CHAPTER 7. Linear Algebra: Matrix Eigenvalue Problems

Sec. 7.1 Eigenvalues, Eigenvectors

Problem Set 7.1. Page 375

1. **Eigenvalues and eigenvectors.** For a diagonal matrix the eigenvalues are the main diagonal entries because the characteristic equation is

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} a_{11} - \lambda & 0 \\ 0 & a_{22} - \lambda \end{vmatrix} = (a_{11} - \lambda)(a_{22} - \lambda) = 0.$$

For the given matrix you obtain from this $\lambda_1 = 4, \lambda_2 = -6$. Now determine an eigenvector of **A** corresponding to $\lambda_1 = 4$. In components, $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{x} = \mathbf{0}$ is

$$\begin{aligned} (a_{11} - \lambda_1)x_1 + \quad a_{12}x_2 &= (4-4)x_1 + 0x_2 = 0 \\ a_{21}x_1 \quad + (a_{22} - \lambda_1)x_2 &= 0x_1 + (-6-4)x_2 = 0. \end{aligned}$$

The first equation gives no condition. The second gives $x_2 = 0$. Hence an eigenvector of **A** corresponding to $\lambda_1 = 4$ is $[x_1 \quad 0]^T$. Since an eigenvector is determined only up to a nonzero constant, you can simply take $[1 \quad 0]^T$ as an eigenvector. For $\lambda_2 = -6$ the procedure is similar and leads to $[0 \quad 1]^T$.

13. **Eigenvalues and eigenvectors.** Ordinarily one would expect that a 3 × 3 matrix has 3 linearly independent eigenvectors. For symmetric, skew-symmetric and many other matrices this is true. A simple example is the 3 × 3 unit matrix, which has but one eigenvalue, 1, but every (nonzero) vector is an eigenvector, so that you can choose, for instance, $[1 \quad 0 \quad 0]^T$, $[0 \quad 1 \quad 0]^T$, $[0 \quad 0 \quad 1]^T$. The given matrix has the characteristic equation

$$\begin{vmatrix} 2-\lambda & 0 & -1 \\ 0 & 1/2-\lambda & 0 \\ 1 & 0 & 4-\lambda \end{vmatrix} = (2-\lambda)(\tfrac{1}{2}-\lambda)(4-\lambda) - (-(\tfrac{1}{2}-\lambda))$$

$$= -\lambda^3 + 6.5\lambda^2 - 12\lambda + 4.5 = 0.$$

The solutions are 1/2 and 3. In product form, the characteristic equation is

$$-(\lambda - 0.5)(\lambda - 3)^2 = 0;$$

hence $\lambda = 3$ has algebraic multiplicity 2. Now determine an eigenvector for $\lambda = 0.5$ from

$$(2-0.5)x_1 - x_3 = 0, \qquad 0 = 0, \qquad x_1 + (4-0.5)x_3 = 0.$$

This gives $x_1 = 0, x_3 = 0, x_2$ arbitrary. Hence you can take $[0 \quad 1 \quad 0]^T$. Similarly for $\lambda = 3$

$$(2-3)x_1 - x_3 = 0, \qquad (0.5-3)x_2 = 0, \qquad x_1 + (4-3)x_3 = 0.$$

Hence $x_1 = -x_3, x_2 = 0$, and you can take as an eigenvector $[-1 \quad 0 \quad 1]^T$, but you cannot obtain another eigenvector such that the three eigenvectors are linearly independent.

19. **Orthogonal projection.** This matrix has no inverse. This is geometrically obvious because all the points (x, y_0) on the horizontal line $y = y_0$ are projected onto the same point $(0, y_0)$ on the y-axis.

Sec. 7.2 Some Applications of Eigenvalue Problems

Problem Set 7.2. Page 379

3. **Elastic membrane.** Problems 1-6 amount to the determination of the eigenvalues (giving the extension or contraction factors) and eigenvectors (giving the principal directions) and a sketch of the latter. The

characteristic equation of the given matrix is

$$\begin{vmatrix} 3.0-\lambda & 1.5 \\ 1.5 & 3.0-\lambda \end{vmatrix} = (3.0-\lambda)^2 - 1.5^2 = (\lambda - 1.5)(\lambda - 4.5) = 0.$$

