CHAPTER 11. Partial Differential Equations

Sec. 11.1 Basic Concepts

Problem Set 11.1. Page 584

15. **Reduction to ordinary differential equations** is possible, for instance, if an equation contains only partial derivatives with respect to x or only with respect to y. The given equation

$$u_y = u$$

with $u = u(x, y)$ can be regarded as $u' = u$ and solved to obtain

$$u = c e^y$$

where c now depends on x, that is, the solution is

$$u(x, y) = c(x) e^y.$$

Check by differentiation that this function with arbitrary $c(x)$ satisfies the given equation.

19. **Substitution $u_x = v$.** This reduces the given equation $u_{xy} = u_x$ to

$$u_y = v.$$

The general solution is

$$v = c(x) e^y$$

where the arbitrary "constant" of integration c depends on x. Integration with respect to x now gives the answer

$$u = \int v\,dx = e^y c_1(x) + c_2(y),$$

where $c_1(x)$ is the integral of $c(x)$ with respect to x and $c_2(y)$ is the corresponding "constant" of integration. Check your result by differentiation to make sure that it is correct.

23. **Boundary value problem.** Verify the solution. Observing the chain rule, you obtain by differentiation

$$u_x = \frac{2ax}{x^2 + y^2}$$

and by another differentiation, with the product rule applied to $2ax$ and $1/(x^2 + y^2)$

$$u_{xx} = \frac{2a}{x^2 + y^2} - \frac{4ax^2}{(x^2 + y^2)^2}. \qquad \text{(A)}$$

Similarly, with y instead of x,

$$u_{yy} = \frac{2a}{x^2 + y^2} - \frac{4ay^2}{(x^2 + y^2)^2}. \qquad \text{(B)}$$

Taking the common denominator $(x^2 + y^2)^2$, you obtain in (A) the numerator

$$2a(x^2 + y^2) - 4ax^2 = -2ax^2 + 2ay^2$$

and in (B)

$$2a(x^2 + y^2) - 4ay^2 = -2ay^2 + 2ax^2.$$

Addition of the two expressions on the right gives 0 and completes the verification.

Now determine a and b in $u(x, y) = a \ln (x^2 + y^2) + b$ from the boundary conditions. For $x^2 + y^2 = 1$ you have $\ln 1 = 0$, so that $b = 0$ from the first boundary condition. From this and the second boundary condition you have $a \ln 4 = 3$. Hence $a = 3/\ln 4$, in agreement with the answer on p. A28 in Appendix 2 of the book.

Sec. 11.3 Separation of Variables. Use of Fourier Series

Problem Set 11.3. Page 594

3. **Vibrating string: simplest solutions.** In (11) a term $\sin px$ is multiplied by $\cos cpt$ because

$$p = \frac{n\pi}{L} \quad \text{in (9) and} \quad \lambda_n = c\frac{n\pi}{L} = cp \quad \text{in } (11^\bullet).$$

p and λ_n differ by a factor c, which is 1 in the problem, by assumption. It follows that for $c = 1$, initial velocity 0, and initial deflection

$$k\left(\sin x - \frac{1}{2}\sin 2x\right)$$

the corresponding solution is

$$k\left(\cos t \sin x - \frac{1}{2}\cos 2t \sin 2x\right).$$

Note that in this problem, $B_1 = k$ and $B_2 = -k/2$ in (11) are given; the initial condition already has the form of a Fourier series, so you need not do anything further.

7. **Use of Fourier series.** Problems 4-9 amount to the determination of the Fourier sine series of the initial deflection. Each term $b_n \sin nx$ is then multiplied by the corresponding $\cos nt$ (since $c = 1$; for arbitrary c it would be $\cos cnt$). The series of the terms thus obtained is the desired solution. For the "triangular" initial deflection in Prob. 7 you obtain the Fourier sine series (see p. 546 with $L = \pi$ and $k = 1/2$)

$$\frac{4}{\pi^2}\left(\sin x - \frac{1}{9}\sin 3x + \frac{1}{25}\sin 5x - + \cdots\right).$$

Multiplying each term $\sin((2n+1)x)$ by the corresponding $\cos((2n+1)t)$, you obtain the answer on p. A28 in Appendix 2 of the book.

