CHAPTER 15. Laurent Series. Residue Integration

Sec. 15.1 Laurent Series

Problem Set 15.1. Page 775

1. Laurent series near a singularity. Examples 4 and 5 in the text illustrate that a function may have different Laurent series in different annuli with the same center. However, practically most important of these is the Laurent series that converges directly near the center at which the given function has a singularity. (In Example 4 this is $z = 0$.) In each of Probs. 1-8 that Laurent series is obtained by using a familiar Maclaurin series or (in Probs. 5 and 7) a series in powers of $1/z$. Thus, in Prob. 1 you consider the Maclaurin series

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + - \dots .$$

Division by z^4 gives the Laurent series

$$z^{-4} \cos z = \frac{1}{z^4} - \frac{1}{2z^2} + \frac{1}{24} - \frac{z^2}{720} + - \dots .$$

The principal part consists of the first two terms on the right. The series converges for all $z \neq 0$.

7. Infinite principal part. Use the familiar Maclaurin series of the exponential funcion,

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + \dots .$$

Substituting $t = -1/z^2$, you obtain

$$\exp(-1/z^2) = \sum \frac{(-1)^n}{z^{2n} n!} = 1 - \frac{1}{z^2} + \frac{1}{2z^4} - \frac{1}{6z^6} + - \dots .$$

Now divide by z^2. You see that the series consists of an infinite principal part and there are no nonnegative powers. The series converges for all $z \neq 0$.

13. Use of the binomial theorem. Develop the numerator z^4 of the given function

$$f(z) = z^4/(z+2i)^4$$

in terms of powers of $z + 2i$ by means of the binomial theorem and then divide by $(z+2i)^4$. (If you have forgotten that theorem, you will find it on p. 1069 of the book.) Since for $(a + b)^4$ the binomial coefficients needed are 1, 4, 6, 4, 1, you obtain

$$\begin{aligned} z^4 &= ([z+2i] - 2i)^4 \\ &= (z+2i)^4 - 4(2i)(z+2i)^3 + 6(2i)^2(z+2i)^2 - 4(2i)^3(z+2i) + (2i)^4 \\ &= (z+2i)^4 - 8i(z+2i)^3 - 24(z+2i)^2 + 32i(z+2i) + 16. \end{aligned}$$

Division by $(z+2i)^4$ gives the Laurent series

$$\frac{z^4}{(z+2i)^4} = 1 - \frac{8i}{z+2i} - \frac{24}{(z+2i)^2} + \frac{32i}{(z+2i)^3} + \frac{16}{(z+2i)^4}.$$

Instead of the binomial theorem you may use the Taylor series (1) in Sec. 14.4, which in the present case reduces to a polynomial because z^4 and its derivatives are z^4, $4z^3$, $12z^2$, $24z$, 24, 0, 0, ... At $z = 2i$ the values of these expressions are

$$(-2i)^4 = 16, \quad 4(-2i)^3 = 32i, \quad 12(-2i)^2 = -48, \quad 24(-2i) = -48i, \quad 24.$$

Division by 0!, 1!, 2!, ... gives the Taylor coefficients

$$16, \quad 32i, \quad -24, \quad -8i, \quad 1, \quad 0, \quad \dots .$$

Hence the development is the same as before, with the terms being in reverse order,

$$z^4 = 16 + 32i(z+2i) - 24(z+2i)^2 - 8i(z+2i)^3 + (z+2i)^4$$

The amount of work was not much more than before, because it would not have been necessary to write down all the intermediate expressions.

Sec. 15.2 Singularities and Zeros. Infinity

Problem Set 15.2. Page 780

1. **Zeros.** Since $\tan z$ is periodic with period π, it follows that $\tan \pi z$ is periodic with period 1. Since $\tan 0 = 0$, you see that $\tan \pi z$ has zeros at $0, \pm 1, \pm 2, \dots$. Determine the order. The derivative is (by the chain rule!)

$$(\tan \pi z)' = \pi / \cos^2 \pi z.$$

Since the cosine is not zero at $z = 0$, the zero of $\tan \pi z$ at 0 is simple. Because of periodicity all those other zeros are simple, too.

Show that $\tan \pi z$ has no further zeros. To have simpler formulas, write $\pi z = s + it$. Then (6b) in Sec. 12.7 becomes

$$\sin \pi z = \sin s \cosh t + i \cos s \sinh t.$$

This is zero if and only if the real part is zero,

$$\sin s \cosh t = 0, \quad \text{hence} \quad \sin s = 0$$

(since $\cosh t \neq 0$, note that s and t are *real*) and the imaginary part is zero,

$$\cos s \sinh t = 0, \quad \text{hence} \quad \sinh t = 0$$

because $\cos s \neq 0$ where $\sin s = 0$ (sin and cos have no zeros in common). Now $\sin s = 0$ gives exactly the zeros at $s = \pi x = 0, \pm\pi, \pm 2\pi, \dots$, that is, at $z = 0, \pm 1, \pm 2, \dots$; these are the zeros discovered before. Furthermore, $\sinh t = 0$ only at $t = 0$ (note again that t is real!); this gives no additional zeros.

