PART F. OPTIMIZATION, GRAPHS

CHAPTER 20. Unconstrained Optimization. Linear Programming

Sec. 20.1 Basic Concepts. Unconstrained Optimization

Problem Set 20.1. Page 993

3. **Cauchy's method of steepest descent.** The given function is

$$f(\mathbf{x}) = 2(x_1^2 + x_2^2) + x_1 x_2 - 5(x_1 + x_2). \qquad \text{(I)}$$

The given starting value is $\mathbf{x}_0 = [1 \quad -2]^T$. Proceed as in Example 1, beginning with the general formulas and using the starting value later. To simplify notations, denote the components of the gradient of f by f_1 and f_2. The gradient of f is

$$\nabla f(x) = [f_1 \quad f_2]^T = [4x_1 + x_2 - 5 \quad 4x_2 + x_1 - 5]^T.$$

In terms of components,

$$f_1 = 4x_1 + x_2 - 5, \quad f_2 = 4x_2 + x_1 - 5. \qquad \text{(II)}$$

Furthermore,

$$\mathbf{z}(t) = [z_1 \quad z_2]^T = \mathbf{x} - t\nabla f(\mathbf{x}) = [x_1 - tf_1 \quad x_2 - tf_2]^T.$$

In terms of components,

$$z_1(t) = x_1 - tf_1, \qquad z_2(t) = x_2 - tf_2. \qquad \text{(III)}$$

Now obtain $g(t) = f(z(t))$ from $f(\mathbf{x})$ in (I) by replacing x_1 with z_1 and x_2 with z_2. This gives

$$g(t) = 2(z_1^2 + z_2^2) + z_1 z_2 - 5(z_1 + z_2).$$

Calculate the derivative of $g(t)$ with respect to t, obtaining

$$g'(t) = 4(z_1 z_1' + z_2 z_2') + z_1' z_2 + z_1 z_2' - 5(z_1' + z_2').$$

From (III) you see that $z_1' = -f_1$ and $z_2' = -f_2$. Substitute this and z_1 and z_2 from (III) into $g'(t)$, obtaining

$$g'(t) = 4[(x_1 - tf_1)(-f_1) + (x_2 - tf_2)(-f_2)] + (-f_1)(x_2 - tf_2) + (x_1 - tf_1)(-f_2) - 5(-f_1 - f_2).$$

Order the terms as follows. Collect the terms containing t and denote their sum by D (suggesting "denominator" in what follows). This gives

$$tD = t[4f_1^2 + 4f_2^2 + f_1 f_2 + f_1 f_2]. \qquad \text{(IV)}$$

Denote the sum of the other terms by N (suggesting "numerator"), obtaining

$$N = -4x_1 f_1 - 4x_2 f_2 - f_1 x_2 - x_1 f_2 + 5f_1 + 5f_2. \qquad \text{(V)}$$

With these notations you have $g'(t) = tD + N$. Solving $g'(t) = 0$ for t gives

$$t = -\frac{N}{D}.$$

Step 1. For the given $\mathbf{x} = \mathbf{x}_0 = [1 \quad -2]^T$ you have $x_1 = 1, x_2 = -2$ and from (II)

$$f_1 = -3, \quad f_2 = -12, \quad tD = 684t, \quad N = -153,$$

so that

$$t = t_0 = -N/D = 0.223684211.$$

From this and (II) and (III) you obtain the next approximation $\mathbf{x}_1$ of the desired solution in the form

$$\mathbf{x}_1 = \mathbf{z}(t_0) = [1 - t_0(-3) \quad -2 - t_0(-12)]^T = [1 + 3t_0 \quad -2 + 12t_0]^T = [1.671052632 \quad 0.68421052$$

This completes the first step.

