25 Structural vibrations

25.1 Introduction

In this chapter, we will commence with discussing the free vibrations of a beam, which will be analysed by traditional methods. This fundamental approach will then be extended to forced vibrations and to damped oscillations, all on beams and by traditional methods.

The main snag with using traditional methods for vibration analysis, however, is that it is extremely difficult to analyse complex structures by this approach. For this reason, the finite element method discussed in the previous chapters will be extended to free vibration analysis, and applications will then be made to a number of simple structures.

Vibrations of structures usually occur due to pulsating or oscillating forces, such as those due to gusts of wind or from the motion of machinery, vehicles etc. If the pulsating load is oscillating at the same natural frequency of the structure, the structure can vibrate dangerously (i.e. resonate). If these vibrations continue for any length of time, the structure can suffer permanent damage.

25.2 Free vibrations of a mass on a beam

We can simplify the treatment of the free vibrations of a beam by considering its mass to be concentrated at the mid-length. Consider, for example, a uniform simply-supported beam of length L and flexural stiffness EI, Figure 25.1.

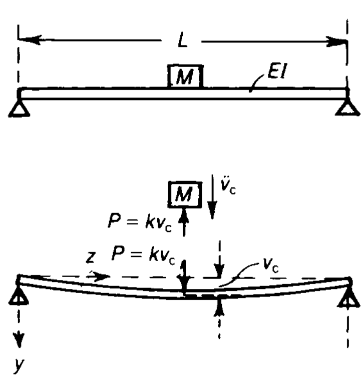


Figure 25.1 Vibrations of a concentrated mass on a beam.

Suppose the beam itself is mass-less, and that a concentrated mass M is held at the mid-span. If we ignore for the moment the effect of the gravitational field, the beam is undeflected when the mass is at rest. Now consider the motion of the mass when the beam is deflected laterally to some position and then released. Suppose, v_c is the lateral deflection of the beam at the mid-span at a time t; as the beam is mass-less the force P on the beam at the mid-span is

$$P = \frac{48EIv_c}{L^3}$$

If $k = 48\,EI/L^3$, then

$$P = kv_c$$

The mass-less beam behaves then as a simple elastic spring of stiffness k. In the deflected position there is an equal and opposite reaction P on the mass. The equation of vertical motion of the mass is

$$M\frac{d^2v_c}{dt^2} = -P = -kv_c$$

Thus

$$\frac{d^2v_c}{dt^2} + \frac{kv_c}{M} = 0$$

The general solution of this differential equation is

$$v_c = A\cos\sqrt{\frac{k}{M}}\,t + B\sin\sqrt{\frac{k}{M}}\,t$$

where A and B are arbitrary constants; this may also be written in the form

$$v_c = C\sin\left(\sqrt{\frac{k}{M}}\,t + \varepsilon\right)$$

where C and ε are also arbitrary constants. Obviously C is the amplitude of a simple-harmonic motion of the beam (Figure 25.2); v_c first assumes its peak value when

$$\sqrt{\frac{k}{M}}\,t_1 + \varepsilon = \frac{\pi}{2}$$

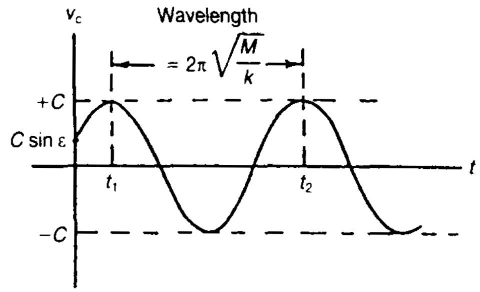


Figure 25.2 Variations of displacement of beam with time.

and again attains this value when

$$\sqrt{\frac{k}{M}}\, t_2 + \varepsilon = \frac{5\pi}{2}$$

This period T of one complete oscillation is then

$$T = t_1 - t_2 = 2\pi\sqrt{\frac{M}{k}} \tag{25.1}$$

The number of complete oscillations occurring in unit time is the frequency of vibrations; this is denoted by n, and is given by

$$n = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{M}} \tag{25.2}$$

The behaviour of the system is therefore directly analogous to that of a simple mass–spring system. On substituting for the value of k we have

$$n = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{48EI}{ML^3}} \tag{25.3}$$

Problem 25.1 A steel I-beam, simply supported at each end of a span of 10 m, has a second moment of area of 10^{-4} m^4. It carries a concentrated mass of 500 kg at the mid-span. Estimate the natural frequency of lateral vibrations.

Solution

In this case

$$EI = (200 \times 10^9)(10^{-4}) = 20 \times 10^6 \text{ Nm}^2$$

Then

$$k = \frac{48EI}{L^3} = \frac{48(20 \times 10^6)}{(10)^3} = 960 \times 10^3 \text{ N/m}$$

The natural frequency is

$$n = \frac{1}{2\pi}\sqrt{\frac{k}{M}} = \frac{1}{2\pi}\sqrt{\frac{960 \times 10^3}{500}} = 6.97 \text{ cycles/sec} = 6.97\ Hz$$

25.3 Free vibrations of a beam with distributed mass

Consider a uniform beam of length L, flexural stiffness EI, and mass m per unit length (Figure 25.3); suppose the beam is simply-supported at each end, and is vibrating freely in the yz-plane, the displacement at any point parallel to the y-axis being v. We assume first that the beam vibrates in a sinusoidal form

$$v = a \sin \frac{\pi z}{L} \sin 2\pi nt \tag{25.4}$$

where a is the lateral displacement, or amplitude, at the mid-length, and n is the frequency of oscillation. The kinetic energy of an elemental length δz of the beam is

$$\frac{1}{2} m\ \delta z \left(\frac{dv}{dt}\right)^2 = \frac{1}{2} m\ \delta z \left[2\pi na \sin\frac{\pi z}{L} \cos 2\pi nt\right]^2$$

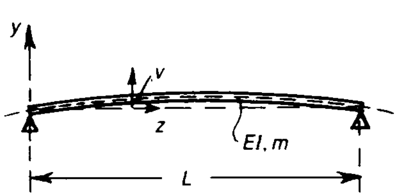


Figure 25.3 Vibrations of a beam having an intrinsic mass.

The bending strain energy in an elemental length is

$$\frac{1}{2}\,EI\left(\frac{d^2v}{dz^2}\right)^2\delta z \;=\; \frac{1}{2}\,EI\left|\frac{a\pi^2}{L^2}\sin\frac{\pi z}{L}\sin 2\pi nt\right|^2\delta z$$

The total kinetic energy at any time t is then

$$\frac{1}{2}m\left[4\pi^2 n^2 a^2\cos^2 2\pi nt\int_0^L\sin^2\frac{\pi z}{L}\,dz\right] \tag{25.5}$$

The total strain energy at time t is

$$\frac{1}{2}EI\,\frac{a^2\pi^4}{L^4}\sin^2 2\pi nt\int_0^L\sin^2\frac{\pi z}{L}\,dz \tag{25.6}$$

For the free vibrations we must have the total energy, i.e. the sum of the kinetic and strain energies, is constant and independent of time. This is true if

$$\frac{1}{2}m\left(4\pi^2 n^2 a^2\right)\cos^2 2\pi nt + \frac{1}{2}EI\left(\frac{\pi^4 a^2}{L^4}\right)\sin^2 2\pi nt \;=\; \text{constant}$$

For this condition we must have

$$\frac{1}{2}m\left(4\pi^2 n^2 a^2\right) \;=\; \frac{1}{2}EI\left(\frac{\pi^4 a^2}{L^4}\right)$$

This gives

$$n^2 \;=\; \frac{\pi^2 EI}{4mL^4} \tag{25.7}$$

Now $mL = M$, say is the total mass of the beam, so that

$$n \;=\; \frac{\pi}{2}\sqrt{\frac{EI}{ML^3}} \tag{25.8}$$

This is the frequency of oscillation of a simply-supported beam in a single sinusoidal half-wave. If we consider the possibility of oscillations in the form

$$v \;=\; a\sin\frac{2\pi z}{L}\sin 2\pi n_2 t$$

then proceeding by the same analysis we find that

$$n_2 = 4n_1 = 2\pi \sqrt{\frac{EI}{ML^3}} \tag{25.9}$$

This is the frequency of oscillations of two sinusoidal half-waves along the length of the beam, Figure 25.4, and corresponds to the second mode of vibration. Other higher modes are found similarly.

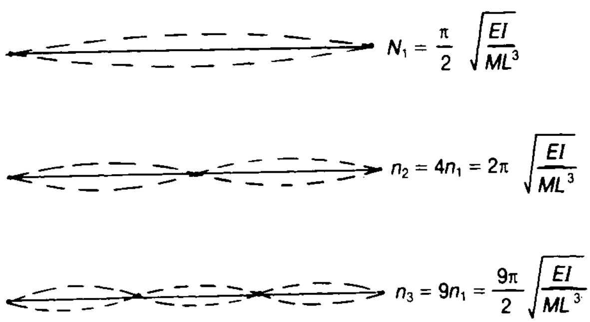


Figure 25.4 Modes of vibration of a simply-supported beam.

As in the case of the beam with a concentrated mass at the mid-length, we have ignored gravitation effects; when the weight of the beam causes initial deflections of the beam, oscillations take place about this deflected condition; otherwise the effects of gravity may be ignored.

The effect of distributing the mass uniformly along a beam, compared with the whole mass being concentrated at the mid-length, is to increase the frequency of oscillations from

$$\frac{1}{2\pi}\sqrt{\frac{48EI}{ML^3}} \quad \text{to} \quad \frac{\pi}{2}\sqrt{\frac{EI}{ML^3}}$$

If

$$n_1 = \frac{1}{2\pi}\sqrt{\frac{48EI}{ML^3}}, \quad \text{and} \quad n_2 = \frac{\pi}{2}\sqrt{\frac{EI}{ML^3}}$$

then

$$\frac{n_2}{n_1} = \left(\frac{\pi}{2}\right) \frac{2\pi}{\sqrt{48}} = \frac{\pi^2}{4\sqrt{3}} = 1.42 \tag{25.10}$$

Problem 25.2 If the steel beam of the Problem 25.1 has a mass of 15 kg per metre run, estimate the lowest natural frequency of vibrations of the beam itself.

