CHAPTER 4. Series Solutions of Differential Equations. Special Functions

Sec. 4.2 Theory of the Power Series Method

Problem Set 4.2. Page 204

1. Power series solution. The equation $y' = -2xy$ can readily be solved by separating variables,

$$dy/y = -2x\,dx, \qquad \ln|y| = -x^2 + C, \qquad y = ce^{-x^2}.$$

If for some reason a Maclaurin series of this solution is wanted, you can obtain it by substituting $-x^2$ for x in the familiar series for e^x. Hence this problem (as well as the others) just serves to explain the techniques in a simple case (in which you would not need them), as a preparation for equations, such as Legendre's, Bessel's, and the hypergeometric equations, to name the most important ones, where you do need these techniques. Start from the series

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots \qquad \text{(a)}$$

and differentiate it termwise, obtaining

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots \,. \qquad \text{(b)}$$

Leave space behind each series on your sheet, so that you can add terms if necessary. From (a) obtain the right side of the given differential equation (write corresponding powers below each other; this will facilitate your further work)

$$-2xy = \quad -2a_0 x - 2a_1 x^2 - 2a_2 x^3 - 2a_3 x^4 - \dots$$

For each power $x^0, x, x^2, x^3, \dots$ equate the two corresponding terms. Denote these equations for determining the coefficients by [0] (constant terms), [1] (first power of x), etc. This looks as follows.

$$a_1 = 0 \qquad [0]$$

$$2a_2 = -2a_0, \quad \text{hence} \quad a_2 = -a_0, \quad a_0 \text{ arbitrary} \qquad [1]$$

$$3a_3 = -2a_1 = 0 \qquad [2]$$

$$4a_4 = -2a_2, \quad \text{hence} \quad a_4 = (-2/4)a_2 = (1/2)a_0 \qquad [3]$$

$$5a_5 = -2a_3 = 0 \qquad [4]$$

$$6a_6 = -2a_4, \quad \text{hence} \quad a_6 = (-2/6)a_4 = -(1/3!)a_0 \qquad [5]$$

and so on. With a little more skill, you may use power series notation and write

$$y' + 2xy = \sum_{n=1}^{\infty} n a_n x^{n-1} + 2x \sum_{m=0}^{\infty} a_m x^m = 0.$$

In the second series multiply each term by $2x$. Then you have the general power x^{m+1}. To get the same power in the first series, set $n = m + 2$; this gives $x^{n-1} = x^{m+1}$ (this was the reason for choosing different summation letters in the two series). Also pull out the first term a_1 of the first series; then both summations begin with $m = 0$ and you can take the two series together, obtaining

$$a_1 + \sum_{m=0}^{\infty} (m+2)\, a_{m+2} x^{m+1} + \sum_{m=0}^{\infty} 2a_m x^{m+1}$$

$$= a_1 + \sum_{m=0}^{\infty} [(m+2)\, a_{m+2} + 2a_m] x^{m+1} = 0.$$

You see now that $a_1 = 0$ and get the recursion

$$(m+2)a_{m+2} + 2a_m = 0 \quad \text{or} \quad a_{m+2} = -2a_m/(m+2), \qquad m = 0, 1, \dots$$

Choosing $m = 0, 1, \dots$, you obtain successively

$$a_2 = -\frac{2}{2}a_0, \quad a_3 = 0, \quad a_4 = -\frac{4}{2}a_2 = \frac{1}{2}a_0, \quad a_5 = 0, \ldots$$

as before.

