Chapter

14

Minimum-Cost Tolerance Allocation

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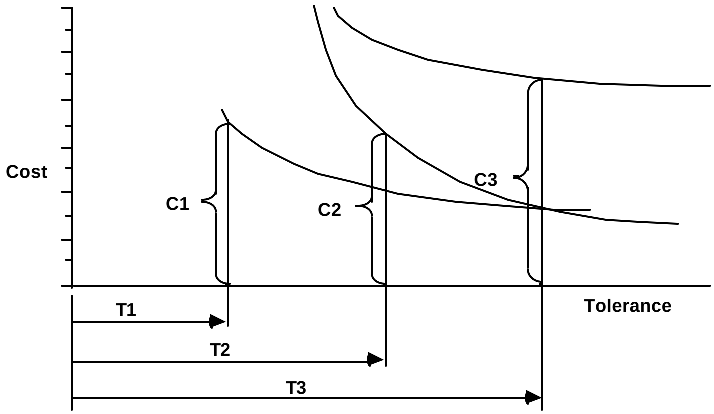
Dr. Chase has taught mechanical engineering at the Brigham Young University since 1968. An advocate of computer technology, he has served as a consultant to industry on numerous projects involving engineering software applications. He served as a reviewer of the Motorola Six Sigma Program at its inception. He also served on an NSF select panel for evaluating tolerance analysis research needs. In 1984, he founded the ADCATS consortium for the development of CAD-based tools for tolerance analysis of mechanical assemblies. More than 30 sponsored graduate theses have been devoted to the development of the tolerance technology contained in the CATS software. Several faculty and students are currently involved in a broad spectrum of research projects and industry case studies on statistical variation analysis. Past and current sponsors include Allied Signal, Boeing, Cummins, FMC, Ford, GE, HP, Hughes, IBM, Motorola, Sandia Labs, Texas Instruments, and the US Navy.

14.1 Tolerance Allocation Using Least Cost Optimization

A promising method of tolerance allocation uses optimization techniques to assign component tolerances that minimize the cost of production of an assembly. This is accomplished by defining a cost-versus-tolerance curve for each component part in the assembly. An optimization algorithm varies the tolerance for each component and searches systematically for the combination of tolerances that minimize the cost.

14.2 1-D Tolerance Allocation

Fig. 14-1 illustrates the concept simply for a three component assembly. Three cost-versus-tolerance curves are shown. Three tolerances (T_1, T_2, T_3) are initially selected. The corresponding cost of production is $C_1 + C_2 + C_3$. The optimization algorithm tries to increase the tolerances to reduce cost; however, the specified assembly tolerance limits the tolerance size. If tolerance T_1 is increased, then tolerance T_2 or T_3 must decrease to keep from violating the assembly tolerance constraint. It is difficult to tell by inspection which combination will be optimum, but you can see from the figure that a decrease in T_2 results in a significant increase in cost, while a corresponding decrease in T_3 results in a smaller increase in cost. In this manner, one could manually adjust tolerances until no further cost reduction is achieved. The optimization algorithm is designed to find the minimum cost automatically. Note that the values of the set of optimum tolerances will be different when the tolerances are summed statistically than when they are summed by worst case.



Total Cost:

$$C_{tot} = C_1 + C_2 + C_3$$

Constraint:

$$T_{tot} = T_1 + T_2 + T_3 \quad \text{[Worst Case]}$$

$$= \sqrt{T_1^2 + T_2^2 + T_3^2} \quad \text{[Statistical]}$$

Figure 14-1 Optimal tolerance allocation for minimum cost

A necessary factor in optimum tolerance allocation is the specification of cost-versus-tolerance functions. Several algebraic functions have been proposed, as summarized in Table 14-1. The Reciprocal Power function: $C = A + B/tol^k$ includes the Reciprocal and Reciprocal Squared rules for integer powers of k. The constant coefficient A represents fixed costs. It may include setup cost, tooling, material, and prior operations. The B term determines the cost of producing a single component dimension to a specified tolerance and includes the charge rate of the machine. Costs are calculated on a per-part basis. When tighter tolerances are called for, speeds and feeds may be reduced and the number of passes increased, requiring more time and higher costs. The exponent k describes how sensitive the process cost is to changes in tolerance specifications.

Table 14-1 Proposed cost-of-tolerance models

Cost Model	Function	Author	Ref
Reciprocal Squared	$A + B/tol^2$	Spotts	Spotts 1973 (Reference 11)
Reciprocal	$A + B/tol$	Chase & Greenwood	Chase 1988 (Reference 3)
Reciprocal Power	$A + B/tol^k$	Chase et al.	Chase 1989 (Reference 4)
Exponential	$A e^{-B(tol)}$	Speckhart	Speckhart 1972 (Reference 10)

Little has been done to verify the form of these curves. Manufacturing cost data are not published since they are so site-dependent. Even companies using the same machines would have different costs for labor, materials, tooling, and overhead.

A study of cost versus tolerance was made for the metal removal processes over the full range of nominal dimensions. This data has been curve fit to obtain empirical functions. The form was found to follow the reciprocal power law. The results are presented in the Appendix to this chapter. The original cost study is decades old and may not apply to modern numerical controlled (N/C) machines.

A closed-form solution for the least-cost component tolerances was developed by Spotts. (Reference 11) He used the method of Lagrange Multipliers, assuming a cost function of the form $C=A+B/tol^2$. Chase extended this to cost functions of the form $C=A+B/tol^k$ as follows: (Reference 4)

$$\frac{\partial}{\partial T_i}(Cost\_\ function) + l\,\frac{\partial}{\partial T_i}(Constraint) = 0 \qquad (i=1,\ldots n)$$

$$\frac{\partial}{\partial T_i}\left(\sum\left(A_j + B_j / T_j^{k_j}\right)\right) + l\,\frac{\partial}{\partial T_i}\left(\sum T_j^2 - T_{asm}^2\right) = 0 \qquad (i=1,\ldots n)$$

$$l = \frac{k_i B_i}{2T_i^{(k_i+2)}} \qquad (i=1,\ldots n)$$

Eliminating l by expressing it in terms of T_1 (arbitrarily selected):

$$T_i = \left(\frac{k_i B_i}{k_1 B_1}\right)^{1/(k_i+2)} T_1^{(k_1+2)/(k_i+2)} \tag{14.1}$$

Substituting for each of the T_i in the assembly tolerance sum:

$$T_{ASM}^2 = T_1^2 + \sum\left(\frac{k_i B_i}{k_1 B_1}\right)^{2/(k_i+2)} T_1^{2(k_1+2)/(k_i+2)} \tag{14.2}$$

The only unknown in Eq. (14.2) is T_1. One only needs to iterate the value of T_1 until both sides of Eq. (14.2) are equal to obtain the minimum cost tolerances. A similar derivation based on a worst case assembly tolerance sum yields:

$$T_{ASM} = T_1 + \sum\left(\frac{k_i B_i}{k_1 B_1}\right)^{1/(k_i+1)} T_1^{(k_1+1)/(k_i+1)} \tag{14.3}$$

A graphical interpretation of this method is shown in Fig. 14-2 for a two-part assembly. Various combinations of the two tolerances may be selected and summed statistically or by worst case. By summing the cost corresponding to any T_1 and T_2, contours of constant cost may be plotted. You can see that cost decreases as T_1 and T_2 are increased. The limiting condition occurs when the tolerance sum equals the assembly requirement T_{ASM}. The worst case limit describes a straight line. The statistical limit is an ellipse. T_1 and T_2 values must not be outside the limit line. Note that as the method of Lagrange Multipliers assumes, the minimum cost tolerance value is located where the constant cost curve is tangent to the tolerance limit curve.

14.3 1-D Example: Shaft and Housing Assembly

The following example is based on the shaft and housing assembly shown in Fig. 14-3. Two bearing sleeves maintain the spacing of the bearings to match that of the shaft. Accumulation of variation in the assembly results in variation in the end clearance. Positive clearance is required.

