Given: Common substances

Tar	Sand
"Silly Putty"	Jello
Modeling clay	Toothpaste
Wax	Shaving cream

Some of these substances exhibit characteristics of solids and fluids under different conditions.

Find: Explain and give examples.

Solution:

Tar, wax, and Jello behave as solids at room temperature or below at ordinary pressures. At high pressures or over long periods, they exhibit fluid characteristics. At higher temperatures, all three liquefy and become viscous fluids.

Modeling clay and silly putty show fluid behavior when sheared slowly. However, they fracture under suddenly applied stress, which is a characteristic of solids.

Toothpaste behaves as a solid when at rest in the tube. When the tube is squeezed hard, toothpaste "flows" out the spout, showing fluid behavior. Shaving cream behaves similarly.

Sand act solid when in repose (a sand "pile"). However, it "flows" from a spout or down a steep incline.

Problem 1.2

Given: Five basic conservation laws stated in Section 1-4.

Write: A word statement of each, as they apply to a system.

Solution: Assume that laws are to be written for a system.

(a) Conservation of mass – The mass of a system is constant by definition.

(b) Newton's second law of motion – The net force acting on a system is directly proportional to the product of the system mass times its acceleration.

(c) First law of thermodynamics – The change in stored energy of a system equals the net energy added to the system as heat and work.

(d) Second law of thermodynamics – The entropy of any isolated system cannot decrease during any process between equilibrium states.

(e) Principle of angular momentum – The net torque acting on a system is equal to the rate of change of angular momentum of the system.

Problem 1.3

Open-Ended Problem Statement: Consider the physics of "skipping" a stone across the water surface of a lake. Compare these mechanisms with a stone as it bounces after being thrown along a roadway.

Discussion: Observation and experience suggest two behaviors when a stone is thrown along a water surface:

(1) If the angle between the path of the stone and the water surface is steep the stone may penetrate the water surface. Some momentum of the stone will be converted to momentum of the water in the resulting splash. After penetrating the water surface, the high drag* of the water will slow the stone quickly. Then, because the stone is heavier than water it will sink.

(2) If the angle between the path of the stone and the water surface is shallow the stone may not penetrate the water surface. The splash will be smaller than if the stone penetrated the water surface. This will transfer less momentum to the water, causing less reduction in speed of the stone. The only drag force on the stone will be from friction on the water surface. The drag will be momentary, causing the stone to lose only a portion of its kinetic energy. Instead of sinking, the stone may skip off the surface and become airborne again.

When the stone is thrown with speed and angle just right, it may skip several times across the water surface. With each skip the stone loses some forward speed. After several skips the stone loses enough forward speed to penetrate the surface and sink into the water.

Observation suggests that the shape of the stone significantly affects skipping. Essentially spherical stones may be made to skip with considerable effort and skill from the thrower. Flatter, more disc-shaped stones are more likely to skip, provided they are thrown with the flat surface(s) essentially parallel to the water surface; spin may be used to stabilize the stone in flight.

By contrast, no stone can ever penetrate the pavement of a roadway. Each collision between stone and roadway will be inelastic; friction between the road surface and stone will affect the motion of the stone only slightly. Regardless of the initial angle between the path of the stone and the surface of the roadway, the stone may bounce several times, then finally it will roll to a stop.

The shape of the stone is unlikely to affect trajectory of bouncing from a roadway significantly.

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* Compared to the negligible aerodynamic drag in air.

Problem 1.4

Open-Ended Problem Statement: The barrel of a bicycle tire pump becomes quite warm during use. Explain the mechanisms responsible for the temperature increase.

Discussion: Two phenomena are responsible for the temperature increase: (1) friction between the pump piston and barrel and (2) temperature rise of the air as it is compressed in the pump barrel.

Friction between the pump piston and barrel converts mechanical energy (force on the piston moving through a distance) into thermal energy as a result of friction. Lubricating the piston helps to provide a good seal with the pump barrel and reduces friction (and therefore force) between the piston and barrel.

Temperature of the trapped air rises as it is compressed. The compression is not adiabatic because it occurs during a finite time interval. Heat is transferred from the warm compressed air in the pump barrel to the cooler surroundings. This raises the temperature of the barrel, making its outside surface warm (or even hot!) to the touch.

Problem 1.5

Given: Tank to contain 15 kg of O_2 at 10 MPa, 35°C.

Find: Tank volume and diameter if spherical.

Solution: Assume ideal gas behavior.

Basic equations: $p = \rho R T$ (p = absolute pressure)

$$\rho = \frac{m}{\forall}$$

Substituting, we obtain $p = \frac{mRT}{\forall}$, so

$$\forall = \frac{mRT}{p}$$

From Table A.6, $R = 259.8\ \text{N}\cdot\text{m/kg}\cdot\text{K}$, so

$$\forall = 15\ \text{kg} \times 259.8\ \frac{\text{N}\cdot\text{m}}{\text{kg}\cdot\text{K}} \times (273+35)\ \text{K} \times \frac{\text{m}^2}{(10\times10^6 + 101\times10^3)\ \text{N}}$$

$$\forall = 0.119\ \text{m}^3$$ $\forall$

For a sphere, $\forall = \frac{4}{3}\pi R^3 = \frac{1}{6}\pi D^3$, so

$$D = \left[\frac{6\forall}{\pi}\right]^{\frac{1}{3}} = \left[\frac{6}{\pi} \times 0.119\ \text{m}\right]^{\frac{1}{3}} = 0.61\ \text{m}$$ D

Problem 1.6

Make a guess at the order of magnitude of the mass (e.g., 0.01, 0.1, 1.0, 10, 100, or 1000 lbm or kg) of standard air that is in a room 10 ft by 10 ft by 8 ft, and then compute this mass in lbm and kg to see how close your estimate was.

Solution

Given: Dimensions of a room.

Find: Mass of air in lbm and kg.

The data for standard air are:

$$R_{air} = 53.33 \cdot \frac{ft \cdot lbf}{lbm \cdot R} \qquad p = 14.7 \cdot psi \qquad T = (59 + 460) \cdot R = 519 \cdot R$$

Then $\rho = \dfrac{p}{R_{air} \cdot T}$

$$\rho = 14.7 \cdot \frac{lbf}{in^2} \times \frac{1}{53.33} \cdot \frac{lbm \cdot R}{ft \cdot lbf} \times \frac{1}{519 \cdot R} \times \left(\frac{12 \cdot in}{1 \cdot ft} \right)^2$$

$$\rho = 0.0765 \frac{lbm}{ft^3} \qquad \text{or} \qquad \rho = 1.23 \frac{kg}{m^3}$$

The volume of the room is $V = 10 \cdot ft \times 10ft \times 8ft \qquad V = 800\,ft^3$

The mass of air is then $m = \rho \cdot V$

$$m = 0.0765 \cdot \frac{lbm}{ft^3} \times 800 \cdot ft^3 \qquad m = 61.2\,lbm \qquad m = 27.8\,kg$$

Problem 1.7

A tank of compressed nitrogen for industrial process use is a cylinder with 6 in. diameter and 4.25 ft length. The gas pressure is 204 atmospheres (gage). Calculate the mass of nitrogen in the tank.

Given: Data on nitrogen tank

Find: Mass of nitrogen

Solution

The given or available data is:

$D = 6 \cdot in$ $\quad L = 4.25 \cdot ft$ $\quad p = 204 \cdot atm$

$T = (59 + 460) \cdot R$ $\quad T = 519\,R$ $\quad R_{N2} = 55.16 \cdot \frac{ft \cdot lbf}{lb \cdot R}$ (Table A.6)

The governing equation is the ideal gas equation

$p = \rho \cdot R_{N2} \cdot T$ and $\rho = \frac{M}{V}$

where *V* is the tank volume $V = \frac{\pi}{4} \cdot D^2 \cdot L$

$$V = \frac{\pi}{4} \times \left(\frac{6}{12} \cdot ft\right)^2 \times 4.25 \cdot ft = 0.834\,ft^3$$

Hence $M = V \cdot \rho = \frac{p \cdot V}{R_{N2} \cdot T}$

$$M = 204 \times 14.7 \cdot \frac{lbf}{in^2} \times \frac{144 \cdot in^2}{ft^2} \times 0.834 \cdot ft^3 \times \frac{1}{55.16} \cdot \frac{lb \cdot R}{ft \cdot lbf} \times \frac{1}{519} \cdot \frac{1}{R} \times 32.2 \cdot \frac{lb \cdot ft}{s^2 \cdot lbf}$$

$M = 12.6\,lb$ $\quad M = 0.391\,slug$

Problem 1.8

Given: Air at standard conditions – $p = 29.9$ in Hg, $T = 59°F$
Uncertainty: in p is ± 0.1 in. Hg, in T is $\pm 0.5°F$
Note that 29.9 in Hg corresponds to 14.7 psia

Find: (a) air density using ideal gas equation of state.
(b) estimate uncertainty in calculated value.

Solution:

$$\rho = \frac{p}{RT} = 14.7\,\frac{\text{lbf}}{\text{in}^2} \times \frac{\text{lb}\,°\text{R}}{53.3\,\text{ft}\cdot\text{lbf}} \times \frac{1}{519°\text{R}} \times 144\,\frac{\text{in}^2}{\text{ft}^2}$$

$$\rho = 0.0765\ \text{lbm/ft}^3 \qquad \rho$$

The uncertainty in density is given by

$$u_\rho = \left[\left(\frac{p}{\rho}\frac{\partial \rho}{\partial p}u_p\right)^2 + \left(\frac{T}{\rho}\frac{\partial \rho}{\partial T}u_T\right)^2\right]^{1/2}$$

$$\frac{p}{\rho}\frac{\partial \rho}{\partial p} = RT\,\frac{1}{RT} = \frac{RT}{RT} = 1\ ; \quad u_p = \frac{\pm 0.1}{29.9} = \pm 0.334\%$$

$$\frac{T}{\rho}\frac{\partial \rho}{\partial T} = \frac{T}{\rho}\left(-\frac{p}{RT^2}\right) = -\frac{p}{\rho RT} = -1\ ; \quad u_T = \frac{\pm 0.5}{460+59} = \pm 0.0963\%$$

Then

$$u_\rho = \left[(u_p)^2 + (-u_T)^2\right]^{1/2} = \pm\left[(0.334)^2 + (-0.0963)^2\right]$$

$$u_\rho = \pm 0.348\% \quad (\pm 2.66 \times 10^{-4}\ \text{lbm/ft}^3) \qquad u_\rho$$

Problem 1.9

Given: Air at pressure, $p = 759 \pm 1$ mm Hg and temperature, $T = -20 \pm 0.5°C$.
Note that 759 mm Hg corresponds to 101 kPa.

Find: (a) air density using ideal gas equation of state
(b) estimate of uncertainty in calculated value

Solution:

$$\rho = \frac{p}{RT} = 101 \times 10^3 \frac{N}{m^2} \times \frac{kg \cdot K}{287 \, N \cdot m} \times \frac{1}{253 \, K} = 1.39 \, kg/m^3$$ ← ρ

The uncertainty in density is given by

$$u_\rho = \left[\left(\frac{p}{\rho} \frac{\partial \rho}{\partial p} u_p \right)^2 + \left(\frac{T}{\rho} \frac{\partial \rho}{\partial T} u_T \right)^2 \right]^{1/2}$$

$$\frac{p}{\rho} \frac{\partial \rho}{\partial p} = RT \frac{1}{RT} = 1 \; ; \quad u_p = \frac{\pm 1}{759} = \pm 0.132\%$$

$$\frac{T}{\rho} \frac{\partial \rho}{\partial T} = \frac{T}{\rho}\left(-\frac{p}{RT^2}\right) = -\frac{p}{\rho RT} = -1 \; ; \quad u_T = \frac{\pm 0.5}{273 - 20} = \pm 0.198\%$$

Then

$$u_\rho = \left[(u_p)^2 + (-u_T)^2 \right]^{1/2} = \pm \left[(0.132)^2 + (-0.198)^2 \right]^{1/2}$$

$$u_\rho = \pm 0.238\% \quad (\pm 3.31 \times 10^{-3} \, kg/m^3)$$ ← u_ρ

Problem 1.10

Given: Standard American golf ball: $m = 1.62 \pm 0.01$ oz (20 to 1)

$D = 1.68 \pm 0.01$ in. (20 to 1)

Find: (a) Density and specific gravity.
(b) Estimate uncertainties in calculated values.

Solution: Density is mass per unit volume, so

$$\rho = \frac{m}{\forall} = \frac{m}{\frac{4}{3}\pi R^3} = \frac{3}{4\pi}\frac{m}{(D/2)^3} = \frac{6}{\pi}\frac{m}{D^3}$$

$$\rho = \frac{6}{\pi} \times 1.62\,\text{oz} \times \frac{1}{(1.68)^3\,\text{in.}^3} \times \frac{0.4536\,\text{kg}}{16\,\text{oz}} \times \frac{\text{in.}^3}{(0.0254)^3\,\text{m}^3} = 1130\ \text{kg/m}^3$$

and

$$SG = \frac{\rho}{\rho_{H_2O}} = 1130\,\frac{\text{kg}}{\text{m}^3} \times \frac{\text{m}^3}{1000\,\text{kg}} = 1.13$$

The uncertainty in density is given by

$$u_\rho = \pm\left[\left(\frac{m}{\rho}\frac{\partial\rho}{\partial m}u_m\right)^2 + \left(\frac{D}{\rho}\frac{\partial\rho}{\partial D}u_D\right)^2\right]^{1/2}$$

$$\frac{m}{\rho}\frac{\partial\rho}{\partial m} = \frac{m}{\rho}\frac{1}{\forall} = \frac{\forall}{\forall} = 1 \; ; \; u_m = \pm\frac{0.01}{1.62} = \pm 0.617 \text{ percent}$$

$$\frac{D}{\rho}\frac{\partial\rho}{\partial D} = \frac{D}{\rho}\left(-3\frac{6}{\pi}\frac{m}{D^4}\right) = \frac{\pi D^4}{6\,m}\left(-3\frac{6}{\pi}\frac{m}{D^4}\right) = -3 \; ; \; u_D = \pm 0.595 \text{ percent}$$

Thus $u_\rho = \pm\left[(u_m)^2 + (-3u_D)^2\right]^{1/2}$

$$= \pm\left\{(0.617)^2 + [-3(0.595)]^2\right\}^{1/2}$$

$u_\rho = \pm 1.89$ percent (± 21.4 kg/m³)

$u_{SG} = u_\rho = \pm 1.89$ percent (± 0.0214)

Finally,

$\rho = 1130 \pm 21.4$ kg/m³ (20 to 1) ← ρ

$SG = 1.13 \pm 0.0214$ (20 to 1) ← SG

Given: Mass flow rate of water determined by collecting discharge over a timed interval is 0.2 kg/s.
Scales can be read to nearest 0.05 kg.
Stopwatch can be read to nearest 0.2 s.

Find: Estimate precision of flow rate calculation for time intervals of (a) 10 s, and (b) 1 min.

Solution: Apply methodology of uncertainty analysis, Appendix F:

Computing equations: $\dot{m} = \dfrac{\Delta m}{\Delta t}$

$$u_{\dot{m}} = \pm\left[\left(\frac{\Delta m}{\dot{m}}\frac{\partial \dot{m}}{\partial \Delta m} u_{\Delta m}\right)^2 + \left(\frac{\Delta t}{\dot{m}}\frac{\partial \dot{m}}{\partial \Delta t} u_{\Delta t}\right)^2\right]^{1/2}$$

Thus

$$\frac{\Delta m}{\dot{m}}\frac{\partial \dot{m}}{\partial \Delta m} = \Delta t\left(\frac{1}{\Delta t}\right) = 1 \quad \text{and} \quad \frac{\Delta t}{\dot{m}}\frac{\partial \dot{m}}{\partial \Delta t} = \frac{\Delta t^2}{\Delta m}\left[(-1)\frac{\Delta m}{\Delta t^2}\right] = -1$$

The uncertainties are expected to be ± half the least counts of the measuring instruments.

Tabulating results:

Time Interval, Δt (s)	Error in Δt (s)	Uncertainty in Δt (percent)	Water Collected, Δm (kg)	Error in Δm (kg)	Uncertainty in Δm (percent)	Uncertainty in $\dot{m}$ (percent)
10	±0.10	±1.0	2.0	±0.025	±1.25	±1.60
60	±0.10	±0.167	12.0	±0.025	±0.208	±0.267

A time interval of about 15 seconds should be chosen to reduce the uncertainty in results to ±1 percent.

Problem 1.12

Given: Pet food can $H = 102 \pm 1\ mm$ (20 to 1)

$D = 73 \pm 1\ mm$ (20 to 1)

$m = 397 \pm 1\ g$ (20 to 1)

Find: Magnitude and estimated uncertainty of pet food density.

Solution: Density is $\rho = \frac{m}{\forall} = \frac{m}{\pi R^2 H} = \frac{4}{\pi}\frac{m}{D^2 H}$ or $\rho = \rho(m, D, H)$

From uncertainty analysis

$$u_\rho = \pm\left[\left(\frac{m}{\rho}\frac{\partial\rho}{\partial m}u_m\right)^2 + \left(\frac{D}{\rho}\frac{\partial\rho}{\partial D}u_D\right)^2 + \left(\frac{H}{\rho}\frac{\partial\rho}{\partial H}u_H\right)^2\right]^{\frac{1}{2}}$$

Evaluating, $\frac{m}{\rho}\frac{\partial\rho}{\partial m} = \frac{m}{\rho}\frac{4}{\pi}\frac{1}{D^2H} = \frac{1}{\rho}\frac{4m}{\pi D^2 H} = 1; \quad u_m = \frac{\pm 1}{397} = \pm\ 0.252\%$

$$\frac{D}{\rho}\frac{\partial\rho}{\partial D} = \frac{D}{\rho}(-2)\frac{4m}{\pi D^3 H} = (-2)\frac{1}{\rho}\frac{4m}{\pi D^2 H} = -2; \quad u_D = \frac{\pm 1}{73} = \pm\ 1.37\%$$

$$\frac{H}{\rho}\frac{\partial\rho}{\partial H} = \frac{H}{\rho}(-1)\frac{4m}{\pi D^2 H^2} = (-1)\frac{1}{\rho}\frac{4m}{\pi D^2 H} = -1; \quad u_H = \frac{\pm 1}{102} = \pm 0.980\%$$

Substituting

$$u_\rho = \pm\left\{[(1)(0.252)]^2 + [(-2)(1.37)]^2 + [(-1)(0.980)]^2\right\}^{1/2}$$

$u_\rho = \pm\ 2.92$ percent ← u_ρ

$$\forall = \frac{\pi}{4}D^2H = \frac{\pi}{4}\times(73)^2\,mm^2 \times 102\ mm \times \frac{m^3}{10^9\,mm^3} = 4.27\times 10^{-4}\ m^3$$

$$\rho = \frac{m}{\forall} = \frac{397\ g}{4.27\times10^{-4}\ m^3}\times\frac{kg}{1000\ g} = 930\ kg/m^3$$

Thus $\rho = 930 \pm 27.2\ kg/m^3$ (20 to 1) ← ρ

Problem 1.13

Given: Standard British golf ball:

$M = 45.9 \pm 0.3\,g$ (20 to 1)

$D = 41.1 \pm 0.3\,mm$ (20 to 1)

Find: (a) Density and specific gravity

(b) Estimate of uncertainties in calculated values.

