CHAPTER 30
THEORY OF VIBRATION ISOLATION

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INTRODUCTION

Vibration isolation concerns means to bring about a reduction in a vibratory effect. A vibration isolator in its most elementary form may be considered as a resilient member connecting the equipment and foundation. The function of an isolator is to reduce the magnitude of motion transmitted from a vibrating foundation to the equipment or to reduce the magnitude of force transmitted from the equipment to its foundation.

CONCEPT OF VIBRATION ISOLATION

The concept of vibration isolation is illustrated by consideration of the single degree-of-freedom system illustrated in Fig. 30.1. This system consists of a rigid body representing an equipment connected to a foundation by an isolator having resilience and energy-dissipating means; it is unidirectional in that the body is constrained to move only in vertical translation. The performance of the isolator may be evaluated by the following characteristics of the response of the equipment-isolator system of Fig. 30.1 to steady-state sinusoidal vibration:

Absolute transmissibility. Transmissibility is a measure of the reduction of transmitted force or motion afforded by an isolator. If the source of vibration is an oscillating motion of the foundation (motion excitation), transmissibility is the ratio of the vibration amplitude of the equipment to the vibration amplitude of the foundation. If the source of vibration is an oscillating force originating within the equipment (force excitation), transmissibility is the ratio of the force amplitude transmitted to the foundation to the amplitude of the exciting force.

Relative transmissibility. Relative transmissibility is the ratio of the relative deflection amplitude of the isolator to the displacement amplitude imposed at the foundation. A vibration isolator effects a reduction in vibration by permitting

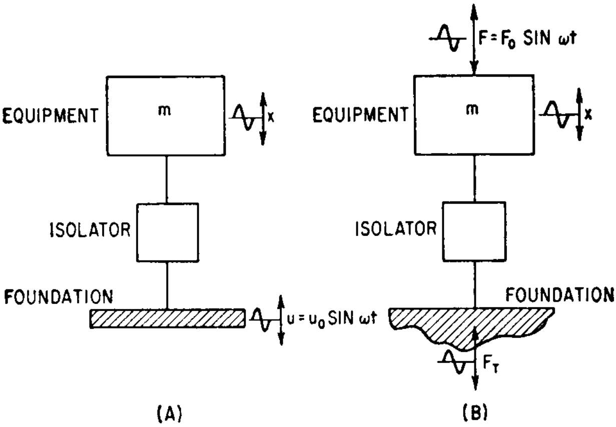


FIGURE 30.1 Schematic diagrams of vibration isolation systems: (*A*) vibration isolation where motion u is imposed at the foundation and motion x is transmitted to the equipment; (*B*) vibration isolation where force F is applied by the equipment and force F_T is transmitted to the foundation.

deflection of the isolator. The relative deflection is a measure of the clearance required in the isolator. This characteristic is significant only in an isolator used to reduce the vibration transmitted from a vibrating foundation.

Motion response. Motion response is the ratio of the displacement amplitude of the equipment to the quotient obtained by dividing the excitation force amplitude by the static stiffness of the isolator. If the equipment is acted on by an exciting force, the resultant motion of the equipment determines the space requirements for the isolator, i.e., the isolator must have a clearance at least as great as the equipment motion.

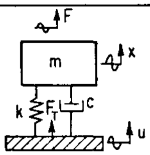
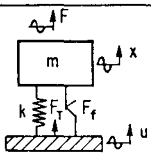
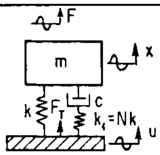
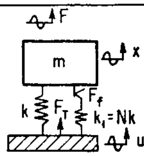
FORM OF ISOLATOR

The essential features of an isolator are resilient load-supporting means and energy-dissipating means. In certain types of isolators, the functions of the load-supporting means and the energy-dissipating means may be performed by a single element, e.g., natural or synthetic rubber. In other types of isolators, the resilient load-carrying means may lack sufficient energy-dissipating characteristics, e.g., metal springs; then separate and distinct energy-dissipating means (dampers) are provided. For purposes of analysis, it is assumed that the springs and dampers are separate elements. In general, the springs are assumed to be linear and massless. The effects of nonlinearity and mass of the load-supporting means upon vibration isolation are considered in later sections of this chapter.

Various types of dampers are shown in combination with ideal springs in the following idealized models of isolators illustrated in Table 30.1. Practical aspects of isolator design are considered in Chap. 32.

Rigidly connected viscous damper. A viscous damper c is connected rigidly between the equipment and its foundation as shown in Table 30.1*A*. The damper has the characteristic property of transmitting a force F_c that is directly proportional to the relative velocity $\dot{\delta}$ across the damper, $F_c = c\dot{\delta}$. This damper sometimes is referred to as a *linear damper.*

TABLE 30.1 Types of Idealized Vibration Isolators

(A) RIGIDLY CONNECTED VISCOUS DAMPER	(B) RIGIDLY CONNECTED COULOMB DAMPER	(C) ELASTICALLY CONNECTED VISCOUS DAMPER	(D) ELASTICALLY CONNECTED COULOMB DAMPER
			
EXCITATION			
$u = u_0 \text{SIN}\omega t$ $F = F_0 \text{SIN}\omega t$	$u = u_0 \text{SIN}\omega t$ * OR $\ddot{u} = \ddot{u}_0 \text{SIN}\omega t$ * $F = F_0 \text{SIN}\omega t$	$u = u_0 \text{SIN}\omega t$ $F = F_0 \text{SIN}\omega t$	$u = u_0 \text{SIN}\omega t$ * OR $\ddot{u} = \ddot{u}_0 \text{SIN}\omega t$ * $F = F_0 \text{SIN}\omega t$
RESPONSE			
$x = x_0 \text{SIN}(\omega t + \theta^{\dagger})$ OR $\delta = \delta_0 \text{SIN}(\omega t + \theta^{\dagger})$ WHERE $\delta = x - u$ $F_T = (F_T)_0 \text{SIN}(\omega t + \theta^{\dagger})$	$x = x_0 \text{SIN}(\omega t + \theta^{\dagger})$ OR $\delta = \delta_0 \text{SIN}(\omega t + \theta^{\dagger})$ WHERE $\delta = x - u$ $F_T = (F_T)_0 \text{SIN}(\omega t + \theta^{\dagger})$	$x = x_0 \text{SIN}(\omega t + \theta^{\dagger})$ OR $\delta = \delta_0 \text{SIN}(\omega t + \theta^{\dagger})$ WHERE $\delta = x - u$ $F_T = (F_T)_0 \text{SIN}(\omega t + \theta^{\dagger})$	$x = x_0 \text{SIN}(\omega t + \theta^{\dagger})$ OR $\delta = \delta_0 \text{SIN}(\omega t + \theta^{\dagger})$ WHERE $\delta = x - u$ $F_T = (F_T)_0 \text{SIN}(\omega t + \theta^{\dagger})$
FREQUENCY PARAMETERS			
$\omega_0 = \sqrt{k/m}$ $(c = 0)$	$\omega_0 = \sqrt{k/m}$ $(F_f = 0)$	$\omega_0 = \sqrt{k/m}$ $(c = 0)$ $\omega_\infty = \sqrt{(N+1)\frac{k}{m}}$ $(c = \infty)$	$\omega_0 = \sqrt{k/m}$ $(F_f = 0)$ $\omega_\infty = \sqrt{(N+1)\frac{k}{m}}$ $(F_f = \infty)$
DAMPING PARAMETERS			
$c_c = 2\sqrt{km}$ $\zeta = c/c_c$	$\eta = \frac{F_f}{k u_0}$ $\xi = \frac{F_f}{m \ddot{u}_0}$ $\xi_F = \frac{F_f}{F_0}$	$c_c = 2\sqrt{km}$ $\zeta = c/c_c$	$\eta = \frac{F_f}{k u_0}$ $\xi = \frac{F_f}{m \ddot{u}_0}$ $\xi_F = \frac{F_f}{F_0}$

* PHYSICALLY, THESE FORMS OF EXCITATION ARE IDENTICAL. THEY ARE EXPRESSED IN TWO DIFFERENT MATHEMATICAL FORMS FOR CONVENIENCE IN DEFINING THE DAMPING PARAMETER FOR COULOMB DAMPING.

† IN VIBRATION ISOLATION, ONLY THE MAGNITUDE OF THE RESPONSE IS OF INTEREST; THUS, THE PHASE ANGLE USUALLY IS NEGLECTED.

TABLE 30.2 Transmissibility and Motion Response for Isolation Systems Defined in Table 30.1

Where the equation is shown graphically, the applicable figure is indicated below the equation. See Table 30.1 for definition of terms.

Type of damper	Absolute transmissibility
Rigidly connected viscous damper	(a) $T_A = \frac{x_0}{u_0} = \frac{F_T}{F_0} = \sqrt{\frac{1 + \left(2\zeta \frac{\omega}{\omega_0}\right)^2}{\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + \left(2\zeta \frac{\omega}{\omega_0}\right)^2}}$ Fig. 30.2
Rigidly connected Coulomb damper (see Note 1)	(d) $(T_A)_D = \frac{x_0}{u_0} = \sqrt{\frac{1 + \left(\frac{4}{\pi}\eta\right)^2 (1 - 2\omega_0^2/\omega^2)}{\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2}}$ Fig. 30.5 (see Note 2)
Elastically connected viscous damper	(g) $T_A = \frac{x_0}{u_0} = \frac{F_T}{F_0} = \sqrt{\frac{1 + 4\left(\frac{N+1}{N}\right)^2 \zeta^2 \frac{\omega^2}{\omega_0^2}}{\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + \frac{4}{N^2}\zeta^2 \frac{\omega^2}{\omega_0^2}\left(N + 1 - \frac{\omega^2}{\omega_0^2}\right)^2}}$ Fig. 30.10 (see Note 3)
Elastically connected Coulomb damper (see Note 1)	(j) $(T_A)_D = \frac{x_0}{u_0} = \sqrt{\frac{1 + \left(\frac{4}{\pi}\eta\right)^2 \left[\left(\frac{N+2}{N}\right) - 2\left(\frac{N+1}{N}\right)\left(\frac{\omega_0}{\omega}\right)^2\right]}{\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2}}$ Fig. 30.13 (see Notes 4 and 5)

NOTE 1: These equations apply only when there is relative motion across the damper.
NOTE 2: This equation applies only when excitation is defined in terms of displacement amplitude.
NOTE 3: These curves apply only for optimum damping [see Eq. (30.15)]; curves for other values of damping are given in Ref. 4.

Rigidly connected Coulomb damper. An isolation system with a rigidly connected Coulomb damper is indicated schematically in Table 30.1*B*. The force F_f exerted by the damper on the mass of the system is constant, independent of position or velocity, but always in a direction that opposes the relative velocity across the damper. In a physical sense, Coulomb damping is approximately attainable from the relative motion of two members arranged to slide one upon the other with a constant force holding them together.

Elastically connected viscous damper. The elastically connected viscous damper is shown in Table 30.1*C*. The viscous damper c is in series with a spring of stiffness k_1; the load-carrying spring k is related to the damper spring k_1 by the parameter $N = k_1/k$. This type of damper system sometimes is referred to as a *viscous relaxation system.*

Elastically connected Coulomb damper. The elastically connected Coulomb damper is shown in Table 30.1*D*. The friction element can transmit only that force which is developed in the damper spring k_1. When the damper slips, the friction force F_f is independent of the velocity across the damper, but always is in a direction that opposes it.

TABLE 30.2 Transmissibility and Motion Response for Isolation Systems Defined in Table 30.1 (*Continued*)

Where the equation is shown graphically, the applicable figure is indicated below the equation. See Table 30.1 for definition of terms.

Relative transmissibility	Motion response
(*b*) $T_R=\frac{\delta_0}{u_0}=\sqrt{\frac{\left(\frac{\omega}{\omega_0}\right)^4}{\left(1-\frac{\omega^2}{\omega_0^2}\right)^2+\left(2\zeta\frac{\omega}{\omega_0}\right)^2}}$ Fig. 30.3	(*c*) $\frac{x_0}{F_0/k}=\sqrt{\frac{1}{\left(1-\frac{\omega^2}{\omega_0^2}\right)^2+\left(2\zeta\frac{\omega}{\omega_0}\right)^2}}$ Fig. 30.14
(*e*) $(T_R)_D=\frac{\delta_0}{u_0}=\sqrt{\frac{\left(\frac{\omega}{\omega_0}\right)^4-\left(\frac{4}{\pi}\eta\right)^2}{\left(1-\frac{\omega^2}{\omega_0^2}\right)^2}}$ Fig. 30.6 (see Note 2)	(*f*) $\frac{x_0}{F_0/k}=\sqrt{\frac{1-\left(\frac{4}{\pi}\xi\right)^2}{\left(1-\frac{\omega^2}{\omega_0^2}\right)^2}}$ (See Note 2)
(*h*) $T_R=\frac{\delta_0}{u_0}=\sqrt{\frac{\frac{\omega^2}{\omega_0^2}+\frac{4}{N^2}\zeta^2\frac{\omega^6}{\omega_0^6}}{\left(1-\frac{\omega^2}{\omega_0^2}\right)^2+\frac{4}{N^2}\zeta^2\frac{\omega^2}{\omega_0^2}\left(N+1-\frac{\omega^2}{\omega_0^2}\right)^2}}$ Fig. 30.11 (see Note 3)	(*i*) $\frac{x_0}{F_0/k}=\sqrt{\frac{1+\frac{4}{N^2}\zeta^2\frac{\omega^2}{\omega_0^2}}{\left(1-\frac{\omega^2}{\omega_0^2}\right)^2+\frac{4}{N^2}\zeta^2\frac{\omega^2}{\omega_0^2}\left(N+1-\frac{\omega^2}{\omega_0^2}\right)^2}}$ Fig. 30.9 (see Note 4)
(*k*) $(T_R)_D=\frac{\delta_0}{u_0}=\sqrt{\frac{\left(\frac{\omega}{\omega_0}\right)^4+\left(\frac{4}{\pi}\eta\right)^2\left[\frac{2}{N}\frac{\omega^2}{\omega_0^2}-\left(\frac{N+2}{N}\right)\right]}{\left(1-\frac{\omega^2}{\omega_0^2}\right)^2}}$ Fig. 30.14 (see Notes 4 and 5)	

NOTE 4: These curves apply only for $N = 3$.

NOTE 5: This equation applies only when excitation is defined in terms of displacement amplitude; for excitation defined in terms of force or acceleration, see Eq. (30.18).

INFLUENCE OF DAMPING IN VIBRATION ISOLATION

The nature and degree of vibration isolation afforded by an isolator is influenced markedly by the characteristics of the damper. This aspect of vibration isolation is evaluated in this section in terms of the single degree-of-freedom concept; i.e., the equipment and the foundation are assumed rigid and the isolator is assumed massless. The performance is defined in terms of absolute transmissibility, relative transmissibility, and motion response for isolators with each of the four types of dampers illustrated in Table 30.1. A system with a rigidly connected viscous damper is discussed in detail in Chap. 2, and important results are reproduced here for completeness; isolators with other types of dampers are discussed in detail here.

The characteristics of the dampers and the performance of the isolators are defined in terms of the parameters shown on the schematic diagrams in Table 30.1. Absolute transmissibility, relative transmissibility, and motion response are defined analytically in Table 30.2 and graphically in the figures referenced in Table 30.2. For the rigidly connected viscous and Coulomb-damped isolators, the graphs generally are explicit and complete. For isolators with elastically connected dampers, typical results are included and references are given to more complete compilations of dynamic characteristics.

RIGIDLY CONNECTED VISCOUS DAMPER

Absolute and relative transmissibility curves are shown graphically in Figs. 30.2 and 30.3, respectively.* As the damping increases, the transmissibility at resonance decreases and the absolute transmissibility at the higher values of the forcing frequency ω increases; i.e., reduction of vibration is not as great. For an undamped isolator, the absolute transmissibility at higher values of the forcing frequency varies inversely as the square of the forcing frequency. When the isolator embodies significant viscous damping, the absolute transmissibility curve becomes asymptotic at high values of forcing frequency to a line whose slope is inversely proportional to the first power of the forcing frequency.

The maximum value of absolute transmissibility associated with the resonant condition is a function solely of the damping in the system, taken with reference to critical damping. For a lightly damped system, i.e., for $\zeta < 0.1$, the maximum absolute transmissibility [see Eq. (2.41)] of the system is[1]

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* For linear systems, the absolute transmissibility $T_A = x_0/u_0$ in the motion-excited system equals F_T/F_0 in the force-excited system. The relative transmissibility $T_R = \delta_0/u_0$ applies only to the motion-excited system.

