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2

Kinematics of Motion

Features

2.1. Introduction

We have discussed in the previous Chapter, that the subject of Theory of Machines deals with the motion and forces acting on the parts (or links) of a machine. In this chapter, we shall first discuss the kinematics of motion *i.e.* the relative motion of bodies without consideration of the forces causing the motion. In other words, kinematics deal with the geometry of motion and concepts like displacement, velocity and acceleration considered as functions of time.

2.2. Plane Motion

When the motion of a body is confined to only one plane, the motion is said to be ***plane motion.*** The plane motion may be either rectilinear or curvilinear.

2.3. Rectilinear Motion

It is the simplest type of motion and is along a straight line path. Such a motion is also known as ***translatory motion.***

2.4. Curvilinear Motion

It is the motion along a curved path. Such a motion, when confined to one plane, is called ***plane curvilinear motion.***

When all the particles of a body travel in concentric circular paths of constant radii (about the axis of rotation perpendicular to the plane of motion) such as a pulley rotating

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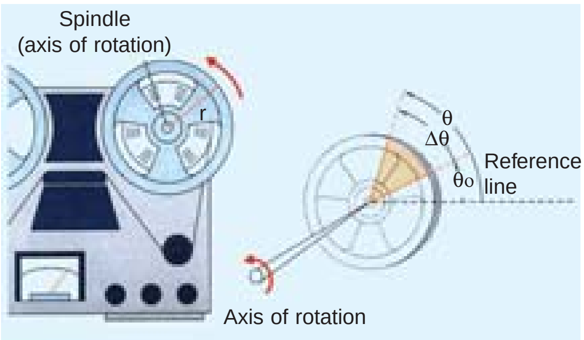
about a fixed shaft or a shaft rotating about its own axis, then the motion is said to be a ***plane rotational motion.***

Note: The motion of a body, confined to one plane, may not be either completely rectilinear nor completely rotational. Such a type of motion is called combined rectilinear and rotational motion. This motion is discussed in Chapter 6, Art. 6.1.

2.5. Linear Displacement

It may be defined as the distance moved by a body with respect to a certain fixed point. The displacement may be along a straight or a curved path. In a reciprocating steam engine, all the particles on the piston, piston rod and cross-head trace a straight path, whereas all particles on the crank and crank pin trace circular paths, whose centre lies on the axis of the crank shaft. It will be interesting to know, that all the particles on the connecting rod neither trace a straight path nor a circular one; but trace an oval path, whose radius of curvature changes from time to time.

The displacement of a body is a vector quantity, as it has both magnitude and direction. Linear displacement may, therefore, be represented graphically by a straight line.



2.6. Linear Velocity

It may be defined as the rate of change of linear displacement of a body with respect to the time. Since velocity is always expressed in a particular direction, therefore it is a vector quantity. Mathematically, linear velocity,

$$v = ds/dt$$

Notes: 1. If the displacement is along a circular path, then the direction of linear velocity at any instant is along the tangent at that point.

2. The speed is the rate of change of linear displacement of a body with respect to the time. Since the speed is irrespective of its direction, therefore, it is a scalar quantity.

2.7. Linear Acceleration

It may be defined as the rate of change of linear velocity of a body with respect to the time. It is also a vector quantity. Mathematically, linear acceleration,

$$a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2 s}{dt^2} \qquad \ldots \left(\because \; v = \frac{ds}{dt}\right)$$

Notes: 1. The linear acceleration may also be expressed as follows:

$$a = \frac{dv}{dt} = \frac{ds}{dt} \times \frac{dv}{ds} = v \times \frac{dv}{ds}$$

2. The negative acceleration is also known as ***deceleration*** or ***retardation.***

2.8. Equations of Linear Motion

The following equations of linear motion are important from the subject point of view:

1. $v = u + a.t$
2. $s = u.t + \frac{1}{2} a.t^2$
3. $v^2 = u^2 + 2a.s$
4. $s = \frac{(u+v)}{2} \times t = v_{av} \times t$

where

u = Initial velocity of the body,

v = Final velocity of the body,

a = Acceleration of the body,

s = Displacement of the body in time t seconds, and

v_{av} = Average velocity of the body during the motion.

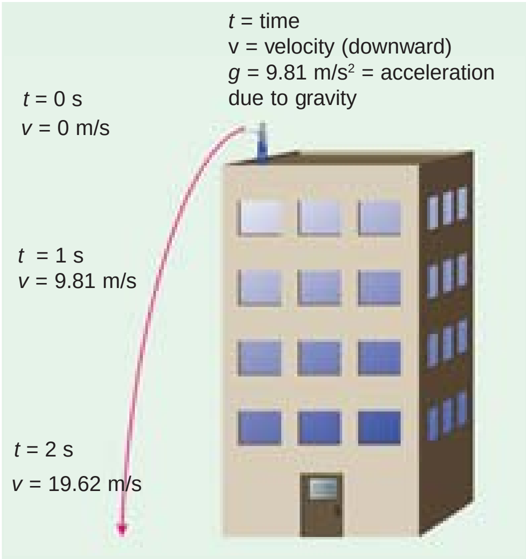
Notes: 1. The above equations apply for uniform acceleration. If, however, the acceleration is variable, then it must be expressed as a function of either t, s or v and then integrated.

2. In case of vertical motion, the body is subjected to gravity. Thus g (acceleration due to gravity) should be substituted for 'a' in the above equations.