17. **Separation of variables** is a relatively general method of solution, particularly when it is used for generating the terms of suitable series developments, as shown in the text.

The given differential equation is $u_{xy} - u = 0$ or $u_{xy} = u$. To solve it, substitute $u = F(x)G(y)$, obtaining

$$\frac{dF}{dx}\cdot\frac{dG}{dy} = FG.$$

Now separate variables. This gives

$$\frac{dF/dx}{F} = \frac{G}{dG/dy}. \tag{A}$$

Both sides must equal a constant, say, k, by the argument used in the text. This gives from the left side of (A), equated to k,

$$\frac{dF}{dx} = kF.$$

The solution is

$$F = c_1 e^{kx}.$$

Equating the right side of (A) to k, you have $G/(dG/dy) = k$; hence

$$\frac{dG}{dy} = \frac{G}{k}.$$

The solution is

$$G = c_2 e^{y/k}.$$

This gives the answer $u = FG = ce^{kx+y/k}$.

Sec. 11.4 D'Alembert's Solution of the Wave Equation

Problem Set 11.4. Page 597

1. Speed. This is a uniform motion (motion with constant speed). Use that in this case, speed is distance divided by time or, equivalently, speed is distance traveled in unit time.

13. Normal form. The given equation

$$u_{xx} + 4u_{xy} + 4u_{yy} = 0$$

is of the form (15) with $A = 1$, $B = 2$, and $C = 4$. Hence $AC - B^2 = 0$, so that the equation is parabolic. Furthermore, the ordinary differential equation in Prob. 11 takes the form

$$y'^2 - 4y' + 4 = 0.$$

You can factor this in the form

$$(y' - 2)^2 = 0. \tag{A}$$

From Prob. 12 you see that for a parabolic equation the new variables leading to the normal form are $v = x$ and $z = \Psi(x, y)$, where $\Psi = const$ is a solution $y = y(x)$ of (A). From (A) you obtain

$$y' - 2 = 0, \quad y - 2x = const, \quad z = 2x - y.$$

($z = y - 2x$ would do it equally well; try it.) Hence the new variables are

$$v = x$$
$$z = 2x - y.$$

The remainder of the work consists of transforming the occurring partial derivatives into partial derivatives with respect to these new variables. This is done by the chain rule. For the first partial derivative with respect to x you obtain

$$u_x = u_v v_x + u_z z_x$$
$$= u_v + 2u_z.$$

The partial derivative of this with respect to x is

$$u_{xx} = (u_v + 2u_z)_v v_x + (u_v + 2u_z)_z z_x$$
$$= u_{vv} + 2u_{zv} + 2(u_{vz} + 2u_{zz}).$$

Assuming the continuity of the partial derivatives involved, you can interchange the order of differentiation and have $u_{zv} = u_{vz}$. Hence u_{xx} becomes

$$u_{xx} = u_{vv} + 4u_{vz} + 4u_{zz}. \tag{B}$$

Now turn to partial differentiation with respect to y. The first partial derivative with respect to y is

$$u_y = u_v v_y + u_z z_y = -u_z$$

because $v_y = 0$ and $z_y = -1$. Take the derivative of this with respect to x, obtaining

$$u_{yx} = u_{xy} = -u_{zv} v_x - u_{zz} z_x \tag{C}$$
$$= -u_{vz} - 2u_{zz}.$$

Finally, taking the partial derivative of u_y with respect to y gives

$$u_{yy} = -u_{zv} v_y - u_{zz} z_y \tag{D}$$
$$= u_{zz}.$$

Substituting the second partial derivatives (B)-(D) into the given equation, you obtain

$$\begin{aligned} u_{xx} + 4u_{xy} + 4u_{yy} &= u_{vv} + 4u_{vz} + 4u_{zz} \\ &\quad - 4u_{vz} - 8u_{zz} \\ &\quad + 4u_{zz} \\ &= u_{vv} = 0. \end{aligned}$$

This is the normal form for parabolic equations.

