CHAPTER 31
THEORY OF SHOCK ISOLATION

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INTRODUCTION

This chapter presents an analytical treatment of the isolation of shock. Two classes of shock are considered: (1) shock characterized by motion of a support or foundation where a shock isolator reduces the severity of the shock experienced by equipment mounted on the support and (2) shock characterized by forces applied to or originating within a machine where a shock isolator reduces the severity of shock experienced by the support. In the simplified concept of shock isolation, the equipment and support are considered rigid bodies, and the effectiveness of the isolator is measured by the forces transmitted through the isolator (resulting in acceleration of equipment if assumed rigid) and by the deflection of the isolator. Linear isolators, both damped and undamped, together with isolators having special types of nonlinear elasticity are considered. When the equipment or floor is not rigid, the deflection of nonrigid members is significant in evaluating the effectiveness of isolators. Analyses of shock isolation are included which consider the response of nonrigid components of the equipment and floor.

IDEALIZATION OF THE SYSTEM

In the application of shock isolators to actual equipments, the locations of the isolators are determined largely by practical mechanical considerations. In general, this results in types of nonsymmetry and coupled modes not well adapted to analysis by simple means. It is convenient in the design of shock isolators to idealize the system to a hypothetical one having symmetry and uncoupled modes of motion.

UNCOUPLED MOTIONS

The first step in idealizing the physical system is to separate the various translational and rotational modes, i.e., to *uncouple* the system. Consider the system of Fig. 31.1

consisting of a *homogeneous* block attached at the corners, by eight identical springs, to a movable rigid frame. The block and frame are constrained to move in the plane of the paper. With the system at rest, the frame is given a sudden vertical translation. Because of the symmetry of both mass and stiffness relative to a vertical plane perpendicular to the paper, the response motion of the block is pure vertical translation. Similarly, a sudden horizontal translation of the frame excites pure horizontal translation of the block. A sudden rotation about an axis through the geometric center of the block produces pure rotation of the block about this axis. This set of response behaviors is characteristic of an *uncoupled* system.

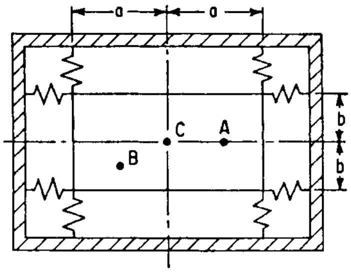

FIGURE 31.1 Schematic diagram of three degree-of-freedom mounting.

If the block of Fig. 31.1 is not homogeneous, the mass center (or center-of-gravity) may be at A or B instead of C. Consider the response to a sudden vertical translation of the frame if the mass center is at A. If the response were pure vertical translation of the block, the dynamic forces induced in the vertical springs would have a resultant acting vertically through C. However, the "inertia force" of the block must act through the mass center at A. Thus, the response cannot be pure vertical translation, but must also include rotation. Then the motions of vertical translation and rotation are said to be *coupled.* A sudden horizontal translation of the frame would still excite only a horizontal translation of the block because A is symmetrical with respect to the horizontal springs; thus this horizontal motion remains *uncoupled.* If the mass center were at B, i.e., in neither the vertical nor the horizontal plane of symmetry, then a sudden vertical translation of the frame would excite both vertical and horizontal translations of the block, together with rotation. In this case, all three motions are said to be *coupled.*

It is not essential that a system have any kind of geometric symmetry in order that its motions be uncoupled, but rather that the resultant of the spring forces be either a force directed through the center-of-gravity of the block or a couple. If the motions are completely uncoupled, there are three mutually orthogonal directions such that translational motion of the base in any one of these directions excites only a translation of the body in the same direction. Similarly there are three orthogonal axes, concurrent at the mass center, having the property that a pure rotation of the base about any one of these axes will excite a pure rotation of the body about the same axis. The idealized systems considered in this chapter are assumed to have uncoupled rigid body motions.

ANALOGY BETWEEN TRANSLATION AND ROTATION

If the motions in translational and rotational modes are uncoupled, motion in the rotational mode may be inferred by analogy from motion in the translational mode, and vice versa. Consider the system of Fig. 31.1. Assume that the mass center is at C and the forces in the four vertical springs have a negligible horizontal component at all times. For horizontal motion the differential equation of motion is

$$m\ddot{\delta} + 4k\delta = -m\ddot{u} \tag{31.1}$$

where δ = horizontal displacement of mass center of block relative to center-of-frame, in.
m = mass of block, lb-sec^2/in.
k = spring stiffness for each spring, lb/in.
u = absolute horizontal displacement of center-of-frame, in. In the equilibrium position the point C lies at the frame center.

Equation (31.1) may be written

$$\ddot{\delta} + \omega_n^2\delta = -\ddot{u} \tag{31.2}$$

where $\omega_n = \sqrt{4k/m}$, rad/sec, is the angular natural frequency in horizontal vibration.
For rotation of the block the corresponding equation of motion is

$$I\ddot{\gamma}_r + 4k(a^2 + b^2)\gamma_r = -I\ddot{\Gamma} \tag{31.3}$$

where I = mass moment of inertia of block about axis through C, perpendicular to plane of paper, lb-in.-sec^2
a, b = distances of spring center lines from mass center (see Fig. 31.1), in.
γ_r = rotation of block relative to frame in plane of paper, rad
Γ = absolute rotation of frame in plane of figure, rad

Equation (31.3) may be written

$$\ddot{\gamma}_r + \omega_{n1}^2\gamma_r = -\ddot{\Gamma} \tag{31.4}$$

where $\omega_{n1} = \sqrt{4k(a^2 + b^2)/I}$ is the angular natural frequency in rotation.
Equations (31.2) and (31.4) are analogous; γ_r corresponds to δ, Γ corresponds to u, and ω_{n1} corresponds to ω_n. Because of this analogy, only the horizontal motion described by Eq. (31.2) is considered in subsequent sections; corresponding results for rotational motion may be determined by analogy.

CLASSIFICATION OF SHOCK ISOLATION PROBLEMS

It is convenient to divide shock isolation problems into two major classifications according to the physical conditions:

Class I. Mitigation of effects of foundation motion

Class II. Mitigation of effects of force generated by equipment

Isolators in the first class include such items as the draft gear on a railroad car, the shock strut of an aircraft landing gear, the mounts on airborne electronic equipment, and the corrugated paper used to package light bulbs. The second class includes the recoil cylinders on gun mounts and the isolators on drop hammers, looms, and reciprocating presses. The objectives in the two classes of problems are allied, but distinct. In Class I the objective is to limit the shock-induced stresses in critical components of the protected equipment. In Class II the purpose is to limit the forces transmitted to the support for the equipment in which the shock originates.

IDEALIZED SYSTEMS—CLASS I

The simplest approach to problems of Class I is through a study of single degree-of-freedom systems (see Chap. 2). Consider the system of Fig. 31.2*A*. The basic elements are a mass and a spring-dashpot unit attached to the mass at one end. The block may be taken to represent the equipment (assumed to be a rigid body), and the spring-dashpot unit to represent the shock isolator. The displacement of the support is u. The equation of motion is

$$m\ddot{\delta} + F(\dot{\delta},\delta) = -m\ddot{u} \tag{31.5}$$

where
m = mass of block, lb-sec^2/in.
δ = deflection of spring ($\delta = x - u$; see Fig. 31.2), in.
$F(\dot{\delta},\delta)$ = force exerted on mass by spring-dashpot unit (positive when tensile), lb
u = absolute displacement of left-hand end of spring-dashpot unit, in.

In the typical shock isolation problem, the system of Fig. 31.2*A* is initially at rest ($\dot{u} = \dot{\delta} = 0$) in an equilibrium position ($u = \delta = 0$). An external shock causes the support to move. The corresponding movement of the left end of the shock isolator is described in terms of the support acceleration $\ddot{u}$. Then Eq. (31.5) may be solved for the resulting extreme values of δ and $F(\dot{\delta},\delta)$, and these values may be compared with the permissible deflection and force transmission limits of the shock isolator. It also is necessary to determine whether the internal stresses developed in the equipment are excessive. If the equipment is sufficiently rigid that all parts have substantially equal accelerations, then the internal stresses are proportional to $\ddot{x}$ where $-m\ddot{x} = F(\dot{\delta},\delta)$.

A critical component of the equipment may be sufficiently flexible to have a substantially different acceleration than that determined by assuming the equipment rigid. If the total mass of such components is small in comparison with the equipment mass, the above analysis may be extended to cover this case. Equation (31.5) is first solved to determine not merely the extreme value of $F(\dot{\delta},\delta)$ but its time-history. Then the acceleration $\ddot{x}$ may be determined from the relation $\ddot{x} = -F(\dot{\delta},\delta)/m$. Now consider the system shown in Fig. 31.2*B* having a component of mass m_c and stiffness-damping characteristics $F_c(\dot{\delta}_c,\delta_c)$. The force $F_c(\dot{\delta}_c,\delta_c)$ transmitted to the mass m_c and the resulting acceleration $\ddot{x}_c = -F_c(\dot{\delta}_c,\delta_c)/m_c$ may be found by solving an equation that is analogous to Eq. (31.5) where $\ddot{x}$ is substituted for $\ddot{u}$, $\ddot{\delta}_c$ for $\ddot{\delta}$, and $F_c(\dot{\delta}_c,\delta_c)$ for $F(\dot{\delta},\delta)$.

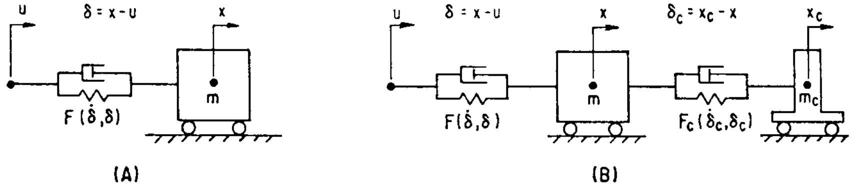

FIGURE 31.2 Idealized systems showing use of isolator with transmitted force $F(\dot{\delta},\delta)$ to protect equipment of mass m from effects of support motion u. In (*A*) the equipment is rigid and in (*B*) there is a flexible component having stiffness-damping characteristics $F_c(\dot{\delta}_c, \delta_c)$ and mass m_c.

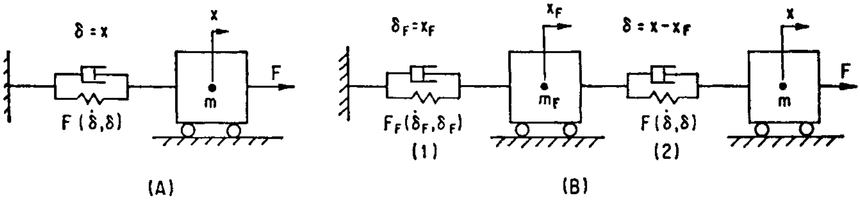

FIGURE 31.3 Idealized systems showing use of isolator with transmitted force $F(\dot{\delta},\delta)$ to reduce force transmitted to foundation when force F is applied to equipment of mass m. In (A) the foundation is rigid and in (B) it has mass m_F and stiffness damping characteristics $F_F(\dot{\delta}_F,\delta_F)$.

IDEALIZED SYSTEMS—CLASS II

Consider the system of Fig. 31.3A to represent the equipment (mass m) attached to its support by the shock isolator (spring-dashpot unit). The left end of the spring-dashpot unit is fixed to the supporting structure and there is a force F applied externally to the mass. The force F may be a real external force or it may be an "inertia force" generated by moving parts of the equipment. The equation of motion may be written as

$$m\ddot{\delta} + F(\dot{\delta},\delta) = F \tag{31.6}$$

where F is the external force applied to the mass in pounds and the relative displacement δ of the ends of the spring-dashpot unit is equal to the absolute displacement x of the mass. Assuming the system to be initially in equilibrium ($\dot{\delta} = 0$, $\delta = 0$), Eq. (31.6) is solved for extreme values of δ and $F(\dot{\delta},\delta)$ since F is a known function of time. These are to be compared with the displacement and force limitations of the shock isolator. Often the supporting structure is sufficiently rigid that the maximum force in the isolator may be considered as a force applied statically to the support. Then the foregoing analysis is adequate for determining the stress in the support.

The load on the floor may be treated as dynamic instead of static by a simple analysis if the displacement and velocity of the support are negligible in comparison with those of the equipment. Consider the system of Fig. 31.3B where the supporting structure is represented as a mass m_F and a spring-dashpot unit in place of the rigid support shown in Fig. 31.3A. The force acting on the supporting structure is a known function of time $F(\dot{\delta},\delta)$ as found from the previous solution of Eq. (31.6). To find the maximum force *within* the support structure requires a solution of an equation analogous to Eq. (31.6) where $\ddot{\delta}_F$ is substituted for $\ddot{\delta}$, m_F for m, $F_F(\dot{\delta}_F,\delta_F)$ for $F(\dot{\delta},\delta)$, and $F(\dot{\delta},\delta)$ for F. For engineering purposes it suffices to find the extreme values of δ_F and $F_F(\dot{\delta}_F,\delta_F)$. The first is needed to verify the assumption that support motion is negligible compared with equipment motion, and can be used to determine the maximum stress in the support. The second is the maximum force applied by the support structure to its base.

MATHEMATICAL EQUIVALENCE OF CLASS I AND CLASS II PROBLEMS

The similarity of shock isolation principles in Class I and Class II is indicated by the similar form of Eqs. (31.5) and (31.6). The right-hand side ($-m\ddot{u}$ or F) is given as a function of time, and the extreme values of δ and $F(\dot{\delta},\delta)$ are desired. When the actual system is represented by two separate single degree-of-freedom systems, as shown in Figs. 31.2*B* and 31.3*B*, the time-history of $F(\dot{\delta},\delta)$ is also required. Figure 31.4 may be considered a generalized form of the applicable system. In Class I, $F = 0$, $F_1(\dot{\delta}_1,\delta_1)$ represents the properties of the isolator, and $m_2,F_2(\dot{\delta}_2,\delta_2)$ represents the component to be protected. In Class II, $u = 0$, $F_2(\dot{\delta}_2,\delta_2)$ represents the properties of the isolator, and $m_1,F_1(\dot{\delta}_1,\delta_1)$ represents the supporting structure.

