CHAPTER 38

TORSIONAL VIBRATION IN RECIPROCATING AND ROTATING MACHINES

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INTRODUCTION

Torsional vibration is an oscillatory angular motion causing twisting in the shaft of a system; the oscillatory motion is superimposed on the steady rotational motion of a rotating/reciprocating machine. Even though the vibration cannot be detected without special measuring equipment, its amplitude can be destructive. For example, gear sets that alter speeds of power transmission systems transmit the vibration to the casing. Similarly, slider crank mechanisms in engines and compressors convert torques to radial forces that are discernable to human perception but are not measurable because of the insensitivity of test equipment and background noise. If gearboxes or reciprocating machines are part of a drive train, excess noise and vibration can indicate trouble. Standards and measurement methods dealing with acceptable magnitudes of radial vibration are provided in Chap. 19.

Motion is rarely a concern with torsional vibration unless it affects the function of a system. It is stresses that affect the structural integrity and life of components and thus determine the allowable magnitude of the torsional vibration. Torsional vibratory motions can produce stress reversals that cause metal fatigue. Components tolerate less reversed stress than steady stress. In addition, stress concentration factors associated with machine members decrease the effectiveness of load-bearing materials.

Figure 38.1 illustrates the twisting of a shaft of an electric motor-compressor system. The torsional mode shape associated with the first torsional natural frequency is shown in Fig. 38.2. A coupling in the power train allows for misalignment in the assembly. The mode shape shows that the stiffness of the coupling is much less than that of other shaft sections. This is indicated by the large slope (change in angular displacement) of the mode shape at the coupling. The coupling will be the predominant component in the motor-compressor system governing the torsional natural frequency associated with the mode.

Torsional vibration is usually a complex vibration having many different frequency components. For example, shock resulting from abrupt start-ups and unload-

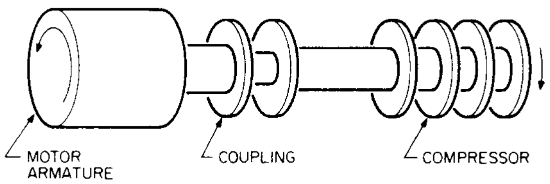


FIGURE 38.1 Schematic drawing illustrating the twisting of the shaft of a motor-compressor system.

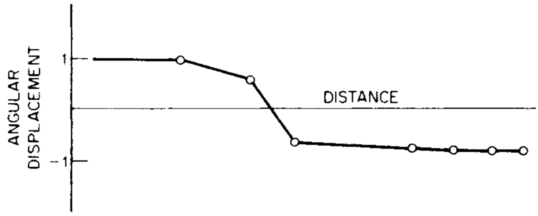


FIGURE 38.2 Torsional mode shape for the motor-compressor system shown in Fig. 38.1.

ing of gear teeth causes transient torsional vibration in some systems; start-up of synchronous electric motor systems may cause torsional resonance. Random torsional vibrations caused by gear inaccuracies and ball bearing defects are relatively common in rotating machines.

MODELING

The torsional elastic system of a drive unit and its associated machinery is a complicated arrangement of mass and elastic distribution. The complete mechanical system can include the drive unit, couplings, gearboxes or other speed-changing devices, and one or more driven units. This complicated system is made amenable to mathematical treatment by representing it as a model—a simpler system that is substantially equivalent dynamically. The equivalent system usually consists of lumped masses which are connected by massless torsionally elastic springs as illustrated in Fig. 38.3. The masses are placed at each crank center and at the center planes of actual flywheels, rotors, propellers, cranks, gears, impellers, and armatures.[1]

The torsional calculation is made not for the drive unit alone but for the complete system, including all driven machinery. On an engine, it is usually possible to consider such parts as camshafts, pumps, and blowers either as detached from the engine (if they are driven elastically) or as additional rigid masses at the point of attachment to the crankshaft (if the driver is relatively rigid). If there is doubt, these parts should be included in the torsional calculation as elastically connected masses and removed if the natural frequencies do not change after the parts are removed from the model.

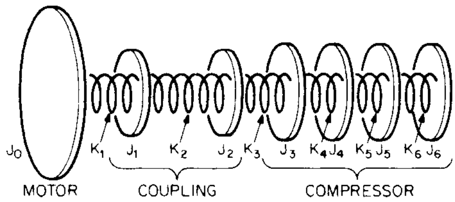


FIGURE 38.3 A model of the motor-compressor system shown in Fig. 38.1, consisting of a series of masses connected by massless torsionally elastic springs (K = stiffness, lb·in./rad; J = polar moment of inertia, lb·in.·sec^2).

CALCULATION OF POLAR MOMENTS OF INERTIA

Circular Disc or Cylinder Rotating about a Perpendicular Axis. The polar moment of inertia, essential in modeling torsional vibration, often is easy to calculate. The general form is $J = \int r^2\, dm$, where r is the instantaneous radius, and dm is the differential mass. The formula for the polar moment of inertia of a circular disc or cylinder rotating about a perpendicular axis is

$$J = \frac{\pi d^4 l \gamma}{32g} \quad \text{lb-in.-sec}^2 \qquad (38.1)$$

where J = polar moment of inertia, lb-in.-sec^2
γ = material density, lb/in.^{3}
d = diameter of disc or cylinder, in.
l = axial length of disc or cylinder, in.
g = acceleration due to gravity, 386.1 in./sec^2

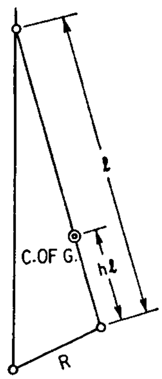


FIGURE 38.4 Schematic diagram of a crank and connecting rod.

Piston and Connecting Rod. The piston and connecting rod shown schematically in Fig. 38.4 introduce a variable-mass problem, the solution of which is complex. The exact solution shows that the effect of the piston and connecting rod can be closely approximated by representing them as a concentrated rotor of polar inertia J defined by

$$J = \left[\frac{W_P}{2} + W_c\left(1 - \frac{h}{2}\right)\right]\frac{R^2}{g} \quad \text{lb-in.-sec}^2 \qquad (38.2)$$

where W_P = weight of piston, piston pin, and cooling fluid, lb
W_c = weight of connecting rod, lb
h = fraction of rod length from crank pin to center-of-gravity
R = crank radius, in.

Crankshaft. The polar inertias of the crank webs (see Fig. 38.5), the crankpin, and the journal sections are added to that given by Eq. (38.2). These polar inertias should be calculated with the best obtainable accuracy. The following procedure is recommended:

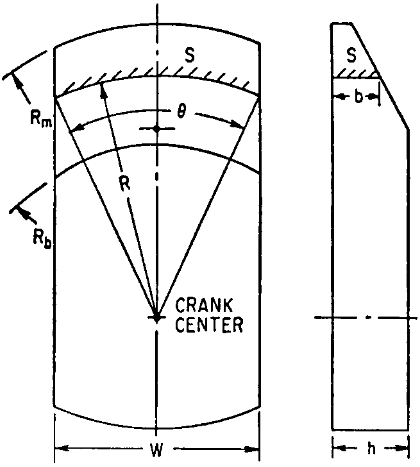


FIGURE 38.5 View of a crank web in a plane normal to the crankshaft axis.

Let the crank web be intercepted by a series of concentric cylinders of radius R. The polar inertia of the crank web is defined by

$$J = \left(\frac{\gamma}{g} \int_{R_b}^{R_m} R^2 S \, dR \right) + J_b \qquad (38.3)$$

where γ = specific weight of the crank web, lb/in.^{3}
R_m = maximum radius of the crank web, in.
R_b = radius of base cylinder (see Fig. 38.5), in.
J_b = polar inertia of portion of crank web within base cylinder, lb-in.-sec^2

The integral in the above expression for J is the area of the R^2S curve between the values of radii R_b and R_m.

For the crank web shown in Fig. 38.5 the area S is defined as $S = bR\theta/57.3$, where θ is measured in degrees. The polar inertia can then be expressed as

$$J = \left(\frac{\gamma}{57.3g} \int_{R_b}^{R_m} bR^3\theta \, dR \right) + J_b \qquad \text{lb-in.-sec}^2 \qquad (38.4)$$

The same procedure is used to calculate the polar inertia of propellers and other irregular parts. In a marine propeller of ogival sections, i.e., flat driving face, circular arc back, and elliptically developed outline (do not use for other shapes), the polar inertia (excluding hub) is given by

$$J = 0.0046 \frac{nD^3bt}{g} \qquad \text{lb-in.-sec}^2$$

where n = number of blades
D = diameter of propeller, in.
b = maximum blade width, in.
t = maximum blade thickness at one-half radius (axis to tip), in.

Propellers. For propellers, pumps, and hydraulic couplings an addition must be made for the virtual inertia of the entrained fluid. For marine propellers this is ordinarily assumed at 26 percent of the propeller inertia. Virtual inertias for pumps are not known accurately, but it can be assumed that half the casing is filled with rotating fluid.

EXPERIMENTAL DETERMINATION OF POLAR MOMENT OF INERTIA

For complex shaft elements such as couplings or small flywheels, it is often easier to determine the polar moment of inertia experimentally than to calculate it. In one experimental technique the element is suspended from three equally spaced vertical wires as shown in Fig. 38.6. The element whose polar moment of inertia is to be measured is hung on the cables and set into torsional motion. Then the period of vibration is measured. The experimentally determined period of torsional vibration, the weight of the element, the length of the suspending cables, and the radius of attachment of the cables are used to determine the polar moment of inertia from the following formula:

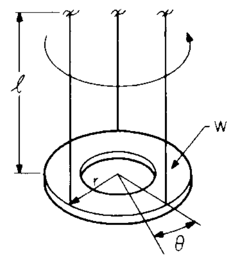


FIGURE 38.6 Experimental determination of the polar moment of inertia. An element of weight W is suspended by three wires and the period of the torsional motion is determined.

$$J = \frac{Wr^2\tau^2}{(6.28)^2 l} \quad \text{lb-in.sec}^2 \tag{38.5}$$

where J = polar moment of inertia
τ = period of vibration, sec/cycle
W = weight of element, lb
l = length of cables, in.
r = radius of suspending cables, in.