17. Radius of convergence. A power series in powers of x may converge for all x (this is the best possible case) or within an interval with the center x_0 as midpoint (in the complex plane: within a disk with center x_0) or only at the center (the practically useless case). In the second case the interval of convergence has length $2R$, where R is called the *radius of convergence* (it is a radius in the complex case, as has just been said) and is given by (7a) or (7b) of Sec. 4.2. Here it is assumed that the limits in these formulas exist. This will be the case in most applications. (For help when this is not the case, see Sec. 14.2.) The convergence radius is important whenever you want to use series for computing values, exploring properties of functions represented by series, or proving relations between functions, tasks of which you will gain a first impression in Secs. 4.3-4.7 and corresponding problems. In Prob. 17 you may set $x^2 = t$. Then you have a power series in t with coefficients of absolute value $|a_m| = 1/|k|^m$. Hence the root in (7a) is $1/|k|$, so that the radius of convergence of the power series in t is $|k|$. That is, the series converges for $|t| < |k|$. This implies $|x| = \sqrt{|t|} < \sqrt{|k|}$. Hence the given series has the convergence radius $\sqrt{|k|}$. Confirm this by using (7b). The quotient in (7b) is

$$|a_{m+1}/a_m| = 1/|k|.$$

This leads to the same result as before. Note that the problem is special; in general, the sequences of those roots and quotients in (7) will not be constant, that is, the terms of such a sequence of quotients (or roots) will not be all the same.

Sec. 4.3 Legendre's Equation. Legendre Polynomials $P_n(x)$

Problem Set 4.3. Page 209

1. Legendre functions for $n = 0$. The power series and Frobenius methods were instrumental in establishing large portions of the very extensive theory of special functions (see, for instance, Refs. [1], [11], [12] in Appendix 1), as needed in engineering, physics (astronomy!), and other areas, simply because many special functions appeared first in the form of power series solutions of differential equations. In general, this concerns properties and relationships between higher transcendental functions. The point of Prob. 1 is an illustration that sometimes such functions may reduce to elementary functions known from calculus. If you set $n = 0$ in (7), it was observed in the problem that $y_1(x)$ becomes $\frac{1}{2}\ln((1+x)/(1-x))$. In this case, the answer suggests using

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - + \ldots .$$

Replacing x by $-x$ and multiplying by -1 on both sides gives

$$-\ln(1-x) = \ln\frac{1}{1-x} = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots .$$

Addition of these two series and division by 2 verifies the last equality sign in the formula of Prob. 1. You are requested to obtain this result directly by solving the Legendre equation (1) with $n = 0$, that is,

$$(1-x^2)y'' - 2xy' = 0 \quad \text{or} \quad (1-x^2)z' = 2xz, \qquad \text{where} \quad z = y'.$$

Separation of variables and integration gives

$$\frac{dz}{z} = \frac{2x}{1-x^2}dx, \qquad \ln|z| = -\ln|1-x^2| + c, \qquad z = C_1/(1-x^2).$$

y is now obtained by another integration, using partial fractions

$$\frac{1}{1-x^2} = \frac{1}{2}\left(\frac{1}{x+1} - \frac{1}{x-1}\right).$$

This gives

$$y = \int z\,dx = \frac{1}{2}C_1(\ln(x+1) - \ln(x-1)) + c = \frac{1}{2}C_1 \ln\frac{x+1}{x-1} + c.$$

Since $y_1(x)$ in (6), Sec. 4.3, reduces to 1 if $n = 0$, you can now readily express your solution obtained in terms of the standard functions y_1 and y_2 in (6) and (7), namely,

$$y = cy_1(x) + C_1y_2(x).$$

7. **Differential equation.** Set $x = az$ and apply the chain rule, according to which

$$\frac{d}{dx} = \frac{d}{dz}\frac{dz}{dx} = \frac{1}{a}\frac{d}{dz} \quad \text{and} \quad \frac{d^2}{dx^2} = \frac{1}{a^2}\frac{d^2}{dz^2}.$$

Substitution now gives

$$(a^2 - a^2z^2)\frac{d^2y}{dz^2}\frac{1}{a^2} - 2az\frac{dy}{dz}\frac{1}{a} + 3\cdot 4y = 0.$$

The factors a cancel and you are left with

$$(1 - z^2)y'' - 2zy' + 3\cdot 4y = 0.$$

Hence the solution is $P_3(z) = P_3(x/a)$, as claimed in Appendix 2.

