CHAPTER 28, PART I
MATRIX METHODS OF ANALYSIS

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INTRODUCTION

The mathematical language which is most convenient for analyzing multiple degree-of-freedom vibratory systems is that of *matrices*. Matrix notation simplifies the preliminary analytical study, and in situations where particular numerical answers are required, matrices provide a standardized format for organizing the data and the computations. Computations with matrices can be carried out by hand or by digital computers. The availability of programs such as MATLAB makes the solution of many complex problems in vibration analysis a matter of routine.

This chapter describes how matrices are used in vibration analysis. It begins with definitions and rules for operating with matrices. The formulation of vibration problems in matrix notation then is treated. This is followed by general matrix solutions of several important types of vibration problems, including free and forced vibrations of both undamped and damped linear multiple degree-of-freedom systems. Part II of this chapter considers finite element models.

MATRICES

Matrices are mathematical entities which facilitate the handling of simultaneous equations. They are applied to the differential equations of a vibratory system as follows:

A single degree-of-freedom system of the type in Fig. 28.1 has the differential equation

$$m\ddot{x} + c\dot{x} + kx = F$$

where m is the mass, c is the damping coefficient, k is the stiffness, F is the applied force, x is the displacement coordinate, and dots denote time derivatives. In Fig. 28.2 a similar three degree-of-freedom system is shown. The equations of motion may be obtained by applying Newton's second law to each mass in turn:

$$\begin{aligned} m\ddot{x}_1 \quad &+ c\dot{x}_1 \quad &&+ 5kx_1 - 2kx_2 \quad &&= F_1 \\ 2m\ddot{x}_2 \quad &&&+ 2c\dot{x}_2 - 2c\dot{x}_3 - 2kx_1 + 3kx_2 - kx_3 = F_2 \\ 3m\ddot{x}_3 \quad &- 2c\dot{x}_2 + 2c\dot{x}_3 \quad &&- kx_2 + kx_3 = F_3 \end{aligned} \tag{28.1}$$

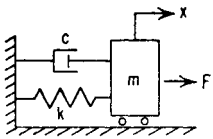

FIGURE 28.1 Single degree-of-freedom system.

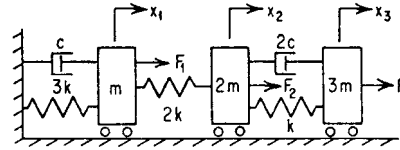

FIGURE 28.2 Three degree-of-freedom system.

The accelerations, velocities, displacements, and forces may be organized into columns, denoted by single boldface symbols:

$$\ddot{\mathbf{x}} = \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} \qquad \dot{\mathbf{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \mathbf{f} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} \tag{28.2}$$

The inertia, damping, and stiffness coefficients may be organized into square arrays:

$$\mathbf{M} = \begin{bmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & 3m \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} c & 0 & 0 \\ 0 & 2c & -2c \\ 0 & -2c & 2c \end{bmatrix} \qquad \mathbf{K} = \begin{bmatrix} 5k & -2k & 0 \\ -2k & 3k & -k \\ 0 & -k & k \end{bmatrix} \tag{28.3}$$

By using these symbols, it is shown below that it is possible to represent the three equations of Eq. (28.1) by the following single equation:

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f} \tag{28.4}$$

Note that this has the same form as the differential equation for the single degree-of-freedom system of Fig. 28.1. The notation of Eq. (28.4) has the advantage that in systems of many degrees-of-freedom it clearly states the physical principle that at every coordinate the external force is the sum of the inertia, damping, and stiffness forces. Equation (28.4) is an abbreviation for Eq. (28.1). It is necessary to develop the rules of operation with symbols such as those in Eqs. (28.2) and (28.3) to ensure that no ambiguity is involved. The algebra of *matrices* is devised to facilitate manipulations of simultaneous equations such as Eq. (28.1). Matrix algebra does not in any way simplify individual operations such as multiplication or addition of numbers, but it is an organizational tool which permits one to keep track of a complicated sequence of operations in an optimum manner. Matrices are essential elements of linear algebra,[1] and are widely employed in structural analysis[2] and vibration analysis.[3]

DEFINITIONS

A *matrix* is an array of elements arranged systematically in rows and columns. For example, a rectangular matrix $\mathbf{A}$, of elements a_{jk}, which has m rows and n columns is

$$\mathbf{A} = [a_{jk}] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

The elements a_{jk} are usually numbers or functions, but, in principle, they may be any well-defined quantities. The first subscript j on the element refers to the row number while the second subscript k refers to the column number. The array is denoted by the single symbol **A**, which can be used as such during operational manipulations in which it is not necessary to specify continually all the elements a_{jk}. When a numerical calculation is finally required, it is necessary to refer back to the explicit specifications of the elements a_{jk}.

A rectangular matrix with m rows and n columns is said to be of order (m,n). A matrix of order (n,n) is a *square matrix* and is said to be simply a square matrix of order n. A matrix of order $(n,1)$ is a *column matrix* and is said to be simply a column matrix of order n. A column matrix is sometimes referred to as a *column vector.* Similarly, a matrix of order $(1,n)$ is a *row matrix* or a *row vector.* Boldface *capital* letters are used here to represent square matrices and *lower-case* boldface letters to represent column matrices or vectors. For example, the matrices in Eq. (28.2) are column matrices of order three and the matrices in Eq. (28.3) are square matrices of order three.

Some special types of matrices are:

1. A *diagonal matrix* is a square matrix **A** whose elements a_{jk} are zero when $j \neq k$. The only nonzero elements are those on the *main diagonal,* where $j = k$. In order to emphasize that a matrix is diagonal, it is often written with small ticks in the direction of the main diagonal:

$$\mathbf{A} = \begin{bmatrix} a_{jj} \end{bmatrix}$$

2. A *unit matrix* or *identity matrix* is a diagonal matrix whose main diagonal elements are each equal to unity. The symbol **I** is used to denote a unit matrix. Examples are

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3. A *null matrix* or *zero matrix* has all its elements equal to zero and is simply written as zero.

4. The *transpose* $\mathbf{A}^T$ of a matrix **A** is a matrix having the same elements but with rows and columns interchanged. Thus, if the original matrix is

$$\mathbf{A} = [a_{jk}]$$

the transpose matrix is

$$\mathbf{A}^T = [a_{jk}]^T = [a_{kj}]$$

For example:

$$\mathbf{A} = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \qquad \mathbf{A}^T = \begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}$$

The transpose of a square matrix may be visualized as the matrix obtained by rotating the given matrix about its main diagonal as an axis.

The transpose of a column matrix is a row matrix. For example,

$$\mathbf{x} = \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix} \qquad \mathbf{x}^T = [3 \quad 4 \quad -2]$$

Throughout this chapter a row matrix is referred to as the transpose of the corresponding column matrix.

5. A *symmetric matrix* is a square matrix whose off-diagonal elements are symmetric with respect to the main diagonal. A square matrix **A** is symmetric if, for all j and k,

$$a_{jk} = a_{kj}$$

A symmetric matrix is equal to its transpose. For example, all three of the matrices in Eq. (28.3) are symmetric. In addition, the matrix **M** is a diagonal matrix.

MATRIX OPERATIONS

Equality of Matrices. Two matrices of the same order are equal if their corresponding elements are equal. Thus two matrices **A** and **B** are equal if, for every j and k,

$$a_{jk} = b_{jk}$$

Matrix Addition and Subtraction. Addition or subtraction of matrices of the same order is performed by adding or subtracting corresponding elements. Thus, $\mathbf{A} + \mathbf{B} = \mathbf{C}$ if for every j and k,

$$a_{jk} + b_{jk} = c_{jk}$$

For example, if

$$\mathbf{A} = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} -1 & 2 \\ 5 & 6 \end{bmatrix}$$

then

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} 2 & 4 \\ 4 & 10 \end{bmatrix} \qquad \mathbf{A} - \mathbf{B} = \begin{bmatrix} 4 & 0 \\ -6 & -2 \end{bmatrix}$$

Multiplication of a Matrix by a Scalar. Multiplication of a matrix by a scalar c multiplies each element of the matrix by c. Thus

$$c\mathbf{A} = c[a_{jk}] = [ca_{jk}]$$

In particular, the negative of a matrix has the sign of every element changed.

Matrix Multiplication. If **A** is a matrix of order (m,n) and **B** is a matrix of order (n,p), then their *matrix product* $\mathbf{AB} = \mathbf{C}$ is defined to be a matrix **C** of order (m,p) where, for every j and k,

$$c_{jk} = \sum_{r=1}^{n} a_{jr} b_{rk} \tag{28.5}$$

The product of two matrices can be obtained only if they are *conformable,* i.e., if the number of columns in **A** is equal to the number of rows in **B**. The symbolic equation

$$(m,n) \times (n,p) = (m,p)$$

indicates the orders of the matrices involved in a matrix product. Matrix products are not commutative, i.e., in general,

$$\mathbf{AB} \neq \mathbf{BA}$$

The matrix products which appear in this chapter are of the following types:

Square matrix × square matrix = square matrix
Square matrix × column vector = column vector
Row vector × square matrix = row vector
Row vector × column vector = scalar
Column vector × row vector = square matrix

In all cases, the matrices must be conformable. Numerical examples are given below.

$$\mathbf{AB} = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} -(3\times 1) + (2\times 5) & (3\times 2) + (2\times 6) \\ (1\times 1) + (4\times 5) & -(1\times 2) + (4\times 6) \end{bmatrix} = \begin{bmatrix} 7 & 18 \\ 21 & 22 \end{bmatrix}$$

$$\mathbf{Ax} = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} (3\times 5) + (2\times 3) \\ -(1\times 5) + (4\times 3) \end{bmatrix} = \begin{bmatrix} 21 \\ 7 \end{bmatrix}$$

$$\mathbf{y}^T\mathbf{A} = [-2 \quad 1] \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} = [-(2\times 3) - (1\times 1) - (2\times 2) + (1\times 4)] = [-7 \quad 0]$$

$$\mathbf{y}^T\mathbf{x} = [-2 \quad 1] \begin{bmatrix} 5 \\ 3 \end{bmatrix} = (-10 + 3) = -7$$

$$\mathbf{x}\mathbf{y}^T = \begin{bmatrix} 5 \\ 3 \end{bmatrix} [-2 \quad 1] = \begin{bmatrix} -(5\times 2) & (5\times 1) \\ -(3\times 2) & (3\times 1) \end{bmatrix} = \begin{bmatrix} -10 & 5 \\ -6 & 3 \end{bmatrix}$$

The last product always results in a matrix with proportional rows and columns.

The operation of matrix multiplication is particularly suited for representing systems of simultaneous linear equations in a compact form in which the coefficients are gathered into square matrices and the unknowns are placed in column matrices. For example, it is the operation of matrix multiplication which gives unambiguous meaning to the matrix abbreviation in Eq. (28.4) for the three simultaneous differential equations of Eq. (28.1). The two sides of Eq. (28.4) are column matrices of order three whose corresponding elements must be equal. On the right, these elements are simply the external forces at the three masses. On the left, Eq. (28.4) states that the resulting column is the sum of three column matrices, each of which results from the matrix multiplication of a square matrix of coefficients defined in Eq. (28.3) into a column matrix defined in Eq. (28.2). The rules of matrix operation just given ensure that Eq. (28.4) is exactly equivalent to Eq. (28.1).

Premultiplication or postmultiplication of a square matrix by the identity matrix leaves the original matrix unchanged; i.e.,

$$\mathbf{IA} = \mathbf{AI} = \mathbf{A}$$

Two symmetrical matrices multiplied together are generally not symmetric. The product of a matrix and its transpose is symmetric.

Continued matrix products such as **ABC** are defined, provided the number of columns in each matrix is the same as the number of rows in the matrix immediately following it. From the definition of matrix products, it follows that the *associative law* holds for continued products:

$$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$$

A square matrix **A** multiplied by itself yields a square matrix which is called the *square of the matrix* **A** and is denoted by $\mathbf{A}^2$. If $\mathbf{A}^2$ is in turn multiplied by **A**, the resulting matrix is $\mathbf{A}^3 = \mathbf{A}(\mathbf{A}^2) = \mathbf{A}^2(\mathbf{A})$. Extension of this process gives meaning to $\mathbf{A}^m$ for any positive integer *power m.* Powers of symmetric matrices are themselves symmetric.

The rule for *transposition* of matrix products is

$$(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$$

Inverse or Reciprocal Matrix. If, for a given square matrix **A**, a square matrix $\mathbf{A}^{-1}$ can be found such that

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I} \tag{28.6}$$

then $\mathbf{A}^{-1}$ is called the *inverse* or *reciprocal* of **A**. Not every square matrix **A** possesses an inverse. If the determinant constructed from the elements of a square matrix is zero, the matrix is said to be *singular* and there is no inverse. Every nonsingular matrix possesses a unique inverse. The inverse of a symmetric matrix is symmetric. The rule for the *inverse of a matrix product* is

$$(\mathbf{AB})^{-1} = (\mathbf{B}^{-1})(\mathbf{A}^{-1})$$

The solution to the set of simultaneous equations

$$\mathbf{Ax} = \mathbf{c}$$

where **x** is the unknown vector and **c** is a known input vector can be indicated with the aid of the inverse of **A**. The formal solution for **x** proceeds as follows:

$$\mathbf{A}^{-1}\mathbf{Ax} = \mathbf{A}^{-1}\mathbf{c}$$

$$\mathbf{Ix} = \mathbf{x} = \mathbf{A}^{-1}\mathbf{c}$$

When the inverse $\mathbf{A}^{-1}$ is known, the solution vector **x** is obtained by a simple matrix multiplication of $\mathbf{A}^{-1}$ into the input vector **c**.

Calculation of inverses and the solutions of simultaneous linear equations are readily performed for surprisingly large values of n by programs such as MATLAB. When $n = 2$ and

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

hand-computation is possible using the following formulas:

$$\mathbf{A}^{-1} = \frac{1}{\Delta}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \qquad x_1 = \frac{\Delta_1}{\Delta} \qquad x_2 = \frac{\Delta_2}{\Delta}$$

where the determinants have the values

$$\Delta = a_{11}a_{22} - a_{12}a_{21} \qquad \Delta_1 = c_1a_{22} - c_2a_{12} \qquad \Delta_2 = c_2a_{11} - c_1a_{21}$$

QUADRATIC FORMS

A general quadratic form Q of order n may be written as

$$Q = \sum_{j=1}^{n} \sum_{k=1}^{n} a_{jk}x_jx_k$$

where the a_{jk} are constants and the x_j are the n variables. The form is quadratic since it is of the second degree in the variables. The laws of matrix multiplication permit Q to be written as

$$Q = [x_1 x_2 \ldots x_n] \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix}$$

which is

$$Q = \mathbf{x}^T\mathbf{A}\mathbf{x}$$

Any quadratic form can be expressed in terms of a symmetric matrix. If the given matrix **A** is not symmetric, it can be replaced by the symmetric matrix

$$\mathbf{B} = \tfrac{1}{2}(\mathbf{A} + \mathbf{A}^T)$$

without changing the value of the form.

As an example of a quadratic form, the *potential energy* V for the system of Fig. 28.2 is given by

$$\begin{aligned} 2V &= 3kx_1^2 \quad + 2k(x_2 - x_1)^2 + k(x_3 - x_2)^2 \\ &= 5kx_1x_1 - 2kx_1x_2 \\ &\quad - 2kx_2x_1 + 3kx_2x_2 - kx_2x_3 \\ &\quad - kx_3x_2 + kx_3x_3 \end{aligned}$$

Using the displacement vector **x** defined in Eq. (28.2) and the stiffness matrix **K** in Eq. (28.3), the potential energy may be written as

$$V = \tfrac{1}{2}\mathbf{x}^T\mathbf{K}\mathbf{x}$$

Similarly, the *kinetic energy* T is given by

$$2T = m\dot{x}_1^2 + 2m\dot{x}_2^2 + 3m\dot{x}_3^2$$

In terms of the inertia matrix **M** and the velocity vector $\dot{\mathbf{x}}$ defined in Eqs. (28.3) and (28.2), the kinetic energy may be written as

$$T = \tfrac{1}{2}\dot{\mathbf{x}}^T\mathbf{M}\dot{\mathbf{x}}$$

The *dissipation function* D for the system is given by

$$\begin{aligned} 2D &= c\dot{x}_1^2 + 2c(\dot{x}_3 - \dot{x}_2)^2 \\ &= c\dot{x}_1\dot{x}_1 \\ &\quad + 2c\dot{x}_2\dot{x}_2 - 2c\dot{x}_2\dot{x}_3 \\ &\quad - 2c\dot{x}_3\dot{x}_2 + 2c\dot{x}_3\dot{x}_3 \end{aligned}$$

In terms of the velocity vector $\dot{\mathbf{x}}$ and the damping matrix $\mathbf{C}$ defined in Eqs. (28.2) and (28.3), the dissipation function may be written as

$$D = \tfrac{1}{2}\dot{\mathbf{x}}^T\mathbf{C}\dot{\mathbf{x}}$$

The dissipation function gives half the rate at which energy is being dissipated in the system.

While quadratic forms assume positive and negative values in general, the three physical forms just defined are intrinsically *positive* for a vibrating system with linear springs, constant masses, and viscous damping; i.e., they can never be negative for a real motion of the system. Kinetic energy is zero only when the system is at rest. The same thing is not necessarily true for potential energy or the dissipation function.

Depending upon the arrangement of springs and dashpots in the system, there may exist motions which do not involve any potential energy or dissipation. For example, in vibratory systems where rigid body motions are possible (crankshaft torsional systems, free-free beams, etc.), no elastic energy is involved in the rigid body motions. Also, in Fig. 28.2, if x_1 is zero while x_2 and x_3 have the same motion, there is no energy dissipated and the dissipation function is zero. To distinguish between these two possibilities, a quadratic form is called *positive definite* if it is never negative and if the only time it vanishes is when all the variables are zero. Kinetic energy is always positive definite, while potential energy and the dissipation function are positive but not necessarily positive definite. It depends upon the particular configuration of a given system whether the potential energy and the dissipation function are positive definite or only positive. The terms positive and positive definite are applied also to the matrices from which the quadratic forms are derived. For example, of the three matrices defined in Eq. (28.3), the matrices $\mathbf{M}$ and $\mathbf{K}$ are positive definite, but $\mathbf{C}$ is only positive. It can be shown that a matrix which is positive but not positive definite is *singular.*

Differentiation of Quadratic Forms. In forming Lagrange's equations of motion for a vibrating system,* it is necessary to take derivatives of the potential energy V, the kinetic energy T, and the dissipation function D. When these quadratic forms are represented in matrix notation, it is convenient to have matrix formulas for differentiation. In this paragraph rules are given for differentiating the slightly more general *bilinear form*

$$F = \mathbf{x}^T\mathbf{A}\mathbf{y} = \mathbf{y}^T\mathbf{A}\mathbf{x}$$

where $\mathbf{x}^T$ is a row vector of n variables x_j, $\mathbf{A}$ is a square matrix of constant coefficients, and $\mathbf{y}$ is a column matrix of n variables y_j. In a quadratic form the x_j are identical with the y_j.