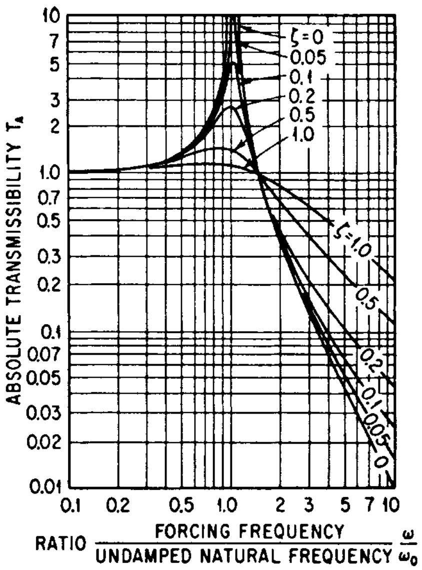

FIGURE 30.2 Absolute transmissibility for the rigidly connected, viscous-damped isolation system shown at *A* in Table 30.1 as a function of the frequency ratio ω/ω_0 and the fraction of critical damping ζ. The absolute transmissibility is the ratio (x_0/u_0) for foundation motion excitation (Fig. 30.1*A*) and the ratio (F_T/F_0) for equipment force excitation (Fig. 30.1*B*).

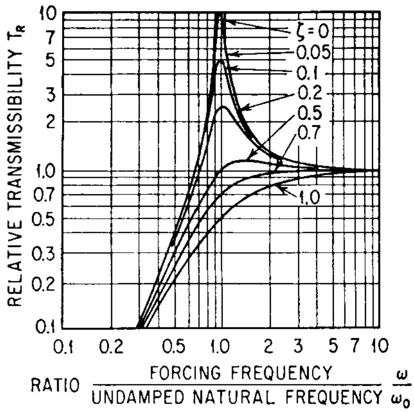

FIGURE 30.3 Relative transmissibility for the rigidly connected, viscous-damped isolation system shown at *A* in Table 30.1 as a function of the frequency ratio ω/ω_0 and the fraction of critical damping ζ. The relative transmissibility describes the motion between the equipment and the foundation (i.e., the deflection of the isolator).

$$T_{\max} = \frac{1}{2\zeta} \tag{30.1}$$

where $\zeta = c/c_c$ is the fraction of critical damping defined in Table 30.1.

The motion response is shown graphically in Fig. 30.4. A high degree of damping limits the vibration amplitude of the equipment at all frequencies, compared to an undamped system. The single degree-of-freedom system with viscous damping is discussed more fully in Chap. 2.

RIGIDLY CONNECTED COULOMB DAMPER

The differential equation of motion for the system with Coulomb damping shown in Table 30.1*B* is

$$m\ddot{x} + k(x - u) \pm F_f = F_0 \sin \omega t \tag{30.2}$$

The discontinuity in the damping force that occurs as the sign of the velocity changes at each half cycle requires a step-by-step solution of Eq. (30.2).[2] An approximate solution based on the equivalence of energy dissipation involves equating the energy dissipation per cycle for viscous-damped and Coulomb-damped systems:[3]

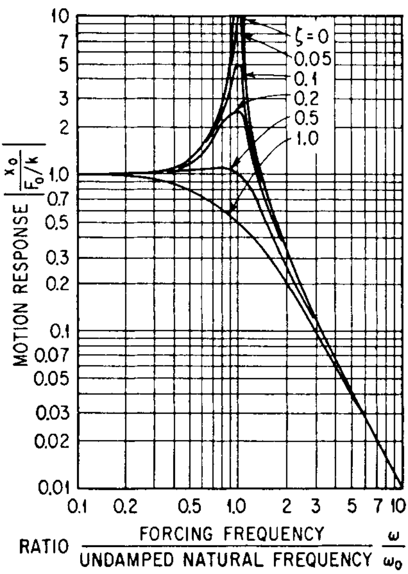

FIGURE 30.4 Motion response for the rigidly connected viscous-damped isolation system shown at *A* in Table 30.1 as a function of the frequency ratio ω/ω_0 and the fraction of critical damping ζ. The curves give the resulting motion of the equipment x in terms of the excitation force F and the static stiffness of the isolator k.

$$\pi c \omega \delta_0^2 = 4F_f \delta_0 \tag{30.3}$$

where the left side refers to the viscous-damped system and the right side to the Coulomb-damped system; δ_0 is the amplitude of relative displacement across the damper. Solving Eq. (30.3) for *c*,

$$c_{eq} = \frac{4F_f}{\pi\omega\delta_0} = j\left(\frac{4F_f}{\pi\dot{\delta}_0}\right) \tag{30.4}$$

where c_{eq} is the *equivalent viscous damping coefficient* for a Coulomb-damped system having equivalent energy dissipation. Since $\dot{\delta}_0 = j\omega\delta_0$ is the relative velocity, the equivalent linearized dry friction damping force can be considered sinusoidal with an amplitude $j(4F_f/\pi)$. Since $c_c = 2k/\omega_0$ [see Eq. (2.12)],

$$\zeta_{eq} = \frac{c_{eq}}{c_c} = \frac{2\omega_0 F_f}{\pi\omega k\delta_0} \tag{30.5}$$

where ζ_{eq} may be defined as the *equivalent fraction of critical damping.* Substituting δ_0 from the relative transmissibility expression [(*b*) in Table 30.2] in Eq. (30.5) and solving for ζ_{eq}^2,

$$\zeta_{eq}^2 = \frac{\left(\frac{2}{\pi}\eta\right)^2\left(1-\frac{\omega^2}{\omega_0^2}\right)^2}{\frac{\omega^2}{\omega_0^2}\left[\frac{\omega^4}{\omega_0^4}-\left(\frac{4}{\pi}\eta\right)^2\right]} \tag{30.6}$$

where η is the Coulomb damping parameter for displacement excitation defined in Table 30.1.

The equivalent fraction of critical damping given by Eq. (30.6) is a function of the displacement amplitude u_0 of the excitation since the Coulomb damping parameter η depends on u_0. When the excitation is defined in terms of the acceleration amplitude $\ddot{u}_0$, the fraction of critical damping must be defined in corresponding terms. Thus, it is convenient to employ separate analyses for displacement transmissibility and acceleration transmissibility for an isolator with Coulomb damping.

Displacement Transmissibility. The absolute displacement transmissibility of an isolation system having a rigidly connected Coulomb damper is obtained by substituting ζ_{eq} from Eq. (30.6) for ζ in the absolute transmissibility expression for viscous damping, (*a*) in Table 30.2. The absolute displacement transmissibility is shown graphically in Fig. 30.5, and the relative displacement transmissibility is shown in Fig. 30.6. The absolute displacement transmissibility has a value of unity when the forcing frequency is low and/or the Coulomb friction force is high. For these conditions, the friction damper is locked in, i.e., it functions as a rigid connection, and there is no relative motion across the isolator. The frequency at which the damper breaks loose,

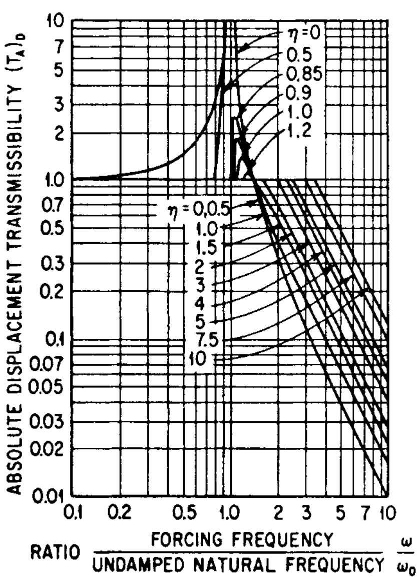

FIGURE 30.5 Absolute displacement transmissibility for the rigidly connected, Coulomb-damped isolation system shown at *B* in Table 30.1 as a function of the frequency ratio ω/ω_0 and the displacement Coulomb-damping parameter η.

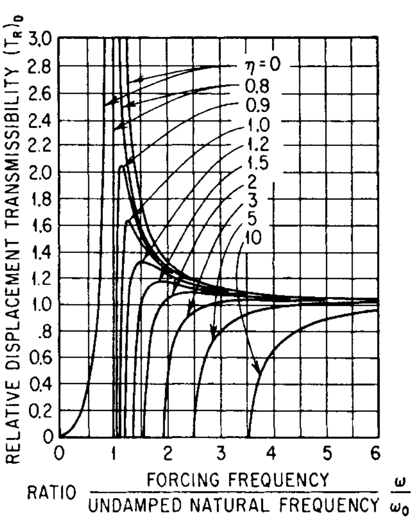

FIGURE 30.6 Relative displacement transmissibility for the rigidly connected, Coulomb-damped isolation system shown at *B* in Table 30.1 as a function of the frequency ratio ω/ω_0 and the displacement Coulomb-damping parameter η.

i.e., permits relative motion across the isolator, can be obtained from the relative displacement transmissibility expression, (*e*) in Table 30.2. The relative displacement is imaginary when $\omega^2/\omega_0^2 \leq (4/\pi)\eta$. Thus, the "break-loose" frequency ratio is*

$$\left(\frac{\omega}{\omega_0}\right)_L = \sqrt{\frac{4}{\pi}\eta} \tag{30.7}$$

The displacement transmissibility can become infinite at resonance, even though the system is damped, if the Coulomb damping force is less than a critical minimum value. The denominator of the absolute and relative transmissibility expressions becomes zero for a frequency ratio ω/ω_0 of unity. If the break-loose frequency is lower than the undamped natural frequency, the amplification of vibration becomes infinite at resonance. This occurs because the energy dissipated by the friction damping force increases linearly with the displacement amplitude, and the energy introduced into the system by the excitation source also increases linearly with the displacement amplitude. Thus, the energy dissipated at resonance is either greater or less than the input energy for *all* amplitudes of vibration. The minimum dry-friction force which prevents vibration of infinite magnitude at resonance is

$$(F_f)_{\min} = \frac{\pi k u_0}{4} = 0.79\, k u_0 \tag{30.8}$$

where k and u_0 are defined in Table 30.1.

As shown in Fig. 30.5, an increase in η decreases the absolute displacement transmissibility at resonance and increases the resonance frequency. All curves intersect at the point $(T_A)_D = 1$, $\omega/\omega_0 = \sqrt{2}$. With optimum damping force, there is no motion across the damper for $\omega/\omega_0 \leq \sqrt{2}$; for higher frequencies the displacement transmissibility is less than unity. The friction force that produces this "resonance-free" condition is

$$(F_f)_{\text{op}} = \frac{\pi k u_0}{2} = 1.57\, k u_0 \tag{30.9}$$

For high forcing frequencies, the absolute displacement transmissibility varies inversely as the square of the forcing frequency, even though the friction damper dissipates energy. For relatively high damping ($\eta > 2$), the absolute displacement transmissibility, for frequencies greater than the break-loose frequency, is approximately $4\eta\omega_0^2/\pi\omega^2$.

Acceleration Transmissibility. The absolute displacement transmissibility $(T_A)_D$ shown in Fig. 30.5 is the ratio of response of the isolator to the excitation, where each is expressed as a displacement amplitude in simple harmonic motion. The damping parameter η is defined with reference to the displacement amplitude u_0 of the excitation. Inasmuch as all motion is simple harmonic, the transmissibility $(T_A)_D$ also applies to acceleration transmissibility when the damping parameter is defined properly. When the excitation is defined in terms of the acceleration amplitude $\ddot{u}_0$ of the excitation,

$$\eta_{\ddot{u}_0} = \frac{F_f \omega^2}{k \ddot{u}_0} \tag{30.10}$$

* This equation is based upon energy considerations and is approximate. Actually, the friction damper breaks loose when the inertia force of the mass equals the friction force, $m u_0 \omega^2 = F_f$. This gives the exact solution $(\omega/\omega_0)_L = \sqrt{\eta}$. A numerical factor of $4/\pi$ relates the Coulomb damping parameters in the exact and approximate solutions for the system.

where ω = forcing frequency, rad/sec
$\ddot{u}_0$ = acceleration amplitude of excitation, in./sec^2
k = isolator stiffness, lb/in.
F_f = Coulomb friction force, lb

For relatively high forcing frequencies, the acceleration transmissibility approaches a constant value $(4/\pi)\xi$, where ξ is the Coulomb damping parameter for acceleration excitation defined in Table 30.1. The acceleration transmissibility of a rigidly connected Coulomb damper system becomes asymptotic to a constant value because the Coulomb damper transmits the same friction force regardless of the amplitude of the vibration.

ELASTICALLY CONNECTED VISCOUS DAMPER

The general characteristics of the elastically connected viscous damper shown at *C* in Table 30.1 may best be understood by successively assigning values to the viscous damper coefficient c while keeping the stiffness ratio N constant. For zero damping, the mass is supported by the isolator of stiffness k. The transmissibility curve has the characteristics typical of a transmissibility curve for an undamped system having the natural frequency

$$\omega_0 = \sqrt{\frac{k}{m}} \tag{30.11}$$

When c is infinitely great, the transmissibility curve is that of an undamped system having the natural frequency

$$\omega_\infty = \sqrt{\frac{k + k_1}{m}} = \sqrt{N+1}\,\omega_0 \tag{30.12}$$

where $k_1 = Nk$. For intermediate values of damping, the transmissibility falls within the limits established for zero and infinitely great damping. The value of damping which produces the minimum transmissibility at resonance is called *optimum damping.*

All curves approach the transmissibility curve for infinite damping as the forcing frequency increases. Thus, the absolute transmissibility at high forcing frequencies is inversely proportional to the square of the forcing frequency. General expressions for absolute and relative transmissibility are given in Table 30.2.

A comparison of absolute transmissibility curves for the elastically connected viscous damper and the rigidly connected viscous damper is shown in Fig. 30.7. A constant viscous damping coefficient of $0.2c_c$ is maintained, while the value of the stiffness ratio N is varied from zero to infinity. The transmissibilities at resonance are comparable, even for relatively small values of N, but a substantial gain is achieved in the isolation characteristics at high forcing frequencies by elastically connecting the damper.

Transmissibility at Resonance. The maximum transmissibility (at resonance) is a function of the damping ratio ζ and the stiffness ratio N, as shown in Fig. 30.8. The maximum transmissibility is nearly independent of N for small values of ζ. However, for $\zeta > 0.1$, the coefficient N is significant in determining the maximum transmissibility. The lowest value of the maximum absolute transmissibility curves corresponds to the conditions of optimum damping.

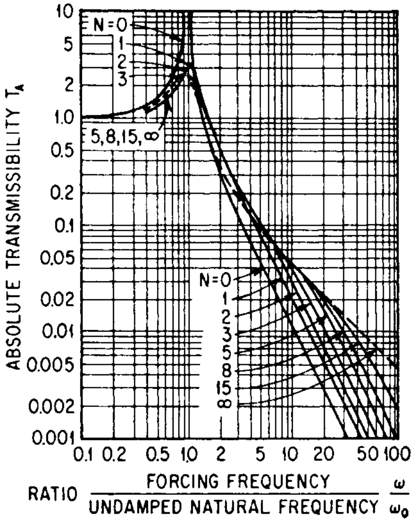

FIGURE 30.7 Comparison of absolute transmissibility for rigidly and elastically connected, viscous damped isolation systems shown at *A* and *C*, respectively, in Table 30.1, as a function of the frequency ratio ω/ω_0. The solid curves refer to the elastically connected damper, and the parameter *N* is the ratio of the damper spring stiffness to the stiffness of the principal support spring. The fraction of critical damping $\zeta = c/c_c$ is 0.2 in both systems. The transmissibility at high frequencies decreases at a rate of 6 dB per octave for the rigidly connected damper and 12 dB per octave for the elastically connected damper.

Motion Response. A typical motion response curve is shown in Fig. 30.9 for the stiffness ratio $N = 3$. For small damping, the response is similar to the response of an isolation system with rigidly connected viscous damper. For intermediate values of damping, the curves tend to be flat over a wide frequency range before rapidly decreasing in value at the higher frequencies. For large damping, the resonance occurs near the natural frequency of the system with infinitely great damping. All response curves approach a high-frequency asymptote for which the attenuation varies inversely as the square of the excitation frequency.