<u>*Solution*</u>

The lowest natural frequency of vibrations is

$$n_1 = \frac{\pi}{2} \sqrt{\frac{EI}{ML^3}}$$

Now

$$EI = 20 \times 10^6 \text{ Nm}^2$$

and

$$ML^3 = (15)(10)(10)^3 = 150 \times 10^3 \text{ kg.m}^3$$

Then

$$\frac{EI}{ML^3} = \frac{20 \times 10^6}{150 \times 10^3} = 133 \ s^{-2}$$

Thus

$$n_1 = \frac{\pi}{2} \sqrt{133} = 18.1 \text{ cycles per sec} = 18.1 \ Hz$$

25.4 Forced vibrations of a beam carrying a single mass

Consider a light beam, simply-supported at each end and carrying a mass M at mid-span, Figure 25.5. Suppose the mass is acted upon by an alternating lateral force

$$P \sin 2\pi N t \tag{25.11}$$

which is applied with a frequency N. If v_c is the central deflection of the beam, then the equation of motion of the mass is

$$M\frac{d^2v_c}{dt^2} + kv_c = P \sin 2\pi Nt$$

where $k = 48\,EI/L^3$. Then

$$\frac{d^2v_c}{dt^2} + \frac{k}{M}v_c = \frac{P}{M}\sin 2\pi Nt$$

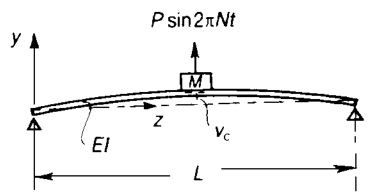


Figure 25.5 Alternating force applied to a beam.

The general solution is

$$v_c = A\cos\sqrt{\frac{k}{M}}t + B\sin\sqrt{\frac{k}{M}}t + \frac{\frac{P}{k}\sin 2\pi Nt}{1 - 4\pi^2N^2\frac{M}{k}} \tag{25.12}$$

in which A and B are arbitrary constants. Suppose initially, i.e. at time $t = 0$, both v_c and dv_c/dt are zero. Then $A = 0$ and

$$B = -\frac{2\pi N.\frac{P}{k}}{1 - 4\pi^2N^2\frac{M}{k}}\frac{1}{\sqrt{\frac{k}{M}}}$$

Then

$$v_c = \frac{P/k}{1 - 4\pi^2 N^2 \dfrac{M}{k}} \left[\sin 2\pi N t - 2\pi N \sqrt{\frac{M}{k}} \sin \sqrt{\frac{k}{M}} t \right] \tag{25.13}$$

Now, the natural frequency of free vibrations of the system is

$$n = \frac{1}{2\pi} \sqrt{\frac{k}{M}}$$

Then

$$\sqrt{k/M} = 2\pi n$$

and

$$v_c = \frac{P/k}{1 - N^2/n^2} \left[\sin 2\pi N t - \frac{N}{n} \sin 2\pi n t \right] \tag{25.14}$$

Now, the maximum value that the term

$$\left(\sin 2\pi N t - \frac{N}{n} \sin 2\pi n t \right)$$

may assume is

$$\left[1 + \frac{N}{n} \right]$$

and occurs when $\sin 2\pi N t = -\sin 2\pi n t = 1$. Then

$$v_{c\max} = \frac{P/k \left(1 + \dfrac{N}{n} \right)}{1 - \dfrac{N^2}{n^2}} = \frac{P/k}{1 - \dfrac{N}{n}} \tag{25.15}$$

Thus, if $N < n$, $v_{c\,max}$ is positive and in phase with the alternating load $P \sin 2\pi N t$. As N approaches n, the values of $v_{c\,max}$ become very large. When $N > n$, $v_{c\,max}$ is negative and out of phase with $P \sin 2\pi N t$. When $N = n$, the beam is in a condition of resonance.

25.5 Damped free oscillations of a beam

The free oscillations of practical systems are inhibited by damping forces. One of the commonest forms of damping is known as velocity, or *viscous*, damping; the damping force on a particle or mass is proportional to its velocity.

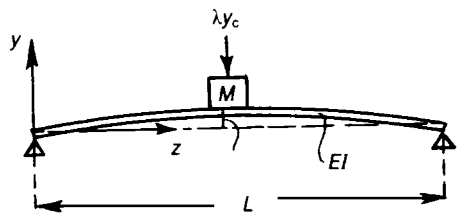


Figure 25.6 Effect of damping on free vibrations.

Suppose in the beam problem discussed in Section 25.2 we have as the damping force $\mu(dv_c/dt)$. Then the equation of motion of the mass is

$$M\frac{d^2v_c}{dt} = -kv_c - \mu\frac{dv_c}{dt}$$

Thus

$$M\frac{d^2v_c}{dt^2} + \mu\frac{dv_c}{dt} + kv_c = 0$$

Hence

$$\frac{d^2v_c}{dt^2} + \frac{\mu}{M}\frac{dv_c}{dt} + \frac{k}{M}v_c = 0$$

The general solution of this equation is

$$v_c = Ae^{\{-\mu/2M + \sqrt{(\mu/2M)^2 - k/M}\}t} + Be^{\{-\mu/2M - \sqrt{(\mu/2M)^2 - k/M}\}t} \tag{25.16}$$

Now (k/M) is usually very much greater than $(\mu/2M)^2$, and so we may write

$$
\begin{aligned}
v_c &= Ae^{\left(-\mu/2M + i\sqrt{k/M}\right)t} + Be^{\left(-\mu/2M - i\sqrt{(k/M)}\right)t} \\
&= e^{-(\mu/2M)t}\left[AE^{i\sqrt{(k/M)}\,t} + Be^{-i\sqrt{(k/M)}\,t}\right] \\
&= e^{-(\mu/2M)t}\left[C\cos\left\{\sqrt{\frac{k}{M}}\,t + \varepsilon\right\}\right]
\end{aligned}
\qquad (25.17)
$$

Thus, when damping is present, the free vibrations given by

$$C\cos\left(\sqrt{\frac{k}{M}}\,t + \varepsilon\right)$$

are damped out exponentially, Figure 25.7. The peak values on the curve of v_c correspond to points of zero velocity.

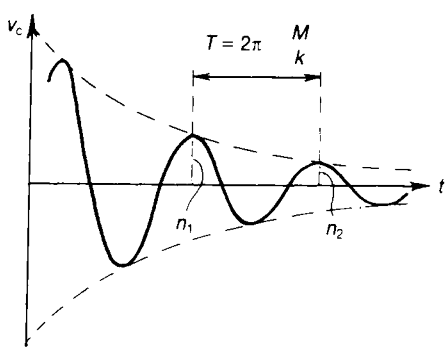


Figure 25.7 Form of damped oscillation of a beam.

These are given by

$$\frac{dv_c}{dt} = 0$$

or

$$\sqrt{\frac{k}{M}}\sin\left(\sqrt{\frac{k}{M}}\,t+\varepsilon\right) - \frac{\mu}{2M}\cos\left(\sqrt{\frac{k}{M}}\,t+\varepsilon\right) = 0$$

Obviously the higher peak values are separated in time by an amount

$$T = 2\pi\sqrt{\frac{M}{k}}$$

We note that successive peak values are in the ratio

$$\frac{v_{c1}}{v_{c2}} = \frac{e^{(-\mu/2M)t}\left[C\cos\left(\sqrt{(k/M)}\,t+\varepsilon\right)\right]}{e^{-(\mu/2M)\left(t+2\pi\sqrt{M/k}\right)}\left[C\cos\left(\sqrt{(k/M)}\,t+\varepsilon\right)\right]} = e^{(\mu/M)\pi\sqrt{M/k}} \tag{25.18}$$

Then

$$\log_e \frac{v_{c1}}{v_{c2}} = \frac{\pi\mu}{M}\sqrt{\frac{M}{k}} \tag{25.19}$$

Now

$$n = \frac{1}{2\pi}\sqrt{\frac{k}{M}}$$

Thus

$$\log_e \frac{v_{c1}}{v_{c2}} = \frac{\mu}{2Mn} \tag{25.20}$$

Hence

$$\mu = 2Mn\,\log_e \frac{v_{c1}}{v_{c2}} \tag{25.21}$$

25.6 Damped forced oscillations of a beam

We imagine that the mass on the beam discussed in Section 25.5 is excited by an alternating force $P \sin 2\pi Nt$. The equation of motion becomes

$$M \frac{d^2 v_c}{dt^2} + \mu \frac{dv_c}{dt} + kv_c = P \sin 2\pi Nt$$

The complementary function is the damped free oscillation; as this decreases rapidly in amplitude we may assume it to be negligible after a very long period. Then the particular integral is

$$v_c = \frac{P \sin 2\pi Nt}{MD^2 + \mu D + k}$$

This gives

$$v_c = \frac{P\left[\left(k - 4\pi^2 N^2 M\right)\sin 2\pi Nt - 2\pi N\mu \cos 2\pi Nt\right]}{\left(k - 4\pi^2 N^2 M\right)^2 + 4\pi^2 N^2 \mu^2} \tag{25.22}$$

If we write

$$n = \frac{1}{2\pi}\sqrt{\frac{k}{M}}$$

then

$$v_c = P\left[\frac{k\left(1 - \frac{N^2}{n^2}\right)\sin 2\pi Nt - 2\pi N\mu \cos 2\pi Nt}{k^2\left(1 - \frac{N^2}{n^2}\right)^2 + 4\pi N^2 \mu^2}\right] \tag{25.23}$$

The amplitude of this forced oscillation is

$$v_{c\max} = \frac{P}{\sqrt{k^2\left(1 - \frac{N^2}{n^2}\right)^2 + 4\pi^2 N^2 \mu^2}} \tag{25.24}$$

25.7 Vibrations of a beam with end thrust

In general, when a beam carries end thrust the period of free undamped vibrations is greater than when the beam carries no end thrust. Consider the uniform beam shown in Figure 25.8; suppose the beam is vibrating in the fundamental mode so that the lateral displacement at any section is given by

$$v = a \sin \frac{\pi z}{L} \sin 2\pi n t \tag{25.25}$$

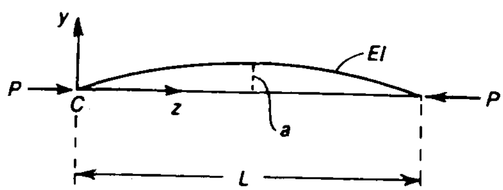


Figure 25.8 Vibrations of a beam carrying a constant end thrust.

