1 Tension and compression: direct stresses

1.1 Introduction

The strength of a material, whatever its nature, is defined largely by the internal stresses, or intensities of force, in the material. A knowledge of these stresses is essential to the safe design of a machine, aircraft, or any type of structure. Most practical structures consist of complex arrangements of many component members; an aircraft fuselage, for example, usually consists of an elaborate system of interconnected sheeting, longitudinal stringers, and transverse rings. The detailed stress analysis of such a structure is a difficult task, even when the loading conditions are simple. The problem is complicated further because the loads experienced by a structure are variable and sometimes unpredictable. We shall be concerned mainly with stresses in materials under relatively simple loading conditions; we begin with a discussion of the behaviour of a stretched wire, and introduce the concepts of direct stress and strain.

1.2 Stretching of a steel wire

One of the simplest loading conditions of a material is that of *tension*, in which the fibres of the material are stretched. Consider, for example, a long steel wire held rigidly at its upper end, Figure 1.1, and loaded by a mass hung from the lower end. If vertical movements of the lower end are observed during loading it will be found that the wire is stretched by a small, but measurable, amount from its original unloaded length. The material of the wire is composed of a large number of small crystals which are only visible under a microscopic study; these crystals have irregularly shaped boundaries, and largely random orientations with respect to each other; as loads are applied to the wire, the crystal structure of the metal is distorted.

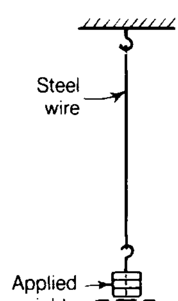


Figure 1.1 Stretching of a steel wire under end load.

For small loads it is found that the extension of the wire is roughly proportional to the applied load, Figure 1.2. This linear relationship between load and extension was discovered by Robert Hooke in 1678; a material showing this characteristic is said to obey *Hooke's law.*

As the tensile load in the wire is increased, a stage is reached where the material ceases to show this linear characteristic; the corresponding point on the load–extension curve of Figure 1.2 is known as the *limit of proportionality.* If the wire is made from a high-strength steel then the load–extension curve up to the *breaking point* has the form shown in Figure 1.2. Beyond the limit of proportionality the extension of the wire increases non-linearly up to the elastic limit and, eventually, the breaking point.

The elastic limit is important because it divides the load–extension curve into two regions. For loads up to the elastic limit, the wire returns to its original unstretched length on removal of the loads; this property of a material to recover its original form on removal of the loads is known as *elasticity*; the steel wire behaves, in fact, as a still elastic spring. When loads are applied above the elastic limit, and are then removed, it is found that the wire recovers only part of its extension and is stretched permanently; in this condition the wire is said to have undergone an *inelastic*, or *plastic*, extension. For most materials, the limit of proportionality and the elastic limit are assumed to have the same value.

In the case of elastic extensions, work performed in stretching the wire is stored as *strain energy* in the material; this energy is recovered when the loads are removed. During inelastic extensions, work is performed in making permanent changes in the internal structure of the material; not all the work performed during an inelastic extension is recoverable on removal of the loads; this energy reappears in other forms, mainly as heat.

The load–extension curve of Figure 1.2 is not typical of all materials; it is reasonably typical, however, of the behaviour of *brittle* materials, which are discussed more fully in Section 1.5. An important feature of most engineering materials is that they behave elastically up to the limit of proportionality, that is, all extensions are recoverable for loads up to this limit. The concepts of linearity and elasticity[1] form the basis of the theory of small deformations in stressed materials.

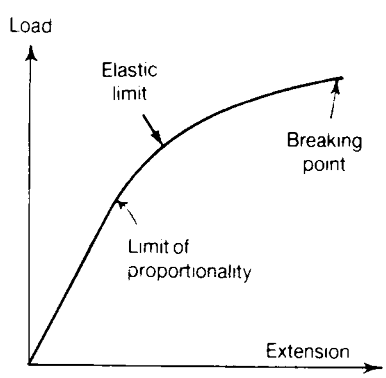


Figure 1.2 Load–extension curve for a steel wire, showing the limit of linear-elastic behaviour (or limit of proportionality) and the breaking point.

[1]The definition of elasticity requires only that the extensions are recoverable on removal of the loads; this does not preclude the possibility of a non-linear relation between load and extension .

1.3 Tensile and compressive stresses

The wire of Figure 1.1 was pulled by the action of a mass attached to the lower end; in this condition the wire is in *tension*. Consider a cylindrical bar *ab*, Figure 1.3, which has a uniform cross-section throughout its length. Suppose that at each end of the bar the cross-section is divided into small elements of equal area; the cross-sections are taken normal to the longitudinal axis of the bar. To each of these elemental areas an equal tensile load is applied normal to the cross-section and parallel to the longitudinal axis of the bar. The bar is then uniformly stressed in tension.

Suppose the total load on the end cross-sections is P; if an imaginary break is made perpendicular to the axis of the bar at the section c, Figure 1.3, then equal forces P are required at the section c to maintain equilibrium of the lengths ac and cb. This is equally true for any section across the bar, and hence on any imaginary section perpendicular to the axis of the bar there is a total force P.

When tensile tests are carried out on steel wires of the same material, but of different cross-sectional area, the breaking loads are found to be proportional approximately to the respective cross-sectional areas of the wires. This is so because the tensile strength is governed by the intensity of force on a normal cross-section of a wire, and not by the total force. This intensity of force is known as *stress*; in Figure 1.3 the *tensile stress* σ at any normal cross-section of the bar is

$$\sigma = \frac{P}{A} \tag{1.1}$$

where P is the total force on a cross-section and A is the area of the cross-section.

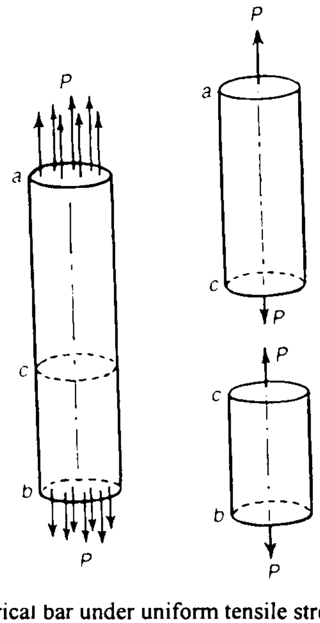


Figure 1.3 Cylindrical bar under uniform tensile stress; there is a similar state of tensile stress over any imaginary normal cross-section.

In Figure 1.3 uniform stressing of the bar was ensured by applying equal loads to equal small areas at the ends of the bar. In general we are not dealing with equal force intensities of this type, and a more precise definition of stress is required. Suppose δA is an element of area of the cross-section of the bar, Figure 1.4; if the normal force acting on this element is δP, then the tensile stress at this point of the cross-section is defined as the limiting value of the ratio $(\delta P/\delta A)$ as δA becomes infinitesimally small. Thus

$$\sigma = \underset{\delta A \to 0}{\text{Limit}} \frac{\delta P}{\delta A} = \frac{dP}{dA} \tag{1.2}$$

This definition of stress is used in studying problems of non-uniform stress distribution in materials.

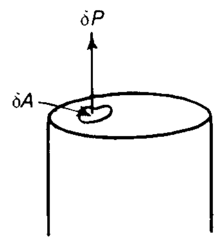
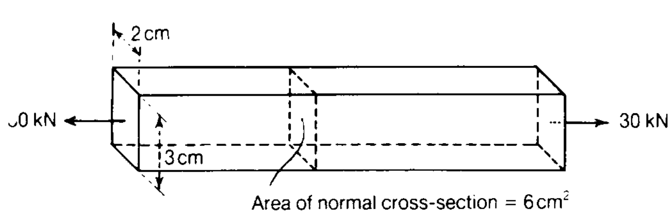


Figure 1.4 Normal load on an element of area of the cross-section.

When the forces P in Figure 1.3 are reversed in direction at each end of the bar they tend to *compress* the bar; the loads then give rise to *compressive stresses*. Tensile and compressive stresses are together referred to as *direct* (or *normal*) *stresses*, because they act perpendicularly to the surface.

Problem 1.1 A steel bar of rectangular cross-section, 3 cm by 2 cm, carries an axial load of 30 kN. Estimate the average tensile stress over a normal cross-section of the bar.



Solution

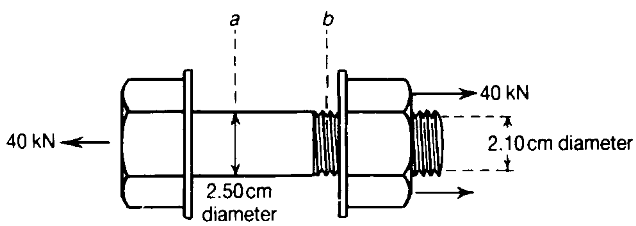
The area of a normal cross-section of the bar is

$$A = 0.03 \times 0.02 = 0.6 \times 10^{-3}\ \text{m}^2$$

The average tensile stress over this cross-section is then

$$\sigma = \frac{P}{A} = \frac{30 \times 10^3}{0.6 \times 10^{-3}} = 50\text{MN/m}^2$$

Problem 1.2 A steel bolt, 2.50 cm in diameter, carries a tensile load of 40 kN. Estimate the average tensile stress at the section a and at the screwed section b, where the diameter at the root of the thread is 2.10 cm.



Solution

The cross-sectional area of the bolt at the section a is

$$A_a = \frac{\pi}{4}\ (0.025)^2 = 0.491 \times 10^{-3}\ \text{m}^2$$

The average tensile stress at A is then

$$\sigma_a = \frac{P}{A_a} = \frac{40 \times 10^3}{0.491 \times 10^{-3}} = 81.4\ \text{MN/m}^2$$

The cross-sectional area at the root of the thread, section b, is

$$A_b = \frac{\pi}{4}\ (0.021)^2 = 0.346 \times 10^{-3}\ \text{m}^2$$

The average tensile stress over this section is

$$\sigma_b = \frac{P}{A_b} = \frac{40 \times 10^3}{0.346 \times 10^{-3}} = 115.6\ \text{MN/m}^2$$

1.4 Tensile and compressive strains

In the steel wire experiment of Figure 1.1 we discussed the extension of the whole wire. If we measure the extension of, say, the lowest quarter-length of the wire we find that for a given load it is equal to a quarter of the extension of the whole wire. In general we find that, at a given load, the ratio of the extension of any length to that length is constant for all parts of the wire; this ratio is known as the *tensile strain*.