CALCULATION OF STIFFNESS

Shaft. The stiffness of a circular shaft is the most common elastic element encountered in the modeling process. Table 38.1 shows some common formulas used to calculate torsional stiffness of a hollow circular shaft, a tapered circular shaft, and two geared shafts. The stiffness is referred to the rotational speed of shaft No. 1. The inertia of geared shafts is obtained in a similar manner.

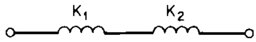
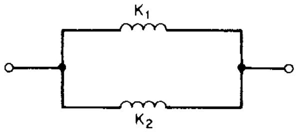
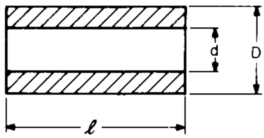
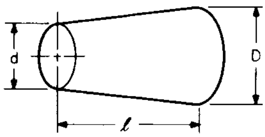
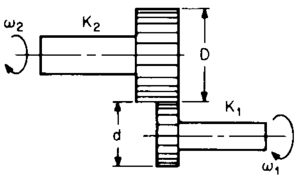
Crankshaft. The crankshaft stiffness is the most uncertain element in a torsional vibration calculation. Shaft stiffness can be measured experimentally either by twisting a shaft with a known torque or from the observed values of the critical speeds in a running engine. Alternatively, it can be calculated from semiempirical formulas such as those given in Ref. 1. Those given by Eqs. (38.6), (38.7), and (38.8) are recommended. Refer to Fig. 38.7 for definitions of the dimensions; l_e is the length of a solid shaft of diameter D_s equal in torsional stiffness to the section of crankshaft between crank centers.

Wilson's formula[2]

$$\frac{l_e}{D_s^4} = \frac{b + 0.4d_s}{D_s^4 - d_s^4} + \frac{a + 0.4D_c}{D_c^4 - d_c^4} + \frac{r - 0.2(D_s + D_c)}{hW^3} \tag{38.6}$$

TABLE 38.1 Formulas for Torsional Stiffness

K = TORSIONAL STIFFNESS, LB-IN./RAD; G = SHEAR MODULUS, LB/IN^2
ω = ROTATIONAL SPEED, RAD/SEC

SPRINGS IN SERIES		$K = \dfrac{1}{1/K_1 + 1/K_2}$
SPRINGS IN PARALLEL		$K = K_1 + K_2$
HOLLOW CIRCULAR SHAFT		$K = \dfrac{\pi}{32}\,\dfrac{G(D^4 - d^4)}{\ell}$
TAPERED CIRCULAR SHAFT		$K = \dfrac{3\pi}{32}\,\dfrac{d^4}{\ell(n + n^2 + n^3)}$ $n = \dfrac{d}{D}$
TWO GEARED SHAFTS (REFERRED TO SHAFT 1)		$K = \dfrac{K_1 K_2}{n^2 K_1 + K_2}$ $n = \dfrac{D}{d} = \dfrac{\omega_1}{\omega_2}$

Ziamanenko's formula

$$\frac{l_e}{D_s^4} = \frac{b + 0.6hD_s/b}{D_s^4 - d_s^4} + \frac{0.8a + 0.2(W/r)D_s}{D_c^4 - d_c^4} + \frac{r^{3/2}}{hW^3D_c^{1/2}} \tag{38.7}$$

Constant's formula

$$\frac{l_e}{D_s^4} = \frac{1}{\alpha_1\alpha_2\alpha_3\alpha_4}\left(\frac{b}{D_s^4 - d_s^4} + \frac{a}{D_c^4 - d_c^4} + \frac{0.94}{hW^3}\right) \tag{38.8}$$

where α_1, α_2, α_3, and α_4 are modifying factors, determined as follows:

$$\alpha_1 = 1 - \frac{0.0825}{\sqrt{\dfrac{W_s - d_s}{2W_s} + \dfrac{W_c - d_c}{2W_c} - 0.32}} \tag{38.9}$$

If the shaft is solid, assume $\alpha_1 = 0.9$. The factor α_2 is a web-thickness modification determined as follows: If $4h/l$ is greater than ⅔, then $\alpha_2 = 1.666 - 4h/l$. If $4h/l <$ ⅔, assume $\alpha_2 = 1$. The factor α_3 is a modification for web chamfering determined as follows: If the webs are chamfered, estimate α_3 by comparison with the cuts on Fig. 38.7:

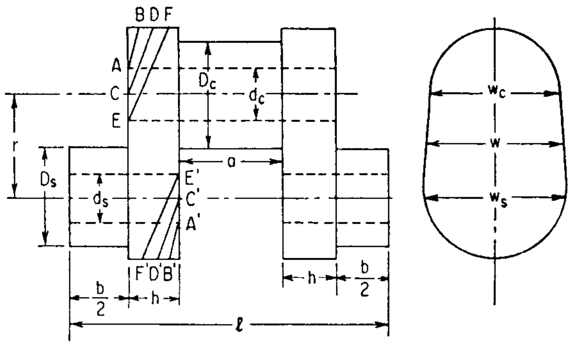


FIGURE 38.7 Schematic diagram of one crank of a crankshaft.

Cut AB and $A'B'$, $\alpha_3 = 1.000$; cut CD alone, $\alpha_3 = 0.965$; cut CD and $C'D'$, $\alpha_3 = 0.930$; cut EF alone, $\alpha_3 = 0.950$; cut EF and $E'F'$, $\alpha_3 = 0.900$; if ends are square, $\alpha_3 = 1.010$. The factor α_4 is a modification for bearing support given by

$$\alpha_4 = \frac{Al^3 w}{D_c^4 - d_c^4} + B \tag{38.10}$$

For marine engine and large stationary engine shafts: $A = 0.0029$, $B = 0.91$
For auto and aircraft engine shafts: $A = 0.0100$, $B = 0.84$
If α_4 as given by Eq. (38.10) is less than 1.0, assume a value of 1.0.

The Constant's formula, Eq. (38.8), is recommended for shafts with large bores and heavy chamfers.

Changes in Section. The shafting of an engine system may contain elements such as changes of section, collars, shrunk and keyed armatures, etc., which require the exercise of judgment in the assessment of stiffness. For a change of section having a fillet radius equal to 10 percent of the smaller diameter, the stiffness can be estimated by assuming that the smaller shaft is lengthened and the larger shaft is shortened by a length λ obtained from the curve of Fig. 38.8. This also may be applied to flanges where D is the bolt diameter. The stiffening effect of collars can be ignored.

Shrunk and Keyed Parts. The stiffness of shrunk and keyed parts is difficult to estimate as the stiffening effect depends to a large extent on the tightness of the shrunk fit and keying. The most reliable values of stiffness are obtained by neglecting the stiffening effect of an armature and assuming that the armature acts as a concentrated mass at the center of the shrunk or keyed fit. Some armature spiders and flywheels have considerable flexibility in their arms; the treatment of these is discussed in the section *Geared and Branched Systems.*

Elastic Couplings. Properties of numerous types of torsionally elastic couplings are available from the manufacturers and are given in Ref. 1.

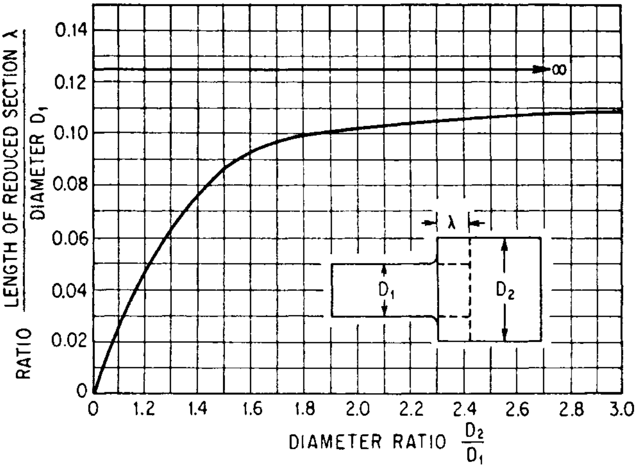


FIGURE 38.8 Curve showing the decrease in stiffness resulting from a change in shaft diameter. The stiffness of the shaft combination is the same as if the shaft having diameter D_1 is lengthened by λ and the shaft having diameter D_2 is shortened by λ. (*F. Porter.*[3])

GEARED AND BRANCHED SYSTEMS

The natural frequencies of a system containing gears can be calculated by assuming a system in which the speed of the driver unit is n times the speed of the driven equipment. Multiply all the inertia and elastic constants on the driven side of the system by $1/n^2$, and calculate the system's natural frequencies as if no gears exist. In any calculations involving damping constants on the driven side, these constants also are multiplied by $1/n^2$. Torques and deflections thus obtained on the driven side of this substitute system, when multiplied by n and $1/n$, respectively, are equal to those in the actual geared system. Alternatively, the driven side can be used as the reference; multiply the driver constants by n^2.

Where two or more drivers are geared to a common load, hydraulic or electrical couplings may be placed between the driver and the gears. These serve as disconnected clutches; they also insulate the gears from any driver-produced vibration. This insulation is so perfect that the driver end of the system can be calculated as if terminating at the coupling gap. The damping effect of such couplings upon the vibration in the driver end of the system normally is quite small and should be disregarded in amplitude calculations.

The majority of applications without hydraulic or electrical couplings involve two identical drivers. For such systems the modes of vibration are of two types:

1. The *opposite-phase* modes in which the drivers vibrate against each other with a node at the gear. These are calculated for a single branch in the usual manner, terminating the calculation at the gear. The condition for a natural frequency is that $\beta = 0$ at the gear.

2. The *like-phase* modes in which the two drivers vibrate in the same direction against the driven machinery. To calculate these frequencies, the inertia and stiffness constants of the driver side of one branch are doubled; then the calculation is made as if there were only a single driver. The condition for a natural frequency is zero residual torque at the end.

If the two identical drivers rotating in the same direction are so phased that the same cranks are vertical simultaneously, all orders of the opposite-phase modes will be eliminated. The two drivers can be so phased as to eliminate certain of the like-phase modes. For example, if the No. 1 cranks in the two branches are placed at an angle of 45° with respect to each other, the fourth, twelfth, twentieth, etc., orders, but no others, will be eliminated. If the drivers are connected with clutches, these phasing possibilities cannot be utilized.

