PART D. COMPLEX ANALYSIS

CHAPTER 12. Complex Numbers and Functions. Conformal Mapping

Sec. 12.1 Complex Numbers. Complex Plane

Example 2. The check is $zz_2 = (\frac{66}{85} + \frac{43}{85}i)(9 - 2i) = 8 + 3i$.

Problem Set 12.1. Page 656

1. **Powers of i.** $i^2 = -1$ and $\frac{1}{i} = -i$ will be used quite frequently. A formal derivation of $i^2 = -1$ from the multiplication formula is shown in the text. $1/i = -i/i(-i) = -i/1 = -i$ follows from (7).

3. **Multiplication.** For $z_1 = 4 + 3i$ and $z_2 = 2 - 5i$ the recipe on p. 654 at the top [which results from (3)] gives

$$\begin{aligned} z_1 z_2 &= (4 + 3i)(2 - 5i) \\ &= 4 \cdot 2 - 4 \cdot 5i + 3i \cdot 2 + i^2 \cdot 3 \cdot (-5) \\ &= 8 - 20i + 6i + 15 = 23 - 14i. \end{aligned}$$

With a little training you can go faster and write down first the two terms of the real part and then the two terms of the imaginary part; thus

$$\begin{aligned} (4 + 3i)(2 - 5i) &= 8 - (-15) + i(-20 + 6) \\ &= 23 - 14i. \end{aligned}$$

5. **Division.** Given $z_1 = 4 + 3i$, find $1/z_1$.

This is a simple special case of (7), which gives

$$\frac{1}{z_1} = \frac{1}{4 + 3i} = \frac{4 - 3i}{(4 + 3i)(4 - 3i)} = \frac{4 - 3i}{16 + 9} = 0.16 - 0.12i.$$

17. **Real part. Complex conjugate.** Let $z = x + iy$. Find Re $(z^2/\bar{z})$.

First determine $z^2/\bar{z}$. According to (7) multiply numerator and denominator by the conjugate of the denominator, which is z, and use that $z\bar{z} = x^2 + y^2$. This gives

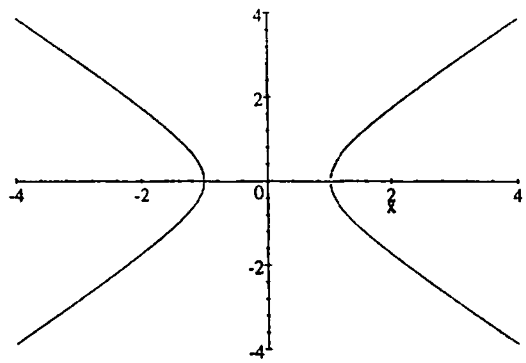
$$\begin{aligned} z^2/\bar{z} = z^3/z\bar{z} &= \frac{(x + iy)^3}{x^2 + y^2} \\ &= \frac{x^3 + 3ix^2y - 3xy^2 - iy^3}{x^2 + y^2}, \end{aligned}$$

where the minus signs come from i^2. The real part of this is obtained by omitting the two terms that have an i. This gives the answer

$$\text{Re}\,(z^2/\bar{z}) = \frac{x^3 - 3xy^2}{x^2 + y^2}.$$

Sec. 12.2 Polar Form of Complex Numbers. Powers and Roots

Generalized triangle inequality (6). Drawing the complex numbers as little arrows and letting each tail coincide with the preceding head, you get a zigzag line of n parts. The left side of (6)



Section 12.3. Problem 7. Hyperbola H bounding the region R between the two branches

11. **Function values.** $z = 7 + 2i$ gives $1 - z = -6 - 2i$, hence

$$\frac{1}{1-z} = \frac{-6+2i}{(-6-2i)(-6+2i)} = \frac{-6+2i}{36+4} = -\frac{3}{20} + \frac{i}{20}.$$

Hence Re $(f) = -3/20$ and Im $(f) = 1/20$.

13. **Continuity.** First of all, the only point where $f(z) = (\text{Im } z)/|z|$ could be discontinuous is the origin because Im $z = y$ is continuous everywhere. Now

$$f(z) = (\text{Im } z)/|z| = y/\sqrt{x^2+y^2}. \qquad \text{(A)}$$

Continuity at 0 would mean that $f(z)$ approaches 0 as z approaches 0 from any direction. But from (A) you see that on the x-axis ($y = 0$) the function has the value 0 for any $x \neq 0$, whereas on the y-axis ($x = 0$) you obtain the value $y/\sqrt{y^2} = y/|y|$, which is $+1$ for positive y and -1 for negative y. Hence $f(z)$ is discontinuous at $z = 0$.

