21 Thick circular cylinders, discs and spheres

21.1 Introduction

Thin shell theory is satisfactory when the thickness of the shell divided by its radius is less than 1/30. When the thickness: radius ratio of the shell is greater than this, errors start to occur and thick shell theory should be used. Thick shells appear in the form of gun barrels, nuclear reactor pressure vessels, and deep diving submersibles.

21.2 Derivation of the hoop and radial stress equations for a thick-walled circular cylinder

The following convention will be used, where all the stresses and strains are assumed to be tensile and positive. At any radius, r

σ_θ = hoop stress

σ_r = radial stress

σ_z = longitudinal stress

ε_θ = hoop strain

ε_r = radial strain

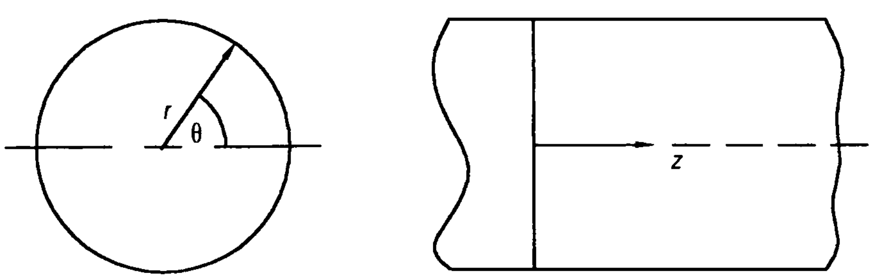

Figure 21.1 Thick cylinder.

ε_z = longitudinal strain (assumed to be constant)

w = radial deflection

From Figure 21.2, it can be seen that at any radius r,

$$\varepsilon_\theta = \frac{2\pi(r + w) - 2\pi r}{2\pi r}$$

or

$$\varepsilon_\theta = w/r \tag{21.1}$$

Similarly,

$$\varepsilon_r = \frac{\delta w}{\delta r} = \frac{\mathrm{d}w}{\mathrm{d}r} \tag{21.2}$$

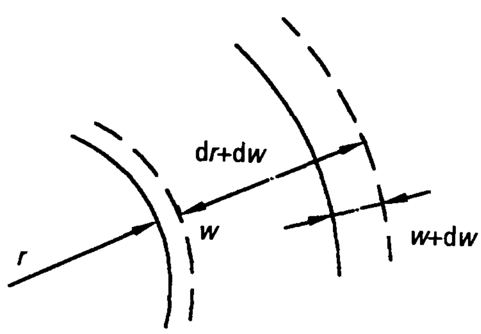

Figure 21.2. Deformation at any radius r.

From the standard stress–strain relationships,

$$E\varepsilon_z = \sigma_z - \nu\,\sigma_\theta - \nu\,\sigma_r = \text{a constant}$$

$$E\varepsilon_\theta = \frac{Ew}{r} = \sigma_\theta - \nu\sigma_z - \nu\sigma_r \tag{21.3}$$

$$E\varepsilon_r = E\frac{dw}{dr} = \sigma_r - \nu\sigma_\theta - \nu\sigma_z \tag{21.4}$$

Multiplying equation (21.3) by r,

$$Ew = \sigma_\theta \times r - \nu\sigma_z \times r - \nu\sigma_r \times r \tag{21.5}$$

and differentiating equation (21.5) with respect to r, we get

$$E\frac{\mathrm{d}w}{\mathrm{d}r} = \sigma_\theta - \nu\sigma_z - \nu\sigma_r + r\left(\frac{\mathrm{d}\sigma_\theta}{\mathrm{d}r} - \nu\frac{\mathrm{d}\sigma_z}{\mathrm{d}r} - \nu\frac{\mathrm{d}\sigma_r}{\mathrm{d}r}\right) \tag{21.6}$$

Subtracting equation (21.4) from equation (21.6),

$$(\sigma_\theta - \sigma_r)(1+\nu) + r\frac{\mathrm{d}\sigma_\theta}{\mathrm{d}r} - \nu r\frac{\mathrm{d}\sigma_z}{\mathrm{d}r} - \nu r\frac{\mathrm{d}\sigma_r}{\mathrm{d}r} = 0 \tag{21.7}$$

As ε_z is constant

$$\sigma_z - \nu\sigma_\theta - \nu\sigma_r = \text{constant} \tag{21.8}$$

Differentiating equation (21.8)with respect to r,

$$\frac{\mathrm{d}\sigma_z}{\mathrm{d}r} - \nu\frac{\mathrm{d}\sigma_\theta}{\mathrm{d}r} - \nu\frac{\mathrm{d}\sigma_r}{\mathrm{d}r} = 0$$

or

$$\frac{\mathrm{d}\sigma_z}{\mathrm{d}r} = \nu\left(\frac{\mathrm{d}\sigma_\theta}{\mathrm{d}r} + \frac{\mathrm{d}\sigma_r}{\mathrm{d}r}\right) \tag{21.9}$$

Substituting equation (21.9) into equation (21.7),

$$(\sigma_\theta - \sigma_r)(1+\nu) + r(1 - \nu^2)\frac{\mathrm{d}\sigma_\theta}{\mathrm{d}r} - \nu r(1+\nu)\frac{\mathrm{d}\sigma_r}{\mathrm{d}r} = 0 \tag{21.10}$$

and dividing equation (21.10) by $(1 + \nu)$, we get

$$\sigma_\theta - \sigma_r + r\,(1 + \nu)\frac{\mathrm{d}\sigma_\theta}{\mathrm{d}r} - \nu r\frac{\mathrm{d}\sigma_r}{\mathrm{d}r} = 0 \tag{21.11}$$

Considering now the radial equilibrium of the shell element, shown in Figure 21.3,

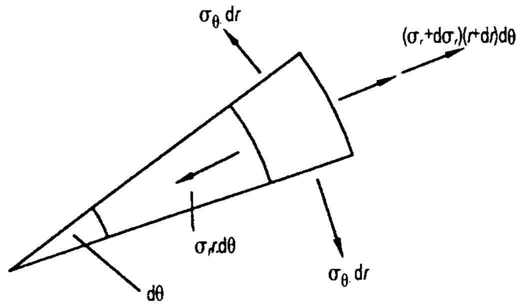

Figure 21.3 Shell element.

$$2\sigma_\theta \quad \delta_r \sin\left(\frac{\delta\theta}{2}\right) + \sigma_r \quad r \quad \delta_\theta - \left(\sigma_r + \delta\sigma_r\right)\left(r + \delta r\right)\delta\theta = 0 \tag{21.12}$$

Neglecting higher order terms in the above, we get

$$\sigma_\theta - \sigma_r - r\frac{d\sigma_r}{dr} = 0 \tag{21.13}$$

Subtracting equation (21.11) from equation (21.12)

$$\frac{d\sigma_\theta}{dr} + \frac{d\sigma_r}{dr} = 0 \tag{21.14}$$

$$\therefore \sigma_\theta + \sigma_r = \text{constant} = 2A \tag{21.15}$$

Subtracting equation (21.13) from equation (21.15),

$$2\sigma_r + r\frac{d\sigma_r}{dr} = 2A$$

or

$$\frac{1}{r}\frac{d\left(\sigma_r r^2\right)}{dr} = 2A$$

$$\frac{d\left(\sigma_r \quad r^2\right)}{dr} = 2Ar$$

Integrating the above,

$$\sigma_r \, r^2 = Ar^2 - B$$

$$\sigma_r = A - \frac{B}{r^2} \qquad (21.16)$$

From equation (21.15),

$$\sigma_\theta = A + \frac{B}{r^2} \qquad (21.17)$$

21.3 Lamé line

If equations (21.16) and (21.17) are plotted with respect to a horizontal axis, where $1/r^2$ is the horizontal axis, the two equations appear as a single straight line, where σ_r lies to the left and σ_θ to the right, as shown by Figure 21.4. For the case shown in Figure 21.4, σ_r is compressive and σ_θ tensile, where

