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DESIGN

Chapter

9

Traditional Approaches to Analyzing Mechanical Tolerance Stacks

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9.1 Introduction

Tolerance analysis is the process of taking known tolerances and analyzing the combination of these tolerances at an assembly level. This chapter will define the process for analyzing tolerance stacks. It will show how to set up a loop diagram to determine a nominal performance/assembly value and four techniques to calculate variation from nominal.

The most important goal of this chapter is for the reader to understand the *assumptions* and *risks* that go along with each tolerance analysis method.

9.2 Analyzing Tolerance Stacks

Fig. 9-1 describes the tolerance analysis process.

9.2.1 Establishing Performance/Assembly Requirements

The first step in the process is to identify the requirements for the system. These are usually requirements that determine the “performance” and/or “assembly” of the system. The system requirements will, either directly, or indirectly, flow down requirements to the mechanical subassemblies. These requirements usually determine what needs to be analyzed. In general, a requirement that applies for most mechanical subassemblies is that parts must fit together. Fig. 9-2 shows a cross section of a motor assembly. In this example, there are several requirements.

- Requirement 1. The gap between the shaft and the inner bearing cap must always be greater than zero to ensure that the rotor is clamped and the bearings are preloaded.
- Requirement 2. The gap between the housing cap and the housing must always be greater than zero to ensure that the stator is clamped.

1. Establish the Performance Requirements

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2. Draw a Loop Diagram

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3. Convert All Dimensions to Mean Dimension with an Equal Bilateral Tolerance

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4. Calculate the Mean Value for the Performance Requirement

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5. Determine the Method of Analysis

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6. Calculate the Variation for the Performance Requirement

Figure 9-1 Tolerance analysis process

- Requirement 3. The mounting surfaces of the rotor and stator must be within $\pm$.005 for the motor to operate.
- Requirement 4. The bearing outer race must always protrude beyond the main housing, so that the bearing stays clamped.
- Requirement 5. The thread of the bearing cap screw must have a minimum thread engagement of .200 inches.
- Requirement 6. The bottom of the bearing cap screw thread must never touch the bottom of the female thread on the shaft.
- Requirement 7. The rotor and stator must never touch. The maximum radial distance between the rotor and stator is .020.

Other examples of performance/assembly requirements are:

- Thermal requirements, such as contact between a thermal plane and a heat sink,
- Amount of “squeeze” on an o-ring
- Amount of “preload” on bearings
- Sufficient “material” for subsequent machining processes
- Aerodynamic requirements
- Interference requirements, such as when pressing pins into holes
- Structural requirements
- Optical requirements, such as alignment of optical elements

The second part of Step 1 is to convert each requirement into an assembly gap requirement. We would convert each of the previous requirements to the following.

- Requirement 1. Gap 1 $\geq$ 0
- Requirement 2. Gap 2 $\geq$ 0

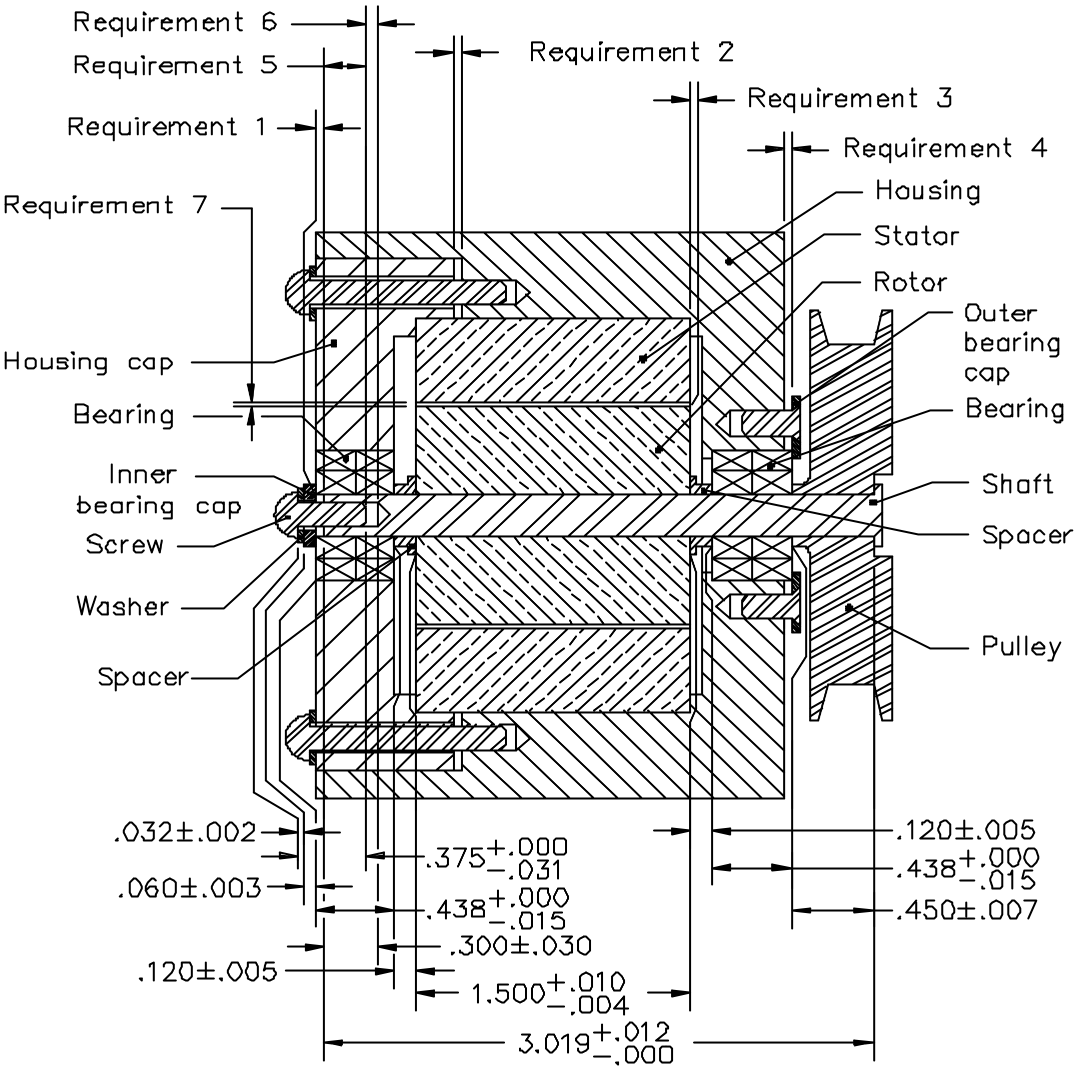

Figure 9-2 Motor assembly

- Requirement 3. Gap 3 ≥ .005
- Requirement 4. Gap 4 ≥ 0
- Requirement 5. Gap 5 ≥ .200
- Requirement 6. Gap 6 ≥ 0
- Requirement 7. Gap 7 ≥ 0 and ≤ .020

9.2.2 Loop Diagram

The loop diagram is a graphical representation of each analysis. Each requirement requires a separate loop diagram. Simple loop diagrams are usually horizontal or vertical. For simple analyses, vertical loop diagrams will graphically represent the dimensional contributors for vertical "gaps." Likewise, horizontal loop diagrams graphically represent dimensional contributors for horizontal “gaps.” The steps for drawing the loop diagram follow.

1. For horizontal dimension loops, start at the surface on the left of the gap. Follow a complete dimension loop, to the surface on the right. For vertical dimension loops, start at the surface on the bottom of the gap. Follow a complete dimension loop, to the surface on the top.
2. Using vectors, create a “closed” loop diagram from the starting surface to the ending surface. Do not include gaps when selecting the path for the dimension loop. Each vector in the loop diagram represents a dimension.
3. Use an arrow to show the direction of each “vector” in the dimension loop. Identify each vector as positive (+), or negative (–), using the following convention.

   For horizontal dimensions:

   Use a + sign for dimensions followed from left to right.

   Use a – sign for dimensions followed from right to left.

   For vertical dimensions:

   Use a + sign for dimensions followed from bottom to top.

   Use a – sign for dimensions followed from top to bottom.
4. Assign a variable name to each dimension in the loop. (For example, the first dimension is assigned the variable name A, the second, B.)

Fig. 9-3 shows a horizontal loop diagram for Requirement 6.

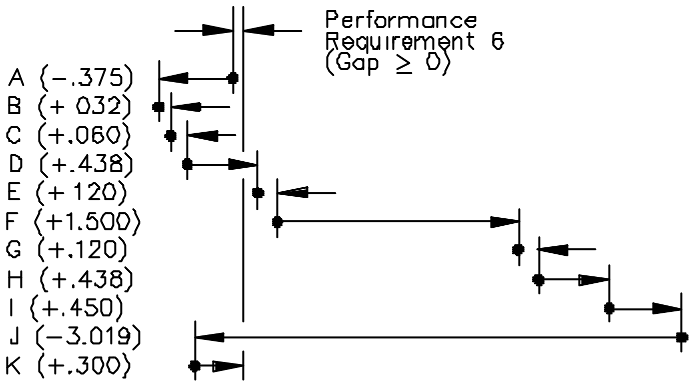

Figure 9-3 Horizontal loop diagram for Requirement 6

5. Record sensitivities for each dimension. The magnitude of the sensitivity is the value that the gap changes, when the dimension changes 1 unit. For example, if the gap changes .001 when the dimension changes .001, then the magnitude of the sensitivity is 1 (.001/.001). On the other hand, if the gap changes .0005 for a .001 change in the dimension, then the sensitivity is .5 (.0005/.001).

   If the dimension vector is positive (pointing to the right for horizontal loops, or up for vertical loops), enter a positive sensitivity. If a dimension with a positive sensitivity increases, the gap will also increase.

   If the vector is negative (pointing to the left for horizontal loops, or down for vertical loops), enter a negative sensitivity. If a dimension with a negative sensitivity increases, the gap will decrease. Note, in Fig. 9-3, all of the sensitivities are equal to ±1.

6. Determine whether each dimension is “fixed” or “variable.” A fixed dimension is one in which we have no control, such as a vendor part dimension. A variable dimension is one that we can change to influence the outcome of the tolerance stack. (This will become important later, because we will be able to “adjust” or “resize” the variable dimensions and tolerances to achieve a desired assembly performance. We are not able to resize fixed dimensions or tolerances.)

9.2.3 Converting Dimensions to Equal Bilateral Tolerances

In Fig. 9-2, there were several dimensions that were toleranced using unilateral tolerances (such as .375 +.000/-.031, 3.019 +.012/-.000 and .438 +.000/-.015) or unequal bilateral tolerances (such as +1.500 +.010/-.004). If we look at the length of the shaft, we see that there are several different ways we could have applied the tolerances. Fig. 9-4 shows several ways we can dimension and tolerance the length of the shaft to achieve the same upper and lower tolerance limits (3.031/3.019). From a design perspective, all of these methods perform the same function. They give a boundary within which the dimension is acceptable.

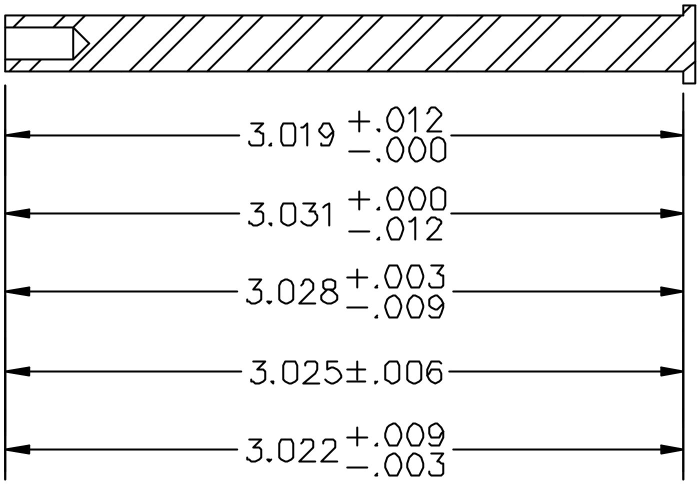

Figure 9-4 Methods to dimension the length of a shaft

The designer might think that changing the nominal dimension has an effect on the assembly. For example, a designer may dimension the part length as 3.019 +.012/-.000. In doing so, the designer may falsely think that this will help minimize the gap for Requirement 1. A drawing, however, doesn’t give preference to any dimension within the tolerance range.