Suppose the initial unstrained length of the wire is L_0, and the e is the extension due to straining; the tensile strain ε is defined as

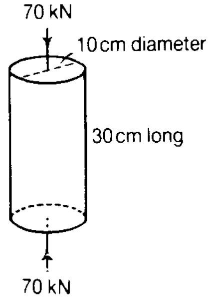
$$\varepsilon = \frac{e}{L_0} \tag{1.3}$$

This definition of strain is useful only for small distortions, in which the extension e is small compared with the original length L_0; this definition is adequate for the study of most engineering problems, where we are concerned with values of ε of the order 0.001, or so.

If a material is compressed the resulting strain is defined in a similar way, except that e is the contraction of a length.

We note that strain is a *non-dimensional* quantity, being the ratio of the extension, or contraction, of a bar to its original length.

Problem 1.3 A cylindrical block is 30 cm long and has a circular cross-section 10 cm in diameter. It carries a total compressive load of 70 kN, and under this load it contracts by 0.02 cm. Estimate the average compressive stress over a normal cross-section and the compressive strain.



<u>*Solution*</u>

The area of a normal cross-section is

$$A = \frac{\pi}{4}(0.10)^2 = 7.85 \times 10^{-3}\ \text{m}^2$$

The average compressive stress over this cross-section is then

$$\sigma = \frac{P}{A} = \frac{70 \times 10^3}{7.85 \times 10^{-3}} = 8.92 \text{ MN/m}^2$$

The average compressive strain over the length of the cylinder is

$$\varepsilon = \frac{0.02 \times 10^{-2}}{30 \times 10^{-2}} = 0.67 \times 10^{-3}$$

1.5 Stress–strain curves for brittle materials

Many of the characteristics of a material can be deduced from the tensile test. In the experiment of Figure 1.1 we measured the extensions of the wire for increasing loads; it is more convenient to compare materials in terms of stresses and strains, rather than loads and extensions of a particular specimen of a material.

The tensile *stress–strain* curve for a high-strength steel has the form shown in Figure 1.5. The stress at any stage is the ratio of the load of the *original* cross-sectional area of the test specimen; the strain is the elongation of a unit length of the test specimen. For stresses up to about 750 MN/m^2 the stress–strain curve is linear, showing that the material obeys Hooke's law in this range; the material is also elastic in this range, and no permanent extensions remain after removal of the stresses. The ratio of stress to strain for this linear region is usually about 200 GN/m^2 for steels; this ratio is known as *Young's modulus* and is denoted by E. The strain at the limit of proportionality is of the order 0.003, and is small compared with strains of the order 0.100 at fracture.

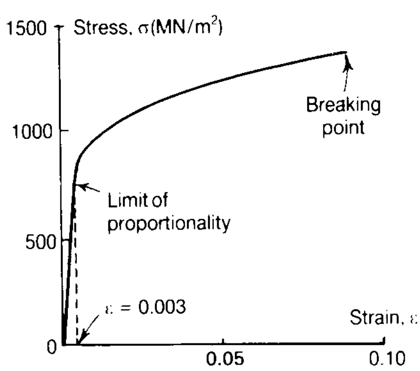


Figure 1.5 Tensile stress–strain curve for a high-strength steel.

We note that *Young's modulus* has the units of a stress; the value of E defines the constant in the linear relation between stress and strain in the elastic range of the material. We have

$$E = \frac{\sigma}{\varepsilon} \tag{1.4}$$

for the linear-elastic range. If P is the total tensile load in a bar, A its cross-sectional area, and L_0 its length, then

$$E = \frac{\sigma}{\varepsilon} = \frac{P / A}{e / L_0} \tag{1.5}$$

where e is the extension of the length L_0. Thus the expansion is given by

$$e = \frac{PL_0}{EA} \tag{1.6}$$

If the material is stressed beyond the linear-elastic range the limit of proportionality is exceeded, and the strains increase non-linearly with the stresses. Moreover, removal of the stress leaves the material with some permanent extension; this range is then both non-linear and inelastic. The maximum stress attained may be of the order of 1500 MN/m^2, and the total extension, or *elongation*, at this stage may be of the order of 10%.

The curve of Figure 1.5 is typical of the behaviour of *brittle* materials—as, for example, area characterized by small permanent elongation at the breaking point; in the case of metals this is usually 10%, or less.

When a material is stressed beyond the limit of proportionality and is then unloaded, permanent deformations of the material take place. Suppose the tensile test-specimen of Figure 1.5 is stressed beyond the limit of proportionality, (point a in Figure 1.6), to a point b on the stress–strain diagram. If the stress is now removed, the stress–strain relation follows the curve bc; when the stress is completely removed there is a residual strain given by the intercept $0c$ on the ε-axis. If the stress is applied again, the stress–strain relation follows the curve cd initially, and finally the curve df to the breaking point. Both the unloading curve bc and the reloading curve cd are approximately parallel to the elastic line $0a$; they are curved slightly in opposite directions. The process of unloading and reloading, bcd, had little or no effect on the stress at the breaking point, the stress–strain curve being interrupted by only a small amount bd, Figure 1.6.

The stress–strain curves of brittle materials for tension and compression are usually similar in form, although the stresses at the limit of proportionality and at fracture may be very different for the two loading conditions. Typical tensile and compressive stress–strain curves for concrete are shown in Figure 1.7; the maximum stress attainable in tension is only about one-tenth of that in compression, although the slopes of the stress–strain curves in the region of zero stress are nearly equal.

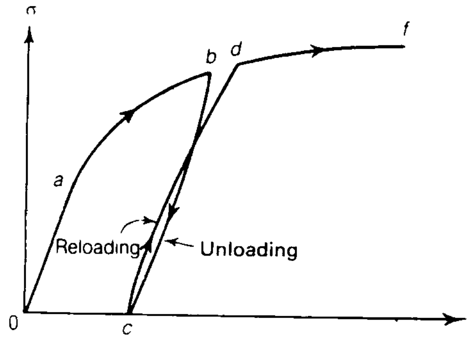


Figure 1.6 Unloading and reloading of a material in the inelastic range; the paths *bc* and *cd* are approximately parallel to the linear-elastic line *oa*.

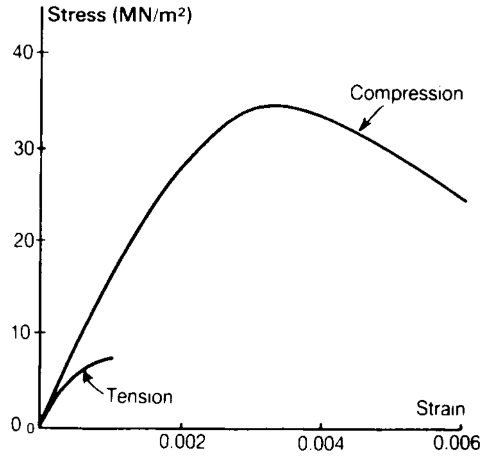


Figure 1.7 Typical compressive and tensile stress–strain curves for concrete, showing the comparative weakness of concrete in tension.

1.6 Ductile materials *(see Section 1.8)*

A brittle material is one showing relatively little elongation at fracture in the tensile test; by contrast some materials, such as mild steel, copper, and synthetic polymers, may be stretched appreciably before breaking. These latter materials are *ductile* in character.

If tensile and compressive tests are made on a mild steel, the resulting stress–strain curves are different in form from those of a brittle material, such as a high-strength steel. If a tensile test specimen of mild steel is loaded axially, the stress–strain curve is linear and elastic up to a point *a*, Figure 1.8; the small strain region of Figure 1.8. is reproduced to a larger scale in Figure 1.9. The ratio of stress to strain, or Young's modulus, for the linear portion 0*a* is usually about 200 GN/m^2, ie, 200 $\times 10^9$ N/m^2. The tensile stress at the point *a* is of order 300 MN/m^2, i.e. 300 $\times$ 10^6 N/m^2. If the test specimen is strained beyond the point *a*, Figures 1.8 and 1.9, the stress must be reduced almost immediately to maintain equilibrium; the reduction of stress, *ab*, takes place rapidly, and the form of the curve *ab* is difficult to define precisely. Continued straining proceeds at a roughly constant stress along *bc*. In the range of strains from *a* to *c* the material is said to *yield*; *a* is the *upper yield point*, and *b* the *lower yield point*. Yielding at constant stress along *bc* proceeds usually to a strain about 40 times greater than that at *a*; beyond the point *c* the material *strain-hardens*, and stress again increases with strain where the slope from *c* to *d* is about 1/50th that from 0 to *a*. The stress for a tensile specimen attains a maximum value at *d* if the stress is evaluated on the basis of the original cross-sectional area of the bar; the stress corresponding to the point *d* is known as the *ultimate stress*, σ_{ult}, of the material. From *d* to *f* there is a reduction in the nominal stress until fracture occurs at *f*. The ultimate stress in tension is attained at a stage when *necking* begins; this is a reduction of area at a relatively weak cross-section of the test specimen. It is usual to measure the diameter of the neck after fracture, and to evaluate a true stress at fracture, based on the breaking load and the reduced cross-sectional area at the neck. Necking and considerable elongation before fracture are characteristics of ductile materials; there is little or no necking at fracture for brittle materials.

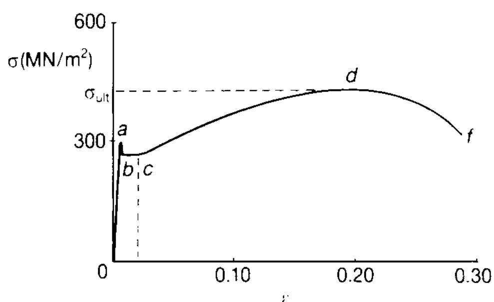


Figure 1.8 Tensile stress–strain curve for an annealed mild steel, showing the drop in stress at yielding from the upper yield point *a* to the lower yield point *b*.

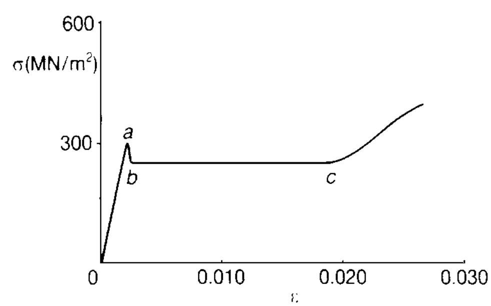


Figure 1.9 Upper and lower yield points of a mild steel.

