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APPLICATIONS

Chapter

21

Predicting Piecepart Quality

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21.1 Introduction

This chapter expands the ideas introduced in the paper, *Statistical Yield Analysis of Geometrically Toleranced Features*, presented at the Second Annual Texas Instruments Process Capability Conference (Nov. 1995). In that paper, we discussed methods to statistically analyze the manufacturing yield (in defects per unit) of part features that are dimensioned using geometric dimensioning and tolerancing (GD&T). That paper specifically discussed features that are located using positional tolerancing.

This chapter expands the prior statistical methods to include features that have multiple tolerancing constraints. The statistical methods presented in this paper:

- Show how to calculate defects per unit (DPU) for part features that have *form* and *orientation* controls in addition to *location* controls.

- Account for *material condition modifiers* (maximum material condition (MMC), least material condition (LMC), and regardless of feature size (RFS)) on *orientation*, and *location* constraints.
- Show how *different manufacturing process distributions* (bivariate normal, univariate normal, and lognormal) impact DPU calculations.

21.2 The Problem

Geometric controls are used to control the size, form, orientation, and location of features. In addition to specifying the ideal or "target" (nominal) dimension, the controls specify how much the feature characteristics can vary from their targets and still meet their functional requirements. The probability that a randomly selected part meets its tolerancing requirements is a function not only of geometric controls, but the amount and nature of the variation in the feature characteristics which result from the manufacturing process used to create the feature. The part-to-part variation in the feature characteristics can be represented by probability distribution functions reflecting the relative frequency that the feature characteristics take on specific values. We can then calculate the probability that a feature is within any one of these specifications by integrating the probability distribution function for that characteristic over the in-specification range of values. For example, if the part-to-part variation in the size of the feature, d, is described by the probability density function $g(d)$, then the probability of generating a part that is within the size upper spec limit and the size lower spec limit is:

$$P(in\_spec) = \int_{SizeLowerSL}^{SizeUpperSL} g(d)\,\mathrm{d}d$$

where SL is the specification limit.

If a feature has several GD&T requirements and we assume that the manufacturing processes that control size, form, orientation, and location are uncorrelated, then the generalized equation for the probability of meeting all of them is:

$$P(in\_spec) = \int_{SizeLowerSL}^{SizeUpperSL} g(d)\,\mathrm{d}d \int_{0}^{FormSL} j(w)\,\mathrm{d}w \int_{0}^{OrientationSL} h(q)\,\mathrm{d}q \int_{0}^{LocationSL} f(r)\,\mathrm{d}r \tag{21.1}$$

where,

$j(w)$ is the form probability distribution function,
$h(q)$ is the orientation probability distribution function, and
$f(r)$ is the location probability distribution function.

The DPU is equal to the probability of *not* being within the specification.

$$P(not\_in\_spec) = 1 - P(in\_spec)$$

$$\mathrm{DPU} = 1 - \int_{SizeLowerSL}^{SizeUpperSL} g(d)\,\mathrm{d}d \int_{0}^{FormSL} j(w)\,\mathrm{d}w \int_{0}^{OrientationSL} h(q)\,\mathrm{d}q \int_{0}^{LocationSL} f(r)\,\mathrm{d}r \tag{21.2}$$

Eq. (21.2) would be complete if there were no relationships between the size, form, orientation, and location limits. As a feature changes *orientation*, however, the amount of allowable *location* tolerance is reduced by the amount that the feature tilts. Therefore, the maximum *location* tolerance zone is a function of the feature's *orientation*. Similarly, sometimes there are relationships between other limits, such as between *size* and *location*, or between *size* and *orientation*. When these relationships are functional, we specify them on a drawing using the maximum material condition modifiers and the least material condition modifiers. If one of these modifiers is used, then, the *orientation* tolerance is a function of the feature *size*, and the *location* tolerance is a function of the feature *size*.

Note: In ASME Y14.5-1994, the tolerance zones for size, form, orientation, and location often overlap each other. For example, the orientation tolerance zone may be inside the location tolerance zone, and the form tolerance zone may be inside the orientation tolerance zone. Since Y14.5 communicates engineering design requirements, this is the correct method to apply tolerance zones.

