17 Energy methods

17.1 Introduction

Energy methods are very useful for analysing structures, especially for those that are statically indeterminate. This chapter introduces the principle of virtual work and applies it to statically determinate and statically indeterminate frameworks. The chapter also shows how the method can be used for the plastic design of beams and rigid-jointed plane frames.

The chapter then introduces strain energy and complementary strain energy, and through the use of worked examples, shows how these methods can be used for analysing structures.

In Chapters 24 and 25, energy methods are used for developing the finite element method, which is one of the most powerful methods for analysing massive and complex structures with the aid of digital computers.

17.2 Principle of virtual work

In its simplest form the *principle of virtual work* is that

> *For a system of forces acting on a particle, the particle is in statical equilibrium if, when it is given any virtual displacement, the net work done by the forces is zero.*

A virtual displacement is any arbitrary displacement of the particle. In the virtual displacement the forces are assumed to remain constant and parallel to their original lines of actions. Consider a particle under the action of three forces, F_1, F_2 and F_3, Figure 17.1.

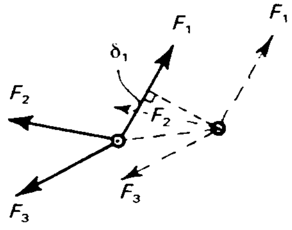


Figure 17.1 System of forces in statical equilibrium acting on a particle.

Imagine the particle to be given a virtual displacement of any magnitude in any direction. Suppose the displacements of the particle along the lines of action of the forces F_1, F_2 and F_3, are δ_1, δ_2 and δ_3, respectively; these are known as *corresponding* displacements. Then the forces form a system in statical equilibrium if

$$F_1\delta_1 + F_2\delta_2 + F_3\delta_3 = 0 \qquad (17.1)$$

On the basis of the principle of virtual work we can show that the resultant of the forces acting on a particle in statical equilibrium is zero. Suppose the forces F_1, F_2 and F_3, acting on the particle of Figure 17.1, have a resultant of magnitude R in some direction; then by giving the particle a suitable virtual displacement, Δ, say, in the direction of R, the net work is

$$R\Delta$$

But by the principle of virtual work the net work is zero, so that

$$R\Delta = 0 \qquad (17.2)$$

As Δ can be non-zero, R must be zero. Hence, by adopting the principle of virtual work as a basic concept, we can show that the resultant of a system of forces in statical equilibrium is zero.

17.3 Deflections of beams

In a pin-jointed frame subjected to loads applied to the joints only the tensile load in any member is constant throughout the length of that member. In the case of a beam under lateral loads the bending moments and shearing forces may vary from one section to another, so that the state of stress is not uniform along the length of the beam. In applying the principle of virtual work to problems of beams we must consider the loading actions on the virtual displacement of an elemental length of the beam.

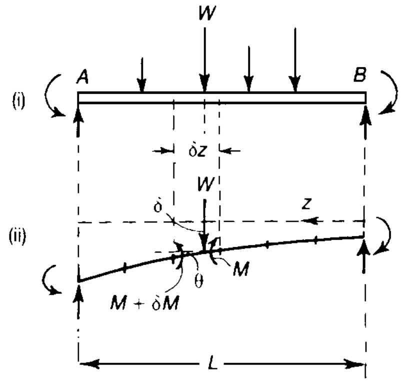


Figure 17.2 Deflections of a straight beam.

Consider a straight beam AB, Figure 17.2, which is in statical equilibrium under the action of a system of external forces and couples. The beam is divided into a number of short lengths; the loading actions on a short length such as δz consist of bending moments M and $(M + \delta M)$, an external lateral load W, and lateral shearing forces at the ends of the short length. Now suppose the short lengths of the beam are given small virtual displacements, θ. If the elements remain connected to each other, then for given values of θ the external forces, such as W, suffer certain displacements, such as δ. Then the values of θ and δ form a *compatible system* of rotations and displacements, and the virtual work of any system of forces and couples in statical in equilibrium on these rotations and displacements is zero. Then

$$\sum \delta M \times \theta + \sum W \times \delta = 0 \tag{17.3}$$

because the net work of the internal shearing forces is zero. The summation $\Sigma \delta M \times \theta$ is carried out for all short lengths of the beam, whereas the summation $\Sigma W \times \delta$ is carried out for all external loads, including couples and force reactions at points of support. If the virtual rotations θ are small, the virtual displacements δ can be found easily. If the lengths δz of the beam are infinitesimally small,

$$\sum \delta M \times \theta = \int_{z=0}^{z=L} \theta \, dM \tag{17.4}$$

where the integration is carried out over the whole length L of the beam. But

$$\int_{z=0}^{z=L} \theta \, dM = \left[M\theta - \int M d\theta \right]_{z=0}^{z=L}$$

Now

$$\left[M\theta \right]_{z=0}^{z=L} = \left[M\theta \right]_{z=L} - \left[M\theta \right]_{z=0}$$

and is the work of the end couples on their respective virtual displacements; this work has already been taken account of in the summation $\Sigma W \times \delta$, so that equation (17.3) becomes

$$\sum W \times \delta = \int_{z=0}^{z=L} M d\theta = \int_0^L M \left(\frac{d\theta}{dz} \right) dz \tag{17.5}$$

Now $(d\theta/dz)$ is the curvature of the beam when it is given the virtual rotations and displacements. If we put

$$\frac{d\theta}{dz} = \frac{1}{R} \tag{17.6}$$

where R is the radius of curvature of the beam, then

$$\sum W \times \delta = \int_0^L M \left(\frac{1}{R} \right) dz \tag{17.7}$$

As an example of the application of equation (17.7), consider the cantilever shown in Figure 17.3; having a uniform flexural stiffness EI. The cantilever carries a vertical load W at the free end; the bending moment at any section due to W is Wz, so that, if the beam remains elastic, the corresponding curvature at any section is

$$\frac{1}{R} = \frac{Wz}{EI}$$

Suppose the corresponding deflection of W is δ, Figure 17.3; then the values of $1/R$ and δ form a system of compatible curvature and displacements.

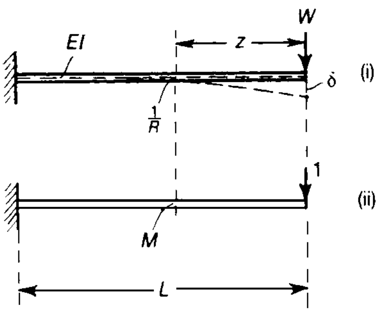


Figure 17.3 Deflections of a cantilever with an end load.