Step 2. Instead of $\mathbf{x}_0$ now use $\mathbf{x}_1$, in terms of components,

$$x_1 = 1.671052632, \quad x_2 = 0.684210526.$$

From this and (II) you obtain $f_1 = 2.368421053, f_2 = -0.592105263$. From this and (IV) it follows that $tD = 21.035318856t$, and (V) gives $N = -5.960006925$. The corresponding solution of $g'(t) = 0$ is

$$t = t_1 = 0.2833333333.$$

From this and (III) calculate

$$\mathbf{x}_2 = \mathbf{z}(t_1) = [x_1 - t_1 f_1 \quad x_2 - t_2 f_2]^T = [1.000000000 \quad 0.851973684]^T.$$

Step 3. Using (II), (IV), (V), and (III), in this order, with $\mathbf{x} = \mathbf{x}_2$, calculate

$$f_1 = -0.148016316, \quad f_2 = -0.592105263, \quad tD = 1.665296316t, \quad N = -0.372500433,$$

hence $t = t_2 = -N/D = 0.223684211$ and

$$\mathbf{x}_3 = \mathbf{z}(t_2) = [1.03311115 \quad 0.984418283]^T.$$

The further steps give

$$\mathbf{x}_4 = [1.000000000 \quad 0.992696070]^T$$
$$\mathbf{x}_5 = [1.001633774 \quad 0.999231165]^T$$
$$\mathbf{x}_6 = [0.999999999 \quad 0.9996396088]^T$$

and so on. The exact solution is $\mathbf{x} = [1 \quad 1]^T$. This can be seen by substituting $x_1 = y_1 + 1$, $x_2 = y_2 + 1$ into $f(\mathbf{x})$, which transforms it into

$$2(y_1^2 + y_2^2) + y_1 y_2 - 5. \qquad \text{(VI)}$$

Except for the -5 this is a quadratic form with the symmetric coefficient matrix

$$\begin{bmatrix} 2 & 1/2 \\ 1/2 & 2 \end{bmatrix}.$$

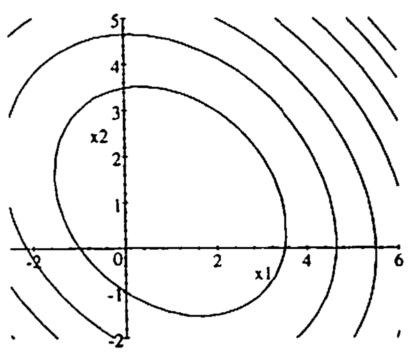
The eigenvalues of this matrix are 2.5 and 1.5, with eigenvectors $[1 \quad 1]^T$ and $[1 \quad -1]^T$, respectively (derive!). Geometrically, this means that the principal axes of the ellipses in the figure make 45 degree angles with the coordinate axes. Hence if you apply a 45-degree rotation (see p. 320 of the book) to (VI), given by

$$y_1 = (X_1 - X_2)/\sqrt{2}$$
$$y_2 = (X_1 + X_2)/\sqrt{2}$$

(note that cos 45 degrees = sin 45 degrees = $1/\sqrt{2}$), you obtain the function

$$h(\mathbf{X}) = 2(X_1^2 + X_2^2) + \frac{1}{2}(X_1^2 - X_2^2) - 5 = \frac{5}{2}X_1^2 + \frac{3}{2}X_2^2 - 5$$

and the curves $h(\mathbf{X}) = c = const$ (with $c > 5$) are ellipses whose principal axes lie in the X_1X_2-coordinate axes.



Section 20.1. Problem 3. Curves $f(x_1, x_2) = const$ (ellipses)

Sec. 20.2 Linear Programming

Problem Set 20.2. Page 997

1. **Position of maximum.** Consider what happens as you move the straight line $z = c = const$, beginning with its position when $c = 0$ (which is shown in Fig. 442) and increasing c continuously.

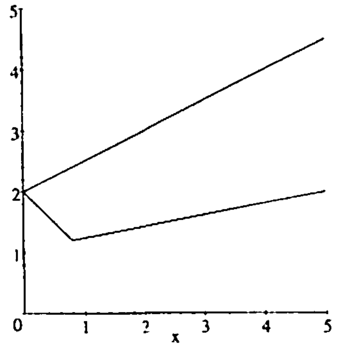
5. **Region, constraints.** The given inequalities are