However, when predicting manufacturing yield for pieceparts, the manufacturing processes are considered. Therefore, we need to separate the tolerance zones for size, form, orientation, and location. Because of this, when we refer to the "allowable" tolerance zone in a statistical analysis, this is different than the "allowable" tolerance zone allowed in Y14.5.

Note: It is difficult to write an equation to show the relationship between *form* and *size* as defined in ASME Y14.5M-1994. It is equally difficult to write relationships for *location* and *orientation* as a function of *form*. In the following equations, we will assume that these relationships are negligible and can be ignored.

21.3 Statistical Framework

21.3.1 Assumptions

Fig. 21-1 shows an example of a feature (a hole) that is toleranced using the following constraints:

- The diameter has an upper spec limit of $D + T_2$.
- The diameter has a lower spec limit of $D - T_1$.
- A perpendicularity control ($\varnothing 2Q$) that is at regardless of feature size.
- A positional control ($\varnothing 2R$) that is at regardless of feature size.

The feature is assumed to have a target location with a tolerance zone defined by a cylinder of radius *R*. In addition, the diameter of the feature also has a target value, *D*. To be within specifications, the

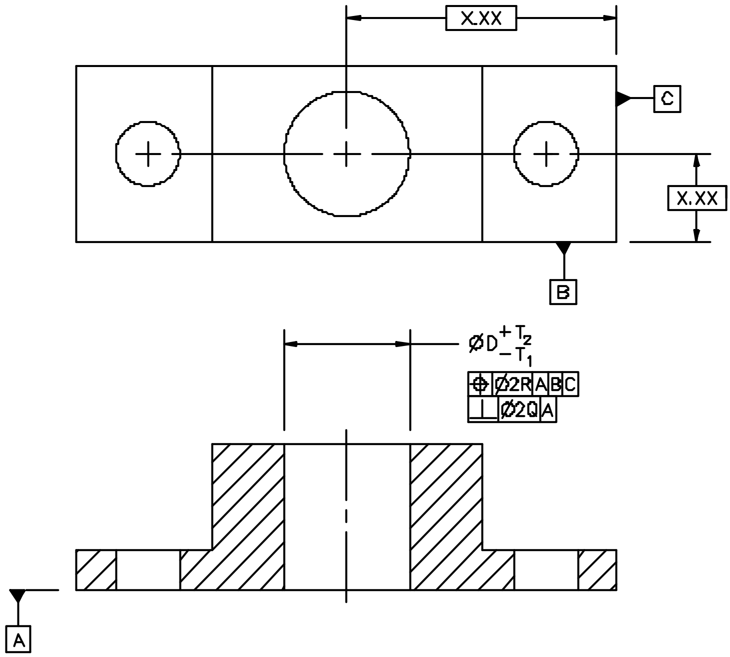

Figure 21-1 Cylindrical (size) feature with orientation and location constraints at RFS

diameter of the feature needs to be between $D - T_1$ and $D + T_2$. The feature is allowed a maximum offset from the vertical of Q.

If the angle between the feature axis and the vertical is given by q, then q has a maximum value of arcsin(2Q/L), where the length of the feature is L (as shown in Fig. 21-2). In addition, as q increases, the amount of the location tolerance available to the feature decreases by the amount of lateral offset from the vertical, L*sin(q)/2. This results in the location tolerance zone having an effective radius of $R - L$*sin(q)/2.

