9 Longitudinal stresses in beams

9.1 Introduction

We have seen that when a straight beam carries lateral loads the actions over any cross-section of the beam comprise a bending moment and shearing force; we have also seen how to estimate the magnitudes of these actions. The next step in discussing the strength of beams is to consider the stresses caused by these actions.

As a simple instance consider a cantilever carrying a concentrated load W at its free end, Figure 9.1. At sections of the beam remote from the free end the upper longitudinal fibres of the beam are stretched, i.e. tensile stresses are induced; the lower fibres are compressed. There is thus a variation of direct stress throughout the depth of any section of the beam. In any cross-section of the beam, as in Figure 9.2, the upper fibres which are stretched longitudinally contract laterally owing to the Poisson ratio effect, while the lower fibres extend laterally; thus the whole cross-section of the beam is distorted.

In addition to longitudinal direct stresses in the beam, there are also shearing stresses over any cross-section of the beam. In most engineering problems shearing *distortions* in beams are relatively unimportant; this is not true, however, of shearing *stresses*.

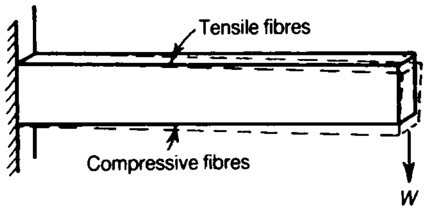


Figure 9.1 Bending strains in a loaded cantilever.

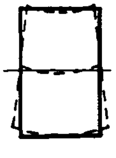


Figure 9.2 Cross-sectional distortion of a bent beam.

9.2 Pure bending of a rectangular beam

An elementary bending problem is that of a rectangular beam under end couples. Consider a straight uniform beam having a rectangular cross-section of breadth b and depth h, Figure 9.3; the axes of symmetry of the cross-section are Cx, Cy.

A long length of the beam is bent in the yz-plane, Figure 9.4, in such a way that the longitudinal centroidal axis, Cz, remains unstretched and takes up a curve of uniform radius of curvature, R.

We consider an elemental length δz of the beam, remote from the ends; in the unloaded condition, AB and FD are transverse sections at the ends of the elemental length, and these sections are initially parallel. In the bent form we assume that planes such as AB and FD remain flat planes; $A'B'$ and $F'D'$ in Figure 9.4 are therefore cross-sections of the bent beam, but are no longer parallel to each other.

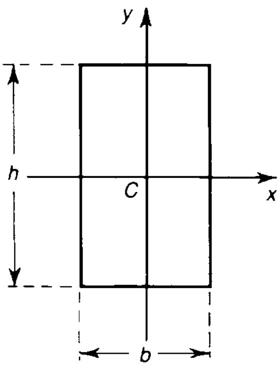


Figure 9.3 Cross-section of a rectangular beam.

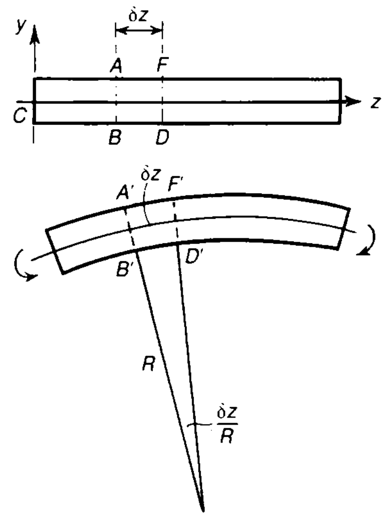


Figure 9.4 Beam bent to a uniform radius of curvature R in the yz-plane.

In the bent form, some of the longitudinal fibres, such as $A'F'$, are stretched, whereas others, such as $B'D'$ are compressed. The unstrained middle surface of the beam is known as the *neutral axis*. Now consider an elemental fibre HJ of the beam, parallel to the longitudinal axis Cz, Figure 9.5; this fibre is at a distance y from the neutral surface and on the tension side of the beam. The original length of the fibre HJ in the unstrained beam is δz; the strained length is

$$H'J' = (R + y)\frac{\delta z}{R}$$

because the angle between $A'B'$ and $F'D'$ in Figure 9.4 and 9.5 is ($\delta z/R$). Then during bending HJ stretches an amount

$$H'J' - HJ = (R + y)\frac{\delta z}{R} - \delta z = \frac{y}{R}\,\delta z$$

The longitudinal strain of the fibre HJ is therefore

$$\varepsilon = \left(\frac{y}{R}\,\delta z\right) / \,\delta z = \frac{y}{R}$$

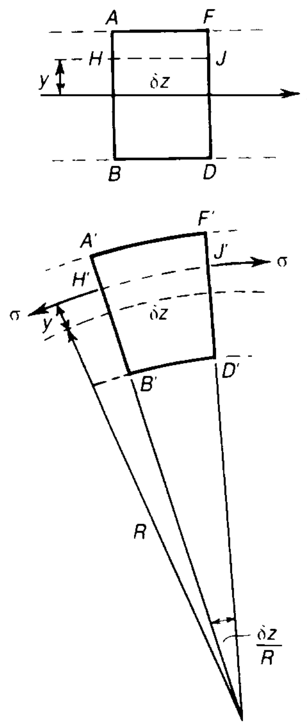


Figure 9.5 Stresses on a bent element of the beam.

Then the longitudinal strain at any fibre is proportional to the distance of that fibre from the neutral surface; over the compressed fibres, on the lower side of the beam, the strains are of course negative.

If the material of the beam remains elastic during bending then the longitudinal stress on the fibre HJ is

$$\sigma = E\varepsilon = \frac{Ey}{R} \tag{9.1}$$

The distribution of longitudinal stresses over the cross-section takes the form shown in Figure 9.6; because of the symmetrical distribution of these stresses about Cx, there is no resultant longitudinal thrust on the cross-section of the beam. The resultant hogging moment is

$$M = \int_{-\frac{1}{2}h}^{+\frac{1}{2}h} \sigma b y \, dy \tag{9.2}$$

On substituting for σ from equation (9.1), we have

$$M = \frac{E}{R}\int_{-\frac{1}{2}h}^{+\frac{1}{2}h} by^2\,dy = \frac{EI_x}{R} \qquad (9.3)$$

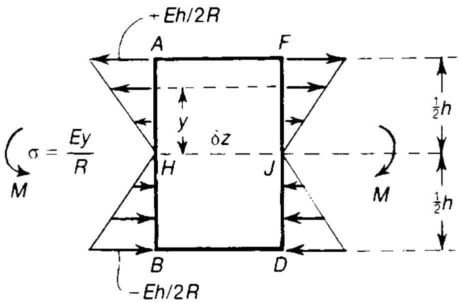


Figure 9.6 Distribution of bending stresses giving zero resultant longitudinal force and a resultant couple M.

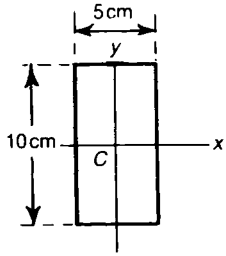
where I_x is the second moment of area of the cross-section about Cx. From equations (9.1) and (9.3), we have

$$\frac{\sigma}{y} = \frac{E}{R} = \frac{M}{I_x} \qquad (9.4)$$

We deduce that a uniform radius of curvature, R, of the centroidal axis Cz can be sustained by end couples M, applied about the axes Cx at the ends of the beam.