More simply, if you use polar coordinates, you have

$$\text{Im } z = r\sin\theta, \quad |z| = r, \quad f(z) = (r\sin\theta)/r = \sin\theta.$$

From the last expression you get 0 on the positive ray of the x-axis ($\theta = 0$), 1 on the positive ray of the y-axis ($\theta = \pi/2$), 0 on the negative ray of the x-axis ($\theta = \pi$), and -1 on the negative ray of the y-axis ($\theta = 3\pi/2$ or $-\pi/2$, etc.). This agrees with the results obtained just before.

19. **Derivative.** The differentiation rules are the same as in calculus. Hence $f(z) = (5+3i)/z^3$ has the derivative

$$f'(z) = -\frac{3(5+3i)}{z^4}. \qquad \text{(A)}$$

For $z = 2 + i$ you obtain by the usual division rule (Sec. 12.1)

$$\frac{1}{z^4} = \frac{1}{(2+i)^4} = \frac{(2-i)^4}{(2+i)^4(2-i)^4} = \frac{((2-i)^2)^2}{(4+1)^4} = \frac{(3-4i)^2}{625} = \frac{9-16-24i}{625}.$$

Multiplying this by the factor $-3(5+3i)$ in (A) finally gives

$$\frac{(-15-9i)(-7-24i)}{625} = \frac{-111+423i}{625}.$$

If you had difficulties with this problem, review a few calculations from Sec. 12.1 and the corresponding problem set because the differentiation as such was done as in calculus.

Sec.12.4 Cauchy-Riemann Equations. Laplace's Equation

Problem Set 12.4. Page 673

1. **Check of analyticity.** The form of the given function, $f(z) = z^6$, shows that in the present case, (7) will be simpler. Indeed, in polar coordinates you simply have

$$f(z) = r^6(\cos 6\theta + i \sin 6\theta).$$

Hence

$$u = r^6 \cos 6\theta, \quad v = r^6 \sin 6\theta.$$

The expressions needed in (7) are obtained by straightforward differentiation. In the first Cauchy-Riemann equation in (7) you need

$$u_r = 6r^5 \cos 6\theta \quad \text{and} \quad v_\theta = 6r^6 \cos 6\theta,$$

with the factor 6 in v_θ resulting from the chain rule. From this you see that this first equation $u_r = v_\theta/r$ is satisfied. In the second Cauchy-Riemann equation you need

$$v_r = 6r^5 \sin 6\theta \quad \text{and} \quad u_\theta = -6r^6 \sin 6\theta.$$

If you divide u_θ by $-r$, you obtain v_r. Hence the second Cauchy-Riemann equation is also satisfied, and you can conclude that z^6 is analytic for all $z \neq 0$.

z^6 is also analytic at $z = 0$. This does not follow from (7), but you have to use (1), which involves more work. (Of course, this will make your work on (7) superfluous.) You can get u and v by using the binomial theorem, obtaining

$$\begin{aligned}(x+iy)^6 &= x^6 + 6x^5(iy) + 15x^4(iy)^2 + 20x^3(iy)^3 + 15x^2(iy)^4 + 6x(iy)^5 + (iy)^6 \\ &= x^6 + 6ix^5y - 15x^4y^2 - 20ix^3y^3 + 15x^2y^4 + 6ixy^5 - y^6.\end{aligned}$$

The terms without an i give the real part

$$u = x^6 - 15x^4y^2 + 15x^2y^4 - y^6.$$

The terms containing i give the imaginary part

$$v = 6x^5y - 20x^3y^3 + 6xy^5.$$

In the first Cauchy-Riemann equation you need the partial derivatives

$$u_x = 6x^5 - 60x^3y^2 + 30xy^4$$

and

$$v_y = 6x^5 - 60x^3y^2 + 30xy^4.$$

Hence the first Cauchy-Riemann equation is satisfied. The second one involves

$$v_x = 30x^4y - 60x^2y^3 + 6y^5$$

and

$$u_y = -30x^4y + 60x^2y^3 - 6y^5.$$

You see that $v_x = -u_y$, so that the second Cauchy-Riemann equation is satisfied, too. This proves analyticity of z^6 for all z.

