22 Introduction to matrix algebra

22.1 Introduction

Since the advent of the digital computer with its own memory, the importance of matrix algebra has continued to grow along with the developments in computers. This is partly because matrices allow themselves to be readily manipulated through skilful computer programming, and partly because many physical laws lend themselves to be readily represented by matrices.

The present chapter will describe the laws of matrix algebra by a methodological approach, rather than by rigorous mathematical theories. This is believed to be the most suitable approach for engineers, who will use matrix algebra as a tool.

22.2 Definitions

A rectangular matrix can be described as a table or array of quantities, where the quantities usually take the form of numbers, as shown be equations (22.1) and (22.2):

$$[\mathbf{A}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \tag{22.1}$$

$$[\mathbf{B}] = \begin{bmatrix} 2 & -1 & 0 \\ 3 & 4 & -2 \\ -3 & 5 & 6 \\ -4 & -5 & 7 \end{bmatrix} \tag{22.2}$$

The matrix [**A**] of equation (22.1) is said to be of order $m \times n$, where

m = number of rows

n = number of columns

A row can be described as a horizontal line of quantities, and a column can be described as a vertical line of quantities, so that the matrix [**B**] of equation (22.2) is of order 4 × 3.

The quantities contained in the third row of [**B**] are −3, 5 and 6, and the quantities contained in the second column of [**B**] are −1, 4, 5 and −5.

A square matrix has the same number of rows as columns, as shown by equation (22.3), which is said to be of order n:

$$[\mathbf{A}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \tag{22.3}$$

A column matrix contains a single column of quantities, as shown by equation (22.4), where it can be seen that the matrix is represented by braces:

$$\{\mathbf{A}\} = \begin{Bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \cdot \\ \cdot \\ \cdot \\ a_{n1} \end{Bmatrix} \tag{22.4}$$

A row matrix contains a single row of quantities, as shown by equation (22.5), where it can be seen that the matrix is represented by the special brackets:

$$\lfloor \mathbf{A} \rfloor = \lfloor a_{11} \quad a_{12} \quad a_{13} \; \cdots \; a_{1n} \rfloor \tag{22.5}$$

The transpose of a matrix is obtained by exchanging its columns with its rows, as shown by equation (22.6):

$$[\mathbf{A}]^T = \begin{bmatrix} 1 & 0 \\ 4 & -3 \\ -5 & 6 \end{bmatrix}^T = [\mathbf{B}] \tag{22.6}$$

$$= \begin{bmatrix} 1 & 4 & -5 \\ 0 & -3 & 6 \end{bmatrix}$$

In equation (22.6), the first row of [A], when transposed, becomes the first column of [**B**]; the second row of [A] becomes the second column of [**B**] and the third row of [A] becomes the third column of [**B**], respectively.

22.3 Matrix addition and subtraction

Matrices can be added together in the manner shown below. If

$$[\mathbf{A}] = \begin{bmatrix} 1 & 0 \\ 4 & -3 \\ -5 & 6 \end{bmatrix}$$

and

$$[\mathbf{B}] = \begin{bmatrix} 2 & 9 \\ -7 & 8 \\ -1 & -2 \end{bmatrix}$$

$$[\mathbf{A}] + [\mathbf{B}] = \begin{bmatrix} (1+2) & (0+9) \\ (4-7) & (-3+8) \\ (-5-1) & (6-2) \end{bmatrix} \tag{22.7}$$

$$\begin{bmatrix} 3 & 9 \\ -3 & 5 \\ -6 & 4 \end{bmatrix}$$

Similarly, matrices can be subtracted in the manner shown below:

$$[\mathbf{A}] - [\mathbf{B}] = \begin{bmatrix} (1 - 2) & (0 - 9) \\ (4 + 7) & (-3 - 8) \\ (-5 + 1) & (6 + 2) \end{bmatrix} = \begin{bmatrix} -1 & -9 \\ 11 & -11 \\ -4 & 8 \end{bmatrix} \tag{22.8}$$

