CHAPTER 22
CONCEPTS IN VIBRATION DATA ANALYSIS

Allan G. Piersol

INTRODUCTION

Vibration data are usually acquired in the form of continuous electrical (analog) signals generated by transducers (see Chap. 12), where each analog signal represents the instantaneous value of a strain, pressure, force, or motion parameter (displacement, velocity, or acceleration) as a function of time. Such a signal is commonly referred to as a *time-history.* A *sample record* is defined as the time-history representing a single vibration measurement $x(t)$ over a finite duration T. Although sample records are usually acquired in the form of time-histories, any other variable of interest can replace time t as the independent variable for analysis purposes. For example, road roughness data are commonly acquired as sample records of road elevation x versus distance d, that is, $x(d); 0 \le d < D$, where D is the length of the record. However, for clarity, all discussions and equations in this chapter are presented in terms of sample time-history records, where it is understood that any other variable can be substituted for time.

CLASSIFICATIONS OF VIBRATION DATA

The appropriate analysis procedures for vibration environments depend heavily upon certain basic characteristics of the vibration. The most important distinctions are defined in Chap. 1 and illustrated in Fig. 22.1. These definitions may be summarized as follows:

1. A *stationary vibration* is one whose basic properties do not vary with time. Stationary vibrations typically occur when the operating and/or environmental condi-

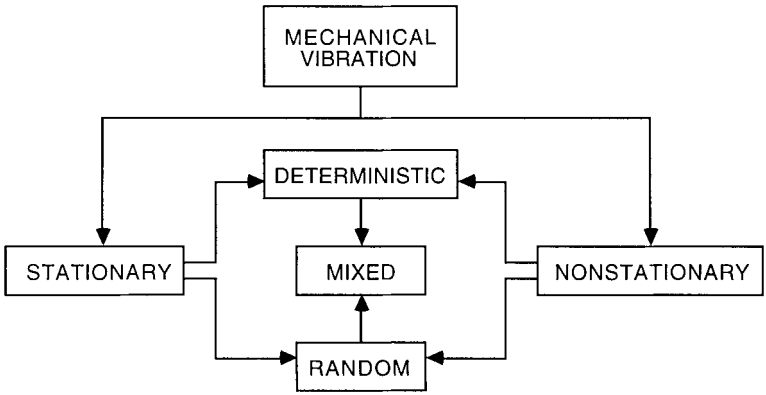

FIGURE 22.1 Classifications of vibration environments.

tions producing the vibration are time invariant. For example, the vibration environment for a motor vehicle driving over a homogeneous road at constant speed and with a constant engine rpm will be stationary.

2. A *nonstationary vibration* is one whose basic properties vary with time, but slowly relative to the lowest frequency of the vibration. For example, the vibration environment for a motor vehicle during acceleration from zero to highway speed will be nonstationary. Those dynamic environments that change rapidly relative to the lowest frequency in the environment are considered transients or shocks, which are addressed in Chap. 23.

3. A *deterministic vibration* is one whose value at any time can be predicted from its value at any other time. It follows that sample records of a deterministic vibration collected repeatedly under similar conditions will have similar time-histories. For example, the vibration environments of rotating machines and reciprocating engines (see Chap. 38) are generally deterministic.

4. A *random vibration* is one whose instantaneous magnitude is not specified at any given time. The instantaneous magnitudes of a random vibration are specified only by probability functions giving the probable fraction of the total time that the magnitude (or some sequence of magnitudes) lies within a specified range. From another viewpoint, a random vibration can be thought of as a single physical realization, $x(t)$, of a random process, which theoretically is described by an ensemble of all possible physical realizations denoted by $\{x(t)\}$.[1] Virtually all stationary random vibrations can be represented by an *ergodic* random process (see Chap. 1), meaning the properties of the random process $\{x(t)\}$ can be described by time averages over a signal sample record $x(t)$. It follows that the sample records of a stationary random vibration collected repeatedly under similar conditions will have time-histories that differ in detail but have the same average properties. For example, the vibrations induced by turbulent flow, wind, and jet noise (see Chaps. 29, Part I; 29, Part II; and 29, Part III) are generally random.

5. A *mixed vibration* is one that includes a combination of deterministic and random components. To some degree, most vibration environments are mixed, although either a deterministic or random component will often dominate.

The next section in this chapter summarizes the quantitative descriptions of vibration environments. This is followed by a discussion of the important preliminary steps in preparing measured vibration data for analysis, and the specific analysis procedures for measured vibration data.

QUANTITATIVE DESCRIPTIONS OF STATIONARY VIBRATIONS

The properties of stationary vibration environments, both deterministic and random, that are of primary interest to engineering applications are defined in Chaps. 11 and 14. Those definitions are now summarized by functional relationships that lead directly to the applied computational algorithms used to compute the desired properties from sample records of measured vibration data.

OVERALL VALUES

The most fundamental descriptions of a stationary vibration with a time-history $x(t)$ are given by overall values. In general, various different overall values might be determined (see Chap. 11), but often the *mean value* μ_x, the *mean-square value* ψ_x^2, and/or the *variance* σ_x^2 are the only overall values of interest. These values for a sample record $x(t)$ with duration T are theoretically given by[1,2]

$$\text{Mean value:} \quad \mu_x = \lim_{T\to\infty} \frac{1}{T}\int_0^T x(t)dt$$

$$\text{Mean-square value:} \quad \psi_x^2 = \lim_{T\to\infty} \frac{1}{T}\int_0^T x^2(t)dt \tag{22.1}$$

$$\text{Variance:} \quad \sigma_x^2 = \lim_{T\to\infty} \frac{1}{T}\int_0^T [x(t) - \mu_x]^2 dt$$

It can be shown[1] that the three quantities defined in Eq. (22.1) are interrelated by

$$\psi_x^2 = \mu_x^2 + \sigma_x^2 \tag{22.2}$$

Hence, a knowledge of any two quantities determines the third. The positive square root of the mean-square value and the variance, ψ_x and σ_x, are called the *root-mean-square* (*rms*) *value* and the *standard deviation,* respectively.

The mean value defines the central tendency (static value) of the vibration, while the standard deviation defines the dispersion of the vibration, each with the same units as the vibration. The rms value is a measure of both the central tendency and dispersion. In many cases, one or more of the following will be true: (*a*) the mean value of the vibration is zero, (*b*) the vibration transducer cannot produce a static (dc) output corresponding to a mean value (e.g., piezoelectric accelerometers), and/or (*c*) a mean value cannot be measured because the data acquisition system is ac coupled, that is, it will not transmit dc. In these cases, the rms value of the vibration is the same as its standard deviation, that is, $\psi_x = \sigma_x$.

FINITE FOURIER TRANSFORMS

Since frequency domain descriptions of vibrations are generally of the greatest engineering value, the Fourier transform plays a major role in both the theoretical definitions of properties and the analysis algorithms for vibration data. The finite Fourier transform of a sample record $x(t)$ is defined as

$$X(f,T) = \int_0^T x(t)e^{-j2\pi ft}dt = \int_0^T x(t)\cos(2\pi ft)dt - j\int_0^T x(t)\sin(2\pi ft)dt \qquad (22.3)$$

where $j = \sqrt{-1}$. Three properties of the definition in Eq. (22.3) should be noted, as follows:

1. The finite Fourier transform is generally a complex number that is defined for both positive and negative frequencies, that is, $X(f,T)$; $-\infty < f < \infty$. However, $X(-f,T) = X^*(f,T)$, where the asterisk denotes the complex conjugate, meaning that values at mathematically negative frequencies are redundant and provide no information beyond that provided by the values at positive frequencies. Since engineers typically think of frequency as a positive value, it is common to present finite Fourier transforms as $2X(f,T)$; $0 < f < \infty$.
2. Fourier transforms are often defined as a function of radial frequency ω in radians/sec, as opposed to cyclical frequency f in Hz, particularly for analytical applications. However, data analysis is usually accomplished in terms of cyclical frequency f, as defined in Eq. (22.3). The two definitions are interrelated by $X(f,T) = 2\pi X(\omega,T)$.
3. The finite Fourier transform $X(f,T)$ is equivalent to the Fourier series of $x(t)$ assumed to have a period T.

STATIONARY DETERMINISTIC VIBRATIONS

Stationary deterministic vibration environments generally fall into one of two categories, namely, periodic vibrations or almost-periodic vibrations.

Periodic Vibrations. Periodic vibrations are those with time-histories that exactly repeat themselves after a time interval T_P, that is, $x(t) = x(t + iT_P)$; $i = 1, 2, 3, \ldots$, where T_P is called the *period* of the vibration. All periodic vibrations can be decomposed into a Fourier series, which consists of a collection of commensurately related sine waves,[1,2] that is,

$$x(t) = a_0 + \sum_k a_k \sin(2\pi kf_1 t + \theta_k) \qquad k = 1, 2, 3, \ldots \qquad (22.4)$$

where a_0 is the mean value, kf_1 is the kth frequency component (harmonic), and a_k and θ_k are the amplitude and phase angle associated with the kth frequency component of the periodic vibration. The $k = 1$ component is called the *fundamental frequency* of the periodic vibration, and is given by $f_1 = 1/T_P$. The magnitude of the frequency components in Eq. (22.4) are given by

$$L_x(f) = \frac{2|X(f,T_P)|}{T_P} \qquad 0 < f \qquad (22.5)$$

where $X(f,T_P)$ is as defined in Eq. (22.3) with $T = T_P$, the period of the vibration. A plot of $L_x(f)$ versus frequency is called a *line spectrum* or a *linear spectrum.* The phase angles, θ_k; $k = 1, 2, 3, \ldots$, are usually ignored, but these phase values should be retained if the time-history is not retained, since both the magnitude and phase values in Eq. (22.4) are required to reconstruct the time-history.

Periodic vibrations are usually produced by the mechanical excitations of rotating machines and reciprocating engines operating with a constant rotational speed. They are also produced by the aerodynamic excitations from large fans and propellers, again operating at a constant rotational speed. An illustration of the time-history and line spectrum for a periodic vibration composed of three harmonic components ($k = 1, 2,$ and 3) is shown in Fig. 22.2.

