CHAPTER 16. Complex Analysis Applied to Potential Theory

Sec. 16.1 Electrostatic Fields

Problem Set 16.1. Page 802

3. **Potential between sloping plates.** $y = x$ and $y = x + 1$ are conducting plates, hence they must be equipotential lines (equipotential planes in xyz-space, with the potential not depending on z; this is commonly called a *two-dimensional problem*). Setting $\Phi_0 = y - x$, the first condition is satisfied. You see that then on the second plate $y = x + 1$ the potential is $\Phi_0 = y - x = 1$. Hence you can satisfy the second condition by multiplying Φ_0 by 110; thus,

$$\Phi = 110(y - x). \qquad \text{(A)}$$

The first condition is still satisfied; hence this gives the answer to the problem as far as the real potential is concerned. The corresponding complex potential $F = \Phi + i\Psi$ can be obtained by the Cauchy-Riemann equations

$$\Phi_x = -110 = \Psi_y, \qquad \text{hence} \qquad \Psi = -110y + h(x),$$

$$\Phi_y = 110 = -\Psi_x = -h'(x), \qquad \text{hence} \qquad h(x) = -110x.$$

Together,

$$F = \Phi + i\Psi = 110(y - x) + i(-110y - 110x) = -110(-y + x + iy + ix).$$

In the parentheses, $x + iy = z$ and $ix - y = iz$. Hence

$$F = -110(z + iz) = -110(1 + i)z.$$

More simply, $-x$ in (A) is the real part of $-z$, and y is the real part of $-iz = -i(x + iy) = -ix + y$. Adding this and multiplying by 110 gives

$$\Phi = 110(\text{Re}\,(-iz) + \text{Re}\,(-z)), \qquad \text{hence} \qquad F = -110(iz + z) = F = -110(1 + i)\,z.$$

11. **Two source lines.** The equipotential lines in Example 7 are

$$|(z - c)/(z + c)| = k = const \qquad (k \text{ and } c \text{ real}).$$

Hence $|z - c| = k|z + c|$. Squaring gives

$$|z - c|^2 = K|z + c|^2 \qquad (K = k^2).$$

Writing this in terms of the real and imaginary parts and taking all the terms to the left, you obtain

$$(x - c)^2 + y^2 - K((x + c)^2 + y^2) = 0.$$

Writing out the squares gives

$$x^2 - 2cx + c^2 + y^2 - K(x^2 + 2cx + c^2 + y^2) = 0. \qquad \text{(B)}$$

For $k = 1$, hence $K = 1$, most terms cancel, and you are left with $-4cx = 0$, hence $x = 0$ (because $c \neq 0$). This is the y-axis. Then

$$|z - c|^2 = |z + c|^2 = y^2 + c^2, \qquad |z - c|/|z + c| = 1, \qquad \text{Ln } 1 = 0.$$

Hence this shows that the y-axis has potential 0. You can now continue with (B), assuming that $K \neq 1$. Collecting terms in (B), you have

$$(1 - K)(x^2 + y^2 + c^2) - 2cx(1 + K) = 0.$$

Division by $1 - K$ ($\neq 0$ because $K \neq 1$) gives

$$x^2 + y^2 + c^2 - 2Lx = 0 \qquad (L = c(1 + K)/(1 - K)).$$

Completing the square in x, you finally obtain

$$(x - L)^2 + y^2 = L^2 - c^2.$$

This is a circle with center at L on the real axis and radius $\sqrt{L^2 - c^2}$. If you insert L and simplify, you see that the radius equals $2kc/(1-k^2)$.

15. **Potential in a sector.** $z^2 = x^2 - y^2 + 2ixy$ gives the potential in sectors of opening $\pi/2$ bounded by the bisecting straight lines of the quadrants because $x^2 - y^2 = 0$ when $y = \pm x$. Similarly, higher powers of z give potentials in sectors of smaller openings on whose boundaries the potential is zero. For $z^3 = x^3 + 3ix^2y - 3xy^2 - iy^3$ the real potential is

$$\Phi_0 = \operatorname{Re} z^3 = x^3 - 3xy^2 = x(x^2 - 3y^2)$$

and $\Phi = 0$ when $y = \pm x/\sqrt{3}$; these are the boundaries given in the problem, the opening of the sector being $\pi/3$, that is, 60 degrees. To satisfy the other boundary condition, multiply Φ_0 by 220.