3. **Cauchy-Riemann equations. Analyticity.** The function

$$f(z) = u + iv = e^x(\cos y + i \sin y) \tag{A}$$

has the real part $u = e^x \cos y$ and the imaginary part $v = e^x \sin y$. The familiar rules for differentiating the exponential function and cosine and sine show that the Cauchy-Riemann equations (1) are satisfied for all $z = x + iy$. Indeed,

$$u_x = e^x \cos y = v_y$$
$$u_y = -e^x \sin y = -v_x.$$

You will see in Sec. 12.6 that (A) defines the complex exponential function. When $y = 0$, so that $z = x$ is real, then $\cos y = \cos 0 = 1$, $\sin y = \sin 0 = 0$, and $f(z)$ becomes e^x, the exponential function known from calculus. More on this follows in Sec. 12.6.

17. Harmonic functions appear as real and imaginary parts of analytic functions. If you remember that the given function u is the real part of $1/z$, you are done; indeed, by the division rule,

$$\frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}.$$

This also shows that a conjugate harmonic of u is $-y/(x^2+y^2)$.

If you don't remember that, you have to work systematically by differentiation, beginning with proving that the Laplace equation (8) is satisfied. Such somewhat lengthy differentiations (as well as other calculations) can often be simplified (and made more reliable) by introducing suitable shorter notations for certain expressions. In the present case you can write

$$u = \frac{x}{F}, \quad \text{where} \quad F = x^2+y^2. \quad \text{Then} \quad F_x = 2x, \quad F_y = 2y. \tag{A}$$

By applying the product rule of differentiation (and the chain rule), not the quotient rule, you obtain the first partial derivative

$$u_x = \frac{1}{F} - \frac{x(2x)}{F^2}. \tag{B}$$

By differentiating this again, using the product and chain rules you obtain the second partial derivative

$$u_{xx} = -\frac{2x}{F^2} - \frac{4x}{F^2} + \frac{8x^3}{F^3}. \tag{C}$$

Similarly, the partial derivative of u with respect to y is obtained from (A) in the form

$$u_y = -\frac{2xy}{F^2}. \tag{D}$$

The partial derivative of this with respect to y is

$$u_{yy} = -\frac{2x}{F^2} + \frac{8xy^2}{F^3}. \tag{E}$$

Adding (C) and (E) and remembering that $F = x^2+y^2$ give

$$u_{xx} + u_{yy} = -\frac{8x}{F^2} + \frac{8x(x^2+y^2)}{F^3} = -\frac{8x}{F^2} + \frac{8x}{F^2} = 0.$$

This shows that $u = x/F = x/(x^2+y^2)$ is harmonic.

Now determine a conjugate harmonic. From (D) and the second Cauchy-Riemann equation you obtain

$$u_y = -\frac{2xy}{F^2} = -v_x.$$

Integration of $2x/F^2$ with respect to x gives $-1/F$, so that integration of v_x with respect to x gives

$$v = -\frac{y}{F} = -\frac{y}{x^2+y^2} + h(y).$$

Now show that $h(y)$ must be a constant (which you can choose to be 0). By differentiation with respect to y and taking the common denominator F^2 you obtain

$$v_y = -\frac{1}{F} + \frac{2y^2}{F^2} = \frac{-x^2+y^2}{F^2} + h'(y).$$

On the other hand, you have from (B)

$$u_x = \frac{1}{F} - \frac{2x^2}{F^2} = \frac{y^2-x^2}{F^2}.$$

By the first Cauchy-Riemann equation, $v_y = u_x$, so that $h'(y) = 0$ and $h(y) = const$, as claimed.

Sec. 12.5 Geometry of Analytic Functions: Conformal Mapping

Problem Set 12.5. Page 678

3. **Mapping** $w = 1/z$. Taking absolute values, you obtain

$$|w| = |1/z| = 1/|z|.$$

This shows that the concentric circles $|z| = 1/3, 1/2, 1, 2, 3$ are mapped onto the concentric circles $|w| = 3, 2, 1, 1/2, 1/3$, respectively, in the w-plane. In particular, the unit circle is mapped onto the unit circle. For this reason this mapping is often called a *reflection in the unit circle*. The points $z = 1$ and $z = -1$ are "mapped onto itself"; this means, they are mapped onto $w = 1$ and $w = -1$, respectively. Other points on the unit circle are not mapped onto itself; for example, $z = i$ is mapped onto $w = 1/i = -i$. Circles inside the unit circle are mapped onto circles outside the unit circle, and conversely.