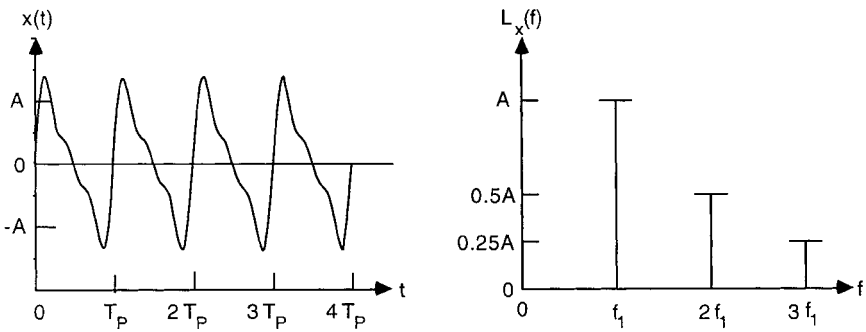

FIGURE 22.2 Time-history and line spectrum for periodic vibration.

Almost-Periodic Vibrations. Although periodic vibrations can be decomposed into a collection of commensurately related sine waves, as given by Eq. (22.4), it does not follow that the sum of two or more independent sinusoidal excitations will produce a periodic vibration. As noted previously in Chap. 1, the sum of such independent sine waves will be periodic only if the ratios of all pairs of frequencies create rational numbers. Those deterministic vibrations that do not have commensurately related frequency components are called *almost-periodic*[1] (also called *quasi-periodic* or *complex*) vibrations. Nevertheless, such vibrations can be described by a line spectrum based upon a relationship similar to Eq. (22.4), except the commensurately related frequencies kf_1 are replaced by independent frequencies f_k; $k = 1, 2, 3, \ldots$. As for periodic vibrations, the magnitude of the frequency components for almost-periodic vibrations can be described by a line spectrum defined in Eq. (22.5), except $T_P \to \infty$.

Almost-periodic vibrations often occur when two or more independent periodic excitations are summed. For example, the vibration produced by two independent rotating machines that are not synchronized or geared together will usually be almost-periodic rather than periodic. An illustration of the time-history and line spectrum for an almost-periodic vibration composed of the sum of two sine waves that are not commensurately related is shown in Fig. 22.3.

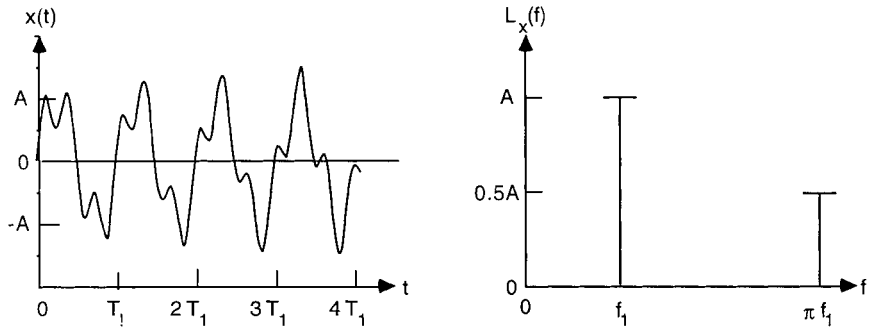

FIGURE 22.3 Time-history and line spectrum for almost-periodic vibration.

STATIONARY RANDOM VIBRATIONS

By definition, random vibrations cannot be described by an explicit mathematical function and, hence, must be described in statistical terms. This can be done (*a*) in the amplitude domain by probability functions, (*b*) in the time domain by correlation functions, and/or (*c*) in the frequency domain by spectral density functions.

Probability Density Functions. From Chap. 11, the *probability density function* of a stationary random vibration $x(t)$ may be defined as

$$p(x) = \lim_{\substack{T \to \infty \\ \Delta x \to 0}} \frac{1}{\Delta x} \frac{T(x,\Delta x)}{T} \tag{22.6}$$

where $T(x,\Delta x)$ is the time that $x(t)$ is within the magnitude interval Δx centered at x during the sample record duration T. The integral of the probability density function between any two magnitudes x_1 and x_2 defines the probability at any future instant that the value of $x(t)$ will fall between x_1 and x_2, that is,

$$\text{Prob}[x_1 < x(t) \le x_2] = \int_{x_1}^{x_2} p(x)dx \tag{22.7}$$

It is noted in Chaps. 11 and 20 that the vibration response of a linear structure to a stationary random excitation tends to be closely approximated by a specific probability density function, namely, the *Gaussian* (*normal*) *probability density function,* which is defined in Eq. (11.14) and plotted in Fig. 22.4. Hence, it is common to omit the computation of probability density functions from the analysis of random vibration data, and to simply assume the probability density function is Gaussian. However, the vibration response of a nonlinear system, even when the excitation is Gaussian, will generally not be Gaussian.[3] For example, the probability density function for the acceleration response to a Gaussian excitation of a single degree-of-freedom system with a stiffness that increases with displacement (often called a hardening spring system as illustrated in Fig. 31.8) is typically as shown in Fig. 22.4. Note that the Gaussian assumption for such data can lead to erroneous conclusions concerning the occurrence of extreme values.

Correlation Functions. *Autocorrelation functions* and *cross-correlation functions* are defined in Eqs. (11.15) through (11.19). They have important theoretical appli-

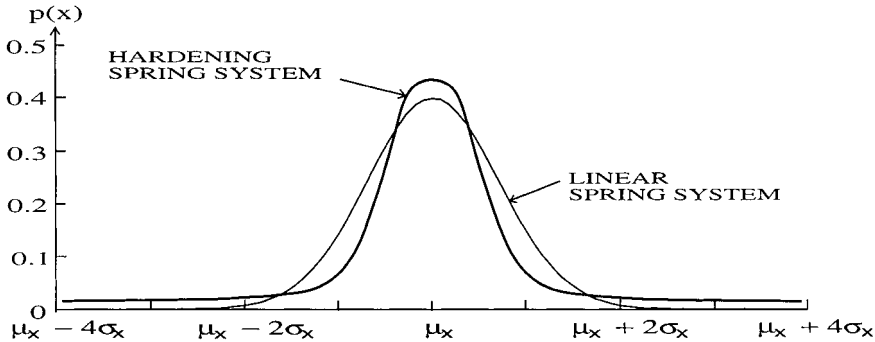

FIGURE 22.4 Probability density functions for the acceleration response of linear and hardening spring systems to stationary random excitation.

cations,[1–6] and a few practical applications to vibration problems.[7] However, the Fourier transform of a correlation function, called a *spectral density function,* is generally of greater interest for practical applications. Furthermore, in those rare cases where a correlation function may be of interest, it can always be computed by taking the inverse Fourier transform of a spectral density function.

Power Spectral Density Functions. The *power spectral density function* (also called the *power spectrum, autospectral density function,* or *autospectrum*) of a stationary random vibration $x(t)$ is often defined as the Fourier transform of the autocorrelation function (see Chap. 11). From a practical viewpoint, however, two equivalent theoretical definitions are more relevant to later data analysis algorithms. First, the power spectrum of $x(t)$ may be defined as[1]

$$W_{xx}(f) = \lim_{T \to \infty} \frac{2}{T} E[|X(f,T)|^2] \qquad f > 0 \tag{22.8}$$

where $E[\ ]$ denotes the expected value of $[\ ]$, which implies an ensemble average, and $X(f,T)$ is defined in Eq. (22.3). Note that the power spectrum $W_{xx}(f)$ in Eq. (22.8) is defined for positive frequencies only, and is often referred to as a *one-sided* spectrum.

The second definition for the power spectrum is more engineering-oriented. Specifically, referring to Fig. 22.5, the random vibration record $x(t)$ is passed through a narrow bandpass filter with a bandwidth B_e and center frequency f to obtain an output $x(f,B_e,t)$. The output is squared and averaged over a duration T to obtain a

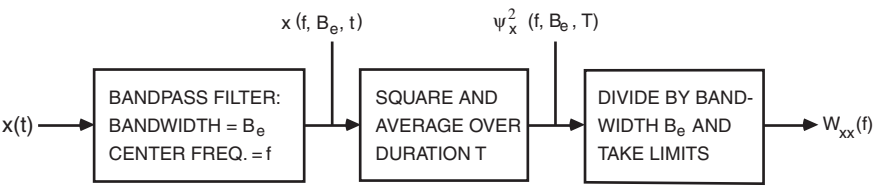

FIGURE 22.5 Definition of power spectrum by filtering, squaring, and averaging operations.

bandwidth-limited mean-square value $\psi_x^2(f,B_e,T)$. Finally, the bandwidth-limited mean-square value is divided by the bandwidth B_e. In the limit as B_e approaches zero and B_eT approaches infinity, the computation illustrated in Fig. 22.5 yields the power spectrum, that is,

$$W_{xx}(f) = \lim_{\substack{T\to\infty \\ B_e\to 0}} \frac{1}{B_eT} \int_0^T x^2(f,B_e,t)dt \qquad f>0 \tag{22.9}$$

It can be shown[1] that Eq. (22.9) produces exactly the same result as Eq. (22.8), as well as the result in Eq. (11.29).

The power spectrum describes the frequency content of the vibration and, hence, is generally the most important and widely used function for engineering applications,[4,7] which are facilitated by three important properties of power spectra, as follows:

1. Given two or more statistically independent vibrations, the power spectrum for the sum of the vibrations is equal to the sum of the power spectra for the individual vibrations, that is,

$$W_{xx}(f) = \sum_i W_{ii}(f) \qquad i=1,2,3,\ldots \tag{22.10}$$

2. The area under the power spectrum between any two frequencies, f_a and f_b, equals the mean-square value of the vibration in the frequency range from f_a to f_b, that is,

$$\psi_x^2(f_a,f_b) = \int_{f_a}^{f_b} W_{xx}(f)df \tag{22.11}$$

3. Given an excitation $x(t)$ to a structural system with a frequency response function $H(f)$ (see Chap. 21), the power spectrum of the response $y(t)$ is given by the product of the power spectrum of the excitation and the squared magnitude of the frequency response function, that is,

$$W_{yy}(f) = |H(f)|^2\, W_{xx}(f) \tag{22.12}$$

Illustrations of the time-histories and autospectra for both wide bandwidth and narrow bandwidth random vibrations are shown in Fig. 22.6.