$\sigma_{\theta 1}$ = internal hoop stress, which can be seen to be the maximum stress

$\sigma_{\theta 2}$ = external hoop stress

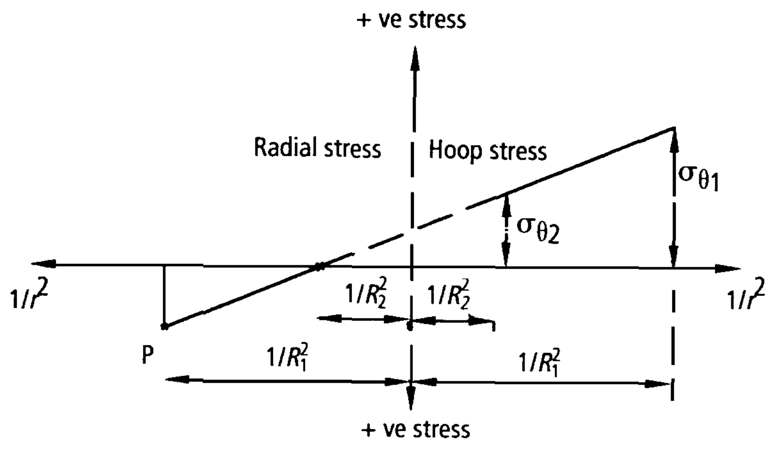

Figure 21.4 Lamé line for the case of internal pressure.

To calculate $\sigma_{\theta 1}$ and $\sigma_{\theta 2}$, equate similar triangles in Figure 21.4,

$$\frac{\sigma_{\theta 1}}{\left(\frac{1}{R_1^2} + \frac{1}{R_2^2}\right)} = \frac{P}{\left(\frac{1}{R_1^2} - \frac{1}{R_2^2}\right)}$$

or

$$\sigma_{\theta 1} = \frac{P\left(\frac{1}{R_1^2} + \frac{1}{R_2^2}\right)}{\left(\frac{1}{R_1^2} - \frac{1}{R_2^2}\right)} \times \frac{R_1^2 R_2^2}{R_1^2 R_2^2} \tag{21.18}$$

$$\sigma_{\theta 1} = \frac{P\left(R_1^2 + R_2^2\right)}{\left(R_2^2 - R_1^2\right)}$$

Similarly, from Figure 21.4

$$\frac{\sigma_{\theta 2}}{\left(\frac{1}{R_2^2} + \frac{1}{R_2^2}\right)} = \frac{P}{\left(\frac{1}{R_1^2} - \frac{1}{R_2^2}\right)}$$

or,

$$\sigma_{\theta 2} = \frac{P\left(\frac{1}{R_2^2}\right) \times 2}{\left(\frac{1}{R_1^2} - \frac{1}{R_2^2}\right)} \times \frac{R_1^2 R_2^2}{R_1^2 R_2^2} \tag{21.19}$$

$$\sigma_{\theta 2} = \frac{2PR_1^2}{\left(R_2^2 - R_1^2\right)}$$

Problem 21.1 A thick-walled circular cylinder of internal diameter 0.2 m is subjected to an internal pressure of 100 MPa. If the maximum permissible stress in the cylinder is limited to 150 MPa, determine the maximum possible external diameter d_2.

Solution

$$\frac{100}{\left(\dfrac{1}{0.2^2} - \dfrac{1}{d_2^2}\right)} = \frac{150}{\left(\dfrac{1}{0.2^2} + \dfrac{1}{d_2^2}\right)}$$

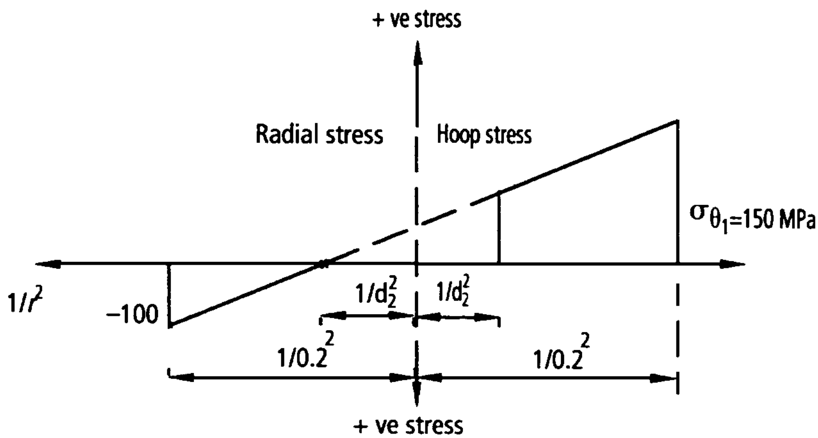

Figure 21.5 Lamé line for thick cylinder.

or

$$\frac{\left(\dfrac{1}{0.2^2} + \dfrac{1}{d_2^2}\right)}{\left(\dfrac{1}{0.2^2} - \dfrac{1}{d_2^2}\right)} \times \left(\frac{0.2^2\ d_2^2}{0.2^2\ d_2^2}\right) = 1.5$$

$$\left(\frac{d_2^2 + 0.2^2}{d_2^2 - 0.2^2}\right) = 1.5$$

or $$d_2^2 + 0.2^2 = 1.5\left(d_2^2 - 0.2^2\right)$$

or $$0.2^2\left(1 + 1.5\right) = d_2^2\left(1.5 - 1\right)$$

$$d_2^2 = 0.2\,\text{m}^2$$

$$d_2 = 0.447\,\text{m}$$

Problem 21.2 If the cylinder in the previous problem were subjected to an external pressure of 100 MPa and an internal pressure of zero, what would be the maximum magnitude of stress.

Solution

Now $\dfrac{1}{d_1^2} = 25$ and $\dfrac{1}{d_2^2} = 5,$

hence the Lamé line would take the form of Figure 21.6.

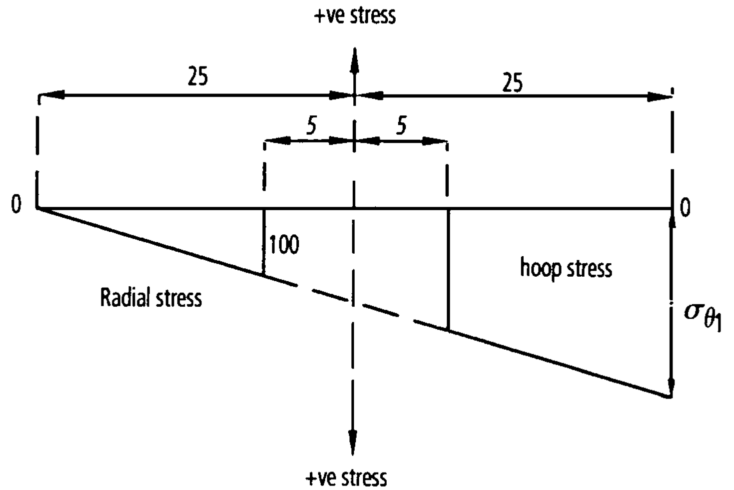

Figure 21.6 Lamé line for external pressure case.