In the general case of nonidentical branches the calculation is made as follows: Reduce the system to a 1:1 gear ratio. Call the branches *a* and *b*. Make the sequence calculation for a branch, with initial amplitude $\beta = 1$, and for the *b* branch, with the initial amplitude the algebraic unknown x. At the junction equate the amplitudes and find x. With this numerical value of the amplitude x substituted, the torques in the two branches and the torque of the gear are added; then the sequence calculation is continued through the last mass.

The branch may consist of a single member elastically connected to the system. Examples of such a branch are a flywheel with appreciable flexibility in its spokes or an armature with flexibility in the spider. Let I be the moment of inertia of the flywheel rim and k the elastic constant of the connection. Then the flexibly mounted flywheel is equivalent to a rigid flywheel of moment of inertia

$$I' = \frac{I}{1 - I\omega^2/k} \tag{38.11}$$

NATURAL FREQUENCY CALCULATIONS

If the model of a system can be reduced to two lumped masses at opposite ends of a massless shaft, the natural frequency is given by

$$f_n = \frac{1}{2\pi}\sqrt{\frac{(J_1 + J_2)k}{J_1 J_2}} \qquad \text{Hz} \tag{38.12}$$

The mode shape is given by $\theta_2/\theta_1 = -J_1/J_2$.

For the three-mass system shown in Fig. 38.9, the natural frequencies are

$$f_n = \frac{1}{2\pi}\sqrt{A \pm (A^2 - B)^{1/2}} \qquad \text{Hz} \tag{38.13}$$

where $$A = \frac{k_{12}(J_1 + J_2)}{2J_1 J_2} + \frac{k_{23}(J_1 + J_2)}{2J_1 J_2}$$

$$B = \frac{(J_1 + J_2 + J_3)k_{12}k_{23}}{J_1 J_2 J_3}$$

In Eqs. (38.12) and (38.13) the ks are torsional stiffness constants expressed in lb-in./rad. The notation k_{12} indicates that the constant applies to the shaft between rotors 1 and 2. The polar inertia J has units of lb-in.-sec^2.

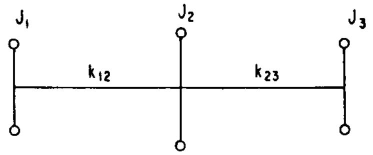


FIGURE 38.9 Schematic diagram of a shaft represented by three masses.

The above formulas and all the developments for multimass torsional systems that follow also apply to systems with longitudinal motion if the polar moments of inertia J are replaced by the masses $m = W/g$ and the torsional stiffnesses are replaced by longitudinal stiffnesses.

TRANSFER MATRIX METHOD

The transfer matrix method[4] is an extended and generalized version of the Holzer method. Matrix algebra is used rather than a numerical table for the analysis of torsional vibration problems. The transfer matrix method is used to calculate the natural frequencies and critical speeds of other eigenvalue problems.

The transfer matrix and matrix iteration (Stodola) methods are numerical procedures. The fundamental difference between them lies in the assumed independent variable. In any eigenvalue problem, a unique mode shape of the system is associated with each natural frequency. The mode shape is the independent variable used in the matrix iteration method. A mode shape is assumed and improved by successive iterations until the desired accuracy is obtained; its associated natural frequency is then calculated.

A frequency is assumed in the transfer matrix method, and the mode shape of the system is calculated. If the mode shape fits the boundary conditions, the assumed frequency is a natural frequency and a critical speed is derived. Determining the correct natural frequencies amounts to a controlled trial-and-error process. Some of the essential boundary conditions (geometrical) and natural boundary conditions (force) are assumed, and the remaining boundary condition is plotted vs. frequency to obtain the natural frequency; the procedure is similar to the Holzer method. For example, if the torsional system shown in Fig. 38.10 were analyzed, the natural boundary conditions would be zero torque at both ends. The torque at station No. 1 is made zero, and the torsional vibration is set at unity. Then M_4 as a function of ω is plotted to find the natural frequencies. This plot is obtained by utilizing the system transfer functions or matrices. These quantities reflect the dynamic behavior of the system.

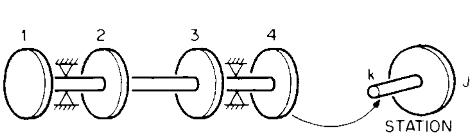


FIGURE 38.10 Typical torsional vibration model.

No accuracy is lost with the transfer matrix method because of coupling of mode shapes. Accuracy is lost with the matrix iteration method, however, because each frequency calculation is independent of the others. A minor disadvantage of the transfer matrix method is the large number of points that must be calculated to obtain an M_4 *vs* ω curve. This problem is overcome if a high-speed digital computer is used.

A typical station (No. 4) from a torsional model is shown in Fig. 38.10. This general station and the following transfer matrix equation, Eq. (38.14), are used in a way similar to the Holzer table to transfer the effects of a given frequency ω across the model.

$$\begin{vmatrix} \theta \\ M \end{vmatrix}_n = \begin{bmatrix} 1 & 1/k \\ -\omega^2 J & -(\omega^2 J/k) + 1 \end{bmatrix}_n \begin{vmatrix} \theta \\ M \end{vmatrix}_{n-1} \tag{38.14}$$

where θ = torsional motion, rad
M = torque, lb-in.
ω = assumed frequency, rad/sec
J = station inertia, lb-in.-sec^2
k = station torsional stiffness, lb-in./rad

The stiffness and polar moment of inertia of each station are entered into the equation to determine the transfer effect of each element of the model. Thus, the calculation begins with station No. 1, which relates to the first spring and inertia in the model of Fig. 38.10. The equation gives the output torque M_1 and output motion θ_1 for given input values, usually 0 and 1, respectively. The equation is used on station No. 2 to obtain M_2 output and θ_2 output as a function of M_1 output and θ_1 output. This process is repeated to find the value of M and θ at the end of the model. This calculation is particularly suited for the digital computer with spreadsheet programs.

FINITE ELEMENT METHOD

The finite element method is a numerical procedure (described in Chap. 28, Part II) to calculate the natural frequencies, mode shapes, and forced response of a discretely modeled structural or rotor system. The complex rotor system is composed of an assemblage of discrete smaller finite elements which are continuous structural members. The displacements (angular) are forced to be compatible, and force (torque) balance is required at the joints (often called *nodes*).

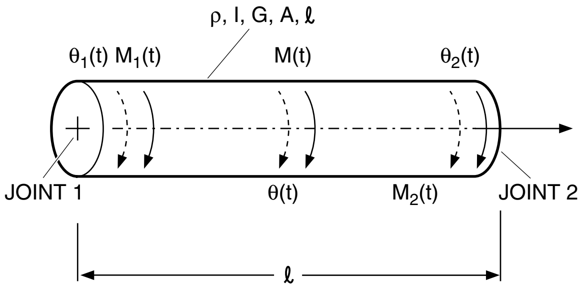


FIGURE 38.11 Finite element for torsional vibration in local coordinates.

Figure 38.11 shows a uniform torsional element in local coordinates. The x axis is taken along the centroidal axis. The physical properties of the element are density (ρ), area (A), shear modulus of elasticity (G), length (l), and polar area moment (I). $M(t)$ are the torsional forcing functions.

The torsional displacement within the element can be expressed in terms of the joint rotations $\theta_1(t)$ and $\theta_2(t)$ as

$$\theta(x,t) = U_1(x)\theta_1(t) + U_2(x)\theta_2(t) \tag{38.15}$$

where $U_1(x)$ and $U_2(x)$ are called *shape functions.* Since $\theta(0,t) = \theta_1(t)$ and $\theta(l,t) = \theta_2(t)$, the shape functions must satisfy the boundary conditions:

$$U_1(0) = 1 \qquad U_1(l) = 0$$

$$U_2(0) = 0 \qquad U_2(l) = 1$$

The shape function for the torsional element is assumed to be a polynomial with two constants of the form

$$U_i(x) = a_i + b_i x \qquad \text{where } i = 1,2 \tag{38.16}$$

Selection of the shape function is performed by the analyst and is a part of the engineering art required to conduct accurate finite element modeling.

Thus with four known boundary conditions the values of a_i and b_i can be determined from Eq. (38.16):

$$U_1(x) = 1 - \frac{x}{l} \qquad U_2(x) = \frac{x}{l}$$

Then from Eq. (38.15)

$$\theta(x,t) = \left(1 - \frac{x}{l}\right)\theta_1(t) + \frac{x}{l}\theta_2(t)$$

The kinetic energy, strain energy, and virtual work are used to formulate the finite element mass and stiffness matrices and the force vectors, respectively. These quantities are used to form the equations of motion. These matrices, derived in Ref. 4, are

$$\{J\} = \frac{\rho I l}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\{K\} = \frac{GI}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\bar{M} = \begin{Bmatrix} M_1(t) \\ M_2(t) \end{Bmatrix} = \begin{Bmatrix} \int_0^l M(x,t)\left(1 - \frac{x}{l}\right)dx \\ \int_0^l M(x,t)\left(\frac{x}{l}\right)dx \end{Bmatrix}$$

where $\{J\}$ = mass matrix
$\{K\}$ = stiffness matrix
$\bar{M}$ = torque vector
ρ = density
I = area polar moment
G = shear modulus
l = length of element

As noted, the previously described finite elements are in local coordinates. Since the system as a whole must be analyzed as a unit, the elements must be transformed into one global coordinate system. Figure 38.12 shows the local element within a global coordinate system. The mass and stiffness matrices and joint force vector of each element must be expressed in the global coordinate system to find the vibration response of the complete system.