Cross-Spectral Density Functions. Given two stationary random vibrations $x(t)$ and $y(t)$, the *cross-spectral density function* (also called the *cross-spectrum*) is defined as

$$W_{xy}(f) = \lim_{T\to\infty} \frac{2}{T}\, E[X^*(f,T)Y(f,T)] \qquad f>0 \tag{22.13}$$

where $E[\ ]$ is the expected value of $[\ ]$, which implies an ensemble average, $X^*(f,T)$ is the complex conjugate of the finite Fourier transform of $x(t)$, as defined in Eq. (22.3), and $Y(f)$ is the finite Fourier transform of $y(t)$, as defined in Eq. (22.3) with $y(t)$ replacing $x(t)$.

The cross-spectrum is generally a complex number that measures the linear relationship between two random vibrations as a function of frequency with a possible phase shift between the vibrations. Specifically, the cross-spectrum can be written as

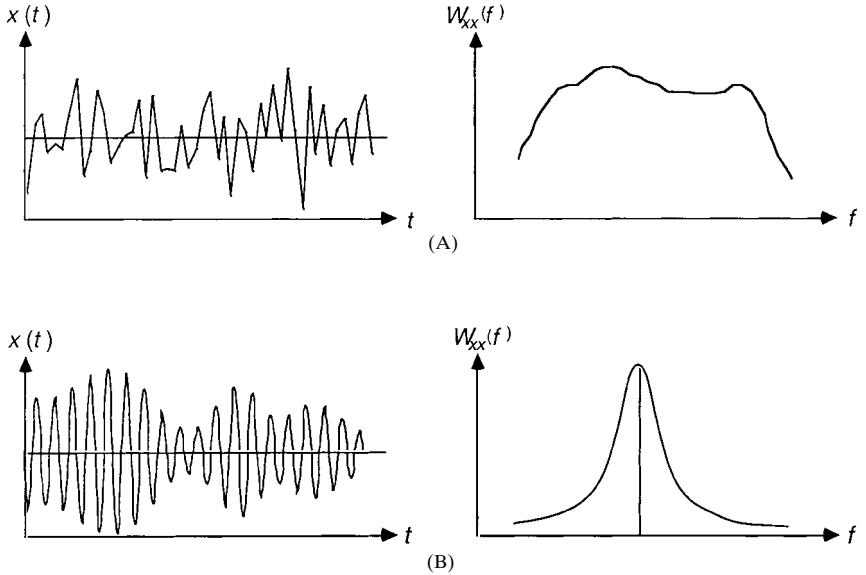

FIGURE 22.6 Time-histories and autospectra for wide-bandwidth (*A*) and narrow-bandwidth (*B*) random vibrations.

$$W_{xy}(f) = |W_{xy}(f)|e^{-j\theta_{xy}(f)} \qquad \theta_{xy}(f) = 2\pi f\tau(f) \tag{22.14}$$

where $\tau(f)$ is the time delay between $x(t)$ and $y(t)$ at frequency f. An important application of the cross-spectrum is as follows. Given a random excitation $x(t)$ to a structure with a frequency response function $H(f)$ (see Chap. 21), the cross-spectrum between the excitation $x(t)$ and the response $y(t)$ is given by the product of the power spectrum of the excitation and the frequency response function, $H(f)$, that is,

$$W_{xy}(f) = H(f)W_{xx}(f) \tag{22.15}$$

Coherence Functions. From Chap. 21, the *coherence function* between two random vibrations $x(t)$ and $y(t)$ is given by

$$\gamma_{xy}^2(f) = \frac{|W_{xy}(f)^2|}{W_{xx}(f)W_{yy}(f)} \qquad f > 0 \tag{22.16}$$

where all terms are as defined in Eqs. (22.8) and (22.13). The coherence function is bounded at all frequencies by zero and unity, where $\gamma_{xy}^2(f) = 0$ means there is no linear relationship between $x(t)$ and $y(t)$ at the frequency f (the two vibrations are uncorrelated) and $\gamma_{xy}^2(f) = 1$ means there is a perfect linear relationship between $x(t)$ and $y(t)$ at the frequency f (one vibration can be exactly predicted from the other). This property leads to an important application of the coherence function. Specifically, given a stationary random vibration $y(t) = x(t) + n(t)$, where $n(t)$ represents extraneous noise, including other vibrations that are not correlated with $x(t)$, then

$$W_{xx}(f) = \gamma_{xy}^2(f)\ W_{yy}(f) \tag{22.17}$$

The result in Eq. (22.17) is referred to as the *coherent output power* relationship.[1] The coherence function is also an important parameter in establishing the statistical sampling errors in various spectral estimates to be discussed later.

Other Functions. There are various other specialized functions that have important applications for certain advanced stationary random data analysis problems, including the following:

1. *Cepstrum functions,* which have important applications to machinery condition monitoring (see Chap. 14).
2. *Hilbert transforms,* which can be used to determine the causality between two measurements[1] and certain properties of modulation processes (Chap. 14).
3. *Conditioned spectral density* and *coherence functions,* which have important applications to the analysis of structural vibration responses to multiple excitations that are partially correlated,[1,7] as well as to the analysis of the vibration responses of nonlinear systems.[3,7]
4. *Higher-order spectral density functions,* such as bi-spectra and tri-spectra, which have applications to the analysis of the vibration responses of nonlinear systems.[3]
5. *Cyclostationary functions,* which have important applications to machinery fault diagnosis procedures.[8]

QUANTITATIVE DESCRIPTIONS OF NONSTATIONARY VIBRATIONS

Unlike stationary vibrations, the properties of nonstationary vibrations must be described as a function of time, which theoretically requires instantaneous averages computed over an ensemble of sample records, $\{x(t)\}$, acquired under statistically similar conditions. In this context, the overall values for stationary vibrations in Eq. (22.1) are given for nonstationary vibrations by

$$\begin{aligned} \text{Mean value:} \quad & \mu_x(t) = E[x(t)] \\ \text{Mean-square value:} \quad & \psi_x^2(t) = E[x^2(t)] \\ \text{Variance:} \quad & \sigma_x^2(t) = E[\{x(t) - \mu_x(t)\}^2] \end{aligned} \tag{22.18}$$

where $E[\ ]$ denotes the expected value of $[\ ]$, which implies an ensemble average. Equation (22.2) applies to the values in Eq. (22.18) at each time t, and the interpretations of these values following Eq. (22.2) apply.

NONSTATIONARY DETERMINISTIC VIBRATIONS

Nonstationary deterministic vibrations are defined here as those vibrations that would be periodic under constant conditions, but where the conditions are time-varying such that the instantaneous magnitude and/or the fundamental frequency of the vibration versus time vary slowly compared to the fundamental frequency of the vibration (often called *phase coherent vibrations*). In other words, the vibration can be described by Eq. (22.4) where the magnitude and phase terms, a_k and θ_k, are replaced by time-varying magnitude and phase terms, $a_k(t)$ and $\theta_k(t)$, and/or the fun-

damental frequency f_1 is replaced by a time-varying fundamental frequency $f_1(t)$, that is,

$$x(t) = a_0(t) + \sum_k a_k(t) \cos [2\pi k f_1(t) + \theta_k(t)] \tag{22.19}$$

A similar nonstationary deterministic vibration is given by Eq. (22.19) with $kf_1(t)$ replaced by $f_k(t)$. Nonstationary deterministic vibrations described by Eq. (22.19) are commonly displayed as a three-dimensional plot of the magnitude of the time-varying coefficients versus time and frequency. Such a plot is often referred to as an *instantaneous line spectrum.* An illustration of the time-history and instantaneous line spectrum for a single instantaneous frequency component with linearly increasing magnitude and frequency is shown in Fig. 22.7.

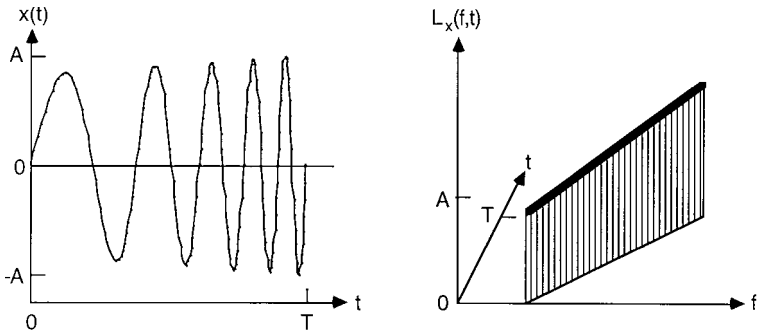

FIGURE 22.7 Time-history and instantaneous line spectrum for sine wave with slowly increasing frequency and amplitude.

Another way to describe the frequency-time characteristics of a nonstationary deterministic vibration is by the *Wigner distribution,* defined as[1,9]

$$WD_{xx}(f,t) = \int_{-\infty}^{\infty} x\left(t - \frac{\tau}{2}\right) x\left(t + \frac{\tau}{2}\right) e^{-j2\pi f\tau}\, d\tau \tag{22.20}$$

The Wigner distribution is similar to the instantaneous power spectrum discussed later in this chapter, and has interesting theoretical properties.[9] However, it often produces negative spectral values, which are difficult to interpret for most engineering applications, and offers few advantages over the instantaneous line spectrum given by Eq. (22.19).