Equation (9.3) implies a linear relationship between M, the applied moment, and (1/R), the curvature of the beam. The constant EI_x in this linear relationship is called the *bending stiffness* or sometimes the *flexural stiffness* of the beam; this stiffness is the product of Young's modulus, E, and the second moment of area, I_x, of the cross-section about the axis of bending.

Problem 9.1 A steel bar of rectangular cross-section, 10 cm deep and 5 cm wide, is bent in the planes of the longer sides. Estimate the greatest allowable bending moment if the bending stresses are not to exceed 150 MN/m^2 in tension and compression.



Solution

The bending moment is applied about Cx. The second moment of area about this axis is

$$I_x = \frac{1}{12}(0.05)(0.10)^3 = 4.16 \times 10^{-6} \text{ m}^2$$

The bending stress, σ, at a fibre a distance y from Cx is, by equation (9.4)

$$\sigma = \frac{My}{I_x}$$

where M is the applied moment. If the greatest stresses are not to exceed 150 MN/m^2, we must have

$$\frac{My}{I_x} \leq 150 \text{ MN/m}^2$$

The greatest bending stresses occur in the extreme fibres where $y = 5$ cm. Then

$$M \leq \frac{(150 \times 10^6) I_x}{(0.05)} = \frac{(150 \times 10^6)(4.16 \times 10^{-6})}{(0.05)}$$

$$= 12500 \text{ Nm}$$

The greatest allowable bending moment is therefore 12 500 Nm. (The second moment of area about Cy is

$$I_y = \frac{1}{12}(0.10)(0.05)^3 = 1.04 \times 10^{-6} \text{ m}^2$$

The greatest allowable bending moment about Cy is

$$M = \frac{(150 \times 10^6) I_y}{(0.025)} = \frac{(150 \times 10^6)(1.04 \times 10^{-6})}{(0.025)}$$

$$= 6250 \text{ Nm}$$

which is only half that about Cx.

9.3 Bending of a beam about a principal axis

In section 9.2 we considered the bending of a straight beam of rectangular cross-section; this form of cross-section has two axes of symmetry. More generally we are concerned with sections having only one, or no, axis of symmetry.

Consider a long straight uniform beam having any cross-sectional form, Figure 9.7; the axes Cx and Cy are *principal axes* of the cross-section. The principal axes of a cross-section are those centroid axes for which the product second moments of area are zero. In Figure 9.7, C is the centroidal of the cross-section; Cz is the longitudinal centroidal axis.

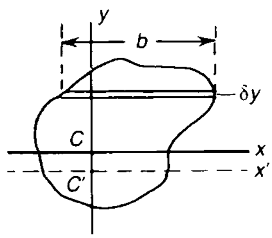


Figure 9.7 General cross-sectional form of a beam.

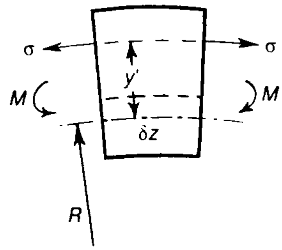


Figure 9.8 Elemental length of a beam.

When end couples M are applied to the beam, we assume as before that transverse sections of the beam remain plane during bending. Suppose further that, if the beam is bent in the yz-plane only, there is a neutral axis $C'x'$, Figure 9.7, which is parallel to Cx and is unstrained; radius of curvature of this neutral surface is R, Figure 9.8. As before, the strain in a longitudinal fibre at a distance y' from $C'x'$ is

$$\varepsilon = \frac{y'}{R}$$

If the material of the beam remains elastic during bending the longitudinal stress on this fibre is

$$\sigma = \frac{Ey'}{R}$$

If there is to be no resultant longitudinal thrust on the beam at any transverse section we must have

$$\int_A \sigma b\,dy' = 0$$

Where b is the breadth of an elemental strip of the cross-section parallel to Cx, and the integration is performed over the whole cross-sectional area, A. But

$$\int_A \sigma b\,dy' = \frac{E}{R}\int_A y' b\,dy'$$

This can be zero only if $C'x'$ is a centroidal axis; now, Cx is a principal axis, and is therefore a centroidal axis, so that $C'x'$ and Cx are coincident, and the neutral axis is Cx in any cross-section of the beam. The total moment about Cx of the internal stresses is

$$M = \int_A \sigma b y \, dy = \frac{E}{R} \int_A b y^2 \, dy$$

But $\int_A by^2\,dy$ is the second moment of area of the cross-section about Cx; if this is denoted by I_x, then

$$M = \frac{EI_x}{R} \tag{9.5}$$

The stress in any fibre a distance y from Cx is

$$\sigma = \frac{Ey}{R} = \frac{My}{I_x} \tag{9.6}$$

No moment about Cy is implied by this stress system, for

$$\int_A \sigma x \, dA = \frac{E}{R} \int_A xy \, dA = 0$$

because Cx and Cy are principal axes for which $\int_A xy\,dA$, or the product second moment of area, is zero; δA is an element of area of the cross-section.

9.4 Beams having two axes of symmetry in the cross-section

Many cross-sectional forms used in practice have two axes of symmetry; examples are the I-section and circular sections, Figure 9.9, besides the rectangular beam already discussed.

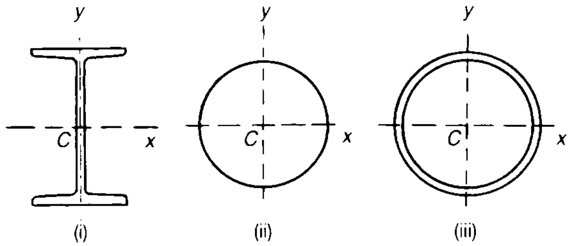


Figure 9.9 (i) I-section beam. (ii) Solid circular cross-section. (iii) Hollow circular cross-section.

An axis of symmetry of a cross-section is also a principal axis; then for bending about the axis Cx we have, from equation (9.6),

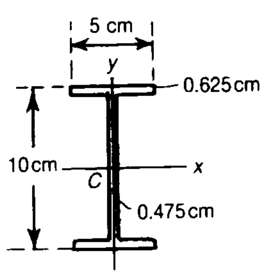
$$\sigma = \frac{Ey}{R_x} = \frac{M_x y}{I_x} \tag{9.7}$$

where R_x is the radius of curvature in the yz-plane, M_x is the moment about Cx, and I_x is the second moment of area about Cx. Similarly for bending by a couple M_y about Cy,

$$\sigma = \frac{Ex}{R_y} = \frac{M_y x}{I_y} \tag{9.8}$$

where R_y is the radius of curvature in the xz-plane, and I_y is the second moment of area about Cy. The longitudinal centroid axis is Cz. From equations (9.7) and (9.8) we see that the greatest bending stresses occur in the extreme longitudinal fibres of the beams.

Problem 9.2 A light-alloy I-beam of 10 cm overall depth has flanges of overall breadth 5 cm and thickness 0.625 cm, the thickness of the web is 0.475 cm. If the bending stresses are not to exceed 150 MN/m^2 in tension and compression estimate the greatest moments which may be applied about the principal axes of the cross-section.



