23 Matrix methods of structural analysis

23.1 Introduction

This chapter describes and applies the matrix displacement method to various problems in structural analysis. The matrix displacement method first appeared in the aircraft industry in the 1940s[7], where it was used to improve the strength-to-weight ratio of aircraft structures.

In today's terms, the structures that were analysed then were relatively simple, but despite this, teams of operators of mechanical, and later electromechanical, calculators were required to implement it. Even in the 1950s, the inversion of a matrix of modest size, often took a few weeks to determine. Nevertheless, engineers realised the importance of the method, and it led to the invention of the finite element method in 1956[8], which is based on the matrix displacement method. Today, of course, with the progress made in digital computers, the matrix displacement method, together with the finite element method, is one of the most important forms of analysis in engineering science.

The method is based on the elastic theory, where it can be assumed that most structures behave like complex elastic springs, the load–displacement relationship of which is linear. Obviously, the analysis of such complex springs is extremely difficult, but if the complex spring is subdivided into a number of simpler springs, which can readily be analysed, then by considering equilibrium and compatibility at the boundaries, or nodes, of these simpler elastic springs, the entire structure can be represented by a large number of simultaneous equations. Solution of the simultaneous equations results in the displacements at these nodes, whence the stresses in each individual spring element can be determined through Hookean elasticity.

In this chapter, the method will first be applied to pin-jointed trusses, and then to continuous beams and rigid-jointed plane frames.

23.2 Elemental stiffness matrix for a rod

A pin-jointed truss can be assumed to be a structure composed of line elements, called rods, which possess only axial stiffness. The joints connecting the rods together are assumed to be in the form of smooth, frictionless hinges. Thus these rod elements in fact behave like simple elastic springs, as described in Chapter 1.

Consider now the rod element of Figure 23.1, which is described by two nodes at its ends, namely, node 1 and node 2.

[7] Levy, S., Computation of Influence Coefficients for Aircraft Structures with Discontinuities and Sweepback, *J. Aero. Sci.*, **14**, 547–560, October 1947.

[8] Turner, M.J., Clough, R.W., Martin, H.C. and Topp, L.J., Stiffness and Deflection Analysis of Complex Structures, *J. Aero. Sci.*, **23**, 805–823, 1956.

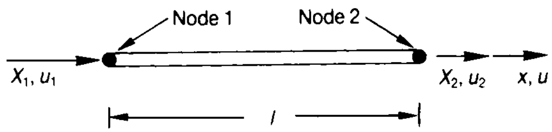


Figure 23.1 Simple rod element.

Let

X_1 = axial force at node 1

X_2 = axial force at node 2

u_1 = axial deflection at node 1

u_2 = axial deflection at node 2

A = cross-sectional area of the rod element

l = elemental length

E = Young's modulus of elasticity

Applying Hooke's law to node 1,

$$\frac{\sigma}{\varepsilon} = E$$

but

$$\sigma = X_1/A$$

and

$$\varepsilon = (u_1 - u_2)/l$$

so that

$$X_1 = AE\,(u_1 - u_2)/l \tag{23.1}$$

From equilibrium considerations

$$X_2 = -X_1 = AE\,(u_2 - u_1)/l \tag{23.2}$$

Rewriting equations (23.1) and (23.2), into matrix form, the following relationship is obtained:

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \frac{AE}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \tag{23.3}$$

or in short form, equation (23.3) can be written

$$\{P_i\} = [\mathbf{k}]\,\{u_i\} \tag{23.4}$$

where,

$$\{P_i\} = \begin{Bmatrix} X_1 \\ \\ X_2 \end{Bmatrix} = \text{a vector of loads}$$

$$\{u_i\} = \begin{Bmatrix} u_1 \\ \\ u_2 \end{Bmatrix} = \text{a vector of nodal displacements}$$

Now, as Force = stiffness × displacement

$$[\mathbf{k}] = \frac{AE}{l} \begin{bmatrix} 1 & -1 \\ \\ -1 & 1 \end{bmatrix} \tag{23.5}$$

= the stiffness matrix for a rod element

23.3 System stiffness matrix [K]

A structure such as pin-jointed truss consists of several rod elements; so to demonstrate how to form the system or structural stiffness matrix, consider the structure of Figure 23.2, which is composed of two in-line rod elements.

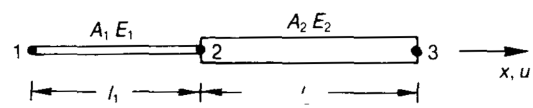


Figure 23.2 Two-element structure.

Consider element 1–2. Then from equation (23.5), the stiffness matrix for the rod element 1–2 is

$$[\mathbf{k}_{1-2}] = \frac{A_1E_1}{l_1}\begin{array}{c} \left[\begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array}\right] \end{array}\begin{array}{c} u_1 \\ u_2 \end{array} \tag{23.6}$$

The element is described as 1–2, which means it points from node 1 to node 2, so that its start node is 1 and its finish node is 2. The displacements u_1 and u_2 are not part of the stiffness matrix, but are used to describe the coefficients of stiffness that correspond to those displacements.

Consider element 2–3. Substituting the values A_2, E_2 and l_2 into equation (23.5), the elemental stiffness matrix for element 2–3 is given by

$$[\mathbf{k}_{2-3}] = \frac{A_2E_2}{l_2}\begin{array}{cc} \begin{array}{cc} u_2 & u_3 \end{array} & \\ \left[\begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array}\right] & \begin{array}{c} u_2 \\ u_3 \end{array} \end{array} \tag{23.7}$$

Here again, the displacements u_2 and u_3 are not part of the stiffness matrix, but are used to describe the components of stiffness corresponding to these displacements.

The system stiffness matrix [K] is obtained by superimposing the coefficients of stiffness of the elemental stiffness matrices of equations (23.6) and (23.7), into a system stiffness matrix of pigeon holes, as shown by equation (23.8):

$$[\mathbf{K}] = \begin{array}{cccc} u_1 & u_2 & u_3 & \\ A_1E_1/l_1 & -A_1E_1/l_1 & 0 & u_1 \\ -A_1E_1/l_1 & A_1E_1/l_1 + A_2E_2/l_2 & -A_2E_2/l_2 & u_2 \\ 0 & -A_2E_2/l_2 & A_2E_2/l_2 & u_3 \end{array} \tag{23.8}$$

It can be seen from equation (23.8), that the components of stiffness are added together with reference to the displacements u_1, u_2 and u_3. This process, effectively mathematically joins together the two springs at their common node, namely node 2.

Let

$$\{q\} = \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix} \tag{23.9}$$

= a vector of known externally applied loads at the nodes, 1, 2 and 3, respectively

$$\{u_i\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \tag{23.10}$$

= a vector of unknown nodal displacements, due to $\{q\}$, at nodes 1, 2 and 3 respectively

Now for the entire structure,

force = stiffness × displacement, or

$$\{q\} = [\mathbf{K}]\ \{u_i\} \tag{23.11}$$

where [**K**] is the system or structural stiffness matrix.

Solution of equation (23.11) cannot be carried out, as [**K**] is singular, i.e. the structure is floating in space and has not been constrained. To constrain the structure of Figure 23.2, let us assume that it is firmly fixed at (say) node 3, so that $u_3 = 0$.

Equation (23.11) can now be partitioned with respect to the free displacements, namely u_1 and u_2, and the constrained displacement, namely u_3, as shown by equation (23.12):

$$\left\{\begin{array}{c} q_F \\ \hline R \end{array}\right\} = \left[\begin{array}{c|c} K_{11} & K_{12} \\ \hline K_{21} & K_{22} \end{array}\right] \left\{\begin{array}{c} u_F \\ \hline u_3 = 0 \end{array}\right\} \tag{23.12}$$

where

$$\{q_F\} = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} \tag{23.13}$$

= a vector of known nodal forces, corresponding to the free displacements, namely u_1 and u_2

$$\{u_F\} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \tag{23.14}$$

= a vector of free displacements, which have to be determined

$$[\mathbf{K}_{11}] = \begin{bmatrix} A_1E_1/l_1 & -AE_1/l_1 \\ -AE_1/l_1 & (A_1E_1/l_1 + A_2E_2/l_2) \end{bmatrix} \tag{23.15}$$

= that part of the system stiffness matrix that corresponds to the free displacements, which in this case is u_1 and u_2

$\{R\}$ = a vector of reactions corresponding to the constrained displacements, which in this case is u_3

$[\mathbf{K}_{22}] = [A_2\,E_2\,/\,l_2]$

$[\mathbf{K}_{21}] = \left[0 - A_2\,E_2\,/\,l_2\right]$

in this case

$$[\mathbf{K}_{12}] = \begin{bmatrix} 0 \\ -A_2\,E_2\,/\,l_2 \end{bmatrix}$$

Expanding the top part of equation (23.12):

$$\begin{aligned} \{q_F\} &= [\mathbf{K}_{11}]\,\{u_F\} \\ \therefore \{u_F\} &= [\mathbf{K}_{11}]^{-1}\,\{q_F\} \end{aligned} \tag{23.16}$$

Once $\{u_F\}$ is determined, the initial stresses can be determined through Hookean elasticity.

For some cases u_3 may not be zero but may have a known value, say u_c. For these cases, equation (23.12) becomes

$$\begin{Bmatrix} q_F \\ \hline R \end{Bmatrix} = \left[\begin{array}{c|c} K_{11} & K_{12} \\ \hline K_{21} & K_{22} \end{array}\right] \begin{Bmatrix} u_F \\ \hline u_c \end{Bmatrix} \tag{23.17}$$

so that

$$\{u_F\} = [\mathbf{K}_{11}]^{-1}\,(\{q_F\} - [\mathbf{K}_{12}]\,\{u_c\}) \tag{23.18}$$

and

$$\{R\} = [K_{21}]\{u_F\} + [K_{22}]\{u_c\} \tag{23.19}$$

23.4 Relationship between local and global co-ordinates

The rod element of Figure 23.1 is not very useful element because it lies horizontally, when in fact a typical rod element may lie at some angle to the horizontal, as shown in Figures 23.3 and 23.4, where the x–$y°$ axes are the global axes and the x–y axes are the local axes.

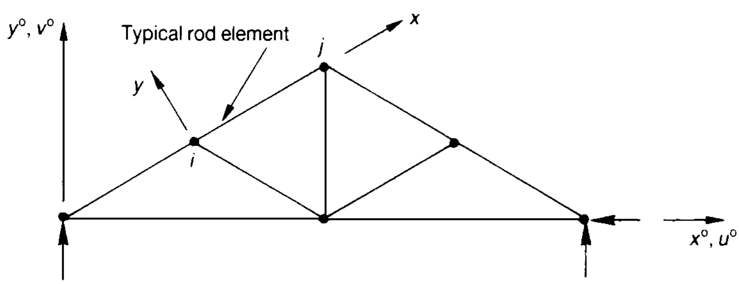


Figure 23.3 Plane pin-jointed truss.

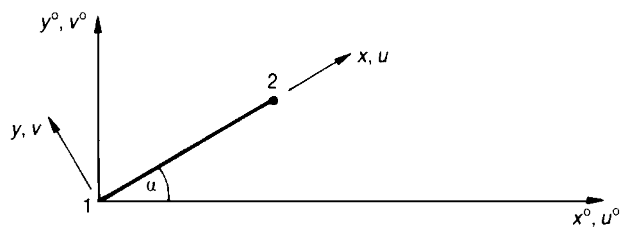


Figure 23.4 Rod element, shown in local and global systems.

From Figure 23.4, it can be seen that the relationships between the local displacements u and v, and the global displacements $u°$ and $v°$, are given by equation (23.20):

$$
\begin{aligned}
u &= u^{\circ}\cos\alpha + v^{\circ}\sin\alpha \\
v &= -u^{\circ}\sin\alpha + v^{\circ}\cos\alpha
\end{aligned}
\tag{23.20}
$$

which, when written in matrix form, becomes:

$$
\left\{\frac{u}{v}\right\} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \left\{\frac{u^{\circ}}{v^{\circ}}\right\}
\tag{23.21}
$$

For node 1,

$$
\left\{\frac{u_1}{v_1}\right\} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{Bmatrix} u_1{}^{\circ} \\ v_1{}^{\circ} \end{Bmatrix}
\tag{23.22}
$$

where,

$$c = \cos\alpha$$

$$s = \sin\alpha$$

Similarly, for node 2

$$
\left\{\frac{u_2}{v_2}\right\} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{Bmatrix} u_2{}^{\circ} \\ v_2{}^{\circ} \end{Bmatrix}
$$

Or, for both nodes,

$$
\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} = \left[\begin{array}{c|c} \zeta & 0_2 \\ \hline 0_2 & \zeta \end{array}\right] \begin{Bmatrix} u_1{}^{\circ} \\ v_1{}^{\circ} \\ u_2{}^{\circ} \\ v_2{}^{\circ} \end{Bmatrix}
\tag{23.23}
$$

where,

$$[\zeta] = \begin{bmatrix} c & s \\ -s & c \end{bmatrix}$$

$$[\mathbf{0_2}] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Equation (23.23) can be written in the form:

$$\{u_i\} = [\mathbf{DC}]\,\{u_i{}^\circ\} \tag{23.24}$$

where,

$$[\mathbf{DC}] = \left[\begin{array}{c|c} \zeta & \mathbf{0_2} \\ \hline \mathbf{0_2} & \zeta \end{array}\right] \tag{23.25}$$

= a matrix of directional cosines

$$\{u_i\} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

$$\{u_i{}^\circ\} = \begin{Bmatrix} u_1{}^\circ \\ v_1{}^\circ \\ u_2{}^\circ \\ v_2{}^\circ \end{Bmatrix}$$

From equation (23.25), it can be seen that [**DC**] is orthogonal, i.e.

$$[\mathbf{DC}]^{-1} = [\mathbf{DC}]^{\mathrm{T}}$$

$$\therefore \{u_i{}^\circ\} = [\mathbf{DC}]^{\mathrm{T}}\,\{u_i\} \tag{23.26}$$

Similarly, it can be shown that

$$\{P_i\} = [\mathbf{DC}]\,\{P_i{}^\circ\} \tag{23.27}$$

and $$\{P_i{}^\circ\} = [\mathbf{DC}]^{\mathrm{T}}\,\{P_i\}$$

where

$$\{P_i\} = \begin{Bmatrix} X_1 \\ Y_1 \\ X_2 \\ Y_2 \end{Bmatrix}$$

and

$$\{P_i{}^\circ\} = \begin{Bmatrix} X_1{}^\circ \\ Y_1{}^\circ \\ X_2{}^\circ \\ Y_2{}^\circ \end{Bmatrix}$$

23.5 Plane rod element in global co-ordinates

For this case, there are four degrees of freedom per element, namely $u_1{}^\circ$, $v_1{}^\circ$, $u_2{}^\circ$ and $v_2{}^\circ$. Thus, the elemental stiffness matrix for a rod in local co-ordinates must be written as a 4 × 4 matrix, as shown by equation (23.28):

$$[\mathrm{k}] = \frac{AE}{l}\begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_2 & v_2 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ u_1 \\ v_1 \\ u_2 \\ v_2 \end{array} \tag{23.28}$$

The reason why the coefficients of the stiffness matrix under v_1 and v_2 are zero, is that the rod only possesses axial stiffness in the local x-direction, as shown in Figure 23.1.

