# III. Control Volume Relations for Fluid Analysis

From consideration of hydrostatics, we now move to problems involving fluid flow with the addition of effects due to fluid motion, e.g. inertia and convective mass, momentum, and energy terms.

We will present the analysis based on a control volume (not differential element) formulation, e.g. similar to that used in thermodynamics for the first law.

## <u>Basic Conservation Laws</u>:

Each of the following basic conservation laws is presented in its most fundamental, **fixed mass** form. We will subsequently develop an equivalent expression for each law that includes the effects of the flow of mass, momentum, and energy (as appropriate) across a control volume boundary. These transformed equations will be the basis for the control volume analyses developed in this chapter.

### Conservation of Mass:

Defining m as the mass of a fixed mass system, the mass for a control volume V is given by

$$m_{sys} = \int_{sys} \rho \, dV$$

The basic equation for conservation of mass is then expressed as

$$\left.\frac{dm}{dt}\right)_{sys} = 0$$

The time rate of change of mass for the control volume is zero since at this point we are still working with a fixed mass system.

### Linear Momentum:

Defining $\overline{P}_{sys}$ as the linear momentum of a fixed mass, the linear momentum of a fixed mass control volume is given by:

$$\overline{P}_{sys} = m\overline{V} = \int_{sys} \overline{V}\rho dV$$

where $\overline{V}$ is the local fluid velocity and $dV$ is a differential volume element in the control volume.

The basic linear momentum equation is then written as

$$\sum \overline{F} = \left.\frac{d\overline{P}}{dt}\right)_{sys} = \left.\frac{d\left(m\overline{V}\right)}{dt}\right)_{sys}$$

**Moment of Momentum:**

Defining $\overline{H}$ as the moment of momentum for a fixed mass, the moment of momentum for a fixed mass control volume is given by

$$\overline{H}_{sys} = \int_{sys} \overline{r} \times \overline{V}\rho dV$$

where $\overline{r}$ is the moment arm from an inertial coordinate system to the differential control volume of interest. The basic equation is then written as

$$\sum \overline{M}_{sys} = \sum \overline{r} \times \overline{F} = \left.\frac{d\overline{H}}{dt}\right)_{sys}$$

**Energy**:

Defining $E_{sys}$ as the total energy of an element of fixed mass, the energy of a fixed mass control volume is given by

$$E_{sys} = \int_{sys} e\rho dV$$

where $e$ is the total energy per unit mass ( includes kinetic, potential, and internal energy ) of the differential control volume element of interest.

The basic equation is then written as

$$\dot{Q} - \dot{W} = \left.\frac{dE}{dt}\right)_{sys} \qquad \text{(Note: written on a rate basis)}$$

It is again noted that each of the conservation relations as previously written applies only to fixed, constant mass systems.

However, since most fluid problems of importance are for open systems, we must transform each of these relations to an equivalent expression for a control volume which includes the effect of mass entering and/or leaving the system.

This is accomplished with the Reynolds transport theorem.

## Reynolds Transport Theorem

We define a general, extensive property ( an extensive property depends on the size or extent of the system) $B_{sys}$ where

$$B_{sys} = \int_{sys} \beta \rho \, dV$$

$B_{sys}$ could be total mass, total energy, total momentum, etc., of a system.

and $B_{sys}$ per unit mass is defined as $\beta$ or $\beta = \frac{dB}{dm}$

Thus, $\beta$ is the intensive equivalent of $B_{sys}$ .

Applying a general control volume formulation to the time rate of change of $B_{sys}$ , we obtain the following (see text for detailed development):

$$\left.\frac{dB}{dt}\right)_{sys} = \frac{\partial}{\partial t}\int_{cv} \beta \rho dV + \int_{A_e} \beta_e \, \rho_e \, V_e \, d\,A_e - \int_{A_i} \beta_i \, \rho_i \, V_i \, d\,A_i$$

| ↓ | ↓ | ↓ | ↓ |
|---|---|---|---|
| **System rate of change of B** | **Rate of change of B in c.v.** | **Rate of B leaving c.v.** | **Rate of B entering c.v.** |
| ↓ | | ↓ | |
| **transient term** | | **convective terms** | |

where B is any conserved quantity, e.g. **mass, linear momentum, moment of momentum, or energy**.

We will now apply this theorem to each of the basic conservation equations to develop their equivalent open system, control volume forms.

