4

Simple Harmonic Motion

4.1. Introduction

Consider a particle moving round the circumference of a circle in an anticlockwise direction, with a constant angular velocity, as shown in Fig. 4.1. Let P be the position of the particle at any instant and N be the projection of P on the diameter XX' of the circle.
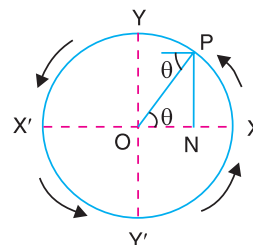

Fig. 4.1. Simple harmonic motion.

It will be noticed that when the point P moves round the circumference of the circle from X to Y, N moves from X to O, when P moves from Y to X', N moves from O to X'. Similarly when P moves from X' to Y', N moves from X' to O and finally when P moves from Y' to X, N moves from O to X. Hence, as P completes one revolution, the point N completes one vibration about the

A clock pendulum executes Simple Harmonic Motion.

point O. This to and fro motion of N is known as *simple harmonic motion* (briefly written as S.H.M.).

4.2. Velocity and Acceleration of a Particle Moving with Simple Harmonic Motion

Consider a particle, moving round the circumference of a circle of radius r, with a uniform angular velocity ω rad/s, as shown in Fig. 4.2. Let P be any position of the particle after t seconds and θ be the angle turned by the particle in t seconds. We know that

$$\theta = \omega.t$$

If N is the projection of P on the diameter $X\,X\,'$, then displacement of N from its mean position O is

$$x = r.\cos\theta = r.\cos\omega.t \qquad ... (i)$$

The velocity of N is the component of the velocity of P parallel to $XX\,'$, *i.e.*

Movements of a ship up and down in a vertical plane about transverse axis (called Pitching) and about longitude (called rolling) are in Simple Harmonic Motion.

$$v_N = v\sin\theta = \omega r\sin\theta = \omega\sqrt{r^2 - x^2} \qquad\qquad ... (ii)$$

$$...\left[\because v = \omega.r,\text{ and } r\sin\theta = NP = \sqrt{r^2 - x^2} \right]$$
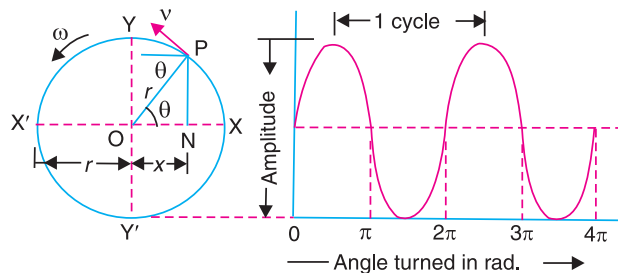

Fig. 4.2. Velocity and acceleration of a particle.

A little consideration will show that velocity is maximum, when $x = 0$, *i.e.* when N passes through O *i.e.*, its mean position.

$$\therefore \qquad v_{max} = \omega.r$$

We also know that the acceleration of P is the centripetal acceleration whose magnitude is $\omega^2.r$. The acceleration of N is the component of the acceleration of P parallel to $XX\,'$ and is directed towards the centre O, *i.e.*,

$$a_N = \omega^2.\,r\cos\theta = \omega^2.x \qquad\qquad ...\ (\because x = r\cos\theta)\ ...(iii)$$

The acceleration is maximum when $x = r$ *i.e.* when P is at X or X'.

$$\therefore \qquad a_{max} = \omega^2.r$$

It will also be noticed from equation **(iii)** that when $x = 0$, the acceleration is zero *i.e.* N passes through O. In other words, the acceleration is zero at the mean position. Thus we see from equation **(iii)** that the acceleration of N is proportional to its displacement from its mean position O, and it is

always directed towards the centre O; so that the motion of N is simple harmonic.

In general, a body is said to move or vibrate with simple harmonic motion, if it satisfies the following two conditions :

1. *Its acceleration is always directed towards the centre, known as point of reference or mean position* ;

2. *Its acceleration is proportional to the distance from that point*.

4.3. Differential Equation of Simple Harmonic Motion

We have discussed in the previous article that the displacement of N from its mean position O is

$$x = r.\cos \theta = r.\cos \omega t \qquad \qquad ... (i)$$

Differentiating equation **(i)**, we have velocity of N,

$$\frac{dx}{dt} = v_N = r.\omega \sin \omega t \qquad \qquad ... (ii)$$

Again differentiating equation **(ii)**, we have acceleration of N,

$$* \frac{d^2x}{dt^2} = a_N = -r.\omega.\omega \cos \omega t = -\omega^2.r \cos \omega t = -\omega^2.x \qquad \qquad ... (iii)$$

$$... (\because r \cos \omega t = x)$$

or $$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

This is the standard differential equation for simple harmonic motion of a particle. The solution of this differential equation is

$$x = A \cos \omega t + B \sin \omega t \qquad \qquad ... (iv)$$

where A and B are constants to be determined by the initial conditions of the motion.

In Fig. 4.2, when $t = 0$, $x = r$ i.e. when points P and N lie at X, we have from equation **(iv)**, $A = r$

Differentiating equation **(iv)**,

$$\frac{dx}{dt} = -A.\omega.\sin \omega t + B.\omega \cos \omega t$$

When $t = 0$, $\frac{dx}{dt} = 0$, therefore, from the above equation, $B = 0$. Now the equation **(iv)** becomes

$$x = r \cos \omega t \qquad \qquad ... \text{[Same as equation (i)]}$$

The equations **(ii)** and **(iii)** may be written as

$$\frac{dx}{dt} = v_N = -\omega.r \sin \omega t = \omega.r \cos (\omega t + \pi/2)$$

and $$\frac{d^2x}{dt^2} = a_N = -\omega^2.r \cos \omega t = \omega^2.r \cos (\omega t + \pi)$$

These equations show that the velocity leads the displacement by 90° and acceleration leads the displacement by 180°.

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* The negative sign shows that the direction of acceleration is opposite to the direction in which x increases, *i.e*. the acceleration is always directed towards the point O.

4.4. Terms Used in Simple Harmonic Motion

The following terms, commonly used in simple harmonic motion, are important from the subject point of view.

1. *Amplitude*. It is the maximum displacement of a body from its mean position. In Fig. 4.2, OX or OX' is the amplitude of the particle P. The amplitude is always equal to the radius of the circle.

2. *Periodic time*. It is the time taken for one complete revolution of the particle.

∴   Periodic time, $t_p = 2\pi/\omega$ seconds

We know that the acceleration,

$$a = \omega^2.x \quad \text{or} \quad \omega^2 = \frac{a}{x} \quad \text{or} \quad \omega = \sqrt{\frac{a}{x}}$$

∴

$$t_p = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{x}{a}} = 2\pi\sqrt{\frac{\text{Displacement}}{\text{Acceleration}}} \text{ seconds}$$

It is thus obvious, that the periodic time is independent of amplitude.

3. *Frequency*. It is the number of cycles per second and is the reciprocal of time period, t_p.

∴   Frequency, $n = \frac{\omega}{2\pi} = \frac{1}{t_p} = \frac{1}{2\pi}\sqrt{\frac{a}{x}}$ Hz

Notes : 1. In S.I. units, the unit of frequency is hertz (briefly written as Hz) which is equal to one cycle per second.