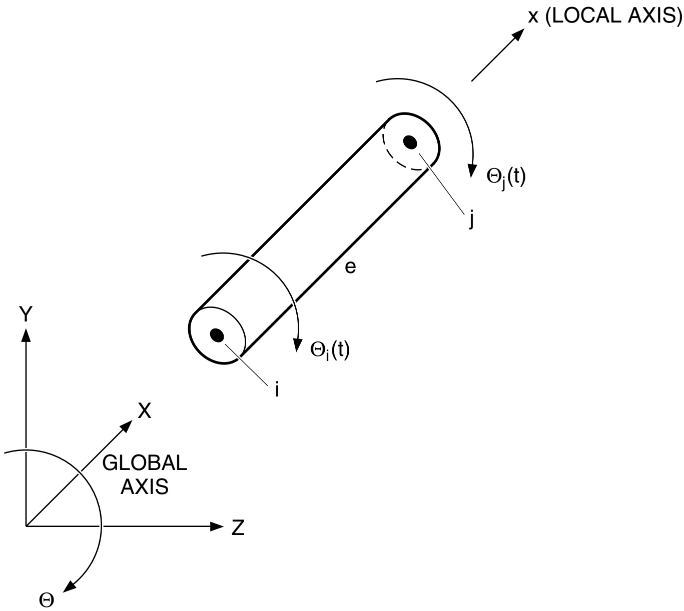


FIGURE 38.12 Local and global joint displacements of element, *l*.

Using transformation matrices,[4] the mass and stiffness matrices and force vectors are used to set up the system equation of motion for a single element in the global coordinates:

$$[J]_e\{\ddot{\Theta}(t)\} + [K]_e\{\ddot{\Theta}(t)\} = \{M_e(t)\}$$

The complete system is an assemblage of the number of finite elements it requires to adequately model its dynamic behavior. The joint displacements of the elements in the global coordinate system are labeled as $\Theta_1(t)$, $\Theta_2(t)$, . . . , $\Theta_m(t)$, or this can be expressed as a column vector:

$$\{\Theta(t)\} = \begin{Bmatrix} \Theta_1(t) \\ \Theta_2(t) \\ \cdot \\ \cdot \\ \cdot \\ \Theta_m(t) \end{Bmatrix}$$

Using global joint displacements, mass and stiffness matrices, and force vectors, the equations of motion are developed:

$$[J]_{nxn}\{\ddot{\Theta}\}_{nx1} + [K]_{nxn}\{\Theta\}_{nx1} = \{M\}_{nx1}$$

where n denotes the number of joint displacements in the system.

In the final step prior to solution, appropriate boundary conditions and constraints are introduced into the global model.

The equations of motion for free vibration are solved for the eigenvalues (natural frequencies) using the matrix iteration method (Chap. 28, Part I). Modal analysis is used to solve the forced torsional response. The finite element method is available in commercially available computer programs for the personal computer. The analyst must select the joints (nodes, materials, shape functions, geometry, torques, and constraints) to model the system for computation of natural frequencies, mode shapes, and torsional response. Similar to other modeling efforts, engineering art and a knowledge of the capabilities of the computer program enable the engineer to provide reasonably accurate results.

CRITICAL SPEEDS

The crankshaft of a reciprocating engine or the rotors of a turbine or motor, and all moving parts driven by them, comprise a torsional elastic system. Such a system has several modes of free torsional oscillation. Each mode is characterized by a natural frequency and by a pattern of relative amplitudes of parts of the system when it is oscillating at its natural frequency. The harmonic components of the driving torque excite vibration of the system in its modes. If the frequency of any harmonic component of the torque is equal to (or close to) the frequency of any mode of vibration, a condition of resonance exists and the machine is said to be running at a *critical speed.* Operation of the system at such critical speeds can be very dangerous, resulting in fracture of the shafting.

The number of complete oscillations of the elastic system per unit revolution of the shaft is called an *order of the operating speed.* It is an order of a critical speed if the forcing frequency is equal to a natural frequency. An order of a critical speed that corresponds to a harmonic component of the torque from the engine as a whole is called a *major order.* A critical speed also can be excited that corresponds to the harmonic component of the torque curve of a single cylinder. The fundamental period of the torque from a single cylinder in a four-cycle engine is 720°; the critical speeds in such an engine can be of ½, 1, 1½, 2, 2½, etc., order. In a two-cycle engine only the critical speeds of 1, 2, 3, etc., order can exist. All critical speeds except those of the major orders are called *minor critical speeds;* this term does not necessarily mean that they are unimportant. Therefore the critical speeds occur at

$$\frac{60f_n}{q} \qquad \text{rpm} \tag{38.17}$$

where f_n is the natural frequency of one of the modes in Hz, and q is the order number of the critical speed. Although many critical speeds exist in the operating range of an engine, only a few are likely to be important.

A dynamic analysis of an engine involves several steps. Natural frequencies of the modes likely to be important must be calculated. The calculation is usually limited to the lowest mode or the two lowest modes. In complicated arrangements, the calculation of additional modes may be required, depending on the frequency of the forces causing the vibration. Vibration amplitudes and stresses around the operating range and at the critical speeds must be calculated. A study of remedial measures is also necessary.

VIBRATORY TORQUES

Torsional vibration, like any other type of vibration, results from a source of excitation. The mechanisms that introduce torsional vibration into a machine system are discussed and quantified in this section. The principal sources of the vibratory torques that cause torsional vibration are engines, pumps, propellers, and electric motors.

GENERAL EXCITATION

Table 38.2 shows some ways by which torsional vibration can be excited. Most of these sources are related to the work done by the machine and thus cannot be entirely removed. Many times, however, adjustments can be made during the design

TABLE 38.2 Sources of Excitation of Torsional Vibration

Source	Amplitude in terms of rated torque	Frequency
	Mechanical	
Gear runout		1 ×, 2 ×, 3 × rpm
Gear tooth machining tolerances		No. gear teeth × rpm
Coupling unbalance		1 × rpm
Hooke's joint		2 ×, 4 ×, 6 × rpm
Coupling misalignment		Dependent on drive elements
	System function	
Synchronous motor start-up	5–10	2 × slip frequency
Variable-frequency induction motors (six-step adjustable frequency drive)	0.04–1.0	6 ×, 12 ×, 18 × line frequency (LF)
Induction motor start-up	3–10	Air gap induced at 60 Hz
Variable-frequency induction motor (pulse width modulated)	0.01–0.2	5 ×, 7 ×, 9 × LF, etc.
Centrifugal pumps	0.10–0.4	No. vanes × rpm and multiples
Reciprocating pumps		No. plungers × rpm and multiples
Compressors with vaned diffusers	0.03–1.0	No. vanes × rpm
Motor- or turbine-driven systems	0.05–1.0	No. poles or blades × rpm
Engine geared systems with soft coupling	0.15–0.3	Depends on engine design and operating conditions; can be 0.5n and n × rpm
Engine geared system with stiff coupling	0.50 or more	Depends on engine design and operating conditions
Shaft vibration		n × rpm

process. For example, certain construction and installation sources—gear runout, unbalanced or misaligned couplings, and gear-tooth machining errors—can be reduced.

In Table 38.2 note that the pulsating torque during start-up of a synchronous motor is equal to twice the slip frequency. The slip frequency varies from twice the line frequency at start-up to zero at synchronous speed. Many mechanical drives exhibit characteristics of pulsating torque during operation due to their design function. Electric motors with variable-frequency drives induce pulsating torques at frequencies that are harmonics of line frequency. Blade-passing excitations can be characterized by the number of blades or vanes on the wheel: The frequency of excitation equals the number of blades multiplied by shaft speed. The amplitude of a pulsating torque is often given in terms of percentage of average torque generated in a system.

ENGINE EXCITATION

In more complex cases, diesel gasoline engines for example, the multiple frequency components depend on engine design and power output. The power output, crankshaft phasing, and relationship between gas torque and inertial torque influence the level of torsional excitation.

Inertia Torque. A harmonic analysis of the inertia torque of a cylinder is closely approximated by[1]

$$M = \frac{W}{g}\Omega^2 r\left(\frac{\lambda}{4}\sin\theta - \frac{1}{2}\sin 2\theta - \frac{3}{4}\lambda\sin 3\theta - \frac{\lambda^2}{4}\sin 4\theta \cdots\right) \tag{38.18}$$

where $W = W_p + hW_c$ [see Fig. 38.4 and Eq.(38.2)]
$\lambda = R/l$ [see Fig. 38.4 and Eq. (38.2)]
Ω = angular speed, rad/sec
R = crank radius, in.
l = connecting rod length, in.
θ = crank angle, radians
W_p = weight of piston, lb
W_c = weight of connecting rod, lb

It is usual to drop all terms above the third order.

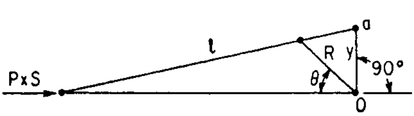


FIGURE 38.13 Schematic diagram of crank and connecting rod used in plotting torque curve.

Gas-Pressure Torque. A harmonic analysis of the turning effort curve yields the gas-pressure components of the exciting torque. The turning effort curve is obtained from the indicator card of the engine by the graphical construction shown in Fig. 38.13.

For a given crank angle θ, let the gas pressure on the piston be P. Erect a perpendicular to the line of action of the piston from the crank center, intersecting the line of the connecting rod. Let the intercept Oa on this perpendicular be y. Then the torque M for angle θ is given by

$$M = PSy \tag{38.19}$$

where S is the piston area. A gas pressure versus rotation curve analyzed to obtain harmonic gas coefficients is required to conduct a gas-pressure torque calibration. Harmonic gas coefficients are often available from engine manufacturers.

FORCED VIBRATION RESPONSE

The torsional vibration amplitude of a modeled system is determined by the magnitude, points of application, and phase relations of the exciting torques produced by engine or compressor gas pressure and inertia and by the magnitudes and points of application of the damping torques. Damping is attributable to a variety of sources, including pumping action in the engine bearings, hysteresis in the shafting and between fitted parts, and energy absorbed in the engine frame and foundation. In a few cases, notably marine propellers, damping of the propeller predominates. When an engine is fitted with a damper, the effects of damping dominate the torsional vibrations.

Techniques available for calculation of vibration amplitudes include the exact solution of differential equations, the energy balance method, the transfer matrix method, and modal analysis. The techniques are implemented on lumped parameter or finite-element models.

EXACT METHOD FOR TWO DEGREE-OF-FREEDOM SYSTEMS

The lowest mode of vibration of some systems, particularly marine installations, can be approximated with a two-mass system; an excitation is applied at one end and damping at the other.