## Conservation of mass

For conservation of mass, we have that

B = m and $\beta = 1$

From the previous statement of conservation of mass and these definitions, Reynolds transport theorem becomes

$$\frac{\partial}{\partial t}\int_{cv} \rho dV + \int_{A_e} \rho_e \, V_e \, d\,A_e - \int_{A_i} \rho_i \, V_i \, d\,A_i = 0$$

or

$$\frac{\partial}{\partial t}\int_{cv} \rho dV + \int_{A_e} \rho_e \, V_e \, d\,A_e - \int_{A_i} \rho_i \, V_i \, d\,A_i = 0$$

| ↓ | ↓ | ↓ |
|---|---|---|
| **Rate of change of mass in c.v.,** | **Rate of mass leaving c.v.,** | **Rate of mass entering c.v.,** |
| ↓ | ↓ | ↓ |
| **= 0 for steady-state** | $\dot{m}_e$ | $\dot{m}_i$ |

This can be simplified to

$$\left.\frac{dm}{dt}\right)_{cv} + \sum \dot{m}_e - \sum \dot{m}_i = 0$$

Note that the exit and inlet velocities $V_e$ and $V_i$ are the local components of fluid velocities at the exit and inlet boundaries **relative to an observer standing on the boundary**. Therefore, if the boundary is moving, the velocity is measured relative to the boundary motion. The location and orientation of a coordinate system for the problem are not considered in determining these velocities.

Also, the result of $\overline{V}_e \cdot d\overline{A}_e$ and $\overline{V}_i \cdot d\overline{A}_i$ is the product of the normal velocity component times the flow area at the exit or inlet, e.g.

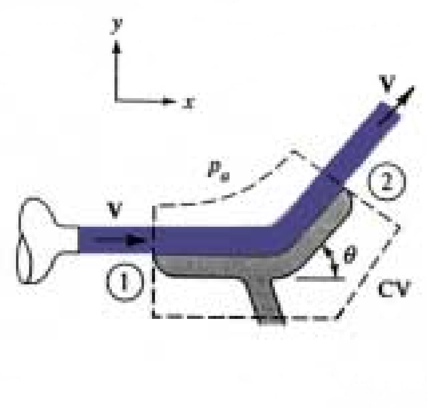
$$V_{e,n}\, dA_e \qquad \text{and} \qquad V_{i,n}\, dA_i$$

**Special Case**: For incompressible flow with a uniform velocity over the flow area, the previous integral expressions simplify to:

$$\dot{m} = \int_{cs} \rho V\, dA = \rho A V$$

## Conservation of Mass Example

Water at a velocity of 7 m/s exits a stationary nozzle with D = 4 cm and is directed toward a turning vane with $\theta = 40^o$, Assume steady-state.




**Determine**:
a. Velocity and flow rate entering the c.v.
b. Velocity and flow rate leaving the c.v.

a. Find $V_1$ and $\dot{m}_1$

Recall that the mass flow velocity is the normal component of velocity measured **relative to the inlet or exit area**.

Thus, relative to the nozzle, V(nozzle) = 7 m/s and since there is no relative motion of point 1 relative to the nozzle, we also have **$V_1$ = 7 m/s ans.**

From the previous equation:

$$\dot{m} = \int_{cs} \rho V \, dA = \rho A V = 998 \text{ kg/m}^3 * 7 \text{ m/s} * \pi * 0.04^2/4$$

$\dot{m}_1$ = **8.78 kg/s ans.**

b. Find $V_2$ and $\dot{m}_2$

Determine the flow rate first.

Since the flow is steady state and no mass accumulates on the vane:

$\dot{m}_1 = \dot{m}_2$ , $\dot{m}_2$ = **8.78 kg/s ans.**

Now: $\dot{m}_2 = 8.78 \text{ kg/s} = \rho A V)_2$

Since $\rho$ and A are constant, **$V_2$ = 7 m/s ans.**

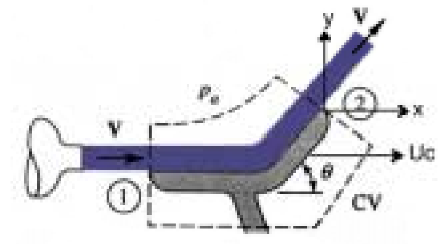
**Key Point**: For steady flow of a constant area, incompressible stream, the flow velocity and total mass flow are the same at the inlet and exit, even though the direction changes.

or alternatively:

**Rubber Hose Concept**: For steady flow of an incompressible fluid, the flow stream can be considered as a rubber hose and if it enters a c.v. at a velocity of V, it exits at a velocity V, even if it is redirected.