By equating similar triangles,

$$\frac{-100}{(25 - 5)} = \frac{\sigma_{\theta i}}{25 + 25}$$

where $\sigma_{\theta i}$ is the internal stress which has the maximum magnitude

$$\therefore \ \sigma_{\theta_i} = \frac{-50 \times 100}{20} = -250 \text{ MPa}$$

Problem 21.3 A steel disc of external diameter 0.2 m and internal diameter 0.1 m is shrunk onto a solid steel shaft of external diameter 0.1 m, where all the dimensions are nominal. If the interference fit, based on diameters, between the shaft and the disc at the common surface is 0.2 mm, determine the maximum stress.

For steel, $E = 2 \times 10^{11}$ N/m^2, $\nu = 0.3$

Solution

Consider the steel disc. In this case the radial stress on the internal surfaces is the unknown P_c. Hence, the Lamé line will take the form shown in Figure 21.7.

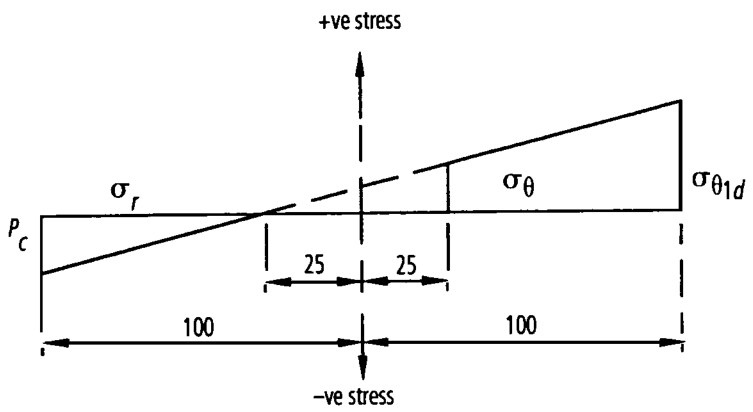

Figure 21.7 Lamé line for steel ring.

Let,

$\sigma_{\theta 1d}$ = hoop stress (maximum stress) on the internal surface of the disc

σ_{r1d} = radial stress on the internal surface of the disc

Equating similar triangles, in Figure 21.7

$$\frac{P_c}{(100 - 25)} = \frac{\sigma_{\sigma 1d}}{100 + 25}$$

$$\therefore \ \sigma_{\theta 1d} = \frac{125 \ P_c}{75} = 1.667 \ P_c$$

Consider now the solid shaft. In this case, the internal diameter of the shaft is zero and as $1/0^2 \to \infty$, the Lamé line must be horizontal or the shaft's hoop stress will be infinity, which is impossible; see Figure 21.8.

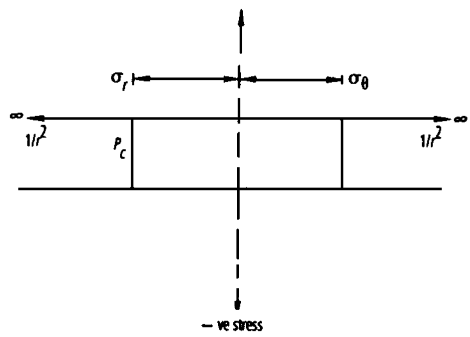

Figure 21.8 Lamé line for a solid shaft.

Let

$$P_c = \text{external pressure on the shaft}$$

$$\therefore \sigma_r = \sigma_\theta = -P_c \text{ (everywhere)} \qquad (21.20)$$

Let,

w_d = increase in the radius of the disc at its inner surface

w_s = increase in the radius of the shaft at its outer surface

Now, applying the expression

$$E\varepsilon_\theta = \frac{w}{r} = \sigma_\theta - \nu\sigma_r - \nu\sigma_x$$

to the inner surface of the disc

$$\frac{EW_d}{5 \times 10^{-2}} = \sigma_{\theta 1d} - \nu\sigma_{r1d}$$

but,

$$\sigma_{r1d} = -P_c$$

therefore

$$\frac{2 \times 10^{11} \times w_d}{5 \times 10^{-2}} = 1.667\ P_c + 0.3\ P_c \qquad (21.21)$$

$$w_d = 4.918 \times 10^{-13}\ P_c$$

Similarly, for the shaft

$$\frac{Ew_s}{5 \times 10^{-2}} = \sigma_{\theta s} - \nu\sigma_{rs}$$

but $\sigma_{\theta s} = \sigma_{rs} = P_c$

$$\therefore \frac{2 \times 10^{11}\ w_s}{5 \times 10^{-2}} = -P_c\ (1 - \nu)$$

$$w_s = -1.75 \times 10^{-13}\ P_c \qquad (21.22)$$

but $w_d - w_s = 2 \times 10^{-3}/2$

$$(4.918 \times 10^{-13} + 1.75 \times 10^{-13})\ P_c = 1 \times 10^{-4}$$

$$\therefore P_c = 150 \text{ MPa}$$

Maximum stress is

$$\sigma_{\theta 1d} = 1.667\ P_c \approx 250 \text{ MPa}$$

21.4 Compound tubes

A compound tube is usually made from two cylinders of different materials where one is shrunk onto the other.

Problem 21.4 A circular steel cylinder of external diameter 0.2 m and internal diameter 0.1 m is shrunk onto a circular aluminium alloy cylinder of external diameter 0.1 m and internal diameter 0.05 m, where the dimensions are nominal.

Determine the radial pressure at the common surface due to shrinkage alone, so that when there is an internal pressure of 300 MPa, the maximum hoop stress in the inner cylinders is 150 Mpa. Sketch the hoop stress distributions.

For steel, $E_s = 2 \times 10^{11}$ N/m², $\nu_s = 0.3$

For aluminium alloy, $E_a = 6.7 \times 10^{10}$ N/m², $\nu_a = 0.32$

Solution

σ_θ^P = the hoop stress due to pressure alone

σ_θ^s = the hoop stress due to shrinkage alone

$\sigma_{\theta,2s}$ = hoop stress in the steel on the 0.2 m diameter

$\sigma_{\theta,1s}$ = hoop stress in the steel on the 0.1 m diameter

$\sigma_{r,2s}$ = radial stress in the steel on the 0.2 m diameter

$\sigma_{r,1s}$ = radial stress in the steel on the 0.1 m diameter

$\sigma_{\theta,1a}$ = hoop stress in the aluminium on the 0.1 m diameter

$\sigma_{r,1a}$ = radial stress in the aluminium on the 0.1 m diameter

$\sigma_{\theta,5a}$ = hoop stress in the aluminum on the 0.05 m diameter

$\sigma_{r,5a}$ = radial stress in the aluminium on the 0.05 m diameter

Consider first the stress due to shrinkage alone, as shown in Figures 21.9 and 21.10.

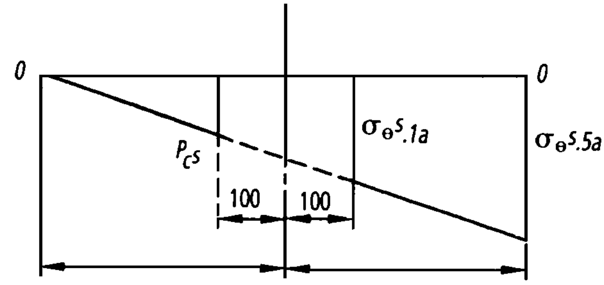

Figure 21.9 Lamé line for aluminium alloy tube.