NONSTATIONARY RANDOM VIBRATIONS

There are several theoretical ways to describe nonstationary random data,[1] including generalized spectra defined for two frequency variables that provide rigorous excitation-response relationships, even for time-varying linear systems. From a data analysis viewpoint, however, the most useful theoretical description for nonstationary random vibrations is provided by the *instantaneous power spectral density function* (also called the *instantaneous power spectrum* or *instantaneous autospectrum*). The instantaneous power spectrum is defined by[1,7]

$$W_{xx}(f,t) = \int E\left[x\left(t-\frac{\tau}{2}\right)\ x\left(t+\frac{\tau}{2}\right)\right]e^{-j2\pi f\tau}\,d\tau \tag{22.21}$$

where $E[\ ]$ denotes the expected value of $[\ ]$, which implies an ensemble average. Note that the instantaneous power spectrum is essentially the Wigner distribution defined in Eq. (22.20), except the product of the values of $x(t)$ at two different times is averaged.

Like the Wigner distribution, the instantaneous power spectrum can have negative values at some frequencies and times.[1] For example, let a nonstationary random process be defined as

$$\{x(t)\} = [\cos 2\pi f_0 t]\{u(t)\} \tag{22.22}$$

where $\{u(t)\}$ is a narrow bandwidth stationary random process with a mean value of zero and a standard deviation of unity, and the cosine term is a modulating function. Substituting Eq. (22.22) for Eq. (22.21) yields

$$W_{xx}(f,t) = \frac{1}{4}\ [W_{uu}(f-f_0) + W_{uu}(f+f_0)] + \frac{1}{2}\cos(4\pi f_0 t)W_{uu}(f) \tag{22.23}$$

where $W_{uu}(f)$ is the power spectrum of the stationary component $\{u(t)\}$. The instantaneous power spectrum given by Eq. (22.23) is plotted in Fig. 22.8. Note that the instantaneous power spectrum consists of two stationary components (often called *sidebands*) that are offset in frequency from the center frequency f_1 of $\{u(t)\}$ by plus and minus the modulating frequency f_0, and a time-varying component at the center

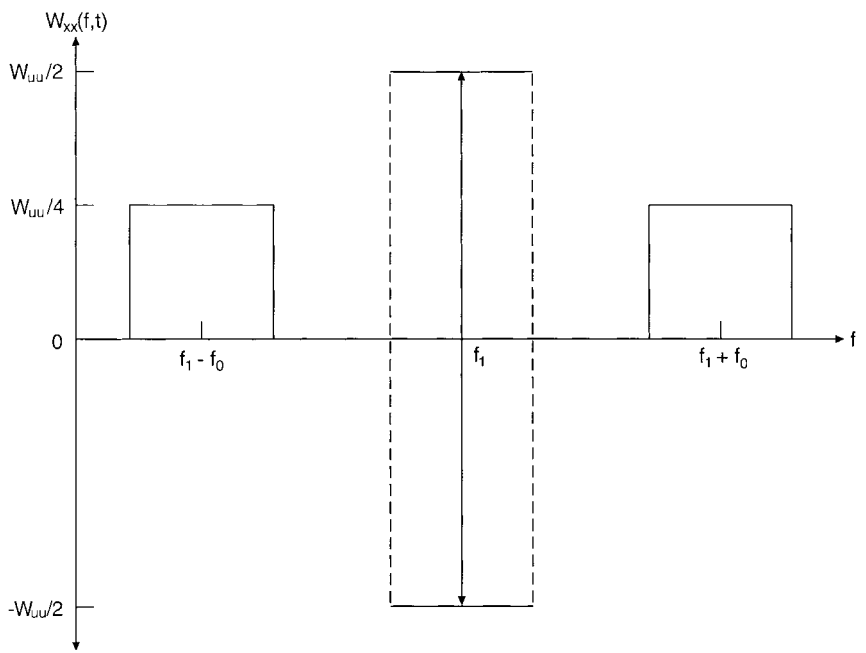

FIGURE 22.8 Instantaneous power spectrum for cosine-modulated, narrow bandwidth random vibration.

frequency f_1 of $\{u(t)\}$ that oscillates between positive and negative values. Further note that for nonstationary vibration environments, as defined in this chapter, a modulating frequency is small compared to the lowest frequency of the stationary component, that is, $f_0 << f_1 - B/2$, where B is the bandwidth of the stationary component. It follows that the stationary and time-varying spectral components of the instantaneous power spectrum will heavily overlap and, hence, eliminate negative spectral values at most times and frequencies.

PRELIMINARY DATA ANALYSIS CONSIDERATIONS

Before the detailed analysis of vibration data is initiated, careful consideration should be given to the following:

1. Final engineering applications of the analyzed data.
2. Stationary sample record durations for the data analysis.
3. Validation and editing of the data.
4. Data storage.
5. Analog-to-digital conversion.

The first two matters should actually be considered prior to the acquisition of the data, but in any case should be carefully reviewed prior to the initiation of the data analysis.

ENGINEERING APPLICATIONS OF DATA ANALYSIS

Numerous possible applications might motivate the acquisition and analysis of vibration data, including the applications in this Handbook summarized in Table 22.1.

TABLE 22.1 Applications of Analyzed Vibration Data

Application	Chapter
Formulation of test criteria and verification of test results	19, 20
Formulation of design criteria	41
Condition monitoring of machinery	16
Modal analysis and testing	21
Assessing the vibration response of structures	24; 29, Part I; 29, Part II; 29, Part III
Assessing the effects of vibration on humans	42
Prediction of structural failures and fatigue damage	11, 34, 35
Calibration of transducers	18
Evaluation of vibration responses of nonlinear systems	4
Balancing of rotating machinery	39, Part I
Input data for mathematical models	11; 28, Part I; 28, Part II

The final application for the data is important in determining which properties of the data should be computed. In most cases, the primary property of interest will be some form of a frequency spectrum. However, there may be applications that require other types of analysis. For example, fatigue damage predictions for random vibration environments generally require some form of amplitude distribution analysis, as detailed in Chaps. 11 and 34. These matters should be thoroughly reviewed prior to initiating data analysis, not only to assure the needed data properties are computed, but also to avoid computing large amounts of unneeded information.

STATIONARY SAMPLE RECORD DURATIONS

It is clear from the descriptions of vibrations in preceding sections that stationary vibrations are much easier to analyze than nonstationary vibrations. It follows that an effort should be made to collect stationary sample records of vibration data for analysis. This is easily accomplished for the vibration data produced by laboratory experiments, since most such experiments are performed under constant conditions that naturally produce stationary results. On the other hand, the vibration data collected from measurements of actual vibration environments are commonly nonstationary. Even in this case, measurement programs can often be designed to produce stationary data for analysis purposes. For example, the vibration environment for a motor vehicle during normal service operations is generally nonstationary. However, if the vehicle is operated over a homogeneous road at constant speed and engine rpm, the resulting vibration levels will be approximately stationary. It follows that the vibration environment of the vehicle under all conditions can be measured and analyzed from a collection of stationary sample records, each representing a specific road condition, vehicle speed, and/or engine rpm, that together cover all the operating conditions for the vehicle. Whether a laboratory experiment or a field experiment, the vibration data acquired for analysis should be forced to be stationary when possible.

Some vibrations are produced by excitations that cannot be forced to be stationary. Examples include the response of structures to wind loads (see Chap. 39, Part I) and ocean waves (see Chap. 39, Part II). Even in these cases, however, it is often possible to identify and select piecewise stationary segments from a long sample record for data analysis purposes. On the other hand, there are some types of vibration environments that are inherently nonstationary, for example, a laboratory vibration test involving a sweep-sine excitation (see Chap. 20) or the vibration environment of a space vehicle during launch. In these situations, some type of nonstationary data analysis procedure must be employed.

DATA VALIDATION AND EDITING

Every effort should be made to acquire accurate vibration data, as outlined in Chap. 15. However, all vibration data collected and stored for later analysis should be validated and, if necessary, edited to remove anomalies prior to analysis. The four most common and serious anomalies in acquired vibration data are as follows:[1,2]

1. *Signal clipping,* which is a limiting on one or both sides of the time-history record, is caused by too high a gain setting on one or more data acquisition instruments.

Severe clipping will reduce the rms value of the data and introduce spurious high-frequency components.

2. *Excessive instrumentation noise,* which appears in the data as broad bandwidth random noise, is caused by too low a gain setting on one or more of the data acquisition instruments. Severe instrumentation noise will sum with random vibration data, increasing the rms value of the data and obscuring the spectral characteristics of the data.
3. *Intermittent noise spikes,* which appear as one or more sharp spikes in the time-history record, are usually caused by a faulty connector in the data acquisition system, but may also occur due to a faulty transmission in telemetry data. Intermittent noise spikes will often severely distort the computed spectral characteristics of the data.
4. *Power-line pickup,* which appears as a sine wave with a frequency of 60 Hz in North America and 50 Hz in many other regions of the world, is caused by faulty shielding and/or grounding of the data acquisition system. Power-line pickup will cause a spectral component in the data at the power-line frequency and, if severe, may saturate one or more of the data acquisition instruments.

These and other anomalies can often be detected by a visual inspection of the time-history record of the measured vibration[1,2] or, for data at frequencies above 50 Hz, by simply listening to the vibration signal with a headset during the data acquisition or the playback of stored sample records. The hearing system of an experienced vibration data analyst can be a powerful detector of data anomalies.

In many cases, the anomalies in acquired vibration data cannot be corrected, but there are important exceptions. For example, power-line pickup can easily be removed from data by interpolation procedures in the frequency domain, assuming the power-line pickup did not saturate a data acquisition instrument and the data do not include an actual periodic component at the power-line frequency.[2] Similarly, intermittent noise spikes can often be removed from the data by interpolation procedures in the time domain.[2] For stationary random vibration data with even the most severe clipping, accurate spectral information can often be recovered by specialized analysis procedures.[1] See the indicated references for details and illustrations.

DATA STORAGE

In some cases, the analysis of sample records of vibration data is accomplished online using real-time data analysis equipment or appropriate online computer programs, but it is more common to input the sample records into some storage medium for later analysis.[2] In either case, since virtually all modern vibration data analysis is accomplished using digital techniques, each analog sample record, $x(t)$; $0 \le t \le T$, is usually converted immediately to a digital sample record, $x(n\Delta t)$; $n = 0, 1, 2, \ldots, (N - 1)$, where Δt is the sampling interval in seconds and $N\Delta t = T$. This translation into a digital format is accomplished using an analog-to-digital converter (see Chaps. 13 and 27). The storage of digital sample records can then be accomplished by directly inputting the data into the random access memory (RAM) or hard disk (HD) on a digital computer or, for long-term storage, a removable storage medium such as a digital tape recorder, digital video disk (DVD), or compact disk/read-only memory (CD/ROM).