Sec. 11.5 Heat Equation: Solution by Fourier Series

Problem Set 11.5. Page 608

3. Single sine terms as initial temperatures lead to solutions of the form (9),

$$B_n \sin \frac{n\pi x}{L} e^{-\lambda_n^2 t},$$

that is, single eigenfunctions (which in more complicated cases would not be sufficient for satisfying the inital condition). The initial condition

$$\sin(0.1\pi x) = \sin(\pi x/10) = \sin(\pi x/L)$$

(where $L = 10$) shows that $n = 1$, that is, the initial condition is such that the solution is given by the first eigenfunction. From the data for $K, \sigma,$ and ρ calculate $c^2 = K/(\sigma\rho) = 1.75202$ (see the very first equation in Sec. 11.5). This gives the answer

$$u = \sin(0.1\pi x) \exp(-1.75202\pi^2 t/100)$$

where $L^2 = 100$ results from

$$\lambda_1^2 = \frac{c^2\pi^2}{L^2}.$$

17. Heat flow in a plate. The problem corresponds to the situation discussed on p. 606 of the text for a rectangle because the boundary conditions are as in the text, with $f(x) = 20 = const.$ Hence you can obtain the solution of the problem from (19) and (20) with $a = b = 24$. You have to begin with (20), which for the problem takes the form

$$\begin{aligned} A_n^* &= \frac{2}{24 \sinh n\pi} \int_0^{24} 20 \sin \frac{n\pi x}{24}\, dx \\ &= \frac{40}{24 \sinh n\pi} \left(-\frac{24}{n\pi}\right) \cos \frac{n\pi x}{24} \Bigg|_{x=0}^{24} \\ &= -\frac{40}{n\pi \sinh n\pi} (\cos n\pi - 1) \\ &= -\frac{40}{n\pi \sinh n\pi} ((-1)^n - 1) \\ &= +\frac{80}{n\pi \sinh n\pi} \qquad \text{if } n = 1, 3, 5, \ldots \end{aligned}$$

and 0 for even n. If you substitute this into (19), you obtain the series

$$u = \frac{80}{\pi} \sum \frac{1}{n \sinh n\pi} \sin \frac{n\pi x}{24} \sinh \frac{n\pi y}{24}$$

where you sum over odd n only because $A_n^* = 0$ for even n. If in this answer you write $2m - 1$ instead of n, then you automatically have the desired summation and can drop the condition "over odd n only". This is the form of the answer given on p. A29 in Appendix 2 of the book.

Sec. 11.6 Heat Equation: Solution by Fourier Integrals and Transforms

Problem Set 11.6. Page 615

3. **Solutions in integral form** are obtained by using the Fourier integral (instead of the Fourier series, whose use is restricted to periodic solutions). These solutions are of the form (6). Here, $A(p)$ and $B(p)$ are given by (8), which shows that they are determined by the initial temperature $f(x)$ in the infinitely long (practically: very long) bar or wire. The given initial temperature in Prob. 3 is

$$f(x) = 1 \text{ if } -1 < x < 1 \quad \text{and } f(x) = 0 \text{ otherwise.} \tag{I}$$

This models a situation when a short portion of the bar is heated, whereas the rest of it is kept at temperature 0.

From (8) and (I) you obtain

$$A(p) = \frac{1}{\pi}\int_{-1}^{1} \cos pv\, dv = \frac{1}{\pi p} \sin pv \Big|_{v=-1}^{1} \tag{II}$$

$$= \frac{1}{\pi p}(\sin p - \sin(-p)) = \frac{2 \sin p}{\pi p}.$$

Furthermore, from (8) and (I),

$$B(p) = \frac{1}{\pi}\int_{-1}^{1} \sin pv\, dv = 0; \tag{III}$$

this follows immediately without calculation by noting that $f(x)$ is an even function, so that $f(x) \sin px$ is odd, and you integrate from -1 to 1, so that the area under the curve from -1 to 0 is minus the area under the curve from 0 to 1. Substituting (II) and (III) into (6) in the text gives the answer shown on p. A29 in Appendix 2 of the book.