For generality it is assumed that the x_j and the y_j are functions of n other variables u_j. In the formulas below, the notation $\mathbf{X}_u$ is used to represent the following square matrix:

* See Chap. 2 for a detailed discussion of Lagrange's equations.

$$\mathbf{X}_u = \begin{bmatrix} \dfrac{\partial x_1}{\partial u_1} & \dfrac{\partial x_2}{\partial u_1} & \cdots & \dfrac{\partial x_n}{\partial u_1} \\ \dfrac{\partial x_1}{\partial u_2} & \dfrac{\partial x_2}{\partial u_2} & \cdots & \dfrac{\partial x_n}{\partial u_2} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial x_1}{\partial u_n} & \dfrac{\partial x_2}{\partial u_n} & \cdots & \dfrac{\partial x_n}{\partial u_n} \end{bmatrix}$$

Now letting $\partial/\partial\mathbf{u}$ stand for the column vector whose elements are the partial differential operators with respect to the u_j, the general differentiation formula is

$$\frac{\partial F}{\partial \mathbf{u}} = \begin{bmatrix} \dfrac{\partial F}{\partial u_1} \\ \dfrac{\partial F}{\partial u_2} \\ \vdots \\ \dfrac{\partial F}{\partial u_n} \end{bmatrix} = \mathbf{X}_u\mathbf{A}\mathbf{y} + \mathbf{Y}_u\mathbf{A}^T\mathbf{x}$$

For a quadratic form $Q = \mathbf{x}^T\mathbf{A}\mathbf{x}$ the above formula reduces to

$$\frac{\partial Q}{\partial \mathbf{u}} = \mathbf{X}_u(\mathbf{A} + \mathbf{A}^T)\mathbf{x}$$

Thus whether $\mathbf{A}$ is symmetric or not, this kind of differentiation produces a *symmetrical* matrix of coefficients $(\mathbf{A} + \mathbf{A}^T)$. It is this fact which ensures that vibration equations in the form obtained from Lagrange's equations always have symmetrical matrices of coefficients. If $\mathbf{A}$ is symmetrical to begin with, the previous formula becomes

$$\frac{\partial Q}{\partial \mathbf{u}} = 2\mathbf{X}_u\mathbf{A}\mathbf{x}$$

Finally, in the important special case where the x_j are identical with the u_j, the matrix $\mathbf{X}_x$ reduces to the identity matrix, yielding

$$\frac{\partial Q}{\partial \mathbf{x}} = 2\mathbf{A}\mathbf{x} \tag{28.7}$$

which is employed in the following section in developing Lagrange's equations.

FORMULATION OF VIBRATION PROBLEMS IN MATRIX FORM

Consider a holonomic linear mechanical system with n degrees-of-freedom which vibrates about a stable equilibrium configuration. Let the motion of the system be described by *n generalized displacements* $x_j(t)$ which vanish in the equilibrium position. The potential energy V can then be expressed in terms of these displacements as a quadratic form. The kinetic energy T and the dissipation function D can be expressed as quadratic forms in the generalized velocities $\dot{x}_j(t)$.

The equations of motion are obtained by applying Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_j}\right) + \frac{\partial D}{\partial \dot{x}_j} + \frac{\partial V}{\partial x_j} = f_j(t) \qquad [j = 1, 2, \ldots, n]$$

The *generalized external force* $f_j(t)$ for each coordinate may be an active force in the usual sense or a force generated by prescribed motion of the coordinates.

If each term in the foregoing equation is taken as the jth element of a column matrix, all n equations can be considered simultaneously and written in matrix form as follows:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\mathbf{x}}}\right) + \frac{\partial D}{\partial \dot{\mathbf{x}}} + \frac{\partial V}{\partial \mathbf{x}} = \mathbf{f}$$

The quadratic forms can be expressed in matrix notation as

$$T = \tfrac{1}{2}(\dot{\mathbf{x}}^T \mathbf{M} \dot{\mathbf{x}})$$

$$D = \tfrac{1}{2}(\dot{\mathbf{x}}^T \mathbf{C} \dot{\mathbf{x}})$$

$$V = \tfrac{1}{2}(\mathbf{x}^T \mathbf{K} \mathbf{x})$$

where the *inertia matrix* $\mathbf{M}$, the *damping matrix* $\mathbf{C}$, and the *stiffness matrix* $\mathbf{K}$ may be taken as symmetric square matrices of order n. Then the differentiation rule (28.7) yields

$$\frac{d}{dt}(\mathbf{M}\dot{\mathbf{x}}) + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f}$$

or simply

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f} \tag{28.8}$$

as the equations of motion in matrix form for a general linear vibratory system with n degrees-of-freedom. This is a generalization of Eq. (28.4) for the three degree-of-freedom system of Fig. 28.2. Equation (28.8) applies to all linear constant-parameter vibratory systems. The specifications of any particular system are contained in the *coefficient matrices* $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$. The type of excitation is described by the column matrix $\mathbf{f}$. The individual terms in the coefficient matrices have the following significance:

m_{jk} is the momentum component at j due to a unit velocity at k.

c_{jk} is the damping force at j due to a unit velocity at k.

k_{jk} is the elastic force at j due to a unit displacement at k.

The general solution to Eq. (28.8) contains $2n$ constants of integration which are usually fixed by the n displacements $x_j(t_0)$ and the n velocities $\dot{x}_j(t_0)$ at some initial time t_0. When the excitation matrix $\mathbf{f}$ is zero, Eq. (28.8) is said to describe the *free vibration* of the system. When $\mathbf{f}$ is nonzero, Eq. (28.8) describes a *forced vibration*. When the time behavior of $\mathbf{f}$ is periodic and steady, it is sometimes convenient to divide the solution into a *steady-state response* plus a *transient response* which decays with time. The steady-state response is independent of the initial conditions.

COUPLING OF THE EQUATIONS

The off-diagonal terms in the coefficient matrices are known as *coupling terms.* In general, the equations have inertia, damping, and stiffness coupling; however, it is often possible to obtain equations that have no coupling terms in one or more of the three matrices. If the coupling terms vanish in all three matrices (i.e., if all three square matrices are diagonal matrices), the system of Eq. (28.8) becomes a set of independent uncoupled differential equations for the n generalized displacements $x_j(t)$. Each displacement motion is a single degree-of-freedom vibration independent of the motion of the other displacements.

The coupling in a system depends on the choice of coordinates used to describe the motion. For example, Figs. 28.3 and 28.4 show the same physical system with two different choices for the displacement coordinates.

The coefficient matrices corresponding to the coordinates shown in Fig. 28.3 are

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \qquad \mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

Here the inertia matrix is uncoupled because the coordinates chosen are the absolute displacements of the masses. The elastic force in the spring k_2 is generated by the relative displacement of the two coordinates, which accounts for the coupling terms in the stiffness matrix.

The coefficient matrices corresponding to the alternative coordinates shown in Fig. 28.4 are

$$\mathbf{M} = \begin{bmatrix} m_1 + m_2 & m_2 \\ m_2 & m_2 \end{bmatrix} \qquad \mathbf{K} = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$$

Here the coordinates chosen relate directly to the extensions of the springs so that the stiffness matrix is uncoupled. The absolute displacement of m_2 is, however, the sum of the coordinates, which accounts for the coupling terms in the inertia matrix.

A fundamental procedure for solving vibration problems in undamped systems may be viewed as the search for a set of coordinates which simultaneously uncouples both the stiffness and inertia matrices. This is always possible. In systems with damping (i.e., with all three coefficient matrices) there exist coordinates which uncouple two of these, but it is not possible to uncouple all three matrices simultaneously, except in the special case, called *proportional damping,* where **C** is a linear combination of **K** and **M**.

The system of Fig. 28.2 provides an example of a three degree-of-freedom system with damping. The coefficient matrices are given in Eq. (28.3). The inertia matrix is uncoupled, but the damping and stiffness matrices are coupled.

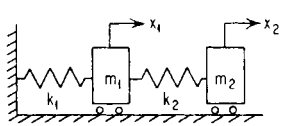

FIGURE 28.3 Coordinates (x_1,x_2) with uncoupled inertia matrix.

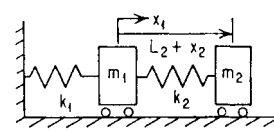

FIGURE 28.4 Coordinates (x_1,x_2) with uncoupled stiffness matrix. The equilibrium length of the spring k_2 is L_2.

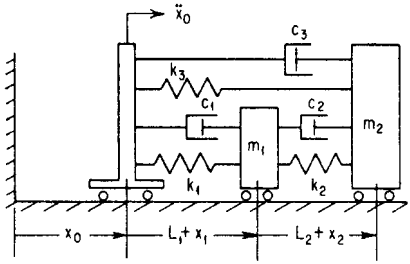

FIGURE 28.5 Two degree-of-freedom vibratory system. The equilibrium length of the spring k_1 is L_1 and the equilibrium length of the spring k_2 is L_2.

Another example of a system with damping is furnished by the two degree-of-freedom system shown in Fig. 28.5. The excitation here is furnished by acceleration $\ddot{x}_0(t)$ of the base. This system is used as the basis for the numerical example at the end of Part I of the chapter. With the coordinates chosen as indicated in the figure, all three coefficient matrices have coupling terms. The equations of motion can be placed in the standard form of Eq. (28.8), where the coefficient matrices and the excitation column are as follows:

$$\mathbf{M} = \begin{bmatrix} m_1 + m_2 & m_2 \\ m_2 & m_2 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} c_1 + c_3 & c_3 \\ c_3 & c_2 + c_3 \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} k_1 + k_3 & k_3 \\ k_3 & k_2 + k_3 \end{bmatrix} \qquad \mathbf{f} = -\ddot{x}_0 \begin{bmatrix} m_1 + m_2 \\ m_2 \end{bmatrix} \tag{28.9}$$

THE MATRIX EIGENVALUE PROBLEM

In the following sections the solutions to both free and forced vibration problems are given in terms of solutions to a specialized algebraic problem known as the matrix eigenvalue problem. In the present section a general theoretical discussion of the matrix eigenvalue problem is given.

The free vibration equation for an undamped system,

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = 0 \tag{28.10}$$

follows from Eq. (28.8) when the excitation **f** and the damping **C** vanish. If a solution for **x** is assumed in the form

$$\mathbf{x} = \Re\{\mathbf{v}e^{j\omega t}\}$$

where **v** is a column vector of unknown amplitudes, ω is an unknown frequency, j is the square root of -1, and $\Re\{\ \}$ signifies "the real part of," it is found on substituting in Eq. (28.10) that it is necessary for **v** and ω to satisfy the following algebraic equation:

$$\mathbf{K}\mathbf{v} = \omega^2\mathbf{M}\mathbf{v} \tag{28.11}$$

This algebraic problem is called the *matrix eigenvalue problem.* Where necessary it is called the *real* eigenvalue problem to distinguish it from the *complex* eigenvalue problem described in the section on *Vibration of Systems with Damping.*

To indicate the formal solution to Eq. (28.11), it is rewritten as

$$(\mathbf{K} - \omega^2\mathbf{M})\mathbf{v} = 0 \tag{28.12}$$

which can be interpreted as a set of n homogeneous algebraic equations for the n elements v_j. This set always has the trivial solution

$$\mathbf{v} = 0$$

It also has nontrivial solutions if the determinant of the matrix multiplying the vector v is zero, i.e., if

$$\det (\mathbf{K} - \omega^2\mathbf{M}) = 0 \tag{28.13}$$

When the determinant is expanded, a polynomial of order n in ω^2 is obtained. Equation (28.13) is known as the *characteristic equation* or *frequency equation.* The restrictions that **M** and **K** be symmetric and that **M** be positive definite are sufficient to ensure that there are n real roots for ω^2. If **K** is singular, at least one root is zero. If **K** is positive definite, all roots are positive. The n roots determine the *n natural frequencies* ω_r $(r = 1, \ldots, n)$ of free vibration. These roots of the characteristic equation are also known as *normal values, characteristic values, proper values, latent roots,* or *eigenvalues.* When a natural frequency ω_r is known, it is possible to return to Eq. (28.12) and solve for the corresponding vector $\mathbf{v}_r$ to within a multiplicative constant. The eigenvalue problem does not fix the absolute amplitude of the vectors **v**, only the relative amplitudes of the n coordinates. There are n independent vectors $\mathbf{v}_r$ corresponding to the n natural frequencies which are known as *natural modes.* These vectors are also known as *normal modes, characteristic vectors, proper vectors, latent vectors,* or *eigenvectors.*

MODAL AND SPECTRAL MATRICES

The complete solution to the eigenvalue problem of Eq. (28.11) consists of n eigenvalues and n corresponding eigenvectors. These can be assembled compactly into matrices. Let the eigenvector $\mathbf{v}_r$ corresponding to the eigenvalue ω_r^2 have elements v_{jr} (the first subscript indicates which row, the second subscript indicates which eigenvector). The n eigenvectors then can be displayed in a single square matrix **V**, each column of which is an eigenvector:

$$\mathbf{V} = [v_{jk}] = \begin{bmatrix} v_{11} & v_{12} & \ldots & v_{1n} \\ v_{21} & v_{22} & \ldots & v_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ v_{n1} & v_{n2} & \ldots & v_{nn} \end{bmatrix}$$

The matrix **V** is called the *modal matrix* for the eigenvalue problem, Eq. (28.11).

The n eigenvalues ω_r^2 can be assembled into a diagonal matrix $\mathbf{\Omega}^2$ which is known as the *spectral matrix* of the eigenvalue problem, Eq. (28.11)

$$\mathbf{\Omega}^2 = [\omega_r^2] = \begin{bmatrix} \omega_1^2 & 0 & \ldots & 0 \\ 0 & \omega_2^2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \omega_n^2 \end{bmatrix}$$

Each eigenvector and corresponding eigenvalue satisfy a relation of the following form:

$$\mathbf{K}\mathbf{v}_r = \mathbf{M}\mathbf{v}_r\omega_r^2$$

By using the modal and spectral matrices it is possible to assemble all of these relations into a single matrix equation

$$\mathbf{KV} = \mathbf{MV}\boldsymbol{\Omega}^2 \tag{28.14}$$

Equation (28.14) provides a compact display of the complete solution to the eigenvalue problem Eq. (28.11).

PROPERTIES OF THE SOLUTION

The eigenvectors corresponding to different eigenvalues can be shown to satisfy the following *orthogonality relations.* When $\omega_r^2 \neq \omega_s^2$,

$$\mathbf{v}_r^T\mathbf{K}\mathbf{v}_s = 0 \qquad \mathbf{v}_r^T\mathbf{M}\mathbf{v}_s = 0 \tag{28.15}$$

In case the characteristic equation has a p-fold multiple root for ω^2, then there is a p-fold infinity of corresponding eigenvectors. In this case, however, it is always possible to choose p of these vectors which mutually satisfy Eq. (28.15) and to express any other eigenvector corresponding to the multiple root as a linear combination of the p vectors selected. If these p vectors are included with the eigenvectors corresponding to the other eigenvalues, a set of n vectors is obtained which satisfies the orthogonality relations of Eq. (28.15) for any $r \neq s$.

The orthogonality of the eigenvectors with respect to **K** and **M** implies that the following square matrices are *diagonal.*

$$\begin{aligned} \mathbf{V}^T\mathbf{K}\mathbf{V} &= \lceil \mathbf{v}_r^T\mathbf{K}\mathbf{v}_r \rfloor \\ \mathbf{V}^T\mathbf{M}\mathbf{V} &= \lceil \mathbf{v}_r^T\mathbf{M}\mathbf{v}_r \rfloor \end{aligned} \tag{28.16}$$

The elements $\mathbf{v}_r^T\mathbf{K}\mathbf{v}_r$ along the main diagonal of $\mathbf{V}^T\mathbf{K}\mathbf{V}$ are called the *modal stiffnesses* k_r, and the elements $\mathbf{v}_r^T\mathbf{M}\mathbf{v}_r$ along the main diagonal of $\mathbf{V}^T\mathbf{M}\mathbf{V}$ are called the *modal masses* m_r. Since **M** is positive definite, all modal masses are guaranteed to be positive. When **K** is singular, at least one of the modal stiffnesses will be zero. Each eigenvalue ω_r^2 is the quotient of the corresponding modal stiffness divided by the corresponding modal mass; i.e.,

$$\omega_r^2 = \frac{k_r}{m_r}$$

In numerical work it is sometimes convenient to normalize each eigenvector so that its largest element is *unity.* In other applications it is common to normalize the eigenvectors so that the modal masses m_r all have the *same* value m, where m is some convenient value such as the total mass of the system. In this case,

$$\mathbf{V}^T\mathbf{M}\mathbf{V} = m\mathbf{I} \tag{28.17}$$

and it is possible to express the inverse of the modal matrix **V** simply as

$$\mathbf{V}^{-1} = \frac{1}{m}\,\mathbf{V}^T\mathbf{M}$$

An interpretation of the modal matrix **V** can be given by showing that it defines a set of generalized coordinates for which both the inertia and stiffness matrices are uncoupled. Let $\mathbf{y}(t)$ be a column of displacements related to the original displacements $\mathbf{x}(t)$ by the following simultaneous equations:

$$\mathbf{y} = \mathbf{V}^{-1}\mathbf{x} \qquad \text{or} \qquad \mathbf{x} = \mathbf{V}\mathbf{y}$$

The potential and kinetic energies then take the forms

$$V = \tfrac{1}{2}\mathbf{x}^T\mathbf{K}\mathbf{x} = \tfrac{1}{2}\mathbf{y}^T(\mathbf{V}^T\mathbf{K}\mathbf{V})\mathbf{y}$$

$$T = \tfrac{1}{2}\dot{\mathbf{x}}^T\mathbf{M}\dot{\mathbf{x}} = \tfrac{1}{2}\dot{\mathbf{y}}^T(\mathbf{V}^T\mathbf{M}\mathbf{V})\dot{\mathbf{y}}$$

where, according to Eq. (28.16), the square matrices in parentheses on the right are *diagonal;* i.e., in the y_j coordinate system there is neither stiffness nor inertia coupling.