Thus, in general, for two $m \times n$ matrices:

$$[\mathbf{A}] + [\mathbf{B}] = \begin{bmatrix} (a_{11} + b_{11})(a_{12} + b_{12}) \ldots (a_{1n} + b_{1n}) \\ (a_{21} + b_{21})(a_{22} + b_{22}) \ldots (a_{2n} + b_{2n}) \\ \cdot \\ \cdot \\ \cdot \\ (a_{m1} + b_{m1})(a_{m2} + b_{m2}) \ldots (a_{mn} + b_{mn}) \end{bmatrix} \tag{22.9}$$

and

$$[\mathbf{A}] - [\mathbf{B}] = \begin{bmatrix} (a_{11} - b_{11})(a_{12} - b_{12}) \ldots (a_{1n} - b_{1n}) \\ (a_{21} - b_{21})(a_{22} - b_{22}) \ldots (a_{2n} - b_{2n}) \\ \cdot \\ \cdot \\ \cdot \\ (a_{m1} - b_{m1})(a_{m2} - b_{m2}) \ldots (a_{mn} - b_{mn}) \end{bmatrix} \tag{22.10}$$

22.4 Matrix multiplication

Matrices can be multiplied together, by multiplying the rows of the premultiplier into the columns of the postmultiplier, as shown by equations (22.11) and (22.12).

If

$$[\mathbf{A}] = \begin{bmatrix} 1 & 0 \\ 4 & -3 \\ -5 & 6 \end{bmatrix}$$

and

$$[\mathbf{B}] = \begin{bmatrix} 7 & 2 & -2 \\ -1 & 3 & -4 \end{bmatrix}$$

$$[\mathbf{A}] \times [\mathbf{B}] = [\mathbf{C}]$$

$$= \begin{bmatrix} (1\times 7+0\times(-1)) & (1\times 2+0\times 3) & (1\times(-2)+0\times(-4)) \\ (4\times 7+(-3)\times(-1)) & (4\times 2+(-3)\times 3) & (4\times(-2)+(-3)\times(-4)) \\ (-5\times 7+6\times(-1)) & (-5\times 2+6\times 3) & (-5\times(-2)+6\times(-4)) \end{bmatrix} \tag{22.11}$$

$$= \begin{bmatrix} (7+0) & (2+0) & (-2+0) \\ (28+3) & (8-9) & (-8+12) \\ (-35-6) & (-10+18) & (10-24) \end{bmatrix}$$

$$[\mathbf{C}] = \begin{bmatrix} 7 & 2 & -2 \\ 31 & -1 & 4 \\ -41 & 8 & -14 \end{bmatrix} \tag{22.12}$$

i.e. to obtain an element of the matrix [**C**], namely C_{ij}, the ith row of the premultiplier [**A**] must be premultiplied into the jth column of the postmultiplier [**B**] to give

$$C_{ij} = \sum_{k=1}^{P} a_{ik} \times b_{kj}$$

where

P = the number of columns of the premultiplier and also, the number of rows of the postmultiplier.

NB The premultiplying matrix [**A**] must have the same number of columns as the rows in the postmultiplying matrix [**B**].

In other words, if [**A**] is of order $(m \times P)$ and [**B**] is of order $(P \times n)$, then the product [**C**] is of order $(m \times n)$.

22.5 Some special types of square matrix

A *diagonal matrix* is a square matrix which contains all its non-zero elements in a diagonal from the top left corner of the matrix to its bottom right corner, as shown by equation (22.13). This diagonal is usually called the main or leading diagonal.

$$[\mathbf{A}] = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & & 0 \\ 0 & 0 & a_{33} & 0 \\ & 0 & & \\ & & & 0 \\ 0 & 0 & 0 & a_{nn} \end{bmatrix} \tag{22.13}$$

A special case of diagonal matrix is where all the non-zero elements are equal to unity, as shown by equation (22.14). This matrix is called a *unit matrix*, as it is the matrix equivalent of unity.

$$[\mathbf{I}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{22.14}$$

A *symmetrical matrix* is shown in equation (22.15), where it can be seen that the matrix is symmetrical about its leading diagonal:

$$[\mathbf{A}] = \begin{bmatrix} 8 & 2 & -3 & 1 \\ 2 & 5 & 0 & 6 \\ -3 & 0 & 9 & -7 \\ 1 & 6 & -7 & 4 \end{bmatrix} \tag{22.15}$$

i.e. for a symmetrical matrix, all

$$a_{ij} = a_{ji}$$

22.6 Determinants

The determinant of the 2×2 matrix of equation (22.16) can be evaluated, as follows:

$$[\mathbf{A}] = \begin{vmatrix} 4 & 2 \\ -1 & 6 \end{vmatrix} \tag{22.16}$$

Determinant of $[\mathbf{A}] = 4 \times 6 - 2 \times (-1) = 24 + 2 = 26$

so that, in general, the determinant of a 2 × 2 matrix, namely **det**[**A**], is given by:

$$\mathbf{det}\ [\mathbf{A}] = a_{11} \times a_{22} - a_{12} \times a_{21} \tag{22.17}$$

where

$$|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \tag{22.18}$$

Similarly, the determinant of the 3×3 matrix of equation (22.19) can be evaluated, as shown by equation (22.20):

$$\mathbf{det}\ |\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \tag{22.19}$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \tag{22.20}$$

For example, the determinant of equation (22.21) can be evaluated, as follows:

$$\begin{aligned} \textbf{det } |\mathbf{A}| &= \begin{vmatrix} 8 & 2 & -3 \\ 2 & 5 & 0 \\ -3 & 0 & 9 \end{vmatrix} \\ &= 8\begin{vmatrix} 5 & 0 \\ 0 & 9 \end{vmatrix} -2\begin{vmatrix} 2 & 0 \\ -3 & 9 \end{vmatrix} + (-3)\begin{vmatrix} 2 & 5 \\ -3 & 0 \end{vmatrix} \\ &= 8\,(45 - 0) - 2(18 - 0) - 3\,(0 + 15) \end{aligned} \tag{22.21}$$

or

$$\textbf{det } |\mathbf{A}| = 279$$

For a determinant of large order, this method of evaluation is unsatisfactory, and readers are advised to consult Ross, C T F, *Advanced Applied Finite Element Methods* (Horwood 1998), or Collar, A R, and Simpson, A, *Matrices and Engineering Dynamics* (Ellis Horwood, 1987) which give more suitable methods for expanding larger order determinants.

22.7 Cofactor and adjoint matrices

The cofactor of a third order matrix is obtained by removing the appropriate columns and rows of the cofactor, and evaluating the resulting determinants, as shown below.

If

$$[\mathbf{A}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

then $[\mathbf{A}]^c$ = the cofactor matrix of [**A**], where,

$$[\mathbf{A}]^c = \begin{bmatrix} a_{11}^c & a_{12}^c & a_{13}^c \\ a_{21}^c & a_{22}^c & a_{23}^c \\ a_{31}^c & a_{32}^c & a_{33}^c \end{bmatrix} \tag{22.22}$$

and the cofactors are evaluated, as follows:

$$a_{11}^c = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$a_{12}^c = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$a_{13}^c = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$a_{21}^c = -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

$$a_{22}^c = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

$$a_{23}^c = -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$a_{31}^{c} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$a_{32}^{c} = -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

$$a_{33}^{c} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

The adjoint or adjugate matrix, $[\mathbf{A}]^{a}$ is obtained by transposing the cofactor matrix, as follows:

ie $$[\mathbf{A}]^{\mathbf{a}} = [\mathbf{A}^{\mathbf{c}}]^{\mathbf{T}} \tag{22.23}$$

22.8 Inverse of a matrix $[\mathbf{A}]^{-1}$

The inverse or reciprocal matrix is required in matrix algebra, as it is the matrix equivalent of a scalar reciprocal, and it is used for division.

The inverse of the matrix [A] is given by equation (22.24):

$$[\mathbf{A}]^{-1} = \frac{[\mathbf{A}]^{\mathbf{a}}}{\mathbf{det}\ |\mathbf{A}|} \tag{22.24}$$

For the 2 × 2 matrix of equation (22.25),

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{22.25}$$

the cofactors are given by

$$a_{11}^{c} = a_{22}$$

$$a_{12}^{c} = -a_{21}$$

$$a_{21}^{c} = -a_{12}$$

$$a_{22}^{c} = a_{11}$$

and the determinant is given by:

$$\mathbf{det}\ |\mathbf{A}| = a_{11} \times a_{22} - a_{12} \times a_{21}$$

so that

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}}{(a_{11} \times a_{22} - a_{12} \times a_{21})} \tag{22.26}$$

In general, inverting large matrices through the use of equation (22.24) is unsatisfactory, and for large matrices, the reader is advised to refer to Ross, C T F, *Advanced Applied Finite Element Methods* (Horwood 1998), where a computer program is presented for solving nth order matrices on a microcomputer.