Fig. 9-5 shows what happens to the manufacturing yield if the manufacturer “aims” for the dimension stated on the drawing and the process follows the normal distribution. In this example, if the manufacturer aimed for 3.019, half of the parts would be outside of the tolerance zone. Since manufacturing shops want to maximize the yield of each dimension, they will aim for the nominal that yields the largest number of good parts. This helps them minimize their costs. In this example, the manufacturer would aim for 3.025. This allows them the highest probability of making good parts. If they aimed for 3.019 or 3.031, half of the manufactured parts would be outside the tolerance limits.

As in the previous example, many manufacturing processes are normally distributed. Therefore, if we put any unilateral, or unequal bilateral tolerances on dimensions, the manufacturer would convert them to a mean dimension with an equal bilateral tolerance. The steps for converting to an equal bilateral tolerance follow.

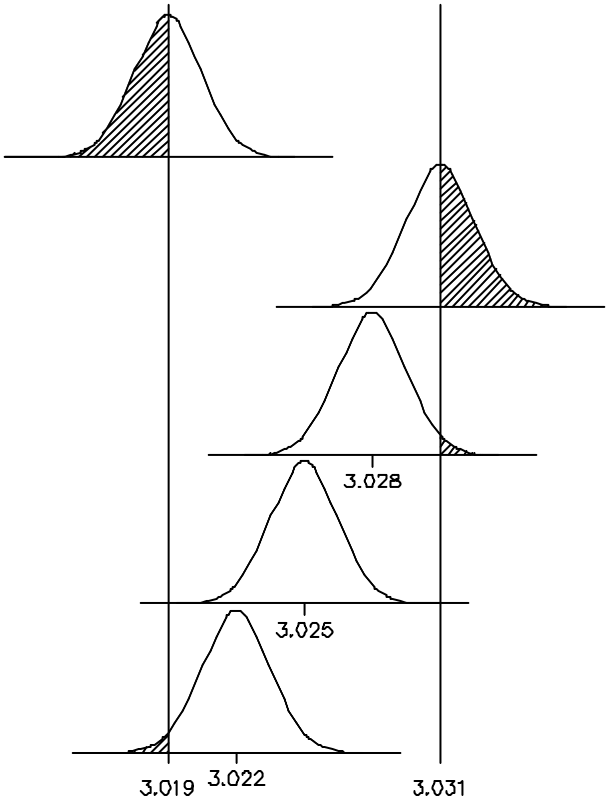

Figure 9-5 Methods of centering manufacturing processes

1. Convert the dimension with tolerances to an upper limit and a lower limit. (For example, 3.028 +.003/-.009 has an upper limit of 3.031 and a lower limit of 3.019.)
2. Subtract the lower limit from the upper limit to get the total tolerance band. (3.031-3.019=.012)
3. Divide the tolerance band by two to get an equal bilateral tolerance. (.012/2=.006)
4. Add the equal bilateral tolerance to the lower limit to get the mean dimension. (3.019 +.006=3.025). Alternately, you could subtract the equal bilateral tolerance from the upper limit. (3.031-.006=3.025)

As a rule, designers should use equal bilateral tolerances. Sometimes, using equal bilateral tolerances may force manufacturing to use nonstandard tools. In these cases, we should not use equal bilateral tolerances. For example, we would not want to convert a drilled hole diameter from ∅.125 +.005/-.001 to ∅.127 ±.003. In this case, we want the manufacturer to use a standard ∅.125 drill. If the manufacturer sees ∅.127 on a drawing, he may think he needs to build a special tool. In the case of drilled holes, we would also want to use an unequal bilateral tolerance because the mean of the drilling process is usually larger than the standard drill size. These dimensions should have a larger plus tolerance than minus tolerance.

As we will see later, when we convert dimensions to equal bilateral tolerances, we don't need to keep track of which tolerances are "positive" and which tolerances are "negative" because the positive tolerances are equal to the negative tolerances. This makes the analysis easier. Table 9-1 converts the necessary dimensions and tolerances to mean dimensions with equal bilateral tolerances.

Table 9-1 Converting to mean dimensions with equal bilateral tolerances

Original Dimension/Tolerance	Mean Dimension with Equal Bilateral Tolerance
.375 +.000/-.031	.3595 +/- .0155
.438 +.000/-.015	.4305 +/- .0075
1.500 +.010/-.004	1.503 +/- .007
3.019 +.012/-.000	3.025 +/- .006

9.2.4 Calculating the Mean Value (Gap) for the Requirement

The first step in calculating the variation at the gap is to calculate the mean value of the requirement. The mean value at the gap is:

$$d_g = \sum_{i=1}^{n} a_i d_i \tag{9.1}$$

where

d_g = the mean value at the gap. If d_g is positive, the mean "gap" has clearance, and if d_g is negative, the mean "gap" has interference.

n = the number of independent variables (dimensions) in the stackup

a_i = sensitivity factor that defines the direction and magnitude for the ith dimension. In a one-dimensional stackup, this value is usually +1 or –1. Sometimes, in a one-dimensional stackup, this value may be +.5 or -.5 if a radius is the contributing factor for a diameter callout on a drawing.

d_i = the mean value of the ith dimension in the loop diagram.

Table 9-2 shows the dimensions that are important to determine the mean gap for Requirement 6. We have assigned Variable Name to each dimension so that we can write a loop equation. We have also added

Table 9-2 Dimensions and tolerances used in Requirement 6

Description	Variable Name	Mean Dimension	Sensitivity	Fixed/ Variable	+/- Equal Bilateral Tolerance
Screw thread length	A	.3595	-1	Fixed	.0155
Washer length	B	.0320	1	Fixed	.0020
Inner bearing cap turned length	C	.0600	1	Variable	.0030
Bearing length	D	.4305	1	Fixed	.0075
Spacer turned length	E	.1200	1	Variable	.0050
Rotor length	F	1.5030	1	Fixed	.0070
Spacer turned length	G	.1200	1	Variable	.0050
Bearing length	H	.4305	1	Fixed	.0075
Pulley casting length	I	.4500	1	Variable	.0070
Shaft turned length	J	3.0250	-1	Variable	.0060
Tapped hole depth	K	.3000	1	Variable	.0300

a column titled Fixed/Variable. This identifies which dimensions and tolerances are "fixed" in the analysis, and which ones are allowed to vary (variable). Typically, we have no control over vendor items, so we treat these dimensions as fixed. As we make adjustments to dimensions and tolerances, we will only change the "variable" dimensions and tolerances.

The mean for Gap 6 is:

$$\text{Gap 6} = a_1d_1 + a_2d_2 + a_3d_3 + a_4d_4 + a_5d_5 + a_6d_6 + a_7d_7 + a_8d_8 + a_9d_9 + a_{10}d_{10} + a_{11}d_{11}$$

$$\text{Gap 6} = (-1)A + (1)B + (1)C + (1)D + (1)E + (1)F + (1)G + (1)H + (1)I + (-1)J + (1)K$$

$$\text{Gap 6} = (-1).3595+(1).0320+(1).0600+(1).4305+(1).1200+(1)1.5030+(1).1200+ (1).4305+(1).4500+(-1)3.0250+(1).0300$$

$$\text{Gap 6} = .0615$$

9.2.5 Determine the Method of Analysis

Eq. (9.1) only calculates the nominal value for the gap. The next step is to analyze the variation at the gap. Historically, mechanical engineers have used two types of tolerancing models to analyze these variations: 1) a "worst case" (WC) model, and 2) a "statistical" model. Each approach offers tradeoffs between piecepart tolerances and assembly "quality." In Chapters 11 and 14, we will see that there are other methods based on the optimization of piecepart and assembly quality and the optimization of total cost.

Fig. 9-6 shows how the assumptions about the pieceparts affect the requirements (gaps), using the worst case and statistical methods. In this figure, the horizontal axis represents the manufactured dimension. The vertical axis represents the number of parts that are manufactured at a particular dimension on the horizontal axis.

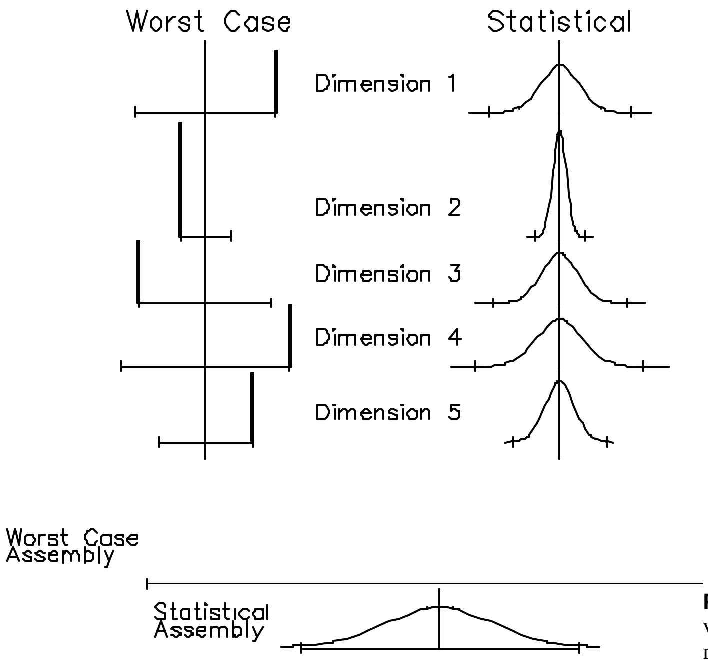

Figure 9-6 Combining piecepart variations using worst case and statistical methods

In the Worst Case Model, we verify that the parts will perform their intended function 100 percent of the time. This is oftentimes a conservative approach. In the statistical modeling approach, we assume that most of the manufactured parts are centered on the mean dimension. This is usually less conservative than a worst case approach, but it offers several benefits which we will discuss later. There are two traditional statistical methods; the Root Sum of the Squares (RSS) Model, and the Modified Root Sum of the Squares (MRSS) Model.

9.2.6 Calculating the Variation for the Requirement

During the design process, the design engineer makes tradeoffs using one of the three classic models. Typically, the designer analyzes the requirements using worst case tolerances. If the worst case tolerances met the required assembly performance, the designer would stop there. On the other hand, if this model did not meet the requirements, the designer increased the piecepart tolerances (to make the parts more manufacturable) at the risk of nonconformance at the assembly level. The designer would make trades, using the RSS and MRSS models.

The following sections discuss the traditional Worst Case, RSS, and MRSS models. Additionally, we discuss the Estimated Mean Shift Model that includes Worst Case and RSS models as extreme cases.

9.2.6.1 Worst Case Tolerancing Model

The Worst Case Model, sometimes referred to as the “Method of Extremes,” is the simplest and most conservative of the traditional approaches. In this approach, the tolerance at the interface is simply the sum of the individual tolerances.

The following equation calculates the expected variation at the gap.

$$t_{wc} = \sum_{i=1}^{n} \left| a_i t_i \right| \tag{9.2}$$

where

t_{wc} = maximum expected variation (equal bilateral) using the Worst Case Model.
t_i = equal bilateral tolerance of the i^{th} component in the stackup.

The variation at the gap for Requirement 6 is:

$$t_{wc} = |(-1).0155|+|(1).0030|+|(1).0050|+|(1).0075|+|(1).0050|+|(1).0070|+|(1).0050| +|(1).0075|+|(1).0070|+|(-1).0060|+|(1).0300|$$

$$t_{wc} = .0955$$

Using the Worst Case Model, the minimum gap is equal to the mean value minus the “worst case” variation at the gap. The maximum gap is equal to the mean value plus the “worst case” variation at the gap.

Minimum gap = $d_g - t_{wc}$
Maximum gap = $d_g + t_{wc}$

The maximum and minimum assembly gaps for Requirement 6 are:

Minimum Gap 6 = $d_g - t_{wc}$ = .0615 - .0955 = -.0340
Maximum Gap 6 = $d_g + t_{wc}$ = .0615 + .0955 = .1570

The requirement for Gap 6 is that the minimum gap must be greater than 0. Therefore, we must increase the minimum gap by .0340 to meet the minimum gap requirement. One way to increase the minimum gap is to modify the dimensions (d_i's) to increase the nominal gap. Doing this will also increase the maximum gap of the assembly by .0340. Sometimes, we can't do this because the maximum requirement may not allow it, or other requirements (such as Requirement 5) won't allow it. Another option is to reduce the tolerance values (t_i's) in the stackup.

Resizing Tolerances in the Worst Case Model

There are two ways to reduce the tolerances in the stackup.

1. The designer could randomly change the tolerances and analyze the new numbers, or
2. If the original numbers were "weighted" the same, then all variable tolerances (those under the control of the designer) could be multiplied by a "resize" factor to yield the minimum assembly gap. This is the correct approach if the designer assigned original tolerances that were equally producible.