An alternative method for obtaining the same interpretation is to start from the eigenvalue problem of Eq. (28.11). Consider the structure of the related eigenvalue problem for **w** where again **w** is obtained from **v** by the transformation involving the modal matrix **V**.

$$\mathbf{w} = \mathbf{V}^{-1}\mathbf{v} \qquad \text{or} \qquad \mathbf{v} = \mathbf{V}\mathbf{w}$$

Substituting in Eq. (28.11), premultiplying by $\mathbf{V}^T$, and using Eq. (28.14),

$$\mathbf{K}\mathbf{v} = \omega^2\mathbf{M}\mathbf{v}$$

$$\mathbf{K}\mathbf{V}\mathbf{w} = \omega^2\mathbf{M}\mathbf{V}\mathbf{w}$$

$$\mathbf{V}^T\mathbf{K}\mathbf{V}\mathbf{w} = \omega^2\mathbf{V}^T\mathbf{M}\mathbf{V}\mathbf{w}$$

$$(\mathbf{V}^T\mathbf{M}\mathbf{V})\boldsymbol{\Omega}^2\mathbf{w} = \omega^2(\mathbf{V}^T\mathbf{M}\mathbf{V})\mathbf{w}$$

Now, since $\mathbf{V}^T\mathbf{M}\mathbf{V}$ is a diagonal matrix of positive elements, it is permissible to cancel it from both sides, which leaves a simple diagonalized eigenvalue problem for **w**:

$$\boldsymbol{\Omega}^2\mathbf{w} = \omega^2\mathbf{w}$$

A modal matrix for **w** is the identity matrix **I**, and the eigenvalues for **w** are the same as those for **v**.

EIGENVECTOR EXPANSIONS

Any set of n independent vectors can be used as a basis for representing any other vector of order n. In the following sections, the eigenvectors of the eigenvalue problem of Eq. (28.11) are used as such a basis. An eigenvector expansion of an arbitrary vector **y** has the form

$$\mathbf{y} = \sum_{r=1}^{n} \mathbf{v}_r a_r \tag{28.18}$$

where the a_r are scalar *mode multipliers.* When **y** and the $\mathbf{v}_r$ are known, it is possible to evaluate the a_r by premultiplying both sides by $\mathbf{v}_s{}^T\mathbf{M}$. Because of the orthogonality relations of Eq. (28.15), all the terms on the right vanish except the one for which $r = s$. Inserting the value of the mode multiplier so obtained, the expansion can be rewritten as

$$\mathbf{y} = \sum_{r=1}^{n} \mathbf{v}_r \frac{\mathbf{v}_r{}^T\mathbf{M}\mathbf{y}}{\mathbf{v}_r{}^T\mathbf{M}\mathbf{v}_r} \tag{28.19}$$

or alternatively as

$$\mathbf{y} = \sum_{r=1}^{n} \frac{\mathbf{v}_r \mathbf{v}_r{}^T \mathbf{M}}{\mathbf{v}_r{}^T \mathbf{M} \mathbf{v}_r} \mathbf{y} \tag{28.20}$$

The form of Eq. (28.19) emphasizes the decomposition into eigenvectors since the fraction on the right is just a scalar. The form of Eq. (28.20) is convenient when a large number of vectors **y** are to be decomposed, since the fractions on the right, which are now square matrices, must be computed only once. The form of Eq. (28.20) becomes more economical of computation time when more than n vectors **y** have to be expanded. A useful check on the calculation of the matrices on the right of Eq. (28.20) is provided by the identity

$$\sum_{r=1}^{n} \frac{\mathbf{v}_r \mathbf{v}_r{}^T \mathbf{M}}{\mathbf{v}_r{}^T \mathbf{M} \mathbf{v}_r} = \mathbf{I} \tag{28.21}$$

which follows from Eq. (28.20) because **y** is completely arbitrary.

An alternative expansion which is useful for expanding the excitation vector **f** is

$$\mathbf{f} = \sum_{r=1}^{n} \omega_r{}^2 \mathbf{M} \mathbf{v}_r a_r = \sum_{r=1}^{n} \mathbf{M} \mathbf{v}_r \frac{\mathbf{v}_r{}^T \mathbf{f}}{\mathbf{v}_r{}^T \mathbf{M} \mathbf{v}_r} \tag{28.22}$$

This may be viewed as an expansion of the excitation in terms of the *inertia force* amplitudes of the natural modes. The mode multiplier a_r has been evaluated by premultiplying by $\mathbf{v}_r{}^T$. A form analogous to Eq. (28.20) and an identity corresponding to Eq. (28.21) can easily be written.

RAYLEIGH'S QUOTIENT

If Eq. (28.11) is premultiplied by $\mathbf{v}^T$, the following scalar equation is obtained:

$$\mathbf{v}^T \mathbf{K} \mathbf{v} = \omega^2 \mathbf{v}^T \mathbf{M} \mathbf{v}$$

The positive definiteness of **M** guarantees that $\mathbf{v}^T \mathbf{M} \mathbf{v}$ is nonzero, so that it is permissible to solve for ω^2.

$$\omega^2 = \frac{\mathbf{v}^T \mathbf{K} \mathbf{v}}{\mathbf{v}^T \mathbf{M} \mathbf{v}} \tag{28.23}$$

This quotient is called "Rayleigh's quotient." It also may be derived by equating time averages of potential and kinetic energy under the assumption that the vibratory system is executing simple harmonic motion at frequency ω with amplitude ratios given by **v** or by equating the maximum value of kinetic energy to the maximum value of potential energy under the same assumption. Rayleigh's quotient has the following interesting properties.

1. When **v** is an eigenvector $\mathbf{v}_r$ of Eq. (28.11), then Rayleigh's quotient is equal to the corresponding eigenvalue $\omega_r{}^2$.
2. If **v** is an approximation to $\mathbf{v}_r$ with an error which is a *first-order* infinitesimal, then Rayleigh's quotient is an approximation to $\omega_r{}^2$ with an error which is a *second-order* infinitesimal; i.e., Rayleigh's quotient is *stationary* in the neighborhoods of the true eigenvectors.
3. As **v** varies through all of n-dimensional vector space, Rayleigh's quotient remains bounded between the smallest and largest eigenvalues.

A common engineering application of Rayleigh's quotient involves simply evaluating Eq. (28.23) for a trial vector $\mathbf{v}$ which is selected on the basis of physical insight. When eigenvectors are obtained by approximate methods, Rayleigh's quotient provides a means of improving the accuracy in the corresponding eigenvalue. If the elements of an approximate eigenvector whose largest element is unity are correct to k decimal places, then Rayleigh's quotient can be expected to be correct to about $2k$ significant decimal places.

Perturbation Formulas. The perturbation formulas which follow provide the basis for estimating the changes in the eigenvalues and the eigenvectors which result from *small* changes in the stiffness and inertia parameters of a system. The formulas are strictly accurate only for infinitesimal changes but are useful approximations for *small* changes. They may be used by the designer to estimate the effects of a proposed change in a vibratory system and may also be used to analyze the effects of minor errors in the measurement of the system properties. Iterative procedures for the solution of eigenvalue problems can be based on these formulas. They are employed here to obtain approximations to the complex eigenvalues and eigenvectors of a lightly damped vibratory system in terms of the corresponding solutions for the same system without damping.

Suppose that the modal matrix $\mathbf{V}$ and the spectral matrix $\mathbf{\Omega}^2$ for the eigenvalue problem

$$\mathbf{KV} = \mathbf{MV\Omega}^2 \tag{28.14}$$

are known. Consider the perturbed eigenvalue problem

$$\mathbf{K}_* \mathbf{V}_* = \mathbf{M}_* \mathbf{V}_* \mathbf{\Omega}_*^2$$

where

$$\mathbf{K}_* = \mathbf{K} + d\mathbf{K} \qquad \mathbf{M}_* = \mathbf{M} + d\mathbf{M}$$

$$\mathbf{V}_* = \mathbf{V} + d\mathbf{V} \qquad \mathbf{\Omega}_*^2 = \mathbf{\Omega}^2 + d\mathbf{\Omega}^2$$

The perturbation formula for the elements $d\omega_r^2$ of the diagonal matrix $d\mathbf{\Omega}^2$ is

$$d\omega_r^2 = \frac{\mathbf{v}_r^T \, d\mathbf{K} \, \mathbf{v}_r - \omega_r^2 \mathbf{v}_r^T \, d\mathbf{M} \, \mathbf{v}_r}{\mathbf{v}_r^T \mathbf{M} \mathbf{v}_r} \tag{28.24}$$

Thus in order to determine the change in a single eigenvalue due to changes in $\mathbf{M}$ and $\mathbf{K}$, it is necessary to know only the corresponding unperturbed eigenvalue and eigenvector. To determine the change in a single eigenvector, however, it is necessary to know *all* the unperturbed eigenvalues and eigenvectors. The following algorithm may be used to evaluate the perturbations of both the modal matrix and the spectral matrix. Calculate

$$\mathbf{F} = \mathbf{V}^T \, d\mathbf{K} \, \mathbf{V} - \mathbf{V}^T \, d\mathbf{M} \, \mathbf{V\Omega}^2$$

and

$$\mathbf{L} = \mathbf{V}^T \mathbf{M} \mathbf{V}$$

The matrix $\mathbf{L}$ is a diagonal matrix of positive elements and hence is easily inverted. Continue calculating

$$\mathbf{G} = \mathbf{L}^{-1}\mathbf{F} = [g_{jk}] \qquad \text{and} \qquad \mathbf{H} = [h_{jk}]$$

where

$$h_{jk} = \begin{cases} 0 & \text{if } \omega_j^2 = \omega_k^2 \\ \dfrac{g_{jk}}{\omega_k^2 - \omega_j^2} & \text{if } \omega_j^2 \neq \omega_k^2 \end{cases}$$

Then, finally, the perturbations of the modal matrix and the spectral matrix are given by

$$d\mathbf{V} = \mathbf{VH} \qquad d\boldsymbol{\Omega}^2 = \left[g_{jj} \right] \tag{28.25}$$

These formulas are derived by taking the total differential of Eq. (28.14), premultiplying each term by $\mathbf{V}^T$, and using a relation derived by taking the transpose of Eq. (28.14). An interesting property of the perturbation approximation is that the change in each eigenvector is orthogonal with respect to $\mathbf{M}$ to the corresponding unperturbed eigenvector; i.e.,

$$\mathbf{v}_j^T \mathbf{M}\, d\mathbf{v}_j = 0$$

VIBRATIONS OF SYSTEMS WITHOUT DAMPING

In this section the damping matrix $\mathbf{C}$ is neglected in Eq. (28.8), leaving the general formulation in the form

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f} \tag{28.26}$$

Solutions are outlined for the following three cases: free vibration ($\mathbf{f} = 0$), steady-state forced sinusoidal vibration ($\mathbf{f} = \Re\{\mathbf{d}e^{j\omega t}\}$, where $\mathbf{d}$ is a column vector of driving-force amplitudes), and the response to general excitation ($\mathbf{f}$ an arbitrary function of time). The first two cases are contained in the third, but for the sake of clarity each is described separately.

FREE VIBRATION WITH SPECIFIED INITIAL CONDITIONS

It is desired to find the solution $\mathbf{x}(t)$ of Eq. (28.26) when $\mathbf{f} = 0$ which satisfies the initial conditions

$$\mathbf{x} = \mathbf{x}(0) \qquad \dot{\mathbf{x}} = \dot{\mathbf{x}}(0) \tag{28.27}$$

at $t = 0$ where $\mathbf{x}(0)$ and $\dot{\mathbf{x}}(0)$ are columns of prescribed initial displacements and velocities. The differential equation to be solved is identical with Eq. (28.10), which led to the matrix eigenvalue problem in the preceding section. Assuming that the solution of the eigenvalue problem is available, the general solution of the differential equation is given by an arbitrary superposition of the natural modes

$$\mathbf{x} = \sum_{r=1}^{n} \mathbf{v}_r (a_r \cos \omega_r t + b_r \sin \omega_r t)$$

where the $\mathbf{v}_r$ are the eigenvectors or natural modes, the ω_r are the natural frequencies, and the a_r and b_r are $2n$ constants of integration. The corresponding velocity is

$$\dot{\mathbf{x}} = \sum_{r=1}^{n} \mathbf{v}_r \omega_r (-a_r \sin \omega_r t + b_r \cos \omega_r t)$$

Setting $t = 0$ in these expressions and substituting in the initial conditions of Eq. (28.27) provides $2n$ simultaneous equations for determination of the constants of integration.

$$\sum_{r=1}^{n} \mathbf{v}_r a_r = \mathbf{x}(0) \qquad \sum_{r=1}^{n} \mathbf{v}_r \omega_r b_r = \dot{\mathbf{x}}(0)$$

These equations may be interpreted as eigenvector expansions of the initial displacement and velocity. The constants of integration can be evaluated by the same technique used to obtain the mode multipliers in Eq. (28.19). Using the form of Eq. (28.20), the solution of the free vibration problem then becomes

$$\mathbf{x}(t) = \sum_{r=1}^{n} \frac{\mathbf{v}_r \mathbf{v}_r{}^T \mathbf{M}}{\mathbf{v}_r{}^T \mathbf{M} \mathbf{v}_r} \left\{ \mathbf{x}(0) \cos \omega_r t + \frac{1}{\omega_r} \dot{\mathbf{x}}(0) \sin \omega_r t \right\} \tag{28.28}$$

STEADY-STATE FORCED SINUSOIDAL VIBRATION

It is desired to find the steady-state solution to Eq. (28.26) for single-frequency sinusoidal excitation **f** of the form

$$\mathbf{f} = \Re \{ \mathbf{d} e^{j\omega t} \}$$

where **d** is a column vector of driving force amplitudes (these may be complex to permit differences in phase for the various components). The solution obtained is a useful approximation for lightly damped systems provided that the forcing frequency ω is not too close to a natural frequency ω_r. For resonance and near-resonance conditions it is necessary to include the damping as indicated in the section which follows the present discussion.

The steady-state solution desired is assumed to have the form

$$\mathbf{x} = \Re \{ \mathbf{a} e^{j\omega t} \}$$

where **a** is an unknown column vector of response amplitudes. When **f** and **x** are inserted in Eq. (28.26), the following set of simultaneous equations for the elements of **a** is obtained:

$$(\mathbf{K} - \omega^2 \mathbf{M}) \mathbf{a} = \mathbf{d} \tag{28.29}$$

If ω is not a natural frequency, the square matrix $\mathbf{K} - \omega^2 \mathbf{M}$ is nonsingular and may be inverted to yield

$$\mathbf{a} = (\mathbf{K} - \omega^2 \mathbf{M})^{-1} \mathbf{d}$$

as a complete solution for the response amplitudes in terms of the driving force amplitudes. This solution is useful if several force amplitude distributions are to be studied while the excitation frequency ω is held constant. The process requires repeated inversions if a range of frequencies is to be studied.

An alternative procedure which permits a more thorough study of the effect of frequency variation is available if the natural modes and frequencies are known. The driving-force vector **d** is represented by the eigenvector expansion of Eq. (28.22), and the response vector **a** is represented by the eigenvector expansion of Eq. (28.18):

$$\mathbf{d} = \sum_{r=1}^{n} \frac{\mathbf{M}\mathbf{v}_r\mathbf{v}_r^T}{\mathbf{v}_r^T\mathbf{M}\mathbf{v}_r}\,\mathbf{d} \qquad \mathbf{a} = \sum_{r=1}^{n} \mathbf{v}_r c_r$$

where the c_r are unknown coefficients. Substituting these into Eq. (28.29), and making use of the fundamental eigenvalue relation of Eq. (28.11), leads to

$$\sum_{r=1}^{n} (\omega_r^2 - \omega^2)\mathbf{M}\mathbf{v}_r c_r = \sum_{r=1}^{n} \frac{\mathbf{M}\mathbf{v}_r\mathbf{v}_r^T}{\mathbf{v}_r^T\mathbf{M}\mathbf{v}_r}\,\mathbf{d}$$

This equation can be uncoupled by premultiplying both sides by $\mathbf{v}_r^T$ and using the orthogonality condition of Eq. (28.15) to obtain

$$(\omega_r^2 - \omega^2)\mathbf{v}_r^T\mathbf{M}\mathbf{v}_r c_r = \mathbf{v}_r^T\mathbf{d}$$

$$c_r = \frac{1}{\omega_r^2 - \omega^2}\,\frac{\mathbf{v}_r^T\mathbf{d}}{\mathbf{v}_r^T\mathbf{M}\mathbf{v}_r}$$

The final solution is then assembled by inserting the c_r back into $\mathbf{a}$ and $\mathbf{a}$ back into $\mathbf{x}$.

$$\mathbf{x} = \Re\left\{\sum_{r=1}^{n} \frac{e^{j\omega t}}{\omega_r^2 - \omega^2}\,\frac{\mathbf{v}_r\mathbf{v}_r^T}{\mathbf{v}_r^T\mathbf{M}\mathbf{v}_r}\,\mathbf{d}\right\} \tag{28.30}$$

This form clearly indicates the effect of frequency on the response.

RESPONSE TO GENERAL EXCITATION

It is now desired to obtain the solution to Eq. (28.26) for the general case in which the excitation $\mathbf{f}(t)$ is an arbitrary vector function of time and for which initial displacements $\mathbf{x}(0)$ and velocities $\dot{\mathbf{x}}(0)$ are prescribed. If the natural modes and frequencies of the system are available, it is again possible to split the problem up into n single degree-of-freedom response problems and to indicate a formal solution.

