CHAPTER 2
MECHANICS OF MATERIALS

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2.1 INTRODUCTION

The fundamental problem of structural analysis is the prediction of the ability of machine components to provide reliable service under its applied loads and temperature. The basis of the solution is the calculation of certain performance indices, such as stress (force per unit area), strain (deformation per unit length), or gross deformation, which can then be compared to allowable values of these parameters. The allowable values of the parameters are determined by the component function (deformation constraints) or by the material limitations (yield strength, ultimate strength, fatigue strength, etc.). Further constraints on the allowable values of the performance indices are often imposed through the application of factors of safety.

This chapter, "Mechanics of Materials," deals with the calculation of performance indices under statically applied loads and temperature distributions. The extension of the theory to dynamically loaded structures, i.e., to the response of structures to shock and vibration loading, is treated elsewhere in this handbook.

The calculations of "Mechanics of Materials" are based on the concepts of force equilibrium (which relates the applied load to the internal reactions, or stress, in the body), material observation (which relates the stress at a point to the internal deformation, or strain, at the point), and kinematics (which relates the strain to the gross deformation of the body). In its simplest form, the solution assumes linear relationships between the components of stress and the components of strain (hookean material models) and that the deformations of the body are sufficiently small that linear relationships exist between the components of strain and the components of deformation. This linear elastic model of structural behavior remains the predominant tool used today for the design analysis of machine components, and is the principal subject of this chapter.

It must be noted that many materials retain considerable load-carrying ability when stressed beyond the level at which stress and strain remain proportional. The modification of the material model to allow for nonlinear relationships between stress and strain is the principal feature of the theory of plasticity. Plastic design allows more effective material utilization at the expense of an acceptable permanent deformation of the structure and smaller (but still controlled) design margins. Plastic design is often used in the design of civil structures, and in the analysis of machine structures under emergency load conditions. Practical introductions to the subject are presented in Refs. 6, 7, and 8.

Another important and practical extension of elastic theory includes a material model in which the stress-strain relationship is a function of time and temperature. This "creep" of components is an important consideration in the design of machines for use in a high-temperature environment. Reference 11 discusses the theory of creep design. The set of equations which comprise the linear elastic structural model do not have a comprehensive, exact solution for a general geometric shape. Two approaches are used to yield solutions:

> The geometry of the structure is simplified to a form for which an exact solution is available. Such simplified structures are generally characterized as being a level surface in the solution coordinate system. Examples of such simplified structures

include rods, beams, rectangular plates, circular plates, cylindrical shells, and spherical shells. Since these shapes are all level surfaces in different coordinate systems, e.g., a sphere is the surface r = constant in spherical coordinates, it is a great convenience to express the equations of linear elastic theory in a coordinate invariant form. General tensor notation is used to accomplish this task.

The governing equations are solved through numerical analysis on a case-by-case basis. This method is used when the component geometry is such that none of the available beam, rectangular plate, etc., simplifications are appropriate. Although several classes of numerical procedures are widely used, the predominant procedure for the solution of problems in the "Mechanics of Materials" is the finite-element method.

2.2 STRESS

2.2.1 Definition[2]

"Stress" is defined as the force per unit area acting on an "elemental" plane in the body. Engineering units of stress are generally pounds per square inch. If the force is normal to the plane the stress is termed "tensile" or "compressive," depending upon whether the force tends to extend or shorten the element. If the force acts parallel to the elemental plane, the stress is termed "shear." Shear tends to deform by causing neighboring elements to slide relative to one another.

2.2.2 Components of Stress[2]

A complete description of the internal forces (stress distributions) requires that stress be defined on three perpendicular faces of an interior element of a structure. In Fig. 2.1 a small element is shown, and, omitting higher-order effects, the stress resultant on any face can be considered as acting at the center of the area.

The direction and type of stress at a point are described by subscripts to the stress symbol σ or τ. The first subscript defines the plane on which the stress acts and the second indicates the direction in which it acts. The plane on which the stress acts is indicated by the normal axis to that plane; e.g., the x plane is normal to the x axis. Conventional notation omits the second subscript for the normal stress and replaces the σ by a τ for the shear stresses. The "stress components" can thus be represented as follows:

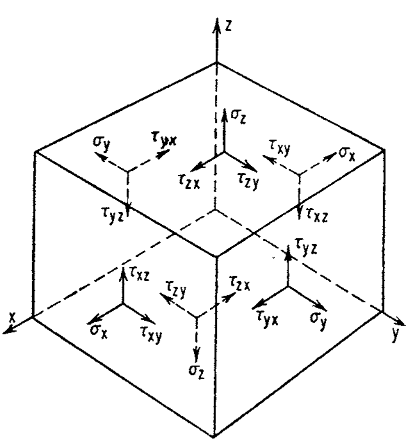


FIG. 2.1 Stress components.

Normal stress:

$$\sigma_{xx} \equiv \sigma_x \qquad \sigma_{yy} \equiv \sigma_y \qquad \sigma_{zz} \equiv \sigma_z \tag{2.1}$$

Shear stress:

$$\sigma_{xy} \equiv \tau_{xy} \qquad \sigma_{yz} \equiv \tau_{yz}$$

$$\sigma_{xz} \equiv \tau_{xz} \qquad \sigma_{zx} \equiv \tau_{zx} \tag{2.2}$$

$$\sigma_{yx} \equiv \tau_{yz} \qquad \sigma_{zy} \equiv \tau_{zy}$$

In tensor notation, the stress components are

$$\sigma_{ij} = \begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{pmatrix} \tag{2.3}$$

Stress is "positive" if it acts in the "positive-coordinate direction" on those element faces farthest from the origin, and in the "negative-coordinate direction" on those faces closest to the origin. Figure 2.1 indicates the direction of all positive stresses, wherein it is seen that tensile stresses are positive and compressive stresses negative.

The total load acting on the element of Fig. 2.1 can be completely defined by the stress components shown, subject only to the restriction that the coordinate axes are mutually orthogonal. Thus the three normal stress symbols σ_x, σ_y, σ_z and six shear-stress symbols τ_{xy}, τ_{xz}, τ_{yx}, τ_{yz}, τ_{zx}, τ_{zy} define the stresses of the element. However, from equilibrium considerations, $\tau_{xy} = \tau_{yx}$, $\tau_{yz} = \tau_{zy}$, $\tau_{xz} = \tau_{zx}$. This reduces the necessary number of symbols required to define the stress state to σ_x, σ_y, σ_z, τ_{xy}, τ_{xz}, τ_{yz}.

2.2.3 Simple Uniaxial States of Stress[1]

Consider a simple bar subjected to axial loads only. The forces acting at a transverse section are all directed normal to the section. The uniaxial normal stress at the section is obtained from

$$\sigma = P/A \tag{2.4}$$

where P = total force and A = cross-sectional area.

"Uniaxial shear" occurs in a circular cylinder, loaded as in Fig. 2.2*a,* with a radius which is large compared to the wall thickness. This member is subjected to a torque distributed about the upper edge:

$$T = \sum Pr \tag{2.5}$$

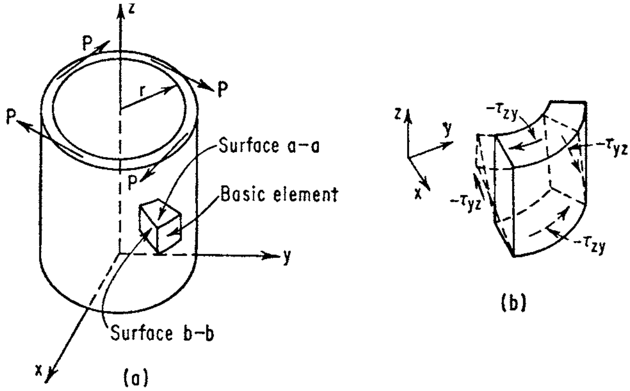


FIG. 2.2 Uniaxial shear basic element.

Now consider a surface element (assumed plane) and examine the stresses acting. The stresses τ which act on surfaces *a-a* and *b-b* in Fig. 2.2*b* tend to distort the original rectangular shape of the element into the parallelogram shown (dotted shape). This type of action of a force along or tangent to a surface produces shear within the element, the intensity of which is the "shear stress."

2.2.4 Nonuniform States of Stress[1]

In considering elements of differential size, it is permissible to assume that the force acts on any side of the element concentrated at the center of the area of that side, and that the stress is the average force divided by the side area. Hence it has been implied thus far that the stress is uniform. In members of finite size, however, a variable stress intensity usually exists across any given surface of the member. An example of a body which develops a distributed stress pattern across a transverse cross section is a simple beam subjected to a bending load as shown in Fig. 2.3*a*. If a section is then taken at *a-a*, F'_1 must be the internal force acting along *a-a* to maintain equilibrium. Forces F_1 and F'_1 constitute a couple which tends to rotate the element in a clockwise direction, and therefore a resisting couple must be developed at *a-a* (see Fig. 2.3*b*). The internal effect at *a-a* is a stress distribution with the upper portion of the beam in tension and the lower portion in compression, as in Fig. 2.3*c*. The line of zero stress on the transverse cross section is the "neutral axis" and passes through the centroid of the area.

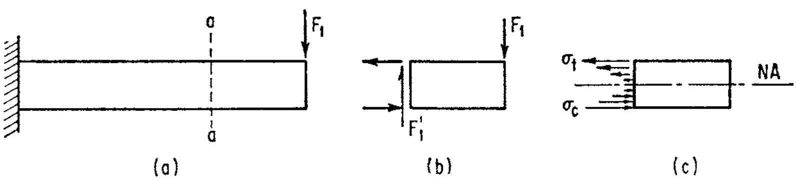


FIG. 2.3 Distributed stress on a simple beam subjected to a bending load.

2.2.5 Combined States of Stress

Tension-Torsion. A body loaded simultaneously in direct tension and torsion, such as a rotating vertical shaft, is subject to a combined state of stress. Figure 2.4*a* depicts such a shaft with end load *W*, and constant torque *T* applied to maintain uniform rotational velocity. With reference to *a-a,* considering each load separately, a force system

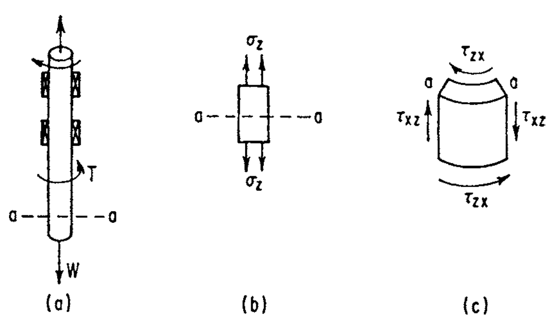


FIG. 2.4 Body loaded in direct tension and torsion.

as shown in Fig. 2.2*b* and *c* is developed at the internal surface *a-a* for the weight load and torque, respectively. These two stress patterns may be superposed to determine the "combined" stress situation for a shaft element.

Flexure-Torsion. If in the above case the load W were horizontal instead of vertical, the combined stress picture would be altered. From previous considerations of a simple beam, the stress distribution varies linearly across section *a-a* of the shaft of Fig. 2.5*a*. The stress pattern due to flexure then depends upon the location of the element in question; e.g., if the element is at the outside (element x) then it is undergoing maximum tensile stress (Fig. 2.5*b*), and the tensile stress is zero if the element is located on the horizontal center line (element y) (Fig. 2.5*c*). The shearing stress is still constant at a given element, as before (Fig. 2.5*d*). Thus the "combined" or "superposed" stress state for this condition of loading varies across the entire transverse cross section.

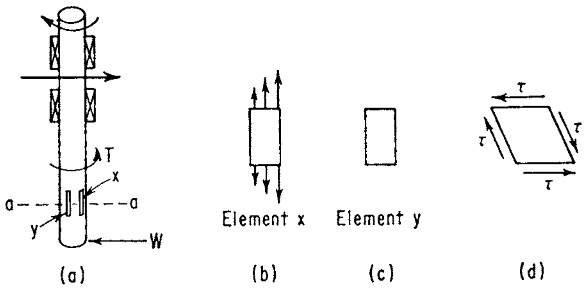


FIG. 2.5 Body loaded in flexure and torsion.

2.2.6 Stress Equilibrium

"Equilibrium" relations must be satisfied by each element in a structure. These are satisfied if the resultant of all forces acting on each element equals zero in each of three mutually orthogonal directions on that element. The above applies to all situations of "static equilibrium." In the event that some elements are in motion an inertia term must be added to the equilibrium equation. The inertia term is the elemental mass multiplied by the absolute acceleration taken along each of the mutually perpendicular axes. The equations which specify this latter case are called "dynamic-equilibrium equations" (see Chap. 4).

Three-Dimensional Case.[5,13] The equilibrium equations can be derived by separately summing all x, y, and z forces acting on a differential element accounting for the incremental variation of stress (see Fig. 2.6). Thus the normal forces acting on areas $dz\,dy$ are $\sigma_x\,dz\,dy$ and $[\sigma_x + (\partial\sigma_x/\partial x)\,dx]\,dz\,dy$. Writing x force-equilibrium equations, and by a similar process y and z force-equilibrium equations, and canceling *higher-order* terms, the following three "cartesian equilibrium equations" result:

$$\partial\sigma_x/\partial x + \partial\tau_{xy}/\partial y + \partial\tau_{xz}/\partial z = 0 \tag{2.6}$$

$$\partial\sigma_y/\partial y + \partial\tau_{yz}/\partial z + \partial\tau_{yx}/\partial x = 0 \tag{2.7}$$

$$\partial\sigma_z/\partial z + \partial\tau_{zx}/\partial x + \partial\tau_{zy}/\partial y = 0 \tag{2.8}$$

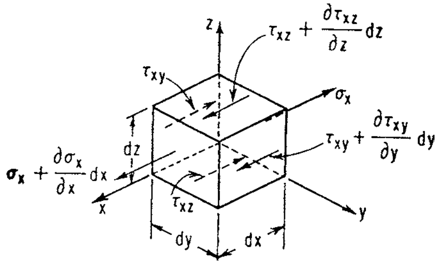


FIG. 2.6 Incremental element (dx, dy, dz) with incremental variation of stress.

or, in cartesian stress-tensor notation,

$$\sigma_{ij,j} = 0 \qquad i,j = x,y,z \tag{2.9}$$

and, in general tensor form,

$$g^{ik}\sigma_{ij,k} = 0 \tag{2.10}$$

where g^{ik} is the contravariant metric tensor.

"Cylindrical-coordinate" equilibrium considerations lead to the following set of equations (Fig. 2.7):

$$\partial\sigma_r/\partial r + (1/r)(\partial\tau_{r\theta}/\partial\theta) + \partial\tau_{rz}/\partial z + (\sigma_r - \sigma_\theta)/r = 0 \tag{2.11}$$

$$\partial\tau_{r\theta}/\partial r + (1/r)(\partial\sigma_\theta/\partial\theta) + \partial\tau_{\theta z}/\partial z + 2\tau_{r\theta}/r = 0 \tag{2.12}$$

$$\partial\tau_{rz}/\partial r + (1/r)(\partial\tau_{\theta z}/\partial\theta) + \partial_{\sigma z}/\partial z + \tau_{rz}/r = 0 \tag{2.13}$$

The corresponding "spherical polar-coordinate" equilibrium equations are (Fig. 2.8)

$$\frac{\partial\sigma_r}{\partial r} + \frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta} + \left(\frac{1}{r\sin\theta}\right)\frac{\partial\tau_{r\phi}}{\partial\phi} + \frac{1}{r}(2\sigma_r - \sigma_\theta - \sigma_\phi + \tau_{r\theta}\cot\theta) = 0 \tag{2.14}$$

$$\frac{\partial\tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta} + \left(\frac{1}{r\sin\theta}\right)\frac{\partial\tau_{\theta\phi}}{\partial\phi} + \frac{1}{r}[(\sigma_\theta - \sigma_\phi)\cot\theta + 3\tau_{r\theta}] = 0 \tag{2.15}$$

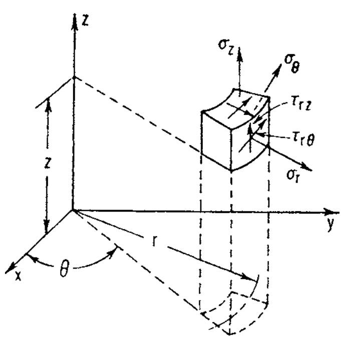


FIG. 2.7 Stresses on a cylindrical element.

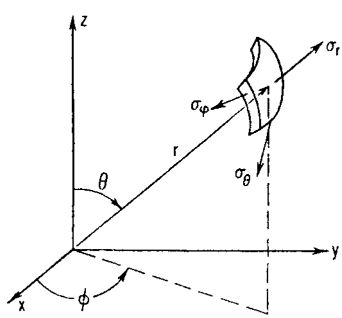


FIG. 2.8 Stresses on a spherical element.

$$\frac{\partial \tau_{r\phi}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\phi}}{\partial \theta} + \left(\frac{1}{r \sin \theta}\right)\frac{\partial \sigma_{\phi}}{\partial \phi} + \frac{1}{r}(3\tau_{r\phi} + 2\tau_{\theta\phi} \cot \theta) = 0 \qquad (2.16)$$

The general orthogonal curvilinear-coordinate equilibrium equations are

$$h_1h_2h_3\left(\frac{\partial}{\partial \alpha}\frac{\sigma_\alpha}{h_2h_3} + \frac{\partial}{\partial \beta}\frac{\tau_{\alpha\beta}}{h_3h_1} + \frac{\partial}{\partial \gamma}\frac{\tau_{\gamma\alpha}}{h_1h_2}\right) + \tau_{\alpha\beta}h_1h_2\frac{\partial}{\partial \beta}\frac{1}{h_1}$$

$$+ \tau_{\gamma\alpha}h_1h_3\frac{\partial}{\partial \gamma}\frac{1}{h_1} - \sigma_\beta h_1h_2\frac{\partial}{\partial \alpha}\frac{1}{h_2} - \sigma_\gamma h_1h_3\frac{\partial}{\partial \alpha}\frac{1}{h_3} = 0 \qquad (2.17)$$

$$h_1h_2h_3\left(\frac{\partial}{\partial \beta}\frac{\sigma_\beta}{h_3h_1} + \frac{\partial}{\partial \gamma}\frac{\tau_{\beta\gamma}}{h_1h_2} + \frac{\partial}{\partial \alpha}\frac{\tau_{\alpha\beta}}{h_2h_3}\right) + \tau_{\beta\gamma}h_2h_3\frac{\partial}{\partial \gamma}\frac{1}{h_2}$$

$$+ \tau_{\alpha\beta}h_2h_1\frac{\partial}{\partial \alpha}\frac{1}{h_2} - \sigma_\gamma h_2h_3\frac{\partial}{\partial \beta}\frac{1}{h_3} - \sigma_\alpha h_2h_1\frac{\partial}{\partial \beta}\frac{1}{h_1} = 0 \qquad (2.18)$$

$$h_1h_2h_3\left(\frac{\partial}{\partial \gamma}\frac{\sigma_\gamma}{h_1h_2} + \frac{\partial}{\partial \alpha}\frac{\tau_{\gamma\alpha}}{h_2h_3} + \frac{\partial}{\partial \beta}\frac{\tau_{\beta\gamma}}{h_3h_1}\right) + \tau_{\gamma\alpha}h_3h_1\frac{\partial}{\partial \alpha}\frac{1}{h_3}$$

$$+ \tau_{\beta\gamma}h_3h_2\frac{\partial}{\partial \beta}\frac{1}{h_3} - \sigma_\alpha h_3h_1\frac{\partial}{\partial \gamma}\frac{1}{h_1} - \sigma_\beta h_3h_2\frac{\partial}{\partial \gamma}\frac{1}{h_2} = 0 \qquad (2.19)$$

where the α, β, γ specify the coordinates of a point and the distance between two coordinate points ds is specified by

$$(ds)^2 = (d\alpha/h_1)^2 + (d\beta/h_2)^2 + (d\gamma/h_3)^2 \qquad (2.20)$$

which allows the determination of h_1, h_2, and h_3 in any specific case.

Thus, in cylindrical coordinates,

$$(ds)^2 = (dr)^2 + (r\,d\theta)^2 + (dz)^2 \qquad (2.21)$$

so that

$$\alpha = r \qquad h_1 = 1$$
$$\beta = \theta \qquad h_2 = 1/r$$
$$\gamma = z \qquad h_3 = 1$$

In spherical polar coordinates,

$$(ds)^2 = (dr)^2 + (r\,d\theta)^2 + (r \sin \theta\, d\phi)^2 \qquad (2.22)$$

so that

$$\alpha = r \qquad h_1 = 1$$
$$\beta = \theta \qquad h_2 = 1/r$$
$$\gamma = \phi \qquad h_3 = 1/(r \sin \theta)$$

All the above equilibrium equations define the conditions which must be satisfied by each interior element of a body. In addition, these stresses must satisfy all surface-stress-boundary conditions. In addition to the cartesian-, cylindrical-, and spherical-coordinate systems, others may be found in the current literature or obtained by reduction from the general curvilinear-coordinate equations given above.

In many applications it is useful to integrate the stresses over a finite thickness and express the resultant in terms of zero or nonzero force or moment resultants as in the beam, plate, or shell theories.

Two-Dimensional Case—Plane Stress.[2] In the special but useful case where the stresses in one of the coordinate directions are negligibly small ($\sigma_z = \tau_{xz} = \tau_{yz} = 0$) the general cartesian-coordinate equilibrium equations reduce to

$$\partial\sigma_x/\partial x + \partial\tau_{xy}/\partial y = 0 \tag{2.23}$$

$$\partial\sigma_y/\partial y + \partial\tau_{yx}/\partial x = 0 \tag{2.24}$$

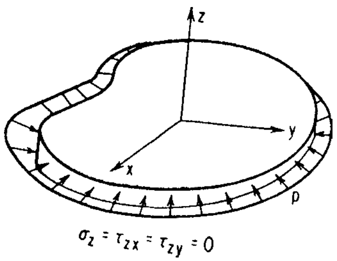


FIG. 2.9 Plane stress on a thin slab.

The corresponding cylindrical-coordinate equilibrium equations become

$$\partial\sigma_r/\partial r + (1/r)(\partial\tau_{r\theta}/\partial\theta) + (\sigma_r - \sigma_\theta)/r = 0 \tag{2.25}$$

$$\partial\tau_{r\theta}/\partial r + (1/r)(\partial\sigma_\theta/\partial\theta) + 2(\tau_{r\theta}/r) = 0 \tag{2.26}$$

This situation arises in "thin slabs," as indicated in Fig. 2.9, which are essentially two-dimensional problems. Because these equations are used in formulations which allow only stresses in the "plane" of the slab, they are classified as "plane-stress" equations.

2.2.7 Stress Transformation: Three-Dimensional Case[4,5]

It is frequently necessary to determine the stresses at a point in an element which is rotated with respect to the x, y, z coordinate system, i.e., in an orthogonal x', y', z' system. Using equilibrium concepts and measuring the angle between any specific original and rotated coordinate by the direction cosines (cosine of the angle between the two axes) the following transformation equations result:

$$\begin{aligned}\sigma_{x'} = {} & [\sigma_x \cos(x'x) + \tau_{xy}\cos(x'y) + \tau_{zx}\cos(x'z)]\cos(x'x)\\ & + [\tau_{xy}\cos(x'x) + \sigma_y\cos(x'y) + \tau_{yz}\cos(x'z)]\cos(x'y)\\ & + [\tau_{zx}\cos(x'x) + \tau_{yz}\cos(x'y) + \sigma_z\cos(x'z)]\cos(x'z)\end{aligned} \tag{2.27}$$

$$\begin{aligned}\sigma_{y'} = {} & [\sigma_x \cos(y'x) + \tau_{xy}\cos(y'y) + \tau_{zx}\cos(y'z)]\cos(y'x)\\ & + [\tau_{xy}\cos(y'x) + \sigma_y\cos(y'y) + \tau_{yz}\cos(y'z)]\cos(y'y)\\ & + [\tau_{zx}\cos(y'x) + \tau_{yz}\cos(y'y) + \sigma_z\cos(y'z)]\cos(y'z)\end{aligned} \tag{2.28}$$

$$\begin{aligned}\sigma_{z'} = {} & [\sigma_x \cos(z'x) + \tau_{xy}\cos(z'y) + \tau_{zx}\cos(z'z)]\cos(z'x)\\ & + [\tau_{xy}\cos(z'x) + \sigma_y\cos(z'y) + \tau_{yz}\cos(z'z)]\cos(z'y)\\ & + [\tau_{zx}\cos(z'x) + \tau_{yz}\cos(z'y) + \sigma_z\cos(z'z)]\cos(z'z)\end{aligned} \tag{2.29}$$

$$\begin{aligned}\tau_{x'y'} = {} & [\sigma_x \cos(y'x) + \tau_{xy}\cos(y'y) + \tau_{zx}\cos(y'z)]\cos(x'x)\\ & + [\tau_{xy}\cos(y'x) + \sigma_y\cos(y'y) + \tau_{yz}\cos(y'z)]\cos(x'y)\\ & + [\tau_{zx}\cos(y'x) + \tau_{yz}\cos(y'y) + \sigma_z\cos(y'z)]\cos(x'z)\end{aligned} \tag{2.30}$$

$$\tau_{y'z'} = [\sigma_x \cos(z'x) + \tau_{xy} \cos(z'y) + \tau_{zx} \cos(z'z)] \cos(y'x)$$
$$+ [\tau_{xy} \cos(z'x) + \sigma_y \cos(z'y) + \tau_{yz} \cos(z'z)] \cos(y'y)$$
$$+ [\tau_{zx} \cos(z'x) + \tau_{yz} \cos(z'y) + \sigma_z \cos(z'z)] \cos(y'z) \tag{2.31}$$

$$\tau_{z'x'} = [\sigma_x \cos(x'x) + \tau_{xy} \cos(x'y) + \tau_{zx} \cos(x'z)] \cos(z'x)$$
$$+ [\tau_{xy} \cos(x'x) + \sigma_y \cos(x'y) + \tau_{yz} \cos(x'z)] \cos(z'y)$$
$$+ [\tau_{zx} \cos(x'x) + \tau_{yz} \cos(x'y) + \sigma_z \cos(x'z)] \cos(z'z) \tag{2.32}$$

In tensor notation these can be abbreviated as

$$\tau_{k'l'} = A_{l'n} A_{k'm} \tau_{mn} \tag{2.33}$$

where $A_{ij} = \cos(ij) \qquad m,n \rightarrow x,y,z \qquad k',l' \rightarrow x',y',z'$

A special but very useful coordinate rotation occurs when the direction cosines are so selected that all the shear stresses vanish. The remaining mutually perpendicular "normal stresses" are called "principal stresses."