We derive a simple system of forces and couples in statical equilibrium by applying a unit vertical load at the end of the cantilever; the bending moment at any section due to this unit load is

$$M = 1 \times z = z$$

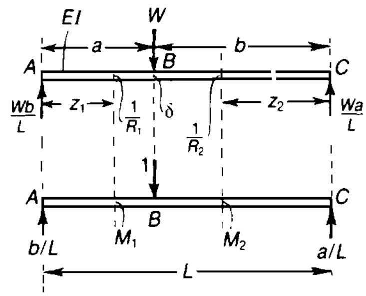
Then, from equation (17.7),

$$1 \times \delta = \int_0^L M\left(\frac{1}{R}\right) dz = \int_0^L \frac{Wz^2}{EI} dz$$

Then

$$\delta = \frac{WL^3}{3EI}$$

Problem 17.1 A simply-supported beam, of uniform flexural stiffness EI, carries a lateral load W at a distance a from the end A. Estimate the vertical deflection of W.



Solution

The bending moment a distance z_1 from A, for the section AB, is

$$\frac{Wbz_1}{L}$$

The curvature for AB if therefore

$$\frac{1}{R_1} = \frac{Wbz_1}{EIL}$$

Similarly, the curvature at any section in BC is

$$\frac{1}{R_2} = \frac{Waz_2}{EIL}$$

Now consider the beam with a unit vertical load at B; the bending moments at sections in AB and BC are, respectively,

$$M_1 = \frac{bz_1}{L}, \qquad M_2 = \frac{az_2}{L}$$

Then, equation (17.7) gives

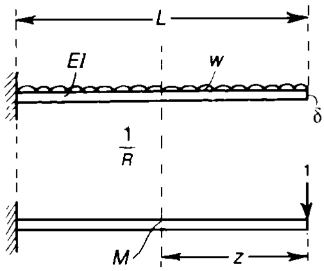
$$\delta = \int_0^a M_1\left(\frac{1}{R_1}\right) dz_1 + \int_0^b M_2\left(\frac{1}{R_2}\right) dz_2$$

$$= \int_0^a \frac{Wb^2}{EIL^2} z_1^2\, dz_1 + \int_0^b \frac{Wa^2}{EIL^2} z_2^2\, dz_2$$

Therefore

$$\delta = \frac{Wa^2b^2}{3EIL^2}(a + b) = \frac{Wa^2b^2}{3EIL}$$

Problem 17.2 A cantilever of uniform flexural stiffness EI carries a uniformly-distributed load of intensity w. Estimate the vertical deflection of the free end.



Solution

Due to the distribution load, the curvature at any section is

$$\frac{1}{R} = \frac{wz^2}{2EI}$$

For a unit vertical load at the free end, the bending moment at any section is

$$M = z$$

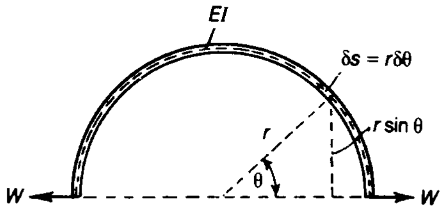
Then equation (17.7) gives

$$\delta = \int_0^L M\left(\frac{1}{R}\right)dz = \int_0^L \frac{wz^3}{2EI}dz$$

Then

$$\delta = \frac{wL^4}{8EI}$$

Problem 17.3 A semicircular thin ring has a radius r and uniform flexural stiffness EI. The ring carries equal and opposite loads W at the ends. Find the increase in distance between the loaded points.



Solution

The bending moment at any angular position θ is

$$M = Wr \sin\theta$$

If the ring is thin, the change of curvature at any section is

$$\frac{1}{R} = \frac{M}{EI}$$

Now consider the virtual work of the forces and couples on their resulting displacements; if δ is the increase in distance between the loaded points

$$W \times \delta = \int_{\theta=0}^{\theta=\pi} M\left(\frac{1}{R}\right) ds = \int_0^{\pi} \frac{M^2 r}{EI} d\theta = \frac{W^2 r^3}{EI} \int_0^{\pi} \sin^2\theta \, d\theta$$

Then

$$\delta = \frac{\pi W r^3}{2EI}$$

17.4 Statically indeterminate beam problems

The principle of virtual work may also be used in solving statically indeterminate beam problems. Consider, for example, the beam of Figure 17.4, which is built-in at A and supported on a roller at B; the beam is of uniform flexural stiffness EI, and carries a uniformly distributed lateral load

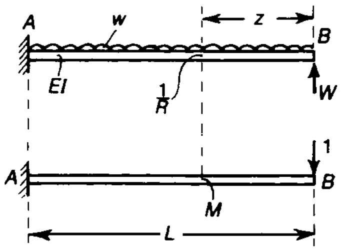


Figure 17.4 Propped cantilever under uniform lateral loading.

of intensity w. Suppose the statically indeterminate reaction at B is W; then the bending moment at any section is

$$\frac{1}{2}wz^2 - Wz$$

and if the beam remains elastic the resulting curvature at an any section is

$$\frac{1}{R} = \frac{1}{EI}\left(\frac{1}{2}wz^2 - Wz\right)$$

The bending moment at any section due to a unit lateral load at B is

$$M = z$$

Then, for no deflection at B in Figure 17.4,

$$1 \times 0 = \int_0^L M\left(\frac{1}{R}\right)dz = \int_0^L \frac{z}{EI}\left(\frac{wz^2}{2} - Wz\right)dz$$

Then

$$\int_0^L \frac{1}{2}wz^3\,dz = \int_0^L Wz^2 dz$$

Thus

$$W = \frac{3wL}{8}$$

17.5 Plastic bending of mild-steel beams

The principle of virtual work is not limited in its application to linear problems of the type discussed in the preceding problems. It is useful, for example, in solving problems of plastic bending; the uniform mild-steel beam of Figure 17.5 has a fully-plastic moment M_P. At collapse of the beam, plastic hinges develop at A and B. Suppose the point B is now given a virtual displacement δ; if δ is small, AB rotates through an angle (δ/a), and BC through an angle $[\delta/(L - a)]$. The work of the system of forces and couples of Figure 17.5(ii) on the virtual displacements and rotations of Figure 17.5(iii) is zero. Then

$$W\delta = M_P\left[\frac{2\delta}{a} + \frac{\delta}{(L - a)}\right]$$

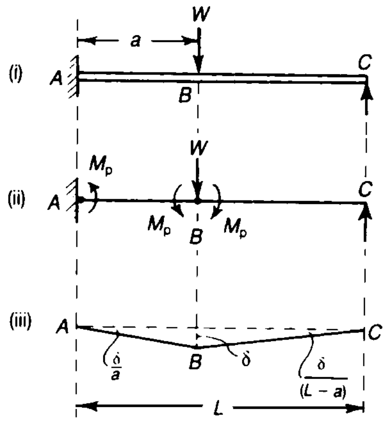


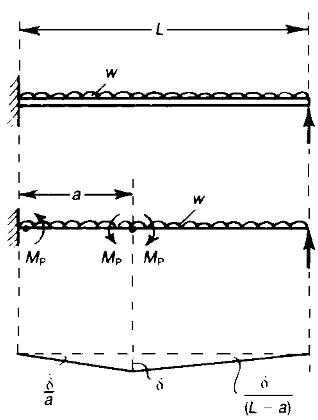
Figure 17.5 Plastic bending of a mild-steel beam.

