CHAPTER 13. Complex Integration

Sec. 13.1 Line Integral in the Complex Plane

Problem Set 13.1. Page 711

1. **Straight-line segment.** The endpoints of the segment are $z_0 = 0$ and $z_1 = 4 - 7i$. Sketch it. If you set

$$z(t) = x(t) + iy(t) = (4 - 7i)t \qquad \text{(A)}$$

you see that for variable real t this represents a straight line because

$$\frac{y(t)}{x(t)} = -\frac{7t}{4t} = -\frac{7}{4} = const;$$

this is the slope of the straight line. Furthermore, from (A) with $t = 0$ and $t = 1$ you obtain

$$z(0) = 0 = z_0, \quad z(1) = 4 - 7i = z_1$$

respectively. This shows that this line passes through the given points. For reasons of continuity, the points on the segment between z_0 and z_1 must correspond to values of t between 0 and 1. Hence the answer is

$$z(t) = (4 - 7i)t, \quad \text{where} \quad 0 \leqq t \leqq 1.$$

7. **Circle.** The given equation

$$|z - (-3 + i)| = 5$$

represents the circle of radius 5 with center at $-3 + i$. Since $z = x + iy$, you can write this as

$$|x + iy + 3 - i| = |x + 3 + i(y - 1)| = \sqrt{(x + 3)^2 + (y - 1)^2} = 5$$

or, squaring this equation, as

$$(x + 3)^2 + (y - 1)^2 = 25.$$

This is the usual nonparametric equation for a circle known from calculus.

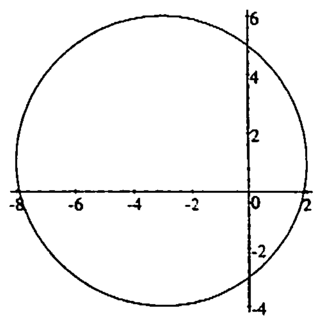
Now for a circle of radius 5 with center at 0 you can write

$$z(t) = x(t) + iy(t) = 5e^{it} = 5(\cos t + i \sin t). \qquad \text{(B)}$$

Note that as t increases the circle is traced in the counterclockwise sense. All you now have to do is moving the center from 0 to $-3 + i$. This can be done simply by adding $-3 + i$ in (B). Using again the notations $z(t)$, $x(t)$, and $y(t)$, for simplicity, you have

$$z(t) = x(t) + iy(t) = -3 + i + 5e^{it}$$
$$= -3 + 5\cos t + i(1 + 5 \sin t),$$

where $0 \leqq t \leqq 2\pi$. This is the desired representation of the given circle.



Section 13.1. Problem 7. Given circle to be represented parametrically

15. **Integration by the use of the path.** The integrand

$$w = u + iv = f(z) = \text{Re}\, z = x$$

is not analytic. Indeed, the first Cauchy-Riemann equation $u_x = v_y$ (Sec 12.4) is not satisfied, $u_x = 1$ but $v = 0$, hence $v_y = 0$. (The second Cauchy-Riemann equation is satisfied, but, of course, this is not enough for analyticity.) Hence you cannot apply the first method (which would be more convenient), but must use the second.

The shortest path from $z_0 = 1 + i$ to $z_1 = 3 + 2i$ is the straight-line segment with these points as endpoints. Sketch the path. The difference of these points is

$$z_1 - z_0 = 3 + 2i - (1 + i) = 2 + i. \tag{I}$$

Set

$$z(t) = z_0 + (z_1 - z_0)t. \tag{II}$$

Then by taking the values $t = 0$ and $t = 1$ you have

$$z(0) = z_0 \quad \text{and} \quad z(1) = z_1$$

(because z_0 cancels when $t = 1$). Hence (II) is a general representation of a segment with given endpoints z_0 and z_1, and t ranges from 0 to 1.

In Prob. 15, Eq. (II) is

$$\begin{aligned} z(t) = x(t) + iy(t) &= 1 + i + (2 + i)t \\ &= 1 + 2t + i(1 + t). \end{aligned} \tag{III}$$

Integrate by using (10) on p. 708. In (10) you need

$$f(z(t)) = x(t) = 1 + 2t,$$

as well as the derivative of $z(t)$ with respect to t, that is,

$$\dot{z}(t) = 2 + i.$$

Both of these expressions are obtained from (III).

You are now ready to integrate. From (10) you obtain

$$\begin{aligned} \int_C f(z)\,dz &= \int_0^1 (1 + 2t)(2 + i)\,dt \\ &= (2 + i)\int_0^1 (1 + 2t)\,dt \\ &= (2 + i)(t + t^2)\Big|_{t=0}^{1} \\ &= 4 + 2i. \end{aligned}$$

17. **Indefinite integration and substitution of limits.** The integrand $f(z) = \sin^2 z$ is analytic. Hence you can apply the first method in the text. This integration is as in calculus. If you have difficulties with this method, review calculus, in particular, integration. The standard trick in this integral is to use the formula

$$\sin^2 z = \frac{1}{2}(1 - \cos 2z).$$

Incidentally, this is easy to remember if you recall what the $\sin^2$-curve looks like. The indefinite integral of the function on the right is

$$\frac{1}{2}\left(z - \frac{1}{2}\sin 2z\right) = \frac{1}{2}z - \frac{1}{4}\sin 2z \tag{IV}$$

because $(\sin 2z)' = 2\cos 2z$, where we used the chain rule.