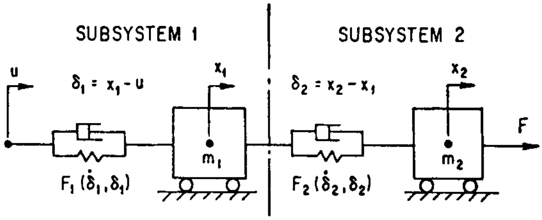

FIGURE 31.4 General two degree-of-freedom system.

The system of Fig. 31.4, with the spring-dashpot units nonlinear, requires the use of a digital computer to investigate performance characteristics. Analytical methods are feasible if the system is linearized by assuming that each spring-dashpot unit has a force characteristic in the form

$$F(\dot{\delta},\delta) = c\dot{\delta} + k\delta \tag{31.7}$$

where c = damping coefficient, lb-sec/in., and k = spring stiffness, lb/in. Even with this simplification, the number of parameters $(m_1,c_1,k_1,m_2,c_2,k_2)$ is so great that it is necessary to confine the analysis to a particular system. If the damping may be neglected [let $c = 0$ in Eq. (31.7)], then it is feasible to obtain equations in a form suitable for routine use. Use of this idealization is described in the section on *Response of Equipment with a Flexible Component.*

A different form of idealization is indicated when the "equipment" is flexible; e.g., a large, relatively flexible aircraft subjected to landing shock. Then it is important to represent the aircraft as a system with several degrees-of-freedom. To find resulting stresses, it is necessary to superimpose the responses in the various modes of motion that are excited.

RESPONSE OF A RIGID BODY SYSTEM TO A VELOCITY STEP

PHYSICAL BASIS FOR VELOCITY STEP

The idealization of a shock motion as a simple change in velocity (velocity step) may form an adequate basis for designing a shock isolator and for evaluating its effectiveness. Consider the two types of acceleration $\ddot{u}$ vs. time t curves illustrated in Fig. 31.5*A*. The solid line represents a rectangular pulse of acceleration and the dashed line represents a half-sine pulse of acceleration. Each pulse has a duration τ. In Fig.

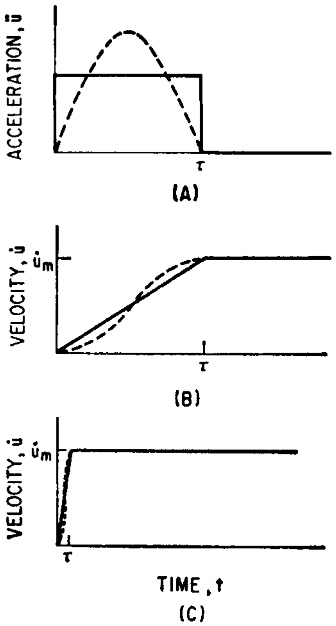

FIGURE 31.5 Acceleration-time curves (*A*) and velocity-time curves (*B*) and (*C*) for rectangular acceleration pulse (solid curves) and half-sine acceleration pulse (dashed curves).

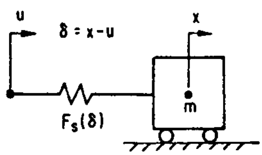

FIGURE 31.6 Idealized system showing use of undamped isolator to protect equipment from effects of support motion *u*.

31.5*B*, the corresponding velocity-time curves are shown. Each of these curves is defined completely by specifying the type of acceleration pulse (rectangular or half-sine), the duration τ, and the velocity change $\dot{u}_m$. The curves of Fig. 31.5*B* are repeated in Fig. 31.5*C* with the time scale shrunk to one-tenth. If τ is *sufficiently short,* the only significant remaining characteristic of the velocity step is the velocity change $\dot{u}_m$. The idealized velocity step, then, is taken to be a discontinuous change of $\dot{u}$ from zero to $\dot{u}_m$. A shock isolator characteristically has a low natural frequency (long period), and this idealization leads to good results even when the pulse duration τ is significantly long.

GENERAL FORM OF ISOLATOR CHARACTERISTICS

The differential equation of motion for the undamped, single degree-of-freedom system shown in Fig. 31.6 is

$$m\ddot{\delta} + F_s(\delta) = -m\ddot{u} \qquad (31.8)$$

where m represents the mass of the equipment considered as a rigid body, u represents the motion of the support which characterizes the condition of shock, and $F_s(\delta)$ is the force developed by the isolator at an extension δ (positive when tensile). Equation (31.8) differs from Eq. (31.5) in that $F_s(\delta)$, which does not depend upon $\dot{\delta}$, replaces $F(\dot{\delta},\delta)$ because the isolator is undamped. The effect of a velocity step of magnitude $\dot{u}_m$ at $t = 0$ is considered by choosing the initial conditions: At $t = 0$, $\delta = 0$ and $\dot{\delta} = \dot{u}_m$. These conditions correspond to a negative velocity step. This choice is made to avoid dealing with negative values of δ and $\dot{\delta}$. If $F_s(\delta)$ is not an odd function of δ, a positive velocity step requires a separate analysis.

A first integration of Eq. (31.8) yields

$$\dot{\delta}^2 = \dot{u}_m{}^2 - \frac{2}{m}\int_0^{\delta} F_s(\delta)d\delta \qquad (31.9)$$

At the extreme value of isolator deflection, $\delta = \delta_m$ and the velocity $\dot{\delta}$ of deflection is zero. Then from Eq. (31.9),

$$\int_0^{\delta_m} F_s(\delta)\,d\delta = \tfrac{1}{2} m \dot{u}_m^2 \tag{31.10}$$

The right side of Eq. (31.10) represents the initial kinetic energy of the equipment relative to the support, and the integral on the left side represents the work done on the isolator. The latter quantity is equal to the elastic potential energy stored in the isolator, since there is no damping.

For the special case of a rigid body mounted on an undamped isolator, Eq. (31.10) suffices to determine all important results. In particular, the quantities of engineering significance are:

1. The maximum deflection of the isolator δ_m
2. The maximum isolator force, $F_m = F_s(\delta_m) = m\ddot{x}_m$
3. The corresponding velocity change $\dot{u}_m$

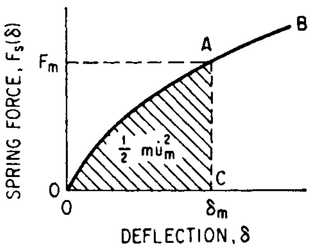

FIGURE 31.7 Typical force-deflection curve for undamped isolator.

The interrelations of these three quantities are shown graphically in Fig. 31.7. The curve *OAB* represents the spring force $F_s(\delta)$ as a function of deflection δ. If point *A* corresponds to the extreme excursion, then its abscissa represents the maximum deflection δ_m. The shaded area *OAC* is proportional to the potential energy stored by the isolator; according to Eq. (31.10), this is equal to the initial kinetic energy $m\dot{u}_m^2/2$. The maximum ordinate (at *A*) represents the maximum spring force F_m. [It is possible to have a spring force $F_s(\delta)$ which attains a maximum value at $\delta = \delta_f < \delta_m$. Then $F_m = F_s(\delta_f)$.]

The design requirements for the isolator usually include as a specification one or more of the following quantities:

1. Maximum allowable deflection δ_a
2. Maximum allowable transmitted force F_a
3. Maximum expected velocity step $\dot{u}_a$

It is important to observe that the limits 1 and 2 establish an upper limit $F_a\delta_a$ on the work done on the mass. It follows that $\dot{u}_a$ must satisfy the relation

$$F_a\delta_a \geq m\dot{u}_a^2/2$$

or the specifications are impossible to meet. The specifications may be expressed mathematically as follows:

$$\delta_m \leq \delta_a \qquad F_m \leq F_a \qquad \dot{u}_m \geq \dot{u}_a \tag{31.11}$$

In many instances it is advantageous to eliminate explicit reference to the mass *m*. Then the allowable absolute acceleration $\ddot{x}_a$ of the mass is specified instead of the allowable force F_a where $F_a = m\ddot{x}_a$. With this substitution the second of Eqs. (31.11) is replaced by

$$\ddot{x}_m \leq \ddot{x}_a \tag{31.12}$$

The acceleration $\ddot{x}$ is determined as a function of time by using $\dot{\delta}$ from Eq. (31.9) and finding the time t corresponding to a given value of δ:

$$t = \int_0^{\delta} \frac{d\delta}{\dot{\delta}} \tag{31.13}$$

From Eq. (31.13) and the relation $\ddot{x} = F_s(\delta)/m$, the acceleration time-history is found.

The integrations required by Eqs. (31.9) and (31.13) sometimes are difficult to perform, and it is necessary to use numerical methods. Then a difficulty arises with the integral in Eq. (31.13). As δ approaches the extreme value δ_m, the velocity $\dot{\delta}$ in the denominator of the integrand approaches zero. The difficulty is circumvented by first using Eq. (31.13) to integrate up to some intermediate displacement δ_b less than δ_m; then the alternative form, Eq. (31.14), may be used in the region of $\delta = \delta_m$:

$$t = t_b + \int_{\dot{\delta}_b}^{\dot{\delta}} \frac{d\dot{\delta}}{\ddot{\delta}} \tag{31.14}$$

where t_b is the time at which $\delta = \delta_b$, as determined from Eq. (31.13).

In the next three sections three different kinds of spring force-deflection characteristics $F_s(\delta)$ are considered. Equation (31.10) is applied to find the relation between $\dot{u}_m$ and δ_m. Curves relating $\dot{u}_m$, δ_m, and $\ddot{x}_m$ in a form useful for design or analysis are presented.

EXAMPLES OF PARTICULAR ISOLATOR CHARACTERISTICS

Linear Spring. The force-deflection characteristic of a linear spring is

$$F_s(\delta) = k\delta \tag{31.15}$$

where k = spring stiffness, lb/in. Using the notation

$$\omega_n = \sqrt{\frac{k}{m}} \qquad \text{rad/sec} \tag{31.16}$$

the maximum acceleration is

$$\ddot{x}_m = \omega_n^2\delta_m \tag{31.17}$$

From Eqs. (31.10) and (31.16), the relation between velocity change $\dot{u}_m$ and maximum deflection δ_m is

$$\dot{u}_m = \omega_n\delta_m \tag{31.18}$$

Combining Eqs. (31.18) and (31.17),

$$\ddot{x}_m = \omega_n\dot{u}_m \tag{31.19}$$

Hardening Spring (Tangent Elasticity). The isolator spring may be nonlinear with a "hardening" characteristic; i.e., the slope of the curve representing spring force vs. deflection increases with increasing deflection. Rubber in compression has this behavior. A representative curve having this characteristic is defined by

$$F_s(\delta) = \frac{2kd}{\pi} \tan \frac{\pi\delta}{2d} \tag{31.20}$$

where the constant k is the *initial* slope of the curve (lb/in.) and a vertical asymptote is defined by $\delta = d$ (in.). Such a curve is shown graphically in Fig. 31.8. Using the notation of Eq. (31.16) and the relation $m\ddot{x}_m = F_s(\delta_m)$, Eq. (31.20) gives the following relation between maximum acceleration and maximum deflection:

$$\frac{\ddot{x}_m}{\omega_n^2 d} = \frac{2}{\pi} \tan \frac{\pi\delta_m}{2d} \tag{31.21}$$

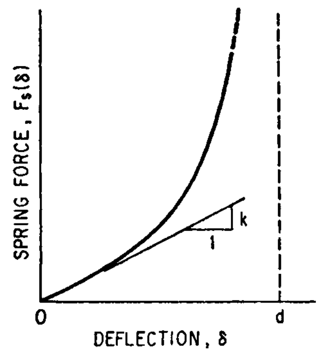

FIGURE 31.8 Typical force-deflection curve for hardening spring (tangent elasticity).

Note that ω_n, the angular natural frequency for a linear system, has the same meaning for small amplitude (small δ_m) motions of the nonlinear system. For large amplitudes the natural frequency depends on δ_m. Using Eq. (31.16), substituting for $F_s(\delta)$ from Eq. (31.20) in Eq. (31.10), and performing the indicated integration, the relation between velocity change and maximum displacement is

$$\frac{\dot{u}_m^2}{\omega_n^2 d^2} = \frac{8}{\pi^2} \log_e \left(\sec \frac{\pi\delta_m}{2d} \right) \tag{31.22}$$

A graphical presentation relating the important variables $\dot{u}_m$, $\ddot{x}_m$, and δ_m is convenient for design and analysis. Such data are presented compactly as relations among the dimensionless parameters δ_m/d, $\dot{u}_m/\omega_n d$, and $\ddot{x}_m\delta_m/\dot{u}_m^2$. The physical significance of the ratio $\ddot{x}_m\delta_m/\dot{u}_m^2$ is interpreted by multiplying both numerator and denominator by m. Then the numerator represents the product of the maximum spring force $F_m (= m\ddot{x}_m)$ and the maximum spring deflection δ_m. This product is the maximum energy that *could* be stored in the spring. The denominator $m\dot{u}_m^2$ is *twice* the energy that *is* stored in the spring. The minimum *possible* value of the ratio $\ddot{x}_m\delta_m/\dot{u}_m^2$ is ½. Actual values of the ratio, always greater than ½, may be considered to be a measure of the departure from optimum capability.

In Fig. 31.9 the solid curve represents $\dot{u}_m/\omega_n d$ as a function of δ_m/d and the dashed curve shows the corresponding result for a linear spring [see Eq. (31.18)]. In Fig. 31.10 the solid curve shows $\ddot{x}_m\delta_m/\dot{u}_m^2$ vs. δ_m/d for an isolator with tangent elasticity. The dashed curve in Fig. 31.10 shows $\ddot{x}_m\delta_m/\dot{u}_m^2$ for a linear spring [see Eqs. (31.17) and (31.18)]; the ratio is constant at a value of unity because a linear spring is 50 percent efficient in storage of energy, independent of the deflection.

Softening Spring (Hyperbolic Tangent Elasticity). A nonlinear isolator also may have a "softening" characteristic; i.e., the slope of the curve representing force

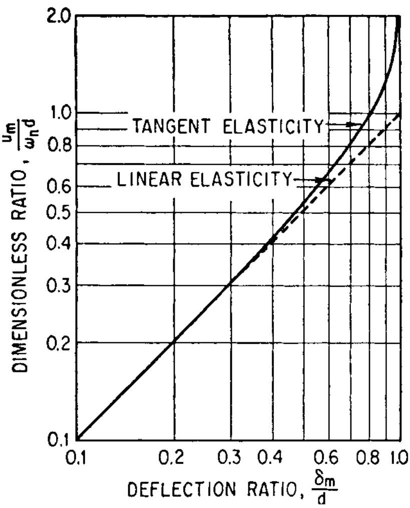

FIGURE 31.9 Dimensionless representation of relation between velocity step $\dot{u}_m$ and maximum isolator deflection δ_m for undamped isolators.