Then

$$W = \frac{M_p(2L - a)}{a(L - a)}$$

This is the value of W at plastic collapse of the beam.

Problem 17.4 A uniform mild-steel beam has a fully-plastic moment M_P. Find the intensity of uniformly distributed loading at collapse of the beam.



<u>*Solution*</u>

Suppose that, at plastic collapse, hinges develop at the built-in end, and at a distance *a* from that end. Then

$$\frac{1}{2}wa\delta + \frac{1}{2}w\,(L - a)\delta = M_P\left[\frac{2\delta}{a} + \frac{\delta}{(L - a)}\right]$$

Thus,

$$w = \frac{2\left(2 - \frac{a}{L}\right)}{\left(\frac{a}{L}\right)\left(1 - \frac{a}{L}\right)}\,\frac{M_P}{L^2}$$

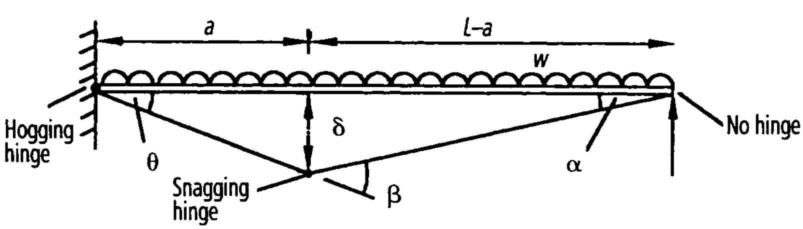
This is a minimum with respect to (*a/L*) when

$$\frac{a}{L} = \left(2 - \sqrt{2}\right)$$

Then the relevant value of *w* is

$$w = \frac{2M_P}{L^2}\left(3 + 2\sqrt{2}\right)$$

An alternative method of solving the above beam problem is to consider rotations of the hinges, as shown in the figure below



$$\delta = \theta a = \alpha\,(L - a) \tag{17.8}$$

$$\therefore\ \alpha = \theta\,.\,a/(L - a)$$

$$\beta = \alpha + \theta = \theta \; a/(L - a) + \theta$$

$$= \theta \; a/(L - a) + \theta \; (L - a) / (L - a)$$

$$= \theta \; (a + L - a) / (L - a)$$

$$\beta = \theta \; L/(L - a) \tag{17.9}$$

Now work done by the hinges

$$= M_p \; \theta + M_p \; \beta$$

$$= M_p \; \theta + M_p \; \theta \; L / (L - a)$$

$$= M_p \; \theta \; (L - a) / (L - a) + M_p \; \theta \, L / (L - a)$$

$$= M_p\theta \; (L - a + L) / (L - a)$$

$$M_p\theta \; (2L - a) / (L - a) \tag{17.10}$$

Work done by the load 'w'

$$w \times L \times \delta/2 = wL \; \theta \; a/2 \tag{17.11}$$

Equating (17.10) and (17.11)

$$M_p \; \theta \; (2L - a) / (L - a) = wL\theta \; a/2$$

$$\text{or } w = 2(2L - a) / [aL \; (L - a)] \; M_p$$

$$= \frac{2L \; (2 - a/L) \; M_p}{aL^2 \; (1 - a/L)}$$

Dividing the top and bottom by L, we get

$$w = \frac{2 \; (2 - a/L) \; M_p}{L^2 \left(\dfrac{a}{L}\right) (1 - a/L)} \tag{17.12}$$

which is the same result as before.

17.6 Plastic design of frameworks

For this case, let us make the following definitions:

λ = load or safety factor

M_P = plastic moment of resistance of the cross-section of a member of the framework

M_Y = the elastic moment of resistance of the cross-section of a member of the framework at first yield

S = shape factor = M_P/M_Y

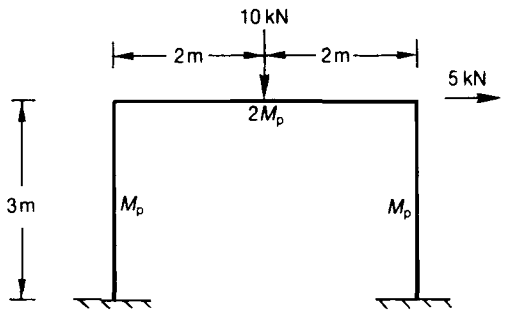
σ_Y = yield stress

Problem 17.5 Obtain a suitable sectional modulus for the portal frame below, given that:

λ = 2.7

S = 1.15

σ_Y = 300 MPa



Solution

Experiments have shown[4] that this framework can fail by any of the following modes:

(a) beam mechanism
(b) sway mechanism
(c) combined beam and sway mechanism.

[4] Baker J F - *A Review of Recent Investigations into the Behaviour of Steel Frames in the Plastic Range*, JICE, 31, 188, 1949, and Baker J F, Horne M R and Heyman J - *The Steel Skeleton*, Cambridge University Press, 1956.