If these displacements are small, the shortening of the beam from the straight configuration is approximately

$$\int_0^L \frac{1}{2}\left(\frac{dv}{dz}\right)^2 dz = \frac{a^2\pi^2}{4L} \sin^2 2\pi n t \tag{25.26}$$

If m is the mass per unit length of the beam, the total kinetic energy at any instant is

$$\int_0^L \frac{1}{2} m \left(2\pi a \sin\frac{\pi z}{L} \cos 2\pi n t\right)^2 dz = m\pi^2 a^2 n^2 \; L \cos^2 2\pi n t \tag{25.27}$$

The total potential energy of the system is the strain energy stored in the strut together with the potential energy of the external loads; the total potential energy is then

$$\left[\frac{1}{2} EIL \left(\frac{a\pi^2}{L^2}\right)^2 - \frac{1}{4} P \left(\frac{a^2\pi^2}{L}\right)\right] \sin^2 2\pi n t \tag{25.28}$$

If the total energy of the system is the same at all instants

$$m\pi^2 a^2 n^2 L = \frac{1}{4} EIL \left(\frac{a\pi^2}{L^2}\right)^2 - \frac{1}{4}P\left(\frac{a^2\pi^2}{L}\right)$$

This gives

$$n^2 = \frac{\pi^2 EI}{4mL^4}\left[1 - \frac{P}{P_e}\right] \tag{25.29}$$

where

$$P_e = \frac{\pi^2 EI}{L^2}$$

and is the Euler load of the column. If we write

$$n_1^2 = \frac{\pi^2 EI}{4mL^4} \tag{25.30}$$

then

$$n = n_1\sqrt{1 - \frac{P}{P_e}}$$

Clearly, as P approaches P_e, the natural frequency of the column diminishes and approaches zero.

25.8 Derivation of expression for the mass matrix

Consider an infinitesimally small element of volume d(vol) and density ρ, oscillating at a certain time t, with a velocity $\dot{u}$.

The kinetic energy ot this element (KE) is given by:

$$\text{KE} = \frac{1}{2}\rho \times d(\text{vol}) \times \dot{u}^2$$

and for the whole body,

$$\text{KE} = \frac{1}{2}\int \rho\ \dot{u}^2\ d(\text{vol}) \tag{25.31}$$

or in matrix form:

$$KE = \frac{1}{2} \int_{vol} \{\dot{u}\}^T \rho \ \{\dot{u}\} \ d(\text{vol}) \tag{25.32}$$

NB The premultiplier of equation (25.32) must be a row and the postmultiplier of this equation must be a column, because KE is a scalar.

Assuming that the structure oscillates with simple harmonic motion, as described in Section 25.2,

$$\{u\} = \{C\}e^{jwt} \tag{25.33}$$

where

$\{C\}$ = a vector of amplitudes

ω = resonant frequency

j = $\sqrt{-1}$

Differentiating $\{u\}$ with respect to t,

$$\{\dot{u}\} = j\omega \ \{C\} \ e^{j\omega t} \tag{25.34}$$

$$= j\omega \ \{u\} \tag{25.35}$$

Substituting equation (25.35) into equation (25.32):

$$KE = -\frac{1}{2} \omega^2 \int_{vol} \{u\}^T \rho \{u\} \ d(\text{vol})$$

but,

$$\{u\} = [\mathbf{N}] \ \{u_i\}$$

$$\therefore KE = -\frac{1}{2} \omega^2 \ \{u_i\}^T \int_{vol} [\mathbf{N}]^{\mathbf{T}} \ \rho \ [\mathbf{N}] \ d(\text{vol}) \ \{u_i\} \tag{25.36}$$

but,

$$KE = \frac{M\dot{u}^2}{2}$$

or in matrix form:

$$KE = \frac{1}{2}\{\dot{u}_i\}^T \ [\mathbf{m}] \ \{\dot{u}_i\}$$

but,

$$\{\dot{u}_i\} = j\omega \ \{u_i\}$$

$$\therefore KE = -\frac{1}{2}\omega^2 \ \{u_i\}^T \ [\mathbf{m}] \ \{u_i\} \tag{25.37}$$

Comparing equation (25.37) with equation (25.36):

$$[\mathbf{m}] = \int_{vol} [N]^T \ \rho \ [\mathbf{N}] \ d(\text{vol}) \tag{25.38}$$

= elemental mass matrix

25.9 Mass matrix for a rod element

The one-dimensional rod element, which has two degree of freedom, is shown in Figure 23.1. As the rod element has two degrees of freedom, it will be convenient to assume a polynomial with two arbitrary constants, as shown in equation (25.39):

$$u = \alpha_1 + \alpha_2 x \tag{25.39}$$

The boundary conditions or boundary values are:

at $x = 0, \ u = u_1$

and

$$\text{at } x = l, \ u = u_2 \tag{25.40}$$

Substituting equations (25.40) into equation (25.39),

$$\alpha_1 = u_1 \tag{25.41}$$

and

$$u_2 = u_1 + \alpha_2 l$$

or

$$\alpha_2 = (u_2 - u_1)/l \tag{25.42}$$

Substituting equations (25.41) and (25.42) into equation (25.39),

$$u = u_1 + (u_2 - u_1)x/l$$

or

$$u = u_1\,(1 - \xi) + u_2\xi \tag{25.43}$$

where,

$$\xi = x/l$$

Rewriting equation (25.43) in matrix form,

$$u = [(1 - \xi) \quad \xi] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= [\mathbf{N}]\,\{u_i\}$$

where

$$[\mathbf{N}] = [(1 - \xi) \quad \xi] \tag{25.44}$$

Substituting equation (25.44) into equation (24.38),

$$[\mathbf{m}] = \int [\mathbf{N}]^{\mathrm{T}}\,\rho\,[\mathbf{N}]\,d(\mathrm{vol})$$

$$= \rho \int_0^1 \begin{bmatrix} (1 - \xi) \\ \xi \end{bmatrix} [(1 - \xi) \quad \xi]\,Al\,d\xi$$

$$= \rho A l \int_0^1 \begin{bmatrix} (1 - 2\xi + \xi^2) & \xi - \xi^2 \\ \xi - \xi^2 & \xi^2 \end{bmatrix} d\xi$$

$$[\mathbf{m}] = \frac{\rho A l}{6} \begin{array}{c} \begin{array}{cc} u_1 & u_2 \end{array} \\ \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \end{array} \begin{array}{c} u_1 \\ u_2 \end{array} \tag{25.45}$$

In two dimensions, it can readily be shown that the elemental mass matrix for a rod is

$$[\mathbf{m}] = \frac{\rho A l}{6} \begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_2 & v_2 \end{array} \\ \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \end{array} \begin{array}{c} u_1 \\ v_1 \\ u_2 \\ v_2 \end{array} \tag{25.46}$$

The expression for the elemental mass matrix in global co-ordinates is given by an expression similar to that of equation (25.35), as shown by equation (25.47):

$$[\mathbf{m}^\circ] = [\mathbf{DC}]^{\mathrm{T}}\,[\mathbf{m}]\,[\mathbf{DC}] \tag{25.47}$$

where,

$$[\mathbf{DC}] = \begin{bmatrix} \zeta & 0_2 \\ 0_2 & \zeta \end{bmatrix}$$

$$[\zeta] = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \tag{25.48}$$

$c = \cos\alpha$

$s = \sin\alpha$

α is defined in Figure 23.4.

Substituting equations (23.25) and (25.46) into equation (25.47):

$$[\mathbf{m}^\circ] = \frac{\rho A l}{6}\begin{array}{c} \begin{array}{cccc} u_1^\circ & v_1^\circ & u_2^\circ & v_2^\circ \end{array} \\ \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \end{array} \begin{array}{c} \\ u_1^\circ \\ v_1^\circ \\ u_2^\circ \\ v_2^\circ \end{array} \tag{25.49}$$

= the elemental mass matrix for a rod in two dimensions, in global co-ordinates.

Similarly, in three dimensions, the elemental mass matrix for a rod in global co-ordinates, is given by:

$$[\mathbf{m}^\circ] = \frac{\rho A l}{6}\begin{array}{c} \begin{array}{cccccc} u_1^\circ & v_1^\circ & w_1^\circ & u_2^\circ & v_2^\circ & w_2^\circ \end{array} \\ \begin{bmatrix} 2 & & & & & \\ 0 & 2 & & & & \\ 0 & 0 & 2 & & & \\ 1 & 0 & 0 & 2 & & \\ 0 & 1 & 0 & 0 & 2 & \\ 0 & 0 & 1 & 0 & 0 & 2 \end{bmatrix} \end{array} \begin{array}{c} \\ u_1^\circ \\ v_1^\circ \\ w_1^\circ \\ u_2^\circ \\ v_2^\circ \\ w_2^\circ \end{array} \tag{25.50}$$

Equations (25.49) and (25.50) show the mass matrix for the self-mass of the structure, but if the effects of an additional concentrated mass are to be included at a particular node, this concentrated mass must be added to the mass matrix at the appropriate node, as follows:

$$M_a \begin{array}{c} \begin{array}{cc} u_i^\circ & v_i^\circ \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ u_i^\circ \\ v_1^\circ \end{array} \quad \text{(in two dimensions)} \tag{25.51}$$

and

$$M_a \begin{array}{c} \begin{array}{ccc} u_i^\circ & v_i^\circ & w_i^\circ \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} u_i^\circ \\ v_1^\circ \\ w_i^\circ \end{array} \quad \text{(in three dimensions)} \tag{25.52}$$

where

M_a = the value of the added mass

i = ith node

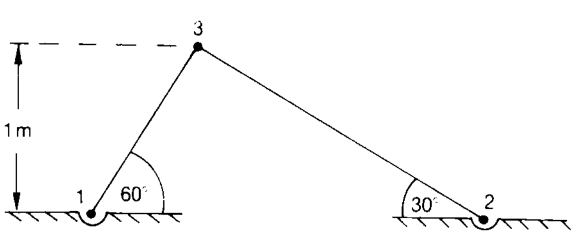
Problem 25.3 Determine the resonant frequencies and eigenmodes for the plane pin-jointed truss, below.