For the inclined rod of Figure 23.4, although the rod only possesses stiffness in the x-direction, it has components of stiffness in the global $x°$- and $y°$-directions.

The elemental stiffness matrix for a rod in global co-ordinates is obtained, as follows. From equation (23.4):

$$\{P_i\} = [\mathbf{k}]\,\{u_i\} \tag{23.29}$$

but

$$\{P_i\} = [\mathbf{DC}]\,\{P_i°\} \tag{23.30}$$

and

$$\{u_i\} = [\mathbf{DC}]\,\{u_i°\} \tag{23.31}$$

Substituting equations (23.30) and (23.31) into equation (23.29), the following is obtained:

$$[\mathbf{DC}]\,\{P_i°\} = [\mathbf{k}]\,[\mathbf{DC}]\,\{u_i°\} \tag{23.32}$$

Premultiplying both sides by $[\mathbf{DC}]^{-1}$,

$$\{P_i°\} = [\mathbf{DC}]^{-1}\,[\mathbf{k}]\,[\mathbf{DC}]\,\{u_i°\}$$

but from equation (22.28),

$$\begin{aligned} [\mathbf{DC}]^{-1} &= [\mathbf{DC}]^{\mathbf{T}} \\ \therefore \{P_i°\} &= [\mathbf{DC}]^{\mathbf{T}}\,[\mathbf{k}]\,[\mathbf{DC}]\,\{u_i°\} \end{aligned} \tag{23.33}$$

Now,

force = stiffness × deflection

$$\therefore \{P_i°\} = [\mathbf{k}°]\,\{u_i°\} \tag{23.34}$$

Comparing equation (23.34) with (23.33),

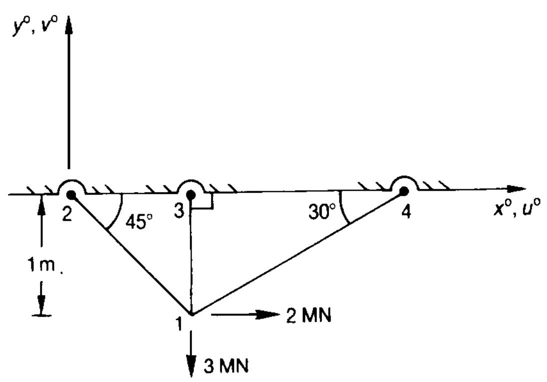
$$[\mathbf{k}^\circ] = [\mathbf{DC}]^T [\mathbf{k}] [\mathbf{DC}] \tag{23.35}$$

= elemental stiffness matrix in global co-ordinates

$$[\mathbf{k}^\circ] = \frac{AE}{l} \begin{array}{c} \begin{array}{cccc} u_1^\circ & v_1^\circ & u_2^\circ & v_2^\circ \end{array} \\ \begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix} \end{array} \begin{array}{c} u_1^\circ \\ v_1^\circ \\ u_2^\circ \\ v_2^\circ \end{array} \tag{23.36}$$

= the elemental stiffness matrix for a rod in global co-ordinates

Problem 23.1 The plane pin-jointed truss below may be assumed to be composed of uniform section members, with the same material properties. If the truss is subjected to the load shown, determine the forces in the members of the truss.

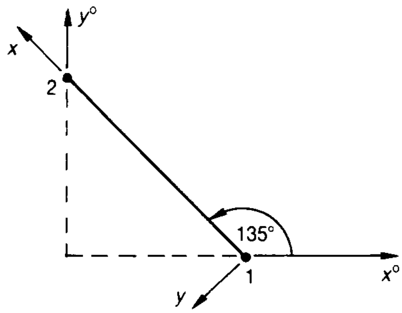


Solution

This truss has two free degrees of freedom, namely, the unknown displacements u_1° and v_1°.

Element 1–2

This element points from 1 to 2, so that its start node is 1 and its end node is 2, as shown:



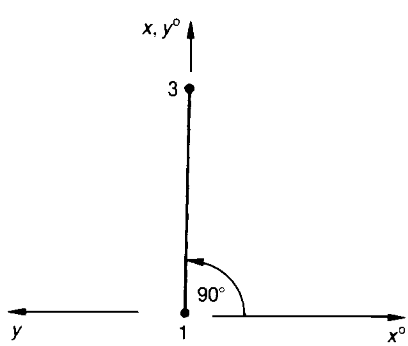
$\alpha = 135°$ $\quad \therefore c = -0.707, \quad s = 0.707, \quad l = 1.414 \text{ m}$

Substituting the above information into equation (23.36), and removing the rows and columns corresponding to the zero displacements, namely $u_2°$ and $v_2°$, the elemental stiffness matrix for element 1–2 is given by

$$\left[\mathbf{k}_{1\text{-}2}°\right] = \frac{AE}{1.414} \begin{array}{c} \begin{array}{cccc} u_1° & v_1° & u_2° & v_2° \end{array} \\ \left[\begin{array}{cccc} 0.5 & -0.5 & & \\ -0.5 & 0.5 & & \\ & & & \\ & & & \end{array}\right] \end{array} \begin{array}{c} \\ u_1° \\ v_1° \\ u_2° \\ v_2° \end{array} \tag{23.37}$$

Element 1–3

This member points from 1 to 3, so that its start node is 1 and its end node is 3, as shown below.



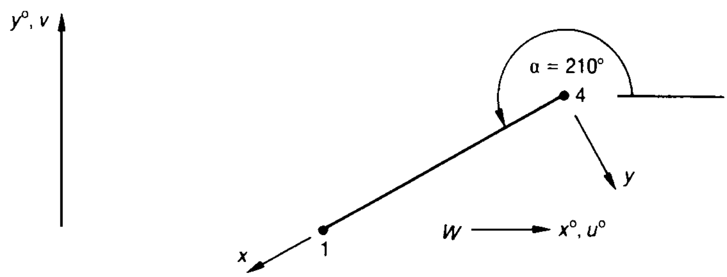
$\alpha = 90°, \quad c = 0, \quad s = 1, \quad l = 1 \text{ m}$

Substituting the above values into equation (23.36) and removing the rows and columns corresponding to the zero displacements, namely u_3° and v_3°, the elemental stiffness matrix for element 1–3 is given by:

$$
\left[\mathbf{k}_{1\text{-}3}^\circ\right] = \frac{AE}{1}
\begin{array}{c}
\begin{array}{cccc} u_1^\circ & v_1^\circ & u_3^\circ & v_3^\circ \end{array} \\
\left[\begin{array}{cccc} 0 & 0 & & \\ 0 & 1 & & \\ & & & \\ & & & \end{array}\right]
\end{array}
\begin{array}{c} u_1^\circ \\ v_1^\circ \\ u_3^\circ \\ v_3^\circ \end{array}
\qquad (23.38)
$$

Element 4 – 1

This element points from 4 to 1, so that its start node is 4 and its end node is 1, as shown:



$$\alpha = 210^\circ$$

or

$$\alpha = -150^\circ$$

$$c = -0.866$$

$$s = -0.5$$

$$l = 2$$

Substituting the above information into equation (23.36), and removing the rows and columns corresponding to the zero displacements, which in this case are u_4° and v_4°, the elemental stiffness matrix is given by

$$[\mathbf{k}_{4-1}{}^{\circ}] = \frac{AE}{2} \begin{array}{c} \begin{array}{cccc} u_4{}^{\circ} & v_4{}^{\circ} & u_1{}^{\circ} & v_1{}^{\circ} \end{array} \\ \left[\begin{array}{cccc} & & & \\ & & & \\ & & 0.75 & 0.433 \\ & & 0.433 & 0.25 \end{array}\right] \end{array} \begin{array}{c} u_4{}^{\circ} \\ v_4{}^{\circ} \\ u_1{}^{\circ} \\ v_1{}^{\circ} \end{array} \tag{23.39}$$

The system stiffness matrix corresponding to the free displacements, namely $u_1{}^{\circ}$ and $v_1{}^{\circ}$, is given by adding together the appropriate coefficients of equations (23.37) to (23.39), as shown by equation (23.40):

$$[\mathbf{K}_{11}] = AE \begin{array}{c} \begin{array}{cc} u_1{}^{\circ} & v_1{}^{\circ} \end{array} \\ \left[\begin{array}{c|c} 0.354 + 0 & -0.354 + 0 \\ +0.375 & +0.217 \\ \hline -0.354 + 0 & 0.354 + 1 \\ +0.217 & +0.125 \end{array}\right] \end{array} \begin{array}{c} u_1{}^{\circ} \\ \\ v_1{}^{\circ} \end{array} \tag{23.40}$$

or

$$[\mathbf{K}_{11}] = AE \begin{array}{c} \begin{array}{cc} u_1{}^{\circ} & v_1{}^{\circ} \end{array} \\ \begin{bmatrix} 0.729 & -0.137 \\ -0.137 & 1.479 \end{bmatrix} \end{array} \begin{array}{c} u_1{}^{\circ} \\ v_1{}^{\circ} \end{array} \tag{23.41}$$

NB $[\mathbf{K}_{11}]$ is of order two, as it corresponds to the two free displacements $u_1{}^{\circ}$ and $v_1{}^{\circ}$, which are unknown.

The vector of external loads $\{q_F\}$, corresponds to the two free displacements $u_1{}^{\circ}$ and $v_1{}^{\circ}$, and can readily be shown to be given by equation (23.42), ie

$$\{qF\} = \begin{Bmatrix} 2 \\ -3 \end{Bmatrix} \begin{array}{c} u_1{}^{\circ} \\ v_1{}^{\circ} \end{array} \tag{23.42}$$

where the load value 2 is in the $u_1{}^{\circ}$ direction, and the load value -3 is in the $v_1{}^{\circ}$ direction.

Substituting equations (23.41) and (23.42) into equation (23.16)

$$\{u_F{}^{\circ}\} = \begin{Bmatrix} u_1{}^{\circ} \\ v_1{}^{\circ} \end{Bmatrix} = [\mathbf{K}_{11}]^{-1} \begin{Bmatrix} 2 \\ -3 \end{Bmatrix}$$

$$= \frac{\dfrac{1}{AE} \begin{bmatrix} 1.479 & 0.137 \\ 0.137 & 0.729 \end{bmatrix} \begin{Bmatrix} 2 \\ -3 \end{Bmatrix}}{(0.729 \times 1.479 - 0.137 \times 0.137)}$$

$$= \frac{1}{AE} \begin{bmatrix} 1.396 & 0.129 \\ 0.129 & 0.688 \end{bmatrix} \begin{Bmatrix} 2 \\ -3 \end{Bmatrix}$$

i.e.

$$\begin{Bmatrix} u_1{}^{\circ} \\ v_1{}^{\circ} \end{Bmatrix} = \frac{1}{AE} \begin{Bmatrix} 2.405 \\ -1.806 \end{Bmatrix} \tag{23.43}$$

These displacements are in global co-ordinates, so it will be necessary to resolve these displacements along the length of each rod element, to discover how much each rod extends or contacts along its length, and then through the use of Hookean elasticity to obtain the internal forces in each element.