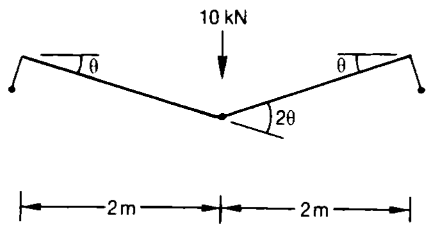
(a) Beam mechanism

This mode of failure, which was discussed in the previous section, is shown below. Applying the principle of virtual work to do this failure mechanism, we get work done by the plastic hinges when rotating = work done by the 10 kN load

or $M_P\,\theta + 2M_P \times 2\theta + M_P\,\theta = 10 \times 2\theta$

$$6M_P = 20\theta$$

$$M_P = 3.33 \text{ kNm}$$



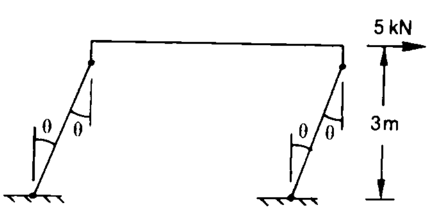
(b) Sway mechanism

This mode of failure is shown below. Applying the principle of virtual work to this failure mechanism, we get

$$M_p\ (\theta + \theta + \theta + \theta) = 5 \times 3\theta$$

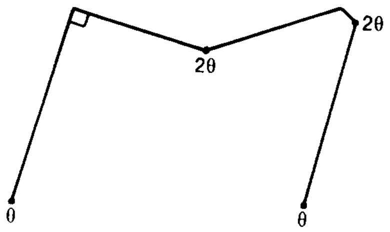
or $4M_P = 15$

$$M_P = 3.75 \text{ kNm}$$



(c) Combined mechanism

This mode of failure is shown below.



From the principle of virtual work,

$$M_P\theta + 2M_P \times 2\theta + M_P \times 2\theta + M_P\theta = 10 \times 2\theta + 5 \times 3\theta$$

or $8M_P = 35$

$$M_P = 4.375 \text{ kNm}$$

The design M_P is obtained from the largest of these values, as this is the value of M_P which will just prevent failure.

$$\therefore \text{design } M_P = 4.375 \times \lambda = 4.375 \times 2.7$$

$$\text{design } M_P = 11.81 \text{ KNm}$$

$$\text{Now } \frac{M_P}{M_Y} = S$$

$$\therefore M_Y = \frac{M_P}{S} = \frac{11.81}{1.15} = 10.27 \text{ kNm}$$

$$Z = \text{sectional modulus} = \frac{M_Y}{\sigma_Y}$$

$$= \frac{10.27 \times 10^3}{300 \times 10^6}$$

$$Z = 3 \times 10^{-5} \text{ m}^3 \text{ (verticals)}$$

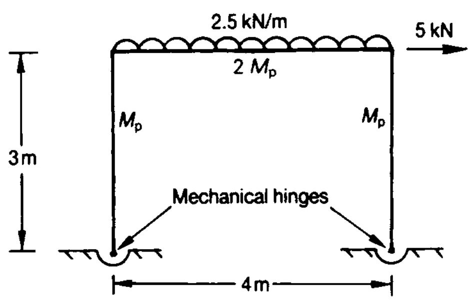
$$Z = 6 \times 10^{-5} \text{ m}^3 \text{ (horizontal beam)}$$

Problem 17.6 Determine a suitable sectional modulus for the portal frame below, assuming that the frame has two mechanical hinges at its base, and that the following apply:

λ = 2.7

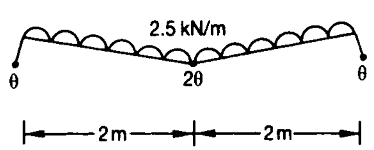
S = 1.15

σ = 300 MPa



Solution

The *beam mechanism* is shown below



For this case

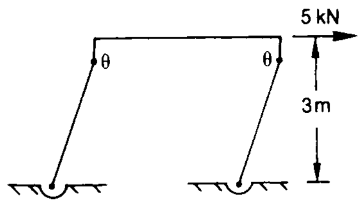
$$M_p\,\theta + 2\,M_p \times 2\theta + M_p\,\theta = 2.5 \times 4 \times 2\theta/2$$

or

$$6M_p = 10$$

$$M_p = 1.67 \text{ kNm}$$

The *sway mechanism* is shown as follows, where it must be noted that the mechanical hinge does no work during failure.



For this case

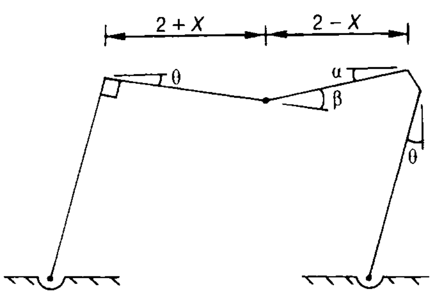
$$M_p(\theta + \theta) = 5 \times 3\theta$$

or

$$2M_P = 15$$

$$M_P = 7.5 \text{ kNm}$$

The *combined mechanism* is shown below, where it can be seen that the sagging hinge on the beam does not necessarily occur at mid-span.



For this case,

$$2M_p\beta + M_p(\alpha + \theta) = 2.5 \times 4 \times \left(\frac{2+X}{2}\right)\theta + 5 \times 3\theta$$

$$= 5(2 + X)\theta + 15\theta \tag{17.13}$$

but

$$(2+X)\,\theta = (2-X)\alpha$$

$$\therefore \alpha = \left(\frac{2+X}{2-X}\right)\theta \tag{17.14}$$

$$\beta = \alpha + \theta = \left(\frac{2+X}{2-X}\right) + \theta$$

$$= \left(\frac{2+X+2-X}{2-X}\right)$$

or $$\beta = \frac{4}{(2-X)}\theta$$

(17.15)

Substituting equations (17.14) and (17.15) into equation (17.13), we get

$$2 \times M_P \times \frac{4}{(2-X)} + M_P\left(\frac{2+X}{2-X}\right) + M_P = 5(2+X) + 15$$

$$\text{or } M_P\,\frac{8+2+X+2-X}{2-X} = 5(2+X) + 15$$

$$\text{or } M_P = [5(2+X)+15]\,\frac{(2-X)}{12}$$

$$= \frac{1}{12}(10+5X+15)(2-X)$$

$$= \frac{1}{12}(25+5X)(2-X)$$

$$= \frac{1}{12}(50-25X+10X-5X^2)$$

$$\text{or } M_P = \frac{1}{12}(50-15X-5X^2) \qquad (17.16)$$

For maximum

$$M_P\,\frac{dM_P}{dX} = 0$$

$$\therefore \frac{dM_P}{dX} = -15-10X$$

(17.17)

or $X = -1.5$ m

Substituting equation (17.17) into equation (17.16)

$$M_P = \frac{1}{12}(50 + 22.5 - 11.25)$$

$$M_P = 5.1 \text{ kNm}$$

Design $\quad M_P = 2.7 \times 5.1$

$$= 13.77 \text{ kNm}$$

$$M_Y = \frac{13.77}{1.15} = 11.97 \text{ kNm}$$

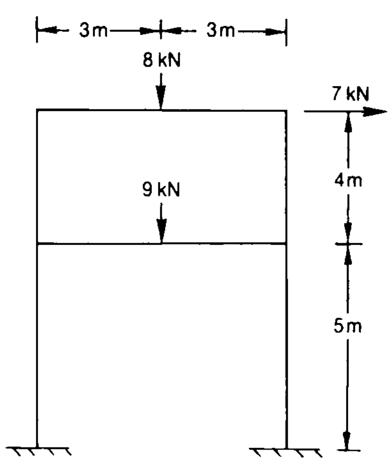
$$Z = \frac{11.97 \times 10^3}{300 \times 10^6}$$

$$Z = 8 \times 10^{-5} \text{ m}^3 \text{ (horizontal beam)}$$

The method will now be applied to *two-storey* and *two-bay* frameworks.

