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Torsional Vibrations

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24.1. Introduction

We have already discussed in the previous chapter that when the particles of a shaft or disc move in a circle about the axis of a shaft, then the vibrations are known as ***torsional vibrations***. In this case, the shaft is twisted and untwisted alternately and torsional shear stresses are induced in the shaft. In this chapter, we shall now discuss the frequency of torsional vibrations of various systems.

24.2. Natural Frequency of Free Torsional Vibrations

Consider a shaft of negligible mass whose one end is fixed and the other end carrying a disc as shown in Fig. 24.1.

Let θ = Angular displacement of the shaft from mean position after time t in radians,

m = Mass of disc in kg,

I = Mass moment of inertia of disc in kg-m^2 = $m.k^2$,

k = Radius of gyration in metres,

q = Torsional stiffness of the shaft in N-m.

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$\therefore$ Restoring force $= q.\theta$ ... **(*i*)**

and accelerating force $= I \times \dfrac{d^2\theta}{dt^2}$ ... **(*ii*)**

Equating equations **(*i*)** and **(*ii*)**, the equation of motion is

$$I \times \frac{d^2\theta}{dt^2} = -q.\theta$$

or
$$I \times \frac{d^2\theta}{dt^2} + q.\theta = 0$$

$\therefore$
$$\frac{d^2\theta}{dt^2} + \frac{q}{I} \times \theta = 0 \quad \text{. . . (iii)}$$

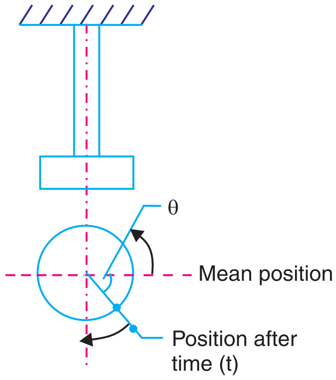


Fig 24.1. Natural frequency of free torsional vibrations.

The fundamental equation of the simple harmonic motion is

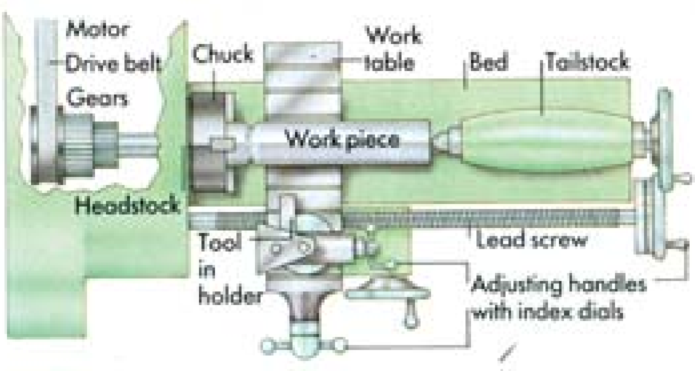
$$\frac{d^2\theta}{dt^2} + \omega^2 . x = 0 \quad \text{. . . (iv)}$$

Comparing equations **(*iii*)** and **(*iv*)**,

$$\omega = \sqrt{\frac{q}{I}}$$

$\therefore$ Time period, $t_p = \dfrac{2\pi}{\omega} = 2\pi \sqrt{\dfrac{I}{q}}$

and natural frequency , $f_n = \dfrac{1}{t_p} = \dfrac{1}{2\pi}\sqrt{\dfrac{q}{I}}$



Note : This picture is given as additional information and is not a direct example of the current chapter.

A modern lathe can create an artificial hip joint from information fed into it by a computer. Accurate drawings of the joint are first made on a computer and the information about the dimensions fed is directly into the lathe.

Note : The value of the torsional stiffness q may be obtained from the torsion equation,

$$\frac{T}{J} = \frac{C.\theta}{l} \quad \text{or} \quad \frac{T}{\theta} = \frac{C.J}{l}$$

$$\therefore \quad q = \frac{C.J}{l} \quad \ldots \left(\because \frac{T}{\theta} = q\right)$$

where C = Modulus of rigidity for the shaft material,

J = Polar moment of inertia of the shaft cross-section,

$= \frac{\pi}{32} d^4$; d is the diameter of the shaft, and

l = Length of the shaft.

Example 24.1. *A shaft of 100 mm diameter and 1 metre long has one of its end fixed and the other end carries a disc of mass 500 kg at a radius of gyration of 450 mm. The modulus of rigidity for the shaft material is 80 GN/m². Determine the frequency of torsional vibrations.*

Solution. Given : d = 100 mm = 0.1 m ; l = 1 m ; m = 500 kg ; k = 450 mm = 0.45 m ; C = 80 GN/m² = 80 × 10^9 N/m²

We know that polar moment of inertia of the shaft,

$$J = \frac{\pi}{32} \times d^4 = \frac{\pi}{32}(0.1)^4 = 9.82 \times 10^{-6} \text{ m}^4$$

∴ Torsional stiffness of the shaft,

$$q = \frac{C.J}{l} = \frac{80 \times 10^9 \times 9.82 \times 10^{-6}}{1} = 785.6 \times 10^3 \text{ N-m}$$

We know that mass moment of inertia of the shaft,

$$I = m.k^2 = 500(0.45)^2 = 101.25 \text{ kg-m}^2$$

∴ Frequency of torsional vibrations,

$$f_n = \frac{1}{2\pi}\sqrt{\frac{q}{I}} = \frac{1}{2\pi}\sqrt{\frac{785.6 \times 10^3}{101.25}} = \frac{88.1}{2\pi} = 14 \text{ Hz}$$ **Ans.**

Example 24.2. *A flywheel is mounted on a vertical shaft as shown in Fig 24.2. The both ends of a shaft are fixed and its diameter is 50 mm. The flywheel has a mass of 500 kg and its radius of gyration is 0.5 m. Find the natural frequency of torsional vibrations, if the modulus of rigidity for the shaft material is 80 GN/m².*

Solution. Given : d = 50 mm = 0.05 m ; m = 500 kg ; k = 0.5m; G = 80 GN/m² = 84 × 10^9 N/m²

We know that polar moment of inertia of the shaft,

$$J = \frac{\pi}{32} \times d^4 = \frac{\pi}{32}(0.05)^4 \text{ m}^4$$

$$= 0.6 \times 10^{-6} \text{ m}^4$$

∴ Torsional stiffness of the shaft for length l_1,

$$q_1 = \frac{C.J}{l_1} = \frac{84 \times 10^9 \times 0.6 \times 10^{-6}}{0.9}$$

= 56 × 10^3 N-m

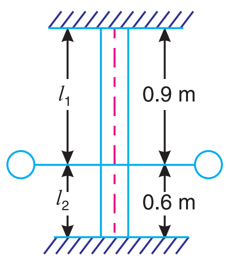


Fig 24.2

Similarly torsional stiffness of the shaft for length l_2,

$$q_2 = \frac{C.J}{l_2} = \frac{84 \times 10^9 \times 0.6 \times 10^{-6}}{0.6} = 84 \times 10^3 \text{ N-m}$$

∴ Total torsional stiffness of the shaft,

$$q = q_1 + q_2 = 56 \times 10^3 + 84 \times 10^3 = 140 \times 10^3 \text{ N-m}$$

We know that mass moment of inertia of the flywheel,

$$I = m.k^2 = 500\,(0.5)^2 = 125 \text{ kg-m}^2$$

∴ Natural frequency of torsional vibration,

$$f_n = \frac{1}{2\pi}\sqrt{\frac{q}{I}} = \frac{1}{2\pi}\sqrt{\frac{140 \times 10^3}{125}} = \frac{33.5}{2\pi} = 5.32 \text{ Hz}$$ **Ans.**

24.3. Effect of Inertia of the Constraint on Torsional Vibrations

Consider a constraint *i.e.* shaft whose one end is fixed and the other end free, as shown in Fig.24.3.

Let ω = Angular velocity of free end,

m = Mass of constraint for unit length,

l = Length of constraint,

m_C = Total mass of constraint = $m.l$,

k = Radius of gyration of constraint,

I_C = Total mass moment of inertia of constraint

$= m_C.k^2 = m.l.k^2.$

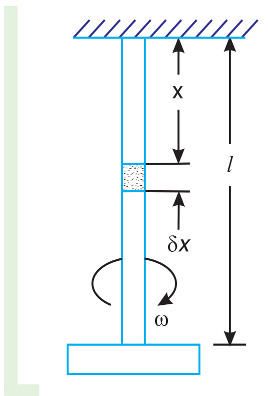


Fig 24.3. Effect of inertia of the constraint on torsional vibrations.

Consider a small element at a distance x from the fixed end and of length δx. Therefore,

Mass moment of inertia of the element

$$= (m.\delta x)\,k^2 = \frac{\delta x}{l} \times m.k^2.l$$

... (Dividing and multiplying by l)

$$= \frac{\delta x}{l} \times I_C$$

... (Substituting $m.k^2.l = I_C$)

and angular velocity of the element

$$= \frac{\omega}{l} \times x$$

Kinetic energy possessed by the element

$$= \frac{1}{2}\left(\frac{\delta x}{l} \times I_C\right)\left(\frac{\omega}{l} \times x\right)^2 = \frac{I_C.\omega^2.x^2}{2\,l^3} \times \delta x$$

∴ Total kinetic energy of the constraint

$$= \int_0^l \frac{I_C.\omega^2}{2l^3} \times x^2\, dx = \frac{I_C.\omega^2}{2l^3}\left[\frac{x^3}{3}\right]_0^l = \frac{1}{2}\left(\frac{I_C}{3}\right)\omega^2 \qquad \ldots (i)$$

If a mass whose mass moment of inertia is equal to $I_C / 3$ is placed at the free end and the constraint is assumed to be of negligible mass, then

Total kinetic energy of the constraint

$$\frac{1}{2}\quad\frac{I_C}{3}\quad{}^{2}$$ . . . [Same as equation (*i*)]

When loads are applied on the above two pulleys, the shaft is subject to torsional vibration

Hence the two systems are dynamically same. Therefore the inertia of the constraint may be allowed for by adding $I_C / 3$ to the mass moment of inertia I of the disc at the free end.

From the above discussion, we find that when the mass moment of inertia of the constraint I_C and the mass moment of inertia of the disc I are known, then natural frequency of vibration,

$$f_n \quad \frac{1}{2}\sqrt{\frac{q}{I \quad I_C/3}}$$

24.4. Free Torsional Vibrations of a Single Rotor System

We have already discussed that for a shaft fixed at one end and carrying a rotor at the free end as shown in Fig. 24.4, the natural frequency of torsional vibration,

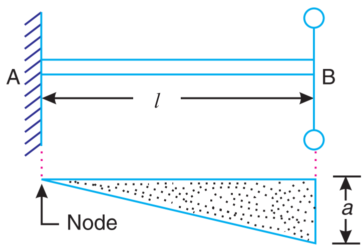


Fig 24.4. Free torsional vibrations of a single rotor system.

$$f_n \quad \frac{1}{2}\sqrt{\frac{q}{I}} \quad \frac{1}{2}\sqrt{\frac{C.J}{l.I}}$$

$$\ldots \quad q \quad \frac{C.J}{l}$$

where C = Modulus of rigidity for shaft material,

J = Polar moment of inertia of shaft

$= \frac{}{32}\; d^4$

d = Diameter of shaft,

l = Length of shaft,

m = Mass of rotor,

k = Radius of gyration of rotor, and

I = Mass moment of inertia of rotor = $m.k^2$

A little consideration will show that the amplitude of vibration is zero at A and maximum at B, as shown in Fig. 24.4. It may be noted that the point or the section of the shaft whose amplitude of torsional vibration is zero, is known as ***node.*** In other words, at the node, the shaft remains unaffected by the vibration.