2. When the particle moves with angular simple harmonic motion, then the periodic time,

$$t_p = 2\pi\sqrt{\frac{\text{Angular displacement}}{\text{Angular acceleration}}} = 2\pi\sqrt{\frac{\theta}{\alpha}} \text{ s}$$

and   frequency, $n = \frac{1}{2\pi}\sqrt{\frac{\alpha}{\theta}}$ Hz

Example 4.1. *The piston of a steam engine moves with simple harmonic motion. The crank rotates at 120 r.p.m. with a stroke of 2 metres. Find the velocity and acceleration of the piston, when it is at a distance of 0.75 metre from the centre.*

Solution. Given : $N = 120$ r.p.m.   or $\omega = 2\pi \times 120/60 = 4\pi$ rad/s ; $2r = 2$ m   or $r = 1$ m; $x = 0.75$ m

Velocity of the piston

We know that velocity of the piston,

$$v = \omega\sqrt{r^2 - x^2} = 4\pi\sqrt{1 - (0.75)^2} = 8.31 \text{ m/s} \textbf{ Ans.}$$

Acceleration of the piston

We also know that acceleration of the piston,

$$a = \omega^2.x = (4\pi)^2\,0.75 = 118.46 \text{ m/s}^2 \textbf{ Ans.}$$

Example 4.2. *A point moves with simple harmonic motion. When this point is 0.75 metre from the mid path, its velocity is 11 m/s and when 2 metres from the centre of its path its velocity is 3 m/s. Find its angular velocity, periodic time and its maximum acceleration.*

Solution. Given : When $x = 0.75$ m, $v = 11$ m/s ; when $x = 2$ m, $v = 3$ m/s

Angular velocity

Let ω = Angular velocity of the particle, and

r = Amplitude of the particle.

We know that velocity of the point when it is 0.75 m from the mid path (v),

$$11 = \omega \sqrt{r^2 - x^2} = \omega \sqrt{r^2 - (0.75)^2} \qquad \ldots (i)$$

Similarly, velocity of the point when it is 2 m from the centre (v),

$$3 = \omega \sqrt{r^2 - 2^2} \qquad \ldots (ii)$$

Dividing equation (*i*) by equation (*ii*),

$$\frac{11}{3} = \frac{\omega \sqrt{r^2 - (0.75)^2}}{\omega \sqrt{r^2 - 2^2}} = \frac{\sqrt{r^2 - (0.75)^2}}{\sqrt{r^2 - 2^2}}$$

Squaring both sides,

$$\frac{121}{9} = \frac{r^2 - 0.5625}{r^2 - 4}$$

$$121\, r^2 - 484 = 9r^2 - 5.06 \quad \text{or} \quad 112\, r^2 = 478.94$$

∴ $r^2 = 478.94 / 112 = 4.276$ or $r = 2.07$ m

Substituting the value of r in equation (*i*),

$$11 = \omega \sqrt{(2.07)^2 - (0.75)^2} = 1.93\, \omega$$

∴ $\omega = 11/1.93 = 5.7$ rad/s **Ans.**

Periodic time

We know that periodic time,

$$t_p = 2\pi / \omega = 2\pi / 5.7 = 1.1 \text{ s } \textbf{Ans.}$$

Maximum acceleration

We know that maximum acceleration,

$$a_{max} = \omega^2.r = (5.7)^2\ 2.07 = 67.25 \text{ m/s}^2 \textbf{ Ans.}$$

4.5. Simple Pendulum

A simple pendulum, in its simplest form, consists of heavy bob suspended at the end of a light inextensible and flexible string. The other end of the string is fixed at O, as shown in Fig. 4.3.

Let L = Length of the string,

m = Mass of the bob in kg,

W = Weight of the bob in newtons

= $m.g$, and

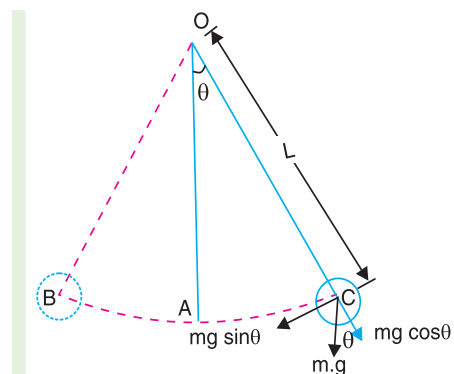θ = Angle through which the string is displaced.

Fig 4.3. Simple pendulum.

When the bob is at A, the pendulum is in equilibrium position. If the bob is brought to B or C and released, it will start oscillating between the two positions B and C, with A as the mean position. It has been observed that if the angle θ is very small (less than $4°$), the bob will have simple harmonic motion. Now, the couple tending to restore the bob to the equilibrium position or restoring torque,

$$T = m.g \sin \theta \times L$$

Since angle θ is very small, therefore $\sin \theta = \theta$ radians.

$\therefore$ $\qquad\qquad\qquad T = m.g.L.\theta$

We know that the mass moment of inertia of the bob about an axis through the point of suspension,

$$I = \text{mass} \times (\text{length})^2 = m.L^2$$

$\therefore$ Angular acceleration of the string,

$$\alpha = \frac{T}{I} = \frac{m.g.L.\theta}{m.L^2} = \frac{g.\theta}{L} \quad \text{or} \quad \frac{\theta}{\alpha} = \frac{L}{g}$$

i.e. $\qquad \dfrac{\text{Angular displacement}}{\text{Angular acceleration}} = \dfrac{L}{g}$

We know that the periodic time,

$$t_p = 2\pi \sqrt{\frac{\text{Displacement}}{\text{Acceleration}}} = 2\pi \sqrt{\frac{L}{g}} \qquad \text{... (i)}$$

and frequency of oscillation,

$$n = \frac{1}{t_p} = \frac{1}{2\pi} \sqrt{\frac{g}{L}} \qquad \text{... (ii)}$$

From above we see that the periodic time and the frequency of oscillation of a simple pendulum depends only upon its length and acceleration due to gravity. The mass of the bob has no effect on it.

Notes : 1. The motion of the bob from one extremity to the other (*i.e.* from B to C or C to B) is known as **beat** or **swing**. Thus one beat = $\frac{1}{2}$ oscillation.

$\therefore$ Periodic time for one beat = $\pi \sqrt{L/g}$

2. A pendulum, which executes one beat per second (*i.e.* one complete oscillation in two seconds) is known as a **second's pendulum.**

4.6. Laws of Simple Pendulum

The following laws of a simple pendulum are important from the subject point of view :

1. *Law of isochronism*. It states, "The time period (t_p) of a simple pendulum does not depend upon its amplitude of vibration and remains the same, provided the angular amplitude (θ) does not exceed $4°$."

2. *Law of mass*. It states, "The time period (t_p) of a simple pendulum does not depend upon the mass of the body suspended at the free end of the string."

3. *Law of length*. It states, "The time period (t_p) of a simple pendulum is directly proportional to $\sqrt{L}$, where L is the length of the string."

4. *Law of gravity*. It states, "The time period (t_p) of a simple pendulum is inversely proportional to $\sqrt{g}$, where g is the acceleration due to gravity."

Note: The above laws of a simple pendulum are true from the equation of the periodic time *i.e.*

$$t_p = 2\pi\sqrt{L/g}$$

4.7. Closely-coiled Helical Spring

Consider a closely-coiled helical spring, whose upper end is fixed, as shown in Fig. 4.4. Let a body be attached to the lower end. Let AA be the equilibrium position of the spring, after the mass is attached. If the spring is stretched up to BB and then released, the mass will move up and down with simple harmonic motion.

Let m = Mass of the body in kg,

W = Weight of the body in newtons = $m.g$,

x = Displacement of the load below equilibrium position in metres,

s = Stiffnes of the spring in N/m *i.e.* restoring force per unit displacement from the equilibrium position,

a = Acceleration of the body in m/s².

We know that the deflection of the spring,

$$\delta = \frac{m.g}{s} \qquad\qquad\qquad ...(i)$$

Then disturbing force = $m.a$

and restoring force = $s.x$ $\qquad\qquad\qquad ...(ii)$

Equating equations (*i*) and (*ii*),

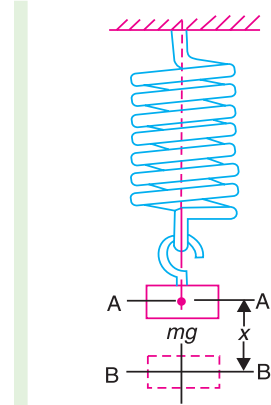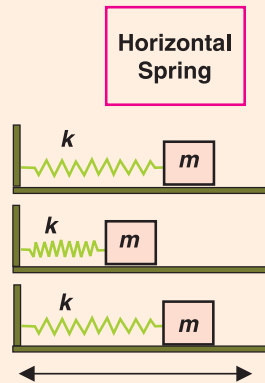$$m.a = s.x* \quad\text{or}\quad \frac{x}{a} = \frac{m}{s}$$

Fig. 4.4. Closely-coiled helical spring.