Following a procedure similar to that just used for steady-state forced sinusoidal vibrations, an eigenvector expansion of the solution is assumed:

$$\mathbf{x}(t) = \sum_{r=1}^{n} \mathbf{y}_r c_r(t)$$

where the c_r are unknown functions of time and the known excitation $\mathbf{f}(t)$ is expanded according to Eq. (28.22). Inserting these into Eq. (28.26) yields

$$\sum_{r=1}^{n} (\mathbf{M}\mathbf{v}_r\ddot{c}_r + \mathbf{K}\mathbf{v}_r c_r) = \sum_{r=1}^{n} \frac{\mathbf{M}\mathbf{v}_r\mathbf{v}_r^T}{\mathbf{v}_r^T\mathbf{M}\mathbf{v}_r}\,\mathbf{f}(t)$$

Using Eq. (28.11) to eliminate $\mathbf{K}$ and premultiplying by $\mathbf{v}_r^T$ to uncouple the equation,

$$\ddot{c}_r + \omega_r^2 c_r^2 = \frac{\mathbf{v}_r^T\mathbf{f}(t)}{\mathbf{v}_r^T\mathbf{M}\mathbf{v}_r} \tag{28.31}$$

is obtained as a single second-order differential equation for the time behavior of the rth mode multiplier. The initial conditions for c_r can be obtained by making eigenvector expansions of $\mathbf{x}(0)$ and $\dot{\mathbf{x}}(0)$ as was done previously for the free vibration case. Formal solutions to Eq. (28.29) can be obtained by a number of methods, including Laplace transforms and variation of parameters. When these mode multipliers are substituted back to obtain $\mathbf{x}$, the general solution has the following appearance:

$$\mathbf{x}(t) = \sum_{r=1}^{n} \frac{\mathbf{v}_r \mathbf{v}_r{}^T \mathbf{M}}{\mathbf{v}_r{}^T \mathbf{M} \mathbf{v}_r} \left\{ \mathbf{x}(0) \cos \omega_r t + \frac{1}{\omega_r} \dot{\mathbf{x}}(0) \sin \omega_r t \right\}$$

$$+ \sum_{r=1}^{n} \frac{\mathbf{v}_r \mathbf{v}_r{}^T}{\omega_r \mathbf{v}_r{}^T \mathbf{M} \mathbf{v}_r} \int_0^t \mathbf{f}(t') \sin \{\omega_r (t - t')\} \, dt' \quad (28.32)$$

The integrals involving the excitation can be evaluated in closed form if the elements $f_i(t)$ of $\mathbf{f}(t)$ are simple (e.g., step functions, ramps, single sine pulses, etc.). When the $f_i(t)$ are more complicated, numerical results can be obtained by using integration software.

VIBRATION OF SYSTEMS WITH DAMPING

In this section solutions to the complete governing equation, Eq. (28.8), are discussed. The results of the preceding section for systems without damping are adequate for many purposes. There are, however, important problems in which it is necessary to include the effect of damping, e.g., problems concerned with resonance, random vibration, etc.

COMPLEX EIGENVALUE PROBLEM

When there is no excitation, Eq. (28.8) becomes

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = 0$$

which describes the free vibration of the system. As in the undamped case, there are $2n$ independent solutions which can be superposed to meet $2n$ initial conditions. Assuming a solution in the form

$$\mathbf{x} = \mathbf{u}e^{pt}$$

leads to the following algebraic problem:

$$(p^2\mathbf{M} + p\mathbf{C} + \mathbf{K})\mathbf{u} = 0 \quad (28.33)$$

for the determination of the vector $\mathbf{u}$ and the scalar p. This is a *complex eigenvalue problem* because the *eigenvalue* p and the elements of the *eigenvector* $\mathbf{u}$ are, in general, complex numbers. The most common technique for solving the nth-order eigenvalue problem, Eq. (28.33), is to transform it to a $2n$th-order problem having the same form as Eq. (28.11). This may be done by introducing the column vector $\tilde{\mathbf{v}}$ of order $2n$ given by

$$\tilde{\mathbf{v}} = \{\mathbf{u} \quad p\mathbf{u}\}^T$$

and the two square matrices of order $2n$ given by

$$\tilde{\mathbf{K}} = \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix} \qquad \tilde{\mathbf{M}} = \begin{bmatrix} C & M \\ M & 0 \end{bmatrix}$$

In terms of these, an eigenvalue problem equivalent to Eq. (28.33) is

$$\tilde{\mathbf{K}}\tilde{\mathbf{v}} = p\tilde{\mathbf{M}}\tilde{\mathbf{v}} \tag{28.34}$$

which is similar to Eq. (28.11) except that $\tilde{\mathbf{M}}$ does not have the positive definite property that $\mathbf{M}$ has. As a result, the eigenvalue p and the eigenvector $\mathbf{v}$ are generally complex. Since the computational time for most eigenvalue problems is proportional to n^3, the computational time for the $2n$th-order system of Eq. (28.34) will be about eight times that for the nth-order system of Eq. (28.11).

If the complex eigenvalue $p = -\alpha + j\beta$ together with the complex eigenvector $\mathbf{u} = \mathbf{v} + j\mathbf{w}$ satisfy the eigenvalue problem of Eq. (28.33), then so also does the complex conjugate eigenvalue $p^C = -\alpha - j\beta$ together with the complex conjugate eigenvector $\mathbf{u}^C = \mathbf{v} - j\mathbf{w}$. There are $2n$ eigenvalues which occur in pairs of complex conjugates or as real negative numbers. When the damping is absent all roots lie on the imaginary axis of the complex p-plane; for small damping the roots lie near the imaginary axis. The corresponding $2n$ eigenvectors $\mathbf{u}_r$ satisfy the following *orthogonality* relations:

$$(p_r + p_s)\mathbf{u}_r^T\mathbf{M}\mathbf{u}_s + \mathbf{u}_r^T\mathbf{C}\mathbf{u}_s = 0$$

$$\mathbf{u}_r^T\mathbf{K}\mathbf{u}_s - p_r p_s \mathbf{u}_r^T\mathbf{M}\mathbf{u}_s = 0$$

whenever $p_r \neq p_s$; they can be made to hold for repeated roots by suitable choice of the eigenvectors associated with a multiple root. When p_s is put equal to p_r^C, the orthogonality relations provide convenient formulas for the real and imaginary parts of the eigenvalues in terms of the eigenvectors

$$2\alpha_r = \frac{\mathbf{u}_r^T\mathbf{C}\mathbf{u}_r^C}{\mathbf{u}_r^T\mathbf{M}\mathbf{u}_r^C} = \frac{\mathbf{v}_r^T\mathbf{C}\mathbf{v}_r + \mathbf{w}_r^T\mathbf{C}\mathbf{w}_r}{\mathbf{v}_r^T\mathbf{M}\mathbf{v}_r + \mathbf{w}_r^T\mathbf{M}\mathbf{w}_r}$$

$$\alpha_r^2 + \beta_r^2 = \frac{\mathbf{u}_r^T\mathbf{K}\mathbf{u}_r^C}{\mathbf{u}_r^T\mathbf{M}\mathbf{u}_r^C} = \frac{\mathbf{v}_r^T\mathbf{K}\mathbf{v}_r + \mathbf{w}_r^T\mathbf{K}\mathbf{w}_r}{\mathbf{v}_r^T\mathbf{M}\mathbf{v}_r + \mathbf{w}_r^T\mathbf{M}\mathbf{w}_r}$$

The complex eigenvalue is often represented in the form

$$p_r = \omega_r(-\zeta_r + j\sqrt{1 - \zeta_r^2}) \tag{28.35}$$

where $\omega_r = \sqrt{\alpha_r^2 + \beta_r^2}$ is called the *undamped natural frequency* of the rth mode, and $\zeta_r = \alpha_r/\omega_r$ is called the *critical damping ratio* of the rth mode.

PERTURBATION APPROXIMATION TO COMPLEX EIGENVALUE PROBLEM

The complex eigenvalue problem of Eq. (28.33) can be solved approximately, when the damping is light, by using the perturbation equations of Eqs. (28.24) and (28.25). When $\mathbf{C} = 0$ in Eq. (28.33) the complex eigenvalue problem reduces to the real eigenvalue problem of Eq. (28.11) with $p^2 = -\omega^2$. Suppose that the real eigenvalue ω_r^2 and the real eigenvector $\mathbf{v}_r$ are known. The perturbation of the rth mode due to the addition of small damping $\mathbf{C}$ can be estimated by considering the damping to be a perturbation of the stiffness matrix of the form

$$d\mathbf{K} = j\omega_r\mathbf{C}$$

In this way it is found that the perturbed solution corresponding to the rth mode consists of a pair of complex conjugate eigenvalues

$$p_r = -\alpha_r + j\omega_r \qquad p_r{}^C = -\alpha_r - j\omega_r$$

and a pair of complex conjugate eigenvectors

$$\mathbf{u}_r = \mathbf{v}_r + j\mathbf{w}_r \qquad \mathbf{u}_r{}^C = \mathbf{v}_r - j\mathbf{w}_r$$

where ω_r and $\mathbf{v}_r$ are taken directly from the undamped system, and α_r and $\mathbf{w}_r$ are small perturbations which are given below. The superscript C is used to denote the complex conjugate. The real part of the eigenvalue, which describes the rate of decay of the corresponding free motion, is given by the following quotient:

$$2\alpha_r = 2\zeta_r\omega_r = \frac{\mathbf{v}_r{}^T\mathbf{C}\mathbf{v}_r}{\mathbf{v}_r{}^T\mathbf{M}\mathbf{v}_r} \tag{28.36}$$

The decay rate α_r for a particular r depends only on the rth mode undamped solution. The imaginary part of the eigenvector $j\mathbf{w}_r$, which describes the perturbations in phase, is more difficult to obtain. All the undamped eigenvalues and eigenvectors must be known. Let $\mathbf{W}$ be a square matrix whose columns are the $\mathbf{w}_r$. The following algorithm may be used to evaluate $\mathbf{W}$ when the undamped modal matrix $\mathbf{V}$ is known. Calculate

$$\mathbf{F} = \mathbf{V}^T\mathbf{C}\mathbf{V}$$

and

$$\mathbf{L} = \mathbf{V}^T\mathbf{M}\mathbf{V}$$

The matrix $\mathbf{L}$ is a diagonal matrix of positive elements and hence is easily inverted. Continue calculating

$$\mathbf{G} = \mathbf{L}^{-1}\mathbf{F} = [g_{jk}] \qquad \text{and} \qquad \mathbf{H} = [h_{jk}]$$

where

$$h_{jk} = \begin{cases} 0 & \text{if } \omega_j{}^2 = \omega_k{}^2 \\ \dfrac{g_{jk}\omega_k}{\omega_k{}^2 - \omega_j{}^2} & \text{if } \omega_j{}^2 \neq \omega_k{}^2 \end{cases}$$

Then, finally, the eigenvector perturbations are given by

$$\mathbf{W} = \mathbf{V}\mathbf{H} \tag{28.37}$$

The individual eigenvector perturbations $\mathbf{w}_r$ obtained in this manner are orthogonal with respect to $\mathbf{M}$ to their corresponding unperturbed eigenvectors $\mathbf{v}_r$; i.e., $\mathbf{w}_r^T\mathbf{M}\mathbf{v}_r = 0$.

FORMAL SOLUTIONS

If the solution to the eigenvalue problem of Eq. (28.33) is available, it is possible to exhibit a general solution to the governing equation

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f} \tag{28.8}$$

for arbitrary excitation $\mathbf{f}(t)$ which meets prescribed initial conditions for $\mathbf{x}(0)$ and $\dot{\mathbf{x}}(0)$ at $t = 0$. The solutions given below apply to the case where the $2n$ eigenvalues occur as n pairs of complex conjugates (which is usually the case when the damping is light). This does, however, restrict the treatment to systems with nonsingular stiffness matrices $\mathbf{K}$ because if $\omega_r^2 = 0$ is an undamped eigenvalue, the corresponding eigenvalues in the presence of damping are real. All quantities in the solutions below are *real*. These forms have been obtained by breaking down complex solutions into real and imaginary parts and recombining. With the notation

$$p_r = -\alpha_r + j\beta_r \qquad \mathbf{u}_r = \mathbf{v}_r + j\mathbf{w}_r$$

for the real and imaginary parts of eigenvalues and eigenvectors, it follows from Eq. (28.35) that

$$\alpha_r = \zeta_r \omega_r \qquad \beta_r = \omega_r \sqrt{1 - \zeta_r^2}$$

The general solution to Eq. (28.8) is then

$$\begin{aligned}\mathbf{x}(t) = {} & \sum_{r=1}^{n} \frac{2}{a_r^2 + b_r^2} \{\mathbf{G}_r\mathbf{M}\dot{\mathbf{x}}(0) + (-\alpha_r\mathbf{G}_r\mathbf{M} + \beta_r\mathbf{H}_r\mathbf{M} + \mathbf{G}_r\mathbf{C})\mathbf{x}(0)\}e^{-\alpha_r t}\cos\beta_r t \\ & + \sum_{r=1}^{n} \frac{2}{a_r^2 + b_r^2} \{\mathbf{H}_r\mathbf{M}\dot{\mathbf{x}}(0) + (-\beta_r\mathbf{G}_r\mathbf{M} - \alpha_r\mathbf{H}_r\mathbf{M} + \mathbf{H}_r\mathbf{C})\mathbf{x}(0)\}e^{-\alpha_r t}\sin\beta_r t \\ & + \sum_{r=1}^{n} \frac{2}{a_r^2 + b_r^2} \mathbf{G}_r \int_0^t \mathbf{f}(t')e^{-\alpha_r(t-t')}\cos\beta_r(t-t')\,dt' \\ & + \sum_{r=1}^{n} \frac{2}{a_r^2 + b_r^2} \mathbf{H}_r \int_0^t \mathbf{f}(t')e^{-\alpha_r(t-t')}\sin\beta_r(t-t')\,dt' \end{aligned} \tag{28.38}$$

where

$$\begin{aligned} a_r &= -2\alpha_r(\mathbf{v}_r^T\mathbf{M}\mathbf{v}_r - \mathbf{w}_r^T\mathbf{M}\mathbf{w}_r) - 4\beta_r\mathbf{v}_r^T\mathbf{M}\mathbf{w}_r + \mathbf{v}_r^T\mathbf{C}\mathbf{v}_r - \mathbf{w}_r^T\mathbf{C}\mathbf{w}_r \\ b_r &= 2\beta_r(\mathbf{v}_r^T\mathbf{M}\mathbf{v}_r - \mathbf{w}_r^T\mathbf{M}\mathbf{w}_r) - 4\alpha_r\mathbf{v}_r^T\mathbf{M}\mathbf{w}_r + 2\mathbf{v}_r^T\mathbf{C}\mathbf{w}_r \\ \mathbf{A}_r &= \mathbf{v}_r\mathbf{v}_r^T - \mathbf{w}_r\mathbf{w}_r^T \qquad \mathbf{B}_r = \mathbf{v}_r\mathbf{w}_r^T + \mathbf{w}_r\mathbf{v}_r^T \\ \mathbf{G}_r &= a_r\mathbf{A}_r + b_r\mathbf{B}_r \qquad \mathbf{H}_r = b_r\mathbf{A}_r - a_r\mathbf{B}_r \end{aligned}$$

The solution of Eq. (28.38) should be compared with the corresponding solution of Eq. (28.32) for systems without damping. When the damping matrix $\mathbf{C} = 0$, Eq. (28.38) reduces to Eq. (28.32).

For the important special case of steady-state forced sinusoidal excitation of the form

$$\mathbf{f} = \Re\{\mathbf{d}e^{j\omega t}\}$$

where $\mathbf{d}$ is a column of driving force amplitudes, the steady-state portion of the response can be written as follows, using the above notation:

$$\mathbf{x}(t) = \Re\left\{\sum_{r=1}^{n} \frac{2e^{j\omega t}}{a_r^2 + b_r^2} \frac{\alpha_r\mathbf{G}_r + \beta_r\mathbf{H}_r + j\omega\mathbf{G}_r}{\omega_r^2 - \omega^2 + j2\zeta_r\omega_r\omega}\,\mathbf{d}\right\} \tag{28.39}$$

This result reduces to Eq. (28.30) when the damping matrix $\mathbf{C}$ is set equal to zero.

APPROXIMATE SOLUTIONS

For a lightly damped system the exact solutions of Eq. (28.38) and Eq. (28.39) can be abbreviated considerably by making approximations based on the smallness of the damping. A systematic method of doing this is to consider the system without damping as a base upon which an infinitesimal amount of damping is superposed as a perturbation. An approximate solution to the complex eigenvalue problem by this method is provided by Eqs. (28.36) and (28.37). This perturbation approximation can be continued into Eqs. (28.38) and (28.39) by simply neglecting all squares and products of the small quantities α_r, ζ_r, $\mathbf{w}_r$, and $\mathbf{C}$. When this is done it is found that the formulas of Eqs. (28.38) and (28.39) may still be used if the parameters therein are obtained from the simplified expressions below.

$$\begin{aligned} \alpha_r &= \zeta_r \omega_r \qquad \beta_r = \omega_r \\ a_r &= -4\omega_r \mathbf{v}_r{}^T \mathbf{M} \mathbf{w}_r \qquad b_r = 2\omega_r \mathbf{v}_r{}^T \mathbf{M} \mathbf{v}_r \\ a_r{}^2 + b_r{}^2 &= 4\omega_r{}^2 (\mathbf{v}_r{}^T \mathbf{M} \mathbf{v}_r)^2 \\ \mathbf{A}_r &= \mathbf{v}_r \mathbf{v}_r{}^T \qquad \mathbf{B}_r = \mathbf{v}_r \mathbf{w}_r{}^T + \mathbf{w}_r \mathbf{v}_r{}^T \\ \mathbf{G}_r &= 2\omega_r (\mathbf{v}_r{}^T \mathbf{M} \mathbf{v}_r)(\mathbf{v}_r \mathbf{w}_r{}^T + \mathbf{w}_r \mathbf{v}_r{}^T) \\ \mathbf{H}_r &= 2\omega_r (\mathbf{v}_r{}^T \mathbf{M} \mathbf{v}_r) \mathbf{v}_r \mathbf{v}_r{}^T \end{aligned} \tag{28.40}$$

For example, the steady-state forced sinusoidal solution of Eq. (28.39) takes the following explicit form in the perturbation approximation:

$$\mathbf{x}(t) = \Re \left\{ \sum_{r=1}^{n} \frac{e^{j\omega t}}{\mathbf{v}_r{}^T \mathbf{M} \mathbf{v}_r} \frac{\mathbf{v}_r \mathbf{v}_r{}^T + \dfrac{j\omega}{\omega_r} \left[\mathbf{v}_r \mathbf{w}_r{}^T + \mathbf{w}_r \mathbf{v}_r{}^T \right]}{\omega_r{}^2 - \omega^2 + j2\zeta_r \omega_r \omega} \mathbf{d} \right\} \tag{28.41}$$

A cruder approximation, which is often used, is based on accepting the complex eigenvalue $p_r = -\alpha_r + j\omega_r$ but completely neglecting the imaginary part $j\mathbf{w}_r$ of the eigenvector $\mathbf{u}_r = \mathbf{v}_r + j\mathbf{w}_r$. It is thus assumed that the undamped mode $\mathbf{v}_r$ still applies for the system with damping. The approximate parameter values of Eq. (28.40) are further simplified by this assumption; e.g., $a_r = 0$, $\mathbf{B}_r = \mathbf{G}_r = 0$. The steady forced sinusoidal response of Eq. (28.41) reduces to

$$\mathbf{x}(t) = \Re \left\{ \sum_{r=1}^{n} \frac{e^{j\omega t}}{\omega_r{}^2 - \omega^2 + j2\zeta_r \omega_r \omega} \frac{\mathbf{v}_r \mathbf{v}_r{}^T}{\mathbf{v}_r{}^T \mathbf{M} \mathbf{v}_r} \mathbf{d} \right\} \tag{28.42}$$

This approximation should be compared with the undamped solution of Eq. (28.30), as well as with the exact solution of Eq. (28.39) and the perturbation approximation of Eq. (28.41).