The magnitudes of the principal stresses $\sigma_{x'}$, $\sigma_{y'}$, $\sigma_{z'}$ are the three roots of the cubic equations associated with the determinant

$$\begin{vmatrix} \sigma_x - \sigma & \tau_{xy} & \tau_{zx} \\ \tau_{xy} & \sigma_y - \sigma & \tau_{yz} \\ \tau_{zx} & \tau_{yz} & \sigma_z - \sigma \end{vmatrix} = 0 \tag{2.34}$$

where $\sigma_x, \ldots, \tau_{xy}, \ldots$ are the general nonprincipal stresses which exist on an element.

The direction cosines of the principal axes x', y' z' with respect to the x, y, z axes are obtained from the simultaneous solution of the following three equations considering separately the cases where $n = x'$, y' z':

$$\tau_{xy} \cos(xn) + (\sigma_y - \sigma_n) \cos(yn) + \tau_{yz} \cos(zn) = 0 \tag{2.35}$$

$$\tau_{zx} \cos(xn) + \tau_{yz} \cos(yn) + (\sigma_z - \sigma_n) \cos(zn) = 0 \tag{2.36}$$

$$\cos^2(xn) + \cos^2(yn) + \cos^2(zn) = 1 \tag{2.37}$$

2.2.8 Stress Transformation: Two-Dimensional Case[2,4]

Selecting an arbitrary coordinate direction in which the stress components vanish, it can be shown, either by equilibrium considerations or by general transformation formulas, that the two-dimensional stress-transformation equations become

$$\sigma_n = [(\sigma_x + \sigma_y)/2] + [(\sigma_x - \sigma_y)/2] \cos 2\alpha + \tau_{xy} \sin 2\alpha \tag{2.38}$$

$$\tau_{nt} = [(\sigma_x - \sigma_y)/2] \sin 2\alpha - \tau_{xy} \cos 2\alpha \tag{2.39}$$

where the directions are defined in Figs. 2.10 and 2.11 ($\tau_{xy} = -\tau_{nt}$, $\alpha = 0$).

The principal directions are obtained from the condition that

$$\tau_{nt} = 0 \qquad \text{or} \qquad \tan 2\alpha = 2\tau_{xy}/(\sigma_x - \sigma_y) \tag{2.40}$$

where the two lowest roots of (first and second quadrants) are taken. It can be easily seen that the first and second principal directions differ by 90°. It can be shown that the *principal* stresses are also the "maximum" or "minimum normal stresses." The "plane of maximum shear" is defined by

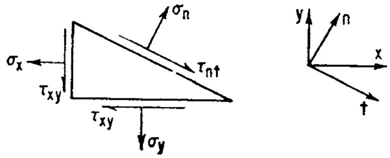


FIG. 2.10 Two-dimensional plane stress.

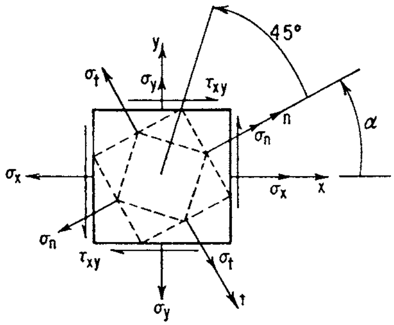


FIG. 2.11 Plane of maximum shear.

$$\tan 2\alpha = -(\sigma_x - \sigma_y)/2\tau_{xy} \tag{2.41}$$

These are also represented by planes which are 90° apart and are displaced from the principal stress planes by 45° (Fig. 2.11).

2.2.9 Mohr's Circle

Mohr's circle is a convenient representation of the previously indicated transformation equations. Considering the *x, y* directions as positive in Fig. 2.11, the stress condition on any elemental plane can be represented as a point in the "Mohr diagram" (clockwise shear taken positive). The Mohr's circle is constructed by connecting the two stress points and drawing a circle through them with center on the σ axis. The stress state of any basic element can be represented by the stress coordinates at the intersection of the circle with an arbitrarily directed line through the circle center. Note that point *x* for positive τ_{xy} is below the σ axis and vice versa. The element is taken as rotated counterclockwise by an angle α with respect to the *x-y* element when the line is rotated counterclockwise an angle 2α with respect to the *x-y* line, and vice versa (Fig. 2.12).

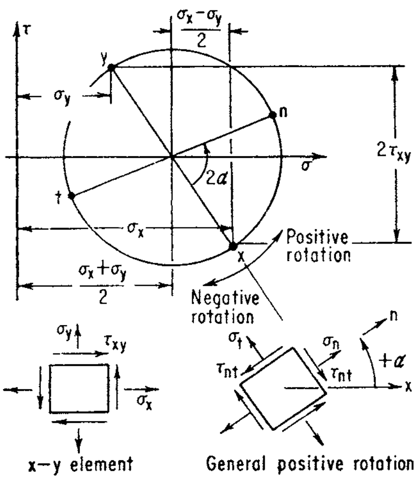


FIG. 2.12 Stress state of basic element.

2.3 STRAIN

2.3.1 Definition[2]

Extensional strain ϵ is defined as the extensional deformation of an element divided by the basic elemental length, $\epsilon = u/l_0$.

In large-strain considerations, l_0 must represent the instantaneous elemental length and the definitions of strain must be given in incremental fashion. In small strain considerations, to which the following discussion is limited, it is only necessary to consider the original elemental length l_0 and its change of length u. Extensional strain is taken positive or negative depending on whether the element increases or decreases in extent. The units of strain are dimensionless (inches/inch).

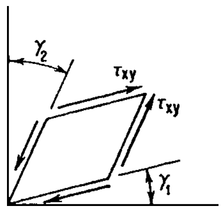


FIG. 2.13 Shear-strain-deformed element.

"Shear strain" γ is defined as the angular distortion of an original right-angle element. The direction of positive shear strain is taken to correspond to that produced by a positive shear stress (and vice versa) (see Fig. 2.13). Shear strain γ is equal to $\gamma_1 + \gamma_2$. The "units" of shear strain are dimensionless (radians).

2.3.2 Components of Strain[2]

A complete description of strain requires the establishment of three orthogonal extensional and shear strains. In cartesian stress nomenclature, the strain components are

Extensional strain:

$$\begin{aligned} \epsilon_{xx} &\equiv \epsilon_x \\ \epsilon_{yy} &\equiv \epsilon_y \\ \epsilon_{zz} &\equiv \epsilon_z \end{aligned} \tag{2.42}$$

Shear strain:

$$\begin{aligned} \epsilon_{xy} &= \epsilon_{yx} \equiv \tfrac{1}{2}\gamma_{xy} \\ \epsilon_{yz} &= \epsilon_{zy} \equiv \tfrac{1}{2}\gamma_{yz} \\ \epsilon_{zx} &= \epsilon_{xz} \equiv \tfrac{1}{2}\gamma_{zx} \end{aligned} \tag{2.43}$$

where positive ϵ_x, ϵ_y, or ϵ_z corresponds to a positive stretching in the x, y, z directions and positive γ_{xy}, γ_{yz}, γ_{zx} refers to positive shearing displacements in the xy, yz, and zx planes. In tensor notation, the strain components are

$$\epsilon_{ij} = \begin{pmatrix} \epsilon_x & \tfrac{1}{2}\gamma_{xy} & \tfrac{1}{2}\gamma_{zx} \\ \tfrac{1}{2}\gamma_{xy} & \epsilon_y & \tfrac{1}{2}\gamma_{yz} \\ \tfrac{1}{2}\gamma_{zx} & \tfrac{1}{2}\gamma_{yz} & \epsilon_z \end{pmatrix} \tag{2.44}$$

2.3.3 Simple and Nonuniform States of Strain[2]

Corresponding to each of the stress states previously illustrated there exists either a simple or nonuniform strain state.

In addition to these, a state of "uniform dilatation" exists when the shear strain vanishes and all the extensional strains are equal in sign and magnitude. Dilatation is defined as

$$\Delta = \epsilon_x + \epsilon_y + \epsilon_z \tag{2.45}$$

and represents the change of volume per increment volume.

In uniform dilatation,

$$\Delta = 3\epsilon_x = 3\epsilon_y = 3\epsilon_z \tag{2.46}$$

2.3.4 Strain-Displacement Relationships[4,5,13]

Considering only small strain, and the previous definitions, it is possible to express the strain components at a point in terms of the associated displacements and their derivatives in the coordinate directions (e.g., *u, v, w* are displacements in the *x, y, z* coordinate system).

Thus, in a "cartesian system" (*x, y, z*),

$$\begin{aligned} \epsilon_x &= \partial u/\partial x & \gamma_{xy} &= \partial v/\partial x + \partial u/\partial y \\ \epsilon_y &= \partial v/\partial y & \gamma_{yz} &= \partial w/\partial y + \partial v/\partial z \\ \epsilon_z &= \partial w/\partial z & \gamma_{zx} &= \partial u/\partial z + \partial w/\partial x \end{aligned} \tag{2.47}$$

or, in stress-tensor notation,

$$2\epsilon_{ij} = u_{i,j} + u_{j,i} \qquad i,j \rightarrow x,y,z \tag{2.48}$$

In addition the dilatation

$$\Delta = \partial u/\partial x + \partial v/\partial y + \partial w/\partial z \tag{2.49}$$

or, in tensor form,

$$\Delta = u_{i,j} \qquad i \rightarrow x,y,z \tag{2.50}$$

Finally, all incremental displacements can be composed of a "pure strain" involving all the above components, plus "rigid-body" rotational components. That is, in general

$$U = \epsilon_x X + \tfrac{1}{2}\gamma_{xy} Y + \tfrac{1}{2}\gamma_{zx} Z - \overline{\omega}_z Y + \overline{\omega}_y Z \tag{2.51}$$

$$V = \tfrac{1}{2}\gamma_{xy} X + \epsilon_y Y + \tfrac{1}{2}\gamma_{yz} Z - \overline{\omega}_x Z + \overline{\omega}_z X \tag{2.52}$$

$$W = \tfrac{1}{2}\gamma_{zx} X + \tfrac{1}{2}\gamma_{yz} Y + \epsilon_z Z - \overline{\omega}_y X + \overline{\omega}_x Y \tag{2.53}$$

where *U, V, W* represent the incremental displacement of the point $x + X$, $y + Y$, $z + Z$ in excess of that of the point *x, y, z* where *X, Y, Z* are taken as the sides of the incremental element. The rotational components are given by

$$\begin{aligned} 2\overline{\omega}_x &= \partial w/\partial y - \partial v/\partial z \\ 2\overline{\omega}_y &= \partial u/\partial z - \partial w/\partial x \\ 2\overline{\omega}_z &= \partial v/\partial x - \partial u/\partial y \end{aligned} \tag{2.54}$$

or, in tensor notation,

$$2\overline{\omega}_{ij} = u_{i,j} - u_{j,i} \qquad i,j = x,y,z \tag{2.55}$$

$$\overline{\omega}_{zy} \equiv \overline{\omega}_x,\ \overline{\omega}_{xz} \equiv \overline{\omega}_y,\ \omega_{yx} \equiv \overline{\omega}_z$$

In *cylindrical coordinates,*

$$\begin{aligned} \epsilon_r &= \partial u_r/\partial r & \gamma_{\theta z} &= (1/r)(\partial u_z/\partial\theta + \partial u_\theta/\partial z \\ \epsilon_\theta &= (1/r)(\partial u_\theta/\partial\theta) + u_r/r & \gamma_{zr} &= \partial u_r/\partial z + \partial u_z/\partial r \\ \epsilon_z &= \partial u_z/\partial z & \gamma_{r\theta} &= \partial u_\theta/\partial r - u_\theta/r + (1/r)(\partial u_r/\partial\theta) \end{aligned} \tag{2.56}$$

The dilatation is

$$\Delta = (1/r)(\partial/\partial r)(ru_r) + (1/r)(\partial u_\theta/\partial\theta) + \partial u_z/\partial z \tag{2.57}$$

and the rotation components are

$$\begin{aligned} 2\overline{\omega}_r &= (1/r)(\partial u_z/\partial\theta) - \partial u_\theta/\partial z \\ 2\overline{\omega}_\theta &= \partial u_r/\partial z - \partial u_z/\partial r \\ 2\overline{\omega}_z &= (1/r)(\partial/\partial r)(ru_\theta) - (1/r)(\partial u_r/\partial\theta) \end{aligned} \tag{2.58}$$

In *spherical polar coordinates,*

$$\begin{aligned} \epsilon_r &= \frac{\partial u_r}{\partial r} & \gamma_{\theta\phi} &= \frac{1}{r}\left[\frac{\partial u_\phi}{\partial\theta} - u_\phi \cot\theta\right] + \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi} \\ \epsilon_\theta &= \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r} & \gamma\,\phi_r &= \frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r} \\ \epsilon_\phi &= \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi} + \frac{u_\theta}{r}\cot\theta + \frac{u_r}{r} & \gamma_{r\theta} &= \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial\theta} \end{aligned} \tag{2.59}$$

The dilatation is

$$\Delta = (1/r^2 \sin\theta)[(\partial/\partial r)(r^2 u_r \sin\theta) + (\partial/\partial\theta)(ru_\theta \sin\theta) + (\partial/\partial\phi)(ru_\phi)] \tag{2.60}$$

The rotation components are

$$\begin{aligned} 2\overline{\omega}_r &= (1/r^2 \sin\theta)[(\partial/\partial\theta)(ru_\phi \sin\theta) - (\partial/\partial\phi)(ru_\theta)] \\ 2\overline{\omega}_\theta &= (1/r\sin\theta)[\partial u_r/\partial\phi - (\partial/\partial r)(ru_\phi \sin\theta)] \\ 2\overline{\omega}_\phi &= (1/r)[(\partial/\partial r)(ru_\theta) - \partial u_r/\partial\theta] \end{aligned} \tag{2.61}$$

In general *orthogonal curvilinear coordinates,*

$$\begin{aligned} \epsilon_\alpha &= h_1(\partial u_\alpha/\partial\alpha) + h_1 h_2 u_\beta(\partial/\partial\beta)(1/h_1) + h_3 h_1 u_\gamma(\partial/\partial\gamma)(1/h_1) \\ \epsilon_\beta &= h_2(\partial u_\beta/\partial\beta) + h_2 h_3 u_\gamma(\partial/\partial\gamma)(1/h_2) + h_1 h_2 u_\alpha(\partial/\partial\alpha)(1/h_2) \\ \epsilon_\gamma &= h_3(\partial u_\gamma/\partial\gamma) + h_3 h_1 u_\alpha(\partial/\partial\alpha)(1/h_3) + h_2 h_3 u_\beta(\partial/\partial\beta)(1/h_3) \\ \gamma_{\beta\gamma} &= (h_2/h_3)(\partial/\partial\beta)(h_3 u_\gamma) + (h_3/h_2)(\partial/\partial\gamma)(h_2 u_\beta) \\ \gamma_{\gamma\alpha} &= (h_3/h_1)(\partial/\partial\gamma)(h_1 u_\alpha) + (h_1/h_3)(\partial/\partial\alpha)(h_3 u_\gamma) \\ \gamma_{\alpha\beta} &= (h_1/h_2)(\partial/\partial\alpha)(h_2 u_\beta) + (h_2/h_1)(\partial/\partial\beta)(h_1 u_\alpha) \end{aligned} \tag{2.62}$$

$$\Delta = h_1 h_2 h_3[(\partial/\partial\alpha)(u_\alpha/h_2 h_3) + (\partial/\partial\beta)(u_\beta/h_3 h_1) + (\partial/\partial\gamma)(u_\gamma/h_1 h_2)] \tag{2.63}$$

$$\begin{aligned} 2\overline{\omega}_\alpha &= h_2 h_3[(\partial/\partial\beta)(u_\gamma/h_3) - (\partial/\partial\gamma)(u_\beta/h_2)] \\ 2\overline{\omega}_\beta &= h_3 h_1[(\partial/\partial\gamma)(u_\alpha/h_1) - (\partial/\partial\alpha)(u_\gamma/h_3)] \\ 2\overline{\omega}_\gamma &= h_1 h_2[(\partial/\partial\alpha)(u_\beta/h_2) - (\partial/\partial\beta)(u_\alpha/h_1)] \end{aligned} \tag{2.64}$$

where the quantities h_1, h_2, h_3 have been discussed with reference to the equilibrium equations.

In the event that one deflection (i.e., w) is constant or zero and the displacements are a function of x, y only, a special and useful class of problems arises termed "plane strain," which are analogous to the "plane-stress" problems. A typical case of plane strain occurs in slabs rigidly clamped on their faces so as to restrict all axial deformation. Although all the stresses may be nonzero, and the general equilibrium equations apply, it can be shown that, after combining all the necessary stress and strain relationships, both classes of *plane* problems yield the same form of equations. From this, one solution suffices for both the related *plane-stress* and *plane-strain* problems, provided that the elasticity constants are suitably modified. In particular the applicable strain-displacement relationships reduce in *cartesian coordinates* to

$$\begin{aligned} \epsilon_x &= \partial u/\partial x \\ \epsilon_y &= \partial v/\partial y \\ \gamma_{xy} &= \partial v/\partial x + \partial u/\partial y \end{aligned} \tag{2.65}$$

and in *cylindrical coordinates* to

$$\begin{aligned} \epsilon_r &= \partial u_r/\partial r \\ \epsilon_\theta &= (1/r)(\partial u_\theta/\partial\theta) + u_r/r \\ \gamma_{r\theta} &= \partial u_\theta/\partial r - u_\theta/r + (1/r)(\partial u_r/\partial\theta) \end{aligned} \tag{2.66}$$

2.3.5 Compatibility Relationships[2,4,5]

In the event that a single-valued continuous-displacement field (u, v, w) is not explicitly specified, it becomes necessary to ensure its existence in solution of the stress, strain, and stress-strain relationships. By writing the strain-displacement relationships and manipulating them to eliminate displacements, it can be shown that the following six equations are both necessary and sufficient to ensure compatibility:

$$\begin{aligned} &\partial^2\epsilon_y/\partial z^2 + \partial^2\epsilon_z/\partial y^2 = \partial^2\gamma_{yz}/\partial y\,\partial z \\ &2(\partial^2\epsilon_x/\partial y\,\partial z) = (\partial/\partial x)(-\partial\gamma_{yz}/\partial x + \partial\gamma_{zx}/\partial y + \partial\gamma_{xy}/\partial z) \end{aligned} \tag{2.67}$$

$$\begin{aligned} &\partial^2\epsilon_z/\partial x^2 + \partial^2\epsilon_x/\partial z^2 = \partial^2\gamma_{zx}/\partial x\,\partial z \\ &2(\partial^2\epsilon_y/\partial z\,\partial x) = (\partial/\partial y)(+\partial\gamma_{yz}/\partial x - \partial\gamma_{zx}/\partial y + \partial y_{xy}/\partial z) \end{aligned} \tag{2.68}$$

$$\begin{aligned} &\partial^2\epsilon_x/\partial y^2 + \partial^2\epsilon_y/\partial x^2 = \partial^2\gamma_{xy}/\partial x\,\partial y \\ &2(\partial^2\epsilon_z/\partial x\,\partial y) = (\partial/\partial z)(+\partial\gamma_{yz}/\partial x + \partial\gamma_{zx}/\partial y - \partial\gamma_{xy}/\partial z) \end{aligned} \tag{2.69}$$

In tensor notation the most general compatibility equations are

$$\epsilon_{ij,kl} + \epsilon_{kl,ij} - \epsilon_{ik,jl} - \epsilon_{jl,ik} = 0 \qquad i,j,k,l = x,y,z \tag{2.70}$$

which represents 81 equations. Only the above six equations are essential.

In addition to satisfying these conditions everywhere in the body under consideration, it is also necessary that all *surface strain* or *displacement boundary conditions* be satisfied.

2.3.6 Strain Transformation[4,5]

As with stress, it is frequently necessary to refer strains to a rotated orthogonal coordinate system (x', y', z'). In this event it can be shown that the stress and strain tensors transform in an identical manner.

$$\sigma_{x'} \rightarrow \epsilon_{x'} \qquad \sigma_x \rightarrow \epsilon_x \qquad \tau_{x'y'} \rightarrow \tfrac{1}{2}\gamma_{x'y'} \qquad \tau_{xy} \rightarrow \tfrac{1}{2}\gamma_{xy}$$

$$\sigma_{y'} \rightarrow \epsilon_{y'} \qquad \sigma_y \rightarrow \epsilon_y \qquad \tau_{y'z'} \rightarrow \tfrac{1}{2}\gamma_{y'z'} \qquad \tau_{yz} \rightarrow \tfrac{1}{2}\gamma_{yz}$$

$$\sigma_{z'} \rightarrow \epsilon_{z'} \qquad \sigma_z \rightarrow \epsilon_z \qquad \tau_{z'x'} \rightarrow \tfrac{1}{2}\gamma_{z'x'} \qquad \tau_{zx} \rightarrow \tfrac{1}{2}\gamma_{zx}$$

In tensor notation the strain transformation can be written as

$$e_{k'l'} = A_{l'n} A_{k'm} \epsilon_{mn} \qquad \begin{matrix} m, n \rightarrow x, y, z \\ l'k' \rightarrow x', y' z' \end{matrix} \tag{2.71}$$

As a result the stress and strain principal directions are coincident, so that all remarks made for the principal stress and maximum shear components and their directions

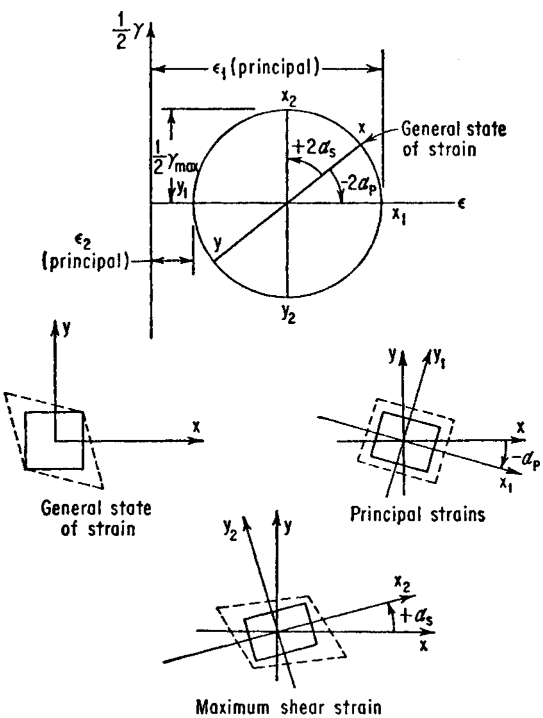


FIG. 2.14 Strain transformation.

apply equally well to strain tensor components. Note that in the use of Mohr's circle in the two-dimensional case one must be careful to substitute $\frac{1}{2}\gamma$ for τ in the *ordinate* and ϵ for σ in the abscissa (Fig. 2.14).

2.4 STRESS-STRAIN RELATIONSHIPS

2.4.1 Introduction[2]

It can be experimentally demonstrated that a one-to-one relationship exists between uniaxial stress and strain during a single loading. Further, if the material is always loaded within its *elastic* or *reversible* range, a one-to-one relationship exists for all loading and unloading cycles.

For stresses below a certain characteristic value termed the "proportional limit," the stress-strain relationship is very nearly linear. The stress beyond which the stress-strain relationship is no longer reversible is called the "elastic limit." In most materials the proportional and elastic limits are identical. Because the departure from linearity is very gradual it is often necessary to prescribe arbitrarily an "apparent" or "offset elastic limit." This is obtained as the intersection of the stress-strain curve with a line parallel to the linear stress-strain curve, but offset by a prescribed amount, e.g., 0.02 percent (see Fig. 2.15*a*). The "yield point" is the value of stress at which continued deformation of the bar takes place with little or no further increase in load, and the "ultimate limit" is the maximum stress that the specimen can withstand.

Note that some materials may show no clear difference between the apparent elastic, inelastic, and proportional limits or may not show clearly defined yield points (Fig. 2.15*b*).

The concept that a useful linear range exists for most materials and that a simple mathematical law can be formulated to describe the relationship between stress and strain in this range is termed "Hooke's law." It is an essential starting point in the "small-strain theory of elasticity" and the associated mechanics of materials. In the above-described tensile specimen, the law is expressed as

$$\sigma = E\epsilon \tag{2.72}$$

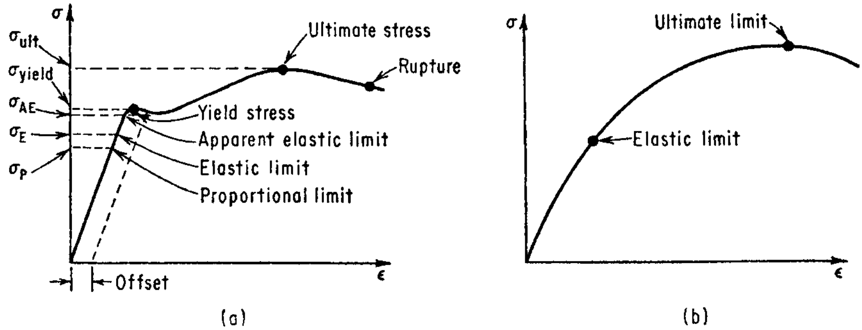


FIG. 2.15 Stress-strain relationship.

as in the analogous torsional specimen

$$\tau = G\gamma \tag{2.73}$$

where E and G are the slope of the appropriate stress-strain diagrams and are called the "Young's modulus" and the "shear modulus" of elasticity, respectively.