Solution

Consider, first, bending about Cx. From equation (8.10), the second moment of area about Cx is

$$I_x = 0.05 \times 0.1^3/12 - (0.05 - 0.00475) \times (0.1 - 2 \times 0.00625)^3/12$$

$$= 4.167 \times 10^{-6} - 0.04525 \times 0.0875^3/12$$

$$= 4.167 \times 10^{-6} - 2.526 \times 10^{-6}$$

$$I_x = 1.641 \times 10^{-6} \text{ m}^4$$

The above calculation has been obtained by taking away the second moments of area of the two inner rectangles from the second moment of area of the outer rectangle, as previously demonstrated in Chapter 8. The allowable moment M_x is

$$M_x = \frac{\sigma I_x}{y} = \frac{(150 \times 10^6)(1.64 \times 10^{-6})}{0.05} = 4926 \text{ Nm}$$

Second, for bending about *Cy*.

$$I_y = (0.1 - 2 \times 0.00625) \times 0.00475^3 / 12 + 2 \times 0.00625 \times 0.05^3 / 12$$

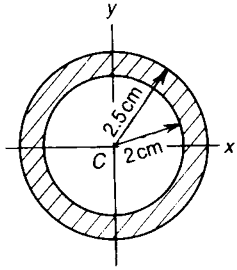
The first term, which is the contribution of the web, is negligible compared with the second. With sufficient accuracy

$$I_y = 2\left(\frac{1}{12}\right)(0.00625)(0.05)^3 = 0.130 \times 10^{-6} \text{ m}^4$$

The allowable moment about *Cy* is

$$M_y = \frac{\sigma I_y}{x} = \frac{(150 \times 10^6)(0.130 \times 10^{-6})}{0.025} = 780 \text{ Nm}$$

Problem 9.3 A steel scaffold tube has an external diameter of 5 cm, and a thickness of 0.5 cm. Estimate the allowable bending moment on the tube if the bending stresses are limited to 100 MN/m^2.



Solution

From equation (8.19), the second moment of area about a centroid axis *Cx* is

$$I_x = \frac{\pi}{4}\left[(0.025)^4 - (0.020)^4\right] = 0.181 \times 10^{-6} \text{ m}^4$$

The allowable bending moment about *Cx* is

$$M_x = \frac{(100 \times 10^6)(0.181 \times 10^{-6})}{0.025} = 724 \text{ Nm}$$

9.5 Beams having only one axis of symmetry

Other common sections in use, as shown in Figure 9.10, have only one axis of symmetry Cx. In each of these, Cx is the axis of symmetry, and Cx and Cy are both principal axes. When bending moments M_x and M_y are applied about Cx and Cy, respectively, the bending stresses are again given by equations (9.7) and (9.8). However, an important feature of beams of this type is that their behaviour in bending when shearing forces are also present is not as simple as that of beams having two axes of symmetry. This problem is discussed in Chapter 10.

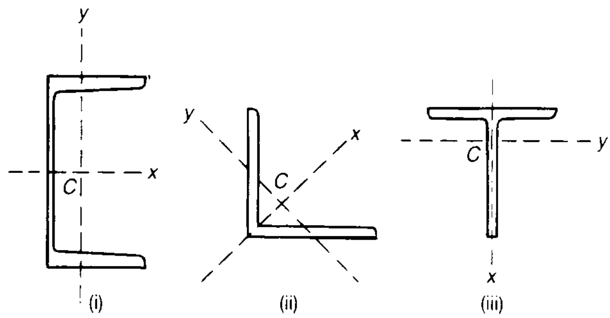


Figure 9.10 (i) Channel section. (ii) Equal angle section. (iii) T–section.

Problem 9.4 A T-section of uniform thickness 1 cm has a flange breadth of 10 cm and an overall depth of 10 cm. Estimate the allowable bending moments about the principal axes if the bending stresses are limited to 150 MN/m^2.

Solution

Suppose $\bar{y}$ is the distance of the principal axis Cx from the remote edge of the flange. The total area of the section is

$$A = (0.10)\ (0.01) + (0.09)\ (0.01) = 1.90 \times 10^{-3}\ \mathrm{m}^2$$

On taking first moments of areas about the upper edge of the flange,

$$A\bar{y} = (0.10)(0.01)(0.005) + (0.09)(0.01)(0.055) = 0.0545 \times 10^{-3}\mathrm{m}^3$$

Then

$$\bar{y} = \frac{0.0545 \times 10^{-3}}{1.9 \times 10^{-3}} = 0.0287 \text{ m}$$

The second moment of area of the flange about Cx is

$$\frac{1}{12}(0.10)(0.01)^3 + (0.10)(0.01)(0.0237)^2 = 0.570 \times 10^{-6} \text{ m}^4$$

The second moment of area of the web about Cx is

$$\frac{1}{12}(0.01)(0.09)^3 + (0.09)(0.01)(0.0263)^2 = 1.230 \times 10^{-6} \text{ m}^4$$

Then

$$I_x = (0.570 + 1.230)\, 10^{-6} = 1.800 \times 10^{-6} \text{ m}^4$$

For bending about Cx, the greatest bending stress occurs at the toe of the web, as shown in the figure. The maximum allowable moment is

$$M_x = \frac{(150 \times 10^6)(1.800 \times 10^{-6})}{0.0713} = 3790 \text{ Nm}$$

The bending stress in the extreme fibres of the flange is only 60.4 MN/m^2 at this bending moment. The second moment of area about Cy is

$$I_y = \frac{1}{12}(0.01)(0.10)^3 + \frac{1}{12}(0.09)(0.01)^3 = 0.841 \times 10^{-6} \text{ m}^4$$

The T-section is symmetrical about Cy, and for bending about this axis equal tensile and compressive stresses are induced in the extreme fibres of the flange; the greatest allowable moment is

$$M_y = \frac{(150 \times 10^6)(0.841 \times 10^{-6})}{0.05} = 2520 \text{ Nm}$$

9.6 More general case of pure bending

In the analysis of the preceding sections we have assumed either that the cross-section has two axes of symmetry, or that bending takes place about a principal axis. In the more general case we are interested in bending stress in the beam when moments are applied about any axis of the cross-section. Consider a long uniform beam, Figure 9.11, having any cross-section; the centroid of a cross-section is C, and Cz is the longitudinal axis of the beam; Cx and Cy are any two mutually perpendicular axes in the cross-section. The axes Cx, Cy and Cz are therefore centroidal axes of the beam.

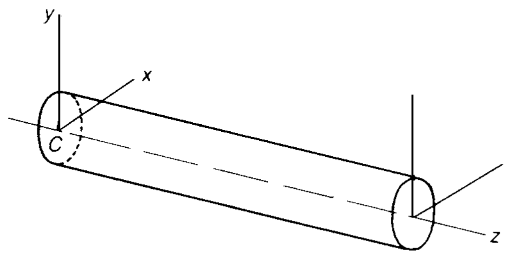


Figure 9.11 Co-ordinate system for a beam of any cross-sectional form.

We suppose first that the beam is bent in the yz-plane only, in such a way that the axis Cz takes up the form of a circular arc of radius R_x, Figure 9.12. Suppose further there is no longitudinal strain of Cx; this axis is then a neutral axis. The strain at a distance y from the neutral axis is

$$\varepsilon = \frac{y}{R_x}$$

If the material of a beam is elastic, the longitudinal stress in this fibre is

$$\sigma = \frac{Ey}{R_x}$$

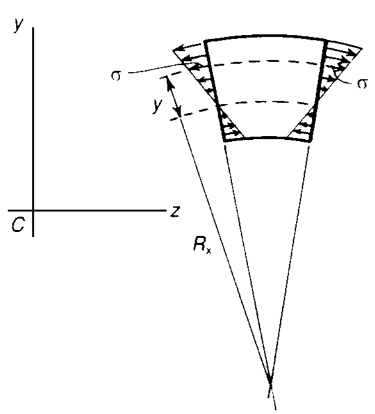


Figure 9.12 Bending in the yz-plane.