It may be assumed that the following apply:

$A = 1 \times 10^{-4}\ \text{m}^2$

$\rho = 7860\ \text{kg/m}^3$

$E = 2 \times 10^{11}\ \text{N/m}^2$



Solution

Element 1–3

$\alpha = 60°, \qquad c = 0.5, \qquad s = 0.866$

$$l_{1\text{-}3} = \frac{1\ \text{m}}{\sin 60} = 1.155\ \text{m} = \text{length of element 1–3}$$

Substituting the above values into equations (23.36) and (25.49), and removing the rows and columns corresponding to the zero displacements, namely $u_1{}^\circ$ and $v_1{}^\circ$, the stiffness and mass matrices for element 1–3 are given by:

$$\left[k_{1-3}{}^\circ\right] = \frac{1\times10^{-4}\times2\times10^{11}}{1.155}\begin{bmatrix}0.25 & 0.433\\ 0.433 & 0.75\end{bmatrix}$$

$$\begin{array}{cc} u_3{}^\circ & v_3{}^\circ \end{array}$$

$$\begin{array}{c} \begin{bmatrix}0.433\times10^7 & 0.75\times10^7\\ 0.75\times10^7 & 1.3\times10^7\end{bmatrix} \end{array}\begin{array}{c} u_3{}^\circ \\ v_3{}^\circ \end{array} \tag{25.53}$$

$$\left[m_{1-3}{}^\circ\right] = \frac{7860\times1\times10^{-4}\times1.155}{6}\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}$$

$$\begin{array}{cc} u_3{}^\circ & v_3{}^\circ \end{array}$$

$$= \begin{bmatrix}0.303 & 0\\ 0 & 0.303\end{bmatrix}\begin{array}{c} u_3{}^\circ \\ v_3{}^\circ \end{array} \tag{25.54}$$

Element 2–3

$\alpha = 150°, \qquad c = -0.866, \qquad s = 0.5$

$$l_{2-3} = \frac{1\ \text{m}}{\sin 30} = 2\ \text{m} = \text{length of element 2–3}$$

Substituting the above values into equations (23.36) and (25.49), and removing the rows and columns corresponding to the zero displacements, namely $u_2{}^\circ$ and $v_2{}^\circ$, the stiffness and mass matrices for element 2–3 are given by:

$$[\mathbf{k}_{2\text{-}3}{}^\circ] = \frac{1 \times 10^{-4} \times 2 \times 10^{11}}{2}\begin{bmatrix} 0.75 & -0.433 \\ -0.433 & 0.25 \end{bmatrix}$$

$$= \begin{array}{c} \begin{array}{cc} u_3{}^\circ & v_3{}^\circ \end{array} \\ \begin{bmatrix} 0.75 \times 10^7 & -0.433 \times 10^7 \\ -0.433 \times 10^7 & 0.25 \times 10^7 \end{bmatrix} \end{array} \begin{array}{c} \\ u_3{}^\circ \\ v_3{}^\circ \end{array} \tag{25.55}$$

$$[\mathbf{m}_{2\text{-}3}{}^\circ] = \frac{7860 \times 1 \times 10^{-4} \times 2}{6}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{array}{c} \begin{array}{cc} u_3{}^\circ & v_3{}^\circ \end{array} \\ \begin{bmatrix} 0.524 & 0 \\ 0 & 0.524 \end{bmatrix} \end{array} \begin{array}{c} \\ u_3{}^\circ \\ v_3{}^\circ \end{array} \tag{25.56}$$

The system stiffness matrix corresponding to the free displacements $u_3{}^\circ$ and $v_3{}^\circ$ is obtained by adding together equations (25.53) and (25.55), as shown by equation (25.57):

$$[\boldsymbol{K}_{11}] = \begin{array}{c} \begin{array}{cc} u_3{}^\circ & v_3{}^\circ \end{array} \\ \left[\begin{array}{c|c} 0.433 \times 10^7 & 0.75 \times 10^7 \\ +0.75 \times 10^7 & -0.433 \times 10^7 \\ \hline 0.75 \times 10^7 & 1.3 \times 10^7 \\ -0.433 \times 10^7 & +0.25 \times 10^7 \end{array}\right] \end{array} \begin{array}{c} \\ u_3{}^\circ \\ \\ v_3{}^\circ \end{array} \tag{25.57}$$

$$[\mathbf{K}_{11}] = \begin{array}{c} \begin{array}{cc} u_3{}^\circ & v_3{}^\circ \end{array} \\ \begin{bmatrix} 1.183 \times 10^7 & 0.317 \times 10^7 \\ 0.317 \times 10^7 & 1.55 \times 10^7 \end{bmatrix} \end{array} \begin{array}{c} \\ u_3{}^\circ \\ v_3{}^\circ \end{array} \tag{25.58}$$

The system mass matrix corresponding to the free displacements $u_3°$ and $v_3°$ is obtained by adding together equations (25.54) and (25.56), as shown by equation (25.59):

$$
\begin{array}{cc}
 & \begin{array}{cc} u_3^\circ & \qquad v_3^\circ \end{array} \\
[\mathbf{M}_{11}] = & \left[\begin{array}{c|c} 0.303 + 0.524 & 0 \\ \hline 0 & 0.303 + 0.524 \end{array}\right] \begin{array}{c} u_3^\circ \\ v_3^\circ \end{array}
\end{array}
\tag{25.59}
$$

$$
= \begin{array}{cc}
 & \begin{array}{cc} u_3^\circ & v_3^\circ \end{array} \\
 & \begin{bmatrix} 0.827 & 0 \\ 0 & 0.827 \end{bmatrix} \begin{array}{c} u_3^\circ \\ v_3^\circ \end{array}
\end{array}
\tag{25.60}
$$

Now, from Section 25.2,

$$\frac{d^2 v_c}{dt^2} + \frac{k v_c}{M} = 0 \tag{25.61}$$

If simple harmonic motion takes place, so that

$$v_c = C e^{j\omega t}$$

then,

$$\frac{d^2 v_c}{dt^2} = -\omega^2 \, C e^{j\omega t} = -\omega^2 v_c \tag{25.62}$$

Substituting equation (25.62) into equation (25.61),

$$-\omega^2 v_c + \frac{k v_c}{M} = 0 \tag{25.63}$$

In matrix form, equation (25.63) becomes

$$\left([\mathbf{K}] - \omega^2 \, [\mathbf{M}]\right) \{u_i\} = 0 \tag{25.64}$$

or, for a constrained structure,

$$\left([\mathbf{K}_{11}] - \omega^2 [\mathbf{M}_{11}]\right) \{u_i\} = 0 \tag{25.65}$$

Now, in equation (25.65), the condition $\{u_i\} = \{0\}$ is not of practical interest, therefore the solution of equation (25.65) becomes equivalent to expanding the determinant of equation (25.66):

$$\left| [\mathbf{K}_{11}] - \omega^2 [\mathbf{M}_{11}] \right| = 0 \tag{25.66}$$

Substituting equations (25.58) and (25.60) into equation (25.66), the following is obtained:

$$\left| \begin{bmatrix} 1.183 \times 10^7 & 0.317 \times 10^7 \\ 0.317 \times 10^7 & 1.55 \times 10^7 \end{bmatrix} - \omega^2 \begin{bmatrix} 0.827 & 0 \\ 0 & 0.827 \end{bmatrix} \right| \tag{25.67}$$

Expanding equation (25.67), results in the quadratic equation (25.68):

$$\left(1.183 \times 10^7 - 0.827\omega^2\right)\left(1.55 \times 10^7 - 0.827\omega^2\right) - \left(0.317 \times 10^7\right)^2 = 0$$

or

$$1.834 \times 10^{14} - 2.26 \times 10^7 \omega^2 + 0.684 \omega^4 - 1 \times 10^{13} = 0$$

or

$$0.684\omega^4 - 2.26 \times 10^7 \omega^2 + 1.734 \times 10^{14} = 0 \tag{25.68}$$

Solving the quadratic equation (25.68), the following are obtained for the roots ω_1^2 and ω_2^2:

$$\omega_1^2 = \frac{2.26 \times 10^7 - 6.028 \times 10^6}{1.368} = 1.211 \times 10^7$$

or

$$\omega_1 = 3480; \; n_1 = 533.9 \text{ Hz}$$

$$\omega_2^2 = \frac{2.26 \times 10^7 + 6.028 \times 10^6}{1.368} = 2.093 \times 10^7$$

or

$$\omega_2 = 4575; \; n_2 = 728 \text{ Hz}$$

To determine the eigenmodes, substitute ω_1^2 into the first row of equation (25.67) and substitute ω_2^2 into the second row of equation (25.67), as follows:

$$\left(1.183 \times 10^7 - 3480^2 \times 0.827\right)u_3^\circ + 0.317 \times 10^7\ v_3^\circ = 0 \qquad (25.69)$$

$$1.815 \times 10^6\ u_3^\circ + 3.17 \times 10^6\ v_3^\circ = 0$$

Let,

$$u_3^\circ = 1$$

$$\therefore v_3^\circ = -0.47$$

so that the first eigenmode is:

$[u_3^\circ \quad v_3^\circ] = [1 \quad -0.47]$ see the figure below at (a).