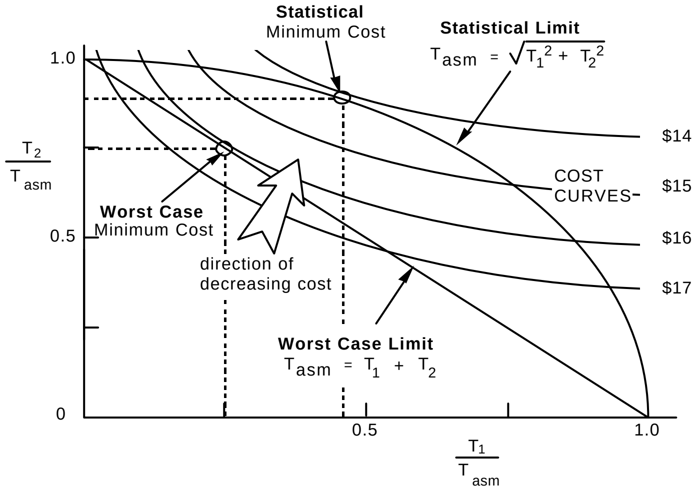


Figure 14-2 Graphical interpretation of minimum cost tolerance allocation

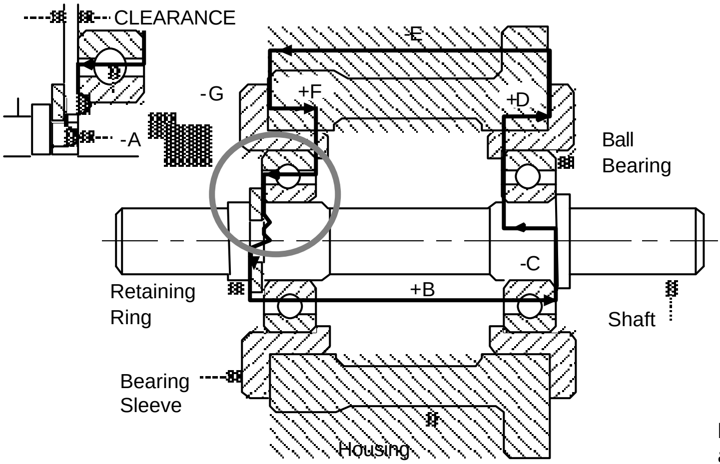


Figure 14-3 Shaft and housing assembly

Initial tolerances for parts B, D, E, and F are selected from tolerance guidelines such as those illustrated in Fig. 14-4. The bar chart shows the typical range of tolerance for several common processes. The numerical values appear in the table above the bar chart. Each row of the numerical table corresponds to a different nominal size range. For example, a turned part having a nominal dimension of .750 inch can be produced to a tolerance ranging from ±.001 to ±.006 inch, depending on the number of passes, rigidity of the machine, and fixtures. Tolerances are chosen initially from the middle of the range for each dimension and process, then adjusted to match the design limits and reduce production costs.

Table 14-2 shows the problem data. The retaining ring (A) and the two bearings (C and G) supporting the shaft are vendor-supplied, hence their tolerances are fixed and must not be altered by the allocation process. The remaining dimensions are all turned in-house. Initial tolerance values for B, D, E, and F were selected from Fig. 14-4, assuming a midrange tolerance. The critical clearance is the shaft end-play, which is determined by tolerance accumulation in the assembly. The vector diagram overlaid on the figure is the assembly loop that models the end-play.

RANGE OF SIZES FROM	THROUGH	TOLERANCES $\pm 3s$								
0.000	0.599	0.00015	0.0002	0.0003	0.0005	0.0008	0.0012	0.002	0.003	0.005
0.600	0.999	0.00015	0.00025	0.0004	0.0006	0.001	0.0015	0.0025	0.004	0.006
1.000	1.499	0.0002	0.0003	0.0005	0.0008	0.0012	0.002	0.003	0.005	0.008
1.500	2.799	0.00025	0.0004	0.0006	0.001	0.0015	0.0025	0.004	0.006	0.010
2.800	4.499	0.0003	0.0005	0.0008	0.0012	0.002	0.003	0.005	0.008	0.012
4.500	7.799	0.0004	0.0006	0.001	0.0015	0.0025	0.004	0.006	0.010	0.015
7.800	13.599	0.0005	0.0008	0.0012	0.002	0.003	0.005	0.008	0.012	0.020
13.600	20.999	0.0006	0.001	0.0015	0.0025	0.004	0.006	0.010	0.015	0.025

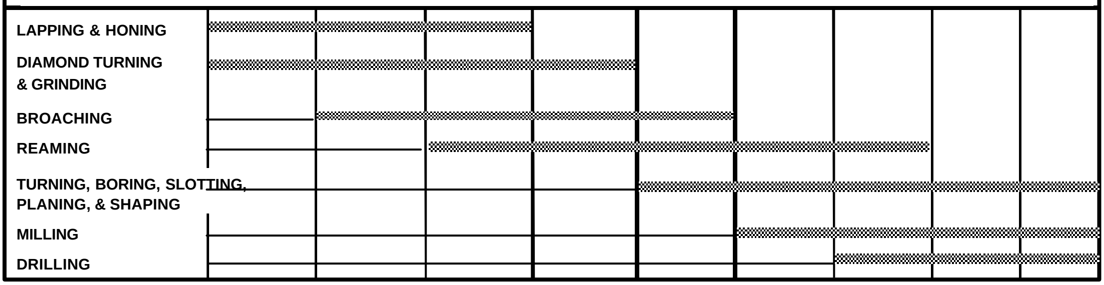


Figure 14-4 Tolerance range of machining processes (Reference 12)

Table 14-2 Initial Tolerance Specifications

Dimension	Nominal	Initial Tolerance	Process Tolerance Limits Min Tol	Max Tol
A	.0505	.0015*	*	*
B	8.000	.008	.003	.012
C	.5093	.0025*	*	*
D	.400	.002	.0005	.0012
E	7.711	.006	.0025	.010
F	.400	.002	.0005	.0012
G	.5093	.0025*	*	*

* Fixed tolerances

The average clearance is the vector sum of the average part dimensions in the loop:

Required Clearance = .020 ± .015

Average Clearance = – A + B – C + D – E + F – G

= – .0505 + 8.000 – .5093 + .400 – 7.711 + .400 – .5093

= .020

The worst case clearance tolerance is obtained by summing the component tolerances:

$$T_{SUM} = T_A + T_B + T_C + T_D + T_E + T_F + T_G$$

$$= +.0015 + .008 + .0025 + .002 + .006 + .002 + .0025$$

$$= .0245 \quad \text{(too large)}$$

To apply the minimum cost algorithm, we must set T_{SUM} = (T_{ASM} - fixed tolerances) and substitute for T_D, T_E, and T_F in terms of T_B, as in Eq. (14.3).

$$T_{ASM} - T_A - T_C - T_G = T_B + \left(\frac{k_D B_D}{k_B B_B}\right)^{1/(k_D+1)} T_B^{(k_B+1)/(k_D+1)} +$$

$$\left(\frac{k_E B_E}{k_B B_B}\right)^{1/(k_E+1)} T_B^{(k_B+1)/(k_E+1)} + \left(\frac{k_F B_F}{k_B B_B}\right)^{1/(k_F+1)} T_B^{(k_B+1)/(k_F+1)}$$

Inserting values into the equation yields:

$$.015 - .0015 - .0025 - .0025 = T_B + \left(\frac{(.46823)(.07202)}{(.43899)(.15997)}\right)^{1/(1.46823)} T_B^{(1.43899)/(1.46823)} +$$

$$\left(\frac{(.46537)(.12576)}{(.43899)(.15997)}\right)^{1/(1.46537)} T_B^{(1.43899)/(1.46537)} + \left(\frac{(.46823)(.07202)}{(.43899)(.15997)}\right)^{1/(1.46823)} T_B^{(1.43899)/(1.46823)}$$

The values of k and B for each nominal dimension were obtained from the fitted cost-tolerance functions for the turning process listed in the Appendix of this chapter. Using a spreadsheet program, calculator with a "Solve" function, or other math utility, the value of T_B satisfying the above expression can be found. T_B can then be substituted into the individual expressions to obtain the corresponding values of T_D, T_E, and T_F, and the predicted cost.

$$T_B = .0025$$

$$T_D = T_F = \left(\frac{(.46823)(.07202)}{(.43899)(.15997)}\right)^{1/(1.46823)} T_B^{(1.43899)/(1.46823)} = .0017$$

$$T_E = \left(\frac{(.46537)(.12576)}{(.43899)(.15997)}\right)^{1/(1.46537)} T_B^{(1.43899)/(1.46537)} = .0025$$

$$C = A_B + B_B(T_B)^{k_B} + A_D + B_D(T_D)^{k_D} + A_E + B_E(T_E)^{k_E} + A_F + B_F(T_F)^{k_F} = \$11.07$$

Numerical results for the example assembly are shown in Table 14-3.