Solution:

Density is mass per unit volume, so

$$\rho = \frac{M}{V} = \frac{M}{\frac{4}{3}\pi R^3} = \frac{3}{4\pi}\frac{M}{(D/2)^3} = \frac{6}{\pi}\frac{M}{D^3}$$

$$\rho = \frac{6}{\pi} \times 0.0459\,kg \times \frac{1}{(0.0411)^3\,m^3} = 1260\,kg/m^3$$

and

$$SG = \frac{\rho}{\rho_{H_2O}} = 1260\,\frac{kg}{m^3} \times \frac{m^3}{1000\,kg} = 1.26$$

The uncertainty in density is given by

$$u_\rho = \pm\left[\left(\frac{M}{\rho}\frac{\partial\rho}{\partial M}u_M\right)^2 + \left(\frac{D}{\rho}\frac{\partial\rho}{\partial D}u_D\right)^2\right]^{1/2}$$

$$\frac{M}{\rho}\frac{\partial\rho}{\partial M} = \frac{M}{\rho}\frac{1}{V} = \frac{V}{V} = 1 \quad ; \quad u_M = \pm\frac{0.3}{45.9} = \pm 0.654\,\%$$

$$\frac{D}{\rho}\frac{\partial\rho}{\partial D} = \frac{D}{\rho}\left(-3\,\frac{6}{\pi}\frac{M}{D^4}\right) = -3\left(\frac{6M}{\pi D^3 \rho}\right) = -3$$

$$u_D = \pm\frac{0.3}{41.1} = 0.730\,\%$$

Thus

$$u_\rho = \pm\left[(u_M)^2 + (-3u_D)^2\right]^{1/2} = \pm\left\{(0.654)^2 + \left[-3(0.730)\right]^2\right\}^{1/2}$$

$$u_\rho = \pm 2.29\,\% \quad (\pm 28.9\,kg/m^3)$$

$$u_{SG} = u_\rho = \pm 2.29\,\% \quad (\pm 0.0289)$$

Summarizing

$\rho = 1260 \pm 28.9\,kg/m^3$ (20 to 1) ← ρ

$SG = 1.26 \pm 0.0289$ (20 to 1) ← SG

Problem 1.14

Given: Nominal mass flow rate of water determined by collecting discharge (in a beaker) over a timed interval is $\dot{m} = 100 \text{ g/s}$

- scales have capacity of 1 kg, with least count of 1 g.
- timer has least count of 0.1 s.
- beakers with volume of 100, 500, 1000 mL are available
  - tare mass of 1000 mL beaker is 500 g.

Find: Estimate (a) time intervals, and (b) uncertainties, in measuring mass flow rate from using each of the three beakers.

Solution: To estimate time intervals assume beaker is filled to maximum volume in case of 100 and 500 mL beakers and to maximum allowable mass of water (500 g) in case of 1000 mL beaker.

Then $\dot{m} = \dfrac{\Delta m}{\Delta t}$ and $\Delta t = \dfrac{\Delta m}{\dot{m}} = \dfrac{\rho \Delta \forall}{\dot{m}}$

Tabulating results

$\Delta \forall =$	100 mL	500 mL	1000 mL
$\Delta t =$	1 s	5 s	5 s

Apply the methodology of uncertainty analysis, Appendix E

Computing equation:

$$u_{\dot{m}} = \pm\left[\left(\frac{\Delta m}{\dot{m}}\frac{\partial \dot{m}}{\partial \Delta m} u_{\Delta m}\right)^2 + \left(\frac{\Delta t}{\dot{m}}\frac{\partial \dot{m}}{\partial \Delta t} u_{\Delta t}\right)^2\right]^{1/2}$$

The uncertainties are expected to be ± half the least counts of the measuring instruments

$$\delta \Delta m = \pm 0.5 \text{ g} \qquad \delta \Delta t = 0.05 \text{ s}$$

$$\frac{\Delta m}{\dot{m}}\frac{\partial \dot{m}}{\partial \Delta m} = \Delta t\left(\frac{1}{\Delta t}\right) = 1 \quad \text{and} \quad \frac{\Delta t}{\dot{m}}\frac{\partial \dot{m}}{\partial \Delta t} = \frac{(\Delta t)^2}{\Delta m}\left[-\frac{\Delta m}{(\Delta t)^2}\right] = -1$$

$$\therefore \; u_{\dot{m}} = \pm\left[(u_{\Delta m})^2 + (-u_{\Delta t})^2\right]^{1/2}$$

Tabulating results:

Beaker Volume $\Delta \forall$ (mL)	Water collected Δm (g)	Error in Δm (g)	Uncertainty in Δm (percent)	Time Interval Δt (s)	Error in Δt (s)	Uncertainty in Δt (percent)	Uncertainty in $\dot{m}$ (percent)
100	100	± 0.50	± 0.50	1.0	± 0.05	± 5.0	± 5.03
500	500	± 0.50	± 0.10	5.0	± 0.05	± 1.0	± 1.0
1000	500	± 0.50	± 0.10	5.0	± 0.05	± 1.0	± 1.0

Since the scales have a capacity of 1 kg and the tare mass of the 1000 mL beaker is 500 g, there is no advantage in using the larger beaker. The uncertainty in $\dot{m}$ could be reduced to ± 0.50 percent by using the large beaker if a scale with greater capacity, the same least count were available

Problem 1.15

Given: Soda can with estimated dimensions $D = 66.0 \pm 0.5$ mm, $H = 110 \pm 0.5$ mm. Soda has $SG = 1.055$

Find: (a) volume of soda in the can (based on measured mass of full and empty can).
(b) estimate average depth to which the can is filled and the uncertainty in the estimate.

Solution:

Measurements on a can of coke give

$$m_f = 386.5 \pm 0.50\,g, \quad m_e = 17.5 \pm 0.50\,g \quad \therefore m = m_f - m_e = 369 \pm u_m\ g$$

$$u_m = \pm\left[\left(\frac{m_f}{m}\frac{\partial m}{\partial m_f}u_{m_f}\right)^2 + \left(\frac{m_e}{m}\frac{\partial m}{\partial m_e}u_{m_e}\right)^2\right]^{1/2}$$

$$u_{m_f} = \pm\frac{0.5g}{386.5g} = \pm 0.00129, \quad u_{m_e} = \pm\frac{0.50}{17.5} = 0.0286$$

$$\therefore u_m = \pm\left\{\left[\frac{386.5}{369}(1)(0.00129)\right]^2 + \left[\frac{17.5}{369}(-1)(0.0286)\right]^2\right\}^{1/2} = 0.0019$$

Density is mass per unit volume and $SG = \rho/\rho_{H_2O}$ so

$$\forall = \frac{m}{\rho} = \frac{m}{\rho_{H_2O}SG} = 369\,g \times \frac{m^3}{1000\,kg} \times \frac{1}{1.055} \times \frac{kg}{1000\,g} = 350 \times 10^{-6}\,m^3 \quad \forall$$

The reference value ρ_{H_2O} is assumed to be precise. Since SG is specified to three places beyond the decimal point, assume $u_{SG} = \pm 0.001$. Then

$$u_\forall = \pm\left[\left(\frac{m}{\forall}\frac{\partial \forall}{\partial m}u_m\right)^2 + \left(\frac{m}{SG}\frac{\partial \forall}{\partial SG}\right)^2\right]^{1/2} = \pm\left\{\left[(1)u_m\right]^2 + \left[(-1)u_{SG}\right]^2\right\}^{1/2}$$

$$u_\forall = \pm\left\{\left[(1)(0.0019)\right]^2 + \left[(-1)(0.001)\right]^2\right\}^{1/2} = 0.0021 \text{ or } 0.21\%$$

$$\forall = \frac{\pi D^2}{4}L \quad \text{or} \quad L = \frac{4\forall}{\pi D^2} = \frac{4}{\pi} \times \frac{350 \times 10^{-6}\,m^3}{(0.066)^2\,m^2} \times \frac{10^3\,mm}{m} = 102\,mm \quad L$$

$$u_L = \pm\left[\left(\frac{\forall}{L}\frac{\partial L}{\partial \forall}u_\forall\right)^2 + \left[\left(\frac{D}{L}\frac{\partial L}{\partial D}u_D\right)^2\right]\right]^{1/2}$$

$$\frac{\forall}{L}\frac{\partial L}{\partial \forall} = \frac{\pi D^2}{4} \cdot \frac{4}{\pi D^2} = 1 \qquad u_D = \pm\frac{0.5\,mm}{66\,mm} = 0.0076$$

$$\frac{D}{L}\frac{\partial L}{\partial D} = D\frac{\pi D^2}{4\forall} \times \frac{4\forall}{\pi}\left(-\frac{2}{D^3}\right) = -2$$

$$u_L = \pm\left\{\left[(1)(0.0021)\right]^2 + \left[(-2)(0.0076)\right]^2\right\}^{1/2} = 0.0153 \text{ or } 1.53\% \quad u_L$$

Note: (1) printing on the can states the content as 355 ml. This suggests that the implied accuracy of the SG value may be over stated.
(2) results suggest that over seven percent of the can height is void of soda.

Problem 1.16

From Appendix A, the viscosity μ (N·s/m^2) of water at temperature T (K) can be computed from $\mu = A10^{B/(T-C)}$, where $A = 2.414 \times 10^{-5}$ N.s/m^2, $B = 247.8$ K, and $C = 140$ K. Determine the viscosity of water at 20°C, and estimate its uncertainty if the uncertainty in temperature measurement is +/- 0.25°C.

Solution

Given: Data on water.

Find: Viscosity and uncertainty in viscosity.

The data provided are:

$$A = 2.414\cdot 10^{-5}\cdot\frac{N\cdot s}{m^2} \qquad B = 247.8\cdot K \qquad C = 140\cdot K \qquad T = 293\cdot K$$

The uncertainty in temperature is $u_T = \frac{0.25\cdot K}{293\cdot K}$ $\qquad u_T = 0.085\,\%$

The formula for viscosity is $\mu(T) = A\cdot 10^{\frac{B}{(T-C)}}$

Evaluating μ $\qquad \mu(T) = 2.414\cdot 10^{-5}\cdot\frac{N\cdot s}{m^2}\times 10^{\frac{247.8\cdot K}{(293\cdot K - 140\cdot K)}}$

$$\mu(T) = 1.005\times 10^{-3}\,\frac{N\cdot s}{m^2}$$

For the uncertainty

$$\frac{d}{dT}\mu(T) \rightarrow -A\cdot 10^{\frac{B}{(T-C)}}\cdot\frac{B}{(T-C)^2}\cdot\ln(10)$$

so

$$u_\mu(T) = \left| \frac{T}{\mu(T)} \cdot \frac{d}{dT} \mu(T) \cdot u_T \right| \rightarrow \ln(10) \cdot \left| T \cdot \frac{B}{(T - C)^2} \cdot u_T \right|$$

Using the given data

$$u_\mu(T) = \ln(10) \cdot \left| 293 \cdot K \cdot \frac{247.8 \cdot K}{(293 \cdot K - 140 \cdot K)^2} \cdot 0.085 \cdot \% \right|$$

$$u_\mu(T) = 0.61\,\%$$

Given: Lateral acceleration, $a = 0.70\ g$, measured on 150-ft diameter skid pad.

Path deviation: ± 2 ft
Vehicle speed: ± 0.5 mph
} measurement uncertainty

Find: (a) Estimate uncertainty in lateral acceleration.
(b) How could experimental procedure be improved?

Solution: Lateral acceleration is given by $a = V^2/R$.

From Appendix F, $u_a = \pm\left[(2u_V)^2 + (u_R)^2\right]^{1/2}$

From the given data,

$$V^2 = aR;\ V = \sqrt{aR} = \left[0.70 \times 32.2\,\frac{ft}{s^2} \times 75\,ft\right]^{1/2} = 41.1\ ft/s$$

Then

$$u_V = \pm\frac{\delta V}{V} = \pm\,0.5\,\frac{mi}{hr} \times \frac{s}{41.1\,ft} \times 5280\,\frac{ft}{mi} \times \frac{hr}{3600\,s} = \pm\,0.0178$$

and

$$u_R = \pm\frac{\delta R}{R} = \pm\,2\,ft \times \frac{1}{75\,ft} = \pm\,0.0267$$

so

$$u_a = \pm\left[(2 \times 0.0178)^2 + (0.0267)^2\right]^{1/2} = \pm\,0.0445$$

$$u_a = \pm\,4.45\ \text{percent}$$

u_a

Experimental procedure could be improved by using a larger circle, assuming the absolute errors in measurement are constant.

For $D = 400$ ft, $R = 200$ ft

$$V = \sqrt{aR} = \left[0.70 \times 32.2\,\frac{ft}{s^2} \times 200\,ft\right]^{1/2} = 67.1\ ft/s = 45.8\ mph$$

$$u_V = \pm\frac{0.5\ mph}{45.8\ mph} = \pm\,0.0109;\quad u_R = \pm\frac{2\,ft}{200\,ft} = \pm\,0.0100$$

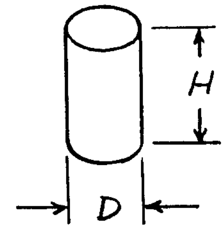
$$u_a = \pm\left[(2 \times 0.0109)^2 + (0.0100)^2\right]^{1/2} = \pm\,0.0240 \text{ or } \pm\,2.4\ \text{percent}$$

Problem 1.18

Given: Dimensions of soda can:

$D = 66\ mm$

$H = 110\ mm$



Find: Measurement precision needed to allow volume to be estimated with an uncertainty of ± 0.5 percent or less.

Solution: Use the methods of Appendix F:

Computing equations: $\forall = \dfrac{\pi D^2 H}{4}$

$$u_\forall = \pm\left[\left(\frac{H}{\forall}\frac{\partial \forall}{\partial H} u_H\right)^2 + \left(\frac{D}{\forall}\frac{\partial \forall}{\partial D} u_D\right)^2\right]^{\frac{1}{2}}$$

Since $\forall = \dfrac{\pi D^2 H}{4}$, then $\dfrac{\partial \forall}{\partial H} = \dfrac{\pi D^2}{4}$ and $\dfrac{\partial \forall}{\partial D} = \dfrac{\pi D H}{2}$

Let $u_D = \pm \dfrac{\delta x}{D}$ and $u_H = \pm \dfrac{\delta x}{H}$, substituting,

$$u_\forall = \pm\left[\left(\frac{4H}{\pi D^2 H}\frac{\pi D^2}{4}\frac{\delta x}{H}\right)^2 + \left(\frac{4D}{\pi D^2 H}\frac{\pi D H}{2}\frac{\delta x}{D}\right)^2\right]^{\frac{1}{2}} = \pm\left[\left(\frac{\delta x}{H}\right)^2 + \left(\frac{2\delta x}{D}\right)^2\right]^{\frac{1}{2}}$$

Solving,

$$u_\forall^2 = \left(\frac{\delta x}{H}\right)^2 + \left(\frac{2\delta x}{D}\right)^2 = (\delta x)^2\left[\left(\frac{1}{H}\right)^2 + \left(\frac{2}{D}\right)^2\right]$$

$$\delta x = \pm\frac{u_\forall}{\left[\left(\frac{1}{H}\right)^2 + \left(\frac{2}{D}\right)^2\right]^{1/2}} = \pm\frac{0.005}{\left[\left(\frac{1}{110\ mm}\right)^2 + \left(\frac{2}{66\ mm}\right)^2\right]^{1/2}} = \pm 0.158\ mm$$

Check: $u_H = \pm \dfrac{\delta x}{H} = \pm \dfrac{0.158\ mm}{110\ mm} = \pm 1.44 \times 10^{-3}$

$u_D = \pm \dfrac{\delta x}{D} = \dfrac{\pm 0.158\ mm}{66\ mm} = \pm 2.39 \times 10^{-3}$

$$u_\forall = \pm\left[(u_H)^2 + (2u_D)^2\right]^{\frac{1}{2}} = \pm\left[(0.00144)^2 + (0.00478)^2\right]^{\frac{1}{2}} = \pm 0.00499$$ ✓✓

If δx represents half the least count, a minimum resolution of about $2\delta x \cong 0.32\ mm$ is needed.

Problem 1.19

Given: American golf ball, $m = 1.62 \pm 0.01$ oz, $D = 1.68$ in.

Find: Precision to which D must be measured to estimate density within uncertainty of ± 1 percent.

Solution: Apply uncertainty concepts

Definition: Density, $\rho \equiv \frac{m}{\forall}$ $\qquad \forall = \frac{4}{3}\pi R^3 = \frac{\pi D^3}{6}$

Computing equation: $u_R = \pm\left[\left(\frac{x_1}{R}\frac{\partial R}{\partial x_1} u_{x_1}\right)^2 + \cdots\right]^{1/2}$

From the definition, $\rho = \frac{m}{\pi D^3/6} = \frac{6m}{\pi D^3} = \rho(m, D)$

Thus $\frac{m}{\rho}\frac{\partial \rho}{\partial m} = 1$ and $\frac{D}{\rho}\frac{\partial \rho}{\partial D} = 3$, so

$$u_\rho = \pm\left[(1\, u_m)^2 + (3 u_D)^2\right]^{1/2}$$

$$u_\rho^2 = u_m^2 + 9 u_D^2$$

Solving, $u_D = \pm \frac{1}{3}\left[u_\rho^2 - u_m^2\right]^{\frac{1}{2}}$

From the data given, $u_\rho = \pm 0.0100$

$$u_m = \frac{\pm 0.01 \text{ oz}}{1.62 \text{ oz}} = \pm 0.00617$$

$$u_D = \pm \frac{1}{3}\left[(0.0100)^2 - (0.00617)^2\right]^{\frac{1}{2}} = \pm 0.00262 \text{ or } \pm 0.262\%$$

Since $u_D = \pm \frac{\delta D}{D}$, then

$$\delta D = \pm D u_D = \pm 1.68 \text{ in.} \times 0.00262 = \pm 0.00441 \text{ in.}$$ δ

The ball diameter must be measured to a precision of ± 0.00441 in. (± 0.112 mm) or better to estimate density within ± 1 percent. A micrometer or caliper could be used.

Problem 1.20

The height of a building may be estimated by measuring the horizontal distance to a point on the ground and the angle from this point to the top of the building. Assuming these measurements are *L* = 100 +/- 0.5 ft and θ = 30 +/- 0.2 degrees, estimate the height *H* of the building and the uncertainty in the estimate. For the same building height and measurement uncertainties, use *Excel*'s *Solver* to determine the angle (and the corresponding distance from the building) at which measurements should be made to minimize the uncertainty in estimated height. Evaluate and plot the optimum measurement angle as a function of building height for 50 < *H* < 1000 ft.

Solution

Given: Data on length and angle measurements.

Find:

The data provided are:

$L = 100 \cdot ft$ $\delta L = 0.5 \cdot ft$ $\theta = 30 \cdot deg$ $\delta\theta = 0.2 \cdot deg$

The uncertainty in *L* is $u_L = \frac{\delta L}{L}$ $u_L = 0.5\%$

The uncertainty in θ is $u_\theta = \frac{\delta\theta}{\theta}$ $u_\theta = 0.667\%$

The height *H* is given by $H = L \cdot \tan(\theta)$ $H = 57.7\,ft$

For the uncertainty

$$u_H = \sqrt{\left(\frac{L}{H} \cdot \frac{\partial}{\partial L} H \cdot u_L\right)^2 + \left(\frac{\theta}{H} \cdot \frac{\partial}{\partial \theta} H \cdot u_\theta\right)^2}$$

and $\frac{\partial}{\partial L}H = \tan(\theta)$ $\frac{\partial}{\partial \theta}H = L\cdot\left(1 + \tan(\theta)^2\right)$

so $u_H = \sqrt{\left(\frac{L}{H}\cdot\tan(\theta)\cdot u_L\right)^2 + \left[\frac{L\cdot\theta}{H}\cdot\left(1 + \tan(\theta)^2\right)\cdot u_\theta\right]^2}$

Using the given data

$$u_H = \sqrt{\left(\frac{100}{57.5}\cdot\tan\left(\frac{\pi}{6}\right)\cdot\frac{0.5}{100}\right)^2 + \left[\frac{100\cdot\frac{\pi}{6}}{57.5}\cdot\left(1 + \tan\left(\frac{\pi}{6}\right)^2\right)\cdot\frac{0.667}{100}\right]^2}$$

$u_H = 0.95\%$ $\delta H = u_H\cdot H$ $\delta H = 0.55\,\text{ft}$

$$H = 57.5 + -0.55\cdot\text{ft}$$

The angle θ at which the uncertainty in *H* is minimized is obtained from the corresponding *Exce* workbook (which also shows the plot of u_H vs θ)

$$\theta_{optimum} = 31.4\cdot\text{deg}$$

Problem 1.20 (In Excel)

The height of a building may be estimated by measuring the horizontal distance to a point on the ground and the angle from this point to the top of the building. Assuming these measurements are L = 100 +/- 0.5 ft and θ = 30 +/- 0.2 degrees, estimate the height H of the building and the uncertainty in the estimate. For the same building height and measurement uncertainties, use *Excel*'s *Solver* to determine the angle (and the corresponding distance from the building) at which measurements should be made to minimize the uncertainty in estimated height. Evaluate and plot the optimum measurement angle as a function of building height for 50 < H < 1000 ft.

Given: Data on length and angle measurements.

Find: Height of building; uncertainty; angle to minimize uncertainty

Given data:

H =	57.7	ft
δL =	0.5	ft
$\delta\theta$ =	0.2	deg

For this building height, we are to vary θ (and therefore L) to minimize the uncertainty u_H.