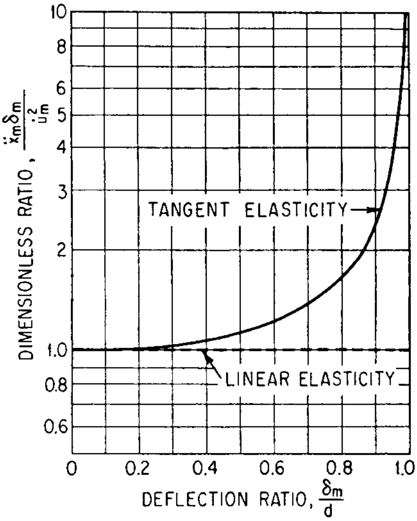

FIGURE 31.10 Dimensionless representation of relation among velocity step $\dot{u}_m$, maximum transmitted acceleration $\ddot{x}_m$, and maximum isolator deflection δ_m for undamped isolators.

vs. deflection decreases with increasing deflection. The force-deflection characteristic for a typical "softening" isolator is

$$F_s(\delta) = kd_1 \tanh \frac{\delta}{d_1} \tag{31.23}$$

where k is the initial slope of the curve. Figure 31.11 shows the form of this curve where the meaning of d_1 is evident from the figure. If $F_s(\delta)$ is replaced by $m\ddot{x}_m$, δ by δ_m, and k by $m\omega_n^2$, Eq. (31.23) becomes

$$\frac{\ddot{x}_m}{\omega_n^2 d_1} = \tanh \frac{\delta_m}{d_1} \tag{31.24}$$

where δ_m and $\ddot{x}_m$ are maximum values of deflection and acceleration, respectively, and ω_n may be interpreted as the angular natural frequency for small values of δ_m. To relate $\dot{u}_m$ to δ_m, substitute $F_s(\delta)$ from Eq. (31.23) in Eq. (31.10), let $\omega_n^2 = k/m$, and integrate:

$$\frac{\dot{u}_m^2}{\omega_n^2 d_1^2} = \log_e \left(\cosh^2 \frac{\delta_m}{d_1} \right) \tag{31.25}$$

A graphical presentation of the relation between $\dot{u}_m/\omega_n d_1$ and δ_m/d_1 is given

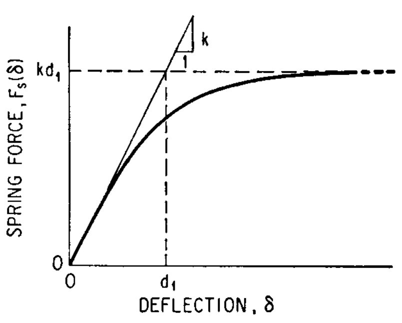

FIGURE 31.11 Typical force-deflection curve for softening spring (hyperbolic tangent elasticity).

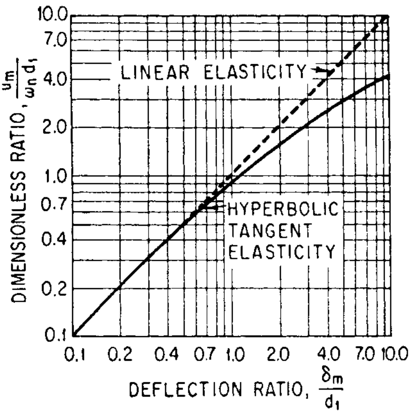

FIGURE 31.12 Dimensionless representation of relation between velocity step $\dot{u}_m$ and maximum isolator deflection δ_m for undamped isolators.

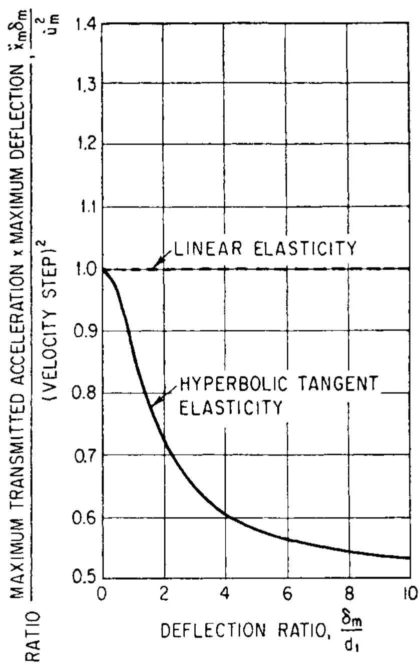

FIGURE 31.13 Dimensionless representation of energy-storage capabilities of undamped isolators.

by the solid curve of Fig. 31.12. The dashed curve shows the corresponding relation for a linear spring. In Fig. 31.13 the solid curve represents $\ddot{x}_m\delta_m/\dot{u}_m^2$ as a function of δ_m/d_1. Note that, for large values of δ_m/d_1, the ordinate approaches the minimum value ½ attainable with an isolator of optimum energy storage efficiency. The dashed curve shows the same relation for a linear spring.

Linear Spring and Viscous Damping. The addition of viscous damping can almost double the energy absorption capability of a linear shock isolator. Consider the system of Fig. 31.2*A,* with both spring and dashpot linear as defined by Eq. (31.7). Substituting $F(\dot{\delta},\delta)$ from Eq. (31.7) in Eq. (31.5) gives the equation of motion. The initial conditions are $\dot{\delta} = \dot{u}_m$, $\delta = 0$, when $t = 0$; for $t > 0$, $\ddot{u} = 0$. Letting $c_c = 2m\omega_n$ and $\zeta = c/c_c$ [see Eq. (2.12)], the equation of motion becomes

$$\ddot{\delta} + 2\zeta\omega_n\dot{\delta} + \omega_n^2\delta = 0 \tag{31.26}$$

Solutions of Eq. (31.26) for maximum deflection δ_m and maximum acceleration $\ddot{x}_m$ as functions of ζ are shown graphically in Figs. 31.14 and 31.15. In Fig. 31.14, the dimensionless ratio $\ddot{x}_m/\dot{u}_m\omega_n$ is plotted as a function of the fraction of critical damping ζ. Note that the presence of small damping reduces the maximum acceleration. As ζ is increased beyond 0.25, the maximum acceleration increases again. For $\zeta > 0.50$, the maximum acceleration occurs at $t = 0$ and exceeds that for no damping; it is accounted for solely by the damping force $c\dot{\delta} = c\dot{u}_m$.

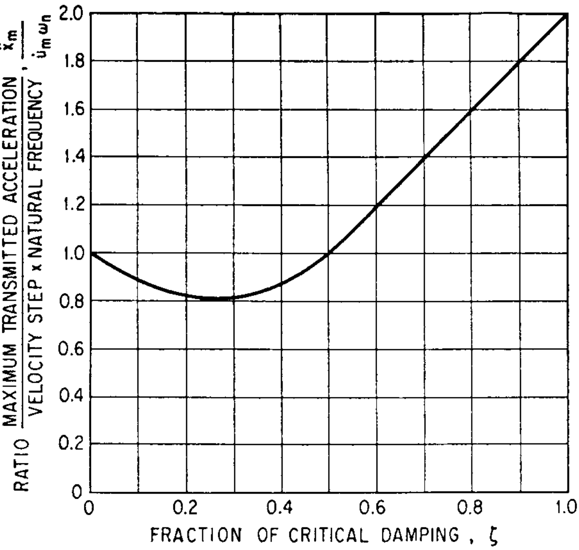

FIGURE 31.14 Dimensionless representation of maximum transmitted acceleration $\ddot{x}_m$ for an isolator having a linear spring and viscous damping.

In Fig. 31.15 the parameter $\ddot{x}_m\delta_m/\dot{u}_m^2$ is plotted as a function of ζ. (As pointed out with reference to Fig. 31.10, $\ddot{x}_m\delta_m/\dot{u}_m^2$ is an inverse measure of shock isolator effectiveness.) Figure 31.15 shows that the presence of damping improves the energy storage effectiveness of the isolator even beyond $\zeta = 0.50$. In the neighborhood $\zeta = 0.40$, the parameter $\ddot{x}_m\delta_m/\dot{u}_m^2$ attains a minimum value of 0.52—only slightly above the theoretical minimum of 0.50. This parameter has the value 1.00 for an undamped linear system, and even higher values for a hardening spring (see Fig. 31.10). On the other hand, $\ddot{x}_m\delta_m/\dot{u}_m^2$ may approach 0.50 when a softening spring is used.

True viscous damping of the type considered above is difficult to attain except in electrical or magnetic form. Fluid dampers which depend upon orifices or other constricted passages to throttle the flow are likely to produce damping forces that vary more nearly as the square of the velocity. Dry friction tends to provide damping forces which are virtually independent of velocity. The analysis of response to a velocity step in the presence of Coulomb friction is similar to that described in the section entitled *General Formulas—No Damping.*

Example 31.1. Equipment weighing 40 lb and sufficiently stiff to be considered rigid is to be protected from a shock consisting of a velocity step $\dot{u}_a = 70$ in./sec. The maximum allowable acceleration is $\ddot{x}_a = 21g$ (g is the acceleration of gravity) and available clearance limits the deflection to $\delta_a = 0.70$ in. Find isolator characteristics for: linear spring, hardening spring, softening spring, and linear spring with viscous damping.

Linear Spring. Taking the maximum velocity $\dot{u}_m$ equal to the expected velocity $\dot{u}_a$ and using Eqs. (31.18) and (31.11),

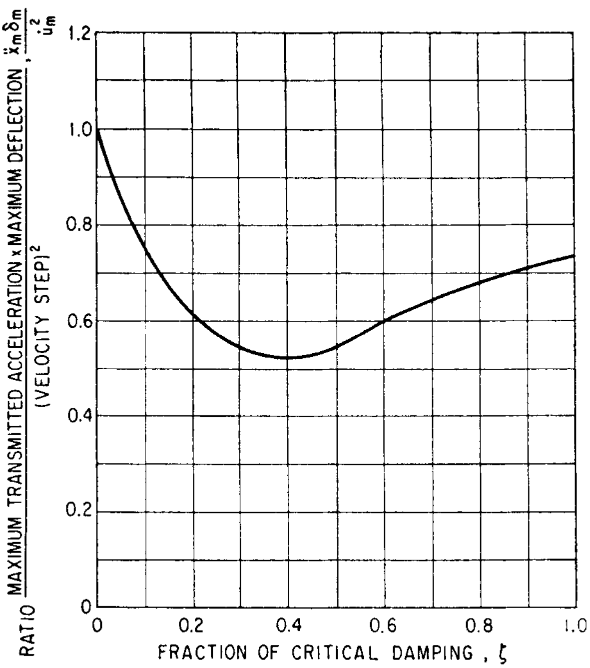

FIGURE 31.15 Dimensionless representation of energy absorption capability of an isolator having a linear spring and viscous damping.

$$\delta_m = \frac{\dot{u}_m}{\omega_n} \le \delta_a \qquad \text{or} \qquad \omega_n \ge \frac{70 \text{ in./sec}}{0.70 \text{ in.}} = 100 \text{ rad/sec}$$

From Eqs. (31.19) and (31.12), $\ddot{x}_m = \omega_n \dot{u}_m \le \ddot{x}_a$. Then

$$\omega_n \le \frac{\ddot{x}_a}{\dot{u}_m} = \frac{21 \times 386 \text{ in./sec}^2}{70 \text{ in./sec}} = 116 \text{ rad/sec}$$

Selecting a value in the middle of the permissible range gives $\omega_n = 108$ rad/sec [17.2 Hz]. The corresponding maximum isolator deflection is $\delta_m = 0.65$ in. and the maximum acceleration of the equipment is $\ddot{x}_m = 7580$ in./sec^2 = 19.6g. The isolator stiffness given by Eq. (31.16) is

$$k = m\omega_n^2 = \frac{40 \text{ lb}}{386 \text{ in./sec}^2} \times (108 \text{ rad/sec})^2 = 1210 \text{ lb/in.}$$

If, as is usually the case, the isolation is provided by several individual isolators in parallel, then the above value of k represents the sum of the stiffnesses of the individual isolators.

Hardening Spring. The tangent elasticity represented by Eq. (31.20) is assumed. Since the linear spring meets the specifications with only a small margin of safety, it is inferred that the poorer energy storage capacity of the hardening spring shown by Fig. 31.10 will severely limit the permissible nonlinearity. Using the specified values as maxima,

$$\frac{\ddot{x}_m \delta_m}{\dot{u}_m^{\,2}} = \frac{\ddot{x}_a \delta_a}{\dot{u}_a^{\,2}} = \frac{(21 \times 386) \times 0.70}{(70)^2} = 1.16$$

From Fig. 31.10:

$$\frac{\delta_m}{d} = 0.54; \text{ thus } d = \frac{0.70}{0.54} = 1.30 \text{ in.}$$

From Fig. 31.9:

$$\frac{\dot{u}_m}{\omega_n d} = 0.58; \text{ thus } \omega_n = \frac{70}{1.30 \times 0.54} = 93 \text{ rad/sec [14.8 Hz]}$$

The initial spring stiffness k from Eq. (31.16) is

$$k = \frac{40}{386}\,(93)^2 = 896 \text{ lb/in.}$$

Because the selected linear spring provides a small margin of safety and the hardening spring provides none, superficial comparison suggests that the former is superior. Various other considerations, such as compactness and stiffness along other axes, may offset the apparent advantage of the linear spring. Moreover, a shock more severe than that specified could cause the linear spring to bottom abruptly and cause much greater acceleration of the equipment.