24.5. Free Torsional Vibrations of a Two Rotor System

Consider a two rotor system as shown in Fig. 24.5. It consists of a shaft with two rotors at its ends. In this system, the torsional vibrations occur only when the two rotors A and B move in opposite directions *i.e.* if A moves in anticlockwise direction then B moves in clockwise direction at the same instant and *vice versa*. It may be noted that the two rotors must have the same frequency.

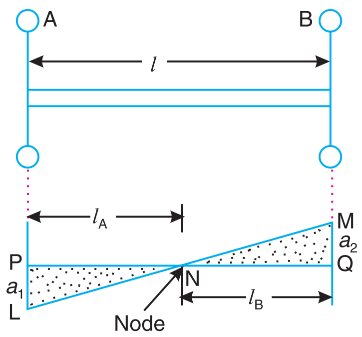


Fig 24.5. Free torsional vibrations of a two rotor system.

We see from Fig. 24.5 that the node lies at point N. This point can be safely assumed as a fixed end and the shaft may be considered as two separate shafts $N P$ and $N Q$ each fixed to one of its ends and carrying rotors at the free ends.

Let l = Length of shaft,

l_A = Length of part NP *i.e.* distance of node from rotor A,

l_B = Length of part NQ, *i.e.* distance of node from rotor B,

I_A = Mass moment of inertia of rotor A,

I_B = Mass moment of inertia of rotor B,

d = Diameter of shaft,

J = Polar moment of inertia of shaft, and

C = Modulus of rigidity for shaft material.

Natural frequency of torsional vibration for rotor A,

$$f_{nA} \quad \frac{1}{2}\sqrt{\frac{C.J}{l_A.I_A}} \qquad \ldots (i)$$

and natural frequency of torsional vibration for rotor B,

$$f_{nB} \quad \frac{1}{2}\sqrt{\frac{C.J}{l_B.I_B}} \qquad \ldots (ii)$$

Since $f_{nA} = f_{nB}$, therefore

$$\frac{1}{2}\sqrt{\frac{C.J}{l_A.I_A}} \quad \frac{1}{2}\sqrt{\frac{C.J}{l_B.I_B}} \quad \text{or} \quad l_A \cdot I_A = l_B \cdot I_B \qquad \ldots (iii)$$

$$l_A \quad \frac{l_B.I_B}{I_A}$$

We also know that

$$l \quad l_A \quad l_B \qquad \ldots (iv)$$

From equations ***(iii)*** and ***(iv)***, we may find the value of l_A and l_B and hence the position of node. Substituting the values of l_A or l_B in equation ***(i)*** or ***(ii)***, the natural frequency of torsional vibration for a two rotor system may be evaluated.

Note : The line *LNM* in Fig.24.5 is known as **elastic line** for the shaft.

24.6. Free Torsional Vibrations of a Three Rotor System

Consider a three rotor system as shown is Fig. 24.6 (*a*). It consists of a shaft and three rotors *A, B* and *C*. The rotors *A* and *C* are attached to the ends of a shaft, whereas the rotor *B* is attached in between *A* and *C*. The torsional vibrations may occur in two ways, that is with either one node or two nodes. In each case, the two rotors rotate in one direction and the third rotor rotates in opposite direction with the same frequency. Let the rotors *A* and *C* of the system, as shown in Fig. 24.6 (*a*), rotate in the same direction and the rotor *B* in opposite direction. Let the nodal points or nodes of such a system lies at N_1 and N_2 as shown in Fig. 24.6 (*b*). As discussed in Art. 24.5, the shaft may be assumed as a fixed end at the nodes.

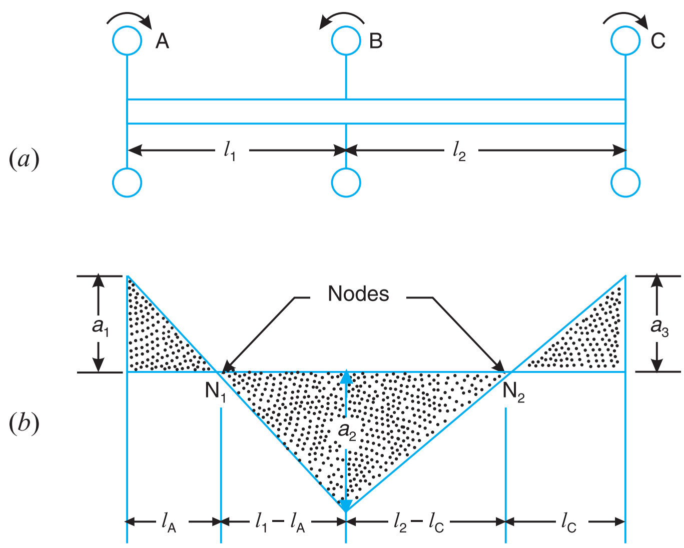


Fig. 24.6. Free torsional vibrations of a three rotor system.

Let
l_1 = Distance between rotors *A* and *B*,
l_2 = Distance between rotors *B* and *C*,
l_A = Distance of node N_1 from rotor *A*,
l_C = Distance of node N_2 from rotor *C*,
I_A = Mass moment of inertia of rotor *A*,
I_B = Mass moment of inertia of rotor *B*,
I_C = Mass moment of inertia of rotor *C*,
d = Diameter of shaft,
J = Polar moment of inertia of shaft, and
C = Modulus of rigidity for shaft material.

Natural frequency of torsional vibrations for rotor A,

$$f_{nA} \quad \frac{1}{2}\sqrt{\frac{C.J}{l_A.I_A}} \qquad \ldots (i)$$

Natural frequency of torsional vibrations for rotor B,

$$*f_{nB}\quad \frac{1}{2}\sqrt{\frac{C.J}{I_B}\quad \frac{1}{l_1\quad l_A}\quad \frac{1}{l_2\quad l_C}} \qquad \ldots (ii)$$

and natural frequency of torsional vibrations for rotor C,

$$f_{nC}\quad \frac{1}{2}\sqrt{\frac{C.J}{l_C.I_C}} \qquad \ldots (iii)$$

Since $f_{nA} = f_{nB} = f_{nC}$, therefore equating equations ***(i)*** and ***(iIi)***

$$\frac{1}{2}\sqrt{\frac{C.J}{l_A.I_A}}\quad \frac{1}{2}\sqrt{\frac{C.J}{l_C.I_C}} \qquad \text{or} \qquad l_A.I_A = l_C.I_C$$

$$l_A\quad \frac{l_C.I_C}{I_A} \qquad \ldots (iv)$$

Now equating equations ***(ii)*** and ***(iii)***,

$$\frac{1}{2}\sqrt{\frac{C.J}{I_B}\quad \frac{1}{l_1\quad l_A}\quad \frac{1}{l_2\quad l_C}}\quad \frac{1}{2}\sqrt{\frac{C.J}{l_C.I_C}}$$

or ... ***(v)***

On substituting the value of l_A from equation ***(iv)*** in the above expression, a quadratic equation in l_C is obtained. Therefore, there are two values of l_C and correspondingly two values of l_A. One value of l_A and the corresponding value of l_C gives the position of two nodes. The frequency obtained by substituting the value of l_A or l_C in equation ***(i)*** or ***(iii)*** is known as ***two node frequency***. But in the other pair of values, one gives the position of single node and the other is beyond the physical limits of the equation. In this case, the frequency obtained is known an ***fundamental frequency*** or ***single node frequency.***

Inside view of a workshop.

Note : This picture is given as additional information and is not a direct example of the current chapter.

* Since the resisting torque of the rotor B is supplied by two lengths $(l_1 - l_A)$ and $(l_2 - l_C)$ between the nodes N_1 and N_2, therefore the each length is twisted through the same angle and the combined torsional stiffness is equal to the sum of the separate stiffness.

We know that torsional stiffness due to $(l_1 - l_A)$ $\quad \frac{C.J}{l_1\quad l_A}$

and torsional stiffness due to $(l_2 - l_C)$ $\quad = \frac{C.J}{l_2\quad l_C}$

Total stiffness of the rotor B $\quad C.J\quad \frac{1}{l_1\quad l_A}\quad \frac{1}{l_2\quad l_C}$

It may be noted that

1. When the rotors *A* and *B* rotate in the same direction and the rotor *C* in the opposite direction, then the torsional vibrations occur with a single node, as shown in Fig. 24.7 (*b*). In this case $l_A > l_1$ *i.e.* the node lies between the rotors *B* and *C*, but it does not give the actual value of the node.
2. When the rotors *B* and *C* rotate in the same direction and the rotor *A* in opposite direction, then the torsional vibrations also occur with a single node as shown in Fig. 24.7 (*c*). In this case $l_C > l_2$ *i.e* the node lies between the rotors *A* and *B*, but it does not give the actual value of the node.

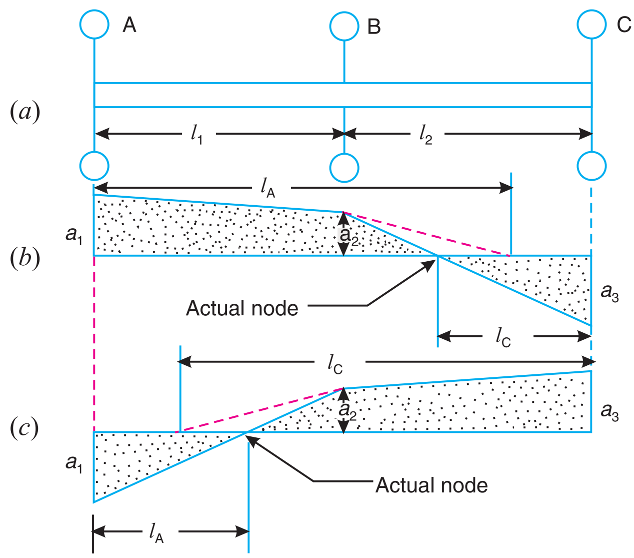


Fig 24.7

3. When the amplitude of vibration for the rotor *A* (a_1) is known, then the amplitude of rotor *B*,

$$a_2 \quad \frac{l_A \quad l_1}{l_A} \quad a_1$$

and amplitude of rotor *C*,

$$a_3 \quad \frac{l_C}{l_C \quad l_2} \quad a_2$$

As there are two values of l_A and l_C, therefore there will be two values of amplitude for one node and two node vibrations.

24.7. Torsionally Equivalent Shaft

In the previous articles, we have assumed that the shaft is of uniform diameter. But in actual practice, the shaft may have variable diameter for different lengths. Such a shaft may, theoretically, be replaced by an equivalent shaft of uniform diameter.