$$-0.5x_1 + x_2 \leqq 2 \qquad \text{(A)}$$

$$x_1 + x_2 \geqq 2 \qquad \text{(B)}$$

$$-x_1 + 5x_2 \geqq 5. \qquad \text{(C)}$$

Consider (A). The equation $-0.5x_1 + x_2 = 2$ gives a straight line. Putting $x_2 = 0$ gives $x_1 = -4$ as the intersection point with the x_1-axis. Putting $x_1 = 0$ gives $x_2 = 2$ as the intersection point with the x_2-axis. Putting $x_1 = x_2 = 0$ in the inequality gives $0 + 0 \leqq 2$, which is true. Hence the region to be determined extends from the line downward. Similarly for the line $x_1 + x_2 = 2$ in (B), which intersects the axes at $x_1 = 2$ and $x_2 = 2$. Since for $x_1 + x_2 = 0$ the inequality $0 + 0 \geqq 2$ is false, the region to be obtained extends from the line upward (away from the origin, leaving the origin outside). Similarly for (C), which gives the line $-x_1 + 5x_2 = 5$, intersecting the axes at $x_1 = -5$ (put $x_2 = 0$) and $x_2 = 1$ (put $x_1 = 0$) and the region extending upward. Hence the region is bounded by a portion of (A) (above), (B) (on the left), and (C) (below), and is unbounded (extends to infinity) on the right. Note that it lies entirely in the first quadrant of the x_1x_2-plane, so that the conditions $x_1 \geqq 0$, $x_2 \geqq 0$ (often imposed by the kind of application, for instance, number of items produced, time or quantity of raw material needed, etc.) are automatically satisfied.



Section 20.2. Problem 5. Region determined by the three inequalities

17. **Maximum profit.** The profit per lamp L_1 is 15 and that per lamp L_2 is 10. Hence if you produce x_1 lamps L_1 and x_2 lamps L_2, the total profit is

$$f(x_1, x_2) = 15x_1 + 10x_2.$$

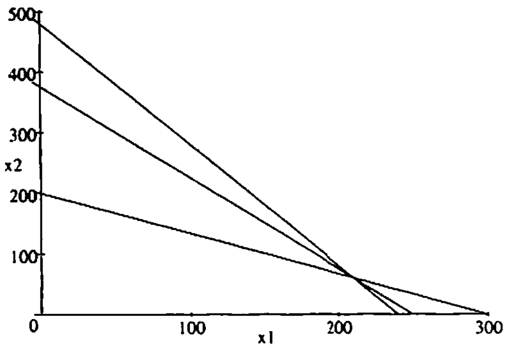
You wish to determine x_1 and x_2 such that the profit $f(x_1, x_2)$ is as large as possible. Limitations arise due to the available work force. For the sake of simplicity the problem is talking about two workers W_1 and W_2, but it is rather obvious how the corresponding constraints could be modified if teams of workers were involved or if additional constraints arose from raw material. The assumption is that for this kind of high-quality work, W_1 is available 100 hours per month and that he or she assembles 3 lamps L_1 per hour or 2 lamps L_2 per hour. Hence W_1 needs 1/3 hour for assembling a lamp L_2 and 1/2 hour for assembling a lamp L_2. For a production of x_1 lamps L_1 and x_2 lamps L_2 this gives the restriction (constraint)

$$\frac{1}{3}x_1 + \frac{1}{2}x_2 \leqq 100. \qquad \text{(A)}$$

(As in other applications, it is essential to measure time or other physical quantities by the same units throughout a calculation.) (A) with equality sign gives a straight line which intersects the x_1-axis at 300 (put $x_2 = 0$) and the x_2-axis at 200 (put $x_1 = 0$); see the figure. If you put both $x_1 = 0$ and $x_2 = 0$, the inequality becomes $0 + 0 \leqq 100$, which is true. This means that the region to be determined extends from that line downward. Worker W_2 paints the lamps, namely, 3 lamps L_1 per hour and 6 lamps L_2 per hour. Hence painting a lamp L_1 takes 1/3 hour, and painting a lamp L_2 takes 1/6 hour. W_2 is available 80 hours per month. Hence if x_1 lamps L_1 and x_2 lamps L_2 are produced per month, his or her availability gives the constraint

$$\frac{1}{3}x_1 + \frac{1}{6}x_2 \leqq 80. \tag{B}$$

(B) with the equality sign gives a straight line which intersects the x_1-axis at 240 (put $x_2 = 0$) and the x_2-axis at 480 (put $x_1 = 0$); see the figure. If you put $x_1 = 0$ and $x_2 = 0$ the inequality (B) becomes $0 + 0 \leqq 80$, which is true. Hence the region to be determined extends from that line downward. And the region must lie in the first quadrant because you must have $x_1 \geqq 0$ and $x_2 \geqq 0$. The intersection of those two lines is at (210, 60). This gives the maximum profit $f = 210 \cdot 15 + 60 \cdot 10 = 3750$. The straight line $f = 3750$ (the middlemost of the three lines in the figure) is given by $x_2 = 375 - 1.5x_1$. And by varying c in the line $f = const$, that is, in $x_2 = c - 1.5x_1$, which corresponds to moving the line up and down, it becomes obvious that (210, 60) does give the maximum profit.