Compressive tests of mild steel give stress–strain curves similar to those for tension. If we consider tensile stresses and strains as positive, and compressive stresses and strains as negative, we can plot the tensile and compressive stress–strain curves on the same diagram; Figure 1.10 shows the stress–strain curves for an annealed mild steel. In determining the stress–strain curves experimentally, it is important to ensure that the bar is loaded axially; with even small eccentricities of loading the stress distribution over any cross-section of the bar is non-uniform, and the upper yield point stress is not attained in all fibres of the material simultaneously. For this reason the lower yield point stress is taken usually as a more realistic definition of yielding of the material.

Some ductile materials show no clearly defined upper yield stress; for these materials the limit of proportionality may be lower than the stress for continuous yielding. The term *yield stress* refers to the stress for continuous yielding of a material; this implies the lower yield stress for a material in which an upper yield point exists; the yield stress is denoted by σ_Y.

Tensile failures of some steel bars are shown in Figure 1.11; specimen (ii) is a brittle material, showing little or no necking at the fractured section; specimens (i) and (iii) are ductile steels showing a characteristic necking at the fractured sections. The tensile specimens of Figure 1.12 show the forms of failure in a ductile steel and a ductile light-alloy material; the steel specimen (i) fails at a necked section in the form of a 'cup and cone'; in the case of the light-alloy bar, two 'cups' are formed. The compressive failure of a brittle cast iron is shown in Figure 1.13. In the case of a mild steel, failure in compression occurs in a 'barrel-like' fashion, as shown in Figure 1.14.

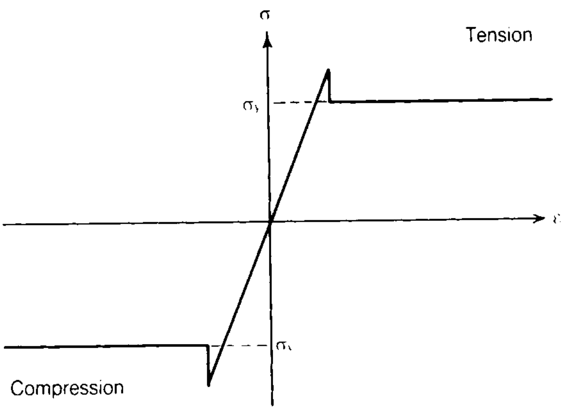
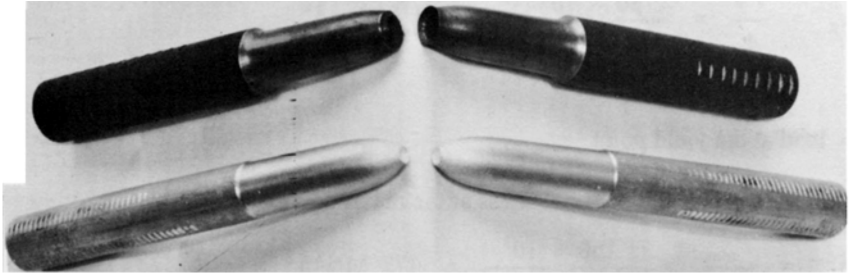


Figure 1.10 Tensile and compressive stress–strain curves for an annealed mild steel; in the annealed condition the yield stresses in tension and compression are approximately equal.

The stress–strain curves discussed in the preceding paragraph refer to static tests carried out at negligible speed. When stresses are applied rapidly the yield stress and ultimate stresses of metallic materials are usually raised. At a strain rate of 100 per second the yield stress of a mild steel may be twice that at negligible speed.

(i)
(ii)
(iii)

Figure 1.11 Tensile failures in steel specimens showing necking in mild steel, (i) and (iii), and brittle fracture in high-strength steel, (ii).



(i)
(ii)

Figure 1.12 Necking in tensile failures of ductile materials.
(i) Mild-steel specimen showing 'cup and cone' at the broken section.
(ii) Aluminium-alloy specimen showing double 'cup' type of failure.

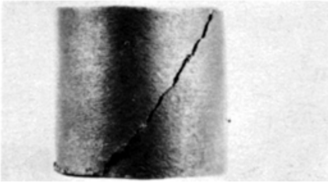


Figure 1.13 Failure in compression of a circular specimen of cast iron, showing fracture on a diagonal plane.

Figure 1.14 Barrel-like failure in a compressed specimen of mild steel.

Problem 1.4 A tensile test is carried out on a bar of mild steel of diameter 2 cm. The bar yields under a load of 80 kN. It reaches a maximum load of 150 kN, and breaks finally at a load of 70 kN.

Estimate:

(i) the tensile stress at the yield point;
(ii) the ultimate tensile stress;
(iii) the average stress at the breaking point, if the diameter of the fractured neck is 1 cm.

Solution

The original cross-section of the bar is

$$A_0 = \frac{\pi}{4}(0.020)^2 = 0.314 \times 10^{-3}\ \text{m}^2$$

(i) The average tensile stress at yielding is then

$$\sigma_Y = \frac{P_Y}{A_0} = \frac{80 \times 10^3}{0.314 \times 10^{-3}} = 254\ \text{MN/m}^2,$$

where P_Y = load at the yield point

(ii) The ultimate stress is the nominal stress at the maximum load, i.e.,

$$\sigma_{\text{ult}} = \frac{P_{\max}}{A_0} = \frac{150 \times 10^3}{0.314 \times 10^{-3}} = 477\ \text{MN/m}^2$$

where $P_{\max}$ = maximum load

(iii) The cross-sectional area in the fractured neck is

$$A_f = \frac{\pi}{4}(0.010)^2 = 0.0785 \times 10^{-3}\ \text{m}^2$$

The average stress at the breaking point is then

$$\sigma_f = \frac{P_f}{A_f} = \frac{70 \times 10^3}{0.0785 \times 10^{-3}} = 892\ \text{MN/m}^2,$$

where P_f = final breaking load.

Problem 1.5 A circular bar of diameter 2.50 cm is subjected to an axial tension of 20 kN. If the material is elastic with a Young's modulus E = 70 GN/m^2, estimate the percentage elongation.

Solution

The cross-sectional area of the bar is

$$A = \frac{\pi}{4}(0.025)^2 = 0.491 \times 10^{-3}\ \text{m}^2$$

The average tensile stress is then

$$\sigma = \frac{P}{A} = \frac{20 \times 10^3}{0.491 \times 10^{-3}} = 40.7\ \text{MN/m}^2$$

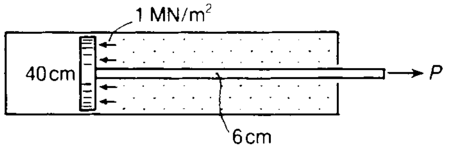
The longitudinal tensile strain will therefore be

$$\varepsilon = \frac{\sigma}{E} = \frac{40.7 \times 10^6}{70 \times 10^9} = 0.582 \times 10^{-3}$$

The percentage elongation will therefore be

$$(0.582 \times 10^{-3})\,100 = 0.058\%$$

Problem 1.6 The piston of a hydraulic ram is 40 cm diameter, and the piston rod 6 cm diameter. The water pressure is 1 MN/m^2. Estimate the stress in the piston rod and the elongation of a length of 1 m of the rod when the piston is under pressure from the piston-rod side. Take Young's modulus as E = 200 GN/m^2.



Solution

The pressure on the back of the piston acts on a net area

$$\frac{\pi}{4}\left[(0.40)^2 - (0.06)^2\right] = \frac{\pi}{4}(0.46)(0.34) = 0.123 \text{ m}^2$$

The load on the piston is then

$$P = (1)(0.123) = 0.123 \text{ MN}$$

Area of the piston rod is

$$A = \frac{\pi}{4}(0.060)^2 = 0.283 \times 10^{-2} \text{ m}^2$$

The average tensile stress in the rod is then

$$\sigma = \frac{P}{A} = \frac{0.123 \times 10^6}{0.283 \times 10^{-2}} = 43.5 \text{ MN/m}^2$$

From equation (1.6), the elongation of a length $L = 1$ m is

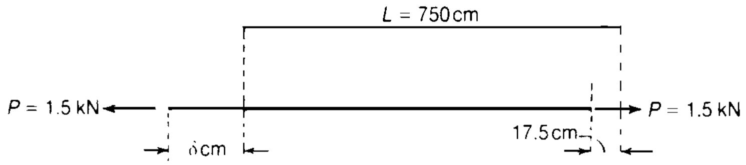
$$e = \frac{PL}{EA} = \frac{P}{A}\left(\frac{L}{E}\right) = \frac{\sigma L}{E}$$

$$= \frac{(43.5 \times 10^6)(1)}{200 \times 10^9}$$

$$= 0.218 \times 10^{-3} \text{ m}$$

$$= 0.0218 \text{ cm}$$

Problem 1.7 The steel wire working a signal is 750 m long and 0.5 cm diameter. Assuming a pull on the wire of 1.5 kN, find the movement which must be given to the signal-box end of the wire if the movement at the signal end is to be 17.5 cm. Take Young's modulus as 200 GN/m^2.



Solution

If δ(cm) is the movement at the signal-box end, the actual stretch of the wire is $e = (\delta - 17.5)$cm

The longitudinal strain is then

$$\varepsilon = \frac{(\delta - 17.5)\ 10^{-2}}{750}$$

Now the cross-sectional area of the wire is

$$A = \frac{\pi}{4}(0.005)^2 = 0.0196 \times 10^{-3}\ \mathrm{m}^2$$

The longitudinal strain can also be defined in terms of the tensile load, namely,

$$\varepsilon = \frac{e}{L} = \frac{P}{EA} = \frac{1.5 \times 10^3}{(200 \times 10^9)(0.0196 \times 10^{-3})}$$

$$= 0.383 \times 10^{-3}$$

On equating these two values of ε,

$$\frac{(\delta - 17.5)\ 10^{-2}}{750} = 0.383 \times 10^{-3}$$

The equation gives

$$\delta = 46.2\ \mathrm{cm}$$

Problem 1.8 A circular, metal rod of diameter 1 cm is loaded in tension. When the tensile load is 5kN, the extension of a 25 cm length is measured accurately and found to be 0.0227 cm. Estimate the value of Young's modulus, E, of the metal.