In the special case of proportional damping, the exact eigenvectors are real and Eq. (28.36) produces the exact decay rate $\alpha_r = \zeta_r \omega_r$, so that the response of Eq. (28.42) is an exact result.

Example 28.1. Consider the system of Fig. 28.5 with the following mass, damping, and stiffness coefficients:

$m_1 = 1$ lb-sec^2/in. $\quad m_2 = 2$ lb-sec^2/in.

$c_1 = 0.10$ lb-sec/in. $\quad c_2 = 0.02$ lb-sec/in. $\quad c_3 = 0.04$ lb-sec/in.

$k_1 = 3$ lb/in. $\quad k_2 = 0.5$ lb/in. $\quad k_3 = 1$ lb/in.

The coefficient matrices of Eq. (28.9) then have the following numerical values:

$$\mathbf{M} = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 0.14 & 0.04 \\ 0.04 & 0.06 \end{bmatrix} \qquad \mathbf{K} = \begin{bmatrix} 4 & 1 \\ 1 & 1.5 \end{bmatrix}$$

Assuming that the numerical values above are exact, the exact solutions to the complex eigenvalue problem of Eq. (28.33) for these values of **M**, **C**, and **K** are, correct to four decimal places,

$$\begin{gathered} p_r = -\alpha_r + j\beta_r \qquad \mathbf{u}_r = \mathbf{v}_r + j\mathbf{w}_r \\ 2\alpha_1 = 0.0279 \qquad \alpha_1 = \zeta_1\omega_1 = 0.0139 \qquad \zeta_1 = 0.0166 \\ \beta_1 = 0.8397 \qquad \omega_1 = 0.8398 \qquad \omega_1^2 = 0.7053 \\ 2\alpha_2 = 0.1221 \qquad \alpha_2 = \zeta_2\omega_2 = 0.0611 \qquad \zeta_2 = 0.0324 \\ \beta_2 = 1.8818 \qquad \omega_2 = 1.8828 \qquad \omega_2^2 = 3.5449 \\ \mathbf{V} = \begin{bmatrix} 0.2179 & -0.9179 \\ 1.0000 & 1.0000 \end{bmatrix} \qquad \mathbf{W} = \begin{bmatrix} 0.0016 & 0.0010 \\ 0 & 0 \end{bmatrix} \end{gathered} \tag{28.43}$$

Note that this is a lightly damped system. The damping ratios in the two modes are 1.66 percent and 3.24 percent, respectively.

For comparison, the solution of the real eigenvalue problem Eq. (28.12) for the corresponding undamped system (i.e., **M** and **K** as above, but $\mathbf{C} = 0$) is, correct to four decimal places,

$$\omega_1^2 = 0.7053 \qquad \omega_2^2 = 3.5447 \qquad \mathbf{V} = \begin{bmatrix} 0.2179 & -0.9179 \\ 1.0000 & 1.0000 \end{bmatrix}$$

Note that, to this accuracy, there is no discrepancy in the real parts of the eigenvectors. There are, however, small discrepancies in the imaginary parts of the eigenvalues. The difference between β_1 for the damped system and ω_1 for the undamped system is 0.0001, and the corresponding difference between β_2 and ω_2 is 0.0009. The imaginary parts of the eigenvectors and the real parts of the eigenvalues for the damped system are completely absent in the undamped system. They may be approximated by applying the perturbation equations of Eqs. (28.36) and (28.37) to the solution of the eigenvalue problem for the undamped system.

The real parts α_r of the eigenvalues obtained from Eq. (28.36) agree, to four decimal places, with the exact values in Eq. (28.43). The imaginary parts $\mathbf{w}_r$ of the eigenvectors obtained from Eq. (28.37) are

$$\mathbf{w}_1 = \begin{Bmatrix} 0.0013 \\ -0.0014 \end{Bmatrix} \qquad \mathbf{w}_2 = \begin{Bmatrix} 0.0002 \\ 0.0009 \end{Bmatrix}$$

These vectors satisfy the orthogonality conditions $\mathbf{v}_r^T \mathbf{M} \mathbf{w}_r = 0$.

In order to compare these values with Eq. (28.43), it is first necessary to normalize the complete eigenvector $\mathbf{v}_r + j\mathbf{w}_r$, so that its second element is unity. For example, this is done in the case of $r = 1$ by dividing both $\mathbf{v}_1$ and $\mathbf{w}_1$ by $1.0000 - j0.0014$. When this is done, it is found that the perturbation approximation to the eigenvectors agrees, to four decimal places, with the exact solution of Eq. (28.43).

To illustrate the application of the formal solutions given above, consider the steady-state forced oscillation of the system shown in Fig. 28.5 at a frequency ω due

to driving force amplitudes d_1 and d_2. Using the exact solution values of Eq. (28.43), the expressions a_r, b_r, $\mathbf{A}_r$, $\mathbf{B}_r$, $\mathbf{G}_r$, and $\mathbf{H}_r$ following Eq. (28.38) are evaluated for $r = 1$ and $r = 2$. With these values, the steady-state response, Eq. (28.39), becomes

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \Re \left\{ \frac{e^{j\omega t} \left\{ \begin{bmatrix} 0.0158 & 0.0723 \\ 0.0723 & 0.3318 \end{bmatrix} + j\omega \begin{bmatrix} 0.0002 & 0.0004 \\ 0.0004 & -0.0011 \end{bmatrix} \right\}}{0.7053 - \omega^2 + 0.0279 j\omega} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \right.$$

$$\left. + \frac{e^{j\omega t} \left\{ \begin{bmatrix} 0.9842 & -1.0724 \\ -1.0724 & 1.1683 \end{bmatrix} + j\omega \begin{bmatrix} -0.0002 & -0.0004 \\ -0.0004 & 0.0011 \end{bmatrix} \right\}}{3.5449 - \omega^2 + 0.1221 j\omega} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \right\}$$

When the approximation in Eq. (28.41) based on the perturbation solution is evaluated, the result is almost identical to this. A few entries differ by one or two units in the fourth decimal place. The crude approximation, Eq. (28.42), is the same as the perturbation approximation except that the terms in the numerators which are multiplied by $j\omega$ are absent. This means that the relative error between the crude approximation and the exact solution can be large at high frequencies. At low frequencies, however, even the crude approximation provides useful results for lightly damped systems. In the present case, the discrepancy between the crude approximation and the exact solution remains under 1 percent as long as ω is less than ω_2 (the highest natural frequency). At higher frequencies the absolute response level decreases steadily, which tends to undercut the significance of the increasing relative discrepancy between approximations.

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CHAPTER 28, PART II
FINITE ELEMENT MODELS

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INTRODUCTION

The *finite element method* (FEM), formally introduced by Clough[1] in 1960, has become a mature engineering discipline during the past forty years. In actual practice, finite element analysis is a systematic applied science, which incorporates (1) the definition of a physical model of a complex system as a collection of building blocks (finite elements), (2) the solution of matrix equations describing the physical model, and (3) the analysis and interpretation of numerical results. The foundations of finite element analysis are (*a*) the design of consistent, robust finite elements[2]; and (*b*) matrix methods of numerical analysis[3,4,5] (see Chap. 28, Part I). Originally developed to address modeling and analysis of complex structures, the finite element approach is now applied to a wide variety of engineering applications including heat transfer, fluid dynamics, and electromagnetics, as well as multiphysics (coupled interaction) applications.

Modern finite element programs include powerful graphical user interface (GUI) driven preprocessors and postprocessors, which automate routine operations required for the definition of models and the interpretation of numerical results, respectively (see Chap. 27). Moreover, finite element analysis, computer-assisted design and optimization, and laboratory/field testing are viewed as an integrated "concurrent engineering" process. Commercially available products, widely used in industry, include MSC/NASTRAN (a product of MSC.Software), ANSYS (a product family of ANSYS Incorporated), and ABAQUS (a product of HKS Incorporated), just to mention a few.

This chapter describes finite element modeling and analysis with an emphasis on its application to the shock and vibration of structures and structures interacting with fluid media. Included are discussions on the theoretical foundations of finite element models, effective modeling guidelines, dynamic system models and analysis strategies, and common industry practice.

THEORETICAL FOUNDATIONS OF FINITE ELEMENT MODELS

APPLICATION OF MINIMAL PRINCIPLES

The matrix equations describing both individual finite elements and complete finite element system models are defined on the basis of minimal principles. In particular,

for structural dynamic systems, Hamilton's Principle or Lagrange's Equations[6] constitute the underlying physical principle. The fundamental statement of *Hamilton's Principle* is

$$\delta \int_{t_0}^{t_1} (T + W)dt = 0 \tag{28.44}$$

where T is the system kinetic energy, W is the work performed by internal and external forces, t represents time, and δ is the variational operator. In the case of statics, Hamilton's Principle reduces to the *Principle of Virtual Work,* stated mathematically as

$$\delta W = 0 \qquad (\text{if } T = 0) \tag{28.45}$$

For most mechanical systems of interest, W may be expressed in terms of a conservative interior elastic potential energy (U), dissipative interior work (W_D), and the work associated with externally applied forces (W_E). Thus Hamilton's Principle is stated as

$$\int_{t_0}^{t_1} (\delta T - \delta U + \delta W_D + \delta W_E)dt = 0 \tag{28.46}$$

The kinematics of a mechanical system of volume, V, are described in terms of the displacement field

$$\{u\} = [N_u \; N_q]\begin{Bmatrix} u_i \\ q \end{Bmatrix} \tag{28.47}$$

where $\{u\}$ is the displacement array at any point in V, $\{u_i\}$ is an array of discrete displacements (typically) on the element surface, and $\{q\}$ is an array of generalized displacement coefficients. The transformation matrix partitions, N_u and N_q, describe assumed shape functions for the particular finite element. The most commonly used elements, namely *H-type* elements, do not include generalized displacement coefficients, $\{q\}$. The more general case element is called a *P-type* element. For simplicity, the subsequent discussion will be limited to H-type elements.

In matrix notation (see Chap. 28, Part I), the strain field within the element volume is related to the assumed displacements by the differential operator matrix $[N_{\varepsilon u}]$ as

$$\{\varepsilon(x,y,z,t)\} = \{\varepsilon\} = [N_{\varepsilon u}]\{u\} \tag{28.48}$$

The stress field within the element volume is expressed as

$$\{\sigma(x,y,z,t)\} = \{\sigma\} = [D]\{\varepsilon\} = [D][N_{\varepsilon u}]\{u\} \tag{28.49}$$

In the case of hybrid finite element formulations, for which there is an assumed element stress field other than simply $[D][N_{\varepsilon u}]$, the situation is more involved.

Using the above general expressions, the kinetic and strain energies associated with a finite element are

$$2T = \int_v \{\dot{u}\}^T[N_u]^T[\rho][N_u]\{\dot{u}\}dV = \{\dot{u}\}^T[M_e]\{\dot{u}\} \tag{28.50}$$

$$2U = \int_v \{u\}^T[N_{\varepsilon u}]^T[D][N_{\varepsilon u}]\{u\}dV = \{u\}^T[K_e]\{u\} \tag{28.51}$$

where $[\rho]$ is the material density matrix, $[D]$ is the material elastic matrix, and $[M_e]$ and $[K_e]$ are the individual element mass and stiffness matrices, respectively. The superscript shown as $\{\ \}^T$ and $[\ ]^T$ denotes the transpose of an array and matrix, respectively. In the case of viscous damping (which is a common yet not necessarily realistic assumption), the *element virtual dissipative work* is

$$\delta W_D = \{\delta u\}^T[B_e]\{\dot{u}\} \tag{28.52}$$

where $[B_e]$ is the symmetric element damping matrix.

In order to assemble the mass, stiffness, and damping matrices associated with a complete finite element system model, the displacement array for the entire system, $\{u_g\}$, must first be defined. The individual element contributions to the system are then allocated (and accumulated) to the appropriate rows and columns of the system matrices. This results in the formation of generally sparse, symmetric matrices. The complete system kinetic and strain energies are, respectively,

$$2T_g = \{\dot{u}_g\}^T[M_{gg}]\{\dot{u}_g\} \tag{28.53}$$

$$2U_g = \{u_g\}^T[K_{gg}]\{u_g\} \tag{28.54}$$

where $[M_{gg}]$ and $[K_{gg}]$ are the system mass and stiffness matrices.

For the case of viscous damping, the complete system virtual dissipative work is

$$\delta W_{Dg} = \{\delta u_g\}^T[B_{gg}]\{\dot{u}_g\} \tag{28.55}$$

Finally, the virtual work associated with externally applied forces on the complete system is defined as

$$\delta W_{Eg} = \{\delta u_g\}^T[\Gamma_{ge}]\{F_e\} \tag{28.56}$$

where $[\Gamma_{ge}]$ represents the allocation matrix for externally applied forces, $\{F_e\}$, including moments, stresses, and pressures if applicable. Substitution of the above expressions for the complete system energies and virtual work into Hamilton's Principle, followed by key manipulations, results in the finite element system differential equations

$$[M_{gg}]\{\ddot{u}_g\} + [B_{gg}]\{\dot{u}_g\} + [K_{gg}]\{u_g\} = [\Gamma_{ge}]\{F_e\} \tag{28.57}$$

The task of defining a finite element model is not yet complete at this point. Constraints and boundary conditions, as required, must now be imposed. The logical sequence of imposed constraint types is (1) *multipoint constraints* (e.g., geometric constraints expressed as algebraic relationships) and (2) *single-point constraints*

(e.g., fixed supports). These constraints are described, in summary, by the linear transformation

$$\{u_g\} = [G_{gf}]\{u_f\} \tag{28.58}$$

where $\{u_f\}$ is the array of "free" displacements. By imposing the constraint transformation, $[G_{gf}]$, in a symmetric manner to the system equations [see Eq. (28.57)], the following constrained system equations are formed:

$$[M_{ff}]\{\ddot{u}_f\} + [B_{ff}]\{\dot{u}_f\} + [K_{ff}]\{u_f\} = [\Gamma_{fe}]\{F_e\} \tag{28.59}$$

where

$$\begin{aligned} [M_{ff}] &= [G_{gf}]^T[M_{gg}][G_{gf}], \qquad [B_{ff}] = [G_{gf}]^T[B_{gg}][G_{gf}] \\ [K_{ff}] &= [G_{gf}]^T[K_{gg}][G_{gf}], \qquad [\Gamma_{fe}] = [G_{gf}]^T[\Gamma_{ge}] \end{aligned} \tag{28.60}$$

TYPICAL FINITE ELEMENTS

Commonly used finite elements in commercial codes may be divided into two primary classes, namely, (1) elements based on technical theories, and (2) elements based on three-dimensional continuum theory. The first class of elements includes one-dimensional beam elements. Truss and bar elements are special cases of the general beam element. A modern beam element permits modeling of the shear deformation and warping associated with general cross-section geometry. Beam elements, which may describe a straight or curved segment, are typically described in terms of nodal displacements (three linear and three angular displacements) at the two extremities as illustrated in Fig. 28.6.

Also within the family of elements based on technical theories are *shell elements.* Membrane and flat plate elements are special cases of the general shell element. Shell elements are typically of triangular or quadrilateral form with straight or

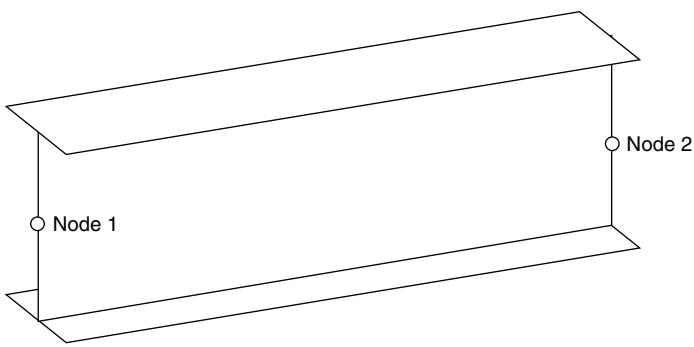

FIGURE 28.6 Typical beam element.

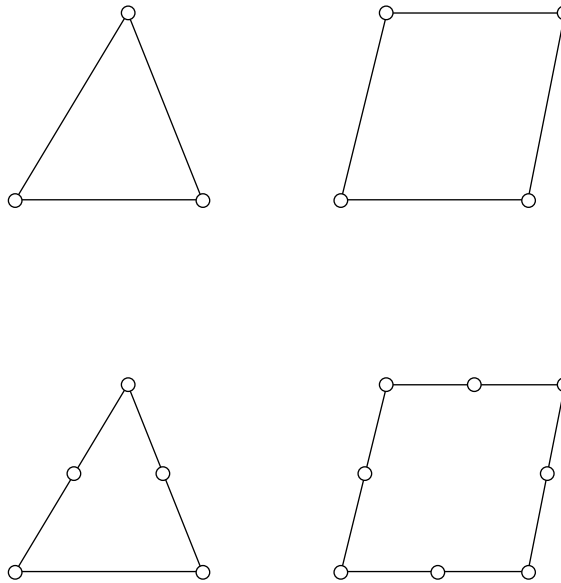

FIGURE 28.7 Typical triangular and quadrilateral shell elements.

curved edges as illustrated in Fig. 28.7. Common H-type shell elements are defined by nodal displacements (three linear and three angular displacements) at the element corners. Shell elements may also be defined in terms of midside nodal displacements. Modern shell elements may include such features as shear deformation, anisotropic elastic materials, and composite layering.