Similarly, to determine the second eigenmode, substitute ω_2^2 into the second row of equation (25.67), as follows:

$$0.317 \times 10^7\ u_3^\circ + \left(1.55 \times 10^7 - 0.827 \times 4575^2\right) v_3^\circ = 0$$

or

$$0.317 \times 10^7\ u_3^\circ - 1.81 \times 10^6\ v_3^\circ = 0$$

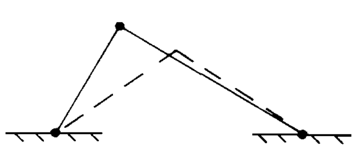
Let,

$$v_3^\circ = 1$$

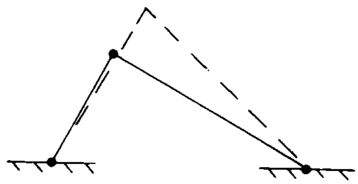
$$\therefore u_3 = 0.57$$

so that the second eigenmode is given by

$[u_3^\circ \quad v_3^\circ] = [0.57 \quad 1]$ see below at (b).



(a) First eigenmode.



(b) Second eigenmode.

Problem 25.4 If the pin-jointed truss of Problem 25.3 had an additional mass of 0.75 kg attached to node 3, what would be the values of the resulting resonant frequencies?

Solution

From equation (25.58):

$$\left[K_{11}\right] = \begin{array}{cc} u_3^\circ & v_3^\circ \\ \left[\begin{array}{cc} 1.183 \times 10^7 & 0.317 \times 10^7 \\ 0.317 \times 10^7 & 1.55 \times 10^7 \end{array}\right] & \begin{array}{c} u_3^\circ \\ v_3^\circ \end{array} \end{array} \tag{25.70}$$

From equation (25.60)

$$\left[\mathbf{M}_{11}\right] = \begin{bmatrix} 0.827 & 0 \\ 0 & 0.827 \end{bmatrix} + \begin{bmatrix} 0.75 & 0 \\ 0 & 0.75 \end{bmatrix}$$

$$= \begin{array}{cc} u_3^\circ & v_3^\circ \\ \left[\begin{array}{cc} 1.577 & 0 \\ 0 & 1.577 \end{array}\right] & \begin{array}{c} u_3^\circ \\ v_3^\circ \end{array} \end{array} \tag{25.71}$$

Substituting equations (25.70) and (25.71) into equations (25.65), the following is obtained:

$$\left|\begin{bmatrix} 1.183 \times 10^7 & 0.317 \times 10^7 \\ 0.317 \times 10^7 & 1.55 \times 10^7 \end{bmatrix} - \omega^2 \begin{bmatrix} 1.577 & 0 \\ 0 & 1.577 \end{bmatrix}\right| = 0 \tag{25.72}$$

Expanding the determinant of equation (25.72), results in the quadratic equation (25.73):

$$\left(1.183 \times 10^7 - 1.577\omega^2\right)\left(1.55 \times 10^7 - 1.577\omega^2\right) - \left(0.317 \times 10^7\right)^2 = 0$$

or

$$1.834 \times 10^{14} - 4.31 \times 10^7\,\omega^2 + 2.487\,\omega^4 - 1 \times 10^{13} = 0$$

or

$$2.487\omega^4 - 4.31 \times 10^7\,\omega^2 + 1.734 \times 10^{14} = 0 \qquad (25.73)$$

The quadratic equation (25.73) has two roots, namely ω_1^2 and ω_2^2, which are obtained as follows:

$$\omega_1^2 = \frac{4.31 \times 10^7 - 1.178 \times 10^7}{4.974} = 6.297 \times 10^6$$

$$\omega_1 = 2509; \; n_1 = 399.3 \text{ Hz}$$

and

$$\omega_2^2 = \frac{4.31 \times 10^7 + 1.178 \times 10^7}{4.974} = 1.103 \times 10^7$$

$$\omega_2 = 3322; \; n_2 = 528.6 \text{ Hz}$$

Problem 25.5 Determine the resonant frequencies and eigenmodes for the pin-jointed space truss of Problem 23.3, given that,

$A = 2 \times 10^{-4}\ \mathrm{m}^2$
$E = 2 \times 10^{11}\ \mathrm{N/m}^2$
$\rho = 7860\ \mathrm{kg/m}^3$

<u>*Solution*</u>

Element 1–4

From Problem 25.3,

$l = 10\ \mathrm{m}$

Substituting this and other values into equation (25.50), and removing the rows and columns corresponding to the zero displacements, namely $u_1{}^\circ$, $v_1{}^\circ$ and $w_1{}^\circ$, the mass matrix for element 1–4 is given by

$$\left[\mathbf{m}_{1\text{-}4}{}^\circ\right] = \frac{7860 \times 2 \times 10^{-4} \times 10}{6}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \tag{25.74}$$

$$= \begin{array}{cc} \begin{array}{ccc} u_4{}^\circ & v_4{}^\circ & w_4{}^\circ \end{array} & \\ \begin{bmatrix} 5.24 & 0 & 0 \\ 0 & 5.24 & 0 \\ 0 & 0 & 5.24 \end{bmatrix} & \begin{array}{c} u_4{}^\circ \\ v_4{}^\circ \\ w_4{}^\circ \end{array} \end{array} \tag{25.75}$$

Element 2–4

From Problem 25.3,

$$l = 10\text{ m}$$

Substituting this and other values into equation (25.50), and removing the rows and columns corresponding to the zero displacements, namely $u_2{}^\circ$, $v_2{}^\circ$ and $w_2{}^\circ$, the mass matrix for element 2–4 is given by

$$\left[\mathbf{m}_{2\text{-}4}{}^\circ\right] = \begin{array}{cc} \begin{array}{ccc} u_4{}^\circ & v_4{}^\circ & w_4{}^\circ \end{array} & \\ \begin{bmatrix} 5.24 & 0 & 0 \\ 0 & 5.24 & 0 \\ 0 & 0 & 5.24 \end{bmatrix} & \begin{array}{c} u_4{}^\circ \\ v_4{}^\circ \\ w_4{}^\circ \end{array} \end{array} \tag{25.76}$$

Element 4–3

From Problem 25.3,

$$l = 10\text{ m}$$

Substituting the above and other values into equation (25.50), and removing the rows and columns corresponding to the zero displacements, namely u_3°, v_3° and w_3°, the mass matrix for element 4–3 is given by

$$\left[\mathbf{m}_{4\text{-}3}^\circ\right] = \begin{array}{c} \begin{array}{ccc} u_4^\circ & v_4^\circ & w_4^\circ \end{array} \\ \begin{bmatrix} 5.24 & 0 & 0 \\ 0 & 5.24 & 0 \\ 0 & 0 & 5.24 \end{bmatrix} \end{array} \begin{array}{c} u_4^\circ \\ v_4^\circ \\ w_4^\circ \end{array} \tag{25.77}$$

To obtain $[\mathbf{M}_{11}]$, the system mass matrix corresponding to the free displacements u_4°, v_4° and w_4°, the elemental mass matrices of equations (25.75) to (25.77), are added together, as shown by equation (25.78):

$$\left[\mathbf{M}_{11}^\circ\right] = \begin{array}{c} \begin{array}{ccc} u_4^\circ & v_4^\circ & w_4^\circ \end{array} \\ \begin{bmatrix} 15.72 & 0 & 0 \\ 0 & 15.72 & 0 \\ 0 & 0 & 15.72 \end{bmatrix} \end{array} \begin{array}{c} u_4^\circ \\ v_4^\circ \\ w_4^\circ \end{array} \tag{25.78}$$

From equation (23.62),

$$\left[\mathbf{K}_{11}\right] = 1 \times 10^6 \begin{array}{c} \begin{array}{ccc} u_4^\circ & v_4^\circ & w_4^\circ \end{array} \\ \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0.832 \\ 0 & 0.832 & 6 \end{bmatrix} \end{array} \begin{array}{c} u_4^\circ \\ v_4^\circ \\ w_4^\circ \end{array} \tag{25.79}$$

Substituting equations (25.78) and (25.79) into equation (25.65), the following determinant is obtained:

$$\left| 1 \times 10^6 \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0.832 \\ 0 & 0.832 & 6 \end{bmatrix} - \omega^2 \begin{bmatrix} 15.72 & 0 & 0 \\ 0 & 15.72 & 0 \\ 0 & 0 & 15.72 \end{bmatrix} \right| \tag{25.80}$$

From the top line of equation (25.80):

$$2 \times 10^6 - 15.72\,\omega^2 = 0$$

or

$$\omega_1^2 = \frac{2 \times 10^6}{15.72} = 1.272 \times 10^5$$

$$\omega_1 = 356.7, \; n = 56.76 \text{ Hz}$$

As the first line of equation (25.80) is uncoupled, this equation can be reduced to the 2 × 2 determinant of equation (25.81):

$$\left| 1 \times 10^6 \begin{bmatrix} 4 & 0.832 \\ 0.832 & 6 \end{bmatrix} - \omega^2 \begin{bmatrix} 15.72 & 0 \\ 0 & 15.72 \end{bmatrix} \right| = 0 \tag{25.81}$$

Expanding equation (25.81), the quadratic equation (25.82) is obtained:

$$\left(4 \times 10^6 - 15.72\omega^2\right)\left(6 \times 10^6 - 15.72\omega^2\right) - \left(0.832 \times 10^6\right)^2 = 0$$

or

$$2.4 \times 10^{13} - 1.572 \times 10^8\omega^2 + 247.12\omega^4 - 6.922 \times 10^{11} = 0$$

or

$$247.12\omega^4 - 1.572 \times 10^8\omega^2 + 2.33 \times 10^{13} = 0 \tag{25.82}$$