Simple Harmonic Motion (SHM)

We know that if we stretch a spring with a mass on the end and let it go, the mass will oscillate back and forth (If there is no friction). This oscillation is called **Simple Harmonic Motion.**

Horizontal Spring

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* The differential equation for the motion of the spring is

$$m\frac{d^2x}{dt^2} = -s.x \quad\text{or}\quad \frac{d^2x}{dt^2} = -\frac{s.x}{m} \qquad\qquad ...\left(\text{Here } \omega^2 = \frac{s}{m}\right)$$

The – ve sign indicates that the restoring force $s.x$ is opposite to the direction of disturbing force.

We know that periodic time,

$$t_p = 2\pi \sqrt{\frac{\text{Displacement}}{\text{Acceleration}}} = 2\pi \sqrt{\frac{x}{a}}$$

$$= 2\pi \sqrt{\frac{m}{s}} = 2\pi \sqrt{\frac{\delta}{g}} \qquad \qquad ... \left(\because \delta = \frac{mg}{s} \right)$$

and   frequency, $n = \dfrac{1}{t_p} = \dfrac{1}{2\pi} \sqrt{\dfrac{s}{m}} = \dfrac{1}{2\pi} \sqrt{\dfrac{g}{\delta}}$

Note: If the mass of the spring (m_1) is also taken into consideration, then the periodic time,

$$t_p = 2\pi \sqrt{\frac{m + m_1/3}{s}} \text{ seconds,}$$

and   frequency, $n = \dfrac{1}{2\pi} \sqrt{\dfrac{s}{m + m_1/3}}$ Hz

Example 4.3. *A helical spring, of negligible mass, and which is found to extend 0.25 mm under a mass of 1.5 kg, is made to support a mass of 60 kg. The spring and the mass system is displaced vertically through 12.5 mm and released. Determine the frequency of natural vibration of the system. Find also the velocity of the mass, when it is 5 mm below its rest position.*

Solution. Given : $m = 60$ kg ; $r = 12.5$ mm $= 0.0125$ m ; $x = 5$ mm $= 0.005$ m

Since a mass of 1.5 kg extends the spring by 0.25 mm, therefore a mass of 60 kg will extend the spring by an amount,

$$\delta = \frac{0.25}{1.5} \times 60 = 10 \text{ mm} = 0.01 \text{ m}$$

Frequency of the system

We know that frequency of the system,

$$n = \frac{1}{2\pi} \sqrt{\frac{g}{\delta}} = \frac{1}{2\pi} \sqrt{\frac{9.81}{0.01}} = 4.98 \text{ Hz } \textbf{Ans.}$$

Velocity of the mass

Let $v = $ Linear velocity of the mass.

We know that angular velocity,

$$\omega^* = \sqrt{\frac{g}{\delta}} = \sqrt{\frac{9.81}{0.01}} = 31.32 \text{ rad/s}$$

and

$$v = \omega \sqrt{r^2 - x^2} = 31.32 \sqrt{(0.0125)^2 - (0.005)^2} = 0.36 \text{ m/s } \textbf{Ans.}$$

4.8. Compound Pendulum

When a rigid body is suspended vertically, and it oscillates with a small amplitude under the action of the force of gravity, the body is known as ***compound pendulum***, as shown in Fig. 4.5.

Let $m = $ Mass of the pendulum in kg,

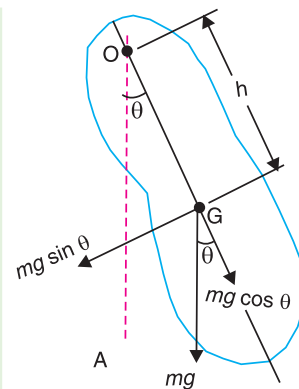$W = $ Weight of the pendulum in newtons $= m.g$,

Fig. 4.5. Compound pendulum.

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*   We know that periodic time,

$t_p = 2\pi / \omega$ or $\omega = 2\pi / t_p = 2\pi \times n = 2\pi \times 4.98 = 31.3$ rad/s   ...($\because n = 1/t_p$)

k_G = Radius of gyration about an axis through the centre of gravity G and perpendicular to the plane of motion, and

h = Distance of point of suspension O from the centre of gravity G of the body.

If the pendulum is given a small angular displacement θ, then the couple tending to restore the pendulum to the equilibrium position OA,

$$T = mg \sin \theta \times h = mgh \sin \theta$$

Since θ is very small, therefore substituting $\sin \theta = \theta$ radians, we get

$$T = mgh \, \theta$$

Now, the mass moment of inertia about the axis of suspension O,

$$I = I_G + m.h^2 = m\left(k_G^2 + h^2\right) \qquad \ldots \text{(By parallel axis theorem)}$$

∴ Angular acceleration of the pendulum,

$$\alpha = \frac{T}{I} = \frac{mgh \, \theta}{m(k_G^2 + h^2)} = \frac{gh \, \theta}{k_G^2 + h^2} = \text{constant} \times \theta$$

We see that the angular acceleration is directly proportional to angular displacement, therefore the pendulum executes simple harmonic motion.

∴ $$\frac{\theta}{\alpha} = \frac{k_G^2 + h^2}{g.h}$$

We know that the periodic time,

$$t_p = 2\pi \sqrt{\frac{\text{Displacement}}{\text{Acceleration}}} = 2\pi \sqrt{\frac{\theta}{\alpha}}$$

$$= 2\pi \sqrt{\frac{k_G^2 + h^2}{g.h}} \qquad \qquad \ldots (i)$$

and frequency of oscillation, $n = \dfrac{1}{t_p} = \dfrac{1}{2\pi} \sqrt{\dfrac{g.h}{k_G^2 + h^2}}$ $\qquad \qquad \ldots (ii)$

Notes : 1. Comparing this equation with equation (**ii**) of simple pendulum, we see that the equivalent length of a simple pendulum, which gives the same frequency as compound pendulum, is

$$L = \frac{k_G^2 + h^2}{h} = \frac{k_G^2}{h} + h$$

2. Since the equivalent length of simple pendulum (*L*) depends upon the distance between the point of suspension and the centre of gravity (*G*), therefore *L* can be changed by changing the position of point of suspension. This will, obviously, change the periodic time of a compound pendulum. The periodic time will be minimum if *L* is minimum. For *L* to be minimum, the differentiation of *L* with respect to *h* must be equal to zero, *i.e.*

$$\frac{dL}{dh} = 0 \quad \text{or} \quad \frac{d}{dh}\left(\frac{k_G^2}{h} + h\right) = 0$$

∴ $$\frac{-k_G^2}{h^2} + 1 = 0 \quad \text{or} \quad k_G = h$$

Thus the periodic time of a compound pendulum is minimum when the distance between the point of suspension and the centre of gravity is equal to the radius of gyration of the body about its centre of gravity.

∴ Minimum periodic time of a compound pendulum,

$$t_{p(min)} = 2\pi \sqrt{\frac{2k_G}{g}} \qquad \qquad \text{... [Substituting } h = k_G \text{ in equation (i)]}$$

4.9. Centre of Percussion

The centre of oscillation is sometimes termed as ***centre of percussion.*** It is defined as that point at which a blow may be struck on a suspended body so that the reaction at the support is zero.