The uncertainty is

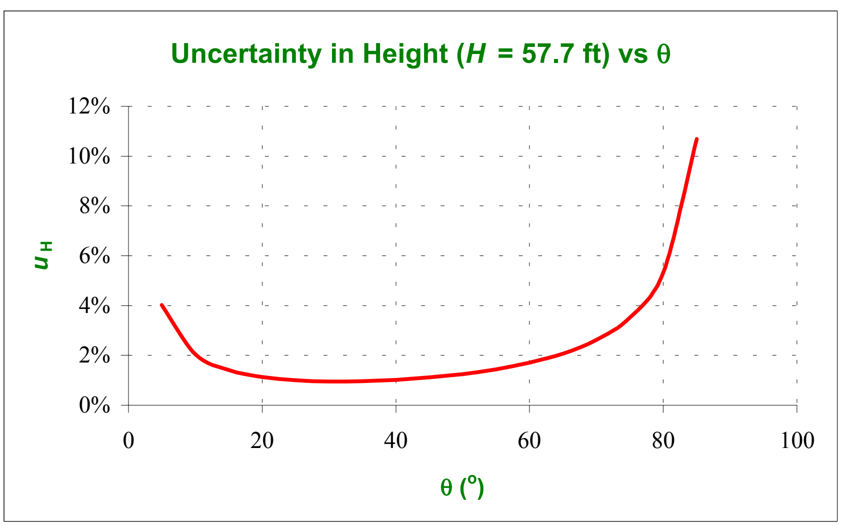
$$u_H = \sqrt{\left(\frac{L}{H}\cdot\tan(\theta)\cdot u_L\right)^2 + \left[\frac{L\cdot\theta}{H}\cdot\left(1+\tan(\theta)^2\right)\cdot u_\theta\right]^2}$$

Expressing u_H, u_L, u_θ and L as functions of θ, (remember that δL and $\delta\theta$ are constant, so as L and θ vary the uncertainties will too!) and simplifying

$$u_H(\theta) = \sqrt{\left(\tan(\theta)\cdot\frac{\delta L}{H}\right)^2 + \left[\frac{\left(1+\tan(\theta)^2\right)}{\tan(\theta)}\cdot\delta\theta\right]^2}$$

Plotting u_H vs θ

θ (deg)	u_H
5	4.02%
10	2.05%
15	1.42%
20	1.13%
25	1.00%
30	0.949%
35	0.959%
40	1.02%
45	1.11%
50	1.25%
55	1.44%
60	1.70%
65	2.07%
70	2.62%
75	3.52%
80	5.32%
85	10.69%

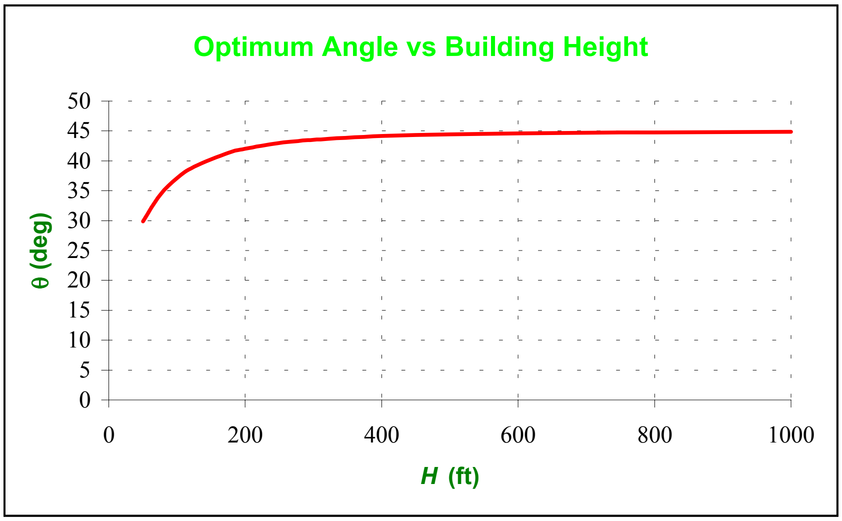


Optimizing using *Solver*

θ (deg)	u_H
31.4	0.95%

To find the optimum θ as a function of building height *H* we need a more complex *Solver*

H (ft)	θ (deg)	u_H
50	29.9	0.99%
75	34.3	0.88%
100	37.1	0.82%
125	39.0	0.78%
175	41.3	0.75%
200	42.0	0.74%
250	43.0	0.72%
300	43.5	0.72%
400	44.1	0.71%
500	44.4	0.71%
600	44.6	0.70%
700	44.7	0.70%
800	44.8	0.70%
900	44.8	0.70%
1000	44.9	0.70%



Use *Solver* to vary ALL θ's to minimize the total u_H!

Total u_H's: 11.32%

Given: Piston-cylinder device to have $\forall = 1\ mm^3$.

Molded plastic parts with dimensional uncertainties, $\delta = \pm 0.002$ in.

Find: (a) Estimate of uncertainty in dispensed volume that results from the dimensional uncertainties.
(b) Determine the ratio of stroke length to bore diameter that minimizes $u_\forall$; plot of the results.
(c) Is this result influenced by the magnitude of δ?

Solution: Apply uncertainty concepts from Appendix F:

Computing equation: $\forall = \frac{\pi D^2 L}{4}$; $u_\forall = \pm\left[\left(\frac{L}{\forall}\frac{\partial \forall}{\partial L}u_L\right)^2 + \left(\frac{D}{\forall}\frac{\partial \forall}{\partial D}u_D\right)^2\right]^{\frac{1}{2}}$

From $\forall$, $\frac{L}{\forall}\frac{\partial \forall}{\partial L} = 1$, and $\frac{D}{\forall}\frac{\partial \forall}{\partial D} = 2$, so $u_\forall = \pm\left[u_L^2 + (2u_D)^2\right]^{\frac{1}{2}}$

The dimensional uncertainty is $\delta = \pm 0.002\ in. \times 25.4\ \frac{mm}{in.} = \pm 0.0508\ mm$

Assume $D = 1\ mm$. Then $L = \frac{4\forall}{\pi D^2} = \frac{4}{\pi} \times 1\ mm^3 \times \frac{1}{(1)^2\ mm^2} = 1.27\ mm$

$u_D = \pm\frac{\delta}{D} = \pm\frac{0.0508}{1} = \pm 5.08$ percent

$u_L = \pm\frac{\delta}{L} = \pm\frac{0.0508}{1.27} = \pm 4.00$ percent

$u_\forall = \pm\left[(4.00)^2 + (2(5.08))^2\right]^{\frac{1}{2}}$

$u_\forall = \pm 10.9$ percent $\qquad u_\forall$

To minimize $u_\forall$, substitute in terms of D:

$$u_\forall = \pm\left[(u_L)^2 + (2u_D)^2\right] = \pm\left[\left(\frac{\delta}{L}\right)^2 + \left(2\frac{\delta}{D}\right)^2\right]^{\frac{1}{2}} = \pm\left[\left(\frac{\pi D^2}{4\forall}\delta\right)^2 + \left(2\frac{\delta}{D}\right)^2\right]^{\frac{1}{2}}$$

This will be minimum when D is such that $\partial[\,]/\partial D = 0$, or

$$\frac{\partial[\,]}{\partial D} = \left(\frac{\pi\delta}{4\forall}\right)^2 4D^3 + (2\delta)^2\left(-2\frac{1}{D^3}\right) = 0;\quad D^6 = 2\left(\frac{4\forall}{\pi}\right)^2;\quad D = 2^{1/6}\left(\frac{4\forall}{\pi}\right)^{1/3}$$

Thus

$$D_{opt} = 2^{1/6}\left(\frac{4}{\pi} \times 1\ mm^3\right)^{1/3} = 1.22\ mm$$

The corresponding L is

$$L_{opt} = \frac{4\forall}{\pi D^2} = \frac{4}{\pi} \times 1\ mm^3 \times \frac{1}{(1.22)^2\ mm^2} = 0.855\ mm$$

The optimum stroke-to-bore ratio is

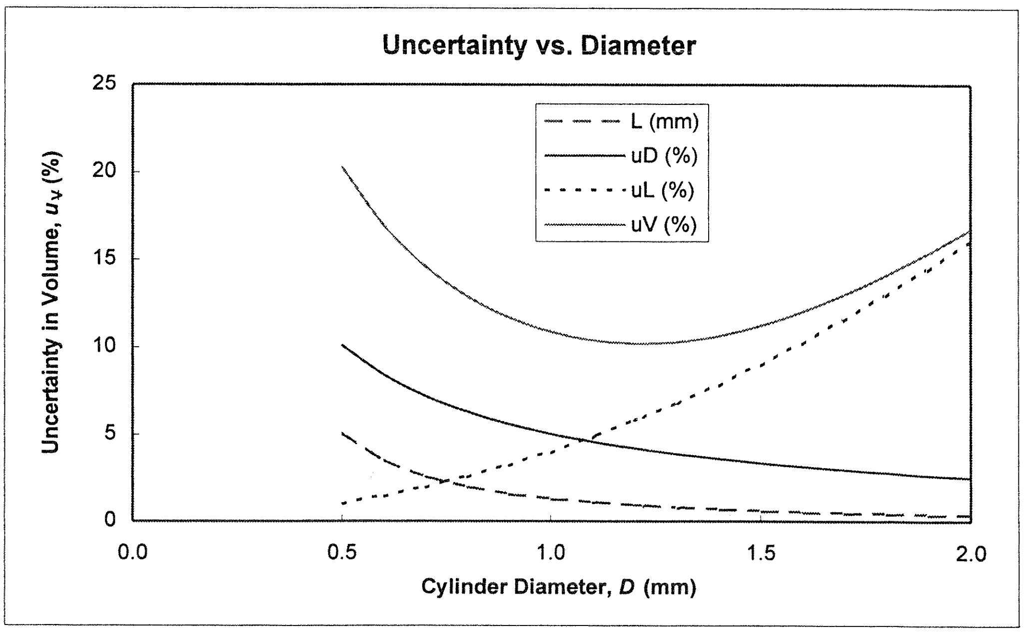
$$L/D)_{opt} = \frac{0.855\ mm}{1.22\ mm} = 0.701$$ (see table and plot on next page) $\qquad L/D)_{opt}$

Note that δ drops out of the optimization equation. This optimum L/D is independent of the magnitude of δ. However, the magnitude of the optimum $u_\forall$ increases as δ increases.

Uncertainty in volume of cylinder:

δ =	0.002	in.	0.0508	mm
V =	1	mm^3		

D (mm)	*L* (mm)	*L/D* (---)	u_D (%)	u_L (%)	u_V (%)
0.5	5.09	10.2	10.2	1.00	20.3
0.6	3.54	5.89	8.47	1.44	17.0
0.7	2.60	3.71	7.26	1.96	14.6
0.8	1.99	2.49	6.35	2.55	13.0
0.9	1.57	1.75	5.64	3.23	11.7
1.0	1.27	1.27	5.08	3.99	10.9
1.1	1.05	0.957	4.62	4.83	10.4
1.2	0.884	0.737	4.23	5.75	10.2
1.22	0.855	0.701	4.16	5.94	10.2
1.3	0.753	0.580	3.91	6.74	10.3
1.4	0.650	0.464	3.63	7.82	10.7
1.5	0.566	0.377	3.39	8.98	11.2
1.6	0.497	0.311	3.18	10.2	12.0
1.7	0.441	0.259	2.99	11.5	13.0
1.8	0.393	0.218	2.82	12.9	14.1
1.9	0.353	0.186	2.67	14.4	15.4
2.0	0.318	0.159	2.54	16.0	16.7
2.1	0.289	0.137	2.42	17.6	18.2
2.2	0.263	0.120	2.31	19.3	19.9
2.3	0.241	0.105	2.21	21.1	21.6
2.4	0.221	0.092	2.12	23.0	23.4
2.5	0.204	0.081	2.03	24.9	25.3

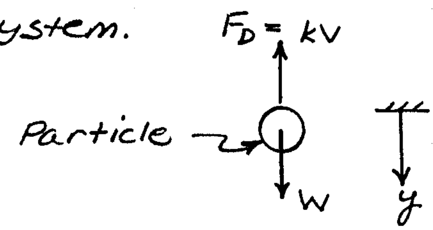


Problem 1.22

Given: Small particle accelerating from rest in a fluid. Net weight is W, resisting force $F_D = kV$, where V is speed.

Find: Time required to reach 95 percent of terminal speed, V_t.

Solution: Consider the particle to be a system. Apply Newton's second law.



Basic equation: $\sum F_y = ma_y$

Assumptions: (1) W is net weight
(2) Resisting force acts opposite to V

Then

$$\sum F_y = W - kV = ma_y = m\frac{dV}{dt} = \frac{W}{g}\frac{dV}{dt}$$

or

$$\frac{dV}{dt} = g\left(1 - \frac{k}{W}V\right)$$

Separating variables,

$$\frac{dV}{1 - \frac{k}{W}V} = g\,dt$$

Integrating, noting that velocity is zero initially,

$$\int_0^V \frac{dV}{1 - \frac{k}{W}V} = -\frac{W}{k}\ln\left(1 - \frac{k}{W}V\right)\Big]_0^V = \int_0^t g\,dt = gt$$

or

$$1 - \frac{k}{W}V = e^{-\frac{kgt}{W}} \quad ; \quad V = \frac{W}{k}\left[1 - e^{-\frac{kgt}{W}}\right]$$

But $V \to V_t$ as $t \to \infty$, so $V_t = \frac{W}{k}$. Therefore

$$\frac{V}{V_t} = 1 - e^{-\frac{kgt}{W}}$$

When $\frac{V}{V_t} = 0.95$, then $e^{-\frac{kgt}{W}} = 0.05$ and $\frac{kgt}{W} = 3$. Thus

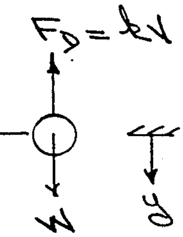
$$t = 3W/gk$$

Problem 1.23

Given: Small particle accelerating from rest in a fluid. Net weight is W, resisting force is $F_D = kV$, where V is speed.

Find: Distance required to reach 95 percent of terminal speed, V_t.

Solution: Consider the particle to be a system. Apply Newton's second law.



Basic equation: $\sum F_y = m a_y$

Assumptions: (1) W is net weight
(2) Resisting force acts opposite to V

Then, $\sum F_y = W - kV = m a_y = m \frac{dV}{dt} = \frac{W}{g} V \frac{dV}{dy}$

or $\quad 1 - \frac{k}{W} V = \frac{V}{g} \frac{dV}{dy}$

At terminal speed, $a_y = 0$ and $V = V_t = \frac{W}{k}$. Then

$$1 - \frac{V}{V_t} = \frac{1}{g} V \frac{dV}{dy}$$

Separating variables

$$\frac{V\,dV}{1 - \frac{1}{V_t} V} = g\,dy$$

Integrating, noting that velocity is zero initially

$$gy = \int_0^{0.95V_t} \frac{V\,dV}{1 - \frac{1}{V_t}V} = \left[-V V_t - V_t^2 \ln\left(1 - \frac{V}{V_t}\right) \right]_0^{0.95V_t}$$

$$gy = -0.95 V_t^2 - V_t^2 \ln(1 - 0.95) - V_t^2 \ln(1) \quad (\ln(1) \to 0)$$

$$gy = -V_t^2 \left[0.95 + \ln 0.05 \right] = 2.05 V_t^2$$

$$\therefore \; y = \frac{2.05}{g} V_t^2 = 2.05 \frac{W^2}{g k^2}$$

Problem 1.24

For a small particle of aluminum (spherical, with diameter $d = 0.025$ mm) falling in standard air at speed V, the drag is given by $F_D = 3\pi\mu Vd$, where μ is the air viscosity. Find the maximum speed starting from rest, and the time it takes to reach 95% of this speed. Plot the speed as a function of time.

Solution

Given: Data on sphere and formula for drag.

Find: Maximum speed, time to reach 95% of this speed, and plot speed as a function of time.

The data provided, or available in the Appendices, are:

$$\rho_{air} = 1.17\cdot\frac{kg}{m^3} \qquad \mu = 1.8\times 10^{-5}\cdot\frac{N\cdot s}{m^2} \qquad \rho_w = 999\cdot\frac{kg}{m^3} \qquad SG_{Al} = 2.64 \qquad d = 0.025\cdot mm$$

Then the density of the sphere is $\rho_{Al} = SG_{Al}\cdot\rho_w$ $\qquad \rho_{Al} = 2637\frac{kg}{m^3}$

The sphere mass is

$$M = \rho_{Al}\cdot\frac{\pi\cdot d^3}{6} = 2637\cdot\frac{kg}{m^3}\times\pi\times\frac{(0.000025\cdot m)^3}{6}$$

$$M = 2.16\times 10^{-11}\,kg$$

Newton's 2nd law for the steady state motion becomes $M\cdot g = 3\cdot\pi\cdot V\cdot d$

so

$$V_{max} = \frac{M\cdot g}{3\cdot\pi\cdot\mu\cdot d} = \frac{1}{3\times\pi}\times\frac{2.16\times 10^{-11}\cdot kg}{s^2}\times 9.81\cdot\frac{m}{s^2}\times\frac{m^2}{1.8\times 10^{-5}\cdot N\cdot s}\times\frac{1}{0.000025\cdot m}$$

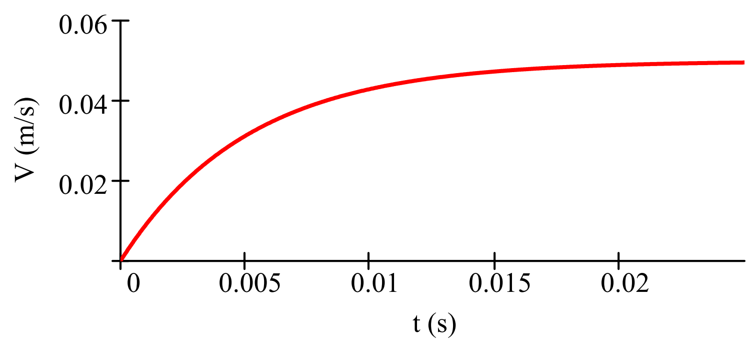
$$V_{max} = 0.0499\frac{m}{s}$$

Newton's 2nd law for the general motion is $M \cdot \frac{dV}{dt} = M \cdot g - 3 \cdot \pi \cdot \mu \cdot V \cdot d$

so
$$\frac{dV}{g - \frac{3 \cdot \pi \cdot \mu \cdot d}{m} \cdot V} = dt$$

Integrating and using limits
$$V(t) = \frac{M \cdot g}{3 \cdot \pi \cdot \mu \cdot d} \cdot \left(1 - e^{\frac{-3 \cdot \pi \cdot \mu \cdot d}{M} \cdot t}\right)$$

Using the given data



The time to reach 95% of maximum speed is obtained from

$$\frac{M \cdot g}{3 \cdot \pi \cdot \mu \cdot d} \cdot \left(1 - e^{\frac{-3 \cdot \pi \cdot \mu \cdot d}{M} \cdot t}\right) = 0.95 \cdot V_{max}$$

so

$$t = -\frac{M}{3 \cdot \pi \cdot \mu \cdot d} \cdot \ln\left(1 - \frac{0.95 \cdot V_{max} \cdot 3 \cdot \pi \cdot \mu \cdot d}{M \cdot g}\right)$$ Substituting values $t = 0.0152\,s$

Problem 1.24 (In Excel)

For a small particle of aluminum (spherical, with diameter d = 0.025 mm) falling in standard air at speed V, the drag is given by $F_D = 3\pi\mu Vd$, where μ is the air viscosity. Find the maximum speed starting from rest, and the time it takes to reach 95% of this speed. Plot the speed as a function of time.

Solution

Given: Data and formula for drag.

Find: Maximum speed, time to reach 95% of final speed, and plot.