Softening Spring. The hyperbolic tangent elasticity represented by Eq. (31.23) is assumed. The softening spring has high energy-storage capacity as shown by Fig. 31.13. By working to sufficiently high values of δ_m/d_1, it is possible to utilize this storage capacity to afford considerable overload capability. Choose $\ddot{x}_m = 20g$ and $\delta_m/d_1 = 3$. From Fig. 31.13, $\ddot{x}_m\delta_m/\dot{u}_m^{\,2} = 0.645$ at $\delta_m/d_1 = 3$. Then

$$\delta_m = 0.645\,\frac{(70)^2}{20 \times 386} = 0.41 \text{ in.} \qquad d_1 = \frac{\delta_m}{3} = 0.137 \text{ in.}$$

From Fig. 31.12, $\dot{u}_m/\omega_n d_1 = 2.15$ at $\delta_m/d_1 = 3$. Then

$$\omega_n = \frac{70}{2.15 \times 0.137} = 238 \text{ rad/sec [37.9 Hz]}$$

The initial spring stiffness k from Eq. (31.16) is

$$k = \frac{40}{386}\,(238)^2 = 5870 \text{ lb/in.}$$

This initial stiffness is much greater than those found for the linear spring and hardening spring. Accordingly, for small shocks (small $\dot{u}_m$) the isolator with the softening spring will induce much higher acceleration of the equipment than will those with linear or hardening springs. This poorer performance for small shocks is unavoidable if the isolator with the softening spring is designed to take advantage of the large energy-storage capability under extreme shocks.

Linear Spring and Viscous Damping. The introduction of viscous damping in combination with a linear spring [Eq. (31.7)] affords the possibility of large energy dissipation capacity without deterioration of performance under small shocks. From Fig. 31.15, the best performance is obtained at the fraction of critical damping $\zeta = 0.40$ where $\ddot{x}_m \delta_m / \dot{u}_m^2 = 0.52$. If the maximum isolator deflection is chosen as $\delta_m = 0.47$ in. (67 percent of δ_a), then

$$\ddot{x}_m = 0.52 \, \frac{\dot{u}_m^{\,2}}{\delta_m} = 5450 \text{ in./sec}^2 = 14.1g$$

This acceleration is 67 percent of $\ddot{x}_a$. From Fig. 31.14:

$$\frac{\ddot{x}_m}{\dot{u}_m \omega_n} = 0.86 \text{ at } \zeta = 0.40$$

Then

$$\omega_n = \frac{5450}{0.86 \times 70} = 90 \text{ rad/sec [14.3 Hz]}$$

The spring stiffness k from Eq. (31.16) is

$$k = \frac{40}{386} \, (90)^2 = 840 \text{ lb/in.}$$

The dashpot constant c is

$$c = 2\zeta m \omega_n = 2 \times 0.40 \times \frac{40}{386} \times 90 = 7.46 \text{ lb-sec/in.}$$

RESPONSE OF RIGID BODY SYSTEM TO ACCELERATION PULSE

The response of a spring-mounted rigid body to various acceleration pulses provides useful information. For example, it establishes limitations upon the use of the velocity step in place of an acceleration pulse and is significant in determining the response of an equipment component when the equipment support is subjected to a velocity step. Additional useful information is afforded by comparing the responses to acceleration pulses of different shapes.

For positive pulses ($\ddot{u} > 0$) having a single maximum value and finite duration, three basic characteristics of the pulse are of importance: maximum acceleration $\ddot{u}_m$, duration τ, and velocity change $\dot{u}_c$. A typical pulse is shown in Fig. 31.16. The relation among acceleration, duration, and velocity change is

$$\dot{u}_c = \int_0^\tau \ddot{u} \, dt \tag{31.27}$$

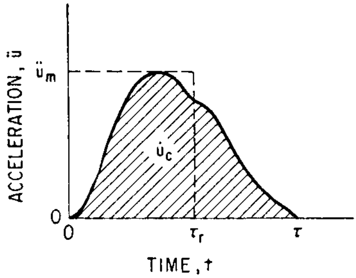

FIGURE 31.16 Typical acceleration pulse with maximum acceleration $\ddot{u}_m$ and duration τ.

where the value of the integral corresponds to the shaded area of the figure. The *equivalent rectangular pulse* is characterized by (*a*) the same maximum acceleration $\ddot{u}_m$ and (*b*) the same velocity change $\dot{u}_c$. In Fig. 31.16, the horizontal and vertical dashed lines outline the equivalent rectangular pulse corresponding to the shaded pulse. From condition (*b*) above and Eq. (31.27), the *effective duration* τ_r of the equivalent rectangular pulse is

$$\tau_r = \frac{1}{\ddot{u}_m} \int_0^\tau \ddot{u}\, dt \qquad (31.28)$$

where τ_r may be interpreted physically as the *average width* of the shaded pulse.

RESPONSE TO A RECTANGULAR PULSE

The rectangular pulse shown in Fig. 31.17 has a maximum acceleration $\ddot{u}_m$ and duration τ; the velocity change is $\dot{u}_c = \ddot{u}_m\tau$. The response of an undamped, linear, single degree-of-freedom system (see Fig. 31.6) to this pulse is found from the differential equation obtained by substituting in Eq. (31.8) $F_s(\delta) = k\delta$ from Eq. (31.15) and $\omega_n^2 = k/m$ from Eq. (31.16):

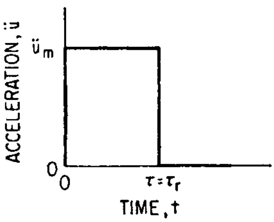

FIGURE 31.17 Rectangular acceleration pulse.

$$\ddot{\delta} + \omega_n^2\delta = -\ddot{u}_m \qquad [0 \le t \le \tau] \qquad (31.29)$$

$$\ddot{\delta} + \omega_n^2\delta = 0 \qquad [t > \tau] \qquad (31.30)$$

Using the initial conditions $\dot{\delta} = 0$, $\delta = 0$ when $t = 0$, the solution of Eq. (31.29) is

$$\delta = \frac{\ddot{u}_m}{\omega_n^2}(\cos \omega_n t - 1) \qquad [0 \le t \le \tau] \qquad (31.31)$$

For the solution of Eq. (31.30), it is necessary to find as initial conditions the values of $\dot{\delta}$ and δ given by Eq. (31.31) for $t = \tau$. Using these values the solution of Eq. (31.30) is

$$\delta = \frac{\ddot{u}_m}{\omega_n^2}[(\cos \omega_n\tau - 1)\cos \omega_n(t - \tau) - \sin \omega_n\tau \sin \omega_n(t - \tau)] \qquad [t > \tau] \qquad (31.32)$$

The motion defined by Eqs. (31.31) and (31.32) is shown graphically in Fig. 31.18 for $\tau = \pi/2\omega_n$, π/ω_n, and $3\pi/2\omega_n$.

In the isolation of shock, the extreme absolute acceleration $\ddot{x}_m$ of the mass is important. Since $\ddot{x}_m = \omega_n^2\delta_m$ [Eq. (31.17)], $\ddot{x}_m$ is found directly from the extreme value of δ. As indicated by Fig. 31.18, for values of τ greater than π/ω_n, the extreme (absolute) value of δ encountered at $t = \pi/\omega_n$ is never exceeded. For values of τ less than π/ω_n, the extreme value occurs after the pulse has ended ($t > \tau$) and is

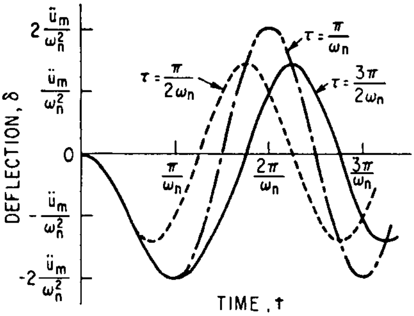

FIGURE 31.18 Response curves for an undamped linear system subjected to rectangular acceleration pulses of height $\ddot{u}_m$ and various durations τ.

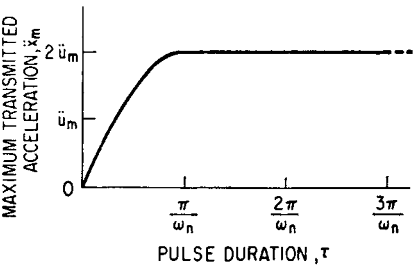

FIGURE 31.19 Maximum acceleration spectrum for a linear system of angular natural frequency ω_n. Support motion is a rectangular acceleration pulse of height $\ddot{u}_m$.

the amplitude of the motion represented by Eq. (31.32). This amplitude may be written

$$\delta_m = 2\frac{\ddot{u}_m}{\omega_n^2}\sin\frac{\omega_n\tau}{2} \tag{31.33}$$

The extreme absolute values of the acceleration $\ddot{x}_m$ are plotted as a function of τ in Fig. 31.19. Note that the extreme value of acceleration is twice that of the acceleration of the rectangular pulse.

HALF-SINE PULSE

Consider the "half-sine" acceleration pulse (Fig. 31.20*A*) of amplitude $\ddot{u}_m$ and duration τ:

$$\begin{aligned} \ddot{u} &= \ddot{u}_m \sin\frac{\pi t}{\tau} \qquad [0 \le t \le \tau] \\ \ddot{u} &= 0 \qquad [t > \tau] \end{aligned} \tag{31.34}$$

From Eq. (31.28), the effective duration is

$$\tau_r = \frac{2}{\pi}\tau \tag{31.35}$$

The response of a single degree-of-freedom system to the half-sine pulse of acceleration, corresponding to Eqs. (31.31) and (31.32) for the rectangular pulse, is defined by Eq. (8.32).

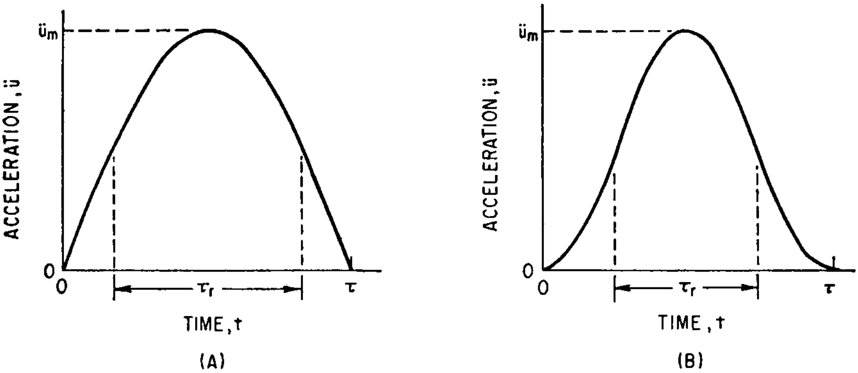

FIGURE 31.20 Half-sine acceleration pulse (*A*) and versed sine acceleration pulse (*B*).

VERSED SINE PULSE

The versed sine pulse (Fig. 31.20*B*) is described by

$$\ddot{u} = \frac{\ddot{u}_m}{2}\left(1 - \cos\frac{2\pi t}{\tau}\right) = \ddot{u}_m \sin^2\frac{\pi t}{\tau} \qquad [0 \le t \le \tau]$$
$$\ddot{u} = 0 \qquad [t > \tau] \tag{31.36}$$

The effective duration τ_r given by Eq. (31.28) is

$$\tau_r = (½)\tau \tag{31.37}$$

The response of a single degree-of-freedom system to a versed sine pulse is defined by Eq. (8.33). The responses to a number of other types of pulse and step excitation also are defined in Chap. 8.

COMPARISON OF MAXIMUM ACCELERATIONS

Velocity Step Approximation. A comparison of values of $\ddot{x}_m$ resulting from various acceleration pulses with that resulting from a velocity step is shown in Fig. 31.21. The maximum acceleration induced by a velocity step is $\omega_n\dot{u}_m$ [see Eq. (31.19)]. The abscissa $\omega_n\tau_r$ is a dimensionless measure of pulse duration. The effect of pulse shape is imperceptible for values of $\omega_n\tau_r < 0.6$. For pulses of duration $\omega_n\tau_r < 1.0$, the effect of pulse shape is small and the maximum possible error resulting from use of the velocity step approximation is of the order of 5 percent.

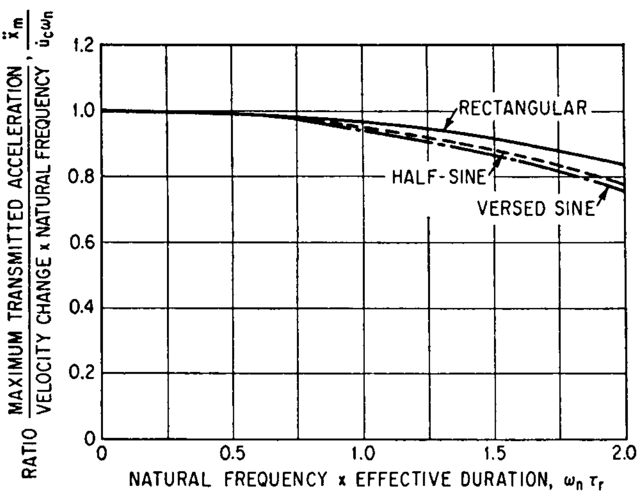

FIGURE 31.21 Dimensionless representation of maximum transmitted acceleration $\ddot{x}_m$ for the undamped linear system of Fig. 31.6.

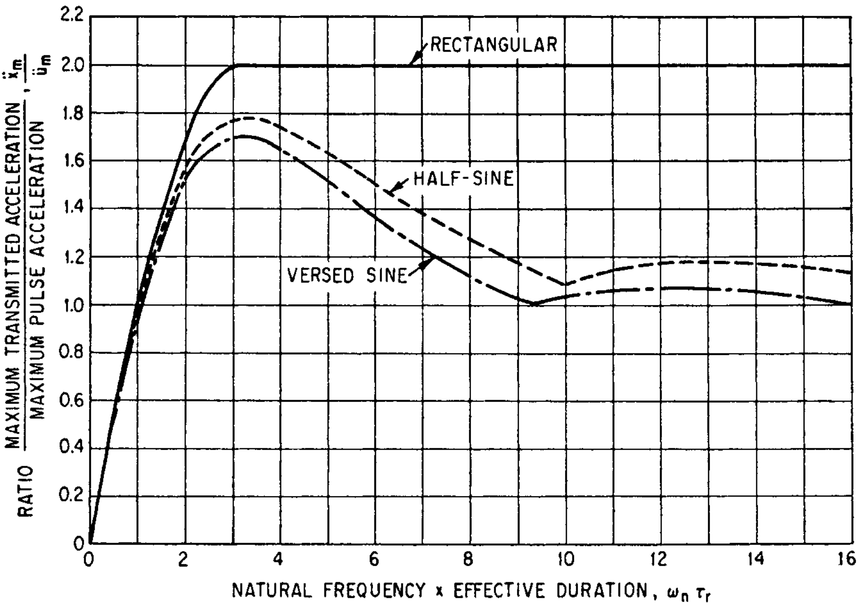

FIGURE 31.22 Shock transmissibility for the undamped linear system of Fig. 31.6 as a function of angular natural frequency ω_n and effective pulse duration τ_r.