Optimum Transmissibility. For a system with optimum damping, maximum transmissibility coincides with the intersections of the transmissibility curves for zero and infinite damping. The frequency ratios $(\omega/\omega_0)_{op}$ at which this occurs are different for absolute and relative transmissibility:

Absolute transmissibility:

$$\left(\frac{\omega}{\omega_0}\right)_{op}^{(A)} = \sqrt{\frac{2(N+1)}{N+2}} \qquad (30.13)$$

Relative transmissibility:

$$\left(\frac{\omega}{\omega_0}\right)_{op}^{(R)} = \sqrt{\frac{N+2}{2}}$$

The optimum transmissibility at resonance, for both absolute and relative motion, is

$$T_{op} = 1 + \frac{2}{N} \qquad (30.14)$$

The optimum transmissibility as determined from Eq. (30.14) corresponds to the minimum points of the curves of Fig. 30.8.

The damping which produces the optimum transmissibility is obtained by differentiating the general expressions for transmissibility [(*g*) and (*h*) in Table 30.2] with respect to the frequency ratio, setting the result equal to zero, and combining it with Eq. (30.13):

Absolute transmissibility:

$$(\zeta_{op})_A = \frac{N}{4(N+1)}\sqrt{2(N+2)} \qquad (30.15a)$$

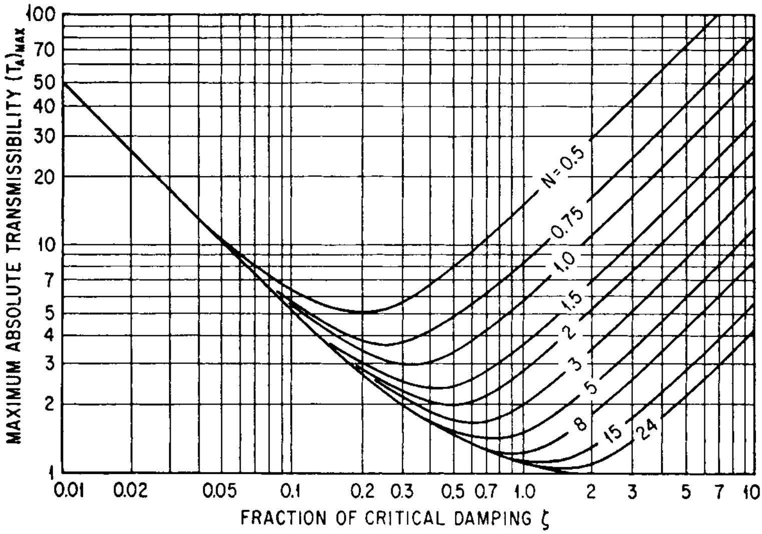

FIGURE 30.8 Maximum absolute transmissibility for the elastically connected, viscous-damped isolation system shown at *C* in Table 30.1 as a function of the fraction of critical damping ζ and the stiffness of the connecting spring. The parameter *N* is the ratio of the damper spring stiffness to the stiffness of the principal support spring.

Relative transmissibility:

$$(\zeta_{op})_R = \frac{N}{\sqrt{2(N+1)(N+2)}} \tag{30.15b}$$

Values of optimum damping determined from the first of these relations correspond to the minimum points of the curves of Fig. 30.8. By substituting the optimum damping ratios from Eqs. (30.15) into the general expressions for transmissibility given in Table 30.2, the optimum absolute and relative transmissibility equations are obtained, as shown graphically by Figs. 30.10 and 30.11, respectively. For low values of the stiffness ratio *N,* the transmissibility at resonance is large but excellent isolation is obtained at high frequencies. Conversely, for high values of *N,* the transmissibility at resonance is lowered, but the isolation efficiency also is decreased.

ELASTICALLY CONNECTED COULOMB DAMPER

Force-deflection curves for the isolators incorporating elastically connected Coulomb dampers, as shown at *D* in Table 30.1, are illustrated in Fig. 30.12. Upon application of the load, the isolator deflects; but since insufficient force has been developed in the spring k_1, the damper does not slide, and the motion of the mass is opposed by a spring of stiffness $(N + 1)k$. The load is now increased until a force is developed in spring k_1 which equals the constant friction force F_f; then the damper begins to slide. When the load is increased further, the damper slides and reduces the effective spring stiffness to k. If the applied load is reduced after reaching its maxi-

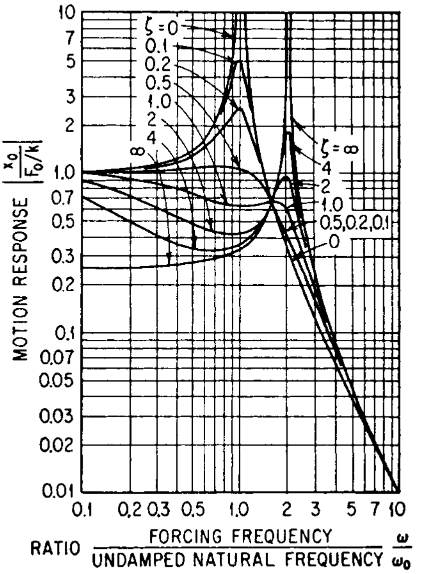

FIGURE 30.9 Motion response for the elastically connected, viscous-damped isolation system shown at *C* in Table 30.1 as a function of the frequency ratio ω/ω_0 and the fraction of critical damping ζ. For this example, the stiffness of the damper connecting spring is 3 times as great as the stiffness of the principal support spring ($N = 3$). The curves give the resulting motion of the equipment in terms of the excitation force F and the static stiffness of the isolator k.

mum value, the damper no longer displaces because the force developed in the spring k_1 is diminished. Upon completion of the load cycle, the damper will have been in motion for part of the cycle and at rest for the remaining part to form the hysteresis loops shown in Fig. 30.12.

Because of the complexity of the applicable equations, the equivalent energy method is used to obtain the transmissibility and motion response functions. Applying frequency, damping, and transmissibility expressions for the elastically connected viscous damped system to the elastically connected Coulomb-damped system, the transmissibility expressions tabulated in Table 30.2 for the latter are obtained.

If the coefficient of the damping term in each of the transmissibility expressions vanishes, the transmissibility is independent of damping. By solving for the frequency ratio ω/ω_0 in the coefficients that are thus set equal to zero, the frequency ratios obtained define the frequencies of optimum transmissibility. These frequency ratios are given by Eqs. (30.13) for the elastically connected viscous damped system and apply equally well to the elastically connected Coulomb damped system because the method of equivalent viscous damping is employed in the analysis. Similarly, Eq. (30.14) applies for optimum transmissibility at resonance.

The general characteristics of the system with an elastically connected Coulomb damper may be demonstrated by successively assigning values to the damping force while keeping the stiffness ratio N constant. For zero and infinite damping, the transmissibility curves are those for undamped systems and bound all solutions. Every transmissibility curve for $0 < F_f < \infty$ passes through the intersection of the two bounding transmissibility curves. For low damping (less than optimum), the damper "breaks loose" at a relatively low frequency, thereby allowing the transmissibility to increase to a maximum value and then pass through the intersection point of the bounding transmissibility curves. For optimum damping, the maximum absolute transmissibility has a value given by Eq. (30.14); it occurs at the frequency ratio $(\omega/\omega_0)_{op}^{(A)}$ defined by Eq. (30.13). For high damping, the damper remains "locked-in" over a wide frequency range because insufficient force is developed in the spring k_1 to induce slip in the damper. For frequencies greater than the break-loose frequency, there is sufficient force in spring k_1 to cause relative motion of the damper. For a further increase in frequency, the damper remains broken loose and the transmissibility is limited to a finite value. When there is insufficient force in spring k_1 to maintain motion across the damper, the damper locks-in and the transmissibility is that of a system with the infinite damping.

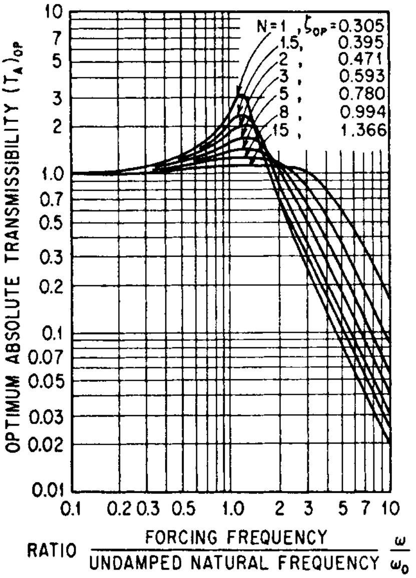

FIGURE 30.10 Absolute transmissibility with optimum damping in elastically connected, viscous-damped isolation system shown at *C* in Table 30.1 as a function of the frequency ratio ω/ω_0 and the fraction of critical damping ζ. These curves apply to elastically connected, viscous-damped systems having optimum damping for absolute motion. The transmissibility $(T_A)_{op}$ is $(x_0/u_0)_{op}$ for the motion-excited system and $(F_T/F_0)_{op}$ for the force-excited system.

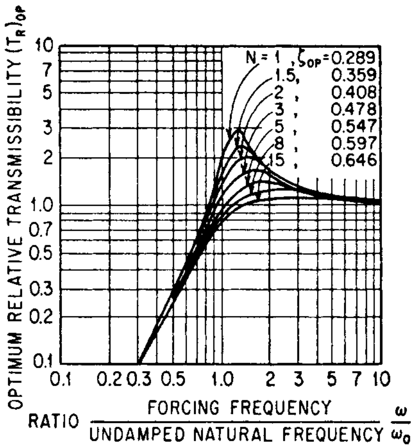

FIGURE 30.11 Relative transmissibility with optimum damping in the elastically connected, viscous-damped isolation system shown at *C* in Table 30.1 as a function of the frequency ratio ω/ω_0 and the fraction of critical damping ζ. These curves apply to elastically connected, viscous-damped systems having optimum damping for relative motion. The relative transmissibility $(T_R)_{op}$ is $(\delta_0/u_0)_{op}$ for the motion-excited system.

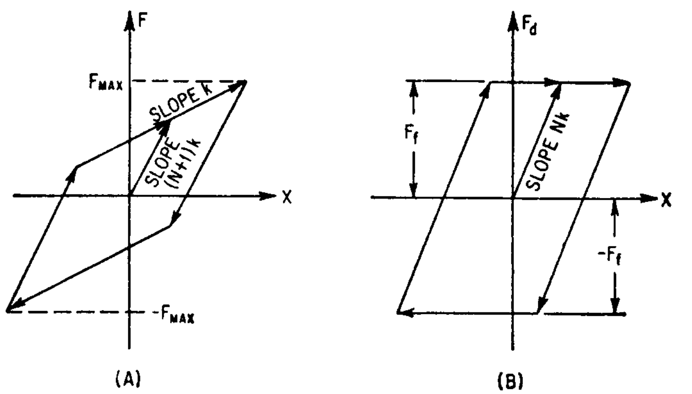

FIGURE 30.12 Force-deflection characteristics of the elastically connected, Coulomb-damped isolation system shown at *D* in Table 30.1. The force-deflection diagram for a cyclic deflection of the complete isolator is shown at *A* and the corresponding diagram for the assembly of Coulomb damper and spring $k_1 = Nk$ is shown at *B*.

The break-loose and lock-in frequencies are determined by requiring the motion across the Coulomb damper to be zero. Then the break-loose and lock-in frequency ratios are

$$\left(\frac{\omega}{\omega_0}\right)_L = \sqrt{\frac{\left(\frac{4}{\pi}\eta\right)(N+1)}{\left(\frac{4}{\pi}\eta\right) \pm N}} \tag{30.16}$$

where η is the damping parameter defined in Table 30.1 with reference to the displacement amplitude u_0. The plus sign corresponds to the break-loose frequency, while the minus sign corresponds to the lock-in frequency. Damping parameters for which the denominator of Eq. (30.16) becomes negative correspond to those conditions for which the damper never becomes locked-in again after it has broken loose. Thus, the damper eventually becomes locked-in only if $\eta > (\pi/4)N$.

Displacement Transmissibility. The absolute displacement transmissibility curve for the stiffness ratio $N = 3$ is shown in Fig. 30.13 where $(T_A)_D = x_0/u_0$. A small decrease in damping force F_f below the optimum value causes a large increase in the transmitted vibration near resonance. However, a small increase in damping force F_f above optimum causes only small changes in the maximum transmissibility. Thus, it is good design practice to have the damping parameter η equal to or greater than the optimum damping parameter η_{op}.

The relative transmissibility for $N = 3$ is shown in Fig. 30.14 where $(T_R)_D = \delta_0/u_0$. All curves pass through the intersection of the curves for zero and infinite damping. For optimum damping, the maximum relative transmissibility has a value given by Eq. (30.14); it occurs at the frequency ratio $\left(\frac{\omega}{\omega_0}\right)_{op}^{(R)}$ defined by Eq. (30.13).

Acceleration Transmissibility. The acceleration transmissibility can be obtained from the expression for displacement transmissibility by substitution of the effective displacement damping parameter in the expression for transmissibility of a system whose excitation is constant acceleration amplitude. If $\ddot{u}_0$ represents the acceleration amplitude of the excitation, the corresponding displacement amplitude is $u_0 = -\ddot{u}_0/\omega^2$. Using the definition of the acceleration Coulomb damping parameter ξ given in Table 30.1, the equivalent displacement Coulomb damping parameter is

$$\eta_{eq} = -\left(\frac{\omega}{\omega_0}\right)^2 \xi \tag{30.17}$$

Substituting this relation in the absolute transmissibility expression given at *j* in Table 30.2, the following equation is obtained for the acceleration transmissibility:

$$(T_A)_A = \frac{\ddot{x}_0}{\ddot{u}_0} = \sqrt{\frac{1 + \left(\frac{4}{\pi}\xi\right)^2 \left(\frac{\omega^2}{\omega_0^2}\right)\left[\left(\frac{N+2}{N}\right)\left(\frac{\omega^2}{\omega_0^2}\right) - 2\left(\frac{N+1}{N}\right)\right]}{\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2}} \tag{30.18}$$

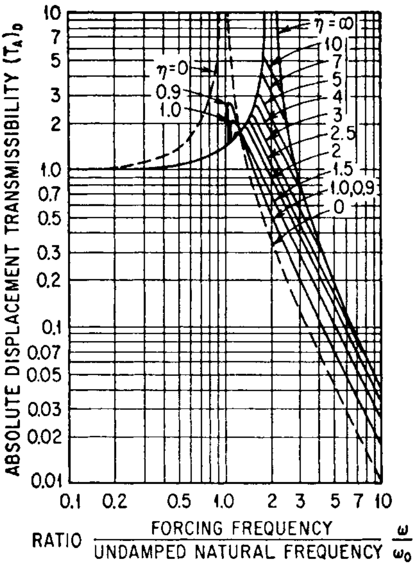

FIGURE 30.13 Absolute displacement transmissibility for the elastically connected, Coulomb-damped isolation system illustrated at D in Table 30.1, for the damper spring stiffness defined by $N = 3$. The curves give the ratio of the absolute displacement amplitude of the equipment to the displacement amplitude imposed at the foundation, as a function of the frequency ratio ω/ω_0 and the displacement Coulomb-damping parameter η.

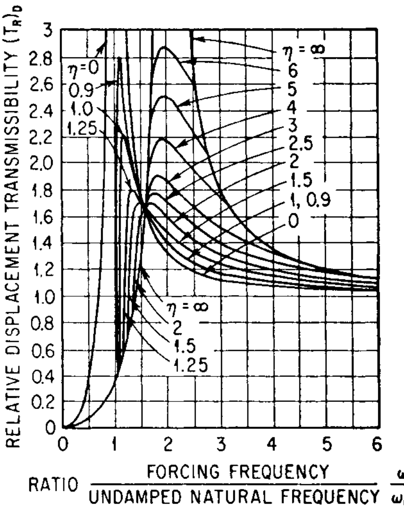

FIGURE 30.14 Relative displacement transmissibility for the elastically connected, Coulomb-damped isolation system illustrated at D in Table 30.1, for the damper spring stiffness defined by $N = 3$. The curves give the ratio of the relative displacement amplitude (maximum isolator deflection) to the displacement amplitude imposed at the foundation, as a function of the frequency ratio ω/ω_0 and the displacement Coulomb-damping parameter η.