ANALOG-TO-DIGITAL CONVERSION

The analog-to-digital (A/D) conversion operation discussed in Chap. 27 introduces two potential errors that must be carefully suppressed, namely, *aliasing errors* and *quantization errors.*

Aliasing Error. The first potential error arises because at least two sample values are needed to define one cycle of a vibration signal. This imposes an upper frequency limit on the digital data given by[1,2]

$$f_A = 1/(2\Delta t) \tag{22.24}$$

where f_A is called the *Nyquist frequency* in Hz. Any signal content in the sample record above the Nyquist frequency f_A will fold back around f_A and sum with the signal content below f_A, often causing a severe distortion of the data referred to as an *aliasing error.* Aliasing can be suppressed by low-pass filtering the analog signals from the transducers before the A/D conversion, where the low-pass filter cut-off frequency is set at $f_c = 0.5\ f_A$ to $0.8\ f_A$, depending on the rolloff rate of the low-pass filter. See Chap. 13 for details.

Quantization Error. The second potential error arises because a continuous analog signal is being converted into a finite set of numbers. This introduces a round-off error commonly referred to as the *quantization error* or *digital noise.* The round-off error is established by the A/D conversion word size, which is the number of binary digits (bits) used to describe each data value. Specifically, a word size of w provides 2^w discrete values (see Chap. 13). Assuming the full range of the A/D converter is used and allowing one bit for sign designation, the peak signal-to-rms noise ratio of the digitized data in dB is given by[1,2]

$$\text{PS/N(dB)} = 6(w - 1) + 10.8 \tag{22.25}$$

The rms signal-to-noise ratio (S/N) for the converter is then given by Eq. (22.25) minus the peak-to-rms value in dB for the signal being converted. For example, if the vibration signal were a sine wave, 3 dB would be subtracted from Eq. (22.25) to obtain the S/N, since the peak-to-rms ratio for a sine wave is $1/\sqrt{2} = -3$ dB. Modern A/D converters typically employ word sizes of $w \geq 12$ bits, corresponding to a PS/N(dB) $\geq$ 76.8 dB. The actual PS/N may be somewhat less than indicated by Eq. (22.25) because of miscellaneous errors in the converter that reduce the effective word size.[1] Nevertheless, if the full range of the converter is used, the digital noise level will usually be sufficiently low for a proper analysis of the data, and often lower than the noise level of the transducer and analog instrumentation preceding the A/D converter. On the other hand, if the full range of the converter is not used, the digital noise could restrict the dynamic range of the analyzed data.

VIBRATION DATA ANALYSIS PROCEDURES

The algorithms for analyzing vibration data evolve directly from the equations for the quantitative descriptions presented earlier, but without the limiting operations. Although usually computed from sample records in the form of a digital time series,

$x(n\Delta t)$; $n = 0, 1, 2 \ldots$, all analysis procedures are presented in terms of both analog equations and digital algorithms for clarity.

THE DISCRETE FOURIER TRANSFORM

Many of the analysis produces for both deterministic and random vibration data require the computation of the finite Fourier transform defined in Eq. (22.3). In digital terms where the sample record $x(t) = x(n\Delta t)$, this finite Fourier transform, often called a *discrete Fourier transform* (DFT), is given by Eq. (14.6) as

$$X(m\Delta f) = \Delta t \sum_{n=0}^{N-1} x(n\Delta t) \exp\,[-j2\pi m\Delta f\, n\Delta t]; m = 0, 1, 2, \ldots, (N-1) \quad (22.26)$$

As discussed in Chap. 14, the DFT can be computed with remarkable efficiency using a *fast Fourier transform* (FFT) algorithm. Note that the DFT defines N discrete frequency values for N discrete time values with an inherent frequency resolution of

$$\Delta f = \frac{1}{N\Delta t} \quad (22.27)$$

However, the Nyquist frequency defined in Eq. (22.24) occurs at $m = (N/2)$. Hence, only the first $[(N/2) + 1]$ frequency components represent unique values; the last $[(N/2) - 1]$ frequency components constitute the redundant values representing the negative frequency components in Eq. (22.3).

PROCEDURES FOR STATIONARY DETERMINISTIC DATA ANALYSIS

The analog equations and digital algorithms for the analysis of stationary deterministic vibration data are summarized in Table 22.2. The hat (^) over the symbol for each computed parameter in Table 22.2 denotes an estimate as opposed to an exact value.

TABLE 22.2 Summary of Algorithms for Stationary Deterministic Vibration Data Analysis

Function	Analog equation	Digital algorithm
Mean value	$\hat{\mu}_x = \frac{1}{T}\int_0^T x(t)dt$	$\hat{\mu}_x = \frac{1}{N}\sum_{n=0}^{N-1} x(n\Delta t)$
Mean-square value	$\hat{\psi}_x^2 = \frac{1}{T}\int_0^T x^2(t)dt$	$\hat{\psi}_x^2 = \frac{1}{N}\sum_{n=0}^{N-1} x^2(n\Delta t)$
Variance	$\hat{\sigma}_x^2 = \frac{1}{T}\int_0^T [x(t) - \hat{\mu}_x]^2$	$\hat{\sigma}_x^2 = \frac{1}{N-1}\sum_{n=0}^{N-1} [x(n\Delta t) - \hat{\mu}_x]^2$
Line spectrum*	$\hat{L}_x(f) = \frac{2}{T}\lvert X(f,T)\rvert; f > 0$	$\hat{L}_x(m\Delta f) = \frac{2}{N\Delta t}\lvert X(m\Delta f)\rvert;$ $m = 1, 2, \ldots, \left(\frac{N}{2} - 1\right)$

*$X(f,T)$ defined in Eq. (22.3), $X(m\Delta f)$ defined in Eq. (22.26).

Overall Values. The mean, mean-square, and variance values for stationary deterministic vibrations are estimated from a sample record using Eq. (22.1) with a finite value for the averaging time T, as shown in Table 22.2. For periodic data, as defined by Eq. (22.4), the averaging time should ideally cover an integer multiple of periods, that is,

$$T = iT_P \qquad i = 1, 2, 3, \ldots \tag{22.28}$$

where T_P is the period of the data. However, since the period of a measured periodic vibration is probably not known prior to estimating its overall values, it is unlikely in practice that the averaging time will comply with Eq. (22.28). This leads to a truncation error that diminishes as the averaging time T increases, and is generally negligible (less than 3 percent) if $T > 10T_P$. For almost-periodic vibration data, there will always be a truncation error, but again it will be negligible if $T > 10T_1$ where T_1 is the period of the lowest frequency in the data.

Line Spectra. The line spectrum for a periodic signal, as defined in Eq. (22.5), will be exact as long as the averaging time complies with Eq. (22.28). Again, compliance with Eq. (22.28) is unlikely in practice for periodic data and is not possible for almost-periodic data, so a line spectrum estimate will generally involve a truncation error. Specifically, rather than a single spectral line at the frequency of each harmonic component of the periodic vibration, as illustrated in Fig. 22.2, spectral lines will occur at all frequencies given by

$$f_k = k/T \qquad k = 1, 2, 3, \ldots \tag{22.29}$$

where $T \neq iT_P$; $i = 1, 2, 3, \ldots$. The largest spectral lines will fall at those frequencies nearest the frequency of the harmonic components of the vibration, but they will underestimate the magnitudes of the harmonic components. Furthermore, the computed spectral lines will fall off about each harmonic frequency as shown in Fig. 14.10. This allows a second type of error, referred to as the *leakage error,* where the magnitude of any one harmonic component can influence the computed values of neighboring harmonic components. Of course, these errors diminish rapidly as $T >> T_P$ for periodic data, or $T >> T_1$ for almost-periodic data where T_1 is the lowest frequency in the data. In addition, sample record-tapering operations (see Chap. 14) or interpolation algorithms[2] can be used to suppress these errors.

PROCEDURES FOR STATIONARY RANDOM DATA ANALYSIS

The analog equations and digital algorithms for the analysis of stationary random vibration data are summarized in Table 22.3. As before, the hat (^) over the symbol for each computed function in Table 22.3 denotes an estimate as opposed to an exact value. Unlike deterministic data, the estimation of parameters for random vibration data will involve statistical sampling errors of two types, namely, (*a*) a random error and (*b*) a bias (systematic) error. It is convenient to present these errors in normalized terms. Specifically, for an estimate $\hat{\phi}$ of a parameter $\phi \neq 0$,

$$\text{Random error: } \varepsilon_r[\hat{\phi}] = \sigma[\hat{\phi}]/\phi \tag{22.30a}$$

$$\text{Bias error: } \varepsilon_b[\hat{\phi}] = (E[\hat{\phi}] - \phi)/\phi \tag{22.30b}$$