Section 20.1. Problem 17. Constraints (A) (lower line) and (B)

Sec. 20.3 Simplex Method

Problem Set 20.3. Page 1001

1. **Maximization by the simplex method.** The objective function to be maximized is

$$z = f(x_1, x_2) = 30x_1 + 20x_2. \tag{A}$$

The constraints are

$$\begin{aligned} -x_1 + x_2 &\leqq 5, \\ 2x_1 + x_2 &\leqq 10. \end{aligned} \tag{B}$$

Begin by writing this in normal form (see (1) and (2) in Sec. 20.3). The inequalities are converted to equations by introducing slack variables, one slack variable per inequality. In (A) and (B) you have the variables x_1 and x_2. Hence denote the slack variables by x_3 (for the first inequality in (B)) and x_4. This gives the normal form (with the objective function written as an equation)

$$\begin{aligned} z - 30x_1 - 20x_2 \qquad\qquad &= 0 \\ -x_1 + x_2 + x_3 \qquad &= 5 \\ 2x_1 + x_2 \qquad + x_4 &= 10. \end{aligned} \tag{C}$$

This is a linear system of equations. The corresponding augmented matrix (a concept you should know!–see Sec. 6.3) is called the *initial simplex table* and is denoted by $\mathbf{T}_0$. Obviously it is

$$\mathbf{T}_0 = \begin{bmatrix} 1 & -30 & -20 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 5 \\ 0 & 2 & 1 & 0 & 1 & 10 \end{bmatrix}. \qquad \text{(D)}$$

Take a look at (4) on p. 999, which has an extra line on top showing z, the variables, and b (denoting the terms on the right side in (C)). Perhaps you may add such a line in (D) and also draw the dashed lines, which separate the first row of $\mathbf{T}_0$ from the others as well as the columns corresponding to z, to the given variables, to the slack variables, and to the right sides. Perform Operation O_1. The first column with a negative entry in Row 1 is Column 2, the entry being −30. This is the column of the first pivot. Perform Operation O_2. Divide the right sides by the corresponding entries of the column just selected. This gives $5/(-1) = -5$ and $10/2 = 5$. The smallest positive of these two quotients is 5. It corresponds to Row 3. Hence select Row 3 as the row of the pivot. (You cannot choose Row 2 because to eliminate −30, you would have to take Row 1 −30 Row 2 as the new Row 1, which would give $z = 0 - 30 \cdot 5 = -150$ as the value of z, which is impossible since $x_1 \geqq 0$ and $x_2 \geqq 0$ by assumption.) Perform Operation O_3, that is, create zeros in Column 2 by the row operations

$$\text{Row}\,1 + 15\,\text{Row}\,3$$
$$\text{Row}\,2 + 0.5\,\text{Row}\,3.$$

This gives the new simplex table (with Row 3 as before)

$$\mathbf{T}_1 = \begin{bmatrix} 1 & 0 & -5 & 0 & 15 & 150 \\ 0 & 0 & 3/2 & 1 & 1/2 & 10 \\ 0 & 2 & 1 & 0 & 1 & 10 \end{bmatrix}.$$

This was the first step. Now comes the second step, which is necessary because of the negative entry −5 in Row 1 of $\mathbf{T}_1$. Hence the column of the pivot is Column 3 of $\mathbf{T}_1$. Calculate 10/(3/2) and 10/1. The first of these is the smallest. Hence the pivot row is Row 2. To create zeros in Column 3 you have to do the row operations Row 1 +5/(3/2) Row 2 Row 3 −(2/3)Row 2 , leaving Row 2 unchanged. This gives the simplex table

$$\mathbf{T}_2 = \begin{bmatrix} 1 & 0 & 0 & 10/3 & 50/3 & 550/3 \\ 0 & 0 & 3/2 & 1 & 1/2 & 10 \\ 0 & 2 & 0 & -2/3 & 2/3 & 10/3 \end{bmatrix}.$$