Problem 17.17 Determine a suitable sectional modulus for the two storey framework below, given that

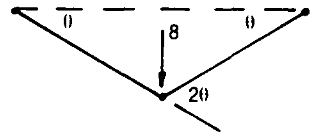
$$\lambda = 3, \quad S = 1.16, \quad \sigma_Y = 316 \text{ MPa}$$



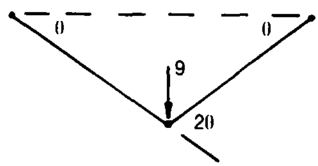
<u>*Solution*</u>

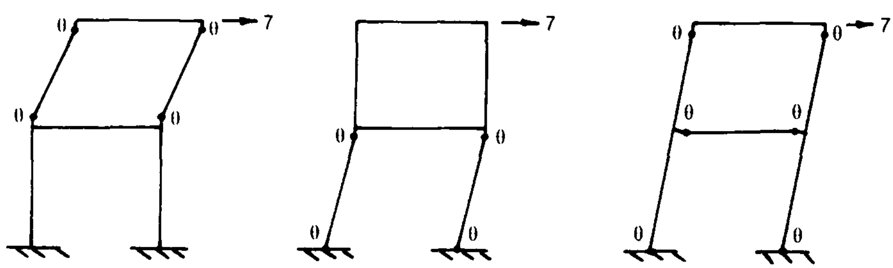
The possible mechanisms are as follows:

(a) Top beam



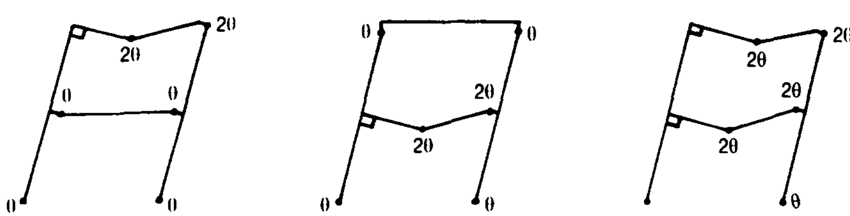
(b) Bottom beam





Top Bottom Top and bottom

(c) Sway mechanisms (3 types)



Top Bottom Top and bottom

(d) Combined mechanisms (3 types)

(a) *Top beam mechanism*

$$M_P(\theta + 2\theta + \theta) = 8 \times 3\theta$$

or

$$4M_P = 24$$

$$M_P = 6 \text{ kNm}$$

(b) *Bottom beam mechanism*

$$M_P(\theta + 2\theta + \theta) = 9 \times 3\theta$$

or

$$4M_P = 27$$

$$M_P = 6.75 \text{ kNm}$$

(c) *Top sway mechanism*

$$M_P(\theta + \theta + \theta + \theta) = 7 \times 4\theta$$

$$M_P = 7 \text{ kNm}$$

(d) *Bottom sway mechanism*

$$M_P(\theta + \theta + \theta + \theta) = 7 \times 5\theta$$

$$M_P = 8.75 \text{ kNm}$$

(e) *Top and bottom sway mechanisms*

$$M_P \times 6\theta = 7 \times 9\theta$$

$$M_P = 10.5 \text{ kNm}$$

(f) *Combined top mechanism*

$$M_P(\theta + \theta + 2\theta + 2\theta + \theta + \theta) = 8 \times 3\theta + 7 \times 9\theta$$

or

$$8M_P = 87$$

$$M_P = 10.88 \text{ kNm}$$

(g) *Combined bottom mechanism*

$$M_P(\theta + \theta + 2\theta + \theta + 2\theta + \theta) = 9 \times 3\theta + 7 \times 9\theta$$

or

$$8M_P = 90$$

$$M_P = 11.25 \text{ kNm}$$

(h) *Combined top and bottom mechanisms*

$$M_P(\theta + 2\theta + 2\theta + 2\theta + 2\theta + 2\theta) = 8 \times 3\theta + 9 \times 3\theta + 7 \times 9\theta$$

or

$$10M_P = 114$$

$$M_P = 11.4 \text{ kNm}$$

Design

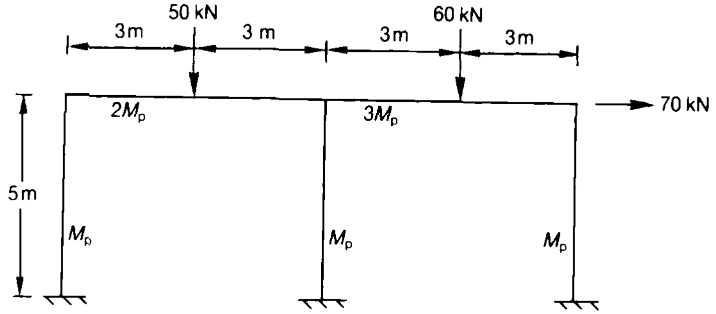
$$M_P = 11.4 \times 3 = 34.2 \text{ kNm}$$