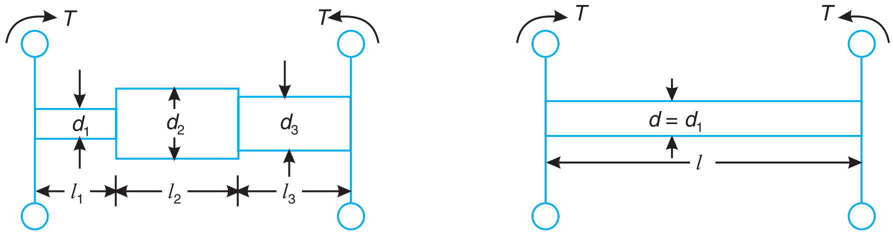
Consider a shaft of varying diameters as shown in Fig. 24.8 (*a*). Let this shaft is replaced by an equivalent shaft of uniform diameter *d* and length *l* as shown in Fig.24.8 (*b*).These two shafts must have the same total angle of twist when equal opposing torques *T* are applied at their opposite ends.

Let d_1, d_2 and d_3 = Diameters for the lengths l_1, l_2 and l_3 respectively,

$_1$, $_2$ and $_3$ = Angle of twist for the lengths l_1, l_2 and l_3 respectively,

= Total angle of twist, and

J_1, J_2 and J_3 = Polar moment of inertia for the shafts of diameters d_1, d_2 and d_3 respectively.



(*a*) Shaft of varying diameters.

(*b*) Torsionally equivalent shaft.

Fig 24.8

Since the total angle of twist of the shaft is equal to the sum of the angle of twists of different lengths, therefore

$$1 \quad 2 \quad 3$$

or
$$\frac{T.l}{C.J} \quad \frac{T.l_1}{C.J_1} \quad \frac{T.l_2}{C.J_2} \quad \frac{T.l_3}{C.J_3}$$

$$\frac{l}{J} \quad \frac{l_1}{J_1} \quad \frac{l_2}{J_2} \quad \frac{l_3}{J_3}$$

$$\frac{l}{\frac{}{32} d^4} \quad \frac{l_1}{\frac{}{32}(d_1)^4} \quad \frac{l_2}{\frac{}{32}(d_2)^4} \quad \frac{l_3}{\frac{}{32}(d_3)^4}$$

$$\frac{l}{d^4} \quad \frac{l_1}{(d_1)^4} \quad \frac{l_2}{(d_2)^4} \quad \frac{l_3}{(d_3)^4}$$

In actual calculations, it is assumed that the diameter d of the equivalent shaft is equal to one of the diameter of the actual shaft. Let us assume that $d = d_1$.

$$\frac{l}{(d_1)^4} \quad \frac{l_1}{(d_1)^4} \quad \frac{l_2}{(d_2)^4} \quad \frac{l_3}{(d_3)^4}$$

or
$$l \quad l_1 \quad l_2 \; \frac{d_1}{d_2}^4 \quad l_3 \; \frac{d_1}{d_3}^4$$

This expression gives the length l of an equivalent shaft.

Example 24.3. *A steel shaft 1.5 m long is 95 mm in diameter for the first 0.6 m of its length, 60 mm in diameter for the next 0.5 m of the length and 50 mm in diameter for the remaining 0.4 m of its length. The shaft carries two flywheels at two ends, the first having a mass of 900 kg and 0.85 m radius of gyration located at the 95 mm diameter end and the second having a mass of 700 kg and 0.55 m radius of gyration located at the other end. Determine the location of the node and the natural frequency of free torsional vibration of the system. The modulus of rigidity of shaft material may be taken as 80 GN/m^2.*

Solution. Given : L = 1.5 m ; d_1 = 95 mm = 0.095 m ; l_1 = 0.6 m ; d_2 = 60 mm = 0.06 m ; l_2 = 0.5 m ; d_3 = 50 mm = 0.05 m ; l_3 = 0.4 m ; m_A = 900 kg ; k_A = 0.85 m ; m_B = 700 kg ; k_B = 0.55 m ; C = 80 GN/m^2 = 80 × 10^9 N/m^2

The actual shaft is shown in Fig. 24.9 (*a*). First of all, let us find the length of the equivalent shaft, assuming its diameter as d_1 = 95 mm as shown in Fig 24.9 (*b*).

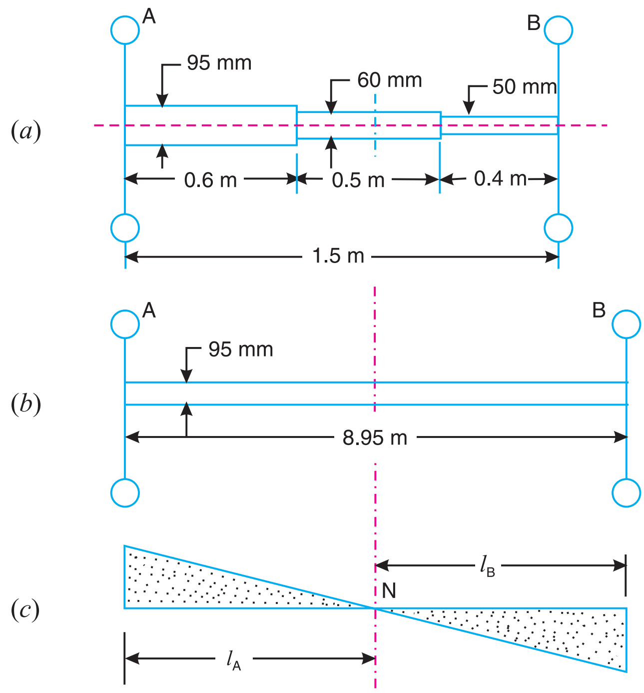


Fig. 24.9

We know that length of the equivalent shaft,

$$l \quad l_1 \quad l_2 \quad \frac{d_1}{d_2}^{4} \quad l_3 \quad \frac{d_1}{d_3}^{4} \quad 0.6 \quad 0.5 \quad \frac{0.095}{0.06}^{4} \quad 0.4 \quad \frac{0.095}{0.05}^{4}$$

$$= 0.6 + 3.14 + 5.21 = 8.95 \text{ m}$$

Location of the node

Suppose the node of the equivalent shaft lies at N as shown in Fig. 24.9 (*c*).

Let l_A = Distance of the node from flywheel A, and

l_B = Distance of the node from flywheel B.

We know that mass moment of inertia of flywheel A,

$$I_A \quad m_A\,(k_A)^2 \quad 900\,(0.85)^2 \quad 650 \ \text{kg-m}^2$$

and mass moment of inertia of flywheel B,

$$I_B \quad m_B\,(k_B)^2 \quad 700\,(0.55)^2 \quad 212 \ \text{kg-m}^2$$

We know that $\quad l_A.I_A \quad l_B.I_B \quad$ or $\quad l_A \quad \dfrac{l_B.I_B}{I_A} \quad \dfrac{l_B \ \ 212}{650} \quad 0.326\, l_B$

Also, $\quad l_A \quad l_B \quad l \quad 8.95$ m $\quad$ or $\quad 0.326\, l_B \quad l_B \quad 8.95 \quad$ or $\quad l_B$ = 6.75 m

and $\quad l_A$ = 8.95 – 6.75 = 2.2 m

Hence the node lies at 2.2 m from flywheel *A* or 6.75 m from flywheel *B* on the equivalent shaft.

∴ Position of node on the original shaft from flywheel *A*

$$l_1 + (l_A - l_1)\left(\frac{d_2}{d_1}\right)^4 = 0.6 + (2.2 - 0.6)\left(\frac{0.06}{0.095}\right)^4 = 0.855 \text{ m} \quad \textbf{Ans.}$$

Natural frequency of free torsional vibrations

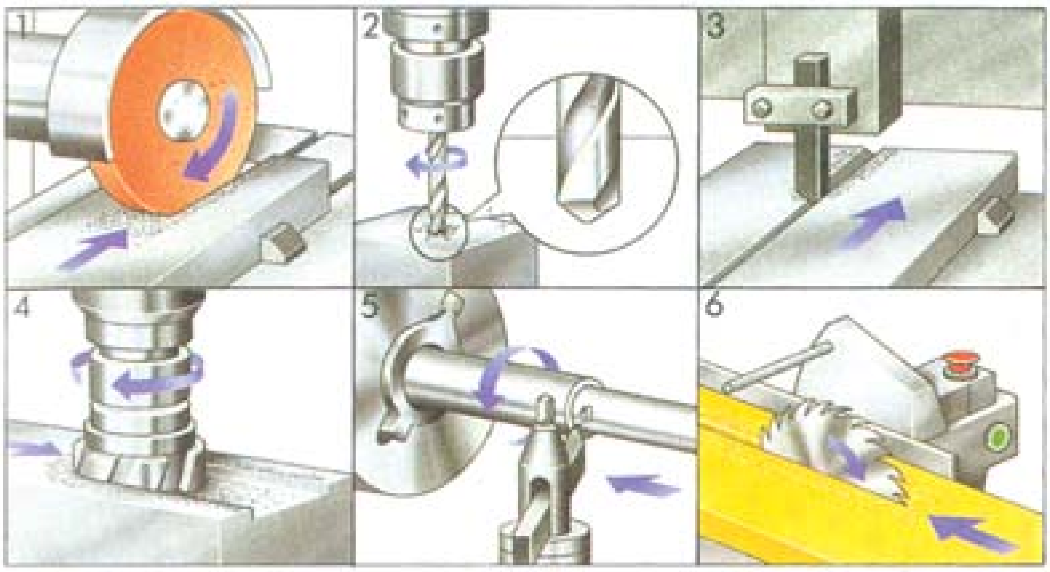
We know that polar moment of inertia of the equivalent shaft,

$$J = \frac{\pi}{32}(d_1)^4 = \frac{\pi}{32}(0.095)^4 = 8 \times 10^{-6} \text{ m}^4$$

Natural frequency of free torsional vibrations,

$$f_n = f_{nA} \text{ or } f_{nB}$$

$$= \frac{1}{2\pi}\sqrt{\frac{C.J}{l_A.I_A}} = \frac{1}{2\pi}\sqrt{\frac{80 \times 10^9 \times 8 \times 10^{-6}}{2.2 \times 650}} = 3.37 \text{ Hz} \qquad \textbf{Ans.}$$



The above machine tools include grinding machine, drill, router, milling machines, lathe and a circular saw.

Note : This picture is given as additional information and is not a direct example of the current chapter.

Example 24.4. *A steel shaft ABCD 1.5 m long has flywheel at its ends A and D. The mass of the flywheel A is 600 kg and has a radius of gyration of 0.6 m. The mass of the flywheel D is 800 kg and has a radius of gyration of 0.9 m. The connecting shaft has a diameter of 50 mm for the portion AB which is 0.4 m long ; and has a diameter of 60 mm for the portion BC which is 0.5 m long ; and has a diameter of d mm for the portion CD which is 0.6 m long. Determine :*

1. *the diameter 'd' of the portion CD so that the node of the torsional vibration of the system will be at the centre of the length BC ; and* ***2.*** *the natural frequency of the torsional vibrations.*

The modulus of rigidity for the shaft material is 80 GN/m².

Solution. Given : L = 1.5 m ; m_A = 600 kg ; k_A = 0.6 m ; m_D = 800 kg ; k_D = 0.9 m ; d_1 = 50 mm = 0.05 m ; l_1 = 0.4 m ; d_2 = 60 mm = 0.06 m ; l_2 = 0.5 m ; $d_3 = d$; l_3 = 0.6 m ; C = 80 GN/m² = 80 × 10^9 N/m²

The actual shaft is shown is Fig. 24.10 (a). First of all, let us find the length of the equivalent shaft, assuming its diameter as d_1 = 50 mm, as shown in Fig. 24.10 (b).