**Problem Extension**:

Let the turning vane (and c.v.) now move to the right at a steady velocity of 2 m/s (other values remain the same); perform the same calculations.




Therefore:

Given: $U_c = 2$ m/s  $V_J = 7$ m/s

For an observer standing at the c.v. inlet (point 1)

$V_1 = V_J - U_c = 7 - 2 = 5$ m/s

$\dot{m}_1 = \rho_1 V_1 A_1 = 998 \text{ kg/m}^3 * 5 \text{ m/s} * \pi * 0.04^2/4 = 6.271$ kg/s

Note: The inlet velocity used to specify the mass flow rate is again measured relative to the inlet boundary but now is given by $V_J - U_c$ .

Exit:

$\dot{m}_1 = \dot{m}_2 = 6.271$ kg/s  Again, since $\rho$ and A are constant, $V_2 = 5$ m/s.

Again, the exit flow is most easily specified by conservation of mass concepts.

**Note**: The coordinate system could either have been placed on the moving cart or have been left off the cart with no change in the results.

**Key Point**: The location of the coordinate system does not affect the calculation of mass flow rate which is calculated relative to the flow boundary. It could have been placed at **Georgia Tech** with no change in the results.

Review material and work examples in the text on conservation of mass.

**Linear Momentum**

For linear momentum, we have that

$$\bar{B} = \bar{P} = m\bar{V} \quad \text{and} \quad \bar{\beta} = \bar{V}$$

From the previous statement of linear momentum and these definitions, Reynolds transport theorem becomes

$$\sum \bar{F} = \left.\frac{d\left(m\bar{V}\right)}{dt}\right)_{sys} = \frac{\partial}{\partial t}\int_{cv} \bar{V}\,\rho\,dV + \int_{A_e} \bar{V}\,\rho_e \bar{V}_e \cdot d\bar{A}_e - \int_{A_i} \bar{V}\,\rho_i \bar{V}_i \cdot d\bar{A}_i$$

or

$$\sum \bar{F} = \frac{\partial}{\partial t}\int_{cv} \bar{V}\,\rho\,dV + \int_{A_e} \bar{V}\,d\dot{m}_e - \int_{A_i} \bar{V}\,d\dot{m}_i$$

**= the $\sum$ of the external forces acting on the c.v.**

**= body + point + distributed, e.g. (pressure) forces**

**= the rate of change of momentum in the c.v.**

**= 0 for steady-state**

**= the rate of momentum leaving the c.v.**

**= the rate of momentum entering the c.v.**

and where $\bar{V}$ is the vector momentum velocity relative to an inertial reference frame.

**Key Point**: Thus, the **momentum velocity has magnitude and direction** and is measured relative to the reference frame (coordinate system) being used for the problem. The **velocities in the mass flow terms** $\dot{m}_i$ and $\dot{m}_e$ **are scalars**, as noted previously, and are **measured relative to the inlet or exit boundary**.

Always clearly define a coordinate system and use it to specify the value of all inlet and exit momentum velocities when working linear momentum problems.

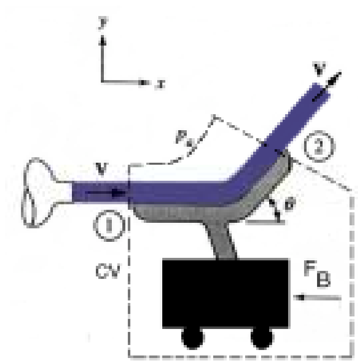
For the 'x' direction, the previous equation becomes

$$\sum \bar{F}_x = \frac{\partial}{\partial t}\int_{cv} \bar{V}_x \, \rho \, dV + \int_{A_e} \bar{V}_{x,e} \, d\dot{m}_e - \int_{A_i} \bar{V}_{x,i} \, d\dot{m}_i$$

Note that the above equation is also valid for control volumes moving at constant velocity with the coordinate system placed on the moving control volume. This is because an inertial coordinate system is a nonaccelerating coordinate system which is still valid for a c.s. moving at constant velocity.

**Example**:

A water jet 4 cm in diameter with a velocity of 7 m/s is directed to a stationary turning vane with $\theta = 40^{o}$. Determine the force F necessary to hold the vane stationary.