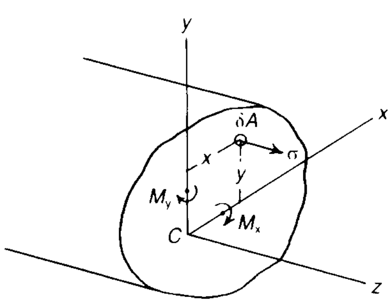


Figure 9.13 Bending moments about the axes C_x and C_y.

Suppose δA is a small element of area of the cross-section of the beam acted upon by the direct stress σ, Figures 9.12 and 9.13. Then the total thrust on any cross-section in the direction Cz is where the integration is performed over the whole area A of the beam. But, as Cx is a centriodal axis, we have

$$\int_A y dA = 0$$

and no resultant longitudinal thrust is implied by the stresses σ. The moment about Cx due to the stresses σ is

$$M_x = \int_A \sigma y dA = \frac{E}{R_x}\int_A y^2 dA = \frac{EI_x}{R_x} \tag{9.9}$$

where I_x is the second moment of area of the cross-section about Cx. For the resultant moment about Cy we have

$$M_y = \int_A \sigma x dA = \frac{E}{R_x}\int_A xy dA = \frac{EI_{xy}}{R_x} \tag{9.10}$$

where I_{xy} is the product second moment of area of the cross-section about Cx and Cy. Unless I_{xy} is zero, in which case Cx and Cy are the principal axes, bending in the yz-plane implies not only a couple M_x about the Cx axis, but also a couple M_y about Cy.

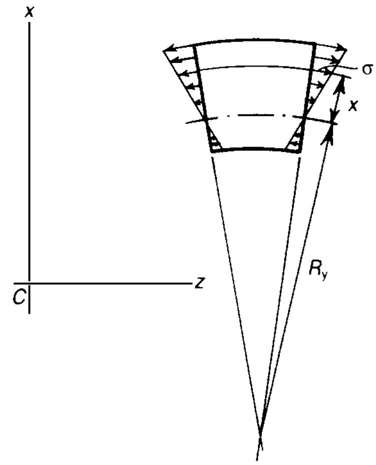


Figure 9.14 Bending in the xz-plane.

When the beam is bent in the xz-plane only, Figure 9.14, so that Cz again lies in the neutral surface, and takes up a curve of radius R_y, the longitudinal stress in a fibre a distance x from the neutral axis is

$$\sigma = \frac{Ex}{R_y}$$

The thrust implied by these stresses is again zero as

$$\int_A \sigma dA = \frac{E}{R_y}\int_A x dA = 0$$

because Cy is a centroidal axis of the cross-section. The bending moment about Cy due to stresses σ is

$$M_y = \int_A \sigma x dA = \frac{E}{R_y}\int_A x^2 \, dA = \frac{EI_y}{R_y} \tag{9.11}$$

where I_y is the second moment of area of the cross-section about Cy. Furthermore,

$$M_x = \int_A \sigma y dA = \frac{E}{R_y}\int_A xy dA = \frac{EI_{xy}}{R_y} \tag{9.12}$$

where I_{xy} is again the product second moment of area.

If we now superimpose the two loading conditions, the total moments about the axes Cx and Cy, respectively, are

$$M_x = \frac{EI_x}{R_x} + \frac{EI_{xy}}{R_y} \tag{9.13}$$

$$M_y = \frac{EI_y}{R_y} + \frac{EI_{xy}}{R_x} \tag{9.14}$$

These equations may be rearranged in the forms

$$\frac{1}{R_x} = \frac{M_x I_y - M_y I_{xy}}{E\left(I_x I_y - I_{xy}^2\right)} \tag{9.15}$$

$$\frac{1}{R_y} = \frac{M_y I_x - M_x I_{xy}}{E\left(I_x I_y - I_{xy}^2\right)} \tag{9.16}$$

where $(1/R_x)$ and $(1/R_y)$ are the curvatures in the yz-and xz-planes caused by any set of moments

M_x and M_y. If C_x and C_y are the principal centroid axes then $I_{xy} = 0$, and equations (9.15) and (9.16) reduce to

$$\frac{1}{R_x} = \frac{M_x}{EI_x}, \quad \frac{1}{R_y} = \frac{M_y}{EI_y} \tag{9.17}$$

In general we require a knowledge of three geometrical properties of the cross-section, namely I_x, I_y and I_{xy}. The resultant longitudinal stress at any point (x, y) of the cross-section of the beam is

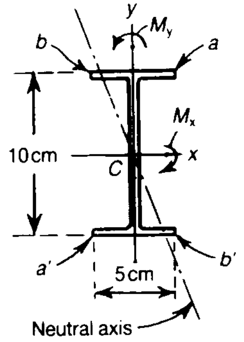
$$\sigma = \frac{Ex}{R_y} + \frac{Ey}{R_x} = \frac{x\left(M_y I_x - M_x I_{xy}\right) + y\left(M_x I_y - M_y I_{xy}\right)}{\left(I_x I_y - I_{xy}^2\right)} \tag{9.18}$$

This stress is zero for points of the cross-section on the line

$$x(M_y I_x - M_x I_{xy}) + y(M_x I_y - M_y I_{xy}) = 0 \tag{9.19}$$

which is the equation of the unstressed fibre, or neutral axis, of the beam.

Problem 9.5 The I-section of Problem 9.2 is bent by couples of 2500 Nm about Cx and 500 Nm about Cy. Estimate the maximum bending stress in the cross-section, and find the equation of the neutral axis of the beam.



<u>*Solution*</u>

From Problem 9.2

$$I_x = 1.641 \times 10^{-6} \text{ m}^4, \quad I_y = 0.130 \times 10^{-6} \text{ m}^4$$

For bending about *Cx* the bending stresses in the extreme fibres of the flanges are

$$\sigma = \frac{M_x y}{I_x} = \frac{(2500)\,(0.05)}{1.641 \times 10^{-6}} = 76.1 \text{ MN/m}^2$$

For bending about *Cy* the bending stresses at the extreme ends of the flanges are

$$\sigma = \frac{M_y x}{I_y} = \frac{(500)\,(0.025)}{0.130 \times 10^{-6}} = 96.1 \text{ MN/m}^2$$

On superposing the stresses due to the separate moments, the stress at the corner *a* is tensile, and of magnitude

$$\sigma_a = (76.1 + 96.1) = 172.2 \text{ MN/m}^2$$

The total stress at the corner *a*′ is also 172.2 MN/m^2, but compressive. The total stress at the corner *b* is compressive, and of magnitude

$$\sigma_b = (96.1 - 76.1) = 20.2 \text{ MN/m}^2$$

The total stress at the corner *b*′ is also 20.0 MN/m^2, but tensile. The equation of the neutral axis is given by

$$xM_y I_x + yM_x I_y = 0$$

Then

$$\frac{y}{x} = -\frac{M_y I_x}{M_x I_y} = -\frac{(500)\,(1.641 \times 10^{-6})}{(2500)\,(0.130 \times 10^{-6})} = -2.53$$

The greatest bending stresses occur at points most remote from the neutral axis; these are the points *a* and *a*′: the greatest bending stresses are therefore ±172.2 MN/m^2.