For arguments you have

$$\arg w = \arg\left(\frac{1}{z}\right) = -\arg z,$$

as follows from (11) in Sec. 12.2 with $z_1 = 1$ and $z_2 = z$. Hence Arg $z = 0$ (the positive ray of the real axis in the z-plane) maps onto Arg $w = 0$. Furthermore, Arg $z = \pi/4$ (the bisecting line of the first quadrant) is mapped onto Arg $w = -\pi/4$ (the bisecting line of the fourth quadrant in the w-plane). And so on.

9. **Mapping of a sector**. The given region in the z-plane is the sector bounded by the positive ray of the y-axis and the bisecting line $y = -x$ of the second quadrant. Since $w = z^2$ doubles angles at the origin, the image of the sector is the sector bounded by the negative ray of the u-axis and the negative ray of the v-axis, where $w = u + iv$, as usual. Thus this image is the third quadrant in the w-plane. More formally,

$$\text{Arg } w = \text{Arg } z^2 = 2 \text{ Arg } z = 2\frac{\pi}{2} \quad \text{and} \quad 2\frac{3\pi}{4},$$

respectively.

17. **Parametric representations of curves** are of great importance in our further work, not only in connection with mappings, but also in integration methods in the complex plane, as we shall see in the next chapter.
A circle of radius r with center at $z = 0$ is given by

$$x^2 + y^2 = r^2. \tag{A}$$

Parametrically represented, you have

$$x = r\cos t, \quad y = r\sin t. \tag{B}$$

Indeed, substituting (B) into (A), you see from $\cos^2 t + \sin^2 t = 1$ that (A) is satisfied. In complex form this can be written

$$z = x + iy = r\cos t + ir\sin t = r(\cos t + i\sin t).$$

For a circle of radius r with center at (a, b), thus at $z = a + ib$ in complex notation, you can write instead of (B)

$$x - a = r\cos t, \quad y - b = r\sin t. \tag{C}$$

Then you obtain instead of (A) more generally the familiar nonparametric representation

$$(x - a)^2 + (y - b)^2 = r^2.$$

From (C) you now have

$$x = a + r\cos t, \quad y = b + r\sin t.$$

or in complex form

$$z = x + iy = a + r\cos t + i(b + r\sin t).$$

In the problem, $a = 3$, $b = -1$, $r^2 = 4$, so that $r = 2$ and the answer is

$$z = 3 + 2\cos t + i(-1 + 2\sin t).$$

Sec. 12.6 Exponential Function

Problem Set 12.6. Page 682

1. Function values. From (1) you obtain

$$e^{2+3\pi i} = e^2(\cos 3\pi + i\sin 3\pi) = e^2(-1 + i\cdot 0) = -e^2 = -7.389.$$

From (10) you obtain the absolute value

$$|e^z| = e^x = e^2 = 7.389.$$

7. Polar form (6). $z = 4 + 3i$ has the absolute value

$$|z| = \sqrt{4^2 + 3^2} = 5$$

and the argument

$$\text{Arg } z = \arctan(3/4).$$

Hence (6) gives the polar form

$$z = 5e^{i\arctan(3/4)}$$
$$= 5e^{0.643501i}$$

You can check this by calculating

$$z = 5(\cos 0.643501 + i\sin 0.643501)$$
$$= 5(0.8 + 0.6i)$$
$$= 4 + 3i.$$

13. Equation. Since $e^x > 0$, the given equation $e^z = -3$ has no real solution. Taking absolute values and using (10) gives

$$|e^z| = e^x = 3, \quad \text{hence} \quad x = \ln 3.$$

From this and (1) you obtain

$$e^z = 3(\cos y + i\sin y) = -3,$$

hence $\cos y = -1$ and $\sin y = 0$. A solution is $y = \pi$. Further solutions are $\pi \pm 2n\pi$, where n is a positive integer. Together,

$$z = x + iy = \ln 3 + i(\pi \pm 2n\pi).$$

17. Conformal mapping. The given domain in the z-plane is a horizontal strip of width $\pi/2$ bounded by the x-axis and the horizontal straight line $y = \pi/2$. From (10) you see that y is the argument of $w = e^z$. Hence that domain is mapped onto the open first quadrant Q, because Q is precisely the domain consisting of all complex w whose principal argument lies between 0 and $\pi/2$.