You see that $\lambda = 1.5$ is an eigenvalue. A corresponding eigenvector is obtained from one of the two equations

$$(3.0-1.5)x_1 + 1.5x_2 = 0, \quad 1.5x_1 + (3.0-1.5)x_2 = 0$$

which both give $x_1 = -x_2$, so that you can take $[1 \quad -1]^T$, the vector from the origin (0, 0) to the point $(1, -1)$ in the fourth quadrant, making a 45 degree angle with the x-axis. In this direction the membrane is stretched by a factor 1.5. Similarly, the other eigenvalue is $\lambda = 4.5$, and an eigenvector is obtained from

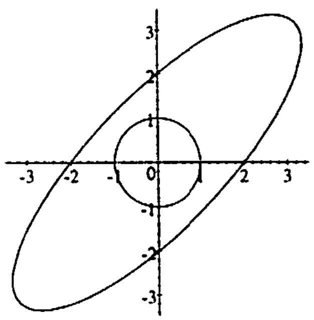
$$(3.5-4.5)x_1 + 1.5x_2 = 0, \quad \text{thus} \quad x_1 = x_2,$$

or from the other equation, which gives the same result. Hence you can take $[1 \quad 1]^T$ as an eigenvector, which you can graph as an arrow from the origin to the point (1, 1) in the first quadrant. In this direction the membrane is stretched by a factor 4.5. The figure shows a circle of radius 1 and its image under stretching, which is an ellipse. A formula of the latter can be obtained by first stretching, leading from $x_1^2 + x_2^2 = 1$ (circle) to $x_1^2/4.5^2 + x_2^2/1.5^2 = 1$ (an ellipse whose axes coincide with the x_1 and x_2 axes) and then applying a 45 degree rotation (rotation through an angle $\alpha = \pi/4$) given by

$$u = x_1 \cos\alpha - x_2 \sin\alpha = (x_1 - x_2)/\sqrt{2},$$

$$v = x_1 \cos\alpha + x_2 \sin\alpha = (x_1 + x_2)/\sqrt{2}.$$

This problem is very similar to Example 1 on p. 376.



Section 7.2. Problem 3. Circular elastic membrane stretched to an ellipse

15. **Open Leontief input-output model.** For reasons explained in the enunciation of the problem you have to solve $\mathbf{x} - \mathbf{Ax} = \mathbf{y}$ for $\mathbf{x}$, where $\mathbf{A}$ and $\mathbf{y}$ are given. With the given data you thus have to solve

$$\mathbf{x} - \mathbf{Ax} = (\mathbf{I} - \mathbf{A})\mathbf{x} = \begin{bmatrix} 1-0.1 & -0.4 & -0.2 \\ -0.5 & 1 & -0.1 \\ -0.1 & -0.4 & 1-0.4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{y} = \begin{bmatrix} 0.1 \\ 0.3 \\ 0.1 \end{bmatrix}.$$

For this you can apply the Gauss elimination to the augmented matrix of the system

$$\begin{bmatrix} 0.9 & -0.4 & -0.2 & 0.1 \\ -0.5 & 1.0 & -0.1 & 0.3 \\ -0.1 & -0.4 & 0.6 & 0.1 \end{bmatrix}.$$

If you use 6 decimals in your calculation, you will obtain the solution

$$x_1 = 0.55, \quad x_2 = 0.64375, \quad x_3 = 0.6875.$$

Sec. 7.3 Symmetric, Skew-Symmetric, and Orthogonal Matrices

Example 1. For a skew-symmetric matrix, $a_{kj} = -a_{jk}$. Hence for the main diagonal entries a_{jj} this gives $a_{jj} = -a_{jj} = 0$.

Problem Set 7.3. Page 384

3. **A common mistake.** This matrix is *NOT* skew-symmetric because a skew-symmetric matrix (which by definition is *real*) must have all diagonal entries zero. Hence you cannot expect its spectrum to lie on the y-axis. You obtain the eigenvalues from the characteristic equation

$$\begin{vmatrix} 1-\lambda & 4 \\ -4 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 16 = \lambda^2 - 2\lambda + 17 = 0.$$

By the usual formula for the solutions of a quadratic equation you obtain

$$\lambda_1 = 1 + \sqrt{1-17} = 1 + 4i, \quad \lambda_2 = 1 - 4i \quad (i = \sqrt{-1}).$$

You see that a real matrix may very well have complex eigenvalues. However, note that if λ is a complex eigenvalue of such a matrix, so is the complex conjugate number; $1 + 4i$ and $1 - 4i$ are complex conjugates. It is interesting that $\mathbf{A} = \mathbf{B} + \mathbf{I}$, where

$$\mathbf{B} = \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}$$

is skew-symmetric. The characteristic equation of $\mathbf{B}$ is $\lambda^2 + 16 = 0$. The roots (eigenvalues of $\mathbf{B}$) are $-4i$ and $4i$; they are pure imaginary. Hence $\mathbf{A} = \mathbf{B} + \mathbf{I}$ must have the eigenvalues $1 + 4i$ and $1 - 4i$, according to the spectral shift explained in Project 16d of Sec. 7.2.

