$

$$D_3 = \begin{vmatrix} 3 & 7 & -13 \\ 2 & 0 & -5 \\ -4 & 1 & 2 \end{vmatrix} = 101.$$

Hence the solution is

$$x = -707/101 = -7, \quad y = 0, \quad z = 101/101 = 1.$$

Sec. 6.7 Inverse of a Matrix. Gauss-Jordan Elimination

Problem Set 6.7. Page 357

9. Inverse. The inverse of a square matrix $\mathbf{A}$ is obtained by the Gauss-Jordan elimination as shown in Example 1 in the text, and this needs hardly any further comments. In particular, Example 1 shows that the entries of the inverse will in general be fractions, even if the entries of $\mathbf{A}$ are integers. If $\mathbf{A}$ is special (symmetric, triangular, etc.), its inverse may be special. The given matrix is upper triangular, and you start from

$$\left[\begin{array}{ccc|ccc} 1 & 8 & -7 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Since the given matrix is upper triangular, the Gauss part of the Gauss-Jordan method is not needed and you can begin immediately with the Jordan elimination of 8, −7, and 3 above the main diagonal, which will reduce the given matrix to the unit matrix. Using Row 2 as the pivot row and working "upward", calculate

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -31 & 1 & -8 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{l} \text{Row 1} - 8\,\text{Row 2} \\ \\ \\ \end{array}$$

Using Row 3 as the pivot row, eliminate 3 and −31, obtaining

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -8 & 31 \\ 0 & 1 & 0 & 0 & 1 & -3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{l} \text{Row 1} + 31\,\text{Row 3} \\ \text{Row 2} - 3\,\text{Row 3} \\ \\ \end{array}$$

The right half of this 3 × 6 matrix is the inverse of the given matrix. Since the latter has 1 1 1 as the main diagonal, you needed no multiplications, as they would usually be necessary (see of the first matrix on p. 353, for example).

We mention that for a (nonsingular) triangular matrix the entries of the inverse can be determined one-by-one, without solving any system of equations. Can you show this for the present problem (which thus merely has the purpose of illustrating the Gauss-Jordan method for determining the inverse)?

13. **Inverse of a symmetric matrix.** Let **A** be symmetric, $\mathbf{A} = \mathbf{A}^T$, and nonsingular. Denote its inverse by **B**. Then by the definition of the inverse,

$$\mathbf{AB} = \mathbf{I}.$$

Transposition and the use of the symmetry of **A** give

$$(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T = \mathbf{B}^T\mathbf{A} = \mathbf{I}^T = \mathbf{I}.$$

Multiply this equation by **B** from the right, obtaining

$$\mathbf{B}^T\mathbf{AB} = \mathbf{IB} = \mathbf{B}.$$

But $\mathbf{AB} = \mathbf{I}$, so that this equation reduces to

$$\mathbf{B}^T = \mathbf{B}.$$

This shows that **B** is symmetric and completes the proof.

By Prob. 12 you would simply have $(\mathbf{A}^{-1})^T = (\mathbf{A}^T)^{-1} = \mathbf{A}^{-1}$.

15. **Formula for the inverse.** Problems 15-20 should aid in understanding the use of minors and cofactors. The given matrix is

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}.$$

Calculate its determinant, $D = \det \mathbf{A} = 1$. Denote the inverse of **A** simply by $\mathbf{B} = [b_{jk}]$. From (4) you thus obtain

$$b_{11} = A_{11} = \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} = 3$$

$$b_{12} = A_{21} = -\begin{bmatrix} 0 & -1 \\ 1 & 3 \end{bmatrix} = -1$$

$$b_{13} = A_{31} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = 1$$

$$b_{21} = A_{12} = -\begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} = -15$$

$$b_{22} = A_{22} = \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix} = 6$$

$$b_{23} = A_{32} = -\begin{bmatrix} 2 & -1 \\ 5 & 0 \end{bmatrix} = -5$$

$$b_{31} = A_{13} = \begin{bmatrix} 5 & 1 \\ 0 & 1 \end{bmatrix} = 5$$

$$b_{32} = A_{23} = -\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = -2$$

$$b_{33} = A_{33} = \begin{bmatrix} 2 & 0 \\ 5 & 1 \end{bmatrix} = 2.$$

Hence, since $\det \mathbf{A} = 1$, the inverse is

$$\mathbf{A}^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}.$$

Check your calculations by verifying that $\mathbf{AA}^{-1} = \mathbf{I}$, the unit matrix.