3. The value of g is taken as + 9.81 m/s^2 for downward motion, and – 9.81 m/s^2 for upward motion of a body.

4. When a body falls freely from a height h, then its velocity v, with which it will hit the ground is given by

$$v = \sqrt{2\, g.h}$$



2.9. Graphical Representation of Displacement with Respect to Time

The displacement of a moving body in a given time may be found by means of a graph. Such a graph is drawn by plotting the displacement as ordinate and the corresponding time as abscissa. We shall discuss the following two cases :

1. *When the body moves with uniform velocity.* When the body moves with uniform velocity, equal distances are covered in equal intervals of time. By plotting the distances on Y-axis and time on X-axis, a displacement-time curve (*i.e.* s-t curve) is drawn which is a straight line, as shown in Fig. 2.1 (a). The motion of the body is governed by the equation $s = u.t$, such that

Velocity at instant $1 = s_1 / t_1$

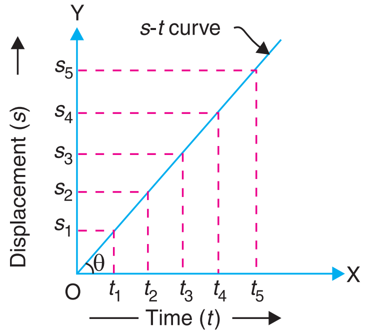
Velocity at instant $2 = s_2 / t_2$

Since the velocity is uniform, therefore

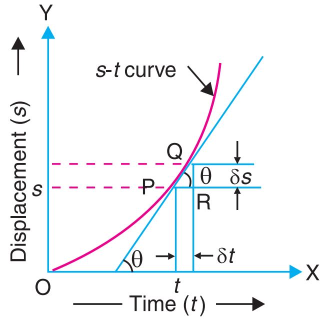
$$\frac{s_1}{t_1} = \frac{s_2}{t_2} = \frac{s_3}{t_3} = \tan\theta$$

where $\tan\theta$ is called the slope of s-t curve. In other words, the slope of the s-t curve at any instant gives the velocity.

2. *When the body moves with variable velocity.* When the body moves with variable velocity, unequal distances are covered in equal intervals of time or equal distances are covered in unequal intervals of time. Thus the displacement-time graph, for such a case, will be a curve, as shown in Fig. 2.1 (*b*).



(*a*) Uniform velocity.



(*b*) Variable velocity.

Fig. 2.1. Graphical representation of displacement with respect to time.

Consider a point *P* on the *s*-*t* curve and let this point travels to *Q* by a small distance δs in a small interval of time δt. Let the chord joining the points *P* and *Q* makes an angle θ with the horizontal. The average velocity of the moving point during the interval *PQ* is given by

$$\tan \theta = \delta s / \delta t \qquad \text{. . . (From triangle } PQR\text{)}$$

In the limit, when δt approaches to zero, the point *Q* will tend to approach *P* and the chord *PQ* becomes tangent to the curve at point *P*. Thus the velocity at *P*,

$$v_p = \tan \theta = ds/dt$$

where tan θ is the slope of the tangent at *P*. Thus the slope of the tangent at any instant on the *s*-*t* curve gives the velocity at that instant.

2.10. Graphical Representation of Velocity with Respect to Time

We shall consider the following two cases :

1. *When the body moves with uniform velocity.* When the body moves with zero acceleration, then the body is said to move with a uniform velocity and the velocity-time curve (*v*-*t* curve) is represented by a straight line as shown by *A B* in Fig. 2.2 (*a*).

We know that distance covered by a body in time *t* second

= Area under the *v*-*t* curve *A B*

= Area of rectangle *OABC*

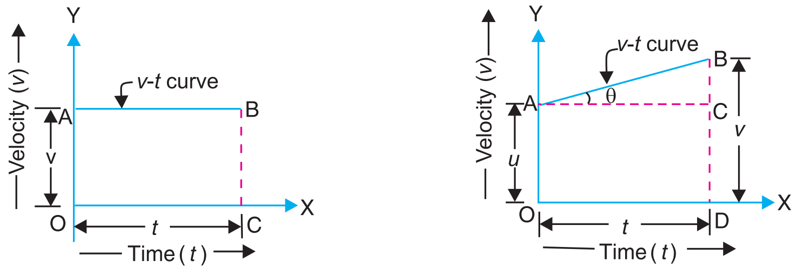
Thus, the distance covered by a body at any interval of time is given by the area under the *v*-*t* curve.

2. *When the body moves with variable velocity.* When the body moves with constant acceleration, the body is said to move with variable velocity. In such a case, there is equal variation of velocity in equal intervals of time and the velocity-time curve will be a straight line *AB* inclined at an angle θ, as shown in Fig. 2.2 (*b*). The equations of motion *i.e.* $v = u + a.t$, and $s = u.t + \frac{1}{2} a.t^2$ may be verified from this *v*-*t* curve.

Let u = Initial velocity of a moving body, and

v = Final velocity of a moving body after time t.

Then,
$$\tan\theta = \frac{BC}{AC} = \frac{v-u}{t} = \frac{\text{Change in velocity}}{\text{Time}} = \text{Acceleration } (a)$$



(*a*) Uniform velocity. (*b*) Variable velocity.

Fig. 2.2. Graphical representation of velocity with respect to time.

Thus, the slope of the v-t curve represents the acceleration of a moving body.