Now comes the second step, the evaluation at the given limits of integration. The given path is of no interest to you; just its endpoints are essential. You have to take the value of (IV) at the terminal point $z = \pi i$ minus the value of (IV) at the initial point $z = -\pi i$. This gives

$$\frac{1}{2}\pi i - \frac{1}{4}\sin(2\pi i) - \left(\frac{1}{2}(-\pi i) - \frac{1}{4}\sin(-2\pi i)\right) \qquad \text{(V)}$$
$$= \frac{1}{2}\pi i + \frac{1}{2}\pi i - \frac{1}{4}\sin(2\pi i) - \frac{1}{4}\sin(2\pi i)$$
$$= \pi i - \frac{1}{2}\sin(2\pi i).$$

You can evaluate the sine either by expressing it in terms of exponential functions (see (1) in Sec. 12.7) or by using (6b) in Sec. 12,7, that is,

$$\sin z = \sin x \cosh y + i \cos x \sinh y,$$

which for $x = 0$ and $y = 2\pi$ gives simply $i \sinh 2\pi$. Your answer thus obtained is $\pi i - \frac{1}{2} i \sinh 2\pi$ and agrees with that on p. A33 in Appendix 2 of the book.

Sec. 13.2 Cauchy's Integral Theorem

Problem Set 13.2. Page 720

7. **Cauchy's integral theorem** applies to the given function, which is an entire function (can you still remember from Sec. 12.6 what this means?). Hence the integral of e^{-z^2} around the unit circle or, as a matter of fact, around any closed path of integration is zero.

11. **Cauchy's integral theorem not applicable. Deformation of path.** You see that $2z - 1 = 0$ at $z = 1/2$. Hence at this point the function $f(z) = 1/(2z-1)$ is not analytic. Since $z = 1/2$ lies inside the contour of integration (the unit circle), Cauchy's theorem is not applicable. Hence you have to integrate by the use of path. However, you can choose a most convenient path by applying the principle of deformation of path. You can move the unit circle to the right by 1/2; that is, you can choose the path C given by

$$z(t) = \frac{1}{2} + e^{it} \quad (0 \leqq t \leqq 2\pi).$$

Note that C is traversed counterclockwise as t increases from 0 to 2π. This is required in the problem. Then

$$f(z(t)) = \frac{1}{2z(t) - 1} = \frac{1}{2e^{it}}$$

and

$$\dot{z}(t) = ie^{it}.$$

With these functions, (10) in Sec. 13.1 gives the desired integral

$$\int_C f(z)\,dz = \int_0^{2\pi} \frac{ie^{it}}{2e^{it}}\,dt.$$

e^{it} cancels, and integration of the remaining $i/2$ from 0 to 2π gives

$$(i/2)2\pi = \pi i.$$

Note that the answer also follows directly from (6) with $m = -1$ and $z_0 = \frac{1}{2}$ without any further calculation.

21. **Use of partial fractions.** $z^2 - 1 = (z+1)(z-1)$ shows that the given function is not analytic at -1 and $+1$. You can write the integrand $f(z) = 1/(z^2 - 1)$ in terms of two partial fractions, namely,

$$\frac{1}{z^2 - 1} = \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right).$$

The integration of the first fraction (together with the factor 1/2) over the right loop gives πi by (6) with $m = -1$ and $z_0 = 1$; and over the left loop it gives 0 by Cauchy's integral theorem because the integrand is analytic inside and on that loop, in particular, at $z = -1$. Note that over the right loop you integrate counterclockwise.

For evaluating the integral of the second fraction (together with the factor 1/2) the idea is the same. Over the left loop you now obtain $-\pi i$ because you integrate clockwise; together with the minus sign in front of the fraction you get $+\pi i$. Over the right loop you now get 0 because the fraction is analytic everywhere on and inside the loop, including at $z = 1$. Adding your two results, you obtain the answer $2\pi i$.

Sec. 13.3 Cauchy's Integral Formula

Problem Set 13.3. Page 724

1. Cauchy's integral formula (1). The given function to be integrated is

$$g(z) = \frac{z^2}{z^4 - 1}.$$

Your first task is to find out where $g(z)$ is not analytic. These are the points where $z^4 = 1$. The solutions of this equation are the four values of the fourth root of 1, namely $1, i, -1, -i$, shown in Fig. 298 in Sec. 12.2.