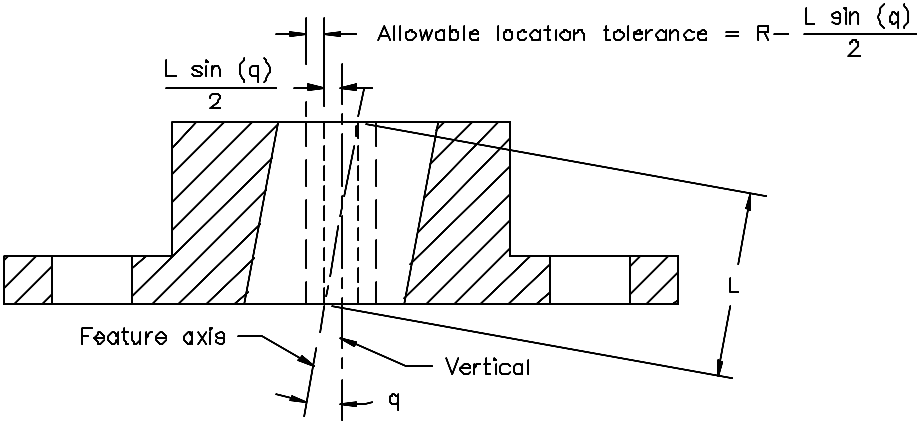

Figure 21-2 Allowable location tolerance as a function of orientation error *(q)*

To account for the variation in the process that generates the feature, the offsets in the X and Y coordinates of the feature location relative to the target location (d_X and d_Y) are assumed to be normally distributed with mean 0 and common standard deviation s. In addition, it is assumed that the X and Y deviations are uncorrelated (independent). The variation in the diameter of the feature, d, is assumed to have a lognormal distribution with mean m_d and standard deviation s_d and the diameter is uncorrelated with either the X or Y deviations. Finally, it is assumed that the variation in the angle of tilt (orientation), q, is lognormally distributed with mean m_q and standard deviation s_q and is also assumed to be uncorrelated with the X and Y deviations and the feature diameter. Note that this analysis assumes that the processes stay centered on the target (nominal dimension). The standard deviations for these processes are generally considered short-term standard deviations. If the means of the processes shift over time, as discussed in Chapters 10 and 11, then the appropriate standard deviations must be inflated to approximate the long-term shift.

If we define $r = \sqrt{d_X^2 + d_Y^2}$ to be the distance from the target location to the location of the feature, then the probability density functions for d, q, and r are given by:

size

$$g(d) = \frac{1}{dg\sqrt{2p}} e^{-\frac{(\ln(d) - q)^2}{2g^2}}$$

where $q = \ln(\mu_d) - \frac{\ln\left(1 + \frac{s_d^2}{m_d^2}\right)}{2}$ and $g = \sqrt{1 + \frac{s_d^2}{m_d^2}}$

orientation
$$h(q) = \frac{1}{qt\sqrt{2p}}\, e^{-\frac{(\ln(q)-n)^2}{2t^2}}$$

where $n = \ln(\mu_q) - \frac{\ln\left(1+\frac{s_q^2}{m_q^2}\right)}{2}$ and $t = \sqrt{1+\frac{s_q^2}{m_q^2}}$

and location
$$f(r) = \frac{r}{s^2}\, e^{-\frac{r^2}{2s^2}}$$

Since d, q, and r are independent, the probability of the feature being simultaneously within specification for size, orientation, and location can be found by taking the product of the density functions and integrating the product over the in-specification range of values for d, q, and r. In the case specified above, where d must be between $D - T_1$ and $D + T_2$, q must be less than arcsin ($2Q/L$), and r must be less than R, this probability is represented by:

$$P(in\_spec) = \int_{D-T_1}^{D+T_2} \int_0^{\arcsin(2Q/L)} \int_0^{(R-L\sin(q)/2)} \frac{1}{dg\sqrt{2p}}\, e^{-\frac{(\ln(d)-q)^2}{2g^2}} \frac{1}{qt\sqrt{2p}}\, e^{-\frac{(\ln(q)-u)^2}{2t^2}} \frac{r}{s^2}\, e^{-\frac{r^2}{2s^2}}\, dd\, dq\, dr$$

$$= \int_{D-T_1}^{D+T_2} \left(\int_0^{\arcsin(2Q/L)} \left(1 - e^{-\frac{(R-L\sin(q)/2)^2}{2s^2}} \right) \frac{1}{qt\sqrt{2p}}\, e^{-\frac{(\ln(q)-n)^2}{2t^2}}\, dq \right) \frac{1}{dg\sqrt{2p}}\, e^{-\frac{(\ln(d)-q)^2}{2g^2}}\, dd$$

where the final integration has to be done using numerical methods. To then calculate the probability of an unacceptable part, or DPU, this value is subtracted from 1.