Element 1– 2

Now,

$$c = -0.707, \quad s = 0.707 \quad \text{and} \quad l = 1.414 \text{ m}$$

Hence, from equation (23.23),

$$u_1 = [c \quad s] \begin{Bmatrix} u_1{}^{\circ} \\ v_1{}^{\circ} \end{Bmatrix}$$

$$= [-0.707 \quad 0.707] \frac{1}{AE} \begin{Bmatrix} 2.405 \\ -1.806 \end{Bmatrix}$$

$$u_1 = -2.977/AE$$

From Hooke's law,

$$F_{1-2} = \text{force in element 1–2}$$

$$= \frac{AE}{l}(u_2 - u_1)$$

$$= \frac{2.977}{1.414}$$

$$F_{1-2} = 2.106 \text{ MN (tension)}$$

Element 1–3

$$c = 0, \qquad s = 1 \quad \text{and} \quad l = 1 \text{ m}$$

From equation (23.23),

$$u_1 = [c \quad s] \begin{Bmatrix} u_1{}^{\circ} \\ v_1{}^{\circ} \end{Bmatrix}$$

$$= [0 \quad 1] \frac{1}{AE} \begin{Bmatrix} 2.405 \\ -1.806 \end{Bmatrix}$$

$$u_1 = -1.806/AE$$

From Hooke's law,

$$F_{1-3} = \text{force in element 1–3}$$

$$= \frac{AE}{l}(u_3 - u_1)$$

$$F_{1-3} = 1.806 \text{ MN (tension)}$$

Element 4–1

$$c = -0.866, \qquad s = 0.5 \quad \text{and} \quad l = 2 \text{ m}$$

From equation (23.23),

$$u_1 = [c \quad s]\begin{Bmatrix} u_1^\circ \\ v_1^\circ \end{Bmatrix}$$

$$= [-0.866 \quad 0.5]\frac{1}{AE}\begin{Bmatrix} 2.405 \\ -1.806 \end{Bmatrix}$$

$$u_1 = -1.1797 \,/\, AE$$

From Hooke's law,

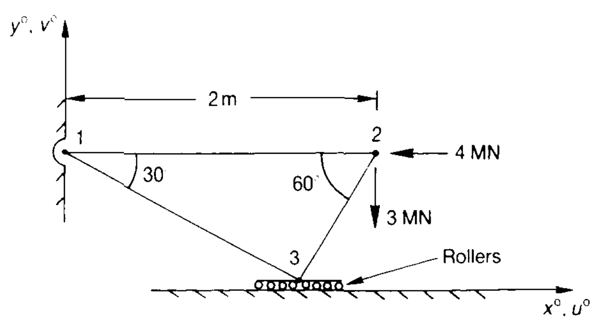
$$F_{4-1} = \text{force in element } 1\text{–}4$$

$$= \frac{AE}{l}(u_1 - u_4)$$

$$= \frac{AE}{2}\,\frac{(-1.1797 - 0)}{AE}$$

$$F_{4-1} = -0.59 \text{ MN (compression)}$$

Problem 23.2 Using the matrix displacement method, determine the forces in the members of the plane pin-jointed truss below, which is free to move horizontally at node 3, but not vertically. It may also be assumed that the truss is firmly pinned at node 1, and that the material and geometrical properties of its members are given in the table below.



Member	A	E
1–2	$2A$	E
1–3	A	$3E$
2–3	$3A$	$2E$

<u>*Solution*</u>

Element 1–2

$$\alpha = 0, \qquad c = 1, \qquad s = 0 \quad \text{and} \quad l = 2\text{ m}$$

Substituting the above values into equation (23.36),

$$\left[\mathbf{k}_{1\text{-}2}{}^{\circ}\right] = \frac{2AE}{2}\;\begin{array}{c} \begin{array}{cccc} u_1{}^{\circ} & v_1{}^{\circ} & u_2{}^{\circ} & v_2{}^{\circ} \end{array} \\ \left[\begin{array}{cccc} & & & \\ & & & \\ & & 1 & 0 \\ & & 0 & 0 \end{array}\right] \end{array}\begin{array}{c} \\ u_1{}^{\circ} \\ v_1{}^{\circ} \\ u_2{}^{\circ} \\ v_2{}^{\circ} \end{array} \tag{23.44}$$

Element 2–3

$$\alpha = 240°, \quad c = -0.5, \quad s = -0.866 \quad \text{and} \quad l = 1\text{ m}$$

$$\left[\mathrm{k}_{1\text{-}3}{}^{\circ}\right] = \frac{3A \times 2E}{1}\;\begin{array}{c} \begin{array}{cccc} u_2{}^{\circ} & v_2{}^{\circ} & u_3{}^{\circ} & v_3{}^{\circ} \end{array} \\ \left[\begin{array}{cccc} 0.25 & 0.433 & -0.25 & \\ 0.433 & 0.75 & -0.433 & \\ -0.25 & -0.433 & 0.25 & \\ & & & \end{array}\right] \end{array}\begin{array}{c} \\ u_2{}^{\circ} \\ v_2{}^{\circ} \\ u_3{}^{\circ} \\ v_3{}^{\circ} \end{array}$$

$$= AE\;\begin{array}{c} \begin{array}{ccc} u_2{}^{\circ} & v_2{}^{\circ} & u_3{}^{\circ} \end{array} \\ \left[\begin{array}{ccc} 1.5 & 2.6 & -1.5 \\ 2.6 & 4.5 & -2.6 \\ -1.5 & -2.6 & 1.5 \end{array}\right] \end{array}\begin{array}{c} \\ u_2{}^{\circ} \\ v_2{}^{\circ} \\ u_3{}^{\circ} \end{array} \tag{23.45}$$

Element 3–1

$\alpha = 150°, \quad c = -0.866, \qquad s = 0.5 \qquad \text{and } l = 1.732\text{ m}$

$$
\left[\mathbf{k}_{3\text{-}1}{}^{\circ}\right] = \frac{A \times 3E}{1.732}
\begin{array}{c}
\begin{array}{cccc} u_3{}^{\circ} & v_3{}^{\circ} & u_1{}^{\circ} & v_1{}^{\circ} \end{array} \\
\left[\begin{array}{cccc} 0.75 & & & \\ & & & \\ & & & \\ & & & \end{array}\right]
\end{array}
\begin{array}{c} \\ u_3{}^{\circ} \\ v_3{}^{\circ} \\ u_1{}^{\circ} \\ v_1{}^{\circ} \end{array}
$$

$$
\begin{array}{c} u_3{}^{\circ} \\ = \left[1.3\right] u_3{}^{\circ} \end{array} \tag{23.46}
$$

The system stiffness matrix $[\mathbf{K}_{11}]$ is obtained by adding together the appropriate components of stiffness, from the elemental stiffness matrices of equations (23.44) to (23.46), with reference to the free degrees of freedom, namely, $u_2{}^{\circ}$, $v_2{}^{\circ}$ and $u_3{}^{\circ}$, as shown by equation (23.47):

$$
[\mathbf{K}_{11}] = AE
\begin{array}{c}
\begin{array}{ccc} u_2{}^{\circ} & v_2{}^{\circ} & u_3{}^{\circ} \end{array} \\
\left[\begin{array}{c|c|c} 1 + 1.5 & 0 + 2.6 & -1.5 \\ \hline 0 + 2.6 & 0 + 4.5 & -2.6 \\ \hline -1.5 & -2.6 & 1.5 + 1.3 \end{array}\right]
\end{array}
\begin{array}{c} \\ u_2{}^{\circ} \\ v_2{}^{\circ} \\ u_3{}^{\circ} \end{array} \tag{23.47}
$$

$$
= AE
\begin{array}{c}
\begin{array}{ccc} u_2{}^{\circ} & v_2{}^{\circ} & u_3{}^{\circ} \end{array} \\
\left[\begin{array}{ccc} 2.5 & 2.6 & -1.5 \\ 2.6 & 4.5 & -2.6 \\ -1.5 & -2.6 & 2.8 \end{array}\right]
\end{array}
\begin{array}{c} \\ u_2{}^{\circ} \\ v_2{}^{\circ} \\ u_3{}^{\circ} \end{array} \tag{23.48}
$$

The vector of loads $\{q_F\}$, corresponding to the free degrees of freedom, namely, $u_2{}^{\circ}$, $v_2{}^{\circ}$ and $u_3{}^{\circ}$ is given by:

$$\{q_F\} = \begin{Bmatrix} -4 \\ -3 \\ 0 \end{Bmatrix} \begin{matrix} u_2{}^\circ \\ v_2{}^\circ \\ u_3{}^\circ \end{matrix} \tag{23.49}$$

Substituting equations (23.48) and (23.49) into equation (23.16) and solving, the vector of free displacements $\{u_F\}$ is given by

$$\begin{Bmatrix} u_2{}^\circ \\ v_2{}^\circ \\ u_3{}^\circ \end{Bmatrix} = \frac{1}{AE} \begin{Bmatrix} -2.27 \\ -0.125 \\ -1.332 \end{Bmatrix} \tag{23.50}$$

The member forces will be obtained by resolving these displacements along the length of each rod element, and then by finding the amount that each rod extends or contracts, to determine the force in each member through Hookean elasticity.

Element 1–2

$$c = 1, \quad s = 0 \quad \text{and} \quad l = 2\text{ m}$$

From equation (23.23),

$$u_2 = \begin{bmatrix} c & s \end{bmatrix} \begin{Bmatrix} u_2{}^\circ \\ v_2{}^\circ \end{Bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{1}{AE} \begin{Bmatrix} -2.27 \\ -0.125 \end{Bmatrix}$$

$$u_2 = -2.27/AE$$

From Hooke's law,

F_{1-2} = force in element 1–2

$$= \frac{2AE}{2}\left(-\frac{2.27}{AE} - 0 \right)$$

$$F_{1-2} = -2.27 \text{ MN (compression)}$$

Element 2–3

$$c = -0.5, \quad s = -0.866 \quad \text{and} \quad l \quad 1\text{ m}$$

From equation (23.23),

$$u_2 = [c \quad s]\begin{Bmatrix} u_2^{\circ} \\ v_2^{\circ} \end{Bmatrix}$$

$$= [-0.5 \quad -0.866]\,\frac{1}{AE}\begin{Bmatrix} -2.27 \\ -0.125 \end{Bmatrix}$$

$$u_2 = 1.243/AE$$

Similarly, from equation (23.23),

$$u_3 = [c \quad s]\begin{Bmatrix} u_3^{\circ} \\ v_3^{\circ} \end{Bmatrix}$$

$$= [-0.5 \quad -0.866]\,\frac{1}{AE}\begin{Bmatrix} -1.332 \\ 0 \end{Bmatrix}$$

$$u_3 = 0.666/AE$$

From Hooke's law,

$$F_{2-3} = \text{force in element 2–3}$$

$$= \frac{3A \times 2E}{1}(u_3 - u_2)$$

$$= 6AE \times \frac{(-0.577)}{AE}$$

$$F_{2-3} = -3.46 \text{ MN (compression)}$$

Element 3–1

$$c = -0.866, \quad s = 0.5 \quad \text{and} \quad l = 1.732 \text{ m}$$

$$u_3 = \begin{bmatrix} c & s \end{bmatrix} \begin{Bmatrix} u_3^{\circ} \\ v_3^{\circ} \end{Bmatrix}$$

$$= \begin{bmatrix} -0.866 & 0.5 \end{bmatrix} \frac{1}{AE} \begin{Bmatrix} -1.332 \\ 0 \end{Bmatrix}$$

$$u_3 = 1.154/AE$$

From Hooke's law,

$$F_{3-1} = \text{force in element 1–3}$$

$$= \frac{A \times 3E}{1.732}\left(0 - \frac{1.154}{AE}\right)$$

$$F_{3-1} = -2 \text{ MN (compression)}$$

23.6 Pin-jointed space trusses

In three dimensions, the relationships between forces and displacements for the rod element of Figure 23.5 are given by equation (23.51):

$$\begin{Bmatrix} X_1 \\ Y_1 \\ Z_1 \\ X_2 \\ Y_2 \\ Z_2 \end{Bmatrix} = \frac{AE}{l} \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \end{Bmatrix} \tag{23.51}$$

where,

X_1 = load in the x direction at node 1

$= AE(u_1 - u_2)/l$

Y_1 = load in the y direction at node 1

$= 0$

Z_1 = load in the z direction at node 1

$= 0$

X_2 = load in the x direction at node 2

$= AE(u_2 - u_1)/l$

Y_2 = load in the y direction at node 2

$= 0$

Z_2 = load in the z direction at node 2

$= 0$

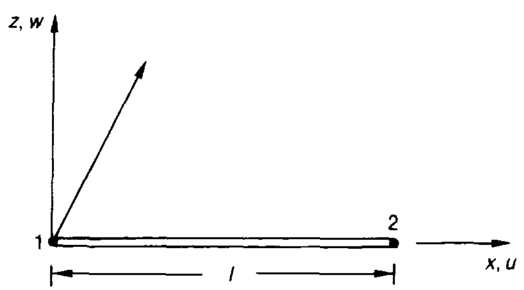


Figure 23.5 Three-dimensional rod in local co-ordinates.

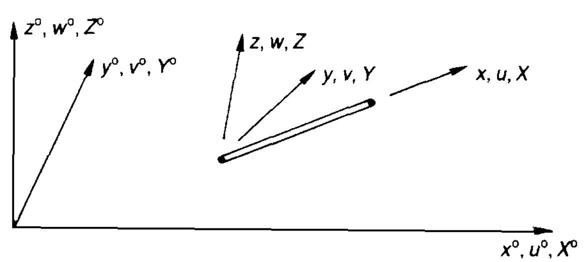


Figure 23.6 Rod in three dimensions.