Sec. 4.4 Frobenius Method

Problem Set 4.4. Page 216

5. **Basis of solutions.** Substitute y, y', and y'', given by (2) and (2*) in Sec. 4.4, into the differential equation $xy'' + 2y' + xy = 0$. This gives

$$\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r-1} + \sum_{m=0}^{\infty}2(m+r)a_m x^{m+r-1} + \sum_{n=0}^{\infty}a_n x^{n+r+1} = 0.$$

The first two series have the same general power, and you can take them together. In the third series set $n = m - 2$ to get the same general power. $n = 0$ then gives $m = 2$. You obtain

$$\sum_{m=0}^{\infty}(m+r)(m+r+1)a_m x^{m+r-1} + \sum_{m=2}^{\infty}a_{m-2}x^{m+r-1} = 0. \qquad \text{(A)}$$

For $m = 0$ this gives the indicial equation

$$r(r+1) = 0.$$

The roots are $r = 0$ and -1. They differ by an integer. This is Case 3 of Theorem 2 in Sec. 4.4. Consider the larger root $r = 0$. Then (A) takes the form

$$\sum_{m=0}^{\infty}m(m+1)a_m x^{m-1} + \sum_{m=2}^{\infty}a_{m-2}x^{m-1} = 0.$$

$m = 1$ gives $2a_1 = 0$. This implies $a_3 = a_5 = \dots = 0$, as is seen by taking $m = 3, 5, \dots$. Furthermore,

$$m = 2 \text{ gives } 2\cdot 3a_2 + a_0 = 0, \text{ hence } a_0 \text{ arbitrary, } a_2 = -a_0/3!$$

$$m = 4 \text{ gives } 4\cdot 5a_4 + a_2 = 0, \text{ hence } a_4 = -a_2/(4\cdot 5) = +a_0/5!$$

and so on. Since you want a basis and a_0 is arbitrary, you can take $a_0 = 1$. Recognize that you then have the Maclaurin series of

$$y_1 = (\sin x)/x.$$

Now determine an independent solution y_2. Since in Case 3 one would have to assume a term involving the logarithm (which may turn out to be zero), reduction of order (Sec. 2.1) seems to be simpler. This begins by writing the equation in standard form (divide by x):

$$y'' + (2/x)y' + y = 0.$$

In (2) of Sec. 2.1 you then have $p = 2/x$, $-\int p\,dx = -2\ln|x| = \ln(1/x^2)$, hence $\exp(-\int p\,dx) = 1/x^2$. Insertion of this and y_1^2 into (9) and cancellation of a factor x^2 gives

$$U = 1/\sin^2 x, \qquad u = \int U\,dx = -\cot x, \qquad y_2 = uy_1 = -\frac{\cos x}{x}.$$

9. Basis of solutions. Try the substitution $x + 2 = t$. Can you see why?

13. Euler-Cauchy equation. The point of this problem is that you should recognize how Euler-Cauchy equations fit into the Frobenius theory.

Sec. 4.5 Bessel's Equation. Bessel Functions $J_\nu(x)$

Problem Set 4.5. Page 226

3. Reduction to Bessel's equation. Bessel's equation gains additional importance by the fact that numerous other differential equations can be reduced to this equation, so that the extensive theory of Bessel functions becomes applicable to the solutions of those other equations and their engineering uses. The corresponding transformations involve applications of the chain rule in transforming derivatives. From $x^2 = z$ you obtain $y' = 2x(dy/dx)$ and y'' by differentiating this; in detail

$$y' = \frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = 2x\frac{dy}{dz} = 2\sqrt{z}\,\frac{dy}{dz},$$

$$y'' = \frac{d^2y}{dx^2} = 2\frac{dy}{dz} + 4x^2\frac{d^2y}{dz^2} = 2\frac{dy}{dz} + 4z\frac{d^2y}{dz^2}.$$

By substitution,

$$x^2y'' + xy' + (4x^4 - \tfrac{1}{4})y = z(4z\frac{d^2y}{dz^2} + 2\frac{dy}{dz}) + 2z\frac{dy}{dz} + (4z^2 - \tfrac{1}{4})y = 0.$$