Resizing is a method of allocating tolerances. (See Chapters 11 and 14 for further discussion on tolerance allocation.) In allocation, we start with a desired assembly performance and determine the piecepart tolerances that will meet this requirement. The resize factor, F_{wc} , scales the original worst case tolerances up or down to achieve the desired assembly performance. Since the designer has no control over tolerances on purchased parts (fixed tolerances), the scaling factor only applies to variable tolerances. Eq. (9.2) becomes:

$$t_{wc} = \sum_{j=1}^{p} \left| a_j t_{jf} \right| + \sum_{k=1}^{q} \left| a_k t_{kf} \right|$$

where,

a_j = sensitivity factor for the j^{th}, fixed component in the stackup
a_k = sensitivity factor for the k^{th}, variable component in the stackup
t_{jf} = equal bilateral tolerance of the j^{th}, fixed component in the stackup
t_{kv} = equal bilateral tolerance of the k^{th}, variable component in the stackup
p = number of independent, fixed dimensions in the stackup
q = number of independent, variable dimensions in the stackup

The resize factor for the Worst Case Model is:

$$F_{wc} = \frac{d_g - g_m - \sum_{j=1}^{p} \left| a_j t_{jf} \right|}{\sum_{k=1}^{q} \left| a_k t_{kv} \right|}$$

where

g_m = minimum value at the (assembly) gap. This value is zero if nointerference or clearance is allowed.

The new variable tolerances $(t_{kv,wc,\ resized})$ are the old tolerances multiplied by the factor F_{wc}.

$$t_{kv,wc,resized} = F_{wc}\ t_{kv}$$

$t_{kv,wc,resized}$ = equal bilateral tolerance of the k^{th}, variable component in the stackup after resizing using the Worst Case Model.

Fig. 9-7 shows the relationship between the piecepart tolerances and the assembly tolerance before and after resizing.

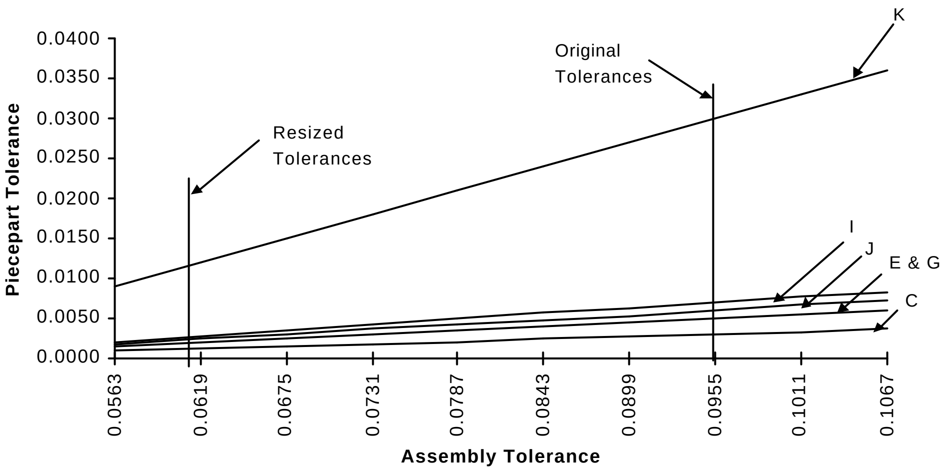

Figure 9-7 Graph of piecepart tolerances versus assembly tolerance before and after resizing using the Worst Case Model

The resize factor for Requirement 6 equals .3929. (For example, .0030 is resized to .3929*.0030 = .0012.) Table 9-3 shows the new (resized) tolerances that would give a minimum gap of zero.

Table 9-3 Resized tolerances using the Worst Case Model

Variable Name	Mean Dimension	Fixed/ Variable	+/- Equal Bilateral Tolerance	Resized +/- Equal Bilateral Tolerance ($t_{iv,wc,resized}$)
A	.3595	Fixed	.0155	
B	.0320	Fixed	.0020	
C	.0600	Variable	.0030	.0012
D	.4305	Fixed	.0075	
E	.1200	Variable	.0050	.0020
F	1.5030	Fixed	.0070	
G	.1200	Variable	.0050	.0020
H	.4305	Fixed	.0075	
I	.4500	Variable	.0070	.0027
J	3.0250	Variable	.0060	.0024
K	.3000	Variable	.0300	.0118

As a check, we can show that the new maximum expected assembly gap for Requirement 6, using the resized tolerances, is:

$$t_{wc,resized} = .0155+.0020+.0012+.0075+.0020+.0070+.0020+.0075+.0027+.0024+.0118$$

$$t_{wc,resized} = .0616$$

The variation at the gap is:

Minimum Gap 6 = $d_g - t_{wc,resized}$ = .0615 - .0616 = -.0001

Maximum Gap 6 = $d_g + t_{wc,resized}$ = .0615 + .0616 = .1231

Assumptions and Risks of Using the Worst Case Model

In the worst case approach, the designer does not make any assumptions about how the individual piecepart dimensions are distributed within the tolerance ranges. The only assumption is that all pieceparts are within the tolerance limits. While this may not always be true, the method is so conservative that parts will probably still fit. This is the method's major advantage.

The major disadvantage of the Worst Case Model is when there are a large number of components or a small "gap" (as in the previous example). In such applications, the Worst Case Model yields small tolerances, which will be costly.

9.2.6.2 RSS Model

If designers cannot achieve producible piecepart tolerances for a given requirement, they can take advantage of probability theory to increase them. This theory is known as the Root Sum of the Squares (RSS) Model.

The RSS Model is based on the premise that it is more likely for parts to be manufactured near the center of the tolerance range than at the ends. Experience in manufacturing indicates that small errors are usually more numerous than large errors. The deviations are bunched around the mean of the dimension and are fewer at points farther from the mean dimension. The number of manufactured pieces with large deviations from the mean, positive or negative, may approach zero as the deviations from the mean increase.

The RSS Model assumes that the manufactured dimensions fit a statistical distribution called a *normal curve*. This model also assumes that it is unlikely that parts in an assembly will be randomly chosen in such a way that the worst case conditions analyzed earlier will occur.

Derivation of the RSS Equation*

We'll derive the RSS equation based on statistical principles of combinations of standard deviations. To make our derivation as generic as possible, let's start with a function of independent variables such as $y=f(x_1,x_2,\ldots,x_n)$. From this function, we need to be able to calculate the standard deviation of y, or σ_y. But how do we find σ_y if all we have is information about the components x_i? Let's start with the definition of σ_y.

$$s_y^2 = \frac{\sum_{i=1}^{r}\left(y_i - m_y\right)^2}{r}$$

*Derived by Dale Van Wyk and reprinted by permission of Raytheon Systems Company where,

m_y = the mean of the random variable y

r = the total number of measurements in the population of interest

Let $\Delta_y = y_i - m_y$

If Δ_y is small, which is usually the case, $\Delta_y \approx dy = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x2} dx_2 + ... + \frac{\partial f}{\partial x_n} dx_n$ (9.3)

Therefore, $s_y^2 = \frac{\sum_{i=1}^{r} dy_i^{\,2}}{r}$ (9.4)

From Eq. (9.3),

$$dy^2 = \left(\frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + ... + \frac{\partial f}{\partial x_n} dx_n \right)^2$$

$$= \left(\frac{\partial f}{\partial x_1} \right)^2 (dx_1)^2 + \left(\frac{\partial f}{\partial x_2} \right)^2 (dx_2)^2 + ... + \left(\frac{\partial f}{\partial x_n} \right)^2 (dx_n)^2$$

$$+ \sum_{j=1}^{n} \sum_{k=1}^{n} \left[\left(\frac{\partial f}{\partial x_j} \right) \left(\frac{\partial f}{\partial x_k} \right) (dx_j)(dx_k) \right]_{j \neq k}$$

If all the variables x_i are independent, $\sum_{j=1}^{n} \sum_{k=1}^{n} \left[\left(\frac{\partial f}{\partial x_j} \right) \left(\frac{\partial f}{\partial x_k} \right) (dx_j)(dx_k) \right]_{j \neq k} = 0$

The same would hold true for all similar terms. As a result,

$$\sum_{i=1}^{r} (dy_i)^2 = \sum_{i=1}^{r} \left[\left(\frac{\partial f}{\partial x_1} \right)^2 (dx_1)^2 + \left(\frac{\partial f}{\partial x_2} \right)^2 (dx_2)^2 + ... + \left(\frac{\partial f}{\partial x_n} \right)^2 (dx_n)^2 \right]_i$$

Each partial derivative is evaluated at its mean value, which is chosen as the nominal. Thus,

$$\frac{\partial f}{\partial x_i} = C_i$$

where C_i is a constant for each x_i,

$$\sum_{i=1}^{r} (dy_i)^2 = \left(\frac{\partial f}{\partial x_1} \right)^2 \sum_{i=1}^{r} (dx_1)_i^2 + \left(\frac{\partial f}{\partial x_2} \right)^2 \sum_{i=1}^{r} (dx_2)_i^2 + ... + \left(\frac{\partial f}{\partial x_n} \right)^2 \sum_{i=1}^{r} (dx_n)_i^2 \qquad (9.5)$$

Using the results of Eq. (9.5) and inserting into Eq. (9.4)

$$s_y^2 = \frac{\left(\frac{\partial f}{\partial x_1}\right)^2 \sum_{i=1}^{r}(dx_1)_i^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 \sum_{i=1}^{r}(dx_2)_i^2 + \ldots + \left(\frac{\partial f}{\partial x_n}\right)^2 \sum_{i=1}^{r}(dx_n)_i^2}{r}$$

$$s_y^2 = \left(\frac{\partial f}{\partial x_1}\right)^2 \frac{\sum_{i=1}^{r}(dx_1)_i^2}{r} + \left(\frac{\partial f}{\partial x_2}\right)^2 \frac{\sum_{i=1}^{r}(dx_2)_i^2}{r} + \ldots + \left(\frac{\partial f}{\partial x_n}\right)^2 \frac{\sum_{i=1}^{r}(dx_n)_i^2}{r} \qquad (9.6)$$

$$s_y^2 = \left(\frac{\partial f}{\partial x_1}\right)^2 s_{x_1}^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 s_{x_2}^2 + \ldots + \left(\frac{\partial f}{\partial x_n}\right)^2 s_{x_n}^2$$

Now, let's apply this statistical principle to tolerance analysis. We'll consider each of the variables x_i to be a dimension, D_i, with a tolerance, T_i. If the nominal dimension, D_i, is the same as the mean of a normal distribution, we can use the definition of a standard normal variable, Z_i, as follows. (See Chapters 10 and 11 for further discussions on Z.)

$$Z_i = \frac{USL_i - D_i}{s_i} = \frac{T_i}{s_i}$$

$$s_i = \frac{T_i}{Z_i} \qquad (9.7)$$

If the pieceparts are randomly selected, this relationship applies for the function y as well as for each T_i.

For one-dimensional tolerance stacks, $y = \sum_{i=1}^{n} a_i D_i$ where each a_i represents the sensitivity.

In this case, $\frac{\partial y}{\partial x_i} = a_i$ and Eq. (9.6) becomes

$$s_y^2 = a_1^2 s_{x_1}^2 + a_2^2 s_{x_2}^2 + \ldots + a_n^2 s_{x_n}^2 \qquad (9.8)$$

When you combine Eq. (9.7) and Eq. (9.8), $\left(\frac{T_y}{Z_y}\right)^2 = \left(\frac{a_1 T_1}{Z_1}\right)^2 + \left(\frac{a_2 T_2}{Z_2}\right)^2 + \ldots + \left(\frac{a_n T_n}{Z_n}\right)^2$ (9.9)

If all of the dimensions are equally producible, for example if all are exactly 3σ tolerances, or all are 6σ tolerances, $Z_y=Z_1=Z_2=\ldots=Z_n$. In addition, let $a_1=a_2=\ldots=a_n=+/-1$.

Eq. (9.9) will then reduce to $T_y^2 = T_1^2 + T_2^2 + \ldots + T_n^2$

or $T_y = \sqrt{T_1^2 + T_2^2 + \ldots + T_n^2}$ (9.10)

which is the classical RSS equation.

Let's review the assumptions that went into the derivation of this equation.

- All the dimensions D_i are statistically independent.
- The mean value of D_i is large compared to s_i. The recommendation is that D_i/σ_i should be greater than five.
- The nominal value is truly the mean of D_i.
- The distributions of the dimensions are Gaussian, or normal.
- The pieceparts are randomly assembled.
- Each of the dimensions is equally producible.
- Each of the sensitivities has a magnitude of 1.
- Z_i equations assume equal bilateral tolerances.

The validity of each of these assumptions will impact how well the RSS prediction matches the reality of production.