For the case of the three dimensional rod in the global co-ordinate system of Figure 23.6, it can be shown through resolution that the relationship between local loads and global loads is given by:

$$\begin{Bmatrix} X_1 \\ Y_1 \\ Z_1 \\ X_2 \\ Y_2 \\ Z_2 \end{Bmatrix} = \begin{bmatrix} \zeta & 0_3 \\ 0_3 & \zeta \end{bmatrix} \begin{Bmatrix} X_1{}^\circ \\ Y_1{}^\circ \\ Z_1{}^\circ \\ X_2{}^\circ \\ Y_2{}^\circ \\ Z_2{}^\circ \end{Bmatrix} \tag{23.52}$$

where

$$[\zeta] = \begin{bmatrix} C_{x,x^\circ} & C_{x,y^\circ} & C_{x,z^\circ} \\ C_{y,x^\circ} & C_{y,y^\circ} & C_{y,z^\circ} \\ C_{z,x^\circ} & C_{z,y^\circ} & C_{z,z} \end{bmatrix} \tag{23.53}$$

x, y, z	=	local axes
$x^\circ, y^\circ, z^\circ$	=	global axes
$C_{x,x}$, $C_{x,y}$, $C_{x,z}$, etc	=	the directional cosines of x with x°, x with y°, x with z°, respectively, etc.
$X_1{}^\circ$	=	force in x° direction at node 1
$Y_1{}^\circ$	=	force in y° direction at node 1
$Z_1{}^\circ$	=	force in z° direction at node 1
$X_2{}^\circ$	=	force in x° direction at node 2
$Y_2{}^\circ$	=	force in y° direction at node 2
$Z_2{}^\circ$	=	force in z° direction at node 2

Now from equation (23.35) the elemental stiffness matrix for a rod in global co-ordinates is given by:

$$[\mathbf{k}^\circ] = [\mathbf{DC}]^{\mathrm{T}} \, [\mathbf{k}] \, [\mathbf{DC}]$$

$$= \begin{bmatrix} \zeta & 0_3 \\ 0_3 & \zeta \end{bmatrix}^T [\mathbf{k}] \begin{bmatrix} \zeta & 0_3 \\ 0_3 & \zeta \end{bmatrix} \tag{23.54}$$

$$[\mathbf{k}^\circ] = \begin{bmatrix} a & -a \\ -a & a \end{bmatrix}$$

where

$$[\mathbf{a}] = \frac{AE}{l} \begin{bmatrix} C^2_{x,x^\circ} & C_{x,x^\circ} C_{x,y^\circ} & C_{x,x^\circ} C_{x,z^\circ} \\ C_{x,x^\circ} C_{x,y^\circ} & C^2_{x,y^\circ} & C_{x,y^\circ} C_{x,z^\circ} \\ C_{x,x^\circ} C_{x,z^\circ} & C_{x,y^\circ} C_{x,z^\circ} & C^2_{x,z^\circ} \end{bmatrix} \tag{23.55}$$

By Pythagoras' theorem in three dimensions:

$$l = \left[(x_2{}^\circ - x_1{}^\circ)^2 + (y_2{}^\circ - y_1{}^\circ)^2 + (z_2{}^\circ - z_1{}^\circ)^2 \right]^{\frac{1}{2}} \tag{23.56}$$

The directional cosines[9] can readily be shown to be given by equation (23.57):

$$\begin{aligned} C_{x,x}{}^\circ &= (x_2{}^\circ - x_1{}^\circ)/l \\ C_{x,y}{}^\circ &= (y_2{}^\circ - y_1{}^\circ)/l \\ C_{x,z}{}^\circ &= (z_2{}^\circ - z_1{}^\circ)/l \end{aligned} \tag{23.57}$$

Problem 23.3 A tripod, with pinned joints, is constructed from three uniform section members, made from the same material. If the tripod is firmly secured to the ground at nodes 1 to 3, and loaded at node 4, as shown below, determine the forces in the members of the tripod, using the matrix displacement method.

[9]Ross, C T F, *Advanced Applied Element Methods*, Horwood, 1998.

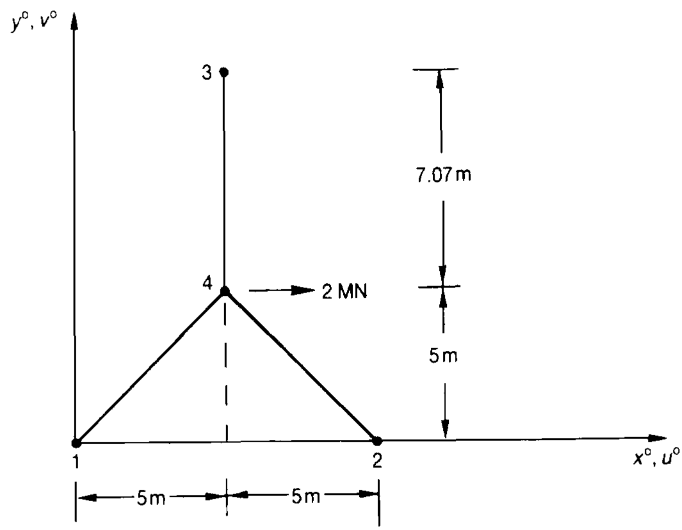
<u>Solution</u>

Element 1–4

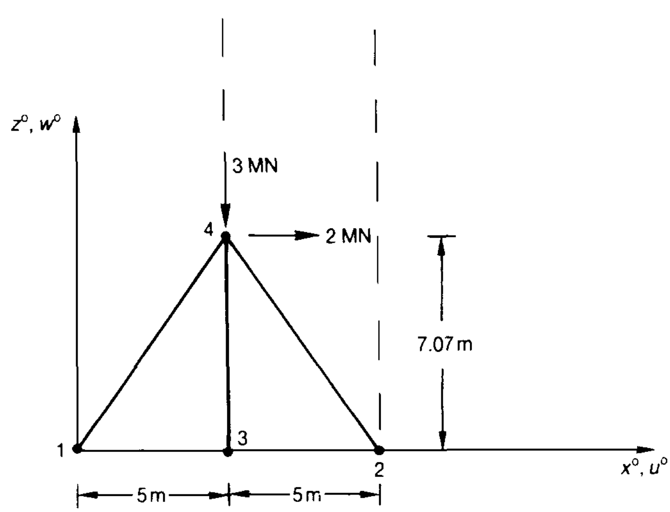
The element points from 1 to 4, so that the start node is 1 and the finish node is 4. From the figure below it can readily be seen that:

$x_1^\circ = 0, \qquad y_1^\circ = 0. \qquad z_1^\circ = 0,$

$z_4^\circ = 5\text{ m}, \qquad y_4^\circ = 5\text{ m}, \qquad z_4^\circ = 7.07\text{ m}$



(a) Plan view of the tripod.



(b) Front view of tripod.

Substituting the above into equation (23.56),

$$l = \left[(5-0)^2 + (5-0)^2 + (7.07-0)^2\right]^{\frac{1}{2}}$$

$$l = 10 \text{ m}$$

Substituting the above into equation (23.57),

$$C_{x,x}{}^\circ = \frac{x_4{}^\circ - x_1{}^\circ}{l} = \frac{5-0}{10} = 0.5$$

$$C_{x,y}{}^\circ = \frac{y_4{}^\circ - y_1{}^\circ}{l} = \frac{5-0}{10} = 0.5$$

$$C_{x,z}{}^\circ = \frac{z_4{}^\circ - z_1{}^\circ}{l} = \frac{7.07-0}{10} = 0.707$$

Substituting the above values into equation (23.54), and removing the coefficients of the stiffness matrix corresponding to the zero displacements, which in this case are $u_1{}^\circ$, $v_1{}^\circ$ and $w_1{}^\circ$, the stiffness matrix for element 1–4 is given by equation 23.58):

$$\left[\mathbf{k}_{1-4}{}^\circ\right] = \frac{AE}{10} \begin{array}{c} \begin{array}{cccccc} u_1{}^\circ & v_1{}^\circ & w_1{}^\circ & u_4{}^\circ & v_4{}^\circ & w_4{}^\circ \end{array} \\ \left[\begin{array}{cccccc} & & & & & \\ & & & & & \\ & & & & & \\ & & & 0.25 & & \\ & & & 0.25 & 0.25 & \\ & & & 0.354 & 0.354 & 0.5 \end{array}\right] \end{array} \begin{array}{c} u_1{}^\circ \\ v_1{}^\circ \\ w_1{}^\circ \\ u_4{}^\circ \\ v_4{}^\circ \\ w_4{}^\circ \end{array} \qquad (23.58)$$

Element 2–4

The member points from 2 to 4, so that the start node is 2 and the finish node is 4. From the above figure,

$$x_2{}^\circ = 10, \qquad y_2{}^\circ = 0, \qquad z_2{}^\circ = 0$$

Substituting the above and $x_4°$, $y_4°$ and $z_4°$ into equation (23.56),

$$l = \left[(5 - 10)^2 + (5 - 0)^2 + (7.07 - 0)^2\right]^{\frac{1}{2}}$$

$$l = 10 \text{ m}$$

From equation (23.57),

$$C_{x,x}° = \frac{x_4° - x_2°}{l} = \frac{5 - 10}{10} = -0.5$$

$$C_{x,y}° = \frac{y_4° - y_2°}{l} = \frac{5 - 0}{10} = 0.5$$

$$C_{x,z}° = \frac{z_4° - z_2°}{l} = \frac{7.07 - 0}{10} = 0.707$$

Substituting the above values into equation (23.54), and removing the rows and columns corresponding to the zero displacements, which in this case are $u_2°$, $v_2°$ and $w_2°$, the stiffness matrix for element 2–4 is given by equation (23.59):

$$\left[\mathbf{k}_{2-4}°\right] = \frac{AE}{10} \begin{array}{c} \begin{array}{cccccc} u_2° & v_2° & w_2° & u_4° & v_4° & w_4° \end{array} \\ \left[\begin{array}{cccccc} & & & & & \\ & & & & & \\ & & & & & \\ & & & 0.25 & & \\ & & & -0.25 & 0.25 & \\ & & & -0.354 & 0.354 & 0.5 \end{array}\right] \end{array} \begin{array}{c} u_2° \\ v_2° \\ w_2° \\ u_4° \\ v_4° \\ w_4° \end{array} \tag{23.59}$$

Element 4–3

The member points from 4 to 3, so that the start node is 4 and the finish node is 3. From the figure at the start of this problem,

$$x_3° = 5 \qquad y_3° = 12.07 \qquad z_3° = 0$$

Substituting the above and x_4°, y_4° and z_4° into equation (23.56),

$$l = \left[(5 - 5)^2 + (12.07 - 5)^2 + (0 - 7.07)^2\right]^{\frac{1}{2}}$$

$$l = 10 \text{ m}$$

From equation (23.57),

$$C_{x,x}^\circ = \frac{x_3^\circ - x_4^\circ}{l} = \frac{5 - 5}{10} = 0$$

$$C_{x,y}^\circ = \frac{y_3^\circ - y_4^\circ}{l} = \frac{12.07 - 5}{10} = 0.707$$

$$C_{x,z}^\circ = \frac{z_3^\circ - z_4^\circ}{l} = \frac{0 - 7.07}{10} = -0.707$$

Substituting the above into equation (23.54), and removing the rows and columns corresponding to the zero displacements, which in this case are u_3°, v_3° and w_3°, the stiffness matrix for element 4–3 is given by equation (23.60):

$$\left[\mathbf{k}_{4-3}^\circ\right] = \frac{AE}{10} \begin{array}{c} \begin{array}{cccccc} u_4^\circ & v_4^\circ & w_4^\circ & u_3^\circ & v_3^\circ & w_3^\circ \end{array} \\ \left[\begin{array}{cccccc} 0 & & & & & \\ 0 & 0.5 & & & & \\ 0 & -0.5 & 0.5 & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{array}\right] \end{array} \begin{array}{c} u_4^\circ \\ v_4^\circ \\ w_4^\circ \\ u_3^\circ \\ v_3^\circ \\ w_3^\circ \end{array} \qquad (23.60)$$

To obtain $[\mathbf{K}_{11}]$, the system stiffness matrix corresponding to the free displacements, namely u_4°, v_4° and w_4°, the appropriate coefficients of the elemental stiffness matrices of equations (23.58) to (23.60) are added together, with reference to these free displacements, as shown by equation (23.61):

$$
[\mathbf{K}_{11}{}^{\circ}] = \frac{AE}{10}
\begin{array}{c}
\begin{array}{ccc} \quad u_4{}^{\circ} \quad & \quad v_4{}^{\circ} \quad & \quad w_4{}^{\circ} \quad \end{array} \\
\left[\begin{array}{c|c|c}
\begin{array}{l} 0.25 \\ +\,0.25 \\ +\,0 \end{array} & & \\
\hline
\begin{array}{l} 0.25 \\ -\,0.25 \\ +\,0 \end{array} & \begin{array}{l} 0.25 \\ +\,0.25 \\ +\,0.5 \end{array} & \\
\hline
\begin{array}{l} 0.354 \\ -\,0.354 \\ +\,0 \end{array} & \begin{array}{l} 0.354 \\ +\,0.354 \\ -\,0.5 \end{array} & \begin{array}{l} 0.5 \\ +\,0.5 \\ +\,0.5 \end{array}
\end{array}\right]
\end{array}
\begin{array}{c} u_4{}^{\circ} \\ \\ \\ v_4{}^{\circ} \\ \\ \\ w_4{}^{\circ} \end{array}
\tag{23.61}
$$

$$
= \frac{AE}{10}
\begin{array}{c}
\begin{array}{ccc} u_4{}^{\circ} & \quad v_4{}^{\circ} \quad & w_4{}^{\circ} \end{array} \\
\begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 1.0 & 0.208 \\ 0 & 0.208 & 1.5 \end{bmatrix}
\end{array}
\begin{array}{c} u_4{}^{\circ} \\ v_4{}^{\circ} \\ w_4{}^{\circ} \end{array}
\tag{23.62}
$$

The vector of loads is obtained by considering the loads in the directions of the free displacements, namely $u_4{}^{\circ}$, $v_4{}^{\circ}$ and $w_4{}^{\circ}$, as shown by equation (23.63):

$$
\{q_F\} = \begin{Bmatrix} 2 \\ 0 \\ -3 \end{Bmatrix} \begin{array}{c} u_4{}^{\circ} \\ v_4{}^{\circ} \\ w_4{}^{\circ} \end{array}
\tag{23.63}
$$

Substituting equations (23.62) and (23.63) into (23.16), the following three simultaneous equations are obtained:

$$
2 = \left(\frac{AE}{10}\right) \times 0.5\ u_4{}^{\circ}
\tag{23.64a}
$$

$$
0 = \left(\frac{AE}{10}\right) \left(v_4{}^{\circ} + 0.208\ w_4{}^{\circ}\right)
\tag{23.64b}
$$

$$
-3 = \left(\frac{AE}{10}\right) \left(0.208\ v_4{}^{\circ} + 1.5\ w_4{}^{\circ}\right)
\tag{23.64c}
$$

From (23.64a)

$$u_4^\circ = 40/AE$$

Hence, from (20.64b) and (23.64c),

$$v_4^\circ = 4.284/AE$$

$$w_4^\circ = -20.594/AE$$

so that,

$$\{u_F\} = \begin{Bmatrix} u_4^\circ \\ v_4^\circ \\ w_4^\circ \end{Bmatrix} = \frac{1}{AE}\begin{Bmatrix} 40 \\ 4.284 \\ -20.594 \end{Bmatrix} \tag{23.65}$$

To determine the forces in the members, the displacements of equation (23.65) must be resolved along the length of each rod, so that the amount the rod contracts or extends can be determined. Then through the use of Hookean elasticity, the internal forces in each member can be obtained.