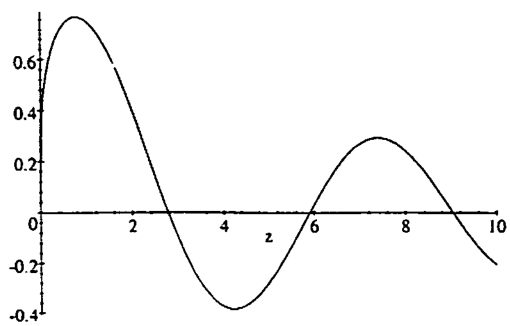
Division by 4 gives the Bessel equation

$$z^2\frac{d^2y}{dz^2} + z\frac{dy}{dz} + (z^2 - \frac{1}{16})y = 0.$$

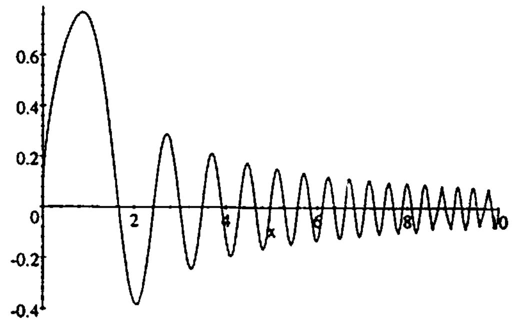
Hence a general solution is

$$y = AJ_{1/4}(z) + BJ_{-1/4}(z) = AJ_{1/4}(x^2) + BJ_{-1/4}(x^2). \qquad \text{(A)}$$

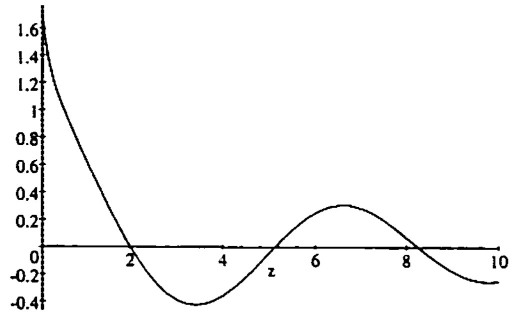
The figures show the two Bessel functions that form the basis in (A), plotted as functions of z (their usual appearance) and as functions of x^2 over the x-axis, in which case the oscillations become more and more rapid with increasing x.



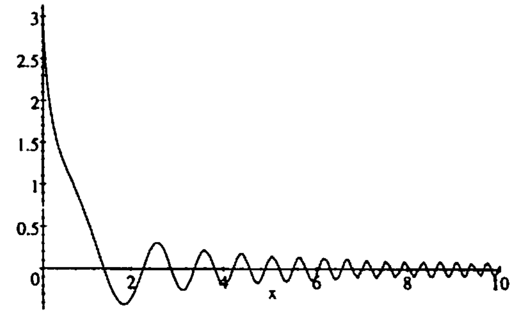
Section 4.5. Problem 3. Bessel function $J_{1/4}(z)$



Section 4.5. Problem 3. Bessel function $J_{1/4}(x^2)$



Section 4.5. Problem 3. Bessel function $J_{-1/4}(z)$



Section 4.5. Problem 3. Bessel function $J_{-1/4}(x^2)$

9. **Reduction to Bessel's equation.** Here you may first transform the dependent variable (the unknown function y) by the given transformation $y = x^{1/3}u$. You need the derivatives

$$y' = \frac{1}{3}x^{-2/3}u + x^{1/3}u',$$

$$y'' = -\frac{2}{9}x^{-5/3}u + \frac{2}{3}x^{-2/3}u' + x^{1/3}u''.$$

Substitute this into the given equation, order terms, and drop a common factor $x^{1/3}$. This gives

$$81x^2u'' + 81xu' + (9x^{2/3} - 1)u = 0. \qquad \text{(B)}$$

Now comes the second step: introduce $z = x^{1/3}$ as the new independent variable. You have to transform the derivatives

$$u' = \frac{du}{dz}\frac{dz}{dx} = \frac{du}{dz}\left(\frac{1}{3}\right)x^{-2/3} = \frac{1}{3}z^{-2}\frac{du}{dz}$$

$$u'' = \frac{d^2u}{dz^2}\left(\frac{1}{9}\right)x^{-4/3} + \frac{du}{dz}\left(-\frac{2}{9}\right)x^{-5/3} = \frac{1}{9}z^{-4}\frac{d^2u}{dz^2} - \frac{2}{9}z^{-5}\frac{du}{dz}.$$