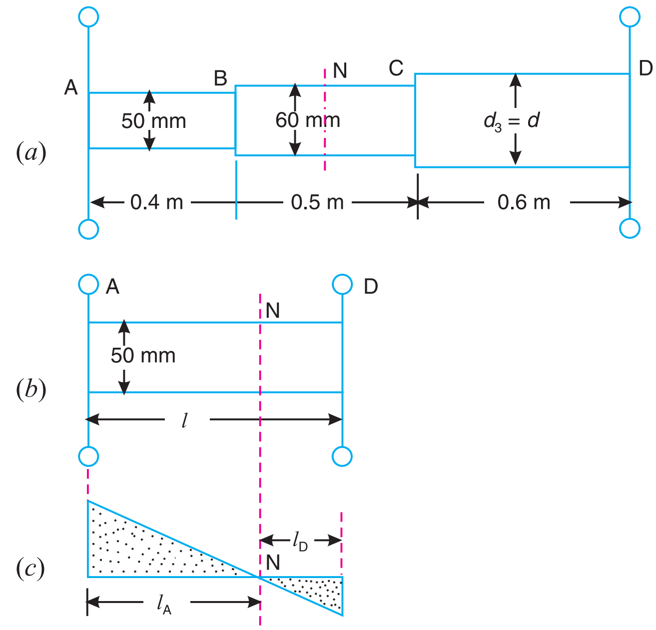


Fig. 24.10

We know that length of the equivalent shaft,

$$l = l_1 + l_2 \left(\frac{d_1}{d_2}\right)^4 + l_3 \left(\frac{d_1}{d_3}\right)^4 = 0.4 + 0.5 \left(\frac{0.05}{0.06}\right)^4 + 0.6 \left(\frac{0.05}{d}\right)^4$$

. . . [Substituting $d_3 = d$]

$$= 0.4 + 0.24 + \frac{3.75 \times 10^{-6}}{d^4} = 0.64 + \frac{3.75 \times 10^{-6}}{d^4} \quad \ldots (i)$$

1. Diameter 'd' of the shaft CD

Suppose the node of the equivalent shaft lies at N as shown in Fig. 24.10 (c).

Let l_A = Distance of the node from flywheel A, and

l_D = Distance of the node from flywheel D.

We know that mass moment of inertia of flywheel A,

$$I_A = m_A (k_A)^2 = 600 (0.6)^2 = 216 \text{ kg-m}^2$$

and mass moment of inertia of flywheel D,

$$I_D = m_D (k_D)^2 = 800 (0.9)^2 = 648 \text{ kg-m}^2$$

A pulley is being processed at a turning centre.

Note : This picture is given as additional information and is not a direct example of the current chapter.

We know that

$$l_A . I_A \quad l_D . I_D$$

or
$$l_D \quad \frac{l_A . I_A}{I_D} \quad \frac{l_A \quad 216}{648} \quad \frac{l_A}{3}$$

Since the node lies in the centre of the length *BC* in an original system, therefore its equivalent length from rotor *A*,

$$l_A \quad l_1 \quad \frac{l_2}{2}\left(\frac{d_1}{d_2}\right)^4 \quad 0.4 \quad \frac{0.5}{2}\left(\frac{0.05}{0.06}\right)^4 \quad 0.52\,\text{m}$$

$$l_D \quad \frac{l_A}{3} \quad \frac{0.52}{3} = 0.173 \text{ m}$$

We know that $l = l_A + l_D$

or
$$0.64 \quad \frac{3.75 \quad 10^{6}}{d^4} \quad 0.52 \quad 0.173 \qquad \text{. . . [From equation } (i)]$$

$$\frac{3.75 \quad 10^{6}}{d^4} \quad 0.52 \quad 0.173 \quad 0.64 \quad 0.053$$

$$d^4 \quad \frac{3.75 \quad 10^{6}}{0.053} \quad 70.75 \quad 10^{6}$$

or $d = 0.0917 \text{ m} = 91.7 \text{ mm}$ **Ans.**

2. Natural frequency of torsional vibrations

We know that polar moment of inertia of the equivalent shaft,

$$J \quad \frac{}{2}(d_1)^4 \quad \frac{}{32}(0.05)^4 \quad 0.614 \quad 10^{6} \text{ m}^4$$

Natural frequency of torsional vibration,

$$f_n \quad f_{nA} \text{ or } f_{nD}$$

$$\frac{1}{2}\sqrt{\frac{C.J}{l_A . I_A}} \quad \frac{1}{2}\sqrt{\frac{80 \quad 10^9 \quad 0.614 \quad 10^{6}}{0.52 \quad 216}} \text{ Hz} = 3.33 \text{ Hz } \textbf{Ans.}$$

Example 24.5. *A single cylinder oil engine drives directly a centrifugal pump. The rotating mass of the engine, flywheel and the pump with the shaft is equivalent to a three rotor system as shown in Fig. 24.11.*

The mass moment of inertia of the rotors A, B and C are 0.15, 0.3 and 0.09 kg-m². Find the natural frequency of the torsional vibration. The modulus of rigidity for the shaft material is 84 kN/mm².

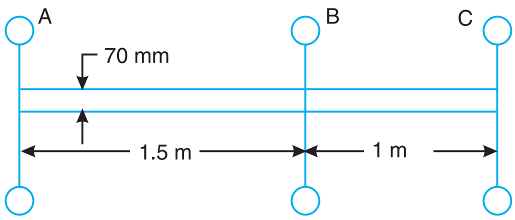


Fig. 24.11

Solution. Given : $I_A = 0.15$ kg-m² ; $I_B = 0.3$ kg-m² ; $I_C = 0.09$ kg-m² ; $d = 70$ mm $= 0.07$ m; $l_1 = 1.5$ m ; $l_2 = 1$ m ; $C = 84$ kN/mm² $= 84 \times 10^9$ N/m²

We know that

$$l_A . I_A = l_C . I_C$$

$$l_A = \frac{l_C \cdot I_C}{I_A} = l_C \times \frac{0.09}{0.15} = 0.6\, l_C$$

Also
$$\frac{1}{l_C . I_C} = \frac{1}{I_B}\left(\frac{1}{l_1 - l_A} + \frac{1}{l_2 - l_C}\right)$$

$$\frac{1}{l_C \times 0.09} = \frac{1}{0.3}\left(\frac{1}{1.5 - l_A} + \frac{1}{1 - l_C}\right)$$

$$\frac{0.3}{l_C \times 0.09} = \frac{1}{1.5 - 0.6\, l_C} + \frac{1}{1 - l_C}$$ . . . (Substituting $l_A = 0.6\, l_C$)

$$\frac{10}{3\, l_C} = \frac{1 - l_C + 1.5 - 0.6\, l_C}{(1.5 - 0.6\, l_C)(1 - l_C)} = \frac{2.5 - 1.6\, l_C}{1.5 - 2.1\, l_C + 0.6 (l_C)^2}$$

On cross-multiplying and re-arranging,

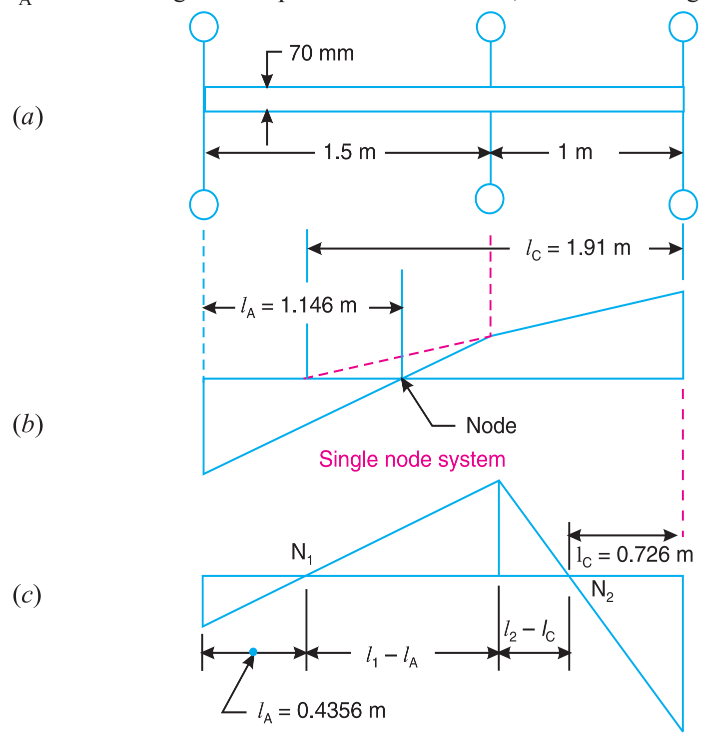
$$10.8\, (l_C)^2 - 28.5\, l_C + 15 = 0$$

$$l_C = \frac{28.5 \pm \sqrt{(28.5)^2 - 4 \times 10.8 \times 15}}{2 \times 10.8} = \frac{28.5 \pm 12.8}{21.6}$$

$= 1.91$ m or 0.726 m

and $l_A = 0.6\, l_C = 1.146$ m or 0.4356 m

Here we see that when $l_C = 1.91$ m, then $l_A = 1.146$ m. This gives the position of single node for $l_A = 1.146$ m, as shown in Fig. 24.12 (*b*). The value of $l_C = 0.726$ m and corresponding value of $l_A = 0.4356$ m gives the position of two nodes, as shown in Fig. 24.12 (*c*).



Two node system.

Fig. 24.12

We know that polar moment of inertia of the shaft,

$$J = \frac{\pi}{32} \times d^4 = \frac{\pi}{32}(0.07)^4 = 2.36 \times 10^{-6}\ \text{m}^4$$

Natural frequency of torsional vibration for a single node system,

$$f_{n1} = \frac{1}{2\pi}\sqrt{\frac{C.J}{l_A.I_A}} = \frac{1}{2\pi}\sqrt{\frac{84 \times 10^9 \times 2.36 \times 10^{-6}}{1.146 \times 0.15}}\ \text{Hz}$$

= 171 Hz **Ans.** . . . (Substituting l_A = 1.146 m)

Similarly, natural frequency of torsional vibration for a two node system,

$$f_{n2} = \frac{1}{2\pi}\sqrt{\frac{C.J}{l_A.I_A}} = \frac{1}{2\pi}\sqrt{\frac{84 \times 10^9 \times 2.36 \times 10^{-6}}{0.4356 \times 0.15}}\ \text{Hz}$$

= 277 Hz **Ans.** . . . (Substituting l_A = 0.4356 m)

Example 24.6. *A motor generator set, as shown in Fig. 24.13, consists of two armatures A and C connected with flywheel between them at B. The modulus of rigidity of the connecting shaft is 84 GN/m^2. The system can vibrate torsionally with one node at 95 mm from A, the flywheel being at antinode. Find : **1.** the position of another node ; **2.** the natural frequency of vibration; and **3.** the radius of gyration of the armature C.*

The other data are given below :

Particulars	*A*	*B*	*C*
Radius of gyration, mm	*300*	*375*	–
Mass, kg	*400*	*500*	*300*

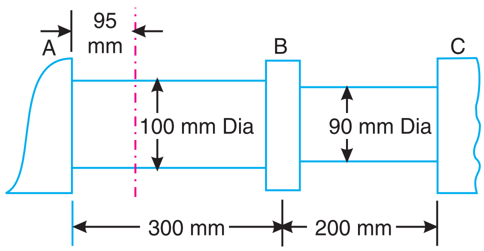


Fig. 24.13

Solution. Given : C = 84 GN/m^2 = 84 × 10^9 N/m^2 ; d_1 = 100 mm = 0.1 m ; d_2 = 90 mm = 0.09 m ; l_1 = 300 mm = 0.3 m ; l_2 = 200 mm = 0.2 m ; l_A = 95 mm = 0.095 m ; m_A = 400 kg; k_A= 300 mm = 0.3 m ; m_B = 500 kg ; k_B = 375 mm = 0.375 m ; m_C = 300 kg

We know that mass moment of inertia of armature A,

$$I_A = m_A (k_A)^2 = 400 (0.3)^2 = 36\ \text{kg-m}^2$$

and mass moment of inertia of flywheel B,

$$I_B = m_B (k_B)^2 = 500 (0.375)^2 = 70.3\ \text{kg-m}^2$$

1. *Position of another node*

First of all, replace the original system, as shown in Fig. 24.14 (*a*), by an equivalent system as shown in Fig. 24.14 (*b*). It is assumed that the diameter of the equivalent shaft is d_1 = 100 mm = 0.1 m because the node lies in this portion. We know that the length of the equivalent shaft,

$$l = l_1 + l_2 \left(\frac{d_1}{d_2}\right)^4 = 0.3 + 0.2 \left(\frac{0.1}{0.09}\right)^4 = 0.605 \text{ m}$$

The first node lies at *E* at a distance 95 mm from rotor *A* *i.e.* l_A = 95 mm = 0.095 m, as shown in Fig. 24.14 (*c*).