Sec. 16.2 Use of Conformal Mapping

Problem Set 16.2. Page 807

3. **Quarter of an elliptical disk.** The given potential in the image domain D^* is $\Phi^*(u,v) = u^2 - v^2$. This is the real part of w^2, where $w = u + iv$. The domain D^* is obtained as the image of the rectangle $D: 0 \leqq x \leqq \pi/2, 0 \leqq y \leqq 1$ under the mapping

$$w = u + iv = \sin z = \sin x \cosh y + i \cos x \sinh y. \qquad \text{(A)}$$

Figure 316 in Sec. 12.7 shows that D^* lies in the first quadrant. Its boundary consists of the segment of the u-axis from 0 to 1 (this is the image of the segment of the x-axis from 0 to $\pi/2$) and from 1 on to $\cosh 1 = 1.5431$ (this is the image of the right vertical edge of D), then along the second largest ellipse in the figure up to the point $\sinh 1 = 1.1752$ on the v-axis (the image of the upper edge of D) and along the v-axis down to the origin from which we started (this is the image of the segment from 1 to 0 on the y-axis) The potential Φ in D is obtained by substituting u and v from (A) into the potential Φ^* in D^*, that is,

$$\Phi(x,y) = \Phi^*(u(x,y), v(x,y)) = u^2(x,y) - v^2(x,y) = \sin^2 x \cosh^2 y - \cos^2 x \sinh^2 y. \qquad \text{(B)}$$

The boundary values are obtained from (B) as follows. On the lower edge ($y = 0$) you have $\Phi = \sin^2 x$, which increases from 0 to 1. On the right edge ($x = \pi/2$) you have $\Phi = \cosh^2 y$, which increases from 1 to $\cosh^2 1 = 2.3811$. On the upper edge ($y = 1$) you obtain

$$\Phi = \sin^2 x \cosh^2 1 - \cos^2 x \sinh^2 1 = 2.3811 \sin^2 x - 1.3811 \cos^2 x,$$

which decreases from 2.3811 to -1.3811. Finally, on the left vertical edge ($x = 0$), you have from (B) the potential $\Phi = -\sinh^2 y$, which increases from -1.3811 to 0.

9. **Angular region.** The potential in the w-plane is $\Phi^*(w) = (6/\pi) \operatorname{Arg} w$. The suggested mapping function is $w = z^2$. Hence $\operatorname{Arg} w = 2 \operatorname{Arg} z$. Substitution gives the potential in the angular region in the form

$$\Phi(z) = \Phi^*(w(z)) = (12/\pi) \operatorname{Arg} z.$$

Indeed, for $\operatorname{Arg} z = \pm\pi/4$ (the boundary of the angular region) you obtain ± 3, as required.

13. **Linear fractional transformation.** Z here plays the role of z in (7), Sec. 12.9, and z plays the role of w. Thus, $z = (Z - i/2)/(-iZ/2 - 1)$. Now multiply both the numerator and the denominator by 2. This gives the answer on p. A36 in Appendix 2.

Sec. 16.3 Heat Problems

Problem Set 16.3. Page 811

3. **Mixed problem.** A potential in a sector (angular region) whose sides are at constant temperatures is, once and for all, of the form

$$T = a + b\,\mathrm{Arg}\,z. \qquad \text{(A)}$$

Here you use the fact that Arg $z = \theta$ = Im (Ln z) is a harmonic function. The two constants a and b can be determined from the given values on the two sides Arg $z = 0$ and Arg $z = \pi/2$. Namely, for Arg $z = 0$ (the x-axis) we have $T = a = 100$. Then for Arg $z = \pi/2$ you have

$$T = 100 + b \cdot \pi/2 = -40.$$

Solving for b gives $b = -280/\pi$. Hence a potential giving the required values on the two sides is

$$T = \frac{280}{\pi}\,\mathrm{Arg}\,z. \qquad \text{(B)}$$

Now comes an important observation. The curved portion of the boundary (a circular arc) is insulated. Hence on this arc the normal derivative of the temperature T must be zero. But the normal direction is the radial direction; so the partial derivative with respect to r must vanish. Now formula (B) shows that T is independent of r, that is, the condition under discussion is automatically satisfied. (If this were not the case, the whole solution would not be valid.) Finally derive the complex potential F. From Sec. 12.8 recall that