2.4.2 General Stress-Strain Relationship[2,4,5]

The one-dimensional concepts discussed above can be generalized for both small and large strain and elastic and nonelastic materials. The following discussion will be limited to small-strain elastic materials consistent with much engineering design. Based upon the above, Hooke's law is expressed as

$$\begin{aligned} \epsilon_x &= (1/E)[\sigma_x - \nu(\sigma_y + \sigma_z)] \qquad & \gamma_{xy} &= \tau_{xy}/G \\ \epsilon_y &= (1/E)[\sigma_y - \nu(\sigma_z + \sigma_x)] \qquad & \gamma_{yz} &= \tau_{yz}/G \\ \epsilon_z &= (1/E)[\sigma_z - \nu(\sigma_x + \sigma_y)] \qquad & \gamma_{zx} &= \tau_{zx}/G \end{aligned} \tag{2.74}$$

where ν is "Poisson's ratio," the ratio between longitudinal strain and lateral contraction in a simple tensile test.

In *cartesian tension form* Eq. (2.74) is expressed as

$$\epsilon_{ij} = [(1 + \nu)/E]\sigma_{ij} - (\nu/E)\delta_{ij}\sigma_{kk} \qquad i,j,k = x,y,z \tag{2.75}$$

where
$$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

The stress-strain laws appear in inverted form as

$$\begin{aligned} \sigma_x &= 2G\epsilon_x + \lambda\,\Delta \\ \sigma_y &= 2G\epsilon_y + \lambda\,\Delta \\ \sigma_z &= 2G\epsilon_z + \lambda\,\Delta \\ \tau_{xy} &= G\gamma_{xy} \\ \tau_{yz} &= G\gamma_{yz} \\ \tau_{zx} &= G\gamma_{zx} \end{aligned} \tag{2.76}$$

where
$$\lambda = \nu/\mathrm{E}/(1 + \nu)(1 - 2\nu)$$
$$\Delta = \epsilon_x + \epsilon_y + \epsilon_z$$
$$G = E/2(1 + \nu)$$

In *cartesian tensor form* Eq. (2.76) is written as

$$\sigma_{ij} = 2G\epsilon_{ij} + \lambda\,\Delta\,\delta_{ij} \qquad i,j = x,y,z \tag{2.77}$$

and in general tensor form as

$$\sigma_{ij} = 2G\epsilon_{ij} + \lambda\,\Delta g_{ij} \tag{2.78}$$

where g_{ij} is the "covariant metric tensor" and these coefficients (stress modulus) are often referred to as "Lamé's constants," and $\Delta = g^{mn}\epsilon_{mn}$.

2.5 STRESS-LEVEL EVALUATION

2.5.1 Introduction[1,6]

The detailed elastic and plastic behavior, yield and failure criterion, etc., are repeatable and simply describable for a simple loading state, as in a tensile or torsional specimen. Under any complex loading state, however, no single stress or strain component can be used to describe the *stress state* uniquely; that is, the *yield, flow,* or *rupture* criterion must be obtained by some combination of all the stress and/or strain components, their derivatives, and loading history. In elastic theory the "yield criterion" is related to an "equivalent stress," or "equivalent strain." It is conventional to treat the *stress* criteria.

An "equivalent stress" is defined in terms of the "stress components" such that plastic flow will commence in the body at any position at which this equivalent stress just exceeds the one-dimensional yield-stress value, for the material under consideration. That is, yielding commences when

$$\sigma_{\text{equivalent}} \geq \sigma_E$$

The "elastic safety factor" at a point is defined as the ratio of the one-dimensional yield stress to the equivalent stress at that position, i.e.,

$$n_i = \sigma_E/\sigma_{\text{equivalent}} \tag{2.79}$$

and the elastic safety factor for the entire structure under any specific loading state is taken as the lowest safety factor of consequence that exists anywhere in the structure.

The "margin of safety," defined as $n - 1$, is another measure of the proximity of any structure to yielding. When $n > 1$, the structure has a *positive* margin of safety and will not yield. When $n = 1$, the margin of safety is zero and the structure just yields. When $n < 1$, the margin of safety is *negative* and the structure is considered unsafe. Note that highly localized yielding is often permitted in ductile materials if it is of such nature as to redistribute stresses without failure, building up a "residual stress state" which allows all subsequent loadings to be accomplished with elastic-stress states. This is the basic concept used in the "autofrettage process" in the strengthening of gun tubes, and it also explains why ductile materials often have low "notch sensitivity" (see Chap. 6).

2.5.2 Effective Stress[1,6]

The concept of effective stress is very closely connected with yield criteria. Geometrically it can be shown that a unique surface can be constructed in stress space in terms of principal stresses (σ_1, σ_2, σ_3) such that all nonyielding states of stress lie within that surface, and yielding states lie on or outside the surface. For *ductile materials* the yield surface may be taken as an infinitely long right cylinder having a center axis defined by $\sigma_1 = \sigma_2 = \sigma_3$. Because the yield criterion is represented by a right cylinder it is adequate to define the yield curve, which is the intersection with a plane normal to the axis of the cylinder. Many yield criteria exist; among these, the "Mises criterion" takes this curve as a circle, and the "Tresca criterion" as a regular hexagon

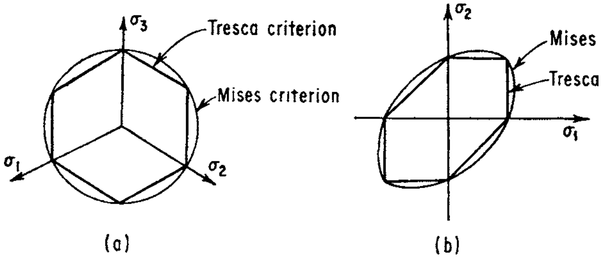


FIG. 2.16 Tresca and Mises criteria. (*a*) General yield criteria. (*b*) One principal stress constant.

(see Fig. 2.16). The former is often referred to by names such as "Hencky," "Hencky-Mises," or "distortion-energy criterion" and the latter by "shear criterion."

Considering any one of the principal stresses as constant, the "yield locus" can be represented by the intersection of the plane σ_i = constant with the cylindrical yield surface, which is represented as an ellipse for the Mises criterion and an elongated hexagon for the Tresca criterion (Fig. 2.16*b*).

In general the yield criteria indicate, upon experimental evidence for a ductile material, that yielding is essentially independent of "hydrostatic compression" or tension for the loadings usually considered in engineering problems. That is, yielding depends only on the "deviatoric" stress component. In general, this can be represented as

$$\sigma'_k = \sigma_k - \tfrac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) \qquad (2.80)$$

where σ' is the principal deviatoric stress component and σ the actual principal stress.

In tensor form

$$\sigma'_{ij} = \sigma_{ij} - \tfrac{1}{3}\sigma_{kk}\delta_{ij} \qquad (2.81)$$

The analytical representation of the yield criteria can be shown to be a function of the "deviatoric stress-tensor invariants." In component form these can be expressed as follows, where σ_0 is the yield stress in simple tension:

Mises:

$$\sigma_0 = (1/\sqrt{2})\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \alpha_1)^2} \qquad (2.82)$$

where σ_1, σ_2, σ_3 are principal stresses.

$$\sigma_0 = (1/\sqrt{2})\sqrt{(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)} \qquad (2.83)$$

or, in tensor form,

$$\tfrac{2}{3}\sigma_0^2 = \sigma'_{ij}\sigma'_{ij} \qquad (2.84)$$

Tresca:

$$\sigma_0 = \sigma_1 - \sigma_3 \qquad (2.85)$$

where $\sigma_1 > \sigma_2 > \sigma_3$, or, in general symmetric terms,

$$[(\sigma_1 - \sigma_3)^2 - \sigma_0^2][(\sigma_2 - \sigma_1)^2 - \sigma_0^2][(\sigma_3 - \sigma_2)^2 - \sigma_0^2] = 0 \qquad (2.86)$$

2.6 FORMULATION OF GENERAL MECHANICS-OF-MATERIAL PROBLEM

2.6.1 Introduction[2,4,5]

Generally the mechanics-of-material problem is stated as follows: Given a prescribed structural configuration, and surface tractions and/or displacements, find the stresses and/or displacements at any, or all, positions in the body. Additionally it is often desired to use the derived stress information to determine the *maximum load-carrying capacity* of the structure, prior to yielding. This is usually referred to as the problem of *analysis.* Alternatively the problem may be inverted and stated: Given a set of surface tractions and/or displacements, find the geometrical configuration for a constraint such as minimum weight, subject to the yield criterion (or some other general stress or strain limitation). This latter is referred to as the *design* problem.

2.6.2 Classical Formulation[2,4,5]

The classical formulation of the equation for the problem of mechanics of materials is as follows: It is necessary to evaluate the six stress components σ_{ij}, six strain components ϵ_{ij}, and three displacement quantities u_i which satisfy the three equilibrium equations, six strain-displacement relationships, and six stress-strain relationships, all subject to the appropriate stress and/or displacement boundary conditions.

Based on the above discussion and the previous derivations, the most general three-dimensional formulation in cartesian coordinates is

$$\left.\begin{aligned} \partial\sigma_x/\partial x + \partial\tau_{xy}/\partial y + \partial\tau_{xz}/\partial z = 0 \\ \partial\sigma_y/\partial y + \partial\tau_{yz}/\partial z + \partial\tau_{yx}/\partial x = 0 \\ \partial\sigma_z/\partial z + \partial\tau_{zx}/\partial x + \partial\tau_{zy}/\partial y = 0 \end{aligned}\right\} \text{(equilibrium)} \qquad (2.87)$$

$$\left.\begin{aligned} \epsilon_x &= \partial u/\partial x & \gamma_{xy} &= \partial v/\partial x + \partial u/\partial y \\ \epsilon_y &= \partial v/\partial y & \gamma_{yz} &= \partial w/\partial y + \partial v/\partial z \\ \epsilon_z &= \partial w/\partial z & \gamma_{zx} &= \partial u/\partial z + \partial w/\partial x \end{aligned}\right\} \text{(strain-displacement)} \qquad (2.47)$$

$$\left.\begin{aligned} \epsilon_x &= (1/E)[\sigma_x - \nu(\sigma_y + \sigma_z)] \\ \epsilon_y &= (1/E)[\sigma_y - \nu(\sigma_z + \sigma_x)] \\ \epsilon_z &= (1/E)[\sigma_z - \nu(\sigma_x + \sigma_y)] \\ \gamma_{xy} &= (1/G)\tau_{xy} \\ \gamma_{yz} &= (1/G)\tau_{yz} \\ \gamma_{zx} &= (1/G)\tau_{xz} \end{aligned}\right\} \text{(stress-strain relationships)} \qquad (2.88)$$

In cartesian tensor form these appear as

$$\sigma_{ij,j} = 0 \qquad \text{(equilibrium)} \qquad (2.89)$$

$$2\epsilon_{ij} = u_{i,j} + u_{j,i} \qquad i,j \rightarrow x,y,z \qquad \text{(strain-displacement)} \qquad (2.48)$$

$$\epsilon_{ij} = [(1 + \nu)/E]\sigma_{ij} - (\nu/E)\delta_{ij}\sigma_{kk} \qquad \text{(stress-strain)} \tag{2.90}$$

All are subject to appropriate boundary conditions.

If the boundary conditions are on displacements, then we can define the displacement field, the six components of strain, and the six components of stress uniquely, using the fifteen equations shown above. If the boundary conditions are on stresses, then the solution process yields six strain components from which three unique displacement components must be determined. In order to assure uniqueness, three constraints must be placed on the strain field. These constraints are provided by the compatibility relationships:

$$\left.\begin{aligned}
\partial^2\epsilon_x/\partial y^2 + \partial^2\epsilon_y/\partial x^2 &= \partial^2\gamma_{xy}/\partial x\,\partial y \\
\partial^2\epsilon_y/\partial z^2 + \partial^2\epsilon_z/\partial y^2 &= \partial^2\gamma_{yz}/\partial y\,\partial z \\
\partial^2\epsilon_z/\partial x^2 + \partial^2\epsilon_x/\partial z^2 &= \partial^2\gamma_{zx}/\partial z\,\partial x \\
2(\partial^2\epsilon_x/\partial y\,\partial z) &= (\partial/\partial x)(-\partial\gamma_{yz}/\partial x + \partial\gamma_{zx}/\partial y + \partial\gamma_{xy}/\partial z) \\
2(\partial^2\epsilon_y/\partial z\,\partial x) &= (\partial/\partial y)(\partial\gamma_{yz}/\partial x - \partial\gamma_{zx}/\partial y + \partial\gamma_{xy}/\partial z) \\
2(\partial^2\epsilon_z/\partial x\,\partial y) &= (\partial/\partial z)(\partial\gamma_{yz}/\partial x + \partial\gamma_{zx}/\partial y - \partial\gamma_{xy}/\partial z)
\end{aligned}\right\} \text{(compatibility)} \tag{2.91}$$

In cartesian tensor form,

$$\epsilon_{ij,kl} + \epsilon_{kl,ij} - \epsilon_{ik,jl} - \epsilon_{jl,ik} = 0 \qquad \text{(compatibility)} \tag{2.92}$$

Of the six compatibility equations listed, only three are independent. Therefore, the system can be uniquely solved for the displacement field.

It is possible to simplify the above sets of equations considerably by combining and eliminating many of the unknowns. One such reduction is obtained by eliminating stress and strain:

$$\begin{aligned}
\nabla^2 u + [1/(1 - 2\nu)](\partial\Delta/\partial x) &= 0 \\
\nabla^2 v + [1/(1 - 2\nu)](\partial\Delta/\partial y) &= 0 \\
\nabla^2 w + [1/(1 - 2\nu)](\partial\Delta/\partial z) &= 0
\end{aligned} \tag{2.93}$$

where ∇^2 is the laplacian operator which in cartesian coordinates is $\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$; and Δ is the dilatation, which in cartesian coordinates is $\partial u/\partial x + \partial v/\partial y + \partial w/\partial z$.

Using the above general principles, it is possible to formulate completely many of the technical problems of mechanics of materials which appear under special classifications such as "beam theory" and "shell theory." These formulations and their solutions will be treated under "Special Applications."

2.6.3 Energy Formulations[2,4,5]

Alternative useful approaches exist for the problem of mechanics of materials. These are referred to as "energy," "extremum," or "variational" formulations. From a strictly formalistic point of view these could be obtained by establishing the analogous integral equations, subject to various restrictions, such that they reduce to a minimum. This is not the usual approach; instead energy functions *U, W* are established so that the stress-strain laws are replaced by

$$\sigma_x = \partial U/\partial \epsilon_x \qquad \tau_{xy} = \partial U/\partial \gamma_{xy}$$
$$\sigma_y = \partial U/\partial \epsilon_y \qquad \tau_{yz} = \partial U/\partial \gamma_{yz}$$
$$\sigma_z = \partial U/\partial \epsilon_z \qquad \tau_{zx} = \partial U/\partial \gamma_{zx}$$

or

$$\sigma_{ij} = \partial U/\partial \epsilon_{ij}$$

and

$$\epsilon_x = \partial W/\partial \sigma_x \qquad \gamma_{xy} = \partial W/\partial \tau_{xy}$$
$$\epsilon_y = \partial W/\partial \sigma_y \qquad \gamma_{yz} = \partial W/\partial \tau_{yz}$$
$$\epsilon_z = \partial W/\partial \sigma_z \qquad \gamma_{zx} = \partial W/\partial \tau_{zx}$$

or

$$\epsilon_{ij} = \partial \mathrm{W}/\partial \sigma_{ij}$$

The energy functions are given by

$$U = \tfrac{1}{2}[2G(\epsilon_x^2 + \epsilon_y^2 + \epsilon_z^2) + \lambda(\epsilon_x + \epsilon_y + \epsilon_z)^2 + G(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{zx}^2)] \tag{2.94}$$

$$W = \tfrac{1}{2}[(1/E)(\sigma_x^2 + \sigma_y^2 + \sigma_z^2) - (2\nu/E)(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x) + (1/G)(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)] \tag{2.95}$$

The variational principle for strains, or theorem of minimum potential energy, is stated as follows: Among all states of strain which satisfy the strain-displacement relationships and displacement boundary conditions the associated stress state, derivable through the stress-strain relationships, which also satisfies the equilibrium equations, is determined by the minimization of Π where

$$\Pi = \int_{\text{volume}} U\,dV - \int_{\text{surface}} (\bar{p}_x u + \bar{p}_y v + \bar{p}_z w)\,dS \tag{2.96}$$

where $\bar{p}_x, \bar{p}_y, \bar{p}_z$ are the x, y, z components of any prescribed surface stresses.

The analogous variational principle for stresses, or principle of least work, is: Among all the states of stress which satisfy the equilibrium equations and stress boundary conditions, the associated strain state, derivable through the stress-strain relationships, which also satisfies the compatibility equations, is determined by the minimization of I, where

$$I = \int_{\text{volume}} W\,dV - \int_{\text{surface}} (p_x \bar{u} + p_y \bar{v} + p_z \bar{w})\,dS \tag{2.97}$$

where $\bar{u}, \bar{v}, \bar{w}$ are the x, y, z components of any prescribed surface displacements and p_x, p_y, p_z are the surface stresses.

In the above theorems $\Pi_{\min}$ and $I_{\min}$ replace the equilibrium and compatibility relationships, respectively. Their most powerful advantage arises in obtaining approximate solutions to problems which are generally intractable by exact techniques. In this, one usually introduces a limited class of assumed stress or displacement functions for minimization, which in themselves satisfy all other requirements imposed in the statement of the respective theorems. Then with the use of these theorems it is possible to find the best solution in that limited class which provides the best minimum to the associated Π or I function. This in reality does not satisfy the missing equilibrium or compatibility equation, but it does it as well as possible for the class of function assumed to describe the stress or strain in the body, within the framework of the principle established above. It has been shown that most reasonable assumptions, regardless of their simplicity, provide useful solutions to most problems of mechanics of materials.

2.6.4 Example: Energy Techniques[2,4,5]

It can be shown that for beams the variational principle for strains reduces to

$$\Pi_{min} = \left\{ \int_0^L [\tfrac{1}{2}EI(y'')^2 - qy]\,dx - \sum P_i y_i \right\}_{min} \tag{2.98}$$

where EI is the flexural rigidity of the beam at any position x, I is the moment of inertia of the beam, y is the deflection of the beam, the y' refers to x derivative of y, q is the distributed loading, the P_i's represent concentrated loads, and L is the span length.

If the minimization is carried out, subject to the restrictions of the variational principle for strains, the beam equation results. However, it is both useful and instructive to utilize the above principle to obtain two approximate solutions to a specific problem and then compare these with the exact solutions obtained by other means.

First a centrally loaded, simple-support beam problem will be examined. The function of minimization becomes

$$\Pi = \int_0^{L/2} EI(y'')^2\,dx - Py_{L/2} \tag{2.99}$$

Select the class of displacement functions described by

$$y = Ax(\tfrac{3}{4}L^2 - x^2) \qquad 0 \le x \le L/2 \tag{2.100}$$

This satisfies the boundary conditions

$$y(0) = y''(0) = y'(L/2) = 0$$

In this A is an arbitrary parameter to be determined from the minimization of Π.

Properly introducing the value of y, y'' into the expression for Π and integrating, then minimizing Π with respect to the open parameter by setting

$$\partial\Pi/\partial A = 0$$

yields

$$y = (Px/12EI)\,(\tfrac{3}{4}L^2 - x^2) \qquad 0 \le x \le L/2 \tag{2.101}$$

It is coincidental that this is the exact solution to the above problem.

A second class of deflection function is now selected

$$y = A \sin(\pi x/L) \qquad 0 \le x \le L \tag{2.102}$$

which satisfies the boundary conditions

$$y(0) = y''(0) = y(L) = y''(L) = 0$$

which is intuitively the expected deflection shape. Additionally, $y(L/2) = A$. Introducing the above information into the expression for Π and minimizing as before yields

$$y = (PL^3/EI)[(2/\pi^4)\sin(\pi x/L)] \tag{2.103}$$

The ratio of the approximate to the exact central deflection is 0.9855, which indicates that the approximation is of sufficient accuracy for most applications.

2.7 FORMULATION OF GENERAL THERMOELASTIC PROBLEM[2,9]

A nonuniform temperature distribution or a nonuniform material distribution with uniform temperature change introduces additional stresses and/or strains, even in the absence of external tractions.

Within the confines of the linear theory of elasticity and neglecting small coupling effects between the *temperature-distribution problem* and the *thermoelastic problem* it is possible to solve the general mechanics-of-material problem as the superposition of the previously defined mechanics-of-materials problem and an initially traction-free thermoelastic problem.

Taking the same consistent definition of stress and strain as previously presented it can be shown that the strain-displacement, stress-equilibrium, and compatibility relationships remain unchanged in the thermoelastic problem. However, because a structural material can change its size even in the absence of stress, it is necessary to modify the stress-strain laws to account for the additional strain due to temperature (αT). Thus Hooke's law is modified as follows:

$$
\begin{aligned}
\epsilon_x &= (1/E)[\sigma_x - \nu(\sigma_y + \sigma_z)] + \alpha T \\
\epsilon_y &= (1/E)[\sigma_y - \nu(\sigma_z + \sigma_x)] + \alpha T \\
\epsilon_z &= (1/E)[\sigma_z - \nu(\sigma_x + \sigma_y)] + \alpha T
\end{aligned}
\tag{2.104}
$$

The shear strain-stress relationships remain unchanged. α is the coefficient of thermal expansion and T the temperature rise above the ambient stress-free state. In uniform, nonconstrained structures this ambient base temperature is arbitrary, but in problems associated with nonuniform material or constraint this base temperature is quite important.

Expressed in cartesian tensor form the stress-strain relationships become

$$\epsilon_{ij} = [(1 + \nu)/E]\sigma_{ij} - (\nu/E)\delta_{ij}\sigma_{kk} + \alpha T\delta_{ij} \tag{2.105}$$

In inverted form the modified stress-strain relationships are

$$
\begin{aligned}
\sigma_x &= 2G\epsilon_x + \lambda\Delta - (3\lambda + 2G)\alpha T \\
\sigma_y &= 2G\epsilon_y + \lambda\Delta - (3\lambda + 2G)\alpha T \\
\sigma_z &= 2G\epsilon_z + \lambda\Delta - (3\lambda + 2G)\alpha T
\end{aligned}
\tag{2.106}
$$

or, in cartesian tensor form,

$$\sigma_{ij} = 2G\epsilon_{ij} + \lambda\Delta\,\delta_{ij} - (3\lambda + 2G)\alpha T\delta_{ij} \tag{2.107}$$

Considering the equilibrium compatibility formulations, it can be shown that the analogous thermoelastic displacement formulations result in

$$
\begin{aligned}
(\lambda + G)(\partial\Delta/\partial x) + G\nabla^2 u - (3\lambda + 2G)\alpha(\partial T/\partial x) &= 0 \\
(\lambda + G)(\partial\Delta/\partial y) + G\nabla^2 v - (3\lambda + 2G)\alpha(\partial T/\partial y) &= 0 \\
(\lambda + G)(\partial\Delta/\partial z) + G\nabla^2 w - (3\lambda + 2G)\alpha(\partial T/\partial z) &= 0
\end{aligned}
\tag{2.108}
$$

A useful alternate stress formulation is

$$(1+\nu)\nabla^2\sigma_x + \frac{\partial^2\Theta}{\partial x^2} + \alpha E\left(\frac{1+\nu}{1-\nu}\nabla^2 T + \frac{\partial^2 T}{\partial x^2}\right) = 0$$

$$(1+\nu)\nabla^2\sigma_y + \frac{\partial^2\Theta}{\partial y^2} + \alpha E\left(\frac{1+\nu}{1-\nu}\nabla^2 T + \frac{\partial^2 T}{\partial y^2}\right) = 0$$

$$(1+\nu)\nabla^2\sigma_z + \frac{\partial^2\Theta}{\partial z^2} + \alpha E\left(\frac{1+\nu}{1-\nu}\nabla^2 T + \frac{\partial^2 T}{\partial z^2}\right) = 0 \tag{2.109}$$

$$(1+\nu)\nabla^2\tau_{xy} + \partial^2\Theta/\partial x\,\partial y + \alpha E(\partial^2 T/\partial x\,\partial y) = 0$$

$$(1+\nu)\nabla^2\tau_{yz} + \partial^2\Theta/\partial y\,\partial z + \alpha E(\partial^2 T/\partial y\,\partial z) = 0$$

$$(1+\nu)\nabla^2\tau_{zx} + \partial^2\Theta/\partial z\,\partial x + \alpha E(\partial^2 T/\partial z\,\partial z) = 0$$

where $\Theta = \sigma_x + \sigma_y + \sigma_z$.

2.8 CLASSIFICATION OF PROBLEM TYPES

In mechanics of materials it is frequently desirable to classify problems in terms of their geometric configurations and/or assumptions that will permit their codification and ease of solution. As a result there exist problems in plane stress or strain, beam theory, curved-beam theory, plates, shells, etc. Although the defining equations can be obtained directly from the general theory together with the associated assumptions, it is often instructive and convenient to obtain them directly from physical considerations. The difference between these two approaches marks one of the principal distinguishing differences between the *theory of elasticity* and *mechanics of materials.*

2.9 BEAM THEORY

2.9.1 Mechanics of Materials Approach[1]

The following assumptions are basic in the development of elementary beam theory:

1. Beam sections, originally plane, remain plane and normal to the "neutral axis."
2. The beam is originally straight and all bending displacements are small.
3. The beam cross section is symmetrical with respect to the loading plane, an assumption that is usually removed in the general theory.
4. The beam material obeys Hooke's law, and the moduli of elasticity in tension and compression are equal.