Equation (30.18) is valid only for the frequency range in which there is relative motion across the Coulomb damper. This range is defined by the break-loose and lock-in frequencies which are obtained by substituting Eq. (30.17) into Eq. (30.16):

$$\left(\frac{\omega}{\omega_0}\right)_L = \sqrt{\frac{\left(\frac{4}{\pi}\xi\right)(N+1) \pm N}{\frac{4}{\pi}\xi}} \tag{30.19}$$

where Eqs. (30.16) and (30.19) give similar results, damping being defined in terms of displacement and acceleration excitation, respectively. For frequencies not included in the range between break-loose and lock-in frequencies, the acceleration transmissibility is that for an undamped system. Equation (30.18) indicates that infinite acceleration occurs at resonance unless the damper remains locked-in beyond a frequency ratio of unity. The coefficient of the damping term in Eq. (30.18) is identical to the corresponding coefficient in the expression for $(T_A)_D$ at j in Table 30.2. Thus, the frequency ratio at the optimum transmissibility is the same as that for displacement excitation.

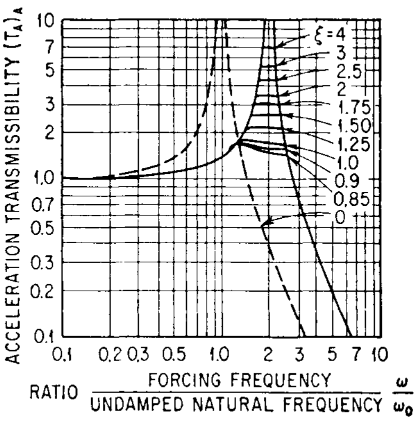

FIGURE 30.15 Acceleration transmissibility for the elastically connected, Coulomb-damped isolation system illustrated at *D* in Table 30.1, for the damper spring stiffness defined by $N = 3$. The curves give the ratio of the acceleration amplitude of the equipment to the acceleration amplitude imposed at the foundation, as a function of the frequency ratio ω/ω_0 and the acceleration Coulomb-damping parameter ξ.

An acceleration transmissibility curve for $N = 3$ is shown by Fig. 30.15. Relative motion at the damper occurs in a limited frequency range; thus, for relatively high frequencies, the acceleration transmissibility is similar to that for infinite damping.

Optimum Damping Parameters. The optimum Coulomb damping parameters are obtained by equating the optimum viscous damping ratio given by Eq. (30.15) to the equivalent viscous damping ratio for the elastically supported damper system and replacing the frequency ratio by the frequency ratio given by Eq. (30.13). The optimum value of the damping parameter η in Table 30.1 is

$$\eta_{op} = \frac{\pi}{2}\sqrt{\frac{N+1}{N+2}} \tag{30.20}$$

To obtain the optimum value of the damping parameter ξ in Table 30.1, Eq. (30.17) is substituted in Eq. (30.20):

$$\xi_{op} = \frac{\pi}{4}\sqrt{\frac{N+2}{N+1}} \tag{30.21}$$

Force Transmissibility. The force transmissibility $(T_A)_F = F_T/F_0$ is identical to $(T_A)_A$ given by Eq. (30.18) if $\xi = \xi_F$, where ξ_F is defined as

$$\xi_F = \frac{F_f}{F_0} \tag{30.22}$$

Thus, the transmissibility curve shown in Fig. 30.15 also gives the force transmissibility for $N = 3$. By substituting Eq. (30.22) into Eq. (30.21), the transmitted force is optimized when the friction force F_f has the following value:

$$(F_f)_{op} = \frac{\pi F_0}{4}\sqrt{\frac{N+2}{N+1}} \tag{30.23}$$

To avoid infinite transmitted force at resonance, it is necessary that $F_f > (\pi/4)F_0$.

Comparison of Rigidly Connected and Elastically Connected Coulomb-Damped Systems. A principal limitation of the rigidly connected Coulomb-damped isolator is the nature of the transmissibility at high forcing frequencies. Because the isolator deflection is small, the force transmitted by the spring is negligible; then the force transmitted by the damper controls the motion experienced by the equipment. The acceleration transmissibility approaches the constant value $(4/\pi)\xi$, independent of frequency. The corresponding transmissibility for an isolator with an elastically connected Coulomb damper is $(N + 1)/(\omega/\omega_0)^2$. Thus, the transmissibility varies inversely as the square of the excitation frequency and reaches a relatively low value at large values of excitation frequency.

MULTIPLE DEGREE-OF-FREEDOM SYSTEMS

The single degree-of-freedom systems discussed previously are adequate for illustrating the fundamental principles of vibration isolation but are an oversimplification insofar as many practical applications are concerned. The condition of unidirectional motion of an elastically mounted mass is not consistent with the requirements in many applications. In general, it is necessary to consider freedom of movement in all directions, as dictated by existing forces and motions and by the elastic constraints. Thus, in the general isolation problem, the equipment is considered as a rigid body supported by resilient supporting elements or isolators. This system is arranged so that the isolators effect the desired reduction in vibration. Various types of symmetry are encountered, depending upon the equipment and arrangement of isolators.

NATURAL FREQUENCIES—ONE PLANE OF SYMMETRY

A rigid body supported by resilient supports with one vertical plane of symmetry has three coupled natural modes of vibration and a natural frequency in each of these modes. A typical system of this type is illustrated in Fig. 30.16; it is assumed to be symmetrical with respect to a plane parallel with the plane of the paper and extending through the center-of-gravity of the supported body. Motion of the supported body in horizontal and vertical translational modes and in the rotational mode, all in the plane of the paper, are coupled. The equations of motion of a rigid body on resilient supports with six degrees-of-freedom are given by Eq. (3.31). By introducing certain types of symmetry and setting the excitation equal to zero, a cubic equation defining the free vibration of the system shown in Fig. 30.16 is derived, as given by Eqs. (3.36). This equation may be solved graphically for the natural frequencies of the system by use of Fig. 3.14.

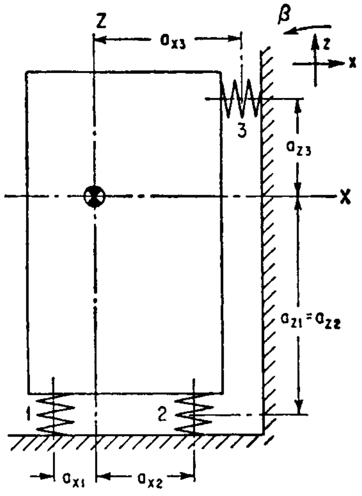

FIGURE 30.16 Schematic diagram of a rigid equipment supported by an arbitrary arrangement of vibration isolators, symmetrical with respect to a plane through the center-of-gravity parallel with the paper.

SYSTEM WITH TWO PLANES OF SYMMETRY

A common arrangement of isolators is illustrated in Fig. 30.17; it consists of an equipment supported by four isolators located adjacent to the four lower corners. It is symmetrical with respect to two coordinate vertical planes through the center-of-gravity of the equipment, one of the planes being parallel with the plane of the paper. Because of this symmetry, vibration in the vertical translational mode is decoupled from vibration in the horizontal and rotational modes. The natural frequency in the vertical translational mode is $\omega_z = \sqrt{\Sigma k_z/m}$, where Σk_z is the sum of the vertical stiffnesses of the isolators.

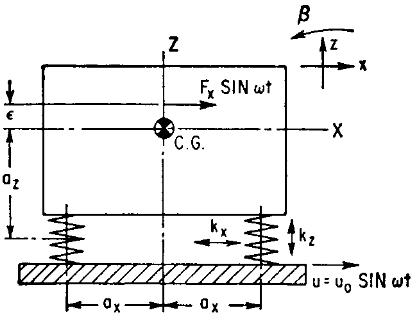

FIGURE 30.17 Schematic diagram in elevation of a rigid equipment supported upon four vibration isolators. The plane of the paper extends vertically through the center-of-gravity; the system is symmetrical with respect to this plane and with respect to a vertical plane through the center-of-gravity perpendicular to the paper. The moment of inertia of the equipment with respect to an axis through the center-of-gravity and normal to the paper is I_y. Excitation of the system is alternatively a vibratory force $F_x \sin \omega t$ applied to the equipment or a vibratory displacement $u = u_0 \sin \omega t$ of the foundation.

Consider excitation by a periodic force $F = F_x \sin \omega t$ applied in the direction of the X axis at a distance ϵ above the center-of-gravity and in one of the planes of symmetry. The differential equations of motion for the equipment in the coupled horizontal translational and rotational modes are obtained by substituting in Eq. (3.31) the conditions of symmetry defined by Eqs. (3.33), (3.34), (3.35), and (3.38). The resulting equations of motion are

$$m\ddot{x} = -4k_x x + 4k_x a\beta + F_x \sin \omega t \qquad (30.24)$$

$$I_y\ddot{\beta} = 4k_x ax - 4k_x a^2\beta - 4k_y b^2\beta - F_x\epsilon \sin \omega t$$

Making the common assumption that transients may be neglected in systems undergoing forced vibration, the translational and rotational displacements of the supported body are assumed to be harmonic at the excitation frequency. The differential equations of motion then are solved simultaneously to give the following expressions for the displacement amplitudes x_0 in horizontal translation and β_0 in rotation:

$$x_0 = \frac{F_x}{4k_z}\left(\frac{A_1}{D}\right) \qquad \beta_0 = \frac{F_x}{4\rho_y k_z}\left(\frac{A_2}{D}\right) \qquad (30.25)$$

where

$$A_1 = \left(\frac{1}{\rho_y^2}\right)(\eta a_z^2 + a_x^2 - \eta\epsilon a_z) - \left(\frac{\omega}{\omega_z}\right)^2$$

$$A_2 = \frac{\epsilon}{\rho_y}\left(\frac{\omega}{\omega_z}\right)^2 + \frac{\eta}{\rho_y}(a_z - \epsilon) \qquad (30.26)$$

$$D = \left(\frac{\omega}{\omega_z}\right)^4 - \left(\eta + \eta\frac{a_z^2}{\rho_y^2} + \frac{a_x^2}{\rho_y^2}\right)\left(\frac{\omega}{\omega_z}\right)^2 + \eta\left(\frac{a_x}{\rho_y}\right)^2$$

In the above equations, $\eta = k_x/k_z$ is the dimensionless ratio of horizontal stiffness to vertical stiffness of the isolators, $\rho_y = \sqrt{I_y/m}$ is the radius of gyration of the supported body about an axis through its center-of-gravity and perpendicular to the paper, $\omega_z = \sqrt{\Sigma k_z/m}$ is the undamped natural frequency in vertical translation, ω is the forcing frequency, a_z is the vertical distance from the effective height of spring (mid-height if symmetrical top to bottom)* to center-of-gravity of body m, and the other parameters are as indicated in Fig. 30.17.

Forced vibration of the system shown in Fig. 30.17 also may be excited by periodic motion of the support in the horizontal direction, as defined by $u = u_0 \sin \omega t$. The differential equations of motion for the supported body are

$$\begin{aligned} m\ddot{x} &= 4k_x(u - x - a_z\beta) \\ I_y\ddot{\beta} &= -4a_zk_x(u - x - a_z\beta) - 4k_za_x^2\beta \end{aligned} \tag{30.27}$$

Neglecting transients, the motion of the mounted body in horizontal translation and in rotation is assumed to be harmonic at the forcing frequency. Equations (30.27) may be solved simultaneously to obtain the following expressions for the displacement amplitudes x_0 in horizontal translation and β_0 in rotation:

$$x_0 = \frac{u_0B_1}{D} \qquad \beta_0 = \frac{u_0B_2}{\rho_yD} \tag{30.28}$$

where the parameters B_1 and B_2 are

$$B_1 = \eta\left(\frac{a_x^2}{\rho_y^2} - \frac{\omega^2}{\omega_z^2}\right) \qquad B_2 = \frac{\eta a_z}{\rho_y}\left(\frac{\omega}{\omega_z}\right)^2 \tag{30.29}$$

and D is given by Eq. (30.26).

Natural Frequencies—Two Planes of Symmetry. In forced vibration, the amplitude becomes a maximum when the forcing frequency is approximately equal to a natural frequency. In an undamped system, the amplitude becomes infinite at resonance. Thus, the natural frequency or frequencies of an undamped system may be determined by writing the expression for the displacement amplitude of the system in forced vibration and finding the excitation frequency at which this amplitude becomes infinite. The denominators of Eqs. (30.25) and (30.28) include the parameter D defined by Eq. (30.26). The natural frequencies of the system in coupled rotational and horizontal translational modes may be determined by equating D to zero and solving for the forcing frequencies:[4]

$$\frac{\omega_{x\beta}}{\omega_z} \times \frac{\rho_y}{a_x} = \frac{1}{\sqrt{2}}\sqrt{\eta\left(\frac{\rho_y}{a_x}\right)^2\left(1 + \frac{a_z^2}{\rho_y^2}\right) + 1 \pm \sqrt{\left[\eta\left(\frac{\rho_y}{a_x}\right)^2\left(1 + \frac{a_z^2}{\rho_y^2}\right) + 1\right]^2 - 4\eta\left(\frac{\rho_y}{a_x}\right)^2}} \tag{30.30}$$

where $\omega_{x\beta}$ designates a natural frequency in a coupled rotational (β) and horizontal translational (x) mode, and ω_z designates the natural frequency in the decoupled

* The distance a_z is taken to the mid-height of the spring to include in the equations of motion the moment applied to the body m by the fixed-end spring. If the spring is hinged to body m, the appropriate value for a_z is the distance from the X axis to the hinge axis.

vertical translational mode. The other parameters are defined in connection with Eq. (30.26). Two numerically different values of the dimensionless frequency ratio $\omega_{x\beta}/\omega_z$ are obtained from Eq. (30.30), corresponding to the two discrete coupled modes of vibration. Curves computed from Eq. (30.30) are given in Fig. 30.18.

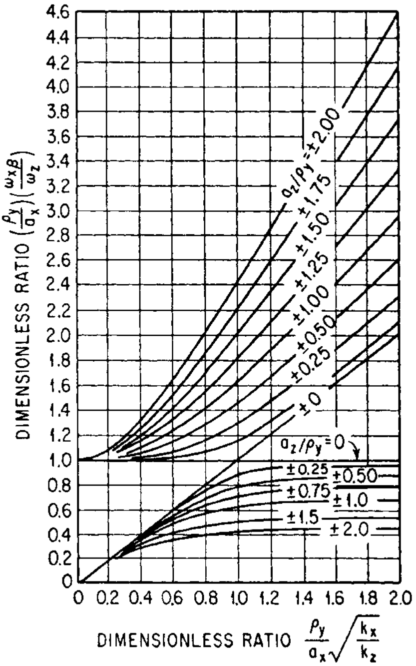

FIGURE 30.18 Curves of natural frequencies $\omega_x\beta$ in coupled modes with reference to the natural frequency in the decoupled vertical translational mode ω_z, for the system shown schematically in Fig. 30.17. The isolator stiffnesses in the X and Z directions are indicated by k_x and k_z, respectively, and the radius of gyration with respect to the Y axis through the center-of-gravity is indicated by ρ_y.

The ratio of a natural frequency in a coupled mode to the natural frequency in the vertical translational mode is a function of three dimensionless ratios, two of the ratios relating the radius of gyration ρ_y to the dimensions a_z and a_x while the third is the ratio η of horizontal to vertical stiffnesses of the isolators. In applying the curves of Fig. 30.18, the applicable value of the abscissa ratio is first determined directly from the constants of the system. Two appropriate numerical values then are taken from the ordinate scale, as determined by the two curves for applicable values of a_z/ρ_y; the ratios of natural frequencies in coupled and vertical translational modes are determined by dividing these values by the dimensionless ratio ρ_y/a_x. The natural frequencies in coupled modes then are determined by multiplying the resulting ratios by the natural frequency in the decoupled vertical translational mode.

The two straight lines in Fig. 30.18 for $a_z/\rho_y = 0$ represent natural frequencies in decoupled modes of vibration. When $a_z = 0$, the elastic supports lie in a plane passing through the center-of-gravity of the equipment. The horizontal line at a value of unity on the ordinate scale represents the natural frequency in a rotational mode. The inclined straight line for the value $a_z/\rho_y = 0$ represents the natural frequency of the system in horizontal translation.

Calculation of the coupled natural frequencies of a rigid body on resilient supports from Eq. (30.30) is sufficiently laborious to encourage the use of graphical means. For general purposes, both coupled natural frequencies can be obtained from Fig. 30.18. For a given type of isolators, $\eta = k_x/k_z$ is a constant and Eq. (30.30) may be evaluated in a manner that makes it possible to select isolator positions to attain optimum natural frequencies.[5] This is discussed under *Space-Plots* in Chap. 3. The convenience of the approach is partially offset by the need for a separate plot for each value of the stiffness ratio k_x/k_z. Applicable curves are plotted for several values of k_x/k_z in Figs. 3.17 to 3.19.