The setup cost is coefficient A in the cost function. Setup cost does not affect the optimization. For this example, the setup costs were all chosen as equal, so they would not mask the effect of the tolerance allocation. In this case, they merely added $4.00 to the assembly cost for each case.

Parts A, C, and G are vendor-supplied. Since their tolerances are fixed, their cost cannot be changed by reallocation, so no cost data is included in the table.

The statistical tolerance allocation results were obtained by a similar procedure, using Eq. (14.2).

Note that in this example the assembly cost increased when worst case allocation was performed. The original tolerances, when summed by worst case, give an assembly variation of .0245 inch. This exceeds the specified assembly tolerance limit of .015 inch. Thus, the component tolerances had to be tightened, driving up the cost. When summed statistically, however, the assembly variation was only .0011 inch. This was less than the spec limit. The allocation algorithm increased the component tolerances, decreasing the cost. A graphical comparison is shown in Fig. 14-5. It is clear from the graph that tolerances for B and E were tightened in the Worst Case Model, while D and F were loosened in the Statistical Model.

Table 14-3 Minimum cost tolerance allocation

Tolerance Cost Data					Allocated Tolerances	
Dimension	Setup A	Coefficient B	Exponent k	Original Tolerance	Worst Case	Stat. $\pm3\sigma$
A		*	*	.0015*	.0015*	.0015*
B	$1.00	.15997	.43899	.008	.00254	.0081
C		*	*	.0025*	.0025*	.0025*
D	1.00	.07202	.46823	.002	.001736	.00637
E	1.00	.12576	.46537	.006	.002498	.00792
F	1.00	.07202	.46823	.002	.001736	.00637
G		*	*	.0025*	.0025*	.0025*
Assembly Variation				.0245(WC) .0111(RSS)	.0150(WC)	.0150(RSS)
Assembly Cost				$9.34	$11.07	$8.06
Acceptance Fraction					1.000	.9973
"True Cost"					$11.07	$8.08

*Fixed tolerances

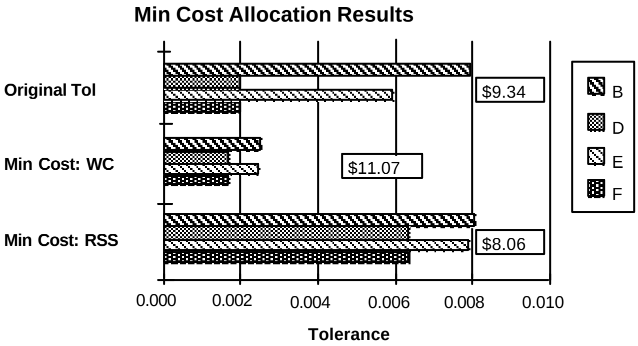


Figure 14-5 Comparison of minimum cost allocation results

14.4 Advantages/Disadvantages of the Lagrange Multiplier Method

The advantages are:

- It eliminates the need for multiple-parameter iterative solutions.
- It can handle either worst case or statistical assembly models.
- It allows alternative cost-tolerance models.

The limitations are:

- Tolerance limits cannot be imposed on the processes. Most processes are only capable of a specified range of tolerance. The designer must check the resulting component tolerances to make sure they are within the range of the process.
- It cannot readily treat the problem of simultaneously optimizing interdependent design specifications. That is, when an assembly has more than one design specification, with common component dimensions contributing to each spec, some iteration is required to find a set of shared tolerances satisfying each of the engineering requirements.

Problems exhibiting multiple assembly requirements may be optimized using nonlinear programming techniques. Manual optimization may be performed by optimizing tolerances for one assembly spec at a time, then choosing the lowest set of shared component tolerance values required to satisfy all assembly specs simultaneously.

14.5 True Cost and Optimum Acceptance Fraction

The "True Cost" in Table 14-4 is defined as the total cost of an assembly divided by the acceptance fraction or yield. Thus, the total cost is adjusted to include a share of the cost of the rejected assemblies. It does not include, however, any parts that might be saved by rework or the cost of rejecting individual component parts.

An interesting exercise is to calculate the optimum acceptance fraction; that is, the rejection rate that would result in the minimum True Cost. This requires an iterative solution. For the example problem, the results are shown in Table 14-4:

Table 14-4 Minimum True Cost

Cost Model	ΣA	$Z_{assembly}$	Optimum Acceptance Fraction	True Cost
$A + B/tol^k$	\$4.00	2.03	.9576	\$7.67
$A + B/tol^k$	\$8.00	2.25	.9756	\$11.82

The results indicate that loosening up the tolerances will save money on production costs, but will increase the cost of rejects. By iterating on the acceptance fraction, it is possible to find the value that minimizes the combined cost of production and rejects. Note, however, that the setup costs were set very low. If setup costs were doubled, as shown in the second row of the table, the cost of rejects would be higher, requiring a higher acceptance level.

In the very probable case where individual process cost-versus-tolerance curves are not available, an optimum acceptance fraction for the assembly could be based instead on more available cost-per-reject data. The optimum acceptance fraction could then be used in conjunction with allocation by proportional scaling or weight factors to provide a meaningful cost-related alternative to allocation by least cost optimization.

14.6 2-D and 3-D Tolerance Allocation

Tolerance allocation may be applied to 2-D and 3-D assemblies as readily as 1-D. The only difference is that each component tolerance must be multiplied by its tolerance sensitivity, derived from the geometry as described in Chapters 9, 11, and 12. The proportionality factors, weight factors, and cost factors are still obtained as described above, with sensitivities inserted appropriately.

14.7 2-D Example: One-way Clutch Assembly

The application of tolerance allocation to a 2-D assembly will be demonstrated on the one-way clutch assembly shown in Fig. 14-6. The clutch consists of four different parts: a hub, a ring, four rollers, and four springs. Only a quarter section is shown because of symmetry. During operation, the springs push the rollers into the wedge-shaped space between the ring and the hub. If the hub is turned counterclockwise, the rollers bind, causing the ring to turn with the hub. When the hub is turned clockwise, the rollers slip, so torque is not transmitted to the ring. A common application for the clutch is a lawn mower starter. (Reference 5)

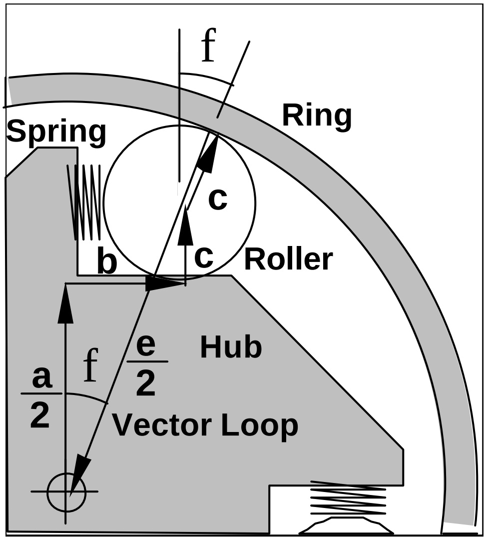


Figure 14-6 Clutch assembly with vector loop

The contact angle ϕ between the roller and the ring is critical to the performance of the clutch. Variable *b*, is the location of contact between the roller and the hub. Both the angle ϕ and length *b* are dependent assembly variables. The magnitude of ϕ and *b* will vary from one assembly to the next due to the variations of the component dimensions *a*, *c*, and *e*. Dimension *a* is the width of the hub; *c* and *e/*2 are the radii of the roller and ring, respectively. A complex assembly function determines how much each dimension contributes to the variation of angle ϕ. The nominal contact angle, when all of the independent variables are at their mean values, is 7.0 degrees. For proper performance, the angle must not vary more than ±1.0 degree from nominal. These are the engineering design limits.

The objective of variation analysis for the clutch assembly is to determine the variation of the contact angle relative to the design limits. Table 14-5 below shows the nominal value and tolerance for the three independent dimensions that contribute to tolerance stackup in the assembly. Each of the independent variables is assumed to be statistically independent (not correlated with each other) and a normally distributed random variable. The tolerances are assumed to be $\pm 3\sigma$.

Table 14-5 Independent dimensions for the clutch assembly

Dimension	Nominal	Tolerance
Hub width - *a*	2.1768 in.	.004 in.
Roller radius - *c*	.450 in.	.0004 in.
Ring diameter - *e*	4.000 in.	.0008 in.