Effects of Pulse Shape. The effects of pulse shape upon the maximum response acceleration $\ddot{x}_m$ for values of $\omega_n\tau_r > 1.0$ are shown in Fig. 31.22. The ordinate $\ddot{x}_m/\ddot{u}_m$ is the ratio of maximum acceleration induced in the responding system to maximum acceleration of the pulse. All three pulses produce the highest value of response acceleration when $\omega_n\tau_r \simeq \pi$. Physically, this corresponds to an effective duration τ_r of one-half of the natural period of the spring-mass system. For longer pulse durations the curves for half-sine and versed sine pulses are similar. For pulse durations beyond the range of Fig. 31.22 ($\omega_n\tau_r > 16$), the half-sine and versed sine curves approach the limiting ordinate $\ddot{x}_m/\ddot{u}_m = 1$. This corresponds physically to approximating a static loading of the spring-mass system. A limiting acceleration ratio $\ddot{x}_m/\ddot{u}_m = 2$ is encountered for all rectangular pulses of duration greater than the half-period of the spring-mass system. A more extensive study of responses to a variety of pulse shapes is included in Chap. 8.

SHOCK RESPONSE SPECTRUM

The abscissa $\omega_n\tau_r$ in Fig. 31.22 may be treated as a measure of pulse duration (proportional to τ_r) for a given spring-mass system with ω_n fixed. Alternatively, the pulse duration may be considered fixed; then the curves show the effect of varying the natural frequency ω_n of the spring-mass system. Each of the curves of Fig. 31.22 shows the maximum acceleration induced by a given acceleration pulse upon spring-mass systems of various natural frequencies ω_n; thus, Fig. 31.22 may be used to determine the required natural frequency of the isolator if $\ddot{x}_m$ and $\ddot{u}_m$ are known, and the pulse shape is defined.

Each curve shown in Fig. 31.22 may be interpreted as a description of a pulse, in terms of the response induced in a system subjected to the pulse. The curve of maximum response as a function of the natural frequency of the responding system is called a shock response spectrum or response spectrum. This concept is discussed more fully in Chap. 23. A pulse is a particular form of a shock motion; thus, each shock motion has a characteristic shock response spectrum. A shock motion has a characteristic effective value of time duration τ_r which need not be defined specifically; instead, the spectra are made to apply explicitly to a given shock motion by using the natural frequency ω_n as a dimensional parameter on the abscissa. By taking the isolator-and-equipment assembly to be the responding system, the natural frequency of the isolator may be chosen to meet any specified maximum acceleration $\ddot{x}_m$ of the equipment supported by isolators. Spectra of maximum isolator deflection δ_m also may be drawn, and are useful in predicting the maximum isolator deflection when the natural frequency of the isolator is known.

When damping is added to the isolator, the analysis of the response becomes much more complex. In general, it is possible to determine the maximum value of the response acceleration $\ddot{x}_m$ only by calculating the time-history of response acceleration over the entire time interval suspected of including the maximum response. A digital computer has been used to find shock response spectra for "half-sine" acceleration pulses with various fractions of critical damping in the responding system, as shown in Fig. 31.23. Similar spectra could be obtained to indicate maximum values of isolator deflection. In selecting a shock isolator for a specified application, it may be necessary to use both maximum acceleration and maximum deflection spectra. This is illustrated in the following example.

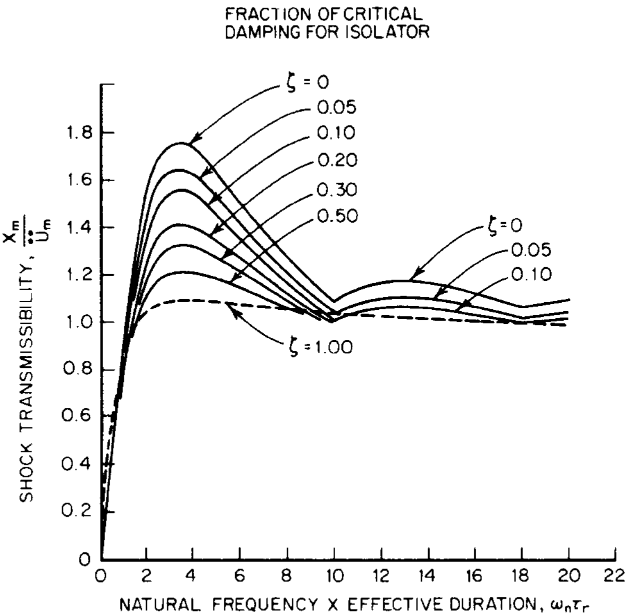

FIGURE 31.23 Shock transmissibility for the system of Fig. 31.2*A* with linear spring and viscous damping.

Example 31.2. A piece of equipment weighing 230 lb is to be isolated from the effects of a vertical shock motion defined by the spectra of acceleration and deflection shown in Fig. 31.24. It is required that the maximum induced acceleration not exceed 7***g*** (2700 in./sec^2). Clearances available limit the isolator deflection to 2.25 in. The curves in Fig. 31.24*A* represent maximum response acceleration $\ddot{x}_m$ as a function of the angular natural frequency ω_n of the equipment supported on the shock isolators. The isolator springs are assumed linear and viscously damped, and separate curves are shown for values of the damping ratio $\zeta = 0, 0.1, 0.2$, and 0.3. The curves in Fig. 31.24*B* represent the maximum isolator deflection δ_m as a function of ω_n for the same values of ζ.

Consider first the requirement that $\ddot{x}_m < 2700$ in./sec^2. In Fig. 31.24*A,* the horizontal dashed line indicates this limiting acceleration. If the damping ratio $\zeta = 0.3$, then the angular natural frequency ω_n may not exceed 38.5 rad/sec on the criterion of maximum acceleration. The dashed horizontal line of Fig. 31.24*B* represents the deflection limit $\delta_m = 2.25$ in. For $\zeta = 0.3$, the minimum natural frequency is 30 rad/sec on the criterion of deflection. Considering both acceleration and deflection criteria, the angular natural frequency ω_n must lie between 30 rad/sec and 38.5 rad/sec. The spectra indicate that both criteria may be just met with $\zeta = 0.2$ if ω_n is 35 rad/sec. Smaller values of damping do not permit the satisfaction of both requirements.

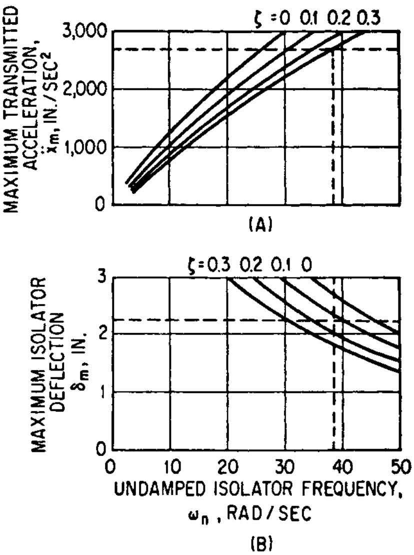

FIGURE 31.24 Shock response spectra: (*A*) maximum acceleration and (*B*) maximum isolator deflection for Example 31.2.

Conservatively, a suitable choice of parameters is $\zeta = 0.3$, $\omega_n = 35$ rad/sec. This limits $\ddot{x}_m$ to 2500 in./sec^2 and δ_m to 2.0 in. The spring stiffness k is

$$k = \omega_n^2 m = (35)^2 \times \frac{230}{386} = 730 \text{ lb/in.}$$

If the equipment is to be supported by four like isolators, then the required stiffness of each isolator is $k/4 = 182.5$ lb/in.

RESPONSE OF EQUIPMENT WITH A FLEXIBLE COMPONENT

IMPACT WITH REBOUND

Consider the system of Fig. 31.4. The block of mass m_1 represents the equipment and m_2 with its associated spring-dashpot unit represents a critical component of the equipment. The left spring-dashpot unit represents the shock isolator. It is assumed here that $m_1 >> m_2$ so that the motion of m_1 is not sensibly affected by m_2; larger values of m_2 are considered in a later section. Consider the entire system to be moving to the left at uniform velocity when the left-hand end of the isolator strikes a fixed support (not shown). The isolator will be compressed until the equipment is brought to rest. Following this the compressive force in the isolator will continue to accelerate the equipment toward the right until the isolator loses contact with the support and the rebound is complete. This type of shock is called *impact with rebound.* Practical examples include the shock experienced by a single railroad car striking a bumper at the end of a siding and that experienced by packaged equipment, shock-mounted inside a container of small mass, when the container is dropped upon a hard surface and then rebounds.

The procedure for finding the maximum acceleration $\ddot{x}_{2m}$ of the component, assuming the component stiffness to be linear and neglecting component damping, is:

1. Using the known striking velocity determine, from velocity step results (Figs. 31.9, 31.10, 31.12 to 31.15), the maximum deflection δ_{1m} of the isolator and the maximum acceleration $\ddot{x}_{1m}$ of the equipment.

2. From Eq. (31.28), find the effective duration τ_r for the acceleration time-history $\ddot{x}_1(t)$ of the equipment.

3. From the shock spectra corresponding to the acceleration pulse $\ddot{x}_1(t)$, find the maximum acceleration $\ddot{x}_{2m}$ of the component.

Details of the procedure using the isolators of Example 31.1 are considered in Example 31.3.

Example 31.3. Let the equipment of Example 31.1 weighing 40 lb have a flexible component weighing 0.2 lb. By vibration testing, this component is found to have an angular natural frequency $\omega_n = 260$ rad/sec and to possess negligible damping. For the isolators of Example 31.1, it is desired to determine the maximum acceleration $\ddot{x}_{2m}$ experienced by the mass m_2 of the component if the equipment, traveling at a velocity of 70 in./sec, is arrested by the free end of the isolator striking a fixed support. The four cases are considered separately. It is assumed that the component has a negligible effect on the motion of the equipment because $m_2 << m_1$.

Linear Spring. From the results of Example 31.1, it is known that $\omega_n = 108$ rad/sec and that the maximum acceleration of the equipment as found from Eq. (31.19) is

$$\ddot{x}_{1m} = 7580 \text{ in./sec}^2 = 19.6g$$

This acceleration occurs at the instant when the isolator deflection has the extreme value $\delta_{1m} = 0.65$ in. [If the equipment (Fig. 31.4) is moving toward the left when the isolator contacts the support, the extreme value of δ_{1m} is negative. It suffices to deal here with absolute values.] Subsequently the isolator spring continues to accelerate the equipment until the isolator force is zero and the rebound is complete. Since there is no damping, the rebound velocity equals the striking velocity (with opposite sign). The velocity change $\dot{x}_{1c}$ is twice the striking velocity and the effective duration τ_r [Eq. (31.28)] is

$$\tau_r = \frac{\dot{x}_{1c}}{\ddot{x}_{1m}} = \frac{2 \times 70}{7580} = 0.0185 \text{ sec}$$

The acceleration time-history of the equipment is a half-sine pulse as represented in Fig. 31.20 (the ordinate is $\ddot{x}_1$ instead of $\ddot{u}$).

Since the equipment is the "support" for the component, the response of the latter may be found from results developed for the response of a rigid body whose support experiences a half-sine pulse of acceleration. The half-sine curve of Fig. 31.22 gives the desired information if the following interpretations are made: For $\ddot{x}_m/\ddot{u}_m$ read $\ddot{x}_{2m}/\ddot{x}_{1m}$; for $\omega_n\tau_r$ read $\omega_{n2}\tau_r$. Now $\omega_{n2}\tau_r = 260 \times 0.0185 = 4.80$. From Fig. 31.22, $\ddot{x}_{2m}/\ddot{x}_{1m} = 1.66$, and $\ddot{x}_{2m} = 1.66 \times 7580 = 12{,}600 \text{ in./sec}^2 = 32.6g$.

Hardening Spring. From Example 31.1, the maximum equipment acceleration is $\ddot{x}_{1m} = 21g = 8100$ in./sec^2. Since the velocity change $\dot{x}_{1c}$ is twice the striking velocity, the effective duration τ_r [Eq. (31.28)] is

$$\tau_r = \frac{\dot{x}_{1c}}{\ddot{x}_{1m}} = \frac{2 \times 70}{8100} = 0.0173 \text{ sec}$$

With a hardening isolator spring, the shape of the acceleration pulse $\ddot{x}_1(t)$ experienced by the equipment varies considerably as the maximum deflection δ_{1m} approaches the upper limit d. Up to $\delta_{1m}/d = 0.5$, the shape is closely approximated by a half-sine pulse. For $\delta_{1m}/d = 0.8$, a symmetric triangular pulse is a good approximation. For higher values of δ_{1m}/d, the pulse is very sharply peaked. The maximum response curve for a half-sine pulse is given in Fig. 31.22. The corresponding curve for a symmetric triangular pulse (Fig. 8.18*b*) is similar to that for the versed sine pulse, though lying generally below the latter. Inasmuch as the curve for the versed sine pulse is below that for the half-sine pulse, it is conservative to use the half-sine pulse for all values of δ_{1m}/d. Accordingly, $\omega_{n2}\tau_r = 260 \times 0.0173 = 4.50$. From the half-sine curve of Fig. 31.22, $\ddot{x}_{2m}/\ddot{x}_{1m} = 1.69$, and $\ddot{x}_{2m} = 1.69 \times 8100 = 13{,}700$ in./sec$^2 = 36.4g$.

Softening Spring. From Example 31.1, the maximum equipment acceleration $\ddot{x}_{1m}$ is

$$\ddot{x}_{1m} = 20g = 7720 \text{ in./sec}^2$$

The effective duration τ_r [Eq. (31.28)] is

$$\tau_r = \frac{\dot{x}_{1c}}{\ddot{x}_{1m}} = \frac{2 \times 70}{7720} = 0.0181 \text{ sec}$$

The shape of the acceleration pulse $\ddot{x}_1(t)$ for the equipment varies markedly as the departure from linearity increases (increasing values of δ_{1m}/d_1). The pulse shape is found by first performing the integration of Eq. (31.9) with $F_s(\delta)$ as given by Eq. (31.23). The result supplies the integrand required for Eq. (31.13). A numerical integration of the latter equation shows that the pulse shape undergoes a rapid transition from the half-sine pulse at very small values of δ_{1m}/d_1 to shapes that are closely approximated by the trapezoidal pulse of Fig. 31.25. Note that the pulse of Fig. 31.25 requires three parameters to fix it completely: the maximum acceleration $\ddot{x}_{1m}$; the effective duration τ_r; and the ratio τ_r/τ, where τ is the actual duration and $\tau_r = \tau - \tau_1$. From results of the numerical integrations of Eq. (31.13), the curve of Fig. 31.26 is constructed to show τ_r/τ as a function of the deflection ratio δ_{1m}/d_1.