Let l_C = Distance of node *F* in an equivalent system from rotor *C*, and

l_3 = Distance between flywheel *B* and armature *C* in an equivalent system = $l - l_1$ = 0.605 – 0.3 = 0.305 m

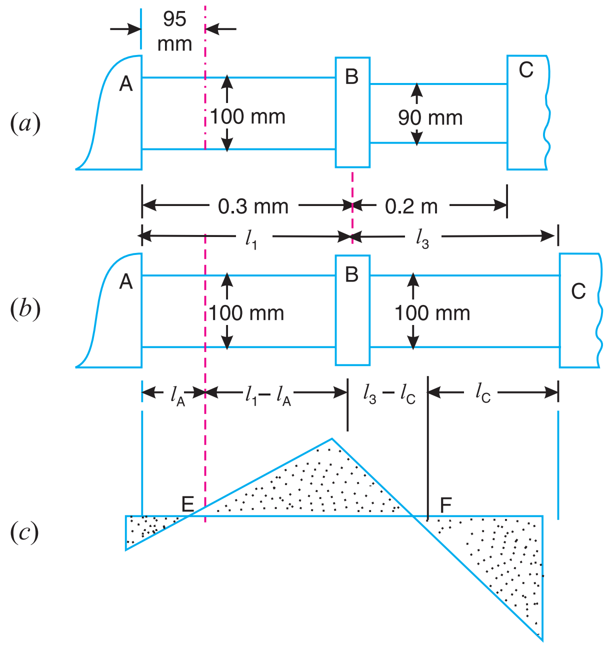


Fig. 24.14

We know that

$$\frac{1}{I_B}\left(\frac{1}{l_1 - l_A} + \frac{1}{l_3 - l_C}\right) = \frac{1}{l_A . I_A}$$

$$\frac{1}{70.3}\left(\frac{1}{0.3 - 0.095} + \frac{1}{0.305 - l_C}\right) = \frac{1}{0.095 \times 36}$$

$$\frac{1}{0.205} + \frac{1}{0.305 - l_C} = \frac{70.3}{0.095 \times 36} = 20.56$$

$$\frac{1}{0.305 - l_C} = 20.56 - \frac{1}{0.205} = 15.68$$

$$0.305 - l_C = 1 / 15.68 = 0.064 \quad \text{or} \quad l_C = 0.21 \text{ m}$$

Corresponding value of l_C in an original system from rotor C

$$0.21 \left(\frac{d_2}{d_1}\right)^4 \quad 0.21 \left(\frac{0.09}{0.1}\right)^4 = 0.13 \text{ m} \quad \textbf{Ans.}$$

2. Natural frequency of vibration

We know that polar moment of inertia of the equivalent shaft,

$$J \quad \frac{}{32}(d_1)^4 \quad \frac{}{32}(0.1)^4 \quad 9.82 \quad 10^{\;6}\,\text{m}^4$$

Natural frequency of vibrations,

$$f_n \quad \frac{1}{2}\sqrt{\frac{C.J}{l_A.I_A}} \quad \frac{1}{2}\sqrt{\frac{84 \quad 10^9 \quad 9.82 \quad 10^{\;6}}{0.095 \quad 36}} \quad \text{Hz}$$

$= 78.1$ Hz **Ans.**

3. Radius of gyration of armature C

Let k_C = Radius of gyration of armature C in metres, and

I_C = Mass moment of inertia of armature $C = m_C\,(k_C)^2$ in kg-m^2.

We know that $\quad l_A.I_A \quad l_C.I_C \quad l_C.m_C\,(k_C)^2$

or $\quad (k_C)^2 \quad \dfrac{l_A.I_A}{l_C.m_C} \quad \dfrac{0.095 \quad 36}{0.21 \quad 300} \quad 0.0543\,\text{m}^2$

$k_C = 0.233$ m **Ans.**

Example 24.7. *A 4-cylinder engine and flywheel coupled to a propeller are approximated to a 3-rotor system in which the engine is equivalent to a rotor of moment of inertia 800 kg-m^2, the flywheel to a second rotor of 320 kg-m^2 and the propeller to a third rotor of 20 kg-m^2. The first and the second rotors being connected by 50 mm diameter and 2 metre long shaft and the second and the third rotors being connected by a 25 mm diameter and 2 metre long shaft.*

Neglecting the inertia of the shaft and taking its modulus of rigidity as 80 GN/m^2, determine: ***1.*** *Natural frequencies of torsional oscillations, and* ***2.*** *The positions of the nodes.*

Solution. Given : $I_A = 800$ kg-m^2 ; $I_B = 320$ kg-m^2 ; $I_C = 20$ kg-m^2 ; $d_1 = 50$ mm $= 0.05$ m; $l_1 = 2$ m ; $d_2 = 25$ mm $= 0.025$ m ; $l_2 = 2$ m ; $C = 80 \times 10^9$ N/m^2

1. Natural frequencies of torsional oscillations

First of all, replace the original system, as shown in Fig. 24.15 (*a*), by an equivalent system as shown in Fig. 24.15 (*b*). It is assumed that the diameter of equivalent shaft is $d_1 = 50$ mm $= 0.05$m.

We know that length of equivalent shaft,

$$l \quad l_1 \quad l_2 \left(\frac{d_1}{d_2}\right)^4 \quad 2 \quad 2 \left(\frac{0.05}{0.025}\right)^4 \quad 34\,\text{m}$$

Now let us find the position of nodes for the equivalent system.

Let l_A = Distance of node N_1 from rotor A, and

l_C = Distance of node N_2 from rotor C.

We know that $\quad l_A.I_A \quad l_C.I_C$

$$l_A \quad \frac{l_C.I_C}{I_A} \quad \frac{l_C \quad 20}{800} \quad 0.025\,l_C$$

Also, $$\frac{1}{l_C . I_C} \quad \frac{1}{I_B} \quad \frac{1}{l_1 \quad l_A} \quad \frac{1}{l_3 \quad l_C}$$

$$\frac{1}{l_C \quad 20} \quad \frac{1}{320} \quad \frac{1}{2 \quad 0.025 l_C} \quad \frac{1}{32 \quad l_C} \qquad \ldots (\quad l_3 = l - l_1)$$

$$\frac{320}{l_C \quad 20} \quad \frac{1}{2 \quad 0.025 l_C} \quad \frac{1}{32 \quad l_C}$$

$$\frac{16}{l_C} \quad \frac{32 \quad l_C \quad 2 \quad 0.025 l_C}{(2 \quad 0.025 l_C)(32 \quad l_C)} \quad \frac{34 \quad 1.025 l_C}{64 \quad 2.8 l_C \quad 0.025 (l_C)^2}$$

or $\quad 1.425\,(l_C)^2 - 78.8\, l_C + 1024 = 0$

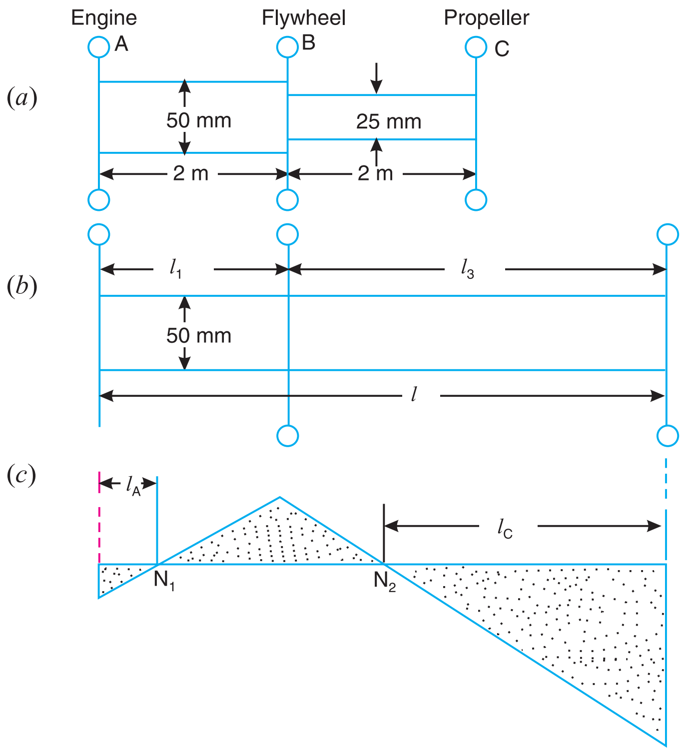


Fig. 24.15

$$l_C \quad \frac{78.8 \quad \sqrt{(78.8)^2 \quad 4 \quad 1.425 \quad 1024}}{2 \quad 1.425} \quad \frac{78.8 \quad 19.3}{2.85}$$

$$= 34.42 \text{ m} \quad \text{or} \quad 20.88 \text{ m}$$

and $\quad l_A = 0.025\, l_C = 0.86 \text{ m} \quad \text{or} \quad 0.52 \text{ m}$

We see that when $l_C = 34.42$ m, then $l_A = 0.86$ m. This gives the position of single node for $l_A = 0.86$ m. The value of $l_C = 20.88$ m and corresponding value of $l_A = 0.52$ m gives the position of two nodes, as shown in Fig. 24.15 (*c*).