14.7.1 Vector Loop Model and Assembly Function for the Clutch

The vector loop method (Reference 2) uses the assembly drawing as the starting point. Vectors are drawn from part-to-part in the assembly, passing through the points of contact. The vectors represent the independent and dependent dimensions that contribute to tolerance stackup in the assembly. Fig. 14-6 shows the resulting vector loop for a quarter section of the clutch assembly.

The vectors pass through the points of contact between the three parts in the assembly. Since the roller is tangent to the ring, both the roller radius *c* and the ring radius *e* are collinear. Once the vector loop is defined, the implicit equations for the assembly can easily be extracted. Eqs. (14.4) and (14.5) shows the set of scalar equations for the clutch assembly derived from the vector loop. h_x and h_y are the sum of vector components in the *x* and *y* directions. A third equation, h_q, is the sum of relative angles between consecutive vectors, but it vanishes identically.

$$h_X = 0 = b + c \sin(\phi) - e \sin(\phi) \tag{14.4}$$

$$h_y = 0 = a + c + c \cos(\phi) - e \cos(\phi) \tag{14.5}$$

Eqs. (14.4) and (14.5) may be solved for ϕ explicitly:

$$f = \cos^{-1}\left(\frac{a+c}{e-c}\right) \tag{14.6}$$

The sensitivity matrix [S] can be calculated from Eq. (14.6) by differentiation or by finite difference:

$$[S] = \begin{bmatrix} \frac{\partial f}{\partial a} & \frac{\partial f}{\partial c} & \frac{\partial f}{\partial e} \\ \frac{\partial b}{\partial a} & \frac{\partial b}{\partial c} & \frac{\partial b}{\partial e} \end{bmatrix} = \begin{bmatrix} -2.6469 & -10.5483 & 2.6272 \\ -103.43 & -440.69 & 104.21 \end{bmatrix}$$

The tolerance sensitivities for $\delta\phi$ are in the top row of [S]. Assembly variations accumulate or stackup statistically by root-sum-squares:

$$\begin{aligned} df &= \sqrt{\Sigma\left(\left(S_{ij}dx_j\right)\right)^2} \\ &= \sqrt{\left(S_{11}da\right)^2 + \left(S_{12}dc\right)^2 + \left(S_{13}de\right)^2} \\ &= \sqrt{\left((-2.6469)(.004)\right)^2 + \left((-10.5483)(.0004)\right)^2 + \left((2.6272)(.0008)\right)^2} \end{aligned}$$

= .01159 radians = .664 degrees

where df is the predicted 3σ variation, dx_j is the set of 3σ component variations.

By worst case:

$$\begin{aligned} df &= \Sigma\left|S_{ij}\right|dx_j \\ &= \left|S_{11}\right|da + \left|S_{12}\right|dc + \left|S_{13}\right|de \\ &= (2.6469)(.004) + (10.5483)(.0004) + (2.6272)(.0008) \end{aligned}$$

= .01691 radians = .9688 degrees

where df is the predicted extreme variation.

14.8 Allocation by Scaling, Weight Factors

Once you have RSS and worst case expressions for the predicted variation $\delta\phi$, you may begin applying various allocation algorithms to search for a better set of design tolerances. As we try various combinations, we must be careful not to exceed the tolerance range of the selected processes. Table 14-6 shows the selected processes for dimensions a, c, and e and the maximum and minimum tolerances obtainable by each, as extracted from the Appendix for the corresponding nominal size.

Table 14-6 Process tolerance limits for the clutch assembly

Part	Dimension	Process	Nominal (inch)	Sensitivity	Minimum Tolerance	Maximum Tolerance
Hub	a	Mill	2.1768	-2.6469	.0025	.006
Roller	c	Lap	.9000	-10.548	.00025	.00045
Ring	e	Grind	4.0000	2.62721	.0005	.0012

14.8.1 Proportional Scaling by Worst Case

Since the rollers are vendor-supplied, only tolerances on dimensions a and e may be altered. The proportionality factor P is applied to da and de, while df is set to the maximum tolerance of ±.017453 radians (±1°).

$$df = \sum \left| S_{ij} \right| dx_j$$

$$.017453 = \left| S_{11} \right| Pda + \left| S_{12} \right| dc + \left| S_{13} \right| Pde$$

$$.017453 = (2.6469)P(.004) + (10.5483)(.0004) + (2.6272)P(.0008)$$

Solving for P:

$$P = 1.0429$$

$$\delta a = (1.0429)(.004) = .00417 \text{ in.}$$

$$\delta e = (1.0429)(.0008) = .00083 \text{ in.}$$

14.8.2 Proportional Scaling by Root-Sum-Squares

$$df = \sqrt{\sum \left(\left(S_{ij} dx_j \right) \right)^2}$$

$$.017453 = \sqrt{(S_{11} Pda)^2 + (S_{12} dc)^2 + (S_{13} Pde)^2}$$

$$.017453 = \sqrt{((-2.6469)P(.004))^2 + ((-10.5483)(.0004))^2 + ((2.6272)P(.0008))^2}$$

Solving for P:

$$P = 1.56893$$

$$\delta a = (1.56893)(.004) = .00628 \text{ in.}$$

$$de = (1.56893)(.0008) = .00126 \text{ in.}$$

Both of these new tolerances exceed the process limits for their respective processes, but by less than .001in each. You could round them off to .006 and .0012. The process limits are not that precise.

14.8.3 Allocation by Weight Factors

Grinding the ring is the more costly process of the two. We would like to loosen the tolerance on dimension e. As a first try, let the weight factors be w_a = 10, w_e = 20. This will change the ratio of the two tolerances and scale them to match the 1.0 degree limit. The original tolerances had a ratio of 5:1. The final ratio will be the product of 1:2 and 5:1, or 2.5:1. The sensitivities do not affect the ratio.

$$df = \sqrt{\sum \left(\left(S_{ij} dx_j \right)^2 \right)}$$

$$.017453 = \sqrt{\left(S_{11}P(10/30)da\right)^2 + \left(S_{12}dc\right)^2 + \left(S_{13}P(20/30)de\right)^2}$$

$$.017453 = \sqrt{\left((-2.6469)P(10/30)(.004)\right)^2 + \left((-10.5483)(.0004)\right)^2 + \left((2.6272)P(20/30)(.0008)\right)^2}$$

Solving for P:

$P = 4.460$

$da = (4.460)(10/30)(.004) = .00595$ in.

$de = (4.460)(20/30)(.0008) = .00238$ in.

Evaluating the results, we see that δa is within the .006in limit, but de is well beyond the .0012 inch process limit. Since da is so close to its limit, we cannot change the weight factors much without causing da to go out of bounds. After several trials, the best design seemed to be equal weight factors, which is the same as proportional scaling. We will present a plot later that will make it clear why it turned out this way.

From the preceding examples, we see that the allocation algorithms work the same for 2-D and 3-D assemblies as for 1-D. We simply insert the tolerance sensitivities into the accumulation formulas and carry them through the calculations as constant factors.

14.9 Allocation by Cost Minimization

The minimum cost allocation applies equally well to 2-D and 3-D assemblies. If sensitivities are included in the derivation presented in Section 14.1, Eqs. (14.1) through (14.3) become:

Table 14-7 Expressions for minimum cost tolerances in 2-D and 3-D assemblies

Worst Case	RSS
$T_i = \left(\dfrac{k_i B_i S_1}{k_1 B_1 S_i}\right)^{1/(k_i+1)} T_1^{(k_1+1)/(k_i+1)}$	$T_i = \left(\dfrac{k_i B_i S^2{}_1}{k_1 B_1 S^2{}_i}\right)^{1/(k_i+2)} T_1^{(k_1+2)/(k_i+2)}$
$T_{ASM} = S_1 T_1 + \sum S_i \left(\dfrac{k_i B_i S_1}{k_1 B_1 S_i}\right)^{1/(k_i+1)} T_1^{(k_1+1)/(k_i+1)}$	$T_{ASM}^2 = S_1^2 T_1^2 + \sum S_i^2 \left(\dfrac{k_i B_i S^2{}_1}{k_1 B_1 S^2{}_i}\right)^{2/(k_i+2)} T_1^{2(k_1+2)/(k_i+2)}$

The cost data for computing process cost is shown in Table 14-8:

Table 14-8 Process tolerance cost data for the clutch assembly

Part	Dimension	Process	Nominal (inch)	Sensitivity	B	k	Minimum Tolerance	Maximum Tolerance
Hub	*a*	Mill	2.1768	-2.6469	.1018696	.45008	.0025	.006
Roller	*c*	Lap	.9000	-10.548	.000528	1.130204	.00025	.00045
Ring	*e*	Grind	4.0000	2.62721	.0149227	.79093	.0005	.0012

14.9.1 Minimum Cost Tolerances by Worst Case

To perform tolerance allocation using a Worst Case Stackup Model, let $T_1 = \delta a$, and $T_i = de$, then $S_1 = S_{11}$, $k_1 = k_a$, and $B_1 = B_a$, etc.

$$T_{ASM} = |S_{11}| da + |S_{12}| dc + |S_{13}| de$$

$$= |S_{11}| da + |S_{12}| dc + |S_{13}| \left(\frac{k_e B_e S_{11}}{k_a B_a S_{13}} \right)^{1/(k_e+1)} da^{(k_a+1)/(k_e+1)}$$

$$.017453 = \ 2.6469\, da + \ 10.5483\, (.0004)\ +\ 2.6272 \left(\frac{(.79093)(.0149227)(2.6469)}{(.45008)(0.1018696)(2.6272)} \right)^{1/(1.79093)} da^{(1.45008)/(1.79093)}$$

The only unknown is da, which may be found by iteration. de may then be found once da is known. Solving for da and de:

da =.00198 in.

$$de = \left(\frac{(.79093)(.0149227)(2.6469)}{(.45008)(0.1018696)(2.6272)} \right)^{1/(1.79093)} .00198^{(1.45008)/(1.79093)} = .00304 \text{ in.}$$

The cost corresponding to holding these tolerances would be reduced from C= \$5.42 to C= \$3.14.

Comparing these values to the process limits in Table 14-6, we see that da is below its lower process limit (.0025< δa <.006), while de is much larger than the upper process limit (.0005< de <.0012). If we decrease δe to the upper process limit, da can be increased until T_{ASM} equals the spec limit. The resulting values and cost are then:

da = .0038 in. de = .0012 in. C = \$4.30

The relationship between the resulting three pairs of tolerances is very clear when they are plotted as shown in Fig. 14-7. Tol *e* and Tol *a* are plotted as points in 2-D tolerance space. The feasible region is bounded by a box formed by the upper and lower process limits, which is cut off by the Worst Case limit curve. The original tolerances of (.004, .0008) lie within the feasible region, nearly touching the WC Limit. Extending a line through the original tolerances to the WC Limit yields the proportional scaling results found in section 14.2 (.00417, .00083), which is not much improvement over the original tolerances. The minimum cost tolerances (OptWC) were a significant change, but moved outside the feasible region. The feasible point of lowest cost (Mod WC) resulted at the intersection of the upper limit for Tol *e* and the WC Limit (.0038, .0012).

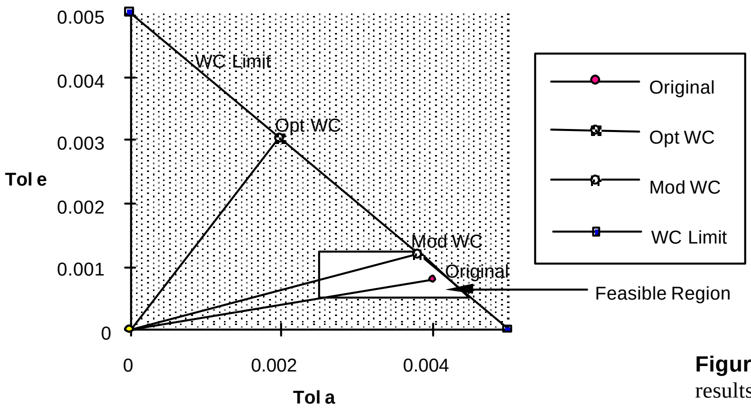


Figure 14-7 Tolerance allocation results for a Worst Case Model

This type of plot really clarifies the relationship between the three results. Unfortunately, it is limited to a 2-D graph, so it is only applicable to an assembly with two design tolerances.

14.9.2 Minimum Cost Tolerances by RSS

Repeating the minimum cost tolerance allocation using the RSS Stackup Model:

$$T_{ASM}^2 = (S_{11}da)^2 + (S_{12}dc)^2 + (S_{13}de)^2$$

$$= (S_{11}da)^2 + (S_{12}dc)^2 + (S_{13})^2 \left(\frac{k_e B_e S_{11}}{k_a B_a S_{13}}\right)^{2/(k_e+2)} da^{2(k_a+2)/(k_e+2)}$$

$$(.017453)^2 = (2.6469\ da)^2 + ((10.5483)(.0004))^2$$

$$+\ 2.6272^2 \left(\frac{(.79093)(.0149227)(2.6469)}{(.45008)(.1018696)(2.6272)}\right)^{2/(2.79093)} da^{2(2.45008)/(2.79093)}$$

Solving for *da* by iteration and *de* as before:
da = .00409 in.

$$de = \left(\frac{(.79093)(.0149227)(2.6469)}{(.45008)(.1018696)(2.6272)}\right)^{1/(2.79093)} (.00409)^{(2.45008)/(2.79093)}$$

= .00495 in.

The cost corresponding to holding these tolerances would be reduced from C= $5.42 to C= $2.20.

Comparing these values to the process limits in Table 14-6, we see that *da* is now safely within its process limits (.0025< *da* <.006), while *de* is still much larger than the upper process limit (.0005< *de* <.0012). If we again decrease *de* to the upper process limit as before, *da* can be increased until it equals the upper process limit. The resulting values and cost are then:

da = .006 in. *de* = .0012 in. C = $4.07

The plot in Fig. 14-8 shows the three pairs of tolerances. The box containing the feasible region is entirely within the RSS Limit curve. The original tolerances of (.004, .0008) lie near the center of the feasible region. Extending a line through the original tolerances to the RSS Limit yields the proportional scaling results found in section 14.2 (.00628, .00126), both of which lie just outside the feasible region. The

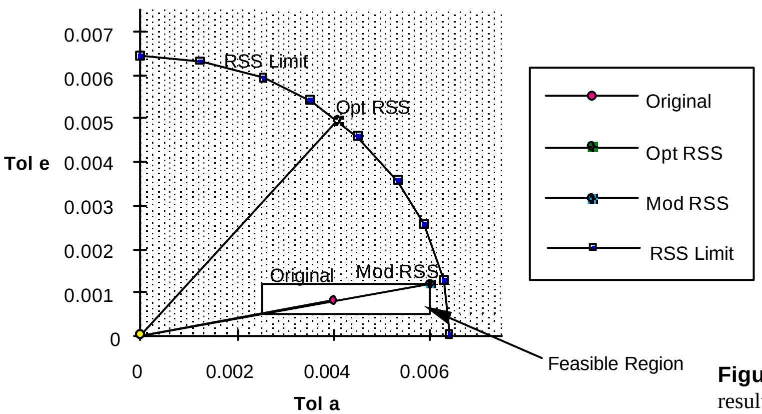


Figure 14-8 Tolerance allocation results for the RSS Model

minimum cost tolerances (OptRSS) were a significant change, but moved far outside the feasible region. The feasible point of lowest cost (ModRSS) resulted at the upper limit corner of the feasible region (.006, .0012).

Comparing Figs. 14-7 and 14-8, we see that the RSS Limit curve intersects the horizontal and vertical axes at values greater than .006 inch, while the WC Limit curve intersects near .005 inch tolerance. The intersections are found by letting Tol *a* or Tol *e* go to zero in the equation for T_{ASM} and solving for the remaining tolerance. The RSS and WC Limit curves do not converge to the same point because the fixed tolerance dc is subtracted from T_{ASM} differently for WC than RSS.

14.10 Tolerance Allocation with Process Selection

Examining Fig. 14-7 further, the feasible region appears very small. There is not much room for tolerance design. The optimization preferred to drive Tol *e* to a much larger value. One way to enlarge the feasible region is to select an alternate process for dimension *e*. Instead of grinding, suppose we consider turning. The process limits change to ($.002 < de < .008$), with $B_e = .118048$ $k_e = -.45747$. Table 14-9 shows the revised data.