Referring to Fig. 38.14, the torque equations for rotors I_1 and I_2 are

$$I_1\omega^2\theta_1 - k(\theta_1 - \theta_2) + M_e = 0$$

$$I_2\omega^2\theta_2 + k(\theta_1 - \theta_2) - jc\omega\theta_2 = 0$$

The natural frequency is given by

$$\omega^2 = \frac{k(I_1 + I_2)}{I_1 I_2}$$

The shaft torque is $M_{12} = k(\theta_1 - \theta_2)$. If the above equations are solved, the amplitude of M_{12} at resonance is

$$|M_{12}| = k|\theta_1 - \theta_2| = M_e \frac{I_2}{I_1} \sqrt{1 + \frac{kI_2(I_1 + I_2)}{I_1 c^2}} \tag{38.20}$$

Since with usual damping the second term under the radical is large compared with unity, Eq. (38.20) reduces to

$$|M_{12}| \simeq \frac{M_e}{c} \frac{I_2}{I_1} \sqrt{(I_1 + I_2)\frac{I_2 k}{I_1}} \tag{38.21}$$

The torsional damping constant c of a marine propeller is a matter of some uncertainty. It is customary to use the "steady-state" value. This is an approximation:

$$c = \frac{4M_{\text{mean}}}{\Omega} \qquad \text{in.-lb/rad/sec}$$

where Ω = angular speed of shaft in radians per second. Considerations of oscillating airfoil theory indicate that this is too high and that a better value would be

$$c = \frac{2.3M_{\text{mean}}}{\Omega} \qquad \text{in.-lb/rad/sec} \quad (38.22)$$

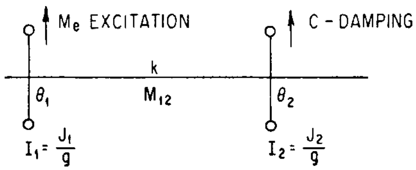


FIGURE 38.14 Schematic diagram of a shaft with two rotors, showing positions of excitation and damping.

Equation (38.21) is applicable only when $I_1/I_2 > 1$. If used outside this range with other types of damping neglected, fictitiously large amplitudes will be obtained. Equation (38.21) gives the resonance amplitude, but the peak may not occur exactly at resonance. The complete amplitude curve is computed by the methods discussed in the following section.

ENERGY BALANCE METHOD

Both rational and empirical formulas for the resonance amplitudes of systems without dampers can be based on the energy balance at resonance. It is assumed that the system vibrates in a normal mode and that the displacement is in a 90° phase relationship to the exciting and damping torques. The energy input by the exciting torques is then equal to the energy output by the damping torques. Unless the damping is extremely large, this assumption gives a very close approximation to the amplitude at resonance.

Figure 38.15 shows a curve of relative amplitude in the first mode of vibration. Assume that a cylinder acts at A. Let the actual amplitude at A be θ_a and the amplitude relative to that of the No. 1 cylinder be β. The β values are taken from the column opposite each rotor number in the sequence calculation for the natural frequency calculation. At a point such as B, where damping may be applied, let the actual amplitude be θ_d and the amplitude relative to the No. 1 cylinder be β_d.

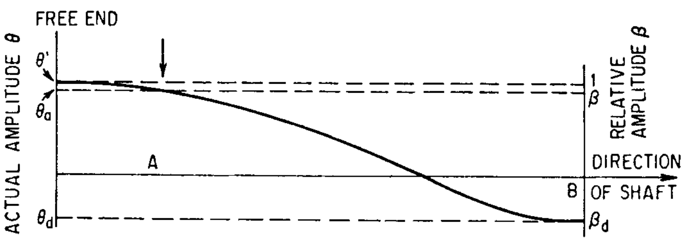


FIGURE 38.15 Diagram of actual amplitude θ and relative amplitude β as a function of position along shaft. Excitation is at A, and B is the position where damping is applied. The No. 1 cylinder is at the free end of the crankshaft.

The energy input to the system from the cylinder acting at A is

$$\pi M_e\theta_a \qquad \text{in.-lb/cycle}$$

and the energy output to the damper is

$$\pi c\omega\theta_d^2 \qquad \text{in.-lb/cycle}$$

where c* is the damping constant action of the damper at B. Equating input to output,

$$M_e\theta_a = c\omega\theta_d^2 \tag{38.23a}$$

Let θ' be the amplitude at the No. 1 cylinder produced by the cylinder acting at A. Then $\theta_e/\theta' = \beta$ and $\theta_d/\theta' = \beta_d$. Substituting in Eq. (38.23*a*) gives

$$\theta' = \frac{M_e\beta}{c\omega\beta_d^2} \tag{38.23b}$$

If all the cylinders act, and if damping is applied at a variety of points, the total amplitude at the No. 1 cylinder is

$$\theta = \Sigma\theta' = \frac{M_e\Sigma\beta}{\omega\Sigma c\beta_d^2} \tag{38.24}$$

where $\Sigma\beta$ is taken over the cylinders and $\Sigma c\beta_d^2$ is taken over the points at which damping is applied. This formula can be applied directly when the magnitude and points of application of the damping torques are known. For the great majority of applications, where the damping is unknown, a number of empirical formulas have been proposed with coefficients based on engine tests. These formulas may give an amplitude varying 30 percent or more from test results if applied to a variety of engines. Better agreement should not be expected, for even identical engines may have amplitudes differing as much as 2 to 1, depending on length of service, bearing fits, mounting, variation in the harmonic excitation because of different combustion rates, and other unknown factors.

Good results have been obtained using the Lewis formula[5]

$$M_m = \Re M_e\Sigma\beta \tag{38.25}$$

The maximum torque at resonance in any part of the system is M_m; the exciting torque per cylinder is M_e. $\Re$ is a constant from Table 38.3. The vector sum over the cylinders of the relative amplitudes as taken from the mode shape for a natural frequency is $\Sigma\beta$. It is determined as follows.

For a four-cycle engine construct a phase diagram, Table 38.4, of the firing sequence in which 720° corresponds to a complete cycle of a single cylinder, or two revolutions. The phase relationship for a critical of order number q is obtained by multiplying the angles in this diagram by $2q$, with the No. 1 crank held fixed. The β values assigned to each direction then are obtained from the values corresponding to each cylinder in the mode shape β. Then $\Sigma\beta$ is the vector sum. The summation extends only to those rotors on which exciting torques act.

In a two-cycle engine the β phase relations are determined by multiplying the crank diagram by q, holding the No. 1 cylinder fixed.

---

* The symbol c is used in this chapter to denote a torsional damping coefficient.

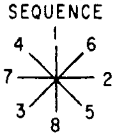
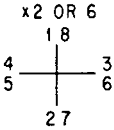
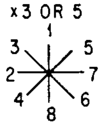
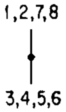
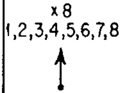
Table 38.4 shows the Σβ phase diagrams and Σβ values for the one-noded mode with a firing sequence 1, 6, 2, 5, 8, 3, 7, 4. The firing sequence is drawn first; then the angles of this diagram are multiplied by 2, 3, 4, etc., in succeeding diagrams. After multiplication by 8 for the fourth order, the diagrams repeat. Diagrams which are equidistant in order number from the 2, 6, 10, etc., orders are mirror images of each other and have the same Σβ. The numerical values of Σβ in Table 38.4 have been obtained by calculation, summing the vertical and horizontal components.

The empirical factor 𝔑 is determined by the measurement of amplitudes in running engines (Table 38.3).

TABLE 38.3 Empirical Factors for Engine Amplitude Calculations

Bore Stroke	𝔑
20 in. × 24 in. or larger	50–60
8 in. × 10 in.	40–50
4 in. × 6 in. or smaller	35

TABLE 38.4 Phase Diagrams and Deflections, β, for a Calculated Torsional Mode

PHASE DIAGRAMS	Σβ	ORDERS
FIRING SEQUENCE 	0.778	1/2, 4½, 8½, -- MIRROR IMAGE FOR 3½, 5½, 7½, --
x2 OR 6 	0.169	1, 5, 9, — MIRROR IMAGE FOR 3, 7, 11
x3 OR 5 	1.549	1½, 5½, 9½, +– MIRROR IMAGE FOR 2½, 6½, 10½, --
x4 	0.4287	2, 6, 10, --
x8 	5.0455	4, 8, 12 MAJOR ORDERS

cn	β
1	1.0000
2	0.9543
3	0.8771
4	0.7709
5	0.6296
6	0.4595
7	0.2742
8	0.0799
	5.0455

The exciting torque per cylinder, M_e in Eq. (38.24) is composed of the sum of the torques produced by gas pressure, inertia force, gravity force, and friction force. The gravity and friction torques are of negligible importance; and the inertia torque is of importance only for first-, second-, and third-order harmonic components.

TRANSFER MATRIX METHOD FOR FORCED RESPONSE

A calculation of the nonresonant or "forced" vibration amplitude is required in some cases to define the range of the more severe critical speeds, particularly with geared drives; it also is required in the design of dampers. The calculation[6] is readily made by an extension of the transfer matrix method. In the calculation the initial amplitude is treated as an algebraic unknown θ. At each station where an exciting torque acts, this torque is added. Assume first that there are no damping torques. Then the residual torque after the last rotor is of the form $a\theta + b$, where a and b are numerical constants resulting from the calculation. Since the residual torque is zero, $\theta = -b/a$.

The amplitude and torque at any point of the system are found by substituting this numerical value of θ at the appropriate point in the calculation. At frequencies well removed from resonance, damping has little effect and can be neglected. Damping can be added to the system by treating it as an exciting torque equal to the imaginary quantity $-jc\omega\theta$, where c is the damping constant and θ is the amplitude at the point of application. Relative damping between two inertias can be treated as a spring of a stiffness constant equal to the imaginary quantity of $+jc\omega$.

For the major critical speeds the exciting torques are all in-phase and are real numbers. For the minor critical speeds the exciting torques are out-of-phase; they must be entered as complex numbers of amplitude and phase as determined from the phase diagram (discussed under *Energy Balance*) for the critical speed of the order under consideration. With damping and/or out-of-phase exciting torques introduced, a and b in the equation $a\theta + b = 0$ are complex numbers, and θ must be entered as a complex number in the calculation in order to determine the angle and torque at any point. The angles and torques are then of the form $r + js$, where r and s are numerical constants and the amplitudes are equal to $\sqrt{r^2 + s^2}$.