We know that polar moment of inertia of the equivalent shaft,

$$J = \frac{\pi}{32}(d_1)^4 = \frac{\pi}{32}(0.05)^4 = 0.614 \times 10^{-6}\ \text{m}^4$$

Natural frequency of torsional vibrations for a single node system,

$$f_{n1} = \frac{1}{2\pi}\sqrt{\frac{C.J}{l_A.I_A}} = \frac{1}{2\pi}\sqrt{\frac{80 \times 10^9 \times 0.614 \times 10^{-6}}{0.86 \times 800}}\ \text{Hz}$$

. . . (Substituting l_A = 0.86 m)

= 1.345 Hz **Ans.**

Similarly natural frequency of torsional vibrations for a two node system,

$$f_{n2} = \frac{1}{2\pi}\sqrt{\frac{C.J}{l_A.I_A}} = \frac{1}{2\pi}\sqrt{\frac{80 \times 10^9 \times 0.614 \times 10^{-6}}{0.52 \times 800}}\ \text{Hz}$$

. . . (Substituting l_A = 0.52 m)

= 1.73 Hz **Ans.**

2. Position of the nodes

We have already calculated that for a two node system on an equivalent shaft, l_C = 20.88 m from the propeller.

Corresponding value of l_C in an original system from the propeller

$$= 20.88\left(\frac{d_2}{d_1}\right)^4 = 20.88\left(\frac{0.025}{0.05}\right)^4 = 1.3\ \text{m}$$

Therefore one node occurs at a distance of l_A = 0.52 m from the engine and the other node at a distance of l_C = 1.3 m from the propeller. **Ans.**

24.8. Free Torsional Vibrations of a Geared System

Consider a geared system as shown in Fig. 24.16 (*a*). It consists of a driving shaft *C* which carries a rotor *A*. It drives a driven shaft *D* which carries a rotor *B*, through a pinion *E* and a gear wheel *F*. This system may be replaced by an equivalent system of continuous shaft carrying a rotor *A* at one end and rotor *B* at the other end, as shown in Fig. 24.16 (*b*). It is assumed that

1. the gear teeth are rigid and are always in contact,

2. there is no backlash in the gearing, and

3. the inertia of the shafts and gears is negligible.

Let d_1 and d_2 = Diameter of the shafts *C* and *D*,

l_1 and l_2 = Length of the shafts *C* and *D*,

I_A and I_B = Mass moment of inertia of the rotors *A* and *B*,

ω_A and ω_B = Angular speed of the rotors *A* and *B*,

$$G = \text{Gear ratio} = \frac{\text{Speed of pinion } E}{\text{Speed of wheel } F} = \frac{\omega_A}{\omega_B}$$

. . . ($\because$ Speeds of *E* and *F* will be same as that of rotors *A* and *B*)

d = Diameter of the equivalent shaft,

l = Length of the equivalent shaft, and

I_B' = Mass moment of inertia of the equivalent rotor B'.

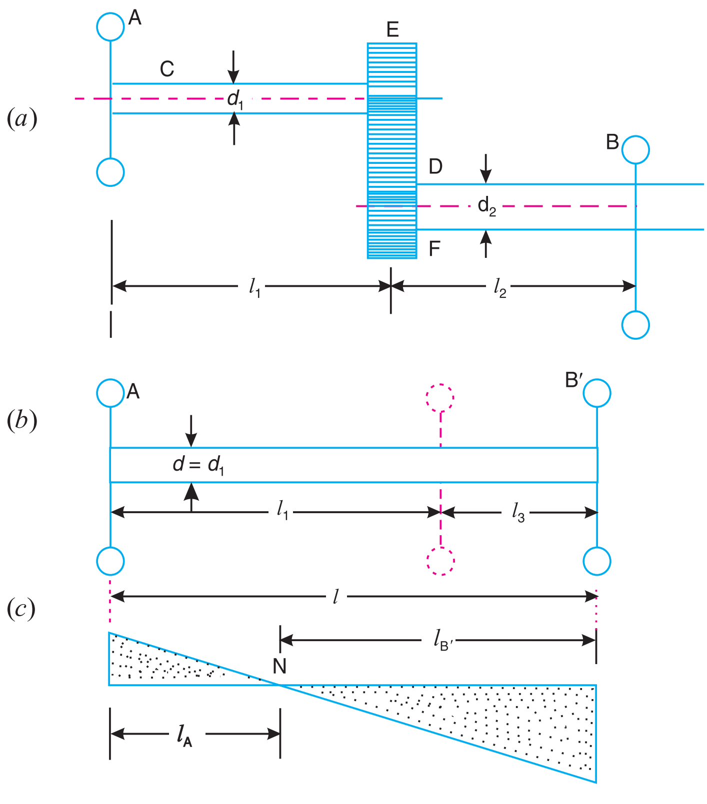


Fig. 24.16

The following two conditions must be satisfied by an equivalent system :

1. The kinetic energy of the equivalent system must be equal to the kinetic energy of the original system.

2. The strain energy of the equivalent system must be equal to the strain energy of the original system.

In order to satisfy the condition **(1)** for a given load,

K.E. of section l_1 + K.E. of section l_3

= K.E. of section l_1 + K. E. of section l_2

K.E. of section l_3 = K.E. of section l_2

or $$\frac{1}{2} \; I_B \; (\;\; _B \;)^2 \quad \frac{1}{2} \; I_B (\;\; _B)^2 \quad \text{or} \quad I_B \; (\;\; _A)^2 \quad I_B (\;\; _B)^2 \quad \ldots (\quad _B \quad _A)$$

$$I_B \quad I_B \; \frac{_B}{_A}^{\;2} \quad \frac{I_B}{G^2} \qquad\qquad G \; \frac{_A}{_B} \qquad \ldots (i)$$

In order to satisfy the condition **(2)** for a given shaft diameter,

Strain energy of l_1 and l_2 = Strain energy of l_1 and l_2

Strain energy of l_3 = Strain energy of l_2

or $$\frac{1}{2} \; T_3 \quad _3 \quad \frac{1}{2} \; T_2 \quad _2 \quad \text{or} \quad \frac{T_3}{T_2} \quad \frac{_2}{_3} \qquad \ldots (ii)$$

where T_2 and T_3 = Torque on the sections l_2 and l_3, and

$_2$ and $_3$ = Angle of twist on sections l_2 and l_3.

Assuming that the power transmitted in the sections l_3 and l_2 is same, therefore

$$T_3 \cdot {}_A \quad T_2 \cdot {}_B \quad \text{or} \quad \frac{T_3}{T_2} \quad \frac{{}_B}{{}_A} \quad \frac{1}{G} \qquad \ldots (iii)$$

Combining equations ***(ii)*** and ***(iii)***,

$$\frac{T_3}{T_2} \quad \frac{{}_2}{{}_3} \quad \frac{1}{G} \qquad \ldots (iv)$$

We know that torsional stiffness,

$$q \quad \frac{T}{} \quad \frac{C.J}{l}$$

where J = Polar moment of inertia of the shaft.

$$\text{For section } l_3, \quad \frac{T_3}{{}_3} \quad \frac{C.J_3}{l_3} \qquad \ldots (v)$$

and

$$\text{For section } l_2, \quad \frac{T_2}{{}_2} \quad \frac{C.J_2}{l_2} \qquad \ldots (vi)$$

Dividing equation ***(v)*** by equation ***(vi)***,

$$\frac{T_3}{T_2} \quad \frac{{}_2}{{}_3} \quad \frac{J_3}{l_3} \quad \frac{l_2}{J_2} \quad \text{or} \quad \frac{T_3}{T_2} \quad \frac{J_3 \cdot {}_3 \, l_2}{J_2 \cdot {}_2 . l_3}$$

or

$$\frac{1}{G} \quad \frac{J_3}{J_2} \quad G \quad \frac{l_2}{l_3} \qquad \text{[From equation } (iv)\text{]}$$

$$l_3 \quad \frac{J_3}{J_2} \quad G^2 \quad l_2 \qquad \ldots (vii)$$

Assuming the diameter of the equivalent shaft as that of shaft C *i.e.* $d = d_1$, therefore

$$J_3 \quad \frac{}{32}(d_1)^4, \qquad \text{and} \qquad J_2 \quad \frac{}{32}(d_2)^4$$

$$\frac{J_3}{J_2} \quad \frac{d_1}{d_2}^{\;4}$$

Now the equation ***(vii)*** may be written as

$$l_3 \quad G^2 \quad l_2 \quad \frac{d_1}{d_2}^{\;4} \qquad \ldots (viii)$$

Thus the single shaft is equivalent to the original geared system, if the mass moment of inertia of the rotor B satisfies the equation ***(i)*** and the additional length of the equivalent shaft l_3 satisfies the equation ***(viii)***.

Length of the equivalent shaft,

$$l = l_1 + \; l_3 = l_1 + G^2.l_2 \quad \frac{d_1}{d_2}^{\;4} \qquad \ldots (ix)$$

Now, the natural frequency of the torsional vibration of a geared system (which have been reduced to two rotor system) may be determined as discussed below :

Let the node of the equivalent system lies at N as shown in Fig. 24.16 (c), then the natural frequency of torsional vibration of rotor A,

$$f_{nA} \quad \frac{1}{2}\sqrt{\frac{C.J}{l_A.I_A}}$$

and natural frequency of the torsional vibration of rotor B ,

$$f_{nB} \quad \frac{1}{2}\sqrt{\frac{C.J}{l_B.I_B}}$$

We know that $f_{nA} \quad f_{nB}$

$$\frac{1}{2}\sqrt{\frac{C.J}{l_A.I_A}} \quad \frac{1}{2}\sqrt{\frac{C.J}{l_B.I_B}}$$

or $\quad l_A.I_A \quad l_B.I_B$ . . . (x)

$$l_A \quad l_B \quad l \qquad \text{. . . }(xi)$$

CNC lathe ata turning centre.

Note : This picture is given as additional information and is not a direct example of the current chapter.

From these two equations (x) and (xi), the value of l_A and l_B may be obtained and hence the natural frequency of the torsional vibrations is evaluated.

Note : When the inertia of the gearing is taken into consideration, then an additional rotor [shown dotted in Fig. 24.16 (b)] must be introduced to the equivalent system at a distance l_1 from the rotor A. This rotor will have a mass moment of inertia $I_E \quad I_E \quad \frac{I_F}{G^2}$, where I_E and I_F are the moments of inertia of the pinion and wheel respectively. The system then becomes a three rotor system and the frequency of such a system may be obtained as discussed in the previous article.

Example 24.8. *A motor drives a centrifugal pump through gearing, the pump speed being one- third that of the motor. The shaft from the motor to the pinion is 60 mm diameter and 300 mm long. The moment of inertia of the motor is 400 kg-m². The impeller shaft is 100 mm diameter and 600 mm long. The moment of inertia of the impeller is 1500 kg-m². Neglecting inertia of the gears and the shaft, determine the frequency of torsional vibration of the system. The modulus of rigidity of the shaft material is 80 GN/m².*

Solution. Given : $G = N_A/N_B = 3$; $d_1 = 60$ mm $= 0.06$ m ; $l_1 = 300$ mm $= 0.3$ m ; $I_A = 400$ kg-m^2; $d^2 = 100$ mm $= 0.1$ m ; $l_2 = 600$ mm $= 0.6$ m ; $I_B = 1500$ kg-m^2 ; $C = 80$ GN/m$^2 = 80 \times 10^9$ N/m^2

The original and the equivalent system, neglecting the inertia of the gears, is shown in Fig. 24.17 (a) and (b) respectively. First of all, let us find the mass moment of inertia of the equivalent rotor B and the additional length of the equivalent shaft, assuming its diameter as $d_1 = 60$ mm.