$$\mathrm{Ln}\,z = \ln|z| + i\,\mathrm{Arg}\,z. \qquad \text{(C)}$$

Hence for Arg z to become the real part (as it must be the case because $F = T + i\Psi$), you must multiply (C) by $-i$. Indeed, then

$$-i\,\mathrm{Ln}\,z = -i\ln|z| + \mathrm{Arg}\,z.$$

Hence from this and (B) you see that the complex potential is

$$F = 100 - \frac{280}{\pi}(-i\,\mathrm{Ln}\,z) = 100 + \frac{280}{\pi}i\,\mathrm{Ln}\,z.$$

13. **Another use of Arg** z. You can proceed similarly as in Prob. 3, starting from

$$T = a + b\,\mathrm{Arg}\,z.$$

For Arg $z = 0$ (the positive ray of the x-axis) you must have

$$T = a = T_0.$$

For Arg $z = \pi/2$ (the positive ray of the y-axis) you must have

$$T = T_0 + b(\pi/2) = T_1.$$

Solving for b gives $b = (2/\pi)(T_1 - T_0)$. Hence the answer is

$$T = T_0 + \frac{2}{\pi}(T_1 - T_0)\,\mathrm{Arg}\,z.$$

Sec. 16.4 Fluid Flow

Example 2. For another interesting application of $w = z + 1/z$ see Sec. 12.5.

Problem Set 16.4. Page 817

1. **Parallel flow.** A flow is completely determined by its complex potential

$$F(z) = \Phi(x, y) + i\Psi(x, y).$$

The stream function Ψ gives the streamlines $\Psi = const$ and is generally more important than the velocity potential Φ, which gives the equipotential lines $\Phi = const$. The flow can best be visualized in terms of the velocity vector V, which is obtained from the complex potential in the form

$$V = V_1 + iV_2 = \overline{F'}(z).$$

(Here we need no special vector notation because a complex function V can always be regarded as a vector function with components V_1 and V_2. Hence for the given

$$F(z) = Kz = K(x + iy) \tag{A}$$

with positive real K you have $\bar{F} = F' = K$; thus,

$$V = V_1 = k.$$

It is essential that K is real (this makes $V_2 = 0$, the velocity vector is parallel to the x-axis) and is positive, so that $V = V_1$ points to the right (in the positive x-direction). Hence you are dealing with a uniform flow (a flow of constant velocity) that is parallel (the streamlines are straight lines parallel to the x-axis) and is flowing to the right (because K is positive). From (A) you see that the equipotential lines are vertical parallel straight lines; indeed,

$$\Phi(x,y) = \text{Re } F(z) = kx = const, \qquad \text{hence} \qquad x = const.$$

9. **Flow around a cylinder.** Since a cylinder of radius r_0 is obtained from a cylinder of radius 1 by a dilatation (a uniform stretch in all directions in the complex plane), it is natural to replace z by az with a real constant a because this corresponds to such a stretch. That is, replace the complex potential

$$z + 1/z \tag{A}$$

in Example 2 by

$$F(z) = \Phi(r,\theta) + i\Psi(r,\theta) = az + \frac{1}{az} = are^{i\theta} + \frac{1}{ar}e^{-i\theta}. \tag{B}$$

The stream function Ψ is the imaginary part of F. Since by Euler's formula, $e^{\pm i\theta} = \cos\theta \pm i \sin\theta$, you obtain

$$\Psi(r,\theta) = \left(ar - \frac{1}{ar}\right)\sin\theta.$$

The streamlines are the curves $\theta = const.$ As in Example 2 of the text, the streamline $\Psi = 0$ consists of the x-axis ($\theta = 0$ and π), where $\sin\theta = 0$, and of the locus where the other factor of Ψ is zero, that is,

$$ar - \frac{1}{ar} = 0, \qquad \text{thus} \quad (ar)^2 = 1 \qquad \text{or} \qquad a = 1/r.$$