This calculation becomes more complicated when material condition modifiers are used. This means that the DPU calculation depends upon whether MMC or LMC is used for the location and orientation specifications and whether the feature is an internal or external feature.

21.3.2 Internal Feature at MMC

Fig. 21-3 shows an example of a feature that is toleranced the same as Fig. 21-1, except that it has a positional control at maximum material condition, and a perpendicularity control at maximum material condition.

In this case, the specified tolerance applies when the feature is at MMC, or the part contains the most material. This means that when the feature is at its smallest allowable size, D-T_1, the tolerance zone for the location of the feature has a radius of R and the orientation (tilt) offset has a maximum of Q. As the feature gets larger, or departs from MMC, the tolerance zones get larger. For each unit of increase in the diameter of the feature, the diameter of the location tolerance zone increases by 1 unit, the radius increases by 1/2 unit, and the maximum orientation tolerance increases by 1 unit. When the feature is at its maximum allowable diameter, D+T_2, the location tolerance zone has a radius of R+ (T_1+T_2)/2 and the orientation

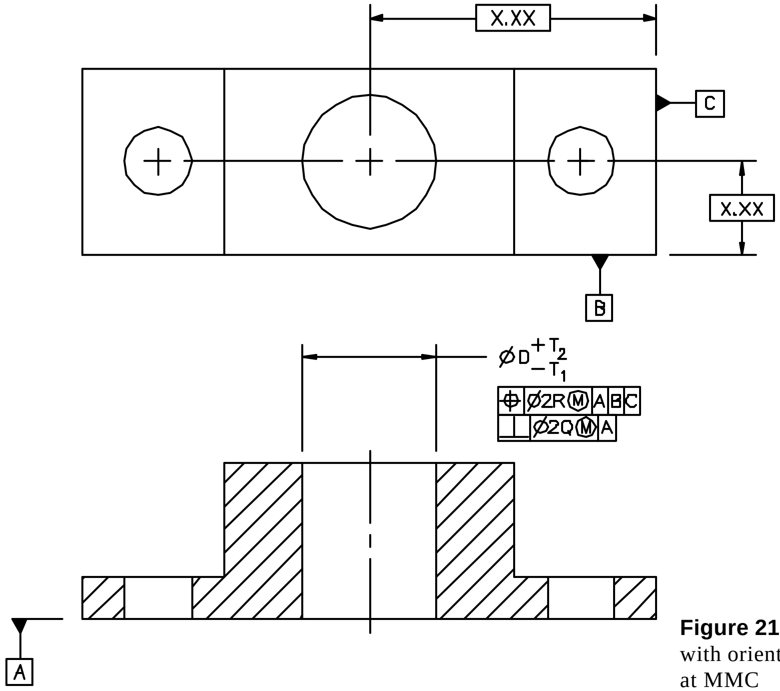

Figure 21-3 Cylindrical (size) feature with orientation and location constraints at MMC

tolerance is $Q + (T_1+T_2)$. As mentioned above, as the orientation increases the radius of the location tolerance zone also decreases by $L*\sin(q)/2$. The radius of the location tolerance zone is therefore a function of d and q:

$$R_M\,(d,q) = R - \frac{D-T_1}{2} + \frac{d}{2} - \frac{L*\sin(q)}{2} = D_1 + \frac{d}{2} - \frac{L*\sin(q)}{2}$$

where $D_1 = R - \dfrac{D-T_1}{2}$

The maximum allowable orientation offset is also a function of d:

$$Q_M\,(d) = Q - (D - T_1) + d$$

The probability that the feature location is within specification is also now a function of d and q. The probability that the feature orientation is within specification is a function of d. If both the location and orientation tolerances are called out at MMC, the probability that the feature is within size, orientation, and location specifications is given by:

$$P(in\_spec) = \int_{D-T_1}^{D+T_2} \left(\int_{0}^{\arcsin\left(\frac{2Q_M\,(d)}{L}\right)} \left(1 - e^{-\frac{(R_M\,(d,q))^2}{2s^2}}\right) \frac{1}{qt\sqrt{2p}} e^{-\frac{(\ln(q)-u)^2}{2t^2}}\, dq \right) \frac{1}{dg\sqrt{2p}} e^{-\frac{(\ln(d)-q)^2}{2g^2}}\, \mathrm{d}d$$

The integration must be done using numerical methods and the DPU for the feature is calculated by subtracting the result from 1.