Solving equation (25.82), the roots ω_2^2 and ω_3^2 are obtained, as follows:

$$\omega_2^2 = \frac{1.572 \times 10^8 - 0.41 \times 10^8}{492.24} = 2.361 \times 10^5$$

$$\omega_2 = 485.9; \; n_2 = 77.32 \text{ Hz}$$

$$\omega_3^2 = \frac{1.572 \times 10^8 + 0.41 \times 10^8}{492.24} = 4.026 \times 10^5$$

$$\omega_3 = 634.5; \; n_3 = 100.98 \text{ Hz}$$

To determine the eigenmodes

By inspection of the first line of equation (25.80),

$$u_4^\circ = 1, \quad v_4^\circ = 0 \quad \text{and} \quad w_4^\circ = 0$$

Therefore, the first eigenmode is

$$[u_4^\circ \quad v_4^\circ \quad w_4^\circ] = [1 \;\; 0 \;\; 0]$$

To obtain the second eigenmode, substitute ω_2^2 into the second line of equation (25.80) to give

$$0 \times u_4^\circ + \left[4 \times 10^6 - \left(485.9^2 \times 15.72\right)\right] v_4^\circ + 0.832 \times 10^6 w_4^\circ = 0$$

or

$$0.289 v_4^\circ + 0.832 w_4^\circ = 0 \tag{25.83}$$

Let,

$$v_4^\circ = 1$$

$$\therefore w_4^\circ = -0.347$$

Therefore, the second eigenmode is

$$[u_4^\circ \quad v_4^\circ \quad w_4^\circ] = [0 \;\; 1 \;\; -0.347]$$

To obtain the third eigenmode, substitute ω_3^2 into the third line of equation (25.80) to give

$$0 \times u_4^\circ + 0.832 \times 10^6 v_4^\circ + (6 \times 10^6 - 634.5^2 \times 15.72) w_4^\circ = 0$$

or

$$0.832 v_4^\circ - 0.329 w_4^\circ = 0 \tag{25.84}$$

Let,

$$w_4^\circ = 1$$

$$\therefore v_4^\circ = 0.395$$

Therefore, the third eigenmode is

$$[u_4^\circ \quad v_4^\circ \quad w_4^\circ] = [0 \;\; 0.395 \;\; 1]$$

Problem 25.6 Determine the resonant frequencies for the tripod of Problem 25.5, if this tripod has a mass of 10 kg added to node 4.

Solution

From equation (25.79),

$$\begin{array}{cc} & \begin{array}{ccc} u_4^\circ & v_4^\circ & w_4^\circ \end{array} \\ [\mathbf{K}_{11}] = 1 \times 10^6 & \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0.832 \\ 0 & 0.832 & 6 \end{bmatrix} \end{array} \begin{array}{c} u_4^\circ \\ v_4^\circ \\ w_4^\circ \end{array} \tag{25.85}$$

From equation (25.78):

$$[\mathbf{M}_{11}] = \begin{bmatrix} 15.72 & 0 & 0 \\ 0 & 15.72 & 0 \\ 0 & 0 & 15.72 \end{bmatrix} + \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix}$$

$$= \begin{array}{c} \begin{array}{ccc} u_4^\circ & v_4^\circ & w_4^\circ \end{array} \\ \begin{bmatrix} 25.72 & 0 & 0 \\ 0 & 25.72 & 0 \\ 0 & 0 & 25.72 \end{bmatrix} \end{array} \begin{array}{c} u_4^\circ \\ v_4^\circ \\ w_4^\circ \end{array} \tag{25.86}$$

Substituting equations (25.85) and (25.86) into equation (25.65), the following determinant is obtained:

$$\left| 1 \times 10^6 \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0.832 \\ 0 & 0.832 & 6 \end{bmatrix} - \omega^2 \begin{bmatrix} 25.72 & 0 & 0 \\ 0 & 25.72 & 0 \\ 0 & 0 & 25.72 \end{bmatrix} \right| = 0 \tag{25.87}$$

From the first line of equation (25.65):

$$\omega_1^2 = \frac{2 \times 10^6}{25.72} = 7.776 \times 10^4$$

$$\omega_1 = 2789; \; n_1 = 44.1 \text{ Hz}$$

As first line is uncoupled, the determinant of equation (25.87) can be reduced to the 2 × 2 determinant of equation (25.88):

$$\left| 1 \times 10^6 \begin{bmatrix} 4 & 0.832 \\ 0.832 & 6 \end{bmatrix} - \omega^2 \begin{bmatrix} 25.72 & 0 \\ 0 & 25.72 \end{bmatrix} \right| = 0 \tag{25.88}$$

Expanding the determinant of equation (25.88), the following quadratic is obtained:

$$(4 \times 10^6 - 25.72\,\omega^2)(6 \times 10^6 - 25.72\,\omega^2) - (0.832 \times 10^6)^2 = 0$$

or

$$2.4 \times 10^{13} - 2.572 \times 10^8\,\omega^2 + 661.5\,\omega^4 - 6.92 \times 10^{11} = 0$$

or

$$661.5\,\omega^4 - 2.572 \times 10^8\,\omega^2 + 2.33 \times 10^{13} = 0 \tag{25.89}$$

Solving equation (25.89),

$$\omega_2^2 = \frac{2.572 \times 10^8 - 0.671 \times 10^8}{1323} = 1.437 \times 10^5$$

$$\omega_2 = 379.1; \; n_2 = 60.3 \text{ Hz}$$

$$\omega_3^2 = \frac{2.572 \times 10^8 + 0.671 \times 10^8}{1323} = 2.451 \times 10^5$$

$$\omega_3 = 495.1; \; n_3 = 78.8 \text{ Hz}$$

25.10 Mass matrix for a beam element

The beam element, which has four degrees of freedom, is shown in Figure 25.9.

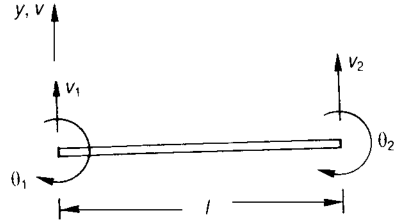


Figure 25.9 Beam element.

A convenient polynomial with which to describe the lateral deflection v is

$$v = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3 \tag{25.90}$$

and

$$\frac{dv}{dx} = \alpha_2 + 2\alpha_3 x + 3\alpha_4 x^2 \tag{25.91}$$

In equation (25.90), it can be seen that the polynomial has four arbitrary constants, and this corresponds to the four degrees of freedom, namely, v_1, θ_1, v_2 and θ_2, i.e.

At $x = 0$, $\quad v = v_1 \quad$ and $\quad \theta_1 = -(dv/dx)_{x=0}$

At $x = l$, $\quad v = v_2 \quad$ and $\quad \theta_2 = -(dv/dx)_{x=l}$

Substituting the first two boundary conditions into equations (25.90) and (25.91):

$$\alpha_1 = v_1$$

and

$$\alpha_2 = -\theta_1$$

Substituting the remaining two boundary conditions into equations (25.90) and (25.91), the following two simultaneous equations are obtained:

$$v_2 = v_1 - \theta_1 l + \alpha_3 l^2 + \alpha_4 l^3 \tag{25.92}$$

and,

$$\theta_2 = \theta_1 - 2\alpha_3 l - 3\alpha_4 l^2 \tag{25.93}$$

Multiplying equation (25.92) by $2/l$, we get:

$$\frac{2}{l}\left(v_2 - v_1\right) = -2\theta_1 + 2\alpha_3 l + 2\alpha_4 l^2 \tag{25.94}$$

Adding equation (25.93) to equation (25.94):

$$\frac{2}{l}\left(v_2 - v_1\right) + \theta_2 = \theta_1 - 2\theta_1 - 3\alpha_4 l^2 + 2\alpha_4 l^2$$

or

$$
\begin{aligned}
-\alpha_4 l^2 &= \frac{2}{l}\left(v_2 - v_1\right) + \theta_2 + \theta_1 \\
\alpha_4 &= -\frac{2}{l^3}\left(v_2 - v_1\right) - \frac{\left(\theta_2 + \theta_1\right)}{l^2}
\end{aligned}
\tag{25.95}
$$

Substituting equation (25.95) into equation (25.92):

$$v_2 - v_1 + \theta_1 l = \alpha_3 l^2 - 2\left(v_2 - v_1\right) - \left(\theta_2 + \theta_1\right) l$$

or

$$\alpha_3 = \frac{3}{l^2}\left(v_2 - v_1\right) + \frac{1}{l}\left(2\theta_1 + \theta_2\right) \tag{25.96}$$

Substituting the above values of α_1 to α_4 into equation (25.90)

$$
\begin{aligned}
v = v_1 - \theta_1 x + 3\xi^2\left(v_2 - v_1\right) + \frac{x^2}{l}\left(2\theta_1 + \theta_2\right) \\
- 2\xi^2\left(v_2 - v_1\right) + \frac{x^3}{l^2}\left(\theta_2 + \theta_1\right)
\end{aligned}
$$

or

$$
\begin{aligned}
v = v_1\left(1 - 3\xi^2 + 2\xi^3\right) + \theta_1 l\left(-\xi + 2\xi^2 - \xi^3\right) \\
+ v_2\left(3\xi^2 - 2\xi^3\right) + \theta_2 l\left(\xi^2 - \xi^3\right)
\end{aligned}
\tag{25.97}
$$

where,

$$\xi = x/l$$

i.e.

$$
\begin{aligned}
v = &\left|\left(1 - 3\xi^2 + 2\xi^3\right) \quad l\left(-\xi + 2\xi^2 - \xi^3\right)\right. \\
&\left.\left(3\xi^2 - 2\xi^3\right) \quad l\left(\xi^2 - \xi^3\right)\right| \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} \\
= &\left[N\right]\left\{u_i\right\}
\end{aligned}
\tag{25.98}
$$

where [N] is a matrix of shape functions for a beam element:

$$[\mathrm{N}] = \left[\left(1-3\xi^2+2\xi^3\right)\ l\left(-\xi+2\xi^2-\xi^3\right)\left(3\xi^2-2\xi^3\right)l\left(\xi^2-\xi^3\right)\right] \tag{25.99}$$

From equation (25.38):

$$[\mathbf{m}] = \int_0^1 [\mathrm{N}]^{\mathrm{T}}\ \rho[\mathrm{N}]\ Al\, \mathrm{d}\,\xi \tag{25.100}$$

Substituting equation (25.99) into equation (25.100), and integrating, the mass matrix for a beam element is given by

$$[\mathbf{m}] = \frac{\rho Al}{420} \begin{array}{c} \begin{array}{cccc} v_1 & \theta_1 & v_2 & \theta_2 \end{array} \\ \left[\begin{array}{cccc} 156 & & & \\ -22l & 4l^2 & & \\ 54 & -13l & 156 & \\ 13l & -3l^2 & 22l & 4l^2 \end{array}\right] \end{array} \begin{array}{c} \\ v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{array} \tag{25.101}$$

Equation (25.101) is the mass matrix of a beam element due to the self-mass of the structure, but if an additional concentrated mass is added to node i, the following additional components of mass must be added to equation (25.102) at the appropriate node.