5. **Use of Sec. 10.8.** That section and its problem set contain integrals that can be used for the present purpose. The initial temperature in Prob. 5 is "wavy" with a relatively fast decreasing maximum amplitude and alternatingly positive and negative temperatures given by $f(x)$.

$\sin x$ is odd. Hence $(\sin x)/x$ is even, so that $B(p)$ in (8) is zero. For $A(p)$ you obtain from (8)

$$A(p) = \frac{1}{\pi}\int_{-1}^{1} \frac{\sin v}{v} \cos pv\, dv$$

$$= \frac{2}{\pi}\int_{0}^{1} \frac{\sin v}{v} \cos pv\, dv. \tag{IV}$$

This is precisely the integral in Prob. 2 of Problem Set 10.8, except for notation. Its value is $\pi/2$ if $0 < x < 1$. Multiplication by $2/\pi$ (the factor in (IV)) gives the values $A(p) = 1$ if $0 < x < 1$ and $A(p) = 0$ if $x > 1$.

(The value at $x = 1$ is $(\pi/4)(2/\pi) = 1/2$; this is of no interest here because we are concerned with an integral, (6).) Substitution of $A(p)$ into (6) gives the integral from 0 to 1 shown on p. A29 in Appendix 2.

Sec. 11.8 Rectangular Membrane. Use of Double Fourier Series

Problem Set 11.8. Page 626

7. **Coefficients of a double Fourier series** can be obtained following the idea in the text. For $f(x, y) = y$ in the square $0 < x < \pi$, $0 < y < \pi$ the calculations are as follows. (Here we use the formula numbers of the text.) The desired series is obtained from (18) with $a = b = \pi$ in the form

$$f(x, y) = y = \sum \left(\sum B_{mn} \sin mx \sin ny\right) \tag{18}$$
$$= \sum K_m(y) \sin mx \qquad \text{(sum over } m\text{)}$$

where the notation

$$K_m(y) = \sum B_{mn} \sin ny \qquad \text{(sum over } n\text{)} \tag{19}$$

was used. Now fix y. Then the second line of (18) is the Fourier sine series of $f(x,y) = y$ considered as a function of x (hence as a constant, but this is not essential). Thus, by (6) in Sec. 10.4 its Fourier coefficients are

$$b_m = K_m(y) = \frac{2}{\pi}\int_0^\pi y \sin mx\,dx. \tag{20}$$

You can pull out y from under the integral sign (since you integrate with respect to x) and integrate, obtaining

$$K_m(y) = \frac{2y}{m\pi}(-\cos m\pi + 1)$$
$$= \frac{2y}{m\pi}(-(-1)^m + 1)$$
$$= \frac{4y}{m\pi} \quad \text{if } m \text{ is odd and 0 for even } m$$

(because $(-1)^m = 1$ for even m). By (6) in Sec. 10.4 (with y instead of x and $L = \pi$) the coefficients of the Fourier series of the function $K_m(y)$ just obtained are

$$B_{mn} = \frac{2}{\pi}\int_0^\pi K_m(y) \sin ny\,dy$$
$$= \frac{2}{\pi}\int_0^\pi \frac{4y \sin ny}{m\pi}\,dy$$
$$= \frac{8}{m\pi^2}\int_0^\pi y \sin ny\,dy.$$

Integration by parts gives

$$B_{mn} = \frac{8}{nm\pi^2}\left(-y\cos ny\Big|_{y=0}^{\pi} + \int_0^\pi \cos ny\,dy\right).$$

The integral gives a sine, which is zero at $y = 0$ and $y = n\pi$. The integral-free part is zero at the lower limit. At the upper limit it gives

$$\frac{8}{nm\pi^2}(-\pi(-1)^n) = \frac{(-1)^{n+1}8}{nm\pi}.$$

Remember that this is the expression when m is odd, whereas for even m these coefficients are zero.