Consider the case of a compound pendulum suspended at *O* as shown in Fig. 4.6. Suppose the pendulum is at rest in the vertical position, and a blow is struck at a distance *L* from the centre of suspension. Let the magnitude of blow is *F* newtons. A little consideration will show that this blow will have the following two effects on the body :

1. A force (*F*) acting at *C* will produce a linear motion with an acceleration *a*, such that

Fig. 4.6. Centre of percusssion.

$$F = m.a \qquad \qquad \text{... (i)}$$

where *m* is the mass of the body.

2. A couple with moment equal to (*F* × *l*) which will tend to produce a motion of rotation in the clockwise direction about the centre of gravity *G*. Let this turning moment (*F* × *l*) produce an angular acceleration (α), such that

$$F \times l = I_G \times \alpha \qquad \qquad \text{... (ii)}$$

where I_G is the moment of inertia of the body about an axis passing through *G* and parallel to the axis of rotation.

From equation (**i**) $\qquad a = F/m$ $\qquad \qquad \qquad \qquad$ ... (***iii***)

and from equation (**ii**), $\qquad \alpha = \dfrac{F.l}{I_G}$
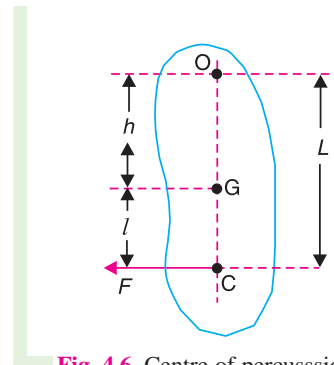
Now corresponding linear acceleration of O,

$$a_0 = \alpha.h = \frac{F.l.h}{I_G} = \frac{F.l.h}{m.k_G^2} \qquad \dots (iv)$$

$$(\because I_G = m.k_G^2)$$

where k_G is the radius of gyration of the body about the centre of gravity G.

Since there is no reaction at the support when the body is struck at the centre of percussion, therefore a should be equal to a_0.

Equating equations **(iii)** and **(iv)**,

$$\frac{F}{m} = \frac{F.l.h}{m.k_G^2}$$

or

$$k_G^2 = l.h, \text{ and } l = \frac{k_G^2}{h} \qquad \dots (v)$$

We know that the equivalent length of a simple pendulum,

$$L = \frac{k_G^2 + h}{h} = \frac{k_G^2}{h} + h = l + h \qquad \dots (vi)$$

From equations **(v)** and **(vi)**, it follows that

1. The centre of percussion is below the centre of gravity and at a distance k_G^2/h.

2. The distance between the centre of suspension and the centre of percussion is equal to the equivalent length of a simple pendulum.

Note: We know that mass moment of inertia of the body about O,

$$I_O = I_G + m.h^2 \quad \text{or} \quad m.k_G^2 = m.k_O^2 + m.h^2$$

$\therefore$

$$k_O^2 = k_G^2 + h^2 = l.h + h^2 = h\,(l + h) = OG \times OC \qquad \dots (\because k_G^2 = l.h)$$

It is thus obvious that the centre of suspension (O) and the centre of percussion (C) are inter-changeable. In other words, the periodic time and frequency of oscillation will be same, whether the body is suspended at the point of suspension or at the centre of percussion.

Example 4.4. *A uniform thin rod, as shown in Fig. 4.7, has a mass of 1 kg and carries a concentrated mass of 2.5 kg at B. The rod is hinged at A and is maintained in the horizontal position by a spring of stiffness 1.8 kN/m at C.*

Find the frequency of oscillation, neglecting the effect of the mass of the spring.
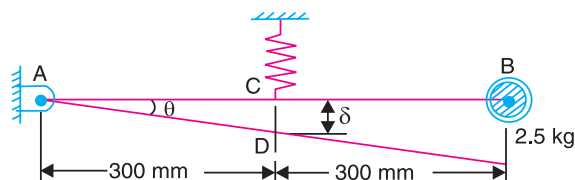
Fig. 4.7

A pendulum clock designed by Galileo. Galileo was the first to deisgn a clock based on the relationship between gravitational force (g), length of the pendulum (l) and time of oscillation (t).

Solution. Given : $m = 1$ kg ; $m_1 = 2.5$ kg ; $s = 1.8$ kN/m $= 1.8 \times 10^3$ N/m

We know that total length of rod,

$$l = 300 + 300 = 600 \text{ mm} = 0.6 \text{ m}$$

∴ Mass moment of inertia of the system about A,

I_A = Mass moment of inertia of 1 kg about A + Mass moment of interia of 2.5 kg about A

$$= \frac{m.l^2}{3} + m_1.l^2 = \frac{1(0.6)^2}{3} + 2.5\,(0.6)^2 = 1.02 \text{ kg-m}^2$$

If the rod is given a small angular displacement θ and then released, the extension of the spring,

$$\delta = 0.3 \sin\theta = 0.3\theta \text{ m}$$

$$\ldots (\because \theta \text{ is very small, therefore substituting } \sin\theta = \theta\,)$$

∴ Restoring force $= s.\delta = 1.8 \times 10^3 \times 0.3\,\theta = 540\,\theta$ N

and restoring torque about A $= 540\,\theta \times 0.3 = 162\,\theta$ N-m ... (*i*)

We know that disturbing torque about A

$$= I_A \times \alpha = 1.02\alpha \text{ N-m} \qquad\qquad \ldots (ii)$$

Equating equations (*i*) and (*ii*),

$$1.02\,\alpha = 162\,\theta \quad \text{or} \quad \alpha/\theta = 162/1.02 = 159$$

We know that frequency of oscillation,

$$n = \frac{1}{2\pi}\sqrt{\frac{\alpha}{\theta}} = \frac{1}{2\pi}\sqrt{159} = 2.01 \text{ Hz } \textbf{Ans.}$$

Example 4.5. *A small flywheel of mass 85 kg is suspended in a vertical plane as a compound pendulum. The distance of centre of gravity from the knife edge support is 100 mm and the flywheel makes 100 oscillations in 145 seconds. Find the moment of inertia of the flywheel through the centre of gravity.*

Solution. Given : $m = 85$ kg ; $h = 100$ mm $= 0.1$ m

Since the flywheel makes 100 oscillations in 145 seconds, therefore frequency of oscillation,

$$n = 100/145 = 0.69 \text{ Hz}$$

Let L = Equivalent length of simple pendulum, and

k_G = Radius of gyration through C.G.

We know that frequency of oscillation (n),

$$0.69 = \frac{1}{2\pi}\sqrt{\frac{g}{L}} = \frac{1}{2\pi}\sqrt{\frac{9.81}{L}} = \frac{0.5}{\sqrt{L}}$$

∴ $\sqrt{L} = 0.5/0.69 = 0.7246$ or $L = 0.525$ m

We also know that equivalent length of simple pendulum (L),

$$0.525 = \frac{k_G^2}{h} + h = \frac{k_G^2}{0.1} + 0.1 = \frac{k_G^2 + (0.1)^2}{0.1}$$

$$k_G^2 = 0.525 \times 0.1 - (0.1)^2 = 0.0425 \text{ m}^2$$

and moment of inertia of the flywheel through the centre of gravity,

$$I = m.k_G^2 = 85 \times 0.0425 = 3.6 \text{ kg-m}^2 \text{ Ans.}$$

Example 4.6. *The connecting rod of an oil engine has a mass of 60 kg, the distance between the bearing centres is 1 metre. The diameter of the big end bearing is 120 mm and of the small end bearing is 75 mm. When suspended vertically with a knife-edge through the small end, it makes 100 oscillations in 190 seconds and with knife-edge through the big end it makes 100 oscillations in 165 seconds. Find the moment of inertia of the rod in kg-m² and the distance of C.G. from the small end centre.*

Solution. Given : $m = 60$ kg ; $h_1 + h_2 = 1$ m ; $d_2* = 102$ mm; $d_1* = 75$ mm

Moment of inertia of the rod

First of all, let us find the radius of gyration of the connecting rod about the centre of gravity (*i.e.* k_G).