Element 1–4

$$C_{x,x}^\circ = 0.5, \qquad C_{x,y}^\circ = 0.5, \qquad C_{x,z}^\circ = 0.707, \qquad l = 10\text{ m}$$

From equation (23.52):

$$u_4 = \begin{bmatrix} C_{x,x}^\circ & C_{x,y}^\circ & C_{x,z}^\circ \end{bmatrix} \begin{Bmatrix} u_4^\circ \\ v_4^\circ \\ w_4^\circ \end{Bmatrix}$$

$$= \begin{bmatrix} 0.5 & 0.5 & 0.707 \end{bmatrix} \frac{1}{AE} \begin{Bmatrix} 40 \\ 4.28 \\ -20.59 \end{Bmatrix}$$

$$u_4 = 7.568/AE$$

From Hooke's law,

$$F_{1-4} = \text{force in member } 1\text{–}4$$

$$= \frac{AE}{10}\left(u_4 - u_1\right) = \frac{AE}{10} \times \frac{7.568}{AE}$$

$$F_{1-4} = 0.757 \text{ MN (tension)}$$

Element 2–4

$$C_{x,x}{}^{\circ} = -0.5, \quad C_{x,y}{}^{\circ} = 0.5, \quad C_{x,z}{}^{\circ} = 0.707, \qquad l = 10 \text{ m}$$

From equation (23.52):

$$u_4 = \begin{bmatrix} C_{x,x}{}^{\circ} & C_{x,y}{}^{\circ} & C_{x,z}{}^{\circ} \end{bmatrix} \begin{Bmatrix} u_4{}^{\circ} \\ v_4{}^{\circ} \\ w_4{}^{\circ} \end{Bmatrix}$$

$$= \begin{bmatrix} -0.5 & 0.5 & 0.707 \end{bmatrix} \frac{1}{AE} \begin{Bmatrix} 40 \\ 4.28 \\ -20.59 \end{Bmatrix}$$

$$u_4 = -32.417/AE$$

From Hooke's law,

$$F_{2-4} = \text{force in member } 2\text{–}4$$

$$= \frac{AE}{10}\left(u_4 - u_2\right) = \frac{AE}{10} \times \left(-32.417/AE\right)$$

$$F_{2-4} = 3.242 \text{ MN (tension)}$$

Element 4–3

$$C_{x,x}{}^{\circ} = 0, \qquad C_{x,y}{}^{\circ} = 0.707, \qquad C_{x,z}{}^{\circ} = -0.707, \qquad l = 10\text{ m}$$

$$u_4 = \begin{bmatrix} C_{x,x}{}^{\circ} & C_{x,y}{}^{\circ} & C_{x,z}{}^{\circ} \end{bmatrix} \begin{Bmatrix} u_4{}^{\circ} \\ v_4{}^{\circ} \\ w_4{}^{\circ} \end{Bmatrix}$$

$$u_4 = \begin{bmatrix} 0 & 0.707 & -0.707 \end{bmatrix} \frac{1}{AE} \begin{Bmatrix} 40 \\ 4.28 \\ -20.59 \end{Bmatrix}$$

$$u_4 = 17.58/AE$$

From Hooke's law,

$$F_{4\text{-}3} = \text{force in member 4–3}$$

$$= \frac{AE}{l}\left(u_3 - u_4\right)$$

$$= \frac{AE}{10}\left(0 - 17.58/AE\right)$$

$$F_{4\text{-}3} = -1.758\text{ MN (compression)}$$

23.7 Beam element

The stiffness matrix for a beam element can be obtained by considering the beam element of Figure 23.7.

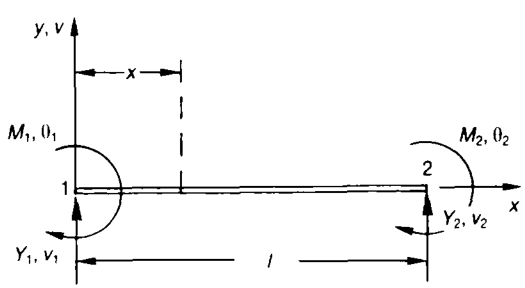


Figure 23.7 Beam element.

From equation (13.4),

$$EI\frac{d^2v}{dx^2} = M = Y_1x + M_1 \tag{23.66}$$

$$EI\frac{dv}{dx} = \frac{Y_1x^2}{2} + M_1x + A \tag{23.67}$$

$$EIv = \frac{Y_1x^3}{6} + \frac{M_1x^2}{2} + Ax + B \tag{23.68}$$

where

Y_1 = vertical reaction at node 1

Y_2 = vertical reaction at node 2

M_1 = clockwise couple at node 1

M_2 = clockwise couple at node 2

v_1 = vertical deflection at node 1

v_2 = vertical deflection at node 2

θ_1 = rotational displacement (clockwise) at node 1

θ_2 = rotational displacement (clockwise) at node 2

There are four unknowns in equation (23.68), namely Y_1, M_1, A and B; therefore, four boundary values will have to be substituted into equations (23.67) and (23.68) to determine these four unknowns, through the solution of four linear simultaneous equations.

These four boundary values are as follows:

$$\text{At } x = 0, \quad v = v_1 \quad \text{and} \quad \theta_1 = -\left(\frac{dv}{dx}\right)_{x=0}$$

$$\text{At } x = l, \quad v = v_2 \quad \text{and} \quad \theta_2 = -\left(\frac{dv}{dx}\right)_{x=l}$$

Substituting these four boundary conditions into equations (23.67) and (23.68), the following are obtained:

$$Y_1 = -\frac{6EI}{l^2}(\theta_1 + \theta_2) + \frac{12EI}{l^3}(v_1 - v_2) \tag{23.69}$$

$$M_1 = \frac{6EI}{l^2}(v_2 - v_1) + \frac{EI}{l}(4\theta_1 + 2\theta_2) \tag{23.70}$$

$$Y_2 = \frac{6EI}{l^2}(\theta_1 + \theta_2) - \frac{12EI}{l^3}(v_1 - v_2) \tag{23.71}$$

$$M_2 = \frac{2EI}{l}\theta_1 + \frac{4EI}{l}\theta_2 - \frac{6EI}{l^2}(v_1 - v_2) \tag{23.72}$$

Equations (23.69) to (23.72) can be put in the form:

$$\{P_i\} = [\mathbf{k}]\{u_i\}$$

where,

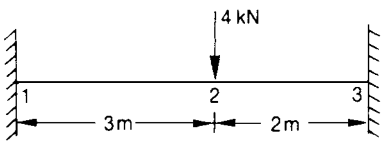
$$[\mathbf{k}] = EI \begin{array}{c} \begin{array}{cccc} v_1 & \theta_1 & v_2 & \theta_2 \end{array} \\ \begin{bmatrix} 12/l^3 & -6/l^2 & -12/l^3 & -6/l^2 \\ -6/l^2 & 4/l & 6/l^2 & 2/l \\ -12/l^3 & 6/l^2 & 12/l^3 & 6/l^2 \\ -6/l^2 & 2/l & 6/l^2 & 4/l \end{bmatrix} \end{array} \begin{array}{c} \\ v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{array} \tag{23.73}$$

= the elemental stiffness matrix for a beam

$$\{P_i\} = \begin{Bmatrix} Y_1 \\ M_1 \\ Y_2 \\ M_2 \end{Bmatrix} = \text{a vector of generalised loads} \tag{23.74}$$

$$\{u_i\} = \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} = \text{a vector of generalised displacements} \tag{23.75}$$

Problem 23.4 Determine the nodal displacements and bending moments for the uniform section beam below, which can be assumed to be fully fixed at its ends.



Solution

Prior to solving this problem, it must be emphasised that the nodes must be numbered in ascending order from left to right, because the beam element has been developed with the assumption that the start node is on the left and the finish node is on the right.

Element 1–2

$$l = 3 \text{ m}$$

Substituting this value of l into equation (23.73), and removing the components of the stiffness matrix corresponding to the zero displacements, namely v_1 and θ_1, the stiffness matrix for element 1–2 is given by equation (23.76):

$$[\mathbf{k}_{1-2}] = EI \begin{array}{c} \begin{array}{cccc} v_1 & \theta_1 & v_2 & \theta_2 \end{array} \\ \left[\begin{array}{cccc} & & & \\ & & & \\ & & 0.444 & 0.667 \\ & & 0.667 & 1.333 \end{array}\right] \end{array} \begin{array}{c} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{array} \tag{23.76}$$

Element 2–3

$$l = 2 \text{ m}$$

Substituting this value of l into equation (23.73), and removing the components of the stiffness matrix corresponding to the zero displacements, namely v_3 and θ_3, the following is obtained for the elemental stiffness matrix 2–3:

$$[\mathbf{k}_{2\text{-}3}] = EI \begin{array}{c} \begin{array}{cccc} v_2 & \theta_2 & v_3 & \theta_3 \end{array} \\ \left[\begin{array}{cccc} 1.5 & -1.5 & & \\ -1.5 & 1.5 & & \\ & & & \\ & & & \end{array}\right] \end{array} \begin{array}{c} \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{array} \tag{23.77}$$

The system stiffness matrix, which corresponds to the free displacements v_2 and θ_2, is obtained by adding together the appropriate components of the elemental stiffness matrices of equations (23.76) and (23.77), as shown by equation (23.78):

$$[\mathbf{K}_{11}] = EI \begin{array}{c} \begin{array}{cc} v_2^{\circ} & \theta_2^{\circ} \end{array} \\ \left[\begin{array}{c|c} 0.444 + 1.5 & 0.667 - 1.5 \\ \hline 0.667 - 1.5 & 1.332 + 2.0 \end{array}\right] \end{array} \begin{array}{c} \\ v_4^{\circ} \\ \theta_2^{\circ} \end{array} \tag{23.78}$$

$$= EI \begin{array}{c} \begin{array}{cc} v_2 & \theta_2 \end{array} \\ \left[\begin{array}{cc} 1.944 & -0.833 \\ -0.833 & 3.333 \end{array}\right] \end{array} \begin{array}{c} \\ v_2 \\ \theta_2 \end{array} \tag{23.79}$$

The vector of generalised loads is obtained by considering the loads in the directions of the free displacements v_2 and θ_2, as follows:

$$\{q_F\} = \begin{Bmatrix} -4 \\ 0 \end{Bmatrix} \begin{matrix} v_2 \\ \theta_2 \end{matrix}$$

From equation (23.11),

$$\begin{Bmatrix} -4 \\ 0 \end{Bmatrix} = EI \begin{bmatrix} 1.944 & -0.833 \\ -0.833 & 3.333 \end{bmatrix} \begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix}$$

or,

$$\begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \frac{\dfrac{1}{EI} \begin{bmatrix} 3.333 & 0.833 \\ 0.833 & 1.944 \end{bmatrix} \begin{Bmatrix} -4 \\ 0 \end{Bmatrix}}{(1.944 \times 3.333 - 0.833^2)}$$

$$= \frac{1}{EI}\begin{bmatrix} 0.576 & 0.144 \\ 0.144 & 0.336 \end{bmatrix}\begin{Bmatrix} -4 \\ 0 \end{Bmatrix} \tag{23.80}$$

$$\begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \frac{1}{EI}\begin{Bmatrix} -2.304 \\ -0.576 \end{Bmatrix} \tag{23.81}$$

NB $\quad v_1 = \theta_1 = v_2 = \theta_2 = 0$

To obtain the nodal bending moments, these values of displacement must be substituted into the slope-deflection equations (23.70) and (23.72), as follows.