Sec. 12.7 Trigonometric Functions. Hyperbolic Functions

Problem Set 12.7. Page 686

1. Real and imaginary parts of cosh z. Use the definition (11), multiply it by 2 (in order not to carry 1/2 along), and set $z = x + iy$ as usual. Because of the definition of the exponential function in Sec. 12.6 this gives

$$2\cosh z = e^z + e^{-z}$$
$$= e^x(\cos y + i\sin y) + e^{-x}(\cos y - i\sin y).$$

Next collect cosine and sine terms, obtaining

$$2\cosh z = (e^x + e^{-x})\cos y + i(e^x - e^{-x})\sin y.$$

The expressions in the parentheses are the real hyperbolic functions $2\cosh x$ and $2\sinh x$, respectively. Division by 2 now gives the expected result

$$\cosh z = \cosh x\,\cos y + i\sinh x\,\sin y.$$

The other formula follows by a similar straightforward calculation.

7. **Function values.** Your CAS may be able to give function values of the complex trigonometric or hyperbolic functions directly. On a calculator you may use the formulas (6) and those in Prob. 1 to calculate those values from values of the real exponential function, cosine, and sine. In the problem,

$$\cosh(-3-6i) = \cosh(-3)\cos(-6) + i\sinh(-3)\sin(-6)$$
$$= \cosh 3\cos 6 + i\sinh 3\sin 6$$

because in the last term you have a product of two minus signs. Expressing cosh and sinh in terms of exponential functions and evaluating them, you obtain from the second line

$$\cosh(-3-6i) = \frac{1}{2}(20.0855 + 0.0498)\cdot 0.960170 + \frac{1}{2}i(20.0855 - 0.0498)\cdot(-0.279415)$$
$$= 9.66667 - 2.79915i.$$

11. **Equation.** $\cosh z = 1/2$ has no real solution because $\cosh x \geqq 1$ for any real x. Use the formula in Prob. 1. For the real parts you have

$$\cosh x\cos y = 1/2 \qquad \text{(A)}$$

and for the imaginary parts

$$\sinh x\sin y = 0. \qquad \text{(B)}$$

From (B) you have $x = 0$ or $y = n\pi$, where n is any integer, positive, zero, or negative.

For $x = 0$ you get from (A) the equation $\cos y = 1/2$. Hence

$$y = \frac{\pi}{3} + 2n\pi \quad \text{or} \quad y = \frac{5\pi}{3} + 2n\pi,$$

where n is any integer. This agrees with the answer on p. A32 in Appendix 2 of the book (which is merely slightly differently written; note that $5\pi/3 - 2\pi = -\pi/3$; this explains it).

For $y = n\pi$ you have in (A)

$$\cosh x\cos n\pi = (-1)^n\cosh x.$$

This is either at least equal to +1 (if n is even), or at most equal to −1 (if n is odd). Hence in none of these two cases it can be equal to 1/2, so that you get no further solutions.

17. **Mapping $w = \cos z$.**

$$w = u + iv = \cos z = \cos x\,\cosh y - i\sin x\,\sinh y. \qquad \text{(A)}$$

For the x-axis ($y = 0$) this becomes $\cos x$ and varies from 1 to −1 as x varies from 0 to π (the lower edge of the rectangle in the z-plane to be mapped). For the y-axis ($x = 0$) the equation (A) gives $\cosh y$, which varies from 1 to cosh 1 as y varies from 0 to 1.

For the vertical line $x = \pi$ the equation (A) gives $-\cosh y$; this varies from −1 to −cosh 1 as y varies from y to 1 along the right boundary of the given rectangle. Hence three edges of the rectangle are mapped into the real axis (the u-axis) of the w-plane.

Finally, for the upper edge $y = 1$ you have in (A)

$$u = \cos x\cosh 1, \qquad v = -\sin x\sinh 1. \qquad \text{(B)}$$

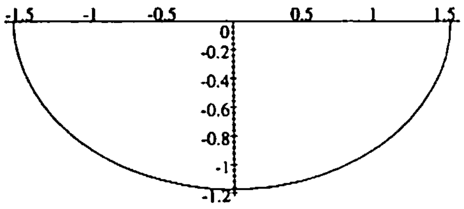
Using $\cos^2 x + \sin^2 x = 1$, you obtain from this