The family of three-dimensional elastic elements includes tetrahedral, pentahedral, wedge, and hexahedral configurations with straight or curved edges as illustrated in Fig. 28.8. H-type continuum elements are defined by nodal displacements (three linear) at the element corners. Three-dimensional H-type elements may also be defined in terms of midside nodal displacements. As in the case of shell elements, anisotropic elastic materials may be employed in element formulations.

Effect of Static Loading—Differential Stiffness. The effective stiffness of structures subjected to static loads may be increased or decreased. For example, the lateral stiffness of a column subjected to axial compression decreases, becoming singular if the fundamental buckling load is imposed. In the case of an inflated balloon, the shell-bending stiffness is almost entirely due to significant membrane tension. In each of these situations, the static load–associated differential stiffness derives from a finite geometric change. Modern commercial finite element codes contain the option to include differential stiffness effects in the model definition.

Fluid-Structure Interaction. Linear dynamic models of oscillating (but otherwise assumed stationary) fluids interacting with elastic structures are employed in vibroacoustics, liquid-filled tank vibratory dynamics, and other applications. One popular approach used to describe the fluid medium employs pressure degrees-of-freedom. On the basis of complementary energy principles,[7] three-dimensional fluid elements (with

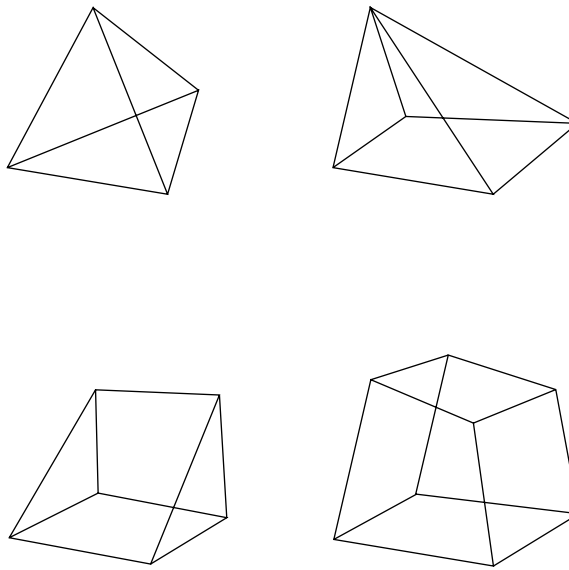

FIGURE 28.8 Typical three-dimensional solid elements.

the geometric configurations illustrated in Fig. 28.8) are defined. The matrix equations describing dynamics of such a fluid interacting with an elastic structure are of the form

$$\begin{bmatrix} C & A^T \\ 0 & M \end{bmatrix} \begin{Bmatrix} \ddot{P} \\ \ddot{u} \end{Bmatrix} + \begin{bmatrix} S & 0 \\ -A & K \end{bmatrix} \begin{Bmatrix} P \\ u \end{Bmatrix} = \begin{bmatrix} \Gamma_Q & 0 \\ 0 & \Gamma_F \end{bmatrix} \begin{Bmatrix} \ddot{Q}_e \\ F_e \end{Bmatrix} \tag{28.61}$$

where $[C]$ is the fluid compliance matrix, $[S]$ is the fluid susceptance matrix (analogous to the inverse of a mass matrix), and $[A]$ is the fluid-structure interface area matrix. The matrix partitions $[\Gamma_Q]$ and $[\Gamma_F]$ are the fluid volumetric source flow $\{\ddot{Q}_e\}$ and the structural applied load $\{F_e\}$ allocation matrices, respectively. The system of equations is unsymmetric due to the fact that it is based on a blend of standard structural displacement and complementary fluid pressure variational principles.

A variety of algebraic manipulations are used to cast the hydroelastic equations in a conventional symmetric form. In many applications involving approximately incompressible (liquid) fluids, the fluid compliance is ignored. The incompressible hydroelastic equations (without source flow excitation) may then be cast in the symmetric form[7]

$$[M + M_f]\{\ddot{u}\} + [K]\{u\} = [\Gamma_F]\{F_e\} \tag{28.62}$$

where the (generally full) fluid mass matrix is

$$[M_f] = [A][S]^{-1}[A]^T \tag{28.63}$$

Specialized constraints are often required to permit the decomposition of the generally singular fluid susceptance matrix.[7] Moreover, specialized eigenvalue analysis procedures are recommended to efficiently deal with the full fluid mass matrix.

For the most general case of a compressible fluid, introduction of the fluid volumetric strain variable

$$\{v\} = [C]\{P\} \tag{28.64}$$

results in the symmetric equation set

$$\begin{bmatrix} S^{-1} & S^{-1}A^T \\ AS^{-1} & M + AS^{-1}A^T \end{bmatrix} \begin{Bmatrix} \ddot{v} \\ \ddot{u} \end{Bmatrix} + \begin{bmatrix} C^{-1} & 0 \\ 0 & K \end{bmatrix} \begin{Bmatrix} v \\ u \end{Bmatrix} = \begin{bmatrix} S^{-1}\Gamma_Q & 0 \\ -AS^{-1}\Gamma_Q & \Gamma_F \end{bmatrix} \begin{Bmatrix} \ddot{Q}_e \\ F_e \end{Bmatrix} \tag{28.65}$$

As for the incompressible, symmetric formulation, a specialized efficient eigenvalue analysis procedure (based on the subspace iteration algorithm[8]) is recommended to efficiently deal with the full hydroelastic mass matrix.

In situations for which the fluid is a lightweight acoustic gas, a decoupling approximation may provide reasonable, approximate dynamic solutions. The approximation assumes that the acoustic medium is driven by a much heavier structure, which is unaffected by fluid interaction. The decoupled approximate dynamic equations are

$$[M]\{\ddot{u}\} + [K]\{u\} = [\Gamma_F]\{F_e\} \tag{28.66}$$

$$[C]\{\ddot{P}\} + [S]\{P\} = -[A^T]\{\ddot{u}\} + [\Gamma_Q]\{\ddot{Q}_e\} \tag{28.67}$$

Uncoupled modal analyses of the structural and acoustic media are used in the computation of the system dynamic response for this approximate formulation.

General Linear System Dynamic Interaction Considerations. In the previous discussion on fluid-structure interaction, a variety of algebraic manipulations, which transform coupled unsymmetric dynamic equations to a conventional symmetric linear formulation, were described. Transformations resulting in symmetric matrix equations, however, are not possible in more general situations involving dynamic interaction.

Linear systems which include complicating effects due to the interaction with general linear subsystems (e.g., control systems, propulsion systems, and perturbed steady fluid flow) are generally appended with nonsymmetric matrix dynamic relationships. The nonconventional linear dynamic formulation incorporates state equations for the interacting subsystem

$$[A_i]\{q_i\} - \{\dot{q}_i\} = [B_i]\{\dot{u}\} + [K_i]\{u\} \tag{28.68}$$

and the forces of interaction with the structural dynamic system

$$[\Gamma_i]\{F_i\} = [\Gamma_i][C_i]\{q_i\} \tag{28.69}$$

where $\{q_i\}$ are subsystem state variables, $[A_i]$ is the subsystem plant matrix, and $[B_i]$, $[K_i]$, and $[C_i]$ are coupling matrices. The complete dynamic system is described by the state equations

$$\begin{bmatrix} -M^{-1}B & -M^{-1}K & M^{-1}\Gamma_i C_i \\ I & 0 & 0 \\ -B_i & -K_i & A_i \end{bmatrix} \begin{Bmatrix} \dot{u} \\ u \\ q_i \end{Bmatrix} - \begin{Bmatrix} \ddot{u} \\ \dot{u} \\ \dot{q}_i \end{Bmatrix} = \begin{bmatrix} -M^{-1}\Gamma_e \\ 0 \\ 0 \end{bmatrix} \{F_e\} \tag{28.70}$$

The above state equations are of the class

$$[A_{\text{sys}}]\{q_{\text{sysi}}\} - \{\dot{q}_{\text{sys}}\} = [\Gamma_{\text{sysi}}]\{F_{\text{sys}}\} \tag{28.71}$$

Nonlinear Dynamic Systems. The most general type of dynamic system includes nonlinear effects, which may be due to large geometric deformations, nonlinear material behavior, stick-slip friction, gapping, and other complicating effects (see Chap. 4). Fortunately, many dynamic systems are approximately linear. A thorough discussion of nonlinear finite element modeling and analysis techniques is beyond the scope of the present discussion. However, two particularly useful classes of models are pointed out herein, namely, (1) linear systems with physically localized nonlinear features, and (2) general nonlinear systems.

A structural dynamic system with physically localized nonlinear features is described by slightly modified linear matrix equations as

$$[M]\{\ddot{u}\} + [B]\{\dot{u}\} + [K]\{u\} = [\Gamma_N]\{F_N(u_N,\dot{u}_N)\} + [\Gamma_F]\{F_e\} \tag{28.72}$$

where $[\Gamma_N]$ is the allocation matrix for nonlinear features and $\{F_N\}$ are the nonlinear forces related to local displacements and velocities. The local displacements and velocities are related to global displacements and velocities as

$$\{u_N\} = [\Gamma_N]^T\{u\}, \{\dot{u}_N\} = [\Gamma_N]^T\{\dot{u}\} \tag{28.73}$$

This type of nonlinear dynamic formulation is useful in that the linear portion of the system may be efficiently treated with modal analysis procedures, to be discussed later.

General situations involving extensively distributed nonlinear behavior are described by equations of the type

$$\{\ddot{u}\} = [M]^{-1}\{F(u,\dot{u},t)\} \tag{28.74}$$

or

$$\begin{Bmatrix} \ddot{u} \\ \dot{u} \end{Bmatrix} = \begin{bmatrix} M^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{Bmatrix} F(u,\dot{u},t) \\ \dot{u} \end{Bmatrix} \tag{28.75}$$

Advanced numerical integration procedures are employed to treat general nonlinear dynamic systems. The procedures fall into two distinct classes, namely, (*a*) implicit methods,[9] and (*b*) explicit methods.[4]

EFFECTIVE MODELING GUIDELINES

CUT-OFF FREQUENCY AND GRID SPACING

In order to develop a relevant dynamic model, general requirements should be addressed based on

TABLE 28.1 Summary of Typical Dynamic Environments

Environment	Chapter or reference
Seismic excitation	Chap. 24
Fluid flow	Chap. 29, Part I
Wind loads	Chap. 29, Part II
Sound	Chap. 29, Part III
Transportation and handling impact	MIL-STD-810E
Transportation and handling vibration	MIL-STD-810E
Shipboard vibration	MIL-STD-167-1

1. Frequency bandwidth $0 < f < f^*$, and intensity (F^*) of anticipated dynamic environments.
2. General characteristics of structural or mechanical components.

Typical dynamic environments are summarized in Table 28.1. Dynamic environments are generally (*a*) harmonic, (*b*) transient, (*c*) impulsive, or (*d*) random. For all categories, the cut-off frequency (f^*) is reliably determined by shock response spectrum analysis (see Chap. 23). The overall intensity level of a dynamic environment is described by a peak amplitude for harmonic, transient, and impulsive events, or by a statistical amplitude (e.g., mean plus a multiple of the standard deviation) for a long-duration random environment (see Chaps. 11 and 22). With the cut-off frequency (f^*) established, the shortest relevant wavelength of a forced vibration for components in a structural assembly may be calculated. For finite element modeling, the quarter wavelength ($L/4$) is of particular interest, since it defines the grid spacing requirement needed to accurately model the dynamics. The guidelines for typical structural components are summarized in Table 28.2.

In addition to the above grid spacing guidelines, the engineer must also consider the limitations associated with beam and plate theories. In particular, if the wavelength-to-thickness ratio (L/h) is less than about 10, a higher-order theory or 3D elasticity modeling should be considered. Moreover, modeling requirements for the capture of stress concentration details may call for a finer grid meshing than suggested by the cut-off frequency. Finally, if the dynamic environment is sufficiently high in amplitude, nonlinear modeling may be required, e.g., if plate deflections are greater than the thickness, h.

MODAL DENSITY AND EFFECTIVENESS OF FINITE ELEMENT MODELS

Finite element modeling is an effective approach for the study of structural and mechanical system dynamics as long as individual vibration modes have sufficient frequency spacing or low modal density. Modal density is typically described as the number of modes within a ⅓ octave frequency band ($f_0 < f < 1.26\,f_0$). When the modal density of a structural component or structural assembly is greater than 10 modes per ⅓ octave band, details of individual vibration modes are not of significance and statistical vibration response characteristics are of primary importance. In such a situation, the Statistical Energy Analysis (SEA) method[10] applies (see Chap. 11). Formulas for modal density[10] as a mathematical derivative, $dn/d\omega$ (n = number of modes, ω = frequency in radians/sec), for typical structural components are summarized in Table 28.3.

TABLE 28.2 Guidelines for Dynamic Finite Element Model Meshing

Component	Mode type	$L/4$	Additional data
String	Lateral	$(\sqrt{T/\rho A})/4f^*$	T = tension, A = area, ρ = mass density
Rod	Axial	$(\sqrt{E/\rho})/4f^*$	E = elastic modulus
Rod	Torsion	$(\sqrt{G/\rho})/4f^*$	G = shear modulus
Beam	Bending	$(\pi/2)(EI/\rho A)^{1/4}/\sqrt{2\pi f^*}$	EI = flexural stiffness
Membrane	Lateral	$(\sqrt{N/\rho h})/4f^*$	N = stress resultant
Plate	Bending	$(\pi/2)(D/\rho h)^{1/4}/\sqrt{2\pi f^*}$	D = plate flexural stiffness, h = plate thickness
3D elastic	Dilatational	$(\sqrt{E/\rho})/4f^*$	
3D elastic	Shear	$(\sqrt{G/\rho})/4f^*$	
Acoustic	Dilatational	$(\sqrt{B/\rho})/4f^*$	B = bulk modulus

DYNAMIC SYSTEM MODELS AND ANALYSIS STRATEGIES

FUNDAMENTAL DYNAMIC FORMULATIONS

finite element dynamic models fall into a variety of classes, which are expressed by the following general equation sets:

1. Linear structural dynamic systems [see Eq. (28.59)]
2. Linear structural dynamic systems interacting with other media [see Eq. (28.70)]
3. Dynamic systems with localized nonlinear features [see Eqs. (28.72) and (28.73)]
4. Dynamic systems with distributed nonlinear features [see Eqs. (28.74) and (28.75)]

TABLE 28.3 Modal Density for Typical Structural Components

Component	Motion	Modal density, $dn/d\omega$	Additional data
String	Lateral	$L/(\pi\sqrt{/T/\rho A})$	T = tension, A = area, ρ = mass density, L = length
Rod	Axial	$L/(\pi\sqrt{E/\rho})$	E = elastic modulus
Rod	Torsion	$L/(\pi\sqrt{G/\rho})$	G = shear modulus
Beam	Bending	$L/(2\pi)(\omega\sqrt{EI/\rho A})^{-1/2}$	EI = flexural stiffness
Membrane	Lateral	$A_s\omega/(2\pi)(N/\rho h)$	N = stress resultant, A_s = surface area
Plate	Bending	$A_s/(4\pi)\sqrt{D/\rho h}$	D = plate flexural stiffness, h = plate thickness
Acoustic	Dilatational	$V_0\omega^2/(2\pi^2)(\sqrt{B/\rho})^3$	B = bulk modulus, V_0 = enclosed volume

The first category represents the type of systems most often dealt with in structural dynamics and mechanical vibration. In the majority of engineering analyses, damping is assumed to be well-distributed in a manner justifying the use of normal mode analysis techniques (see Chaps. 21 and 28, Part I). Systems in the first and second categories having more general damping features may be treated by complex modal analysis procedures (see Chap. 28, Part I). When localized nonlinear features are present, normal or complex mode analysis procedures may also be applied. The final class, namely dynamic systems with distributed nonlinear features, must be treated using numerical integration procedures. When a nonlinear system is subjected to a slowly applied or moderately low frequency environment, implicit numerical integration is often the preferred numerical integration strategy. Alternatively, when the dynamic environment is suddenly applied, high-frequency and/or short-lived explicit numerical integration is often advantageous.

APPLICATION OF NORMAL MODES IN TRANSIENT DYNAMIC ANALYSIS

The homogeneous form for the conventional linear structural dynamic formulation [see Eq. (28.59)], with damping ignored, defines the real eigenvalue problem, that is,

$$[K]\{\Phi_n\} - [M]\{\Phi_n\}\omega_n^2 = \{0\} \tag{28.76}$$

where

$$\{u\} = \{\Phi_n\} \sin(\omega_n t) \tag{28.77}$$

There are as many distinct eigenvectors or modes, $\{\Phi_n\}$, as set degrees-of-freedom for a well-defined undamped dynamic system. The eigenvalues, ω_n^2 (ω_n = natural frequency of mode n), however, are not necessarily all distinct. Individual modes or mode shapes represent displacement patterns of arbitrary amplitude. It is convenient to normalize the mode shapes (to unit modal mass) as follows:

$$\{\Phi_n\}^T[M]\{\Phi_n\} = 1 \tag{28.78}$$

The assembly of all or a truncated set of normalized modes into a modal matrix, $[\Phi]$, defines the (orthonormal) modal transformation

$$\{U\} = [\Phi]\{q\} \tag{28.79}$$

where

$$\begin{aligned} [\Phi]^T[M][\Phi] &= [OR] = [I] = \text{diagonal identity matrix} \\ [\Phi]^T[K][\Phi] &= [\Lambda] = [\omega_n^2] = \text{diagonal eigenvalue matrix} \end{aligned} \tag{28.80}$$

The modal transformation produces the mathematically diagonal matrix

$$[\Phi]^T[B][\Phi] = [2\zeta_n\omega_n] = \text{diagonal modal damping matrix} \tag{28.81}$$

only for special forms of the damping matrix. One such form, known as proportional damping, is

$$[B] = \alpha[M] + \beta[K] \tag{28.82}$$

In reality, proportional damping is a mathematical construction that bears little resemblance to physical reality. It is experimentally observed in many situations, however, that the diagonal modal damping matrix is a valid approximation.

Application of the modal transformation to the dynamic equations [see Eq. (28.59)] results in the uncoupled single degree-of-freedom dynamic equations

$$\ddot{q}_n + 2\zeta_n\omega_n\dot{q}_n + \omega_n^2 q_n = [\Phi_n^T\Gamma]\{F(t)\} = [\Gamma_{qn}]\{F(t)\} = Q_n(t) \tag{28.83}$$

The symbol ζ_n is the critical damping ratio and $[\Gamma_{qn}] = [\Phi_n^T\Gamma]$ is the modal excitation gain array.