Let h_1 and h_2 = Distance of centre of gravity from the small and big end centres respectively,

L_1 and L_2 = Equivalent length of simple perdulum when the axis of oscillation coincides with the small and big end centres respectively

Connecting rod

When the axis of oscillation coincides with the small end centre, then frequency of oscillation,

$$n_1 = 100/190 = 0.526 \text{ Hz}$$

When the axis of oscillation coincides with the big end centre, the frequency of oscillation,

$$n_2 = 100/165 = 0.606 \text{ Hz}$$

We know that for a simple pendulum,

$$n_1 = \frac{1}{2\pi} \sqrt{\frac{g}{L_1}} \text{ Hz}$$

∴ $$L_1 = \frac{g}{(2\pi n_1)^2} = \frac{9.81}{(2\pi \times 0.526)^2} = 0.9 \text{ m}$$

Similarly $$L_2 = \frac{g}{(2\pi n_2)^2} = \frac{9.81}{(2\pi \times 0.606)^2} = 0.67 \text{ m}$$

We know that $$L_1 = \frac{k_G^2 + (h_1)^2}{h_1} \quad \text{or} \quad k_G^2 = L_1.h_1 - (h_1)^2 \qquad \text{... (i)}$$

Similarly $$k_G^2 = L_2.h_2 - (h_2)^2 \qquad \text{... (ii)}$$

From equations (*i*) and (*ii*), we have

$$L_1.h_1 - (h_1)^2 = L_2.h_2 - (h_2)^2$$

——————————

* Superfluous data.

$$0.9 \times h_1 - (h_1)^2 = 0.67 (1 - h_1) - (1 - h_1)^2 \qquad \ldots (\because h_1 + h_2 = 1 \text{ m})$$

$$= 0.67 - 0.67 h_1 - 1 - (h_1)^2 + 2h_1$$

$$0.9 \, h_1 + 0.67 \, h_1 - 2 \, h_1 = -0.33 \quad \text{or} \quad -0.43 \, h_1 = -0.33$$

$$\therefore \qquad\qquad h_1 = 0.33/0.43 = 0.767 \text{ m}$$

Substituting the value of h_1 in equation (*i*), we have

$$k_G^2 = 0.9 \times 0.767 - (0.767)^2 = 0.69 - 0.59 = 0.1 \, \text{m}^2$$

We know that mass moment of inertia of the rod,

$$I = m.k_G^2 = 60 \times 0.1 = 6 \text{ kg-m}^2 \text{ **Ans.**}$$

Distance of C.G. from the small end centre

We have calculated above that the distance of C.G. from the small end centre,

$$h_1 = 0.767 \text{ m } \textbf{Ans.}$$

Example 4.7. *A uniform slender rod 1.2 m long is fitted with a transverse pair of knife-edges, so that it can swing in a vertical plane as a compound pendulum. The position of the knife edges is variable. Find the time of swing of the rod, if **1.** the knife edges are 50 mm from one end of the rod, and **2.** the knife edges are so placed that the time of swing is minimum.*

*In case (**1**) find also the maximum angular velocity and the maximum angular acceleration of the rod if it swings through 3° on either side of the vertical.*

Solution. Given : $l = 1.2$ m ; $\theta = 3° = 3 \times \pi /180 = 0.052$ rad

1. *Time of swing of the rod when knife edges are 50 mm*

Since the distance between knife edges from one end of the rod is 50 mm = 0.05 m, therefore distance between the knife edge and C.G. of the rod,

$$h = \frac{1.2}{2} - 0.05 = 0.55 \, \text{m}$$

We know that radius of gyration of the rod about C.G.,

$$k^*_G = \frac{l}{\sqrt{12}} = \frac{1.2}{\sqrt{12}} = 0.35 \text{ m}$$

$\therefore$ Time of swing of the rod,

$$t_p = 2\pi \sqrt{\frac{k_G^2 + h^2}{g.h}} = 2\pi \sqrt{\frac{(0.35)^2 + (0.55)^2}{9.81 \times 0.55}}$$

$$= 1.76 \text{ s } \textbf{Ans.}$$

2. *Minimum time of swing*

We know that minimum time of swing,

$$t_{p(min)} = 2\pi \sqrt{\frac{2k_G}{g}} = 2\pi \sqrt{\frac{2 \times 0.35}{9.81}} = 1.68 \text{ s } \textbf{Ans.}$$

---

* We know that mass moment of inertia of the rod about an axis through C.G.

$$I = m. \, l^2/12$$

Also $\qquad I = m.k^2 \quad$ or $\qquad k^2 = I/m = m.l^2/12 \times m = l^2/12 \quad$ or $\quad k = l/\sqrt{12}$

Maximum angular velocity

In case (1), the angular velocity,

$$\omega = 2\pi / t_p = 2\pi /1.76 = 3.57 \text{ rad/s}$$

We know that maximum angular velocity,

$$\omega_{max} = \omega.\theta = 3.57 \times 0.052 = 0.1856 \text{ rad/s } \textbf{Ans.}$$

Maximum angular acceleration

We know that maximum angular acceleration,

$$\alpha_{max} = \omega^2.\theta = (3.57)^2 \times 0.052 = 0.663 \text{ rad/s}^2 \textbf{ Ans.}$$

Example 4.8. *The pendulum of an Izod impact testing machine has a mass of 30 kg. Its centre of gravity is 1.05 m from the axis of suspension and the striking knife is 150 mm below the centre of gravity. The time for 20 small free oscillations is 43.5 seconds. In making a test the pendulum is released from an angle of 60° to the vertical. Determine :*

1. the position of the centre of percussion relative to the striking knife and the striking velocity of the pendulum, and 2. the impulse on the pendulum and the sudden change of axis reaction when a specimen giving an impact value of 55 N-m is broken.

Solution. Given : m = 30 kg ; $OG = h$ = 1.05 m ; AG = 0.15 m

Since the time for 20 small free oscillations is 43.5 s, therefore frequency of oscillation,

$$n = \frac{20}{43.5} = 0.46 \text{ Hz}$$

1. *The position of centre of percussion relative to the striking knife and the striking velocity of the pendulum*

Let L = Equivalent length of simple pendulum,

k_G = Radius of gyration of the pendulum about the centre of gravity, and

k_O = Radius of gyration of the pendulum about O.

We know that the frequency of oscillation,

$$n = \frac{1}{2\pi} \sqrt{\frac{g}{L}}$$

or

$$L = \frac{g}{(2\pi n)^2} = \frac{9.81}{(2\pi \times 0.46)^2}$$

$$= 1.174 \text{ m}$$

∴ Distance of centre of percussion (C) from the centre of gravity (G),

$$CG = OC - OG = L - OG$$

$$= 1.174 - 1.05 = 0.124 \text{ m}$$

and distance of centre of percussion (C) from knife edge A,

$$AC = AG - CG = 0.15 - 0.124 = 0.026 \text{ m } \textbf{Ans.}$$

We know that $k_O^2 = (h + l)\ h = L.h = 1.174 \times 1.05 = 1.233 \text{ m}^2$

A little consideration will show that the potential energy of the pendulum is converted into kinetic energy of the pendulum before it strikes the test piece. Let v and ω be the linear and angular velocity of the pendulum before it strikes the test piece.