$$M_Y = \frac{34.2}{1.16} = 29.48 \text{ kNm}$$

$$Z = \frac{29.48 \times 10^3}{316 \times 10^6} = 9 \times 10^{-5} \text{ m}^3$$

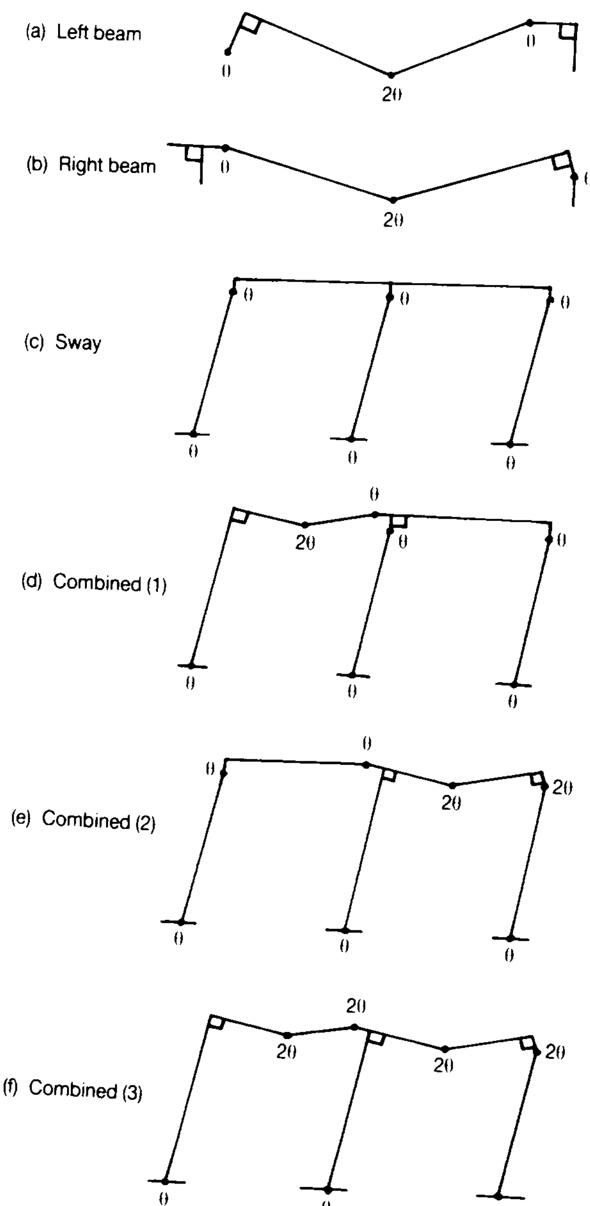
Problem 17.18 Determine suitable sectional moduli for the two-bay framework below, given that

$\lambda = 3 \quad S = 1.15 \quad \sigma_Y = 316 \text{ MPa}$



Solution

The various possible mechanisms are given below:



(a) *Left beam*

$$M_P\,(\theta + 4\theta + 2\theta) = 50 \times 3\theta$$

$$7M_P = 150$$

$$M_P = 21.4 \text{ kNm}$$

(b) *Right beam*

$$M_P(3\theta + 6\theta + \theta) = 60 \times 3\theta$$

$$10M_P = 180$$

$$M_P = 18 \text{ kNm}$$

(c) *Sway*

$$M_P \times 6\theta = 70 \times 5\theta$$

$$6M_P = \text{`}350$$

$$M_P = 58.3 \text{ kNm}$$

(d) *Combined (1)*

$$M_P(\theta + 4\theta + 2\theta + \theta + \theta + \theta + \theta) = 70 \times 5\theta + 50 \times 3\theta$$

$$11M_P = 500$$

$$M_P = 45.5 \text{ kNm}$$

(e) *Combined (2)*

$$M_P(\theta + \theta + 2\theta + \theta + 6\theta + 2\theta + \theta)$$

$$= 70 \times 5\theta + 60 \times 3\theta$$

or

$$14M_P = 530$$

$$M_P = 37.86 \text{ kNm}$$

(f) *Combined (3)*

$$M_P(\theta + 4\theta + 4\theta + \theta + 6\theta + 2\theta + \theta)$$

$$= 70 \times 5\theta + 50 \times 3\theta + 60 \times 3\theta$$

or

$$19M_P = 680$$

$$M_P = 35.8 \text{ kNm}$$

Design $M_P = 58.3 \times 3 = 174.9\ kNm$

$$M_Y = \frac{174.9}{1.15} = 152.1\ kNm$$

$$Z = \frac{152.1 \times 10^3}{316 \times 10^6} = 4.8 \times 10^{-4}\ \mathrm{m}^3 \text{ (verticals)}$$

$$Z = 9.6 \times 10^{-4}\ \mathrm{m}^3 \text{ (left beam)}$$

$$Z = 1.44 \quad 10^{-3}\ \mathrm{m}^3 \text{ (right beam)}$$

17.7 Complementary energy

The principle of virtual work leads also to a concept of wider application in stress–strain analysis than that of strain energy; this other property of a structure is known as *complementary energy*.

Consider the statically determinate pin-jointed frame shown in Figure 17.6; the frame is pinned to a rigid foundation at A and B, and carries external loads W_1 and W_2 at joints C and D, respectively. Suppose the corresponding displacements of the joints C and D are δ_1, and δ_2, respectively; the tensile force induced in a typical member, such as BC, is P, and its resulting extension is e. The forces W_1, W_2, P etc. are a system of forces in statical equilibrium, whereas the extensions, e, etc., are compatible with the displacements δ_1 and δ_2 of the joints. Thus by the principle of virtual work

$$W_1\delta_1 + W_2\delta_2 = \sum_m Pe \tag{17.18}$$

where the summation is carried out for all member of the frame.

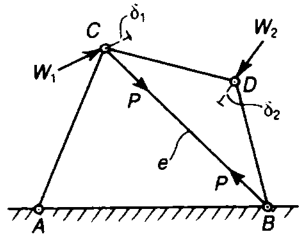


Fig. 17.6 Statically determinate plane frame under any system of external load.

Now suppose the external load W_1 is increased in magnitude by a small amount δW_1, the external load W_2 remaining unchanged; due to change in W_1 small changes occur in the forces in the members of the frame P, for example, increasing to $(P + \delta P)$. Now consider the virtual work of the modified system of forces on the original set of displacements and extensions; we have

$$(W_1 + \delta W_1)\delta_1 + W_2\delta_2 = \sum_m (P + \delta P)e$$

where the summation is carried out for all members of the frame. Now suppose the external load W_2 is increased in magnitude by a small amount δW_1, the external load W_2 remaining unchanged; due to change in W_1 small changes occur in the forces in the members of the frame, P, for example, increasing to $(P + \delta P)$.

Now consider the virtual work of the modified system of forces on the original set of displacement and extensions; we have

$$(W_1 + \delta W_1)\delta_1 + W_2\delta_2 = \sum_m (P + \delta P)e \tag{17.19}$$

On subtracting equations (17.18) and (17.19), we have

$$\delta_1 \times \delta W_1 = \sum_m e\delta P \tag{17.20}$$

The quantity $e\delta P$ for a member is the shaded elemental area shown on the load-extension diagram of Figure 17.7, this is an element of the area C shown in Figure 17.8.