9.7 Elastic section modulus

For bending of a section about a principal axis Cx, the longitudinal bending stress at a fibre a distance y from Cx, due to a moment M_x, is from equation (9.18) (in which we put $I_{xy} = 0$ and $M_y = 0$),

$$\sigma = \frac{M_x y}{I_x}$$

where I_x is the second moment of area about Cx. The greatest bending stress occurs at the fibre most remote from Cx. If the distance to the extreme fibre is y_{max}, the maximum bending stress is

$$\sigma_{max} = \frac{M_x y_{max}}{I_x}$$

The allowable moment for a given value of σ_{max} is therefore

$$M_x = \frac{I_x \sigma_{max}}{y_{max}} \tag{9.20}$$

The geometrical quantity (I_x/y_{max}) is the *elastic section modulus*, and is denoted by Z_e.

Then

$$M_x = Z_e \, \sigma_{max} \tag{9.21}$$

The allowable bending moment is therefore the product of a geometrical quantity, Z_e, and the maximum allowable stress, σ_{max}. The quantity $Z_e \, \sigma_{max}$ is frequently called the *elastic moment of resistance.*

Problem 9.6 A steel I-beam is to be designed to carry a bending moment of 10^5 Nm, and the maximum bending stress is not to exceed 150 MN/m^2. Estimate the required elastic section modulus, and find a suitable beam.

<u>*Solution*</u>

The required elastic section modulus is

$$Z_e = \frac{M}{\sigma} = \frac{10^5}{150 \times 10^6} = 0.667 \times 10^{-3} \text{ m}^3$$

The elastic section modulus of a 22.8 cm by 17.8 cm standard steel I-beam about its axis of greatest bending stiffness is 0.759×10^{-3} m^3, which is a suitable beam.

9.8 Longitudinal stresses while shearing forces are present

The analysis of the proceeding paragraphs deals with longitudinal stresses in beams under uniform bending moment. No shearing forces are present at cross-sections of the beam in this case.

When a beam carries lateral forces, bending moments may vary along the length of the beam. Under these conditions we may assume with sufficient accuracy in most engineering problems that the longitudinal stresses at any section are dependant only on the bending moment at that section, and are unaffected by the shearing force at that section.

Where a shearing force is present at the section of a beam, an elemental length of the beam undergoes a slight shearing distortion; these shearing distortions make a negligible contribution to the total deflection of the beam in most engineering problems.

Problem 9.7 A 4 m length of the I-beam of Problem 9.2 is simply-supported at each end. What maximum central lateral load may be applied if the bending stresses are not to exceed 150 MN/m^2?

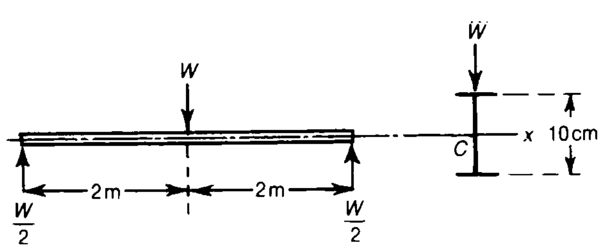
Solution

Suppose W is the central load. If this is applied in the plane of the web, then bending takes place about Cx. The maximum bending moment is

$$M_x = \frac{1}{2}W(2) = W \text{ Nm}$$

From Problem 9.2,

$$I_x = 1.641 \times 10^{-6} \text{ m}^4$$



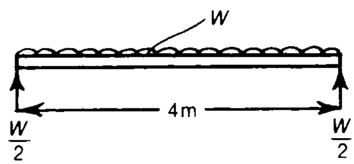
Then, the greatest bending stress is

$$\sigma = \frac{M_x \, y_{max}}{I_x} = \frac{(W)\,(0.05)}{1.641 \times 10^{-6}}$$

If this is equal to 150 MN/m², then

$$W = \frac{(150 \times 10^6)\,(1.641 \times 10^{-6})}{0.05} = 4920 \text{ N}$$

Problem 9.8 If the bending stresses are again limited to 150 MN/m², what total uniformly-distributed load may be applied to the beam of Problem 9.7?



Solution

The maximum bending moment occurs at mid-span, and has the value

$$M_x = \frac{WL}{8} = \frac{1}{2} W \text{ Nm}$$

Then

$$\frac{1}{2} W = \frac{(150 \times 10^6)\,(1.641 \times 10^{-6})}{0.05} = 4920 \text{ N}$$

and

$$W = 9840 \text{ N}$$

9.9 Calculation of the principal second moments of area

In problems of bending involving beams of unsymmetrical cross-section we have frequently to find the principal axes of the cross-section.

Suppose Cx and Cy are any two centroidal axes of the cross-section of the beam, Figure 9.15.

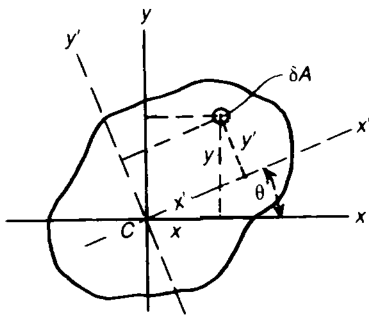


Figure 9.15 Derivation of the principal axes of a section.

If δA is an elemental area of the cross-section at the point (x, y), then the property of the axes Cx and Cy is that

$$\int_A x dA = \int_A y dA = 0$$

The second moments of area about the axes Cx and Cy, respectively, are

$$I_x = \int_A y^2 dA, \quad I_y = \int_A x^2 dA \tag{9.22}$$

The product second moment of area is

$$I_{xy} = \int_A xy dA \tag{9.23}$$

Now consider two mutually perpendicular axes Cx' and Cy', which are the principal axes of bending, inclined at an angle θ to the axes Cx and Cy. A point having co-ordinates (x, y) in the xy-system, now has co-ordinates (x', y') in the $x'y'$-system. Further, we have

$$x' = x \cos\theta + y \sin\theta$$

$$y' = y \cos\theta - x \sin\theta$$

The second moment of area of the cross-section about Cx' is

$$I_{x'} = \int_A y'1^2 dA$$

which becomes

$$I_{x'} = \int_A (y\cos\theta - x\sin\theta)^2\, dA$$

This may be written

$$I_{x'} = \cos^2\theta \int_A y^2 dA - 2\cos\theta\sin\theta \int_A xy dA + \sin^2\theta \int_A x^2 dA$$

But

$$\int_A y^2\, dA = I_x, \int_A x^2\, dA = I_y, \text{ and } \int_A xy dA = I_{xy}$$

Then

$$I_{x'} = I_x\cos^2\theta - 2I_{xy}\cos\theta\sin\theta + I_y\sin^2\theta \qquad (9.24)$$

Similarly, the second moment of area about Cy' is

$$I_{y'} = \int_A x'^2 dA = \int_A (x\cos\theta + y\sin\theta)^2\, dA$$

Then

$$I_{y'} = I_y\cos^2\theta + 2I_{xy}\cos\theta\sin\theta + I_x\sin^2\theta \qquad (9.25)$$

Finally, the product second moment of area about Cx' and Cy' is

$$I_{x'y'} = \int_A x'y' dA \int_A (x\cos\theta + y\sin\theta)(y\cos\theta - x\sin\theta) dA$$

Then

$$I_{x'y'} = I_x\sin\theta\cos\theta + I_{xy}\left(\cos^2\theta - \sin^2\theta\right) - I_y\cos\theta\sin\theta \qquad (9.26)$$

We note from equations (9.24) and (9.25), that

$$I_{x'} + I_{y'} = I_x + I_y \qquad (9.27)$$

that is, the sum of the second moments of area about any perpendicular axes is independent of θ. The sum is in fact the polar second moment of area, or the second moment of area about an axis through C, perpendicular to the xy-plane.