We know that mass moment of the equivalent rotor B ,

$$I_B \quad I_B / G^2 \quad 1500/3^2 \quad 166.7 \text{ kg–m}^2$$

and additional length of the equivalent shaft,

$$l_3 \quad G^2.l_2 \quad \frac{d_1}{d_2}^4 \quad 3^2 \quad 0.6 \quad \frac{0.06}{0.1}^4 \quad 0.7\text{m} \quad 700 \text{ mm}$$

Total length of the equivalent shaft,

$$l = l_1 + l_3 = 300 + 700 = 1000 \text{ mm} = 1 \text{ m}$$

Let the node of the equivalent system lies at *N*, as shown in Fig. 24.17 (*c*). We know that

$$l_A . I_A = l_B . I_B \quad \text{or} \quad l_A \times 400 = (1 - l_A)166.7 \quad \ldots (\because l_B = l - l_A)$$

$$\therefore \quad l_A = 0.294 \text{ m} = 294 \text{ mm}$$

We know that polar moment of inertia of the equivalent shaft,

$$J = \frac{\pi}{32}(d_1)^4 = \frac{\pi}{32}(0.06)^4 = 1.27 \times 10^{-6} \text{ m}^4$$

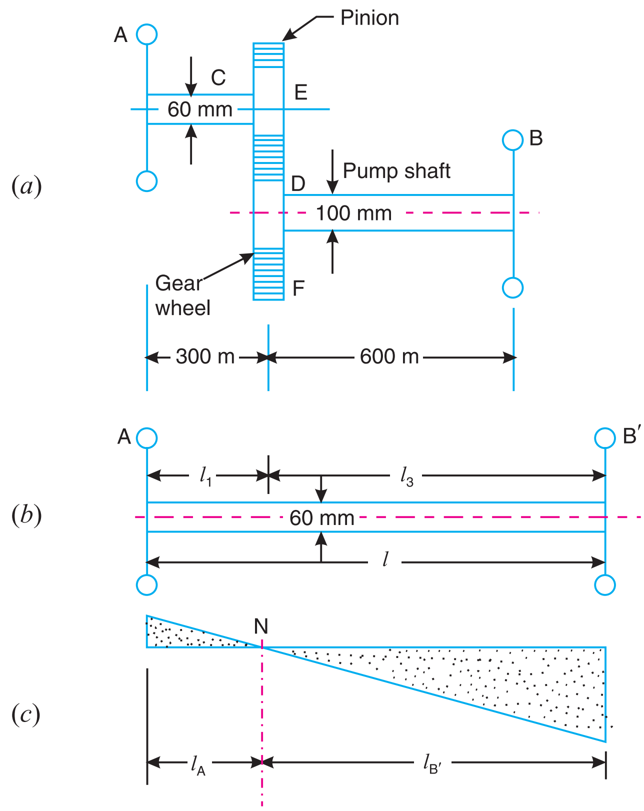


Fig. 24.17

Frequency of torsional vibration,

$$f_n = \frac{1}{2\pi}\sqrt{\frac{C.J}{l_A . I_A}} = \frac{1}{2\pi}\sqrt{\frac{80 \times 10^9 \times 1.27 \times 10^{-6}}{0.294 \times 400}} \text{ Hz}$$

$$= 4.7 \text{ Hz} \quad \textbf{Ans.}$$

Example 24.9. *An electric motor is to drive a centrifuge, running at four times the motor speed through a spur gear and pinion. The steel shaft from the motor to the gear wheel is 54 mm diameter and L metre long ; the shaft from the pinion to the centrifuge is 45 mm diameter and 400 mm long. The masses and radii of gyration of motor and centrifuge are respectively 37.5 kg, 100 mm ; 30 kg and 140 mm.*

Neglecting the inertia effect of the gears, find the value of L if the gears are to be at the node for torsional oscillation of the system and hence determine the frequency of torsional oscillation. Assume modulus of rigidity for material of shaft as 84 GN/m².

Solution. Given : $G = N_A/N_B = 1/4 = 0.25$; $d_1 = 54$ mm $= 0.054$ m ; $l_1 = L$ m ; $d_2 = 45$ mm $= 0.045$ m ; $l_2 = 400$ mm $= 0.4$ m ; $m_A = 37.5$ kg ; $k_A = 100$ mm $= 0.1$ m ; $m_B = 30$ kg ; $k_B = 140$ mm $= 0.14$ m; $C = 84$ GN/m² $= 84 \times 10^9$ N/m²

Value of L

We know that mass moment of inertia of the motor,

$$I_A \quad m_A (k_A)^2 \quad 37.5(0.1)^2 \quad 0.375 \text{ kg-m}^2$$

and mass moment of inertia of the centrifuge,

$$I_B \quad m_B (k_B)^2 \quad 30(0.14)^2 \quad 0.588 \text{ kg-m}^2$$

The original and the equivalent system, neglecting the inertia effect of the gears, is shown in Fig. 24.18 (*a*) and (*b*) respectively.

First of all, let us find the mass moment of inertia of the equivalent rotor B and the additional length of the equivalent shaft, keeping the diameter of the equivalent shaft as $d_1 = 54$ mm.

We know that mass moment of inertia of the equivalent rotor B ,

$$I_B \quad I_B / G^2 \quad 0.588/(0.25)^2 \quad 9.4 \text{ kg-m}^2$$

and additional length of equivalent shaft,

$$l_3 \quad G^2 . l_2 \quad \frac{d_1}{d_2} \quad ^4 \quad (0.25)^2 \, 0.4 \quad \frac{0.054}{0.045} \quad ^4 \quad 0.0518 \text{ m}$$

Since the node N for torsional oscillation of the system lies at the gears, as shown in Fig. 24.18 (*c*), therefore

$$l_A \quad L, \quad \text{and} \quad l_B \quad l_3 \quad 0.0518 \text{ m}$$

Grinding is a commonly used method for removing the excess material from the cestings, forgings and weldments

Note : This picture is given as additional information and is not a direct example of the current chapter.

We know that $l_A . I_A = l_B . I_B$

$L \times 0.375 = 0.0518 \times 9.4 = 0.487$ or $L = 1.3$ m **Ans.**

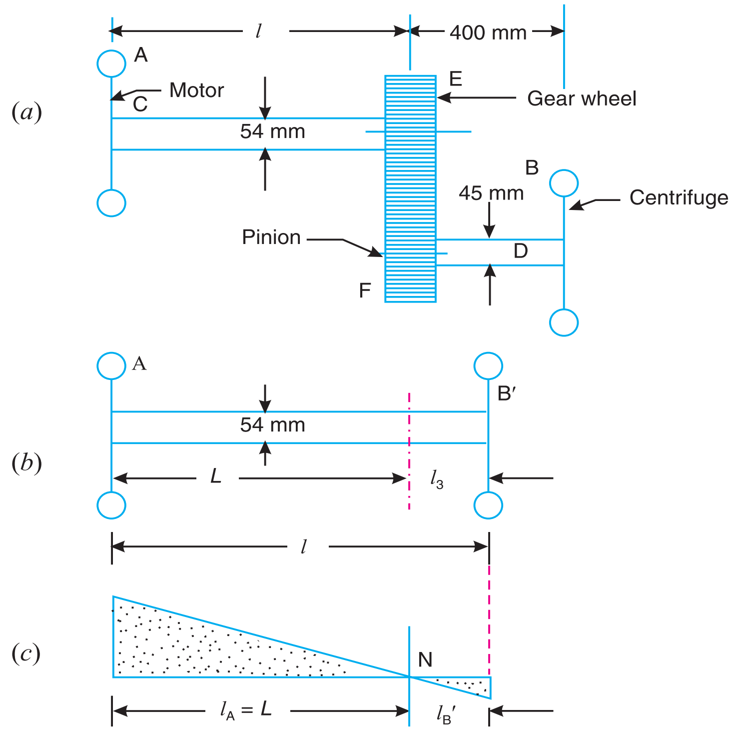


Fig. 24.18

Frequency of torsional oscillations

We know that polar moment of inertia of the equivalent shaft,

$$J = \frac{\pi}{32}(d_1)^4 = \frac{\pi}{32}(0.054)^4 = 0.835 \times 10^{-6}\ \text{m}^4$$

Frequency of torsional oscillations,

$$f_n = \frac{1}{2\pi}\sqrt{\frac{C.J}{l_A.I_A}} = \frac{1}{2\pi}\sqrt{\frac{84 \times 10^9 \times 0.835 \times 10^{-6}}{1.3 \times 0.375}} = 60.4 \text{ Hz } \textbf{Ans.}$$

Example 24.10. *Determine the natural frequencies of torsional oscillation for the following system. The system is a reciprocating I.C. engine coupled to a centrifugal pump through a pair of gears. The shaft from the flywheel of the engine to the gear wheel is of 60 mm diameter and 950 mm length. The shaft from the pinion to the pump is of 40 mm diameter and 300 mm length. The engine speed is* $\frac{1}{4}$ *th of the pump speed.*

Moment of inertia of the flywheel = 800 kg-m²

Moment of inertia of the gear wheel = 15 kg-m²

Moment of inertia of the pinion = 4 kg-m²

Moment of inertia of the pump = 17 kg-m²

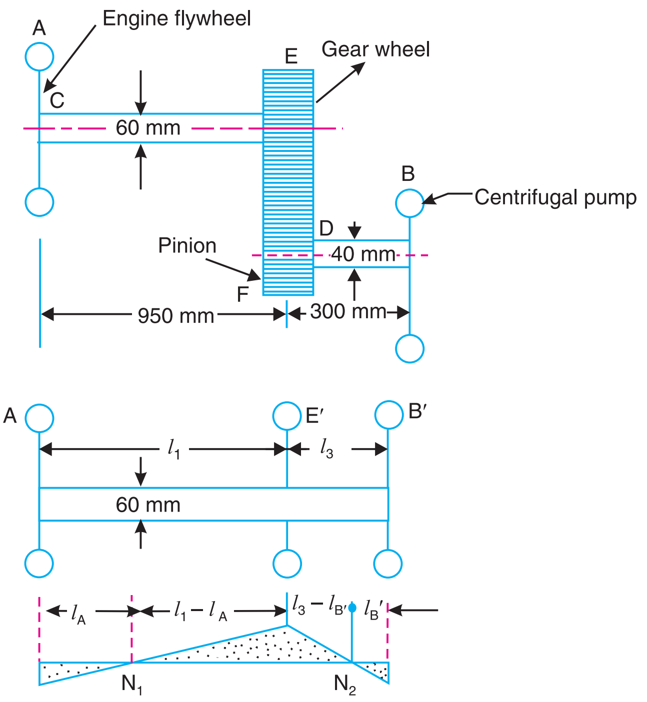
Modulus of rigidity for shaft material is 84 GN/m².

Solution. Given : d_1 = 60 mm = 0.06 m ; l_1 = 950 mm = 0.95 m ; d_2 = 40 mm = 0.04 m ; l_2 = 300 mm = 0.3 m ; G = 1/4 = 0.25 ; I_A = 800 kg-m^2 ; I_E = 15 kg-m^2 ; I_F = 4 kg-m^2 ; I_B = 17 kg-m^2; C = 84 GN/m^2 = 84 × 10^9 N/m^2

The original and the equivalent system is shown in Fig. 24.19 (*a*) and (*b*) respectively. First of all, let us find the mass moment of inertia of the equivalent gearing E , the equivalent pump B and the additional length of the equivalent shaft keeping its diameter as d_1 = 60 mm.