TABLE 22.3 Summary of Algorithms for Stationary Random Vibration Data Analysis

Function	Analog equation*	Digital algorithm*
Mean, mean-square, and variance values	Same as in Table 22.2	Same as in Table 22.2
Probability density function	$\hat{p}(x) = \dfrac{T(x,\Delta x)}{\Delta x\, T}$	$\hat{p}(x) = \dfrac{N(x,\Delta x)}{\Delta x\, N}$
Power spectrum, via ensemble averaging	$\hat{W}_{xx}(f) = \dfrac{2}{n_d T}\sum_{i=1}^{n_d} \lvert X_i(f,T)\rvert^2; f>0$	$\hat{W}_{xx}(m\Delta f) = \dfrac{2}{n_d N\Delta t}\sum_{i=1}^{n_d} \lvert X_i(m\Delta f)\rvert^2;$ $m = 1,2,\ldots,\left(\dfrac{N}{2}-1\right)$
Power spectrum via bandpass filtering	$\hat{W}_{xx}(f) = \dfrac{1}{B_e T}\int_0^T x^2(f,B_e,T)dt;$ $f>0$	$\hat{W}_{xx}(m\Delta f) = \dfrac{1}{B_e N\Delta t}\sum_{n=0}^{N-1} x^2(B_e,m\Delta f,n\Delta t);$ $m = 1,2,\ldots,\left(\dfrac{N}{2}-1\right)$
Cross-spectrum via ensemble averaging	$\hat{W}_{xy}(f) = \dfrac{2}{n_d T}\sum_{i=1}^{n_d} X^*(f,T)Y(f,T);$ $f>0$	$\hat{W}_{xy}(m\Delta f) = \dfrac{2}{n_d N\Delta t}\sum_{i=1}^{n_d} \lvert X_i^*(m\Delta f)Y_i(m\Delta f)\rvert;$ $m = 1,2,\ldots,\left(\dfrac{N}{2}-1\right)$
Coherence function	$\hat{\gamma}_{xy}^2(f) = \dfrac{\lvert\hat{W}_{xy}(f)\rvert^2}{\hat{W}_{xx}(f)\hat{W}_{yy}(f)}; f>0$	$\hat{\gamma}_{xy}^2(m\Delta f) = \dfrac{\lvert\hat{W}_{xy}(m\Delta f)\rvert^2}{\hat{W}_{xx}(m\Delta f)\hat{W}_{yy}(m\Delta f)}$ $m = 1,2,\ldots,\left(\dfrac{N}{2}-1\right)$
Frequency response function	$\hat{H}_{xy}(f) = \dfrac{\hat{W}_{xy}(f)}{\hat{W}_{xx}(f)}; f>0$	$\hat{H}_{xy}(m\Delta f) = \dfrac{\hat{W}_{xy}(m\Delta f)}{\hat{W}_{xx}(m\Delta f)};$ $m = 1,2,\ldots,\left(\dfrac{N}{2}-1\right)$
Coherent output power function	$\hat{W}_{xx}(f) = \hat{\gamma}_{xy}(f)\hat{W}_{yy}(f); f>0$	$\hat{W}_{xx}(m\Delta f) = \hat{\gamma}_{xy}^2(m\Delta f)\hat{W}_{yy}(m\Delta f);$ $m = 1,2,\ldots,\left(\dfrac{N}{2}-1\right)$

*$X(f,T)$ defined in Eq. (22.3), $X(m\Delta f)$ defined in Eq. (22.26).

where $\sigma[\hat{\phi}]$ is the standard deviation of the estimate $\hat{\phi}$ and $E[\ ]$ denotes the expected value. For example, if the random error for an estimate $\hat{\phi}$ is $\varepsilon_r[\hat{\phi}] = 0.1$, this means that the estimate $\hat{\phi}$ is a random variable with a standard deviation that is 10 percent of the value of the parameter ϕ being estimated. If the bias error is $\varepsilon_b[\hat{\phi}] = -0.1$, this means the estimate $\hat{\phi}$ is systematically 10 percent less than the value of the parameter ϕ being estimated; note that the bias error can be either positive or negative. The random and bias errors for the various estimates in Table 22.3 are summarized in Table 22.4.

TABLE 22.4 Statistical Sampling Errors for Stationary Random Vibration Data Analysis

Function	Random error	Bias error
Mean value	$\varepsilon_r[\hat{\mu}_x] = \frac{1}{\sqrt{2BT}}\left(\frac{\sigma_x}{\mu_x}\right)$	None
Mean-square value	$\varepsilon_r[\hat{\psi}_x] = \frac{1}{\sqrt{BT}}\left(\frac{\sigma_x^2}{\psi_x^2}\right) + \frac{\sqrt{2}}{\sqrt{BT}}\left(\frac{\mu_x\sigma_x}{\psi_x^2}\right)$	None
Variance	$\varepsilon_r[\hat{\sigma}_x^2] = \frac{1}{\sqrt{BT}}$	None
Probability density function	$\varepsilon_r[\hat{p}(x)] \le \frac{1}{\sqrt{2BT\,\Delta x\, p(x)}}$	$\varepsilon_b[\hat{p}(x)] = \frac{(\Delta x)^2 d^2[p(x)]/dx^2}{24\, p(x)}$
Power spectrum*	$\varepsilon_r[\hat{W}_{xx}(f)] = \frac{1}{\sqrt{n_d}}$	$\varepsilon_b[\hat{W}_{xx}(f)] = -\frac{1}{3}\left(\frac{B_e}{2\zeta f_r}\right)^2$
Cross-spectrum magnitude*	$\varepsilon_r[\lvert\hat{W}_{xy}(f)\rvert] = \frac{1}{\lvert\gamma_{xy}(f)\rvert\sqrt{n_d}}$	$\varepsilon_b[\hat{W}_{xy}(f)] = \frac{B_e d^2\lvert W_{xy}(f)\rvert/df^2}{24\, W_{xy}(f)}$
Cross-spectrum phase*	$\sigma_r[\lvert\hat{\theta}_{xy}(f)\rvert] = \frac{[1-\gamma_{xy}^2(f)]^{1/2}}{\lvert\gamma_{xy}(f)\rvert\sqrt{2n_d}}$	†
Coherence function*	$\varepsilon_r[\lvert\hat{\gamma}_{xy}^2(f)\rvert] = \frac{\sqrt{2}[1-\gamma_{xy}^2(f)]}{\lvert\gamma_{xy}(f)\rvert\sqrt{n_d}}$	$\varepsilon_b[\hat{\gamma}_{xy}^2(f)] = \frac{[1-\gamma_{xy}^2(f)]^2}{\gamma_{xy}^2(f) n_d}$
Frequency response function magnitude*	$\varepsilon_r[\lvert\hat{H}_{xy}(f)\rvert] = \frac{[1-\gamma_{xy}^2(f)]^{1/2}}{\lvert\gamma_{xy}(f)\rvert\sqrt{2n_d}}$	†
Frequency response function phase*	$\sigma_r[\lvert\hat{\phi}_{xy}(f)\rvert] = \frac{[1-\gamma_{xy}^2(f)]^{1/2}}{\lvert\gamma_{xy}(f)\rvert\sqrt{2n_d}}$	†
Coherent output power spectrum*	$\varepsilon_r[\hat{\gamma}_{xy}(f)\hat{W}_{xy}(f)] = \frac{[2-\gamma_{xy}^2(f)]^{1/2}}{\lvert\gamma_{xy}(f)\rvert\sqrt{n_d}}$	†

* n_d can be replaced by $B_e T_r$ when frequency-averaging or digital filtering is employed.

† There are several sources of bias errors,[1,9] but they usually will be small if the bias error for the power spectral density estimate is small.

Overall Values. The mean, mean-square, and variance values for a stationary random vibration are estimated from a sample record using Eq. (22.1) with a finite value for the averaging time T in the same way as for stationary deterministic vibration data, as shown in Table 22.2. For random data, however, truncation errors are replaced by the random errors given in Table 22.4, where it is assumed that the data have a uniform power spectrum over a frequency range with a bandwidth B. Since vibration data rarely have uniform power spectra, the error formulas for the overall values provide only coarse approximations for the random errors to be expected. However, for sample records of adequate duration to provide reasonably accurate power spectra estimates, to be detailed shortly, the random error in overall value estimates will generally be negligible.

Probability Density Functions. The probability density function for a stationary random vibration is estimated from a sample record using Eq. (22.6) with finite values for the averaging time T and an amplitude window width Δx, as shown in Table 22.3. In this table, $T(x,\Delta x)$ is the total time the analog record $x(t)$ falls within the amplitude window Δx centered at x, and $N(x,\Delta x)$ is the total number of values of the digital record $x(n\Delta t)$, $n = 0, 1, 2, \ldots$, that fall within the amplitude window Δx centered at x. Probability density estimates for random vibration data will involve both a bias error and a random error, as summarized in Table 22.4. The bias error is a function of the second derivative of the probability density versus amplitude, which generally is not known prior to the analysis. However, if the probability density function is relatively smooth and the analysis is performed with an amplitude window width of $\Delta x \le 0.1\ \sigma_x$, experience suggests the bias error will typically be less than 5 percent for all values of x. The random error shown in Table 22.4 is only a bound; the actual random error depends on the power spectrum of the data,[1] but in most cases will be small if the sample record duration is adequate to provide accurate power spectra estimates.

Power Spectra. Referring to Table 22.3, there are two basic ways to estimate the power spectrum from a sample record of a stationary random vibration, as follows:

Ensemble Averaging Procedure. The first approach to the estimation of a power spectrum, identified as "ensemble averaging" in Table 22.3, is based upon the definition in Eq. (22.8), and involves the following primary steps:[1]

1. Given a sample record of total duration $T_r = n_d\, N\Delta t$, divide the record into an ensemble of n_d contiguous segments, each of duration $T = N\Delta t$.
2. Apply an appropriate tapering operation to each segment of duration $T = N\Delta t$ to suppress side-lobe leakage (see Chap. 14).
3. Compute a "raw" power spectrum from each segment of duration $T = N\Delta t$, which will produce $N/2$ spectral values at positive frequencies with a resolution of $\Delta f = 1/T = 1/(N\Delta t)$.
4. Average the "raw" power spectra values from the n_d segments to obtain a power spectrum estimate with n_d averages and a frequency resolution of $B_e = \Delta f$.

The averaging operation over the ensemble of n_d estimates simulates the expected value operation in Eq. (22.8), and determines the random error in the estimate given in Table 22.4. The resolution bandwidth $B_e = 1/(N\Delta t)$ determines the maximum bias error in the estimate given in Table 22.4, which for structural vibration data typically occurs at peaks and notches in the power spectrum caused by the resonant response of the structure at a frequency f_r with a damping ratio ζ. See Chap. 14 for details on

the computation of power spectra for random data, including overlapped processing and "zoom" transform procedures.