Added mass matrix at node i

$$= \begin{array}{c} \begin{array}{cc} v_i & \theta_i \end{array} \\ \left[\begin{array}{cc} M_a & 0 \\ 0 & MMI \end{array}\right] \end{array} \begin{array}{c} \\ v_i \\ \theta_i \end{array} \tag{25.102}$$

where MMI is the mass moment of inertia and M_a is the mass.

Problem 25.7 Determine the resonant frequencies for the beam of the figure in Problem 23.4, assuming that the 4 kN load is not present, and that

$E = 2 \times 10^{11}\ \mathrm{N/m^2}$, $\rho = 7860\ \mathrm{kg/m^3}$

$A = 1 \times 10^{-4}\ \mathrm{m^2}$, $I = 1 \times 10^{-7}\ \mathrm{m^4}$

Solution

Element 1–2

$$l = 3\text{ m}$$

Substituting the above value of l into equation (25.101), together with the other properties of this element, and removing the columns and rows corresponding to the zero displacements v_1 and θ_1, the elemental mass matrix is given by

$$[\mathbf{m}_{1\text{-}2}] = \frac{7860 \times 1 \times 10^{-4} \times 3}{420} \begin{array}{c} \begin{array}{cc} v_2 & \theta_2 \end{array} \\ \begin{bmatrix} 156 & 66 \\ 66 & 36 \end{bmatrix} \end{array} \begin{array}{c} \\ v_2 \\ \theta_2 \end{array}$$

$$= \begin{array}{c} \begin{array}{cc} v_2 & \theta_2 \end{array} \\ \begin{bmatrix} 0.876 & 0.371 \\ 0.371 & 0.202 \end{bmatrix} \end{array} \begin{array}{c} \\ v_2 \\ \theta_2 \end{array} \qquad (25.103)$$

Element 2–3

$$l = 2\text{ m}$$

Substituting the above value of l into equation (25.101), together with the other properties of this element, and removing the columns and rows corresponding to the zero displacements v_3 and θ_3, the elemental mass matrix is given by:

$$[\mathbf{m}_{2\text{-}3}] = \begin{array}{c} \begin{array}{cc} v_2 & \theta_2 \end{array} \\ \begin{bmatrix} 0.584 & -0.165 \\ -0.165 & 0.0599 \end{bmatrix} \end{array} \begin{array}{c} \\ v_2 \\ \theta_2 \end{array} \qquad (25.104)$$

The system mass matrix $[\mathbf{M}_{11}]$ is obtained by adding together the elemental mass matrices of equations (25.103) and (25.104):

$$[\mathbf{M}_{11}] = \begin{array}{c} \begin{array}{cc} v_2 & \theta_2 \end{array} \\ \begin{bmatrix} 1.46 & 0.206 \\ 0.206 & 0.262 \end{bmatrix} \end{array} \begin{array}{c} \\ v_2 \\ \theta_2 \end{array} \qquad (25.105)$$

From equation (25.84),

$$
[\mathbf{K}_{11}] = \begin{array}{cc} v_2 & \theta_2 \\ \left[\begin{array}{cc} 38\ 880 & -16\ 660 \\ -16\ 660 & 66\ 660 \end{array}\right] \end{array} \begin{array}{c} \\ v_2 \\ \theta_2 \end{array} \qquad (25.106)
$$

Substituting equations (25.105) and (25.106) into equation (25.65), the following determinant is obtained:

$$
\left| \begin{bmatrix} 38\ 880 & -16\ 660 \\ -16\ 660 & 66\ 660 \end{bmatrix} - \omega^2 \begin{bmatrix} 1.46 & 0.206 \\ 0.206 & 0.262 \end{bmatrix} \right| = 0 \qquad (25.107)
$$

Expanding the determinant of equation (25.107), the following quadratic equation is obtained:

$$
(38\ 880 - 1.46\omega^2)(66\ 660 - 0.262\omega^2) - (-\ 16\ 660 - 0.206\omega^2)^2 = 0
$$

or,

$$
2592 \times 10^6 - 0.107 \times 10^6\,\omega^2 + 0.383\,\omega^4
$$

$$
- 278 \times 10^6 - 6864\,\omega^2 - 0.042\,\omega^4 = 0
$$

$$
0.341\,\omega^4 - 0.1139 \times 10^6\,\omega^2 + 2.314 \times 10^9 = 0 \qquad (25.108)
$$

The roots of equation (25.108), namely, ω_1^2 and ω_2^2, can readily be shown to be:

$$
\omega_1^2 = \frac{0.1139 \times 10^6 - 99\ 080}{0.682} = 2.173 \times 10^4
$$

or

$$
\omega_1 = 147.4; \; n_1 = 23.45 \text{ Hz}
$$

$$
\omega_2^2 = \frac{0.1139 \times 10^6 + 99\ 080}{0.682} = 3.123 \times 10^5
$$

and,

$$
\omega_2 = 558.8; \; n_2 = 88.93 \text{ Hz}
$$

To obtain the first eigenmode, substitute ω_1^2 into the first line of equation (25.107), to give

$$(38\,880 - 1.46 \times 147.4^2)\, v_2 + (-\,16\,660 + 0.206 \times 147.4^2)\, \theta_2 = 0$$

or

$$7159\, v_2 - 21\,136\, \theta_2 = 0 \tag{25.109}$$

i.e.

$[v_2 \;\; \theta_2] = [1 \;\; 0.339]$ – see the figure below at (a).

To obtain the second eigenmode, substitute ω_2^2 into the second line of equation (25.107) to give:

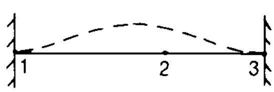
$$(-\,16\,660 - 0.206 \times 558.8^2)\, v_2 + (66\,660 - 0.262 \times 558.8^2)\, \theta_2 = 0$$

or,

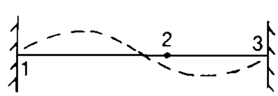
$$-\,80\,985\, v_2 - 15\,150\, \theta_2 = 0 \tag{25.110}$$

i.e.

$[v_2 \qquad \theta_2] = [-\,0.187 \;\; 1]$ – see the figure below at (b).



(a) First eigenmode



(b) Second eigenmode

Problem 25.8 If the beam of Problem 25.7 has a mass of 1 kg, with a mass moment of inertia of 0.1 kg m^2 added to node 2, determine the resonant frequencies of the beam.

<u>*Solution*</u>

From equation (25.105)

$$[M_{11}] = \begin{bmatrix} 1.46 & 0.206 \\ 0.206 & 0.262 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0.1 \end{bmatrix}$$

$$= \begin{array}{cc} v_2 & \theta_2 \end{array} \qquad \begin{array}{c} \\ \begin{bmatrix} 2.46 & 0.206 \\ 0.206 & 0.362 \end{bmatrix} \begin{array}{c} v_2 \\ \theta_2 \end{array} \end{array} \tag{25.111}$$

From equation (25.101),

$$[\mathbf{K}_{11}] = \begin{bmatrix} 38880 & -16660 \\ -16660 & 66660 \end{bmatrix} \tag{25.112}$$

Substituting equations (25.111) and (25.112) into equation (25.65),

$$\left| \begin{bmatrix} 38\,880 & -16\,660 \\ -16\,660 & 66\,660 \end{bmatrix} - \omega^2 \begin{bmatrix} 2.46 & 0.206 \\ 0.206 & 0.362 \end{bmatrix} \right| = 0 \tag{25.113}$$

$$(38\,880 - 2.46\,\omega^2)\,(66\,660 - 0.362\,\omega^2) - (16\,660 + 0.206\,\omega^2)^2 = 0$$

or

$$0.259 \times 10^{10} - 0.178 \times 10^6\,\omega^2 + 0.891\,\omega^4 - 2.776 \times 10^8 - 6864\,\omega^2 - 0.042\,\omega^4 = 0$$

or

$$0.849\,\omega^4 - 0.1849 \times 10^6\,\omega^2 + 0.231 \times 10^{10} = 0 \tag{25.114}$$

Solution of the quadratic equation (25.114) results in the roots $\omega_1{}^2$ and $\omega_2{}^2$, as follows:

$$\omega_1^2 = \frac{0.1849 \times 10^6 - 0.162 \times 10^6}{1.698} = 1.394 \times 10^4$$

or

$$\omega_1 = 116.1;\; n_1 = 18.48 \text{ Hz}$$

and,

$$\omega_2^2 = \frac{0.1849 \times 10^6 + 0.162 \times 10^6}{1.698} = 2.043 \times 10^5$$

or

$$\omega_2 = 452;\; n_2 = 71.93 \text{ Hz}$$

25.11 Mass matrix for a rigid-jointed plane frame element

Prior to obtaining the mass matrix for an element of a rigid-jointed plane frame, it will be necessary to obtain the mass matrix for the inclined beam of Figure 25.10.

The mass matrix for an inclined beam element in global co-ordinates is

$$[m_b{}^\circ] = [DC]^T [m] [DC] \tag{25.115}$$

where,

[DC] is given equation (25.85) and [m] is given by equation (25.101).