Solution

The cross-sectional area is

$$A = \frac{\pi}{4}(0.01)^2 = 0.0785 \times 10^{-3}\ \mathrm{m}^2$$

The tensile stress is then

$$\sigma = \frac{P}{A} = \frac{5 \times 10^3}{0.0785 \times 10^{-3}} = 63.7\ \mathrm{MN/m^2}$$

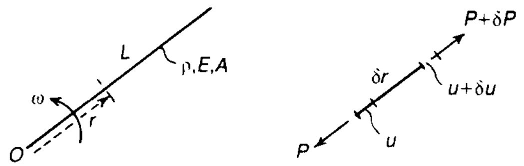
The measured tensile strain is

$$\varepsilon = \frac{e}{L} = \frac{0.0227 \times 10^{-2}}{25 \times 10^{-2}} = 0.910 \times 10^{-3}$$

Then Young's modulus is defined by

$$E = \frac{\sigma}{\varepsilon} = \frac{63.7 \times 10^6}{0.91 \times 10^{-3}} = 70\ \mathrm{GN/m^2}$$

Problem 1.9 A straight, uniform rod of length L rotates at uniform angular speed ω about an axis through one end and perpendicular to its length. Estimate the maximum tensile stress generated in the rod and the elongation of the rod at this speed. The density of the material is ρ and Young's modulus is E.



Solution

Suppose the radial displacement of any point a distance r from the axis of rotation is u. The radial displacement a distance $r + \delta r$) from 0 is then $(u + \delta u)$, and the elemental length δr of the rod is stretched therefore an amount δu. The longitudinal strain of this element is therefore

$$\varepsilon = \underset{\delta r \to 0}{\text{Limit}} \frac{\delta u}{\delta r} = \frac{du}{dr}$$

The longitudinal stress in the elemental length is then

$$\sigma = E\varepsilon = E \frac{du}{dr}$$

If A is the cross-sectional area of the rod, the longitudinal load at any radius r is then

$$P = \sigma A = EA \frac{du}{dr}$$

The centrifugal force acting on the elemental length δr is

$$(\rho A \delta r)\ \omega^2 r$$

Then, for radial equilibrium of the elemental length,

$$\delta P + \rho A \omega^2 r\, \delta r = 0$$

This gives

$$\frac{dP}{dr} = -\rho A \omega^2 r$$

On integrating, we have

$$P = -\frac{1}{2}\rho A \omega^2 r^2 + C$$

where C is an arbitrary constant; if $P = 0$ at the remote end, $r = L$, of the rod, then

$$C = \frac{1}{2}\rho A \omega^2 L^2$$

and

$$P = \frac{1}{2}\rho A\omega^2 L^2\left(1-\frac{r^2}{L^2}\right)$$

The tensile stress at any radius is then

$$\sigma = \frac{P}{A} = \frac{1}{2}\rho\omega^2 L^2\left(1-\frac{r^2}{L^2}\right)$$

This is greatest at the axis of rotation, $r = 0$, so that

$$\sigma_{\max} = \frac{1}{2}\rho\omega^2 L^2$$

The longitudinal stress, σ, is defined by

$$\sigma = E\frac{du}{dr}$$

so

$$\frac{du}{dr} = \frac{\sigma}{E} = \frac{\rho\,\omega^2 L^2}{2E}\left(1 - \frac{r^2}{L^2}\right)$$

On integrating,

$$u = \frac{\rho\,\omega^2 L^2}{2E}\left(r - \frac{r^3}{3L^2} + D\right)$$

where D is an arbitrary constant; if there is no radial movement at 0, then $u = 0$ at $= r = 0$, and we have $D = 0$.

Thus

$$u = \frac{\rho\,\omega^2 L^2}{2E}\left[r\left(1 - \frac{r^2}{3L^2}\right)\right]$$

At the remote end, $r = L$,

$$u_L = \frac{\rho\,\omega^2 L^2}{2E}\left[L\left(\frac{2}{3}\right)\right] = \frac{\rho\,\omega^2 L^3}{3E}$$

1.7 Proof stresses

Many materials show no well-defined yield stresses when tested in tension or compression. A typical stress–strain curve for an aluminium alloy is shown in Figure 1.15.

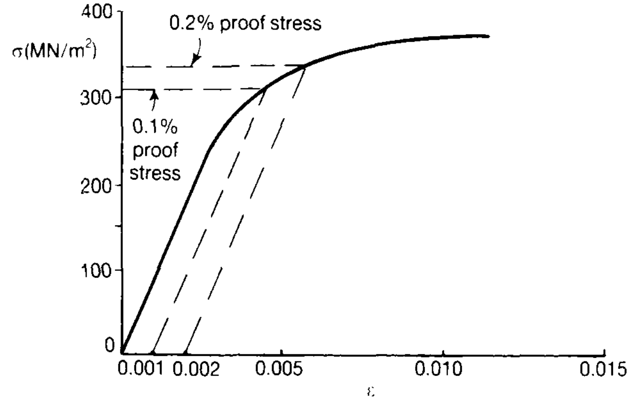


Figure 1.15 Proof stresses of an aluminium-alloy material; the proof stress is found by drawing the line parallel to the linear-elastic line at the appropriate proof strain.

The limit of proportionality is in the region of 300 MN/m^2, but the exact position of this limit is difficult to determine experimentally. To overcome this problem a *proof stress* is defined; the 0.1% proof stress required to produce a permanent strain of 0.001 (or 0.1%) on removal of the stress. Suppose we draw a line from the point 0.001 on the strain axis, Figure 1.15, parallel to the elastic line of the material; the point where this line cuts the stress–strain curve defines the proof stress. The 0.2% proof stress is defined in a similar way.

1.8 Ductility measurement

The Ductility value of a material can be described as the ability of the material to suffer plastic deformation while still being able to resist applied loading. The more ductile a material is the more it is said to have the ability to deform under applied loading.

The ductility of a metal is usually measured by its percentage reduction in cross-sectional area or by its percentage increase in length, i.e.

$$\text{percentage reduction in area} = \frac{(A_I - A_F)}{A_I} \times 100\%$$

and

$$\text{percentage increase in length} = \frac{(L_I - L_F)}{L_I} \times 100\%$$

where

A_I = initial cross-sectional area of the tensile specimen

A_F = final cross-sectional area of the tensile specimen

L_I = initial gauge length of the tensile specimen

L_F = final gauge length of the tensile specimen

It should be emphasised that the shape of the tensile specimen plays a major role on the measurement of the ductility and some typical relationships between length and character for tensile specimens i.e. given in Table 1.1

Materials such as copper and mild steel have high ductility and brittle materials such as bronze and cast iron have low ductility.

Table 1.1 Circular cylindrical tensile specimens

Place	L_I	L_I/D_I*
UK	4√area	3.54
USA	4.51√area	4.0
Europe	5.65√area	5.0

area = cross-sectional area

* D_I = initial diameter of the tensile specimen

1.9 Working stresses

In many engineering problems the loads sustained by a component of a machine or structure are reasonably well-defined; for example, the lower stanchions of a tall building support the weight of material forming the upper storeys. The stresses which are present in a component, under normal working conditions, are called the *working stresses*; the ratio of the yield stress, σ_Y, of a material to the largest working stress, σ_W, in the component is the *stress factor* against yielding. The stress factor on yielding is then

$$\frac{\sigma_Y}{\sigma_W} \tag{1.7}$$

If the material has no well-defined yield point, it is more convenient to use the *proof stress*, σ_p; the stress factor on proof stress is then

$$\frac{\sigma_P}{\sigma_W} \tag{1.8}$$

Some writers refer to the stress factor defined above as a 'safety factor'. It is preferable, however, to avoid any reference to 'safe' stresses, as the degree of safety in any practical problem is difficult to define. The present writers prefer the term 'stress factor' as this defines more precisely that the working stress is compared with the yield, or proof stress of the material. Another reason for using 'stress factor' will become more evident after the reader has studied Section 1.10.

1.10 Load factors

The *stress factor* in a component gives an indication of the working stresses in relation to the yield, or proof, stress of the material. In practical problems working stresses can only be estimated approximately in stress calculations. For this reason the stress factor may give little indication of the degree of safety of a component.

A more realistic estimate of safety can be made by finding the extent to which the working loads on a component may be increased before collapse or fracture occurs. Consider, for example, the continuous beam in Figure 1.16, resting on three supports. Under working conditions the beam carries lateral loads P_1, P_2 and P_3, Figure 1.16(i). If all these loads can be increased simultaneously by a factor n before collapse occurs, the load factor against collapse is n. In some complex structural systems, as for example continuous beams, the collapse loads, such as nP_1, nP_2 and nP_3, can be estimated reasonably accurately; the value of the load factor can then be deduced to give working loads P_1, P_2 and P_3.

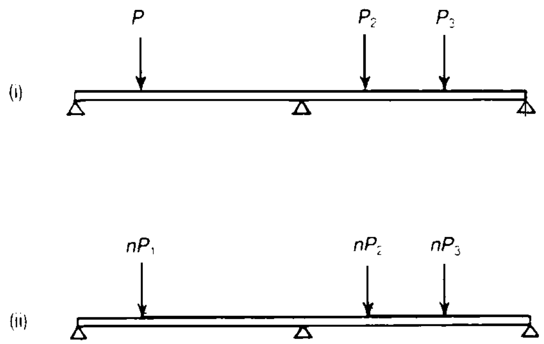


Figure 1.16 Factored loads on a continuous beam.
(i) Working loads. (ii) Factored working loads leading to collapse.

1.11 Lateral strains due to direct stresses

When a bar of a material is stretched longitudinally—as in a tensile test—the bar extends in the direction of the applied load. This longitudinal extension is accompanied by a lateral contraction of the bar, as shown in Figure 1.17. In the linear-elastic range of a material the lateral strain is proportional to the longitudinal strain; if ε_x is the longitudinal strain of the bar, then the lateral strain is

$$\varepsilon_y = \nu\varepsilon_x \tag{1.9}$$

The constant ν in this relationship is known as *Poisson's ratio*, and for most metals it has a value of about 0.3 in the linear-elastic range; it cannot exceed a value of 0.5. For concrete it has a value of about 0.1. If the longitudinal strain is tensile, the lateral strain is a contraction; for a compressed bar there is a lateral expansion.

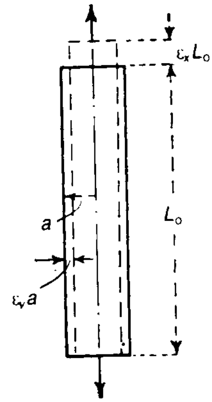


Figure 1.17 The Poisson ratio effect leading to lateral contraction of a bar in tension.