We may write equation (9.26) in the form

$$I_{x'y'} = \frac{1}{2}\left(I_x - I_y\right) \sin 2\theta + I_{xy} \cos 2\theta \tag{9.28}$$

The principal axes Cx' and Cy' are defined as those for which $I_{x'y'} = 0$; then for the principal axes

$$\frac{1}{2}\left(I_x - I_y\right) \sin 2\theta + I_{xy} \cos 2\theta = 0$$

or

$$\tan 2\theta = \frac{2I_{xy}}{I_y - I_x} \tag{9.29}$$

This relationship gives two values of θ differing by 90°. On making use of equation (9.27), we may write equations (9.24) and (9.25) in the forms

$$I_{x'} = \frac{1}{2}\left(I_x + I_y\right) + \frac{1}{2}\left(I_x - I_y\right)\cos 2\theta + I_{xy} \sin 2\theta$$

$$I_{y'} = \frac{1}{2}\left(I_x + I_y\right) - \frac{1}{2}\left(I_x - I_y\right)\cos 2\theta + I_{xy} \sin 2\theta \tag{9.30}$$

Now

$$I_{x'} I_{y'} = \left[\frac{1}{2}\left(I_x + I_y\right) + \frac{1}{2}\left(I_x - I_y\right) \cos 2\theta - I_{xy} \sin 2\theta\right]$$

$$\times \left[\frac{1}{2}\left(I_x + I_y\right) - \frac{1}{2}\left(I_x - I_y\right) \cos 2\theta + I_{xy} \sin 2\theta\right]$$

$$= \frac{1}{4}\left(I_x + I_y\right)^2 - \frac{1}{4}\left(I_x + I_y\right)\left(I_x - I_y\right) \cos 2\theta$$

$$+ \frac{1}{2}\left(I_x + I_y\right) . I_{xy} \sin 2\theta$$

$$+ \frac{1}{4}\left(I_x + I_y\right) \cos 2\theta . \left(I_x - I_y\right)$$

$$-\frac{1}{4}\left(I_x - I_y\right)^2 \cos^2 2\theta + \frac{1}{2}\left(I_x - I_y\right)\cos 2\theta \,.\, I_{xy} \,.\, \sin 2\theta$$

$$-\frac{1}{2} I_{xy}\left(I_x + I_y\right)\sin 2\theta + \frac{1}{2} I_{xy}\left(I_x - I_y\right)\sin 2\theta \cos 2\theta - I_{xy}^2 \sin^2 2\theta$$

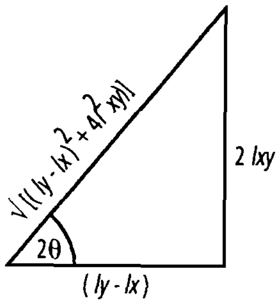
$$= \frac{1}{4}\left(I_x + I_y\right)^2 - \frac{1}{4}\left(I_x - I_y\right)^2 \cos^2 2\theta - I_{xy}^2 \sin^2 2\theta$$

$$+ I_{xy}\left(I_x - I_y\right)\sin 2\theta \cos 2\theta$$

or

$$I_{x'} I_{y'} = \left[\frac{1}{2}\left(I_x + I_y\right)\right]^2 - \left\{\frac{1}{2}\left[\left(I_x - I_y\right)\cos 2\theta - 2 I_{xy}\sin 2\theta\right]\right\}^2 \qquad (9.31)$$

From equation (9.29), the mathematical triangle of the figure below is obtained:



From the mathematical triangle

$$\cos 2\theta = \frac{\left(I_y - I_x\right)}{\sqrt{\left(I_y - I_x\right)^2 + 4 I_{xy}^2}}$$

and

$$\sin 2\theta = \frac{2 I_{xy}}{\sqrt{\left(I_y - I_x\right)^2 + 4 I_{xy}^2}}$$

$$\therefore I_{x'} I_{y'} = \left[\frac{1}{2}(I_x + I_y)\right]^2 - \left\{\frac{\frac{1}{2}(I_x - I_y) \cdot (I_y - I_x)}{\sqrt{(I_y - I_x)^2 + 4 I_{xy}^2}} - \frac{2 I_{xy} \cdot 2 I_{xy}}{\sqrt{(I_y - I_x)^2 + 4 I_{xy}^2}}\right\}^2$$

$$= \left[\frac{1}{2}(I_x + I_y)\right]^2 - \left\{\frac{1}{2} \cdot \frac{-[I_y - I_x]^2 - 4 I_{xy}^2}{\sqrt{(I_y - I_x)^2 + 4 I_{xy}^2}}\right\}^2$$

$$= \left[\frac{1}{2}(I_x + I_y)\right]^2 - \left\{\frac{1}{2}\sqrt{(I_y - I_x)^2 + 4 I_{xy}^2}\right\}^2$$

$$= \frac{1}{4}\left(I_x^2 + I_y^2 + 2 I_x I_y\right) - \frac{1}{4}\left[(I_y - I_x)^2 + 4 I_{xy}^2\right]$$

$$= \frac{1}{4}\left(I_x^2 + I_y^2 + 2 I_x I_y\right) - \frac{1}{4}\left(I_y^2 + I_x^2 - 2 I_x I_y + 4 I_{xy}^2\right)$$

or $$I_{x'} I_{y'} = I_x I_y - I_{xy}^2 \tag{9.32}$$

Substituting equation (9.27) into equation (9.32) we get

$$(I_x + I_y - I_{y'}) I_{y'} = I_x I_y - I_{xy}^2$$

or $$I_{y'}^2 - (I_x + I_y) I_{y'} + (I_x I_y - I_{xy}^2) \tag{9.33a}$$

Similarly,

$$I_{x'}^2 - (I_x + I_y) I_{x'} + (I_x I_y - I_{xy}^2), \tag{9.33b}$$

which are both quadratic equations.