21.3.3 Internal Feature at LMC

Fig. 21-4 shows an example of a feature that is toleranced the same as Fig. 21-1, except that it has a positional control at least material condition, and a perpendicularity control at least material condition.

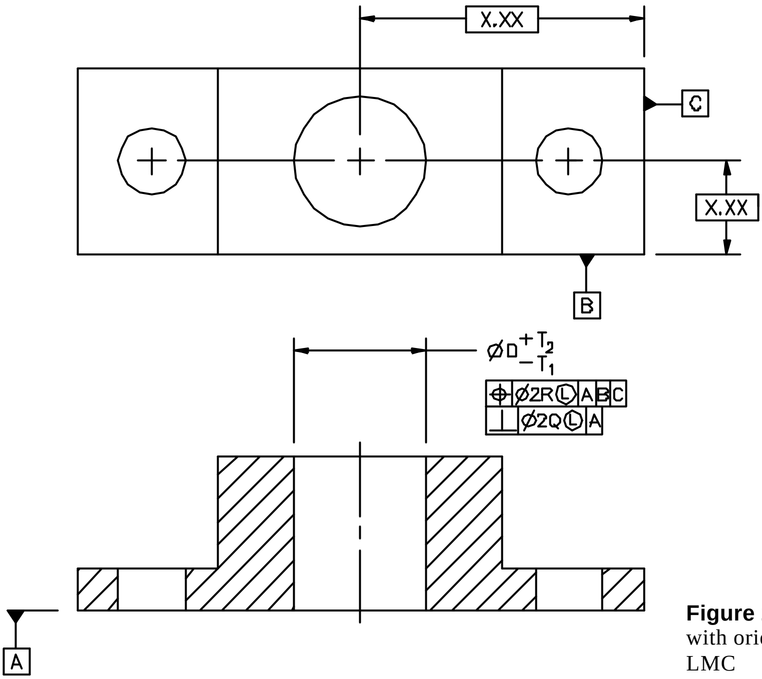

Figure 21-4 Cylindrical (size) feature with orientation and location constraints at LMC

In this case, the specified location tolerance applies when the feature is at LMC, or the part contains the least material. This means that when the feature is at its largest allowable size, $D+T_2$, the tolerance zone for the location of the feature has a radius of R. As the feature gets smaller, or departs from LMC, the tolerance zone gets larger. This means that when the feature is at its largest allowable size, $D+T_2$, the tolerance zone for the location of the feature has a radius of R and the tolerance for the orientation offset is Q. For each unit of decrease in the diameter of the feature, the diameter of the tolerance zone and the orientation offset tolerance each increases by 1 unit. When the feature is at its minimum allowable diameter, $D - T_1$, the location tolerance zone has a radius of $R+(T_1 + T_2)/2$ and the orientation tolerance is $Q + (T_1 + T_2)$. As before, as the orientation increases, the radius of the location tolerance zone decreases by $L*\sin(q)/2$. The radius of the location tolerance zone is therefore a function of d and q:

$$R_L(d,q) = R + \frac{D+T_2}{2} - \frac{d}{2} - \frac{L*\sin(q)}{2} = D_2 - \frac{d}{2} - \frac{L*\sin(q)}{2}$$

where $D_2 = R + \frac{D+T_2}{2}$

The maximum allowable orientation offset is also a function of d:

$$Q_L(d) = Q + (D + T_2) - d$$

The probability that the feature location is within specification is also now a function of d and q. The probability that the feature orientation is within specification is a function of d. If both the location and orientation tolerances are called out at LMC, the probability that the feature is within the size, orientation, and location specifications is given by:

$$P(inspec) = \int_{D-T_1}^{D+T_2} \left(\int_0^{\arcsin\left(\frac{2Q_L(d)}{L}\right)} \left(1 - e^{-\frac{(R_L(d,q))^2}{2s^2}} \right) \frac{1}{qt\sqrt{2p}} e^{-\frac{(\ln(q)-u)^2}{2t^2}} \mathrm{d}q \right) \frac{1}{dg\sqrt{2p}} e^{-\frac{(\ln(d)-q)^2}{2g^2}} \mathrm{d}d$$

The integration must be done using numerical methods and the DPU for the feature is calculated by subtracting the result from 1.