The data given or availabke from the Appendices is

$\mu =$ 1.80E-05 $\mathrm{Ns/m^2}$

$\rho =$ 1.17 $\mathrm{kg/m^3}$

$SG_{Al} =$ 2.64

$\rho_w =$ 999 $\mathrm{kg/m^3}$

$d =$ 0.025 mm

Data can be computed from the above using the following equations

$$\rho_{Al} = SG_{Al} \cdot \rho_w$$

$$M = \rho_{Al} \cdot \frac{\pi \cdot d^3}{6}$$

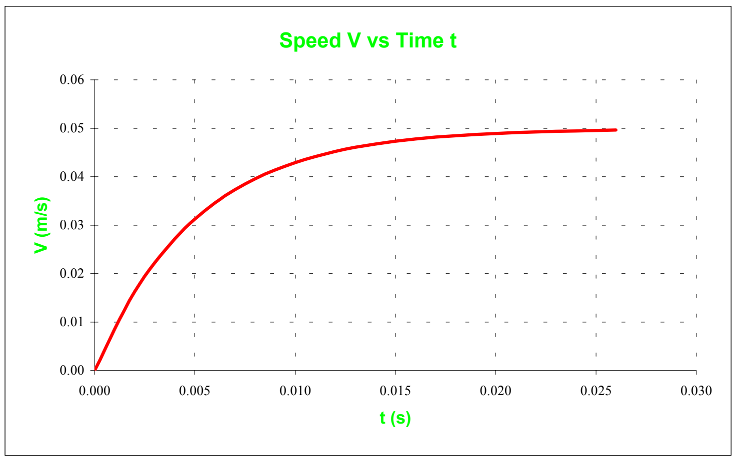
$$V_{max} = \frac{M \cdot g}{3 \cdot \pi \cdot \mu \cdot d}$$

$$V(t) = \frac{M \cdot g}{3 \cdot \pi \cdot \mu \cdot d} \cdot \left(1 - e^{\frac{-3 \cdot \pi \cdot \mu \cdot d}{M} \cdot t}\right)$$

$\rho_{Al} =$ 2637 $\mathrm{kg/m^3}$

M = 2.16E-11 kg

Vmax = 0.0499 m/s



t (s)	V (m/s)
0.000	0.0000
0.002	0.0162
0.004	0.0272
0.006	0.0346
0.008	0.0396
0.010	0.0429
0.012	0.0452
0.014	0.0467
0.016	0.0478
0.018	0.0485
0.020	0.0489
0.022	0.0492
0.024	0.0495
0.026	0.0496

For the time at which $V(t) = 0.95V_{max}$, use *Goal Seek*:

t (s)	V (m/s)	$0.95V_{max}$	Error (%)
0.0152	0.0474	0.0474	0.04%

Problem 1.25

For small spherical water droplets, diameter *d*, falling in standard air at speed *V*, the drag is given by $F_D = 3\pi\mu Vd$, where μ is the air viscosity. Determine the diameter *d* of droplets that take 1 second to fall from rest a distance of 1 m. (Use *Excel*'s *Goal Seek*.)

Solution

Given: Data on sphere and formula for drag.

Find: Diameter of water droplets that take 1 second to fall 1 m.

The data provided, or available in the Appendices, are:

$$\mu = 1.8 \times 10^{-5} \cdot \frac{N \cdot s}{m^2} \qquad \rho_w = 999 \cdot \frac{kg}{m^3}$$

Newton's 2nd law for the sphere (mass M) is $M \cdot \frac{dV}{dt} = M \cdot g - 3 \cdot \pi \cdot \mu \cdot V \cdot d$

so

$$\frac{dV}{g - \frac{3 \cdot \pi \cdot \mu \cdot d}{m} \cdot V} = dt$$

Integrating and using limits $V(t) = \frac{M \cdot g}{3 \cdot \pi \cdot \mu \cdot d} \cdot \left(1 - e^{\frac{-3 \cdot \pi \cdot \mu \cdot d}{M} \cdot t}\right)$

Integrating again

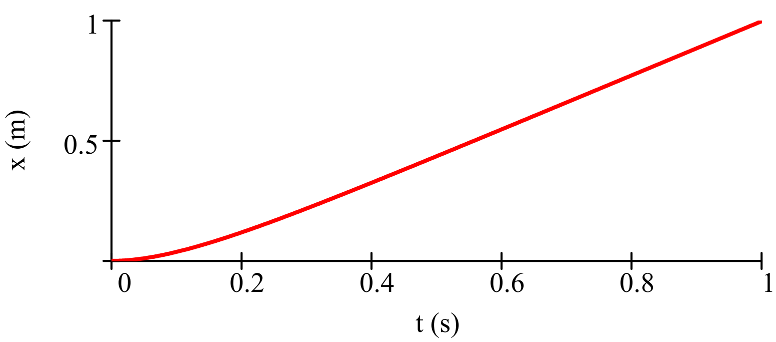
$$x(t) = \frac{M \cdot g}{3 \cdot \pi \cdot \mu \cdot d} \cdot \left[t + \frac{M}{3 \cdot \pi \cdot \mu \cdot d} \cdot \left(e^{\frac{-3 \cdot \pi \cdot \mu \cdot d}{M} \cdot t} - 1 \right) \right]$$

Replacing M with an expression involving diameter d $M = \rho_w \cdot \frac{\pi \cdot d^3}{6}$

$$x(t) = \frac{\rho_w \cdot d^2 \cdot g}{18 \cdot \mu} \cdot \left[t + \frac{\rho_w \cdot d^2}{18 \cdot \mu} \cdot \left(e^{\frac{-18 \cdot \mu}{\rho_w \cdot d^2} \cdot t} - 1 \right) \right]$$

This equation must be solved for d so that $x(1 \cdot s) = 1 \cdot m$. The answer can be obtained from manual iteration, or by using *Excel*'s *Goal Seek*.

$d = 0.193 \cdot mm$



Problem 1.25 (In Excel)

For small spherical water droplets, diameter d, falling in standard air at speed *V*, the drag is given by $F_D = 3\pi\mu V d$, where μ is the air viscosity. Determine the diameter *d* of droplets that take 1 second to fall from rest a distance of 1 m. (Use *Excel*'s *Goal Seek*.) speed. Plot the speed as a function of time.

Solution

Given: Data and formula for drag.

Find: Diameter of droplets that take 1 s to fall 1 m.

The data given or availabke from the Appendices is

$\mu =$ 1.80E-05 Ns/m^2

$\rho_w =$ 999 kg/m^3

Make a guess at the correct diameter (and use *Goal Seek* later):
(The diameter guess leads to a mass.)

d = 0.193 mm

M = 3.78E-09 kg

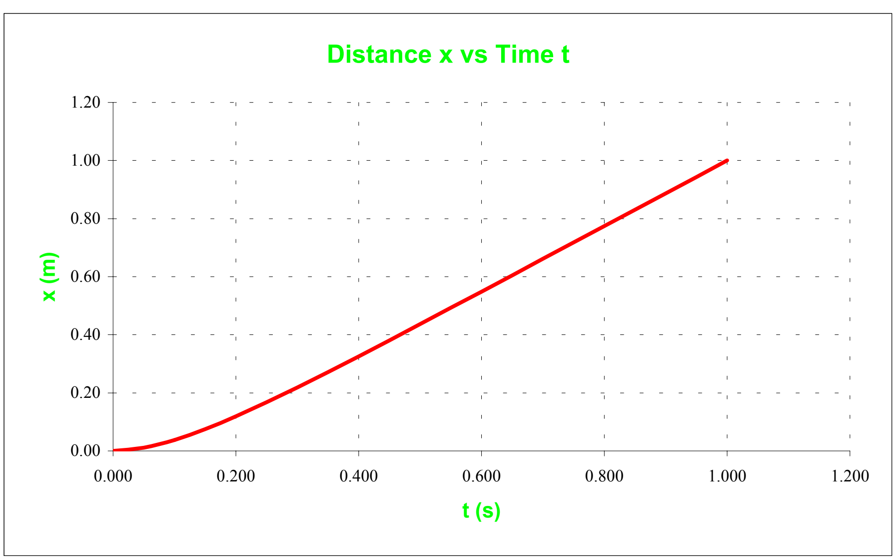
Data can be computed from the above using the following equations:

$$M = \rho_w \cdot \frac{\pi \cdot d^3}{6}$$

$$x(t) = \frac{M \cdot g}{3 \cdot \pi \cdot \mu \cdot d} \cdot \left[t + \frac{M}{3 \cdot \pi \cdot \mu \cdot d} \cdot \left(e^{\frac{-3 \cdot \pi \cdot \mu \cdot d}{M} \cdot t} - 1 \right) \right]$$

Use *Goal Seek* to vary *d* to make *x* (1s) = 1 m:

t (s)	x (m)
1.000	1.000



t (s)	x (m)
0.000	0.000
0.050	0.011
0.100	0.037
0.150	0.075
0.200	0.119
0.250	0.167
0.300	0.218
0.350	0.272
0.400	0.326
0.450	0.381
0.500	0.437
0.550	0.492
0.600	0.549
0.650	0.605
0.700	0.661
0.750	0.718
0.800	0.774
0.850	0.831
0.900	0.887
0.950	0.943
1.000	1.000

Given: Sky diver with $m = 75$ kg and $F_D = kV^2$; $k = 0.228 \frac{N \cdot s^2}{m^2}$

Find: (a) Maximum speed in free fall
(b) Speed reached in fall of 100 m

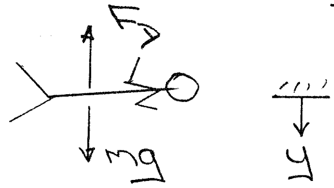
Plot: (a) Speed $V = V(t)$ and (b) $V = V(y)$

Solution:

Treat the sky diver as a system; apply Newton's 2nd law

Basic equation: $\Sigma F_y = m a_y$

Assumptions: $F_D = kV^2$ acts opposite to V
Initial conditions $V = 0$ at $t = 0$ and $y = 0$



Then

$$\Sigma F_y = mg - kV^2 = ma \quad \text{.....(1)}$$

At terminal speed, $a_y = 0$ and $V = V_t$, so $mg - kV_t^2 = 0$. Thus

$$V_t = \sqrt{\frac{mg}{k}} = \left[75\,\text{kg} \times 9.81\,\frac{m}{s^2} \times \frac{m^2}{0.228\,N \cdot s^2} \times \frac{N \cdot s^2}{kg \cdot m}\right]^{1/2} = 56.8\ m/s \qquad V_t$$

(b) To solve for V at $y = 100$ m, we need an expression for $V(y)$.

Note that $a_y = \frac{dV}{dt} = \frac{dV}{dy}\frac{dy}{dt} = \frac{dV}{dy}V = V\frac{dV}{dy}$

Then substituting into Eq. 1,

$$mg - kV^2 = mV\frac{dV}{dy} \quad \text{or} \quad 1 - \frac{kV^2}{mg} = \frac{V}{g}\frac{dV}{dy}$$

Separating variables and integrating

$$\int_0^V \frac{V\,dV}{1 - kV^2/mg} = \int_0^y g\,dy$$

$$-\frac{mg}{2k}\ln\left(1 - \frac{kV^2}{mg}\right)\Bigg]_0^V = gy \quad \text{or} \quad \ln\left(1 - \frac{kV^2}{mg}\right) = -\frac{2k}{m}y$$

Thus, $1 - \frac{kV^2}{mg} = e^{-2ky/m}$ and $V^2 = \frac{mg}{k}\left(1 - e^{-2ky/m}\right)$

$$V = V_t\left[1 - e^{-2ky/m}\right]^{1/2} \quad \text{---(2)}$$

At $y = 100$ m, $V = 56.8\,\frac{m}{s} \times \left[1 - e^{-2 \times 0.228\,\frac{N \cdot s}{m^2} \times 100\,m \times \frac{1}{75\,kg} \times \frac{kg \cdot m}{N \cdot s^2}}\right]^{1/2}$

$V = 38.3$ m/s $\longleftarrow$ $V_{y=100m}$

From Eq. 2, we can plot $V(y) = V_t\left[1-e^{-2ky/m}\right]^{1/2}$

or $V/V_t = \left[1-e^{-2ky/m}\right]^{1/2}$ . . . . (2a)

To obtain an expression for $V = V(t)$ we write

$$\sum F_y = mg - kV^2 = ma = m\frac{dV}{dt}$$

Separating variables and integrating

$$\int_0^t dt = \int_0^V \frac{m\,dV}{mg-kV^2} = \frac{m}{k}\int_0^V \frac{dV}{\frac{mg}{k}-V^2}$$

$$t = \frac{1}{2}\sqrt{\frac{m}{kg}}\ \ln\left|\frac{\sqrt{\frac{mg}{k}}+V}{\sqrt{\frac{mg}{k}}-V}\right| = \frac{1}{2}\sqrt{\frac{m}{kg}}\ \ln\frac{|V_t+V|}{|V_t-V|}$$

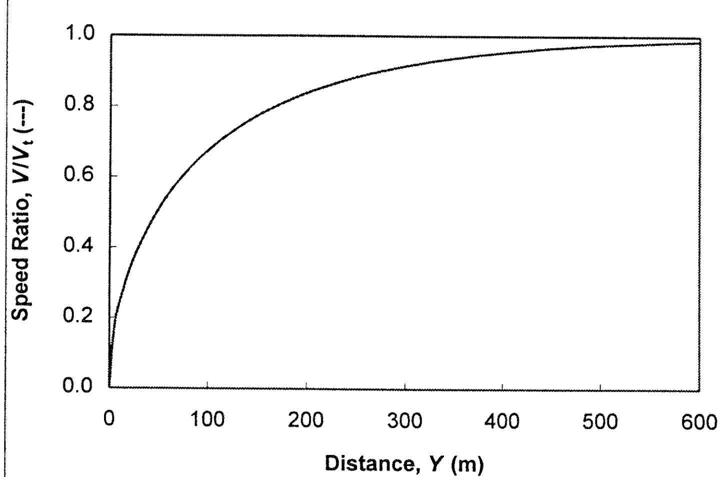
Then,

$$\frac{V_t+V}{V_t-V} = e^{2\sqrt{\frac{kg}{m}}\,t}$$, and

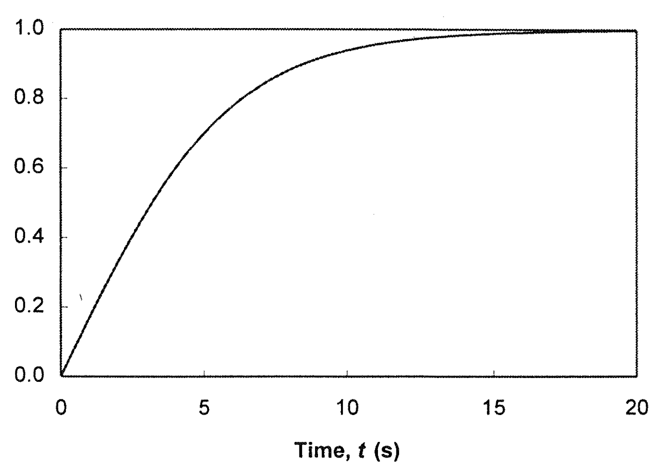
$$\frac{V}{V_t} = \frac{\left(e^{2\sqrt{\frac{kg}{m}}\,t}-1\right)}{\left(e^{2\sqrt{\frac{kg}{m}}\,t}+1\right)} = \tanh\left(V_t\frac{k}{m}t\right) \text{ - - - - - (3)}$$

Eqs. 2a and 3 are plotted below

Eq. 2a: Speed Ratio vs. Distance



Eq. 3: Speed Ratio vs. Time



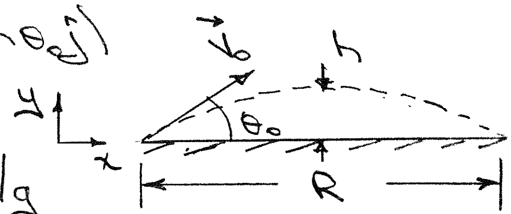
Given: Long bow at range, $R = 100\ m$. Maximum height of arrow is $h = 10\ m$. Neglect air resistance.

Find: Estimate of (a) speed, and (b) angle, of arrow leaving the bow.

Plot: (a) release speed, and (b) angle, as a function of h

Solution:

Let $\vec{V}_o = u_o\hat{i} + v_o\hat{j} = V_o(\cos\theta_o\hat{i} + \sin\theta_o\hat{j})$



$\Sigma F_y = m\frac{dv}{dt} = -mg$, so

$v = v_o - gt$, and $t_f = 2t_{v=0} = 2v_o/g$

Also, $mv\frac{dv}{dy} = -mg$, $v\,dv = -g\,dy$, $0 - \frac{v_o^2}{2} = -gh$

Thus $h = v_o^2/2g$ (1)

$\Sigma F_x = m\frac{du}{dt} = 0$, so $u = u_o = \text{const}$, and $R = u_o t_f = \frac{2u_o v_o}{g}$ (2)

From (1) $v_o^2 = 2gh$ .... (3)

(2) $u_o = \frac{gR}{2v_o} = \frac{gR}{2\sqrt{2gh}}$ $\therefore u_o^2 = \frac{gR^2}{8h}$

Then

$V_o^2 = u_o^2 + v_o^2 = \frac{gR^2}{8h} + 2gh$ and $V_o = \left[2gh + \frac{gR^2}{8h}\right]^{1/2}$ ...(4)

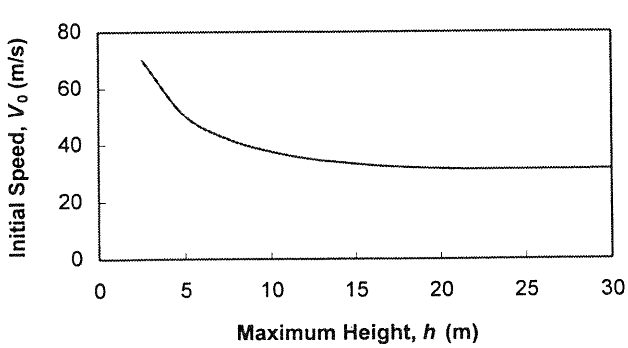
$V_o = \left[2\times 9.81\frac{m}{s^2}\times 10\,m + \frac{9.81}{8}\frac{m}{s^2}\times(100)^2 m^2 \times \frac{1}{10\,m}\right]^{1/2} = 37.7\ m/s$ ← V_o

From Eq. 3 $v_o = \sqrt{2gh} = V_o\sin\theta$, $\theta = \sin^{-1}\frac{\sqrt{2gh}}{V_o}$ ....(5)

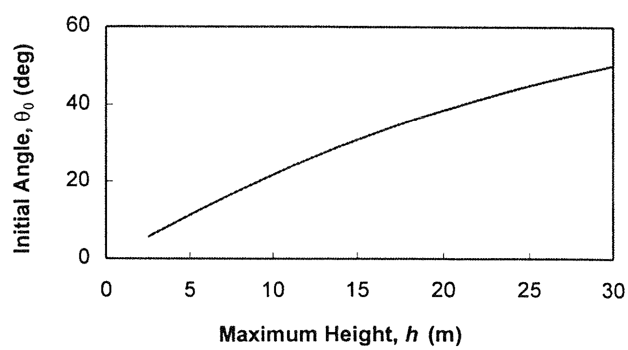
$\theta = \sin^{-1}\left[\left(2\times 9.81\frac{m}{s}\times 10\,m\right)^{1/2}\frac{s}{37.7\,m}\right] = 21.8^\circ$ ← θ_o

Plots of $V_o = V_o(h)$ {Eq 4} and $\theta_o = \theta_o(h)$ {Eq. 5} are presented below

Eq. 4: Initial Speed vs. Max. Height



Eq. 5: Initial Angle vs. Max. Height



Given: Basic dimensions M, L, t and T.

Find: Dimensional representation of quantities below, and typical units in SI and English systems.

Solution:

(a) Power $= \frac{\text{Energy}}{\text{Time}} = \frac{\text{Force} \times \text{Distance}}{\text{Time}}$

From Newton's second law, Force = Mass x Acceleration

$\therefore$ Power $= \frac{\text{Mass} \times \text{Acc.} \times \text{Dist.}}{\text{Time}} = \left[\frac{M \frac{L}{t^2} L}{t}\right] = \left[\frac{ML^2}{t^3}\right]$; $\frac{kg \cdot m^2}{s^3}$ or $\frac{slug \cdot ft}{s^3}$

(b) Pressure $= \frac{\text{Force}}{\text{Area}} = \left[\frac{F}{L^2}\right] = \left[\frac{ML}{t^2}\frac{1}{L^2}\right] = \left[\frac{M}{Lt^2}\right]$; $\frac{kg}{m \cdot s^2}$ or $\frac{slug}{ft \cdot s^2}$

(c) Modulus of elasticity $= \frac{\text{Force}}{\text{Area}} = \left[\frac{M}{Lt^2}\right]$; $\frac{kg}{m \cdot s^2}$ or $\frac{slug}{ft \cdot s^2}$

(d) Angular velocity $= \frac{\text{Radians}}{\text{Time}} = \left[\frac{1}{t}\right]$; $\frac{1}{s}$ or $\frac{1}{s}$

(e) Energy = Force x Distance $= \left[\frac{ML}{t^2} L\right] = \left[\frac{ML^2}{t^2}\right]$; $\frac{kg \cdot m^2}{s^2}$ or $\frac{slug \cdot ft^2}{s^2}$

(f) Momentum = Mass x Velocity $= \left[M \frac{L}{t}\right] = \left[\frac{ML}{t}\right]$; $\frac{kg \cdot m}{s}$ or $\frac{slug \cdot ft}{s}$

(g) Shear stress $= \frac{\text{Force}}{\text{Area}} = \left[\frac{M}{Lt^2}\right]$; $\frac{kg}{m \cdot s^2}$ or $\frac{slug}{ft \cdot s^2}$

(h) Specific heat $= \frac{\text{Energy}}{\text{Mass} \times \text{Temperature}} = \left[\frac{\frac{ML^2}{t^2}}{M \quad T}\right] = \left[\frac{L^2}{t^2 T}\right]$; $\frac{m^2}{s^2 \cdot K}$ or $\frac{ft^2}{s^2 \cdot {}^\circ R}$

(i) Thermal expansion coefficient $= \frac{\text{Change in length / Length}}{\text{Temperature}} = \left[\frac{1}{T}\right]$;

$\frac{1}{K}$ or $\frac{1}{{}^\circ R}$

(j) Angular momentum = momentum x distance

= mass x velocity x distance

$= \left[M \frac{L}{t} L\right] = \left[\frac{ML^2}{t}\right]$; $\frac{kg \cdot m^2}{s}$ or $\frac{slug \cdot ft^2}{s}$

Given: Basic dimensions F, L, t and T.