Governing equation:

$$\sum \bar{F}_x = \frac{\partial}{\partial t}\int_{cv} \bar{V}_x \, \rho \, dV + \int_{A_e} \bar{V}_{x,e} \, d\dot{m}_e - \int_{A_i} \bar{V}_{x,i} \, d\dot{m}_i$$

Since the flow is steady and the c.v. is stationary, the time rate of change of momentum within the c.v. is zero. Also with uniform velocity at each inlet and exit and a constant flow rate, the momentum equation becomes

$$-F_b = \dot{m}_e V_e - \dot{m}_i V_i$$

Note that the braking force, $F_b$, is written as negative since it is **assumed** to be in the negative x direction relative to positive x for the coordinate system.

From the previous example for conservation of mass, we can again write

$$\dot{m} = \int_{cs} \rho V \, dA = \rho A V = 998 \text{ kg/m}^3 * 7 \text{ m/s} * \pi * 0.04^2/4$$

$\dot{m}_1$ = **8.78 kg/s** and $V_1$ = **7 m/s**

and for the exit:

$\dot{m}_2$ = **8.78 kg/s** and $V_2$ = **7 m/s** inclined 40° above the horizontal.

Substituting in the momentum equation, we obtain

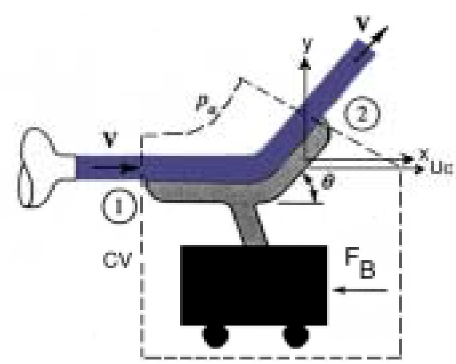
$$-F_b = 8.78 \text{ kg/s} * 7 \text{ m/s} * \cos 40^o - 8.78 \text{ kg/s} * 7 \text{ m/s}$$

and $-F_b = -14.4 \text{ kg m/s}^2$ or $F_b$ = **14.4 N** ← **ans.**

**Note**: Since our final answer is positive, our original assumption of the applied force being to the left was correct. Had we assumed that the applied force was to the right, our answer would be negative, meaning that the direction of the applied force is opposite to what was assumed.

**Modify Problem**:

Now consider the same problem but with the cart moving to the right with a velocity $U_c$ = 2 m/s. Again solve for the value of braking force $F_b$ necessary to maintain a constant cart velocity of 2 m/s.




Note: The coordinate system for the problem has now been placed on the moving cart.

The transient term in the momentum equation is still zero. With the coordinate system on the cart, the momentum of the cart relative to the coordinate system is still zero. The fluid stream is still moving relative to the coordinate system, however, the flow is steady with constant velocity and the time rate of change of momentum of the fluid stream is therefore also zero. Thus

**The momentum equation has the same form as for the previous problem** (However the value of individual terms will be different.)

$$-F_b = \dot{m}_e V_e - \dot{m}_i V_i$$

$\dot{m}_1 = \rho_1 V_1 A_1 = 998 \text{ kg/m}^3 * 5 \text{ m/s} * \pi * 0.04^2/4 = 6.271 \text{ kg/s} = \dot{m}_2$

Now we must determine the momentum velocity at the inlet and exit. With the coordinate system on the moving control volume, the values of momentum velocity are

$V_1 = V_J - U_c = 7 - 2 = 5$ m/s and $V_2 = 5$ m/s inclined $40^o$

The momentum equation ( x - direction ) now becomes

$$-F_b = 6.271 \text{ kg/s} * 5 \text{ m/s} * \cos 40^o - 6.271 \text{ kg/s} * 5 \text{ m/s}$$

and $-F_b = -7.34 \text{ kg m/s}^2$ or $F_b =$ **7.34 N** ← **ans.**

**Question**: What would happen to the braking force $F_b$ if the turning angle had been > $90^o$, e.g., $130^o$? Can you explain based on your understanding of change in momentum for the fluid stream?

Review and work examples for linear momentum with fixed and non-accelerating (moving at constant velocity) control volumes.

## Accelerating Control Volume

The previous formulation applies only to an inertial coordinate system, i.e., fixed or moving at constant velocity (non-accelerating).

We will now consider problems with accelerating control volumes. For these problems we will again place the coordinate system on the accelerating control volume, thus making it a non-inertial coordinate system.

For coordinate systems placed on an accelerating control volume, we must account for the acceleration of the c.s. by correcting the momentum equation for this acceleration. This is accomplished by including the term as shown below:

$$\sum \bar{F} - \int_{cv} \bar{a}_{cv}\, d\, m_{cv} = \frac{\partial}{\partial t}\int_{cv} \bar{V} \rho\, dV + \int_{A_e} \bar{V}\, d\, \dot{m}_e - \int_{A_i} \bar{V}\, d\, \dot{m}_i$$

↓

integral sum of
the local c.v. (c.s.)
acceleration * the c.v. mass

The added term accounts for the acceleration of the control volume and allows the problem to be worked with the coordinate system placed on the accelerating c.v.