The preceding analysis of the dynamics of a rigid body on resilient supports includes the assumption that the principal axes of inertia of the rigid body are, respectively, parallel with the principal elastic axes of the resilient supports. This makes it possible to neglect the products of inertia of the rigid body. The coupling

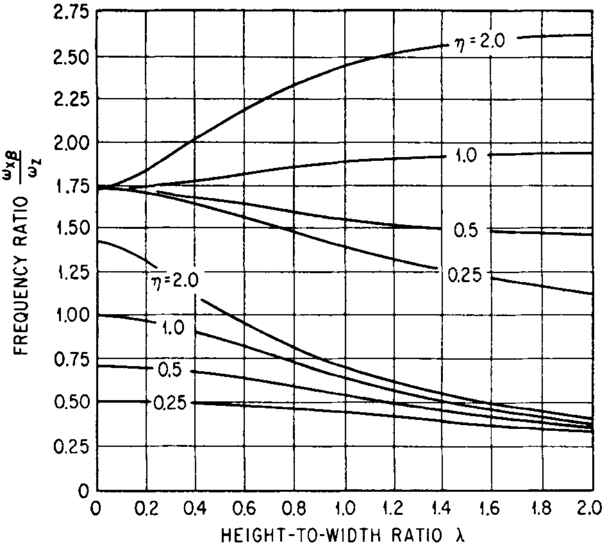

FIGURE 30.19 Curves indicating the natural frequencies $\omega_x\beta$ in coupled rotational and horizontal translational modes with reference to the natural frequency ω_z in the decoupled vertical translational mode, for the system shown in Fig. 30.17. The ratio of horizontal to vertical stiffness of the isolators is η, and the height-to-width ratio for the equipment is λ. These curves are based upon the assumption that the mass of the equipment is uniformly distributed and that the isolators are attached precisely at the extreme lower corners thereof.

introduced by the product of inertia is not strong unless the angle between the above-mentioned inertia and elastic axes is substantial. It is convenient to take the coordinate axes through the center-of-gravity of the supported body, parallel with the principal elastic axes of the isolators. If the moments of inertia with respect to these coordinate axes are used in Eqs. (30.24) to (30.30), the calculated natural frequencies usually are correct within a few percent without including the effect of product of inertia. When it is desired to calculate the natural frequencies accurately or when the product of inertia coupling is strong, a calculation procedure is available that may be used for certain conventional arrangements using four isolators.[6]

The procedure for determining the natural frequencies in coupled modes summarized by the curves of Fig. 30.18 represents a rigorous analysis where the assumed symmetry exists. The procedure is somewhat indirect because the dimensionless ratio ρ_y/a_x appears in both ordinate and abscissa parameters and because it is necessary to determine the radius of gyration of the equipment. The relations may be approximated in a more readily usable form if (1) the mounted equipment can be considered a cuboid having uniform mass distribution, (2) the four isolators are attached precisely at the four lower corners of the cuboid, and (3) the height of the isolators may be considered negligible. The ratio of the natural frequencies in the coupled rotational and horizontal translational modes to the natural frequency in the vertical translational mode then becomes a function of only the dimensions of the cuboid and the stiffnesses of the isolators in the several coordinate directions. Making these assumptions and substituting in Eq. (30.30),

$$\frac{\omega_{x\beta}}{\omega_z} = \frac{1}{\sqrt{2}} \sqrt{\frac{4\eta\lambda^2 + \eta + 3}{\lambda^2 + 1} \pm \sqrt{\left(\frac{4\eta\lambda^2 + \eta + 3}{\lambda^2 + 1}\right)^2 - \frac{12\eta}{\lambda^2 + 1}}} \tag{30.31}$$

where $\eta = k_x/k_z$ designates the ratio of horizontal to vertical stiffness of the isolators and $\lambda = 2a_z/2a_x$ indicates the ratio of height to width of mounted equipment. This relation is shown graphically in Fig. 30.19. The curves included in this figure are useful for calculating approximate values of natural frequencies and for indicating trends in natural frequencies resulting from changes in various parameters as follows:

1. Both of the coupled natural frequencies tend to become a minimum, for any ratio of height to width of the mounted equipment, when the ratio of horizontal to vertical stiffness k_x/k_z of the isolators is low. Conversely, when the ratio of horizontal to vertical stiffness is high, both coupled natural frequencies also tend to be high. Thus, when the isolators are located underneath the mounted body, a condition of low natural frequencies is obtained using isolators whose stiffness in a horizontal direction is less than the stiffness in a vertical direction. However, low horizontal stiffness may be undesirable in applications requiring maximum stability. A compromise between natural frequency and stability then may lead to optimum conditions.

2. As the ratio of height to width of the mounted equipment increases, the lower of the coupled natural frequencies decreases. The trend of the higher of the coupled natural frequencies depends on the stiffness ratio of the isolators. One of the coupled natural frequencies tends to become very high when the horizontal stiffness of the isolators is greater than the vertical stiffness and when the height of the mounted equipment is approximately equal to or greater than the width. When the ratio of height to width of mounted equipment is greater than 0.5, the spread between the coupled natural frequencies increases as the ratio k_x/k_z of horizontal to vertical stiffness of the isolators increases.

Natural Frequency—Uncoupled Rotational Mode. Figure 30.20 is a plan view of the body shown in elevation in Fig. 30.17. The distances from the isolators to the principal planes of inertia are designated by a_x and a_y. The horizontal stiffnesses of the isolators in the directions of the coordinate axes X and Y are indicated by k_x and k_y, respectively. When the excitation is the applied couple $M = M_0 \sin \omega t$, the differential equation of motion is

$$I_z\ddot{\gamma} = -4\gamma a_x^2 k_y - 4\gamma a_y^2 k_x + M_0 \sin \omega t \tag{30.32}$$

where I_z is the moment of inertia of the body with respect to the Z axis. Neglecting transient terms, the solution of Eq. (30.32) gives the displacement amplitude γ_0 in rotation:

$$\gamma_0 = \frac{M_0}{4(a_x^2 k_y + a_y^2 k_x) - I_z\omega^2} \tag{30.33}$$

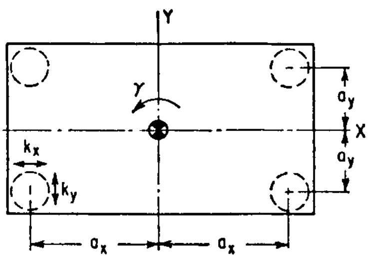

FIGURE 30.20 Plan view of the equipment shown schematically in Fig. 30.17, indicating the uncoupled rotational mode specified by the rotation angle γ.

where the natural frequency ω_γ in rotation about the Z axis is the value of ω that makes the denominator of Eq. (30.33) equal to zero:

$$\omega_\gamma = 2\sqrt{\frac{a_x{}^2k_y + a_y{}^2k_x}{I_z}} \tag{30.34}$$

VIBRATION ISOLATION IN COUPLED MODES

When the equipment and isolator system has several degrees-of-freedom and the isolators are located in such a manner that several natural modes of vibration are coupled, it becomes necessary in evaluating the isolators to consider the contribution of the several modes in determining the motion transmitted from the support to the mounted equipment or the force transmitted from the equipment to the foundation. Methods for determining the transmissibility under these conditions are best illustrated by examples.

For example, consider the system shown schematically in Fig. 30.21 wherein a machine is supported by relatively long beams which are in turn supported at their opposite ends by vibration isolators. The isolators are assumed to be undamped, and the excitation is considered to be a force applied at a distance $\epsilon = 4$ in. above the center-of-gravity of the machine-and-beam assembly. Alternatively, the force is (1) $F_x = F_0 \cos \omega t$, $F_z = F_0 \sin \omega t$ in a plane normal to the Y axis or (2) $F_y = F_0 \cos \omega t$, $F_z = F_0 \sin \omega t$ in a plane normal to the X axis. This may represent an unbalanced weight rotating in a vertical plane. A force transmissibility at each of the four isolators is determined by calculating the deflection of each isolator, multiplying the

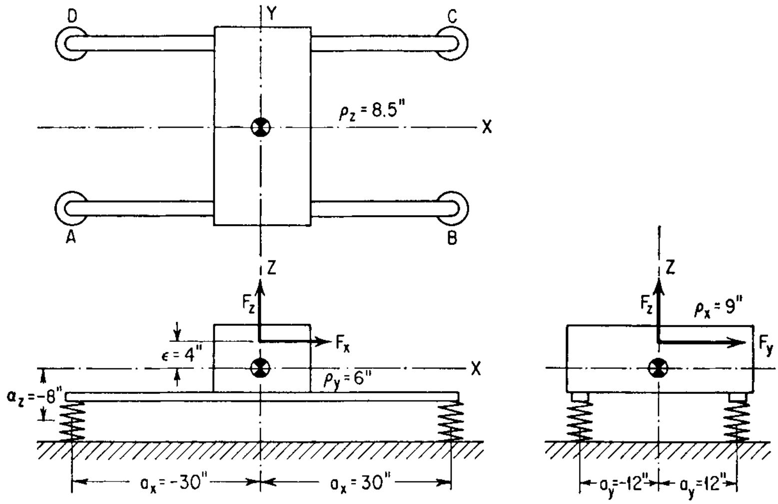

FIGURE 30.21 Schematic diagram of an equipment mounted upon relatively long beams which are in turn attached at their opposite ends to vibration isolators. Excitation for the system is alternatively (1) the vibratory force $F_x = F_0 \cos \omega t$, $F_Z = F_0 \sin \omega t$ in the XZ plane or (2) the vibratory force $F_y = F_0 \cos \omega t$, $F_Z = F_0 \sin \omega t$ in the YZ plane.

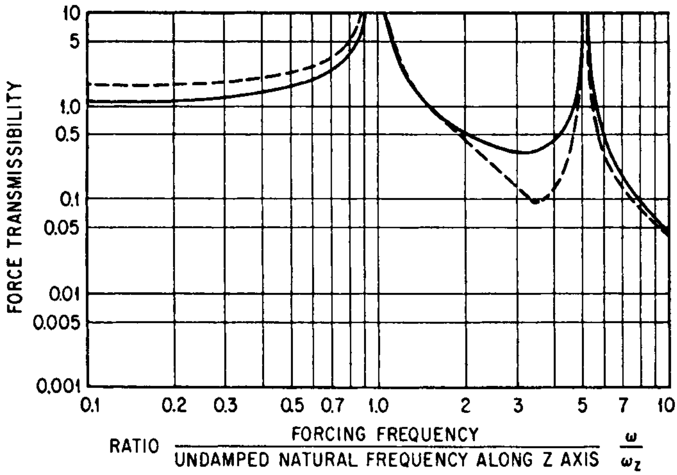

FIGURE 30.22 Transmissibility curves for the system shown in Fig. 30.21 when the excitation is in a plane perpendicular to the *Y* axis. The solid line indicates the transmissibility at each of isolators *B* and *C*, whereas the dotted line indicates the transmissibility at each of isolators *A* and *D*.

deflection by the appropriate isolator stiffness to obtain transmitted force, and dividing it by $F_0/4$.

When the system is viewed in a vertical plane perpendicular to the *Y* axis, the transmissibility curves are as illustrated in Fig. 30.22. The solid line defines the transmissibility at each of isolators *B* and *C* in Fig. 30.21, and the dotted line defines the transmissibility at each of isolators *A* and *D*. Similar transmissibility curves for a plane perpendicular to the *X* axis are shown in Fig. 30.23 wherein the solid line indicates the transmissibility at each of isolators *C* and *D*, and the dotted line indicates the transmissibility at each of isolators *A* and *B*.

Note the comparison of the transmissibility curves of Figs. 30.22 and 30.23 with the diagram of the system in Fig. 30.21. Figure 30.23 shows the three resonance conditions which are characteristic of a coupled system of the type illustrated. The transmissibility remains equal to or greater than unity for all excitation frequencies lower than the highest resonance frequency in a coupled mode. At greater excitation frequencies, vibration isolation is attained, as indicated by values of force transmissibility smaller than unity.

The transmissibility curves in Fig. 30.22 show somewhat similar results. The long horizontal beams tend to spread the resonance frequencies by a substantial frequency increment and merge the resonance frequency in the vertical translational mode with the resonance frequency in one of the coupled modes. A low transmissibility is again attained at excitation frequencies greater than the highest resonance frequency. Note that the transmissibility drops to a value slightly less than unity over a small frequency interval between the predominant resonance frequencies. This is a force reduction resulting from the relatively long beams, and it constitutes an acceptable condition if the magnitude of the excitation force in this direction is relatively small. Thus, the natural frequencies of the isolators could be somewhat higher with a consequent gain in stability; it is necessary, however, that the excitation frequency be substantially constant.

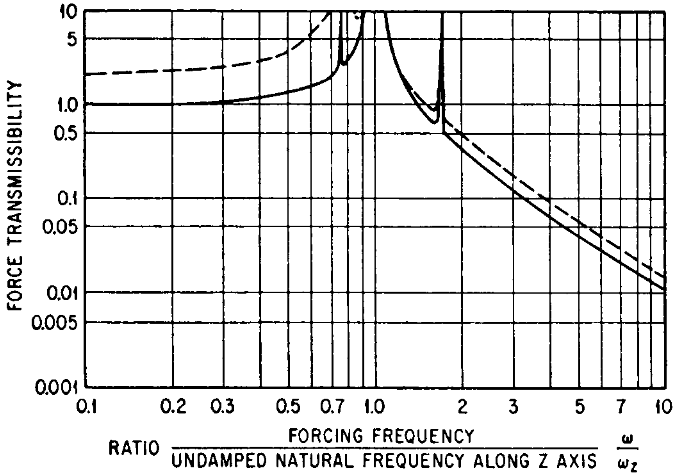

FIGURE 30.23 Transmissibility curves for the system illustrated in Fig. 30.21 when the excitation is in a plane perpendicular to the X axis. The solid line indicates the transmissibility at each of isolators C and D, whereas the dotted line indicates the transmissibility at each of isolators A and B.

Consider the equipment illustrated in Fig. 30.24 when the excitation is horizontal vibration of the support. The effectiveness of the isolators in reducing the excitation vibration is evaluated by plotting the displacement amplitude of the horizontal vibration at points A and B with reference to the displacement amplitude of the support. Transmissibility curves for the system of Fig. 30.24 are shown in Fig. 30.25. The solid line in Fig. 30.25 refers to point A and the dotted line to point B. Note that there is no significant reduction of amplitude except when the forcing frequency exceeds the maximum resonance frequency of the system.

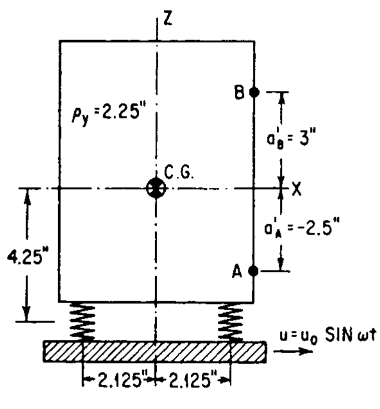

FIGURE 30.24 Schematic diagram of an equipment supported by vibration isolators. Excitation is a vibratory displacement $u = u_0 \sin \omega t$ of the foundation.

A general rule for the calculation of necessary isolator characteristics to achieve the results illustrated in Figs. 30.22, 30.23, and 30.25 is that the forcing frequency should be not less than 1.5 to 2 times the maximum natural frequency in any of six natural modes of vibration. In exceptional cases, such as illustrated in Fig. 30.22, the forcing frequency may be interposed between resonance frequencies if the forcing frequency is a constant.