Consider the beam portion loaded as shown in Fig. 2.17*a*.

For static equilibrium, the internal actions required at section B which are supplied by the immediately adjacent section to the right must consist of a vertical shearing force V and an internal moment M, as shown in Fig. 2.17*b*.

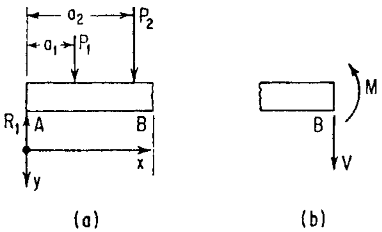


FIG. 2.17 Internal reactions due to externally applied loads. (*a*) External loading of beam segment. (*b*) Internal moment and shear.

The evaluation of the shear V is accomplished by noting, from equilibrium $\sum F_y = 0$,

$$V = R - P_1 - P_2 \qquad \text{(for this example)} \tag{2.110}$$

The algebraic sum of all the shearing forces at one side of the section is called the shearing force at that section. The moment M is obtained from $\sum M = 0$:

$$M = R_1 x - P_1(x - a_1) - P_2(x - a_2) \tag{2.111}$$

The algebraic sum of the moments of all external loads to one side of the section is called the bending moment at the section.

Note the sign conventions employed thus far:

1. Shearing force is positive if the right portion of the beam tends to shear downward with respect to the left.
2. Bending moment is positive if it produces bending of the beam concave upward.
3. Loading w is positive if it acts in the positive direction of the y axis.

In Fig. 2.18*a* a portion of one of the beams previously discussed is shown with the bending moment M applied to the element.

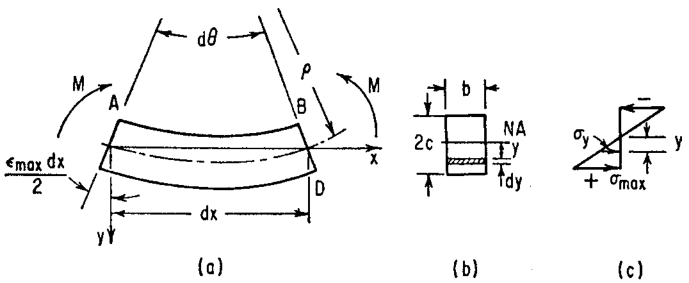


FIG. 2.18 Beam bending with externally applied load. (*a*) Beam element. (*b*) Cross section. (*c*) Bending-stress pattern at section *B–D*.

Equilibrium conditions require that the sum of the normal stresses σ on a cross section must equal zero, a condition satisfied only if the "neutral axis," defined as the plane or axis of zero normal stress, *is also the centroidal axis* of the cross section.

$$\int_{-c}^{+c} \sigma b \, dy = \frac{\sigma}{y} \int_{-c}^{+c} by \, dy = 0 \tag{2.112}$$

where $\sigma/y = (\epsilon/y)E = (\epsilon_{max}/y_{max})E = \text{const.}$

Further, if the moments of the stresses acting on the element dy of the figure are summed over the height of the beam,

$$M = \int_{-c}^{+c} \sigma by \, dy = \frac{\sigma}{y} \int_{-c}^{+c} by^2 \, dy = \frac{\sigma}{y} I \tag{2.113}$$

where y = distance from neutral axis to point on cross section being investigated, and

$$I = \int_{-c}^{+c} by^2 \, dy$$

is the area moment of inertia about the centroidal axis of the cross section. Equation (2.113) defines the flexural stress in a beam subject to moment *M:*

$$\sigma = My/I \tag{2.114}$$

Thus

$$\sigma_{max} = Mc/I \tag{2.115}$$

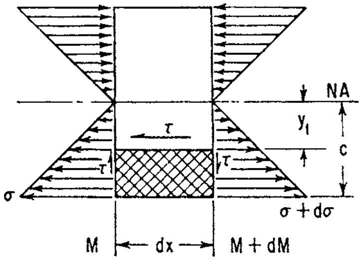


FIG. 2.19 Shear-stress diagram for beam subjected to varying bending moment.

To develop the equations for shear stress τ, the general case of the element of the beam subjected to a varying bending moment is taken as in Fig. 2.19.

Applying axial-equilibrium conditions to the shaded area of Fig. 2.19 yields the following general expression for the horizontal shear stress at the lower surface of the shaded area:

$$\tau = \frac{dM}{dx}\frac{1}{Ib}\int_{y_1}^{c} y\,dA \tag{2.116}$$

or, in familiar terms,

$$\tau = \frac{V}{Ib}\int_{y_1}^{c} y\,dA = \frac{V}{Ib}Q \tag{2.117}$$

where Q = moment of area of cross section about neutral axis for the shaded area above the surface under investigation
V = net vertical shearing force
b = width of beam at surface under investigation

Equilibrium considerations of a small element at the surface where τ is computed will reveal that this value represents both the vertical and horizontal shear.

For a rectangular beam, the vertical shear-stress distribution across a section of the beam is parabolic. The maximum value of this stress (which occurs at the neutral axis) is 1.5 times the average value of the stress obtained by dividing the shear force V by the cross-sectional area.

For many typical structural shapes the maximum value of the shear stress is approximately 1.2 times the average shear stress.

To develop the governing equation for bending deformations of beams, consider again Fig. 2.18. From geometry,

$$\frac{(\epsilon/2)\,dx}{y} = \frac{dx/2}{\rho} \tag{2.118}$$

Combining Eqs. (2.118), (2.114), and (2.72) yields

$$1/\rho = M/EI \tag{2.119}$$

Since

$$1/\rho \approx -\,d^2y/dx^2 = -y'' \tag{2.120}$$

Therefore

$$y'' = -M/EI \qquad \text{(Bernoulli-Euler equation)} \tag{2.121}$$

In Fig. 2.20 the element of the beam subjected to an arbitrary load $w(x)$ is shown together with the shears and bending moments as applied by the adjacent cross sections of the beam. Neglecting higher-order terms, moment summation leads to the following result for the moments acting on the element:

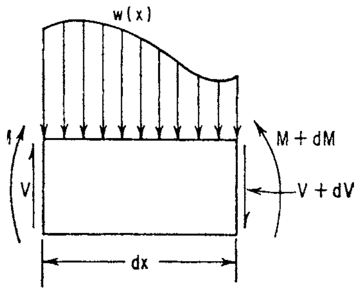


FIG. 2.20 Shear and bending moments for a beam with load $w(x)$ applied.

$$dM/dx = V \tag{2.122}$$

Differentiation of the Bernoulli-Euler equation yields

$$y''' = -V/EI \tag{2.123}$$

In similar manner, the summation of transverse forces in equilibrium yields

$$dV/dx = -w(x) \tag{2.124}$$

or

$$y^{\text{IV}} = \frac{w(x)}{EI} \tag{2.125}$$

where due attention has been given to the proper sign convention.

See Table 2.1 for typical shear, moment, and deflection formulas for beams.

2.9.2 Energy Considerations

The total strain energy of bending is

$$U_b = \int_0^L \frac{M^2}{2EI}\,dx \tag{2.126}$$

The strain energy due to shear is

$$U_s = \int_0^L \frac{V^2}{2GA}\,dx \tag{2.127}$$

In calculating the deflections by the energy techniques, shear-strain contributions need not be included unless the beam is short and deep.

The deflections can then be obtained by the application of Castigliano's theorem, of which a general statement is: The partial derivative of the total strain energy of any structure with respect to any one generalized load is equal to the generalized deflection at the point of application of the load, and is in the direction of the load. The generalized loads can be forces or moments and the associated generalized deflections are displacements or rotations:

$$Y_a = \partial U/\partial P_a \tag{2.128}$$

$$\theta_a = \partial U/\partial M_a \tag{2.129}$$

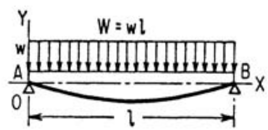
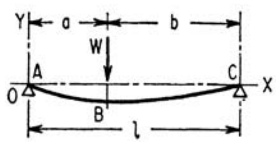
where U = total strain energy of bending of the beam
P_a = load at point a
M_a = moment at point a
Y_a = deflection of beam at point a
θ_a = rotation of beam at point a

Thus

$$Y_a = \frac{\partial U}{\partial P_a} = \frac{\partial}{\partial P_a}\int_0^L M^2 \frac{dx}{2EI} = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial P_a}\,dx \tag{2.130}$$

TABLE 2.1 Shear, Moment, and Deflection Formulas for Beams[1,12]

Notation: W = load (lb); w = unit load (lb/linear in). M is positive when clockwise; V is positive when upward; y is positive when upward. Constraining moments, applied couples, loads, and reactions are positive when acting as shown. All forces are in pounds, all moments in inch-pounds, all deflections and dimensions in inches. θ is in radians and $\tan\theta = \theta$.

Loading, support, and reference number	Reactions R_1 and R_2, vertical shear V	Deflection y, maximum deflection, and end slope θ
End supports Uniform load  (1)	$R_1 = +\frac{1}{2}W \quad R_2 = +\frac{1}{2}W$ $V = \frac{1}{2}W\left(1 - \frac{2x}{l}\right)$	$y = -\frac{1}{24}\frac{Wx}{EIl}(l^3 - 2lx^2 + x^3)$ Max $y = -\frac{5}{384}\frac{Wl^3}{EI}$ at $x = \frac{1}{2}l$ $\theta = -\frac{1}{24}\frac{Wl^2}{EI}$ at $A \quad \theta = +\frac{1}{24}\frac{Wl^2}{EI}$ at B
End supports Intermediate load  (2)	$R_1 = +W\frac{b}{l} \quad R_2 = +W\frac{a}{l}$ (A to B) $V = +W\frac{b}{l}$ (B to C) $V = -W\frac{a}{l}$	(A to B) $y = -\frac{Wbx}{6EIl}[2l(l-x) - b^2 - (l-x)^2]$ (B to C) $y = -\frac{Wa(l-x)}{6EIl}[2lb - b^2 - (l-x)^2]$ Max $y = -\frac{Wab}{27EIl}(a+2b)\sqrt{3a(a+2b)}$ at $x = \sqrt{\frac{1}{3}a(a+2b)}$ when $a > b$ $\theta = -\frac{1}{6}\frac{W}{EI}\left(bl - \frac{b^3}{l}\right)$ at A; $\quad \theta = +\frac{1}{6}\frac{W}{EI}\left(2bl + \frac{b^3}{l} - 3b^2\right)$ at C

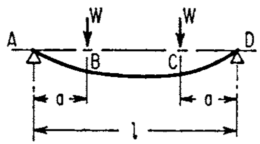
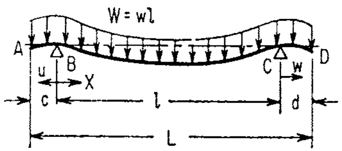
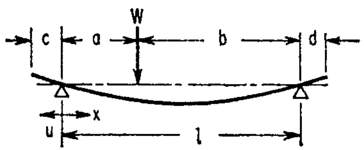
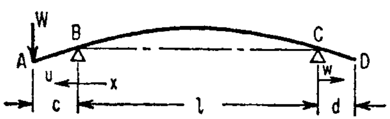
Supported at both ends Two symmetrical loads  (3)	$R_1 = R_2 = W$ $(A \text{ to } B)$ and $(C \text{ to } D)$ $V = W$ $(B \text{ to } C)$ $V = 0$	$(A \text{ to } B)$ and $(C \text{ to } D)$ $y = -\frac{Wx}{6EI}(3\,la - 3a^2 - x^2)$ $(B \text{ to } C)$ $y = -\frac{Wa}{6EI}(3lx - 3x^2 - a^2)$ $\text{Max } y = -\frac{Wa}{24EI}(3l^2 - 4a^2)$ $\theta_A = -\frac{Pa}{2EI}(l - a)$
Both ends overhanging, supports unsymmetrical Uniform load  (4)	$R_1 = \frac{W}{2l}(l^2 - d^2 + c^2)$ $R_2 = \frac{W}{2l}(l^2 + d^2 - c^2)$	$(A \text{ to } B)$ $y = -\frac{Wu}{24EIl}[2l(d^2 + 2c^2) + 6c^2u - u^2(4c - u) - l^3]$ $(B \text{ to } C)$ $y = -\frac{Wx(l - x)}{24EIl}\left\{x(l - x) + l^2 - 2(d^2 + c^2) - \frac{2}{l}[d^2x + c^2(l - x)]\right\}$ Deflection at end: $y = -\frac{Wc}{24EIl}[2l(d^2 + 2c^2) + 3c^3 - l^3]$

TABLE 2.1 Shear, Moment, and Deflection Formulas for Beams[1,12] (*Continued*)

Loading, support, and reference number	Reactions R_1 and R_2, vertical shear V	Deflection y, maximum deflection, and end slope θ
Both ends overhanging supports, load at any point between  (5)	$R_1 = \frac{Wb}{l}$ $R_2 = \frac{Wa}{l}$	Between supports same as case 1; for overhang $y = \frac{Wabu}{6EIl}(l + b)$ Max y same as case 1 Max y at end Max $y = \frac{Wabc}{6EIl}(l + b)$
Both ends overhanging supports, single overhanging load  (6)	$R_1 = \frac{W(c + l)}{l}$ $R_2 = \frac{Wc}{l}$	(A to B) $y = -\frac{Wu}{6EI}(3cu - u^2 + 2cl)$ (B to C) $y = -\frac{Wcx}{6EIl}(l - x)(2l - x)$ (C to D) $y = -\frac{Wclw}{6EI}$ $y_A = -\frac{Wc^2}{3EI}(c + l)$ $y_D = -\frac{Wcld}{6EI}$ Max $y = \frac{Wcl^2}{15.55EI}$ at $x = 0.42265L$

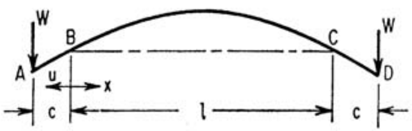
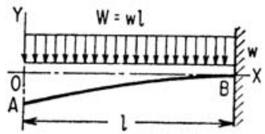
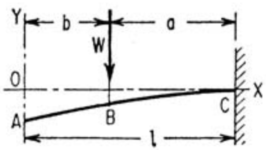
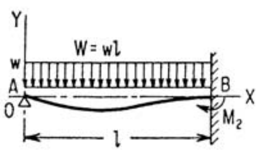
Both ends overhanging supports, symmetrical overhanging loads  (7)	$R_1 = R_2 = W$	(A to B) and (C to D) $y = -\frac{Wu}{6EI}[3c(l+u) - u^2]$ (B to C) $y = -\frac{Wc^2}{2EI}(l - x)$ $y_A = y_D = -\frac{Wc^2}{6EI}(2c + 3l)$ Max $y = \frac{Wcl^2}{8EI}$
Cantilever Uniform load  (8)	$R_2 = +W$ $V = -\frac{W}{l}x$	$y = -\frac{1}{24}\frac{W}{EIl}(x^4 - 4l^3x + 3l^4)$ Max $y = -\frac{1}{8}\frac{Wl^3}{EI}$ $\theta = +\frac{1}{6}\frac{Wl^2}{EI}$ at A

TABLE 2.1 Shear, Moment, and Deflection Formulas for Beams[1,12] (*Continued*)

Loading, support, and reference number	Reactions R_1 and R_2, vertical shear V	Deflection y, maximum deflection, and end slope θ
Cantilever Intermediate load  (9)	$R_2 = +W$ $(A \text{ to } B)\ V = 0$ $(B \text{ to } C)\ V = -W$	$(A \text{ to } B)\ y = -\frac{1}{6}\frac{W}{EI}(-a^3 + 3a^2l - 3a^2x)$ $(B \text{ to } C)\ y = -\frac{1}{6}\frac{W}{EI}[(x-b)^3 - 3a^2(x-b) + 2a^3]$ $\text{Max } y = -\frac{1}{6}\frac{W}{EI}(3a^2l - a^3)$ $\theta = +\frac{1}{2}\frac{Wa^2}{EI}\ (A \text{ to } B)$
One end fixed, one end supported Uniform load  (10)	$R_1 = \frac{3}{8}W \qquad R_2 = \frac{5}{8}W$ $M_2 = \frac{1}{8}Wl$ $V = W\left(\frac{3}{8} - \frac{x}{l}\right)$	$y = \frac{1}{48}\frac{W}{EIl}(3lx^3 - 2x^4 - l^3x)$ $\text{Max } y = -0.0054\frac{Wl^3}{EI} \text{ at } x = 0.4215l$ $\theta = -\frac{1}{48}\frac{Wl^2}{EI} \text{ at } A$

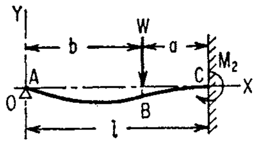
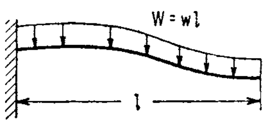
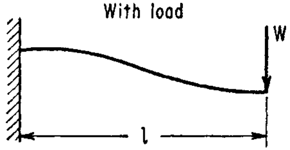
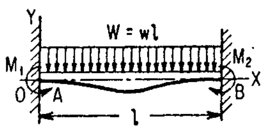
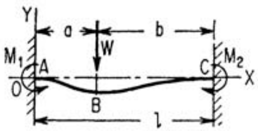
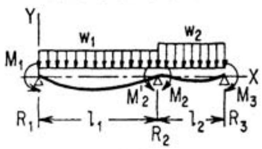
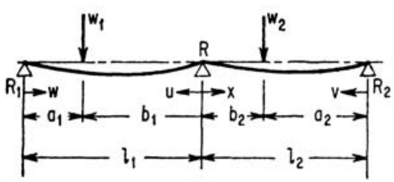
One end fixed, one end supported Intermediate load  (11)	$R_1 = \frac{1}{2} W \left(\frac{3a^2 l - a^3}{l^3} \right)$ $R_2 = W - R_1$ $M_2 = \frac{1}{2} W \left(\frac{a^3 + 2al^2 - 3a^2 l}{l^2} \right)$ (A to B) $V = +R_1$ (B to C) $V = R_1 - W$	(A to B) $y = \frac{1}{6EI} [R_1(x^3 - 3l^2 x) + 3Wa^2 x]$ (B to C) $y = \frac{1}{6EI} \{R_1(x^3 - 3l^2 x) + W[3a^2 x - (x - b)^3]\}$ If $a < 0.586l$, max y is between A and B at: $x = l \sqrt{1 - \frac{2l}{3l - a}}$ If $a > 0.586l$, max y is at: $x = \frac{l(l^2 + b^2)}{3l^2 - b^2}$ If $a = 0.586l$, max y is at B and $= -0.0098 \frac{Wl^3}{EI}$, max possible deflection $\theta = \frac{1}{4} \frac{W}{EI} \left(\frac{a^3}{l} - a^2 \right)$ at A
Fixed at one end, free but guided at the other Uniform load  (12)	$R_1 = W$	$y = -\frac{Wx^2}{24EIl} (2l - x)^2$ Max $y = -\frac{Wl^3}{24EI}$

TABLE 2.1 Shear, Moment, and Deflection Formulas for Beams[1,12] (*Continued*)

Loading, support, and reference number	Reactions R_1 and R_2, vertical shear V	Deflection y, maximum deflection, and end slope θ
Fixed at one end, free but guided at the other With load  (13)	$R_1 = W$	$y = -\frac{Wx^2}{12EI}(3l - 2x)$ $\text{Max } y = -\frac{Wl^3}{12EI}$
Both ends fixed Uniform load  (14)	$R_1 = \frac{1}{2}W \qquad R_2 = \frac{1}{2}W$ $M_1 = \frac{1}{12}Wl \qquad M_2 = \frac{1}{12}Wl$ $V = \frac{1}{2}W\left(\frac{1 - 2x}{l}\right)$	$y = \frac{1}{24}\frac{Wx^2}{EIl}(2lx - l^2 - x^2)$ $\text{Max } y = -\frac{1}{384}\frac{Wl^3}{EI} \text{ at } x = \frac{1}{2}l$

Both ends fixed Intermediate load  (15)	$R_1 = \frac{Wb^2}{l^3}(3a+b)$ $\quad R_2 = \frac{Wa^2}{l^3}(3b+a)$ $M_1 = W\frac{ab^2}{l^2}$ $\quad M_2 = W\frac{a^2b}{l^2}$ $(A \text{ to } B)\ V = R_1$ $(B \text{ to } C)\ V = R_1 - W$	$(A \text{ to } B)\ y = -\frac{1}{6}\frac{Wb^2x^2}{EIl^3}(3ax + bx - 3al)$ $(B \text{ to } C)\ y = -\frac{1}{6}\frac{Wa^2(l-x)^2}{EIl^3}[(3b+a)(l-x) - 3bl]$ $\text{Max } y = \frac{2}{3}\frac{W}{EI}\frac{a^3b^2}{(3a+b)^2} \text{ at } x = \frac{2al}{3a+b} \text{ if } a > b$ $\text{Max } y = \frac{2}{3}\frac{W}{EI}\frac{a^2b^3}{(3b+a)^2} \text{ at } x = l - \frac{2bl}{3b+a} \text{ if } a < b$
Continuous beam, each span uniformly loaded; spans, loads, and sections different  (16)	$\frac{M_1l_1}{I_1} + 2M_2\left[\frac{l_1}{I_1} + \frac{l_2}{I_2}\right] + \frac{M_3l_2}{I_2} = \frac{W_1l_1^3}{4I_1} + \frac{W_2l_2^3}{4I_2}$ (Theorem of Three Moments: I_1 and I_2 refer to 1st and 2d spans. Equation gives M_2 when M_1 and M_3 are known, or can be written for each pair of spans of a continuous beam and resulting equations solved. M_2 acts on span 1, M_2' on span 2.)	Superpose cases 1 and simple beam with end moment
Continuous beam with two unequal spans Unequal loads at any point of each  (17)	$R_1 = \frac{W_1b - m}{l_1}$ $R_2 = \frac{W_1a_1 + m}{l_1} + \frac{W_2a_2 + m}{l_2}$ $R^3 = \frac{W_2b_2 - m}{l_2}$ $m = \frac{1}{2(l_1+l_2)}\left[\frac{W_1a_1b_1}{l_1}(l_1+a_1) + \frac{W_2a_2b_2}{l_2}(l_2+a_2)\right]$	$(R_1 \text{ to } W_1)\quad y = \frac{w}{6EI}\left[(l_1 - w)(l_1 + w)R_1 - \frac{W_1b_1^3}{l_1}\right]$ $(R \text{ to } W_1)\quad y = \frac{u}{6EIl_1}[W_1a_1b_1(l_1+a_1) - W_1a_1u^2 - m(2l_1 - u)(l_1 - u)]$ $(R \text{ to } W_2)\quad y = \frac{x}{6EIl_2}[W_2a_2b_2(l_2+a_2) - W_2a_2x^2 - m(2l_2 - x)(l_2 - x)]$ $(R_2 \text{ to } W_2)\quad y = \frac{v}{6EI}\left[(l_2 - v)(l_2 + v)R_2 - \frac{W_2b_2^3}{l_2}\right]$ $y_{W_1} = \frac{a_1b_1}{6EIl_1}[2a_1b_1W_1 - m(l_1 + a_1)]$ $y_{W_2} = \frac{a_2b_2}{6EIl_2}[2a_2b_2W_2 - m(l_2 + a_2)]$

$$\theta_a = \frac{\partial U}{\partial M_a} = \frac{\partial}{\partial M_a} \int_0^L M^2 \frac{dx}{2EI} = \int_0^L \frac{M}{EI} \frac{\partial M}{\partial M_a} dx \tag{2.131}$$

An important restriction on the use of this theorem is that the deflection of the beam or structure must be a linear function of the load; i.e., geometrical changes and other nonlinear effects must be neglected.

A second theorem of Castigliano states that

$$P_a = \partial U/\partial Y_a \tag{2.132}$$

$$M_a = \partial U/\partial \theta_a \tag{2.133}$$

and is just the inverse of the first theorem. Because it does not have a "linearity" requirement, it is quite useful in special problems.

To illustrate, the deflection y at the center of wire of length $2L$ due to a central load P will be found.

From geometry, the extension δ of each half of the wire is, for small deflections,

$$\delta \approx y^2/2L \tag{2.134}$$

The strain energy absorbed in the system is

$$U = 2\tfrac{1}{2}(AE/L)\delta^2 = (AE/4L^3)y^4 \tag{2.135}$$

Then, by the second theorem,

$$P = \partial U/\partial y = (AE/L^3)y^3 \tag{2.136}$$

or the deflection is

$$y = L\sqrt[3]{P/AE} \tag{2.137}$$

Among the other useful energy theorems are:

Theorem of Virtual Work. If a beam which is in equilibrium under a system of external loads is given a small deformation ("virtual deformation"), the work done by the load system during this deformation is equal to the increase in internal strain energy.

Principle of Least Work. For beams with statically indeterminate reactions, the partial derivative of the total strain energy with respect to the unknown reactions must be zero.

$$\partial U/\partial P_i = 0 \qquad \partial U/\partial M_i = 0 \tag{2.138}$$

depending on the type of support. (This follows directly from Castigliano's theorems.) The magnitudes of the reactions thus determined are such as to minimize the strain energy of the system.

2.9.3 Elasticity Approach[2]

In developing the conventional equations for beam theory from the basic equations of elastic theory (i.e., stress equilibrium, strain compatibility, and stress-strain relations) the beam problem is considered a plane-stress problem. The equilibrium equations for plane stress are

$$\partial\sigma_x/\partial x + \partial\tau_{xy}/\partial y = 0 \tag{2.139}$$

$$\partial\sigma_y/\partial y + \partial\tau_{xy}/\partial x = 0 \tag{2.140}$$

By using an "Airy stress function" ψ, defined as follows:

$$\sigma_x = \partial^2\psi/\partial y^2 \qquad \sigma_y = \partial^2\psi/\partial x^2 \qquad \tau_{xy} = -\partial^2\psi/\partial x\,\partial y \tag{2.141}$$

and the compatibility equation for strain, as set forth previously, the governing equations for beams can be developed.