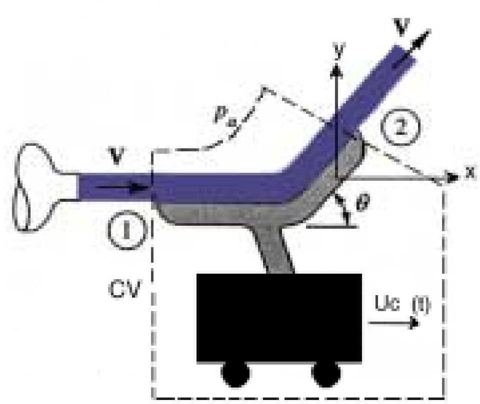
Note: Thus, all vector (momentum) velocities are then measured relative to an observer (coordinate system) on the accelerating control volume. For example, the velocity of a rocket as seen by an observer (c.s.) standing on the rocket is zero and the time rate of change of momentum is zero in this reference frame even if the rocket is accelerating.

### Accelerating Control Volume Example

A turning vane with $\theta = 60^{o}$ accelerates from rest due to a jet of water ($V_J = 35$ m/s, $A_J = 0.003$ m$^2$). Assuming the mass of the cart $m_c$, is 75 kg and neglecting drag and friction effects, find:

a. Cart acceleration at t = 0.

b. $U_c$ as a f(t)



Starting with the general equation shown above, we can make the following assumptions:

1. $\sum F_x = 0$, no friction or body forces.
2. The jet has uniform velocity and constant properties.
3. The entire cart accelerates uniformly over the entire control volume.
4. Neglect the relative momentum change of the jet stream that is within the control volume.

With these assumptions, the governing equation simplifies to

$$-a_c\, m_c = \dot{m}_e\, V_{x,e} - \dot{m}_i\, V_{x,i}$$

We thus have terms that account for the acceleration of the control volume, for the exit momentum, and for the inlet momentum (both of which change with time.)

**Mass flow**:

As with the previous example for a moving control volume, the mass flow terms are given by:

$$\dot{m}_i = \dot{m}_e = \dot{m} = \rho\, A_J\, (V_J - U_c)$$

Note that since the cart accelerates, $U_c$ is not a constant but rather changes with time.

**Momentum velocities:**

$$U_{x,i} = V_J - U_c \qquad U_{x,e} = (V_J - U_c)\cos\theta$$

Substituting, we now obtain

$$- a_c\, m_c = \rho\, A_J\, (V_J - U_c)^2 \cos\theta - \rho\, A_J\, (V_J - U_c)$$

Solving for the cart acceleration, we obtain

$$a_c = \frac{\rho A_J\, (1 - \cos\theta)(V_J - U_c)^2}{m_c}$$

Substituting for the given values at t = 0, i.e., $U_c = 0$, we obtain

$$a_c\,(t = 0) = 24.45 \text{ m/s}^2 = 2.49 \text{ g's}$$

Note: The acceleration at any other time can be obtained once the cart velocity $U_c$ at that time is known.

To determine the equation for cart velocity as a function of time, the equation for the acceleration must be written in terms of $U_c$ (t) and integrated.

$$\frac{dU_c}{dt} = \frac{\rho A_J (1 - \cos\theta)(V_J - U_c)^2}{m_c}$$

Separating variables, we obtain

$$\int_0^{U_c(t)} \frac{dU_c}{(V_J - U_c)^2} = \int_0^t \frac{\rho A_J (1 - \cos\theta)}{m_c}\, dt$$

Completing the integration and rearranging the terms, we obtain a final expression of the form

$$\frac{U_c}{V_J} = \frac{V_J\, b\, t}{1 + V_J\, b\, t} \qquad \text{where} \qquad b = \frac{\rho A_J (1 - \cos\theta)}{m_c}$$