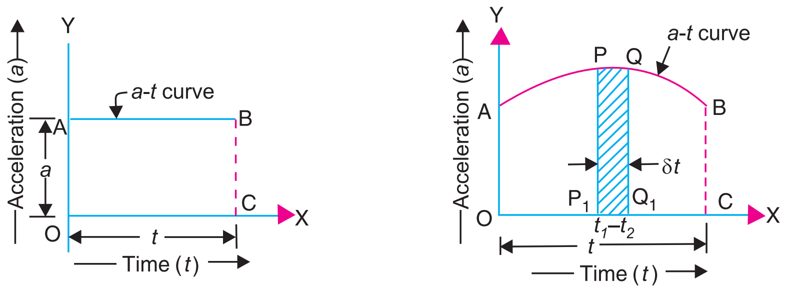
Now
$$a = \tan\theta = \frac{BC}{AC} = \frac{v-u}{t} \quad \text{or} \quad v = u + a.t$$

Since the distance moved by a body is given by the area under the v-t curve, therefore distance moved in time (t),

$$s = \text{Area } OABD = \text{Area } OACD + \text{Area } ABC$$

$$= u.t + \frac{1}{2}(v-u)t = u.t + \frac{1}{2}a.t^2 \qquad \ldots (\because v - u = a.t)$$

2.11. Graphical Representation of Acceleration with Respect to Time



(*a*) Uniform velocity. (*b*) Variable velocity.

Fig. 2.3. Graphical representation of acceleration with respect to time.

We shall consider the following two cases :

1. *When the body moves with uniform acceleration.* When the body moves with uniform acceleration, the acceleration-time curve (a-t curve) is a straight line, as shown in Fig. 2.3(a). Since the change in velocity is the product of the acceleration and the time, therefore the area under the a-t curve (*i.e.* $OABC$) represents the change in velocity.

2. *When the body moves with variable acceleration.* When the body moves with variable acceleration, the a-t curve may have any shape depending upon the values of acceleration at various instances, as shown in Fig. 2.3(b). Let at any instant of time t, the acceleration of moving body is a.

Mathematically, $a = dv / dt$ or $dv = a.dt$

Integrating both sides,

$$\int_{v_1}^{v_2} dv = \int_{t_1}^{t_2} a.dt \quad \text{or} \quad v_2 - v_1 = \int_{t_1}^{t_2} a.dt$$

where v_1 and v_2 are the velocities of the moving body at time intervals t_1 and t_2 respectively.

The right hand side of the above expression represents the area (PQQ_1P_1) under the *a-t* curve between the time intervals t_1 and t_2 . Thus the area under the *a-t* curve between any two ordinates represents the change in velocity of the moving body. If the initial and final velocities of the body are *u* and *v*, then the above expression may be written as

$$v - u = \int_0^t a.d\,t = \text{ Area under } a\text{-}t \text{ curve } AB = \text{Area } OABC$$

Example 2.1. *A car starts from rest and accelerates uniformly to a speed of 72 km. p.h. over a distance of 500 m. Calculate the acceleration and the time taken to attain the speed.*

If a further acceleration raises the speed to 90 km. p.h. in 10 seconds, find this acceleration and the further distance moved. The brakes are now applied to bring the car to rest under uniform retardation in 5 seconds. Find the distance travelled during braking.

Solution. Given : $u = 0$; $v = 72$ km. p.h. = 20 m/s ; $s = 500$ m

First of all, let us consider the motion of the car from rest.

Acceleration of the car

Let a = Acceleration of the car.

We know that $v^2 = u^2 + 2\, a.s$

$\therefore$ $(20)^2 = 0 + 2a \times 500 = 1000\, a$ or $a = (20)^2 / 1000 = 0.4$ m/s^2 **Ans.**

Time taken by the car to attain the speed

Let t = Time taken by the car to attain the speed.

We know that $v = u + a.t$

$\therefore$ $20 = 0 + 0.4 \times t$ or $t = 20/0.4 = 50$ s **Ans.**

Now consider the motion of the car from 72 km.p.h. to 90 km.p.h. in 10 seconds.

Given : * $u = 72$ km.p.h. = 20 m/s ; $v = 96$ km.p.h. = 25 m/s ; $t = 10$ s

Acceleration of the car

Let a = Acceleration of the car.

We know that $v = u + a.t$

$25 = 20 + a \times 10$ or $a = (25 - 20)/10 = 0.5$ m/s^2 **Ans.**

Distance moved by the car

We know that distance moved by the car,

$$s = u.t + \frac{1}{2} a.t^2 = 20 \times 10 + \frac{1}{2} \times 0.5 (10)^2 = 225 \text{ m} \quad \textbf{Ans.}$$

* It is the final velocity in the first case.

Now consider the motion of the car during the application of brakes for brining it to rest in 5 seconds.