Fig. 4.8
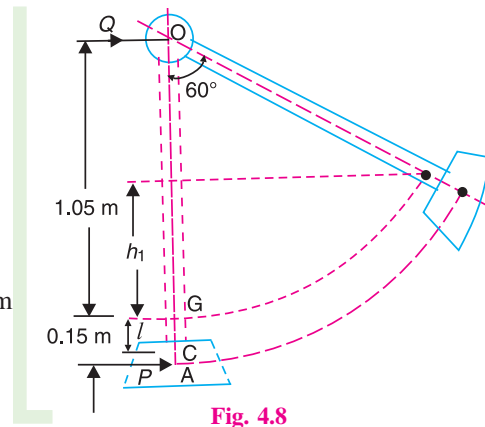
$$\therefore \qquad m.g.h_1 = \frac{1}{2}\,m.v^2 = \frac{1}{2}\,m.k_O^2.\omega^2 \qquad\qquad \dots\ (\because v = k_O.\omega)$$

$$30 \times 9.81 \times 1.05\,(1 - \cos 60°) = \frac{1}{2} \times 30 \times 1.233\,\omega^2 \quad\text{or}\quad 154.5 = 18.5\,\omega^2$$

$$\therefore \qquad\qquad \omega^2 = 154.5/18.5 = 8.35 \quad\text{or}\quad \omega = 2.9 \text{ rad/s}$$

$\therefore$ Velocity of striking $= \omega \times OA = 2.9\,(1.05 + 0.15) = 3.48$ m/s **Ans.**

2. *Impulse on the pendulum and sudden change of axis reaction*

It is given that the impact value of the specimen (*i.e.* the energy used for breaking the specimen) is 55 N-m. Let ω_1 be the angular velocity of the pendulum immediately after impact. We know that

$$\text{Loss of kinetic energy} = \frac{1}{2}\,I\,(\omega^2 - \omega_1^2) = \frac{1}{2}\,m.k_O^2\,(\omega^2 - \omega_1^2) = 55 \text{ N-m}$$

$$\therefore \quad \frac{1}{2} \times 30 \times 1.233\,(2.9^2 - \omega_1^2) = 55$$

$$18.5\,(8.41 - \omega_1^2) = 55 \quad\text{or}\quad \omega_1^2 = 8.41 - 55/18.5 = 5.44$$

$$\therefore \qquad\qquad \omega_1 = 2.33 \text{ rad/s}$$

Let P and Q be the impulses at the knife edge A and at the pivot O respectively as shown in Fig. 4.8.

$$\therefore \qquad\qquad P + Q = \text{Change of linear momentum}$$
$$= m.h\,(\omega - \omega_1) = 30 \times 1.05\,(2.9 - 2.33) = 17.95 \qquad \dots\ (i)$$

Taking moments about G,

$$0.15\,P - 1.05\,Q = \text{Change of angular momentum}$$
$$= m.k_G^2\,(\omega - \omega_1) = m\,(k_O^2 - h^2)\,(\omega - \omega_1)$$
$$= 30\,(1.233 - 1.05^2)\,(2.9 - 2.33) = 2.27 \qquad \dots\ (ii)$$

From equations (*i*) and (*ii*),

$$P = 17.6 \text{ N-s; and } Q = 0.35 \text{ N-s } \textbf{Ans.}$$

$\therefore$ Change in axis reaction when pendulum is vertical

$$= \text{Change in centrifugal force}$$

$$= m\,(\omega^2 - \omega_1^2)\,h = 30\,(2.9^2 - 2.33^2)\,1.05 = 94 \text{ N } \textbf{Ans.}$$

4.10. Bifilar Suspension

The moment of inertia of a body may be determined experimentally by an apparatus called *bifilar suspension*. The body whose moment of inertia is to be determined (say $A\,B$) is suspended by two long parallel flexible strings as shown in Fig. 4.9. When the body is twisted through a small angle θ about a vertical axis through the centre of gravity G, it will vibrate with simple harmonic motion in a horizontal plane.

Let $\qquad m$ = Mass of the body,

$\qquad\qquad W$ = Weight of the body in newtons $= m.g$,

$\qquad\qquad k_G$ = Radius of gyration about an axis through the centre of gravity,

I = Mass moment of inertia of the body about a vertical axis through $G = m.k_G^2$,

l = Length of each string,

x = Distance of A from G (*i.e. AG*),

y = Distance of B from G (*i.e. BG*),

θ = Small angular displacement of the body from the equilibrium position in the horizontal plane,

ϕ_A and ϕ_B = Corresponding angular displacements of the strings, and

α = Angular acceleration towards the equilibrium position.

Fig. 4.9. Bifilar suspension.

When the body is stationary, the tension in the strings are given by

$$T_A = \frac{m.g.y}{x+y}, \quad \text{and} \quad T_B = \frac{m.g.x}{x+y} \quad \text{...(Taking moments about } B \text{ and } A \text{ respectively,)}$$

When the body is displaced from its equilibrium position in a horizontal plane through a small angle θ, then the angular displacements of the strings are given by

$$AA' = \phi_A.l = x.\theta \text{ ; and } BB' = \phi_B.l = y.\theta$$

$\therefore$

$$\phi_A = \frac{x.\theta}{l}; \text{ and } \phi_B = \frac{y.\theta}{l}$$

Component of tension T_A in the horizontal plane, acting normal to $A'B'$ at A' as shown in Fig. 4.9

$$= T_A.\phi_A = \frac{m.g.y}{x+y} \times \frac{x.\theta}{l} = \frac{m.g.x.y.\theta}{l(x+y)}$$

Component of tension T_B in the horizontal plane, acting normal to $A'B'$ at B' as shown in Fig. 4.9

$$= T_B.\phi_B = \frac{m.g.x}{x+y} \times \frac{y.\theta}{l} = \frac{m.g.x.y.\theta}{l(x+y)}$$

These components of tensions T_A and T_B are equal and opposite in direction, which gives rise to a couple. The couple or torque applied to each string to restore the body to its initial equilibrium position, *i.e.* restoring torque

$$= T_A.\phi_A.x + T_B.\phi_B.y$$

$$= \frac{m.g.x.y.\theta}{l(x+y)}(x+y) = \frac{m.g.x.y.\theta}{l} \qquad \qquad ... \text{(}i\text{)}$$

and accelerating (or disturbing) torque

$$= I.\alpha = m.k_G^2.\alpha \qquad \qquad ... \text{(}ii\text{)}$$

Equating equations (*i*) and (*ii*),

$$\frac{m.g.x.y.\theta}{l} = m.k_G^2.\alpha \quad \text{or} \quad \frac{\theta}{\alpha} = \frac{k_G^2.l}{g.x.y}$$
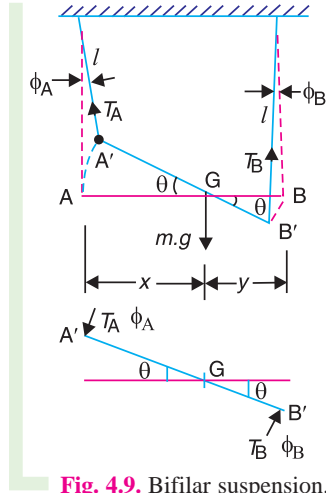
i.e. $\dfrac{\text{Angular displacement}}{\text{Angular acceleration}} = \dfrac{k_G^2 \, l}{g.x.y}$

We know that periodic time,

$$t_p = 2\pi \sqrt{\dfrac{\text{Angular displacement}}{\text{Angular acceleration}}} = 2\pi \sqrt{\dfrac{k_G^2 \, l}{g.x.y}}$$

$$= 2\pi k_G \sqrt{\dfrac{l}{g.x.y}}$$

and   frequency, $n = \dfrac{1}{t_p} = \dfrac{1}{2\pi \, k_G} \sqrt{\dfrac{g.x.y}{l}}$

Note : The bifilar suspension is usually used for finding the moment of inertia of a connecting rod of an engine. In this case, the wires are attached at equal distances from the centre of gravity of the connecting rod (*i.e.* $x = y$) so that the tension in each wire is same.

Example 4.9. *A small connecting rod of mass 1.5 kg is suspended in a horizontal plane by two wires 1.25 m long. The wires are attached to the rod at points 120 mm on either side of the centre of gravity. If the rod makes 20 oscillations in 40 seconds, find the radius of gyration and the mass moment of inertia of the rod about a vertical axis through the centre of gravity.*

Solution. Given : $m = 1.5$ kg ; $l = 1.25$ m ; $x = y = 120$ mm $= 0.12$ m

Since the rod makes 20 oscillations in 40 s, therefore frequency of oscillation,

$n = 20/40 = 0.5$ Hz

Radius of gyration of the connecting rod

Let k_G = Radius of gyration of the connecting rod.