$$\frac{u^2}{\cosh^2 1} + \frac{v^2}{\sinh^2 1} = 1.$$

This represents an ellispse with semiaxes $\cosh 1 = 1.54308$ and $\sinh 1 = 1.17520$. Since part of the u-axis is part of the boundary of the image, the image must be the upper or lower half of the interior of this ellipse. To find out, calculate the image of the midpoint of the upper edge of the rectangle, which has the coordinates $x = \pi/2$, $y = 1$. As the image of this point you obtain from (A)

$$\cos(\pi/2 + i) = 0 - i \sin \pi/2 \sinh 1 = -i \sinh 1.$$

Since this is a point in the lower half-plane, the image of the rectangle must lie in the lower half-plane, not in the upper. More simply: $\sin x$ in (B) is positive (except at $x = 0$ and $x = \pi$), hence v in (B) is negative.



Section 12.7. Problem 17. Image of the given rectangle under $w = \cos z$

Sec. 12.8 Logarithm. General Power

Problem Set 12.8. Page 691

3. **Analyticity.** Use $\operatorname{Ln} z = \ln|z| + i \operatorname{Arg}(z) = \ln r + i\theta$. Then show that (7) in Sec. 12.4 is satisfied everywhere except at $z = 0$ and on the negative ray of the x-axis.

5. **Principal value.** Note that the real logarithm of a negative number is undefined. The principal value $\operatorname{Ln} z$ of $\ln z$ is defined by (2), where $\operatorname{Arg} z$ is the principal value of $\arg z$. Now recall from Sec. 12.2 that the principal value of the argument is defined by

$$-\pi < \operatorname{Arg}\theta \leqq \pi.$$

In particular, for a negative real number you always have $\operatorname{Arg}\theta = +\pi$, as you should keep in mind. From this and (2) you obtain the answer

$$\operatorname{Ln}(-5) = \ln 5 + i\pi.$$

13. **All values of a complex logarithm.** You need the absolute value and the argument of $-e^{-i}$ because by (1) and (2),

$$\ln(-e^{-i}) = \ln|-e^{-i}| + i \arg(-e^{-i}) = \ln|-e^{-i}| + i \operatorname{Arg}(-e^{-i}) \pm 2n\pi i.$$

Now the absolute value of the exponential function e^z with a pure imaginary exponent always equals 1, as you should memorize.; the derivation is

$$|e^{iy}| = |\cos y + i \sin y| = \sqrt{\cos^2 y + \sin^2 y} = 1.$$

(Can you see where this calculation would break down if y were not real?) In our case,

$$|-e^{-i}| = 1, \quad \text{hence} \quad \ln|-e^{-i}| = 0. \qquad \text{(A)}$$

The argument of $-e^{-i}$ is obtained from (10) in Sec. 12.6, that is,

$$\arg(e^z) = \operatorname{Arg}(e^z) \pm 2n\pi = y \pm 2n\pi.$$

In Prob. 13 you have $z = -i$, hence $y = -1$, and, therefore,

$$\arg(e^{-i}) = -1 \pm 2n\pi. \tag{B}$$

Finally, by (9) in Sec. 12.2, the argument of a product is the sum of the arguments of the factors, up to integer multiples of 2π. Hence multiplying e^{-i} by -1 corresponds to adding π in the argument. From (B) you thus obtain

$$\arg(-e^{-i}) = \pi - 1 \pm 2n\pi. \tag{C}$$

From (A) and (C) you obtain the answer

$$\ln(-e^{-i}) = (\pi - 1)i \pm 2n\pi i.$$

21. General power. You first have

$$3^{4-i} = 3^4 3^{-i} = 81 \cdot 3^{-i}. \tag{A}$$

To the last factor apply (8) with $a = 3$ and $z = -i$. This gives

$$3^{-i} = e^{-i\ln 3}.$$

On the right now use the definition of the exponential function (Sec. 12.6). This gives

$$3^{-i} = \cos(\ln 3) - i\sin(\ln 3).$$

Substituting this into (A) gives the answer

$$\begin{aligned} 3^{4-i} &= 81(\cos(\ln 3) - i\sin(\ln 3)) \\ &= 36.841 - 72.137i. \end{aligned}$$

Sec. 12.9 Linear Fractional Transformations. *Optional*

Problem Set 12.9. Page 698

1. Inverse. Write (1) as $(cz + d)w = az + b$ and and take the z-terms to the left and the other terms to the right,

$$z(cw - a) = b - dw. \tag{A}$$

Now divide. Note that the result is determined only up to a common factor in the numerator and the denominator. For instance, to obtain (4) from (A), multiply (A) by -1 on both sides.