Given : *u = 25 m/s ; v = 0 ; t = 5 s

We know that the distance travelled by the car during braking,

$$s = \frac{u+v}{2} \times t = \frac{25+0}{2} \times 5 = 62.5 \text{ m} \quad \textbf{Ans.}$$

Example 2.2. *The motion of a particle is given by $a = t^3 - 3t^2 + 5$, where a is the acceleration in m/s^2 and t is the time in seconds. The velocity of the particle at t = 1 second is 6.25 m/s, and the displacement is 8.30 metres. Calculate the displacement and the velocity at t = 2 seconds.*

Solution. Given : $a = t^3 - 3t^2 + 5$

We know that the acceleration, $a = dv/dt$. Therefore the above equation may be written as

$$\frac{dv}{dt} = t^3 - 3t^2 + 5 \quad \text{or} \quad dv = (t^3 - 3t^2 + 5)dt$$

Integrating both sides

$$v = \frac{t^4}{4} - \frac{3t^3}{3} + 5t + C_1 = \frac{t^4}{4} - t^3 + 5t + C_1 \qquad \text{...}(i)$$

where C_1 is the first constant of integration. We know that when t = 1 s, v = 6.25 m/s. Therefore substituting these values of t and v in equation (*i*),

$$6.25 = 0.25 - 1 + 5 + C_1 = 4.25 + C_1 \quad \text{or} \quad C_1 = 2$$

Now substituting the value of C_1 in equation (*i*),

$$v = \frac{t_4}{4} - t^3 + 5t + 2 \qquad \text{...}(ii)$$

Velocity at t = 2 seconds

Substituting the value of t = 2 s in the above equation,

$$v = \frac{2^4}{4} - 2^3 + 5 \times 2 + 2 = 8 \text{ m/s} \quad \textbf{Ans.}$$

Displacement at t = 2 seconds

We know that the velocity, $v = ds/dt$, therefore equation (*ii*) may be written as

$$\frac{ds}{dt} = \frac{t^4}{4} - t^3 + 5t + 2 \quad \text{or} \quad ds = \left(\frac{t^4}{4} - t^3 + 5t + 2\right)dt$$

Integrating both sides,

$$s = \frac{t^5}{20} - \frac{t^4}{4} + \frac{5t^2}{2} + 2\,t + C_2 \qquad \text{...}(iii)$$

where C_2 is the second constant of integration. We know that when t = 1 s, s = 8.30 m. Therefore substituting these values of t and s in equation (*iii*),

$$8.30 = \frac{1}{20} - \frac{1}{4} + \frac{5}{2} + 2 + C_2 = 4.3 + C_2 \quad \text{or} \quad C_2 = 4$$

* It is the final velocity in the second case.

Substituting the value of C_2 in equation ***(iii)***,

$$s = \frac{t^5}{20} - \frac{t^4}{4} + \frac{5t^2}{2} + 2t + 4$$

Substituting the value of t = 2 s, in this equation,

$$s = \frac{2^5}{20} - \frac{2^4}{4} + \frac{5 \times 2^2}{2} + 2 \times 2 + 4 = 15.6 \text{ m} \quad \textbf{Ans.}$$

Example 2.3. *The velocity of a train travelling at 100 km/h decreases by 10 per cent in the first 40 s after application of the brakes. Calculate the velocity at the end of a further 80 s assuming that, during the whole period of 120 s, the retardation is proportional to the velocity.*

Solution. Given : Velocity in the beginning (*i.e.* when t = 0), v_0 = 100 km/h

Since the velocity decreases by 10 per cent in the first 40 seconds after the application of brakes, therefore velocity at the end of 40 s,

$$v_{40} = 100 \times 0.9 = 90 \text{ km/h}$$

Let v_{120} = Velocity at the end of 120 s (or further 80s).

Since the retardation is proportional to the velocity, therefore,

$$a = -\frac{dv}{dt} = k.v \quad \text{or} \quad \frac{dv}{v} = -k.dt$$

where k is a constant of proportionality, whose value may be determined from the given conditions. Integrating the above expression,

$$\log_e v = -k.t + C \qquad \ldots (i)$$

where C is the constant of integration. We know that when t = 0, v = 100 km/h. Substituting these values in equation ***(i)***,

$$\log_e 100 = C \quad \text{or} \quad C = 2.3 \log 100 = 2.3 \times 2 = 4.6$$

We also know that when t = 40 s, v = 90 km/h. Substituting these values in equation ***(i)***,

$$\log_e 90 = -k \times 40 + 4.6 \qquad \ldots (\because C = 4.6)$$

$$2.3 \log 90 = -40k + 4.6$$

or $$k = \frac{4.6 - 2.3 \log 90}{40} = \frac{4.6 - 2.3 \times 1.9542}{40} = 0.0026$$

Substituting the values of k and C in equation ***(i)***,

$$\log_e v = -0.0026 \times t + 4.6$$

or $$2.3 \log v = -0.0026 \times t + 4.6 \qquad \ldots (ii)$$

Now substituting the value of t equal to 120 s, in the above equation,

$$2.3 \log v_{120} = -0.0026 \times 120 + 4.6 = 4.288$$

or $$\log v_{120} = 4.288 / 2.3 = 1.864$$

$\therefore$ $$v_{120} = 73.1 \text{ km/h} \quad \textbf{Ans.} \qquad \ldots \text{(Taking antilog of 1.864)}$$

Example 2.4. *The acceleration (a) of a slider block and its displacement (s) are related by the expression, $a = k\sqrt{s}$, where k is a constant. The velocity v is in the direction of the displacement and the velocity and displacement are both zero when time t is zero. Calculate the displacement, velocity and acceleration as functions of time.*