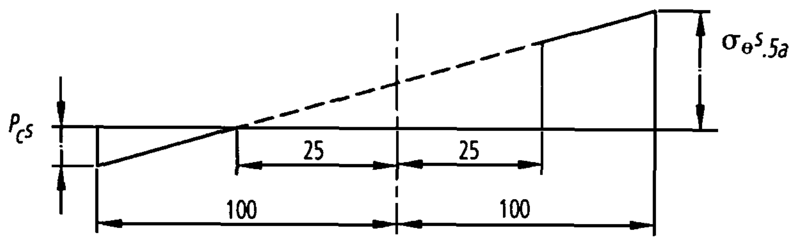

Figure 21.10 Lamé line for steel tube, due to shrinkage with respect to e.

Equating similar triangles in Figure 21.9.

$$\frac{\sigma^{s}_{\theta,5a}}{400 + 400} = \frac{-P^{s}_{c}}{400 - 100} \qquad (21.23)$$

$$\sigma^{s}_{\theta,5a} = -2.667 \; P^{s}_{c}$$

Similarly, from figure 21.9,

$$\frac{\sigma^{s}_{\theta,1a}}{400 + 100} = \frac{-P^{s}_{c}}{400 - 100} \qquad (21.24)$$

$$\sigma^{s}_{\theta,1a} = -1.667 \; P^{s}_{c}$$

Equating similar triangles in Figure 21.10.

$$\frac{\sigma^{s}_{\theta,1s}}{100 + 25} = \frac{P^{s}_{c}}{100 - 25} \qquad (21.25)$$

$$\sigma^{s}_{\theta,1s} = 1.667 \; P^{s}_{c}$$

Consider the stresses due to pressure alone

P_c = internal pressure

P_c^P = pressure at the common surface due to pressure alone

The Lamé lines will be as shown in Figures 21.11 and 21.12.

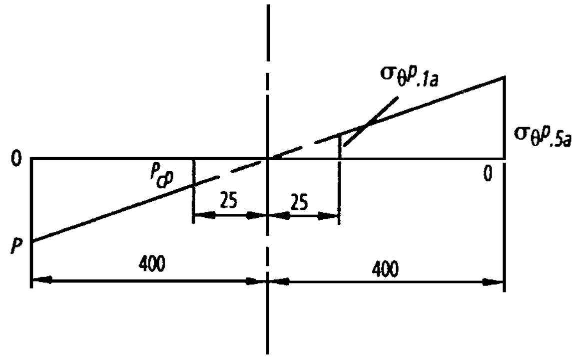

Figure 21.11 Lamé line in aluminium alloy, due to pressure alone.

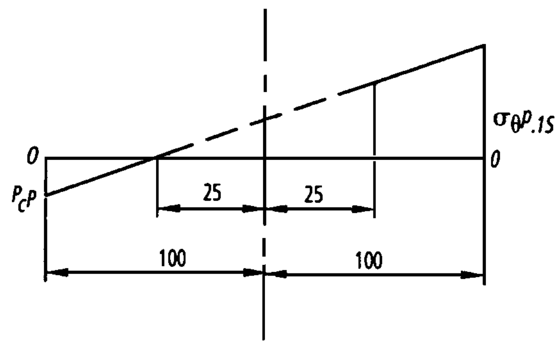

Figure 21.12 Lamé line for steel, due to pressure alone.

Equating similar triangles in Figure 21.11.

$$\frac{P - P_c^P}{400 - 100} = \frac{\sigma_{\theta,1a}^P + P}{400 + 100}$$

or
$$\frac{300 - P_c^P}{300} = \frac{\sigma_{\theta,1a}^P + 300}{500} \tag{21.26}$$

or
$$\sigma_{\theta.1a}^P = 200 - 1.667 P_c^P$$

Similarly, from Figure 21.11,

$$\frac{P - P_c^P}{300} = \frac{\sigma_{\theta,5a}^P + P}{800}$$

$$\frac{300-P_c^P}{300}=\frac{\sigma_{\theta,5a}^P+300}{800} \tag{21.27}$$

or
$$\sigma_{\theta,5a}^P=\frac{8}{3}\left(300-P_c^P\right)-300$$

$$\sigma_{\theta,5a}^P=500-2.667P_c^P$$

Similarly , from Figure 21.12,

$$\frac{\sigma_{\theta,1s}^P}{100+25}=\frac{P_c^P}{100-25} \tag{21.28}$$

$$\sigma_{\theta,1s}^P=1.667\,P_c^P$$

Owing to pressure alone, there is *no interference fit*, so that

$$w_a^P=w_s^P$$

Now

$$\frac{E_s\,w_s^P}{0.05}=\sigma_{\theta,1s}^P+\nu_s\,P_c^P$$

or
$$w_s^P=\frac{0.05}{2\times 10^{11}}\left(1.667\,P_c^P+0.3\,P_c^P\right)$$

or
$$w_s=4.917\times10^{-13}\,P_c^P \tag{21.29}$$

Similarly

$$\frac{E_a\,w_a^P}{0.05}=\sigma_{\theta,1a}^P+\nu_a\,P_c^P$$

or
$$w_a^P=\frac{0.05}{6.7\times10^{10}}\left(\sigma_{\theta,1a}^P+0.32P_c^P\right)$$

$$= \frac{0.05}{6.7 \times 10^{10}}\left(200 - 1.667 P_c^P + 0.32 P_c^P\right) \tag{21.30}$$

$$w_a^P = 1.493 \times 10^{-10} - 1.0 \times 10^{-12} P_c^P$$

Equating (21.29) and (21.30)

$$4.917 \times 10^{-13} P_c^P = 1.493 \times 10^{-10} - 1.0 \times 10^{-12} \tag{21.31}$$

$$\therefore P_c^P = 100\,MPa$$

Substituting equation (21.31) into equations (21.26) and (21.27)

$$\sigma_{\theta,5a} = 500 - 2.667 \times 100 = 233.3 \text{ MPa} \tag{21.32}$$

$$\sigma_{\theta,1a}^P = 200 - 1.667 \times 100 = 33.3 \text{ MPa} \tag{21.33}$$

Now the maximum hoop stress in the inner tube lies either on its internal surface or its external surface, so that either

$$\sigma_{\theta,1a}^P + \sigma_{\theta,1a}^s = 150 \tag{21.34}$$

or

$$\sigma_{\theta,5a}^P + \sigma_{\theta,5a}^s = 150 \tag{21.35}$$

Substituting equations (21.32) and (21.24) into equation (21.34), we get

$$33.3 - 1.667 P_c^s = 150$$

$$\text{or } P_c^s = -70 \text{ MPa}$$

Substituting equations (21.33) and (21.23) into equation (21.35), we get

$$233.3 - 2.667 P_c^s = 150$$

$$\therefore P_c^s = 31.2 \text{ MPa}$$

i.e. $P_c^s = 31.2$ MPa, as P_c^s cannot be negative!