Element 1–2

Substituting v_1, θ_1, v_2 and θ_2 into equations (23.70) and (23.72):

$$M_1 = \frac{6EI}{9}\left(\frac{-2.304}{EI} - 0\right) + \frac{EI}{3}\left(4 \times 0 - \frac{2 \times 0.576}{EI}\right)$$

$$= -1.536 - 0.384$$

$$M_1 = -1.92 \text{ kNm}$$

and,

$$M_2 = \frac{2EI}{3} \times 0 + \frac{4EI}{3} \times \left(\frac{-0.576}{EI}\right) - \frac{6EI}{9}\left(0 + \frac{2.304}{EI}\right)$$

$$= -0.768 - 1.536$$

$$M_2 = -2.304 \text{ kNm}$$

Element 2–3

Substituting v_2, θ_2, v_3 and θ_3 into equations (23.70) and (23.72), and remembering that the first node is node 2 and the second node is node 3, the following is obtained for M_2 and M_3:

$$M_2 = \frac{4EI}{2} \times \left(\frac{-0.576}{EI}\right) + 0 - \frac{6EI}{4}\left(\frac{-2.304}{EI}\right)$$

$$= -1.152 + 3.456$$

$$M_2 = 2.304 \text{ kNm}$$

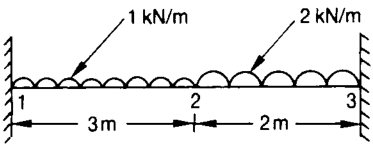
and,

$$M_3 = \frac{2EI}{2}\left(\frac{-0.576}{EI}\right) + 0 - \frac{6EI}{4}\left(\frac{-2.304}{EI} - 0\right)$$

$$= -0.576 + 3.456$$

$$M_3 = 2.88 \text{ kNm}$$

Problem 23.5 Determine the nodal displacements and bending moments for the encastré beam:



Solution

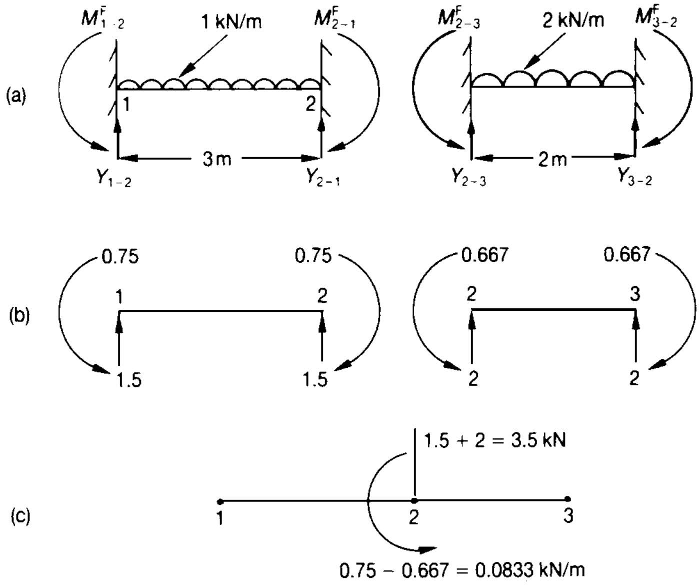
Now the matrix displacement method is based on applying the loads at the nodes, but for the above beam, the loading on each element is between the nodes. It will therefore be necessary to adopt the following process, which is based on the principle of superposition:

1. Fix the beam at its nodes and determine the end fixing forces, as shown in the following figure at (a) and (b) and as calculated below.

2. The beam in condition (1) is not in equilibrium at node 2, hence, it will be necessary to subject the beam to the negative resultants of the end fixing forces at node 2 to achieve equilibrium, as shown in the figure at (c). It should be noted that, as the beam is firmly fixed at nodes 1 and 3, any load or couple applied to these ends will in fact be absorbed by these walls.

3. Using the matrix displacement method, determine the nodal displacements due to the loads of the figure at (c) and, hence, the resulting bending moments.

4. To obtain the final values of nodal bending moments, the bending moments of condition (1) must be superimposed with those of condition (3).



End-fixing forces

Element 1–2

$$M^F_{1-2} = -\frac{wl^2}{12} = -\frac{1 \times 3^2}{12} = -0.75 \text{ kNm}$$

$$M^F_{2-1} = \frac{wl^2}{12} = 0.75 \text{ kNm}$$

$$Y_{1-2} = Y_{2-1} = \frac{1 \times 3}{2} = 1.5 \text{ kN}$$

Element 2–3

$$M^F_{2-3} = -\frac{wl^2}{12} = -\frac{2 \times 2^2}{12} = -0.667 \text{ kNm}$$

$$M^F_{3-2} = \frac{wl^2}{12} = 0.667 \text{ kNm}$$

$$Y_{2-3} = Y_{3-2} = \frac{wl}{2} = \frac{2 \times 2}{2} = 2 \text{ kN}$$

From the figure above, at (c), the vector of generalised loads is obtained by considering the free degrees of freedom, which in this case, are v_2 and θ_2.

$$\{q_F\} = \begin{Bmatrix} -3.5 \\ -0.0833 \end{Bmatrix} \begin{matrix} v_2 \\ \theta_2 \end{matrix} \tag{23.82}$$

From equation (23.80),

$$[\mathbf{K}_{11}]^{-1} = \frac{1}{EI}\begin{bmatrix} 0.576 & 0.144 \\ 0.144 & 0.336 \end{bmatrix}$$

and from equation (23.16),

$$\{u_F\} = \begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \frac{1}{EI}\begin{bmatrix} 0.576 & 0.144 \\ 0.144 & 0.336 \end{bmatrix}\begin{Bmatrix} -3.5 \\ -0.0833 \end{Bmatrix}$$

or,

$$\begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \frac{1}{EI}\begin{Bmatrix} -2.028 \\ -0.532 \end{Bmatrix} \tag{23.83}$$

NB $\quad v_1 = \theta_1 = v_3 = \theta_3 = 0$

To determine the nodal bending moments, the nodal bending moments obtained from the equations (23.70) and (20.72) must be superimposed with the end-fixing bending moment of the figure above, as follows.

Element 1–2

Substituting equation (23.83) into equation (23.70) and adding the end-fixing bending moment from the figure above (b),

$$M_1 = \frac{6EI}{9}\left(\frac{-2.028}{EI} - 0\right) + \frac{EI}{3}\left(4 \times 0 - \frac{2 \times 0.532}{EI}\right) - 0.75$$

$$= -1.352 - 0.355 - 0.75$$

$$M_1 = -2.457 \text{ kNm}$$

Similarly, substituting equation (23.83) into equation (23.72) and adding the end-fixing bending moment of the above figure at (b),

$$M_2 = -\frac{6EI}{3^2}\left(0 + \frac{2.028}{EI}\right) + \frac{EI}{3}\left(0 - \frac{4 \times 0.532}{EI}\right) + 0.75$$

$$= -1.352 - 0.709 + 0.75$$

$$M_2 = 1.311 \text{ kN/m}$$

Element 2–3

Substituting equation (23.83) into equations (23.70) and (23.72) and remembering that node 2 is the first node and node 3 is the second node, and adding the end fixing moments from the above figure at (b),

$$M_2 = \frac{6EI}{4}\left(\frac{2.028}{EI} + 0\right) + \frac{EI}{2}\left(-\frac{4 \times 0.532}{EI}\right) - 0.667$$

$$= 3.042 - 1.064 - 0.667$$

$$M_2 = 1.311 \text{ kNm}$$

$$M_3 = \frac{6EI}{4}\left(\frac{2.028}{EI} + 0\right) + \frac{EI}{2}\left(-\frac{2 \times 0.532}{EI}\right) + 0.667$$

$$= 3.042 - 0.532 + 0.667$$

$$M_3 = 3.177 \text{ kNm}$$

23.8 Rigid-jointed plane frames

The elemental stiffness matrix for a rigid-jointed plane frame element in local co-ordinates, can be obtained by superimposing the elemental stiffness matrix for the rod element of equation (23.28) with that of the beam element of equation (23.73), as shown by equation (23.84):

$$[\mathbf{k}] = EI\begin{bmatrix} (A/lI) & 0 & 0 & (-A/lI) & 0 & 0 \\ 0 & 12/l^3 & -6/l^2 & 0 & -12/l^3 & -6/l^2 \\ 0 & -6/l^2 & 4/l & 0 & 6/l^2 & 2/l \\ (-A/lI) & 0 & 0 & (A/lI) & 0 & 0 \\ 0 & -12/l^3 & 6/l^2 & 0 & 12/l^3 & 6/l^2 \\ 0 & -6/l^2 & 2/l & 0 & 6/l^2 & 4/l \end{bmatrix} \tag{23.84}$$

= the elemental stiffness matrix for a rigid-jointed plane frame element, in local co-ordinates

Now the stiffness matrix of equation (23.84) is of little use in that form, as most elements for a rigid-jointed plane frame will be inclined at some angle to the horizontal, as shown by Figure 23.8.

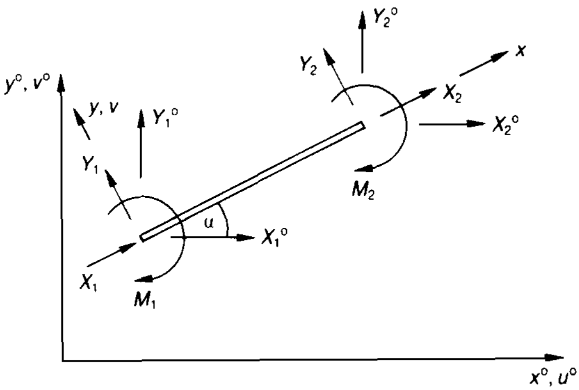


Figure 23.8 Rigid-jointed plane frame element.

It can readily be shown that the relationships between the local and global forces for the element are:

$$\begin{Bmatrix} X_1 \\ Y_1 \\ M_1 \\ X_2 \\ Y_2 \\ M_2 \end{Bmatrix} = \begin{bmatrix} c & s & 0 & & & \\ -s & c & 0 & & 0_3 & \\ 0 & 0 & 1 & & & \\ & & & c & s & 0 \\ & 0_3 & & -s & c & 0 \\ & & & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} X_1^\circ \\ Y_1^\circ \\ M_1^\circ \\ X_2^\circ \\ Y_2^\circ \\ M_2^\circ \end{Bmatrix} \tag{23.85}$$

or,

$$\{P_i\} = [\mathbf{DC}]\{P_i^\circ\}$$

where,

$$[\mathbf{DC}] = \begin{bmatrix} \zeta & 0_3 \\ 0_3 & \zeta \end{bmatrix}$$

$$[\zeta] = \begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now, from equation (23.35):

$$\begin{aligned}[\mathbf{k}^\circ] &= [\mathbf{DC}]^{\mathbf{T}}\ [\mathbf{k}]\ [\mathbf{DC}] \\ &= [\mathbf{k_r}^\circ] + [\mathbf{k_b}^\circ]\end{aligned} \tag{23.86}$$

where,

$$[\mathbf{k_r}^\circ] = \frac{AE}{l} \begin{array}{c} \begin{array}{cccccc} u_1^\circ & v_1^\circ & \theta_1 & u_2^\circ & v_2^\circ & \theta_2 \end{array} \\ \begin{bmatrix} c^2 & cs & 0 & -c^2 & -cs & 0 \\ cs & s^2 & 0 & -cs & -s^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -c^2 & -cs & 0 & c^2 & cs & 0 \\ -cs & -s^2 & 0 & cs & s^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} u_1^\circ \\ v_1^\circ \\ \theta_1 \\ u_2^\circ \\ v_2^\circ \\ \theta_2 \end{array} \tag{23.87}$$

$$[\mathbf{k}_b^\circ] = EI \begin{array}{c} \begin{array}{cccccc} u_1^\circ & v_1^\circ & \theta_1 & u_2^\circ & v_2^\circ & \theta_2 \end{array} \\ \begin{bmatrix} \frac{12}{l^3}s^2 & & & & & \\ \frac{12}{l^3}cs^2 & \frac{12}{l^3}c^2 & & & & \\ \frac{6}{l^2}s & -\frac{6}{l^2}c & \frac{4}{l} & & & \\ -\frac{12}{l^3}s^2 & \frac{12}{l^3}cs & -\frac{6}{l^2}s & \frac{12}{l^3}s^2 & & \\ \frac{12}{l^3}cs & -\frac{12}{l^3}c^2 & \frac{6}{l^2}c & -\frac{12}{l^3}cs & \frac{12}{l^3}c^2 & \\ \frac{6}{l^2}s & -\frac{6}{l^2}c & \frac{2}{l} & -\frac{6}{l^2}s & \frac{6}{l^2}c & \frac{4}{l} \end{bmatrix} \end{array} \begin{array}{c} u_1^\circ \\ v_1^\circ \\ \theta_1^\circ \\ u_2^\circ \\ v_2^\circ \\ \theta_2 \end{array} \tag{23.88}$$

c = cos α

s = sin α

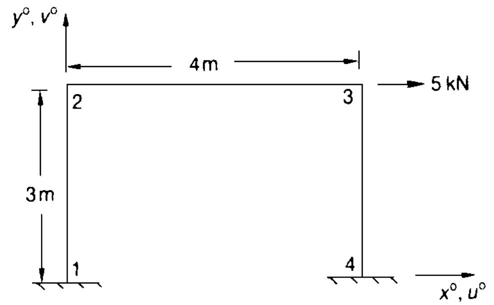
A = cross-sectional area

I = second moment of area of the element's cross-section

l = elemental length

E = Young's modulus of elasticity

Problem 23.6 Using the matrix displacement method, determine the nodal bending moments for the rigid-jointed plane frame shown in the figure below. It may be assumed that the axial stiffness of each element is very large compared to the flexural stiffness, so that $v_2^\circ = v_3^\circ = 0$, and $u_2^\circ = u_3^\circ$.



Solution

As the axial stiffness of the elements are large compared with their flexural stiffness, the effects of $[\mathbf{k}_r^\circ]$ can be ignored.