Note that while Eq. (9.10) is the classical RSS equation, we should generally write it as follows so that we don't lose sensitivities.

$$t_{rss} = \sqrt{a_1^2 t_1^2 + a_2^2 t_2^2 + \ldots + a_n^2 t_n^2} \tag{9.11}$$

Historically, Eq. (9.11) assumed that all of the component tolerances (t_i) represent a $3\sigma_i$ value for their manufacturing processes. Thus, if all the component distributions are assumed to be normal, then the probability that a dimension is between $\pm t_i$ is 99.73%. If this is true, then the assembly gap distribution is normal and the probability that it is $\pm t_{rss}$ between is 99.73%.

Although most people have assumed a value of $\pm 3\sigma$ for piecepart tolerances, the RSS equation works for "equal σ" values. If the designer assumed that the input tolerances were $\pm 4\sigma$ values for the piecepart manufacturing processes, then the probability that the assembly is between $\pm t_{rss}$ is 99.9937 (4σ).

The 3σ process limits using the RSS Model are similar to the Worst Case Model. The minimum gap is equal to the mean value minus the RSS variation at the gap. The maximum gap is equal to the mean value plus the RSS variation at the gap.

Minimum 3σ process limit = $d_g - t_{rss}$
Maximum 3σ process limit = $d_g + t_{rss}$

Using the original tolerances for Requirement 6, t_{rss} is:

$$t_{rss} = \left[\begin{matrix} (-1)^2\ .0155^2 + (1)^2\ .0020^2 + (1)^2\ .0030^2 + (1)^2\ .0075^2 + (1)^2\ .0050^2 + (1)^2\ .0070^2 + \\ (1)^2\ .0050^2 + (1)^2\ .0075^2 + (1)^2\ .0070^2 + (-1)^2\ .0060^2 + (1)^2\ .0300^2 \end{matrix}\right]^{\frac{1}{2}}$$

$$t_{rss} = .0381$$

The three sigma variation at the gap is:

Minimum 3σ process variation for Gap 6 = $d_g - t_{rss}$ = .0615 - .0381 = .0234
Maximum 3σ process variation for Gap 6 = $d_g + t_{rss}$ = .0615 + .0381 = .0996

Resizing Tolerances in the RSS Model

Using the RSS Model, the minimum gap is greater than the requirement. As in the Worst Case Model, we can resize the variable tolerances to achieve the desired assembly performance. As before, the scaling factor only applies to variable tolerances.

The resize factor, F_{rss}, for the RSS Model is:

$$F_{rss} = \sqrt{\frac{\left(d_g - g_m\right)^2 - \sum_{j=1}^{p}\left(a_j t_{jf}\right)^2}{\sum_{k=1}^{q}\left(a_k t_{kv}\right)^2}}$$

The new variable tolerances ($t_{kv,rss,\ resized}$) are the old tolerances multiplied by the factor F_{rss}.

$$t_{kv,rss,resized} = F_{rss}\, t_{kv}$$

$t_{kv,rss,resized}$ = equal bilateral tolerance of the k^{th}, variable component in the stackup after resizing using the RSS Model.

Fig. 9-8 shows the relationship between the piecepart tolerances and the assembly tolerance before and after resizing.

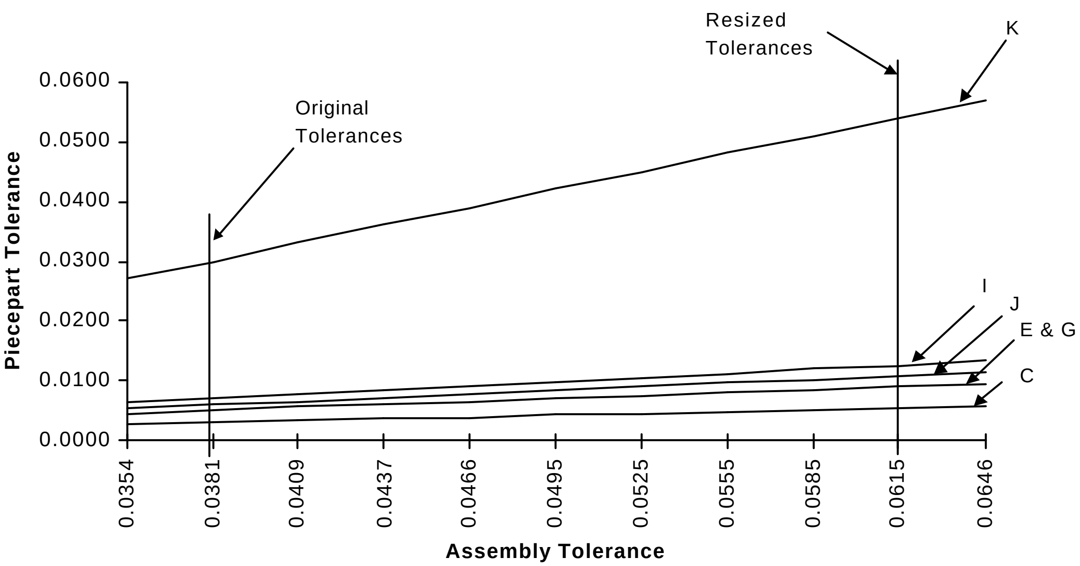

Figure 9-8 Graph of piecepart tolerances versus assembly tolerance before and after resizing using the RSS Model

The new variable tolerances are the old tolerances multiplied by the factor F_{rss}.
The resize factor for Requirement 6 is 1.7984. (For example, .0030 is resized to 1.7984*.0030 = .0054.)

Table 9-4 shows the new tolerances that would give a minimum gap of zero.

Table 9-4 Resized tolerances using the RSS Model

Variable Name	Mean Dimension	Fixed/ Variable	Original +/- Equal Bilateral Tolerance	Resized +/- Equal Bilateral Tolerance ($t_{iv,rss,resized}$)
A	.3595	Fixed	.0155	
B	.0320	Fixed	.0020	
C	.0600	Variable	.0030	.0054
D	.4305	Fixed	.0075	
E	.1200	Variable	.0050	.0090
F	1.5030	Fixed	.0070	
G	.1200	Variable	.0050	.0090
H	.4305	Fixed	.0075	
I	.4500	Variable	.0070	.0126
J	3.0250	Variable	.0060	.0108
K	.3000	Variable	.0300	.0540

As a check, we can show that the new maximum expected assembly gap for Requirement 6, using the resized tolerances, is:

$$t_{rss,resized} = \begin{bmatrix} (-1)^2\,.0155^2 + (1)^2\,.0020^2 + (1)^2\,.0054^2 + (1)^2\,.0075^2 + (1)^2\,.0090^2 + (1)^2\,.0070^2 + \\ (1)^2\,.0090^2 + (1)^2\,.0075^2 + (1)^2\,.0126^2 + (-1)^2\,.0108^2 + (1)^2\,.0540^2 \end{bmatrix}^{\frac{1}{2}}$$

$$t_{rss,resized} = .0615$$

The variation at the gap is:

Minimum $3s$ process variation for Gap 6 = $d_g - t_{rss,resized}$ = .0615 - .0615 = 0
Maximum $3s$ process variation for Gap 6 = $d_g + t_{rss,resized}$ = .0615 + .0615 = .1230

Assumptions and Risks of Using the RSS Model

The RSS Model yields larger piecepart tolerances for a given assembly gap, but the risk of defects at assembly is higher. The RSS Model assumes:

a) Piecepart tolerances are tied to process capabilities. This model assumes that when the designer changes a tolerance, the process capabilities will also change.

b) All process distributions are centered on the midpoint of the dimension. It does not allow for mean shifts (tool wear, etc.) or for purposeful decentering.

c) All piecepart dimensions are independent (covariance equals zero).

d) The bad parts are thrown in with the good in the assembly. The RSS Model does not take into account part screening (inspection).

e) The parts included in any assembly have been thoroughly mixed and the components included in any assembly have been selected at random.

f) The RSS derivation assumes equal bilateral tolerances.

Remember that by deriving the RSS equation, we made the assumption that all tolerances (t_i's) were equally producible. This is usually not the case. The only way to know if a tolerance is producible is by understanding the process capability for each dimension. The traditional assumption is that the tolerance (t_i) is equal to 3σ, and the probability of a defect at the gap will be about .27%. In reality, it is very unlikely to be a 3σ value, but rather some unknown number.

The RSS Model is better than the Worst Case Model because it accounts for the tendency of pieceparts to be centered on a mean dimension. In general, the RSS Model is not used if there are less than four dimensions in the stackup.

9.2.6.3 Modified Root Sum of the Squares Tolerancing Model

In reality, the probability of a worst case assembly is very low. At the other extreme, empirical studies have shown that the RSS Model does not accurately predict what is manufactured because some (or all) of the RSS assumptions are not valid. Therefore, an option designers can use is the RSS Model with a "correction" factor. This model is called the Modified Root Sum of the Squares Method.

$$t_{mrss} = C_f \sqrt{a_1^2 t_1^2 + a_2^2 t_2^2 + \ldots + a_n^2 t_n^2}$$

where

C_f = correction factor used in the MRSS equation.
t_{mrss} = expected variation (equal bilateral) using the MRSS model.

Several experts have suggested correction factors (C_f) in the range of 1.4 to 1.8 (References 1,4,5 and 6). Historically, the most common factor is 1.5.

The variation at the gap is:

Minimum gap = d_g - t_{mrss}
Maximum gap = $d_g + t_{mrss}$

In our example, we will use the correction factor suggested in Reference 2.

$$C_f = \frac{0.5\left(t_{wc} - t_{rss}\right)}{t_{rss}\left(\sqrt{n} - 1\right)} + 1$$

This correction factor will always give a t_{mrss} value that is less than t_{wc}. In our example, C_f is:

$$C_f = \frac{0.5(.0955 - .0381)}{.0381\left(\sqrt{11} - 1\right)} + 1$$

C_f = 1.3252

Using the original tolerances for Requirement 6, t_{mrss} is:

$$t_{mrss} = 1.3252 \begin{bmatrix} (-1)^2\,.0155^2 + (1)^2\,.0020^2 + (1)^2\,.0030^2 + (1)^2\,.0075^2 + (1)^2\,.0050^2 + (1)^2\,.0070^2 + \\ (1)^2\,.0050^2 + (1)^2\,.0075^2 + (1)^2\,.0070^2 + (-1)^2\,.0060^2 + (1)^2\,.0300^2 \end{bmatrix}^{\frac{1}{2}}$$

$$t_{mrss} = .0505$$

The variation at the gap is:

Minimum Gap 6 = d_g - t_{mrss} = .0615 - .0505 = .0110
Maximum Gap 6 = d_g t_{mrss} = .0615 + .0505 = .1120

Resizing Tolerances in the RSS Model

Similar to the RSS Model, the minimum gap using the MRSS Model is greater than the requirement. Like the other models, we can resize the variable tolerances to achieve the desired assembly performance. The equation for the resize factor, F_{mrss}, is much more complex for this model. The value of F_{mrss} is a root of the following quadratic equation.

$$aF_{mrss}^{\,2} + bF_{mrss} + c = 0$$

where

$$a = 0.25\left(\sum_{k=1}^{q} a_k t_{kv}\right)^2 - 2.25\sum_{k=1}^{q}\left(a_k t_{kv}\right)^2 + 3\sqrt{n}\sum_{k=1}^{q}\left(a_k t_{kv}\right)^2 - n\sum_{k=1}^{q}\left(a_k t_{kv}\right)^2$$

$$b = 0.5\sum_{k=1}^{q}\left(a_k t_{kv}\right)\sum_{j=1}^{p}\left(a_j t_{jf}\right) + \left(\sum_{k=1}^{q} a_k t_{kv}\right)\left(d_g - g_m\right) - \sqrt{n}\left(\sum_{k=1}^{q} a_k t_{kv}\right)\left(d_g - g_m\right)$$

$$c = 0.25\left(\sum_{j=1}^{p} a_j t_{jf}\right)^2 + \left(d_g - g_m\right)^2 - 2\sqrt{n}\left(d_g - g_m\right)^2 + n\left(d_g - g_m\right)^2 + \left(\sum_{j=1}^{p} a_j t_{jf}\right)\left(d_g - g_m\right)$$
$$-\sqrt{n}\left(\sum_{j=1}^{p} a_j t_{jf}\right)\left(d_g - g_m\right) - 2.25\sum_{j=1}^{p}\left(a_j t_{jf}\right)^2 + 3\sqrt{n}\sum_{j=1}^{p}\left(a_j t_{jf}\right)^2 - n\sum_{j=1}^{p}\left(a_j t_{jf}\right)^2$$

Therefore,

$$F_{mrss} = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Fig. 9-9 shows the relationship between the piecepart tolerances and the assembly tolerance before and after resizing.