Since no more negative entries appear in Row 1, you are finished. From Row 1 you see that $f_{max} = 550/3$. Row 3 gives the corresponding x_1-value $(10/3)/2 = 5/3$. Row 2 gives the corresponding x_2-value $10/(3/2) = 20/3$. Hence the maximum value of $z = f(x_1, x_2)$ is reached at the point $P : (5/3, 20/3)$ in the x_1x_2-plane.

Draw a sketch of the region determined by the constraints and convince yourself that the maximum value is taken at one of the vertices of the quadrangle determined by the constraints, with vertices at (0, 0), (5, 0), (0, 5), and P.

7. **Minimization by the simplex method.** The given problem in normal form (with $z = f(x_1, x_2)$ written as an equation) is

$$\begin{aligned} z - 5x_1 + 20x_2 \qquad\quad &= 0 \\ -2x_1 + 10x_2 + x_3 \qquad &= 5 \\ 2x_1 + 5x_2 + \qquad x_4 &= 10. \end{aligned}$$

From this you see that the initial simplex table is

$$T_0 = \begin{bmatrix} 1 & -5 & 20 & 0 & 0 & 0 \\ 0 & -2 & 10 & 1 & 0 & 5 \\ 0 & 2 & 5 & 0 & 1 & 10 \end{bmatrix}.$$

Since you minimize (instead of maximizing), consider the columns whose first entry is positive (instead of negative). There is only one such column, namely, Column 3. The quotients are $5/10 = 1/2$ (from Row 2) and $10/5 = 2$ (from Row 3). The smallest of these is 1/2. Hence you have to choose Row 2 as pivot row and 10 as the pivot. Create zeros by the row operations Row 1 – 2 Row 2 (this gives the new Row 1) and Row 3 – (1/2) Row 2 (this gives the new Row 3), leaving Row 2 unchanged. The result is

$$\mathbf{T}_1 = \begin{bmatrix} 1 & -1 & 0 & -2 & 0 & -10 \\ 0 & -2 & 10 & 1 & 0 & 5 \\ 0 & 3 & 0 & -1/2 & 1 & 15/2 \end{bmatrix}.$$

Since in the first row there are no further positive entries, you are done. From Row 1 of $\mathbf{T}_1$ you see that $f_{\text{min}} = -10$. From Row 2 (and columns 3 and 6) you see that $x_2 = 5/10 = 1/2$. From Row 3 (and columns 5 and 6) you see that $x_4 = (15/2)/1 = 15/2$. Now x_4 appears in the second constraint, written as equation, that is,

$$2x_1 + 5x_2 + x_4 = 10.$$

Inserting $x_2 = 1/2$ and $x_4 = 15/2$ gives $2x_1 + 10 = 10$, hence $x_1 = 0$. Hence the minimum -10 of $z = f(x_1, x_2)$ occurs at the point (0, 1/2). Since this problem involves only two variables (not counting the slack variables), as a control and to better understand the problem, you can plot the constraints, which determine a quadrangle, and calculate the values of f at the four vertices, obtaining 0 at (0, 0), 25 (at (5, 0), −7.5 at (2.5, 1), and −10 at (0, 0.5). This confirms your result.

Sec. 20.4 Simplex Method: Degeneracy, Difficulties in Starting

Problem Set 20.4. Page 1007

3. **Degeneracy.** The given problem is

$$z = x_1 + x_2$$
$$2x_1 + 3x_2 \leqq 130$$
$$3x_1 + 8x_2 \leqq 300$$
$$4x_1 + 3x_2 \leqq 140.$$

Its normal form (with $z = f(x_1, x_2)$ written as an equation) is

$$\begin{aligned} z - x_1 - x_2 &= 0 \\ 2x_1 + 3x_2 + x_3 &= 130 \\ 3x_1 + 8x_2 + x_4 &= 300 \\ 4x_1 + 2x_2 + x_5 &= 140. \end{aligned}$$