Example 30.1. Consider the machine illustrated in Fig. 30.21. The force that is to be isolated is harmonic at the constant frequency of 8 Hz; it is assumed to result from the rotation of an unbalanced member whose plane of rotation is alternatively (1) a plane perpendicular to the Y axis and (2) a plane perpendicular to the X axis. The distance

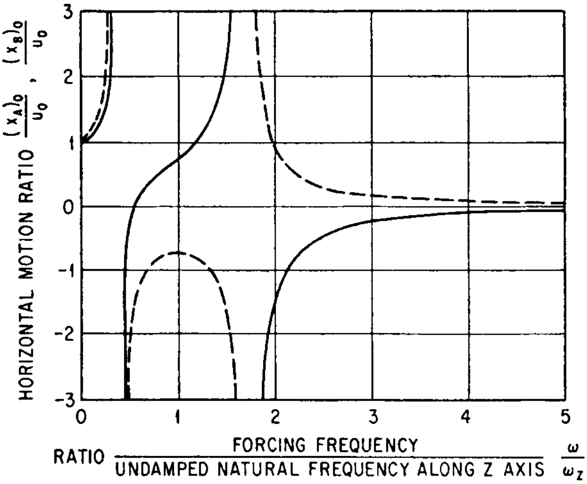

FIGURE 30.25 Displacement transmissibility curves for the system of Fig. 30.24. Transmissibility between the foundation and point *A* is shown by the solid line; transmissibility between the foundation and point *B* is shown by the dotted line.

between isolators is 60 in. in the direction of the *X* axis and 24 in. in the direction of the *Y* axis. The center of coordinates is taken at the center-of-gravity of the supported body, i.e., at the center-of-gravity of the machine-and-beams assembly. The total weight of the machine and supporting beam assembly is 100 lb, and its radii of gyration with respect to the three coordinate axes through the center-of-gravity are $\rho_x = 9$ in., $\rho_z = 8.5$ in., and $\rho_y = 6$ in. The isolators are of equal stiffnesses in the directions of the three coordinate axes:

$$\eta = \frac{k_x}{k_z} = \frac{k_y}{k_z} = 1$$

The following dimensionless ratios are established as the initial step in the solution:

$$a_z/\rho_y = -1.333 \qquad a_z/\rho_x = -0.889$$

$$a_x/\rho_y = \pm 5.0 \qquad a_y/\rho_x = \pm 1.333$$

$$(a_z/\rho_y)^2 = 1.78 \qquad (a_z/\rho_x)^2 = 0.790$$

$$(a_x/\rho_y)^2 = 25.0 \qquad (a_y/\rho_x)^2 = 1.78$$

$$\eta(\rho_y/a_x)^2 = 0.04 \qquad \eta(\rho_x/a_y)^2 = 0.561$$

The various natural frequencies are determined in terms of the vertical natural frequency ω_z. Referring to Fig. 30.18, the coupled natural frequencies for vibration in a plane perpendicular to the *Y* axis are determined as follows:

First calculate the parameter

$$\frac{\rho_y}{a_x}\sqrt{\frac{k_x}{k_z}} = 0.2$$

For $a_z/\rho_y = -1.333$, $(\omega_{x\beta}/\omega_z)(\rho_y/a_x) = 0.19$; 1.03. Note the signs of the dimensionless ratios a_z/ρ_y and a_x/ρ_y. According to Eq. (30.30), the natural frequencies are independent of the sign of a_z/ρ_y. With regard to the ratio a_x/ρ_y, the sign chosen should be the same as the sign of the radical on the right side of Eq. (30.30). The frequency ratio $(\omega_{x\beta}/\omega_z)$ then becomes positive. Dividing the above values for $(\omega_{x\beta}/\omega_z)(\rho_y/a_x)$ by $\rho_y/a_x = 0.2$, $\omega_{x\beta}/\omega_z = 0.96$; 5.15.

Vibration in a plane perpendicular to the X axis is treated in a similar manner. It is assumed that exciting forces are not applied concurrently in planes perpendicular to the X and Y axes; thus, vibration in these two planes is independent. Consequently, the example entails two independent but similar problems and similar equations apply for a plane perpendicular to the X axis:

$$\frac{\rho_x}{a_y}\sqrt{\frac{k_z}{k_y}} = 0.75$$

For $a_z/\rho_x = 0.889$, $(\omega_{y\alpha}/\omega_z)(\rho_x/a_y) = 0.57$; 1.29. Dividing by $\rho_x/a_y = 0.75$, $\omega_{y\alpha}/\omega_z = 0.76$; 1.72.

The natural frequency in rotation with respect to the Z axis is calculated from Eq. (30.34) as follows, taking into consideration that there are two pairs of springs and that $k_x = k_y = k_z$:

$$\omega_\gamma = \sqrt{\left(\frac{a_x^2 + a_y^2}{\rho_z^2}\right)\left(\frac{4k_z g}{W}\right)} = 3.8\omega_z$$

The six natural frequencies are as follows:

1. Translational along Z axis: ω_z
2. Coupled in plane perpendicular to Y axis: $0.96\omega_z$
3. Coupled in plane perpendicular to Y axis: $5.15\omega_z$
4. Coupled in plane perpendicular to X axis: $0.76\omega_z$
5. Coupled in plane perpendicular to X axis: $1.72\omega_z$
6. Rotational with respect to Y axis: $3.8\omega_z$

Considering vibration in a plane perpendicular to the Y axis, the two highest natural frequencies are the natural frequency ω_y in the translational mode along the Z axis and the natural frequency $5.15\omega_z$ in a coupled mode. In a similar manner, the two highest natural frequencies in a plane perpendicular to the X axis are the natural frequency ω_z in translation along the Z axis and the natural frequency $1.72\omega_z$ in a coupled mode. The natural frequency in rotation about the Z axis is $3.80\omega_z$. The widest frequency increment which is void of natural frequencies is between $1.72\omega_z$ and $3.80\omega_z$. This increment is used for the forcing frequency which is taken as $2.5\omega_z$. Inasmuch as the forcing frequency is established at 8 Hz, the vertical natural frequency is 8 divided by 2.5, or 3.2 Hz. The required vertical stiffnesses of the isolators are calculated from Eq. (30.11) to be 105 lb/in. for the entire machine, or 26.2 lb/in. for each of the four isolators.

INCLINED ISOLATORS

Advantages in vibration isolation sometimes result from inclining the principal elastic axes of the isolators with respect to the principal inertia axes of the equipment, as illustrated in Fig. 30.26. The coordinate axes X and Z are, respectively, parallel with the principal inertia axes of the mounted body, but the center of coordinates is taken at the elastic axis. The location of the elastic axis is determined by the elastic properties of the system. If a force is applied to the body along a line extending through the elastic axis, the body is displaced in translation without rotation; if a couple is applied to the body, the body is displaced in rotation without translation.

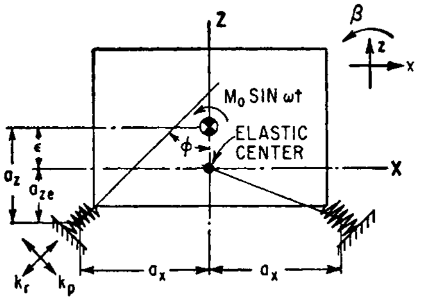

FIGURE 30.26 Schematic diagram of an equipment supported by isolators whose principal elastic axes are inclined to the principal inertia axes of the equipment.

The principal elastic axes r, p of the isolators are parallel with the paper and inclined with respect to the coordinate axes, as indicated in Fig. 30.26. The stiffness of each isolator in the direction of the respective principal axis is indicated by k_r, k_p. The principal elastic axis of an isolator is the axis along which a force must be applied to cause a deflection colinear with the applied force (see the section *Properties of a Biaxial Stiffness Isolator*).

Assume the excitation for the system shown in Fig. 30.26 to be a couple $M_0 \sin \omega t$ acting about an axis normal to the paper. The equations of motion for the body in the horizontal translational and rotational modes may be written by noting that the displacement of the center-of-gravity in the direction of the X axis is $x - \epsilon\beta$; thus, the corresponding acceleration is $\ddot{x} - \epsilon\ddot{\beta}$. A translational displacement x produces only an external force $-k_x x$, whereas a rotational displacement β produces only an external couple $-k_\beta \beta$. The equations of motion are

$$m(\ddot{x} - \epsilon\ddot{\beta}) = -k_x x$$
$$m\rho_e^2\ddot{\beta} - m\epsilon\ddot{x} = -k_\beta \beta + M_0 \sin \omega t \tag{30.35}$$

where ρ_e is the radius of gyration of the mounted body with respect to the elastic axis. The radius of gyration ρ_e is related to the radius of gyration ρ_y with respect to a line through the center-of-gravity by $\rho_e = \sqrt{\rho_y^2 + \epsilon^2}$, where ϵ is the distance between the elastic axis and a parallel line passing through the center-of-gravity. In the equations of motion, k_x and k_β represent the translational and rotational stiffness of the isolators in the x and β coordinate directions, respectively.

By assuming steady-state harmonic motion for the horizontal translation x and rotation β, the following displacement amplitudes are obtained by solving Eqs. (30.35):

$$x_0 = \frac{-M_0\epsilon\omega^2}{m[\rho_e^2(\omega^2 - \omega_\beta^2)(\omega^2 - \omega_x^2) - \epsilon^2\omega^4]}$$

$$\beta_0 = \frac{-M_0}{m\left[\rho_e^2(\omega^2 - \omega_\beta^2) - \dfrac{\epsilon^2\omega^4}{\omega^2 - \omega_x^2}\right]} \tag{30.36}$$

where $\omega_x = \sqrt{k_x/m}$ and $\omega_\beta = \sqrt{k_\beta/m\rho_e^2}$ are hypothetical natural frequencies defined for convenience. The natural frequencies $\omega_{x\beta}$ in the coupled x,β modes are determined by equating the denominator of Eqs. (30.36) to zero and solving for ω (now identical to $\omega_{x\beta}$):

$$\frac{\omega_{x\beta}}{\omega_x} = \sqrt{\frac{1 + \lambda_1^2 \pm \sqrt{(1 + \lambda_1^2)^2 - 4\lambda_1^2[1 - (\epsilon/\rho_e)^2]}}{2[1 - (\epsilon/\rho_e)^2]}} \tag{30.37}$$

where λ_1 is a dimensionless ratio given by

$$\lambda_1 = \frac{(a_x/\rho_e)\sqrt{k_r/k_p}}{\cos^2\phi + (k_r/k_p)\sin^2\phi} \tag{30.38}$$

The hypothetical natural frequency ω_x is

$$\omega_x = \sqrt{\frac{4k_p}{m}\left[\cos^2\phi + \frac{k_r}{k_p}\sin^2\phi\right]} \tag{30.39}$$

The relation given by Eq. (30.37) is shown graphically by Fig. 30.27. The parameters needed to evaluate the natural frequencies by using this graph are calculated from the physical properties of the system and the relations of Eqs. (30.38) and (30.39). In addition, the distance ϵ between a parallel line passing through the center-of-gravity and the elastic axis must be known. The distance ϵ is determined by effecting a small horizontal displacement of the equipment in the X direction and equating the resulting summation of elastic couples to zero:

$$\epsilon = a_z - \frac{a_x(1 - k_p/k_r)\cot\phi}{(k_p/k_r)\cot^2\phi + 1} \tag{30.40}$$

where a_z is the distance between the parallel planes passing through the center-of-gravity of the body and the mid-height of the isolators, as shown in Fig. 30.26.

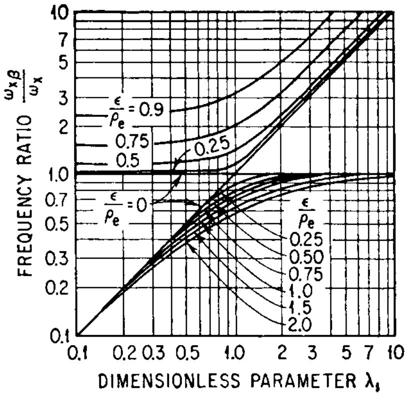

FIGURE 30.27 Curves indicating the natural frequencies $\omega_{x\beta}$ in coupled modes with reference to the natural frequency in the decoupled (fictitious) horizontal translational mode ω_x for the system shown schematically in Fig. 30.26. The radius of gyration with respect to the elastic axis is indicated by ρ_e, and the distance between the center-of-gravity and the elastic center is ϵ. The dimensionless parameter λ_1 is defined by Eq. (30.38) and ω_x is defined by Eq. (30.39).

DECOUPLING OF MODES

The natural modes of vibration of a body supported by isolators may be decoupled one from another by proper orientation of the isolators. Each mode of vibration then exists independently of the others, and vibration in one mode does not excite vibration in other modes. The necessary conditions for decoupling may be stated as follows: The resultant of the forces applied to the mounted body by the isolators when the mounted body is displaced in translation must be a force directed through the center-of-gravity; or, the resultant of the couples applied to the mounted body by the isolators when the mounted body is displaced in rotation must be a couple about an axis through the center-of-gravity.

In general, the natural frequencies of a multiple degree-of-freedom system can be made equal only by decoupling the natural modes of vibration, i.e., by making $a_z = 0$ in Fig. 30.17. The natural frequencies in decoupled modes are indicated by the two straight lines in Fig. 30.18 marked $a_z/\rho_y = 0$. The natural frequencies in translation along the X axis and in rotation about the Y axis become equal at the intersection of these lines; i.e., when $a_z/\rho_y = 0$, $k_x/k_z = 1$ and $\rho_y/a_x = 1$. The physical significance of these mathematical conditions is that the isolators be located in a plane passing through the center-of-gravity of the equipment, that the distance between isolators be twice the radius of gyration of the equipment, and that the stiffness of each isolator in the directions of the X and Z axes be equal.

When the isolators cannot be located in a plane which passes through the center-of-gravity of the equipment, decoupling can be achieved by inclining the isolators, as illustrated in Fig. 30.26. If the elastic axis of the system is made to pass through the center-of-gravity, the translational and rotational modes are decoupled because the inertia force of the mounted body is applied through the elastic center and introduces no tendency for the body to rotate. The requirements for a decoupled system are established by setting $\epsilon = 0$ in Eq. (30.40) and solving for k_r/k_p:

$$\frac{k_r}{k_p} = \frac{(a_x/a_z) + \cot \phi}{(a_x/a_z) - \tan \phi} \tag{30.41}$$

The conditions for decoupling defined by Eq. (30.41) are shown graphically in Figs. 30.28 and 3.23. The decoupled natural frequencies are indicated by the straight lines $\epsilon/\rho_e = 0$ in Fig. 30.27. The horizontal line refers to the decoupled natural frequency ω_x in translation in the direction of the X axis, while the inclined line refers to the decoupled natural frequency ω_β in rotation about the Y axis.

PROPERTIES OF A BIAXIAL STIFFNESS ISOLATOR

A biaxial stiffness isolator is represented as an elastic element having a single plane of symmetry; all forces act in this plane and the resultant deflections are limited by symmetry or constraints to this plane. The characteristic elastic properties of the isolator may be defined alternatively by sets of influence coefficients as follows:

1. If the two coordinate axes in the plane of symmetry are selected arbitrarily, three stiffness parameters are required to define the properties of the isolator. These are the axial influence coefficients* along the two coordinate axes, and a characteristic coupling influence coefficient* between the coordinate axes.

* The influence coefficient κ is a function only of the isolator properties and not of the constraints imposed by the system in which the isolator is used. Both positive and negative values of the influence coefficient κ are permissible.

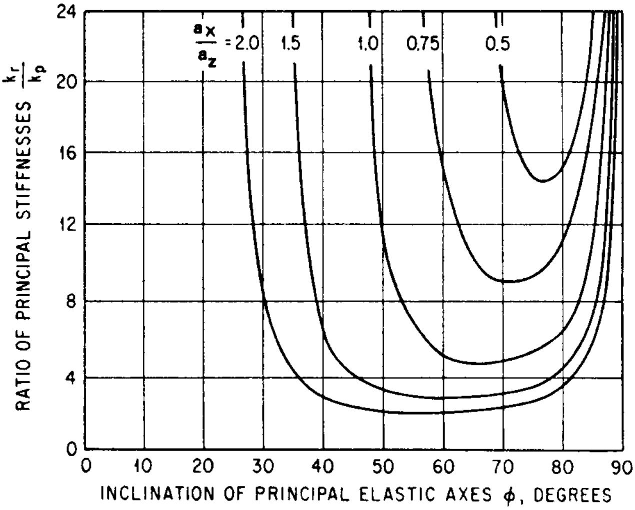

FIGURE 30.28 Ratio of stiffnesses k_r/k_p along principal elastic axes required for decoupling the natural modes of vibration of the system illustrated in Fig. 30.26.

2. If the two coordinate axes in the plane of symmetry are selected to coincide with the principal elastic axes of the isolator, two influence coefficients are required to define the properties of the isolator. These are the principal influence coefficients. If the isolator is used in a system, a third parameter is required to define the orientation of the principal axes of the isolator with the coordinate axes of the system.