With a knowledge of the lateral contraction of a stretched bar it is possible to calculate the change in volume due to straining. The bar of Figure 1.17 is assumed to have a square cross-section of side a; L_0 is the unstrained length of the bar. When strained longitudinally an amount ε_x, the corresponding lateral strain of contractions is ε_y. The bar extends therefore an amount $\varepsilon_x L_0$, and each side of the cross-section contracts an amount $\varepsilon_y a$. The volume of the bar before stretching is

$$V_0 = a^2 L_0$$

After straining the volume is

$$V = (a - \varepsilon_y a)^2 \, (L_0 + \varepsilon_x L_0)$$

which may be written

$$V = a^2 L_0 (1 - \varepsilon_y)^2 (1 + \varepsilon_x) = V_0 (1 - \varepsilon_y)^2 (1 + \varepsilon_y)$$

If ε_x and ε_y are small quantities compared to unit, we may write

$$(1 - \varepsilon_y)^2 (1 + \varepsilon_x) = (1 - 2\varepsilon_y)(1 + \varepsilon_x) = 1 + \varepsilon_x - 2\varepsilon_y$$

ignoring squares and products of ε_x and ε_y. The volume after straining is then

$$V = V_0 (1 + \varepsilon_x - 2\varepsilon_y)$$

The *volumetric strain* is defined as the ratio of the change of volume to the original volume, and is therefore

$$\frac{V - V_0}{V_0} = \varepsilon_x - 2\varepsilon_y \qquad (1.10)$$

If $\varepsilon_y = v\,\varepsilon_{xy}$ then the volumetric strain is $\varepsilon_x(1-2v)$. Equation (1.10) shows why v cannot be greater than 0.5; if it were, then under *compressive* hydrostatic stress a *positive* volumetric strain will result, which is impossible.

Problem 1.10 A bar of steel, having a rectangular cross-section 7.5 cm by 2.5 cm, carries an axial tensile load of 180 kN. Estimate the decrease in the length of the sides of the cross-section if Young's modulus, E, is 200 GN/m^2 and Poisson's ratio, v, is 0.3.

<u>*Solution*</u>

The cross-sectional area is

$$A = (0.075)(0.025) = 1.875 \times 10^{-3}\ \text{m}^2$$

The average longitudinal tensile stress is

$$\sigma = \frac{P}{A} = \frac{180 \times 10^3}{1.875 \times 10^{-3}} = 96.0\ \text{MN/m}^2$$

The longitudinal tensile strain is therefore

$$\varepsilon = \frac{\sigma}{E} = \frac{96.0 \times 10^6}{200 \times 10^9} = 0.48 \times 10^{-3}$$

The lateral strain is therefore

$$v\varepsilon = 0.3(0.48 \times 10^{-3}) = 0.144 \times 10^{-3}$$

The 7.5 cm side then contracts by an amount

$$(0.075)(0.144 \times 10^{-3}) = 0.0108 \times 10^{-3}\ \text{m} = 0.00108\ \text{cm}$$

The 2.5 cm side contracts by an amount

$$(0.025)(0.144 \times 10^{-3}) = 0.0036 \times 10^{-3}\ \text{m} = 0.00036\ \text{cm}$$

1.12 Strength properties of some engineering materials

The mechanical properties of some engineering materials are given in Table 1.2. Most of the materials are in common engineering use, including a number of relatively new and important materials; namely glass-fibre composites, carbon-fibre composites and boron composites. In the case of some brittle materials, such as cast iron and concrete, the ultimate stress in tension is considerably smaller than in compression.

Composite materials, such as glass fibre reinforced plastics, (GRP), carbon-fibre reinforced plastics (CFRP), boron-fibre reinforced plastics, 'Kevlar' and metal-matrix composites are likely to revolutionise the design and construction of many structures in the 21st century. The glass fibres used in GRP are usually made from a borosilicate glass, similar to the glass used for cooking utensils. Borosilicate glass fibres are usually produced in 'E' glass or glass that has good electrical resistance. A very strong form of borosilicate glass fibre appears in the form of 'S' glass which is much more expensive than 'E' glass.

Some carbon fibres, namely high modulus (HM) carbon fibres, have a tensile modulus much larger than high strength steels, whereas other carbon fibres have a very high tensile strength (HS) much larger than high tensile steels.

Currently 'S' glass is some eight times more expensive than 'E' glass and HS carbon is about 50 times more expensive than 'E' glass. HM carbon is some 250 times more expensive than 'E' glass while 'Kevlar' is some 15 times more expensive than 'E' glass.

1.13 Weight and stiffness economy of materials

In some machine components and structures it is important that the weight of material should be as small as possible. This is particularly true of aircraft, submarines and rockets, for example, in which less structural weight leads to a larger pay–load. If σ_{ult} is the ultimate stress of a material in tension and ρ is its density, then a measure of the strength economy is the ratio

$$\frac{\sigma_{ult}}{\rho}$$

The materials shown in Table 1.2 are compared on the basis of strength economy in Table 1.3 from which it is clear that the modern fibre-reinforced composites offer distinct savings in weight over the more common materials in engineering use.

In some engineering applications, stiffness rather than strength is required of materials; this is so in structures likely to buckle and components governed by deflection limitations. A measure of the stiffness economy of a material is the ratio

$$\frac{E}{\rho},$$

some values of which are shown in Table 1.2. Boron composites and carbon-fibre composites show outstanding stiffness properties, whereas glass-fibre composites fall more into line with the best materials already in common use.

Table 1.2 Approximate strength properties of some engineering materials

Material	Limit of proportionality (MN/m^2)	Ultimate stress σ_{ult} (MN/m^2)	Elongation at tensile fracture (as a fraction of the original length)	Young's modulus E (GN/m^2)	Density ρ (kg/m^3)	σ_{ult}/ρ $(m/s)^2$	E/ρ $(m/s)^2$	Coefficient of linear expansion α (per °C)
Medium-strength mild steel	280	370	0.30	200	7840	47×10^3	25×10^6	1.2×10^{-5}
High-strength steel	770	1550	0.10	200	7840	198×10^3	25×10^6	1.3×10^{-5}
Medium-strength aluminium alloy	230	430	0.10	70	2800	154×10^3	25×10^6	2.3×10^{-5}
Titanium alloy	385	690	0.15	120	4500	153×10^3	27×10^6	0.9×10^{-5}
Magnesium alloy	155	280	0.08	45	1800	156×10^3	25×10^6	2.7×10^{-5}
Wrought iron	185	310	—	190	7670	40×10^3	25×10^6	1.2×10^{-5}
Cast iron } tension	—	155	—	140	7200	—	20×10^6	1.1×10^{-5}
} compression	—	700	—	140	7200	$97^* \times 10^3$	20×10^6	1.1×10^{-5}
Concrete } tension	—	3.0	—	14	2410	—	6×10^6	1.2×10^{-5}
} compression	—	30.0	—	14	2410	$12^* \times 10^3$	6×10^6	1.2×10^{-5}
Nylon (polyamide)	77	90	1.00	2		79×10^3	1.8×10^6	10×10^{-5}
Polystyrene	46	60	0.03	3.5	1050	57×10^3	3.3×10^6	10×10^{-5}
Fluon (tetrafluoroethylene)	8	15	2.00	0.4	2220	7×10^3	0.2×10^6	11×10^{-5}
Polythene (ethylene)	6	12	5.00	0.2	915	13×10^3	0.2×10^6	28×10^{-5}
High-strength glass-fibre composite	—	1600	—	60	2000	800×10^3	30×10^6	—
Carbon-fibre composite	—	1400	—	170	1600	875×10^3	105×10^6	—
Boron composite	—	1300	—	270	2000	650×10^3	135×10^6	—

* Evaluate on the compressive value of σ_{ult}.

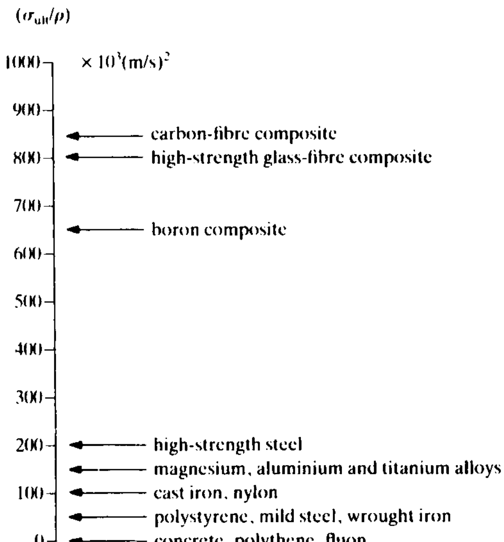


Table 1.3 Strength economy of some engineering materials

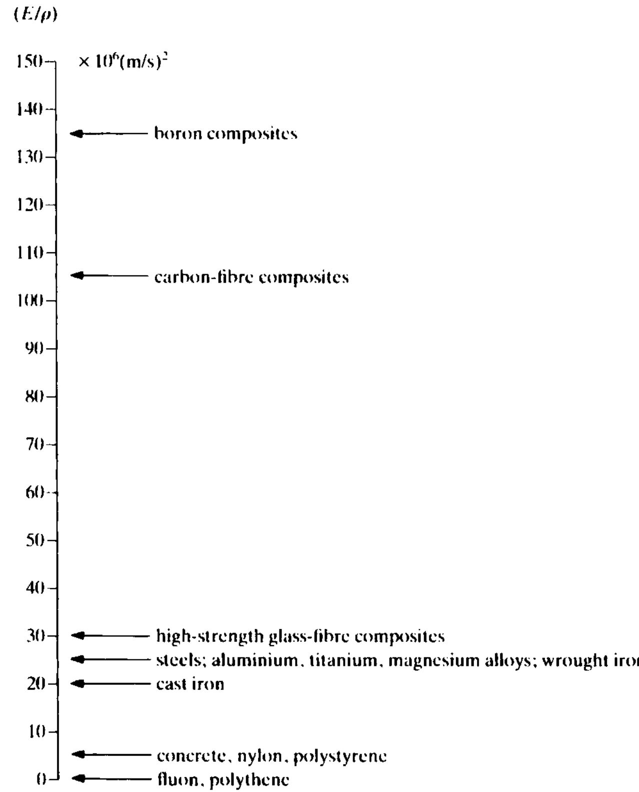


Table 1.4 Stiffness economy of some engineering materials

1.14 Strain energy and work done in the tensile test

As a tensile specimen extends under load, the forces applied to the ends of the test specimen move through small distances. These forces perform work in stretching the bar. If, at a tensile load P, the bar is stretched a small additional amount δe, Figure 1.18, then the work done on the bar is approximately $P\delta e$

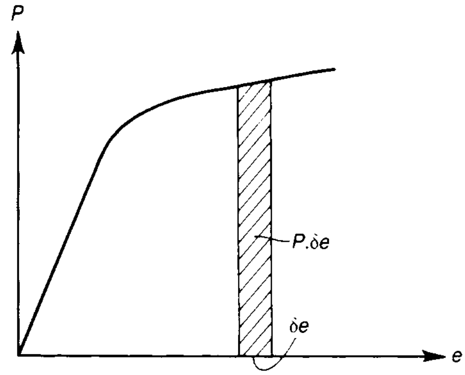


Figure 1.18 Work done in stretching a bar through a small extension, δe.