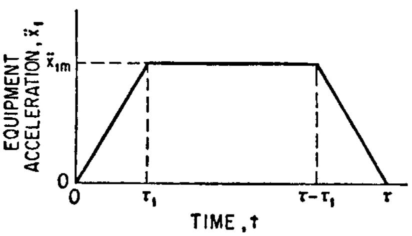

FIGURE 31.25 Symmetric trapezoidal acceleration pulse.

To find the maximum acceleration $\ddot{x}_{2m}$ of the component, the maximum response curves (shock spectra) of Fig. 31.27 are used. These curves are constructed for symmetric trapezoidal pulses (Fig. 31.25). The top curve ($\tau_r/\tau = 1.0$) corresponds to the limiting (rectangular) form. The dashed curve ($\tau_r/\tau = 0.64$) represents response to a half-sine pulse.

The value of δ_{1m}/d_1 corresponding to the maximum acceleration $\ddot{x}_{1m}$ of the equipment is (from Example 31.1) $\delta_{1m}/d_1 = 3$. From Fig. 31.26: $\tau_r/\tau = 0.88$. Now $\omega_n\tau_r = 260 \times 0.0181 = 4.7$. Using Fig. 31.27, linear interpolation between the curves for $\tau_r/\tau = 0.8$ and $\tau_r/\tau = 0.9$ gives $\ddot{x}_{2m}/\ddot{x}_{1m} = 1.98$ and $\ddot{x}_{2m} = 1.98 \times 7720 = 15{,}300$ in./sec^2 = 39.6*g*.

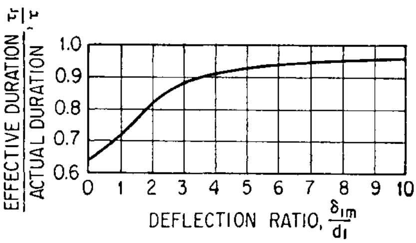

FIGURE 31.26 Dimensionless representation of effective duration τ_r of acceleration pulse experienced by equipment during impact with rebound.

Linear Spring and Viscous Damping. The presence of damping in the isolator adds several complications: (1) the rebound velocity is no longer equal to the striking velocity; (2) the acceleration pulse of the equipment is not symmetrical and returns to zero before the isolator deformation δ_{1m} returns to zero; and (3) the pulse shape varies greatly with damping ratio ζ_1. Shock response spectra for acceleration pulse shapes corresponding to damping ratios of particular interest ($0.10 < \zeta_1 < 0.40$) are not available. However, for single acceleration pulses which do not change sign, it is conservative to assume that the maximum acceleration $\ddot{x}_{2m}$ of the component is twice the maximum acceleration $\ddot{x}_{1m}$ of the equipment. Using the results of Example 31.1, the maximum acceleration of the component is $\ddot{x}_{2m} = 2\ddot{x}_{1m} = 2 \times 5450 = 10{,}900$ in./sec^2 = 28.2*g*.

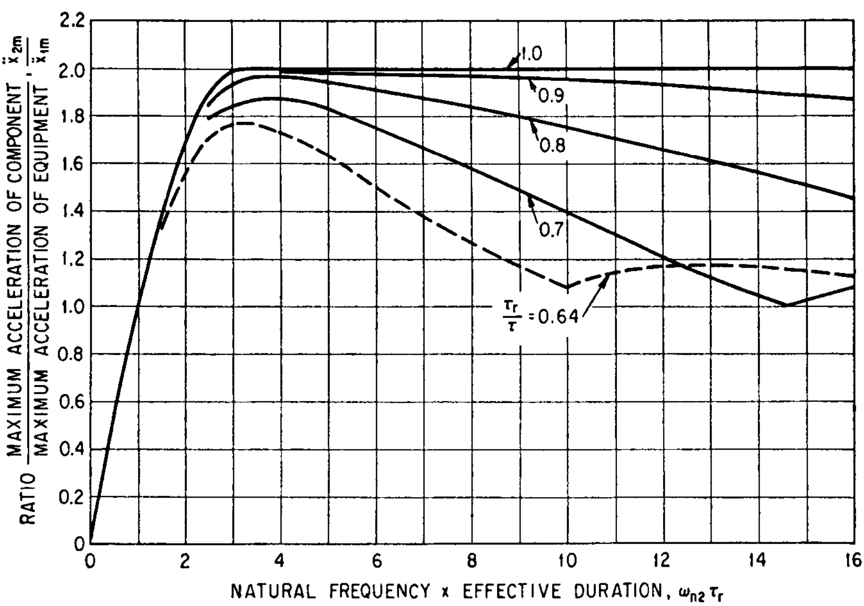

FIGURE 31.27 Shock response spectra for component having undamped linear elasticity with angular natural frequency ω_{n2}.

IMPACT WITHOUT REBOUND

When impact of the isolator occurs without rebound, it must be recognized that the equipment-isolator system continues to oscillate until the initial kinetic energy is dissipated. Consider the system of Fig. 31.4; it consists of equipment m_1, shock isolator (left spring-dashpot unit), and flexible component (subsystem 2). The system is initially at rest. The left end of the shock isolator is attached to a support (not shown) which is given a velocity step of magnitude $\dot{u}_m$ at $t = 0$. The subsequent motion of the support is $u = \dot{u}_m t$. Determine the maximum force F_{1m} transmitted by the isolator, the maximum isolator deflection δ_{1m}, and the maximum acceleration $\ddot{x}_{2m}$ of the component.

Solutions are available only for linear systems, i.e., linear springs and viscous damping. Two such simplified analyses of this problem are included in the following sections: (1) The influence of damping is considered, but the component mass m_2 is assumed of negligible size relative to m_1 and (2) damping is neglected but the effect of the mass m_2 of the component upon the motion of the system is considered.

Component Mass Negligible. Assume that $m_1 >> m_2$ so that the motion x_1 of the equipment may be determined by neglecting the effect of the component. Then the extreme value of the force F_{1m} transmitted by the isolator and the extreme deflection δ_{1m} of the isolator occur during the first quarter-cycle of the equipment motion; they may be found from Figs. 31.14 and 31.15 in the section on *Response of a Rigid Body System to a Velocity Step.* The subsequent motion of the equipment is an exponentially decaying sinusoidal oscillation or, if there is no damping in the isolator, a constant-amplitude oscillation. If the component also is undamped, an analytic determination of the component response is not difficult. The motion consists of harmonic oscillation at the frequency ω_{n1} of the equipment oscillation and a superposed oscillation at the frequency ω_{n2} of the component system. Since the oscillations are assumed to persist indefinitely in the absence of damping, the extreme acceleration of the component is the sum of the absolute values of the maximum accelerations associated with the oscillations at frequencies ω_{n1} and ω_{n2}. In the particular case of resonance ($\omega_{n1} = \omega_{n2}$), the vibration amplitude of the component increases indefinitely with time. Because actual systems always possess damping (usually a considerable amount in the isolator), solutions of this type tend to be unduly conservative for engineering applications.

The equation of motion for the viscous damped component is a special case of Eq. (31.5) with $F(\dot{\delta},\delta)$ as given by Eq. (31.7). If appropriate subscripts are supplied and customary substitutions are made, the equation is

$$\ddot{\delta}_2 + 2\zeta_2\omega_{n2}\dot{\delta}_2 + \omega_{n2}^2\delta_2 = -\ddot{x}_1 \tag{31.38}$$

Analytic solutions of Eq. (31.38) to find the acceleration $\ddot{x}_2 = \ddot{x}_1 + \ddot{\delta}_2$ of the component are too laborious to be practical. However, computer-generated results are shown in Fig. 31.28. The ordinate is the ratio of the maximum acceleration $\ddot{x}_{2m}$ of the component to the maximum acceleration $\dot{u}_m\omega_{n2}$ [see Eq. (31.19)] that the component would experience if the shock isolator were rigid. The abscissa is the ratio of the undamped natural frequency ω_{n2} of the component to the undamped natural frequency ω_{n1} of the equipment on the isolator spring. Curves are given for several different values of the fraction of critical damping ζ_1 for the isolator. For all curves the fraction of critical damping for the component is $\zeta_2 = 0.01$. The effect of isolator damping in reducing the maximum acceleration $\ddot{x}_{1m}$ of the component is great in the neighborhood of $\omega_{n2}/\omega_{n1} = 1$. Above $\omega_{n2}/\omega_{n1} = 2$, small damping ($\zeta_1 \leq 0.1$) in the isolator has little effect and large damping may significantly increase the maximum acceleration of the component.

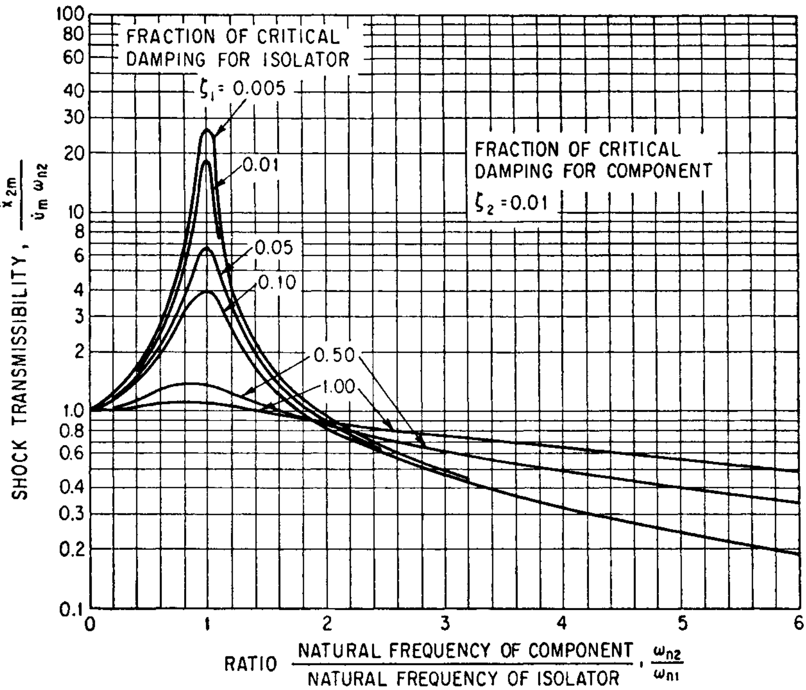

FIGURE 31.28 Shock transmissibility for a component of a viscously damped system with linear elasticity, where the effect of the component on the equipment motion is neglected.

The ordinate in Fig. 31.28 represents the ratio of the maximum acceleration of the component to that which would be experienced with the isolator rigid (absent); thus, it may properly be called *shock transmissibility.* If shock transmissibility is less than unity, the isolator is beneficial (for the component considered). An isolator must have a natural frequency significantly less than that of the critical component in order to reduce the transmitted acceleration. If there are several critical components having different natural frequencies ω_{n2}, each must be considered separately and the natural frequency of the isolator must be significantly lower than the lowest natural frequency of a component.

Two Degrees-of-Freedom—No Damping. This section includes an analysis of the transient response of the two degree-of-freedom system shown in Fig. 31.4, neglecting the effects of damping but assuming the equipment mass m_1 and the component mass m_2 to be of the same order of magnitude. The equations of motion are

$$\begin{aligned} m_1\ddot{\delta}_1 + k_1\delta_1 &= k_2\delta_2 - m_1\ddot{u} \\ m_2\ddot{\delta}_2 + k_2\delta_2 &= -m_2\ddot{\delta}_1 - m_2\ddot{u} \end{aligned} \tag{31.39}$$

where k_1 = stiffness of isolator spring, lb/in., and k_2 = stiffness of component, lb/in. The system is initially in equilibrium; at time $t = 0$, the left end of the isolator spring is given a velocity step of magnitude $\dot{u}_m$. Initial conditions are: $\dot{\delta}_1 = \dot{u}_m$, $\dot{\delta}_2 = 0$, $\delta_1 = \delta_2 = 0$. Equations (31.39) may be solved simultaneously for maximum values of the acceleration $\ddot{x}_{2m}$ of the component and maximum deflection δ_{1m} of the isolator:

$$\ddot{x}_{2m} = \frac{\dot{u}_m \omega_{n2}}{\left[\left(\frac{\omega_{n2}}{\omega_{n1}} - 1\right)^2 + \frac{m_2}{m_1}\left(\frac{\omega_{n2}}{\omega_{n1}}\right)^2\right]^{1/2}} \tag{31.40}$$

$$\delta_{1m} = \frac{\dot{u}_m}{\omega_{n1}} \frac{1 + \frac{\omega_{n2}}{\omega_{n1}}\left(1 + \frac{m_2}{m_1}\right)}{\left[\left(\frac{\omega_{n2}}{\omega_{n1}} + 1\right)^2 + \frac{m_2}{m_1}\left(\frac{\omega_{n2}}{\omega_{n1}}\right)^2\right]^{1/2}} \tag{31.41}$$

where $\ddot{x}_{2m}$ = maximum absolute acceleration of component mass, in./sec^2; δ_{1m} = maximum deflection of isolator spring, in.; ω_{n1}* = angular natural frequency of isolator $(k_1/m_1)^{1/2}$, rad/sec; and ω_{n2}* = angular natural frequency of component $(k_2/m_2)^{1/2}$, rad/sec. (The natural frequencies ω_{n1} and ω_{n2} are hypothetical in the sense that they do not consider the coupling between the subsystems.) Equation (31.40) is shown graphically in Fig. 31.29. The dimensionless ordinate is the ratio of maximum acceleration $\ddot{x}_{2m}$ of the component to the maximum acceleration $\dot{u}_m\omega_{n2}$ which the component would experience with no isolator present. The abscissa is the ratio of component natural frequency ω_{n2} to isolator natural frequency ω_{n1}. Separate curves are given for mass ratios $m_2/m_1 = 0.01, 0.1, 0.3$, and 1.0. Equation (31.41) is shown graphically in Fig. 31.30. The ordinate is the ratio of the maximum isolator deflection δ_{1m} to the deflection $\dot{u}_m(1 + m_2/m_1)^{1/2}/\omega_{n1}$ which would occur if component stiffness k_2 were infinite. The abscissa is the ratio of natural frequencies ω_{n2}/ω_{n1}, and curves are given for values of $m_2/m_1 = 0.1$ and 1.0.