PROPERTIES OF ISOLATOR WITH RESPECT TO ARBITRARILY SELECTED AXES

A schematic representation of a linear biaxial stiffness element is shown in Fig. 30.29 where the X and Y axes are arbitrarily chosen to define a plane to which all forces and motions are restricted. In general, the deflection of an isolator resulting from an applied load is not in the same direction as the load, and a coupling influence coefficient is required to define the properties of the isolator in addition to the influence coefficients along the X and Y axes. The three characteristic stiffness coefficients that uniquely describe the load-deflection properties of a biaxial stiffness element are:

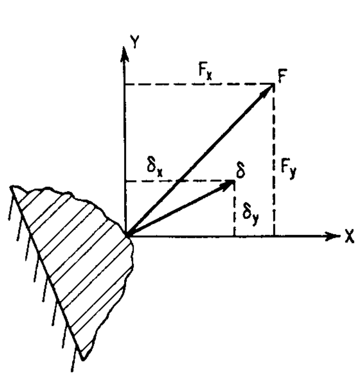

FIGURE 30.29 Schematic diagram of a linear biaxial stiffness element.

1. The influence coefficient of the element in the X coordinate direction is κ_x. It is the ratio of the component of the applied force in the X direction to the resulting deflection when the isolator is constrained to deflect in the X direction.

2. The influence coefficient of the element in the Y coordinate direction is κ_y. It is the ratio of the component of the applied force in the Y direction to the resulting deflection when the isolator is constrained to deflect in the Y direction.
3. The coupling influence coefficient is κ_{xy}. It represents the force required in the X direction to produce a unit displacement in the Y direction when the isolator is constrained to deflect only in the Y direction. (By Maxwell's reciprocity principle, the same force is required in the Y direction to produce a unit displacement in the X direction; i.e., $\kappa_{xy} = \kappa_{yx}$.)

Consider the isolator shown in Fig. 30.29 where the applied force F has components F_x and F_y; the resulting displacement has components δ_x and δ_y. From the above definitions of influence coefficients, the forces in the X and Y coordinate directions required to effect a displacement δ_x are

$$F_{xx} = \kappa_x \delta_x \qquad F_{yx} = \kappa_{yx} \delta_x \tag{30.42}$$

The forces required to effect a displacement δ_y in the Y direction are

$$F_{xy} = \kappa_{xy} \delta_y \qquad F_{yy} = \kappa_y \delta_y \tag{30.43}$$

The force components F_x and F_y required to produce the deflection having components δ_x, δ_y are the sums from Eqs. (30.42) and (30.43):

$$\begin{aligned} F_x &= \kappa_x \delta_x + \kappa_{xy} \delta_y \\ F_y &= \kappa_{yx} \delta_x + \kappa_y \delta_y \end{aligned} \tag{30.44}$$

If the three influence stiffness coefficients κ_x, κ_y, and $\kappa_{xy} = \kappa_{yx}$ are known for a given stiffness element, the load-deflection properties are given by Eq. (30.44).

The deflections of the isolator in response to forces F_x, F_y are determined by solving Eqs. (30.44) simultaneously:

$$\begin{aligned} \delta_x &= \frac{F_x \kappa_y - F_y \kappa_{xy}}{\kappa_x \kappa_y - \kappa_{xy}^2} \\ \delta_y &= \frac{F_y \kappa_x - F_x \kappa_{xy}}{\kappa_x \kappa_y - \kappa_{xy}^2} \end{aligned} \tag{30.45}$$

These expressions give the orthogonal components of the displacement δ for any load having the components F_x and F_y applied to a biaxial stiffness isolator. By substituting the relations of Eqs. (30.45) into Eq. (30.44), the following alternate forms of the force-deflection equations are obtained:

$$\begin{aligned} F_x &= \left(\kappa_x - \frac{\kappa_{xy}^2}{\kappa_y} \right) \delta_x + \frac{\kappa_{xy}}{\kappa_y} F_y \\ F_y &= \left(\kappa_y - \frac{\kappa_{xy}^2}{\kappa_x} \right) \delta_y + \frac{\kappa_{xy}}{\kappa_x} F_x \end{aligned} \tag{30.46}$$

The specific force-deflection equations for a given situation are obtained from these general load-deflection expressions by applying the proper constraint conditions.

Unconstrained Motion. The general force-deflection equations can be used to obtain the effective stiffness coefficients when the forces F_x and F_y shown in Fig. 30.29 are applied independently. The resulting deflection of the isolator is unconstrained motion, i.e., the isolator is free to deflect out of the line of force application. The force divided by that component of deflection along the line of action of the force is the effective stiffness k. When $F_y = 0$, the effective stiffness k_x resulting from the applied force F_x is obtained from Eq. (30.46):

$$k_x = \frac{F_x}{\delta_x} = \left(\kappa_x - \frac{\kappa_{xy}^2}{\kappa_y} \right) \tag{30.47}$$

When $F_x = 0$, the effective stiffness k_y in response to the applied force F_y is

$$k_y = \frac{F_y}{\delta_y} = \left(\kappa_y - \frac{\kappa_{xy}^2}{\kappa_x} \right) \tag{30.48}$$

For unconstrained motion, $k_x/k_y = \kappa_x/\kappa_y$; i.e., the ratio of the effective stiffnesses in two mutually perpendicular directions is equal to the ratio of the corresponding influence coefficients for the same directions.

Constrained Motion. When the isolator is constrained either by the symmetry of a system or by structural constraints to deflect only along the line of the applied force, the effective stiffness is obtained directly by letting appropriate deflections be zero in Eq. (30.44):

$$k_x = \frac{F_x}{\delta_x} = \kappa_x \qquad k_y = \frac{F_y}{\delta_y} = \kappa_y \tag{30.49}$$

The force required to maintain constrained motion is found by letting appropriate deflections be zero in Eqs. (30.46). For example, the force that must be applied in the X direction to ensure that the isolator deflects in the Y direction in response to a force F_y is

$$F_x = \frac{\kappa_{xy}}{\kappa_y} F_y \tag{30.50}$$

INFLUENCE COEFFICIENT TRANSFORMATION

Assume the influence coefficients κ_x, κ_y, and κ_{xy} are known in the X, Y coordinate system. It may be convenient to work with isolator influence coefficients in the X', Y' coordinate system as shown in Fig. 30.30. The X', Y' coordinate system is obtained by rotating the coordinate axes counterclockwise through an angle θ from the X, Y system. The influence coefficients with respect to the X', Y' axes are related to the influence coefficients with respect to the X, Y axes as follows:

$$\kappa_x' = \frac{\kappa_x + \kappa_y}{2} + \frac{\kappa_x - \kappa_y}{2} \cos 2\theta + \kappa_{xy} \sin 2\theta$$

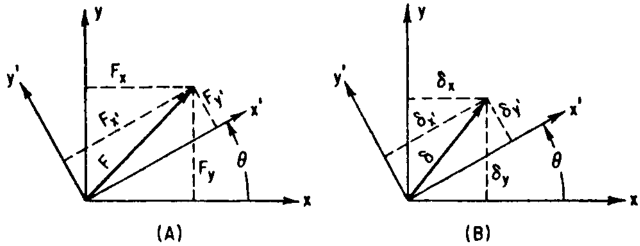

FIGURE 30.30 (*A*) Force and (*B*) displacement transformation diagrams for a linear biaxial stiffness element.

$$\kappa_{x'y'} = \frac{\kappa_y - \kappa_x}{2} \sin 2\theta + \kappa_{xy} \cos 2\theta \tag{30.51}$$

$$\kappa_y{}' = \frac{\kappa_x + \kappa_y}{2} - \frac{\kappa_x - \kappa_y}{2} \cos 2\theta - \kappa_{xy} \sin 2\theta$$

The influence coefficient transformation of a biaxial stiffness isolator from one set of arbitrarily chosen coordinate axes to another arbitrarily chosen set of coordinate axes is described by the two-dimensional Mohr circle.[7] Since the influence coefficient is a tensor quantity, the following invariants of the influence coefficient tensor give additional relations between the influence coefficients in the *X, Y* and the *X′*, *Y′* set of axes:

$$\begin{aligned} \kappa_x + \kappa_y &= \kappa_{x'} + \kappa_{y'} \\ \kappa_x\kappa_y - \kappa_{xy}{}^2 &= \kappa_{x'}\kappa_{y'} - \kappa_{x'y'}{}^2 \end{aligned} \tag{30.52}$$

PRINCIPAL INFLUENCE COEFFICIENTS

The set of axes for which there exists no coupling influence coefficient are the principal axes of stiffness (*principal elastic axes*). These axes can be found by requiring $\kappa_{x'y'}$ to be zero in Eq. (30.51) and solving for the rotation angle corresponding to this condition. Letting θ' represent the angle of rotation for which $\kappa_{x'y'} = 0$:

$$\tan 2\theta' = \frac{2\kappa_{xy}}{\kappa_x - \kappa_y} \tag{30.53}$$

By substituting this value of the angle of rotation into the general influence coefficient expressions, Eqs. (30.51), the following relation is obtained for the principal influence coefficients:

$$\kappa_p, \kappa_q = \frac{\kappa_x + \kappa_y}{2} \pm \sqrt{\left(\frac{\kappa_x - \kappa_y}{2}\right)^2 + \kappa_{xy}{}^2} \tag{30.54}$$

where *p* and *q* represent the principal axes of stiffness. The principal influence coefficients are the maximum and minimum influence coefficients that exist for a linear biaxial stiffness isolator. In Eq. (30.54), the plus sign gives the maximum influence coefficient whereas the minus sign gives the minimum influence coefficient. Either κ_p or κ_q can be the maximum influence coefficient, depending on the degree of axis rotation and the relative values of κ_x, κ_y, and κ_{xy}.

INFLUENCE COEFFICIENT TRANSFORMATION FROM THE PRINCIPAL AXES

The influence coefficient transformation from the principal axes *p, q* is of practical interest. The influence coefficients in the *XY* frame of reference are determined from Eq. (30.51) by setting $\kappa_{x'y'} = \kappa_{pq} = 0$, $\kappa_x' = \kappa_p$, $\kappa_{y'} = \kappa_q$, and $\theta = \theta'$. The influence coefficients in the *XY* frame-of-reference may be expressed in terms of the principal influence coefficients as follows:

$$\kappa_x = \kappa_p \cos^2 \theta' + \kappa_q \sin^2 \theta' = \frac{\kappa_p + \kappa_q}{2} + \frac{\kappa_p - \kappa_q}{2} \cos 2\theta'$$

$$\kappa_{xy} = (\kappa_p - \kappa_q) \sin \theta' \cos \theta' = \frac{\kappa_p - \kappa_q}{2} \sin 2\theta' \tag{30.55}$$

$$\kappa_y = \kappa_p \sin^2 \theta' + \kappa_q \cos^2 \theta' = \frac{\kappa_p + \kappa_q}{2} - \frac{\kappa_p - \kappa_q}{2} \cos 2\theta'$$

The transformation from the principal axes in the form of a two-dimensional Mohr's circle is shown by Fig. 30.31. This circle provides quick graphical determination of

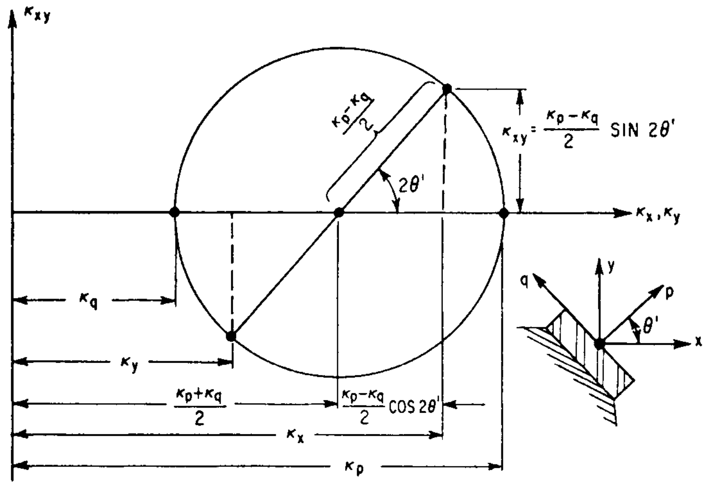

FIGURE 30.31 Mohr-circle representation of the stiffness transformation from the principal axes of stiffness of a biaxial stiffness element. The *p, q* axes represent the principal stiffness axes and the *X, Y* axes are any arbitrary set of axes separated from the *p, q* axes by a rotation angle θ′.

the three influence coefficients κ_x, κ_y, and κ_{xy} for any angle θ' between the P and X axes, where θ' is positive in the sense shown in the inset to Fig. 30.31.

Example 30.2. Consider the system shown schematically by Fig. 30.26. The transformation theory for the influence coefficient of a biaxial stiffness element may be applied to develop the effective stiffness coefficients for this system. The center of coordinates for the XZ axes is at the elastic center of the system. The principal elastic axes of the isolators p, r are oriented at an angle ϕ with the coordinate axes X, Z, respectively.* The position of the elastic center is determined by effecting a small horizontal displacement δ_x of the body, letting δ_z be zero and equating the summation of couples resulting from the isolator forces. The forces F_x and F_z are determined from Eqs. (30.44):

$$F_x = \kappa_x \delta_x = \kappa_x \delta_x \qquad F_z = \kappa_{zx} \delta_x = \kappa_{zx} \delta_x$$

Each of the forces F_x acts at a distance $-a_{ze}$ from the elastic center; the force F_z at the right-hand isolator is positive and acts at a distance a_x from the elastic center whereas the force F_z at the left-hand isolator is negative and acts at a distance $-a_x$ from the elastic center. Taking a summation of the moments:

$$-2a_{ze}F_x + 2a_xF_z = 0$$

Substituting the above relations between the forces F_x, F_z and the influence coefficients κ_z, κ_{zz} into Eqs. (30.55), and noting that $\theta' = 90° - \phi$ (compare Figs. 30.30 and 30.26), the following result is obtained in terms of principal stiffnesses:

$$\frac{a_{ze}}{a_x} = \frac{F_z}{F_x} = \frac{\kappa_{zx}}{\kappa_x} = \frac{(k_r - k_p)\sin\phi\cos\phi}{k_r \sin^2\phi + k_p \cos^2\phi}$$

Substituting $\epsilon = a_z - a_{ze}$ in the preceding equation, the relation for ϵ given by Eq. (30.40) is obtained.

Since the equations of motion are written in a coordinate system passing through the elastic center, all displacements in this frame-of-reference are constrained. Therefore, the effective stiffness coefficients for a single isolator may be obtained from Eq. (30.55) as follows [see Eq. (30.49)]:

$$k_x = \kappa_x = k_r \sin^2\phi + k_p \cos^2\phi$$

$$k_z = \kappa_z = k_r \cos^2\phi + k_p \sin^2\phi$$

These effective stiffness coefficients define the hypothetical natural frequency ω_x given by Eq. (30.39) as well as the uncoupled vertical natural frequency ω_z. Since four isolators are used in the problem represented by Fig. 30.26, the translational stiffnesses given by the above expressions for k_x and k_z must be multiplied by 4 to obtain the total translational stiffness.

The effective rotational stiffness of a single isolator k_β can be obtained by determining the sum of the restoring moments for a constrained rotation β. When the body is rotated through an angle β, the displacements at the right isolator are $\delta_x = -a_{ze}\beta$ and $\delta_z = a_x\beta$, where a_{ze} is a negative distance since it is measured in the negative Z direction. The sum of the restoring moments is $(F_za_x - F_xa_{ze})$, where F_x and F_z

* The properties of a biaxial stiffness element may be defined with respect to any pair of coordinate axes. In Fig. 30.26, the principal elastic axis q is parallel with the coordinate axis Y; then the analysis considers the principal elastic axes p, r which lie in the plane defined by the XZ coordinate axes.

are the forces acting on the right isolator in Fig. 30.26. The forces F_x and F_z may be written in terms of the influence coefficients and the displacements δ_x and δ_z by use of Eq. (30.44) to produce the following moment equation:

$$M_\beta = k_\beta \beta = \beta[k_x a_{ze}^2 - 2k_{xz} a_{ze} a_x + k_z a_x^2]$$

where the effective rotational stiffness k_β of a single isolator is

$$k_\beta = k_x a^2 - 2k_{xz} a_{ze} a_x + k_z a_x^2$$

The distance a_{ze} can be eliminated from the expression for rotational stiffness by substituting $a_{ze} = a_x F_z / F_x$ obtained from the summation of couples about the elastic center:

$$k_\beta = a_x^2 \left(\frac{k_x k_z - k_{xz}^2}{k_x} \right)$$

The numerator of this expression can be replaced by $k_r k_p$ [see Eq. (30.52)] where the r, p axes are the principal elastic axes of the isolator and $k_{rp} = 0$. Also, k_x can be replaced by its equivalent form given by Eq. (30.55). Making these substitutions, the effective rotational stiffness for one isolator in terms of the principal stiffness coefficients of the isolator becomes

$$k_\beta = \frac{a_x^2 k_p}{\sin^2 \phi + (k_p/k_r) \cos^2 \phi}$$

Since four isolators are used in the problem represented by Fig. 30.26, the rotational stiffness given by the above expression for k_β must be multiplied by 4 to obtain the total rotational stiffness of the system.