Table 14-9 Revised process tolerance cost data for the clutch assembly

Part	**Dimension**	**Process**	**Nominal (inch)**	**Sensitivity**	**B**	**k**	**Minimum Tolerance**	**Maximum Tolerance**
Hub	*a*	Mill	2.1768	-2.6469	.1018696	.45008	.0025	.006
Roller	*c*	Lap	.9000	-10.548	.000528	1.130204	.00025	.00045
Ring	*e*	Turn	4.0000	2.62721	.118048	.45747	.002	.008

Milling and turning are processes with nearly the same precision. Thus, B_e and B_a are nearly equal as are k_e and k_a. The resulting RSS allocated tolerances and cost are:

da =.00434 in. de = .00474 in. C = $2.54

The new optimization results are shown in Fig. 14-9. The feasible region is clearly much larger and the minimum cost point (Mod Proc) is on the RSS Limit curve on the region boundary. The new optimum point has also changed from the previous result (Opt RSS) because of the change in B_e and k_e for the new process.

The resulting WC allocated tolerances and cost are:

da = .00240 in. de = .00262 in. C = $3.33

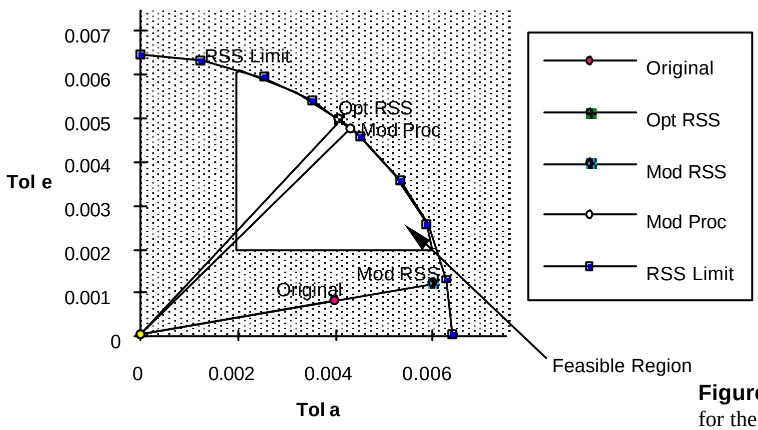


Figure 14-9 Tolerance allocation results for the modified RSS Model

The modified optimization results are shown in Fig. 14-10. The feasible region is the smallest yet due to the tight Worst Case (WC) Limit. The minimum cost point (Mod Proc) is on the WC Limit curve on the region boundary.

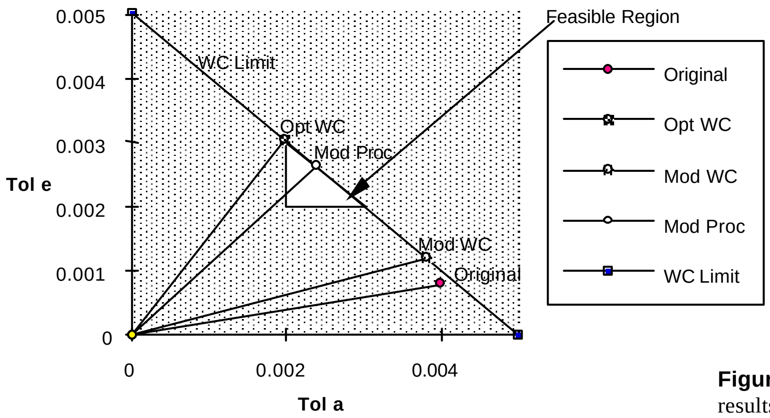


Figure 14-10 Tolerance allocation results for the modified WC Model

Cost reductions can be achieved by comparing cost functions for alternate processes. If cost-versus-tolerance data are available for a full range of processes, process selection can even be automated. A very systematic and efficient search technique, which automates this task, has been published. (Reference 4) It compares several methods for including process selection in tolerance allocation and gives a detailed description of the one found to be most efficient.

14.11 Summary

The results of WC and RSS cost allocation of tolerances are summarized in the two bar charts, Figs. 14-11 and 14-12. The changes in magnitude of the tolerances are readily apparent. Costs have been added for comparison.

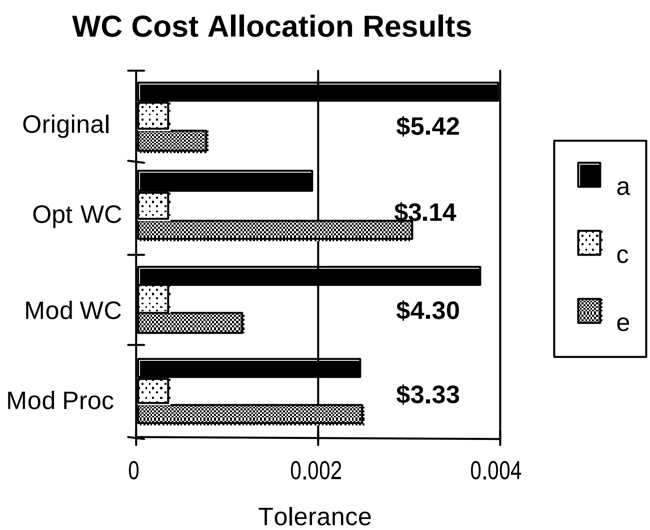


Figure 14-11 Tolerance allocation results for the WC Model

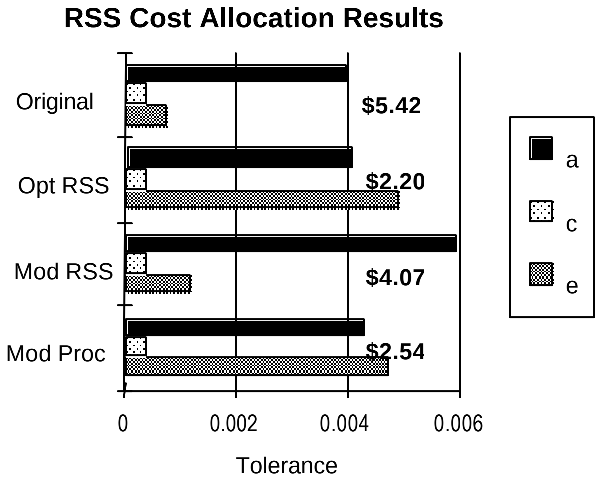


Figure 14-12 Tolerance allocation results for the RSS Model

Summarizing, the original tolerances for both WC and RSS were safely within tolerance constraints, but the costs were high. Optimization reduced the cost dramatically; however, the resulting tolerances exceeded the recommended process limits. The modified WC and RSS tolerances were adjusted to conform to the process limits, resulting in a moderate decrease in cost, about 20%. Finally, the effect of changing processes was illustrated, which resulted in a cost reduction near the first optimization. Only the allocated tolerances remained in the new feasible region.

A designer would probably not attempt all of these cases in a real design problem. He would be wise to rely on the RSS solution, possibly trying WC analysis for a case or two for comparison. Note that the clutch assembly only had three dimensions contributing to the tolerance stack. If there had been six or eight, the difference between WC and RSS would have been much more significant.

It should be noted that tolerances specified at the process limit may not be desirable. If the process is not well controlled, it may be difficult to hold it at the limit. In such cases, the designer may want to back off from the limits to allow for process uncertainties.

14.12 References

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14.13 Appendix

Cost-Tolerance Functions for Metal Removal Processes

Although it is well known that tightening tolerances increases cost, adjusting the tolerances on several components in an assembly and observing its effect on cost is an impossible task. Until you have a mathematical model, you cannot effectively optimize the allocation of tolerance in an assembly. Elegant tools for minimum cost tolerance allocation have been developed over several decades. However, they require empirical functions describing the relationship between tolerance and cost.

Cost-versus-tolerance data is very scarce. Very few companies or agencies have attempted to gather such data. Companies who do, consider it proprietary, so it is not published. The data is site and machine-specific and subject to obsolescence due to inflation. In addition, not all processes are capable of continuously adjustable precision.

Metal removal processes have the capability to tighten or loosen tolerances by changing feeds, speeds, and depth of cut or by modifying tooling fixtures, cutting tools and coolants. The workpiece may also be modified, switching to a more machinable alloy or modifying geometry to achieve greater rigidity.