The character and content of an individual normal mode, $[\Phi_n]$, is described fundamentally by the geometric distribution of the displacement degrees-of-freedom. Utilizing the mass matrix, $[M]$, the modal momentum distribution is

$$\{P_n\} = [M]\{\Phi_n\} \tag{28.84}$$

and the modal kinetic energy distribution is

$$\{E_n\} = \{P_n\} \otimes \{\Phi_n\} = ([M]\{\Phi_n\}) \otimes \{\Phi_n\} \tag{28.85}$$

where $\otimes$ denotes term-by-term multiplication. The sum of the terms in the modal kinetic energy vector, $\{E_n\}$, is 1.0 when the mode is normalized to unit modal mass.

Internal structural loads and stresses, relative displacements, strains, and other user-defined terms are calculated as recovery variables. In many cases the recovery variables, $\{S\}$, are related to the physical displacements, $\{u\}$, through a load transformation matrix, $[K_S]$, specifically,

$$\{S\} = [K_S]\{u\} \tag{28.86}$$

A modal (displacement-based) load transformation matrix, defined by substitution of the modal transformation, is

$$\{S\} = [\Phi_{KS}]\{q\} \tag{28.87}$$

where

$$[\Phi_{KS}] = [K_S][\Phi]$$

The dynamic response of a structural dynamic system, described in terms of normal modes, is computed as follows:

Step 1. Calculate the modal responses numerically with, for example, the Duhamel integral (see Chap. 8) given by

$$q_n(t) = \int_0^t h_n(t-\tau) Q_n(\tau) d\tau \tag{28.88}$$

where

$$h_n(t-\tau) = \frac{\omega_n}{\sqrt{1-\zeta_n^2}} \; e^{-\zeta_n \omega_n (t-\tau)} \sin\left((\omega_n \sqrt{1-\zeta_n^2})(t-\tau)\right) \tag{28.89}$$

Similar relationships exist for modal velocity and acceleration.

Step 2. Calculate the physical displacement, velocity, and acceleration responses by modal superposition using Eq. (28.79) and calculate loads using Eq. (28.87).

It should be noted that the calculation of modal responses to harmonic and random excitation environments follows strategies paralleling steps 1 and 2. These matters will be discussed at the end of this chapter.

MODAL TRUNCATION

A common practice in structural dynamics analysis is to describe a system response in terms of a truncated set of lowest-frequency modes. The selection of an appropriate truncated mode set is accomplished by a normalized displacement, shock response spectrum analysis (see Chap. 23) of each force component in the excitation environment, $\{F(t)\}$, and establishment of the cut-off frequency, ω^*. All modal responses for systems with a natural frequency, $\omega_n > \omega^*$, will respond quasi-statically. Therefore, the dynamic response will be governed by the truncated set of modes, $[\Phi_L]$, with natural frequencies below ω^*. The remaining set of high-frequency modes is denoted as $[\Phi_H]$. Therefore, the partitioned modal relationships are

$$\{u\} = [\Phi_L]\{q_L\} + [\Phi_H]\{q_H\}$$

$$\{\ddot{q}_L\} + [2\zeta_L \omega_L]\{\dot{q}_L\} + [\omega_L^2]\{q_L\} = [\Phi_L^T \Gamma]\{F(t)\} \tag{28.90}$$

$$[\omega_H^2]\{q_H\} \approx [\Phi_H^T \Gamma]\{F(t)\}$$

Since the high-frequency modal equations are algebraic, the modal transformation becomes

$$\{u\} = [\Phi_L]\{q_L\} + [\Psi_\rho]\{F(t)\} \tag{28.91}$$

where $[\Psi_\rho]$ is the residual flexibility matrix defined as

$$[\Psi_\rho] = [\Phi_H][\omega_H^2]^{-1}[\Phi_H]^T[\Gamma] \tag{28.92}$$

The computation of structural dynamic response employing a truncated set of modes often is inaccurate if the quasi-static response associated with the high-

frequency modes is not accounted for. This being the case, it appears that all modes must be computed as indicated in Eq. (28.92). Such a requirement results in an excessive computational burden for large-order finite element models.

Residual Mode Vectors and Mode Acceleration. The significance of residual flexibility (quasi-static response of high-frequency modes) is well established,[11] as are methods for the efficient definition of residual vectors.[12] The basic definition for residual flexibility, using all of the high-frequency modal vectors, is computationally inefficient for large-order models. Therefore, procedures that do not explicitly require knowledge of the high-frequency modes have been developed.

The most fundamental procedure for deriving residual vectors forms residual shape vectors as the difference between a complete static solution and a static solution based on the low-frequency mode subset. The complete static solution for unit-applied loads, using a shifted stiffness (allowing treatment of an unconstrained structure), is

$$[\Psi_S] = [K + \lambda_S M]^{-1}[\Gamma] \tag{28.93}$$

where λ_S is a small "shift" used for singular stiffness matrices. For nonsingular stiffness, the shift is not required. The corresponding truncated, low-frequency mode static solution is

$$[\Psi_L] = [\Phi_L][\omega_L^2 + \lambda_S]^{-1}[\Phi_L]^T[\Gamma] \tag{28.94}$$

Therefore, the residual vectors are

$$[\Psi_\rho] = [\Psi_S] - [\Psi_L] = [K + \lambda_S M]^{-1}[\Gamma] - [\Phi_L][\omega_L^2 + \lambda_S]^{-1}[\Phi_L]^T[\Gamma] \tag{28.95}$$

Note that the high-frequency modes are not explicitly required in this formulation. Therefore the excessive computational burden for large-order finite element models is mitigated.

An alternative strategy, which automatically compensates for modal truncation, is the mode acceleration method.[13] The basis for this strategy is the substitution of truncated expressions for acceleration and velocity in the system dynamic equations, which results in

$$[K]\{u\} = [\Gamma]\{F\} - [M][\Phi_L]\{\ddot{q}_L\} - [B][\Phi_L]\{\dot{q}_L\} \tag{28.96}$$

In most applications, the term with modal velocity is ignored. The static solution of the above equation, at each time point, produces physical displacements, which include the quasi-static effects of all high-frequency modes.

Load Transformation Matrices. Recovery of structural loads is often organized by a definition of the *load transformation matrices* (LTMs).[14] When residual mode vectors are employed, Eqs. (28.91) and (28.86) are combined to define the displacement LTM relationship

$$\{S\} = [\mathrm{LTM}_q]\{q\} + [\mathrm{LTM}_F]\{F\} \tag{28.97}$$

where

$$[\text{LTM}_q] = [K_S][\Phi_L], [\text{LTM}_F] = [K_S][\Psi_\rho] \tag{28.98}$$

When the mode acceleration method is employed, Eqs. (28.96) and (28.86) are combined to define the mode acceleration LTM relationship

$$\{S\} = [\text{LTM}_A]\{\ddot{q}\} + [\text{LTM}_V]\{\dot{q}\} + [\text{LTM}_{AF}]\{F\} \tag{28.99}$$

where

$$\begin{aligned} [\text{LTM}_A] &= -[K_S][K^{-1}M\Phi_L] \\ [\text{LTM}_V] &= -[K_S][K^{-1}B\Phi_L] \\ [\text{LTM}_F] &= [K_S][K^{-1}\Gamma] \end{aligned} \tag{28.100}$$

In practice, $[\text{LTM}_V]$ is generally ignored. Mode acceleration LTMs are used extensively in the aeronautical and space vehicle industries, while their mode displacement (and residual vector)–based counterpart is rarely applied.

APPLIED LOADS AND ENFORCED MOTIONS

Dynamic excitation environments sometimes are described in terms of specified foundation or boundary motions, for example, in the study of structural dynamic response to seismic excitations (see Chap. 24). In such situations, the physical displacement array is partitioned into two subsets as follows:

$$\{u\} = \begin{Bmatrix} u_i \\ u_b \end{Bmatrix} = \begin{Bmatrix} \text{interior motions} \\ \text{boundary motions} \end{Bmatrix} \tag{28.101}$$

The conventional linear structural dynamic formulation is expressed in partitioned form as

$$\begin{bmatrix} M_{ii} & M_{ib} \\ M_{bi} & M_{bb} \end{bmatrix}\begin{Bmatrix} \ddot{u}_i \\ \ddot{u}_b \end{Bmatrix} + \begin{bmatrix} B_{ii} & B_{ib} \\ B_{bi} & B_{bb} \end{bmatrix}\begin{Bmatrix} \dot{u}_i \\ \dot{u}_b \end{Bmatrix} + \begin{bmatrix} K_{ii} & K_{ib} \\ K_{bi} & K_{bb} \end{bmatrix}\begin{Bmatrix} u_i \\ u_b \end{Bmatrix} = \begin{Bmatrix} F_i \\ F_b \end{Bmatrix} \tag{28.102}$$

Using the partitioned stiffness matrix, the transformation from absolute to relative response displacements is

$$\begin{Bmatrix} u_i \\ u_b \end{Bmatrix} = \begin{bmatrix} I_{ii} & -K_{ii}^{-1}K_{ib} \\ 0_{bi} & I_{bb} \end{bmatrix}\begin{Bmatrix} u_{ir} \\ u_b \end{Bmatrix} = \begin{bmatrix} I_{ii} & \Psi_{ib} \\ 0_{bi} & I_{bb} \end{bmatrix}\begin{Bmatrix} u_{ir} \\ u_b \end{Bmatrix} \tag{28.103}$$

Moreover, this transformation may be expressed in modal form by substituting the lowest-frequency modes associated with the interior eigenvalue problem, which follows the relationships already discussed in Eqs. (28.76) through (28.81), that is,

$$[K_{ii}]\{\Phi_{in}\} = [M_{ii}]\{\Phi_{in}\}\omega_{in}^2, \{u_i\} = [\Phi_i]\{q_i\} \tag{28.104}$$

By combining Eqs. (28.103) and (28.104), the modal reduction transformation is

$$\begin{Bmatrix} u_i \\ u_b \end{Bmatrix} = \begin{bmatrix} \Phi_i & \Psi_{ib} \\ 0_{bi} & I_{bb} \end{bmatrix} \begin{Bmatrix} q_i \\ u_b \end{Bmatrix} \tag{28.105}$$

Substitution of this transformation into the partitioned dynamic equation set, Eq. (28.102), results in

$$\begin{bmatrix} I_{ii} & P_{ib} \\ P_{bi} & M'_{bb} \end{bmatrix} \begin{Bmatrix} \ddot{q}_i \\ \ddot{u}_b \end{Bmatrix} + \begin{bmatrix} 2\zeta_i\omega_i & 0_{ib} \\ 0_{bi} & B'_{bb} \end{bmatrix} \begin{Bmatrix} \dot{q}_i \\ \dot{u}_b \end{Bmatrix} + \begin{bmatrix} \omega_i^2 & 0_{ib} \\ 0_{bi} & K'_{bb} \end{bmatrix} \begin{Bmatrix} q_i \\ u_b \end{Bmatrix} = \begin{Bmatrix} \Phi_i^T F_i \\ \Psi_{ib}^T F_i + F_b \end{Bmatrix} \tag{28.106}$$

The terms in the above equation set have the following significance:

1. $[P_{ib}]$ is the *modal participation factor matrix.* Its terms express the degree of excitation delivered by individual foundation accelerations. Moreover, its transpose describes the degree of foundation reaction loads associated with individual modal accelerations. The term-by-term product $[P_{ib}] \otimes [P_{ib}]$, called the *modal effective mass matrix,* is often used to evaluate the completeness of a truncated set of modes.
2. $[M'_{bb}]$ is the boundary mass matrix. When the boundary motions are sufficient to impose all six rigid body motions (in a statically determinate or redundant manner), this matrix expresses the complete rigid body mass properties of the modeled system.
3. $[K'_{bb}]$ is the boundary stiffness matrix. When the boundary motions are sufficient to impose all six rigid body motions in a statically determinate manner, this matrix is null. If the boundary is statically indeterminate, the boundary stiffness matrix will have six singularities associated with the six rigid body motions. In rare situations, additional singularities will (correctly) be present if the structural system includes mechanisms.
4. Critical evaluation of the properties of $[M'_{bb}]$ and $[K'_{bb}]$ is an effective means for model verification.
5. In most situations, damping is not explicitly modeled. Therefore the boundary damping matrix, $[B'_{bb}]$, will not be computed.

When the dynamic excitation environment consists entirely of prescribed boundary motions, ($\{F_i\} = \{0\}$), Eq. (28.106) may be expressed in the following convenient form:

$$\begin{aligned} \{\ddot{q}_i\} + [2\zeta_i\omega_i]\{\dot{q}_i\} + [\omega_i^2]\{q_i\} &= -[P_{ib}]\{\ddot{u}_b\} \qquad \text{(modal response)} \\ \{F_b\} &= [M'_{bb}]\{\ddot{u}_b\} + [K'_{bb}]\{u_b\} + [P_{bi}]\{\ddot{q}_i\} \qquad \text{(boundary reactions)} \end{aligned} \tag{28.107}$$

The accurate recovery of structural loads is preferably accomplished with the mode acceleration method. The load transformation matrix relationship for this situation takes the following form (ignoring damping):

$$\{S\} = [\text{LTM}_{\ddot{q}}]\{\ddot{q}\} + [\text{LTM}_{\ddot{u}_b}]\{\ddot{u}_b\} + [\text{LTM}_{u_b}]\{u_b\} + [\text{LTM}_{F_i}]\{F_i\} \tag{28.108}$$

The above relationships are commonly used in seismic structural analysis and equipment shock response analysis.

STRATEGIES FOR DEALING WITH LARGE-ORDER MODELS

The capabilities of computer resources and commercial finite element software have continually increased making very large-order (~10^6 degrees-of-freedom or more) finite element models a practical reality. A variety of numerical analysis strategies have been introduced to efficiently deal with these large-order models.

In 1965, what is popularly known as the *Guyan reduction method*[15] was introduced. This method employs a static reduction transformation based on the model stiffness matrix to consistently reduce the mass matrix. By subdividing the model displacements into analysis (a) and omitted (o) subsets, the static reduction transformation is

$$\begin{Bmatrix} u_a \\ u_o \end{Bmatrix} = \begin{bmatrix} I_{aa} \\ -K_{oo}^{-1}K_{oa} \end{bmatrix} \{u_a\} \tag{28.109}$$

By applying this transformation to the dynamic system, an approximate reduced dynamic system for modal analysis is defined as

$$[M_{aa}]\{\ddot{u}_a\} + [K_{aa}]\{u_a\} = \{0\} \tag{28.110}$$

where

$$\begin{aligned} [M_{aa}] &= \begin{bmatrix} I_{aa} \\ -K_{oo}^{-1}K_{oa} \end{bmatrix}^T \begin{bmatrix} M_{aa,o} & M_{ao} \\ M_{oa} & M_{oo} \end{bmatrix} \begin{bmatrix} I_{aa} \\ -K_{oo}^{-1}K_{oa} \end{bmatrix} \\ [K_{aa}] &= \begin{bmatrix} I_{aa} \\ -K_{oo}^{-1}K_{oa} \end{bmatrix}^T \begin{bmatrix} K_{aa,o} & K_{ao} \\ K_{oa} & K_{oo} \end{bmatrix} \begin{bmatrix} I_{aa} \\ -K_{oo}^{-1}K_{oa} \end{bmatrix} \end{aligned} \tag{28.111}$$

The reduced approximate mass and stiffness matrices are generally fully populated, in spite of the fact that the original system matrices are typically quite sparse. The effective selection of an appropriate analysis set, $\{u_a\}$, is a process requiring good physical intuition. A recently introduced two-step procedure[16] automatically identifies an appropriate analysis set. The Guyan reduction method is no longer a favored strategy for dealing with large-order dynamic systems due to the development of powerful numerical procedures for very large-order sparse dynamic systems. It continues to be employed, however, for the definition of *test-analysis models* (TAMs) which are used for modal test planning and test-analysis correlation analyses (see Chap. 41). Numerical procedures, which are currently favored for dealing with modern large-order dynamic system modal (eigenvalue) analyses, are (1) the *Lanczos method*[17] (refined and implemented by many other developers) and (2) *subspace iteration.*[8]

Segmentation of Large-Order Dynamic Systems. Many dynamic systems, such as aircraft, launch vehicle–payload assemblies, spacecraft, and automobiles, naturally lend themselves to substructure segmentation (see Fig. 28.9). Numerical analysis strategies, which exploit substructure segmentation, were originally introduced to improve the computational efficiency of large-order dynamic system analysis. However, advances in numerical analysis of very large-order dynamic systems have reduced the need for substructure segmentation. The enduring utilization of substructure segmentation, especially in the aerospace industry, is a result of the fact that substructure models provide cooperating organizations with a standard means for sharing and integrating subsystem data. It should also be noted that some research efforts in the area of parallel processing are utilizing mature substructure

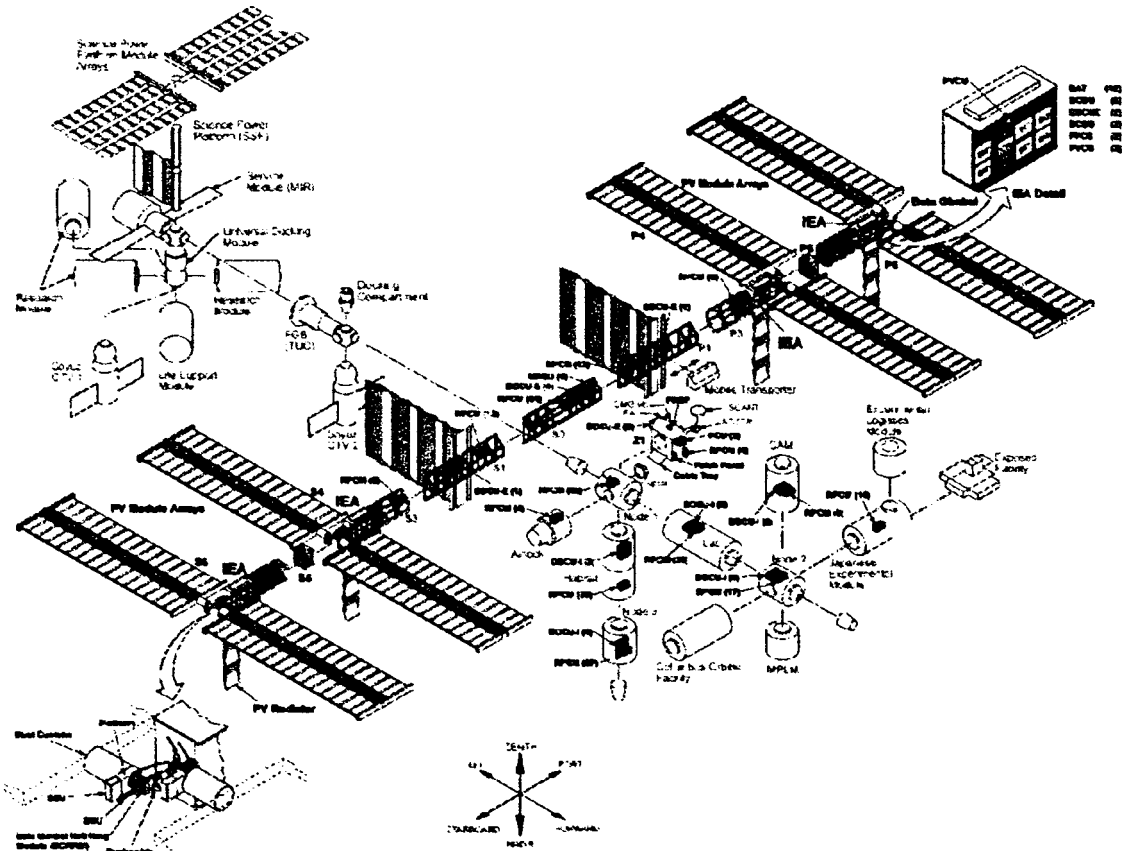

FIGURE 28.9 International space station substructure segmentation.

analysis concepts. Each designated substructure (which also may be termed a superelement) is defined in terms of interior, $\{u_i\}$, and boundary, $\{u_b\}$, displacement subsets. Specific types of modal analysis strategies are employed to reduce or condense the individual substructures to produce *modal components*.