$$P_c = P_c^s + P_c^P = 31.2 + 100 = 131.2 \text{ MPa} \tag{21.36}$$

$$\frac{\sigma_{\theta,2s}}{25 + 25} = \frac{P_c^s + P_c^P}{100 - 25}$$

$$\sigma_{\theta,2s} = 87.5 \text{ MPa}$$

$$\sigma_{\theta,1s} = 1.667\left(P_c^s + P_c^P\right) = 218.7 \text{ MPa}$$

$$\sigma_{\theta,1a} = 200 - 1.667\left(P_c^s + P_c^P\right) = -18.7 \text{ MPa}$$

$$\sigma_{\theta,5a} = 500 - 2.667\left(P_c^s + P_c^P\right) = 150 \text{ MPa}$$

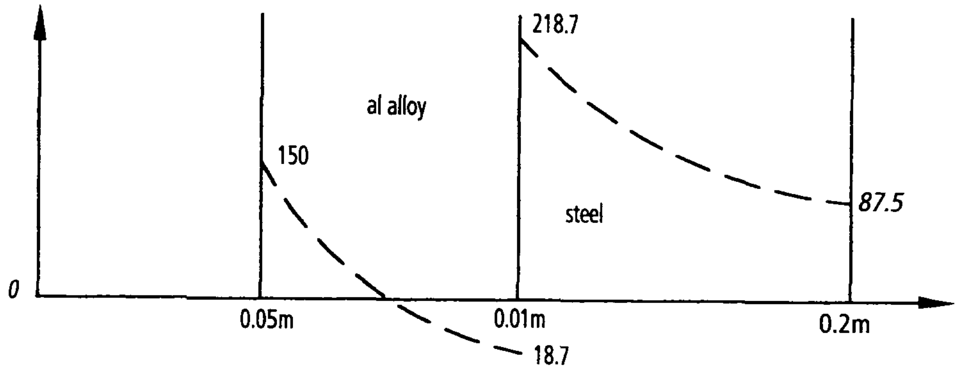

Figure 21.13 Hoop stress distribution.

21.5 Plastic deformation of thick tubes

The following assumptions will be made in this theory:

1. Yielding will take place according to the maximum shear stress theory, (Tresca).
2. The material of construction will behave in an ideally elastic-plastic manner.
3. The longitudinal stress will be the 'minimax' stress in the three-dimensional system of stress.

For this case, the equilibrium considerations of equation (21.13) apply, so that

$$\sigma_\theta - \sigma_r - r\frac{d\sigma_r}{dr} = 0 \tag{21.37}$$

Now, according to the maximum shear stress criterion of yield,

$$\begin{aligned} \sigma_\theta - \sigma_r &= \sigma_{yp} \\ \sigma_\theta &= \sigma_{yp} + \sigma_r \end{aligned} \tag{21.38}$$

Substituting equation (21.38) into equation (21.37),

$$\sigma_{yp} + \sigma_r - \sigma_r - r\frac{d\sigma_r}{dr} = 0$$

$$\begin{aligned} d\sigma_r &= \sigma_{yp}\frac{dr}{r} \\ \sigma_r &= \sigma_{yp}\ln r + C \end{aligned} \tag{21.39}$$

For the case of the partially plastic cylinder shown in Figure 21.14,

at $r = R_2$, $\sigma_r = -P_2$

Substituting this boundary condition into equation (21.39), we get

$$-P_2 = \sigma_{yp}\ln R_2 + C$$

therefore

$$C = -\sigma_{yp}\ln R_2 - P_2$$

and,

$$\sigma_r = \sigma_{yp}\ln\left(\frac{r}{R_2}\right) - P_2 \tag{21.40}$$

Similarly, from equation (21.38),

$$\sigma_\theta = \sigma_{yp}\left\{1 + \ln\left(\frac{r}{R_2}\right)\right\} - P_2 \tag{21.41}$$

where,

R_1 = internal radius

R_2 = outer radius of plastic section of cylinder

R_3 = external radius

P_1 = internal pressure

P_2 = external pressure

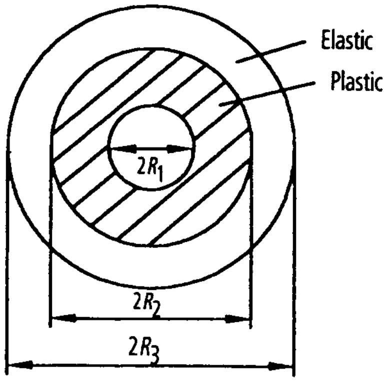

Figure 21.14 Partially plastic cylinder.

The vessel can be assumed to behave as a compound cylinder, with the internal portion behaving plastically, and the external portion elastically. The Lamé line for the elastic portion of the cylinder is shown in Figure 21.15.

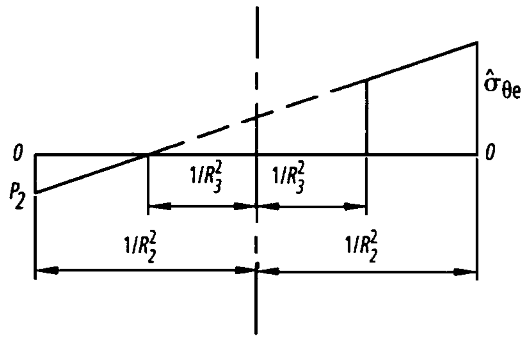

Figure 21.15 Lamé line for elastic zone.

In Figure 21.15,

$$\hat{\sigma}_{\theta e} = \text{elastic hoop stress at } r = R_2$$

so that according to the maximum shear stress criterion of yield on this radius,

$$\sigma_{yp} = \hat{\sigma}_{\theta e} + P_2 \tag{21.42}$$

From Figure 21.15

$$\frac{P_2}{\left(\dfrac{1}{R_2^2} - \dfrac{1}{R_3^2}\right)} = \frac{\hat{\sigma}_{\theta e}}{\left(\dfrac{1}{R_2^2} + \dfrac{1}{R_3^2}\right)}$$

therefore

$$\hat{\sigma}_{\theta e} = \frac{P_2\left(R_3^2 + R_2^2\right)}{\left(R_3^2 - R_2^2\right)} \tag{21.43}$$

Substituting equation (21.43) into equation (21.42),

$$P_2 = \sigma_{yp}\left(R_3^2 - R_2^2\right) / \left(2R_3^2\right) \tag{21.44}$$

Consider now the portion of the cylinder that is plastic. Substituting equation (21.44) into equation (21.41), the stress distributions in the plastic zone are given by:

$$\sigma_r = -\sigma_{yp}\left\{\ln\left(\frac{R_2}{r}\right) + \frac{\left(R_3^2 - R_2^2\right)}{2R_3^2}\right\} \tag{21.45}$$

$$\sigma_\theta = \sigma_{yp}\left\{\frac{\left(R_3^2 + R_2^2\right)}{2R_3^2} - \ln\left(\frac{R_2}{r}\right)\right\} \tag{21.46}$$

To find the *pressure to just cause yield*, put

$$\sigma_r = -P_1 \quad \text{when} \quad r = R_1$$

where P_l is the internal pressure that causes the onset of yield. Therefore,

$$P_1 = \sigma_{yp} \left\{ \ln \left(\frac{R_2}{R_1} \right) + \left(\frac{R_3^2 - R_2^2}{2R_3^2} \right) \right\} \tag{21.47}$$

but, if yield is only on the inside surface,

$$R_1 = R_2$$

in (21.61), so that,

$$P_1 = \sigma_{yp} \left\{ \left(R_3^2 - R_1^2 \right) / \left(2R_3^2 \right) \right\} \tag{21.48}$$

To determine the *plastic collapse pressure* P_p, put $R_2 = R_3$ in equation (21.47), to give

$$P_P = \sigma_{yp} \ln \left(\frac{R_3}{R_1} \right) \tag{21.49}$$

To determine the hoop stress distribution in the plastic zone, $\sigma_{\theta p}$, it must be remembered that

$$\sigma_{yp} = \sigma_\theta - \sigma_r$$

therefore

$$\sigma_{\theta P} = \sigma_{yp} \left\{ 1 + \ln \left(R_3 / R_1 \right) \right\} \tag{21.50}$$

Plots of the stress distributions in a partially plastic cylinder, under internal pressure, are shown in Figure 21.16.