APPLICATION OF MODAL ANALYSIS TO ROTOR SYSTEMS

Classical modal analysis of vibrating systems (see Chap. 21) can be used to obtain the forced response of multistation rotor systems in torsional motion. The natural frequencies and mode shapes of the system are found using the transfer matrix method. The response of the rotor to periodic phenomena (not necessarily a harmonic or shaft frequency) is determined as a linear weighted combination of the mode shapes of the system. Heretofore with this technique, damping has been entered in modal form; the damping forces are a function of the various modal velocities. The formation of equivalent viscous damping constants that are some percentage of critical damping is required. The critical damping factor is formed from the system modal inertia.[7]

The modal analysis technique can be used for a torsional distributed mass model of engine systems using modal damping; nonsynchronous speed excitations are allowed. The shaft sections of the modeled rotor have distributed mass properties and lumped end masses (including rotary inertia). A transfer matrix analysis is performed to obtain a finite number of natural frequencies. The number required depends on the range of forcing frequencies used in the problem. The natural frequencies are substituted back into the transfer matrices to obtain the mode shapes. A function consisting of a weighted average of the mode shapes is formed and substituted into

$$\theta(x, t) = \sum_{n=1}^{N} a_n(x) f_n(t)$$

where θ = torsional response
a_n = normal modes
f_n = periodic time-varying weighting factors

The function $f_n(t)$ is determined from the ordinary differential equations of motion and is a function of the forcing functions, rotor speed, modal damping constants, and mode shapes of the system.

DIRECT INTEGRATION

Direct integration of equations of motion of a system utilize first- or second-order differential equations. The method is fundamental for linear and nonlinear response problems.[8] Any digitally describable vibration or shock excitation can be carried out with this method.

Direct integration can be used on nonlinear models and arbitrary excitation, so it is one of the most general techniques available for response calculation. However, large computer storage is required, and large computer costs are usually incurred because small time- or space-step sizes are needed to maintain numerical stability. An adjustable step integration routine such as predictor-corrector helps to alleviate this problem. Such a numerical integration must be started with another routine such as Runge-Kutta.

Direct integration is particularly useful when nonlinear components such as elastomeric couplings are involved or when the excitation force varies in frequency and magnitude. Direct integration is used for analysis of synchronous motor start-ups in which the magnitude of the torque varies with rotor speed and the frequency is 2 times the slip frequency—starting at twice the line frequency and ending at zero when the rotor is locked on synchronous speed. Examples of this type of analysis are given in Refs. 8 and 9.

PERMISSIBLE AMPLITUDES

Failure caused by torsional vibration invariably initiates in fatigue cracks that start at points of stress concentration—e.g., at the ends of keyway slots, at fillets where there is a change of shaft size, and particularly at oil holes in a crankshaft. Failures can also start at corrosion pits, such as occur in marine shafting. At the shaft oil holes the cracks begin on lines at 45° to the shaft axis and grow in a spiral pattern until failure occurs. Theoretically the stress at the edges of the oil holes is 4 times the mean shear stress in the shaft, and failure may be expected if this concentrated stress exceeds the fatigue limit of the material. The problem of estimating the stress required to cause failure is further complicated by the presence of the steady stress from the mean driving torque and the variable bending stresses.

In practice the severity of a critical speed is judged by the maximum nominal torsional stress

$$\tau = \frac{16M_m}{\pi d^3}$$

where M_m is the torque amplitude from torsional vibration and d is the crankpin diameter. This calculated nominal stress is modified to include the effects of increased stress and is compared to the fatigue strength of the material.

U.S. MILITARY STANDARD

A military standard[10] issued by the U.S. Navy Department states that the limit of acceptable nominal torsional stress within the operating range is

$$\tau = \frac{\text{ultimate tensile strength}}{25} \qquad \text{for steel}$$

$$\tau = \frac{\text{torsional fatigue limit}}{6} \qquad \text{for cast iron}$$

If the full-scale shaft has been given a fatigue test, then

$$\tau = \frac{\text{torsional fatigue limit}}{2} \qquad \text{for either material}$$

Such tests are rarely, if ever, possible.

For critical speeds below the operating range which are passed through in starting and stopping, the nominal torsional stress shall not exceed 1¾ times the above values.

Crankshaft steels which have ultimate tensile strengths between 75,000 and 115,000 lb/in.^{2} usually have torsional stress limits of 3000 to 4600 lb/in.^{2}

For gear drives the vibratory torque across the gears, at any operating speed, shall not be greater than 75 percent of the driving torque at the same speed or 25 percent of full-load torque, whichever is smaller.

AMERICAN PETROLEUM INSTITUTE

Sources of torsional excitation considered by American Petroleum Institute[11] (API) include but are not limited to the following: gear problems such as unbalance, pitch line runout, and eccentricity; start-up conditions resulting from inertial impedances; and torsional transients from synchronous and induction electric motors.

Torsional natural frequencies of the machine train shall be at least 10 percent above or below any possible excitation frequency within the specified operating speed range. Torsional critical speeds at integer multiples of operating speeds (e.g., pump vane pass frequencies) should be avoided or should be shown to have no adverse effect where excitation frequencies exist. Torsional excitations that are nonsynchronous to operating speeds are to be considered. Identification of torsional excitations is the mutual responsibility of the purchaser and the vendor.

When torsional resonances are calculated to fall within the ±10 percent margin and the purchaser and vendor have agreed that all efforts to remove the natural frequency from the limiting frequency range have been exhausted, a stress analysis shall be performed to demonstrate the lack of adverse effect on any portion of the machine system.

In the case of synchronous motor driven units, the vendor is required to perform a transient torsional vibration analysis with the acceptance criteria mutually agreed upon by the purchaser and the vendor.

TORSIONAL MEASUREMENT

Torsional vibration is more difficult to measure than lateral vibration because the shaft is rotating. Procedures for signal analysis are similar to those used for lateral vibration. Torsional response—both strains and motions—can be measured at intermediate points in a system. But sensors cannot be placed at a nodal point; for this reason the transfer matrix method is valuable for calculating mode shapes prior to sensor location selection.

SENSORS

Strain gauges, described in Chap. 17, are available in a variety of sizes and sensitivities and can be placed almost anywhere on a shaft. They can be calibrated to indicate instantaneous torque by using static torque loads on drive shafts. If calibration is not possible, stresses and torques can be calculated from strength of materials theory. Strain gauges are usually mounted at 45° angles so that shaft bending does not influence torque measurements. The signal must be processed by a bridge-amplifier unit that can be arranged to compensate for temperature. Because strain gauge signals are difficult to take from a rotating shaft, such techniques are not common diagnostic tools.

Slip rings can be used to obtain a vibration signal from a shaft. Wireless telemetry is also available. A small transmitter mounted on the rotating shaft at a convenient location broadcasts a signal to a nearby receiver. Commercial torque transducers are available for torsional measurement. However, they must be inserted in the drive line and thus may change the dynamic characteristics of the system. If the natural frequency of the system is changed, the vibration response will not accurately reflect the properties of the system.

The velocity of torsional vibration is measured using a toothed wheel and a fixed sensor.[12] The signal generated by the teeth of the wheel passing the fixed sensor has a frequency equal to the number of teeth multiplied by shaft speed. If the shaft is undergoing torsional vibration, the carrier frequency will exhibit frequency modulation (change in frequency) because the time required for each tooth to pass the fixed pickup varies.

DATA ACQUISITION

The frequency change (velocity) is converted to a voltage change by a demodulator and integrated to obtain angular displacement. Angular displacement can be measured at the end of a shaft with encoders or at intermediate points with a gear-magnetic pickup or proximity probe arrangement. The frequency of the carrier signal (e.g., number of teeth on a gear × rpm) must be at least 4 times the highest frequency to be measured. In most cases, the raw torsional signal is tape recorded prior to processing and analysis. Because the output of the magnetic pickup is speed dependent and the gap between the magnetic pickup and the toothed wheel is less than 0.025 in. the proximity probe is preferred—especially in synchronous motor startups.

TORSIONAL ANALYSIS

A torsional signal must be analyzed for frequency components using a spectrum analyzer, described in Chap. 14. Figure 38.16 shows a torsional response spectrum for a variable-frequency motor-driven pump. The pump ran at 408 rpm. The torsional vibration response excited by the variable frequency motor is 0.23° at a frequency of 38 Hz.

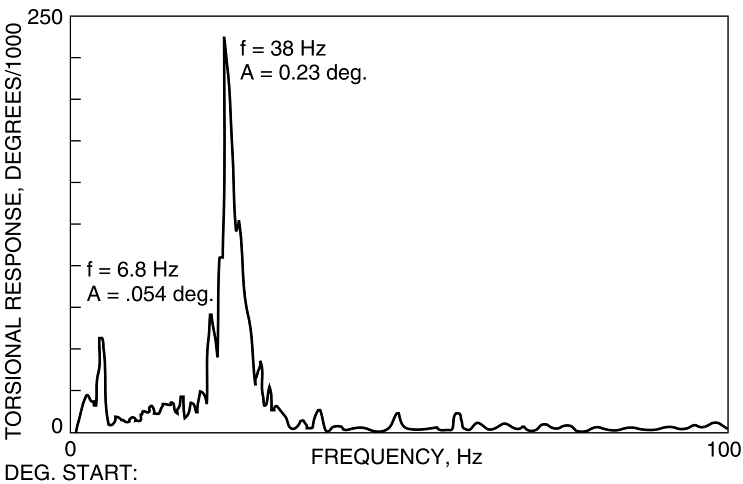


FIGURE 38.16 Torsional response of a variable-frequency motor-driven pump at 408 rpm. There are significant peaks at 6.8 and 38.0 Hz.