3. Occurrence of infinity. For the given data the left side of (6) takes the form

$$\frac{w - (-1)}{w - 1} \cdot \frac{-i - 1}{-i - (-1)}. \tag{B}$$

The rule of complex division in Sec. 12.1 shows that the second quotient has the value $-i$. Hence (B) reduces to

$$\frac{-i(w + 1)}{w - 1}. \tag{C}$$

Infinity occurs on the right side of (6), which for the gives data becomes

$$\frac{z - 0}{z - \infty} \cdot \frac{1 - \infty}{1 - 0} \tag{D}$$

By Theorem 2 you have to replace the quotient of $1 - \infty$ divided by $z - \infty$ by the value 1. Hence the whole expression (D) reduces to z. From this and (C) you have by multiplying by $w - 1$

$$-i(w + 1) = z(w - 1).$$

Reshuffling terms gives

$$w(-i-z) = -z+i$$

Division by $-i-z$ and multiplying both the numerator and the denominator of the result by -1 you obtain

$$w = \frac{-z+i}{-i-z} = \frac{z-i}{z+i},$$

in agreement with Example 2.

13. Determination of a linear fractional transformation. Problems 7-14 can be solved by a straightforward use of (6), a formula not to be remembered, but to be looked up when needed. For the data in Prob. 13 the left side of (6) is

$$\frac{w-0}{w-(-1)} \cdot \frac{\infty-(-1)}{\infty-0}.$$

Replacing the quotient containing the infinities by 1, there remains

$$\frac{w}{w+1}.$$

The right side of (6) is

$$\frac{z-i}{z-0} \cdot \frac{-i-0}{-i-i} = \frac{z-i}{2z}.$$

Equating the two results and multiplying through by the two denominators, you obtain

$$w \cdot 2z = (w+1)(z-i).$$

Collecting the w-terms on the left and the others on the right, you have

$$w(2z-(z-i)) = z-i.$$

Simplification and division finally gives

$$w = \frac{z-i}{z+i}.$$

You may also get the result by using the given data one after another, as follows, starting from (1). $z = i$ maps onto $w = 0$. Hence in the numerator of (2) you have $ai+b = 0$, $b = -ia$. $z = -i$ maps onto $w = \infty$. Hence in the denominator of (2) you must have $c(-i)+d = 0$. Hence, so far you have

$$w = \frac{az-ia}{cz+ic}. \tag{E}$$

$z = 0$ maps onto $w = -1$. This gives $-1 = -ia/(ic)$, hence $c = a$, so that the quotient in (E) becomes $(az-ia)/(az+ia)$. Now divide by a.

17. Matrices. It is clear that there must be a condition on the coefficients a, b, c, d in (1) and (4), as stated in the problem, because these coefficients are determined only up to a constant factor.

Sec. 12.10 Riemann Surfaces. *Optional*

Problem Set 12.10. Page 700

1. Square root. If z moves on the unit circle, it has absolute value $|z| = 1$. The image under $w = \sqrt{z}$ also has absolute value $|w| = 1$, as you can see from

$$z = re^{i\theta}, \quad w = \sqrt{r}\, e^{i\theta/2}. \tag{A}$$

Hence the image point also moves around the unit circle (in the w-plane). Since (A) shows that

$$\operatorname{Arg} w = \frac{1}{2} \arg z,$$

it follows that as z begins its motion and moves once around the unit circle, w will move from $w = 1$ to

$w = -1$ in the upper half-plane. When z continues and moves once more around the unit circle, w will move from -1 to 1 in the lower half-plane.

5. **Logarithm.** Use Fig. 318 in Sec. 12.8 as a guide to visualizing the answer on p. A32 in Appendix 2. Note that on the unit circle $z = 1$ you have $\ln |z| = \ln 1 = 0$, which gives the indicated motion of w on the imaginary axis (the v-axis).

9. **Branch points and sheets.** The radicand $2z + i$ is zero at $z = -i/2$. This is the location of a branch point, the only one. The Riemann surface of a cube root has 3 sheets and looks as shown in Fig 323b. In the figure the branch point is at 0, whereas in the present problem it is at $-i/2$. The corresponding function value is $w = 3$, whereas in that figure it is $w = 0$.