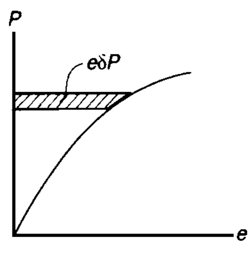


Figure 17.7 Increment of complementary energy of a single member.

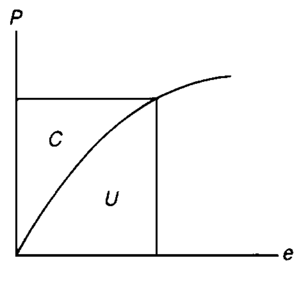


Figure 17.8 Strain energy and complementary energy of a single member.

When a bar is extended the work done on the bar is the area below the P–e curve of Figure 17.7, for a conservative structural member this work is stored as strain energy, which we have already referred to as U. We define the area above the P–e curve of Figure 17.7 as the *complementary energy, C*, of the member; we have that

$$U + C = Pe \tag{17.21}$$

and

$$\delta C = e\delta P \tag{17.22}$$

In equation (17.18) we may write, therefore,

$$\delta_1 \times \delta W_1 = \delta C \tag{17.23}$$

where C is the complementary energy of all members of the frame. If δW_1 is infinitesimally small

$$\frac{\partial C}{\partial W_1} = \delta_1 \tag{17.24}$$

Then the partial derivative of the complementary energy function C with respect to the external load W_1 gives the corresponding displacement δ_1 of that load.

17.8 Complementary energy in problems of bending

The complementary energy method may be used to considerable advantage in the solution of problems of bending of straight and thin curved beams. In general we suppose that the moment–curvature relationship for an element of a beam is of the form shown in Figure 17.9. The complementary energy of bending of an elemental length δs due to a bending moment M is

$$\int_0^M \left(\frac{\delta s}{R}\right) dM$$

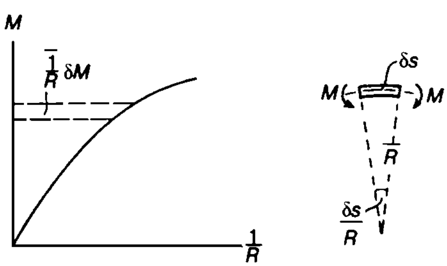


Figure 17. 9 Complementary energy of bending of the element of a beam.

For a linear-elastic beam of flexural stiffness EI

$$\frac{1}{R} = \frac{M}{EI}$$

and so the complementary energy is

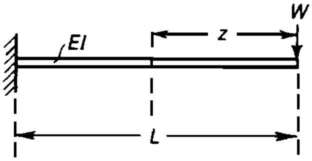
$$\int_0^M \frac{M}{EI}\, dM\delta s = \frac{M^2\delta s}{2EI} \tag{17.25}$$

For a length L of the beam, the complementary energy is therefore

$$C = \int_0^L \frac{M^2 ds}{2EI} \tag{17.26}$$

As in the case of pin-jointed frames, the partial derivative of C with respect to any external load is the corresponding displacement of that load. For statically indeterminate beams, the partial derivative of the complementary energy with respect to a redundant force or couple is zero.

Problem 17.9 Estimate the vertical displacement of the free end of the uniform cantilever shown.



Solution

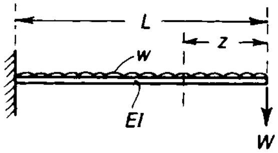
The complementary energy of bending is

$$C = \int_0^L \frac{M^2 dz}{2EI} = \int_0^L \frac{W^2 z^2 dz}{2EI} = \frac{W^2 L^3}{6EI}$$

The corresponding displacement of W is

$$\delta_w = \frac{\partial C}{\partial W} = \frac{WL^3}{3EI}$$

Problem 17.10 A cantilever has a uniform flexural stiffness EI. Estimate the vertical deflection at the free end if the cantilever carries a uniformly distributed lateral load of intensity w.



Solution

Introduce a vertical load W at the free end; the bending moment at any section is then

$$M = \frac{1}{2}wz^2 + Wz$$

The complementary energy of bending is

$$C = \frac{1}{2EI}\int_0^L \left(\frac{1}{2}wz^2 + Wz\right)^2 dz$$

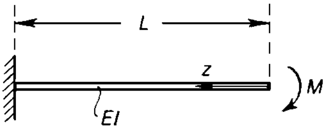
The corresponding displacement of W is

$$\delta_w = \frac{\partial C}{\partial W} = \frac{1}{EI}\int_0^L \left(\frac{1}{2}wz^2 + Wz\right) z dz$$

Now put $W = 0$; then

$$\delta_w \quad \frac{1}{EI}\int_0^L \frac{1}{2}wz^3 dz = \frac{wL^4}{8EI}$$

Problem 17.11 A cantilever of uniform flexural stiffness EI carries a moment M at the remote end. Estimate the angle of rotation at that end of the beam.



Solution

All sections of the beam carry the same bending moment M, so the complementary energy is

$$C = \int_0^L \frac{M^2 dz}{2EI} = \frac{M^2 L}{2EI}$$

The corresponding displacement of M is

$$\theta_M = \frac{ML}{EI}$$

which is the angle of rotation at the remote end.

Problem 17.12 Solve the problem discussed in Section 17.4, using complementary energy.

Solution

The bending moment at any section in terms of w and the redundant force W is $\frac{1}{2}wz^2 - Wz$. Then

$$C = \int_0^L \left(\frac{1}{2}wz^2 - Wz\right)^2 \frac{dz}{2EI}$$

The property $\partial C/\partial W = 0$ gives

$$\int_0^L \frac{1}{2}wz^3\,dz = \int_0^L Wz^2\,dz$$

Then

$$W = \frac{3wL}{8}$$

Problem 17.13 Solve Problem 17.3 using complementary energy.

Solution

The bending moment at any angular position θ is

$$M = Wr\sin\theta$$

Then

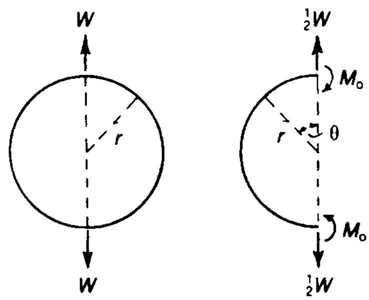
$$C = \int_0^\pi \frac{M^2}{2EI}\,r\,d\theta$$

Thus

$$\delta_w = \frac{\partial C}{\partial W} = \frac{\partial C}{\partial M}\frac{\partial M}{\partial W} = \int_0^\pi M\frac{\partial M}{\partial W}\frac{r\,d\theta}{EI}$$

$$= \int_0^\pi \frac{Wr^3\sin^2\theta}{EI}\,d\theta = \frac{\pi Wr^3}{2EI}$$

Problem 17.14 A thin circular ring of radius r and uniform flexural stiffness carries two radial loads W applied along a diameter. Estimate the maximum bending moment in the ring.