Solution. Given : $a = k\sqrt{s}$

We know that acceleration,

$$a = v \times \frac{dv}{ds} \quad \text{or} \quad k\sqrt{s} = v \times \frac{dv}{ds} \qquad \ldots \left[\because \frac{dv}{dt} = \frac{ds}{dt} \times \frac{dv}{ds} = v \times \frac{dv}{ds}\right]$$

$$\therefore \qquad v \times dv = k.s^{1/2}\, ds$$

Integrating both sides,

$$\int_0^v v.dv = k\int s^{1/2} ds \quad \text{or} \quad \frac{v^2}{2} = \frac{k.s^{3/2}}{3/2} + C_1 \qquad \ldots (i)$$

where C_1 is the first constant of integration whose value is to be determined from the given conditions of motion. We know that $s = 0$, when $v = 0$. Therefore, substituting the values of s and v in equation (*i*), we get $C_1 = 0$.

$$\therefore \qquad \frac{v^2}{2} = \frac{2}{3}k.s^{3/2} \quad \text{or} \quad v = \sqrt{\frac{4k}{3}} \times s^{3/4} \qquad \ldots (ii)$$

Displacement, velocity and acceleration as functions of time

We know that
$$\frac{ds}{dt} = v = \sqrt{\frac{4k}{3}} \times s^{3/4} \qquad \ldots \text{[From equation } (ii)]$$

$$\therefore \qquad \frac{ds}{s^{3/4}} = \sqrt{\frac{4k}{3}}\, dt \quad \text{or} \quad s^{-3/4}\, ds = \sqrt{\frac{4k}{3}}\, dt$$

Integrating both sides,

$$\int_0^s s^{-3/4}\, ds = \sqrt{\frac{4k}{3}} \int_0^t dt$$

$$\frac{s^{1/4}}{1/4} = \sqrt{\frac{4k}{3}} \times t + C_2 \qquad \ldots (iii)$$

where C_2 is the second constant of integration. We know that displacement, $s = 0$ when $t = 0$. Therefore, substituting the values of s and t in equation (*iii*), we get $C_2 = 0$.

$$\therefore \qquad \frac{s^{1/4}}{1/4} = \sqrt{\frac{4k}{3}} \times t \quad \text{or} \quad s = \frac{k^2.t^4}{144} \quad \textbf{Ans.}$$

We know that velocity,

$$v = \frac{ds}{dt} = \frac{k^2}{144} \times 4t^3 = \frac{k^2.t^3}{36} \quad \textbf{Ans.} \qquad \ldots \left(\text{Differentiating } \frac{k^2.t^4}{144}\right)$$

and acceleration,
$$a = \frac{dv}{dt} = \frac{k^2}{36} \times 3\,t^2 = \frac{k^2.t^2}{12} \quad \textbf{Ans.} \qquad \ldots \left(\text{Differentiating } \frac{k^2.t^3}{36}\right)$$

Example 2.5. *The cutting stroke of a planing machine is 500 mm and it is completed in 1 second. The planing table accelerates uniformly during the first 125 mm of the stroke, the speed remains constant during the next 250 mm of the stroke and retards uniformly during the last 125 mm of the stroke. Find the maximum cutting speed.*

Planing Machine.

Solution. Given : s = 500 mm ; t = 1 s ; s_1 = 125 mm ; s_2 = 250 mm ; s_3 = 125 mm

Fig. 2.4 shows the acceleration-time and velocity-time graph for the planing table of a planing machine.

Let

v = Maximum cutting speed in mm/s.

Average velocity of the table during acceleration and retardation,

$$v_{av} = (0 + v)/2 = v/2$$

Time of uniform acceleration $t_1 = \dfrac{s_1}{v_{av}} = \dfrac{125}{v/2} = \dfrac{250}{v}$ s

Time of constant speed, $t_2 = \dfrac{s_2}{v} = \dfrac{250}{v}$ s

and time of uniform retardation, $t_3 = \dfrac{s_3}{v_{av}} = \dfrac{125}{v/2} = \dfrac{250}{v}$ s

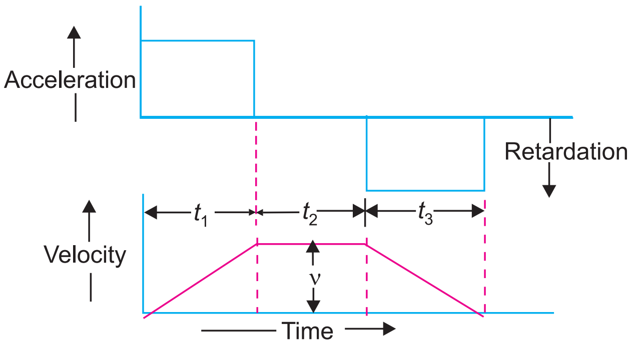


Fig. 2.4

Since the time taken to complete the stroke is 1 s, therefore

$$t_1 + t_2 + t_3 = t$$

$$\frac{250}{v} + \frac{250}{v} + \frac{250}{v} = 1 \quad \text{or } v = 750 \text{ mm/s } \textbf{Ans.}$$

2.12. Angular Displacement

It may be defined as the angle described by a particle from one point to another, with respect to the time. For example, let a line OB has its inclination θ radians to the fixed line OA, as shown in

Fig. 2.5. If this line moves from *OB* to *OC*, through an angle $\delta\theta$ during a short interval of time δt, then $\delta\theta$ is known as the ***angular displacement*** of the line *OB*.

Since the angular displacement has both magnitude and direction, therefore it is also a ***vector quantity.***

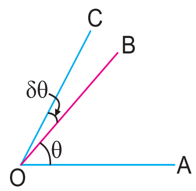


Fig. 2.5. Angular displacement.