Element 1–2

$\alpha = 90^\circ \qquad c = 0 \qquad s = 1 \qquad l = 3\text{ m}$

Substituting the above into equation (23.88), and removing the rows and columns corresponding to the zero displacements, which in this case are u_1°, v_1°, θ_1 and v_2°, the elemental stiffness matrix for member 1–2 becomes

$$
\left[\mathbf{k}_{1-2}^{\circ}\right] = EI
\begin{array}{c}
\begin{array}{cccccc} u_1^{\circ} & v_1^{\circ} & \theta_1 & u_2^{\circ} & v_2^{\circ} & \theta_2 \end{array} \\
\left[\begin{array}{cccccc}
 & & & & & \\
 & & & & & \\
 & & & & & \\
 & & & 0.444 & & \\
 & & & -0.667 & 1.333 &
\end{array}\right]
\end{array}
\begin{array}{c} u_1^{\circ} \\ v_1^{\circ} \\ \theta_1 \\ u_2^{\circ} \\ v_2^{\circ} \\ \theta_2 \end{array}
\tag{23.89}
$$

Element 2–3

$\alpha = 0, \qquad c = 1, \qquad s = 0, \qquad l = 4 \text{ m}$

Substituting the above into equation (23.88), and removing the columns and rows corresponding to zero displacements, which in this case are v_2° and v_3°, the elemental stiffness matrix for member 2–3 is given by

$$
\left[\mathbf{k}_{2-3}^{\circ}\right] = EI
\begin{array}{c}
\begin{array}{cccccc} u_2^{\circ} & v_2^{\circ} & \theta_2 & u_3^{\circ} & v_3^{\circ} & \theta_3 \end{array} \\
\left[\begin{array}{cccc}
0 & & & \\
0 & 1 & & \\
0 & 0 & & \\
0 & 0.5 & 0 & 1
\end{array}\right]
\end{array}
\begin{array}{c} u_2^{\circ} \\ v_2^{\circ} \\ \theta_2 \\ u_3^{\circ} \\ v_3^{\circ} \\ \theta_3 \end{array}
\tag{23.90}
$$

Element 3–4

$\alpha = -90^{\circ}, \qquad c = 0, \qquad s = -1, l = 3 \text{ m}$

Substituting the above into equation (23.88), and removing the columns and rows corresponding to zero displacements, namely v_3°, u_4°, v_4° and θ_4, the elemental stiffness matrix for member 3–4 is given by

$$[\mathbf{k}_{3\text{-}4}{}^{\circ}] = EI\;\begin{array}{c} \begin{array}{cccccc} u_3{}^{\circ} & v_3{}^{\circ} & \theta_3 & u_4{}^{\circ} & v_4{}^{\circ} & \theta_4 \end{array} \\ \left[\begin{array}{cccccc} 0.444 & & & & & \\ & & & & & \\ -0.667 & & 1.333 & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{array}\right] \end{array} \begin{array}{c} u_3{}^{\circ} \\ v_3{}^{\circ} \\ \theta_3 \\ u_4{}^{\circ} \\ v_4{}^{\circ} \\ \theta_4 \end{array} \tag{23.91}$$

Superimposing the stiffness influence coefficients, corresponding to the free displacements, $u_2{}^{\circ}$, θ_2, $u_3{}^{\circ}$ and θ_3, the system stiffness matrix $[\mathbf{K}_{11}]$ is obtained, as shown by equation (23.92):

$$[\mathbf{K}_{11}{}^{\circ}] = EI\;\begin{array}{c} \begin{array}{cccc} u_2{}^{\circ} & \theta_2 & u_3{}^{\circ} & \theta_3 \end{array} \\ \left[\begin{array}{c|c|c|c} 0.444 + 0 & & & \\ \hline -0.667 + 0 & 1.333 + 1 & & \\ \hline & & 0.444 & \\ & 0.5 & -0.667 & 1+1.333 \end{array}\right] \end{array} \begin{array}{c} u_2{}^{\circ} \\ \theta_2 \\ u_3{}^{\circ} \\ \theta_3 \end{array} \tag{23.92}$$

$$[\mathbf{K}_{11}{}^{\circ}] = EI\;\begin{array}{c} \begin{array}{cccc} u_2{}^{\circ} & \theta_2 & u_3{}^{\circ} & \theta_3 \end{array} \\ \left[\begin{array}{cccc} 0.444 & -0.667 & 0 & 0 \\ -0.667 & 2.333 & 0 & 0.5 \\ 0 & 0 & 0.444 & -0.667 \\ 0 & 0.5 & -0.667 & 2.333 \end{array}\right] \end{array} \begin{array}{c} u_2{}^{\circ} \\ \theta_2 \\ u_3{}^{\circ} \\ \theta_3 \end{array} \tag{23.93}$$

The vector of loads corresponding to these free displacements is given by

$$\{q_F\} = \begin{Bmatrix} 0 \\ 0 \\ 5 \\ 0 \end{Bmatrix} \begin{matrix} u_2^\circ \\ \theta_2 \\ u_3^\circ \\ \theta_3 \end{matrix} \tag{23.94}$$

Rewriting equations (23.93) and (23.94) in the form of four linear simultaneous equations, and noting that the 5 kN load is shared between members 1–2 and 3–4, the following is obtained:

$$\begin{aligned} 2.5 &= EI(0.444\,u_2^\circ - 0.667\,\theta_2) \\ 0 &= EI(-0.667\,u_2^\circ + 2.333\,\theta_2 + 0.5\,\theta_3) \\ 2.5 &= EI(0.444\,u_3^\circ - 0.667\,\theta_3) \\ 0 &= EI(0.5\,\theta_2 - 0.667\,u_3^\circ + 2.333\,\theta_3) \end{aligned} \tag{23.95}$$

Now for this case

$$\theta_2 = \theta_3 \tag{23.96}$$

and

$$u_2^\circ = u_3^\circ$$

Hence, equation (23.95) can be reduced to the form shown in equation (23.97):

$$\begin{aligned} 2.5 &= 0.444\ EIu_2^\circ - 0.667\ EI\theta_2 \\ 0 &= -0.667\ EIu_2^\circ + 2.833\ EI\theta_2 \end{aligned} \tag{23.97}$$

Solving the above

$$u_2^\circ = u_3^\circ = 8.707/EI$$

and

$$\theta_2 = \theta_3 = 2.049/EI \tag{23.98}$$

To determine the nodal bending moments, the displacements in the local v and θ directions will have to be calculated, prior to using equations (23.70) and (23.72).

Element 1–2

$$c = 0, \qquad s = 1, \qquad l = 3\text{ m}$$

From equation (23.23):

$$v_2 = \begin{bmatrix} -s & c \end{bmatrix} \begin{Bmatrix} u_2^\circ \\ v_2^\circ \end{Bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \end{bmatrix} \frac{1}{EI} \begin{Bmatrix} 8.707 \\ 0 \end{Bmatrix}$$

$$v_2 = -8.707/EI$$

By inspection,

$$v_1 = \theta_1 = 0 \quad \text{and} \quad \theta_2 = 2.049/EI$$

Substituting the above values into the slope–deflection equations (23.70) and (23.72)

$$M_{1\text{-}2} = 0 + \frac{2EI}{3} \times \frac{2.049}{EI} - \frac{6EI}{9}\left(0 + \frac{8.707}{EI}\right)$$

$$= 1.366 - 5.805$$

$$M_{1\text{-}2} = -4.43\ \text{kNm}$$

$$M_{2\text{-}1} = 0 + \frac{4EI}{3} \times \frac{2.049}{EI} - \frac{6EI}{9}\left(0 + \frac{8.707}{EI}\right)$$

$$= 2.732 - 5.805$$

$$M_{2\text{-}1} = -3.07\ \text{kNm}$$

Element 2–3

$$l = 4\text{ m}$$

By inspection,

$$v_2 = v_3 = 0$$

and

$$\theta_2 = \theta_3 = 2.049/EI$$

Substituting the above values into the slope–deflection equations (23.70) an (23.72):

$$M_{2\text{-}3} = \frac{4EI}{4} \times \frac{2.049}{EI} + \frac{2EI}{4} \times \frac{2.049}{EI}$$

$$M_{2\text{-}3} = 3.07 \text{ kNm}$$

Element 3–4

$$c = 0, \qquad s = -1, \qquad l = 3 \text{ m}$$

From equation (23.23):

$$v_3 = \begin{bmatrix} -s & c \end{bmatrix} \begin{Bmatrix} u_3{}^\circ \\ v_3{}^\circ \end{Bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{1}{EI} \begin{Bmatrix} 8.707 \\ 0 \end{Bmatrix}$$

$$v_3 = 8.707/EI$$

By inspection,

$$v_4 = \theta_4 = 0 \qquad \text{and } \theta_3 = 2.049/EI$$

Substituting the above values into equations (23.70) and (23.72),

$$M_{3\text{-}4} = \frac{4EI}{3} \times \frac{2.049}{EI} + 0 - \frac{6EI}{9}\left(\frac{8.707}{EI} - 0\right)$$

$$= 2.732 - 5.805$$

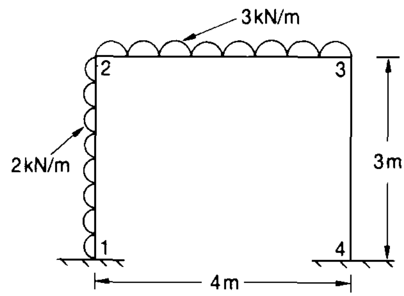
$$M_{3\text{-}4} = -3.07 \text{ kNm}$$

$$M_{4\text{-}3} = \frac{2EI}{3} \times 2.049 + 0 - \frac{6EI}{9}\left(\frac{8.707}{EI} - 0\right)$$

$$= 1.366 - 5.805$$

$$M_{4\text{-}3} = -4.44 \text{ kNm}$$

Problem 23.7 Using the matrix displacement method, determine the nodal bending moments for the rigid-jointed plane frame shown below.

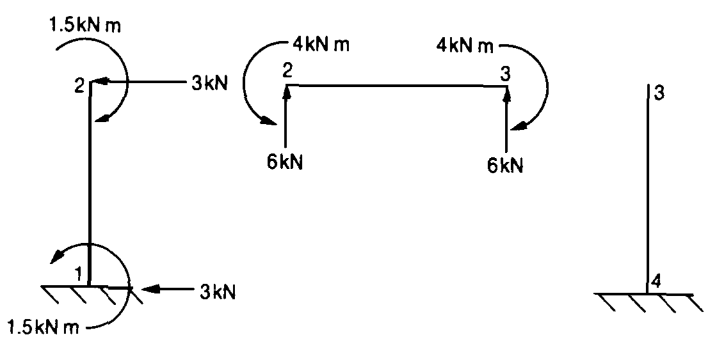


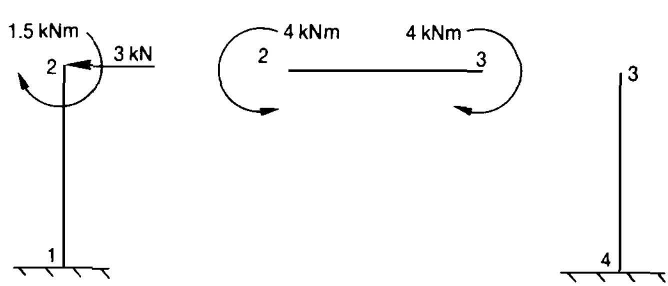
<u>*Solution*</u>

As this frame has distributed loading between some of the nodes, it will be necessary to treat the problem in a manner similar to that described in the solution of Problem 23.5.

There are four degrees of freedom for this structure, namely, $u_2°$, θ_2, $u_3°$ and θ_3, hence $\{q_F\}$ will be of order 4×1.

To determine $\{q_F\}$, it will be necessary to fix the structure at its nodes, and calculate the end fixing forces, as shown and calculated below.





End fixing forces

$$M_{1-2}^{F} = -\frac{wl^2}{12} = -\frac{2 \times 3^2}{12} = -1.5 \text{ kNm}$$

$$M_{2-1}^{F} = \frac{wl^2}{12} = 1.5 \text{ kNm}$$

Horizontal reaction at node 1 $= \dfrac{wl}{2} = \dfrac{2 \times 3}{2} = 3\,\text{kN}$

Horizontal reaction at node 2 $= \dfrac{wl}{2} = \dfrac{2 \times 3}{2}$

$= 3$ kN

$$M_{2-3}^{F} = -\frac{wl^2}{12} = -\frac{3 \times 4^2}{12} = -4 \text{ kNm}$$

$$M_{3-2}^{F} = -M_{2-3}^{F} = 4 \text{ kNm}$$

Vertical reaction at node 2 $= \dfrac{wl}{2} = \dfrac{3 \times 4}{2} = 6\,\text{kN}$

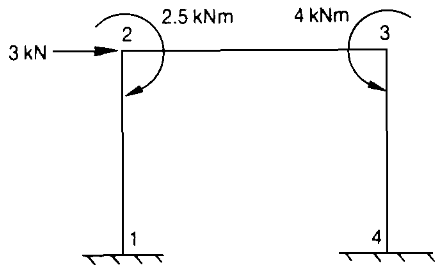
Vertical reaction at node 3 $= \dfrac{wl}{2} = \dfrac{3 \times 4}{2}$

$= 6$ kN

Now, for this problem, as

$$u_1^{\circ} = v_1^{\circ} = \theta_1 = v_2^{\circ} = v_3^{\circ} = u_4^{\circ} = v_4^{\circ} = \theta_4 = 0$$

the only components of the end-fixing forces required for calculating $\{q_F\}$ are shown below:



The negative resultants of the end-fixing forces are shown below, where it can be seen that

$$q_F = \begin{Bmatrix} 3 \\ 2.5 \\ 0 \\ -4 \end{Bmatrix} \begin{matrix} u_2^\circ \\ \theta_2 \\ u_3^\circ \\ \theta_3 \end{matrix} \tag{23.99}$$

From equation (23.93),

$$[K_{11}] = EI \begin{array}{c} \begin{matrix} u_2^\circ & \theta_2 & u_3^\circ & \theta_3 \end{matrix} \\ \begin{bmatrix} 0.444 & -0.667 & 0 & 0 \\ -0.667 & 2.333 & 0 & 0.5 \\ 0 & 0 & 0.444 & -0.667 \\ 0 & 0.5 & -0.667 & 2.333 \end{bmatrix} \end{array} \begin{matrix} u_2^\circ \\ \theta_2 \\ u_3^\circ \\ \theta_3 \end{matrix} \tag{23.100}$$

Rewriting equations (23.99) and (23.100) in the form of four simultaneous equations,

$$3 = 0.444u_2^\circ / EI - 0.667\theta_2 / EI \tag{23.101a}$$

$$2.5 = -0.667u_2^\circ / EI + 2.333\theta_2 / EI + 0.5\theta_3 / EI \tag{23.101b}$$

$$0 = 0.444u_3^\circ / EI - 0.667\theta_3 / EI \tag{23.101c}$$

$$-4 = 0.5\theta_2 / EI - 0.667u_3^\circ / EI + 2.333\theta_3 / EI \tag{23.101d}$$

Now, as the 2.5 kN load is shared between elements 1–2 and 3–4, equation (23.101a) *must be added* to equation (23.101c), as shown by equation (23.102):

$$3 = 0.888u_2^\circ / EI - 0.667\theta_2 / EI - 0.667\theta_3 / EI \tag{23.102}$$

Putting $u_2^\circ = u_3^\circ$, the simultaneous equations (23.101) now become:

$$\begin{aligned} 3 &= 0.888\,u_2^\circ/EI - 0.667\,\theta_2/EI - 0.667\,\theta_3/EI \\ 2.5 &= -0.667\,u_2^\circ/EI + 2.333\,\theta_2/EI + 0.5\,\theta_3/EI \\ -4 &= -0.667\,u_2^\circ/EI + 0.5\,\theta_2/EI + 2.333\,\theta_3/EI \end{aligned} \tag{23.103}$$

Solving the above,

$$u_2^\circ = u_3^\circ = 4.61/EI$$

$$\theta_2 = 2.593/EI$$

$$\theta_3 = -0.953/EI$$

To determine the nodal bending moments, the end fixing moments will have to be added to the moments obtained from the slope–deflection equations.