The new variable tolerances $(t_{kv,mrss,\ resized})$ are the old tolerances multiplied by the factor F_{mrss}.

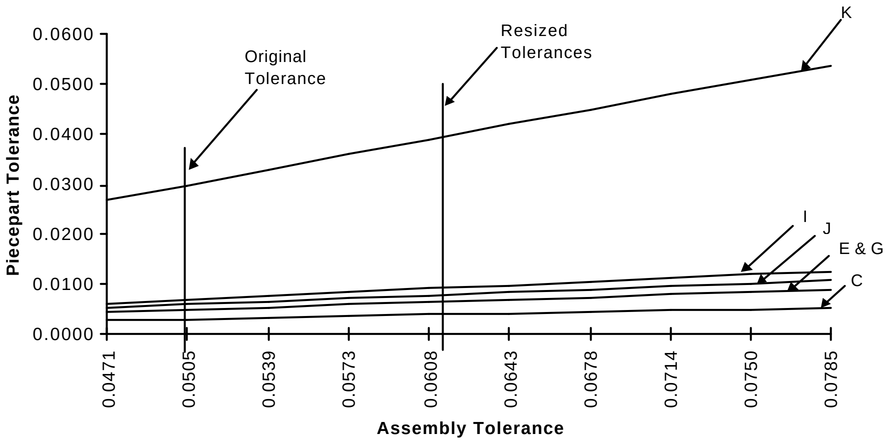

Figure 9-9 Graph of piecepart tolerances versus assembly tolerance before and after resizing using the MRSS Model

$$t_{kv,mrss,resized} = F_{mrss}\, t_{kv}$$

$t_{kv,mrss,resized}$ = equal bilateral tolerance of the k^{th}, variable component in the stackup after resizing using the MRSS Model.

The resize factor for Requirement 6 is 1.3209. (For example, .0030 is resized to 1.3209*.0030 = .0040.) Table 9-5 shows the new tolerances that would give a minimum gap of zero.

Table 9-5 Resized tolerances using the MRSS Model

Variable Name	Mean Dimension	Fixed/ Variable	Original +/- Equal Bilateral Tolerance	Resized +/- Equal Bilateral Tolerance ($t_{iv,mrss,resized}$)
A	.3595	Fixed	.0155	
B	.0320	Fixed	.0020	
C	.0600	Variable	.0030	.0040
D	.4305	Fixed	.0075	
E	.1200	Variable	.0050	.0066
F	1.5030	Fixed	.0070	
G	.1200	Variable	.0050	.0066
H	.4305	Fixed	.0075	
I	.4500	Variable	.0070	.0092
J	3.0250	Variable	.0060	.0079
K	.3000	Variable	.0300	.0396

As a check, we show the following calculations for the resized tolerances.

$$t_{wc,\ resized} = .0155+.0020+.0040+.0075+.0066+.0070+.0066+.0075+.0092+.0079+.0396$$

$$t_{wc,\ resized} = .1134$$

$$t_{rss,\ resized} = \left[\begin{matrix}(-1)^2\,.0155^2 + (1)^2\,.0020^2 + (1)^2\,.0040^2 + (1)^2\,.0075^2 + (1)^2\,.0066^2 + (1)^2\,.0070^2 + \\ (1)^2\,.0066^2 + (1)^2\,.0075^2 + (1)^2\,.0092^2 + (-1)^2\,.0079^2 + (1)^2\,.0396^2\end{matrix}\right]^{\frac{1}{2}}$$

$$t_{rss,\ resized} = .0472$$

$$C_{f,\ resized} = \frac{0.5(.1134 - .0472)}{.0472\,(\sqrt{11} - 1)} + 1$$

$$C_{f,\ resized} = 1.3032$$

$$t_{mrss,\ resized} = 1.3032\left[\begin{matrix}(-1)^2\,.0155^2 + (1)^2\,.0020^2 + (1)^2\,.0040^2 + (1)^2\,.0075^2 + (1)^2\,.0066^2 + (1)^2\,.0070^2 + \\ (1)^2\,.0066^2 + (1)^2\,.0075^2 + (1)^2\,.0092^2 + (-1)^2\,.0079^2 + (1)^2\,.0396^2\end{matrix}\right]^{\frac{1}{2}}$$

$$t_{mrss,\ resized} = .0615$$

As a check, we can show that the expected assembly gap for Requirement 6, using the resized tolerances, is:

Minimum Gap 6 = $d_g - t_{mrss,resized}$ = .0615 - .0615 = .0000
Maximum Gap 6 = $d_g + t_{mrss,resized}$ = .0615 + .0615 = .1230

Assumptions and Risks of Using the MRSS Model

The uncertainty associated with the MRSS Model is that there is no mathematical reason for the factor C_f. The correction factor can be thought of as a "safety" factor. The more the RSS assumptions depart from reality, the higher the safety factor should be.

The MRSS Model also has other problems.

a) It applies the same "safety" factor to all the tolerances, even though they don't deviate from the RSS assumptions equally.

b) If fixed correction factors proposed in the literature are used, the MRSS tolerance can be larger than the worst case stackup. This problem is eliminated with the use of the calculated C_f shown here.

c) If the tolerances are equal and there are only two of them, the MRSS assembly tolerance will always be larger than the worst case assembly tolerance when using the calculated correction factor.

The MRSS Model is generally considered better than the RSS and Worst Case models because it tries to model what has been measured in the real world.

9.2.6.4 Comparison of Variation Models

Table 9-6 summarizes the Worst Case, RSS, and MRSS models for Requirement 6. The "Resized" columns show the tolerances that will give a minimum expected gap value of zero, and a maximum expected gap value of .1230 inch. As expected, the worst case tolerance values are the smallest. In this example, the resized RSS tolerance values are approximately three times greater than the worst case tolerances. It is obvious that the RSS tolerances will yield more pieceparts. The MRSS resized tolerance values fall between the worst case (most conservative) and RSS (most risk of assembly defects) values.

Table 9-6 Comparison of results using the Worst Case, RSS, and MRSS models

			Tolerance Analysis					
			Worst Case		RSS		MRSS	
Mean Dim.	**Sens.**	**Dim. Type**	Original	Resized	Original	Resized	Original	Resized
.3595	-1.0000	Variable	.0155		.0155		.0155	
.0320	1.0000	Fixed	.0020		.0020		.0020	
.0600	1.0000	Variable	.0030	.0012	.0030	.0054	.0030	.0040
.4305	1.0000	Fixed	.0075		.0075		.0075	
.1200	1.0000	Variable	.0050	.0020	.0050	.0090	.0050	.0066
1.5030	1.0000	Fixed	.0070		.0070		.0070	
.1200	1.0000	Variable	.0050	.0020	.0050	.0090	.0050	.0066
.4305	1.0000	Fixed	.0075		.0075		.0075	
.4500	1.0000	Variable	.0070	.0027	.0070	.0126	.0070	.0092
3.0250	-1.0000	Variable	.0060	.0024	.0060	.0108	.0060	.0079
.3000	1.0000	Variable	.0300	.0118	.0300	.0540	.0300	.0396
Nominal Gap			.0615	.0615	.0615	.0615	.0615	.0615
Minimum Gap			-.0340	.0001	.0234	.0000	.0110	.0000
Expected Variation			.0955	.0616	.0381	.0615	.0505	.0615

Table 9-7 summarizes the tradeoffs for the three models. All the models have different degrees of risk of defects. The worst case tolerances have the least amount of risk (i.e. largest number of assemblies within the expected assembly requirements). Because of the tight tolerances we will reject more pieceparts. Worst case also implies that we are doing 100% inspection. Since we have to tighten up the tolerances to meet the assembly specification, the number of rejected pieceparts increases. Therefore, this model has the highest costs associated with it. The RSS tolerances will yield the least piecepart cost at the expense of a lower probability of assembly conformance. The MRSS Model tries to take the best of both of these models. It gives a higher probability of assembly conformance than the RSS Model, and lower piecepart costs than the Worst Case Model.

Within their limitations, the traditional tolerancing models have worked in the past. The design engineer, however, could not quantify how well they worked. He also could not quantify how cost effective the tolerance values were. Obviously, these methods cannot consistently achieve quality goals. One way to achieve quality goals is to eliminate the assumptions that go along with the classical tolerancing models. By doing so, we can quantify (sigma level, defects per million opportunities (dpmo)) the tolerances and optimize tolerances for maximum producibility. These issues are discussed in Chapter 11, Predicting Assembly Quality.

Table 9-7 Comparison of analysis models

Consideration	Worst Case Model	RSS Model	MRSS Model
Risk of Defect	Lowest	Highest	Middle
Cost	Highest	Lowest	Middle
Assumptions about component processes	None	The process follows a normal distribution. The mean of the process is equal to the nominal dimension. Processes are independent.	The process follows a normal distribution. The mean of the process distribution is not necessarily equal to the nominal dimension.
Assumptions about drawing tolerances	Dimensions outside the tolerance range are screened out.	The tolerance is related to a manufacturing process capability. Usually the tolerance range is assumed to be the +/- 3 sigma limit of the process.	The tolerance is related to a manufacturing process capability. Usually the tolerance range is assumed to be the +/- 3 sigma limit of the process.
Assumptions about expected assembly variation	100% of the parts are within the maximum and minimum performance range.	The assembly distribution is normal. Depending on the piecepart assumptions, a percentage of the assemblies will be between the minimum and maximum gap. Historically, this has been 99.73%. Some out of specification parts reach assembly.	99.73% of the assemblies will be between the minimum and maximum gap. The correction factor *(C_f)* is a safety factor.

9.2.6.5 Estimated Mean Shift Model

Generally, if we don't have knowledge about the processes for manufacturing a part, such as a vendor part, we are more inclined to use the Worst Case Model. On the other hand, if we have knowledge about the processes that make the part, we are more inclined to use a statistical model. Chase and Greenwood proposed a tolerancing model that blends the Worst Case and RSS models. (Reference 6) This *Estimated Mean Shift Model* is:

$$t_{ems} = \sum_{i=1}^{n} \left| m_i a_i t_i \right| + \sum_{i=1}^{n} \sqrt{\left((1 - m_i)^2 a_i^2 t_i^2 \right)}$$

where

m_i = the mean shift factor for the ith component

In this model, the mean shift factor is a number between 0 and 1.0 and represents the amount that the midpoint is estimated to shift as a fraction of the tolerance range. If a process were closely controlled, we would use a small mean shift, such as .2. If we know less about the process, we would use higher mean shift factors.

Using a mean shift factor of .2 for the variable components and .8 for the fixed components, the expected variation for Requirement 6 is:

$$t_{ems} = |.8(-1).0155| + |.8(1).0020| + |.2(1).0030| + |.8(1).0075| + |.2(1).0050| + |.8(1).0070| + |.2(1).0050| + |.8(1).0075| + |.2(1).0070| + |.2(-1).0060| + |.2(1).0300| +$$

$$\left[\begin{array}{l}(.2(-1).0155) + (.2^2(1)^2.0020^2) + (.8^2(1)^2.0030^2) + (.2^2(1)^2.0075^2) + \\ (.8^2(1)^2.0050^2) + (.2^2(1)^2.0070^2) + (.8^2(1)^2.0050^2) + (.2^2(1)^2.0075^2) + \\ (.8^2(1)^2.0070^2) + (.8^2(-1)^2.0060^2) + (.8^2(1)^2.0300^2)\end{array}\right]^{\frac{1}{2}}$$

$t_{ems} = .0690$

The first part of the Estimated Mean Shift Model is the sum of the mean shifts and is similar to the Worst Case Model. Notice if we set the mean shift factor to 1.0 for all the components, t_{ems} is equal to .0955, which is the same as t_{wc}. The second part of the model is the sum of the statistical components. Notice if we used a mean shift factor of zero for all of the components, t_{ems} is equal to .0381, which is the same as t_{rss}.

The two major advantages of the Estimated Mean Shift Model are:

- It allows flexibility in the design. Some components may be modeled like worst case, and some may be modeled statistically.
- The model can be used to estimate designs (using conservative shift factors), or it can accept manufacturing data (if it is available).

9.3 Analyzing Geometric Tolerances

The previous discussions have only included tolerances associated with dimensions in the tolerance analysis. We have not yet addressed how to model geometric tolerances in the loop diagram.