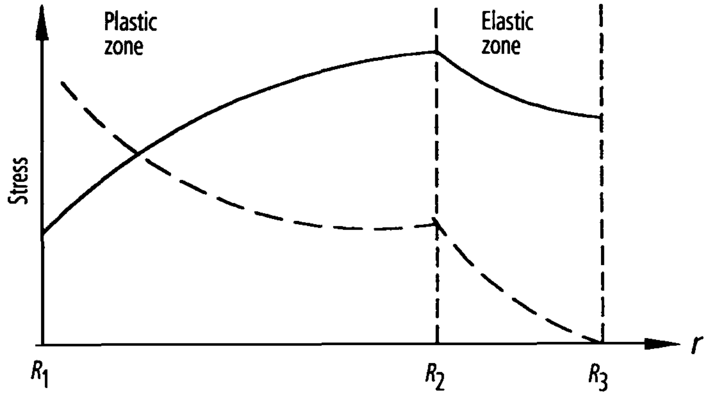

Figure 21.16 Stress distribution plots.

Problem 21.5 A circular cylinder of 0.2 m external diameter and of 0.1 m internal diameter is shrunk onto another circular cylinder of external diameter 0.1 m and of bore 0.05 m, where the dimensions are nominal. If the interference fit is such that when an internal pressure of 10 MPa is applied to the inner face of the inner cylinder, the inner face of the inner cylinder is on the point of yielding. What internal pressure will cause plastic penetration through half the thickness of the inner cylinder. It may be assumed that the Young's modulus and Poisson's ratio for both cylinders is the same, but that the outer cylinder is made of a higher grade steel which will not yield under these conditions. The yield stress of the inner cylinder may be assumed to be 160 MPa.

<u>Solution</u>

The Lamé line for the compound cylinder at the onset of yield is shown in Figure 21.17.

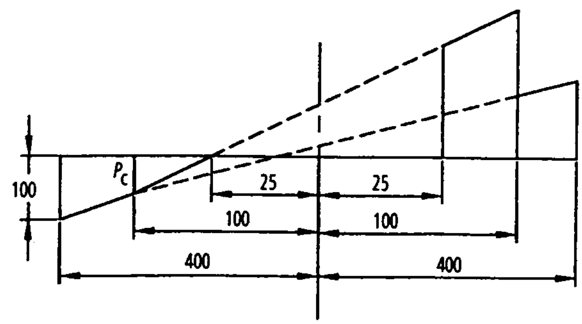

Figure 21.17 Lamé line for compound cylinder.

In Figure 21.17,

σ_1 = hoop stress on inner surface of inner cylinder.

σ_2 = hoop stress on outer surface of inner cylinder.

σ_3 = hoop stress on inner surface of outer cylinder.

As yield occurs on the inner surface of the inner surface when an internal pressure of 50 MPa is applied,

$$\sigma_1 - (-100) = 160$$

$$\therefore \sigma_1 = 60 \text{ MPa}$$

Equating similar triangles in Figure 21.17, we get

$$\frac{\sigma_1 + 100}{400 + 400} = \frac{100 - P_c}{400 - 100}$$

$$\frac{160 \times 300}{800} = 100 - P_c$$

$$\therefore P_c = 40 \text{ MPa} \qquad (21.51)$$

Similarly from Figure 21.17

$$\frac{\sigma_2 + 100}{400 + 100} = \frac{100 - P_c}{400 - 100}$$

$$\sigma_2 = 0 \qquad (21.52)$$

Also from Figure 21.17,

$$\frac{\sigma_3}{100 + 25} = \frac{P_c}{100 - 25}$$

$$\therefore \sigma_3 = \frac{400 \times 125}{75} = 66.7 \text{ MPa} \qquad (21.53)$$

Consider, now, plastic penetration of the inner cylinder to a diameter 0.075. The Lamé line in the elastic zones will be as shown in Figure 21.17. From Figure 21.18,

$$\sigma_6 + P_3 = 160$$

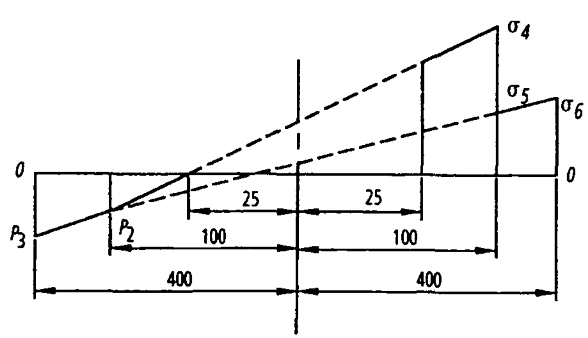

Figure 21.18 Lamé line in elastic zones.

therefore

$$\therefore \sigma_6 = 160 - P_3 \tag{21.54}$$

Similarly

$$\frac{P_3 - P_2}{400 - 100} = \frac{\sigma_6 + P_3}{400 + 400} = \frac{160}{800} \tag{21.55}$$

$$\therefore P_3 = 60 + P_2 \tag{21.56}$$

Also from Figure 21.18

$$\frac{\sigma_4}{100 + 25} = \frac{P_2}{100 - 25} \tag{21.57}$$

or

$$\sigma_4 = 1.667\, P_2 \tag{21.58}$$

Substituting equation (21.56) into equation (21.58), we get

$$\sigma_4 = 1.667\, (P_3 - 60)$$

or

$$\sigma_4 = 1.667\, P_3 - 100$$

Also from equation (21.55)

$$\frac{\sigma_5 + P_3}{100 + 400} = \frac{P_3 - P_2}{400 - 100} = \frac{160}{800}$$

$$\therefore \sigma_5 = 100 - P_3 \tag{21.59}$$

Now,

$$w = \frac{r}{E}\left(\sigma_\theta - \nu\sigma_r\right)$$

which will be the same for both cylinders at the common surface, i.e.,

$$\frac{1}{E}\left\{(\sigma_5 - \sigma_2) - \nu(P_2 - P_c)\right\} = \left\{\frac{1}{E}(\sigma_4 - \sigma_3) - \nu(P_2 - P_c)\right\}$$

or

$$\sigma_5 - \sigma_2 = \sigma_4 - \sigma_3$$

Substituting equations (21.52), (21.53), (21.58) and (21.59) into the above, we get

$$100 - P_3 - 0 = 1.667\ P_3 - 100 - 66.7$$

or
$$2.667\ P_3 = 100 + 100 + 66.7$$

$$P_3 = 100$$

Consider now the yielded portion

$$\sigma_r = \sigma_{yp} \ln r + c$$

$$\sigma_{yp} = 160$$

at $r = 0.0375$ m,

$$\sigma_r = -P_3 = -100$$

or
$$-100 = 160 \ln (0.0375) + C$$

$$C = -100 + 525.3$$

$$\therefore C = 425.3$$

Now, at $r = 0.025$ m,

$$-P = 160 \ln (0.025) + 425.3$$

$$= -590.2 + 425.3$$

$$P = 164.9 \text{ MPa}$$

which is the pressure to cause plastic penetration.