Find: Dimensional representation of quantities below, and typical units in SI and English systems.

Solution:

(a) Power $= \frac{\text{Energy}}{\text{Time}} = \frac{\text{Force} \times \text{Distance}}{\text{Time}} = \left[\frac{FL}{t}\right]$; $\frac{N \cdot m}{s}$ or $\frac{lbf \cdot ft}{s}$

(b) Pressure $= \frac{\text{Force}}{\text{Area}} = \left[\frac{F}{L^2}\right]$; $\frac{N}{m^2}$ or $\frac{lbf}{ft^2}$

(c) Modulus of elasticity $= \frac{\text{Force}}{\text{Area}} = \left[\frac{F}{L^2}\right]$; $\frac{N}{m^2}$ or $\frac{lbf}{ft^2}$

(d) Angular velocity $= \frac{\text{Radians}}{\text{Time}} = \left[\frac{1}{t}\right]$; $\frac{1}{s}$ or $\frac{1}{s}$

(e) Energy = Force × Distance $= [FL]$; $N \cdot m$ or $lbf \cdot ft$

(f) Moment of a force = Force × Distance $= [FL]$; $N \cdot m$ or $lbf \cdot ft$

(g) Momentum = Mass × Velocity $= \left[\frac{ML}{t}\right]$

From Newton's second law, $F = ma$, so $m = \frac{F}{a}$

$\therefore$ Momentum $= \frac{\text{Force} \times \text{Velocity}}{\text{Acceleration}} = \left[\frac{F\frac{L}{t}}{\frac{L}{t^2}}\right] = [Ft]$; $N \cdot s$ or $lbf \cdot s$

(h) Shear stress $= \frac{\text{Force}}{\text{Area}} = \left[\frac{F}{L^2}\right]$; $\frac{N}{m^2}$ or $\frac{lbf}{ft^2}$

(i) Strain $= \frac{\text{Change in length}}{\text{Length}} = \left[\frac{L}{L}\right] = [-]$; $(-)$ or $(-)$

(j) Angular momentum = momentum × distance

= Mass × Velocity × distance

$= \left[M \frac{L}{t} L\right] = \left[\frac{Ft^2}{L} \frac{L^2}{t}\right]$

$= [FLt]$; $N \cdot m \cdot s$ or $lbf \cdot ft \cdot s$

Problem 1.30

Derive the following conversion factors:
(a) Convert a pressure of 1 psi to kPa.
(b) Convert a volume of 1 liter to gallons.
(c) Convert a viscosity of 1 lbf.s/ft^2 to N.s/m^2.

Solution

Using data from tables (e.g. Table G.2)

(a) $$1 \cdot psi = 1 \cdot psi \times \frac{6895 \cdot Pa}{1 \cdot psi} \times \frac{1 \cdot kPa}{1000 \cdot Pa} = 6.89 \cdot kPa$$

(b) $$1 \cdot liter = 1 \cdot liter \times \frac{1 \cdot quart}{0.946 \cdot liter} \times \frac{1 \cdot gal}{4 \cdot quart} = 0.264 \cdot gal$$

(c) $$1 \cdot \frac{lbf \cdot s}{ft^2} = 1 \cdot \frac{lbf \cdot s}{ft^2} \times \frac{4.448 \cdot N}{1 \cdot lbf} \times \left(\frac{\frac{1}{12} \cdot ft}{0.0254 \cdot m} \right)^2 = 47.9 \cdot \frac{N \cdot s}{m^2}$$

Problem 1.31

Derive the following conversion factors:

(a) Convert a viscosity of 1 m^2/s to ft^2/s.

(b) Convert a power of 100 W to horsepower.

(c) Convert a specific energy of 1 kJ/kg to Btu/lbm.

Solution

Using data from tables (e.g. Table G.2)

(a) $$1\cdot\frac{m^2}{s} = 1\cdot\frac{m^2}{s}\times\left(\frac{\frac{1}{12}\cdot ft}{0.0254\cdot m}\right)^2 = 10.76\cdot\frac{ft^2}{s}$$

(b) $$100\cdot W = 100\cdot W\times\frac{1\cdot hp}{746\cdot W} = 0.134\cdot hp$$

(c) $$1\cdot\frac{kJ}{kg} = 1\cdot\frac{kJ}{kg}\times\frac{1000\cdot J}{1\cdot kJ}\times\frac{1\cdot Btu}{1055\cdot J}\times\frac{0.454\cdot kg}{1\cdot lbm} = 0.43\cdot\frac{Btu}{lbm}$$

Problem 1.32

Given: Density of mercury is $\rho = 26.3\ slug/ft^3$.

Acceleration of gravity on moon is $g_m = 5.47\ ft/s^2$.

Find: (a) Specific gravity of mercury.
(b) Specific volume of mercury, in m^3/kg.
(c) Specific weight on Earth.
(d) Specific weight on moon.

Solution: Apply definitions: $\gamma \equiv \rho g$, $v = 1/\rho$, $SG \equiv \rho/\rho_{H_2O}$

Thus $SG = 26.3 \frac{slug}{ft^3} \times \frac{ft^3}{1.94\ slug} = 13.6$ ← SG

$$v = \frac{ft^3}{26.3\ slug} \times \frac{(0.3048)^3 m^3}{ft^3} \times \frac{slug}{32.2\ lbm} \times \frac{lbm}{0.4536\ kg} = 7.37 \times 10^{-5}\ m^3/kg$$ ← v

On Earth,

$$\gamma_E = 26.3 \frac{slug}{ft^3} \times 32.2 \frac{ft}{s^2} \times \frac{lbf \cdot s^2}{slug \cdot ft} = 847\ lbf/ft^3$$ ← γ_E

On the moon,

$$\gamma_m = 26.3 \frac{slug}{ft^3} \times 5.47 \frac{ft}{s^2} \times \frac{lbf \cdot s^2}{slug \cdot ft} = 144\ lbf/ft^3$$ ← γ_m

{Note that the mass-based quantities (SG and v) are independent of gravity.}

Problem 1.33

Derive the following conversion factors:

(a) Convert a volume flow rate in in.^{3}/min to mm^3/s.
(b) Convert a volume flow rate in cubic meters per second to gpm (gallons per minute).
(c) Convert a volume flow rate in liters per minute to gpm (gallons per minute).
(d) Convert a volume flow rate of air in standard cubic feet per minute (SCFM) to cubic meters per hour. A standard cubic foot of gas occupies one cubic foot at standard temperature and pressure (T = 15°C and p = 101.3 kPa absolute).

Solution

Using data from tables (e.g. Table G.2)

(a) $$1\cdot\frac{in^3}{min} = 1\cdot\frac{in^3}{min}\times\left(\frac{0.0254\cdot m}{1\cdot in}\times\frac{1000\cdot mm}{1\cdot m}\right)^3\times\frac{1\cdot min}{60\cdot s} = 273\cdot\frac{mm^3}{s}$$

(b) $$1\cdot\frac{m^3}{s} = 1\cdot\frac{m^3}{s}\times\frac{1\cdot quart}{0.000946\cdot m^3}\times\frac{1\cdot gal}{4\cdot quart}\times\frac{60\cdot s}{1\cdot min} = 15850\cdot gpm$$

(c) $$1\cdot\frac{liter}{min} = 1\cdot\frac{liter}{min}\times\frac{1\cdot quart}{0.946\cdot liter}\times\frac{1\cdot gal}{4\cdot quart}\times\frac{60\cdot s}{1\cdot min} = 0.264\cdot\frac{gal}{min}$$

(d) $$1\cdot SCFM = 1\cdot\frac{ft^3}{min}\times\left(\frac{0.0254\cdot m}{\frac{1}{12}\cdot ft}\right)^3\times\frac{60\cdot min}{hr} = 1.70\cdot\frac{m^3}{hr}$$

Problem 1.34

Given: In European usage, 1 kgf is the force exerted on 1 kg mass in standard gravity.

Find: Convert 32 psi to units of kgf/cm².

Solution: Apply Newton's second law.

Basic equation: $F = ma$

The force exerted on 1 kg in standard gravity is

$$F = 1\,kg \times 9.81\,\frac{m}{s^2} \times \frac{N \cdot s^2}{kg \cdot m} = 9.81\,N = 1\,kgf$$

Setting up a conversion from psi to kgf/cm²,

$$1\,\frac{lbf}{in.^2} = 1\,\frac{lbf}{in.^2} \times 4.448\,\frac{N}{lbf} \times \frac{in.^2}{(2.54)^2\,cm^2} \times \frac{kgf}{9.81\,N} = 0.0703\,\frac{kgf}{cm^2}$$

or

$$1 \equiv \frac{0.0703\,kgf/cm^2}{psi}$$

Thus

$$32\,psi = 32\,psi \times \frac{0.0703\,kgf/cm^2}{psi}$$

$$32\,psi = 2.25\,kgf/cm^2$$

Problem 1.35

Sometimes "engineering" equations are used in which units are present in an inconsistent manner. For example, a parameter that is often used in describing pump performance is the specific speed, NScu, given by

$$N_{Scu} = \frac{N(\mathrm{rpm}) \cdot Q(\mathrm{gpm})^{\frac{1}{2}}}{H(\mathrm{ft})^{\frac{3}{4}}}$$

What are the units of specific speed? A particular pump has a specific speed of 2000. What will be the specific speed in SI units (angular velocity in rad/s)?

Solution

Using data from tables (e.g. Table G.2)

$$N_{Scu} = 2000 \cdot \frac{\mathrm{rpm} \cdot \mathrm{gpm}^{\frac{1}{2}}}{\mathrm{ft}^{\frac{3}{4}}} = 2000 \times \frac{\mathrm{rpm} \cdot \mathrm{gpm}^{\frac{1}{2}}}{\mathrm{ft}^{\frac{3}{4}}} \times \frac{2 \cdot \pi \cdot \mathrm{rad}}{1 \cdot \mathrm{rev}} \times \frac{1 \cdot \mathrm{min}}{60 \cdot \mathrm{s}} \times ..$$

$$\left(\frac{4 \cdot \mathrm{quart}}{1 \cdot \mathrm{gal}} \cdot \frac{0.000946 \cdot \mathrm{m}^3}{1 \cdot \mathrm{quart}} \cdot \frac{1 \cdot \mathrm{min}}{60 \cdot \mathrm{s}}\right)^{\frac{1}{2}} \times \left(\frac{\frac{1}{12} \cdot \mathrm{ft}}{0.0254 \cdot \mathrm{m}}\right)^{\frac{3}{4}} = 4.06 \cdot \frac{\frac{\mathrm{rad}}{\mathrm{s}} \cdot \left(\frac{\mathrm{m}^3}{\mathrm{s}}\right)^{\frac{1}{2}}}{\mathrm{m}^{\frac{3}{4}}}$$

Problem 1.36

A particular pump has an "engineering" equation form of the performance characteristic equatio given by H (ft) = 1.5 - 4.5 x 10^{-5} [Q (gpm)]^2, relating the head H and flow rate Q. What are the units of the coefficients 1.5 and 4.5 x 10^{-5}? Derive an SI version of this equation.

Solution

Dimensions of "1.5" are ft.

Dimensions of "4.5 x 10^{-5}" are ft/gpm^2.

Using data from tables (e.g. Table G.2), the SI versions of these coefficients can be obtained

$$1.5\cdot\text{ft} = 1.5\cdot\text{ft}\times\frac{0.0254\cdot\text{m}}{\frac{1}{12}\cdot\text{ft}} = 0.457\cdot\text{m}$$

$$4.5\times10^{-5}\cdot\frac{\text{ft}}{\text{gpm}^2} = 4.5\cdot10^{-5}\cdot\frac{\text{ft}}{\text{gpm}^2}\times\frac{0.0254\cdot\text{m}}{\frac{1}{12}\cdot\text{ft}}\times\left(\frac{1\cdot\text{gal}}{4\cdot\text{quart}}\cdot\frac{1\text{quart}}{0.000946\cdot\text{m}^3}\cdot\frac{60\cdot\text{s}}{1\text{min}}\right)^2$$

$$4.5\cdot10^{-5}\cdot\frac{\text{ft}}{\text{gpm}^2} = 3450\cdot\frac{\text{m}}{\left(\frac{\text{m}^3}{\text{s}}\right)^2}$$

The equation is $$H(\text{m}) = 0.457 - 3450\cdot\left(Q\left(\frac{\text{m}^3}{\text{s}}\right)\right)^2$$

Given: Empty container weighing 3.5 lbf when empty, has a mass of 2.5 slug when filled with water at 90°F.

Find: (a) Weight of water in the container
(b) Container volume in ft^3.

Solution:

Basic equation: $F = ma$

Weight is the force of gravity on a body, $W = mg$

Then $W_t = W_{H_2O} + W_c$

$$W_{H_2O} = W_t - W_c = mg - W_c$$

$$W_{H_2O} = 2.5\ \text{slug} \times 32.2 \frac{\text{ft}}{\text{s}^2} \times \frac{\text{lbf} \cdot \text{s}^2}{\text{slug} \cdot \text{ft}} - 3.5\ \text{lbf} = 77.0\ \text{lbf} \quad W_{H_2O}$$

The volume is given by

$$\forall = \frac{M_{H_2O}}{\rho} = \frac{M_{H_2O}\, g}{\rho g} = \frac{W_{H_2O}}{\rho g}$$

From Table A.7, $\rho = 1.93\ \text{slug}/\text{ft}^3$ at $T = 90°F$

$$\therefore \forall = 77.0\ \text{lbf} \times \frac{\text{ft}^3}{1.93\ \text{slug}} \times \frac{\text{s}^2}{32.2\ \text{ft}} \times \frac{\text{slug} \cdot \text{ft}}{\text{lbf} \cdot \text{s}^2} = 1.24\ \text{ft}^3 \quad \forall$$

Problem 2.1

For the velocity fields given below, determine:
(a) whether the flow field is one-, two-, or three-dimensional, and why.
(b) whether the flow is steady or unsteady, and why.
(The quantities a and b are constants.)

(1) $\vec{V} = [ax^2e^{-bt}]\hat{i}$
(2) $\vec{V} = ax\hat{i} - by\hat{j}$
(3) $\vec{V} = ax^2\hat{i} + bx\hat{j} + c\hat{k}$
(4) $\vec{V} = ax^2\hat{i} + bxz\hat{j} + cz\hat{k}$
(5) $\vec{V} = [ae^{-bx}]\hat{i} + bx^2\hat{j}$
(6) $\vec{V} = axy\hat{i} - byzt\hat{j}$
(7) $\vec{V} = a(x^2 + y^2)^{1/2}(1/z^3)\hat{k}$
(8) $\vec{V} = (ax + t)\hat{i} - by^2\hat{j}$

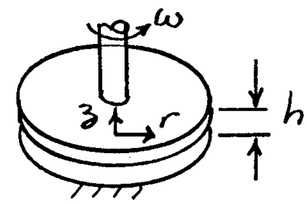
Solution

(1)	$\vec{V} = \vec{V}(x)$	1D	$\vec{V} = \vec{V}(t)$	Unsteady
(2)	$\vec{V} = \vec{V}(x,y)$	2D	$\vec{V} \neq \vec{V}(t)$	Steady
(3)	$\vec{V} = \vec{V}(x)$	1D	$\vec{V} \neq \vec{V}(t)$	Steady
(4)	$\vec{V} = \vec{V}(x,z)$	2D	$\vec{V} \neq \vec{V}(t)$	Steady
(5)	$\vec{V} = \vec{V}(x)$	1D	$\vec{V} \neq \vec{V}(t)$	Steady
(6)	$\vec{V} = \vec{V}(x,y,z)$	3D	$\vec{V} = \vec{V}(t)$	Unsteady
(7)	$\vec{V} = \vec{V}(x,y,z)$	3D	$\vec{V} \neq \vec{V}(t)$	Steady
(8)	$\vec{V} = \vec{V}(x,y)$	2D	$\vec{V} = \vec{V}(t)$	Unsteady

Given: Viscous liquid sheared between parallel disks.

Upper disk rotates, lower fixed.

Velocity field is $\vec{V} = \hat{e}_\theta r\omega z/h$.



Find: (a) Dimensions of velocity field.
(b) Satisfy physical boundary conditions.

Solution: To find dimensions, compare to $\vec{V} = \vec{V}(x, y, z)$ form.

The given field is $\vec{V} = \vec{V}(r, z)$. Two space coordinates are included, so field is 2-D. ← 2-D

Flow must satisfy the no-slip condition:

(1) At lower disk, $\vec{V} = 0$, since stationary.

$z = 0$, so $\vec{V} = \hat{e}_\theta r\omega(0)/h = 0$ ∴ satisfied ← $z =$

(2) At upper disk, $\vec{V} = \hat{e}_\theta r\omega$, since it rotates as a solid body.

$z = h$, so $\vec{V} = \hat{e}_\theta r\omega(h)/h = \hat{e}_\theta r\omega$ ∴ satisfied ← $z =$

Given: Velocity field, $\vec{V} = ax\hat{i} - by\hat{j}$ $(a = b = 1\,sec^{-1})$

Find: Equation for the flow streamlines, and

Plot: Representative streamlines for $x \geq 0$ and $y \geq 0$

Solution:

The slope of the streamlines in the x-y plane is given by

$$\frac{dy}{dx} = \frac{v}{u}$$

For $\vec{V} = ax\hat{i} - by\hat{j}$, then $u = ax$, $v = -by$. Hence

$$\frac{dy}{dx} = \frac{v}{u} = -\frac{b}{a}\frac{y}{x}$$

To solve the differential equation, separate variables and integrate

$$\int \frac{dy}{y} = -\int \frac{b}{a}\frac{dx}{x}$$

$$\ln y = -\frac{b}{a}\ln x + \text{constant}$$

$$\ln y = \ln x^{-\frac{b}{a}} + \ln c \qquad \text{where constant} = \ln c$$

Then

$$y = c\,x^{-\frac{b}{a}}$$ y(x)

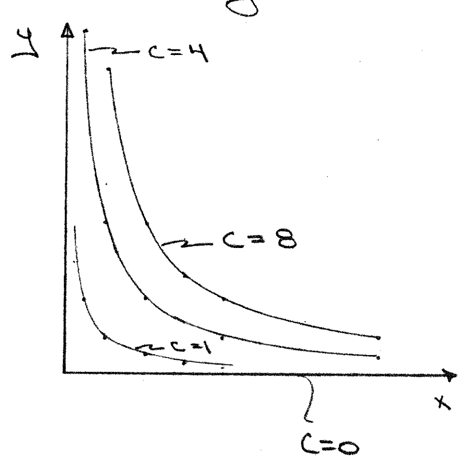
or alternately $x = \left(\frac{y}{c}\right)^{-\frac{a}{b}} = \left(\frac{c}{y}\right)^{\frac{a}{b}}$

For a given velocity field, the constants a and b are fixed. Different streamlines are obtained by assigning different values to the constant of integration, c.

Since $a = b = 1\,sec^{-1}$, then $a/b = 1$, and the streamlines are given by the equation

$$y = cx^{-1} = \frac{c}{x} \qquad \text{or} \qquad x = \frac{c}{y}$$

For $c = 0$ $y = 0$ for all x and $x = 0$ for all y.



The equation $y = \frac{c}{x}$ is the equation of a hyperbola.

Curves are shown for different values of c

Problem 2.4

A velocity field is given by

$$\vec{V} = ax\hat{i} - bty\hat{j}$$

where $a = 1\ s^{-1}$ and $b = 1\ s^{-2}$. Find the equation of the streamlines at any time t. Plot several streamlines in the first quadrant at $t = 0$ s, $t = 1$ s, and $t = 20$ s.