Element 1–2

$$c = 0 \qquad s = 1 \qquad l = 3\text{ m}$$

From equation (23.23)

$$v_2 = \begin{bmatrix} -s & c \end{bmatrix} \begin{Bmatrix} u_2^\circ \\ v_2^\circ \end{Bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{Bmatrix} 4.61/EI \\ 0 \end{Bmatrix}$$

$$v_2 = -4.61/EI$$

By inspection,

$$v_l = \theta_1 = 0 \qquad \text{and} \qquad \theta_2 = -0.953/\mathrm{EI}$$

Substituting the above into the slope–deflection equations (23.70) and (23.72), and adding the end fixing moments,

$$M_{1\text{-}2} = 0 + \frac{2EI}{3} \times (2.593/EI) - \frac{6EI}{9}(0 + 4.61/EI) - 1.5$$

$$= 1.729 - 3.07 - 1.5$$

$$M_{1\text{-}2} = -2.84 \text{ kNm}$$

and

$$M_{2\text{-}1} = \frac{4EI}{3} \times \frac{2.593}{EI} - 3.07 + 1.5$$

$$M_{2\text{-}1} = 1.89 \text{ kNm}$$

Element 2–3

By inspection,

$$v_2 = v_3 = 0$$

and

$$\theta_2 = 2.593/\mathrm{EI}, \qquad \theta_3 = -0.953/\mathrm{EI}$$

Substituting the above into equations (23.70) and (23.72), adding the end-fixing moments for this element, and remembering that node 2 is the first node and node 3 the second node,

$$M_{2\text{-}3} = \frac{4EI}{4} \times \frac{2.593}{EI} + \frac{2EI}{4} \times \left(\frac{-0.953}{EI}\right) - 4$$

$$M_{2\text{-}3} = -1.88 \text{ kNm}$$

$$M_{3\text{-}2} = \frac{2EI}{4} \times \frac{2.593}{EI} + \frac{4EI}{4} \times \left(\frac{-0.953}{EI}\right) + 4$$

$$M_{3\text{-}2} = 4.34 \text{ kNm}$$

Element 3–4

$$c = 0, \qquad s = 1, \qquad l = 3 \text{ m}$$

From equation (23.23),

$$v_3 = \begin{bmatrix} -s & c \end{bmatrix} \begin{Bmatrix} u_3{}^\circ \\ v_3{}^\circ \end{Bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{Bmatrix} 4.61/EI \\ 0 \end{Bmatrix}$$

$$v_3 = 4.61/EI$$

By inspection,

$$u_3 = u_4 = v_4 = \theta_4 = 0$$

and

$$\theta_3 = -0.953/EI$$

$$M_{3\text{-}4} = \frac{4EI}{3} \times \left(\frac{-0.953}{EI} \right) + 0 - \frac{6EI}{9} (4.61/EI)$$

$$M_{3\text{-}4} = -4.34 \text{ kNm}$$

$$M_{4\text{-}3} = \frac{2EI}{3} \times \left(\frac{-0.953}{EI} \right) + 0 - \frac{6EI}{9} (4.61/EI)$$

$$M_{4\text{-}3} = -3.71 \text{ kNm}$$

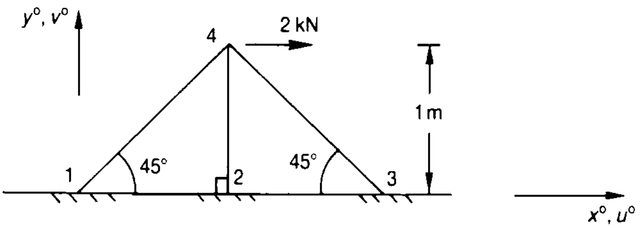
Further problems *(answers on page 697)*

23.8 Determine the forces in the members of the framework of the figure below, under the following conditions:

(a) all joints are pinned;
(b) all joints are rigid (i.e. welded).

The following may be assumed:

AE = $100\,EI$
A = cross-sectional area
I = second moment of area
E = Young's modulus
$[\mathbf{k}]^{\circ}$ = the elemental stiffness matrix
= $[\mathbf{k}_b{}^{\circ}] + [\mathbf{k}_r{}^{\circ}]$



(Portsmouth, 1987, Standard level)

23.9 Determine the displacements at node 5 for the framework shown below under the following conditions:

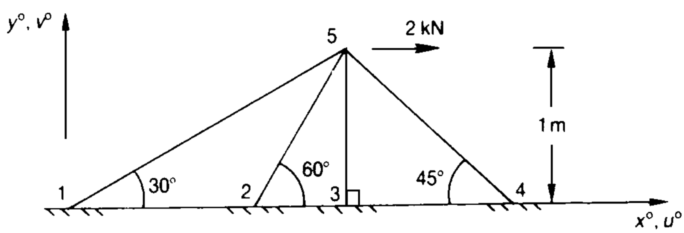
(a) all joints are pinned;
(b) all joints are rigid (i.e. welded).

It may be assumed, for all members of the framework,

A = $100\,EI$

where

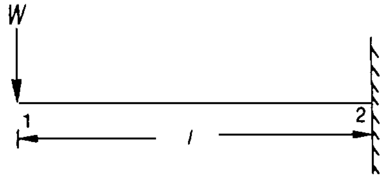
A = cross-sectional area
I = second moment of area
E = Young's modulus
$[\mathbf{k}]^{\circ}$ = the stiffness matrix
= $[\mathbf{k}_b{}^{\circ}] + [\mathbf{k}_r{}^{\circ}]$



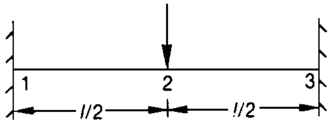
(Portsmouth, 1987, Honours level)

23.10 Determine the nodal displacements and moments for the beams shown below, using the *matrix displacement* method.

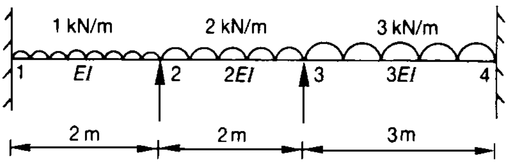
(a)



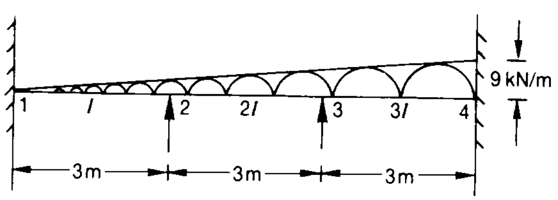
(b)



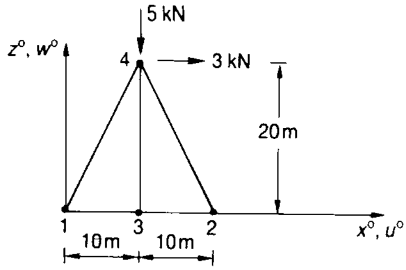
23.11 Determine the nodal bending moments in the continuous beam below, using the *matrix displacement* method.



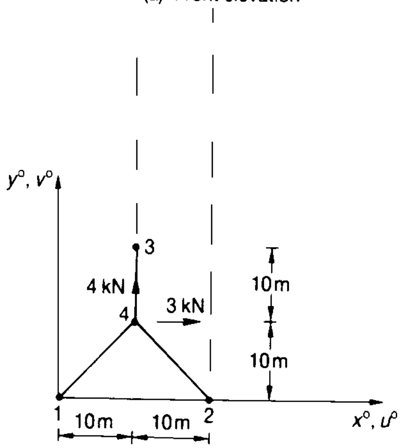
23.12 A ship's bulkhead stiffener is subjected to the hydrostatic loading shown below. If the stiffener is firmly supported at nodes 2 and 3, and fixed at nodes 1 and 4, determine the nodal displacements and moments.



23.13 Using the *matrix displacement* method, determine the forces in the pin-jointed space trusses shown in the following figures. It may be assumed that AE = a constant.

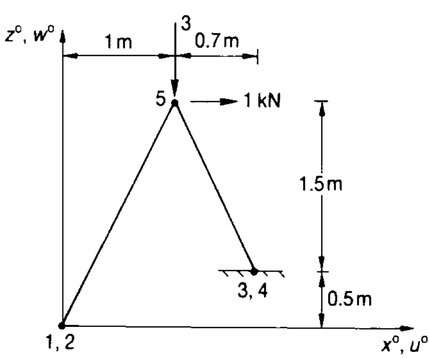


(a) Front elevation

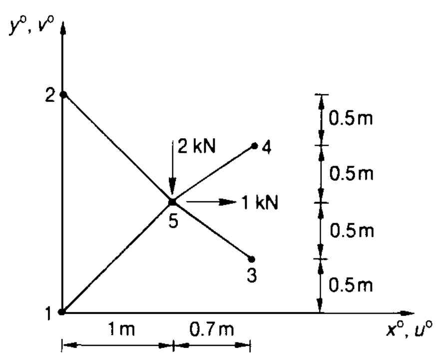


(b) Plan

(Portsmouth, 1989)

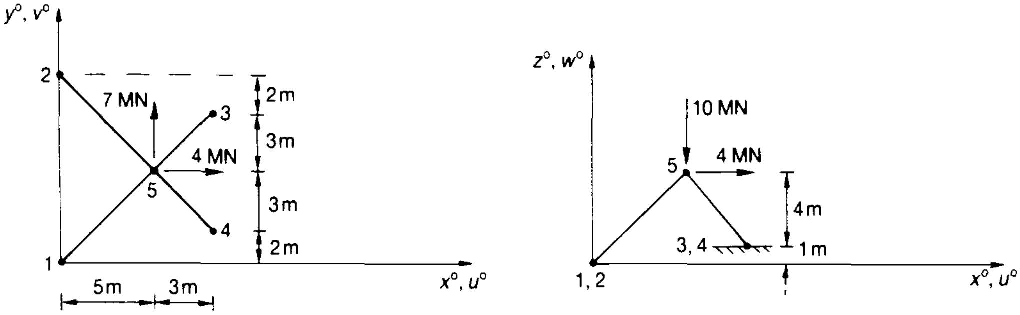


(a) Front elevation



(b) Plan

(Portsmouth, 1983)



(a) Plan

(b) Front elevation

(Portsmouth, 1989)

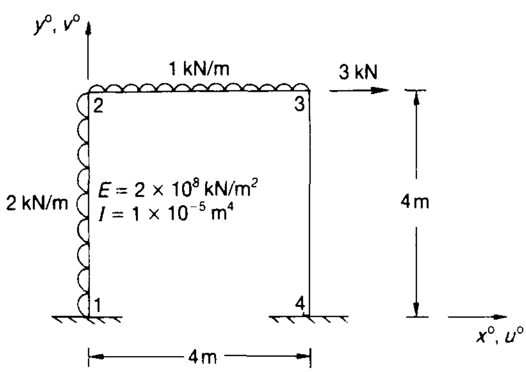
23.14 Determine the nodal displacements and moments for the uniform section rigid-jointed plane frames shown in the two figures below.

It may be assumed that the axial stiffness of each member is large compared with its flexural stiffness, so that,

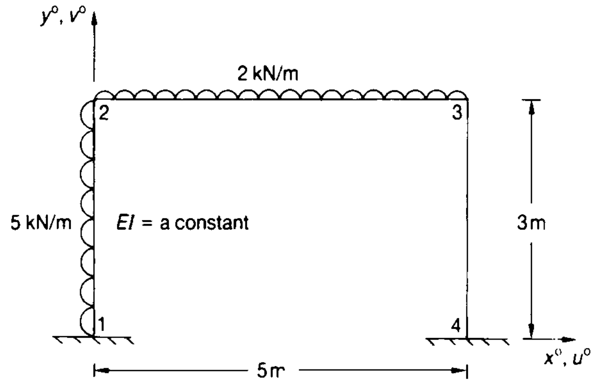
$$v_2^{\circ} = v_3^{\circ} = 0$$

and

$$u_2^{\circ} = u_3^{\circ}$$



(Portsmouth, 1984)



y^o, v^o
2 kN/m
2
3
5 kN/m
EI = a constant
3m
1
4
5m
x^o, u^o