11. **Inverse of a skew-symmetric matrix.** Let $\mathbf{A}$ be skew-symmetric, that is,

$$\mathbf{A}^T = -\mathbf{A}. \tag{1}$$

Let $\mathbf{A}$ be nonsingular. Let $\mathbf{B}$ be its inverse. Then

$$\mathbf{AB} = \mathbf{I}. \tag{2}$$

Transposition of (2) and the use of the skew symmetry (1) of $\mathbf{A}$ give

$$\mathbf{I} = \mathbf{I}^T = (\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T = \mathbf{B}^T(-\mathbf{A}) = -\mathbf{B}^T\mathbf{A}. \tag{3}$$

Now multiply (3) by $\mathbf{B}$ from the right and use (2), obtaining

$$\mathbf{B} = -\mathbf{B}^T\mathbf{AB} = -\mathbf{B}^T.$$

This proves that $\mathbf{B} = \mathbf{A}^{-1}$ is skew-symmetric.

Sec. 7.4 Complex Matrices: Hermitian, Skew-Hermitian, Unitary

Example 1. In $\mathbf{A}$ the diagonal entries are real, hence equal to their conjugates. $a_{21} = \bar{1} + 3i$ is the complex conjugate of $a_{12} = 1 - 3i$, as it should be for a Hermitian matrix. In $\mathbf{B}$ you have $\bar{b}_{11} = -3i = -b_{11}$, $\bar{b}_{22} = i = -b_{22}$, $\bar{b}_{21} = -2 - i = -b_{12}$. Hence $\mathbf{B}$ is skew-Hermitian. The complex conjugate transpose of $\mathbf{C}$ is

$$\begin{bmatrix} -i/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -i/2 \end{bmatrix}.$$

Multiplying this by **C**, you obtain the unit matrix. This verifies the defining relation of a unitary matrix.

Problem Set 7.4. Page 390

3. Complex matrix. The determination of eigenvalues and eigenvectors is the same in principle as for a real matrix. The matrix

$$\mathbf{B} = \begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix}$$

is skew-Hermitian, as has just been shown. The characteristic equation is

$$\begin{vmatrix} 3i-\lambda & 2+i \\ -2+i & -i-\lambda \end{vmatrix} = (3i-\lambda)(-i-\lambda)-(2+i)(-2+i)$$

$$= \lambda^2 + (-3i+i)\lambda + 3 - (-4+2i-2i-1)$$

$$= \lambda^2 - 2i\lambda + 8 = 0.$$

The roots (eigenvalues of **B**) are obtained by the usual formula for solving a quadratic equation

$$\lambda_1 = i + \sqrt{-1-8} = i + 3i = 4i, \qquad \lambda_2 = i - 3i = -2i,$$

as given on p. 387 of the book. Observe that the eigenvalues need no longer be complex conjugates because the matrix is no longer real. To determine an eigenvector corresponding to $\lambda_1 = 4i$, substitute $\lambda = \lambda_1$ into the two equations, obtaining

$$(3i-4i)x_1 + (2+i)x_2 = 0, \qquad (-2+i)x_1 + (-i-4i)x_2 = 0.$$

Simplification gives

$$-ix_1 + (2+i)x_2 = 0,$$

$$(-2+i)x_1 - 5ix_2 = 0.$$

The first equation suggests choosing $x_1 = 2+i$, $x_2 = i$. Check your result as follows. For that choice the second equation gives

$$(-2+i)(2+i) - 5ii = -4 - 2i + 2i - 1 + 5 = 0,$$

as expected. An eigenvector corresponding to $\lambda_2 = -2i$ is obtained in the same way from

$$(3i+2i)x_1 + (2+i)x_2 = 0, \quad \text{that is,} \quad 5ix_1 + (2+i)x_2 = 0.$$

You see that you can choose $x_1 = 2+i$, $x_2 = -5i$, as indicated in the solution on p. A19 in Appendix 2.