The ensemble averaging procedure can be replaced by a frequency-averaging procedure, as follows:[1]

1. Given a sample record of total duration $T_r = n_d N\Delta t$, compute a raw power spectrum over the entire duration of the sample record, which will produce $n_d N/2$ spectral estimates at positive frequencies with a resolution of $B_e = 1/T_r = 1/(n_d N\Delta t)$.
2. Divide the frequency range of the spectral components into a collection of contiguous frequency segments, each containing n_d spectral components.
3. Average the spectral components in each of the frequency segments to obtain the power spectrum estimate.

The averaging over n_d spectral components in a frequency segment produces the same random error in Table 22.4 as averaging over n_d raw power spectra estimates in the ensemble-averaging procedure. In addition, for the same values of N and n_d, the frequency resolution is the same as for the ensemble-averaging procedure, meaning the bias error in Table 22.4 is essentially the same. However, the bandwidth for the various frequency segments need not be a constant. Any desired variation in the bandwidth can be introduced, including a bandwidth that increases linearly with its center frequency (commonly referred to as a *constant percentage* frequency resolution).

Bandpass Filtering Procedure. The second approach to the estimation of a power spectrum, identified as "bandpass filtering" in Table 22.3, uses the definition given by Eq. (22.9), as illustrated in Fig. 22.5, and involves the following primary steps:

1. Using digital filters discussed in Chap. 14, pass the sample record of total duration T_r through a collection of contiguous bandpass filters, each centered at frequency f_i with a bandwidth of B_i; $i = 1, 2, 3, \ldots$.
2. Square and average the output of each bandpass filter over the total sample record duration T_r to obtain the mean-square value of the sample record within each filter bandwidth B_i.
3. Divide the mean-square value from each bandpass filter by the filter bandwidth to obtain a power spectrum estimate at the center frequency of each filter.

It can be shown[1] that the product of the bandwidth B_i and the averaging time T_r in the above procedure is equivalent to n_d in the ensemble-averaging procedure. Hence, the bandpass filtering procedure produces the same random and bias errors shown in Table 22.4 with $n_d = B_i T_r$ and $B_i = B_e$.

Optimum Resolution Bandwidth Selections. A common problem in the estimation of power spectra from sample records of stationary random vibration data is the selection of an appropriate resolution bandwidth, $B_e = 1/T = 1/(N\Delta t)$. One approach to this problem is to select that resolution bandwidth that will minimize the total *mean-square error* in the estimate given by

$$\varepsilon^2 = \varepsilon_r^2 + \varepsilon_b^2 \tag{22.31}$$

where ε_r and ε_b are defined in Eq. (22.30). From Table 22.4, the maximum mean-square error for power-spectral density estimates of structural vibration data is approximated by

$$\varepsilon^2[\hat{W}_{xx}(f)] = \frac{1}{B_e T_r} + \frac{1}{9}\left(\frac{B_e}{2\zeta f_r}\right)^4 \tag{22.32}$$

where ζ is the damping ratio of the structure at the resonance frequency f_r. Taking the derivative of Eq. (22.32) with respect to B_e and equating to zero yields the resolution bandwidth that will minimize the mean-square error as

$$B_0(f) = 2\frac{(\zeta f_r)^{4/5}}{T_r^{4/5}} \tag{22.33}$$

Note in Eq. (22.33) that the optimum resolution bandwidth $B_0(f)$ is a function of the $-\frac{1}{5}$ power of the sample record duration, T_r, meaning the optimum resolution bandwidth is relatively insensitive to the sample record duration. Further, the optimum resolution bandwidth $B_0(f)$ is proportional to the $\frac{1}{5}$ power of the product ζf. Assuming all structural resonances have approximately the same damping, this means a constant percentage resolution bandwidth will provide near-optimum results in terms of a minimum mean-square error in the power-spectrum estimate. For example, assume the vibration response of a structure exposed to a random excitation is measured with a total sample record duration of $T_r = 10$ sec. Further assume all resonant modes of the structure have a damping ratio of $\zeta = 0.05$. From Eq. (22.33), the optimum resolution bandwidth for the computation of a power spectrum of the structural vibration is $B_0(f) = 0.115f^{4/5}$. Hence, if the frequency range of the analysis is, say, 10 Hz to 1000 Hz, the optimum resolution bandwidth for the analysis increases from $B_0 = 0.726$ Hz at $f = 10$ Hz [$B_0(f) = 0.0726f$] to $B_0 = 28.9$ Hz at $f = 1000$ Hz [$B_0(f) = 0.0280\ f$]. It follows that a $\frac{1}{12}$ octave bandwidth resolution, which is equivalent to $B_e(f) = 0.058f$, will provide relative good spectral estimates over the frequency range of interest.

Cross-Spectra. Referring to Table 22.3 and Eq. (22.13), the computational approach for estimating the cross-spectrum between two sample records $x(t)$ and $y(t)$ is the same as described for power spectra, except $|X(f)|^2$ is replaced by $X^*(f)Y(f)$. Referring to Table 22.4, the random errors in the magnitude and phase of a cross-spectrum estimate are heavily dependent on the coherence function, as defined in Eq. (22.16). Specifically, if the coherence at any frequency is unity, this means the two sample records, $x(t)$ and $y(t)$, are linearly related and the normalized random error in the estimate is the same as for a power-spectrum estimate. On the other hand, if the coherence is zero, then $x(t)$ and $y(t)$ are unrelated and the normalized random error in any estimate that may be computed is infinite. In practice, the true value of the coherence is not known, so sample estimates of the coherence, to be discussed shortly, would be used in the error formula shown in Table 22.4. There are several sources of bias errors for cross-spectra estimates,[1,10] but these bias errors will generally be minor if the bias errors in the power-spectra estimates for the two sample records are small and there is no major time delay between the two sample records.

Other Spectral Functions. Referring to Table 22.3, the frequency response, coherence, and coherent output power functions defined in Eqs. (22.15) through (22.17) are estimated from sample records using the appropriate estimates for the power spectra, cross-spectra, and coherence functions of the data. From Table 22.4, as for the cross-spectrum, the random errors for estimates of these functions are heavily dependent on the coherence function. There are several sources of bias errors in the estimates of these functions,[1,10] but the bias errors will generally be

minor if the bias errors in the power spectra estimates used to compute the functions is small and there is no major time delay between the two sample records.

PROCEDURES FOR NONSTATIONARY DATA ANALYSIS

As noted earlier, nonstationary vibration data are defined here as those whose basic properties vary slowly relative to the lowest frequency in the vibration time-history. Under this definition, the analog equations and digital algorithms for the analysis of nonstationary vibration data from a single sample record, $x(t)$, are summarized in Table 22.5. These procedures are essentially the same as summarized in Tables 22.2 and 22.3, except the computations are performed over each of a sequence of short, contiguous segments of the data where each segment is sufficiently short not to smooth out the nonstationary characteristics of the data. In other words, given a nonstationary sample record $x(t)$ of total duration T_r the record is assumed to be a sequence of piecewise stationary segments, each covering the interval

$$iT \text{ to } (i+1)T = iN\Delta t \text{ to } (i+1)N\Delta t \qquad i = 0, 1, 2, \ldots \tag{22.34}$$

In many cases, rather than computing the estimates over the contiguous segments defined in Eq. (22.34), a new segment is initiated every digital increment Δt such that each covers the interval

$$i\Delta t \text{ to } (i+N)\Delta t \qquad i = 0, 1, 2, \ldots \tag{22.35}$$

The computation of estimates over the intervals defined in either Eq. (22.34) or (22.35) is commonly referred to as a *running average* (also called a *moving average*). Whether the averaging is performed over segments given by Eq. (22.34) or (22.35), the primary problem is to select an appropriate averaging time, $T = N\Delta t$, for the estimates.

Overall Average Values for Deterministic Data. Referring to Table 22.5, the optimum averaging time for the computation of time-varying mean, mean-square, and variance values for nonstationary deterministic vibration data is bounded as follows. At the lower end, the averaging time must be at least as long as the period for periodic data or the period of the lowest frequency component for almost-periodic data. At the upper end, the averaging time must be sufficiently short to not smooth out the time-varying properties in the data. This selection is usually accomplished by trial-and-error procedures, as illustrated shortly.

Overall Average Values for Random Data. The optimum averaging time for the computation of time-varying mean, mean-square, and variance values for nonstationary random vibration data is bounded as for nonstationary deterministic data with one difference, namely, the computations for random data will involve a statistical sampling (random) error, as summarized in Table 22.4. To minimize these random errors, an averaging time that is as close as feasible to the upper bound noted for deterministic data is desirable. Analytical procedures to select an optimum averaging time that will minimize the mean-square error of the resulting time-varying average value have been formulated,[1] but they require a knowledge of the power spectrum of the data, which is normally not available when overall average values are being estimated. Hence, it is more common to select an averaging time by trial-and-error procedures, as follows:

TABLE 22.5 Summary of Algorithms for Nonstationary Vibration Data Analysis

Function	Analog equation	Digital algorithm
Mean value	$\hat{\mu}_x(t) = \frac{1}{T}\int_{t-T/2}^{t+T/2} x(\tau)d\tau$	$\hat{\mu}_x(k\Delta t) = \frac{1}{N}\sum_{n=k-N/2}^{k+N/2} x(n\Delta t)$
Mean-square value	$\hat{\psi}_x^2(t) = \frac{1}{T}\int_{t-T/2}^{t+T/2} x^2(\tau)d\tau$	$\hat{\psi}_x^2(k\Delta t) = \frac{1}{N}\sum_{n=k-N/2}^{k+N/2} x^2(n\Delta t)$
Variance	$\hat{\sigma}_x^2(t) = \frac{1}{T}\int_{t-T/2}^{t+T/2} [x(\tau)-\hat{\mu}_x]^2 d\tau$	$\hat{\sigma}_x^2(k\Delta t) = \frac{1}{N-1}\sum_{n=k-N/2}^{k+N/2} [x(n\Delta t)-\hat{\mu}_x]^2$
Instantaneous line spectrum via FFT for deterministic data*	$\hat{L}_x(f,t_i) = \frac{2}{T}\lvert X_i(f,T)\rvert; f>0;$ $i = 1,2,3,\ldots;$ and $X_i(f,T)$ computed over $t_i \pm T/2$	$\hat{L}_x(m\Delta f,t_i) = \frac{2}{N_i\Delta t}\lvert X(m\Delta f,t_i)\rvert;$ $m = 1,2,\ldots,[(N/2)-1]$ and $X(m\Delta f,t_i)$ computed over $t_i \pm (N_i\Delta t/2)$
Instantaneous power spectrum via bandpass filtering for random data	$\hat{W}_{xx}(f_k,t_i) = \frac{1}{B_k T_i}\int_{t_i-T_i/2}^{t_i+T_i/2} x^2(f_k,B_k,\tau)d\tau;$ $i = 1,2,3,\ldots,$ and $k = 1,2,3,\ldots$	$\hat{W}_{xx}(f_k,n_i\Delta t) = \frac{1}{B_k N_i\Delta t}\sum_{n=n_i-(N_i/2)}^{n_i+(N_i/2)} x^2(f_k,B_k,n\Delta t);$ $i = 1,2,3,\ldots,$ and $k = 1,2,3,\ldots$

* $X(f,T)$ defined in Eq. (22.3), $X(m\Delta f)$ defined in Eq. (22.26).