The only compatibility equation not identically satisfied in this case is

$$\partial^2\epsilon_x/\partial y^2 + \partial^2\epsilon_y/\partial x^2 = \partial^2\gamma_{xy}/\partial x\,\partial y \tag{2.142}$$

Substituting the stress-strain relationships into the compatibility equations and introducing the Airy stress function yields

$$\partial^4\psi/\partial x^4 + 2\partial^4\psi/\partial x^2\,\partial y^2 + \partial^4\psi/\partial y^4 = \nabla^4\psi = 0 \tag{2.143}$$

which is the "biharmonic" equation where ∇^2 is the Laplace operator.

To illustrate the utility of this equation consider a uniform-thickness cantilever beam (Fig. 2.21) with end load P. The boundary conditions are $\sigma_y = \tau_{xy} = 0$ on the surfaces $y = \pm c$, and the summation of shearing forces must be equal to the external load P at the loaded end,

$$\int_{-c}^{+c} \tau_{xy} b\,dy = P$$

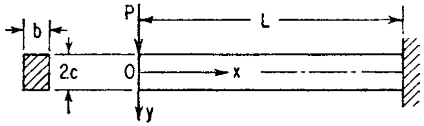


FIG. 2.21 Cantilever beam with end load P.

The solution for σ_x is

$$\sigma_x = \partial^2\psi/\partial y^2 = cxy \tag{2.144}$$

Introducing $b(2c)^3/12 = I$, the final expressions for the stress components are

$$\begin{aligned}\sigma_x &= -Pxy/I = -My/I\\ \sigma_y &= 0\\ \tau_{xy} &= -P(c^2 - y^2)/2I\end{aligned} \tag{2.145}$$

To extend the theory further to determine the displacements of the beam, the definitions of the strain components are

$$\begin{aligned}\epsilon_x &= \partial u/\partial x = \sigma_x/E = -Pxy/EI\\ \epsilon_y &= \partial v/\partial y = -\nu\sigma_x/E = \nu Pxy/EI\\ \gamma_{xy} &= \partial u/\partial y + \partial v/\partial x = [2(1+\nu)/E]\tau_{xy} = [(1+\nu)P/EI](c^2 - y^2)\end{aligned} \tag{2.146}$$

Solving explicitly for the u and v subject to the boundary conditions

$$u = v = \partial u/\partial x = 0 \qquad \text{at } x = L \text{ and } y = 0$$

there results

$$v = \nu Pxy^2/2EI + Px^3/6EI - PL^2x/2EI + PL^3/3EI \tag{2.147}$$

The equation of the deflection curve at $y = 0$ is

$$(v)_{y=0} = (P/6EI)(x^3 - 3L^2x + 2L^3) \tag{2.148}$$

The curvature of the deflection curve is therefore the Bernoulli-Euler equation

$$1/\rho \approx -(\partial^2 v/\partial^2 x)_{y=0} = -Px/EI = M/EI = -y'' \tag{2.149}$$

EXAMPLE 1 The moment at any point x along a simply supported uniformly loaded beam (w lb/ft) of span L is

$$M = wLx/2 - wx^2/2 \tag{2.150}$$

Integrating Eq. (2.121) and employing the boundary conditions $y(0) = y(L) = 0$, the solution for the elastic or deflection curve becomes

$$y = (wL^4/24EI)(x/L)[1 - 2(x/L)^2 + (x/L)^3] \tag{2.151}$$

EXAMPLE 2 In order to obtain the general deflection curve, a fictitious load P_a is placed at a distance a from the left support of the previously described uniformly loaded beam.

$$\begin{aligned} M &= -wx^2/2 + wLx/2 + [P_a x(L - a)]/L \qquad 0 < x < a \\ M &= -wx^2/2 + wLx/2 + [P_a a(L - x)]/L \qquad a < x < L \end{aligned} \tag{2.152}$$

From Castigliano's theorem,

$$y_a = \frac{\partial U}{\partial P_a} = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial P_a}\,dx \tag{2.153}$$

and therefore

$$y_a = (1/EI)(\tfrac{1}{24}wa^4 - \tfrac{1}{12}wLa^3 + \tfrac{1}{24}waL^3) \tag{2.154}$$

The elastic curve of the beam is obtained by substituting x for a in Eq. (2.154), resulting in the same expression as obtained by the double-integration technique.

EXAMPLE 3 It can be shown, considering stresses away from the beam ends, and essentially considering temperature variations only in the direction perpendicular to the beam axis (or in two or more directions by superposition) that the traction-free thermoelastic stress distribution is given by

$$\sigma_x = -\alpha ET + \frac{1}{2c}\int_{-c}^{+c} \alpha ET\,dy + \frac{3y}{2c^3}\int_{-c}^{+c} \alpha ETy\,dy \tag{2.155}$$

Since this stress distribution results in a net axial force-free, and moment-free, distribution but nonzero bending displacements and slopes, it would be necessary to superpose any additional stresses associated with actual boundary constraints.

Because the stress distribution is independent of any linear temperature gradient, it is always possible to add an arbitrary linear distribution T', such that $T_{tot} = T + T'$ and $\int T_{tot}\, dy = \int T_{tot} y\, dy = 0$.

Thus

$$\sigma_x = -\alpha E T_{tot} \tag{2.156}$$

In general T' can be chosen with sufficient accuracy by *visual examination* of the temperature distribution to make the "total-temperature integral" and its "first moment" equal zero. Thus the simplified formula together with its interpretation presents a useful graphical thermoelastic solution for beam and slab problems.

For the beam with temperature distribution $T = a(c^2 - y^2)$ and a is an arbitrary constant, superimpose a temperature distribution T' such that $A_T = -A_{T'}$, where A_T refers to the area under the temperature distribution curve. Evidently

$$T' = -\tfrac{2}{3}ac^2$$

and

$$T_{tot} = T + T' = a(c^2 - y^2) - \tfrac{2}{3}ac^2$$

and $\int T_{tot}\, dy = \int T_{tot} y\, dy = 0$, evident from the selection of T' and symmetry. The stress is therefore

$$\sigma = -\alpha E T_{tot} = -\alpha E[a(c^2 - y^2) - \tfrac{2}{3}ac^2] \tag{2.157}$$

2.10 CURVED-BEAM THEORY[1]

A "curved beam" (see Fig. 2.22) is defined as a beam in which the line joining the centroid of the cross sections (hereafter referred to as the "center line") is a curve. In the standard developments of the equations for the stresses and deflections for curved beams the following *assumptions* are usually made and represent the *restrictions* on the applicability of curved-beam theory:

The sections of the beam originally plane and normal to the center line of the beam remain so after bending.

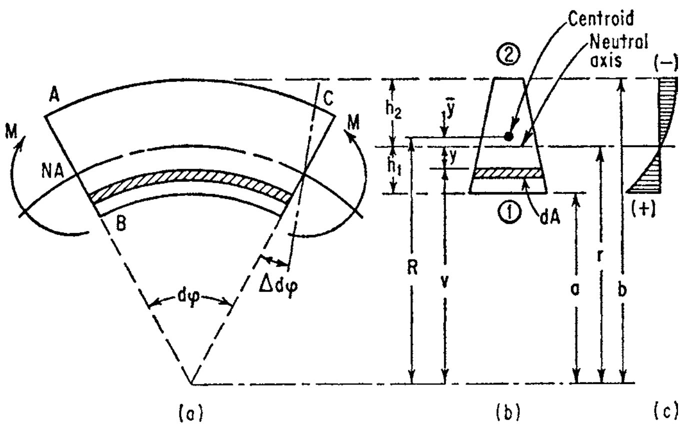


FIG. 2.22 Bending of curved beam element. (*a*) Beam element. (*b*) Cross section. (*c*) Bending-stress pattern at section *C–D*.

All cross sections have an axis of symmetry in the plane of the center line.

The beam is subjected to forces and moments acting in the plane of symmetry.

2.10.1 Equilibrium Approach

In Fig. 2.22 a "positive bending" moment is taken as one which tends to *decrease* the *curvature* of the beam. If R denotes the curvature of the beam at the centroid of a section, then it can be shown that the neutral axis is displaced from R a distance toward the center of curvature. As with straight beams, the "neutral axis" is defined as that axis about which the integrated tangential force is zero, when the external traction is restricted to a bending moment. This distance $\bar{y}$ may be computed from

$$\bar{y} = R - \frac{A}{\int dA/v} \tag{2.158}$$

where A = cross-section area of beam, v the distance from the center of curvature to the incremental area; and the integration $\int dA/v$ is carried out over the entire section of the beam.

The flexural stress at any point a distance y from the neutral axis is

$$\sigma = E\epsilon = Ey(\Delta d\phi)/(r - y)\, d\phi \tag{2.159}$$

which shows that the normal stress distribution over a cross section is not linear, as would be the case in simple beam theory, but hyperbolic in shape. Equating the sum of the moments of each of these segments of normal stress to the applied bending moment on the element,

$$\int \sigma y\, dA = \frac{E(\Delta\, d\phi)}{d\phi} \int \frac{y^2\, dA}{r - y} = M \tag{2.160}$$

yields the resulting expression for the stress

$$\sigma = My/A\bar{y}(r - y) \tag{2.161}$$

with the stresses at the extreme fibers, points 1 and 2, expressed by

$$\sigma_1 = Mh_1/A\bar{y}a \qquad \text{(tensile)} \qquad \sigma_2 = Mh_2/A\bar{y}b \qquad \text{(compressive)} \tag{2.162}$$

The above-described stresses result from pure bending only. If a more general loading condition is given, it must be reduced to the statically equivalent couple and a normal force through the centroid of the section in question.* Then the extensional stresses resulting from the normal force are superposed on the flexural stresses due to the couple.

The development of the governing equation for the displacement of a curved beam, for small deflections with a circular center line, depends on the differential quantities shown in Fig. 2.23, where $1/r$ and $1/r_1$ are the respective curvatures of the undeflected and the deflected beam.

Note that the radial displacement u is taken positive inward in these relations. A comparison of the changes in length $\Delta\, ds$ and central angle $\Delta\, d\phi$ due to deformation leads to

$$1/r_1 - 1/r = u/r^2 + d^2u/ds^2 \tag{2.163}$$

*This also applies in the general case when a shearing force also may be acting on this section.

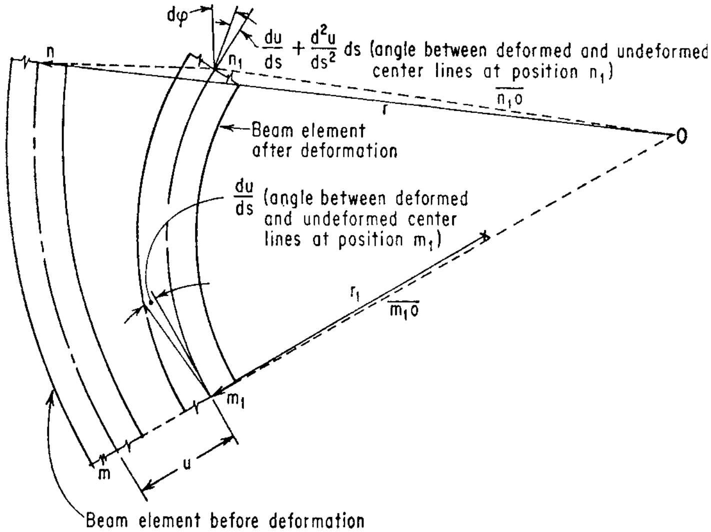


FIG. 2.23 Curved beam with circular center line.

If the thickness of the beam is small compared with the curvature, then

$$1/r_1 - 1/r = -M/EI \tag{2.164}$$

and the differential equation for the deflection curve of a curved beam, which is entirely analogous to that of simple-beam theory, becomes

$$d^2u/ds^2 + u/r^2 = -M/EI \tag{2.165}$$

For an infinitely large r this reduces to the Bernoulli-Euler equation for simple beams.

2.10.2 Energy Approach

A second and more powerful approach, which does not require that the center line of the beam be circular, is essentially the application of Castigliano's theorem. The expression for the strain energy in bending of a curved beam is similar to that of simple-beam theory,

$$U = \int_0^s \frac{M^2\, ds}{2EI} \tag{2.166}$$

where the integration is over the entire length of the beam. The deflection (or rotation) of the beam at a point under a concentrated load (or moment), is

$$\delta_i = \partial U/\partial P_i \tag{2.167}$$

The utility of Eqs. (2.166) and (2.167) in calculating the deflection of curved beams can best be shown by example.

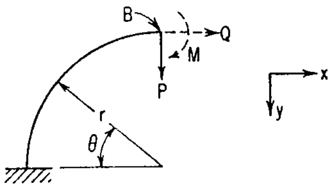


FIG. 2.24 Simple curved beam with vertical load *P*.

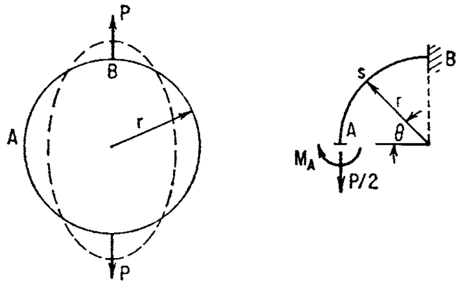


FIG. 2.25 Circular ring.

EXAMPLE 1 The curved beam shown in Fig. 2.24 has a uniform cross section. To determine the horizontal and vertical deflections and rotations, at point *B* fictitious load *Q* and *M* are assumed to act. At any point *s* on the beam,

$$M_s = -[Pr\cos\theta + Qr(1-\sin\theta) + M] \tag{2.168}$$

$$\partial M_s/\partial P = -r\cos\theta \qquad \partial M_s/\partial Q = -r(1-\sin\theta) \qquad \partial M_s/\partial M = -1 \tag{2.169}$$

Substituting $Q = 0$, $M = 0$ in the expression M_s we have, for the three deformations with $ds = r\,d\theta$,

$$\delta_y = \left.\frac{\partial U}{\partial P}\right|_{\substack{Q\to 0\\ M\to 0}} = \int_0^s \frac{M_s}{EI}\frac{\partial M_s}{\partial P}\,ds = \int_0^{\pi/2} \frac{Pr^3}{EI}\cos^2\theta\,d\theta = \frac{\pi}{4}\frac{Pr^3}{EI} \tag{2.170}$$

$$\delta_x = \left.\frac{\partial U}{\partial Q}\right|_{\substack{Q\to 0\\ M\to 0}} = \int_0^s \frac{M_s}{EI}\frac{\partial M}{\partial Q}\,ds = \int_0^{\pi/2} \frac{Pr^3}{EI}\cos\theta\,(1-\sin\theta)\,d\theta = \frac{Pr^3}{2EI} \tag{2.171}$$

$$\theta = \left.\frac{\partial U}{\partial M}\right|_{\substack{Q\to 0\\ M\to 0}} = \int_0^s \frac{M_s}{EI}\frac{\partial M_s}{\partial M}\,ds = \int_0^{\pi/2} \frac{Pr^2}{EI}\cos\theta\,d\theta = \frac{Pr^2}{EI} \tag{2.172}$$

EXAMPLE 2 *Circular Ring-Energy Approach.* The circular ring is subjected to equal and opposite forces *P* as shown in Fig. 2.25. From symmetry considerations an equivalent model may be constructed where the load on the horizontal section is denoted by moment M_A and force $P/2$. From symmetry, there is no rotation of the horizontal section at point *A*. Therefore, by Castigliano's theorem,

$$\theta_A = \partial U/\partial M_A = 0 \tag{2.173}$$

The moment at any point *s* is given by

$$M_s = M_A - (P/2)r(1-\cos\theta) \qquad \text{and} \qquad \partial M_s/\partial M_A = 1 \tag{2.174}$$

Substituting in Eq. (2.173) and imposing the condition of zero rotation,

$$\theta_A = 0 = \frac{\partial U}{\partial M_A} = \int_0^s \frac{M_s}{EI}\frac{\partial M_s}{\partial M_A}\,ds = \int_0^{\pi/2} \frac{1}{EI}\left[M_A - \frac{Pr}{2}(1-\cos\theta)\right] r\,d\theta \tag{2.175}$$

which leads to

$$M_A = (Pr/2)(1 - 2/\pi) \tag{2.176}$$

which yields the moment expression

$$M_s = (Pr/2)(\cos\theta - 2/\pi) \qquad (2.177)$$

The total strain energy for the entire ring is four times that of the quadrant considered. To obtain the total increase in the vertical diameter, the following steps are taken:

$$U = 4\int_0^{\pi/2} \frac{M_s^2}{2EI} r\, d\theta \qquad (2.178)$$

$$\delta_y = \frac{\partial U}{\partial P} = \frac{4}{EI}\int_0^{\pi/2} M_s\left(\frac{\partial M_s}{\partial P}\right) r\, d\theta = \frac{Pr^3}{EI}\int_0^{\pi/2}\left(\cos\theta - \frac{2}{\pi}\right)^2 d\theta \qquad (2.179)$$

$$\delta_y = \frac{Pr^3}{EI}\left(\frac{\pi}{4} - \frac{2}{\pi}\right) \qquad (2.180)$$

Equilibrium Approach: The basic equation for this problem is

$$d^2u/ds^2 + u/r^2 = -M_s/EI \qquad (2.181)$$

Substituting Eq. (2.177) yields

$$r^2\, d^2u/ds^2 + u = (Pr^3/2EI)(2/\pi - \cos\theta) \qquad (2.182)$$

The boundary conditions, derived from the symmetry of the ring, are

$$u'(\theta = 0) = u'(\theta = \pi/2) = 0$$

The general solution of Eq. (2.182) is

$$u = A\cos\theta + B\sin\theta + Pr^3/EI\pi - (Pr^3/4EI)\,\theta\sin\theta \qquad (2.183)$$

$$A = -Pr^3/4EI \qquad \text{and} \qquad B = 0 \qquad (2.184)$$

Then $$u = Pr^3/EI\pi - (Pr^3/4EI)\cos\theta - (Pr^3/4EI)\theta\sin\theta$$

The increase in the vertical radius (point B at $\theta = \pi/2$) becomes

$$u(\theta = \pi/2) = (Pr^3/EI)(1/\pi - \pi/8) \qquad (2.185)$$

Thus the total increase in the vertical diameter δ_v is

$$\delta_v = -2u(\theta = \pi/2) = (Pr^3/EI)(\pi/4 - 2/\pi) \qquad (2.186)$$

which is in agreement with Eq. (2.180).

2.11 THEORY OF COLUMNS[1]

The equilibrium approach for slender symmetrical columns is based on two sets of assumptions:

1. All the basic assumptions inherent in the derivation of the Bernoulli-Euler equation apply to columns also.
2. The transverse deflection of the column at the point of load application is *not* small when compared with the eccentricity of the applied load.

This theory is best described with the aid of a typical column, taken with a built-in support and subjected to an eccentric load, as illustrated in Fig. 2.26.

The moment at any section a distance x from the base is

$$M = -P(\delta + e - y) \tag{2.187}$$

where the negative sign is in accordance with the sign convention for simple beams. Writing the Bernoulli-Euler equation for the bending deflection of this member with the aid of the substitution $p^2 = P/EI$ gives

$$y'' + p^2 y = p^2(\delta + e) \tag{2.188}$$

The general solution of Eq. (2.188) is

$$y = A \sin px + B \cos px + \delta + e \tag{2.189}$$

From the boundary conditions for a built-in end

$$y(0) = y'(0) = 0$$

the equation for the deflection curve is

$$y = (\delta + e)(1 - \cos px) \tag{2.190}$$

The deflection at the end of the column at $x = L$ is seen to be

$$\delta = e(1 - \cos pL)/(\cos pL) \tag{2.191}$$

The complete description of the deflection curve for any point in the column thus becomes

$$y = e(1 - \cos px)/(\cos pL) \tag{2.192}$$

These results can easily be extended to cover a column hinged at both ends by redefining terms as indicated in Fig. 2.27. Thus, from symmetry, the relation of the deflection at the mid-span can be written directly from the previous results:

$$\delta = e\left[1 - \cos\left(\frac{pL}{2}\right)\right] \Big/ \cos\left(\frac{pL}{2}\right) \tag{2.193}$$

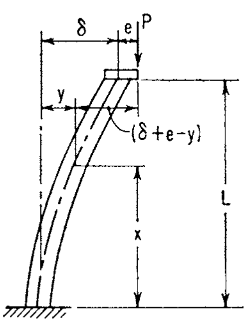


FIG. 2.26 Column subjected to an eccentric load.

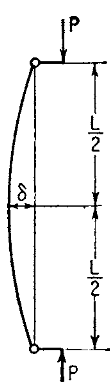


FIG. 2.27 Column hinged at both ends.

In applying these equations note that the deflection is not proportional to the compressive load *P*. Hence the method of superposing a compressive deflection due to *P* and a bending deflection due to the couple *Pe* cannot be used for column action.

In considering column action the concept of "critical load" is of fundamental importance. As the argument of the cosine in the equations for the maximum deflection approaches a value of $\pi/2$, the deflection δ increases without bound and in actual practice the column will fail in a buckling *regardless of the eccentricity e.* Substituting the value $pL = \pi/2$ and $pL/2 = \pi/2$ back into $p^2 = P/EI$ will yield

$$P_{CR} = \pi^2 EI/4L^2 \quad \text{for the single built-in support} \tag{2.194}$$

$$P_{CR} = \pi^2 EI/L^2 \quad \text{for the hinged ends} \tag{2.195}$$

$$P_{CR} = 4\pi^2 EI/L^2 \quad \text{for the both ends built-in} \tag{2.196}$$

The above equations for the critical load (Euler's loads) depend only on the dimensions of the column (I/L^2) and the modulus of the material *E*. If the moment of inertia is written $I = Ak^2$, where *k* is the radius of gyration, the critical-load expressions take the form

$$P_{CR} = \frac{C_1 \pi^2 AE}{(L/k)^2} \tag{2.197}$$

where the dependence is now on the material and a slenderness ratio L/k.

The "critical stress" for a column with hinged ends is given by

$$\sigma_{CR} = \frac{P_{CR}}{A} = \frac{\pi^2 E}{(L/k)^2} \tag{2.198}$$

A plot of σ_{CR} vs. L/k is hyperbolic in form, as shown in Fig. 2.28.

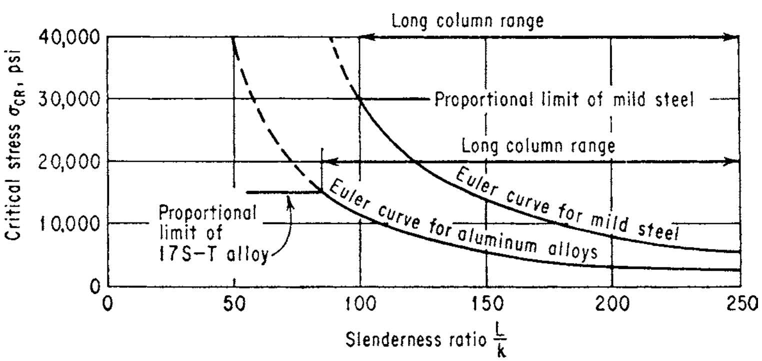


FIG. 2.28 Curve of critical stress vs. slenderness ratio for a column with hinged ends.

The horizontal line in Fig. 2.28 indicates the compressive yield stress of a typical structural steel. For analysis purposes, one uses the compressive yield stress as the design criterion for *small* slenderness ratios and the Euler curve for higher ratios.

Much column design, especially in heavy structural engineering, is accomplished by means of the application of empirical formulas developed as a result of experimental work and practical experience. Several of these formulas are presented below, for hinged bars. Straight-line formulas for structural-steel bars:

$$\sigma_{CR} = P_{CR}/A = 48{,}000 - 210(L/k) \tag{2.199}$$

Parabolic formula for structural steels:

$$\sigma_{CR} = 40{,}000 - 1.35(L/k)^2 \tag{2.200}$$

Gordon-Rankine formula for main members with $120 < L/k < 200$:

$$\sigma_\omega = \frac{18{,}000}{1 + L^2/18{,}000k^2}\left(1.6 - \frac{L}{200k}\right) \tag{2.201}$$

where σ_ω is a working stress.

2.12 SHAFTS, TORSION, AND COMBINED STRESS[1]

2.12.1 Torsion of Solid Circular Shafts

When a solid circular shaft is subjected to a pure torsional load, the reaction of the shaft for small angles of twist is assumed to be subject to the following restraints:

1. Circular cross sections remain circular and their diameters remain unchanged.
2. The axial distances between adjacent cross sections do not change.
3. A lateral surface element of the shaft is in a state of pure shear.

The shearing stresses acting on an element a distance r from the axis are

$$\tau = Gr\theta \tag{2.202}$$

where θ = angle of twist per unit length of shaft
G = modulus of rigidity = $E/2(1 + \nu)$

The shear stress, maximum at the surface is

$$\tau_{max} = \tfrac{1}{2}G\theta D \tag{2.203}$$

where D = diameter of the shaft.

The total torque acting on the shaft can be expressed as

$$T = \int_A r(\tau\, dA) = \int_A G\theta r^2\, dA = G\theta J \tag{2.204}$$

$$T = K\theta \tag{2.205}$$

where J = polar moment of inertia of the circular cross section and K = torsional rigidity = GJ for circular shafts. Therefore, the most useful relations in dealing with circular-shaft torsional problems are

$$\tau_{max} = TD/2J = 16T/\pi D^3 \tag{2.206}$$

$$\phi = TL/GJ = TL/K \tag{2.207}$$

where ϕ = total angle of twist for the shaft of length L.