Sec. 11.9 Laplacian in Polar Coordinates

Problem Set 11.9. Page 628

3. **Laplacian in polar coordinates.** The alternative form is sometimes more practical than (4), which is obtained from it simply by performing the indicated product differentiation.

5. **Solution depending only on r.** If r is the only independent variable involved, you have to write the usual derivatives instead of partial derivatives. Hence (4), with the last term absent, now gives

$$u'' + u'/r = 0.$$

This can be solved for u' by separating variables and subsequent exponentiation, giving $u' = a/r$, where a is an arbitrary constant. Now integrate.

9. **Temperature in a circular disk.** The disk is bounded by the circle $x^2 + y^2 = r^2 = 1$. The boundary temperature means that the left half of the circle is kept at temperature 0. At $x = 0$, $y = -1$ (the lowest point of the circle) the temperature jumps from 0 to $-\pi/2$. Then it increases steadily to 0 (this temperature is reached where the circle crosses the x-axis) and on to $\pi/2$ (reached at the uppermost point $x = 0$, $y = 1$ of the circle), where it jumps down to 0.

The temperature inside the disk is obtained from (6) (see Team Project 6). For this you need the Fourier series of the boundary temperature (sketch it over the θ-axis from $-\pi$ to in π the usual fashion). This is an odd function; hence that series is a Fourier sine series. Because it has period 2π, the Fourier coefficients are obtained from (6*) in Sec. 10.4 (or from the original Euler formulas in Sec. 10.2). To evaluate the integral, use integration by parts. The result is

$$u = \frac{2}{\pi}\sin\theta + \frac{1}{2}\sin 2\theta - \frac{2}{9\pi}\sin 3\theta - \frac{1}{4}\sin 4\theta + \cdots,$$

in agreement with the answer to Prob. 15 in Problem Set 10.2. From this the temperature in the disk is obtained by multiplying each term by a power of r, namely, $\sin n\theta$ is multiplied by r^n; thus

$$u = \frac{2r}{\pi}\sin\theta + \frac{r^2}{2}\sin 2\theta - \frac{2r^3}{9\pi}\sin 3\theta - \frac{r^4}{4}\sin 4\theta + \cdots.$$

Try to sketch the isotherms by noting the following. You know that the left half of the boundary circle is at temperature 0. So is the x-axis because it corresponds to $\theta = 0$ and $\theta = \pi$, for which the sine is 0. All positive isotherms must begin at the corresponding boundary points of the circle in the first quadrant and must end at the jump at $y = 1$ on the y-axis, and they must remain in the upper half of the disk. Similarly in the lower half of the disk, for the negative isotherms between $-\pi/2$ and 0. Since the boundary temperature is symmetric with respect to the x-axis, so are the isotherms, that is, a negative isotherm is obtained by reflecting the corresponding positive isotherm in the x-axis. At the upper jump, isotherms, such as $u = 0, \pi/8, \pi/4, 3\pi/8$, must make equal angles with one another ($\pi/4$ for these four isotherms). Why?

This discussion is typical. It shows that in many cases, one can obtain substantial qualitative information without actual computation.

Sec. 11.10 Circular Membrane. Use of Fourier-Bessel Series

Problem Set 11.10. Page 634

7. **Nonzero initial velocity.** Differentiating (12) with respect to t, you obtain

$$u_t(r,t) = \sum_{m=1}^{\infty}\left(-a_m\lambda_m \sin\lambda_m t + b_m\lambda_m\cos\lambda_m t\right)J_0\left(\frac{\alpha_m r}{R}\right)\Big).$$

For $t = 0$ this must equal $g(r)$, the given inital velocity in (4). Since $\sin 0 = 0$ and $\cos 0 = 1$, you have

$$u_t(r,0) = \sum_{m=1}^{\infty} b_m\lambda_m J_0\left(\frac{\alpha_m r}{R}\right) = g(r).$$

Hence the $b_m\lambda_m$ are the coefficients of the Fourier-Bessel series of $g(r)$. Accordingly, you obtain their form from (14) with a_m replaced by $b_m\lambda_m$ on the left and $f(r)$ replaced by $g(r)$ under the integral sign. Division by $\lambda_m = ck_m = c\alpha_m/R$ (see before) gives (15) because a factor R cancels.