21.3.4 External Features

In the case of an external feature called out at MMC, the specified tolerance applies when the feature is at its largest allowable size, $D+T_2$. As the feature gets smaller, or departs from MMC, the tolerance zones get larger. This is the same situation as for the internal feature at LMC, so the probability of the feature being within size, orientation, and location specification is calculated using the same formula.

In the case of an external feature called out at LMC, the specified tolerance applies when the feature is at its smallest allowable size, D-T_1. As the feature gets larger, the tolerance zones get larger. This is the same situation as for the internal feature at MMC, so the probability of the feature being within size, orientation, and location specification is calculated using the same formula.

21.3.5 Alternate Distribution Assumptions

Traditionally, the feature diameter has been assumed to have a normal, or Gaussian, distribution. In order to compare the results of GD&T specifications with traditional tolerancing methods, it may be necessary to calculate the DPU with this distribution assumption. Also, when the feature is formed by casting, as opposed to machining, the normal distribution assumption is applicable. In these cases, the probability distribution function for d, $g(d)$, is given by:

$$g(d) = \frac{1}{s_d\sqrt{2p}} e^{-\frac{(d-\mu_d)^2}{2s_d^2}}$$

In the case where the feature location is constrained only in one direction, such as when the feature is a slot, then r is usually assumed to have a normal distribution with a mean of 0 and a standard deviation of σ. See Fig. 21-5.

The probability that the feature is in location specification is given by

$$P(in\_spec) = \int_{-(R-L\sin(q)/2)}^{R-L\sin(q)/2} \frac{1}{s\sqrt{2p}} e^{-\frac{r^2}{2s^2}} \mathrm{d}r$$

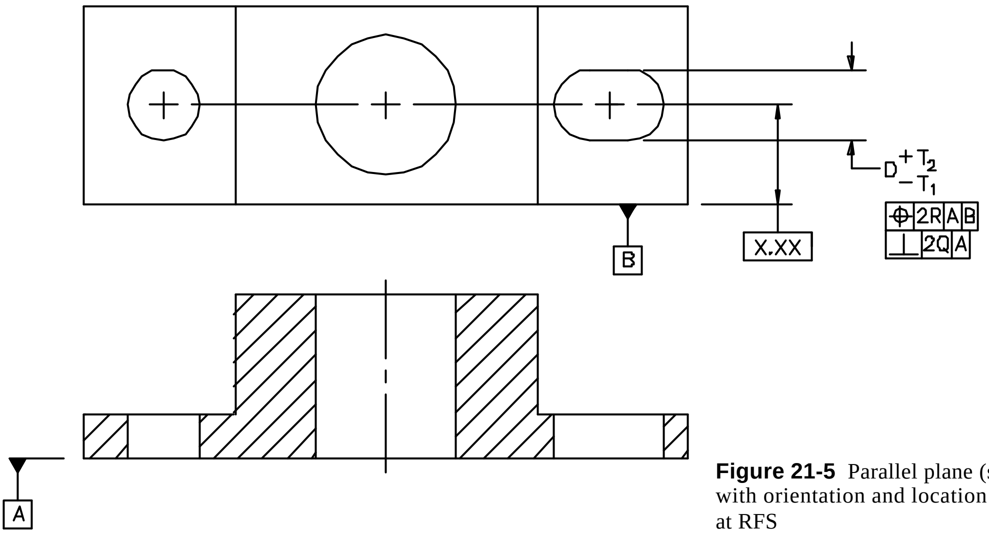

Figure 21-5 Parallel plane (size) feature with orientation and location constraints at RFS

In this case, q is the orientation angle between the center plane of the feature and a plane orthogonal to datum A. If an internal feature is toleranced at MMC, or an external feature is toleranced at LMC, $R - L*\sin(q)/2$ is replaced by R_M . It is replaced by R_L when an internal feature is toleranced at LMC or an external feature is toleranced at MMC.