A noteworthy study by the US Army in the 1940s experimentally determined the natural tolerance range for the most common metal removal processes. (Reference 13) They also compared the cost of the various processes and the relative cost of tightening tolerances. Relative costs were used to eliminate the effects of inflation. The resulting chart, Table 14A-1, appears in References 7 and 8. Least squares curve fits were performed at Brigham Young University and are presented here for the first time. The Reciprocal Power equation, $C = A + B/T^k$, presented in Chapter 14, was used as the empirical function. Fig. 14A-1 shows a typical plot of the original data and the fitted data. The curve fit procedure was a standard nonlinear method described in Reference 9, which uses weighted logarithms of the data to convert to a linear regression problem. Results are tabulated in Table 14A-2 and plotted in Figs. 14A-2 and 14A-3.

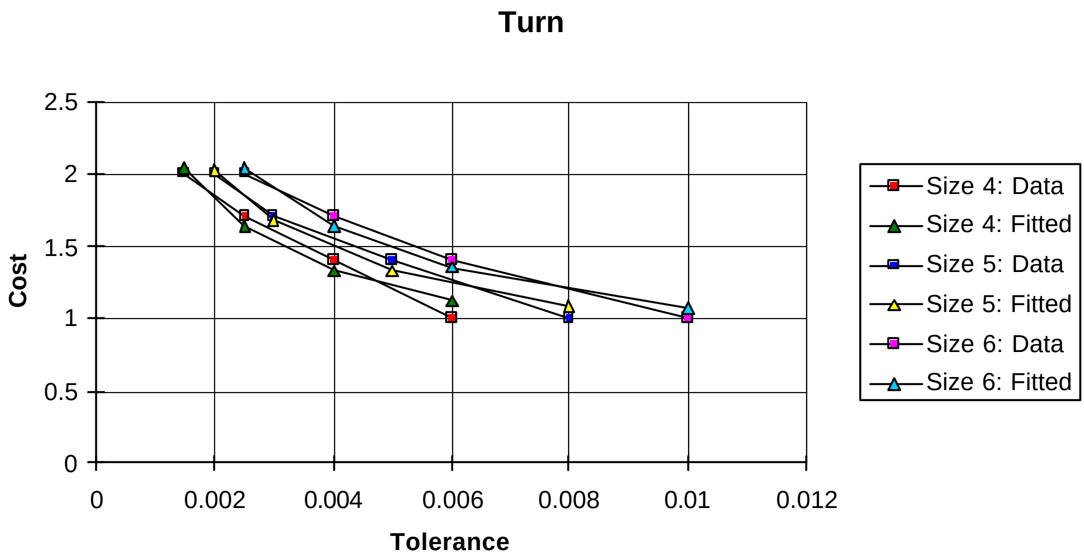


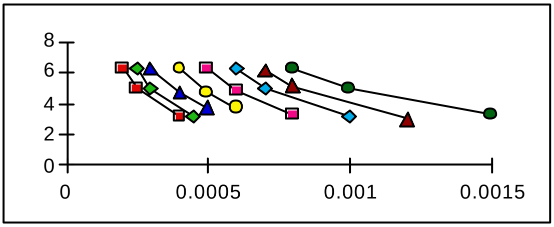
Figure 14A-1 Plot of cost-versus-tolerance for fitted and raw data for the turning process

Table 14A-1 Relative cost of obtaining various tolerance levels

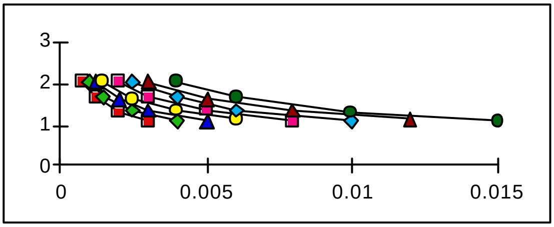
Table 14A-2 Cost-tolerance functions for metal removal processes

Size Range	A	B	k	Min Tol	Max Tol
Lap / Hone					
0.000-0.599		0.00189378	0.9508781	0.0002	0.0004
0.600-0.999		0.00052816	1.1302036	0.00025	0.00045
1.000-1.499		0.00220173	0.9808618	0.0003	0.0005
1.500-2.799		0.00033129	1.2590875	0.0004	0.0006
2.800-4.499		0.00026156	1.3269297	0.0005	0.0008
4.500-7.799		0.00038119	1.3073528	0.0006	0.001
7.800-13.599		0.00059824	1.2716314	0.0007	0.0012
13.600-20.999		0.00427422	1.0221757	0.0008	0.0015
Grind / Diamond turn					
0.000-0.599		0.02484363	0.6465727	0.0002	0.0005
0.600-0.999		0.01525616	0.7221989	0.00025	0.0006
1.000-1.499		0.0205072	0.7039047	0.0003	0.0008
1.500-2.799		0.0133561	0.7827624	0.0004	0.001
2.800-4.499		0.01492268	0.790932	0.0005	0.0012
4.500-7.799		0.02467047	0.7413291	0.0006	0.0015
7.800-13.599		0.05119944	0.6548091	0.0007	0.002
13.600-20.999		0.08317908	0.6017646	0.0008	0.0025
Broach					
0.000-0.599		0.0438552	0.548619	0.00025	0.0008
0.600-0.999		0.04670538	0.55230115	0.0003	0.001
1.000-1.499		0.04071362	0.58686634	0.0004	0.0012
1.500-2.799		0.048524	0.579761	0.0005	0.0015
2.800-4.499		0.0637591	0.559608	0.0006	0.002
4.500-7.799		0.0922923	0.521758	0.0007	0.0025
7.800-13.599		0.144046	0.46957	0.0008	0.003
13.600-20.999		0.171785	0.45907	0.001	0.004
Ream					
0.000-0.599		0.03245261	0.6000163	0.0005	0.0012
0.600-0.999		0.04682158	0.565492	0.0006	0.0015
1.000-1.499		0.04204992	0.6021191	0.0008	0.002
1.500-2.799		0.04809684	0.6021191	0.001	0.0025
2.800-4.499		0.06929088	0.565492	0.0012	0.003
4.500-7.799		0.09203907	0.5409254	0.0015	0.004
Turn / bore / shape					
0.000-0.599		0.07201641	0.46822793	0.0008	0.003
0.600-0.999		0.085969502	0.45747142	0.001	0.004
1.000-1.499		0.101233386	0.44723008	0.0012	0.005
1.500-2.799		0.11800302	0.4389869	0.0015	0.006
2.800-4.499		0.11804756	0.45747142	0.002	0.008
4.500-7.799		0.12576137	0.46536684	0.0025	0.01
7.800-13.599		0.15997103	0.4389869	0.003	0.012
13.600-20.999		0.15300611	0.46822793	0.004	0.015
Mill					
0.000-0.599		0.0862308	0.4259173	0.0012	0.003
0.600-0.999		0.10878812	0.4044547	0.0015	0.004
1.000-1.499		0.09544417	0.4431399	0.002	0.005
1.500-2.799		0.10186958	0.4500798	0.0025	0.006
2.800-4.499		0.14399071	0.4044547	0.003	0.008
4.500-7.799		0.12976209	0.4431399	0.004	0.01
7.800-13.599		0.13916564	0.4500798	0.005	0.012
13.600-20.999		0.17114563	0.4259173	0.006	0.015
Drill					
0.000-0.599		0.00301435	1.0955124	0.003	0.005
0.600-0.999		0.00085791	1.3801824	0.004	0.006
1.000-1.499		0.00318631	1.1906627	0.005	0.008
1.500-2.799		0.00644133	1.0955124	0.006	0.01
2.800-4.499		0.00223316	1.3801824	0.008	0.012