Solution

For streamlines $\frac{v}{u} = \frac{dy}{dx} = \frac{-b \cdot t \cdot y}{a \cdot x}$

So, separating variables $\frac{dy}{y} = \frac{-b \cdot t}{a} \cdot \frac{dx}{x}$

Integrating $\ln(y) = \frac{-b \cdot t}{a} \cdot \ln(x)$

The solution is $y = c \cdot x^{\frac{-b}{a} \cdot t}$

For $t = 0$ s $y = c$

For $t = 1$ s $y = \frac{c}{x}$

For $t = 20$ s $y = c \cdot x^{-20}$

See the plots in the corresponding *Excel* workbook

Problem 2.4 (In Excel)

A velocity field is given by

$$\vec{V} = ax\hat{i} - bty\hat{j}$$

where a = 1 s^{-1} and b = 1 s^{-2}. Find the equation of the streamlines at any time t.
Plot several streamlines in the first quadrant at t = 0 s, t =1 s, and t =20 s.

Solution

The solution is $$y = c \cdot x^{\frac{-b}{a} \cdot t}$$

For t = 0 s $$y = c$$

For t = 1 s $$y = \frac{c}{x}$$

For t = 20 s $$y = c \cdot x^{-20}$$

t = 0

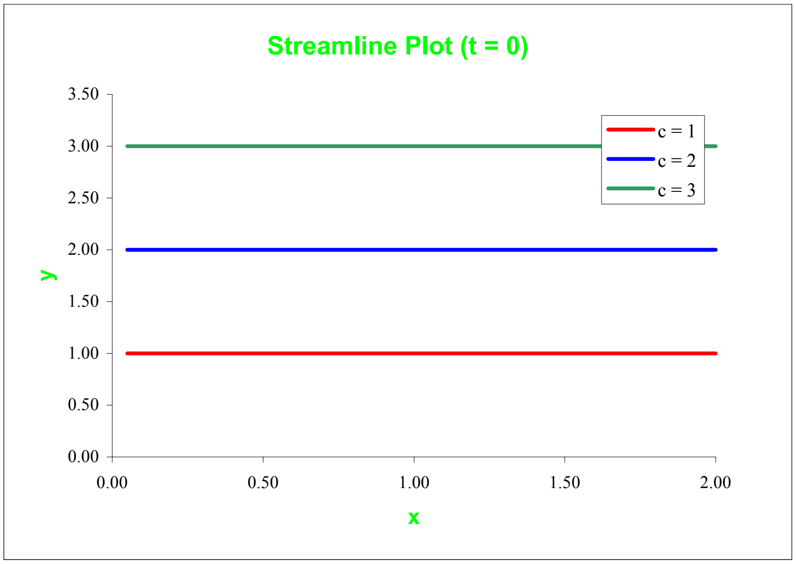
	c = 1	c = 2	c = 3
x	**y**	**y**	**y**
0.05	1.00	2.00	3.00
0.10	1.00	2.00	3.00
0.20	1.00	2.00	3.00
0.30	1.00	2.00	3.00
0.40	1.00	2.00	3.00
0.50	1.00	2.00	3.00
0.60	1.00	2.00	3.00
0.70	1.00	2.00	3.00
0.80	1.00	2.00	3.00
0.90	1.00	2.00	3.00
1.00	1.00	2.00	3.00
1.10	1.00	2.00	3.00
1.20	1.00	2.00	3.00
1.30	1.00	2.00	3.00
1.40	1.00	2.00	3.00
1.50	1.00	2.00	3.00
1.60	1.00	2.00	3.00
1.70	1.00	2.00	3.00
1.80	1.00	2.00	3.00
1.90	1.00	2.00	3.00
2.00	1.00	2.00	3.00

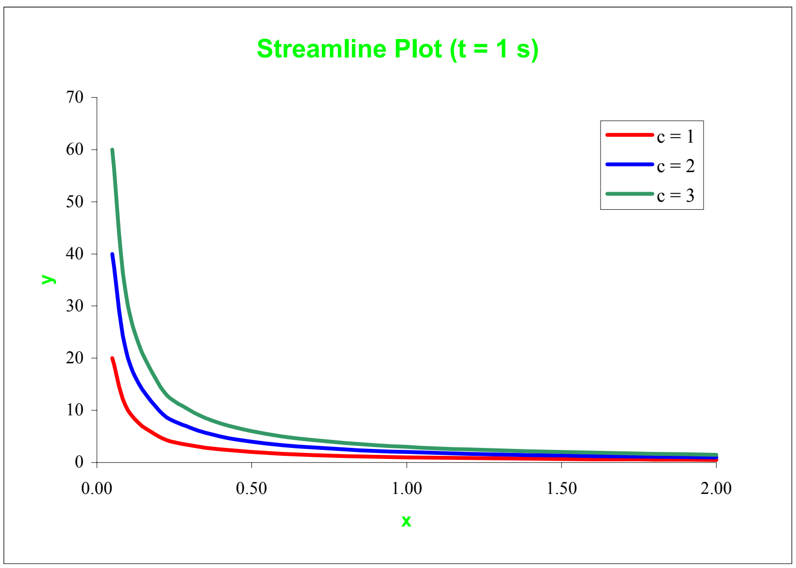
t =1 s
(### means too large to view)

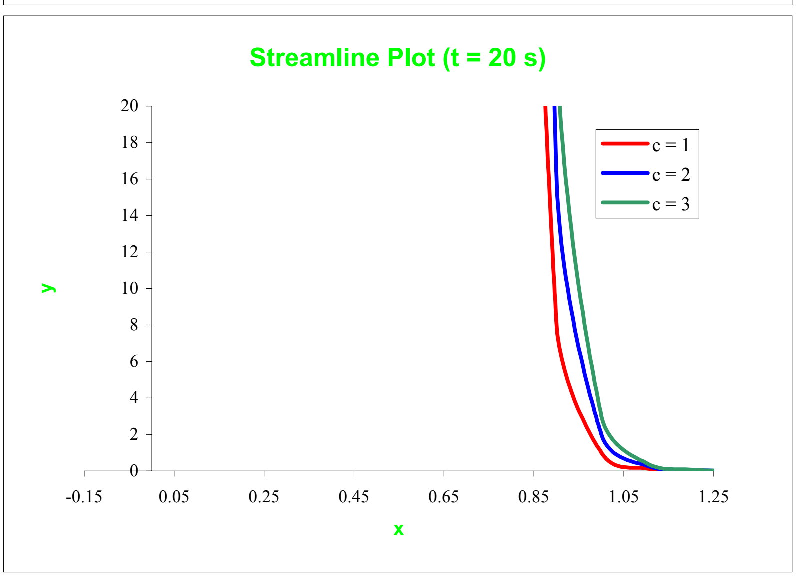
	c = 1	c = 2	c = 3
x	**y**	**y**	**y**
0.05	20.00	40.00	60.00
0.10	10.00	20.00	30.00
0.20	5.00	10.00	15.00
0.30	3.33	6.67	10.00
0.40	2.50	5.00	7.50
0.50	2.00	4.00	6.00
0.60	1.67	3.33	5.00
0.70	1.43	2.86	4.29
0.80	1.25	2.50	3.75
0.90	1.11	2.22	3.33
1.00	1.00	2.00	3.00
1.10	0.91	1.82	2.73
1.20	0.83	1.67	2.50
1.30	0.77	1.54	2.31
1.40	0.71	1.43	2.14
1.50	0.67	1.33	2.00
1.60	0.63	1.25	1.88
1.70	0.59	1.18	1.76
1.80	0.56	1.11	1.67
1.90	0.53	1.05	1.58
2.00	0.50	1.00	1.50

t = 20 s

	c = 1	c = 2	c = 3
x	**y**	**y**	**y**
0.05	#####	#####	#####
0.10	#####	#####	#####
0.20	#####	#####	#####
0.30	#####	#####	#####
0.40	#####	#####	#####
0.50	#####	#####	#####
0.60	#####	#####	#####
0.70	#####	#####	#####
0.80	86.74	#####	#####
0.90	8.23	16.45	24.68
1.00	1.00	2.00	3.00
1.10	0.15	0.30	0.45
1.20	0.03	0.05	0.08
1.30	0.01	0.01	0.02
1.40	0.00	0.00	0.00
1.50	0.00	0.00	0.00
1.60	0.00	0.00	0.00
1.70	0.00	0.00	0.00
1.80	0.00	0.00	0.00
1.90	0.00	0.00	0.00
2.00	0.00	0.00	0.00


Streamline Plot (t = 0)
3.50
3.00
2.50
2.00
1.50
1.00
0.50
0.00
y
0.00
0.50
1.00
1.50
2.00
x
c = 1
c = 2
c = 3


Streamline Plot (t = 1 s)
70
60
50
40
30
20
10
0
y
0.00
0.50
1.00
1.50
2.00
x
c = 1
c = 2
c = 3


Streamline Plot (t = 20 s)
20
18
16
14
12
10
8
6
4
2
0
y
-0.15
0.05
0.25
0.45
0.65
0.85
1.05
1.25
x
c = 1
c = 2
c = 3

Problem 2.5

Given: Velocity field, $\vec{V} = Axy\,\hat{i} + By^2\,\hat{j}$

$A = 1\ \text{m}^{-1}\text{s}^{-1}$, $B = -0.5\ \text{m}^{-1}\text{s}^{-1}$; coordinates in meters

Find: Equation for flow streamlines

Plot: several streamlines in upper half plane

Solution:

Streamlines are tangent to the velocity vector, so

$$\left.\frac{dy}{dx}\right)_{\text{streamline}} = \frac{v}{u} = \frac{By^2}{Axy} = \frac{By}{Ax} = \frac{-0.5}{\text{m}\cdot\text{s}} \times \frac{\text{m}\cdot\text{s}}{1.0}\,\frac{y}{x} = -\frac{y}{2x}$$

Separating variables,

$$\frac{dx}{x} = -\frac{2\,dy}{y} \quad \text{or} \quad \frac{dx}{x} + 2\frac{dy}{y} = 0$$

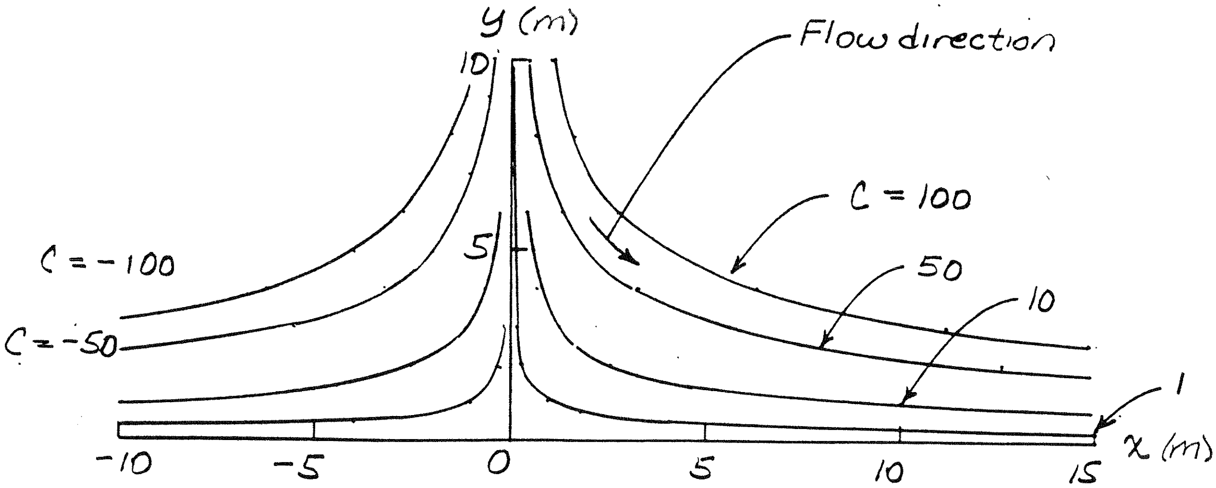
Integrating,

$$\ln x + 2\ln y = c_1 = \ln c \quad \text{or} \quad \ln x + \ln y^2 = \ln c$$

Taking antilogarithms,

$$xy^2 = c \quad \longleftarrow \text{(Equation for streamlines)}$$

Plotting:



Problem 2.6

A velocity field is specified as

$$\vec{V} = ax^2\hat{i} + bxy\hat{j}$$

where $a = 2\ m^{-1}s^{-1}$ and $b = -6\ m^{-1}s^{-1}$, and the coordinates are measured in meters. Is the flow field one-, two-, or three-dimensional? Why? Calculate the velocity components at the point (2, 1/2). Develop an equation for the streamline passing through this point. Plot several streamlines in the first quadrant including the one that passes through the point (2, 1/2).

Solution

The velocity field is a function of *x* and *y*. It is therefore 2D

At point (2,1/2), the velocity components are

$$u = a\cdot x^2 = 2\cdot\frac{1}{m\cdot s}\times(2\cdot m)^2 \qquad u = 8\cdot\frac{m}{s}$$

$$v = b\cdot x\cdot y = -6\cdot\frac{1}{m\cdot s}\times 2\cdot m\times\frac{1}{2}\cdot m \qquad v = -6\cdot\frac{m}{s}$$

For streamlines $\quad \frac{v}{u} = \frac{dy}{dx} = \frac{b\cdot x\cdot y}{a\cdot x^2} = \frac{b\cdot y}{a\cdot x}$

So, separating variables $\quad \frac{dy}{y} = \frac{b}{a}\cdot\frac{dx}{x}$

Integrating $\quad \ln(y) = \frac{b}{a}\cdot\ln(x) \qquad y = c\cdot x^{\frac{b}{a}} = c\cdot x^{-3}$

The solution is $\quad y = \frac{c}{x^3}$

See the plot in the corresponding *Excel* workbook

Problem 2.6 (In Excel)

A velocity field is specified as

$$\vec{V} = ax^2\hat{i} + bxy\hat{j}$$

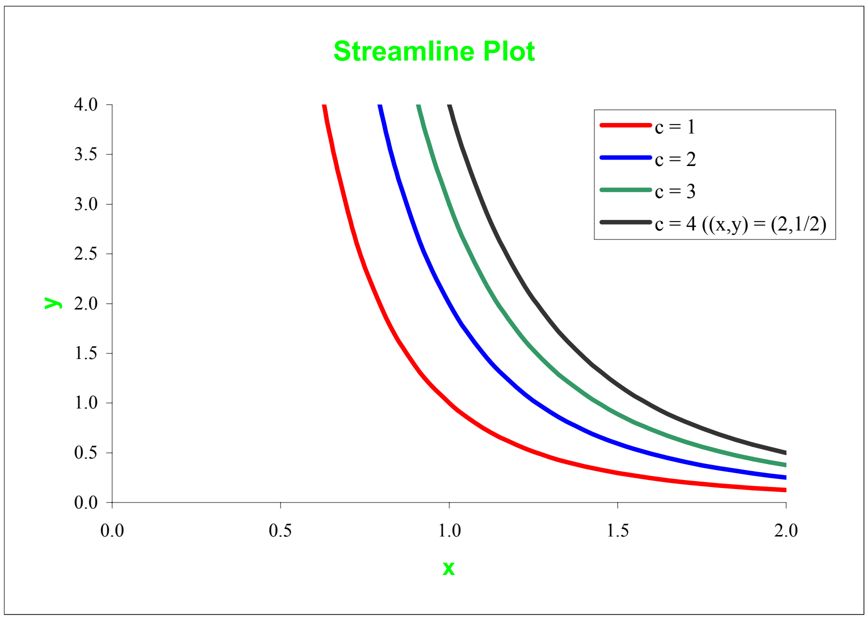
where a = 2 $m^{-1}s^{-1}$, b = - 6 $m^{-1}s^{-1}$, and the coordinates are measured in meters.
Is the flow field one-, two-, or three-dimensional? Why?
Calculate the velocity components at the point (2, 1/2). Develop an equation
for the streamline passing through this point. Plot several streamlines in the first
quadrant including the one that passes through the point (2, 1/2).

Solution

The solution is $y = \frac{c}{x^3}$

c =

	1	2	3	4
x	y	y	y	y
0.05	8000	16000	24000	32000
0.10	1000	2000	3000	4000
0.20	125	250	375	500
0.30	37.0	74.1	111.1	148.1
0.40	15.6	31.3	46.9	62.5
0.50	8.0	16.0	24.0	32.0
0.60	4.63	9.26	13.89	18.52
0.70	2.92	5.83	8.75	11.66
0.80	1.95	3.91	5.86	7.81
0.90	1.37	2.74	4.12	5.49
1.00	1.00	2.00	3.00	4.00
1.10	0.75	1.50	2.25	3.01
1.20	0.58	1.16	1.74	2.31
1.30	0.46	0.91	1.37	1.82
1.40	0.36	0.73	1.09	1.46
1.50	0.30	0.59	0.89	1.19
1.60	0.24	0.49	0.73	0.98
1.70	0.20	0.41	0.61	0.81
1.80	0.17	0.34	0.51	0.69
1.90	0.15	0.29	0.44	0.58
2.00	0.13	0.25	0.38	0.50



Problem 2.7

A flow is described by the velocity field $\vec{V} = (Ax + B)\hat{i} + (-Ay)\hat{j}$, where $A = 10$ ft/s/ft and $B = 20$ ft/s. Plot a few streamlines in the xy plane, including the one that passes through the point $(x, y) = (1, 2)$.

Solution

Streamlines are given by $$\frac{v}{u} = \frac{dy}{dx} = \frac{-A \cdot y}{A \cdot x + B}$$

So, separating variables $$\frac{dy}{-A \cdot y} = \frac{dx}{A \cdot x + B}$$

Integrating $$-\frac{1}{A}\ln(y) = \frac{1}{A}\cdot\ln\left(x + \frac{B}{A}\right)$$

The solution is $$y = \frac{C}{x + \frac{B}{A}}$$

For the streamline that passes through point $(x,y) = (1,2)$

$$C = y\cdot\left(x + \frac{B}{A}\right) = 2\cdot\left(1 + \frac{20}{10}\right) = 6$$

$$y = \frac{6}{x + \frac{20}{10}}$$

$$y = \frac{6}{x + 2}$$

See the plot in the corresponding *Excel* workbook

Problem 2.7 (In Excel)

A flow is described by the velocity field $\vec{V} = (Ax + B)\hat{i} + (-Ay)\hat{j}$, where A = 10 ft/s/ft and B = 20 ft/s. Plot a few streamlines in the xy plane, including the one that passes through the point $(x, y) = (1, 2)$.

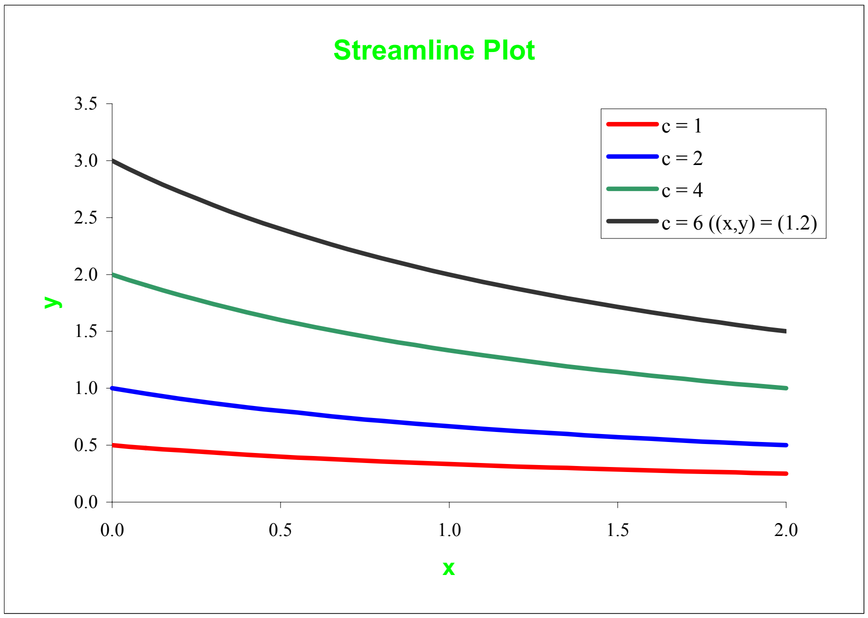
Solution

The solution is $$y = \frac{C}{x + \frac{B}{A}}$$

A = **10**
B = **20**

C =

	1	2	4	6
x	**y**	**y**	**y**	**y**
0.00	0.50	1.00	2.00	3.00
0.10	0.48	0.95	1.90	2.86
0.20	0.45	0.91	1.82	2.73
0.30	0.43	0.87	1.74	2.61
0.40	0.42	0.83	1.67	2.50
0.50	0.40	0.80	1.60	2.40
0.60	0.38	0.77	1.54	2.31
0.70	0.37	0.74	1.48	2.22
0.80	0.36	0.71	1.43	2.14
0.90	0.34	0.69	1.38	2.07
1.00	0.33	0.67	1.33	2.00
1.10	0.32	0.65	1.29	1.94
1.20	0.31	0.63	1.25	1.88
1.30	0.30	0.61	1.21	1.82
1.40	0.29	0.59	1.18	1.76
1.50	0.29	0.57	1.14	1.71
1.60	0.28	0.56	1.11	1.67
1.70	0.27	0.54	1.08	1.62
1.80	0.26	0.53	1.05	1.58
1.90	0.26	0.51	1.03	1.54
2.00	0.25	0.50	1.00	1.50



Problem 2.8

A velocity field is given by $\vec{V} = ax^3\hat{i} + bxy^3\hat{j}$, where a = 1 m^{-2} s^{-1} and b = 1 m^{-3} s^{-1}. Find the equation of the streamlines. Plot several streamlines in the first quadrant.