Solution

By symmetry the loading action on a half-ring are ½W and M_0. The bending moment at any angular position θ is

$$M = M_0 - \frac{1}{2}Wr\sin\theta$$

Then

$$C = \int_0^\pi \left(M_0 - \frac{1}{2}Wr\ \sin\theta\right)^2 \frac{r d\theta}{2EI}$$

But

$$\partial C/\partial M_0 = 0, \text{ so that}$$

$$\int_0^\pi M_0 d\theta = \frac{1}{2}Wr \int_0^\pi \sin\theta\, d\theta$$

Then

$$M_0 = Wr/\pi$$

17.9 The Raleigh–Ritz method

This method is also known as the *method of minimum potential*, and in Chapters 24 and 25, it is used in the finite element method.

In mathematical terms, it can be stated, as follows:

$$\frac{\partial \pi_p}{\partial W} = 0$$

where

π_p = total potential = $U_e + WD$

U_e = strain energy

WD = the potential of the load system

W = load

The method will be applied to problem 17.12 to determine an expression for δ_w.

Now

$$U_e = \int \frac{M^2}{2EI} dz = \text{the bending strain energy of a beam}$$

As

$$M = Wz = \text{bending moment at } z,$$

$$U_e = \frac{1}{2EI} \int_0^l W^2 z^2 dz$$

or

$$U_e = \frac{W^2 l^3}{6EI}$$

By inspection

$$WD = \text{potential of the load system}$$

$$= -W \delta_w$$

$$\therefore \pi_p = \frac{W^2 l^3}{6EI} - W\delta_w$$

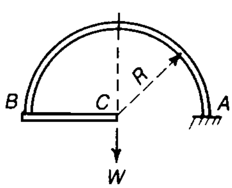
Now,

$$\frac{\partial \pi_p}{\partial W} = 0 = \frac{Wl^3}{3EI} - \delta_w$$

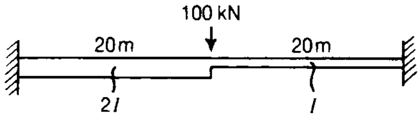
$$\therefore \delta_w = \frac{Wl^3}{3EI} \text{ as required}$$

Further problems *(answers on page 693)*

17.15 A thin semicircular bracket, AB, of radius R is built-in at A, and has at B a rigid horizontal arm BC of length R. the arm carries a vertical load W at C. Show that the vertical deflection at C is $\pi WR^3/2EI$, where EI is the flexural rigidity of the strip, and determine the horizontal deflection. *(Nottingham)*



17.16 A beam has a second moment of area of $2I$ over one-half of the span and I over the other half. Find the fixed-end moments when a load of 100 kN is carried at the mid-length.

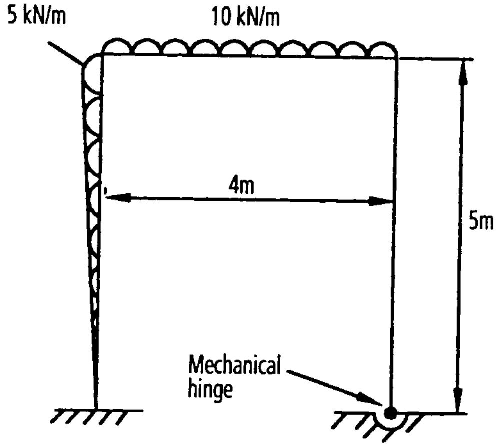


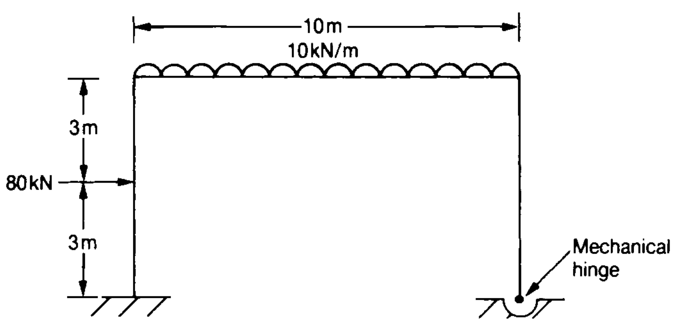
17.17 A ring radius R and uniform cross-section hangs from a single support. Find the position and magnitude of the maximum bending moment due to its own weight. *(London)*

17.18 An 'S' hook follows part of the outline of two equal circles of radius R that just touch. It embraces 5/6ths of one circle and 2/3rds of the other. If the ends are pulled apart by a force, P, by how much will they be moved if the hook has a constant rigidity EI? *(London)*

17.19 Using the plastic hinge theory determine a suitable sectional modulus for the rigid-jointed framework shown below. The following may be assumed to apply to the framework

$\lambda = 4$ $\quad\quad \sigma_Y = 300$ MPa $\quad\quad S = 1.15$





17.20 A portal frame of uniform section is subjected to the loading above. Using the plastic hinge theory, determine a suitable section modulus for the frame, based on a load factor of 4, a shape factor of 1.15 and a yield stress of 275 MPa. *(Portsmouth, Standard 1989)*

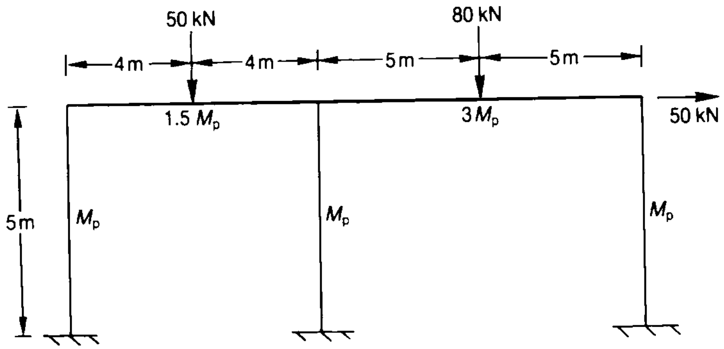
17.21 Using the plastic hinge theory, determine a suitable section modulus for the two bay rigid-jointed plane frame below.

The following assumptions should be made:-

load factor = 4

shape factor = 1.15

yield stress = 275 MPa



(Portsmouth, Honours 1989)