19. **Pole, essential singularity.** Since $\sinh z$ is an entire function, the only singularity the given function

$$f(z) = (z - \pi i)^{-2} \sinh z$$

can have in the finite complex plane (see p. 693) is at $z = \pi i$. It seems to be a pole of second order, but you must be cautious because $\sinh z$ may perhaps be zero at that point. Now, indeed, by the definition of sin and sinh in Sec. 12.7 you obtain

$$\sinh \pi i = (e^{\pi i} - e^{-\pi i})/2 = i(e^{\pi i} - e^{-\pi i})/(2i) = i \sin \pi = 0.$$

Fortunately, this zero is simple because the derivative is $\cosh z$, and at πi,

$$\cosh \pi i = (e^{\pi i} + e^{-\pi i})/2 = \cos \pi \neq 0.$$

Hence the given function still has a pole, albeit a simple one, due to the occurrence of that zero. Indeed, for $\sinh z$ to have a simple zero at $z = \pi i$, the Taylor series of $\sinh z$ with center πi must be of the form

$$a_1(z - \pi i) + a_2(z - \pi i)^2 + \dots,$$

so that

$$\frac{\sinh z}{(z - \pi i)^2} = \frac{a_1}{z - \pi i} + a_2 + a_3(z - \pi i) + \dots .$$

By definition this is the Laurent series near a simple pole. (If $\sinh z$ had a double zero or a zero of still higher order at πi, the given function would be analytic at πi.)

Furthermore, $f(z)$ has an essential singularity at infinity, for reasons given in Example 5 in the text. To see this directly, consider

$$g(w) = f(1/w) = \frac{\sinh(1/w)}{(1/w - \pi i)^2}$$

at $w = 0$. The function

$$\frac{1}{(1/w - \pi i)^2} = \frac{w^2}{(1 - \pi i w)^2}$$

is analytic at $w = 0$. The other factor of $g(w)$, the function sinh $(1/w)$ has near $w = 0$ the Laurent series

$$\sinh\left(\frac{1}{w}\right) = \frac{1}{w} + \frac{1}{3!\,w^3} + \frac{1}{5!\,w^5} + \ldots,$$

as obtained from the familiar Maclaurin series of sinh s by setting $s = 1/w$. Since this series has an infinite series as its principal part, $w = 0$ is an essential singularity of $g(w)$, by definition. Again by definition, this means that $f(z)$ has an essential singularity at infinity.

Sec. 15.3 Residue Integration Method

Problem Set 15.3. Page 786

1. **Simple poles.** $1 + z^2 = 0$ at $z^2 = -1$, $z = i$ and $-i$. Hence the given function $f(z) = 4/(1 + z^2)$ has simple poles at $z = i$ and $-i$. For $z = z_0 = i$, using $(1 + z^2)' = 2z$ and $1/i = -i$, you obtain from (4)

$$\operatorname*{Res}_{z=i} f(z) = 4/2i = -2i. \qquad \text{(A)}$$

Similarily, for $z = -i$ you obtain

$$\operatorname*{Res}_{z=-i} f(z) = 4/(-2i) = 2i.$$

Formula (3) gives the same answers. In (3) you need for $z = z_0 = i$

$$(z - i) \cdot 4/(z^2 + 1) = 4/(z + i).$$

At $z = i$ this has the value $4/(2i) = -2i$, in agreement with (A). Similarly for the pole at $z = -i$.

3. **Use of the Laurent series.** In Prob. 1 you have used formulas that gave the residue directly, without reference to the whole Laurent series. For the function $f(z) = (\sin 2z)/z^6$ you may use the familiar Maclaurin series of the sine function and find the coefficient a_5 of the power z^5 because $a_5 z^5/z^6 = a_5/z$, which shows that a_5 is the residue of $f(z)$ at $z = 0$, where $f(z)$ has a pole of fifth order (not sixth because sin $2z$ has a simple zero at $z = 0$). You obtain

$$\frac{\sin 2z}{z^6} = \frac{1}{z^6}\left(2z - \frac{(2z)^3}{3!} + \frac{(2z)^5}{5!} - + \ldots\right).$$

Hence $a_5 = 2^5/5! = 32/120 = 4/15$.

13. **Residue theorem.** $\tan \pi z = (\sin \pi z)/(\cos \pi z)$ is singular where $\cos \pi z = 0$, that is, at $\pi z = \pm\pi/2, \pm 3\pi/2, \ldots$, hence at $z = \pm 1/2, \pm 3/2, \ldots$. These are simple zeros of $\cos \pi z$, hence simple poles of $\tan \pi z$, as follows from Theorem 4 in Sec. 15.2. Here you have used that $\sin \pi z \neq 0$ at points where $\cos \pi z = 0$. Hence $\tan \pi z$ has infinitely many simple poles. But only those at $z = 1/2$ and $z = -1/2$ lie inside the contour of integration, which is the unit circle $|z| = 1$.

You can apply (4) to

$$\tan \pi z = p(z)/q(z) = (\sin \pi z)/(\cos \pi z).$$

In (4) you need

$$p(z)/q'(z) = (\sin \pi z)/(\cos \pi z)' = (\sin \pi z)/(-\pi \sin \pi z) = -1/\pi$$

where the factor π results from the chain rule. From this and the residue theorem (Theorem 1) you obtain the answer $2\pi i(-1/\pi - 1/\pi) = -4i$.