In general, equations (9.33a) and (9.33b) can be written as the following quadratic equation, where I = a *principal* second moment of area

$$I^2 - (I_x + I_y) I + (I_x I_y - I_{xy}^2) = 0 \tag{9.34}$$

Then

$$I = \frac{1}{2}\left(I_x + I_y\right) \pm \sqrt{\frac{1}{4}\left(I_x + I_y\right)^2 - \left(I_x\, I_y - I_{xy}^2\right)} \tag{9.35}$$

which may be written

$$I = \frac{1}{2}\left(I_x - I_y\right) \pm \sqrt{\frac{1}{4}\left(I_x - I_y\right)^2 + I_{xy}^2} \tag{9.36}$$

Equations (9.30) and (9.26) may be written in the forms

$$I_{x'} - \frac{1}{2}\left(I_x + I_y\right) = \frac{1}{2}\left(I_x - I_y\right)\cos 2\theta - I_{xy}\sin 2\theta$$

$$I_{x'y'} = \frac{1}{2}\left(I_x - I_y\right)\sin 2\theta + I_{xy}\cos 2\theta \tag{9.37}$$

Square each equation, and then add; we have

$$\left[I_{x'} - \tfrac{1}{2}\left(I_x - I_y\right)\right]^2 + \left[I_{x'y'}\right]^2 = \left[\tfrac{1}{2}\left(I_x - I_y\right)\right]^2 + \left[I_{xy}\right]^2 \tag{9.38}$$

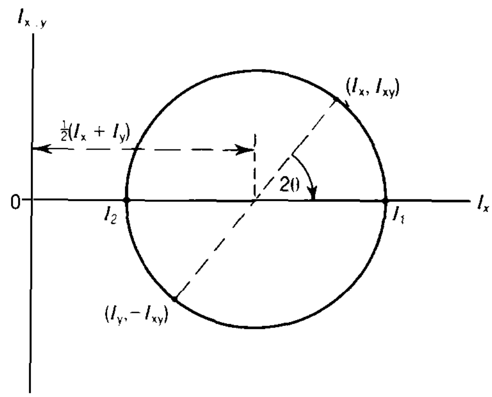


Figure 9.16 Graphical representation of the second moments of area.

Then I_x, I_{xy} lie on a circle of radius

$$\left\{\left[\frac{1}{2}\left(I_x - I_y\right)\right]^2 + \left[I_{xy}^2\right]\right\}^{\frac{1}{2}} \tag{9.39}$$

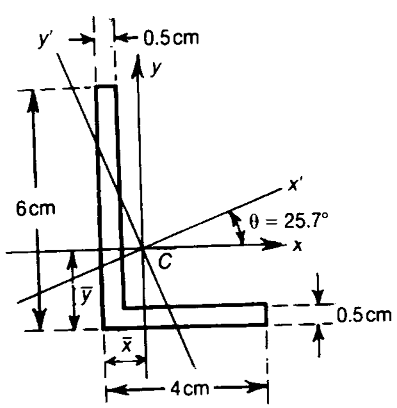
and centre

$$\left[\frac{1}{2}\left(I_x + I_y\right),\ 0\right] \tag{9.40}$$

in the $I_{x'}$, $I_{x'y'}$ diagram.

Suppose $OI_{x'}$ and $OI_{x'y'}$ are mutually perpendicular axes; then equation (9.38) has the graphical representation shown in Figure 9.16. To find the principal second moments of area, locate the points (I_x, I_{xy}) and $(I_y, -I_{xy})$ in the $(I_{x'}, I_{x'y'})$ plane. With the line joining these points as a diameter construct a circle. The principal second moments of area, I_1 and I_2, are given by the points where the circle cuts the axis $OI_{x'}$. Figure 9.16 might be referred to as the *circle of second moments of area*.

Problem 9.9 An unequal angle section of uniform thickness 0.5 cm has legs of lengths 6 cm and 4 cm. Estimate the positions of the principal axes, and the principal second moments of area.



<u>*Solution*</u>

Firstly, find the position of the centroid of the cross-section. Total area is

$$A = (0.06)\,(0.005) + (0.035)\,(0.005)$$

$$= 0.475 \times 10^{-3}\ \mathrm{m}^2$$

Now

$$A\bar{x} = (0.055)\,(0.005)\,(0.0025) + (0.04)\,(0.005)\,(0.02)$$

$$= 4.69 \times 10^{-6}\ \mathrm{m}^3$$

Then

$$\bar{x} = \frac{4.69 \times 10^{-6}}{0.475 \times 10^{-3}} = 9.86 \times 10^{-3} \text{ m}$$

Again

$$A\bar{y} = (0.035)\,(0.005)\,(0.0025) + (0.06)\,(0.005)\,(0.03) = 9.44 \times 10^{-6} \text{ m}^3$$

Then

$$\bar{y} = \frac{9.44 \times 10^{-6}}{0.475 \times 10^{-3}} = 19.85 \times 10^{-3} \text{ m}$$

Now

$$I_x = \frac{1}{3}\,(0.005)\,(0.06)^3 + \frac{1}{3}\,(0.035)\,(0.005)^3 - (0.475 \times 10^{-3})\,(0.01985)^2$$

$$= 0.174 \times 10^{-6} \text{ m}^4$$

and

$$I_y = \frac{1}{3}(0.005)(0.04)^3 + \frac{1}{3}(0.055)(0.005)^3 - (0.475 \times 10^{-3})(0.00986)^2$$

$$= 0.063 \times 10^{-6} \text{ m}^4$$

With the axes Cx and Cy having the positive directions shown,

$$I_{xy} = \int\int xy\,dx\,dy$$

$$= \int_{-\bar{x}}^{0.04-\bar{x}} x\,dx \int_{-\bar{y}}^{0.005-\bar{y}} y\,dy + \int_{\bar{x}}^{0.005-\bar{x}} x\,dx \int_{-\bar{y}+0.005}^{0.06-\bar{y}} y\,dy$$

$$= \frac{1}{4}\left\{\left[(0.04-\bar{x})^2 - (-\bar{x})^2\right]\left[(0.005-\bar{y})^2 - (-\bar{y})^2\right]\right.$$

$$\left. + \left[(0.005-\bar{x})^2 - (-\bar{x})^2\right]\left[(0.06-\bar{y})^2 - (-\bar{y}+0.005)^2\right]\right\}$$

$$= -0.06 \times 10^{-6} \text{ m}^4$$

From equation (9.29),

$$\tan 2\theta = \frac{2 \times (-0.06)}{0.063 - 0.174} = 1.08$$

Then

$$2\theta = 47.2^\circ$$

and

$$\theta = 23.6^\circ$$

From equations (9.36) the principal second moments of area are

$$\frac{1}{2}\left(I_x + I_y\right) \pm \left[\left\{\frac{1}{2}\left(I_x - I_y\right)\right\}^2 + I_{xy}^2\right]^{\frac{1}{2}} = (0.1185 \pm 0.0988)\ 10^{-6}$$

$$= 0.2173 \text{ or } 0.0197 \times 10^{-6}\ \mathrm{m}^4$$

9.10 Elastic strain energy of bending

As couples are applied to a beam, strain energy is stored in the fibres. Consider an elemental length δz of a beam, which is bent about a principal axis Cx by a moment M_x, Figure 9.17. During bending, the moments M_x at each end of the element are displaced with respect to each other an angular amount

$$\theta = \frac{\delta z}{R_x} \qquad (9.41)$$

where R_x is the radius of curvature in the yz-plane. But from equation (9.6)

$$M_x = \frac{EI_x}{R_x}$$

and thus

$$\theta = \frac{M_x \delta z}{EI_x} \tag{9.42}$$

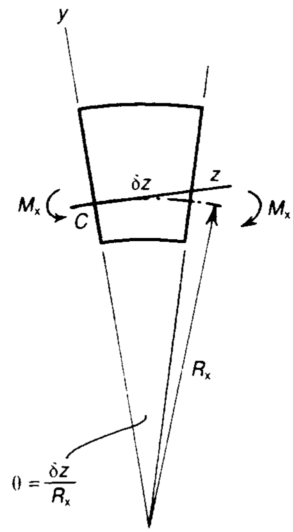


Figure 9.17 Bent form of an elemental length of beam.