Problem 21.6 Determine the internal pressure that will cause complete plastic collapse of the compound cylinder given that the yield stress for the material of the outer cylinder is 700 MPa.

Solution

Now,

$$P_p = \sigma_{yp} \ln\left(\frac{R_3}{R_1}\right) \tag{21.60}$$

$$= \sigma_{yp2} \ln\left(\frac{R_3}{R_2}\right) + \sigma_{ypl} \ln\left(\frac{R_2}{R_1}\right)$$

$$= 700 \ln\left(\frac{0.1}{0.05}\right) + 160 \ln\left(\frac{0.05}{0.0375}\right)$$

$$= 485 + 46$$

$$P_n = 531 \text{ MPa}$$

which is the plastic collapse pressure of the compound cylinder.

21.6 Thick spherical shells

Consider a thick hemispherical shell element of radius r, under a compressive radial stress P, as shown in Figure 21.19.

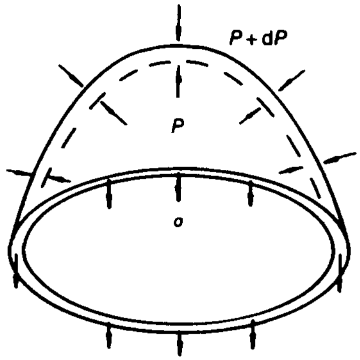

Figure 21.19 Thick hemispherical shell element.

Let w be the radial deflection at any radius r,

so that

$$\text{hoop strain} = w/r$$

and

$$\text{radial strain} = \frac{dw}{dr}$$

From three-dimensional stress–strain relationships,

$$E\,\frac{w}{r} = \sigma - \nu\sigma + \nu P \tag{21.61}$$

and

$$E\,\frac{\mathrm{d}w}{\mathrm{d}r} = -P - \nu\sigma - \nu\sigma \tag{21.62}$$

$$= -P - 2\nu\sigma$$

Now

$$Ew = \sigma\ r - \nu\sigma\ r + \nu P\ r$$

which, on differentiating with respect to r, gives

$$\begin{aligned} E\frac{\mathrm{d}w}{\mathrm{d}r} &= \sigma + r\frac{\mathrm{d}\sigma}{\mathrm{d}r} - \nu\sigma - \nu r\frac{\mathrm{d}\sigma}{\mathrm{d}r} + \nu P + \nu r\frac{\mathrm{d}P}{\mathrm{d}r} \\ &= (1-\nu)\left(\sigma - r\frac{\mathrm{d}\sigma}{\mathrm{d}r}\right) + \nu\left(P + r\frac{\mathrm{d}P}{\mathrm{d}r}\right) \end{aligned} \tag{21.63}$$

Equating (21.62) and (21.63),

$$-P - 2\nu\sigma = (1-\nu)\left(\sigma - r\,\frac{\mathrm{d}\sigma}{\mathrm{d}r}\right) + \nu\left(P + r\,\frac{\mathrm{d}P}{\mathrm{d}r}\right)$$

or

$$(1+\nu)\,(\sigma + P) + r\,(1-\nu)\,\frac{\mathrm{d}\sigma}{\mathrm{d}r} + \nu r\,\frac{\mathrm{d}P}{\mathrm{d}r} = 0 \tag{21.64}$$

Considering now the equilibrium of the hemispherical shell element,

$$\sigma \times 2\pi r \times dr = P \times \pi r^2 - \left(P + dP\right) \times \pi \times \left(r + dr\right)^2 \tag{21.65}$$

Neglecting higher order terms, equation 21.65 becomes

$$\sigma + P = (-r/2)\frac{dP}{dr} \tag{21.66}$$

Substituting equation (21.66) into equation (21.64),

$$-(r/2)\,(dP/dr)\,(1 + \nu) + r\,(1 - \nu)\,(d\sigma/dr) + \nu r\,(dP/dr) = 0$$

or

$$\frac{d\sigma}{dr} - \frac{1}{2}\frac{dP}{dr} = 0 \tag{21.67}$$

which on integrating becomes,

$$\sigma - P/2 = A \tag{21.68}$$

Substituting equation (21.68) into equation (21.66)

$$3P/2 + A = (-r/2)\,(dP/dr)$$

or

$$-\frac{1}{r^2}\frac{d(P \times r^3)}{dr} = 2A$$

or

$$\frac{d(P \times r^3)}{dr} = -2Ar^2$$

which on integrating becomes,

$$P \times r^3 = -2Ar^3/3 + B$$

or

$$P = -2A/3 + B/r^3 \tag{21.69}$$

and $\sigma = 2A/3 + B/(2r^3)$ (21.70)

21.7 Rotating discs

These are of much importance in engineering components that rotate at high speeds. If the speed is high enough, such components can shatter when the centrifugal stresses become too large. The theory for thick circular cylinders can be extended to deal with problems in this category.

Consider a uniform thickness disc, of density ρ, rotating at a constant angular velocity ω.

From

$$E\frac{dw}{dr} = \sigma_r - v\sigma_\theta \tag{21.71}$$

and,

$$E\frac{w}{r} = \sigma_\theta - v\sigma_r \tag{21.72}$$

or,

$$Ew = \sigma_\theta \times r - v\sigma_r \times r \tag{21.73}$$

Differentiating equation (21.73) with respect to r,

$$E\frac{dw}{dr} = \sigma_\theta + r\frac{d\sigma_\theta}{dr} - v\sigma_r - vr\frac{d\sigma_r}{dr} \tag{21.74}$$

Equating (21.71) and (21.74),

$$(\sigma_\theta - \sigma_r)(1 + v) + r\frac{d\sigma_\theta}{dr} - vr\frac{d\sigma_r}{dr} = 0 \tag{21.75}$$

Considering radial equilibrium of an element of the disc, as shown in Figure 21.20,

$$2\sigma_\theta \times dr \times \sin\left(\frac{d\theta}{2}\right) + \sigma_r \times r \times d\theta$$

$$- (\sigma_r + d\sigma_r)(r + dr)d\theta = \rho \times \omega^2 \times r^2 \times dr \times d\theta$$

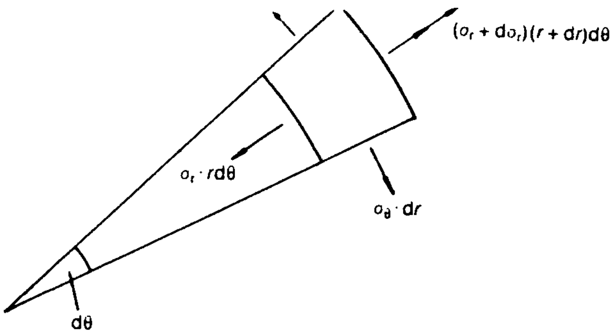

Figure 21.20 Element of disc.