The total work done in extending the bar to the extension e is then

$$W = \int_0^e P\,de, \tag{1.11}$$

which is the area under the P–e curve up to the stretched condition. If the limit of proportionality is not exceeded, the work done in extending the bar is stored as *strain energy*, which is directly recoverable on removal of the load. For this case, the strain energy, U, is

$$U = W = \int_0^e P\,de \tag{1.12}$$

But in the linear-elastic range of the material, we have from equation (1.6) that

$$e = \frac{PL_0}{EA}$$

where L_0 is the initial length of the bar, A is its cross-sectional area and E is Young's modulus. Then equation (1.12) becomes

$$U = \int_0^e \frac{EA}{L_0}\,e\,de = \frac{EA}{2L_0}\left(e^2\right) \tag{1.13}$$

In terms of P

$$U = \frac{EA}{2L_0}\left(e^2\right) = \frac{L_0}{2EA}\left(P^2\right) \tag{1.14}$$

Now (P/A) is the tensile stress σ in the bar, and so we may write

$$U = \frac{AL_0}{2E}\left(\sigma^2\right) = \frac{\sigma^2}{2E} \times \text{ the volume} \tag{1.15}$$

Moreover, as AL_0 is the original volume of the bar, the strain energy per unit volume is

$$\frac{\sigma^2}{2E} \tag{1.16}$$

When the limit of proportionality of a material is exceeded, the work done in extending the bar is still given by equation (1.11); however, not all this work is stored as strain energy; some of the work done is used in producing permanent distortions in the material, the work reappearing largely in the form of heat. Suppose a mild-steel bar is stressed beyond the yield point, Figure 1.19, and up to the point where strain-hardening begins; the strain at the limit of proportionality is small compared with this large inelastic strain; the work done per unit volume in producing a strain ε is approximately

$$W = \sigma_Y \, \varepsilon \tag{1.17}$$

in which σ_Y is the yield stress of the material. This work is considerably greater than that required to reach the limit of proportionality. A ductile material of this type is useful in absorbing relatively large amounts of work before breaking.

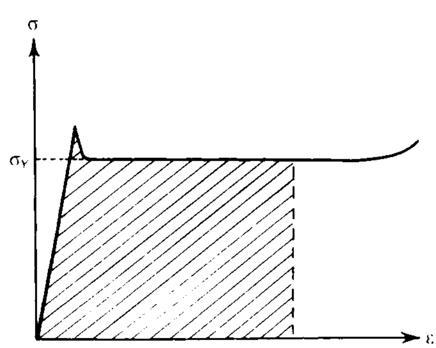
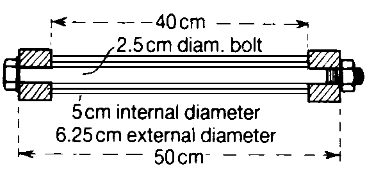


Figure 1.19 Work done in stretching a mild-steel bar; the work done during plastic deformation is very considerable compared with the elastic strain energy.

1.15 Initial stresses

It frequently happens that, before any load is applied to some part of a machine or structure, it is already in a state of stress. In other words, the component is *initially stressed* before external forces are applied. Bolted joints and connections, for example, involve bolts which are pre-tensioned; subsequent loading may, or may not, affect the tension in a bolt. Most forms of welded connections introduce initial stresses around the welds, unless the whole connection is stress relieved by a suitable heat treatment; in such cases, the initial stresses are not usually known with any real accuracy. Initial stresses can also be used to considerable effect in strengthening certain materials; for example, concrete can be made a more effective material by precompression in the form of prestressed concrete. The problems solved below are *statically indeterminate* (see Chapter 2) and therefore require *compatibility* considerations as well as *equilibrium* considerations.

Problem 1.11 A 2.5 cm diameter steel bolt passes through a steel tube 5 cm internal diameter, 6.25 cm external diameter, and 40 cm long. The bolt is then tightened up onto the tube through rigid end blocks until the tensile force in the bolts is 40 kN. The distance between the head of the bolt and the nut is 50 cm. If an external force of 30 kN is applied to the end blocks, tending to pull them apart, estimate the resulting tensile force in the bolt.



<u>*Solution*:</u>

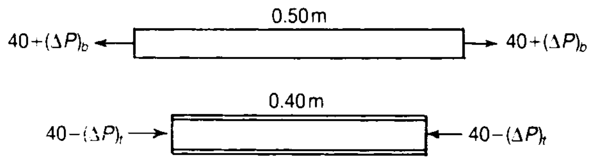
The cross-sectional area of the bolt is

$$\frac{\pi}{4}(0.025)^2 = 0.491 \times 10^{-3}\ \mathrm{m}^2$$

The cross-sectional are of the tube is

$$\frac{\pi}{4}\left[(0.0625)^2 - (0.050)^2\right] = \frac{\pi}{4}(0.1125)(0.0125) = 0.110 \times 10^{-2}\ \mathrm{m}^2$$

Before the external load of 30 kN is applied, the bolt and tube carry internal loads of 40 kN. When the external load of 30 kN is applied, suppose the tube and bolt are each stretched by amounts δ; suppose further that the *change* of load in the bolt is $(\Delta P)_b$, tensile, and the *change* of load in the tube is $(\Delta P)_t$, tensile.



Then for *compatibility*, the elastic stretch of each component due to the additional external load of 30 kN is

$$\delta = \frac{(\Delta P)_b\,(0.50)}{\left(0.491 \times 10^{-3}\right) E} = \frac{(\Delta P)_t\,(0.40)}{\left(0.110 \times 10^{-2}\right) E}$$

where E is Young's modulus. Then

$$(\Delta P)_b = 0.357\,(\Delta P)_t$$

But for *equilibrium* of internal and external forces,

$$(\Delta P)_b + (\Delta P)_t = 30 \text{ kN}$$

These two equations give

$$(\Delta P)_b = 7.89 \text{ kN}, \qquad (\Delta P)_t = 22.11 \text{ kN}$$

The resulting tensile force in the bolt is

$$40 + (\Delta P)_b = 47.89 \text{ kN}$$

1.16 Composite bars in tension or compression

A composite bar is one made of two materials, such as steel rods embedded in concrete. The construction of the bar is such that constituent components extend or contract equally under load. To illustrate the behaviour of such bars consider a rod made of two materials, 1 and 2, Figure 1.20; A_1, A_2 are the cross-sectional areas of the bars, and E_1, E_2 are the values of Young's modulus. We imagine the bars to be rigidly connected together at the ends; then for *compatibility*, the longitudinal strains to be the same when the composite bar is stretched we must have

$$\varepsilon = \frac{\sigma_1}{E_1} = \frac{\sigma_2}{E_2} \tag{1.18}$$

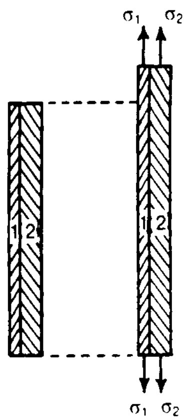


Figure 1.20 Composite bar in tension; if the bars are connected rigidly at their ends, they suffer the same extensions.

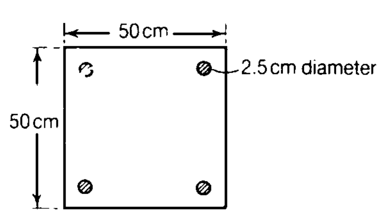
where σ_1 and σ_2 are the stresses in the two bars. But from *equilibrium* considerations,

$$P = \sigma_1 A_1 + \sigma_2 A_2 \qquad (1.19)$$

Equations (1.18) and (1.19) give

$$\sigma_1 = \frac{PE_1}{A_1 E_1 + A_2 E_2}, \quad \sigma_2 = \frac{PE_2}{A_1 E_1 + A_2 E_2} \qquad (1.20)$$

Problem 1.12 A concrete column, 50 cm square, is reinforced with four steel rods, each 2.5 cm in diameter, embedded in the concrete near the corners of the square. If Young's modulus for steel is 200 GN/m^2 and that for concrete is 14 GN/m^2, estimate the compressive stresses in the steel and concrete when the total thrust on the column is 1 MN.



<u>*Solution*</u>

Suppose subscripts *c* and *s* refer to concrete and steel, respectively. The cross-sectional area of steel is

$$A_s = 4\left[\frac{\pi}{4}(0.025)^2\right] = 1.96 \times 10^{-3}\ m^2$$

and the cross-sectional area of concrete is

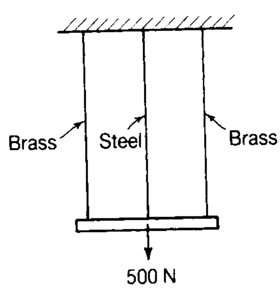
$$A_c = (0.50)^2 - A_s = 0.248 \text{ m}^2$$

Equations (1.20) then give

$$\sigma_c = \frac{10^6}{(0.248) + (1.96 \times 10^{-3})\left(\dfrac{200}{14}\right)} = 3.62 \text{ MN/m}^2$$

$$\sigma_s = \frac{10^6}{(0.248)\left(\dfrac{14}{200}\right) + (1.96 \times 10^{-3})} = 51.76 \text{ MN/m}^2$$

Problem 1.13 A uniform beam weighing 500 N is held in a horizontal position by three vertical wires, one attached to each end of the beam, and one at the mid-length. The outer wires are brass of diameter 0.125 cm, and the central wire is of steel of diameter 0.0625 cm. If the beam is rigid and the wires are of the same length, and unstressed before the beam is attached, estimate the stresses in the wires. Young's modulus for brass is 85 GN/m^2 and for steel is 200 GN/m^2.