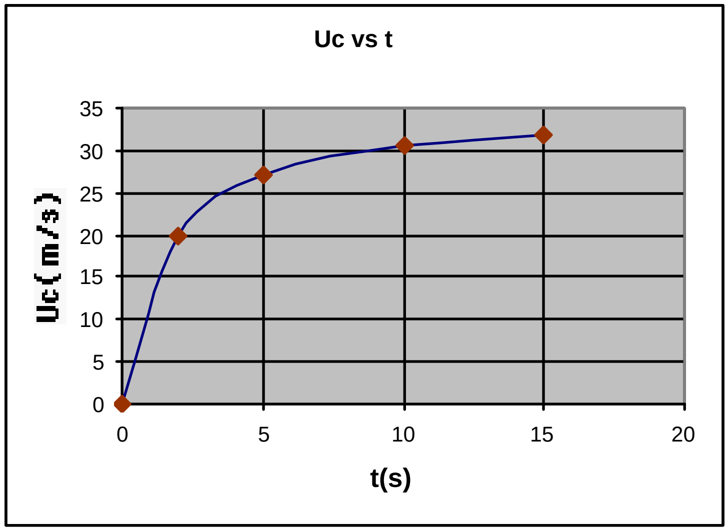
Substituting for known values, we obtain $V_J\, b = 0.699 \text{ s}^{-1}$

Thus the final equation for $U_c$ is give by

$$\frac{U_c}{V_J} = \frac{0.699\, t}{1 + 0.699\, t}$$

The final results are now given as shown below:

| t (s) | $U_c/V_J$ | $U_c$ (m/s) | $a_c$ ($m/s^2$) |
|---|---|---|---|
| 0 | 0.0 | 0.0 | 24.45 |
| 2 | 0.583 | 20.0 | 4.49 |
| 5 | 0.757 | 27.2 | 1.22 |
| 10 | 0.875 | 30.6 | 0.39 |
| 15 | 0.912 | 31.9 | 0.192 |
| ∞ | 1.0 | 35 | 0.0 |




Note that the limiting case occurs when the cart velocity reaches the jet velocity. At this point, the jet can impart no more momentum to the cart, the acceleration is now zero, and the terminal velocity has been reached.

Review the text example on accelerating control volumes.

**Moment of Momentum (angular momentum)**

For moment of momentum we have that

$$\overline{B} = \overline{H} = \overline{r} \times \left( m\,\overline{V} \right) \quad and \quad \overline{\beta} = \overline{r} \times \overline{V}$$

From the previous equation for moment of momentum and these definitions, Reynolds transport theorem becomes

$$\sum \bar{M} = \frac{\partial}{\partial t}\int_{cv} \bar{r} \times \bar{V} \rho dV + \int_{A_e} \bar{r} \times \bar{V} d\dot{m}_e - \int_{A_i} \bar{r} \times \bar{V} d\dot{m}_i$$

↓ ↓ ↓ ↓

| | | | |
|---|---|---|---|
| **= the $\sum$ of all external moments acting on the c.v.** | **= the rate of change of moment of momentum in the c.v. = 0 for steady state** | **= the rate of moment of momentum leaving the c.v.** | **= the rate of moment of momentum entering the c.v.** |

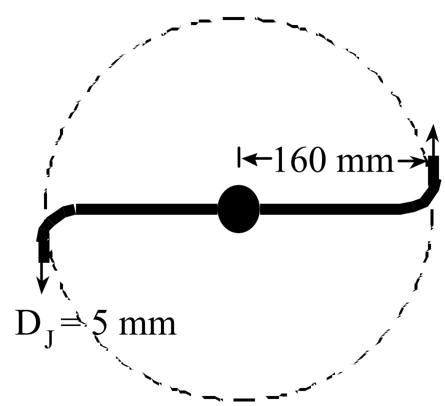
For the special case of steady-state, steady-flow and uniform properties at any exit or inlet, the equation becomes

$$\sum \bar{M} = \sum \dot{m}_e \, \bar{r} \times V_e - \sum \dot{m}_i \, \bar{r} \times V_i$$

For moment of momentum problems, we must be careful to correctly evaluate the moment of all applied forces and all inlet and exit momentum flows, with particular attention to the signs.

**Moment of Momentum Example**:

A small lawn sprinkler operates as indicated. The inlet flow rate is 9.98 kg/min with an inlet pressure of 30 kPa. The two exit jets direct flow at an angle of 40° above the horizontal.