As there is a linear relation between θ and M_x, the total work done by the moments M_x during bending of the element is

$$\frac{1}{2} M_x \theta = \frac{M_x^2 \delta z}{2EI_x} \tag{9.43}$$

which is equal to the strain energy of bending of the element. For a uniform beam of length L under a moment M_x, constant throughout its length, the bending strain energy is then

$$U = \frac{M_x^2 L}{2EI_x} \tag{9.44}$$

When the bending moment varies along the length, the total bending strain energy is

$$U = \int_L \frac{M_x^2 \, dz}{2EI_x} \tag{9.45}$$

where the integration is carried out over the whole length L of the beam.

9.11 Change of cross-section in pure bending

In Section 9.1 we pointed out the change which takes place in the shape of the cross-section when a beam is bent. This change involves infinitesimal lateral strains in the beam. The upper and lower edges of a cross-section which was originally rectangular, are strained into concentric circular arcs with their centre on the opposite side of the beam to the axis of bending. The upper and lower surfaces of the beam then have *anticlastic curvature*, the general nature of the strain being as shown in Figure 9.18. The anticlastic curvature effect can be readily observed by bending a flat piece of india-rubber. If the beam is bent to a mean radius R, we find that cross-sections are bent to a mean radius (R/v).

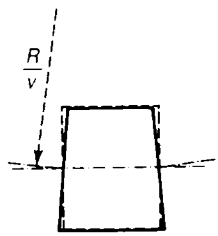


Figure 9.18 Anticlastic curvature in the cross-section of a bent rectangular beam.

Problem 9.10 What load can a beam 4 m long carry at its centre, if the cross-section is a hollow square 30 cm by 30 cm outside and 4 cm thick, the permissible longitudinal stress being 75 MN/m^2?

<u>Solution</u>

We must find the second moment of area of cross-section about its neutral axis. The inside is a square 22 cm by 22 cm. Then

$$\frac{1}{12}\left(0.3^4 - 0.22^4\right) = 0.47 \times 10^{-3} \text{ m}^4$$

The length of the beam is 4 m; therefore if W N be a concentrated load at the middle, the maximum bending moment is

$$M_x = \frac{WL}{4} = W \text{ Nm}$$

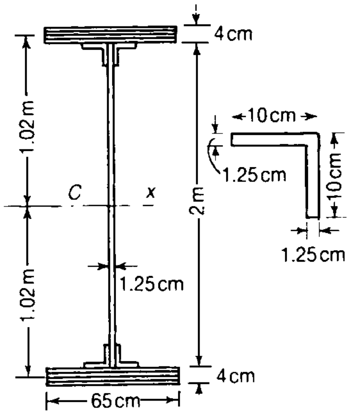
Hence the maximum stress is

$$\sigma = \frac{M_x y}{I_x} = \frac{W(0.15)}{0.47 \times 10^{-3}}$$

If $\sigma = 75$ MN/m^2 we must therefore have

$$W = \frac{(75 \times 10^6)(0.47 \times 10^{-3})}{0.15} = 235 \text{ kN}$$

Problem 9.11 Estimate the elastic section modulus and the maximum longitudinal stress in a built-up I-girder, with equal flanges carrying a load of 50 kN per metre run, with a clear span of 20 m. The web is of thickness 1.25 cm and the depth between flanges 2 m. Each flange consists of four 1 cm plates 65 cm wide, and is attached to the web by angle iron sections 10 cm by 10 cm by 1.25 cm thick. *(Cambridge)*



<u>*Solution*</u>

The second moment of area of each flange about Cx is

$$(0.04)(0.65)(1.02)^2 = 0.0270 \text{ m}^4$$

The second moment of area of the web about Cx is

$$\frac{1}{12}(0.0125)(2)^3 = 0.0083 \text{ m}^4$$

The horizontal part of each angle section has an area 0.00125 m^2, and its centroid is 0.944 m from the neutral axis. Therefore the corresponding second moment of area is approximately

$$(0.00125)(0.994)^2 = 0.0012 \text{ m}^4$$

The area of the vertical part of each angle section is 0.001093 m^2, and its centroid is 0.944 m from the neutral axis. Therefore the corresponding second moment of area is approximately

$$(0.001093)(0.944)^2 = 0.00097 \text{ m}^4$$

The second moment of area of the whole section of the angle section about Cx is then

$$0.0012 + 0.00097 = 0.0022 \text{ m}^4$$

The second moment of area of the whole cross-section of the beam is then

$$I_x = 2(0.0270) + (0.0083) + 4(0.0022)$$

$$= 0.0711 \text{ m}^4$$

The elastic section modulus is therefore

$$Z_e = \frac{0.0711}{1.04} = 0.0684 \text{ m}^3$$

The bending moment at the mid-span is

$$M_x = \frac{wL^2}{8} = \frac{(50)(20)^2}{8} = 2.50 \text{ MN.m}$$

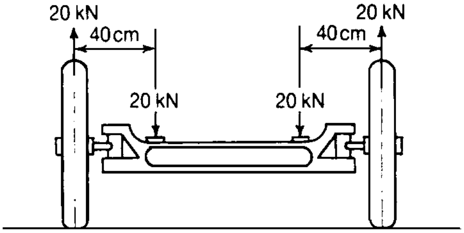
The greatest longitudinal stress is then

$$\sigma = \frac{M_x}{Z_e} = \frac{2.50 \times 10^6}{0.0684} = 36.6 \text{ MN/m}^2$$

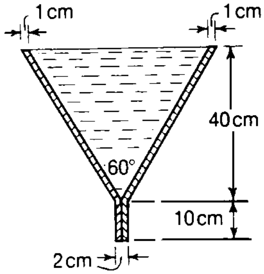
Further problems *(answers on page 692)*

9.12 A beam of I-section is 25 cm deep and has equal flanges 10 cm broad. The web is 0.75 cm thick and the flanges 1.25 cm thick. If the beam may be stressed in bending to 120 MN/m^2, what bending moment will it carry? *(Cambridge)*

9.13 The front-axle beam of a motor vehicle carries the loads shown. The axle is of I-section: flanges 7.5 cm by 2.5 cm, web 5 cm by 2.5 cm. Calculate the tensile stress at the bottom of the axle beam. *(Cambridge)*



9.14 A water trough 8 m long, is simply-supported at the ends. It is supported at its extremities and is filled with water. If the metal has a density 7840 kg/m^3, and the water a density 1000 kg/m^3, calculate the greatest longitudinal stress for the middle cross-section of the trough. *(Cambridge)*



9.15 A built-up steel I-girder is 2 m deep over the flanges, each of which consists of four 1 cm plates, 1 m wide, riveted together. The web is 1 cm thick and is attached to the flanges by four 9 cm by 9 cm by 1 cm angle sections. The girder has a clear run of 30 m between the supports and carries a superimposed load of 60 kN per metre. Find the maximum longitudinal stress. *(Cambridge)*

9.16 A beam rests on supports 3 m apart carries a load of 10 kN uniformly distributed. The beam is rectangular in section 7.5 cm deep. How wide should it be if the skin-stress must not exceed 60 MN/m^2? *(RNEC)*