We know that mass moment of inertia of the equivalent gearing E ,

$$I_E \quad I_E \quad I_F / G^2 \quad 15 \quad 4/(0.25)^2 \quad 79 \text{ kg-m}^2$$



All dimensions in mm.

Fig. 24.19

Mass moment of inertia of the equivalent pump B ,

$$I_B \quad I_B / G^2 \quad 17/(0.25)^2 \quad 272 \text{ kg-m}^2$$

and additional length of the equivalent shaft,

$$l_3 \quad G^2 . l_2 \left(\frac{d_1}{d_2}\right)^4 \quad (0.25)^2 \, 0.3 \left(\frac{0.06}{0.04}\right)^4 \quad 0.095 \text{ m}$$

The original system is thus reduced to a three rotor system, as shown in 24.19 (*b*). Let us find the position of nodes for the equivalent system.

Let $\qquad l_A$ = Distance of node N_1 from rotor A, and

l_B = Distance of node N_2 from rotor B .

We know that $l_A . I_A = l_B . I_B$ or $l_A = l_B \times \frac{I_B}{I_A} = l_B \times \frac{272}{800} = 0.34 l_B$

Also

$$\frac{1}{l_B . I_B} = \frac{1}{I_E}\left[\frac{1}{l_1 - l_A} + \frac{1}{l_3 - l_B}\right]$$

$$\frac{1}{l_B \times 272} = \frac{1}{79}\left[\frac{1}{0.95 - 0.34 l_B} + \frac{1}{0.095 - l_B}\right]$$

$$\frac{79}{l_B \times 272} = \frac{(0.095 - l_B + 0.95 - 0.34 l_B)}{(0.95 - 0.34 l_B)(0.095 - l_B)}$$

$$\frac{0.29}{l_B} = \frac{1.045 - 1.34 l_B}{0.09 - 0.98 l_B + 0.34 (l_B)^2}$$

$$0.026 - 0.28 l_B + 0.1 (l_B)^2 = 1.045 l_B - 1.34 (l_B)^2$$

$$1.44 (l_B)^2 - 1.325 l_B + 0.026 = 0$$

$$l_B = \frac{1.325 \pm \sqrt{(1.325)^2 - 4 \times 1.44 \times 0.026}}{2 \times 1.44} = \frac{1.325 \pm 1.267}{2.88}$$

$$= 0.9 \text{ m or } 0.02 \text{ m}$$

and $l_A = 0.34 l_B = 0.306 \text{ m}$ or 0.0068 m

We see that when $l_B = 0.9$ m, then $l_A = 0.306$ m. This gives the position of single node for $l_A = 0.306$ m or 306 mm. When $l_B = 0.02$ m, the corresponding value of $l_A = 0.0068$ m or 6.8 mm. This gives the position of two nodes as shown in Fig. 24.19 (*c*).

We know that polar moment of inertia of the equivalent shaft,

$$J = \frac{\pi}{32}(d_1)^4 = \frac{\pi}{32}(0.06)^4 = 1.27 \times 10^{-6} \text{ m}^4$$

Natural frequency of torsional oscillations for a single node system,

$$f_{n1} = \frac{1}{2\pi}\sqrt{\frac{C.J}{l_A . I_A}} = \frac{1}{2\pi}\sqrt{\frac{84 \times 10^9 \times 1.27 \times 10^{-6}}{0.306 \times 800}} \text{ Hz}$$

$= 3.32$ Hz **Ans.** . . . (Substituting $l_A = 0.306$ m)

Similarly natural frequency of torsional oscillations for a two node system,

$$f_{n2} = \frac{1}{2\pi}\sqrt{\frac{C.J}{l_A . I_A}} = \frac{1}{2\pi}\sqrt{\frac{84 \times 10^9 \times 1.27 \times 10^{-6}}{0.0068 \times 800}} \text{ Hz}$$

$= 22.3$ Hz **Ans.** . . . (Substituting $l_A = 0.0068$ m)

EXERCISES

1. A shaft of 100 mm diameter and 1 metre long is fixed at one end and the other end carries a flywheel of mass 1 tonne. The radius of gyration of the flywheel is 0.5 m. Find the frequency of torsional vibrations, if the modulus of rigidity for the shaft material is 80 GN/m^2. **[Ans. 8.9 Hz]**
2. The flywheel of an engine driving a dynamo has a mass of 180 kg and a radius of gyration of 30 mm. The shaft at the flywheel end has an effective length of 250 mm and is 50 mm diameter. The armature mass is 120 kg and its radius of gyration is 22.5 mm. The dynamo shaft is 43 mm diameter and 200 mm effective length. Calculate the position of node and frequency of torsional oscillation. C = 83 kN/mm^2. **[Ans. 205 mm from flywheel, 218 Hz]**
3. The two rotors A and B are attached to the end of a shaft 500 mm long. The mass of the rotor A is 300 kg and its radius of gyration is 300 mm. The corresponding values of the rotor B are 500 kg and 450 mm respectively. The shaft is 70 mm in diameter for the first 250 mm ; 120 mm for the next 70 mm and 100 mm diameter for the remaining length. The modulus of rigidity for the shaft material is 80 GN/m^2. Find : 1. The position of the node, and 2. The frequency of torsional vibration. **[Ans. 225 mm from A ; 27.3 Hz]**
4. Three rotors A, B and C having moment of inertia of 2000 ; 6000 ; and 3500 kg-m^2 respectively are carried on a uniform shaft of 0.35 m diameter. The length of the shaft between the rotors A and B is 6 m and between B and C is 32 m. Find the natural frequency of the torsional vibrations. The modulus of rigidity for the shaft material is 80 GN/m^2. **[Ans. 6.16 Hz ; 18.27 Hz]**
5. A motor generator set consists of two armatures P and R as shown in Fig. 24.20, with a flywheel between them at Q. The modulus of rigidity of the material of the shaft is 84 GN/m^2. The system can vibrate with one node at 106.5 mm from P, the flywheel Q being at antinode. Using the data of rotors given below, find: 1. The position of the other node, 2. The natural frequency of the free torsional vibrations, for the given positions of the nodes, and 3. The radius of gyration of the rotor R.

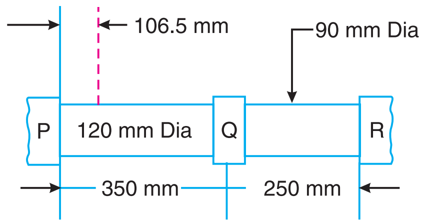


Fig. 24.20

Data of rotors

Rotor	*P*	*Q*	*R*
Mass, kg	450	540	360
Radius of gyration, mm	250	300	

[Ans. 225 mm from R ; 120 Hz ; 108 mm]

6. An electric motor rotating at 1500 r.p.m. drives a centrifugal pump at 500 r.p.m. through a single stage reduction gearing. The moments of inertia of the electric motor and the pump impeller are 400 kg-m^2 and 1400 kg-m^2 respectively. The motor shaft is 45 mm in diameter and 180 mm long. The pump shaft is 90 mm in diameter and 450 mm long.
Determine the frequency of torsional oscillations of the system, neglecting the inertia of the gears. The modulus of rigidity for the shaft material is 84 GN/m^2. **[Ans. 4.2 Hz]**
7. Two parallel shafts A and B of diameters 50 mm and 70 mm respectively are connected by a pair of gear wheels, the speed of A being 4 times that of B. The flywheel of mass moment of inertia 3 kg-m^2 is mounted on shaft A at a distance of 0.9 m from the gears. The shaft B also carries a flywheel of mass moment of inertia 16 kg-m^2 at a distance of 0.6 m from the gears. Neglecting the effect of the shaft and gear masses, calculate the fundamental frequency of free torsional oscillations and the position of node . Assume modulus of rigidity as 84 GN/m^2.
[Ans. 22.6 Hz ; 0.85 m from the flywheel on shaft A]
8. A centrifugal pump is driven through a pair of spur wheels from an oil engine. The pump runs at 4 times the speed of the engine. The shaft from the engine flywheel to the gear is 75 mm diameter and 1.2 m long, while that from the pinion to the pump is 50 mm diameter and 400 mm long. The moment of inertia are as follows:
Flywheel = 1000 kg-m^2, Gear = 25 kg m^2, Pinion = 10 kg-m^2, and Pump impeller = 40 kg-m^2.
Find the natural frequencies of torsional oscillations of the system. Take C = 84 GN/m^2.
[Ans. 3.4 Hz ; 19.7 Hz]

DO YOU KNOW ?

1. Derive an expression for the frequency of free torsional vibrations for a shaft fixed at one end and carrying a load on the free end.
2. Discuss the effect of inertia of a shaft on the free torsional vibrations.
3. How the natural frequency of torsional vibrations for a two rotor system is obtained ?
4. Describe the method of finding the natural frequency of torsional vibrations for a three rotor system.
5. What is meant by torsionally equivalent length of a shaft as referred to a stepped shaft? Derive the expression for the equivalent length of a shaft which have several steps.
6. Establish the expression to determine the frequency of torsional vibrations of a geared system.

OBJECTIVE TYPE QUESTIONS

1. The natural frequency of free torsional vibrations of a shaft is

   (*a*) $2\sqrt{\frac{q}{I}}$ (b) $2\sqrt{q.I}$ (c) $\frac{1}{2}\sqrt{\frac{q}{I}}$ (d) $\frac{1}{2}\sqrt{q.I}$

   where q = Torsional stiffness of the shaft, and
   I = Mass moment of inertia of the disc attached at the end of the shaft.
2. At a nodal point in a shaft, the amplitude of torsional vibration is
   (*a*) zero (*b*) minimum (*c*) maximum
3. Two shafts *A* and *B* are shown in Fig. 24.21. The length of an equivalent shaft *B* is given by

   (*a*) $l = l_1 + l_2 + l_3$ (*b*) $l \quad l_1 \quad \frac{d_2}{d_3}^{4}$

   (*c*) $l \quad l_1 \quad l_2 \quad \frac{d_2}{d_2}^{4}$ (*d*) $l \quad l_1 \quad l_2 \quad \frac{d_1}{d_2}^{4} \quad l_3 \quad \frac{d_1}{d_3}^{4}$

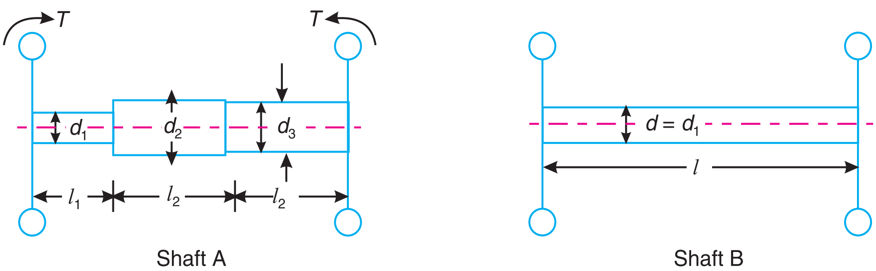


Fig. 24.21

4. A shaft carrying two rotors as its ends will have
   (*a*) no node (*b*) one node (*c*) two nodes (*d*) three nodes
5. A shaft carrying three rotors will have
   (*a*) no node (*b*) one node (*c*) two nodes (*d*) three nodes

ANSWERS

1. (*c*) **2.** (*a*) **3.** (*d*) **4.** (*b*) **5.** (*c*)

GO To FIRST