Figure 31.29 shows that the effect of the mass ratio m_2/m_1 upon the maximum component acceleration $\ddot{x}_{2m}$ is very great near resonance ($\omega_{n2}/\omega_{n1} \simeq 1$). As ω_{n2}/ω_{n1} increases above resonance, the effect of finite component mass steadily decreases. Figure 31.30 shows that except for small values of ω_{n2}/ω_{n1} the effect of finite component mass on the maximum isolator deflection δ_{1m} is slight. As ω_{n2}/ω_{n1} increases, the curves for all mass ratios asymptotically approach the ordinate 1.0.

The factor $(1 + m_2/m_1)^{1/2}$ in the ordinate parameter of Fig. 31.30 is introduced because the total equipment mass is $m_1 + m_2$. For the limiting case of rigid equipment (k_2 infinite), the natural frequency ω_n is given by

$$\omega_n^2 = \frac{k_1}{m_1 + m_2} \qquad \omega_n = \frac{\omega_{n1}}{(1 + m_2/m_1)^{1/2}}$$

Substituting this relation in Eq. (31.18) and solving for δ_{1m}:

$$\delta_{1m} = \dot{u}_m(1 + m_2/m_1)^{1/2}/\omega_{n1}$$

This is in agreement with the result given by Eq. (31.41) as ω_{n2}/ω_{n1} approaches infinity.

Example 31.4. Equipment weighing 152 lb has a flexible component weighing 3 lb. The angular natural frequency of the component is $\omega_{n2} = 130$ rad/sec. The equipment is mounted on a shock isolator with a linear spring $k_1 = 2400$ lb/in. and having a fraction of critical damping $\zeta_1 = 0.10$. Find the maximum isolator deflection δ_{1m} and

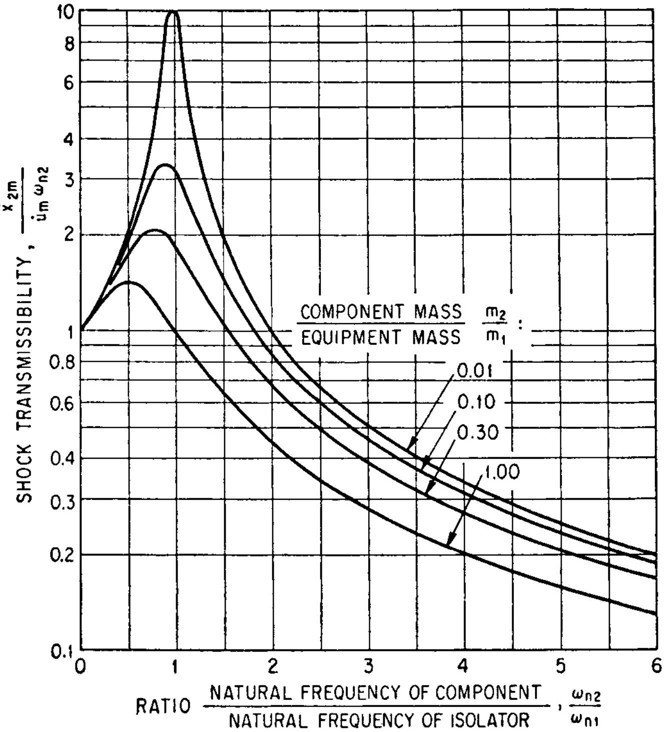

FIGURE 31.29 Shock transmissibility for component of system of Fig. 31.4 under impact at velocity $\dot{u}_m$ without rebound, where component and isolator have undamped linear elasticity.

the maximum component acceleration $\ddot{x}_{2m}$ which result when the base experiences a velocity step $\dot{u}_m = 55$ in./sec.

Consider first a solution assuming that m_2 has a negligible effect on the equipment motion:

$$m_1 = \frac{152 \text{ lb}}{386 \text{ in./sec}^2} = 0.393 \text{ lb-sec}^2\text{/in.}$$

$$\omega_{n1} = \sqrt{\frac{k_1}{m_1}} = \sqrt{\frac{2400}{0.393}} = 78.1 \text{ rad/sec [12.4 Hz]}$$

Figure 31.14 gives $\ddot{x}_{1m}/\dot{u}_m\omega_{n1} = 0.88$ and Fig. 31.15 gives $\ddot{x}_{1m}\delta_{1m}/\dot{u}_m^2 = 0.76$ for $\zeta_1 = 0.1$. Then

$$\delta_{1m} = \frac{0.76}{0.88} \times \frac{\dot{u}_m}{\omega_{n1}} = \frac{0.76 \times 55}{0.88 \times 78.1} = 0.61 \text{ in.}$$

In finding $\ddot{x}_{2m}$ it is assumed that damping of the component has the typical value $\zeta_2 = 0.01$. Using $\omega_{n1}/\omega_{n2} = 130/78.1 = 1.67$, Fig. 31.28 gives $\ddot{x}_{2m}/\dot{u}_m\omega_{n2} = 1.15$; then $\ddot{x}_{2m} = 1.15 \times 55 \times 130 = 8230$ in./sec$^2 = 21.3g$.

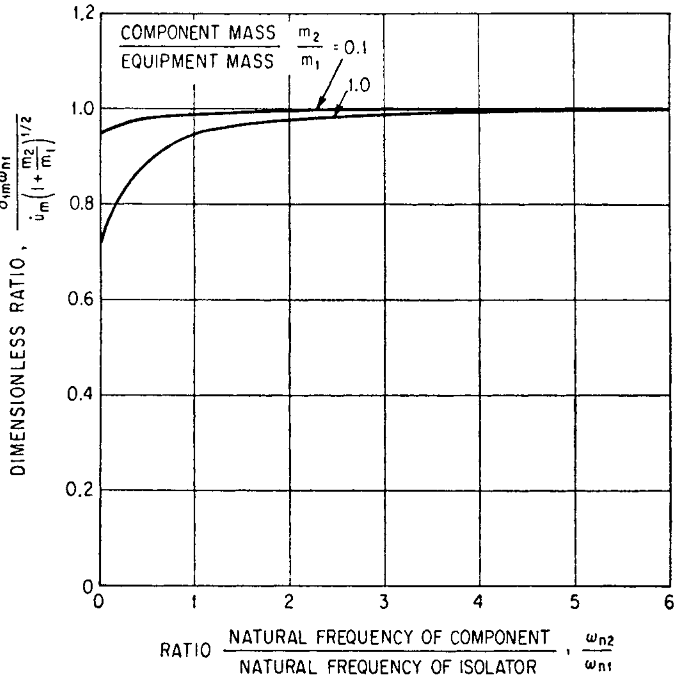

FIGURE 31.30 Dimensionless representation of maximum isolator deflection in system of Fig. 31.4 under impact at velocity $\dot{u}_m$ without rebound, where component and isolator have undamped linear elasticity.

A second solution, taking into consideration the mass m_2 of the component, may be obtained if the damping is neglected. From Eq. (31.41),

$$\delta_{1m} = \frac{\dot{u}_m}{\omega_{n1}} \frac{1 + \dfrac{\omega_{n2}}{\omega_{n1}}\left(1 + \dfrac{m_2}{m_1}\right)}{\left[\left(\dfrac{\omega_{n2}}{\omega_{n1}} + 1\right)^2 + \dfrac{m_2}{m_1}\left(\dfrac{\omega_{n2}}{\omega_{n1}}\right)^2\right]^{1/2}}$$

$$= \frac{55}{78.1} \times \frac{1 + 1.67(1 + 3/152)}{[(2.67)^2 + 3/152(1.67)^2]^{1/2}} = 0.71 \text{ in.}$$

From Eq. (31.40):

$$\ddot{x}_{2m} = \dot{u}_m\omega_{n2}\left[\left(\frac{\omega_{n2}}{\omega_{n1}} - 1\right)^2 + \frac{m_2}{m_1}\left(\frac{\omega_{n2}}{\omega_{n1}}\right)^2\right]^{-1/2}$$

$$= 55 \times 130[(0.67)^2 + 3/152(1.67)^2]^{-1/2}$$

$$= 10{,}070 \text{ in./sec}^2 = 26.1g$$

This example is too complex for a practicable solution when damping and the mass effects are considered together. However, the two above solutions may be taken conservatively as limiting conditions; it is unlikely that the actual acceleration and deflection would exceed the maxima of the limiting conditions.

SUPPORT PROTECTION

This section considers conditions in which the shock originates within the equipment (e.g., guns and drop hammers). Attention is first given to determining the response of the support for such equipment in the absence of a shock isolator. The effect of a shock isolator introduced to protect the support from excessive loads is considered later.

EQUIPMENT RIGIDLY ATTACHED TO SUPPORT

If the equipment is rigidly attached to the support, the support and equipment may be idealized as a single degree-of-freedom system for purposes of a simplified analysis. Consider the system of Fig. 31.3*B* with the spring-dashpot unit 2 assumed to be rigid. The mass m represents the equipment, and the mass m_F represents, with spring and dashpot assembly (1), the support. The force F, applied externally to the equipment, is taken to be a known function of time. The equation of motion is

$$(m_F + m)\ddot{\delta} + F(\dot{\delta},\delta) = F$$

Considering only force-time relations $F(t)$ in the form of a single pulse, the analogous mathematical relations of Eqs. (31.5) and (31.6) are used by defining the impulse J applied by the force F as

$$J = \int_0^\tau F\,dt \tag{31.42}$$

where τ is the duration of the pulse.

Short-Duration Impulses. If τ is short compared with the half-period of free oscillation of the system, then the results derived in the section on *Response of a Rigid Body System to a Velocity Step* may be applied directly. An impulse J of negligible duration acting on the mass m produces a velocity change $\dot{u}_m$ given by

$$\dot{u}_m = \frac{J}{m} \tag{31.43}$$

The subsequent relative motion of the system is identical with that resulting from a velocity step of magnitude $\dot{u}_m$.

If the damping capacity of the support is small, then velocity step results derived for linear springs, hardening springs, and softening springs are applicable. If the damping of the support may be represented as viscous and the stiffness as linear, then the linear-spring viscous damping results apply. In most installations it is sufficiently accurate to consider the support an undamped linear system.

A structure used to support an equipment generally has distributed mass and elasticity; thus the application of an impulse tends to excite the structure to vibrate not only in its fundamental mode but also in higher modes of vibration. The mass-spring-dashpot system shown in Fig. 31.3*B* to represent the structure would have equivalent mass and stiffness suitable to simulate only the fundamental mode of vibration. In many applications, such simulation is adequate because the displacements and strains are greater in the fundamental mode than in higher modes. The vibration of members having distributed mass is discussed in Chap. 7, and the formulation of models suitable for use in the analysis of systems subjected to shock is discussed in Chap. 28, Part II.

Long-Duration Impulses. If the duration τ of the applied impulse exceeds about one-third of the natural period of the equipment-support system, application of velocity step results may be unduly conservative. Then the results developed in the section on *Response of Rigid Body System to Acceleration Pulse* are applicable. The mathematical equivalence of Eqs. (31.5) and (31.6) is based on identifying $-m\ddot{u}$ in the former with F in the latter. Accordingly, if the shape of the force F vs. time curve is similar to the shape of the curve of acceleration $\ddot{u}$ vs. time, then the response of a system to an acceleration pulse may be used by analogy to find the response to a force pulse by making the following substitutions:

$$\ddot{u}_m = \frac{F_m}{m} \qquad \tau_r = \frac{J}{F_m}$$

where F_m is the maximum value of F, $\ddot{u}_m$ is the maximum value of $\ddot{u}$, and τ_r is the effective duration. If the mathematical equivalence is literally applied, F_m/m is analogous to $-\ddot{u}_m$, not $\ddot{u}_m$. Since acceleration pulse results are given in terms of extreme *absolute* values, the sign is not important.

EQUIPMENT SHOCK ISOLATED

Idealized System. When a shock isolator is used to reduce the magnitude of the force transmitted to the support, the idealized system is as shown in Fig. 31.4. Subsystem 2 represents the equipment (mass m_2) mounted on the shock isolator (right-hand spring-dashpot unit). Subsystem 1 is an idealized representation of the support with effective mass m_1 and with stiffness and damping capacity represented by the left spring-dashpot unit. The free end of the latter unit is taken to be fixed ($u = 0$).

It is assumed that the system is initially in equilibrium ($\dot{\delta}_1 = \dot{\delta}_2 = 0$; $\delta_1 = \delta_2 = 0$) and that force F (positive in the $+X$ direction) applies an impulse J to m_2. Analysis is simplified by treating the duration τ of impulse J as negligible. This assumption, always conservative, usually is warranted if the natural frequency of the shock isolator is small relative to the natural frequency of the support.

System Separable. In many applications the support motion $x_1 (= \delta_1)$ is sufficiently small compared with the equipment motion x_2 that the equipment acceleration $\ddot{x}_2$ is closely approximated by $\ddot{\delta}_2$ where $\ddot{x}_2 = \ddot{\delta}_2 + \ddot{x}_1$. Using this approximation, the analysis is resolved into two separate parts, each dealing with a single degree-of-freedom system.

If the system consists only of linear elements as defined by Eq. (31.7), the equation of motion of the equipment mounted on the shock isolator (subsystem 2 of Fig. 31.4) is

$$\ddot{\delta}_2 + 2\zeta_2\omega_{n2}\dot{\delta}_2 + \omega_{n2}^2\delta_2 = 0 \tag{31.44}$$

where $\omega_{n2}^2 = k_2/m_2$ and $\zeta_2 = c_2/2m_2\omega_{n2}$. The initial conditions are: $\delta_2 = 0$, $\dot{\delta}_2 = \dot{u}_m = J/m_2$ when $t = 0$. Because of the similarity of Eqs. (31.26) and (31.44), and the respective initial conditions, the maximum equipment acceleration $\ddot{x}_{2m}$ and the maximum isolator deflection δ_{2m} may be found from Figs. 31.14 and 31.15. The differential equation for the motion of the support in Fig. 31.4 is

$$\ddot{\delta}_1 + 2\zeta_1\omega_{n1}\dot{\delta}_1 + \omega_{n1}^2\delta_1 = -\frac{m_2}{m_1}\ddot{x}_2 \tag{31.45}$$

where $\omega_{n1}^2 = k_1/m_1$ and $\zeta_1 = c_1/2m_1\omega_{n1}$. The initial conditions are $\dot{\delta}_1 = 0$, $\delta_1 = 0$.