Substitution into (B) and collection of terms gives

$$9\frac{z^6}{z^4}\frac{d^2u}{dz^2} - 18\frac{z^6}{z^5}\frac{du}{dz} + 81\frac{z^3}{z^2}\left(\frac{1}{3}\right)\frac{du}{dz} + (9z^2 - 1)u = 0.$$

Dividing this by 9 gives the Bessel equation with parameter 1/3 and unknown function $u(z)$, whose solution is

$$u(z) = AJ_{1/3}(z) + BJ_{-1/3}(z).$$

Replacing z by $x^{1/3}$ and multiplying by $x^{1/3}$ gives $y = x^{1/3}u(x^{1/3})$, as shown in Appendix 2.

Sec. 4.6 Bessel Functions of the Second Kind $Y_\nu(x)$

Problem Set 4.6. Page 232

7. Reduction to Bessel's equation. You have to transform the independent variable x by setting $z = kx^2/2$ as well as the unknown function y by setting $y = \sqrt{x}\,u$. Using the chain rule, you can perform the two transformations one after another – this would be similar to Prob. 9 in Problem Set 4.5 – or simultaneously, as we shall now explain. You will need $dz/dx = kx$. Differentiation with respect to x gives

$$\frac{dy}{dx} = \frac{1}{2}x^{-1/2}u + x^{1/2}\frac{du}{dz}kx$$
$$= \frac{1}{2}x^{-1/2}\,u + kx^{3/2}\frac{du}{dz}.$$

Differentiating this again, you obtain the second derivative

$$\frac{d^2y}{dx^2} = -\frac{1}{4}x^{-3/2}u + \frac{1}{2}x^{-1/2}\frac{du}{dz}kx + \frac{3}{2}\,kx^{1/2}\frac{du}{dz} + kx^{3/2}\frac{d^2u}{dz^2}kx$$
$$= -\frac{1}{4}x^{-3/2}u + 2kx^{1/2}\frac{du}{dz} + k^2x^{5/2}\frac{d^2u}{dz^2}.$$

Substituting this expression for y'' as well as y into the given equation and dividing the whole equation by $k^2x^{5/2}$ gives

$$\frac{d^2u}{dz^2} + \frac{2}{kx^2}\frac{du}{dz} + \left(1 - \frac{1}{4k^2x^4}\right)u = 0.$$

Now recall that $kx^2/2 = z$. Hence $kx^2 = 2z$. Substitute this into the last equation to get

$$\frac{d^2u}{dz^2} + \frac{1}{z}\frac{du}{dz} + \left(1 - \frac{1}{16z^2}\right)u = 0.$$

This is Bessel's equation with parameter $\nu = 1/4$. Hence a general solution of the given equation is

$$y = x^{1/2}u(z) = x^{1/2}(AJ_{1/4}(z) + BY_{1/4}(z)) = x^{1/2}(AJ_{1/4}(kx^2/2) + BY_{1/4}(kx^2/2)).$$

11. Y_0 for small x. Use (6). Neglect the series in (6), which is 0 for $x = 0$; hence solve $\ln(x/2) = -\gamma$. This gives $x = 1.1$, approximately. The actual 2S-value of the zero is 0.89; see Ref. [1], p. 410.