MEASURES OF CONTROL

The various methods which are available for avoiding a critical speed or reducing the amplitude of vibration at the critical speed may be classified as:

1. Shifting the values of critical speeds by changes in mass and elasticity
2. Vector cancellation methods
3. Change in mass distribution to utilize the inherent damping in the system
4. Addition of dampers of various types

SHIFTING OF CRITICAL SPEEDS

If the stiffness of all the shafting to a system is increased in the ratio *a,* then all the frequencies will increase in the ratio *a,* provided that there is no corresponding increase in the inertia. It is rarely possible to increase the crankshaft diameters on modern engines; in order to reduce bearing pressures, bearing diameters usually are made as large as practical. If bearing diameters are increased, the increase in the critical speed will be much smaller than indicated by the a ratio because a considerable increase in the inertia will accompany the increase in diameter. Changes in the stiffness of a system made near a nodal point will have maximum effect. Changes in inertia near a loop will have maximum effect, while those near a node will have little effect.

By the use of elastic couplings it may be possible to place certain critical speeds below the operating speed where they are passed through only in starting and stopping; this leaves a clear range above the critical speed. This procedure must be used with caution because some critical speeds, for example the fourth order in an eight-cylinder, four-cycle engine, are so violent that it may be dangerous to pass through them. If the acceleration through the critical speed is sufficiently high, some reduction in amplitude may be attained, but with a practical rate the reduction may not be large. The rate of deceleration when stopping is equally important. In some cases mechanical clutches disconnect the driven machinery from the engine until the engine has attained a speed above dangerous critical speeds. Elastic couplings may take many forms including helical springs arranged tangentially, flat leaf springs arranged longitudinally or radially, various arrangements using rubber, or small-diameter shaft sections of high tensile steel.[1]

VECTOR CANCELLATION METHODS

Choice of Crank Arrangement and Firing Order. The amplitude at certain minor critical speeds sometimes can be reduced by a suitable choice of crank arrangement and firing order (i.e., firing sequence). These fix the value of the vector sum $\Sigma\beta$ in Eq (38.25), $M_m = \Re M_e \Sigma\beta$. But considerations of balance, bearing pressures, and internal bending moments restrict this freedom of choice. Also, an arrangement which decreases the amplitude at one order of critical speed invariably increases the amplitude at others. In four-cycle engines with an even number of cylinders, the amplitude at the half-order critical speeds is fixed by the firing order because this determines the $\Sigma\beta$ value. Tables 38.5 and 38.6 list the torsional-vibration characteristics for the crank arrangements and firing orders, for eight-cylinder two- and four-cycle engines having the most desirable properties.

The values of $\Sigma\beta$ are calculated by assuming $\beta = 1$ for the cylinder most remote from the flywheel, assuming $\beta = 1/n$ for the cylinder adjacent to the flywheel (where n is the number of cylinders), and assuming a linear variation of β there between. In any actual installation $\Sigma\beta$ must be calculated by taking β from the relative modal curve; however, if the $\Sigma\beta$ as determined above is small, it also will be small for the actual β distribution. These arrangements assume equal crank angles and firing intervals. The reverse arrangements (mirror images) have the same properties.

V-Type Engines. In V-type engines, it may be possible to choose an angle of the V which will cancel certain criticals. Letting ϕ be the V angle between cylinder banks, and q the order number of the critical, the general formula is

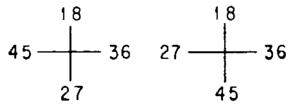
$$q\phi = 180°, 540°, 1080°, \text{etc.} \tag{38.26}$$

For example, in an eight-cylinder engine the eighth order is canceled at angles of 22½°, 67½°, 112½°, etc.

TABLE 38.5 Torsional-Vibration Characteristics for Eight-Cylinder, Four-Cycle Engine Having 90° Crank Spacing

	$\Sigma\beta$ OF ORDERS*				
FIRING ORDER	1/2, 7½, 8½, 3½, 4½	1½, 5½, 6½	2, 6, 10	1, 3, 5, 8, 7	4,8
1,6,2,5,8,3,7,4	0.745	1.44	0	0	4.5
1,6,2,4,8,3,7,5	0.686	1.48	0	0	4.5
1,3,2,5,8,6,7,4	1.48	0.686	0	0	4.5
6,3,5,7,8,6,4,2	1.74	0.176	0	0	4.5
1,7,4,3,8,2,5,6	0.176	1.74	0	0	4.5

CRANK ARRANGEMENT #1,2,3 #4,5



* Values of 0 in the $\Sigma\beta$ column indicate small but not necessarily 0 values for actual β distribution.

TABLE 38.6 Torsional-Vibration Characteristics for Eight-Cylinder, Two-Cycle Engine Having 45° Crank Spacing

	$\Sigma\beta$ of orders				
Firing order	1, 7, 9	2, 6, 10	3, 5, 11	4, 12	8, 16
1, 8, 2, 6, 4, 5, 3, 7	0.056	0	0.79	2.0	4.5
1, 7, 4, 3, 8, 2, 5, 6	0.175	0	1.61	0	4.5
1, 6, 5, 2, 7, 4, 3, 8	0.112	0	1.58	0.5	4.5

In four-cycle engines, ϕ is to be taken as the actual bank angle if the second-bank cylinders fire directly after the first and as $360° + \phi$ if the second-bank cylinders omit a revolution before firing. In the latter case the cancellation formula is

$$(\phi + 360°)q = n \times 180° \tag{38.27}$$

where $n = 1, 3, 5$, etc. For example, to cancel a 4.5-order critical the bank angle should be

$$\phi = \frac{180°}{4.5} = 40° \qquad \text{for direct firing}$$

or

$$\phi = \frac{11 \times 180°}{4.5} - 360° = 80° \qquad \text{for the 360° delay}$$

Cancellation by Shift of the Node. If an engine can be arranged with approximately equal flywheel (or other rotors) at each end so that the node of a particular mode is at the center of the engine, $\Sigma\beta$ will cancel for the major orders of that mode. This procedure must be used with caution because the double flywheel arrangement may reduce the natural frequency in such a manner that low-order minor criticals of large amplitudes take the place of the canceled major criticals.

Reduction by Use of Propeller Damping in Marine Installations. From Eq. (38.21) it is evident that the torque amplitude in the shaft can be reduced below any desired level by making the flywheel moment of inertia I_1 of sufficient magnitude. The ratio of the propeller amplitude to the engine amplitude increases as the flywheel becomes larger; thus the effectiveness of the propeller as a damper is increased.

DAMPERS

Many arrangements of dampers can be employed (see Chap. 6). In each type there is a loose flywheel or inertia member which is coupled to the shaft by:

1. Coulomb friction (Lanchester damper)
2. Viscous fluid friction
3. Coulomb or viscous friction plus springs
4. Centrifugal force, equivalent to a spring having a constant proportional to the square of the speed (pendulum damper) (see Chap. 6)

Each of these types acts by generating torques in opposition to the exciting torques.

The Lanchester damper illustrated in Fig. 6.35 has been entirely superseded by designs in which fluid friction is utilized. In the Houdaille damper, Fig. 38.17, a flywheel is mounted in an oiltight case with small clearances; the case is filled with silicone fluid. The damping constant is

$$c = 2\pi\mu\left[\frac{r_2^3 b}{h_2} + \frac{1}{2}\,\frac{r_2^4 - r_1^4}{h_1}\right] \qquad \text{in.-lb-sec} \tag{38.28}$$

where μ is the viscosity of the fluid and r_1, r_2, b, h_1, and h_2 are dimensions indicated in Fig. 38.17.

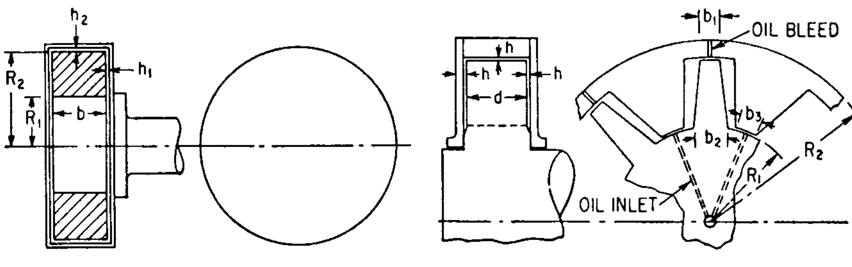


FIGURE 38.17 Schematic diagram of dampers. (*A*) Houdaille type. (*B*) Paddle type.

The paddle-type damper illustrated in Fig. 38.17 utilizes the engine lubricating oil supplied through the crankshaft. It has the damping constant

$$c = \frac{3\mu d^2(r_2^2 - r_1^2)^2 n}{h^3\left[\dfrac{d}{b_1} + \dfrac{d}{b_3} + \dfrac{4(r_2 - r_1)}{b_1 + b_2}\right]} \quad \text{in.-lb-sec} \tag{38.29}$$

where n is the number of paddles, μ is the viscosity of the fluid, and b_1, b_2, r_1, r_2, and d are dimensions indicated in Fig. 38.17. Other types of dampers are described in Ref. 2.

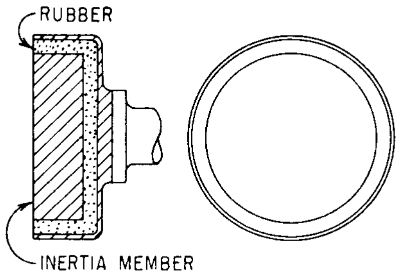


FIGURE 38.18 Schematic diagram of bonded rubber damper.

The effectiveness of these dampers may be increased somewhat by connecting the flywheel to the engine by a spring of proper stiffness, in addition to the fluid friction. In one form, Fig. 38.18, the connection is by rubber bonded between the flywheel and the shaft member. The rubber acts both as the spring and by hysteresis as the energy absorbing member. See Chaps. 32 and 34 for discussions of damping in rubber. Dampers without and with springs are defined here as *untuned* and *tuned viscous dampers*, respectively.

In many cases the mode of vibration to be damped is essentially internal to the engine. Then the damper is located at the end of the engine remote from the flywheel. If the mode to be damped is essentially one between driven masses, other locations may be desirable or necessary.

Design of the Untuned Viscous Damper, Exact Procedure. The first step in the design procedure is to make a tentative assumption of the polar moment of inertia of the floating inertia member. If the damper is attached to the forward end of the crankshaft with the primary purpose of damping vibration in the engine, the size should be from 5 to 25 per cent, depending on the severity of the critical to be damped, of the total inertia in the engine part of the system, excluding the flywheel.