We know that frequency of oscillation (n),

$$0.5 = \dfrac{1}{2\pi \, k_G} \sqrt{\dfrac{g.x.y}{l}} = \dfrac{1}{2\pi \, k_G} \sqrt{\dfrac{9.81 \times 0.12 \times 0.12}{1.25}} = \dfrac{0.0535}{k}$$

∴ $k_G = 0.0535/0.5 = 0.107$ m $= 107$ mm **Ans.**

Mass moment of inertia of the connecting rod

We know that mass moment of inertia,

$I = m \, (k_G)^2 = 1.5 \, (0.107)^2 = 0.017$ kg-m^2 **Ans.**

4.11. Trifilar Suspension (Torsional Pendulum)

It is also used to find the moment of inertia of a body experimentally. The body (say a disc or flywheel) whose moment of inertia is to be determined is suspended by three long flexible wires *A, B* and *C*, as shown in Fig. 4.10. When the body is twisted about its axis through a small angle θ and then released, it will oscillate with simple harmonic motion.

Let m = Mass of the body in kg,

W = Weight of the body in newtons $= m.g$,

k_G = Radius of gyration about an axis through c.g.,

Fig. 4.10. Trifilar suspension.

I = Mass moment of inertia of the disc about an axis through O and perpendicular to it $= m.k^2$,

l = Length of each wire,

r = Distance of each wire from the axis of the disc,

θ = Small angular displacement of the disc,

ϕ = Corresponding angular displacement of the wires, and

α = Angular acceleration towards the equilibrium position.

Then, for small displacements,

$$r.\,\theta = l.\,\phi \quad \text{or} \quad \phi = r.\theta/l$$

Since the three wires are attached symmetrically with respect to the axis, therefore the tension in each wire will be one-third of the weight of the body.

∴ Tension in each wire $= m.g/3$

Component of the tension in each wire perpendicular to r

$$= \frac{m.g.\sin\phi}{3} = \frac{m.g.\phi}{3} = \frac{m.g.r.\theta}{3l} \qquad \ldots (\because \phi \text{ is a small angle, and } \phi = r.\theta/l)$$

∴ Torque applied to each wire to restore the body to its initial equilibrium position *i.e.* restoring torque

$$= \frac{m.g.r.\theta}{3l} \times r = \frac{m.g.r^2.\theta}{3l}$$

Total restoring torque applied to three wires,

$$T = 3 \times \frac{m.g.r^2.\theta}{3l} = \frac{m.g.r^2.\theta}{l} \qquad \qquad \text{... } (i)$$

We know that disturbing torque

$$= I.\alpha = m.k_G^2.\alpha \qquad \qquad \text{... } (ii)$$

Equating equations (*i*) and (*ii*),

$$\frac{m.g.r^2.\theta}{l} = m.k_G^2.\alpha \quad \text{or} \quad \frac{\theta}{\alpha} = \frac{l.k_G^2}{g.r^2}$$

i.e. $\dfrac{\text{Angular displacement}}{\text{Angular acceleration}} = \dfrac{l.k_G^2}{g.r^2}$

We know that periodic time,

$$t_p = 2\pi \sqrt{\frac{\text{Angular displacement}}{\text{Angular acceleration}}} = 2\pi \sqrt{\frac{l.k_G^2}{g.r^2}} = \frac{2\pi k_G}{r} \sqrt{\frac{l}{g}}$$

and frequency, $n = \dfrac{1}{t_p} = \dfrac{r}{2\pi k_G} \sqrt{\dfrac{g}{l}}$

Example 4.10. *In order to find the radius of gyration of a car, it is suspended with its axis vertical from three parallel wires 2.5 metres long. The wires are attached to the rim at points spaced 120° apart and at equal distances 250 mm from the axis.*

It is found that the wheel makes 50 torsional oscillations of small amplitude about its axis in 170 seconds. Find the radius of gyration of the wheel.

Solution. Given : $l = 2.5$ m ; $r = 250$ mm $= 0.25$ m ;

Since the wheel makes 50 torsional oscillations in 170 seconds, therefore frequency of oscillation,

$$n = 50/170 = 5/17 \text{ Hz}$$

Let k_G = Radius of gyration of the wheel

We know that frequency of oscillation (n),

$$\frac{5}{17} = \frac{r}{2\pi\,k_G}\sqrt{\frac{g}{l}} = \frac{0.25}{2\pi\,k_G}\sqrt{\frac{9.81}{2.5}} = \frac{0.079}{k_G}$$

∴ $k_G = 0.079 \times 17/5 = 0.268$ m $= 268$ mm **Ans.**

Example 4.11. *A connecting rod of mass 5.5 kg is placed on a horizontal platform whose mass is 1.5 kg. It is suspended by three equal wires, each 1.25 m long, from a rigid support. The wires are equally spaced round the circumference of a circle of 125 mm radius. When the c.g. of the connecting rod coincides with the axis of the circle, the platform makes 10 angular oscillations in 30 seconds. Determine the mass moment of inertia about an axis through its c.g.*

      Solution. Given : $m_1 = 5.5$ kg ; $m_2 = 1.5$ kg ; $l = 1.25$ m ; $r = 125$ mm $= 0.125$ m

Since the platform makes 10 angular oscillations in 30 s, therefore frequency of oscillation,

                 $n = 10/30 = 1/3$ Hz

Let k_G = Radius of gyration about an axis through the c.g.

We know that frequency of oscillation (n)

$$\frac{1}{3} = \frac{r}{2\pi k_G}\sqrt{\frac{g}{l}} = \frac{0.125}{2\pi\,k_G}\sqrt{\frac{9.81}{1.25}} = \frac{0.056}{k_G}$$

∴ $k_G = 0.056 \times 3 = 0.168$ m

and mass moment of inertia about an axis through its c.g.,

$$I = m.k_G^2 = (m_1 + m_2)\,k_G^2 = (5.5 + 1.5)\,(0.168)^2 \text{ kg-m}^2$$

$$= 0.198 \text{ kg-m}^2 \textbf{ Ans.}$$

EXERCISES

1.   A particle, moving with simple harmonic motion, performs 10 complete oscillations per minute and its speed, when at a distance of 80 mm from the centre of oscillation is 3/5 of the maximum speed. Find the amplitude, the maximum acceleration and the speed of the particle, when it is 60 mm from the centre of the oscillation.   **[Ans. 100 mm ; 109.6 mm/s^2 ; 83.76 mm/s]**

2.   A piston, moving with a simple harmonic motion, has a velocity of 8 m/s, when it is 1 metre from the centre position and a velocity of 4 m/s, when it is 2 metres from the centre. Find : 1. Amplitude, 2. Periodic time, 3. Maximum velocity, and 4. Maximum acceleration.

                                             [Ans. 2.236 m ; 1.571 s ; 8.94 m/s ; 35.77 m/s^2]

3.   The plunger of a reciprocating pump is driven by a crank of radius 250 mm rotating at 12.5 rad/s. Assuming simple harmonic motion, determine the maximum velocity and maximum acceleration of the plunger.   **[Ans. 3.125 m/s ; 39.1 m/s^2]**

4.   A part of a machine of mass 4.54 kg has a reciprocating motion which is simple harmonic in character. It makes 200 complete oscillations in 1 minute. Find : 1. the accelerating force upon it and its velocity when it is 75 mm, from midstroke ; 2. the maximum accelerating force, and 3. the maximum velocity if its total stroke is 225 mm *i.e.* if the amplitude of vibration is 112.5 mm.

                                       [Ans. 149.5 N ; 1.76 m/s ; 224 N ; 2.36 m/s]

5.   A helical spring of negligible mass is required to support a mass of 50 kg. The stiffness of the spring is 60 kN/m. The spring and the mass system is displaced vertically by 20 mm below the equilibrium position and then released. Find : 1. the frequency of natural vibration of the system ; 2. the velocity and acceleration of the mass when it is 10 mm below the rest position.