2.12.2 Shafts of Rectangular Cross Section

For a shaft of rectangular cross section,

$$\tau_{max} = (T/ht^2)[3 + 1.8(t/h)] \qquad (h > t) \tag{2.208}$$

$$\phi = TL/K \tag{2.209}$$

where $K = \beta ht^3G$ and the constant β is 0.141, 0.249, and 0.333 for various ratios of h/t of 1.0, 2.5, and ∞, respectively.

2.12.3 Single-Cell Tubular-Section Shaft

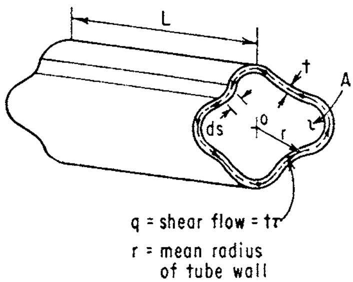


FIG. 2.29 Shaft with general tubular shape.

In Fig. 2.29 a general tubular shape is subjected to a pure torque load; the resultant angular rotations will be about point 0. Considering a slice of the tube of length L, the "shear flow" q can be shown to be constant around the tube. In thin-walled-tube problems the shear is expressed in terms of force per unit length and is thought of as "flowing" from *source* to *sink* in much the same manner as in hydrodynamics problems. Thus it can be shown that

$$q = T/2A \tag{2.210}$$

where A = area enclosed by the median line of the section = πr^2 for a circular tubular section of mean radius r. The total strain energy and total angle of twist (with no warping) for the tube is

$$U = \oint (q^2L/2tG)\, ds \tag{2.211}$$

$$\phi = \partial U/\partial T = (TL/4A^2G) \oint ds/t$$

$$= (L/2AG) \oint (q/t)\, ds \tag{2.212}$$

The unit angle of twist is

$$\theta = \phi/L = (1/2AG) \oint (q/t)\, ds \tag{2.213}$$

The torsional rigidity of the tubular section may then be defined as

$$K = \frac{4A^2}{\oint(ds/t)} G \tag{2.214}$$

For a circular pipe of mean diameter D_m, the value of the rigidity becomes

$$K = (\pi D_m^3\, t/4)G \tag{2.215}$$

2.12.4 Combined Stresses

Frequently problems in shafts involve combined torsion and bending. If the weight of the shaft is neglected relative to load P, there are two major components contributing to the maximum stress:

1. Torsional stress τ, which is a maximum at any point on the surface of the shaft
2. Flexural stress σ, which is a maximum at the built-in end on an element most remote from the shaft axis

The direct stresses due to shear are not significant, since they are a maximum at the axis of the shaft where the flexural stress is zero. The loads on the element at the built-in end are $T = Pr$ and $M = -PL$, and the stresses may be evaluated as follows:

$$\tau = TD/2J = PrD/2J \tag{2.216}$$

$$\sigma = Mc/I = \pm\, PLD/2I \tag{2.217}$$

where r = load offset distance
D = shaft diameter
L = shaft length

The maximum principle stress on the element in tension is

$$\sigma_{max} = \sigma/2 + \tfrac{1}{2}\sqrt{\sigma^2 + 4\tau^2} \tag{2.218}$$

Noting that for a circular shaft the polar moment is twice the area moment about a diameter ($J = 2I$):

$$\sigma_{max} = D/4I(M + \sqrt{M^2 + T^2}) = (PLD/4I)[1 + \sqrt{1 + (r/L)^2}] \tag{2.219}$$

where $$I = \pi D^4/64$$

The maximum shear stress on the element is

$$\tau_{max} = D/4I\sqrt{M^2 + T^2} = (PLD/4)\sqrt{1 + (r/L)^2} \tag{2.220}$$

EXAMPLE 1 For a hollow circular shaft the loads and stresses are

$$T = Pr \qquad \tau = PrD/2J$$

$$M = PL \qquad \sigma = PLD/2I$$

$$\sigma_{max} = \frac{16D_0PL}{\pi(D_0^4 - D_i^4)}\left[1 + \sqrt{1+\left(\frac{r}{L}\right)^2}\right]$$

$$\tau_{max} = \frac{16D_0PL}{\pi(D_0^4 - D_i^4)}\left[\sqrt{1+\left(\frac{r}{L}\right)^2}\right] \tag{2.221}$$

For the angular deflection,

$$K = \pi D^3 tG/4$$

$$\theta = T/K = 4Pr/\pi D^3 tG \tag{2.222}$$

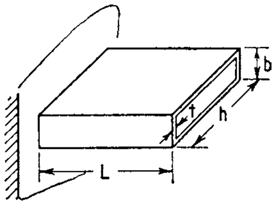


FIG. 2.30 Rectangular thin-walled section.

$$\phi = \theta L = 4PrL/\pi D^3 tG$$

where $D = (D_0 + D_i)/2$.

EXAMPLE 2 The shear stresses and angular deflection of a box as shown in Fig. 2.30 will now be investigated.

The shear flow is given by

$$q = T/2A \tag{2.223}$$

where A = plane area $(b-t)(h-t)$. The shear stress

$$\tau = q/t = T/2t(b-t)(h-t) \tag{2.224}$$

The steps in obtaining the torsional rigidity are

$$\oint ds/t \approx 1/t[2(b-t) + 2(h-t)] \tag{2.225}$$

$$K = \frac{4A^2G}{\oint ds/t} = \frac{2Gt[(b-t)(h-t)]^2}{(b-t)+(h-t)} \tag{2.226}$$

$$\phi = \theta L = \frac{TL}{K} = \frac{TL[(b-t)+(h-t)]}{2Gt[(b-t)(h-t)]^2} \tag{2.227}$$

2.13 PLATE THEORY[3,10]

2.13.1 Fundamental Governing Equation

In deriving the first-order differential equation for a plate under action of a transverse load, the following assumptions are usually made:

1. The plate material is homogeneous, isotropic, and elastic.
2. The least lateral dimension (length or width) of the plate is at least 10 times the thickness h.
3. At the boundary, the edges of the plate are unrestrained in the plane of the plate; thus the reactions at the edges are taken transverse to the plate.
4. The normal to the original middle surface remains normal to the distorted middle surface after bending.
5. Extensional strain in the middle surface is neglected.

The deflected shape of a simply supported plate due to a normal load q is illustrated in Fig. 2.31, which also defines the coordinate system. The positive shears, twists, and moments which act on an element of the plate are depicted in Fig. 2.32.

Application of the equations of equilibrium and Hooke's law to the differential element leads to the following relations:

$$\begin{aligned} &\partial M_x/\partial x + \partial M_{yx}/\partial y = Q_x \qquad M_{xy} = D(1-\nu)(\partial^2 w/\partial x\,\partial y) = -M_{yx} \\ &\partial M_y/\partial y + \partial M_{yx}/\partial x = Q_y \qquad M_x = -D(\partial^2 w/\partial x^2 + \nu\partial^2 w/\partial y^2) \\ &\partial Q_x/\partial x + \partial Q_y/\partial y = -q \qquad M_y = -D(\partial^2 w/\partial y^2 + \nu\partial^2 w/\partial x^2) \end{aligned} \tag{2.228}$$

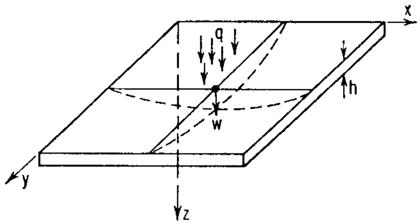


FIG. 2.31 Simply supported plate subjected to normal load *q*.

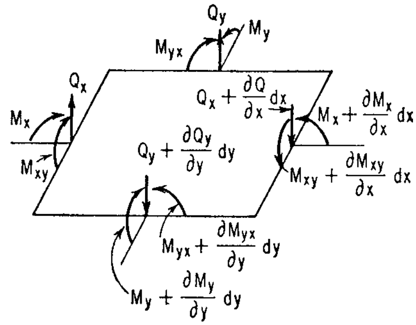


FIG. 2.32 Shears, twists, and moments on a plate element.

where $D = Eh^3/12(1 - \nu^2)$ = plate stiffness and is analogous to the flexural rigidity per unit width (EI) of beam theory.

Properly combining Eqs. (2.228) leads to the basic differential equation of plate theory

$$\nabla^4 w = \partial^4 w/\partial x^4 + 2(\partial^4 w/\partial^2 x\, \partial^2 y) + \partial^4 w/\partial y^4 = q/D \tag{2.229}$$

This may be compared with the similar equation for beams

$$d^4 y/dx^4 = q/EI \tag{2.230}$$

The solution of a specific plate problem involves finding a function w which satisfies Eq. (2.229) and the boundary conditions. With w known, the stresses may be evaluated by employing

$$\begin{aligned} \sigma_{max} &= \pm\, 6M_x/h^2 \\ \sigma_{max} &= \pm\, 6M_y/h^2 \\ \tau_{max} &= 6M_{xy}/h^2 \end{aligned} \tag{2.231}$$

2.13.2 Boundary Conditions

The usual support conditions and the associated boundary conditions are as follows: Simply supported plate:

$w = 0$ (zero deflection)

$M = 0$ (zero moment)

Built-in plate:

$w = 0$ (zero deflection)

$w' = 0$ (zero slope)

Free boundary:

$$M = 0 \qquad \text{(zero moment)}$$
$$V = 0 \qquad \text{(zero reactive shear)}$$

where it is noted that

$$\begin{aligned} V_x &= Q_x - \partial M_{xy}/\partial y = -D[\partial^3 w/\partial x^3 + (2-\nu)(\partial^3 w/\partial x\,\partial y^2)] \\ V_y &= Q_y - \partial M_{xy}/\partial x = -D[\partial^3 w/\partial y^3 + (2-\nu)(\partial^3 w/\partial y\,\partial x^2)] \end{aligned} \tag{2.232}$$

where positive V is directed similar to positive Q.

Note that two boundary conditions are necessary and sufficient to solve the problem of bending for plates with transverse loads.

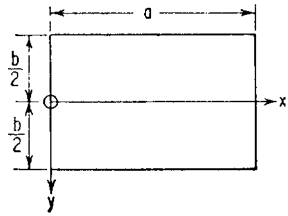


FIG. 2.33 Rectangular plate, simply supported.

EXAMPLE 1 For this problem the loading on a simply supported plate is taken to be distributed uniformly over the surface of that plate, and the coordinate axes are redefined as indicated in Fig. 2.33.

The deflection solution must satisfy Eq. (2.229) and the boundary conditions

$$w = 0 \qquad \text{and} \qquad \partial^2 w/\partial x^2 = 0 \qquad \text{at } x = 0, a$$
$$w = 0 \qquad \text{and} \qquad \partial^2 w/\partial y^2 = 0 \qquad \text{at } y = \pm b/2$$

A series solution of the form

$$w = \sum_{m=1}^{\infty} Y_m \sin \frac{m\pi x}{a} \tag{2.233}$$

is assumed, and after manipulation, the following general expression results for the deflection surface:

$$w = \frac{4qa^4}{\pi^5 D} \sum_{m=1,3,\ldots}^{\infty} \frac{1}{m^5}\left(1 - \frac{\alpha_m \tanh \alpha_m + 2}{2\cosh \alpha_m} \cosh \frac{2\alpha_m y}{b} + \frac{\alpha_m}{2\cosh \alpha_m} \frac{2y}{b} \sinh \frac{2\alpha_m y}{b}\right) \sin \frac{m\pi x}{a} \tag{2.234}$$

where $\alpha_m = m\pi b/a$. The maximum deflection occurs at $x = a/2$ and $y = 0$:

$$w_{\max} = \frac{4qa^4}{\pi^5 D} \sum_{m=1,3,\ldots}^{\infty} \frac{(-1)^{(m-1)/2}}{m^5}\left(1 - \frac{\alpha_m \tanh \alpha_m + 2}{2\cosh \alpha_m}\right) = \frac{\alpha q a^4}{D} \tag{2.235}$$

where $\alpha = \alpha(b/a)$. The maximum moments $M_{x\,\max}$ and $M_{y\,\max}$ may be expressed by $\beta q a^2$ and $\beta_1 q a^2$, respectively, where β and β_1 are functions of b/a.

The final group of plate problems deals with edge reactions which are distributed along the edge and the concentrated forces at the corners to keep the corners of the plate from rising, as shown in Fig. 2.34.

Note that the V_x and V_y are negative in sign, whereas R is positive and directed down in accordance with the adopted sign convention for plates.

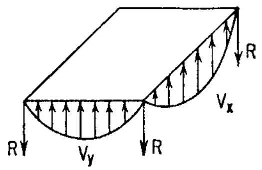


FIG. 2.34 Edge reactions of simply supported plate.

The maximum value of V_x, V_y, and R may be expressed by δqa, $\delta_1 qa$, and ηqa, respectively, where δ, δ_1, and η have the same basic functional relation as the earlier defined coefficients.

Figure 2.35 presents curves of the important plate coefficients.

EXAMPLE 2 The solution for the simple-support uniform-load problem as illustrated above is used as the basis for the solution for a built-in-edge problem. Transposing coordinates as in Fig. 2.36,

$$w = \frac{4qa^4}{\pi^5 D} \sum_{m=1,3,\ldots}^{\infty} \frac{(-1)^{(m-1)/2}}{m^5} \cos \frac{m\pi x}{a} \left(1 - \frac{\alpha_m \tanh \alpha_m + 2}{2 \cosh \alpha_m} \cosh \frac{m\pi y}{a} + \frac{1}{2 \cosh \alpha_m} \frac{m\pi y}{a} \sinh \frac{m\pi y}{a}\right) \quad (2.236)$$

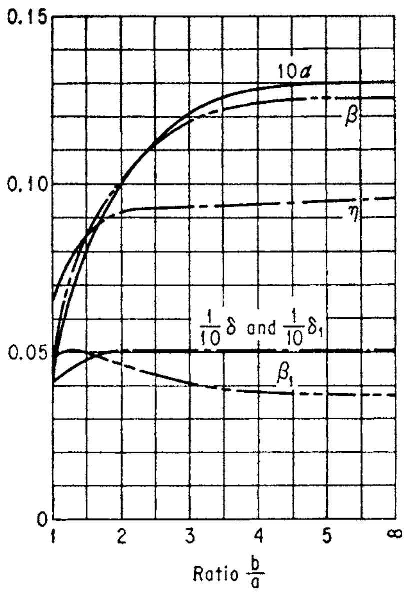


FIG. 2.35 Deflection and moment coefficients for rectangular plate on simple supports.

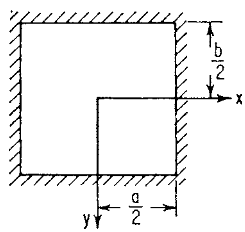


FIG. 2.36 Rectangular plate with built-in edges.

To satisfy the edge restrictions imposed by built-in supports, the deflection of a plate with moments distributed along the edges is superposed on this simple-support solution. These moments are then adjusted to satisfy the built-in boundary condition $\partial w/\partial n = 0$. The procedure finally results in a solution for the maximum deflection at the center of the plate of the form

$$w_{\max} = \alpha' qa^4/D \quad (2.237)$$

The coefficient α' can be evaluated as a function of b/a.

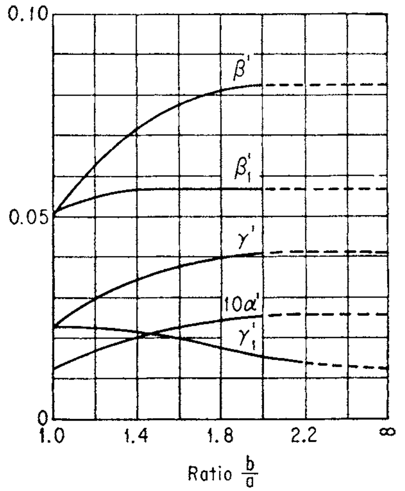


FIG. 2.37 Deflection and moment coefficients for rectangular plate with built-in edges.

Similar expressions for the moments at the center of the plate and at two edges are as follows:

Edge: $M_x = -\beta' qa^2 \qquad M_y = -\beta_1' qa^2$ (2.238*a*)

Center: $M_x = \gamma' qa^2 \qquad M_y = \gamma_1' qa^2$ (2.238*b*)

Figure 2.37 presents α', β', β_1', γ', γ_1', in graphical form.

EXAMPLE 3 We shall use a numerical method to solve the simple-support uniform-load problem. The following expressions define the finite-difference form for the second-order partial derivatives in question (see Fig. 2.38).

$$\left.\frac{\partial^2 w}{\partial x^2}\right|_i = \frac{w_{i+1} - 2w_i + w_{i-1}}{h^2} \qquad \left.\frac{\partial^2 w}{\partial y^2}\right|_i = \frac{w_{i'+1} - 2w_i + w_{i'-1}}{k^2} \tag{2.239}$$

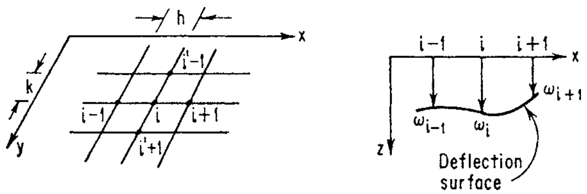


FIG. 2.38 Grid of rectangular plate, simply supported.

For the grid, $h = k$,

$$\nabla^2 w = (1/h^2)(w_{i+1} + w_{i'+1} + w_{i-1} + w_{i'-1} - 4w_i) \tag{2.240}$$

Utilizing

$$\phi = (\nabla^2 w)D \tag{2.241}$$

and

$$\nabla^2 \phi = (1/h^2)(\phi_{i+1} + \phi_{i'+1} + \phi_{i-1} + \phi_{i'-1} - 4\phi_i) \tag{2.242}$$

The governing equation is

$$\nabla^2 \phi = q \tag{2.243}$$

Consider now a square plate coarsely divided into four segments. Then evidently $h = k = a/2$ and $w_{i+1} = w_{i'+1} = w_{i-1} = w_{w'-1} = 0$ since these points are on the boundary. Also $\nabla^2 w_{i+1} = \nabla^2 w_{i'+1} = 0$ and, using the last of the above equations, we find

$$\phi_{i+1} = \phi_{i'+1} = \phi_{i-1} = \phi_{i'-1} = 0$$

Therefore,
$$\nabla^2\phi = -4\phi_i/h^2 = q \tag{2.244}$$

Substituting back into the original equation yields

$$\nabla^2 w_i = \phi_i/D = -4w_i/h^2 = -qa^2/16D \tag{2.245}$$

$$w_i = (1/256)(qa^4/D) \tag{2.246}$$

This is approximately 3.5 percent below the maximum deflection obtained by analytical means:

$$w_{max} = 0.00406(qa^4/D) \tag{2.247}$$

Thus, even with this crude grid, the center deflection can be obtained with reasonable accuracy. Since the stress is obtained by combinations of higher derivatives, however, a finer grid is required for a correspondingly accurate stress evaluation.

2.14 SHELL THEORY[3,10]

2.14.1 Membrane Theory: Basic Equation

The general problem of determining stresses in shells may be subdivided into two distinct categories. The first of these is a moment-free "membrane" state of stress, which often predominates over a major portion of the shell. The second, associated with bending effects, is the "discontinuity stress," which affects the shell for a limited segment in the vicinity of a load or profile discontinuity. The final solution often consists of the membrane solution, corrected locally in the regions of the boundaries for the discontinuity effects.

The "membrane solution" for shells in the form of surfaces of revolution, and loaded symmetrically with respect to the axis, often requires the following simplifying assumptions:

1. The thickness h of the shell is small compared with other dimensions of the shell, and the radii of curvature.
2. Linear elements which are normal to the undisturbed middle surface remain straight and normal to the deflected middle surface of the shell.
3. Bending stresses are small and can be neglected; so that only the direct stresses due to strains in the middle surface need be considered.

The basic element for a shell of revolution when subjected to axisymmetric loading is shown in Fig. 2.39. From conditions of static equilibrium the following equations may be derived:

$$(d/d\phi)(N_\varphi r_0) - N_\theta r_1 \cos\phi + Yr_1 r_0 = 0 \tag{2.248}$$

$$N_\varphi r_0 + N_\theta r_1 \sin\phi + Zr_1 r_0 = 0 \tag{2.249}$$

where Y and Z are the components of the external load parallel to the two-coordinate axis, respectively. Thus, in general, if these components of the load are known and the geometry of the shell is specified, the forces N_ϕ and N_θ can be calculated.

If now the conditions of static equilibrium are applied to a portion of a shell above a parallel circle instead of an element as indicated in Fig. 2.40, then the following equations result:

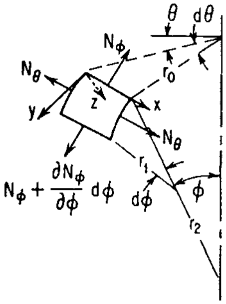


FIG. 2.39 Element of shell of revolution.

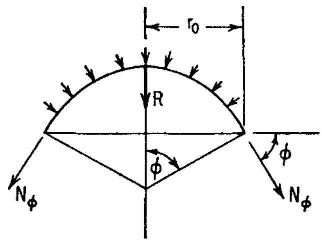


FIG. 2.40 Shell portion above parallel circle.

$$2\pi r_0 N_\phi \sin \phi + R = 0 \tag{2.250}$$

$$N_\phi / r_1 + N_\theta / r_2 = -Z \tag{2.251}$$

where R is the resultant of the total load on that part of the shell in question. Again these two equations suffice for the determination of the two forces N_ϕ and N_θ.

The associated principal stresses for these members are then

$$\sigma_\phi = N_\phi / h \qquad \text{and} \qquad \sigma_\theta = N_\theta / h \tag{2.252}$$

To complete the consideration of membrane action for shells with axisymmetric loading, the concomitant displacements must be computed. The procedure indicated below applies for symmetrical deformations, in which the displacement (along a meridian) is indicated by v and the displacement in the radial direction is w (with an inward displacement taken as positive).

From geometry and Hooke's law,

$$dv/d\phi - v \cot \phi = f(\phi) \tag{2.253}$$

where $$f(\phi) = (1/Eh)[N_\phi(r_1 + \nu r_2) - N_\theta(r_2 + \nu r_1)]$$

The solution of Eq. (2.253) is

$$v = \sin \phi \{\textstyle\int [f(\phi)/\sin \phi] d\phi + C\} \tag{2.254}$$

The constant C is evaluated from the boundary conditions of the problem. The radial displacement is then determined from the following:

$$w = v \cot \phi - (r_2/EH)(N_\theta - \nu N_\phi) \tag{2.255}$$

A similar analysis on a cylindrical shell without the restriction of symmetric loading will yield the basic equations

$$\partial N_x/\partial x + (1/r)(\partial N_{x\phi}/\partial \phi) = -X \tag{2.256}$$

$$\partial N_{x\phi}/\partial x + (1/r)(\partial N_\phi/\partial \phi) = -Y \tag{2.257}$$

$$N_\phi = -Zr \tag{2.258}$$

where the coordinate axes are redefined in Fig. 2.41 to be consistent with usual practice and Eqs. (2.254) and (2.255) are suitably modified.

2.14.2 Example of Spherical Shell Subjected to Internal Pressure

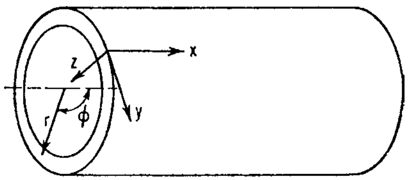


FIG. 2.41 Cylindrical shell.

For a hemispherical section of radius a, the forces N_ϕ distributed along the edges are required to maintain equilibrium. These may be determined from the previous equations. Now

$$2\pi a N_\phi \sin\phi + R = 0 \tag{2.259}$$

where $R = -\pi a^2 p \qquad \sin\phi = \sin(\pi/2) = 1$

so that

$$N_\phi = pa/2 \tag{2.260}$$

From the other equation of equilibrium,

$$N_\phi/a + N_\theta/a = -Z = p \tag{2.261}$$

where $N_\theta = pa/2$

The stresses may now be written as

$$\sigma_\phi = \sigma_\theta = pa/2h \tag{2.262}$$

From loading symmetry, it is concluded that $v = 0$ and the increase in the radial direction is given by

$$w = (-a/Eh)(N_\theta - \nu N_\phi) \tag{2.263}$$

$$w = (-pa^2/2Eh)(1 - \nu) \tag{2.264}$$

2.14.3 Example of Cylindrical Shell Subjected to Internal Pressure

Using a procedure similar to that outlined above, the following relationships are derived:
For open-ended cylinder:

$$\sigma_\phi = pa/h \tag{2.265}$$

$$w = -pa^2/Eh \tag{2.266}$$

For closed-ended cylinder:

$$\sigma_\phi = pa/h \tag{2.267}$$

$$\sigma_x = pa/2h \tag{2.268}$$

$$w = -(pa^2/Eh)(1 - \nu/2) \tag{2.269}$$

2.14.4 Discontinuity Analysis

The membrane solution does not usually satisfy all “edge” conditions, and it is therefore often necessary to superpose a second, “edge-loaded” shell in order to obtain a *complete* solution.

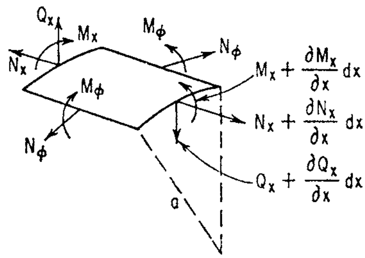


FIG. 2.42 Cylindrical-shell element.

To develop the general equations for the cylindrical shell with axisymmetric load, consider an element as shown in Fig. 2.42, where *bending moments are assumed acting*. The coordinate system is as defined in Fig. 2.41. All forces and moments are shown in their positive directions. Referring to Fig. 2.42, Q_x is the shear force per unit length, M_x is the axial moment per unit length, M_ϕ is the circumferential moment per unit length, and N_x, N_ϕ are the normal forces defined in the discussion of membrane theory.