2.13. Representation of Angular Displacement by a Vector

In order to completely represent an angular displacement, by a vector, it must fix the following three conditions :

1. *Direction of the axis of rotation*. It is fixed by drawing a line perpendicular to the plane of rotation, in which the angular displacement takes place. In other words, it is fixed along the axis of rotation.

2. *Magnitude of angular displacement*. It is fixed by the length of the vector drawn along the axis of rotation, to some suitable scale.

3. *Sense of the angular displacement.* It is fixed by a right hand screw rule. This rule states that if a screw rotates in a fixed nut in a clockwise direction, *i.e.* if the angular displacement is clockwise and an observer is looking along the axis of rotation, then the arrow head will point away from the observer. Similarly, if the angular displacement is anti-clockwise, then the arrow head will point towards the observer.

2.14. Angular Velocity

It may be defined as the rate of change of angular displacement with respect to time. It is usually expressed by a Greek letter ω (omega). Mathematically, angular velocity,

$$\omega = d\theta / dt$$

Since it has magnitude and direction, therefore, it is a vector quantity. It may be represented by a vector following the same rule as described in the previous article.

Note : If the direction of the angular displacement is constant, then the rate of change of magnitude of the angular displacement with respect to time is termed as ***angular speed.***

2.15. Angular Acceleration

It may be defined as the rate of change of angular velocity with respect to time. It is usually expressed by a Greek letter α (alpha). Mathematically, angular acceleration,

$$\alpha = \frac{d\omega}{dt} = \frac{d}{dt}\left(\frac{d\theta}{dt}\right) = \frac{d^2\theta}{dt^2} \qquad \ldots \left(\because \omega = \frac{d\theta}{dt}\right)$$

It is also a vector quantity, but its direction may not be same as that of angular displacement and angular velocity.

2.16. Equations of Angular Motion

The following equations of angular motion corresponding to linear motion are important from the subject point of view :

1. $\omega = \omega_0 + \alpha.t$

2. $\theta = \omega_0.t + \frac{1}{2}\alpha.t^2$

3. $\omega^2 = (\omega_0)^2 + 2\alpha.\theta$

4. $\theta = \frac{(\omega_0 + \omega)t}{2}$

where ω_0 = Initial angular velocity in rad/s,
ω = Final angular velocity in rad/s,

t = Time in seconds,

θ = Angular displacement in time t seconds, and

α = Angular acceleration in rad / s^2.

Note : If a body is rotating at the rate of N r.p.m. (revolutions per minute), then its angular velocity,

$$\omega = 2\pi N / 60 \text{ rad/s}$$

2.17. Relation between Linear Motion and Angular Motion

Following are the relations between the linear motion and the angular motion :

Particulars	*Linear motion*	*Angular motion*
Initial velocity	u	ω_0
Final velocity	v	ω
Constant acceleration	a	α
Total distance traversed	s	θ
Formula for final velocity	$v = u + a.t$	$\omega = \omega_0 + \alpha.t$
Formula for distance traversed	$s = u.t + \frac{1}{2} a.t^2$	$\theta = \omega_0 . t + \frac{1}{2} \alpha.t^2$
Formula for final velocity	$v^2 = u^2 + 2\,a.s$	$\omega = (\omega_0)^2 + 2\,\alpha.\theta$

2.18. Relation between Linear and Angular Quantities of Motion

Consider a body moving along a circular path from A to B as shown in Fig. 2.6.

Let r = Radius of the circular path,

θ = Angular displacement in radians,

s = Linear displacement,

v = Linear velocity,

ω = Angular velocity,

a = Linear acceleration, and

α = Angular acceleration.

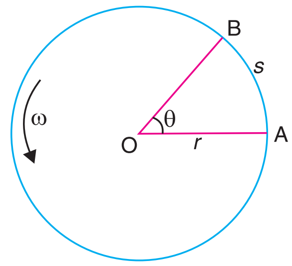


Fig. 2.6. Motion of a body along a circular path.

From the geometry of the figure, we know that

$$s = r\,.\,\theta$$

We also know that the linear velocity,

$$v = \frac{ds}{dt} = \frac{d(r.\theta)}{dt} = r \times \frac{d\theta}{dt} = r.\omega \qquad \text{... (i)}$$

and linear acceleration,

$$a = \frac{dv}{dt} = \frac{d(r.\omega)}{dt} = r \times \frac{d\omega}{dt} = r.\alpha \qquad \text{... (ii)}$$

Example 2.6. *A wheel accelerates uniformly from rest to 2000 r.p.m. in 20 seconds. What is its angular acceleration? How many revolutions does the wheel make in attaining the speed of 2000 r.p.m.?*

Solution. Given : $N_0 = 0$ or $\omega = 0$; $N = 2000$ r.p.m. or $\omega = 2\pi \times 2000/60 = 209.5$ rad/s ; $t = 20$s

Angular acceleration

Let α = Angular acceleration in rad/s^2.

We know that

$$\omega = \omega_0 + \alpha.t \quad \text{or} \quad 209.5 = 0 + \alpha \times 20$$

$\therefore$ $\alpha = 209.5 / 20 = 10.475$ rad/s^2 **Ans.**

Number of revolutions made by the wheel

We know that the angular distance moved by the wheel during 2000 r.p.m. (*i.e.* when $\omega = 209.5$ rad/s),

$$\theta = \frac{(\omega_0 + \omega)t}{2} = \frac{(0 + 209.5)20}{2} = 2095 \text{ rad}$$

Since the angular distance moved by the wheel during one revolution is 2π radians, therefore number of revolutions made by the wheel,

$$n = \theta / 2\pi = 2095/2\pi = 333.4 \quad \textbf{Ans.}$$

2.19. Acceleration of a Particle along a Circular Path

Consider A and B, the two positions of a particle displaced through an angle $\delta\theta$ in time δt as shown in Fig. 2.7 (*a*).