Solution

Streamlines are given by $\frac{v}{u} = \frac{dy}{dx} = \frac{b \cdot x \cdot y^3}{a \cdot x^3}$

So, separating variables $\frac{dy}{y^3} = \frac{b \cdot dx}{a \cdot x^2}$

Integrating $-\frac{1}{2 \cdot y^2} = \frac{b}{a} \cdot \left(-\frac{1}{x}\right) + C$

The solution is $y = \frac{1}{\sqrt{2 \cdot \left(\frac{b}{a \cdot x} + C\right)}}$

Note: For convenience the sign of C is changed.

See the plot in the corresponding *Excel* workbook

Problem 2.8 (In Excel)

A velocity field is given by $\vec{V} = ax^3\hat{i} + bxy^3\hat{j}$, where $a = 1$ m^{-2} s^{-1} and $b = 1$ m^{-3} s^{-1}. Find the equation of the streamlines. Plot several streamlines in the first quadrant.

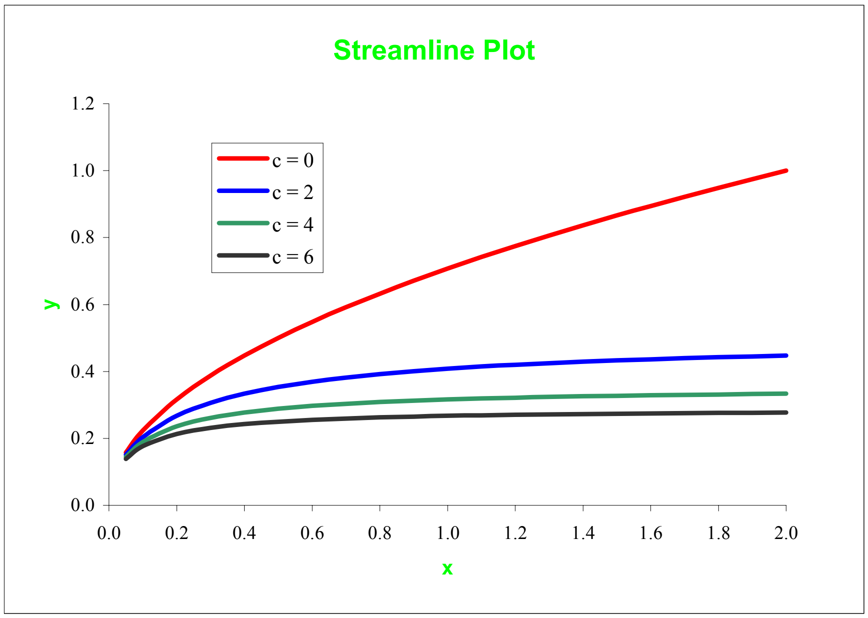
Solution

The solution is $$y = \frac{1}{\sqrt{2 \cdot \left(\frac{b}{a \cdot x} + C\right)}}$$

a = **1**
b = **1**

C =

	0	**2**	**4**	**6**
x	**y**	**y**	**y**	**y**
0.05	0.16	0.15	0.14	0.14
0.10	0.22	0.20	0.19	0.18
0.20	0.32	0.27	0.24	0.21
0.30	0.39	0.31	0.26	0.23
0.40	0.45	0.33	0.28	0.24
0.50	0.50	0.35	0.29	0.25
0.60	0.55	0.37	0.30	0.26
0.70	0.59	0.38	0.30	0.26
0.80	0.63	0.39	0.31	0.26
0.90	0.67	0.40	0.31	0.27
1.00	0.71	0.41	0.32	0.27
1.10	0.74	0.41	0.32	0.27
1.20	0.77	0.42	0.32	0.27
1.30	0.81	0.42	0.32	0.27
1.40	0.84	0.43	0.33	0.27
1.50	0.87	0.43	0.33	0.27
1.60	0.89	0.44	0.33	0.27
1.70	0.92	0.44	0.33	0.28
1.80	0.95	0.44	0.33	0.28
1.90	0.97	0.44	0.33	0.28
2.00	1.00	0.45	0.33	0.28



Problem 2.9

Given: Steady, incompressible flow in xy plane with

$$\vec{V} = \frac{A}{x}\hat{i} + \frac{Ay}{x^2}\hat{j} \quad \text{where } A = 2\,m^2/s$$

and coordinates are in meters.

Find: (a) Equation for streamline through $(x,y) = (1,3)$.
(b) Time required for a fluid particle to move from $x = 1\,m$ to $x = 3\,m$.

Solution: The velocity field is $\vec{V} = u\hat{i} + v\hat{j}$, so $u = \frac{A}{x}$, $v = \frac{Ay}{x^2}$

Computing equations: $\left.\frac{dy}{dx}\right)_{streamline} = \frac{v}{u}$; $u_p = \frac{dx}{dt}$

Substituting, $\frac{dy}{dx} = \frac{Ay}{x^2}\frac{x}{A} = \frac{y}{x}$ so $\frac{dx}{x} = \frac{dy}{y}$

Integrating, $\ln x = \ln y + c^* = \ln y + \ln c$ or $x = cy$

For point $(x,y) = (1,3)$, $c = \frac{x}{y} = \frac{1}{3}$

Thus $x = \frac{y}{3}$ is equation (a)

For a particle, $u_p = \frac{dx}{dt} = \frac{A}{x}$ or $x\,dx = A\,dt$

Integrating, $\int_{x_0}^{x} x\,dx = \frac{x^2 - x_0^2}{2} = At$ so $t = \frac{x^2 - x_0^2}{2A}$

$$t = \frac{1}{2} \times \left[(3)^2 m^2 - (1)^2 m^2\right] \times \frac{s}{2\,m^2} = 2\,s$$ (b)

Problem 2.10

Given: Velocity field $\vec{V} = ax\hat{i} - by\hat{j}$, where $a = b = 1\ s^{-1}$.

Find: (a) Show that particle motion is described by the parametric equations $x_p = c_1 e^{at}$ and $y_p = c_2 e^{-bt}$

(b) Obtain equation of pathline for particle located at (1,2) at $t=0$.

(c) Compare pathline with streamline through same point

Solution

(a) A particle moving in the velocity field $\vec{V} = ax\hat{i} - by\hat{j}$ will have velocity components $u = ax$, $v = -by$

Thus $u_p = \frac{dx}{dt} = ax$ or $\frac{dx}{x} = a\,dt$ and $\int \frac{dx}{x} = \int a\,dt$ ...(1)

$v_p = \frac{dy}{dt} = -by$ or $\frac{dy}{y} = -b\,dt$ and $\int \frac{dy}{y} = -\int b\,dt$ ...(2)

Integrating Eqs. (1) and (2) we obtain

$\ln x = at + \ln c_1$ or $\frac{x}{c_1} = e^{at}$ and $x = c_1 e^{at}$

$\ln y = -bt + \ln c_2$ or $\frac{y}{c_2} = e^{-bt}$ and $y = c_2 e^{-bt}$ ← Q.E.D.

(b) To obtain the equation of the pathline we eliminate t from the parametric equations.

$x = c_1 e^{at}$ $\therefore \ln \frac{x}{c_1} = at$ or $t = \frac{1}{a} \ln \frac{x}{c_1}$

$y = c_2 e^{-bt}$ $\therefore \ln \frac{y}{c_2} = -bt$ or $t = -\frac{1}{b} \ln \frac{y}{c_2}$

Equating expressions for t, we obtain

$\frac{1}{a} \ln \frac{x}{c_1} = -\frac{1}{b} \ln \frac{y}{c_2}$ or $-\frac{b}{a} \ln \frac{x}{c_1} = \ln \frac{y}{c_2}$

Thus $\left(\frac{x}{c_1}\right)^{-b/a} = \frac{y}{c_2}$ or $y\left(\frac{x}{c_1}\right)^{b/a} = c_2$

At $t=0$ $x = 1 = c_1$, $y = 2 = c_2$. Since $a = b$, then the pathline of the particle is $xy = 2$. ← Pathline

(c) The streamline in the x-y plane has slope $\frac{dy}{dx} = \frac{v}{u} = -\frac{b}{a}\frac{y}{x}$

Thus $\frac{dy}{y} + \frac{b}{a}\frac{dx}{x} = 0$. This can be integrated to obtain

$\ln y + \frac{b}{a} \ln x = \text{constant} = \ln c$

Simplifying we obtain $y x^{b/a} = c$. With $b = a$, the equation of the streamline through point (1,2) is then

$xy = 2$ ← Streamline

Problem 2.11

A velocity field is given by $\vec{V} = ayt\hat{i} - bx\hat{j}$, where $a = 1\ \text{s}^{-2}$ and $b = 4\ \text{s}^{-1}$. Find the equation of the streamlines at any time t. Plot several streamlines at $t = 0$ s, $t = 1$ s, and $t = 20$ s.

Solution

Streamlines are given by $\frac{v}{u} = \frac{dy}{dx} = \frac{-b \cdot x}{a \cdot y \cdot t}$

So, separating variables $a \cdot t \cdot y \cdot dy = -b \cdot x \cdot dx$

Integrating $\frac{1}{2} \cdot a \cdot t \cdot y^2 = -\frac{1}{2} \cdot b \cdot x^2 + C$

The solution is $y = \sqrt{C - \frac{b \cdot x^2}{a \cdot t}}$

For $t = 0$ s $x = c$

For $t = 1$ s $y = \sqrt{C - 4 \cdot x^2}$

For $t = 20$ s $y = \sqrt{C - \frac{x^2}{5}}$

See the plots in the corresponding *Excel* workbook

Problem 2.11 (In Excel)

A velocity field is given by $\vec{V} = ayt\hat{i} - bx\hat{j}$, where $a = 1\ s^{-2}$ and $b = 4\ s^{-1}$. Find the equation of the streamlines at any time t. Plot several streamlines at $t = 0$ s, $t = 1$ s, and $t = 20$ s.

Solution

The solution is $$y = \sqrt{C - \frac{b \cdot x^2}{a \cdot t}}$$

For $t = 0$ s $$x = c$$

For $t = 1$ s $$y = \sqrt{C - 4 \cdot x^2}$$

For $t = 20$ s $$y = \sqrt{C - \frac{x^2}{5}}$$

t = 0

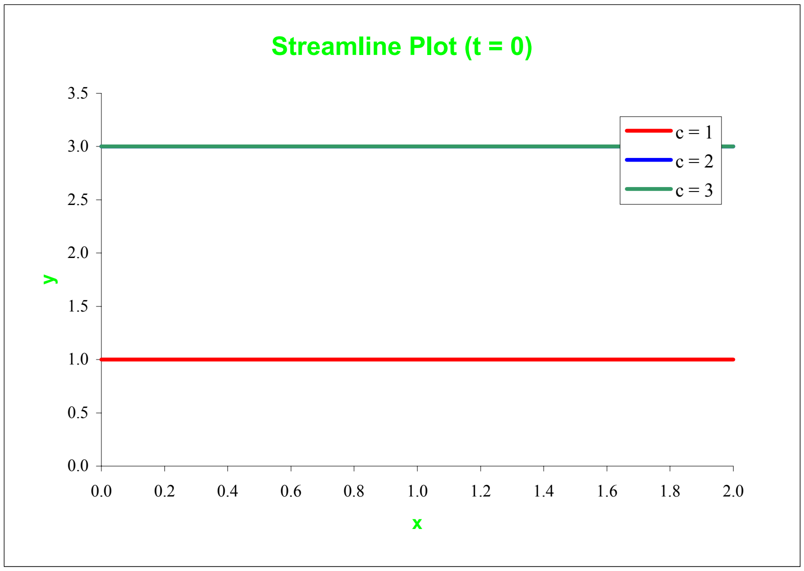
	C = 1	C = 2	C = 3
x	y	y	y
0.00	1.00	2.00	3.00
0.10	1.00	2.00	3.00
0.20	1.00	2.00	3.00
0.30	1.00	2.00	3.00
0.40	1.00	2.00	3.00
0.50	1.00	2.00	3.00
0.60	1.00	2.00	3.00
0.70	1.00	2.00	3.00
0.80	1.00	2.00	3.00
0.90	1.00	2.00	3.00
1.00	1.00	2.00	3.00
1.10	1.00	2.00	3.00
1.20	1.00	2.00	3.00
1.30	1.00	2.00	3.00
1.40	1.00	2.00	3.00
1.50	1.00	2.00	3.00
1.60	1.00	2.00	3.00
1.70	1.00	2.00	3.00
1.80	1.00	2.00	3.00
1.90	1.00	2.00	3.00
2.00	1.00	2.00	3.00

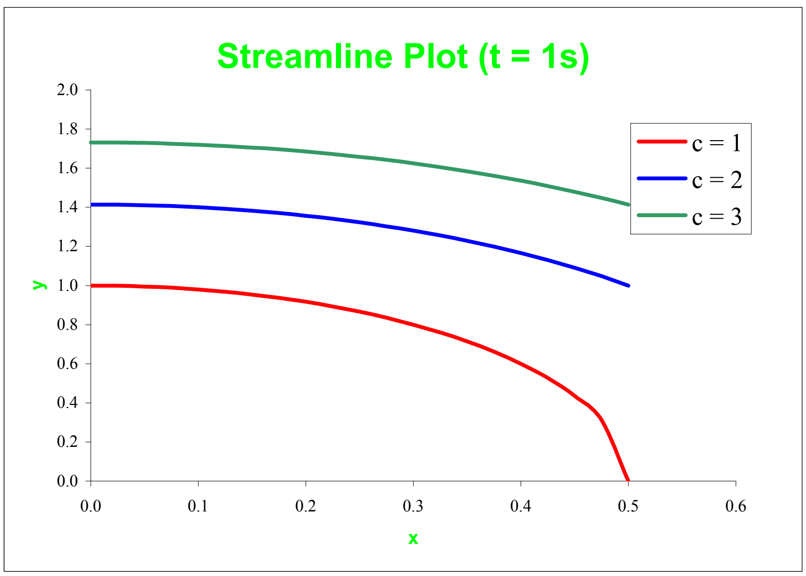
t =1 s

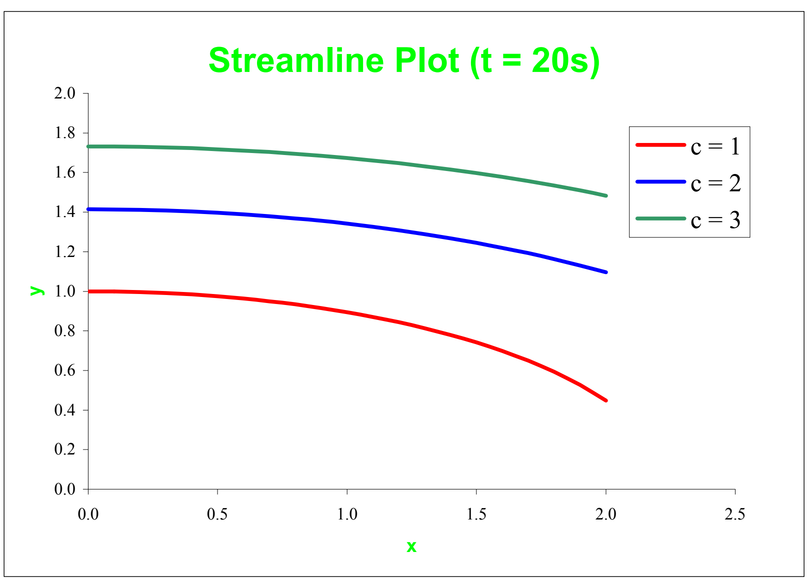
	C = 1	C = 2	C = 3
x	y	y	y
0.000	1.00	1.41	1.73
0.025	1.00	1.41	1.73
0.050	0.99	1.41	1.73
0.075	0.99	1.41	1.73
0.100	0.98	1.40	1.72
0.125	0.97	1.39	1.71
0.150	0.95	1.38	1.71
0.175	0.94	1.37	1.70
0.200	0.92	1.36	1.69
0.225	0.89	1.34	1.67
0.250	0.87	1.32	1.66
0.275	0.84	1.30	1.64
0.300	0.80	1.28	1.62
0.325	0.76	1.26	1.61
0.350	0.71	1.23	1.58
0.375	0.66	1.20	1.56
0.400	0.60	1.17	1.54
0.425	0.53	1.13	1.51
0.450	0.44	1.09	1.48
0.475	0.31	1.05	1.45
0.500	0.00	1.00	1.41

t = 20 s

	C = 1	C = 2	C = 3
x	y	y	y
0.00	1.00	1.41	1.73
0.10	1.00	1.41	1.73
0.20	1.00	1.41	1.73
0.30	0.99	1.41	1.73
0.40	0.98	1.40	1.72
0.50	0.97	1.40	1.72
0.60	0.96	1.39	1.71
0.70	0.95	1.38	1.70
0.80	0.93	1.37	1.69
0.90	0.92	1.36	1.68
1.00	0.89	1.34	1.67
1.10	0.87	1.33	1.66
1.20	0.84	1.31	1.65
1.30	0.81	1.29	1.63
1.40	0.78	1.27	1.61
1.50	0.74	1.24	1.60
1.60	0.70	1.22	1.58
1.70	0.65	1.19	1.56
1.80	0.59	1.16	1.53
1.90	0.53	1.13	1.51
2.00	0.45	1.10	1.48


Streamline Plot (t = 0)
c = 1
c = 2
c = 3
y
x
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0.0 0.2 0.4 0.6 0.8 1.0 1.2 1.4 1.6 1.8 2.0


Streamline Plot (t = 1s)
c = 1
c = 2
c = 3
y
x
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0.0 0.1 0.2 0.3 0.4 0.5 0.6


Streamline Plot (t = 20s)
c = 1
c = 2
c = 3
y
x
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0.0 0.5 1.0 1.5 2.0 2.5

Problem 2.12

Given: Velocity field $\vec{V} = (ax\hat{i} - ay\hat{j})(2 + \cos\omega t)$

where $a = 3\ s^{-1}$ and $\omega = \pi\ s^{-1}$; x and y measured in m

Find: (a) Algebraic equation for streamline at $t = 0$

(b) Plot streamline through point $(x, y) = (2, 4)$ at $t = 0$

(c) Will the streamline change with time? Explain.

(d) Show velocity vector at same point, time. Tangent? Explain

<u>Solution</u>: For a streamline, $\frac{dy}{v} = \frac{du}{x}$. From the given field, at $t = 0$,

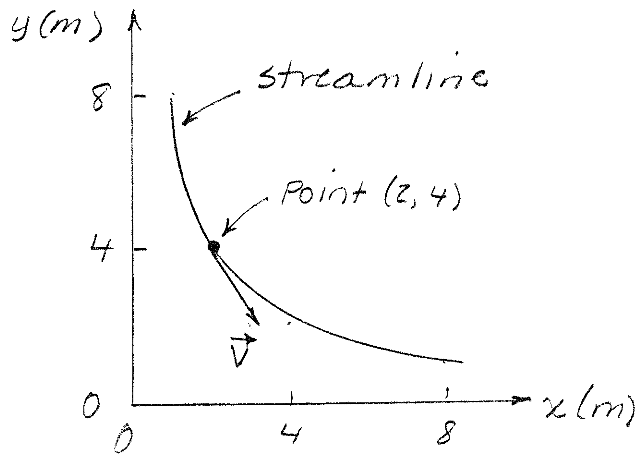
$$u = 2ax \text{ and } v = -2ay, \text{ so } \frac{dy}{v} = -\frac{dy}{2ay} = \frac{dx}{u} = \frac{dx}{2ax}$$

or

$$\frac{dx}{x} + \frac{dy}{y} = 0$$

Integrating, $\ln x + \ln y = \ln c$ or $xy = c$ ← streamline $(t = 0)$

For point $(x, y) = (2, 4)$, $xy = (2)(4) = c = 8$, or $xy = 8$ ← Thru $(x,y) = (2,4)$



Streamline pattern will not change with time, since $\frac{dy}{dx} \neq f(t)$. ← Time

At point $(2, 4)$ at $t = 0$, $u = 2ax = (2)(3\ s^{-1})(2\ m) = 12\ m/s$

$$v = -2ay = -(2)(3\ s^{-1})(4\ m) = -24\ m/s$$

The velocity vector <u>is</u> tangent to the streamline. ← Tangent

Given: Velocity field $\vec{V} = A\hat{i} + Bt\hat{j}$, where $A = 2$ m/s, $B = 0.6$ m/s^2, and coordinates are in meters.