The Craig-Bampton Modal Component. The most popularly employed modal component type, the *Craig-Bampton*[18] (or *Hurty*[19]) component, is defined by Eqs. (28.101) through (28.106) and (28.108). The undamped key dynamic equations describing this component are as follows:

1. The Craig-Bampton reduction transformation (boundary-fixed interior modes and boundary deflection shapes) is identical to Eq. (28.105), that is,

$$\begin{Bmatrix} u_i \\ u_b \end{Bmatrix} = \begin{bmatrix} \Phi_i & \Psi_{ib} \\ 0_{bi} & I_{bb} \end{bmatrix} \begin{Bmatrix} q_i \\ u_b \end{Bmatrix} \tag{28.112}$$

2. The Craig-Bampton mass and stiffness matrices, from Eq. (28.106), are

$$\begin{bmatrix} I_{ii} & P_{ib} \\ P_{bi} & M'_{bb} \end{bmatrix} \begin{Bmatrix} \ddot{q}_i \\ \ddot{u}_b \end{Bmatrix} + \begin{bmatrix} \omega_i^2 & 0_{ib} \\ 0_{bi} & K'_{bb} \end{bmatrix} \begin{Bmatrix} q_i \\ u_b \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{28.113}$$

The MacNeal-Rubin Modal Component. The *MacNeal-Rubin*[12,20] component reduction transformation consists of a truncated set of free boundary modes and quasi-static residual vectors associated with unit loads applied at the boundary degrees-of-freedom. The key dynamic equations describing this component are as follows:

1. The MacNeal-Rubin reduction transformation (boundary-free component modes and residual vectors) is

$$\begin{Bmatrix} u_i \\ u_b \end{Bmatrix} = \begin{bmatrix} \Phi_{ii} & \Psi_{i\rho} \\ \Phi_{bi} & \Psi_{b\rho} \end{bmatrix} \begin{Bmatrix} q_i \\ u_\rho \end{Bmatrix} \tag{28.114}$$

Noting that there are as many residual vectors as boundary degrees-of-freedom, the above transformation may be expressed in terms of the modal and boundary degrees-of-freedom, that is,

$$\begin{Bmatrix} u_i \\ u_b \end{Bmatrix} = \begin{bmatrix} \Phi_{ii} - \Psi_{i\rho}\Psi_{b\rho}^{-1}\Phi_{bi} & \Psi_{i\rho}\Psi_{b\rho}^{-1} \\ 0_{bi} & I_{bb} \end{bmatrix} \begin{Bmatrix} q_i \\ u_b \end{Bmatrix} \tag{28.115}$$

2. The MacNeal-Rubin mass and stiffness matrices: Using the first reduction transformation form [see Eq. (28.114)], the undamped component mode equations are of the form

$$\begin{bmatrix} I_{ii} & 0_{i\rho} \\ 0_{\rho i} & M_{\rho\rho} \end{bmatrix} \begin{Bmatrix} \ddot{q}_i \\ \ddot{u}_\rho \end{Bmatrix} + \begin{bmatrix} \omega_i^2 & 0_{i\rho} \\ 0_{\rho i} & K_{\rho\rho} \end{bmatrix} \begin{Bmatrix} q_i \\ u_\rho \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{28.116}$$

When the second reduction transformation form [see Eq. (28.115)] is employed, the component mode equations are of the fully coupled form

$$\begin{bmatrix} M'_{ii} & M'_{ib} \\ M'_{bi} & M'_{bb} \end{bmatrix} \begin{Bmatrix} \ddot{q}_i \\ \ddot{u}_b \end{Bmatrix} + \begin{bmatrix} K'_{ii} & K'_{ib} \\ K'_{bi} & K'_{bb} \end{bmatrix} \begin{Bmatrix} q_i \\ u_b \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{28.117}$$

The second form of the MacNeal-Rubin mass and stiffness matrices is preferred for automated assembly of modal components.

The Mixed Boundary Modal Component. A more general type of modal component may be defined employing fixed- and free-boundary degree-of-freedom subsets.[21] The reduced component mass and stiffness matrices associated with this component are fully coupled, having a form similar to Eq. (28.117).

Each of the above three modal component types employs a truncated set of subsystem modes. The frequency band, which determines an adequate set of subsystem modes, is related to the base frequency band of the expected dynamic environment. In particular, a generally accepted standard for the modal frequency band defines the cut-off frequency as $1.4f^*$ (see the discussion on *Cut-Off Frequency and Grid Spacing* f^*).

COMPONENT MODE SYNTHESIS STRATEGIES

Two alternative strategies for component mode synthesis are generally accepted in industry. The first strategy views all substructures as appendages. The second alternative views substructures as appendages, which attach to a common main body.

General Method 1: Assembly of Appendage Substructures. The boundary degrees-of-freedom for each component of a complete structural assembly map onto an assembled structure boundary (collector, c) array, that is,

$$\{u_b\} = [T_{bc}]\{u_c\} \tag{28.118}$$

Therefore, each component's reduction transformation is expressed in the assembled (collector) degrees-of-freedom as

$$\begin{Bmatrix} u_i \\ u_b \end{Bmatrix} = \begin{bmatrix} \Psi_{ii} & \Psi_{ib}T_{bc} \\ 0_{bi} & T_{bc} \end{bmatrix} \begin{Bmatrix} q_i \\ u_c \end{Bmatrix} \tag{28.119}$$

where Ψ_{ii} represents the upper left modal transformation partition for the particular modal component type. Application of this transformation to Eq. (28.113) or (28.117) results in

$$\begin{bmatrix} M'_{ii} & M'_{ic} \\ M'_{ci} & M'_{cc} \end{bmatrix} \begin{Bmatrix} \ddot{q}_i \\ \ddot{u}_c \end{Bmatrix} + \begin{bmatrix} K'_{ii} & K'_{ic} \\ K'_{ci} & K'_{cc} \end{bmatrix} \begin{Bmatrix} q_i \\ u_c \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{28.120}$$

The format of the assembled system dynamic equations, shown here for an assembly of three components denoted as 1, 2, and 3, is

$$\begin{bmatrix} M'_{11} & & & M'_{1C} \\ & M'_{22} & & M'_{2C} \\ & & M'_{33} & M'_{3C} \\ M'_{C1} & M'_{C2} & M'_{C3} & M'_{CC} \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \\ \ddot{u}_C \end{Bmatrix} + \begin{bmatrix} K'_{11} & & & K'_{1C} \\ & K'_{22} & & K'_{2C} \\ & & K'_{33} & K'_{3C} \\ K'_{C1} & K'_{C2} & K'_{C3} & K'_{CC} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ u_C \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{28.121}$$

The system normal modes are calculated from the above equation where the final system mode transformation (which decouples the system mass and stiffness matrices) is

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ u_C \end{Bmatrix} = [\Phi_{\text{sys}}]\{q_{\text{sys}}\} \tag{28.122}$$

General Method 2: Attachment of "Appendage" Substructures to a Main Body. This method of component mode synthesis differs from General Method 1 in that all components are not considered appendages. A simple way to view this approach is to first follow General Method 1 for all appendage substructures up to Eq. (28.121). The boundary collector degrees-of-freedom, in this case, correspond to those associated with a main body, which is described in terms of main body mass and stiffness matrices $[M_m]$ and $[K_m]$, respectively. The assembled system of appendages and main body are described as

$$\begin{bmatrix} M'_{11} & & & M'_{1C} \\ & M'_{22} & & M'_{2C} \\ & & M'_{33} & M'_{3C} \\ M'_{C1} & M'_{C2} & M'_{C3} & M'_m \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \\ \ddot{u}_C \end{Bmatrix} + \begin{bmatrix} K'_{11} & & & K'_{1C} \\ & K'_{22} & & K'_{2C} \\ & & K'_{33} & K'_{3C} \\ K'_{C1} & K'_{C2} & K'_{C3} & K'_m \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ u_C \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{28.123}$$

where the boundary-loaded main body mass and stiffness matrices are

$$[M'_m] = [M'_{cc}] + [M_m], [K'_m] = [K'_{cc}] + [K_m] \tag{28.124}$$

The truncated set of modes associated with the boundary-loaded main body define the intermediate transformation

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ u_c \end{Bmatrix} = \begin{bmatrix} I_1 & 0 & 0 & 0 \\ 0 & I_2 & 0 & 0 \\ 0 & 0 & I_3 & 0 \\ 0 & 0 & 0 & \Phi_m \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_m \end{Bmatrix} \tag{28.125}$$

Application of the above transformation to Eq. (28.124) results in the following modal equations for the system

$$\begin{bmatrix} M'_{11} & & & M''_{1C} \\ & M'_{22} & & M''_{2C} \\ & & M'_{33} & M''_{3C} \\ M''_{C1} & M''_{C2} & M''_{C3} & I_m \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \\ \ddot{q}_m \end{Bmatrix} + \begin{bmatrix} K'_{11} & & & K''_{1C} \\ & K'_{22} & & K''_{2C} \\ & & K'_{33} & K''_{3C} \\ K''_{C1} & K''_{C2} & K''_{C3} & \omega_m^2 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_m \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{28.126}$$

If the appendages are all of the Craig-Bampton type, the above equation set reduces to the following *Benfield-Hruda*[22] form

$$\begin{bmatrix} I_1 & & & P_{1C} \\ & I_2 & & P_{2C} \\ & & I_3 & P_{3C} \\ P_{C1} & P_{C2} & P_{C3} & I_m \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \\ \ddot{q}_m \end{Bmatrix} + \begin{bmatrix} \omega_1^2 & & & \\ & \omega_2^2 & & \\ & & \omega_3^2 & \\ & & & \omega_m^2 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_m \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{28.127}$$

The mass coupling terms (P_{1C}, etc.) are modal participation factor matrices, which indicate the relative level of excitation delivered to the appendages by main body modal accelerations. This feature of the Benfield-Hruda form is the primary reason for the enduring popularity of the method. Uncoupled system modes are finally computed from the eigenvalue solution of Eq. (28.127). Component mode synthesis procedures are also applied in multilevel cascades when such a strategy is warranted.

DYNAMIC RESPONSE RESULTING FROM VARIOUS ENVIRONMENTS

The response of linear structural dynamic systems to dynamic environments may be computed by either modal or direct methods. Modal methods tend to be computationally efficient when the required number of system modes addressing the dynamic environment frequency band are significantly smaller than the order of the system finite element model. When this is not the case, direct methods may be more efficient. In addition, when transient environments are brief or impulsive, direct integration may be more efficient than modal strategies. The following discussion provides an overview of strategies for the computation of dynamic response to various environments.

Transient Environments. General relationships detailing the modal method of transient dynamic analysis are presented in the section entitled *Application of Normal Modes in Transient Dynamic Analysis.* Enhancement of the modal solution accuracy with residual vectors and the mode acceleration method was discussed in the sections entitled *Residual Mode Vectors and Mode Acceleration* and *Load Transformation Matrices,* respectively. Direct integration methods employing implicit[9] or explicit[4] numerical strategies may be advantageous when environments are of wide bandwidth and short-lived.

Brief or Impulsive Environments. Brief or impulsive dynamic environments are often described in terms of shock response spectra (see Chap. 23). Peak dynamic responses and structural loads are estimated by employing approximate modal superposition methods utilizing shock response spectra as modal weighting functions.[23] A systematic approach to this process, which incorporates positive and negative spectra and quasi-static residual vectors, is presented in Ref. 11. Approximate shock response spectra–based modal superposition methods are employed in earthquake engineering, equipment (e.g., naval shipboard subsystems) shock survivability prediction, and related applications. This approach is especially appropriate when standard dynamic environments are specified as shock response spectra.

Simple Harmonic Excitation. Computation of the structural dynamic response due to simple harmonic excitation is either an end in itself or a key intermediate step in the computation of the response to random or transient environments. In the case of transient environments, the time-history response may be calculated through application of Fourier transform techniques (see Chap. 23). The applied force and displacement response, respectively, are conveniently expressed in terms of complex exponential functions by

$$\{F\} = F_o(\omega)e^{i\omega t}, \quad \{u\} = \{U(\omega)\}e^{i\omega t}, \quad \{\dot{u}\} = i\omega\{U(\omega)\}e^{i\omega t}, \quad \{\ddot{u}\} = -\omega^2\{U(\omega)\}e^{i\omega t} \tag{28.128}$$

where ω is the forcing frequency in radians per second. Upon substitution of the above relationships into the linear structural dynamic equations [see Eq. (28.59)], the following algebraic matrix equation is defined.

$$[K + i\omega B - \omega^2 M]\{U(\omega)\} = \{\Gamma_F\}F_o(\omega) \tag{28.129}$$

When $F_o(\omega) = 1$, the response quantities are called frequency response functions (see Chap. 21). If the normal mode substitution is employed, the above equation set is diagonalized (assuming modal viscous damping) as follows:

$$\{U(\omega)\} = [\Phi]\{q(\omega)\} \qquad \{\dot{U}(\omega)\} = i\omega[\Phi]\{q(\omega)\} \qquad \{\ddot{U}(\omega)\} = -\omega^2[\Phi]\{q(\omega)\}$$
$$(\omega_n^2 + 2i\zeta_n\omega_n\omega - \omega^2)q_n(\omega) = \{\Phi_n\}^T[\Gamma_F]\{F(\omega)\} \qquad 1 \le n \le n_{\max} \tag{28.130}$$

When the modal method is used, it is recommended that a quasi-static residual vector be employed to mitigate modal truncation errors. This is not required if the direct method, namely, the solution of Eq. (28.129), is employed.

The modal approach to simple harmonic or frequency response analysis is computationally more efficient than the direct method if the number of modes required in a frequency band of interest ($0 \le \omega \le \omega_{\max}$) is much less than the number of finite element model degrees-of-freedom. When this is not the case, the direct method becomes more efficient since the direct solution for $\{U(\omega)\}$ involves decomposition of a sparse coefficient matrix at each forcing frequency.

When the direct solution procedure is employed, it is most convenient to describe modal damping as complex structural damping (see Chap. 2). In this situation the linear, frequency domain, structural dynamic equations are

$$[(1 + i\eta)K + i\omega B_L - \omega^2 M]\{U(\omega)\} = \{\Gamma_F\}F_o(\omega) \tag{28.131}$$

where the well-known approximate equivalence of structural damping loss factor, η, and (viscous) modal damping ratio, ζ, is $\eta \approx 2\zeta$. The advantage associated with structural damping is that the modes need not be explicitly determined in order to account for modal damping effects. The matrix $[B_L]$ is included in the above equation to account for any known discrete viscous damping features.

An important aspect of effective frequency response analysis, regardless of whether the modal or direct method is used, is the selection of a frequency grid for the clear definition of harmonic response peaks. It is generally recommended that solutions be calculated at frequency points capturing at least four points within a modal half-power bandwidth, that is,

$$\Delta\omega = \zeta_n\omega_n/2 = \eta\omega_n \tag{28.132}$$

This guideline suggests a logarithmic frequency grid ($\Delta\omega$ increases with increasing frequency) is desirable.

Random Excitation. In the most common situations, random environments are assumed to be associated with ergodic (see Chap. 1) processes.[24] The computation of structural dynamic response to random excitation, in such a situation, utilizes numerical results from the response to a simple harmonic excitation. If a random environment is imposed at several discrete structural degrees-of-freedom or as several geometric load patterns, the frequency responses associated with the individual loads are denoted as

$$H_{ij}(\omega) = U_i(\omega)/F_{oj}(\omega) \tag{28.133}$$

where these functions are computed either by the modal or direct method. Therefore, the frequency-domain response associated with several excitations is

$$U_i(\omega) = \sum_j H_{ij}(\omega) \cdot F_{oj}(\omega) \tag{28.134}$$

or in matrix form

$$U(\omega) = [H(\omega)]\{F_o(\omega)\} \tag{28.135}$$

Describing the correlated random excitations in terms of the input cross-spectral density matrix, $[G_{FF}(\omega)]$, the response autospectral density is

$$W_{uu}(\omega) = [H(\omega)] \cdot [G_{FF}(\omega)] \cdot [H(\omega)]^{T*} \tag{28.136}$$

where the asterisk $[\,]^{T*}$ denotes the complex conjugate transpose of a matrix. Finally, the mean square of response is calculated as the integral

$$\Psi_u^2 = \overline{u_i(t)^2} = \int_{\omega_1}^{\omega_2} W_{uu}(\omega) d\omega \tag{28.137}$$

In order to assure the accurate computation of a mean-square response, this integral must be evaluated with a frequency grid with refinement consistent with Eq. (28.132). If too coarse a frequency grid is used, the mean-square response may be severely underestimated.

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