Generally, geometric controls will restrain one or several of the following attributes:

- Location of the feature
- Orientation of the feature
- Form of the feature

The most difficult task when modeling geometric tolerances is determining which of the geometric controls contribute to the requirement and how these controls should be modeled in the loop diagram. Because the geometric controls are interrelated, there are no hard and fast rules that tell us how to include geometric controls in tolerance analyses. Since there are several modeling methods, sometimes we include GD&T in the model, and sometimes we do not.

Generally, however, if a feature is controlled with geometric tolerances, the following apply.

- If there is a location control on a feature in the loop diagram, we will usually include it in the analysis.
- If there is an orientation control on a feature in the loop diagram, we may include it in the analysis as long as the location of the feature is not a contributor to the requirement.

- If there is a form control on a feature in the loop diagram, we may include it in the analysis as long as the location, orientation, or size of the feature is not a contributor to the requirement. Any time parts come together, however, we have surface variations that introduce variations in the model.
- Geometric form and orientation controls on datum features are usually not included in loop diagrams. Since datums are the "starting points" for measurements, and are defined as the geometric counterparts (high points) of the datum feature, the variations in the datum features usually don't contribute to the variation analysis.

There is a difference between a GD&T control (such as a form control) and a feature variation (such as form variation). If we add a GD&T control to a stack, we add to the output. Therefore, we should only include the GD&T controls that add to the output.

GD&T controls are generally used only in worst case analyses. Previously we said that the Worst Case Model assumes 100% inspection. Since GD&T controls are the specification limits for inspection, it makes sense to use them in this type of analysis. In a statistical analysis, however, we either make assumptions about the manufacturing processes (as shown previously), or use real data from the manufacturing processes (as shown in Chapter 11). Since the manufacturing processes are sources of variation, they should be inputs to the statistical analyses. Since GD&T controls are not sources of variation, they should not be used in a statistical analysis.

The following sections show examples of how to model geometric tolerances. The examples are single part stacks, but the concepts can be applied to stacks with multiple components.

9.3.1 Form Controls

Form controls should seldom be included in a variation analysis. For nonsize features, the location, or orientation tolerance usually controls the extent of the variation of the feature. The form tolerance is typically a refinement of one of these controls. If a form control is applied to a size feature (and the Individual Feature of Size Rule applies from ASME Y14.5), the size tolerance is usually included in the variation analysis. In these cases, the form tolerance boundary is inside the size tolerance boundary, the location tolerance boundary, or the orientation tolerance boundary, so the form control is not modeled.

If form tolerances are used in the loop diagram, they are modeled with a nominal dimension equal to zero, and an equal bilateral tolerance equal to the form tolerance. (Depending on the application, sometimes the equal bilateral tolerance is equal to half the form tolerance.)

Fig. 9-10 shows an assembly with four parts. In this example, the requirement is for the Gap to be greater than zero. For this requirement, the following applies to the form controls.

- Flatness of .001 on the substrate is not included in the loop diagram because it is a datum.
- Flatness of .002 on the heatsink is included in the loop diagram.
- Flatness of .002 on the housing is not included in the loop diagram because it is a refinement of the location tolerance.
- Flatness of .004 on the housing is not included in the loop diagram because it is a datum.
- Flatness of .006 on the housing is not included in the loop diagram because it is a refinement of the location.

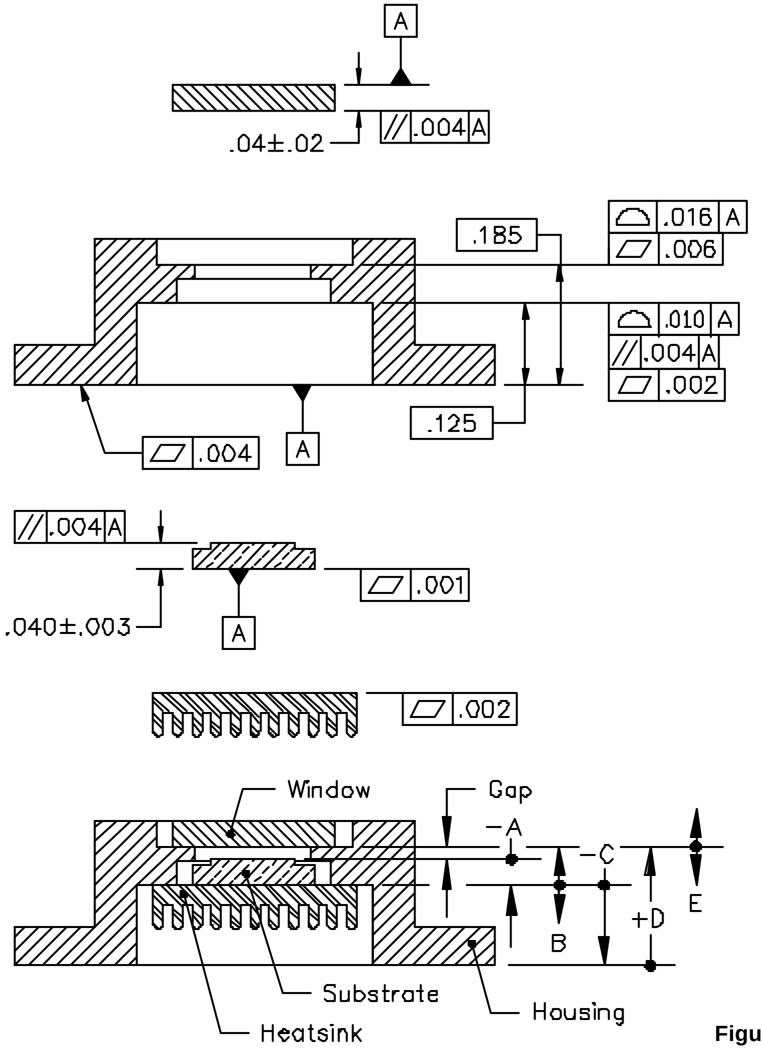

Figure 9-10 Substrate package

9.3.2 Orientation Controls

Like form controls, we do not often include orientation controls in a variation analysis. Typically we determine the feature's worst-case tolerance boundary using the location or size tolerance.

If orientation tolerances are used in the loop diagram, they are modeled like form tolerances. They have a nominal dimension equal to zero, and an equal bilateral tolerance equal to the orientation tolerance. (Depending on the application, sometimes the equal bilateral tolerance is equal to half the orientation tolerance.)

In Fig. 9-10, the following describes the application of the orientation controls to the Gap analysis.

- Parallelism of .004 to datum A on the Substrate is not included in the loop diagram because it is a refinement of the size dimension (.040 ±.003).
- Parallelism of .004 to datum A on the Housing is not included in the loop diagram because it is a refinement of the location tolerance.
- Parallelism of .004 to datum A on the Window is included in the loop diagram.

Therefore, the equation for the Gap in Fig. 9-10 is: Gap = -A+B-C+D+E
where

A = .040 ±.003

B = 0 ±.002

C = .125 ±.005

D = .185 ±.008

E = 0 ±.004

9.3.3 Position

There are several ways to model a position geometric constraint. When we use position at regardless of feature size (RFS), the size of the feature, and the location of the feature are treated independently. When we use position at maximum material condition (MMC) or at least material condition (LMC), the size and location dimensions cannot be treated independently. The following sections show how to analyze these situations.

9.3.3.1 Position at RFS

Fig. 9-11 shows a hole positioned at RFS.

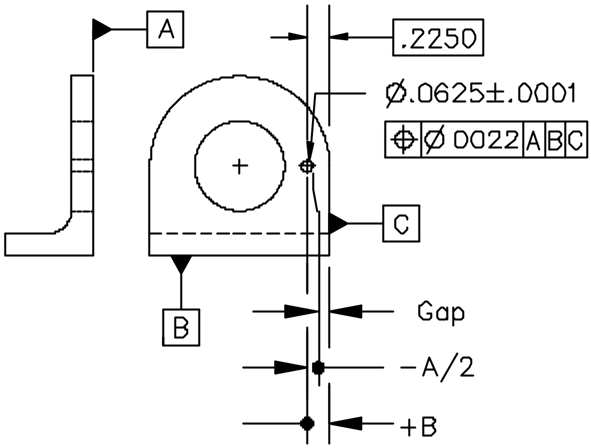

Figure 9-11 Position at RFS

The equation for the Gap in Fig. 9-11 is: Gap = –A/2+B
where

A = .0625 ±.0001

B = .2250 ±.0011

9.3.3.2 Position at MMC or LMC

As stated earlier, when we use position at MMC or LMC, the size and location dimensions should be combined into one component in the loop diagram. We can do this using the following method.

1) Calculate the largest “outer” boundary allowed by the dimensions and tolerances.
2) Calculate the smallest “inner” boundary allowed by the dimensions and tolerances.
3) Convert the inner and outer boundary into a nominal diameter with an equal bilateral tolerance.

9.3.3.3 Virtual and Resultant Conditions

When calculating the internal and external boundaries for features of size, it is helpful to understand the following definitions from ASME Y14.5M-1994.

Virtual Condition: A constant boundary generated by the collective effects of a size feature's specified MMC or LMC and the geometric tolerance for that material condition.

- The virtual condition (outer boundary) of an external feature, called out at MMC, is equal to its maximum material condition plus its tolerance at maximum material condition.
- The virtual condition (inner boundary) of an internal feature, called out at MMC, is equal to its maximum material condition minus its tolerance at maximum material condition.
- The virtual condition (inner boundary) of an external feature, called out at LMC, is equal to its least material condition minus its tolerance at least material condition.
- The virtual condition (outer boundary) of an internal feature, called out at LMC, is equal to its least material condition plus its tolerance at least material condition.

Resultant Condition: The variable boundary generated by the collective effects of a size feature's specified MMC or LMC, the geometric tolerance for that material condition, the size tolerance, and the additional geometric tolerance derived from its specified material condition.

- The smallest resultant condition (inner boundary) of an external feature, called out at MMC, is equal to its least material condition minus its tolerance at least material condition.
- The largest resultant condition (outer boundary) of an internal feature, called out at MMC, is equal to its least material condition plus its tolerance at least material condition.
- The largest resultant condition (outer boundary) of an external feature, called out at LMC, is equal to its maximum material condition plus its tolerance at maximum material condition.
- The smallest resultant condition (inner boundary) of an internal feature, called out at LMC, is equal to its maximum material condition minus its tolerance at maximum material condition.

9.3.3.4 Equations

We can use the following equations to calculate the inner and outer boundaries.

For an external feature at MMC
outer boundary = VC = MMC + Geometric Tolerance at MMC
inner boundary = (smallest) RC = LMC – Tolerance at LMC

For an internal feature at MMC
inner boundary = VC = MMC - Geometric Tolerance at MMC
outer boundary = (largest) RC = LMC + Tolerance at LMC

For an external feature at LMC
inner boundary = VC = LMC - Geometric Tolerance at LMC
outer boundary = (largest) RC = MMC + Tolerance at MMC

For an internal feature at LMC
outer boundary = VC = LMC + Geometric Tolerance at LMC
inner boundary = (smallest) RC = MMC – Tolerance at MMC

Converting an Internal Feature at MMC to a Nominal Value with an Equal Bilateral Tolerance

Fig. 9-12 shows a hole that is positioned at MMC.

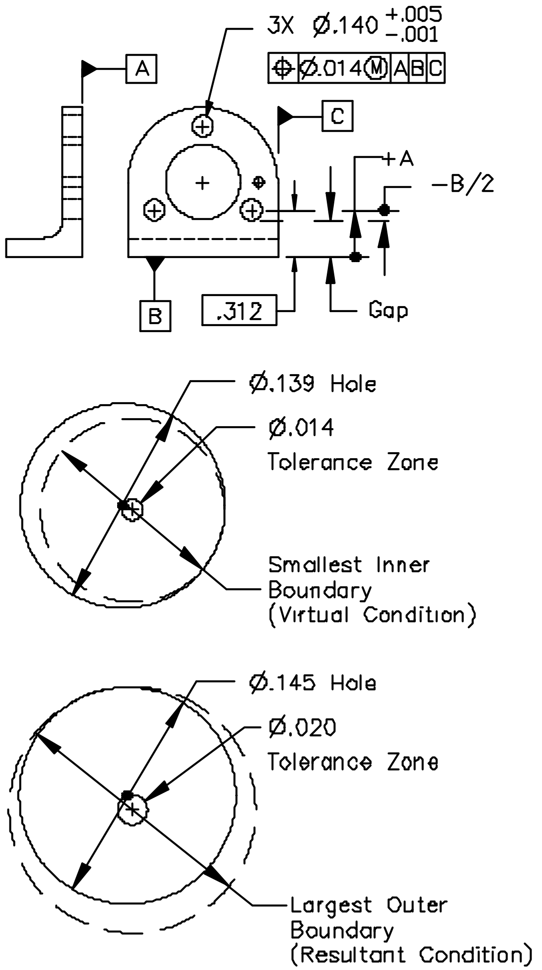

Figure 9-12 Position at MMC—internal feature

The value for B in the loop diagram is:

- Largest outer boundary = ∅.145 + ∅.020 = ∅.165
- Smallest inner boundary = ∅.139 – ∅.014 = ∅.125
- Nominal diameter = (∅.165 + ∅.125)/2 = ∅.145
  Equal bilateral tolerance = ∅.020

For position at MMC, an easier way to convert this is:

LMC ± (total size tolerance + tolerance in the feature control frame)

= ∅.145 ± (.006+.014) = .145±.020

The equation for the Gap in Fig. 9-12 is: Gap = A-B/2

where

A = .312 ±0 and B = .145 ±.020

Converting an External Feature at MMC to a Nominal Value with an Equal Bilateral Tolerance

Fig. 9-13 shows a pin positioned at MMC.