In the limit, this reduces to

$$\sigma_\theta - \sigma_r - r\frac{d\sigma_r}{dr} = \rho\omega^2 r^2 \tag{21.76}$$

Substituting equation (21.76) into equation (21.75),

$$\left(r\frac{d\sigma_r}{dr} + \rho\omega^2 r^2\right)(1 + \nu) + r\frac{d\sigma_\theta}{dr} - \nu r\frac{d\sigma_r}{dr} = 0$$

or,

$$\frac{d\sigma_\theta}{dr} + \frac{d\sigma_r}{dr} = -\rho\omega^2 r^2 (1 + \nu)$$

which on integrating becomes,

$$\sigma_\theta + \sigma_r = -\left(\rho\omega^2 r^2/2\right)(1 + \nu) + 2A \tag{21.77}$$

Subtracting equation (21.76) from equation (21.77),

$$2\sigma_r + r\frac{d\sigma_r}{dr} = -\left(\rho\omega^2 r^2/2\right)(3 + \nu) + 2A$$

or,

$$\frac{1}{r}\frac{d\left(\sigma_r \times r^2\right)}{dr} = -\frac{\rho\omega^2 r^2 (3 + \nu)}{2} + 2A$$

which on integrating becomes,

$$\sigma_r \, r^2 = -\left(\rho\omega^2 r^4/8\right)(3+\nu) + Ar^2 - B \tag{21.78}$$

or

$$\sigma_r = A - B/r^2 - (3+\nu)\left(\rho\omega^2 r^2/8\right)$$

and,

$$\sigma_\theta = A + B/r^2 - (1+3\nu)\left(\rho\omega^2 r^2/8\right) \tag{21.79}$$

Problem 21.7 Obtain an expression for the variation in the thickness of a disc, in its radial direction, so that it will be of constant strength when it is rotated at an angular velocity ω.

Solution

Let,

t_0 = thickness at centre

t = thickness at a radius r

$t + dt$ = thickness at a radius $r + dr$

σ = stress = constant (everywhere)

Consider the radial equilibrium of an element of this disc at any radius r as shown in Figure 21.21.

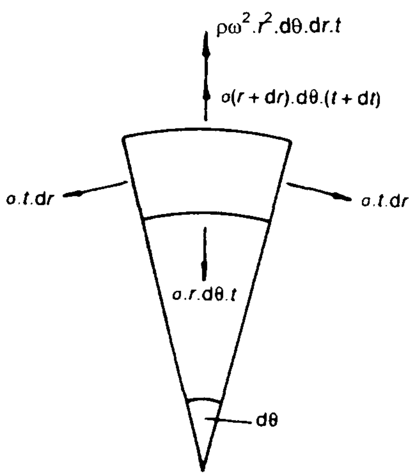

Figure 21.21 Element of constant strength disc.

Resolving forces radially

$$2\sigma \times t \times \mathrm{d}r \sin\left(\frac{\mathrm{d}\theta}{2}\right) + \sigma t r \,\mathrm{d}\theta = \sigma(r + \mathrm{d}r)(t + \mathrm{d}t)\,\mathrm{d}\theta + \rho\omega^2 r^2 t \,\mathrm{d}\theta\,\mathrm{d}r$$

Neglecting higher order terms, this equation becomes

$$\sigma t \mathrm{d}t = \sigma r \mathrm{d}t + \sigma t \mathrm{d}r + \rho\omega^2 r t \mathrm{d}r$$

or

$$\frac{\mathrm{d}t}{\mathrm{d}r} = -\rho\omega^2 r t/\sigma$$

which on integrating becomes,

$$\ln t = -\rho\omega^2 r^2 t/(2\sigma) + \ln C$$

or

$$t = Ce^{(-\rho\omega^2 r^2/2\sigma)}$$

Now, at $r = 0,\ t = t_0 \therefore C = t_0$

Hence,

$$t = t_0 e^{(-\rho\omega^2 r^2/2\sigma)} \tag{21.80}$$

21.7.1 Plastic collapse of rotating discs

Assume that $\sigma_\theta > \sigma_r$, and that plastic collapse occurs when

$$\sigma_\theta = \sigma_{yp}$$

where σ_{yp} is the yield stress.

Let R be the external radius of the disc. Then,

from *equilibrium considerations*,

$$\sigma_{yp} - \sigma_r - r\frac{\mathrm{d}\sigma_r}{\mathrm{d}r} = \rho\omega^2 r^2$$

or,

$$\int r\,d\sigma_r = \int \left\{\sigma_{yp} - \sigma_r - \rho\omega^2 r^2\right\} dr$$

Integrating the left-hand side of the above equation by parts,

$$r\ \sigma_r - \int \sigma_r\ dr = \sigma_{yp}\ r - \int \sigma_r\ dr\ \ \rho\omega^2 r^3/3 + A$$

therefore

$$\sigma_r = \sigma_{yp} - \rho\omega^2 r^2/3 + A/r \tag{21.81}$$

For a solid disc, at $r = 0$, $\sigma_r \neq \infty$, or the disc will collapse at small values of ω. Therefore

$$A = 0$$

and

$$\sigma_r = \sigma_{yp} - \rho\omega^2 r^2/3$$

at $r = R$, $\sigma_r = 0$; therefore

$$0 = \sigma_{yp} - \rho\omega^2 R^2/3$$

$$\therefore\ \omega = \frac{1}{R}\sqrt{\frac{3\sigma_{yp}}{\rho}} \tag{21.82}$$

where, ω is the angular velocity of the disc, which causes plastic collapse of the disc.

For an *annular disc*, of internal radius R_1 and external radius R_2, suitable boundary conditions for equation (21.81) are:

at $r = R_2$, $\sigma_r = 0$; therefore

$$A = \left(\rho\omega^2 R_1^2/3 - \sigma_{yp}\right)R_1$$

$$\therefore \sigma_r = \sigma_{yp} - \rho\omega^2 r^2/3 + \left(\rho\omega^2 R_1^2/3 - \sigma_{yp}\right)\left(R_1/r\right) \qquad (21.83)$$

at $r = R_2$, $\sigma_r = 0$; therefore

$$0 = \sigma_{yp} - \rho\omega^2 R_2^2/3 + \left(\rho\omega^2 R_1^2/3 - \sigma_{yp}\right)\left(R_1/R_2\right)$$

$$\text{Hence,} \quad \omega = \sqrt{\left\{\left(\frac{3\sigma_{yp}}{\rho}\right)\frac{\left(R_2 - R_1\right)}{\left(R_2^3 - R_1^3\right)}\right\}} \qquad (21.84)$$

21.8 Collapse of rotating rings

Consider the radial equilibrium of the thin semicircular ring element shown in Figure 21.21.

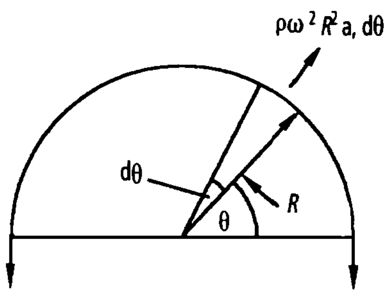

Figure 21.21 Ring element.

Let,

a = cross-sectional area of ring

R = mean radius of ring

Resolving forces vertically

$$\sigma_\theta \times a \times 2 = \int_0^\pi \rho\omega^2 R^2 a \, d\theta \quad \sin\theta$$

$$= \rho\omega^2 R^2 a \left[-\cos\theta\right]_0^\pi$$

$$= 2\rho\omega^2 R^2 a$$

$$\therefore \sigma_\theta = \rho\omega^2 R^2$$

at collapse,

$$\sigma_\theta = \sigma_{yp}$$

$$\therefore \omega = \frac{1}{R}\sqrt{\left(\frac{\sigma_{yp}}{\rho}\right)} \tag{21.85}$$

where ω is the angular velocity required to fracture the ring.