Sec. 6.8 Vector Spaces, Inner Product Spaces, Linear Transformations. *Optional*

Problem Set 6.8. Page 364

1. Vector space. To see whether the vectors satisfying

$$v_1 - 3v_2 + 2v_3 = 0 \qquad \text{(A)}$$

form a vector space V, you have to find out whether for any two vectors $\mathbf{v}$ and $\mathbf{w}$ satisfying (A) and

$$w_1 - 3w_2 + 2w_3 = 0 \qquad \text{(B)}$$

a linear combination

$$\mathbf{u} = a\mathbf{v} + b\mathbf{w} = [u_1 \quad u_2 \quad u_3] = [av_1 + bw_1 \quad av_2 + bw_2 \quad av_3 + bw_3]$$

also satisfies the corresponding relation

$$u_1 - 3u_2 + 2u_3 = 0. \qquad \text{(C)}$$

This is true, as you can see by inserting the components of **u** into (C), obtaining

$$av_1 + bw_1 - 3(av_2 + bw_2) + 2(av_3 + bw_3) = a(v_1 - 3v_2 + 2v_3) + b(w_1 - 3w_2 + 2w_3)$$

because from (A) and (B) it follows that each of the expressions in the two pairs of parentheses on the right is zero.

V has dimension 2 because R^3 has dimension 3 and the vectors of V satisfy one relation, namely, (A), which may be written

$$v_1 = 3v_2 - 2v_3. \qquad \text{(A*)}$$

You may use (A*) in determining a basis **p**, **q** as follows. **p** in V with $p_2 = 1$ and $p_3 = 0$ must have $p_1 = 3$, because of (A*). Hence $\mathbf{p} = [3 \quad 1 \quad 0]^T$. A vector **q** in V with $q_2 = 0$ and $q_3 = 1$ must have $q_1 = -2$, again because of (A*). Hence $\mathbf{q} = [-2 \quad 0 \quad 1]^T$. The answer in Appendix 2 gives **p** and $-\mathbf{q}$. Recall that you have infinitely many choices for a basis for V. Indeed, another basis is $\mathbf{u} = a\mathbf{p} + b\mathbf{q}$ and $\mathbf{w} = c\mathbf{p} + k\mathbf{q}$ with any scalars a, b, c, k such that **u** and **w** are linearly independent.

5. **Vector space.** A 2 × 2 skew-symmetric matrix has main diagonal 0 0 and $a_{21} = -a_{12}$. This gives dimension 1 and explains the form of the basis given in the answer, consisting of a single element (a single matrix).

21. **Orthogonality** is a concept of basic importance; for instance, the choice of orthogonal vectors for a basis simplifies many calculations. The inner product of the given $\mathbf{a} = [1 \quad 2 \quad 0]^T$ and any $\mathbf{v} = [v_1 \quad v_2 \quad v_3]^T$ is

$$\mathbf{a} \bullet \mathbf{v} = \mathbf{a}^T\mathbf{v} = v_1 + 2v_2$$

and is zero if and only if $v_2 = -v_1/2$ and v_3 is arbitrary. Geometrically, these are all the vectors in R^3 whose orthogonal projections in the xy-plane are orthogonal to $[1 \quad 2]^T$ (perpendicular to this vector or zero). Make a sketch. These vectors **v** form a vector space. This can be proved as in Prob. 1 above.

25. **Linear transformation.** In vector form you have $\mathbf{y} = \mathbf{A}\mathbf{x}$, where

$$\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}.$$

The inverse is $\mathbf{x} = \mathbf{A}^{-1}\mathbf{y}$. Hence Probs. 25-30 are solved by determining the inverse of the coefficient matrix **A** of the given transformation (if it exists, that is, if **A** is nonsingular).