Applying the equations of equilibrium and Hooke's law and expressing the curvature as a function of moment (as was done in developing the plate equations), the following basic shell differential equation is obtained:

$$d^4w/dx^4 + 4\beta^4 w = Z/D \qquad (2.270)$$

where $D = Eh^3/12(1 - \nu^2)$
$\beta^4 = 3(1 - \nu^2)/a^2h^2$
w = radial deflection of shell (positive inward)

The solution of the above equation in any particular case depends on the specific boundary conditions at the ends of the cylinder.

One very useful solution is for the case of a long cylinder, without radial pressure, subjected to uniformly distributed forces and moments along the edge, $x = 0$. The assumed positive directions of these loads are as shown in Fig. 2.43. The resultant expression for the deflection is

$$w = (e^{-\beta x}/2\beta^3 D)[\beta M_0(\sin \beta x - \cos \beta x) - Q_0 \cos \beta x] \qquad (2.271)$$

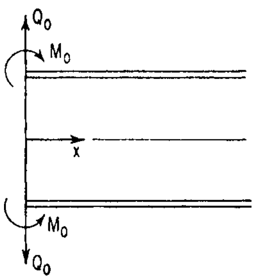


FIG. 2.43 Long-cylinder section.

The maximum deflection occurs at the loaded end and is evaluated as

$$w_{x=0} = -(1/2\beta^3 D)(\beta M_0 + Q_0) \qquad (2.272)$$

The accompanying slope at the loaded end is

$$w'_{x=0} = (1/2\beta^2 D)(2\beta M_0 + Q_0) = (dw/dx)_{x=0} \qquad (2.273)$$

The successive derivatives of the above expression for deflection can be written in the following simplified form:

$$w = -(1/2\beta^3 D)[\beta M_0 \psi(\beta x) + Q_0 \theta(\beta x)]$$
$$w' = (1/2\beta^2 D)[2\beta M_0 \theta(\beta x) + Q_0 \phi(\beta x)] \qquad (2.274)$$
$$w'' = -(1/2\beta D)[2\beta M_0 \phi(\beta x) + 2Q_0 \xi(\beta x)]$$
$$w''' = (1/D)[2\beta M_0 \xi(\beta x) - Q_0 \psi(\beta x)]$$

where $\phi(\beta x) = e^{-\beta x}(\cos \beta + \sin \beta x)$

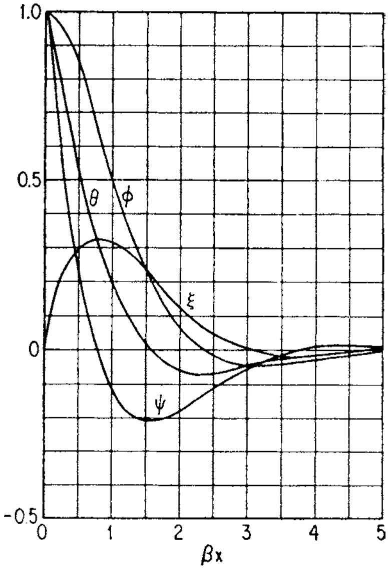


FIG. 2.44 Slope and deflection functions.

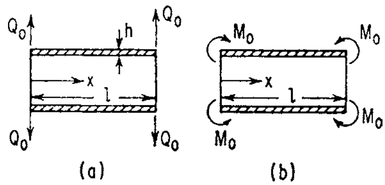


FIG. 2.45 Short shell. (*a*) Bending by shears. (*b*) Bending by moments.

$$\psi(\beta x) = e^{-\beta x}(\cos \beta x - \sin \beta x)$$

$$\theta(\beta x) = e^{-\beta x} \cos \beta x$$

$$\zeta(\beta x) = e^{-\beta x} \sin \beta x$$

Figure 2.44 is a plot of ϕ, ψ, θ, ζ as a function of βx. Because each function decreases in absolute magnitude with increasing βx, in most engineering applications the effect of edge loads may be neglected at locations for which $\beta x > \pi$.

For the short shell, for which opposite end conditions interact, the following results are obtained. For the case of bending by uniformly distributed shearing forces, as shown in Fig. 2.45*a*, the slope and deflection are given by

$$w_{x = 0,l} = -(2Q_0 \beta a^2/Eh)\chi_1(\beta l) \tag{2.275}$$

$$w'_{x = 0,l} = \pm(2Q_0 \beta^2 a^2/Eh)\chi_2(\beta l) \tag{2.276}$$

where $\chi_1(\beta l) = (\cosh \beta l + \cos \beta l)/(\sinh \beta l + \sin \beta l)$

$$\chi_2(\beta l) = (\sinh \beta l - \sin \beta l)/(\sinh \beta l + \sin \beta l)$$

For the case of bending by uniformly distributed moments M_0 (as shown in Fig. 2.45*b*), the slope and deflection are given by

$$w_{x = 0,l} = -(2M_0 \beta^2 a^2/Eh)\chi_2(\beta l) \tag{2.277}$$

$$w'_{x = 0,l} = \pm(4M_0 \beta^3 a^2/Eh)\chi_3(\beta l) \tag{2.278}$$

where $\chi_3(\beta l) = (\cosh \beta l - \cos \beta l)/(\sinh \beta + \sin \beta l)$

Figure 2.46 is a plot of the functions χ_1, χ_2, and χ_3 as a function of βl. The axial bending moment at any location is given by

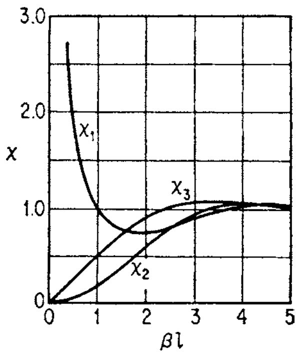


FIG. 2.46 Slope and deflection functions.

$$M_x = -Dw'' \tag{2.279}$$

and the maximum stress which occurs at the inside and outside surfaces of the shell,

$$\sigma_{x,\text{bending}} = \pm(6M_x/h^2) \tag{2.280}$$

Likewise the shear force at any point is given by

$$Q_x = -Dw''' \tag{2.281}$$

and the associated maximum shear stress is $\tau_x = 3Q_x/2h$, which occurs midway between the inner and outer surfaces; the shear stress at the surface is zero.

EXAMPLE Examine the case of a long cylinder subjected to an internal pressure and fixed at the ends as depicted in Fig. 2.47*a*; axial pressure is taken to be zero.

The stress and deformation of this shell can be obtained by the superposition of two distinct problems, the membrane and edge-loaded cylinders. The first presupposes free ends and a membrane action as indicated in Fig. 2.47*b*. The built-in ends resist this membrane deflection at the edges through a system of forces Q_0 and moments M_0 which are required to enforce the boundary conditions of zero deflection and rotation, as shown in Fig. 2.47*c*.

The increase in the radius due to membrane action, as a result of pressure, is then obtained from the membrane solution

$$-w_p = \delta = pa^2/Eh \tag{2.282}$$

The boundary conditions for the edge-loaded problem, based on the actual built-in ends, become

$$w_x = \delta \qquad \text{and} \qquad w'_{x=0} = 0$$

Hence

$$\delta = (1/2\beta^3 D)(\beta M_0 + Q_0) \tag{2.283}$$

$$0 = (1/2\beta^2 D)(2\beta M_0 + Q_0) \tag{2.284}$$

Solving Eqs. (2.283) and (2.284), using Eq. (2.282)

$$M_0 = p/2\beta^2 \qquad \text{and} \qquad Q_0 = -p/\beta \tag{2.285}$$

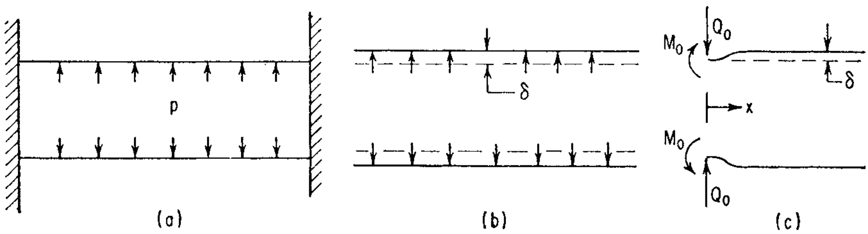


FIG. 2.47 Long cylinder with fixed ends. (*a*) Action of internal pressure. (*b*) Membrane action. (*c*) Discontinuity forces for boundary conditions.

Thus the *complete* solution for the deflection is

$$w = -(1/4\beta^4 D)[p\psi(\beta x) - 2p\theta(\beta x)] - pa^2/Eh \tag{2.286}$$

The axial stresses are given by

$$\sigma_x = \pm(6M_x/h^2) \tag{2.287}$$

where $$M_x = -Dw'' = (p/2\beta^2)[\phi(\beta x) - 2\xi(\beta x)]$$

The mean circumferential stress can be evaluated from

$$\sigma_{\phi,\text{direct}} = -Ew/a \tag{2.288}$$

and the added component of flexural stress due to the Poisson effect is

$$\sigma_{\phi,\text{bending}} = -\nu\sigma_{x,\text{bending}} \tag{2.289}$$

so that $$\sigma_{\phi,\text{total}} = \sigma_{\phi,\text{direct}} + \sigma_{\phi,\text{bending}} \tag{2.290}$$

2.15 CONTACT STRESSES: HERTZIAN THEORY[2]

As discussed in the writings of Hertz (the contact stresses presented here are often termed hertzian stresses), the maximum pressure q due to a compressive force P is given by

$$q = 3P/2\pi a^2 \tag{2.291}$$

and is taken to have a spherical distribution as shown in Fig. 2.48, where

$$a = \sqrt[3]{\frac{3P}{4}\frac{R_1 R_2}{R_1 + R_2}\left(\frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2}\right)} \tag{2.292}$$

These expressions may be simplified, if both spheres are composed of identical materials. For Poisson's ratio ν of approximately 0.3, which is common to steel, iron, aluminum, and most structural materials, there results

$$a = 1.11\sqrt[3]{(P/E)[R_1R_2/(R_1 + R_2)]} \tag{2.292a}$$

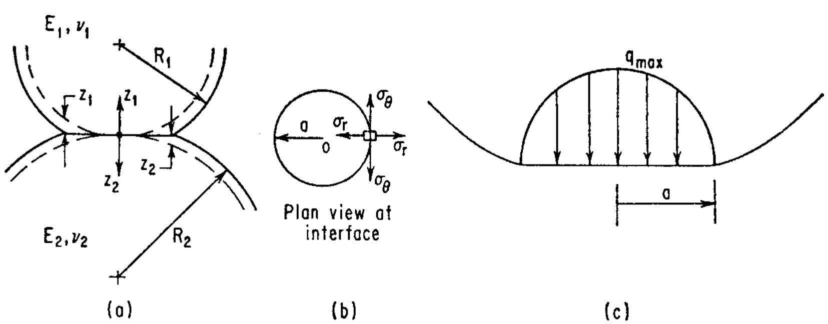


FIG. 2.48 Two spheres in contact. (*a*), (*b*), (*c*) Contact pressure distribution.

and
$$q = 0.388\sqrt[3]{PE^2[(R_1 + R_2)/R_1R_2]^2} \tag{2.292b}$$

The general stress levels in the spheres can now be presented based on the above relations. Maximum compressive stress, which occurs at point O, is

$$\sigma_z = -q \tag{2.293}$$

The maximum tensile stress in the radial direction, which occurs on the periphery of the surface of contact at radius a, is

$$\sigma_r = [(1-2\nu)/3]q \tag{2.294}$$

Maximum shear stress, which occurs under point O of Fig. 2.48b, at a depth

$$z_1 = 0.47a$$

is approximately

$$\tau = \tfrac{1}{3}q \tag{2.295}$$

and is in a plane inclined to the z axis. This latter stress is usually the governing criterion in the design for bodies in contact, fabricated from ductile materials. A compilation of important contact-stress cases is given in Table 2.2. Other important cases, associated with rolling-element bearings, are discussed in Chap. 15.

2.16 FINITE-ELEMENT NUMERICAL ANALYSIS[14,15,16]

2.16.1 Introduction

The chapter thus far has dealt with exact solutions to sets of equations which predict the deformation and internal stress distribution of particular bodies such as beams, columns, thin plates, and thin shells. These sets of equations are derived from the same concepts (equilibrium, kinematics, material observation) used to derive the general equations of elastic theory [Eqs. (2.47), (2.76), and (2.87)]. The particular equations for beams, plates, and shells are, in fact, derivable from the general equation by imposing the appropriate limitations with respect to thickness, etc.

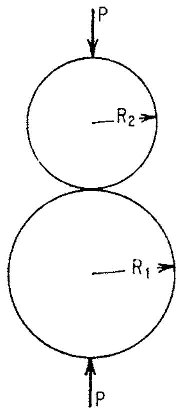
In dealing with the more complex structural shapes typical of actual machinery it becomes increasingly difficult to derive appropriate sets of equations for which exact solutions may be found. For such structures predictions of deformation and stress can be effected through numerical solutions of the general equations. Three general numerical techniques are widely used to effect numerical solutions in structural mechanics: transfer-matrix techniques, finite-difference techniques, and finite-element techniques.

Transfer-matrix approaches are typically used to effect solutions in one-dimensional structures. These approaches are widely used in the solution of vibrating shafts and turbine foils. Typical transfer-matrix approaches include, among others, the Holzer method for torsional vibration and the Prohl-Miklestad (Chap. 4) procedure for lateral shaft vibration.

Finite-difference methods[17] are widely used to effect deformation and stress solutions to multidimensional structures. In this technique, the differential operators of the governing equations are replaced with difference operators which relate the values of the unknowns at a gridwork of points in the structure. Example 3 of Sec. 2.13.2 presents a simple illustration of a finite-difference method. The well-known relaxation

TABLE 2.2 Contact Stresses[2,5,12]

Sphere on a sphere,
P = total load



$$a = 0.9085 \sqrt[3]{PR_2 \frac{1}{1 + R_2/R_1} \left(\frac{1 - \nu_1^2}{E_1} + \frac{1 - \nu_2^2}{E_2} \right)}$$

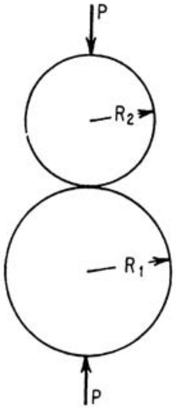
$$\max \sigma_z = 0.580 \sqrt[3]{\frac{P \left(\frac{1 + R_2/R_1}{R_2} \right)^2}{\left(\frac{1 - \nu_1^2}{E_1} + \frac{1 - \nu_2^2}{E_2} \right)^2}}$$

$$y = 0.8255 \sqrt[3]{\frac{P^2}{R_2} \left(1 + \frac{R_2}{R_1} \right) \left(\frac{1 - \nu_1^2}{E_1} + \frac{1 - \nu_2^2}{E_2} \right)^2}$$

For a flat surface $R_1 = \infty$
For a concave surface R_1 is negative

y is the decrease in center-to-center distance between the two spheres.

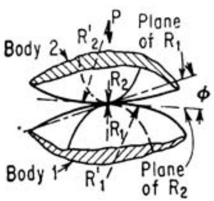
Cylinder on cylinder axes, parallel, P = load/linear in.



$$b = 1.13\sqrt{PR_2\frac{1}{1+R_2/R_1}\left(\frac{1-\nu_1^2}{E_1}+\frac{1-\nu_2^2}{E_2}\right)}$$

$$\max \sigma_z = 0.564\sqrt{\frac{(P/R_2)(1+R_2/R_1)}{\frac{1-\nu_1^2}{E_1}+\frac{1-\nu_2^2}{E_2}}}$$

General case of two bodies in contact, P = total pressure



At point of contact minimum and maximum radii of curvature are R_1 and R_1' for body 1, R_2 and R_2' for body 2. Then $1/R_1$ and $1/R_1'$ are *principal curvatures* of body 1, and $1/R_2$ and $1/R_2'$ of body 2, and in each body the principal curvatures are mutually perpendicular. The plane containing curvature $1/R_1$ in body 1 makes with the plane containing curvature $1/R_2$ in body 2 the angle ϕ. Then

$\max \sigma_z = 1.5P/\pi cd$, $c = \alpha\sqrt[3]{P\delta/K}$, $d = \beta\sqrt[3]{P\delta/K}$, and $y = \lambda\sqrt[3]{P^2/K^2\delta}$, where $\delta = \dfrac{4}{1/R_1 + 1/R_2 + 1/R_1' + 1/R_2'}$

and $K = \dfrac{8}{3}\dfrac{E_1E_2}{E_2(1-\nu_1^2)+E_1(1-\nu_2^2)}$

α and β are given by the following table, where

$$\theta = \arccos \tfrac{1}{4}\,\delta\sqrt{(1/R_1 - 1/R_1')^2 + (1/R_2 - 1/R_2')^2 + 2(1/R_1 - 1/R_1')(1/R_2 - 1/R_2')\cos 2\phi}$$

θ	0°	10°	20°	30°	35°	40°	45°	50°	55°	60°	65°	70°	75°	80°	85°	90°
α	∞	6.612	3.778	2.731	2.397	2.136	1.926	1.754	1.611	1.486	1.378	1.284	1.202	1.128	1.061	1.00
β	0	0.319	0.408	0.493	0.530	0.567	0.604	0.641	0.678	0.717	0.759	0.802	0.846	0.893	0.944	1.00
λ	—	0.851	1.220	1.453	1.550	1.637	1.709	1.772	1.828	1.875	1.912	1.944	1.967	1.985	1.996	2.00

Note: y is the decrease in center-to-center distance of the two cylinders. b is the half-width of the contact surface.

method for the solution of the governing equation of multidimensional heat-transfer analysis is an example of a finite-difference solution.

Since the 1960s finite-element methods have become the preeminent tool for the numerical solution of deformation and stress problems in structural mechanics. This popularity arises from the ease with which the most general of structural geometries can be considered. Finite-element analysis replaces the exact structure to be considered with a set of simple structural elements (blocks, plates, shells, etc.) interconnected at a finite set of node points. The set of governing equations for this approximate structure can be solved exactly.

Finite-element analysis deals with the spatial approximation of complex structural shapes. It can be used directly to yield solutions in static elasticity or combined with other numerical techniques to obtain the response of structures with nonlinear material properties (plasticity, creep, relaxation), undergoing finite deformation, or subject to shock and vibration excitation. The finite-element technique has also found a wide application in the analysis of heat transfer and fluid flow in complex multidimensional applications.

Many users of the finite-element method do so through the application of large-scale, general-purpose computer codes.[18–20] These codes are widely available, highly user-oriented, and simple to use. They are also easy to misuse. The consequences of misuse are excess expense and, more important, invalid predictions of the state of stress and deformation of the structure due to the applied loading.

The discussion herein introduces the process of finite-element analysis to enable the prospective user to have a suitable understanding of the calculations being performed. The discussion is limited to the *stiffness* approach, which is the most widely used basis of finite elements. Further, for ease of understanding, the presentation deals with structures of two dimensions. The generalization to three dimensions follows directly.

2.16.2 The Concept of Stiffness

The governing equations of the theory of elasticity relate the loads applied externally to a body to the resulting deformation of that body, using the stress equilibrium equations to relate external forces to internal forces, i.e., stresses. The stress-strain relations relate internal forces to internal strains. The strain-displacement equations relate internal strains to observed deformations. The whole solution process can be stated by the relationship

$$F = k\delta \tag{2.296}$$

where F = externally applied forces
δ = observed deformation
k = stiffness of the structure

Thus, all the material and geometric information for the structure is contained in the stiffness term.

Numerical procedures involve relating the observed deformation at a discrete number of points of the body to the forces applied at these points. The relationship between force and displacement is then expressed most effectively in terms of matrix notation.

$$\{F\} = [k]\{\delta\} \tag{2.297}$$

Thus F becomes the vector of applied forces, δ the vector of displacement response, and k the stiffness matrix of the structure.

Stiffness of Simple Discrete Elements. For simple structures, relationships of the type of Eq. (2.297) can be derived directly.

EXAMPLE 1 The governing equation for the simple spring in Fig. 2.49 is

$$F_1 = k(\delta_1 - \delta_2)$$

$$F_2 = k(\delta_2 - \delta_1)$$

which in matrix notation becomes

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \end{Bmatrix} \tag{2.298}$$

where k is the stiffness of the spring, F_1 and F_2 the applied forces at nodes 1 and 2, and δ_1 and δ_2 the resulting displacements at nodes and 1 and 2.

EXAMPLE 2 The truss element in Fig. 2.50 is limited to stretching-compression response under its applied loads. The general governing equation is

$$F = (AE/L)\,\Delta L$$

where A is the cross-sectional area, E is Young's modulus, and L is the length. ΔL is the change in L under the action of the forces. A series of relationships of the form

$$F_i = k_{ij}\,\delta_j$$

may be developed where F_i is the force at node i and δ_j is the displacement of node j. The resulting set of relationships is

$$\begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} \cos^2\alpha & \cos\alpha\sin\alpha & -\cos^2\alpha & -\cos\alpha\sin\alpha \\ \cos\alpha\sin\alpha & \sin^2\alpha & -\cos\alpha\sin\alpha & -\sin^2\alpha \\ -\cos^2\alpha & -\cos\alpha\sin\alpha & \cos^2\alpha & \cos\alpha\sin\alpha \\ -\cos\alpha\sin\alpha & -\sin^2\alpha & \cos\alpha\sin\alpha & \sin^2\alpha \end{bmatrix} \begin{Bmatrix} \delta_{1x} \\ \delta_{1y} \\ \delta_{2x} \\ \delta_{2y} \end{Bmatrix} \tag{2.299}$$

which is the form of Eq. (2.297). In general terms, Eq. (2.299) has the form

$$\begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \begin{Bmatrix} \delta_{1x} \\ \delta_{1y} \\ \delta_{2x} \\ \delta_{2y} \end{Bmatrix} \tag{2.300}$$

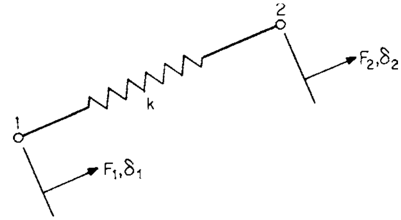


FIG. 2.49 Simple spring finite element.

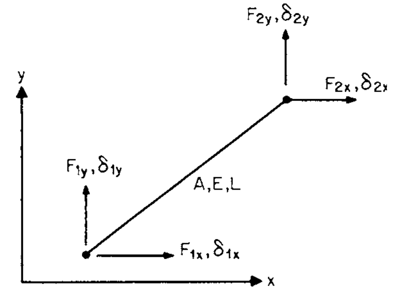


FIG. 2.50 Tension-compression finite element.

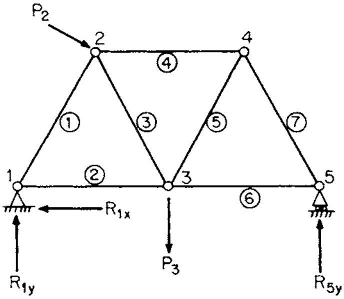


FIG. 2.51 Finite-element model of a simple truss.

Stiffness of a Complex Structure. The simple structures of these examples often comprise the elements of a more complex structure. Thus, the governing equation of the truss of Fig. 2.51 can be developed by combining the relationships of the individual truss elements from Example 2. One simple procedure is to insert the stiffness contribution from each row and column of each truss element to the stiffness of the appropriate row and column of the complex structure. For the truss of Fig. 2.51 the resulting relationship is shown in Eq. (2.301).

$$\begin{Bmatrix} R_{1x} \\ R_{1y} \\ P_{2x} \\ P_{2y} \\ 0 \\ P_{3y} \\ 0 \\ 0 \\ 0 \\ R_{5y} \end{Bmatrix} = \begin{bmatrix} k^1_{11}+k^2_{11} & k^1_{12}+k^2_{12} & k^1_{13} & k^1_{14} & k^2_{13} & k^2_{14} & 0 & 0 & 0 & 0 \\ k^1_{21}+k^2_{21} & k^1_{22}+k^2_{22} & k^1_{23} & k^1_{24} & k^2_{23} & k^2_{24} & 0 & 0 & 0 & 0 \\ k^1_{31} & k^1_{32} & k^1_{33}+k^3_{11}+k^{14}_{11} & k^1_{34}+k^3_{12}+k^4_{12} & k^3_{13} & k^3_{14} & k^4_{13} & k^4_{14} & 0 & 0 \\ k^1_{41} & k^1_{42} & k^1_{43}+k^3_{21}+k^4_{21} & k^1_{44}+k^3_{22}+k^4_{22} & k^3_{23} & k^3_{24} & k^4_{23} & k^4_{24} & 0 & 0 \\ k^2_{31} & k^2_{32} & k^3_{31} & k^3_{32} & k^2_{33}+k^3_{33}+k^5_{11}+k^6_{11} & k^2_{34}+k^3_{34}+k^5_{12}+k^6_{12} & k^5_{13} & k^5_{14} & k^6_{13} & k^6_{14} \\ k^2_{41} & k^2_{42} & k^3_{41} & k^3_{42} & k^2_{43}+k^3_{43}+k^5_{21}+k^6_{21} & k^2_{44}+k^3_{44}+k^5_{22}+k^6_{22} & k^5_{23} & k^5_{24} & k^6_{23} & k^6_{24} \\ 0 & 0 & k^4_{31} & k^4_{32} & k^5_{31} & k^5_{32} & k^4_{33}+k^5_{33}+k^7_{11} & k^4_{34}+k^5_{34}+k^7_{12} & k^7_{13} & k^7_{14} \\ 0 & 0 & k^4_{41} & k^4_{42} & k^5_{41} & k^5_{42} & k^4_{43}+k^5_{43}+k^7_{21} & k^4_{44}+k^5_{44}+k^7_{22} & k^7_{23} & k^7_{24} \\ 0 & 0 & 0 & 0 & k^6_{31} & k^6_{32} & k^7_{31} & k^7_{32} & k^6_{33}+k^7_{33} & k^6_{34}+k^7_{34} \\ 0 & 0 & 0 & 0 & k^6_{11} & k^6_{92} & k^7_{11} & k^7_{42} & k^6_{43}+k^7_{43} & k^6_{44}+k^7_{44} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \delta_{2x} \\ \delta_{2y} \\ \delta_{3x} \\ \delta_{3y} \\ \delta_{4x} \\ \delta_{4y} \\ \delta_{5x} \\ 0 \end{Bmatrix} \tag{2.301}$$

2.16.3 Basic Procedure of Finite-Element Analysis

Equation (2.301) comprises 10 linear algebraic equations in 10 unknowns. The unknowns include the forces of reaction R_{1x}, R_{1y}, R_{5y} and the displacements δ_{2x}, δ_{2y}, δ_{3x}, δ_{3y}, δ_{4x}, δ_{4y}, δ_{5x}. Many procedures for the solution of sets of simultaneous, linear algebraic equations are available. One well-known approach is Gauss-Jordan elimination.[17]

Once the nodal displacements are known, the forces acting on each truss element can be determined by solution of the set of equations (2.300) applicable to that element.