Let r = Radius of curvature of the circular path,

v = Velocity of the particle at A, and

$v + dv$ = Velocity of the particle at B.

The change of velocity, as the particle moves from A to B may be obtained by drawing the vector triangle *oab*, as shown in Fig. 2.7 (*b*). In this triangle, *oa* represents the velocity v and *ob* represents the velocity $v + dv$. The change of velocity in time δt is represented by *ab*.

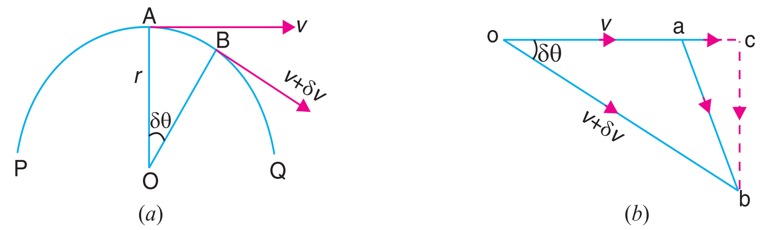


Fig. 2.7. Acceleration of a particle along a circular path.

Now, resolving *ab* into two components *i.e.* parallel and perpendicular to *oa*. Let *ac* and *cb* be the components parallel and perpendicular to *oa* respectively.

$\therefore$ $\quad ac = oc - oa = ob \cos \delta\theta - oa = (v + \delta v) \cos \delta\theta - v$

and $\quad cb = ob \sin \delta\theta = (v + \delta v) \sin \delta\theta$

Since the change of velocity of a particle (represented by vector *ab*) has two mutually perpendicular components, therefore the acceleration of a particle moving along a circular path has the following two components of the acceleration which are perpendicular to each other.

1. *Tangential component of the acceleration*. The acceleration of a particle at any instant moving along a circular path in a direction tangential to that instant, is known as tangential component of acceleration or tangential acceleration.

$\therefore$ Tangential component of the acceleration of particle at A or tangential acceleration at A,

$$a_t = \frac{ac}{\delta t} = \frac{(v + \delta v)\cos \delta\theta - v}{\delta t}$$

In the limit, when δt approaches to zero, then

$$a_t = dv/dt = \alpha.r \qquad \text{... (i)}$$

2. *Normal component of the acceleration*. The acceleration of a particle at any instant moving along a circular path in a direction normal to the tangent at that instant and directed towards the centre of the circular path (*i.e.* in the direction from A to O) is known as normal component of the acceleration or normal acceleration. It is also called ***radial*** or ***centripetal*** acceleration.

∴ Normal component of the acceleration of the particle at A or normal (or radial or centripetal) acceleration at A,

$$a_n = \frac{cb}{\delta t} = \frac{(v + \delta v)\sin\theta}{\delta t}$$

In the limit, when δt approaches to zero, then

$$a_n = v \times \frac{d\theta}{dt} = v.\omega = v \times \frac{v}{r} = \frac{v^2}{r} = \omega^2.r \quad \text{... } (ii)$$

$$\text{... } [\because \; d\theta / dt = \omega, \text{ and } \; \omega = v / r]$$

Since the tangential acceleration (a_t) and the normal acceleration (a_n) of the particle at any instant A are perpendicular to each other, as shown in Fig. 2.8, therefore total acceleration of the particle (a) is equal to the resultant acceleration of a_t and a_n.

∴ Total acceleration or resultant acceleration,

$$a = \sqrt{(a_t)^2 + (a_n)^2}$$

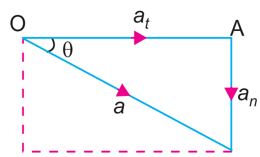


Fig. 2.8. Total acceleration of a particle.

and its angle of inclination with the tangential acceleration is given by

$$\tan\theta = a_n/a_t \text{ or } \theta = \tan^{-1}(a_n/a_t)$$

The total acceleration or resultant acceleration may also be obtained by the vector sum of a_t and a_n.

Notes : 1. From equations ***(i)*** and ***(ii)*** we see that the tangential acceleration (a_t) is equal to the rate of change of the magnitude of the velocity whereas the normal or radial or centripetal acceleration (a_n) depends upon its instantaneous velocity and the radius of curvature of its path.

2. When a particle moves along a straight path, then the radius of curvature is infinitely great. This means that v^2/r is zero. In other words, there will be no normal or radial or centripetal acceleration. Therefore, the particle has only tangential acceleration (in the same direction as its velocity and displacement) whose value is given by

$$a_t = dv/dt = \alpha.r$$

3. When a particle moves with a uniform velocity, then dv/dt will be zero. In other words, there will be no tangential acceleration; but the particle will have only normal or radial or centripetal acceleration, whose value is given by

$$a_n = v^2/r = v.\omega = \omega^2 r$$

Example 2.7. A *horizontal bar 1.5 metres long and of small cross-section rotates about vertical axis through one end. It accelerates uniformly from 1200 r.p.m. to 1500 r.p.m. in an interval of 5 seconds. What is the linear velocity at the beginning and end of the interval ? What are the normal and tangential components of the acceleration of the mid-point of the bar after 5 seconds after the acceleration begins* ?