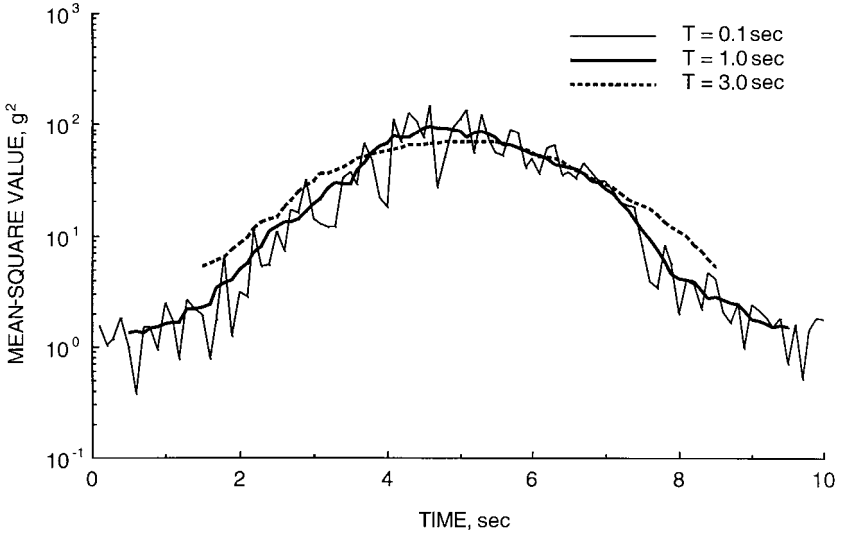

FIGURE 22.9 Running mean-square value estimates for nonstationary vibration data.

1. Compute a running average for the overall value of interest using either Eq. (22.34) or (22.35) with an averaging time, $T = N\Delta t$, that is too short to smooth out the variations with time in the overall value being estimated.
2. Continuously recompute the running average with an increasing averaging time until it is clear that the averaging time is smoothing out variations with time in the overall value being estimated.
3. Choose that averaging time for the analysis that is just short of the averaging time that clearly smoothes out variations with time in the overall value being estimated.

This procedure is illustrated in Fig. 22.9, which shows running average estimates for the time-varying mean-square value of a nonstationary random vibration record computed with averaging times of $T = 0.1$, 1.0, and 3.0 sec. Note that the running average estimates with $T = 0.1$ sec reveal substantial random variations from one estimate to the next, indicative of excessive random estimation errors, while the estimates with $T = 3$ sec reveal a clear smoothing of the nonstationary trend in the data, indicative of an excessive time interval bias error. The averaging time of $T = 1$ sec provides a good compromise between the suppression of random and bias errors in the data analysis.

Instantaneous Line Spectrum for Deterministic Data. Again referring to Table 22.5, the most common way to analyze the spectral characteristics of nonstationary deterministic vibration data is to estimate the instantaneous line spectrum defined in Eq. (22.19) by a sequence of line spectra computed over the time intervals defined in Eq. (22.34) or (22.35). The resulting collection of line spectra is commonly referred to as a *waterfall plot* or a *cascade plot.* An illustration of a waterfall plot computed from a sample record of nonstationary deterministic vibration data is shown in Fig. 14.25.

For a spectral analysis using Fourier transforms, the averaging time $T = N\Delta t$ and the frequency resolution $\Delta f = 1/T = 1/(N\Delta t)$ are obviously interrelated. It follows that there must always be a compromise between these two analysis parameters. On the

one hand, the averaging time must be longer than the period of the lowest instantaneous frequency component in the data at any time covered by the sample record. On the other hand, the frequency resolution must be narrower than the minimum frequency separation of any two instantaneous frequency components in the data at any time covered by the sample record. This compromise will generally be achievable for nonstationary deterministic vibration data that would be periodic if they were stationary. In this case, assuming the maximum period at any time covered by the sample record is T_P, it follows that $\Delta f < 1/T_P$ if $T > T_P$. However, for almost-periodic deterministic vibration data, there may be two spectral components that, at some instant, might be separated by less than $\Delta f = 1/T$ where $T > T_1$. See Chap. 14 for further details on the computation of waterfall plots and other procedures for the analysis of nonstationary deterministic vibration data.

Instantaneous Power Spectra for Random Data. Referring to Table 22.5, the instantaneous power spectrum for nonstationary random vibration data requires an averaging operation to suppress the statistical sampling errors associated with all random data analysis, as suggested by the expected value operation in Eq. (22.21). This averaging operation can be accomplished in several ways. For example, the sample record could be divided into a sequence of contiguous time intervals of appropriate durations and a power spectrum for the data in each time interval computed using the ensemble-averaging procedure detailed in Table 22.3. However, the most straightforward way is to compute the instantaneous power spectrum using the bandpass filtering approach in Fig. 22.5, and computing a running average of the squared output of each bandpass filter centered at frequency f_i with an averaging time of $T_i = N_i\Delta t$; $i = 1, 2, 3, \ldots$, as shown in Table 22.5. For reasons to be discussed shortly, a fixed averaging time of $T = N\Delta t$ commonly can be used for all frequency bands with good results.

A straightforward but time-consuming way to select an appropriate averaging time for an instantaneous power spectrum estimate with bandpass digital filters is to use the trial-and-error procedure illustrated for nonstationary mean-square value estimates in Fig. 22.9, except now the optimum averaging time would have to be determined separately for each frequency resolution bandwidth B_i. On the other hand, the problem can also be approached analytically by determining the averaging time and resolution bandwidth that will minimize the total mean-square error in the estimate, similar to the procedure given in Eqs. (22.31) through (22.33) for stationary random vibration data. In this case, however, there is a third error that must be included in the total mean-square error, namely, a time resolution bias error caused by smoothing through the time-varying values of the instantaneous power spectrum. A maximum value for the normalized time resolution bias error can be approximated by[1]

$$\varepsilon_{bt}[\hat{W}_{xx}(f)] = \frac{T_i^2}{24}\left[\frac{2\pi}{3T_{Di}}\right]^2 \tag{22.36}$$

where T_{Di} is the half-power point duration about the maximum power-spectral density value in the ith resolution bandwidth, that is, the time interval between the time t_1 before and the time t_2 after that time t_m when the maximum value occurs such that $W_{xx}(f_i,t_1) = W_{xx}(f_i,t_2) = W_{xx}(f_i,t_m)/2$. Ideally, this time duration should be determined individually for each frequency resolution bandwidth, but it will often suffice to use a single value for T_D determined from the estimate for the time-varying mean-square value of the data, as illustrated in Fig. 22.9. Adding Eq. (22.36) with a constant value T_D to Eq. (22.32), taking partial derivatives with respect to T and B_e, equating to zero, and solving the two simultaneous equations, yields the optimum averaging time and resolution bandwidth as[1]

$$T_0(f) = 1.31\ T_D^{5/6}/(\zeta f)^{1/6} \qquad B_0(f) = 1.94(\zeta f)^{5/6}/T_D^{1/6} \tag{22.37}$$

Note in Eq. (22.37) that the averaging time $T_0(f)$ is a function of the $-\frac{1}{6}$ power of the product ζf, while the resolution bandwidth $B_0(f)$ is a function of the $\frac{5}{6}$ power of the product ζf. Assuming all structural resonances have approximately the same damping ratio, this means a fixed averaging time and a constant percentage resolution bandwidth will provide near-optimum results in terms of a minimum mean-square error in the instantaneous power-spectrum estimate. For example, assume the measured vibration response of a structure exposed to a nonstationary random excitation has a time-varying mean-square value similar to that shown in Fig. 22.9, where the half-power duration is about $T_D \approx 2.5$ sec. Further assume all resonant modes of the structure have a damping ratio of $\zeta = 0.05$. From Eq. (22.37), the optimum averaging time for the computation of an instantaneous power spectrum of the nonstationary structural vibration is $T_0(f) = 4.63f^{-1/6}$, while the optimum resolution bandwidth is $B_0(f) = 0.137f^{5/6}$. Hence, if the frequency range of the analysis is, say, 10 Hz to 1000 Hz, the optimum averaging time for the analysis decreases from $T_0 = 3.15$ sec at 10 Hz to $T_0 = 1.46$ sec at 1000 Hz, while the optimum resolution bandwidth increases from $B_0 = 0.933$ Hz at $f = 10$ Hz [$B_0(f) = 0.0933f$] to $B_0 = 43.3$ Hz at $f = 1000$ Hz [$B_0(f) = 0.0433f$]. It follows that an analysis with a fixed averaging time of about $T = 2.5$ sec and a constant percentage resolution bandwidth of $\frac{1}{12}$ octave, which is equivalent to $B_e(f) = 0.058f$, will provide relative good instantaneous spectral estimates over the entire frequency range of interest. See Ref. 1 for details on specialized procedures for analyzing special cases of nonstationary random vibration data.

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