Your next task is to find out which of those four values lie inside the contour and to make sure that none of them lies on the contour (a case we would not yet be able to handle). The contour is the circle (sketch it!)

$$|z + 1| = 1.$$

Its center is $z_0 = -1$. Hence for this contour you have to set in (1)

$$g(z) = \frac{z^2}{z^4 - 1} = \frac{f(z)}{z - z_0} = \frac{f(z)}{z + 1} \tag{A}$$

Since

$$\begin{aligned} z^4 - 1 &= (z^2 + 1)(z^2 - 1) \\ &= (z^2 + 1)(z + 1)(z - 1) \end{aligned}$$

you see that (A) gives

$$f(z) = \frac{z^2}{(z^2 + 1)(z - 1)}. \tag{B}$$

Alternatively, you can obtain the denominator of $f(z)$ by the division

$$(z^4 - 1)/(z + 1) = z^3 - z^2 + z - 1.$$

(Verify!) From (1) and (B) you thus obtain the answer

$$2\pi i f(z_0) = 2\pi i f(-1) = 2\pi i(-1)^2/(-4) = -\pi i/2.$$

7. Another application of (1). The given function $g(z) = (\cosh 3z)/(2z)$ is not analytic at $z = z_0 = 0$. Hence in (1) you have $g(z) = f(z)/z$

$$f(z) = zg(z) = \frac{z \cosh 3z}{2z} = \frac{\cosh 3z}{2}.$$

This calculation was simpler than that in Prob. 1. Do you see why? Since $\cosh 0 = 1$, the answer is

$$2\pi i f(z_0) = 2\pi i/2 = \pi i.$$

19. Partial fractions were used in Prob. 21 of the previous section because the integrand had two points at which it was not analytic. In the present problem the situation is similar as long as $z_1 \neq z_2$

$$\frac{1}{(z - z_1)(z - z_2)} = K\left(\frac{1}{z - z_1} - \frac{1}{z - z_2}\right),$$

where

$$K = 1/(z_1 - z_2).$$

Now apply (6) in Sec. 13.2 with $m = -1$ to each of the two fractions separately. For the first fraction [with z_1 instead of z_0 in (6)] this gives $K2\pi i$. For the second fraction [with z_2 instead of z_0 in (6)] this gives $-K2\pi i$. Hence the answer is 0, as claimed.

If $z_1 = z_2$, the integrand is $1/(z - z_2)^2$. In this case the result also follows from (6) in Sec. 13.2, but this time you have to use (6) with $m = -2$, one of the infinitely many cases in which (6) gives 0.

Sec. 13.4 Derivatives of Analytic Functions

Example 1. Evaluate $\sin(\pi i)$ by (1) or (6b) in Sec. 12.7.

Example 2. The factor $\pi i = 2\pi i/2$ results from (1) since $n = 2$ (second derivative).

Problem Set 13.4. Page 729

1. **Use of a third derivative.** The given function $(\sinh 2z)/z^4$ is of the form of the integrand in (1) with $z_0 = 0$, $n + 1 = 4$, hence $n = 3$, and $f(z) = \sinh 2z$. For these data you conclude from (1) that the integral around the unit circle in the counterclockwise sense equals $2\pi i/3!$ times the third derivative of $f(z) = \sinh 2z$, which is $8 \cosh 2z$, taken at $z = 0$; this gives 8; here the factor 8 comes from the chain rule of differentiation. Together this gives the answer $2\pi i 8/6 = 8\pi i/3$.

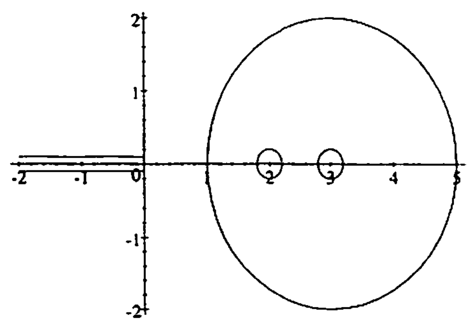
5. **Use of a second derivative.** $(\tan z)/(z - \frac{\pi}{4})^3$ requires the evaluation of the second derivative of $\tan z$ at $z_0 = \pi/4$, as can be seen from (1). You obtain

$$(\tan z)'' = (1/\cos^2 z)' = (-2/\cos^3 z)(-\sin z). \qquad \text{(A)}$$

Since $\cos(\pi/4) = \sin(\pi/4) = 1/\sqrt{2}$, the value of (A) at $\pi/4$ is 4. Hence, by (1) the integral equals $2\pi i \cdot 4/2! = 4\pi i$.

13. **First derivative. Logarithm.** The given integrand is $\mathrm{Ln}(z)/(z - 2)^2$. The contour of integration is the circle of radius 2 with center at 3. At 0 and at the points on the negative ray of the real axis the function $\mathrm{Ln}\, z$ is not analytic, and it is essential that these points lie outside that circle. Otherwise, that is, if that ray intersected or touched the contour, we would not be able to integrate.

Furthermore, the integrand is not analytic at $z = z_0 = 2$, which lies inside the contour. Thus, according to (1) with $n + 1 = 2$, hence $n = 1$, and $z_0 = 2$, the integral equals $2\pi i$ times the value of the first derivative of $\mathrm{Ln}\, z$, which is $1/z$, at $z = z_0 = 2$; this gives a factor $1/2$. Hence the answer is πi.



Section 13.4. Problem 13. Behavior of the integrand and path of integration