For these conditions, determine the following:

a. Jet velocity relative to the nozzle.

b. Torque required to hold the arm stationary.

c. Friction torque if the arm is rotating at 35 rpm.

d. Maximum rotational speed if we neglect friction.

a. R = 160 mm, $D_J$ = 5 mm, Therefore, for each of the two jets:

$Q_J = 0.5 * 9.98\ kg/min/998\ kg/m^3 = 0.005\ m^3/min$

$A_J = \pi\ \pi\ 0.0025^2 = 1.963*10^{-5}\ m^2$

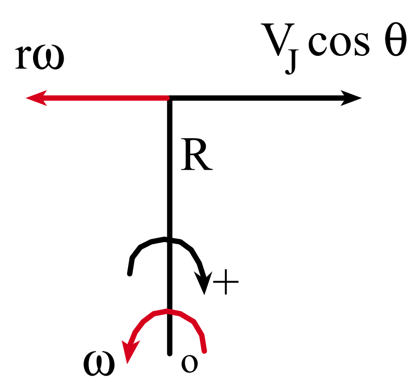
$V_J = 0.005\ m^3/min\ /\ 1.963*10^{-5}\ m^2\ /60\ s/min$

$V_J = 4.24$ m/s relative to the nozzle exit **ans.**

b. Torque required to hold the arm stationary.

First develop the governing equations and analysis for the general case of the arm rotating.

With the coordinate system at the center of rotation of the arm, a general velocity diagram for the case when the arm is rotating is shown in the adjacent schematic.




Taking the moment about the center of rotation, the moment of the inlet flow is zero since the moment arm is zero for the inlet flow.

The basic equation then becomes

$$T_0 = 2\dot{m}_e R\left(V_J \cos\alpha - R\,\omega\right)$$

Note that the net momentum velocity is the difference between the tangential component of the jet exit velocity and the rotational speed of the arm. Also note that the direction of positive moments was taken as the same as for $V_J$ and opposite to the direction of rotation.

For a stationary arm $R\,\omega = 0$. We thus obtain for the stationary torque

$T_o = 2\,\rho\, Q_J\, R\, V_J \cos\alpha$

$$T_o = 2 * 998 \frac{kg}{m^3} .005 \frac{m^3}{\min} \frac{1\min}{60} m * 4.24 \frac{m}{s} \cos 4.160^o$$

$T_o = 0.0864$ N m clockwise. **ans.**

A resisting torque of 0.0864 N m must be applied in the clockwise direction to keep the arm from rotating in the counterclockwise direction.

c. At $\omega = 30$ rpm, calculate the friction torque $T_f$

$$\omega = 30 \frac{rev}{\min} 2\pi \frac{rad}{rev} \frac{1\min}{60} = \pi \frac{rad}{s}$$

$$T_o = 2 * 998 \frac{kg}{m^3} 0.005 \frac{m^3}{\min} \frac{1\min}{60} 0.16m \left[ 4.24 \frac{m}{s} \cos 40^o - .16m * \pi \frac{rad}{s} \right]$$

**ans.**

Note; The resisting torque decreases as the speed increases.

d. Find the maximum rotational speed.

The maximum rotational speed occurs when the opposing torque is zero and all the moment of momentum goes to the angular rotation. For this case,

$$V_J \cos\theta - R\omega = 0$$

$$\omega = \frac{V_J \cos\theta}{R} = \frac{4.2 \frac{rad}{s} = 193.8 rpm 4 m/s \bullet \cos 40}{0.16\, m} = 20.3 \frac{rad}{s} = 193.8\, rpm$$ **ans.**

Review material and examples on moment of momentum.

**Energy Equation (Extended Bernoulli Equation)**

For energy, we have that

$$B = E = \int_{cv} e\,\rho\,dV \qquad \text{and} \qquad \beta = e = u + \frac{1}{2}V^2 + g\,z$$

From the previous statement of conservation of energy and these definitions, Reynolds transport theorem becomes:

$$\dot{Q} - \dot{W} = \left.\frac{dE}{dt}\right)_{sys} = \frac{\partial}{\partial t}\int_{cv} e\,\rho\,dV + \int_{A_e} e_e\,\rho_e\,\bar{V}_e \cdot d\,A_e - \int_{A_i} e_i\,\rho_i\,\bar{V}_i \cdot d\,A_i$$

After extensive algebra and simplification (see text for detailed development), we obtain:

$$\frac{P_1 - P_2}{\rho\,g} = \frac{V_2^2 - V_1^2}{2\,g} + Z_2 - Z_1 + h_{f,1-2} - h_p$$

| $\frac{P_1 - P_2}{\rho g}$ ↓ | $\frac{V_2^2 - V_1^2}{2g}$ ↓ | $Z_2 - Z_1$ ↓ | $h_{f,1-2}$ ↓ | $h_p$ ↓ |
|---|---|---|---|---|
| **Pressure drop from 1 – 2, in the flow direction** | **Pressure drop due to acceleration of the fluid** | **Pressure drop due to elevation change** | **Pressure drop due to frictional head loss** | **Pressure drop due to mechanical work on fluid** |