The inverse of a unit matrix is another unit matrix of the same order, and the inverse of a diagonal matrix is obtained by finding the reciprocals of its leading diagonal.

The inverse of an orthogonal matrix is equal to its transpose. A typical orthogonal matrix is shown in equation (22.27):

$$[\mathbf{A}] = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \tag{22.27}$$

where

$$c = \cos\alpha$$

$$s = \sin\alpha$$

The cofactors of [**A**] are:

$$a_{11}^{c} = c$$

$$a_{12}^{c} = s$$

$$a_{21}^{c} = -s$$

$$a_{22}^{c} = c$$

and

$$\mathbf{det}\ |\mathbf{A}| = c^2 + s^2 = 1$$

so that

$$[\mathbf{A}]^{-1} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$$

i.e. for an orthogonal matrix

$$[\mathbf{A}]^{-1} = [\mathbf{A}]^{\mathrm{T}} \tag{22.28}$$

22.9 Solution of simultaneous equations

The inverse of a matrix can be used for solving the set of linear simultaneous equations shown in equation (22.29). If,

$$[\mathbf{A}]\{x\} = \{c\} \tag{22.29}$$

where [A] and {c} are known and $\{x\}$ is a vector of unknowns, then $\{x\}$ can be obtained from equation (22.30), where $[\mathbf{A}]^{-1}$ has been pre-multiplied on both sides of this equation:

$$\{x\} = [\mathbf{A}]^{-1}\{c\} \tag{22.30}$$

Another method of solving simultaneous equations, which is usually superior to inverting the matrix, is by triangulation. For this case, the elements of the matrix below the leading diagonal are eliminated, so that the last unknown can readily be determined, and the remaining unknowns obtained by back-substitution.

Further problems *(answers on page 695)*

If

$$[\mathbf{A}] = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad [\mathbf{B}] = \begin{bmatrix} -1 & 0 \\ 2 & -4 \end{bmatrix}$$

Determine:

22.1 [A] + [B]

22.2 [A] - [B]

22.3 $[\mathbf{A}]^T$

22.4 $[\mathbf{B}]^T$

22.5 $[\mathbf{A}] \times [\mathbf{B}]$

22.6 $[\mathbf{B}] \times [\mathbf{A}]$

22.7 det $[\mathbf{A}]$

22.8 det $[\mathbf{B}]$

22.9 $[\mathbf{A}]^{-1}$

22.10 $[\mathbf{B}]^{-1}$

If

$$[\mathbf{C}] = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 1 & -2 \\ 0 & -2 & 1 \end{bmatrix}$$

and

$$[\mathbf{D}] = \begin{bmatrix} 9 & 1 & -2 \\ -1 & 8 & 3 \\ -4 & 0 & 6 \end{bmatrix}$$

determine:

22.11 $[\mathbf{C}] + [\mathbf{D}]$

22.12 $[\mathbf{C}] - [\mathbf{D}]$

22.13 $[\mathbf{C}]^{\mathrm{T}}$

22.14 $[\mathbf{D}]^{\mathrm{T}}$

22.15 $[\mathbf{C}] \times [\mathbf{D}]$

22.16 $[\mathbf{D}] \times [\mathbf{C}]$

22.17 $\det [\mathbf{C}]$

22.18 $\det [\mathbf{D}]$

22.19 $[\mathbf{C}]^{-1}$

22.20 $[\mathbf{D}]^{-1}$

If

$$[\mathbf{E}] = \begin{bmatrix} 2 & 4 \\ -3 & 1 \\ 5 & 6 \end{bmatrix}$$

and

$$[\mathbf{F}] = \begin{bmatrix} 0 & 7 & -1 \\ 8 & -4 & -5 \end{bmatrix}$$

determine:

22.21 $[\mathbf{E}]^{\mathrm{T}}$

22.22 $[\mathbf{F}]^{\mathrm{T}}$

22.23 $[\mathbf{E}] \times [\mathbf{F}]$

22.24 $[\mathbf{F}]^T \times [\mathbf{E}]^T$

22.25 If

$$\begin{aligned} x_1 - 2x_2 + 0 &= -2 \\ -x_1 + x_2 - 2x_3 &= 1 \\ 0 - 2x_2 + x_3 &= 3 \end{aligned}$$

determine

x_1, x_2 and x_3