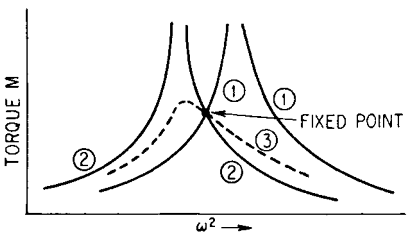


FIGURE 38.19 Resonance curves for various conditions of auxiliary mass dampers: (1) damper free, $c = 0$; (2) damper locked, $c = \infty$; (3) auxiliary mass coupled to shaft by damping.

Usually it is advantageous to minimize the torque in a particular shaft section. This may be done as follows: For a series of frequencies plot the resonance curve of this torque, first without the floating damper mass and then with the damper mass locked to the damper hub. Plot the curves with all ordinates positive. The nature of such a plot is shown in Fig. 38.19. The point of intersection is called the *fixed point*. The plot is shown as if there were only one resonant frequency. Usually only one is of interest, and the curves are plotted in its vicinity. If the plot were extended, there would be a series of fixed points.

If a damping constant is assigned to the damper and the new resonance curve plotted, it will be similar to curve 3 in Fig. 38.19 and will pass through the fixed point. If there is no other damping in the system except that in the damper, all of the resonance curves will pass through the fixed points, independent of the value assigned to the damping constant.[13] Therefore, the amplitude at the fixed point is the lowest that can be obtained for the assumed damper size. If this amplitude is too large, it will be necessary to increase the damper size; if the amplitude is unnecessarily small, the damper size can be decreased. When a satisfactory size of damper has been selected, it is necessary to find the damping constant which will put the resonance curve through the fixed point with a zero slope. Assume a value of ω^2 slightly lower than its value at the fixed point, and compute the amplitude at that value of ω^2 with the damping constant c entered as an algebraic unknown. Equating this amplitude to that at the fixed point, the unknown damping constant c can be calculated. Repeat the calculation with a value of ω^2 higher than the fixed point value by the same increment. The mean of the two values of c thus obtained will be as close to the optimum value as construction of the damper will permit. In constructing these resonance curves, it is not necessary to construct complete curves over a wide range of frequencies but only over a short interval in the vicinity of the fixed point.

Two-Mass Approximation. If the system is replaced by a two-mass system in the manner utilized to make a first estimate (see the section *Natural Frequency Calculations*) of the one-noded mode, the results are further approximated by the following formulas:

For such a two-mass plus damper system the amplitude at the fixed point is given by[13]

$$M_{12} = M_e \Sigma\beta \left(\frac{2I_2 + I_d}{I_d} \right) \tag{38.30}$$

where $M_e = Srh$ is the exciting torque per cylinder. The optimum damping is

$$c = \left[\frac{KI_2 I_d^2 (2I_1 + 2I_2 + I_d)}{I_1 (I_2 + I_d)(2I_3 + I_d)} \right]^{1/2} \quad \text{in.-lb/rad/sec} \tag{38.31}$$

where I_1 = polar moment of inertia for flywheel or generator
I_2 = 40 percent of engine polar moment of inertia taken up to flywheel
I_d = polar moment of inertia of damper floating element
k = stiffness from No. 1 crank to flywheel

Tuned Viscous Dampers. The procedure for the design of a tuned viscous damper is as follows:

1. Assume a polar inertia and a spring constant for the damper. As a first assumption, adjust the spring constant so that if f is the frequency of the mode to be suppressed and f_n is the natural frequency of the damper, assuming the hub as a fixed point,

$$\frac{f_n}{f} = 0.8$$

2. Plot the resonance curves of M for a particular section, first for the damper locked, then with zero damping but the damper spring in place. All ordinates are plotted positive. The curves have the general form of those shown in Fig. 38.20. They intersect in two fixed points through which all resonance curves pass, irrespective of the damping constant in the damper. If the fixed point a is higher than b, assume a lower constant for the damper spring and recalculate the M curve. If a is lower than b, do the reverse. Thus adjust the damper spring constant until a and b are of equal height. If this amplitude M is higher than desired, it is necessary to repeat the calculation with a larger damper.

With the spring and damper mass adjusted, a direct calculation (similar to that for the untuned damper) can be made to determine the damping constant c_r which will give the resonance curve the same ordinate at an intermediate frequency indicated by point c as at a and b. Figure 38.20*B* shows the resonance curve of an ideally adjusted damper.

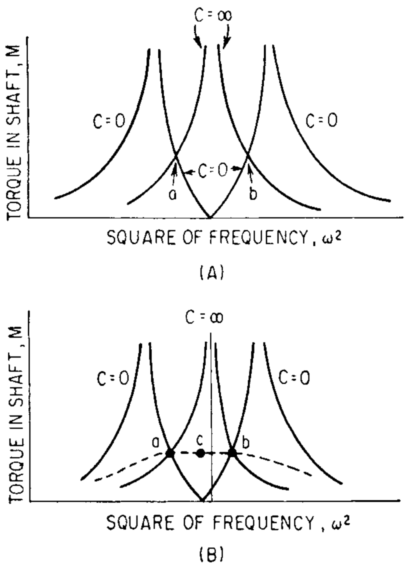


FIGURE 38.20 Curves of torque vs. square of frequency for auxiliary mass damper.

3. For a range of frequencies, using the inertia, spring, and damping constants as determined above, compute the amplitude of the damper mass relative to its hub by a forced-vibration calculation. In this calculation the damper spring constant becomes the complex number $(k + jc\omega)$. The load for which the damper springs must be designed is k times the relative amplitude of the damper mass to its hub. The torque on the damper is approximately $M_e\Sigma\beta$. For a discussion on an untuned viscous damper, see Ref. 6.

Pendulum Dampers. The principle of a pendulum damper is shown in Fig. 38.21*A*. (Also see Chap. 6.) The hole-pin construction usually used, which is equivalent to that of Fig. 38.21*A*, is shown in Fig. 38.21*B*. It is undesirable to have any friction in the damper. The damper produces an effect equivalent to a fixed flywheel, and the inertia of this flywheel is different for each order of vibration.

The design formulas for the pendulum damper are as follows:[1] If the length L is made equal to

$$L = \frac{R}{1 + q_0^2} \tag{38.32}$$

the damper is said to be tuned to order q_0. For excitation of q_0 cycles per revolution, it will act as an infinite flywheel, keeping the shaft at its point of attachment to uniform rotation insofar as q_0 order vibrations are concerned. But other orders of vibration may exist in the shaft.

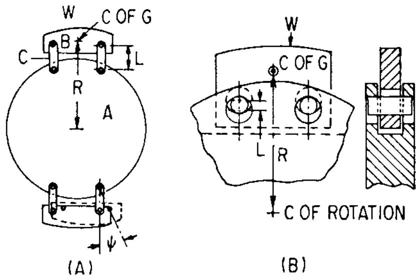


FIGURE 38.21 Pendulum-type damper. The arrangement is shown in principle at *A*, and the Chilton construction is shown schematically at *B*.

If the shaft at the point of attachment of the damper is vibrating with order q and amplitude θ, the maximum link angle ψ (see Fig. 38.21) is

$$\psi = \frac{\theta q^2(1 + q_0^2)}{q_0^2 - q^2} \quad \text{rad} \qquad (38.33)$$

The torque exerted by a single element of the damper is

$$M = \left[\frac{WR^2}{1 - q^2/q_0^2 + J}\right]\frac{q^2\Omega^2\theta}{g} \quad \text{in.-lb} \qquad (38.34)$$

where W is the weight of an element and J is the polar inertia of an element about its own center-of-gravity. The J term is equivalent to an addition to the damper hub. Dropping this term, the damper is equivalent to a flywheel of polar inertia

$$J_d = \frac{WR^2}{1 - q^2/q_0^2} \quad \text{in.}^2\text{-lb} \qquad (38.35)$$

For $q < q_0$ this is a positive flywheel, for $q = q_0$ an infinite flywheel, and for $q > q_0$ a negative flywheel. Omitting the J term and eliminating θ between Eqs. (38.33) and (38.34),

$$\psi = \frac{M(1 + q_0^2)g}{q_0^2 WR^2\Omega^2} \quad \text{rad} \qquad (38.36)$$

In-Line Diesel Engine. As applied to a diesel engine, the above procedure is much more difficult. The exciting torques in diesel engines are nearly independent of speed. Hence from Eq. (38.36) it is evident that ψ will be inversely proportional to Ω^2. Thus for a variable-speed engine the damper size is fixed by the low-speed end of the range; if ψ is kept in the 20 to 30° limit, the size may be excessive. This difficulty usually can be overcome by tuning the damper as a negative flywheel, thus acting to raise the undesired critical above the operating range while keeping ψ to a reasonable limit at low speed. The procedure is as follows:

Assuming a damper size and a q/q_0 ratio, a forced-vibration calculation is made starting at the flywheel end, for the maximum speed of the engine. In this calculation the damper is treated as a fixed flywheel of polar inertia $n\{[WR^e(1 - q^2/q_0^2)^{-1}] + J\}$ plus the inertia of the fixed carrier which supports the moving weights, where n is the number of weights. This calculation will yield θ, the amplitude at the damper hub, and the maximum torque in the engine shaft. Then ψ is given by Eq. (38.33). If either the shaft torque or the damper amplitude ψ is too large, it is necessary to increase the damper size and possibly adjust the q/q_0 ratio as well. A similar check for ψ is made at the low-speed end of the range with further adjustment of WR^e and q/q_0 if necessary.

With a pendulum damper fitted, the equivalent inertia is different for each order of vibration so that each order has a different frequency. A damper tuned as a negative flywheel for one order becomes a positive flywheel for lower orders; thus, it reduces the frequencies of those orders, with possibly unfortunate results.

In in-line engines the application of a pendulum damper may be further complicated by the necessity of suppressing several orders of vibration, thus requiring several sets of damper weights. Alternatively, both a pendulum- and viscous-type damper may be fitted to an engine.

In general, the pendulum-type dampers are more expensive than the viscous types. Wear in the pins and their bushings changes the properties of the damper, thus requiring replacement of these parts at intervals.

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