Note: this formulation must be written in the flow direction from 1 - 2 to be consistent with the sign of the mechanical work term and so that $h_{f,1\text{-}2}$ is always a positive term. Also note the following:

- The points 1 and 2 must be specific points along the flow path
- Each term has units of linear dimension, e.g., ft or meters, and $z_2 - z_1$ is positive for $z_2$ above $z_1$
- The term $h_{f,1\text{-}2}$ is always positive when written in the flow direction and for internal, pipe flow includes pipe or duct friction losses and fitting or piping component (valves, elbows, etc.) losses,

- The term $h_p$ is positive for pumps and fans ( i.e., pumps increase the pressure in the flow direction) and negative for turbines (turbines decrease the pressure in the flow direction)
- For pumps:

$$h_p = \frac{w_s}{g}$$ where $w_s$ = the useful work per unit mass to the fluid

Therefore: $w_s = g\, h_p$ and $\dot{W}_f = \dot{m} w_s = \rho Q g\, h_p$

where $\dot{W}_f$ = the useful power delivered to the fluid

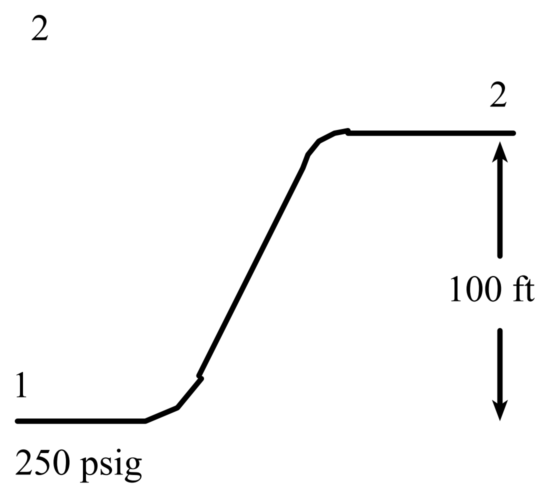
and $\dot{W}_p = \dfrac{\dot{W}_f}{\eta_p}$ where $\eta_p$ is the pump efficiency

**Example**

Water flows at 30 ft/s through a 1000 ft length of 2 in diameter pipe. The inlet pressure is 250 psig and the exit is 100 ft higher than the inlet.

Assuming that the frictional loss is given by 18 $V^2/2g$,

Determine the exit pressure.




Given: $V_1 = V_2 = 30$ ft/s, L = 1000 ft, $Z_2 - Z_1 = 100$ ft, $P_1 = 250$ psig

Also, since there is no mechanical work in the process, the energy equation simplifies to

$$\frac{P_1 - P_2}{\rho g} = Z_2 - Z_1 \quad + \quad h_f$$

$$\frac{P_1 - P_2}{\rho g} = 100\ ft + 18\frac{30^2\ ft^2/s^2}{64.4\ ft/s^2} \quad = \quad 351.8\ ft$$

$P_1 - P_2 = 62.4$ lbf/ft$^3$ 351.8 ft = 21,949 psf = 152.4 psi

$P_2 = 250 - 152.4 = 97.6$ psig **ans.**

**Problem Extension**

A pump driven by an electric motor is now added to the system. The motor delivers 10.5 hp. The flow rate and inlet pressure remain constant and the pump efficiency is 71.4 %, determine the new exit pressure.

$$Q = AV = \pi\ \pi\ (1/12)^2\ ft^2 * 30\ ft/s = 0.6545\ ft^3/s$$

$$W_f = \eta_p\ W_p = \rho\ Q\ g\ h_p$$

$$h_p = \frac{0.714 * 10.5\,hp * 550\,ft - lbf/s/hp}{62.4\,lbm/ft^3 * 0.6545\,ft^3/s} \quad = 101\,ft$$

The pump adds a head increase equal to 101 ft to the system and the exit pressure should increase.

Substituting in the energy equation, we obtain

$$\frac{P_1 - P_2}{\rho g} = 100\ ft + 18\frac{30^2\ ft^2/s^2}{64.4\ ft/s^2} \quad - 101\ ft = \quad 250.8\ ft$$

$P_1 - P_2 = 62.4$ lbf/ft$^3$ 250.8 ft = 15,650 psf = 108.7 psi

$P_2 = 250 - 108.7 = 141.3$ psig **ans.**

Review examples for the use of the energy equation