13. **Periodicity.** This follows from the fact that θ is an angular variable (an angular coordinate), which is determined only up to integer multiples of 2π, so that its increase or decrease by 2π, 4π, etc. does not change the point to which it belongs.

To express (20) in terms of the usual Bessel equation, set $kr = s$, thus $r = s/k$, $d/dr = k\,d/ds$, so that by the chain rule,

$$(s^2/k^2)k^2\ddot{W} + (s/k)k\dot{W} + (s^2 - n^2)W = 0, \qquad s^2\ddot{W} + s\dot{W} + (s^2 - n^2)W = 0,$$

where dots denote derivatives with respect to s.

15. **Membrane vibrations also depending on angle.** These solutions are given by (22). This formula is a consequence of the steps of separation in Probs. 11-14. To understand it completely, it is worth thinking about how this case relates to that of independence of angle. In the latter case, the Bessel equation for the radial coordinate r is (7*), which is Bessel's equation with $\nu = 0$. The latter is obtained from Bessel's equation (7) involving a constant k (not to be confused with the parameter ν; k slips into the argument, whereas ν specifies the whole function and distinguishes it from a Bessel function with another value of ν). That arbitrary k is later determined by the boundary condition that the membrane be fixed on the boundary. In this case of independence of the angle you are concerned exclusively with the Bessel function J_0. And there is no differential equation for the angle–of course not because the angle does not appear. In the case of angular dependence you do obtain Bessel functions with ν different from 0 (hence, greater than zero; Bessel's equation involves the square ν^2, so you can always assume ν to be positive or 0). And you have (19), the equation for the angle, which for reasons of periodicity leads to the integer values $\nu = n = 1, 2, 3, \ldots$. Instead of the zeros α_m of J_0 resulting from fixing the membrane along the boundary circle you now have the zeros α_{mn} of J_n resulting for the very same reason.

For $n = 0$ the solutions u_{mn} in the first line of (22) reduce to those in (11), and the solutions u^*_{mn} in the second line of (22) are no longer present when $n = 0$ because then $\sin n\theta$ is identically zero.

Sec. 11.11 Laplace's Equation in Cylindrical and Spherical Coordinates. Potential

Problem Set 11.11. Page 641

7. **Potential between two spheres.** Since the region is bounded by two concentric spheres, and since these spheres are kept at constant potentials, the potential between the spheres will be spherically symmetric, that is, the equipotential surfaces will be concentric spheres. Now a spherically symmetric solution of the three-dimensional Laplace equation is

$$u(r) = c/r + k$$

(see Prob. 6). The constants c and k can be determined from the two boundary conditions $u(2) = 220$ and $u(4) = 140$; thus,

$$u(2) = c/2 + k = 220 \tag{A}$$

and $u(4) = c/4 + k = 140$ or, multiplied by 2,

$$2u(4) = c/2 + 2k = 280. \tag{B}$$

(B) minus (A) gives $k = 60$. From this and (A) you have $c/2 = 160$, hence $c = 320$. Check the result.