Solution. Given : r = 1.5 m ; N_0 = 1200 r.p.m. or $\omega_0 = 2\pi \times 1200/60 = 125.7$ rad/s ; N = 1500 r.p.m. or $\omega = 2\pi \times 1500/60 = 157$ rad/s ; t = 5 s

Linear velocity at the beginning

We know that linear velocity at the beginning,

$$v_0 = r.\omega_0 = 1.5 \times 125.7 = 188.6 \text{ m/s} \quad \textbf{Ans.}$$

Linear velocity at the end of 5 seconds

We also know that linear velocity after 5 seconds,

$$v_5 = r.\omega = 1.5 \times 157 = 235.5 \text{ m/s} \quad \textbf{Ans.}$$

Tangential acceleration after 5 seconds

Let α = Constant angular acceleration.

We know that $\omega = \omega_0 + \alpha . t$

$157 = 125.7 + \alpha \times 5$ or $\alpha = (157 - 125.7)/5 = 6.26 \text{ rad/s}^2$

Radius corresponding to the middle point,

$r = 1.5/2 = 0.75$ m

$\therefore$ Tangential acceleration $= \alpha . r = 6.26 \times 0.75 = 4.7 \text{ m/s}^2$ **Ans.**

Radial acceleration after 5 seconds

Radial acceleration $= \omega^2 . r = (157)^2 \; 0.75 = 18\,487 \text{ m/s}^2$ **Ans.**

EXERCISES

1. A winding drum raises a cage through a height of 120 m. The cage has, at first, an acceleration of 1.5 m/s^2 until the velocity of 9 m/s is reached, after which the velocity is constant until the cage nears the top, when the final retardation is 6 m/s^2. Find the time taken for the cage to reach the top. **[Ans. 17.1s]**

2. The displacement of a point is given by $s = 2t^3 + t^2 + 6$, where s is in metres and t in seconds. Determine the displacement of the point when the velocity changes from 8.4 m/s to 18 m/s. Find also the acceleration at the instant when the velocity of the particle is 30 m/s. **[Ans. 6.95 m ; 27 m/s^2]**

3. A rotating cam operates a follower which moves in a straight line. The stroke of the follower is 20 mm and takes place in 0.01 second from rest to rest. The motion is made up of uniform acceleration for 1/4 of the time, uniform velocity for $\frac{1}{2}$ of the time followed by uniform retardation. Find the maximum velocity reached and the value of acceleration and retardation.

[Ans. 2.67 m/s ; 1068 m/s^2 ; 1068 m/s^2]

4. A cage descends a mine shaft with an acceleration of 0.5 m/s^2. After the cage has travelled 25 metres, a stone is dropped from the top of the shaft. Determine : 1. the time taken by the stone to hit the cage, and 2. distance travelled by the cage before impact. **[Ans. 2.92 s ; 41.73 m]**

5. The angular displacement of a body is a function of time and is given by equation :

$\theta = 10 + 3\,t + 6\,t^2$, where t is in seconds.

Determine the angular velocity, displacement and acceleration when t = 5 seconds. State whether or not it is a case of uniform angular acceleration. **[Ans. 63 rad/s ; 175 rad ; 12 rad/s^2]**

6. A flywheel is making 180 r.p.m. and after 20 seconds it is running at 140 r.p.m. How many revolutions will it make, and what time will elapse before it stops, if the retardation is uniform ?

[Ans. 135 rev. ; 90 s]

7. A locomotive is running at a constant speed of 100 km / h. The diameter of driving wheels is 1.8 m. The stroke of the piston of the steam engine cylinder of the locomotive is 600 mm. Find the centripetal acceleration of the crank pin relative to the engine frame. **[Ans. 288 m/s^2]**

DO YOU KNOW ?

1. Distinguish clearly between speed and velocity. Give examples.

2. What do you understand by the term ‘acceleration’ ? Define positive acceleration and negative acceleration.

3. Define ‘angular velocity’ and ‘angular acceleration’. Do they have any relation between them ?

4. How would you find out the linear velocity of a rotating body ?

5. Why the centripetal acceleration is zero, when a particle moves along a straight path ?

6. A particle moving with a uniform velocity has no tangential acceleration. Explain clearly.

OBJECTIVE TYPE QUESTIONS

1. The unit of linear acceleration is
(*a*) kg-m (*b*) m/s (*c*) m/s^2 (*d*) rad/s^2

2. The angular velocity (in rad/s) of a body rotating at *N* r.p.m. is
(*a*) $\pi N/60$ (*b*) $2\pi N/60$ (*c*) $\pi N/120$ (*d*) $\pi N/180$

3. The linear velocity of a body rotating at ω rad/s along a circular path of radius *r* is given by
(*a*) $\omega.r$ (*b*) ω/r (*c*) $\omega^2.r$ (*d*) ω^2/r

4. When a particle moves along a straight path, then the particle has
(*a*) tangential acceleration only (*b*) centripetal acceleration only
(*c*) both tangential and centripetal acceleration

5. When a particle moves with a uniform velocity along a circular path, then the particle has
(*a*) tangential acceleration only (*b*) centripetal acceleration only
(*c*) both tangential and centripetal acceleration

ANSWERS

1. (*c*) **2.** (*b*) **3.** (*a*) **4.** (*a*) **5.** (*b*)

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