Since the cylinder has radius $r = r_0$, you must have $a = 1/r_0$. With this, the answer is

$$F(z) = az + \frac{1}{az} = \frac{z}{r_0} + \frac{r_0}{z}.$$

Sec.16.5 Poisson's Integral Formula

Problem Set 16.5. Page 822

7. **Sinusoidal boundary values** lead to a series (7) that reduces to finitely many terms (a "trigonometric polynomial"). In Prob. 7 the given boundary function $\Phi(1,\theta) = 4\sin^3\theta$ is not immediately one of the terms in (7), but can be expressed in terms of sine functions of multiple angles. The corresponding formula is

$$\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta.$$

Hence the boundary value can be written

$$\Phi(1,\theta) = 3\sin\theta - \sin 3\theta.$$

From (7) you now see immedately that the potential in the unit disk satisfying the given boundary conditionis

$$\Phi(r,\theta) = 3r\sin\theta - r^3\sin 3\theta.$$

11. **Piecewise linear boundary values** lead to a series (7) whose coefficients can be found from Chap. 11. For instance, the Fourier series of the present boundary values is a special case of the odd periodic extension in

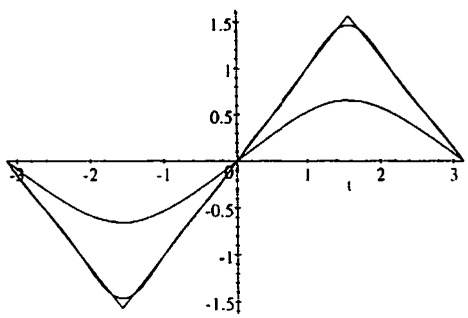
Example 3 of Sec. 10.4 with $k = \pi/2$ and $L = \pi$, that is,

$$f(x) = \frac{4}{\pi}(\sin x - \frac{1}{9}\sin 3x + \frac{1}{25}\sin 5x - + \dots).$$

From this with $x = \theta$ you obtain the potential (7) in the disk in the form

$$\Phi(r, \theta) = \frac{4}{\pi}(r\sin\theta - \frac{1}{9}r^3\sin 3\theta + \frac{1}{25}r^5\sin 5\theta - + \dots). \quad \text{(A)}$$

The figure shows the given boundary potential, an approximation of it (the sum of the first three terms of the series (A)), which is rather good, and an approximation of the potential on the circle of radius $r = 1/2$ (the sum of those three terms for $r = 1/2$)). The latter is practically a sine curve because the terms in (A) with $r = 1/2$ have coefficients $4/\pi$ times 1/2, 1/72, 1/800, etc, which decrease very fast. This illustrates that the partial sums of the series (A) give good approximations of the potential in the disk.



Section 16.5. Problem 11. Boundary potential and approximations for $r = 1$ and $r = 1/2$

13. Terms of (7). Use the Laplacian in polar coordinates, given by (5) in Sec. 11.11 (without the z-term) .

Sec. 16.6 General Properties of Harmonic Functions

Problem Set 16.6. Page 825

1. Mean value of an analytic function. Verify Theorem 1 for

$$F(z) = (z-1)^2, \qquad z_0 = 1/2, \qquad |z - 1/2| = 1.$$

The last of these formulas arises from the requirement that you integrate around a circle of radius 1 and that the center of the circle be z_0. Since $F(z_0) = F(1/2) = (-1/2)^2 = 1/4$, your task is to verify that the integral in (2) in the proof of Theorem 1 has the value 1/4. Use (2) with $z_0 = 1/2$ and $r = 1$ (since the circle of integration has radius 1). The path of integration is

$$z = z_0 + e^{i\alpha} = \frac{1}{2} + e^{i\alpha}.$$

Hence on this path the integrand is

$$F(z_0 + e^{i\alpha}) = \left(\frac{1}{2} + e^{i\alpha} - 1\right)^2 = \left(-\frac{1}{2} + e^{i\alpha}\right)^2.$$

Squaring as indicated, you obtain

$$F(z_0 + e^{i\alpha}) = \frac{1}{4} + e^{i\alpha} + e^{2i\alpha}.$$

Indefinite integration over α gives

$$\frac{\alpha}{4} - \frac{1}{i}e^{i\alpha} + \frac{1}{2i}e^{2i\alpha}.$$

(the factor $1/2\pi$ in front of the integral in (2) will not be carried along, but introduced at the end of the calculation). Now insert the limits of integration 0 and 2π. Then the first term gives

$$\frac{1}{4}2\pi = \frac{1}{2}\pi. \tag{A}$$

Since $e^{2\pi i} = 1$, the next term gives $1 - 1 = 0$. Similarly for the last term. Multiplying (A) by the factor $1/2\pi$ (which you did not carry along in (2), in order to have simpler formulas), you obtain $(\pi/2)/(2\pi) = 1/4$. This completes your verification.