The solution of Eq. (31.45) is formally identical with that of Eq. (31.38) because the equations differ only by the interchange of the numerical subscripts and the presence of the factor m_2/m_1 on the right-hand side of Eq. (31.45). The solutions of Eq. (31.45) as obtained by a computer are shown in Fig. 31.31. The ordinate is the ratio of the maximum force F_{1m} in the support to the quantity $J\omega_{n1}$. The latter quantity is the maximum force which would be developed in an undamped, linear, single degree-of-freedom support of mass m_1 and stiffness k_1 if the impulse J were applied directly to m_1. The abscissa in Fig. 31.31 is the ratio of the undamped support natural frequency ω_{n1} to the undamped isolator natural frequency ω_{n2}. Curves are drawn for

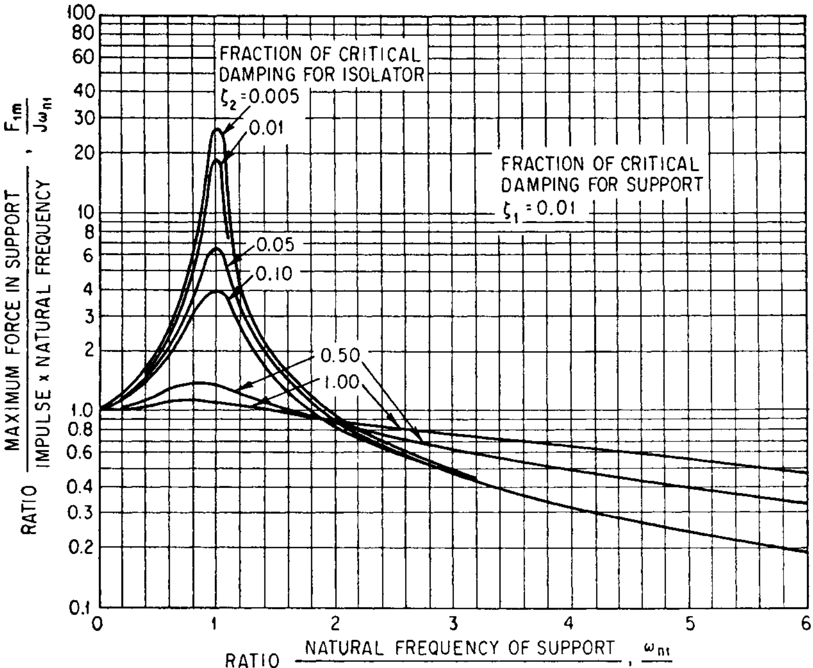

FIGURE 31.31 Dimensionless representation of maximum force in support F_{1m} resulting from action of impulse J on equipment.

various values of the fraction of critical damping ζ_2 for the isolator, assuming that the fraction of critical damping ζ_1 for the support is constant at $\zeta_1 = 0.01$.

Figure 31.31 appears to show that the presence of an isolator increases the maximum force F_{1m} transmitted by the support if the natural frequencies of isolator and support are nearly equal. This conclusion is misleading because the analysis assumes that the support deflection δ_1 is small compared with the isolator deflection δ_2, a condition which is not met in the neighborhood of unity frequency ratio. A more realistic analysis involves the two degree-of-freedom system discussed in the next section.

Two Degree-of-Freedom Analysis. This section includes an analysis of the system of Fig. 31.4 considered as a coupled two degree-of-freedom system where both the support and isolator are linear and undamped $[F_1(\dot{\delta}_1,\delta_1) = k_1\delta_1,\ F_2(\dot{\delta}_2,\delta_2) = k_2\delta_2]$. This analysis makes it possible to consider the effect of deflection of the support on the motion of the equipment. Fixing the support base ($u = 0$), the equations of motion may be written

$$\begin{aligned} \ddot{\delta}_1 + \omega_{n1}{}^2\delta_1 &= \frac{m_2}{m_1}\,\omega_{n2}{}^2\delta_2 \\ \ddot{\delta}_2 + \omega_{n2}{}^2\delta_2 &= -\ddot{\delta}_1 \end{aligned} \tag{31.46}$$

Assuming that the impulse J has negligible duration, the initial conditions are: $\dot{\delta}_1 = 0$, $\dot{\delta}_2 = J/m_2$, $\delta_1 = \delta_2 = 0$. The solution of Eqs. (31.46) parallels that of Eqs. (31.39); the resulting expressions for the maximum isolator deflection δ_{2m} and force F_{1m} applied to the support are

$$\delta_{2m} = \frac{J}{m_2\omega_{n2}}\left[1 + \frac{m_2/m_1}{(1 + \omega_{n1}/\omega_{n2})^2}\right]^{-1/2} \tag{31.47}$$

$$F_{1m} = J\omega_{n1}\left[\left(1 - \frac{\omega_{n1}}{\omega_{n2}}\right)^2 + \frac{m_2}{m_1}\right]^{-1/2} \tag{31.48}$$

The maximum deflection of the isolator given in Eq. (31.47) is shown graphically in Fig. 31.32. For small values of the ratio of support natural frequency to isolator natural frequency, the flexibility of the support may significantly reduce the maximum isolator deflection, especially if the mass of the support is small relative to the mass of the equipment. For large values of the frequency ratio, the effect of the mass ratio is small.

Maximum values of force in the support, given by Eq. (31.48), are shown in Fig. 31.33. The maximum deflection of the floor is the maximum force F_{1m} divided by the stiffness of the floor. The effect of mass ratio is profound for small values of the frequency ratio. The curves of Figs. 31.31 and 31.33 show corresponding results, the former including damping and the latter including the coupling effect between the two systems. The analysis which ignores the coupling effect may grossly overestimate the maximum force applied to the support at low values of the frequency ratio. At high values of the frequency ratio, the two analyses yield like results if the fraction of critical damping in the isolator is less than about $\zeta_2 = 0.10$. The two methods are compared in Example 31.5.

Example 31.5. A forging machine weighs 7000 lb exclusive of the 600-lb hammer. It is mounted at the center of a span formed by two 12-in., 50 lb/ft I beams hav-

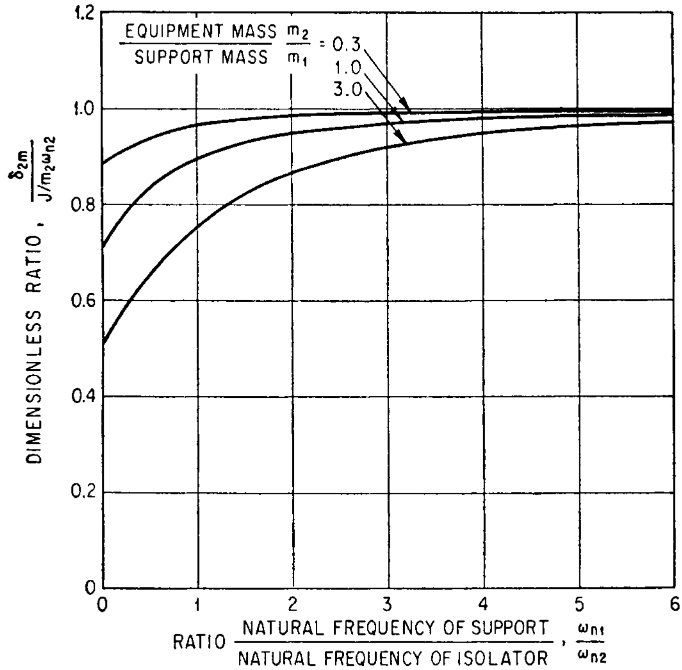

FIGURE 31.32 Dimensionless representation of maximum isolator deflection δ_{2m} resulting from action of impulse J on equipment. Isolator and support have undamped linear elasticity.

ing hinged ends and a span $l = 18$ ft. The hammer falls freely from a height of 60 in. before striking the work. Determine:

a. Maximum force F_{1m} in the beams and maximum deflection δ_{1m} of the beams if the machine is rigidly bolted to the beams.

b. The maximum force F_{1m} in the beams and the maximum deflection δ_{2m} of an isolator interposed between machine and beams.

Solution

a. When the machine is bolted rigidly to the beams, the system may be considered to have only a single degree-of-freedom. The mass is that of the machine, plus the hammer, plus the effective mass of the beams. For the machine: $m_2 = (7000 + 600)/386 = 19.2$ lb-sec²/in. The effective mass of the beams is taken as one-half of the actual mass:

$$m_1 = \frac{2(0.5)(18)(50)}{386} = 2.33 \text{ lb-sec}^2\text{/in.}$$

$$m = m_1 + m_2 = 21.5 \text{ lb-sec}^2\text{/in.}$$

The stiffness of the beams is

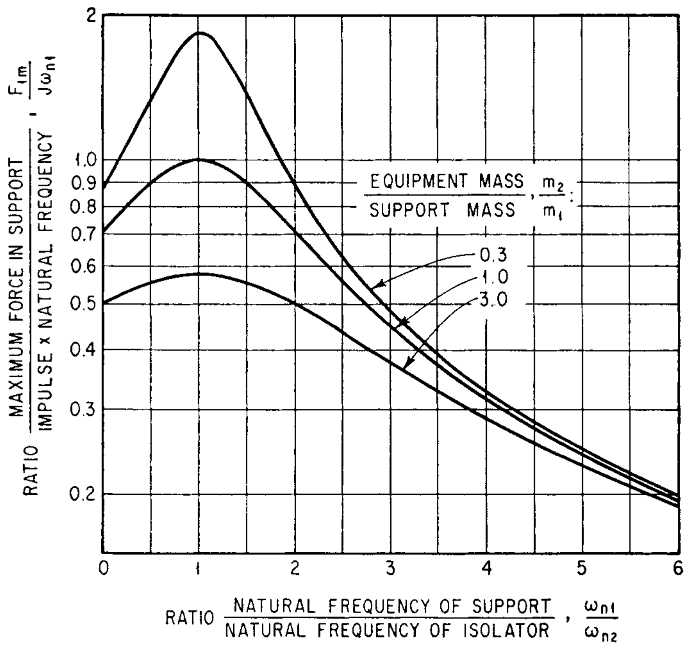

FIGURE 31.33 Dimensionless representation of maximum force in support F_{1m} resulting from action of impulse J on equipment.

$$k = 2\,\frac{48EI}{l^3} = 2\,\frac{48 \times (30 \times 10^6) \times 302}{(18 \times 12)^3} = 123{,}000 \text{ lb/in.}$$

The natural frequency of the machine-and-beams system is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{123{,}000}{21.5}} = 75.6 \text{ rad/sec [12.0 Hz]}$$

If the impact between the hammer and work is inelastic and its duration is negligible, the resulting velocity $\dot{u}_m$ of the machine may be found from conservation of momentum. The impulse J is the product of weight of hammer and time of fall:

$$J = (600)\left(\frac{2 \times 60}{386}\right)^{1/2} = 335 \text{ lb-sec}$$

Then $\dot{u}_m = J/m = 335/21.5 = 15.6$ in./sec. If the damping of the beams is neglected, the maximum beam deflection is found from Eq. (31.18):

$$\delta_{1m} = \frac{\dot{u}_m}{\omega_n} = \frac{15.6}{75.6} = 0.21 \text{ in.}$$

The maximum force in the beams is the product of beam stiffness and maximum deflection:

$$F_{1m} = k\delta_{1m} = 25{,}300 \text{ lb}$$

b. An isolator having a stiffness $k_2 = 36{,}000$ lb/in. and a fraction of critical damping $\zeta_2 = 0.10$ is interposed between the machine and beams. The "uncoupled natural frequencies" defined in connection with Eqs. (31.40) and (31.41) are

$$\omega_{n2} = \sqrt{\frac{k_2}{m_2}} = \sqrt{\frac{36{,}000}{19.2}} = 43.3 \text{ rad/sec [6.9 Hz]}$$

$$\omega_{n1} = \sqrt{\frac{k_1}{m_1}} = \sqrt{\frac{123{,}000}{2.33}} = 230 \text{ rad/sec [36.6 Hz]}$$

Consider first that the system is separable. Figures 31.14 and 31.15 give, respectively: $\ddot{x}_{2m}/\dot{u}_m\omega_{n2} = 0.88$; $\ddot{x}_{2m}\delta_{2m}/\dot{u}_m^{\,2} = 0.76$. Substituting $\dot{u}_m = J/m_2 = 17.4$ in./sec and solving for δ_{2m},

$$\delta_{2m} = \frac{0.76 \times 17.4}{0.88 \times 43.3} = 0.35 \text{ in.}$$

Entering Fig. 31.31 at $\omega_{n1}/\omega_{n2} = 5.3$, $F_{1m}/J\omega_{n1} = 0.23$. Then

$$F_{1m} = 17{,}700 \text{ lb}$$

Thus, the effect of the isolator is to reduce the maximum load in the beams from 25,300 lb to 17,700 lb. An isolator with less stiffness would permit a further reduction of this force at the expense of greater machine motion.

Consider now that the floor and machine-isolator systems are coupled, and use the two degree-of-freedom analysis which neglects damping. From Eq. (31.47):

$$\delta_{2m} = \frac{J}{m_2\omega_{n2}}\left[1 + \frac{m_2/m_1}{\left(1 + \dfrac{\omega_{n1}}{\omega_{n2}}\right)^2}\right]^{-1/2}$$

$$= \frac{335}{19.2 \times 43.3}\left[1 + \frac{19.2/2.33}{(1 + 5.3)^2}\right]^{-1/2} = 0.37 \text{ in.}$$

From Eq. (31.48):

$$F_{1m} = J\omega_{n1}\left[\left(1 - \frac{\omega_{n1}}{\omega_{n2}}\right)^2 + \frac{m_2}{m_1}\right]^{-1/2}$$

$$= 335 \times 230\left[(1 - 5.3)^2 + \frac{19.2}{2.33}\right]^{-1/2} = 14{,}900 \text{ lb}$$

Thus, the two results for the isolator deflection δ_{2m} differ only slightly, but the two degree-of-freedom analysis gives a maximum load in the beams about 16 percent smaller than that obtained by assuming the systems to be separable.