13. **Special Fourier-Legendre series.** These series were introduced in Example 2 of Sec. 4.8. They are of the form

$$a_0 P_0 + a_1 P_1 + a_2 P_2 + \ldots .$$

Since x is one of our coordinates in space, you must choose another notation. Choose w or use ϕ obtained by setting $w = \cos\phi$. The Legendre polynomials $P_n(w)$ involve powers of w; thus $P_n(\cos\phi)$ involves powers of $\cos\phi$. Accordingly, you have to transform $\cos 2\phi$ into powers of $\cos\phi$. This gives

$$\Phi = 2\cos^2\phi - 1 = 2w^2 - 1.$$

This must now be expressed in terms of Legendre polynomials. From the occurring powers you see that you need P_2 and P_0. The result is

$$2w^2 - 1 = \frac{4}{3}P_2(w) - \frac{1}{3}P_0(w). \tag{C}$$

Check this by using the definitions of P_2 and P_0 in Sec. 4.3.

From (C) and (16) you finally see that the answer is

$$u(r, \phi) = -\frac{1}{3}P_0(\cos\phi) + \frac{4}{3}P_2(\cos\phi).$$

Note that in the present case the coefficient formulas (18) or (18*) were not needed because the boundary condition was so simple that the coefficients were already known to us. Note further that $P_0 = 1 = const$, but our notations $P_0(w)$ and $P_0(\cos\phi)$ are correct because a constant is a special case of a function of any variable.

21. **Transformation of the Laplacian.** Set $r = 1/s$, hence $s = 1/r$. You have to transform the partial derivatives occurring in the Laplacian in polar coordinates. By the chain rule you obtain for the first partial derivative

$$u_r = u_s s_r = u_s\left(-\frac{1}{r^2}\right) = -s^2 u_s.$$

From $u_r = u_s s_r$ you obtain for the second partial derivative

$$u_{rr} = u_{ss}s_r^2 + u_s s_{rr} = u_{ss}\left(-\frac{1}{r^2}\right)^2 + u_s\frac{2}{r^3} = s^4 u_{ss} + 2s^3 u_s.$$

Substituting these expressions into the Laplacian gives

$$u_{rr} + (1/r)u_r + (1/r^2)u_{\theta\theta} = s^4 u_{ss} + 2s^3 u_s + s(-s^2 u_s) + s^2 u_{\theta\theta}.$$

On the right side take the second and third terms together. Then you see that this is s^4 times the Laplacian in $s = 1/r$ and θ. Hence if $u(r,\theta)$ satisfies Laplace's equation, so does $v(r,\theta) = u(1/r,\theta)$, as claimed.

Sec. 11.12 Solution by Laplace Transforms

Problem Set 11.12. Page 646

5. **First-order differential equation.** The boundary conditions mean that $u(x,t)$ vanishes on the positive parts of the coordinate axes in the xt-plane. Let U be the Laplace transform of $u(x,t)$ considered as a function of t; write $U = U(x, s)$. The derivative u_t has the transform sU because $u(x,0) = 0$. The transform of t on the right is $1/s^2$. Hence you first have

$$xU_x + sU = \frac{x}{s^2}.$$

This is a first-order linear differential equation with the independent variable x. Division by x gives

$$U_x + \frac{sU}{x} = \frac{1}{s^2}.$$

Its solution is given by the integral formula (4) in Sec. 1.6. Using the notation in that section, you obtain

$$p = s/x, \qquad h = \int p\,dx = s\ln x, \qquad e^h = x^s, \qquad e^{-h} = 1/x^s.$$

Hence (4) in section 1.6, with the "constant" of integration depending on s, gives, since $1/s^2$ does not depend on x,

$$U(x, s) = \frac{1}{x^s}\left(\int \frac{x^s}{s^2}\,dx + c(s)\right) = \frac{c(s)}{x^s} + \frac{x}{s^2(s+1)}$$

(note that x^s cancels in the second term, leaving the factor x). Here you must have $c(s) = 0$ for $U(x, s)$ to be finite at $x = 0$. Then

$$U(x,s) = \frac{x}{s^2(s+1)}.$$

Now

$$\frac{1}{s^2(s+1)} = \frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}.$$

This has the inverse Laplace transform $t - 1 + e^{-t}$ and gives the solution $u(x, t) = x(t - 1 + e^{-t})$.