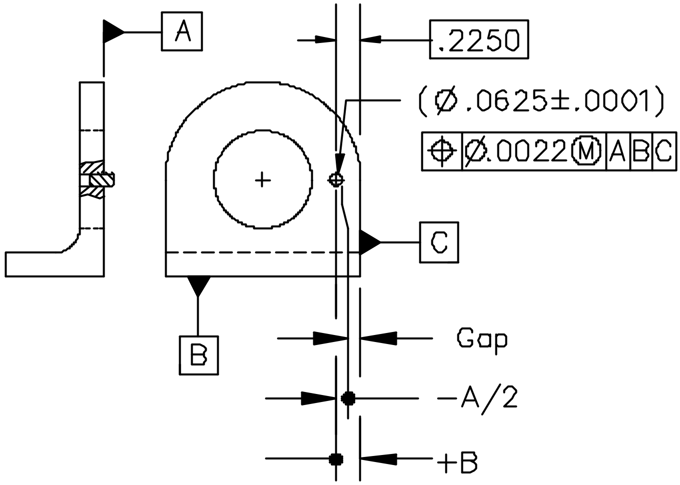

Figure 9-13 Position at MMC—external feature

The value for B in the loop diagram is:

- Largest outer boundary = ∅.0626 + ∅.0022 = ∅.0648
- Smallest inner boundary = ∅.0624 – ∅.0024 = ∅.0600
- Nominal diameter = (∅.0648 + ∅.0600)/2 = ∅.0624
  Equal bilateral tolerance = ∅.0024

As shown earlier, the easier conversion for position at MMC, is:

LMC ±(total size tolerance + tolerance in the feature control frame)

= ∅.0624 ±(.0002+.0022) = .0624+/-.0024

The equation for the Gap in Fig. 9-13 is: Gap = -A/2+B

where

A = .0624 ±.0024

B = .2250 ±0

Converting an Internal Feature at LMC to a Nominal Value with an Equal Bilateral Tolerance

Fig. 9-14 shows a hole that is positioned at LMC.

The value for B in the loop diagram is:

- Largest outer boundary = ∅.52+∅.03 = ∅.55
- Smallest inner boundary = ∅.48-∅.07 = ∅.41
- Nominal diameter = (∅.55+∅.41)/2= ∅.48
  Equal bilateral tolerance = ∅.07

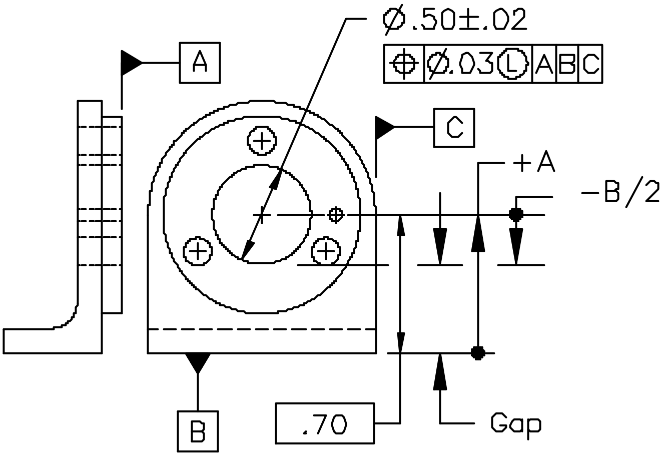

Figure 9-14 Position at LMC—internal feature

For position at LMC, an easier way to convert this is:

MMC ±(total size tolerance + tolerance in the feature control frame)

= ∅.48 ± (04+.03) = .48 ±.07

The equation for the Gap in Fig. 9-14 is: Gap = A – B/2

where

A = .70 ±0

B = .48 ±.07

Converting an External Feature at LMC to a Nominal Value with an Equal Bilateral Tolerance

Fig. 9-15 shows a “boss” that is positioned at LMC.

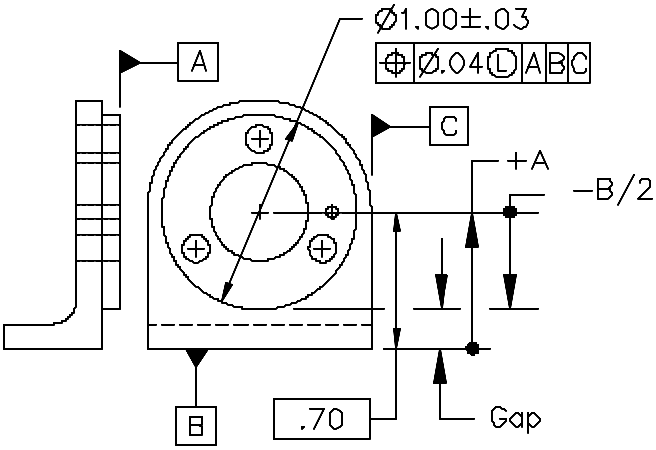

Figure 9-15 Position at LMC—external feature

The value for B in the loop diagram is:

- Largest outer boundary = ∅1.03 + ∅.10 = ∅1.13
- Smallest inner boundary = ∅.97 – ∅.04 = ∅.93
- Nominal diameter = (∅1.13 + ∅.93)/2 = ∅1.03

  Equal bilateral tolerance = ∅.10

As shown earlier, the easier conversion for position at LMC is:

MMC $\pm$(total size tolerance + tolerance in the feature control frame)

$= \varnothing 1.03 \pm (.06+.04) = 1.03 +/-.10$

The equation for the Gap in Fig. 9-15 is: Gap = A-B/2

where

A = .70 $\pm$ 0

B = 1.03 $\pm$.10

9.3.3.5 Composite Position

Fig. 9-16 shows an example of composite positional tolerancing.

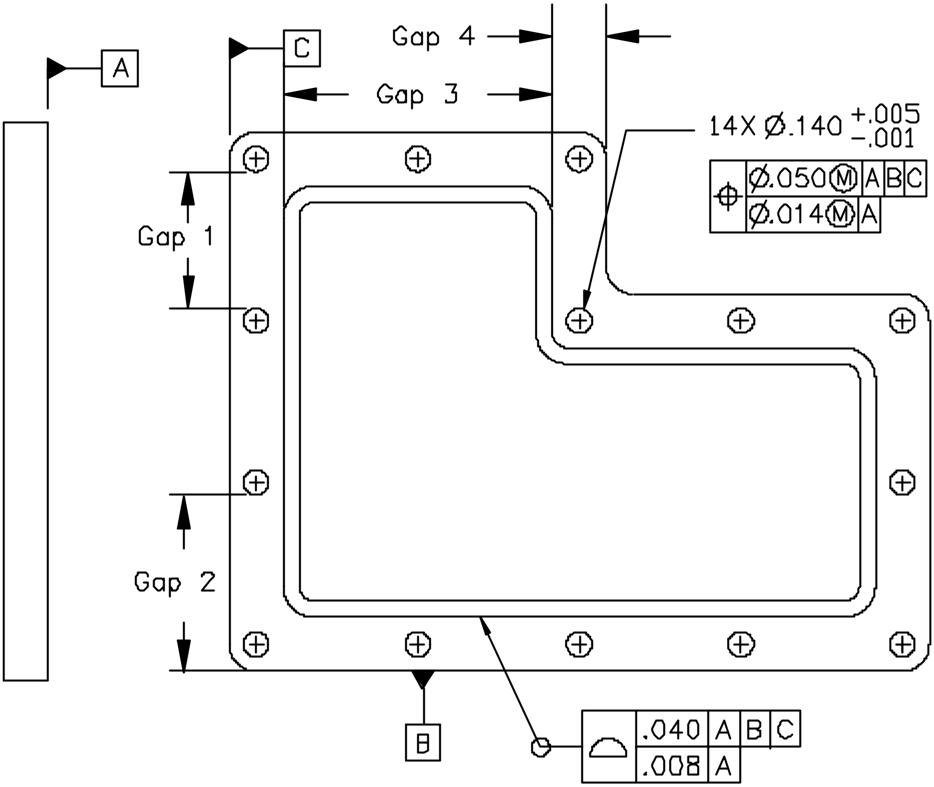

Figure 9-16 Composite position and composite profile

Composite positional tolerancing introduces a unique element to the variation analysis; an understanding of which tolerance to use. If a requirement only includes the pattern of features and nothing else on the part, we use the tolerance in the lower segment of the feature control frame. Since Gap 1 in Fig. 9-16 is controlled by two features within the pattern, we use the tolerance of ∅.014 to calculate the variation for Gap 1.

Gap 2, however, includes variations of the features back to the datum reference frame. In this situation, we use the tolerance in the upper segment of the feature control frame (∅.050) to calculate the variation for Gap 2.

9.3.4 Runout

Analyzing runout controls in tolerance stacks is similar to analyzing position at RFS. Since runout is always RFS, we can treat the size and location of the feature independently. We analyze total runout the same as circular runout, because the worst-case boundary is the same for both controls.

Fig. 9-17 shows a hole that is positioned using runout.

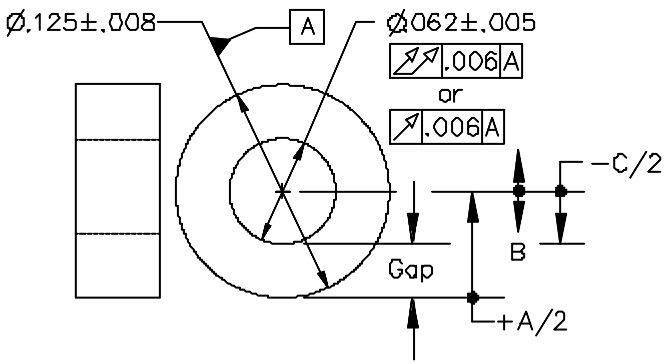

Figure 9-17 Circular and total runout

We model the runout tolerance with a nominal dimension equal to zero, and an equal bilateral tolerance equal to half the runout tolerance.

The equation for the Gap in Fig. 9-17 is: Gap = + A/2 + B – C/2

where

A = .125 ±.008

B = 0 ±.003

C = .062 ±.005

9.3.5 Concentricity/Symmetry

Analyzing concentricity and symmetry controls in tolerance stacks is similar to analyzing position at RFS and runout.

Fig. 9-18 is similar to Fig. 9-17, except that a concentricity tolerance is used to control the ∅.062 feature to datum A.

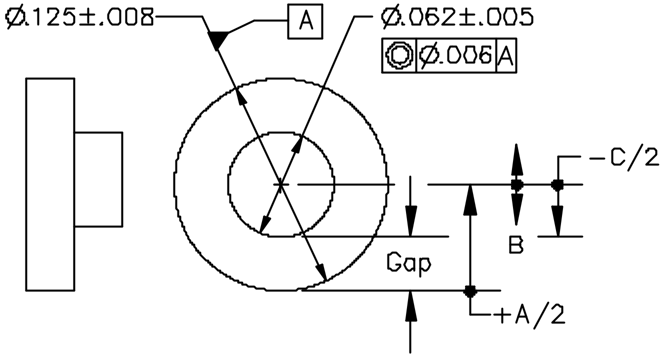

Figure 9-18 Concentricity

The loop diagram for this gap is the same as for runout. The equation for the Gap in Fig. 9-18 is:
Gap = + A/2 + B – C/2
where

A = .125 ±.008

B = 0 ±.003

C = .062 ±.005

Symmetry is analogous to concentricity, except that it is applied to planar features. A loop diagram for symmetry would be similar to concentricity.

9.3.6 Profile

Profile tolerances have a basic dimension locating the true profile. The tolerance is depicted either equal bilaterally, unilaterally, or unequal bilaterally. For equal bilateral tolerance zones, the profile component is entered as a nominal value. The component is equal to the basic dimension, with an equal bilateral tolerance that is half the tolerance in the feature control frame.