21.4 Non-Size Feature Applications

The examples shown thus far were features of size (hole, pins, slots, etc.). This methodology can be expanded to include features that do not have size, such as profiled features. For features that do not have size, the material condition modifiers no longer impact the equation. Therefore, the only relationship that we should account for is between *location* and *orientation*. In these cases, Eq. (21.2) reduces to:

$$DPU = 1 - \int_0^{LocationSpecLimit} f(r)\,dr \int_0^{OrientationSpecLimit} h(q)\,dq \int_0^{FormSpecLimit} j(w)\,dw$$

21.5 Example

Table 21-1 compares the predicted dpmo's for various tolerancing scenarios. Cases 1, 2, and 3 are the same, except for the material condition modifiers. Case 2 (MMC) and Case 3 (LMC) estimate the same dpmo, as expected. Both cases predict a much lower dpmo than Case 1 (RFS). Cases 4, 5, and 6 are similar to Cases 1, 2, and 3, respectively, except that the tolerance limits are less. As expected, the number of defects increased.

Table 21-1 Comparison of tolerancing scenarios

		Case 1	**Case 2**	**Case 3**	**Case 4**	**Case 5**	**Case 6**
Feature Type		Internal	Internal	Internal	Internal	Internal	Internal
Length	L	.500	.500	.500	.500	.500	.500
Size	D	.1273	.1273	.1273	.1273	.1273	.1273
	T_1	.0010	.0010	.0010	.0007	.0007	.0007
	T_2	.0010	.0010	.0010	.0007	.0007	.0007
	μ_d	.1273	.1273	.1273	.1273	.1273	.1273
	σ_d	.00025	.00025	.00025	.00025	.00025	.00025
	Distribution type	Lognormal	Lognormal	Lognormal	Lognormal	Lognormal	Lognormal
Orientation	2Q	.0008	.0008	.0008	.0004	.0004	.0004
	μ_q	.00003	.00003	.00003	.00003	.00003	.00003
	σ_q	.00013	.00013	.00013	.00013	.00013	.00013
	Material condition	RFS	MMC	LMC	RFS	MMC	LMC
	Distribution type	Log-normal	Log-normal	Log-normal	Log-normal	Log-normal	Log-normal
Location	2R	.0064	.0064	.0064	.0032	.0032	.0032
	μ	0	0	0	0	0	0
	σ	.0005	.0005	.0005	.0005	.0005	.0005
	Material condition	RFS	MMC	LMC	RFS	MMC	LMC
	Distribution type	Normal	Normal	Normal	Normal	Normal	Normal
Figure		21-1	21-3	21-4	21-1	21-3	21-4
dpmo		838	111	111	14134	6195	6204

21.6 Summary

The equations presented in this chapter can predict the probability that a feature on a part will meet the constraints imposed by geometric tolerancing. Notice how Eq. (21.1) is similar to, but not exactly the same as the "four fundamental levels of control" in Chapter 5 (see section 5.6). Chapter 5 discusses how these levels of control should be added as demanded by the functional requirements of the feature. It is possible (and often likely) to add GD&T constraints that "function" with little or no insight to the manufacturability of the applied tolerances. The equations in this chapter help predict the *cost* of manufacturing in terms of defective features.

Although these equations are generic, they do not encompass all combinations of GD&T feature control frames. These equations do, however, provide a framework for expansion to include all GD&T relationships.

21.7 References

1. Drake, Paul, Dale Van Wyk, and Dan Watson. 1995. Statistical Yield Analysis of Geometrically Toleranced Features. Paper presented at Second Annual Texas Instruments Process Capability Conference. Nov. 1995. Plano, Texas.
2. The American Society of Mechanical Engineers. 1995. *ASME Y14.5M-1994, Dimensioning and Tolerancing.* New York, New York: The American Society of Mechanical Engineers.