Solution

On considering the two outer brass wires together, we may take the system as a composite one consisting of a single brass member and a steel member. The area of the steel member is

$$A_s = \frac{\pi}{4}(0.625 \times 10^{-3})^2 = 0.306 \times 10^{-6} \text{ m}^2$$

The total area of the two brass members is

$$A_b = 2\left[\frac{\pi}{4}(1.25 \times 10^{-3})^2\right] = 2.45 \times 10^{-6} \text{ m}^2$$

Equations (1.20) then give, for the steel wire

$$\sigma_s = \frac{500}{\left(0.306 \times 10^{-6}\right) + \left(2.45 \times 10^{-6}\right)\left(\frac{85}{200}\right)} = 370 \text{ MN/m}^2$$

and for the brass wires

$$\sigma_b = \frac{500}{\left(0.306 \times 10^{-6}\right)\left(\frac{200}{85}\right) + \left(2.45 \times 10^{-6}\right)} = 158 \text{ MN/m}^2$$

1.17 Temperature stresses

When the temperature of a body is raised, or lowered, the material expands, or contracts. If this expansion or contraction is wholly or partially resisted, stresses are set up in the body. Consider a long bar of a material; suppose L_0 is the length of the bar at a temperature θ_0, and that α is the coefficient of linear expansion of the material. The bar is now subjected to an increase θ in temperature. If the bar is completely free to expand, its length increases by $\alpha L_0 \theta$, and the length becomes $L_0 (1 + \alpha\theta)$ were compressed to a length L_0; in this case the compressive strain is

$$\varepsilon = \frac{\alpha L_0 \theta}{L_0 (1 + \alpha\theta)} = \alpha\theta$$

since $\alpha\theta$ is small compared with unity; the corresponding stress is

$$\sigma = E\varepsilon = \alpha\theta E \qquad (1.21)$$

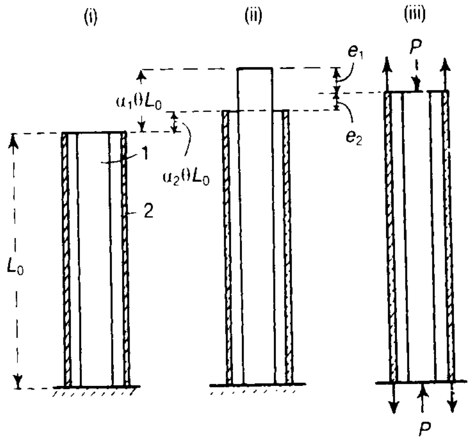
By a similar argument the tensile stress set up in a constrained bar by a fall θ in temperature is $\alpha\theta E$. It is assumed that the material remains elastic.

In the case of steel $\alpha = 1.3 \times 10^{-5}$ per °C; the product αE is approximately 2.6 MN/m^2 per °C, so that a change in temperature of 4°C produces a stress of approximately 10 MN/m^2 if the bar is completely restrained.

1.18 Temperature stresses in composite bars

In a component or structure made wholly of one material, temperature stresses arise only if external restraints prevent thermal expansion or contraction. In composite bars made of materials with different rates of thermal expansion, internal stresses can be set up by temperature changes; these stresses occur independently of those due to external restraints.

Consider, for example, a simple composite bar consisting of two members—a solid circular bar, 1, contained inside a circular tube, 2, Figure 1.21. The materials of the bar and tube have different coefficients of linear expansion, α_1 and α_2, respectively. If the ends of the bar and tube are attached rigidly to each other, longitudinal stresses are set up by a change of temperature. Suppose firstly, however, that the bar and tube are quite free of each other; if L_0 is the original length of each bar, Figure 1.21, the extensions due to a temperature increase θ are $\alpha_1\ \theta L_0$ and $\alpha_2\ \theta L_0$, Figure 1.21(ii). The difference in lengths of the two members is $(\alpha_1 - \alpha_2)\ \theta L_0$; this is now eliminated by compressing the inner bar with a force P, and pulling the outer tube with an equal force P, Figure 1.21(iii).



(i) (ii) (iii)

Figure 1.21 Temperature stress in a composite bar.

If A_1 and E_1 are the cross-sectional area and Young's modulus, respectively, of the inner bar, and A_2 and E_2 refer to the outer tube, then the contraction of the inner bar to P is

$$e_1 = \frac{PL_0}{E_1\ A_1}$$

and the extension of the outer tube due to P is

$$e_2 = \frac{PL_0}{E_2\ A_2}$$

Then from *compatibility* considerations, the difference in lengths $(\alpha_1 - \alpha_2)\ \theta L$, is eliminated completely when

$$(\alpha_1 - \alpha_2)\ \theta L_0 = e_1 + e_2$$

On substituting for $e_1 + e_2$, we have

$$(\alpha_1 - \alpha_2)\theta L_0 = PL\left(\frac{1}{E_1 A_1} + \frac{1}{E_2 A_2}\right) \tag{1.22}$$

The force P is induced by the temperature change θ if the ends of the two members are attached rigidly to each other; from equation (1.22), P has the value

$$P = \frac{(\alpha_1 - \alpha_2)\theta}{\left(\frac{1}{E_1 A_1} + \frac{1}{E_2 A_2}\right)} \tag{1.23}$$

An internal load is only set up if α_1 is different from α_2.

Problem 1.14 An aluminium rod 2.2 cm diameter is screwed at the ends, and passes through a steel tube 2.5 cm internal diameter and 0.3 cm thick. Both are heated to a temperature of 140°C, when the nuts on the rod are screwed lightly on to the ends of the tube. Estimate the stress in the rod when the common temperature has fallen to 20°C. For steel, E = 200 GN/m^2 and α = 1.2×10^{-5} per °C, and for aluminium, E = 70 GN/m^2 and α = 2.3×10^{-5} per °C, where E is Young's modulus and α is the coefficient of linear expansion.

<u>*Solution*</u>

Let subscript a refer to the aluminium rod and subscript s to the steel tube. The problem is similar to the one discussed in Section 1.17, except that the composite rod has its temperature lowered, in this case from 140°C to 20°C. From equation (1.23), the common force between the two components is

$$P = \frac{(\alpha_a - \alpha_s)\,\theta}{\frac{1}{(EA)_a} + \frac{1}{(EA)_s}}$$

The stress in the rod is therefore

$$\frac{P}{A_a} = \frac{(\alpha_a - \alpha_s)\,\theta}{\frac{1}{E_a} + \frac{A_a}{E_s A_s}} = \frac{(\alpha_a - \alpha_s)\,E_a\,\theta}{1 + \frac{E_a A_a}{E_s A_s}}$$

Now

$$(EA)_a = (70 \times 10^9)\left[\frac{\pi}{4}(0.022)^2\right] = 26.6 \text{ MN}$$

Again

$$(EA)_s = (200 \times 10^9)\,[\pi\,(0.028)\,(0.003)] = 52.8 \text{ MN}$$

Then

$$\frac{P}{A_a} = \frac{[(2.3 - 1.2)\,10^{-5}]\,(70 \times 10^9)\,(120)}{1 + \left(\dfrac{26.6}{52.8}\right)} = 61.4 \text{ MN/m}^2$$

1.19 Circular ring under radial pressure

When a thin circular ring is loaded radially, a circumferential force is set up in the ring; this force extends the circumference of the ring, which in turn leads to an increase in the radius of the ring. Consider a thin ring of mean radius r, Figure 1.22(i), acted upon by an internal radial force of intensity p per unit length of the boundary. If the ring is cut across a diameter, Figure 1.22(ii), circumferential forces P are required at the cut sections of the ring to maintain equilibrium of the half-ring. For equilibrium

$$2P = 2pr$$

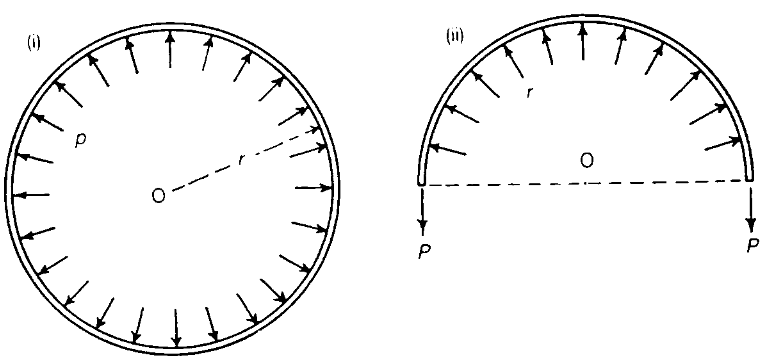
so that

$$P = pr \tag{1.24}$$

A section may be taken across any diameter, leading to the same result; we conclude, therefore, that P is the circumferential tension in all parts of the ring.

If A is the cross-sectional area of the ring at any point of the circumference, then the tensile circumferential stress in the ring is

$$\sigma = \frac{P}{A} = \frac{pr}{A} \tag{1.25}$$



(i) (ii)

Figure 1.22 Thin circular ring under uniform radial loading, leading to a uniform circumferential tension.

If the cross-section is a rectangle of breadth b, (normal to the plane of Figure 1.22), and thickness t, (in the plane of Figure 1.22), then

$$\sigma = \frac{pr}{bt} \tag{1.26}$$

Circumferential stresses of a similar type are set up in a circular ring rotating about an axis through its centre. We suppose the ring is a uniform circular one, having a cross-sectional area A at any point, and that it is rotating about its central axis at uniform angular velocity ω. If ρ is the density of the material of the ring, then the centrifugal force on a unit length of the circumference is

$$\rho A \omega^2 r$$

In equation (1.25) we put this equal to p; thus, the circumferential tensile stress in the ring is

$$\sigma = \frac{pr}{A} = \rho \omega^2 r^2 \tag{1.27}$$

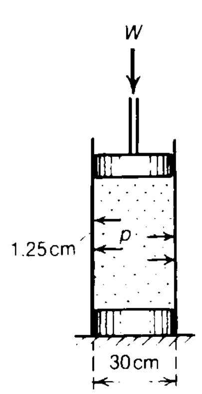
which we see is independent of the actual cross-sectional area. Now, ωr is the circumferential velocity, V (say), of the ring, so

$$\sigma = \rho V^2 \tag{1.28}$$

For steel we have $\rho = 7840\ \text{kg/m}^3$; to produce a tensile stress of $10\ \text{MN/m}^2$, the circumferential velocity must be

$$V = \sqrt{\frac{\sigma}{\rho}} = \sqrt{\frac{(10 \times 10^6)}{7840}} = 35.7\ \text{m/s}$$

Problem 1.15 A circular cylinder, containing oil, has an internal bore of 30 cm diameter. The cylinder is 1.25 cm thick. If the tensile stress in the cylinder must not exceed $75\ \text{MN/m}^2$, estimate the maximum load W which may be supported on a piston sliding in the cylinder.