From this you see that the initial simplex table is

$$T_0 = \begin{bmatrix} 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 1 & 0 & 0 & 130 \\ 0 & 3 & 8 & 0 & 1 & 0 & 300 \\ 0 & 4 & 2 & 0 & 0 & 1 & 140 \end{bmatrix}.$$

The first pivot must be in Column 2 because of the entry −1 in this column. Determine the row of the first pivot by calculating

$$130/2 = 65 \qquad \text{(from Row 2)}$$
$$300/3 = 100 \qquad \text{(from Row 3)}$$
$$140/4 = 35 \qquad \text{(from Row 4).}$$

Since 35 is smallest, Row 4 is the pivot row and 4 the pivot. With this the next simplex table becomes

$$\mathbf{T}_1 = \begin{bmatrix} 1 & 0 & -0.5 & 0 & 0 & 0..25 & 35 \\ 0 & 0 & 2 & 1 & 0 & -0.5 & 60 \\ 0 & 0 & 6.5 & 0 & 1 & -0.75 & 195 \\ 0 & 4 & 2 & 0 & 0 & 1 & 140 \end{bmatrix} \begin{matrix} \text{Row 1} + 0.25\,\text{Row 4} \\ \text{Row 2} - 0.5\,\text{Row 4} \\ \text{Row 3} - 0.75\,\text{Row 4} \\ \text{Row 4} \end{matrix}.$$

You have reached a point at which $z = 35$. To find the point, calculate

$$x_1 = 140/4 = 35 \qquad \text{(from Row 4 and Column 2)}$$
$$x_3 = 60/1 = 60 \qquad \text{(from Row 2 and Column 4).}$$

From this and the first constraint you obtain

$$2x_1 + 3x_2 + x_3 = 70 + 3x_2 + 60 = 130, \qquad \text{hence} \quad x_2 = 0.$$

(More simply: x_1, x_3, x_4 are basic. x_2, x_5 are nonbasic. Equating the latter to zero gives $x_2 = 0, x_5 = 0$.) Thus $z = 35$ at the point (35, 0) on the x_1-axis.

Column 3 of $\mathbf{T}_1$ contains the negative entry –0.5. Hence this column is the column of the next pivot. To obtain the row of the pivot, calculate

$$60/2 = 30 \qquad \text{(from Row 2 and Column 3)}$$
$$195/6.5 = 30 \qquad \text{(from Row 3 and Column 3)}$$
$$140/2 = 70 \qquad \text{(from Row 4 and Column 3).}$$

Hence you could take 2 or 6.5. Take the first of the two, so that Row 2 is the pivot row. With this calculate the next simplex table

$$\mathbf{T}_2 = \begin{bmatrix} 1 & 0 & 0 & 0.25 & 0 & 0.125 & 50 \\ 0 & 0 & 2 & 1 & 0 & -0.5 & 60 \\ 0 & 0 & 0 & -3.25 & 1 & 0.875 & 0 \\ 0 & 4 & 0 & -1 & 0 & 1.5 & 80 \end{bmatrix} \begin{matrix} \text{Row 1} + 0.25\,\text{Row 2} \\ \text{Row 2} \\ \text{Row 3} - 3.25\,\text{Row 2} \\ \text{Row 4} - \text{Row 2} \end{matrix}.$$

There are no more negative entries in Row 1. Hence you have reached the maximum $z_{max} = 50$. You see that x_1, x_2, x_4 are basic, and x_3, x_5 are nonbasic. z_{max} occurs at (20, 30) because $x_1 = 80/4 = 20$ (from Row 4 and Column 2) and $x_2 = 60/2 = 30$ (from Row 2 and Column 3). The point (20, 30) corresponds to a degenerate solution because $x_4 = 0/1 = 0$ from Row 3 and Column 5, in addition to $x_3 = 0$ and $x_5 = 0$. Geometrically, this means that the straight line $3x_1 + 8x_2 = 300$ resulting from the second constraint, also passes through (20, 30) because $3 \cdot 20 + 8 \cdot 30 = 300$. Now in Example 1 of Sec. 20.4 we reached a degenerate solution before we reached the maximum (the optimal solution), and for this reason we had to do an additional step (Step 2). In contrast, in the present problem you reached the maximum when you reached a degenerate solution. Hence no additional work is necessary.