Sec. 4.7 Sturm-Liouville Problems. Orthogonal Functions

Example 5 and Theorem 2. In (13), n is fixed. The smallest n is $n = 0$. Hence then (13) concerns J_0. It then takes the form

$$\int_0^R xJ_0(k_{m0}x)J_0(k_{j0}x)\,dx = 0 \qquad (j \neq m, \text{ both integer}).$$

If $n = 0$ were the only possible value of n, you could simply write k_m and k_j instead of k_{m0} and k_{j0}; write it down for youself to see what (13) then looks like. Recall that k_{m0} is related to the zero α_{m0} of J_0 by $k_{m0} = \alpha_{m0}/R$. In applications (vibrating drumhead in Sec. 11.10, for instance) the number R can have any value depending on the problem (in Sec. 11.10 it is the radius of the drumhead); this is the reason for introducing the arbitrary k near the beginning of the example; it gives us the flexibility needed in practice.

Problem Set 4.7. Page 238

1. **Case 3 of Theorem 1.** In this case the proof runs as follows. By assumption, $r(a) = 0$ and $r(b) \neq 0$. The starting point of the proof is (8), as before. Since $r(a) = 0$, you see that (8) reduces to

$$r(b)[y_n'(b)y_m(b) - y_m'(b)y_n(b)]$$

and you have to show that this is zero. Now from (2b) you have (we write L instead of l, to avoid confusion with the number 1)

$$L_1y_n(b) + L_2y_n'(b) = 0$$

$$L_1y_m(b) + L_2y_m'(b) = 0.$$

At least one of the two coefficients is different from zero, by assumption, say, $L_2 \neq 0$. Now multiply the first equation by $y_m(b)$ and the second by $-y_n(b)$ and add, obtaining

$$L_2[y_n'(b)y_m(b) - y_m'(b)y_n(b)] = 0.$$

Since L_2 is not zero, the expression in the brackets must be zero. But this expression is identical with that in the brackets in the first line of (8), which we have written above. The second line of (8) is zero because of the assumption $r(a) = 0$. Hence (8) is zero, and from (7) you obtain the relationship (9) to be proved. (For $L_1 \neq 0$ the proof is similar. Supply the details; this will show you whether you really understand the present proof.)

3. **Sturm-Liouville problem.** The given equation and boundary conditions do constitute a Sturm-Liouville problem. The equation is of the form (1) with $r = 1, q = 0, p = 1$. The interval in which solutions are sought is given by $a = 0$ and $b = 1$ as its endpoints. In the boundary conditions, $k_1 = 1, k_2 = 0, l_1 = 0, l_2 = 1$. You first have to find a general solution. In Prob. 3 it is

$$y = A\cos kx + B\sin kx \qquad \text{where} \quad k = \sqrt{\lambda}. \tag{A}$$

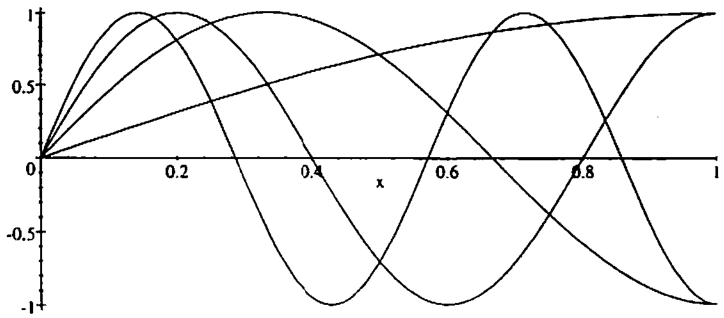
You obtain eigenvalues and functions by using the boundary conditions. The first condition gives $y(0) = A = 0$. Differentiation of the remaining part of equation (A) gives

$$y'(x) = kB\cos kx, \qquad \text{hence} \quad y'(1) = kB\cos k = 0,$$

thus $\cos k = 0$. This yields $k = k_n = (2n+1)\pi/2$, where $n = 0, 1, \ldots$; these are the positive values for which the cosine is zero. You need not consider negative values of n because the cosine is even, so that you would get the same eigenfunctions. The eigenvalues are $\lambda = \lambda_n = k_n^2$. The corresponding eigenfunctions are

$$y(x) = y_n(x) = \sin(k_nx) = \sin((2n+1)\pi x/2).$$

The figure shows the first few eigenfunctions. All of them start at 0 and have a horizontal tangent at the other end of the interval from 0 to 1. This is the geometric meaning of the boundary conditions. y_1 has no zero in the interior of this interval. Its graph shown corresponds to 1/4 of the period of the cosine. y_2 has one such zero (at 2/3), and its graph shown corresponds to 3/4 of that period. y_3 has two such zeros (at 0.4 and 0.8). y_4 has three, and so on.