11. Decomposition. Let $\mathbf{A} = [a_{jk}]$ be arbitrary complex. Then $\bar{\mathbf{A}}^T = [\bar{a}_{kj}]$. The sum multiplied by 1/2 is

$$\mathbf{H} = \frac{1}{2}(\mathbf{A} + \bar{\mathbf{A}}^T) = \frac{1}{2}[a_{jk} + \bar{a}_{kj}].$$

Its conjugate transpose is

$$\bar{\mathbf{H}}^T = \frac{1}{2}[\bar{a}_{kj} + a_{jk}] = \mathbf{H}.$$

Hence **H** is Hermitian. Similarly, the difference multiplied by 1/2 is

$$\mathbf{S} = \frac{1}{2}(\mathbf{A} - \bar{\mathbf{A}}^T) = \frac{1}{2}[a_{jk} - \bar{a}_{kj}].$$

Its conjugate transpose is

$$\bar{\mathbf{S}}^T = \frac{1}{2}[\bar{a}_{kj} - a_{jk}] = -\mathbf{S}.$$

Hence **S** is skew-Hermitian. The sum **H** + **S** equals the given matrix **A**. This completes the derivation of this representation.

Sec. 7.5 Similarity of Matrices. Basis of Eigenvectors. Diagonalization

Problem Set 7.5. Page 397

1. **Similar matrices.** The solutions of Probs. 1-6 are obtained by calculating the inverse and performing straightforward matrix multiplication. The importance of similarity transformations, for instance, in designing numerical methods for eigenvalue problems, justifies these problems. Given

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{P} = \begin{bmatrix} 1 & 3 \\ 3 & 6 \end{bmatrix}.$$

First calculate the inverse of **P**, which is best done by (4*) in Sec. 6.7. Then calculate

$$\hat{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} -2 & 1 \\ 1 & -1/3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 3 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 1 \\ 1 & -1/3 \end{bmatrix}\begin{bmatrix} 7 & 15 \\ 14 & 30 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 7/3 & 5 \end{bmatrix}.$$

The eigenvalues of both **A** and $\hat{\mathbf{A}}$ are 0 and 5. Eigenvectors **y** of $\hat{\mathbf{A}}$ are $[15 \quad -7]^T$ and $[0 \quad 1]^T$, respectively. Multiplying these by **P** yields the vectors **x** given in the answer on p. A19 of Appendix 2.

15. **Diagonalization** is done by (5). For this you need the matrix **X** whose columns are eigenvectors of **A**. The characteristic equation of **A** gives the eigenvalues 15, −15, and 0. Eigenvectors are $[1 \quad 1 \quad 0]^T$, $[0 \quad 1 \quad 1]^T$, and $[2 \quad 0 \quad 1]^T$, respectively. Using these, form **X** and calculate its inverse. This gives

$$\mathbf{X} = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad \mathbf{X}^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ -1 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix}.$$

Now obtain the diagonal matrix **D** from (5) with main diagonal 15, −15, 0. This differs from the answer on p. A20 in Appendix 2 because the columns of **X** were chosen in a different order, which corresponds to the order of the eigenvalues on the main diagonal.

17. **Principal axes transformation.** The symmetric coefficient matrix of the given form is

$$\mathbf{A} = \begin{bmatrix} 7 & 3 \\ 3 & 7 \end{bmatrix}.$$

The eigenvalues 4 and 10 of **A** are obtained from the characteristic equation

$$(7-\lambda)(7-\lambda) - 9 = \lambda^2 - 14\lambda + 40 = (\lambda - 4)(\lambda - 10) = 0.$$

Eigenvectors are $[1 \quad -1]^T$ and $[1 \quad 1]^T$, respectively. From (10) you thus obtain the principal axes form

$$Q = 4y_1^2 + 10y_2^2 = 200.$$

This is an ellipse. The orthonormal matrix **X** in (9) is

$$\mathbf{X} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

It gives the transformation $\mathbf{x} = \mathbf{X}\mathbf{y}$ shown in the answer on p. A20 in Appendix 2. Note that the eigenvectors (the columns of **X**) are determined only up to a minus sign; hence another **X** is obtained if you take $[-1 \quad 1]^T$ instead of $[1 \quad -1]^T$, giving another correct answer.