The procedure used to determine the displacements and internal forces of the truss of Fig. 2.51 illustrates the procedure used to determine the response to applied load inherent in the stiffness finite-element procedure. These steps include:

1. Divide the structure into an appropriate number of discrete (or finite) elements connected only at a finite set of points in the structure.
2. Develop a load-deflection relationship of the form of Eq. (2.300) for each finite element.
3. Sum up the load-deflection relationships for each element to obtain the load-deflection relationship for the entire structure, as in Eq. (2.301).
4. Obtain the deformation pattern for the entire structure using conventional procedures.
5. Determine the internal force distribution for each element from the known deformations using the element force-deflection relationships.

The key steps in finite-element analysis are the discretization of the structure and the development of load-deflection relationships for the finite element. The subsequent assembly of the structure load-deflection relationship, the solution of the resulting set of simultaneous algebraic equations, and the subsequent determination of internal forces are straightforward mechanical procedures. Thus, it remains to illustrate the approximations associated with developing finite elements to the analysis of complex structures.

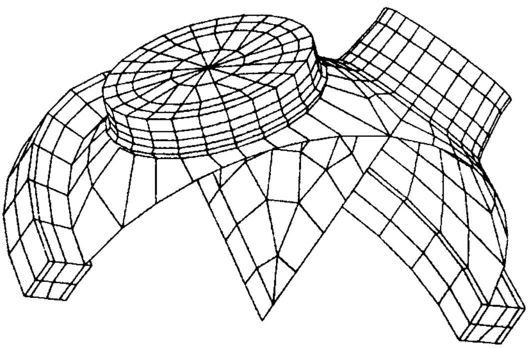


FIG. 2.52 Finite-element model of pressure-vessel head. (*Courtesy of Imo Industries Inc.*)

The truss represents a simplified structure relative to those for which solutions are usually required. A structure such as the pressure-vessel head modeled in Fig. 2.52 is more typical of the component analysis associated with finite-element modeling.

Finite Elements by the Direct Approach. The direct approach to the development of finite elements requires that a complete set of relationships between the internal and externally applied forces be known a priori. For many structural analyses this is not readily available; i.e., the available equilibrium equations are not sufficient. Therefore, the applicability of the direct approach is limited.

The procedure for development of the load-deflection relationships includes:

1. Define the internal displacement field of the element in terms of the nodal displacements. This requires the assumption of a relationship. Usually polynomial expansions are used.

2. Relate the internal displacement field to the internal force field through the strain-displacement and stress-strain equations.
3. Relate the internal force field to the external forces through the force-equilibrium relations.
4. Combine the results of steps 1–4 to obtain a relationship of the form of Eq. (2.300).

EXAMPLE 1 The beam of Fig. 2.53 has length L, Young's modulus E, and area moment of inertia I. At nodes 1 and 2 it is acted upon by external forces and moments F_1, F_2, $\overline{M}_1$, $\overline{M}_2$. As a result, the nodal displacements and rotations are w_1, w_2, θ_1, θ_2.

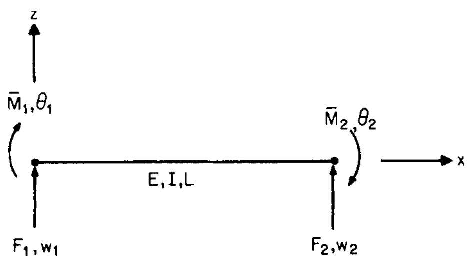


FIG. 2.53 Beam finite element.

Within the beam the deformation pattern is characterized by lateral deflection $w(x)$ and rotation $\theta(x)$, where

$$\theta = -\frac{dw}{dx}$$

Assume that the internal displacement field is governed by the polynomial

$$w(x) = \alpha_1 x^3 + \alpha_2 x^2 + \alpha_3 x + \alpha_4 \tag{2.302}$$

The α's are determined from the boundary conditions on $w(x)$, namely

$$w(0) = w_1$$

$$\theta(0) = \theta_1 = -\left.\frac{dw}{dx}\right|_{x=0}$$

$$w(L) = w_2$$

$$\theta(L) = \theta_2 = -\left.\frac{dw}{dx}\right|_{x=1}$$

The number of terms in the polynomial expansion for $w(x)$ is, in general, limited to the number of nodal degrees of freedom. With the α's known, Eq. (2.300) becomes

$$w(x) = \frac{1}{L^3}[x^3\ x^2\ x\ 1]\begin{bmatrix} 2 & -L & -2 & -L \\ -3L & 2L^2 & 3L & L^2 \\ 0 & -L^3 & 0 & 0 \\ L^3 & 0 & 0 & 0 \end{bmatrix} \tag{2.303}$$

For the special case of a beam, the internal moments are related to the internal displacement field by the Bernoulli-Euler equation

$$M(x) = EI\,(d^2w/dx^2) \tag{2.304}$$

Further, at the nodes

$$M(0) = M_1 = \overline{M}_1 \tag{2.305}$$

$$M(L) = M_2 = -\overline{M}_2 \tag{2.306}$$

For the beam to be in equilibrium under the applied forces and moments it is necessary that

$$F_2L = \overline{M}_1 + \overline{M}_2 \tag{2.307}$$

$$F_1L = -\overline{M}_2 - \overline{M}_1 \tag{2.308}$$

Therefore

$$F_2L = M_1 - M_2 \tag{2.309}$$

$$F_1L = M_2 - M_1 \tag{2.310}$$

Expressing Eqs. (2.305), (2.306), (2.309), and (2.310), in matrix format

$$\begin{Bmatrix} \mathbf{F}_1 \\ M_1 \\ \mathbf{F}_2 \\ M_2 \end{Bmatrix} = \begin{bmatrix} -1/L & 1/L \\ 1 & 0 \\ 1/L & -1/L \\ 0 & -1 \end{bmatrix} \begin{Bmatrix} M_1 \\ M_2 \end{Bmatrix} \tag{2.311}$$

Combining Eqs. (2.303), (2.304), and (2.311) yields

$$\begin{Bmatrix} \mathbf{F}_r \\ M_1 \\ \mathbf{F}_2 \\ M_1 \end{Bmatrix} = \frac{2EI}{L^3} \begin{bmatrix} 6 & -3L & -6 & -3L \\ -3L & 2L^2 & 3L & L^2 \\ -6 & 3L & 6 & 3L \\ -3L & L^2 & 3L & 2L^2 \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix} \tag{2.312}$$

which is the required load/deflection relationship.

Finite Elements by Energy Minimization. The principle of stationary potential energy states that, for equilibrium to be ensured, the total potential energy must be stationary with respect to variations of admissible displacement fields. An "admissible displacement field" is one which satisfies the natural boundary conditions of the structure, typically those boundary conditions that constrain displacements and slopes. The exact displacement field will result in the minimum value of potential energy.

This energy principle allows the development of a general load-deflection relationship which, in turn, allows the development of a wide variety of finite elements directly from the assumed displacement field. The total potential energy is, in general, defined by

$$\Pi(u, v, w) = U(u, v, w) - V(u, v, w) \tag{2.313}$$

where Π = total potential energy
U = strain energy of deformation
V = work done by applied loads
u, v, w = components of displacement field within the element

For Π to be stationary it is necessary that

$$\frac{\partial \Pi}{\partial u_i} = 0 \qquad \frac{\partial \Pi}{\partial v_i} = 0 \qquad \frac{\partial \Pi}{\partial w_i} = 0 \qquad i = 1, r \tag{2.314}$$

where the subscript i denotes the ith node of the finite element, and r is the number of nodes. Further, the energy over the volume of the element is

$$U = \int_{\text{vol}} U_0 \, d\bar{v} \tag{2.315}$$

$$V = \{\delta\}^T\{F\} \tag{2.316}$$

where U_0 = strain energy of a unit volume of material
$\{F\}$ = matrix of nodal forces on the element
$\{\delta\}$ = matrix of nodal displacements

If we further express the stress-strain and strain-displacement equations [Eqs. (2.76) and (2.47)] in matrix format:

$$\{\sigma\} = [D]\{\epsilon\} \tag{2.317}$$

$$\{\epsilon\} = [B]\{\delta\} \tag{2.318}$$

where the element $[B]$ are differential operators, then

$$U_0 = \tfrac{1}{2}\{\epsilon\}[D]\{\epsilon\} \tag{2.319}$$

Combining Eqs. (2.313) to (2.319) and performing the indicated operations leads to a relationship of the form

$$\{F\} = \int_{\text{vol}} [B]^T[D][B] \, d\bar{v} \, \{\delta\} \tag{2.320}$$

Equation (2.320) constitutes a general load-deflection relationship which can be particularized to define a wide variety of finite elements.

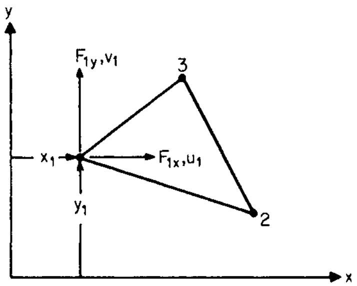


FIG. 2.54 Planar finite element.

EXAMPLE 1 The displacement field within the triangular element in Fig. 2.54 is assumed to be

$$u = \alpha_1 + \alpha_2 x + \alpha_3 y$$
$$v = \alpha_4 + \alpha_5 x + \alpha_6 y \tag{2.321}$$

The six α's may be determined in terms of the six nodal displacement components as was done for the beam element, whence

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \{\delta\} \tag{2.322}$$

$$\text{where } \{\delta\}^T = \{u_1 \; v_1 \; u_1 \; v_2 \; u_3 \; v_3\}^T \tag{2.323}$$

$$N_i = (a_i + b_i x + c_i y)/2\Delta \tag{2.324}$$

The factors a_i, b_i, c_i, and Δ are constants which evolve from the algebraic manipulations. Continuing, for the two-dimensional case

$$\{\epsilon\} = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \end{Bmatrix} = \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{Bmatrix} \tag{2.325}$$

Substituting Eq. (2.322) into Eq. (2.325) yields

$$\{\epsilon\} = [B]\{\delta\} \tag{2.326}$$

where

$$[B] = \frac{1}{2\Delta}\begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix} \tag{2.327}$$

Finally, for the element of Fig. 2.58

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [D]\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{2.328}$$

where, for plane strain

$$[D] = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1 & \nu/(1-\nu) & 0 \\ \nu/(1-\nu) & 1 & 0 \\ 0 & 0 & (1-2\nu)/2(1-\nu) \end{bmatrix} \tag{2.329}$$

Therefore, all the terms in Eq. (2.320) have been defined and so the load-deflection relationship for this element is established.

Since all the terms under the integral in Eq. (2.320) are constants, the integral may be evaluated exactly. Note that the resulting matrix equation contains six simultaneous algebraic equations, corresponding to the six degrees of freedom associated with the triangular element of Fig. 2.54.

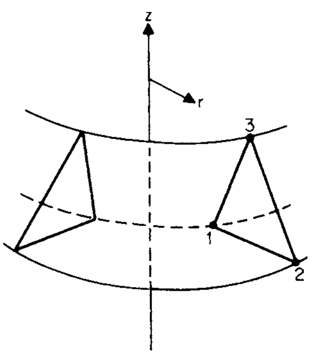


FIG. 2.55 Axisymmetric finite element.

EXAMPLE 2 The displacement function for the axisymmetric element of Fig. 2.55 is

$$\begin{aligned} u &= \alpha_1 + \alpha_2 r + \alpha_3 z \\ v &= \alpha_4 + \alpha_5 r + \alpha_6 z \end{aligned} \tag{2.330}$$

Following the same procedure as in Example 1 we find

$$\{\epsilon\} = \begin{Bmatrix} \epsilon_z \\ \epsilon_r \\ \epsilon_\theta \\ \gamma_{rz} \end{Bmatrix} = \begin{Bmatrix} \partial v/\partial z \\ \partial u/\partial r \\ u/r \\ \partial u/\partial z + \partial v/\partial r \end{Bmatrix} \tag{2.331}$$

whence

$$[B] = \begin{bmatrix} 0 & c_1 & 0 & c_2 & 0 & c_3 \\ b_1 & 0 & b_2 & 0 & b_3 & 0 \\ e_1 & 0 & e_2 & 0 & e_3 & 0 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix} \tag{2.332}$$

where $e_i = a_i/r + b_i + c_i(z/r)$. Further,

$$[D] = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1 & \nu/(1-\nu) & \nu/(1-\nu) & 0 \\ \nu/(1-\nu) & 1 & \nu/(1-\nu) & 0 \\ \nu/(1-\nu) & \nu/(1-\nu) & 1 & 0 \\ 0 & 0 & 0 & (1-2\nu)/2(1-\nu) \end{bmatrix} \quad (2.333)$$

The integral in Eq. (2.320) now has the form

$$2\pi \int_{\text{vol}} [B]^T[D][B] r\, dr\, dz$$

However, $[B]$ is no longer a constant array, i.e., $[B] = [B(r,z)]$ so that integration is a complex process. For many elements, the integrand is sufficiently complex that the integration must be carried out numerically. This numerical integration is a wholly different problem from the numerical analysis that is the finite-element method.

The three-dimensional analog to the triangle element of Example 1 is a four-node tetrahedron. A basic feature of these elements is that the strain field within the element is constant. Thus, to model a structure in which the strains vary considerably throughout the body, a large number of elements are required. Constant-strain elements are most useful for modeling thick-walled bodies in which the main action is stretching. Analysis of more flexible bodies in which bending is significant requires elements in which the strain can vary. These higher-order elements contain higher-order terms in the polynomial displacement expressions, e.g., Eq. (2.321).

Higher-Order Elements. The key ingredient in the development of a finite element is the selection of the shape function, that function which relates the internal-element displacement field to the nodal displacement field, e.g., Eq. (2.322). The remainder of the development is a mechanical process.

The shape function may be selected directly to establish some desired element characteristics or it may evolve from the selection of the displacement function as in the elements developed above. If the displacement function approach is used then the size of the polynomial is limited by the number of nodal degrees of freedom of the element, since the $\mathscr{L}$'s must be uniquely expressed in terms of the nodal degrees of freedom. Thus, in the examples above, the beam element is limited to a cubic polynomial, the triangular plane elements to linear polynomials. Higher-order polynomials require the insertion of additional nodes in the elements or of additional degrees of freedom at the existing nodes. Some typical higher-order elements involving additional nodes are shown in Fig. 2.56. An element involving additional nodal degrees of freedom is shown in Fig. 2.57. This latter type is commonly used to model shell- and plate-type structures.

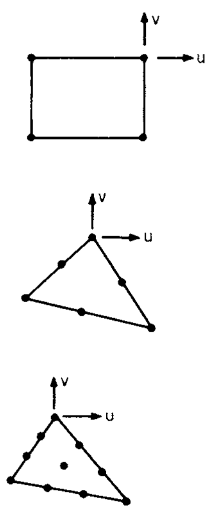


FIG. 2.56 Higher-order finite elements.

A widely used class of elements in which the shape function is chosen directly is the isoparametric elements. The key feature

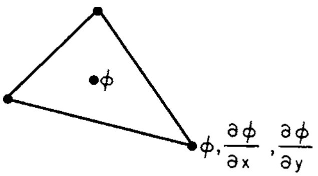


FIG. 2.57 Shell-type finite element.

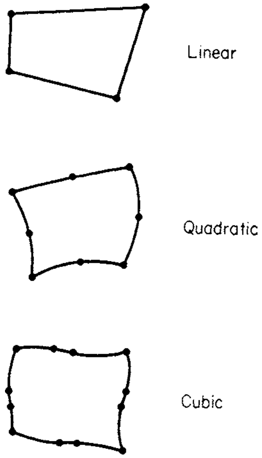


FIG. 2.58 Isoparametric finite elements.

of isoparametric elements is that the elements can have curved sides (Fig. 2.58). This feature allows the element to follow the flow of the structure more readily so that significantly fewer elements are needed to achieve a successful model.

2.16.4 Nature of the Solution

Unless the displacement function used constitutes the exact solution, the equilibrium equation applied within the finite element, or to the total structure, will not be satisfied, i.e., only the exact solution satisfies the equilibrium equations. Further, equilibrium is not satisfied across element boundaries. For example, two adjacent constant-strain (and hence constant-stress) elements cannot correctly represent a continuously varying strain field. Given the approximate nature of the solution, it is appropriate to question whether the response to applied load is at least approximately correct. It can, in fact, be shown that, subject to certain conditions on the finite elements, that the solution will converge to the exact solution with increasing grid refinement. Thus, if questions of accuracy in the analysis of a structure exist, one need only subdivide the critical areas into successively finer element grids. The solutions from these refined analyses will converge toward the correct answer.

The conditions on the elements to assure convergence can be satisfied if the displacement functions used are continuous polynomials of at least the first order within the element and if the elements are compatible. Compatibility requires that at least the nodal variables vary continuously along the boundary between adjacent elements, e.g., the displacement along edge 1-2 of the triangle element of Fig. 2.54 must be the same as along the edge of any other similar element attached to nodes 1 and 2. For the triangular element, since the displacements along edges are straight lines, compatibility is assured.

The above discussion does not preclude the successful use of nonpolynomial displacement functions or nonconverging elements or incompatible elements. However, such elements must be used with great care.

2.16.5 Finite-Element Modeling Guidelines

General rules for finite-element modeling do not exist. However, some reasonable guidelines have evolved to aid the analyst in developing a model which will yield accurate results with a reasonable effort. The more important of these guidelines include:

1. If at all possible, use converging, compatible elements.
2. Grids can be relatively coarse in regions where the state of strain varies slowly. In regions where strains change rapidly, e.g., strain concentrations and structural discontinuities, the grid should be refined.
3. Quadrilateral elements should be used wherever possible in place of triangular elements.
4. Accurate determination of forces and displacements can be accomplished with a more coarse grid than needed for accurate determination of strains and stresses.
5. Prediction of modes of vibration requires a more refined grid than that needed for prediction of natural frequencies.
6. Higher-order elements are generally preferable to constant strain elements.
7. Aspect ratios of multisided two- or three-dimensional elements should be kept below 5.
8. When the accuracy of the solution from a grid is in doubt, the grid should be refined in the critical regions and the analysis rerun.

2.16.6 Generalizations of the Applications

The finite-element method has applications in mechanics of materials beyond the static, linear elastic, isothermal, small-strain class of analyses discussed herein. The generalizations can be classified as related to generalizations of the stress-strain equations and generalizations of the equilibrium equations.

Generalizations of the Stress-Strain Relations. A more general statement of the stress-strain relations of linear elasticity (Eq. 2.74) is

$$\begin{aligned}
\epsilon_x - \epsilon_x^0 - \alpha(T - T_0) &= (1/E)\{(\sigma_x - \sigma_x^0) - \nu[(\sigma_y - \sigma_y^0) + (\sigma_z - \sigma_z^0)]\} \\
\epsilon_y - \epsilon_y^0 - \alpha(T - T_0) &= (1/E)\{(\sigma_y - \sigma_y^0) - \nu[(\sigma_z - \sigma_z^0) + (\sigma_x - \sigma_x^0)]\} \\
\epsilon_z - \epsilon_z^0 - \alpha(T - T_0) &= (1/E)\{(\sigma_z - \sigma_z^0) - \nu[(\sigma_x - \sigma_x^0) + (\sigma_y - \sigma_y^0)]\} \\
\gamma_{xy} - \gamma_{xy}^0 &= (1/G)(\tau_{xy} - \tau_{xy}^0) \\
\gamma_{yz} - \gamma_{yz}^0 &= (1/G)(\tau_{yz} - \tau_{yz}^0) \\
\gamma_{zx} - \gamma_{zx}^0 &= (1/G)(\tau_{zx} - \tau_{zx}^0)
\end{aligned} \tag{2.334}$$

The strain terms with a superscript 0 represent a possible general state of initial strain. The stress terms with a superscript 0 represent a possible general state of initial stress. The strain terms $\alpha(T - T_0)$ represent a possible state of temperature-induced strain. Inclusion of these terms in the development of the finite-element results in a set of additional terms in the load-deflection relationship, which takes on the form

$$\{F\}_{\sigma 0} + \{F\}_{\epsilon 0} + \{F\}_{\alpha T} + \{F\} = \int_{\text{vol}} [B]^T[D][B]\, dv\, \{\delta\} \tag{2.335}$$

where
$$\{F\}_{\sigma 0} = \int_{\text{vol}} [B]^T\{\sigma^0\}\, dv$$
$$\{F\}_{\epsilon 0} = -\int_{\text{vol}} [B]^T[D]\{\epsilon^0\}\, d\bar{v}$$

and similarly for $\{F\}_{\alpha T}$.

Nonlinear stress-strain relationships, i.e.,

$$[\sigma] = F[\epsilon] \tag{2.336}$$

are generally incorporated into finite-element analysis in terms of the incremental plasticity formulation (see Refs. 6 to 8). Solutions are effected by applying load to the structure in additive increments. For each load increment a modified linear analysis is performed. Thus the numerical analysis in the space defined by the finite-element model is supplemented by a numerical analysis in the load dimension to yield an analysis of the total problem.

Similarly, creep problems, for which the stress-strain relation is of the form

$$[\sigma] = f([\epsilon], [\partial\epsilon/\partial t]) \tag{2.337}$$

are solved using a numerical analysis in the time domain to supplement the finite-element models in space.

Generalizations of the Equilibrium Equations. The equilibrium equations, with the addition of body force terms, such as gravitational or inertia, have the form

$$\begin{aligned} \partial\sigma_x/\partial x + \partial\tau_{xy}/\partial y + \partial\tau_{xz}/\partial z + \bar{F}_x &= 0 \\ \partial\sigma_y/\partial y + \partial\tau_{yz}/\partial z + \partial\tau_{yx}/\partial x + \bar{F}_y &= 0 \\ \partial\sigma_z/\partial z + \partial\tau_{zx}/\partial x + \partial\tau_{zy}/\partial y + \bar{F}_z &= 0 \end{aligned} \tag{2.338}$$

With these terms, the load-deflection relationship now has the form

$$\{\bar{F}\}_{BF} + \{F\} = \int_{\text{vol}} [B]^T[D][B]\, d\bar{v}\, \{\delta\} \tag{2.339}$$

where
$$\{\bar{F}\}_{BF} = -\int_{\text{vol}} [N]^T\{\bar{F}\}\, d\bar{v}$$

$[N]^T$ = shape-function matrix, analogous to Eq. (2.322)

If $\{\bar{F}\}_{BF}$ represents an acceleration force per unit volume, then

$$\{\bar{F}\}_{BF} = \rho[N](\partial^2/\partial t^2)\{\delta\} \tag{2.340}$$

where ρ is the mass per unit volume.

If we further define

$$\begin{aligned} [M] &= \rho \int_{\text{vol}} [N]^T[N]\, d\bar{v} \\ [k] &= \int_{\text{vol}} [B]^T[D][B]\, d\bar{v} \end{aligned} \tag{2.341}$$

then Eq. (2.297) takes the form

$$[M]\{\delta\} + [k]\{\delta\} = \{F\} \tag{2.342}$$

which is the matrix statement of the general vibration problem discussed in Chap. 4. Therefore, all the solution techniques noted therein are applicable to the spatial finite-element model. A damping force vector can also be developed for Eq. (2.342).

2.16.7 Finite-Element Codes

Structural analysis by the finite-element method contains two major engineering steps: the design of the grid and the use of an appropriate finite element. Finite elements have been developed to represent a broad range of structural configurations, including constant strain and higher-order two- and three-dimensional solids, shells, plates, beams, bars, springs, masses, damping elements, contact elements, fracture mechanics elements, and many others. Elements have been designed for static and dynamic analysis, linear and nonlinear material models, linear and nonlinear deformations.

The finite element depends upon the selection of an appropriate shape or displacement function. The remainder of the analysis is a mechanical process. The element stiffness matrix calculations, including any numerical integrations required, the assembly of the structural load-deflection relationship, the solution of the structure equations for loads and deflections, and the back substitution into the individual element relationships to obtain stress and strain fields require a huge number of calculations but no engineering judgment. The calculation procedure is clearly suited to the "number crunching" digital computer. Effective use of the computer for models of any substantial size requires that efficient computer-oriented numerical integration and simultaneous equation solvers be incorporated into the solution process.

To this end many large, general-purpose, finite-element-based computer codes have been developed[18,19,20] and are available in the marketplace. These codes feature large element libraries, extremely efficient solution algorithms, and a broad range of applications. The code developers strive to make these codes "user friendly" to minimize the effort required to assemble the computer input once the engineering decisions of grid design and element selection from the element library have been made.

Many special-purpose codes with unique finite elements are available to solve problems beyond the range of the general-purpose codes. Beyond the contents of the marketplace, the creation of a finite-element program for any particular application is a relatively simple process once the required finite element has been designed.

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