Section 4.7. Problem 3. First four eigenfunctions of the Sturm-Liouville problem

13. **Change of** x. Equate $ct + k$ to the endpoints of the given interval and solve for t to get the new interval on which you can prove orthogonality.

Sec. 4.8 Orthogonal Eigenfunction Expansions

Example 2. Answers to the questions near the end. $a_{13}P_{13}$ is the next term. $a_2 = a_4 = \dots = 0$ because $\sin \pi x$ is odd. $P_3(x)$ resembles $-\sin \pi x$ more closely than any other term does; see Fig.101 in Sec. 4.3.

Problem Set 4.8. Page 246

1. **Fourier-Legendre series.** In Example 2 of the text we had to determine the coefficients by integration. In the present case this would be possible, but the method of undetermined coefficients is much simpler. The given function

$$f(x) = 70x^4 - 84x^2 + 30$$

is of degree 4, hence you need only $P_0, P_1, \dots, P_4$. Since f is an even function, you actually need only P_0, P_2, P_4. Write

$$f(x) = a_4P_4(x) + a_2P_2(x) + a_0P_0(x) = 70x^4 - 84x^2 + 30.$$

Begin by determining a_4 so that the x^4-terms on both sides agree. Since $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$ (see Sec. 4.3), you have the condition

$$a_4(35/8) = 70, \quad \text{hence} \quad a_4 = 70 \cdot 8/35 = 16.$$

Calculate the remaining function

$$f_1(x) = f(x) - 16P_4(x) = 70x^4 - 84x^2 + 30 - (16/8)(35x^4 - 30x^2 + 3)$$
$$= -84x^2 + 30 + 60x^2 - 6 = -24x^2 + 24.$$

Now determine a_2 by the same process so that the x^2-terms on both sides of the last equation agree. Using $P_2(x) = (1/2)(3x^2 - 1)$, you obtain

$$a_2(3/2) = -24, \quad \text{hence} \quad a_2 = -24 \cdot 2/3 = -16.$$

Calculate the remaining function (a constant!)

$$f_2(x) = f_1(x) - (-16)P_2(x) = -24x^2 + 24 - (-16/2)(3x^2 - 1) = 16.$$

Hence $a_0 = 16$ because $P_0(x) = 1$. The answer is

$$f = 16(P_0 - P_2 + P_4).$$

5. **Fourier-Legendre series.** The coefficients are given by (7) in the form

$$a_m = (m + 1/2)\int_{-1}^{1} \cos\left(\frac{\pi x}{2}\right) P_m(x)\,dx.$$

For $m = 0$ this gives $a_0 = 2/\pi = 0.6366$ by calculus. For even m you obtain by two successive integrations by parts

$$\int_{-1}^{1} x^m \cos\left(\frac{\pi x}{2}\right) dx = \frac{4}{\pi} - \frac{4m(m-1)}{\pi^2}\int_{-1}^{1} x^{m-2} \cos\left(\frac{\pi x}{2}\right) dx.$$

Thus for $m = 2$ this gives $4/\pi - (8/\pi^2)(4/\pi)$, where $4/\pi$ $(= 2a_0)$ is the value of the integral of $\cos(\pi x/2)$ just calculated. From this you further obtain

$$a_2 = \frac{5}{2}\int_{-1}^{1} \cos\left(\frac{\pi x}{2}\right) \frac{1}{2}(3x^2 - 1)\,dx$$

$$= \frac{5}{4}\left(3\left(\frac{4}{\pi} - \frac{32}{\pi^3}\right) - \frac{4}{\pi}\right) = -0.687085,$$

and similarly for the further coefficients as given in the answer in Appendix 2 (which were calculated by a CAS).