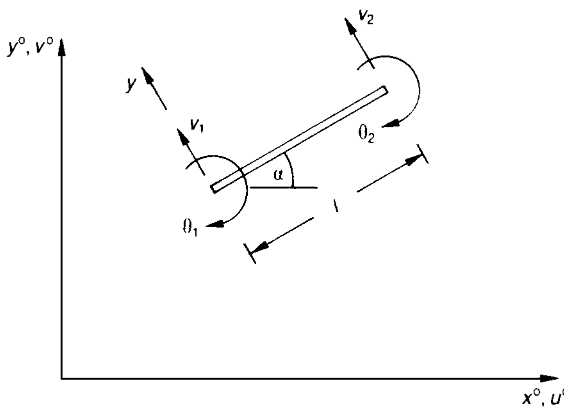


Figure 25.10 Inclined beam element.

$$[m_b{}^\circ] = \frac{\rho A l}{420} \begin{array}{c} \begin{array}{cccccc} u_1{}^\circ & v_1{}^\circ & \theta_1 & u_2{}^\circ & v_2{}^\circ & \theta_2 \end{array} \\ \left[\begin{array}{cccccc} 156s^2 & & & & & \\ -156cs^2 & 156c^2 & & & & \\ 22ls & -22lc & 4l^2 & & & \\ 54s^2 & -54cs & 13ls & 156s^2 & & \\ -54cs & 54c^2 & -13cl & -156cs & 156c^2 & \\ -13ls & 13lc & -3l^2 & -22ls & 22lc & 4l^2 \end{array}\right] \end{array} \begin{array}{c} u_1{}^\circ \\ v_1{}^\circ \\ \theta_1{}^\circ \\ u_2{}^\circ \\ v_2{}^\circ \\ \theta_2 \end{array} \tag{25.116}$$

For the element of a rigid-jointed plane frame, the elemental mass matrix in global co-ordinates is given by

$$[\mathbf{m}^\circ] = [\mathbf{m}_b{}^\circ] + [\mathbf{m}_r{}^\circ] \qquad (25.117)$$

where $[\mathbf{m}_r{}^\circ]$ is the axial part of the mass matrix of a rod element:

$$\left[\mathbf{m}_r{}^\circ\right] = [\mathbf{DC}]^{\mathrm{T}} \frac{\rho A l}{6} \begin{bmatrix} 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} [\mathbf{DC}] \qquad (25.118)$$

where, in equation (25.118), the components of mass in the v displacement direction have been removed, because they have already been included in $[\mathbf{m}_b{}^\circ]$.

Substituting $[\mathbf{DC}]$ from equation (25.85) into equation (25.118):

$$\left[\mathbf{m}_r{}^\circ\right] = \frac{\rho A l}{6} \begin{array}{c} \begin{array}{cccccc} u_1{}^\circ & v_1{}^\circ & \theta_1 & u_2{}^\circ & v_2{}^\circ & \theta_2 \end{array} \\ \begin{bmatrix} 2c^2 & & & & & \\ 2cs & 2s^2 & & & & \\ 0 & 0 & 0 & & & \\ c^2 & cs & 0 & 2c^2 & & \\ cs & s^2 & 0 & 2cs & 2s^2 & \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} u_1{}^\circ \\ v_1{}^\circ \\ \theta_1 \\ u_2{}^\circ \\ v_2{}^\circ \\ \theta_2 \end{array} \qquad (25.119)$$

From equations (25.116) and (25.118), it can be seen that application of these elemental mass matrices, together with the elemental stiffness matrix of equation (25.85), to a realistic rigid-jointed plane frame will be extremely difficult without the aid of a computer.

Equation (25.117) shows the mass matrix for the self-mass of an element of a rigid-jointed plane frame, but if the effects of an additional concentrated mass are to be included at a particular node, the concentrated mass must be added to the appropriate node, as follows:

$$
\begin{array}{c}
\begin{array}{ccc} u_1^\circ & v_1^\circ & \theta_1 \end{array} \\
\begin{bmatrix} M_a & 0 & 0 \\ 0 & M_a & 0 \\ 0 & 0 & \mathbf{MMI} \end{bmatrix}
\end{array}
\begin{array}{c} \\ u_1^\circ \\ v_1^\circ \\ \theta_1 \end{array}
\qquad (25.120)
$$

where

M_a = the value of the mass

MMI = the mass moment of inertia of this mass

25.12 Units in structural dynamics

Considerable care should be taken in choosing suitable units in structural dynamics. Recommended units are as follows:

(i) *Imperial*

Mass (lbf s^2/in); density (lbf s^2/in^4); E (lbf/in^2); time(s); length (in); Force(lbf); second moment of area (in^4); cross-sectional area (in^2).

(ii) *SI*

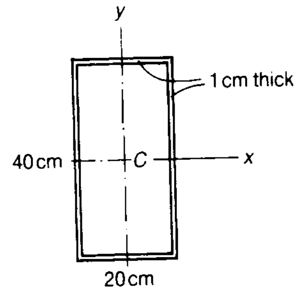
Mass (kg); density (kg/m^3); E (N/m^2); time (s); length (m); Force(N); second moment of area (m^4); cross-sectional area (m^2).

(iii) *Derived SI*

Mass (kg); density (kg/mm^3); E (mN/mm^2); time (s); length (mm); force(mN); second moment of area (mm^4); cross-sectional area (mm^2).

Further problems *(answers on page 698)*

25.9 A doubly symmetrical beam consists of a hollow rectangular steel section, having the cross-section shown, and of length 10 m. It is simply-supported in bending about both axes Cx, Cy at the ends. Estimate the lowest few natural frequencies of lateral vibrations of the beam about the axes Cx and Cy. Take E = 200 GN/m^2.



25.10 If the beam of Problem 25.7 carries an axial thrust of 10^3 kN, what is the lowest natural frequency of the beam?

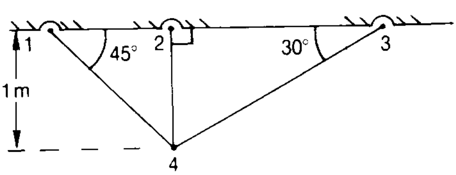
25.11 A light, uniform cantilever, of length L and uniform flexural stiffness EI, carries a mass M at the free end. Estimate the natural frequency of vibrations.

25.12 Determine the resonant frequencies for the plane pin-jointed truss shown below, assuming that the truss is loaded with a mass of 1 kg at node 4, and that the following apply:

$A = 1 \times 10^{-4}\ \text{m}^2$

$E = 2 \times 10^{11}\ \text{N/m}^2$

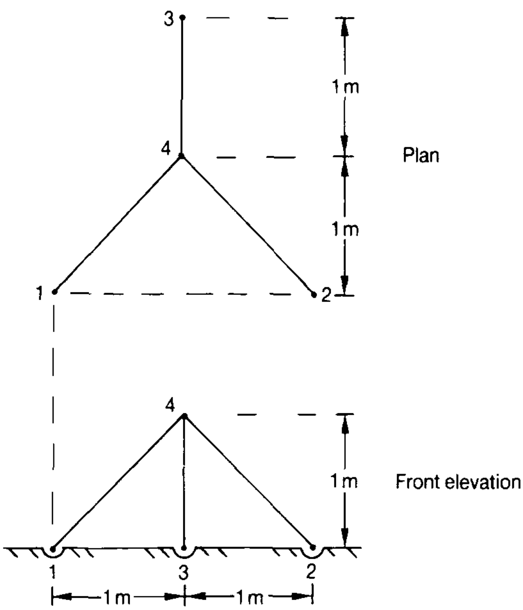
$\rho = 7860\ \text{kg/m}^3$



(*Portsmouth 1989*)

25.13 Determine the resonant frequencies for the pin-jointed tripod, below, given that the following apply:

Element	A (m²)	E(N/m²)	ρ(kg/m³)
1–4	1×10^{-3}	2×10^{11}	7860
2–4	2×10^{-3}	2×10^{11}	7860
3–4	1×10^{-3}	2×10^{11}	7860



(*Portsmouth 1983*)

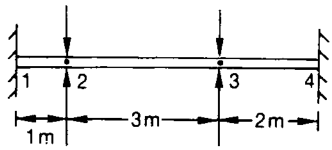
25.14 A continuous beam is fixed at the nodes 1 and 4, and simply-supported at the nodes 2 and 3, as shown in the figure below.

Determine the two lowest resonant frequencies of vibration, given the following:

$E = 2 \times 10^{11}$ N/m^2

$\rho = 7860$ kg/m^3

Element	A (m^2)	I(m^4)
1–2	1×10^{-4}	1×10^{-7}
2–3	2×10^{-4}	2×10^{-7}
3–4	1×10^{-4}	2×10^{-7}



(*Portsmouth 1987*)

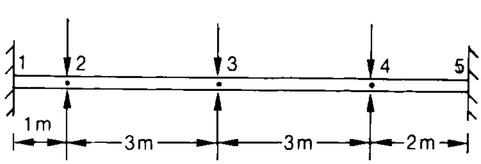
25.15 A continuous beam is fixed at the nodes 1 and 5, and simply-supported at the nodes 2, 3 and 4, as shown below.

Determine the two lowest resonant frequencies of vibration given the following:

$E = 2 \times 10^{11}$ N/m^2

$\rho = 7860$ kg/m^3

Element	A (m^2)	I (m^4)
1–2	1×10^{-4}	1×10^{-7}
2–3	2×10^{-4}	2×10^{-7}
3–4	2×10^{-4}	2×10^{-7}
4–5	1×10^{-4}	1×10^{-7}



(*Portsmouth 1987, Honours*)

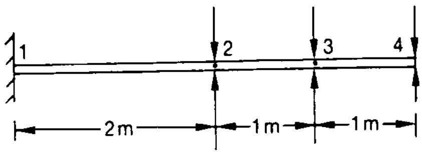
25.16 Calculate the three lowest natural frequencies of vibration for the continuous beam below, where

$A = 0.001$ m^2

$I = 1 \times 10^{-6}$ m^4

$E = 2 \times 10^{11}$ N/m^2

$\rho = 7860$ kg/m^3


1
2
3
4
2m
1m
1m