7. **Mean values of harmonic functions (Theorem 2).** The two formulas in the proof of Theorem 2 give the mean values to be calculated in order to verify Theorem 2. The given function, point (x_0, y_0), and circle are

$$\Phi(x, y) = (x-1)(y-1), \qquad (x_0, y_0) = (3, -3), \qquad z = 3 - 3i + e^{i\alpha}. \tag{B}$$

Φ is harmonic; verify this by differentiation. Note that $z_0 = x_0 + iy_0 = 3 - 3i$ is the center of the circle in (B). In terms of the real and imaginary parts it is (by Euler's formula (5) in Sec. 12.7)

$$x = 3 + \cos\alpha, \qquad y = -3 + \sin\alpha. \tag{C}$$

This is the representation you need, since Φ is a real function of the two real variables x and y. You see that

$$\Phi(z_0, y_0) = \Phi(3, -3) = (x_0 - 1)(y_0 - 1) = (3-1)(-3-1) = -8.$$

Hence you have to show that each of the two mean values equals -8. Substituting (C) into (B) (which is a completely schematical process) gives

$$\begin{aligned} \Phi(3+\cos\alpha, -3+\sin\alpha) &= (3+\cos\alpha - 1)(-3+\sin\alpha - 1) \\ &= (2+\cos\alpha)(-4+\sin\alpha) \\ &= -8 + 2\sin\alpha - 4\cos\alpha + \cos\alpha\sin\alpha. \end{aligned} \tag{D}$$

Consider the mean value over the circle. Integrate each of the four terms in the last line of (D) over α from 0 to 2π. The first of them gives -16π. The second term gives 0, and so does the third. The fourth term equals $(1/2)\sin 2\alpha$, and its integral is 0, too. Multiplication by $1/(2\pi)$ (the factor in front of the first integral in the proof of Theorem 2) gives $-16\pi/(2\pi) = -8$. This is the mean value of the given harmonic function over the circle considered and completes the verification of the first part of the theorem for our given data. Now calculate the mean value over the disk of radius 1 and center (3, − 3). The integrand of the double integral (second formula in the proof of Theorem 2) is similar to that in (D), but in (D) you had $r = 1$ (the circle over which you integrated), whereas you now have r variable and integrate over it from 0 to 1, and you also have a factor r resulting from the element of area in polar coordinates, which is $r\,dr\,d\theta$. Hence instead of $(2+\cos\alpha)(-4+\sin\alpha)$ in (D) you now have

$$(2 + r\cos\alpha)(-4 + r\sin\alpha)r = -8r + 2r^2\sin\alpha - 4r^2\cos\alpha + r^3\cos\alpha\sin\alpha.$$

The factors of r have no influence on the integration over α from 0 to 2π. Accordingly, the present four terms on the right give upon integration over α the values $-8r\cdot 2\pi = -16r\pi$, 0, 0, and 0. Integration of r from 0 to 1 gives 1/2, so that the double integral equals -8π. In front of the double integral you have the factor $1/(\pi r_0^2) = 1/\pi$, because the circle of integration has radius 1. Hence your second result is $-8\pi/\pi = -8$. This completes the verification.

15. **Location of maxima.** Look for a counterexample, as simple as possible. $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$ are harmonic in any region, say, to have a simple situation, in the square $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$. Then you have $\max x = 1$ at all points on the right boundary and $\max y = 1$ at all points of the upper boundary. Hence there is a point, (1, 1), that is, $z = 1 + i$, where both functions have a maximum. But this should give you the idea: omit the point (1, 1) from the region. Or take a rectangle, a triangle, a square with vertices ± 1, $\pm i$, and so on.