NONLINEAR VIBRATION ISOLATORS

In vibration isolation, the vibration amplitudes generally are small and linear vibration theory usually is applicable with sufficient accuracy.* However, the static effects of nonlinearity should be considered. Even though a nonlinear isolator may have approximately constant stiffness for small incremental deflections, the nonlinearity becomes important when large deflections of the isolator occur due to the effects of equipment weight and sustained acceleration. A vibration isolator often exhibits a stiffness that increases with applied force or deflection. Such a nonlinear stiffness is characteristic, for example, of rubber in compression or a conical spring.

In Eq. (30.11) for natural frequency, the stiffness k for a linear stiffness element is a constant. However, for a nonlinear isolator, the stiffness k is the slope of the force-deflection curve and Eq. (30.11) may be written

$$\omega_n = 2\pi f_n = \sqrt{\frac{g(dF/d\delta)}{W}} \tag{30.56}$$

where W is the total weight supported by the isolator, g is the acceleration of gravity, and $dF/d\delta$ is the slope of the line tangent to the force-deflection curve at the static equilibrium position. Vibration is considered to be small variations in the posi-

* If the vibration amplitude is large, nonlinear vibration theory as discussed in Chap. 4 is applicable.

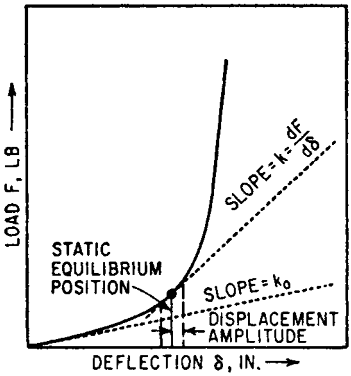

FIGURE 30.32 Typical force-deflection characteristic of a tangent hardening isolator.

tion of the supported equipment above and below the static equilibrium position, as indicated in Fig. 30.32. Thus, the natural frequency is determined solely by the stiffness characteristics in the region of the isolator deflection.

NATURAL FREQUENCY

In determining the natural frequency of a nonlinear isolator, it is important to note whether or not all the load results from the dead weight of a massive body. The force F on the isolator may be greater than the weight W because of a belt pull or sustained acceleration of a missile. Then the load on the isolator is

$$F = n_g W \tag{30.57}$$

where n_g is some multiple of the acceleration of gravity. For example, n_g may indicate the absolute value of the sustained acceleration of a missile measured in "number of g's."

Characteristic of Tangent Isolator. It is convenient to define the force-deflection characteristics of a nonlinear isolator having increasing stiffness (hardening characteristic) by a tangent function:[8]

$$F = \frac{2k_0 h_c}{\pi} \tan\left(\frac{\pi\delta}{2h_c}\right) \tag{30.58}$$

where F is the total force applied to the isolator, k_0 is the stiffness of the isolator at zero deflection, δ is the deflection of the isolator, and h_c is the characteristic height of the isolator. The force-deflection characteristic defined by Eq. (30.58) is shown graphically in Fig. 30.33*A*. The characteristic height h_c represents a height or thickness characteristic of the isolator which may be adjusted empirically to obtain optimum agreement, over the deflection range of interest, between Eq. (30.58) and the actual force-deflection curve for the isolator.

The stiffness of the tangent isolator is obtained by differentiation of Eq. (30.58) with respect to δ:

$$k = \frac{dF}{d\delta} = k_0 \sec^2\left(\frac{\pi\delta}{2h_c}\right) = k_0\left[1 + \left(\frac{F\pi}{2k_0 h_c}\right)^2\right] \tag{30.59}$$

The stiffness-deflection relation defined by Eq. (30.59) is shown graphically in Fig. 30.33*B*.

Replacing the load F by $n_g W$ in Eq. (30.59) and substituting the resulting stiffness relation into Eq. (30.56):

$$f_n \sqrt{h_c} = 3.13 \sqrt{2.46 n_g^2 \left(\frac{W}{k_0 h_c}\right) + \left(\frac{k_0 h_c}{W}\right)} \tag{30.60}$$

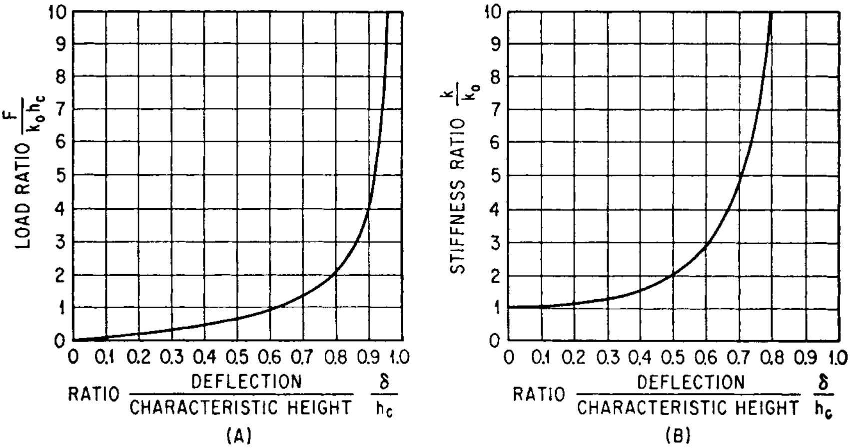

FIGURE 30.33 Elastic properties of a tangent isolator in terms of its characteristic height h_c and stiffness k_0 at zero deflection: (*A*) dimensionless force-deflection curve; (*B*) dimensionless stiffness-deflection curve.

The relation defined by Eq. (30.60) is shown graphically in Fig. 30.34. The ordinate is the natural frequency f_n (Hz) times the square root of the characteristic height of the isolator (in.). The theoretical and experimental force-deflection curves for the isolator are matched to establish the numerical value of the characteristic height. For a given value of the acceleration parameter n_g, the natural frequency of the isolation system is determined by h_c and W/k_0h_c.

The deflection of the isolator under a sustained acceleration loading is obtained by substituting Eq. (30.57) into the general force-deflection expression, Eq. (30.58), and solving for the dimensionless ratio δ/h_c:

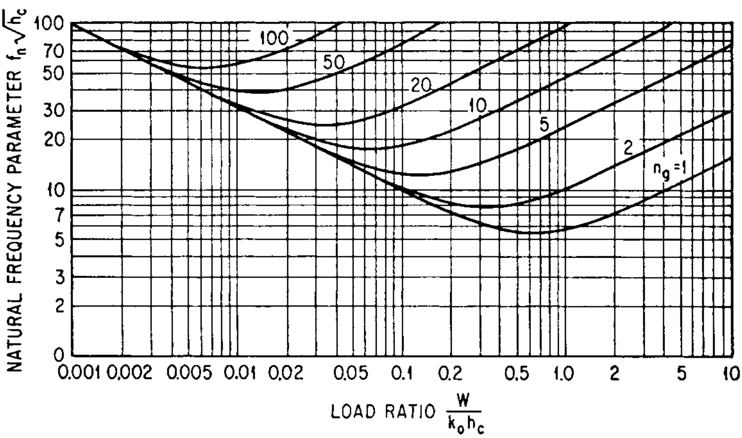

FIGURE 30.34 Natural frequency f_n of a tangent isolator system when a portion of the total load applied to the isolator is nonmassive. The weight carried by the isolator is W and the sustained acceleration parameter is n_g, a multiple of the gravitational acceleration. The characteristic height is h_c and the stiffness at zero deflection is k_0.

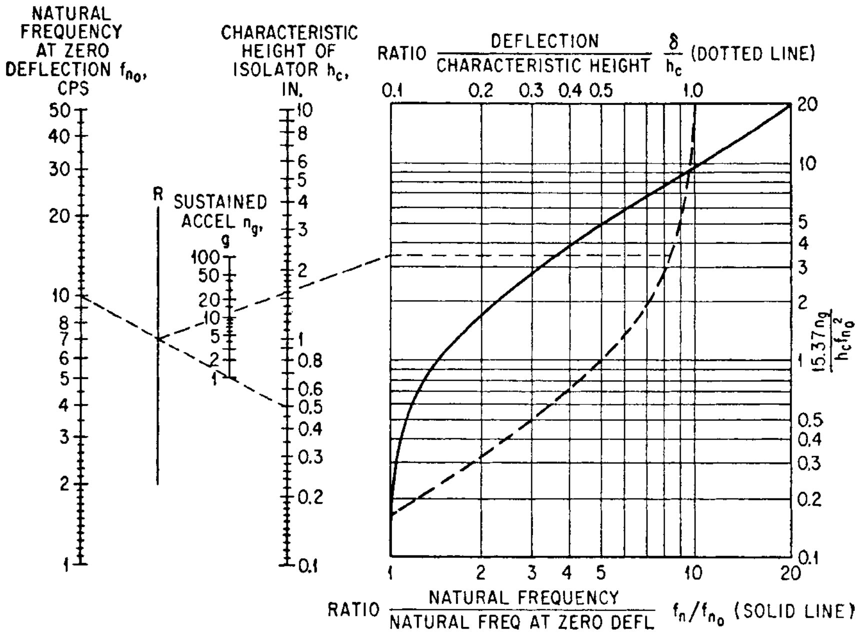

FIGURE 30.35 Nomograph and curve for determining the natural frequency and deflection of an isolation system incorporating a tangent isolator when a portion of the total load applied to the isolator is nonmassive.

$$\frac{\delta}{h_c} = \frac{2}{\pi} \tan^{-1}\left(\frac{\pi n_g}{2} \cdot \frac{W}{k_0 h_c}\right) = \frac{2}{\pi} \tan^{-1}\left[15.37\left(\frac{n_g}{h_c f_{n_0}^{\,2}}\right)\right] \tag{30.61}$$

A reference natural frequency f_{n_0} is the natural frequency that occurs when the isolator is not deflected by the dead-weight load; i.e., $n_g = 0$. The nomograph of Fig. 30.35 gives the deflection ratio δ/h_c and the frequency ratio f_n/f_{n_0}.[9] The value of the parameter $15.37(n_g/h_c f_{n_0}^{\,2})$ is transferred by a horizontal projection to the coordinate system for the curves. Values for the natural frequency ratio f_n/f_{n_0} are read from the lower abscissa scale and values for the deflection ratio δ/h_c are read from the upper abscissa scale.

Example 30.3. A rubber isolator having a characteristic height $h_c = 0.5$ in. (determined experimentally for the particular isolator design) has a natural frequency $f_n = 10$ Hz for small deflections and a fraction of critical damping $\zeta = 0.2$. The equipment supported by the isolator is subjected to a sustained acceleration of $11g$. It is desired to determine the absolute transmissibility of the isolation system when the forcing frequency is 100 Hz, and to determine the deflection of the isolator under the sustained acceleration.

Referring to the nomograph of Fig. 30.35, a straight line is drawn from a value of 10 on the f_{n_0} scale to 0.5 on the h_c scale. A second straight line is drawn from the intersection of the first line with the R scale through the value $n_g = 11$. The second line intersects the left side of the coordinate system and is extended horizontally so that it intersects the solid and dotted curves. The intersection points indicate that the natural frequency ratio $f_n/f_{n_0} = 3.5$ and the deflection ratio $\delta/h_c = 0.81$. The deflection of the isolator at equilibrium as a result of the sustained acceleration is $0.81h_c = 0.405$ in. The undamped natural frequency for the sustained acceleration of $11g$ is $f_n = 3.5 \times 10 = 35$ Hz. The natural frequency also can be obtained from Fig. 30.34 by noting that $W/k_0h_c = (g/h_c)/(2\pi f_{n_0})^2 = 0.196$ [see Eq. (30.60) when $n_g = 0$]. Then for $n_g = 11$, $f_n = 24.5/\sqrt{0.5} = 35$ Hz.

From Fig. 30.2 the transmissibility for $\zeta = 0.2$, $f/f_n = 100/35 = 2.88$ is 0.22. In the absence of the sustained acceleration, the corresponding transmissibility would be 0.042 as obtained from Fig. 30.2 at $f/f_n = 100/10 = 10$. Thus, the transmissibility at 100 Hz under a sustained acceleration of $11g$ is 5 times as great as that which would exist for a dead-weight loading of the isolator.

Minimum Natural Frequency. The weight W_0 for which a given tangent isolator has a minimum natural frequency is

$$W_0 = \frac{2k_0h_c}{\pi n_g} = \frac{k_0 g}{2\pi^2 (f_n)_{\min}^2} \qquad [f_n = \text{minimum}] \tag{30.62}$$

where the minimum natural frequency $(f_n)_{\min}$ is defined by

$$(f_n)_{\min} = \frac{1}{2}\sqrt{\frac{n_g g}{\pi h}} \tag{30.63}$$

The minimum natural frequency is shown graphically in Fig. 30.36 as a function of the characteristic height h_c and the sustained acceleration parameter n_g. The weight W_0 required to produce the minimum natural frequency $(f_n)_{\min}$ is shown graphically in Fig. 30.37 as a function of the initial stiffness k_0 and the minimum natural frequency $(f_n)_{\min}$. When the isolator is loaded to produce the minimum natural frequency, the isolator deflection is one-half the characteristic height ($\delta = h_c/2$) and the stiffness under load is twice the initial stiffness ($k = 2k_0$).

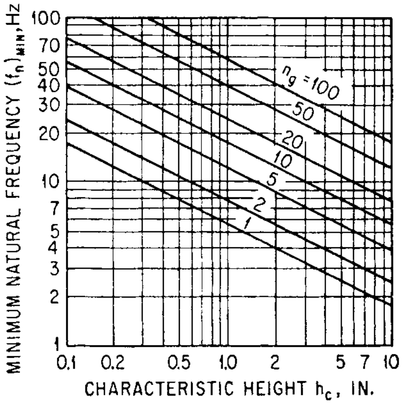

FIGURE 30.36 Minimum natural frequency $f_{n(\min)}$ of a tangent isolator system as a function of (1) the characteristic height h_c of the isolator and (2) the sustained acceleration n_g expressed as a multiple of the gravitational acceleration.

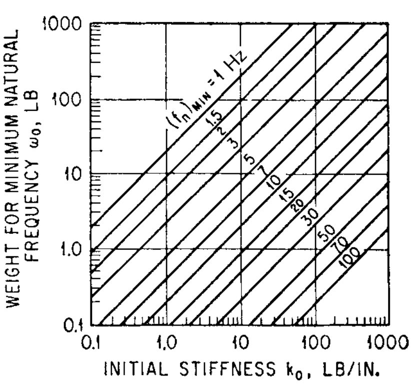

FIGURE 30.37 Weight loading W_0 required to cause a tangent isolator to have a minimum natural frequency $f_{n(\min)}$, as a function of the stiffness k_0 at zero deflection.

ISOLATION OF RANDOM VIBRATION

In random vibration, all frequencies exist concurrently, and the amplitude and phase relations are distributed in a random manner. A trace of random vibration is illustrated in Fig. 11.1*A*. The equipment-isolator assembly responds to the random vibration with the substantially single-frequency pattern shown in Fig. 11.1*B*. This response is similar to a sinusoidal motion with a continuously and irregularly varying envelope; it is described as narrow-band random vibration or a random sine wave.

The characteristics of random vibration are defined by a frequency spectrum of power spectral density (see Chaps. 11 and 22). This is a generic term used to designate the mean-square value of some magnitude parameter passed by a filter, divided by the bandwidth of the filter, and plotted as a spectrum of frequency. The magnitude is commonly measured as acceleration in units of g; then the particular expression to use in place of power spectral density is mean-square acceleration density, commonly expressed in units of g^2/Hz. When the spectrum of mean-square acceleration density is substantially flat in the frequency region extending on either side of the natural frequency of the isolator, the response of the isolator may be determined in terms of (1) the mean-square acceleration density of the isolated equipment and (2) the deflection of the isolator at successive cycles of vibration.

The mean-square acceleration densities of the foundation and the isolated equipment are related by the absolute transmissibility that applies to sinusoidal vibration:

$$W_r(f) = W_e(f)T_A{}^2 \tag{30.64}$$

where $W_r(f)$ and $W_e(f)$ are the mean-square acceleration densities of the equipment and the foundation, respectively, in units of g^2/Hz and T_A is the absolute transmissibility for the vibration-isolation system.

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