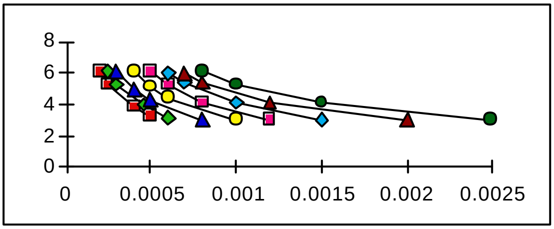
Lap / Hone



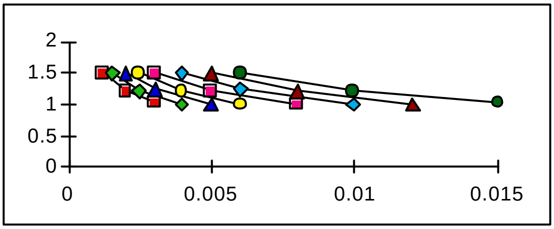
Turn / bore / shape



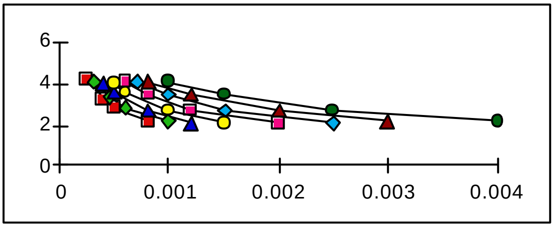
Grind / Diamond turn



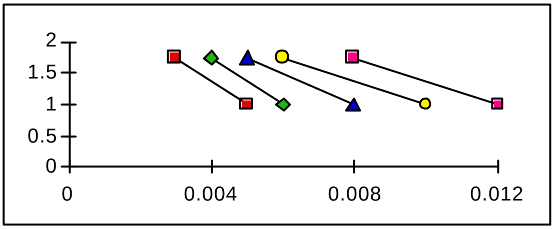
Mill



Broach



Drill



Ream

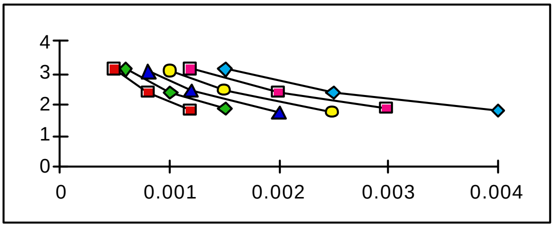


Figure 14A-2 Plot of fitted cost versus tolerance functions

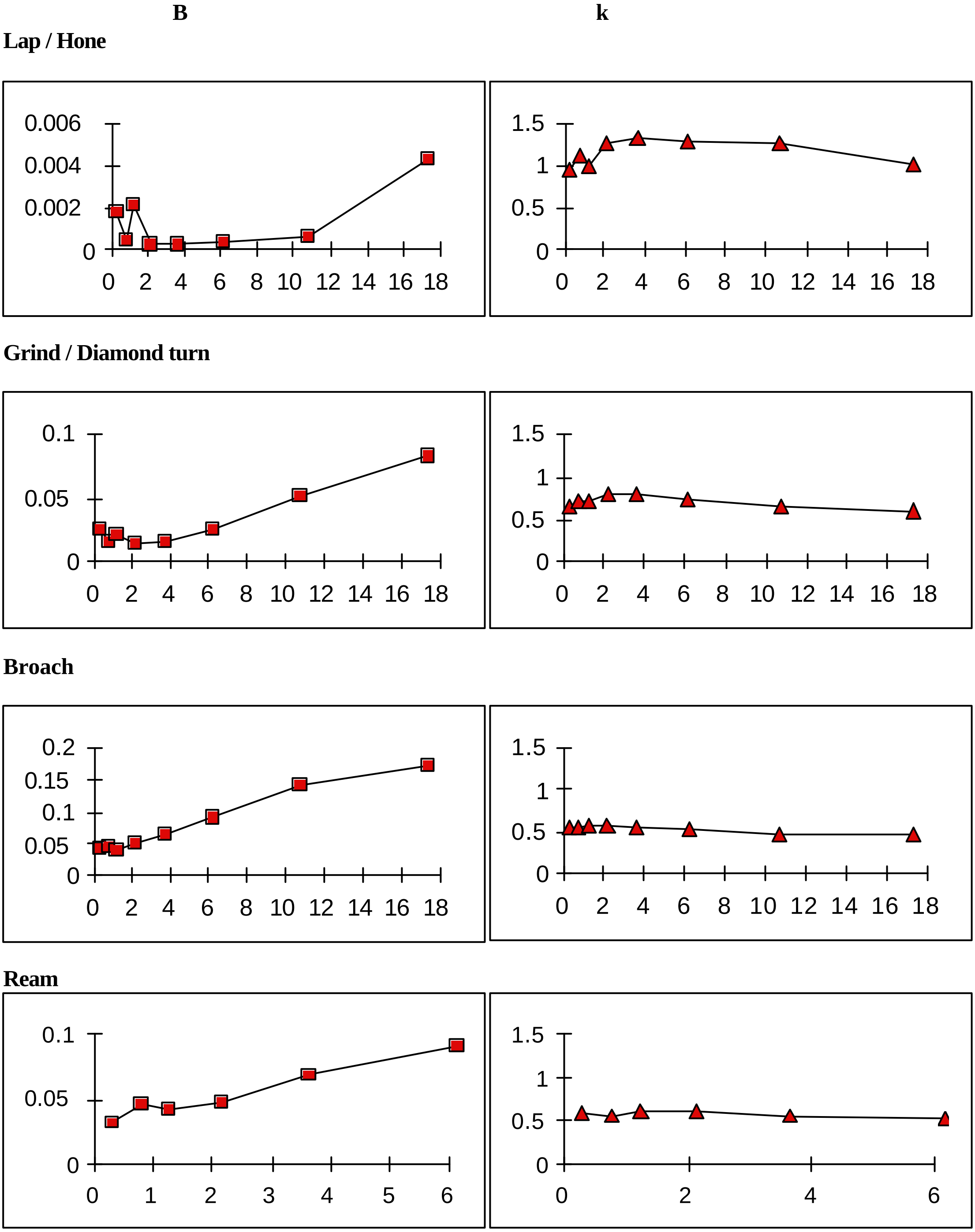
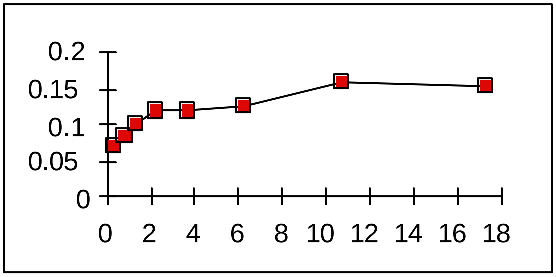
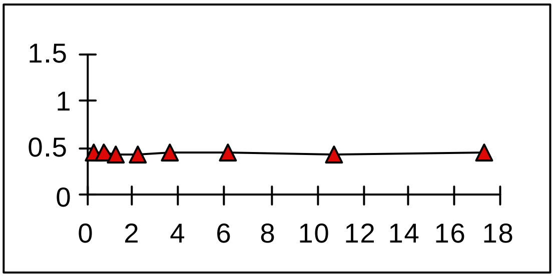


Figure 14A-3 Plot of coefficients versus size for cost-tolerance functions

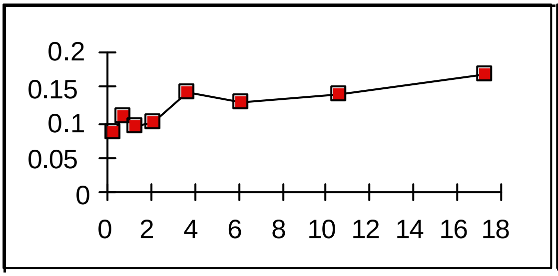
B **k**

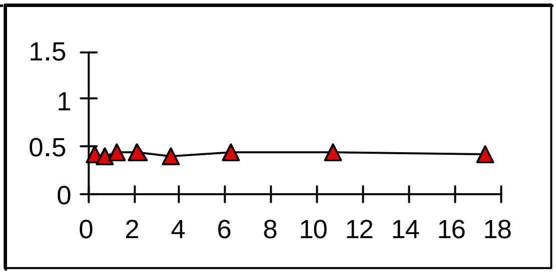
Turn / bore / shape



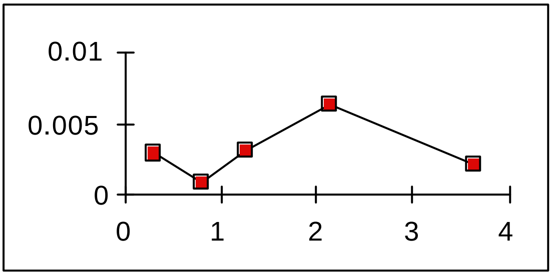


Mill





Drill



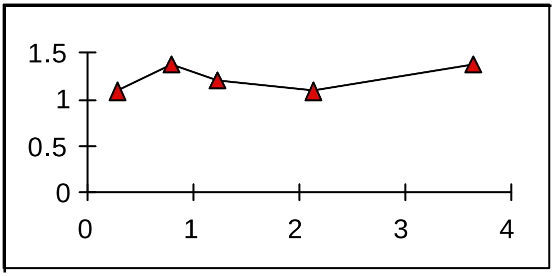


Figure 14A-3 continued Plot of coefficients versus size for cost-tolerance functions