                                     [Ans. 5.5 Hz ; 0.6 m/s ; 11.95 m/s^2]

6.   A spring of stiffness 2 kN/m is suspended vertically and two equal masses of 4 kg each are attached to the lower end. One of these masses is suddenly removed and the system oscillates. Determine : 1. the amplitude of vibration, 2. the frequency of vibration, 3. the velocity and acceleration of the mass when

passing through half amplitude position, and 4. kinetic energy of the vibration in joules.

[Ans. 0.019 62 m ; 3.56 Hz ; 0.38 m/s , 4.9 m/s^2 ; 0.385 J]

7. A vertical helical spring having a stiffness of 1540 N/m is clamped at its upper end and carries a mass of 20 kg attached to the lower end. The mass is displaced vertically through a distance of 120 mm and released. Find : 1. Frequency of oscillation ; 2. Maximum velocity reached ; 3. Maximum acceleration; and 4. Maximum value of the inertia force on the mass.

[Ans. 1.396 Hz ; 1.053 m/s ; 9.24 m/s^2 ; 184.8 N]

8. A small flywheel having mass 90 kg is suspended in a vertical plane as a compound pendulum. The distance of centre of gravity from the knife edge support is 250 mm and the flywheel makes 50 oscillations in 64 seconds. Find the moment of inertia of the flywheel about an axis through the centre of gravity. **[Ans. 3.6 kg-m^2]**

9. The connecting rod of a petrol engine has a mass 12 kg. In order to find its moment of inertia it is suspended from a horizontal edge, which passes through small end and coincides with the small end centre. It is made to swing in a vertical plane, such that it makes 100 oscillations in 96 seconds. If the point of suspension of the connecting rod is 170 mm from its c.g., find : 1. radius of gyration about an axis through its c.g., 2. moment of inertia about an axis through its c.g., and 3. length of the equivalent simple pendulum. **[Ans. 101 mm ; 0.1224 kg-m^2 ; 0.23 m]**

10. A connecting rod of mass 40 kg is suspended vertically as a compound pendulum. The distance between the bearing centres is 800 mm. The time for 60 oscillations is found to be 92.5 seconds when the axis of oscillation coincides with the small end centre and 88.4 seconds when it coincides with the big end centre. Find the distance of the centre of gravity from the small end centre, and the moment of inertia of the rod about an axis through the centre of gravity. **[Ans. 0.442 m ; 2.6 kg-m^2]**

11. The following data were obtained from an experiment to find the moment of inertia of a pulley by bifilar suspension :
Mass of the pulley = 12 kg ; Length of strings = 3 m ; Distance of strings on either side of centre of gravity = 150 mm ; Time for 20 oscillations about the vertical axis through c.g. = 46.8 seconds
Calculate the moment of inertia of the pulley about the axis of rotation.

[Ans. 0.1226 kg-m^2]

12. In order to find the moment of inertia of a flywheel, it is suspended in the horizontal plane by three wires of length 1.8 m equally spaced around a circle of 185 mm diameter. The time for 25 oscillations in a horizontal plane about a vertical axis through the centre of flywheel is 54 s. Find the radius of gyration and the moment of inertia of the flywheel if it has a mass of 50 kg.

[Ans. 74.2 mm; 0.275 kg-m^2]

DO YOU KNOW ?

1. Explain the meaning of S.H.M. and give an example of S.H.M.
2. Define the terms amplitude, periodic time, and frequency as applied to S.H.M.
3. Show that when a particle moves with simple harmonic motion, its time for a complete oscillation is independent of the amplitude of its motion.
4. Derive an expression for the period of oscillation of a mass when attached to a helical spring.
5. What is a simple pendulum ? Under what conditions its motion is regarded as simple harmonic?
6. Prove the formula for the frequency of oscillation of a compound pendulum. What is the length of a simple pendulum which gives the same frequency as compound pendulum ?
7. Show that the minimum periodic time of a compound pendulum is

$$t_{p(min)} = 2\pi \sqrt{\frac{2k_G}{g}}$$

where k_G is the radius of gyration about the centre of gravity.

8. What do you understand by centre of percussion ? Prove that it lies below the centre of gravity of the body and at a distance k_G^2/h, where k_G is the radius of gyration about c.g. and h is the distance between the centre of suspension and centre of gravity.

9. Describe the method of finding the moment of inertia of a connecting rod by means of bifilar suspension. Derive the relations for the periodic time and frequency of oscillation.

10. What is a torsional pendulum ? Show that periodic time of a torsional pendulum is

$$t_p = \frac{2\pi k_G}{r} \sqrt{\frac{l}{g}}$$

where k_G = Radius of gyration,

l = Length of each wire, and

r = Distance of each wire from the axis of the disc.

OBJECTIVE TYPE QUESTIONS

1. The periodic time (t_p) is given by
   (a) $\omega / 2\pi$ (b) $2\pi / \omega$ (c) $2\pi \times \omega$ (d) π/ω

2. The velocity of a particle moving with simple harmonic motion is . . . . at the mean position.
   (a) zero   (b) minimum   (c) maximum

3. The velocity of a particle (v) moving with simple harmonic motion, at any instant is given by
   (a) $\omega \sqrt{r^2 - x^2}$ (b) $\omega \sqrt{x^2 - r^2}$ (c) $\omega^2 \sqrt{r^2 - x^2}$ (d) $\omega^2 \sqrt{x^2 - r^2}$

4. The maximum acceleration of a particle moving with simple harmonic motion is
   (a) ω (b) $\omega.r$ (c) $\omega^2.r$ (d) ω^2/r

5. The frequency of oscillation for the simple pendulum is
   (a) $\frac{1}{2\pi} \sqrt{\frac{L}{g}}$ (b) $\frac{1}{2\pi} \sqrt{\frac{g}{L}}$ (c) $2\pi \sqrt{\frac{L}{g}}$ (d) $2\pi \sqrt{\frac{g}{L}}$

6. When a rigid body is suspended vertically and it oscillates with a small amplitude under the action of the force of gravity, the body is known as
   (a) simple pendulum   (b) torsional pendulum
   (c) compound pendulum   (d) second's pendulum

7. The frequency of oscillation of a compound pendulum is
   (a) $\frac{1}{2\pi} \sqrt{\frac{g.h}{k_G^2 + h^2}}$ (b) $\frac{1}{2\pi} \sqrt{\frac{k_G^2 + h^2}{g.h}}$ (c) $2\pi \sqrt{\frac{g.h}{k_G^2 + h^2}}$ (d) $2\pi \sqrt{\frac{k_G^2 + h^2}{g.h}}$

   where k_G = Radius of gyration about the centroidal axis, and
          h = Distance between the point of suspension and centre of gravity of the body.

8. The equivalent length of a simple pendulum which gives the same frequency as the compound pendulum is
   (a) $\frac{h}{k_G^2 + h^2}$ (b) $\frac{k_G^2 + h^2}{h}$ (c) $\frac{h^2}{k_G^2 + h^2}$ (d) $\frac{k_G^2 + h^2}{h^2}$

9. The centre of percussion is below the centre of gravity of the body and is at a distance equal to
   (a) h / k_G (b) $h.k_G$ (c) h^2/k_G (d) k_G^2 / h

10. The frequency of oscillation of a torsional pendulum is
    (a) $\frac{2\pi k_G}{r} \sqrt{\frac{g}{l}}$ (b) $\frac{r}{2\pi k_G} \sqrt{\frac{g}{l}}$ (c) $\frac{2\pi k_G}{r} \sqrt{\frac{l}{g}}$ (d) $\frac{r}{2\pi k_G} \sqrt{\frac{l}{g}}$

ANSWERS

1. (b)	**2.** (c)	**3.** (a)	**4.** (c)	**5.** (b)
6. (c)	**7.** (a)	**8.** (b)	**9.** (d)	**10.** (b)