Solution

A load W on the piston generates an internal pressure p given by

$$W = \pi r^2 p$$

where r is the radius of the cylinder. In this case

$$p = \frac{W}{\pi r^2} = \frac{W}{\pi (0.150)^2}$$

A unit length of the cylinder is equivalent to a circular ring subjected to an internal load of p per unit length of circumference. The circumferential load set up by p in this ring is, from equation (1.24),

$$P = pr = p\,(0.150)$$

The circumferential stress is, therefore,

$$\sigma = \frac{P}{1 \times t} = \frac{P}{0.0125} = 80P$$

where t is the thickness of the wall of the cylinder. If σ is limited to 75 MN/m^2, then

$$80P = 75 \times 10^6$$

But

$$80P = 80\,[p\,(0.150)] = 12p = \frac{12W}{\pi (0.150)^2}$$

Then

$$\frac{12W}{\pi (0.150)^2} = 75 \times 10^6$$

giving

$$W = 441 \text{ kN}$$

Problem 1.16 An aluminium-alloy cylinder of internal diameter 10.000 cm and wall thickness 0.50 cm is shrunk onto a steel cylinder of external diameter 10.004 cm and wall thickness 0.50 cm. If the values of Young's modulus for the alloy and the steel are 70 GN/m^2 and 200 GN/m^2, respectively, estimate the circumferential stresses in the cylinders and the radial pressure between the cylinders.

Solution

We take unit lengths of the cylinders as behaving like thin circular rings. After the shrinking operation, we suppose p is the radial between the cylinders. The mean radius of the steel tube is

$$\tfrac{1}{2}\ [10.004 - 0.50] \quad = \quad 4.75 \text{ cm}$$

The compressive circumferential stress in the steel tube is then

$$\sigma_s \quad = \quad \frac{pr}{t} \quad = \quad \frac{p\,(0.0475)}{0.0050} \quad = \quad 9.5p$$

The circumferential strain in the steel tube is then

$$\varepsilon_s \quad = \quad \frac{\sigma_s}{E_s} \quad = \quad \frac{9.50p}{200 \times 10^9}$$

The mean radius of the alloy tube is

$$\tfrac{1}{2}\ [10.000 + 0.50] \quad = \quad 5.25 \text{ cm}$$

The tensile circumferential stress in the alloy tube is then

$$\sigma_a \quad = \quad \frac{pr}{t} \quad = \quad \frac{p(0.0525)}{(0.0050)} \quad = \quad 10.5p$$

The circumferential strain in the alloy tube is then

$$\varepsilon_a \quad = \quad \frac{\sigma_a}{E_a} \quad = \quad \frac{10.5p}{70 \times 10^9}$$

The circumferential expansion of the alloy tube is

$$2\pi\ r\ \varepsilon_a$$

so the mean radius increases effectively by an amount

$$\delta_a \quad = \quad r\ \varepsilon_a \quad = \quad 0.0525\ \varepsilon_a$$

Similarly, the mean radius of the steel tube contracts by an amount

$$\delta_s \quad = \quad r\ \varepsilon_s \quad = \quad 0.0475\ \varepsilon_s$$

For the shrinking operation to be carried out we must have that the initial lack of fit, δ, is given by

$$\delta = \delta_a + \delta_s$$

Then

$$\delta_a + \delta_s = 0.002 \times 10^{-2}$$

On substituting for δ_a and δ_s, we have

$$0.0525 \left[\frac{10.5p}{70 \times 10^9}\right] + 0.0475 \left[\frac{9.50p}{200 \times 10^9}\right] = 0.002 \times 10^{-2}$$

This gives

$$p = 1.97 \text{ MN/m}^2$$

The compressive circumferential stress in the steel cylinder is then

$$\sigma_s = 9.50p = 18.7 \text{ MN/m}^2$$

The tensile circumferential stress in the alloy cylinder is

$$\sigma_a = 10.5p = 20.7 \text{ MN/m}^2$$

1.20 Creep of materials under sustained stresses

At ordinary laboratory temperatures most metals will sustain stresses below the limit of proportionality for long periods without showing additional measurable strains. At these temperatures metals deform continuously when stressed above the elastic range. This process of continuous inelastic strain is called *creep*. At high temperatures metals lose some of their elastic properties, and creep under constant stress takes place more rapidly.

When a tensile specimen of a metal is tested at a high temperature under a constant load, the strain assumes instantaneously some value ε_0, Figure 1.23. If the initial strain is in the inelastic range of the material then creep takes place under constant stress. At first the creep rate is fairly rapid, but diminishes until a point *a* is reached on the strain–time curve, Figure 1.23; the point *a* is a point of inflection in this curve, and continued application of the load increases the creep rate until fracture of the specimen occurs at *b*.

At ordinary temperatures concrete shows creep properties; these may be important in pre-stressed members, where some of the initial stresses in the concrete may be lost after a long period due to creep. Composites are also vulnerable to creep and this must be considered when using them for construction.

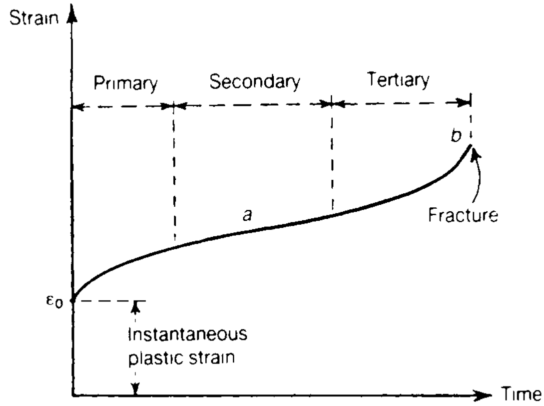


Figure 1.23 Creep curve for a material in the inelastic range; ε_0 is the instantaneous plastic strain.

1.21 Fatigue under repeated stresses

When a material is subjected to repeated cyclic loading, it can fail at a stress which may be much less than the material's yield stress. The problem that occurs here, is that the structure might have minute cracks in it or other stress raisers. Under repeated cyclic loading the large stresses that occur at these stress concentrations cause the cracks to grow, until fracture eventually occurs. Materials likely to suffer fatigue include aluminium alloys and composites; see Figure 1.24.

Failure of a material after a large number of cycles of tensile stress occurs with little, or no, permanent set; fractures show the characteristics of brittle materials. Fatigue is primarily a problem of repeated tensile stresses; this is due probably to the fact that microscopic cracks in a material can propagate more easily when the material is stressed in tension. In the case of steels it is found that there is a critical stress—called the *endurance limit*—below which fluctuating stresses cannot cause a fatigue failure; titanium alloys show a similar phenomenon. No such endurance limit has been found for other non-ferrous metals and other materials.

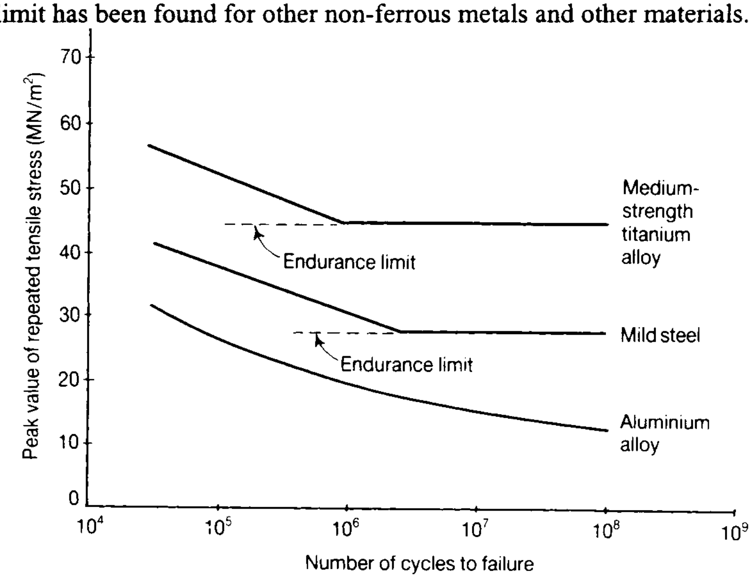


Figure 1.24 Comparison of the fatigue strengths of metals under repeated tensile stresses.

Further problems *(answers on page 691)*

1.17 The piston rod of a double-acting hydraulic cylinder is 20 cm diameter and 4 m long. The piston has a diameter of 40 cm, and is subjected to 10 MN/m^2 water pressure on one side and 3 MN/m^2 on the other. On the return stroke these pressures are interchanged. Estimate the maximum stress occurring in the piston-rod, and the change of length of the rod between two strokes, allowing for the area of piston-rod on one side of the piston. Take $E = 200$ GN/m^2. *(RNC)*

1.18 A uniform steel rope 250 m long hangs down a shaft. Find the elongation of the first 125 m at the top if the density of steel is 7840 kg/m^3 and Young's modulus is 200 GN/m^2. *(Cambridge)*

1.19 A steel wire, 150 m long, weighs 20 N per metre length. It is placed on a horizontal floor and pulled slowly along by a horizontal force applied to one end. If this force measures 600 N, estimate the increase in length of the wire due to its being towed, assuming a uniform coefficient of friction. Take the density of steel as 7840 kg/m^3 and Young's modulus as 200 GN/m^2. *(RNEC)*

1.20 The hoisting rope for a mine shaft is to lift a cage of weight W. The rope is of variable section so that the stress on every section is equal to σ when the rope is fully extended. If ρ is the density of the material of the rope, show that the cross-sectional area A at a height z above the cage is

$$A = \left(\frac{W}{\sigma}\right) e^{\rho g z/\sigma}$$

1.21 To enable two walls, 10 m apart, to give mutual support they are stayed together by a 2.5 cm diameter steel tension rod with screwed ends, plates and nuts. The rod is heated to 100°C when the nuts are screwed up. If the walls yield, relatively, by 0.5 cm when the rod cools to 15°C, find the pull of rod at that temperature. The coefficient of linear expansion of steel is $\alpha = 1.2 \times 10^{-5}$ per °C, and Young's modulus $E = 200$ GN/m^2. *(RNEC)*

1.22 A steel tube 3 cm diameter, 0.25 cm thick and 4 m long, is covered and lined throughout with copper tubes 0.2 cm thick. The three tubes are firmly joined together at their ends. The compound tube is then raised in temperature by 100°C. Find the stresses in the steel and copper, and the increase in length of the tube, will prevent its expansion? Assume $E = 200$ GN/m^2 for steel and $E = 110$ GN/m^2 for copper; the coefficients of linear expansion of steel and copper are 1.2×10^{-5} per °C and 1.9×10^{-5} per °C, respectively.