9.3.6.1 Profile Tolerancing with an Equal Bilateral Tolerance Zone

Fig. 9-19 shows an application of profile tolerancing with an equal bilateral tolerance zone.

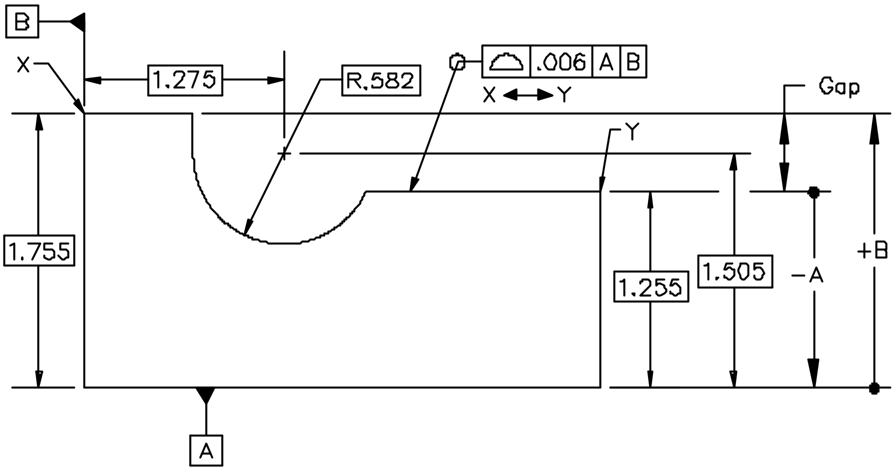

Figure 9-19 Equal bilateral tolerance profile

The equation for the Gap in Fig. 9-19 is: Gap = -A+B
where

A = 1.255 ±.003

B = 1.755 ±.003

9.3.6.2 Profile Tolerancing with a Unilateral Tolerance Zone

Fig. 9-20 shows a figure similar to Fig. 9-19 except the equal bilateral tolerance was changed to a unilateral tolerance zone.

The equation for the Gap is the same as Fig. 9-19: Gap = – A + B

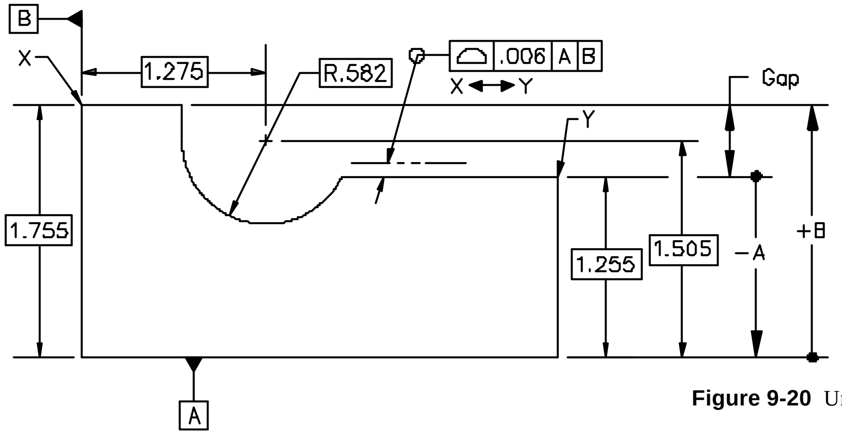

Figure 9-20 Unilateral tolerance profile

In this example, however, we need to change the basic dimensions and unilateral tolerances to mean dimensions and equal bilateral tolerances.

Therefore,

A = 1.258 $\pm$.003

B = 1.758 $\pm$.003

9.3.6.3 Profile Tolerancing with an Unequal Bilateral Tolerance Zone

Fig. 9-21 shows a figure similar to Fig. 9-19 except the equal bilateral tolerance was changed to an unequal bilateral tolerance zone.

The equation for the Gap is the same as Fig. 9-19: Gap = – A + B

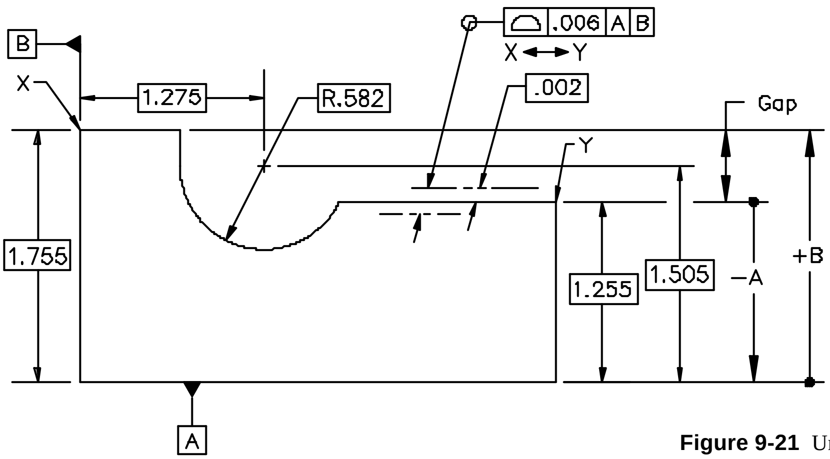

Figure 9-21 Unequal bilateral tolerance profile

As we did in Fig. 9-20, we need to change the basic dimensions and unequal bilateral tolerances to mean dimensions and equal bilateral tolerances.
Therefore,

A = 1.254 ±.003

B = 1.754 ±.003

9.3.6.4 Composite Profile

Composite profile is similar to composite position. If a requirement only includes features within the profile, we use the tolerance in the lower segment of the feature control frame. If the requirement includes variations of the profile back to the datum reference frame, we use the tolerance in the upper segment of the feature control frame.

Fig. 9-16 shows an example of composite profile tolerancing. Gap 3 is controlled by features within the profile, so we would use the tolerance in the lower segment of the profile feature control frame (∅.008) to calculate the variation for Gap 3.

Gap 4, however, includes variations of the profiled features back to the datum reference frame. In this situation, we would use the tolerance in the upper segment of the profile feature control frame (∅.040) to calculate the variation for Gap 4.

9.3.7 Size Datums

Fig. 9-22 shows an example of a pattern of features controlled to a secondary datum that is a feature of size.

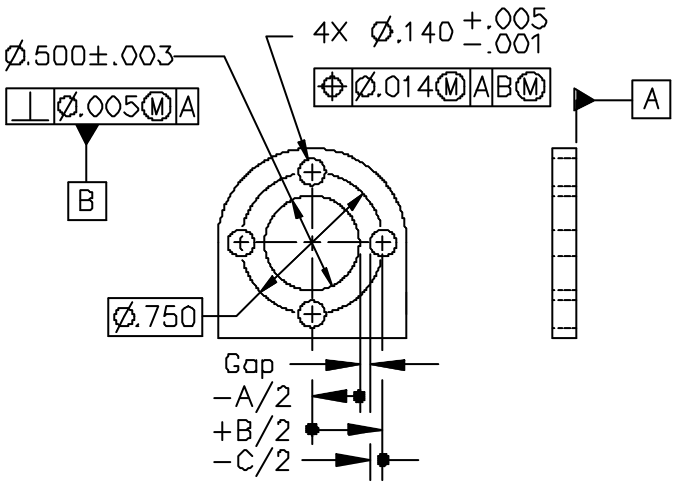

Figure 9-22 Size datum

In this example, ASME Y14.5 states that the datum feature applies at its virtual condition, even though it is referenced in its feature control frame at MMC. (Note, this argument also applies for secondary and tertiary datums invoked at LMC.) In the tolerance stack, this means that we will get an additional "shifting" of the datum that we need to include in the loop diagram.

The way we handle this in the loop diagram is the same way we handled features controlled with position at MMC or LMC. We calculate the virtual and resultant conditions, and convert these boundaries into a nominal value with an equal bilateral tolerance.

The value for A in the loop diagram is:

- Largest outer boundary = ∅.503 + ∅.011 = ∅.514
- Smallest inner boundary = ∅.497 – ∅.005 = ∅.492
- Nominal diameter = (∅.514 + ∅.492)/2 = ∅.503
- Equal bilateral tolerance = ∅.011

An easier way to convert to this radial value is:

LMC ±(total size tolerance + tolerance in the feature control frame)

= ∅.503 ±(.006+.005) = .503±.011

The value for C in the loop diagram is:

- Largest outer boundary = ∅.145 + ∅.020 = ∅.165
- Smallest inner boundary = ∅139 – ∅.014 = ∅.125
- Nominal diameter = (∅.165 + ∅.125)/2 = ∅.145
- Equal bilateral tolerance = ∅.020

An easier way to convert to this radial value is:

LMC ±(total size tolerance + tolerance in the feature control frame)

= ∅.145 ±(.006+.014) = .145 ±.020

The equation for the Gap in Fig. 9-22 is: Gap = – A/2 + B/2 – C/2

where

A = .503 ±.011

B = .750 ±0

C = .145 ±.020

9.4 Abbreviations

Variable	**Definition**
a_i	sensitivity factor that defines the direction and magnitude for the ith dimension. In a one-dimensional stackup, this value is usually +1 or -1. Sometimes, in a one-dimensional stackup, this value may be +.5 or -.5 if a radius is the contributing factor for a diameter callout on a drawing.
a_j	sensitivity factor for the jth, fixed component in the stackup
a_k	sensitivity factor for the kth, variable component in the stackup
C_f	correction factor used in the MRSS equation
$C_{f,resized}$	correction factor used in the MRSS equation, using resized tolerances
$\frac{\partial f}{\partial x_i}$	partial derivative of function y with respect to x_i
d_g	the mean value at the gap. If d_g is positive, the mean "gap" has clearance, and if d_g is negative, the mean "gap" has interference
d_i	the mean value of the ith dimension in the loop diagram

D_i	dimension associated with i^{th} random variable x_i
F_{wc}	resize factor that is multiplied by the original tolerances to achieve a desired assembly performance using the Worst Case Model
F_{mrss}	resize factor that is multiplied by the original tolerances to achieve a desired assembly performance using the MRSS Model
F_{rss}	resize factor that is multiplied by the original tolerances to achieve a desired assembly performance using the RSS Model
g_m	minimum value at the (assembly) gap. This value is zero if no interference or clearance is allowed.
m_y	mean of random variable y
n	number of independent variables (dimensions) in the equation (stackup)
p	number of independent, fixed dimensions in the stackup
q	number of independent, variable dimensions in the stackup
r	the total number of measurements in the population of interest
s_y	standard deviation of function y
t_i	equal bilateral tolerance of the ith component in the stackup
T_i	tolerance associated with ith random variable x_i
t_{jf}	equal bilateral tolerance of the jth, fixed component in the stackup
t_{kv}	equal bilateral tolerance of the kth, variable component in the stackup
$t_{kv,wc,resized}$	equal bilateral tolerance of the kth, variable component in the stackup after resizing, using the Worst Case Model
$t_{kv,rss,resized}$	equal bilateral tolerance of the kth, variable component in the stackup after resizing, using the RSS Model
$t_{kv,mrss,resized}$	equal bilateral tolerance of the kth, variable component in the stackup after resizing, using the MRSS Model
t_{mrss}	expected assembly gap variation (equal bilateral) using the MRSS Model
$t_{mrss,resized}$	the expected variation (equal bilateral) using the MRSS Model and resized tolerances
t_{rss}	the expected variation (equal bilateral) using the RSS Model
$t_{rss,resized}$	the expected variation (equal bilateral) using the RSS Model and resized tolerances
t_{wc}	maximum expected variation (equal bilateral) using the Worst Case Model
$t_{wc,resized}$	maximum expected variation (equal bilateral) using the Worst Case Model and resized tolerances
USL_i	upper specification limit of the ith dimension
x_i	ith independent variable
y	function consisting of n independent variables ($x_1,\ldots,x_n$)
Z_i	standard normal transform of ith dimension
Z_y	standard normal transform of y

9.5 Terminology

MMC = Maximum Material Condition: The condition in which a feature of size contains the maximum amount of material within the stated limits of size.

LMC = Least Material Condition: The condition in which a feature of size contains the least amount of material within the stated limits of size.

VC = Virtual Condition: A constant boundary generated by the collective effects of a size feature's specified MMC or LMC material condition and the geometric tolerance for that material condition.

RC = Resultant Condition: The variable boundary generated by the collective effects of a size feature's specified MMC or LMC material condition, the geometric tolerance for that material condition, the size tolerance, and the additional geometric tolerance derived from the feature's departure from its specified material condition.

9.6 References

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