Find: (a) position functions for particle located at $(x_0, y_0) = 1,1$ at time $t = 0$
(b) algebraic expression for pathline of particle of part (a).

Plot: the pathline and compare with streamline through the same point at $t = 0, 1, 2$ s.

Solution:

For a particle $u = \frac{dx}{dt}$ and $v = \frac{dy}{dt}$

Then,

$u = A = dx/dt$, $\int_{x_0}^{x} dx = \int_0^t A\,dt$ and $x = x_0 + At$ (1a)

$v = Bt = dy/dt$, $\int_{y_0}^{y} dy = \int_0^t Bt\,dt$ and $y = y_0 + \frac{1}{2}Bt^2$ (1b)

Substituting values for A, B, x_0, and y_0, then

$x = 1 + 2t$ and $y = 1 + 0.30\,t^2$ ← x, y

(b) To determine the pathline for the particle, we eliminate t from the paremetric equations of part (a).

From Eq. 1a, $t = (x - x_0)/A$. Substituting into Eq. (1b), then

$$y - y_0 = \frac{B(x - x_0)^2}{2A^2} \qquad (2)$$

Substituting numerical values,

$y = 1 + 0.075\,(x - 1)^2$ ← pathline

(c) The steamline is found (at given t) from $\left.\frac{dy}{dx}\right)_s = \frac{v}{u}$

$$\left.\frac{dy}{dx}\right)_{\text{streamline}} = \frac{v}{u} = \frac{Bt}{A}$$

$\therefore\ y = \frac{Bt}{A}x + C$

Through point (1,1)

$C = 1 - \frac{0.6}{2}t = 1 - 0.3t$

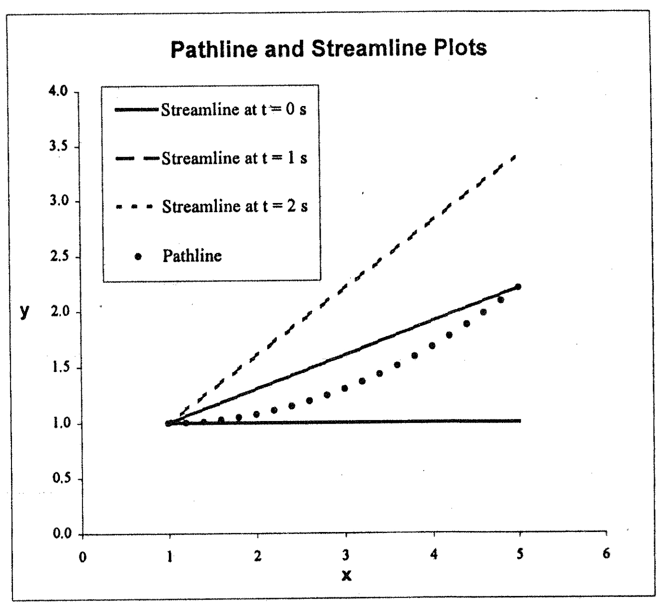
$y = 1 + 0.3\,t(x - 1)$ ←

Streamline through (1,1)

@ $t = 0$, $y = 1$

$t = 1$ s, $y = 1 + 0.3(x - 1)$

$t = 2$ s, $y = 1 + 0.6\,(x - 1)$



Problem 2.14

Given: Velocity field $\vec{V} = Bx(1+At)\hat{i} + Cy\hat{j}$, with $A = 0.5\,s^{-1}$, $B = C = 1\,s^{-1}$; coordinates measured in meters.

Plot: the pathline of the particle that passed through the point (1,1,0) at time t=0.
Compare with the streamlines through the same point at the instants t = 0, 1, and 2s

Solution:

For a particle, $u = \frac{dx}{dt}$ and $v = dy/dt$

then
$$u = Bx(1+At) = \frac{dx}{dt}, \quad \int_{x_0}^{x} \frac{dx}{x} = \int_0^t B(1+At)\,dt$$
$$\ln \frac{x}{x_0} = B\left[t + \tfrac{1}{2}At^2\right]_0^t = B\left[t + \tfrac{1}{2}At^2\right] \quad \therefore\ x = x_0 e^{B(t+\frac{1}{2}At^2)}$$
$$v = Cy = dy/dt, \quad \int_0^t C\,dt = \int_{y_0}^{y} \frac{dy}{y} \quad \therefore\ y = y_0 e^{ct}$$

x, y

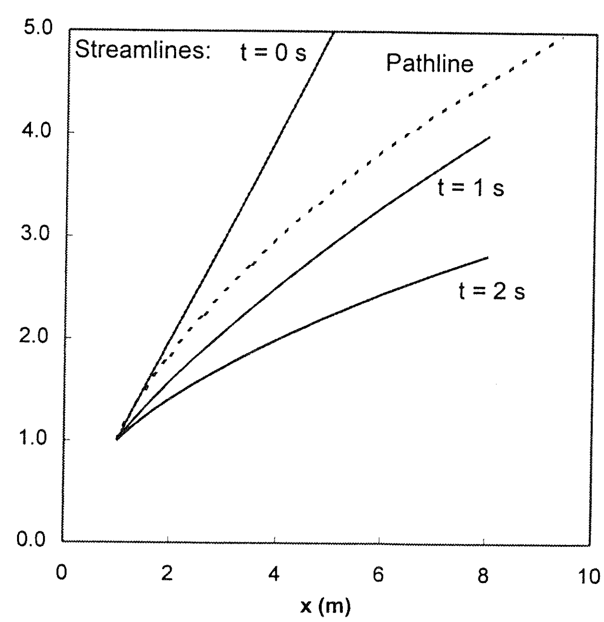
The pathline may be plotted by varying t as shown below

The streamline is found (at given t) from $\left.\frac{dy}{dx}\right)_{streamline} = \frac{v}{u}$

then
$$\frac{dy}{dx} = \frac{Cy}{Bx(1+At)} \quad \text{and} \quad (1+At)\frac{dy}{y} = \frac{C}{B}\frac{dx}{x}$$
and
$$(1+At)\ln y = \frac{C}{B}\ln x + \ln c_1, \quad c_1 x^{C/B} = y^{(1+At)}$$
Streamline through point (1,1,0) gives $c_1 = 1$. Then on substituting for A, B, and C we obtain
$$x = y^{(1+0.5t)}$$

Streamline

At $t = 0$, $x = y$
$t = 1s$, $x = y^{1.5}$
$t = 2s$, $x = y^2$



Problem 2.15

A velocity field is given by $\vec{V} = axt\hat{i} - by\hat{j}$, where $a = 0.1\ s^{-2}$ and $b = 1\ s^{-1}$. For the particle that passes through the point $(x, y) = (1, 1)$ at instant $t = 0$ s, plot the pathline during the interval from $t = 0$ to $t = 3$ s. Compare with the streamlines plotted through the same point at the instants $t = 0$, 1, and 2 s.

Solution

Pathlines are given by $\frac{dx}{dt} = u = a \cdot x \cdot t$ $\qquad$ $\frac{dy}{dt} = v = -b \cdot y$

So, separating variables $\frac{dx}{x} = a \cdot t \cdot dt$ $\qquad$ $\frac{dy}{y} = -b \cdot dt$

Integrating $\ln(x) = \frac{1}{2} \cdot a \cdot t^2 + c_1$ $\qquad$ $\ln(y) = -b \cdot t + c_2$

For initial position (x_0,y_0) $\qquad$ $x = x_0 \cdot e^{\frac{a}{2} \cdot t^2}$ $\qquad$ $y = y_0 \cdot e^{-b \cdot t}$

Using the given data, and IC $(x_0,y_0) = (1,1)$ at $t = 0$

$$x = e^{0.05 \cdot t^2} \qquad y = e^{-t}$$

Problem 2.15 (In Excel)

A velocity field is given by $\vec{V} = axt\hat{i} - by\hat{j}$, where $a = 0.1\ \mathrm{s}^{-2}$ and $b = 1\ \mathrm{s}^{-1}$. For the particle that passes through the point $(x, y) = (1, 1)$ at instant $t = 0$ s, plot the pathline during the interval from $t = 0$ to $t = 3$ s. Compare with the streamlines plotted through the same point at the instants $t = 0$, 1, and 2 s.

Solution

Using the given data, and IC $(x_0,y_0) = (1,1)$ at t = 0, the pathline is $x = e^{0.05 \cdot t^2}$ $y = e^{-t}$

The streamline at (1,1) at t = 0 s is $x = 1$

The streamline at (1,1) at t = 1 s is $y = x^{-10}$

The streamline at (1,1) at t = 2 s is $y = x^{-5}$

Pathline

t	x	y
0.00	1.00	1.00
0.25	1.00	0.78
0.50	1.01	0.61
0.75	1.03	0.47
1.00	1.05	0.37
1.25	1.08	0.29
1.50	1.12	0.22
1.75	1.17	0.17
2.00	1.22	0.14
2.25	1.29	0.11
2.50	1.37	0.08
2.75	1.46	0.06
3.00	1.57	0.05
3.25	1.70	0.04
3.50	1.85	0.03
3.75	2.02	0.02
4.00	2.23	0.02
4.25	2.47	0.01
4.50	2.75	0.01
4.75	3.09	0.01
5.00	3.49	0.01

Streamlines

t = 0

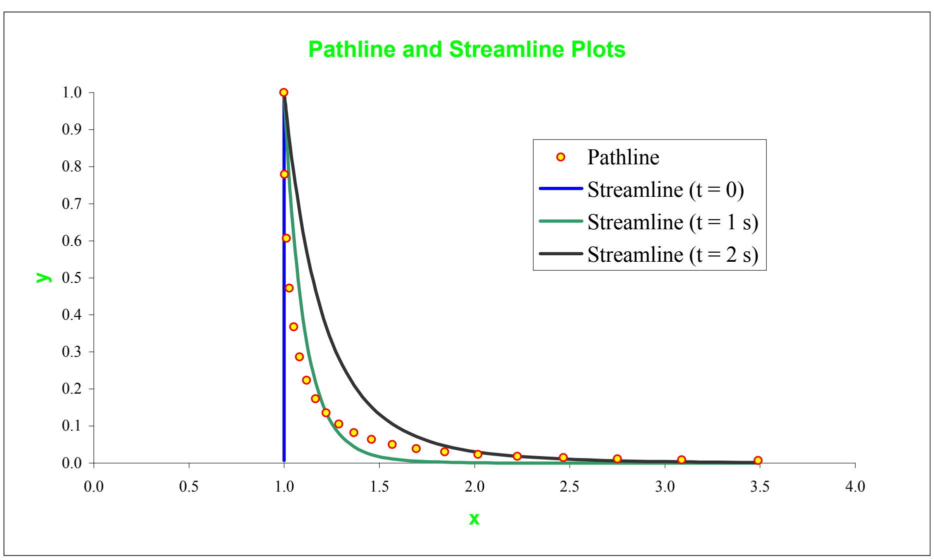
x	y
1.00	1.00
1.00	0.78
1.00	0.61
1.00	0.47
1.00	0.37
1.00	0.29
1.00	0.22
1.00	0.17
1.00	0.14
1.00	0.11
1.00	0.08
1.00	0.06
1.00	0.05
1.00	0.04
1.00	0.03
1.00	0.02
1.00	0.02
1.00	0.01
1.00	0.01
1.00	0.01
1.00	0.01

t = 1 s

x	y
1.00	1.00
1.00	0.97
1.01	0.88
1.03	0.75
1.05	0.61
1.08	0.46
1.12	0.32
1.17	0.22
1.22	0.14
1.29	0.08
1.37	0.04
1.46	0.02
1.57	0.01
1.70	0.01
1.85	0.00
2.02	0.00
2.23	0.00
2.47	0.00
2.75	0.00
3.09	0.00
3.49	0.00

t = 2 s

x	y
1.00	1.00
1.00	0.98
1.01	0.94
1.03	0.87
1.05	0.78
1.08	0.68
1.12	0.57
1.17	0.47
1.22	0.37
1.29	0.28
1.37	0.21
1.46	0.15
1.57	0.11
1.70	0.07
1.85	0.05
2.02	0.03
2.23	0.02
2.47	0.01
2.75	0.01
3.09	0.00
3.49	0.00



Problem 2.16

Given: Velocity field $\vec{V} = axt\hat{i} + b\hat{j}$ where $a = 0.2 s^{-2}$, $b = 3 m/s$ and coordinates are measured in meters.

Plot: the pathline (during the interval $0 \le t \le 3 s$) of the particle that passed through the point $(x_0, y_0) = (3, 1)$ at time $t = 0$.
Compare with the streamline plotted through the same point at $t = 1, 2,$ and $3 s$.

Solution:

For a particle, $u = dx/dt$ and $v = dy/dt$

Then, $u = axt = dx/dt$, $\int_{x_0}^{x} \frac{dx}{x} = \int_0^t at\,dt$

$\ln \frac{x}{x_0} = \frac{1}{2}at^2$ and $x = x_0 e^{\frac{1}{2}at^2}$ $\therefore x = 3e^{0.1t^2}$

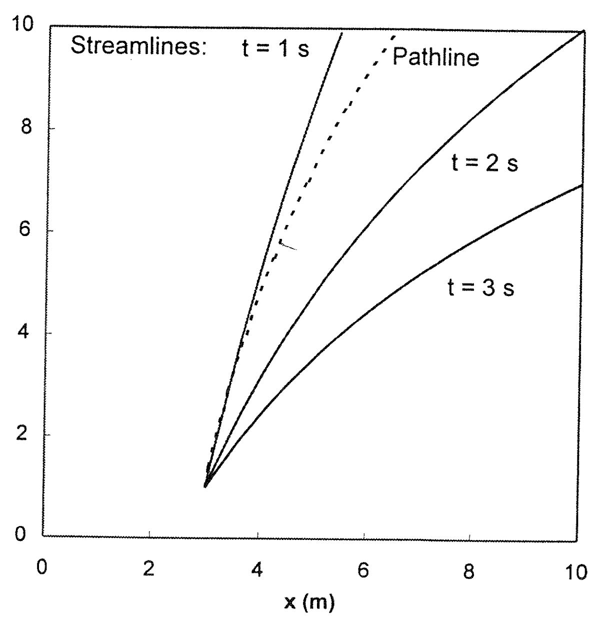
Also, $v = dy/dt = b$, $\int_{y_0}^{y} dy = \int_0^t b\,dt$, $y = y_0 + bt$ $\therefore y = 1 + 3t$ } (x, y) pathline

The pathline may be plotted by varying t as shown below.
The streamline is found (at given t) from $\left.\frac{dy}{dx}\right)_s = \frac{v}{u}$.

Then $\frac{dy}{dx} = \frac{b}{axt}$ and the streamline through (x_0, y_0) at time is $\int_{y_0}^{y} dy = \int_{x_0}^{x} \frac{b}{at}\frac{dx}{x}$ or $y = y_0 + \frac{b}{at}\ln\frac{x}{x_0}$

Substituting for a, b, x_0, and y_0, $y = 1 + \frac{15}{t}\ln\frac{x}{3}$ streamline

At $t = 1$, $y = 1 + 15 \ln x/3$
$t = 2$, $y = 1 + 7.5 \ln x/3$
$t = 3$, $y = 1 + 5 \ln x/3$



Problem 2.17

Given: Velocity field $\vec{V} = ax\hat{i} + by(1+ct)\hat{j}$, where $a = b = 2\,s^{-1}$, $c = 0.4\,s^{-1}$, and coordinates are measured in meters

Plot: the pathline (during the interval $0 \le t \le 1.5\,s$) of the particle that passed through the point $(x_0, y_0) = (1,1)$ at time $t = 0$.
Compare with the streamline plotted through the same point at $t = 0$, 1, and 1.5 s

Solution:

For a particle, $u = dx/dt$ and $v = dy/dt$

then

$$u = dx/dt = ax \quad , \quad \int_{x_0}^{x} \frac{dx}{x} = \int_0^t a\,dt \quad , \quad \ln\frac{x}{x_0} = at \quad , \quad x = x_0 e^{at}$$

Also

$$v = dy/dt = by(1+ct) \quad , \quad \int_{y_0}^{y} \frac{dy}{y} = \int_0^t b(1+ct)\,dt$$

$$\ln y/y_0 = b\left(t + \tfrac{1}{2}ct^2\right) \qquad y = y_0 e^{b\left(t + \frac{1}{2}ct^2\right)}$$

Substituting for a, b, c, x_0, and y_0

$$x = e^{2t} \quad , \quad y = e^{(2t + 0.4t^2)} \qquad \longleftarrow (x,y) \text{ pathline}$$

The streamline is found (at given t) from $dy/dx)_s = v/u$

then

$$\frac{dy}{dx} = \frac{by(1+ct)}{ax} \quad , \quad \int_{y_0}^{y} \frac{dy}{y} = \int_{x_0} \frac{b(1+ct)}{a}\frac{dx}{x} \quad , \quad \ln\frac{y}{y_0} = \frac{b(1+ct)}{a}\ln\frac{x}{x_0}$$

$$y = y_0\left(\frac{x}{x_0}\right)^{\frac{b(1+ct)}{a}}$$

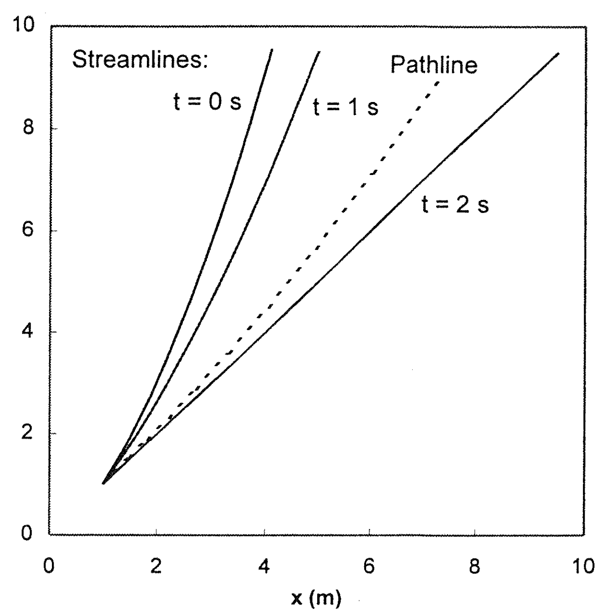
Substituting for a, b, c, x_0, and y_0

$$y = x^{(1+0.4t)} \qquad \longleftarrow \text{streamline}$$

At $t = 0$, $y = x$

$t = 1\,s$, $y = x^{1.4}$

$t = 1.5\,s$, $y = x^{1.6}$



Given: Velocity field $\vec{V} = Bx(1+At)\hat{i} + Cy\hat{j}$, with $A = 0.5\,s^{-1}$, $B = C = 1\,s^{-1}$; coordinates measured in meters

Plot: the streakline formed by particles that passed through point $(x_0, y_0, z_0) = (1,1,0)$ during interval from $t=0$ to $t = 3s$.
Compare with streamlines through point at $t =$ 0, 1, and 2 s

<u>Solution</u>

Streakline at $t = 3s$ connects particles that passed through point (1,1,0) at earlier times $t_0 = 0, 1,$ and $2s$

For a particle, $u = \frac{dx}{dt}$ and $v = dy/dt$

then

$$u = Bx(1+At) = \frac{dx}{dt}, \quad \int_{x_0}^{x} \frac{dx}{x} = \int_{t_0}^{t} B(1+At)\,dt$$

$$\therefore \ln\frac{x}{x_0} = B\left[t + \tfrac{1}{2}At^2\right]_{t_0}^{t} = B\left[(t-t_0) + \tfrac{1}{2}AB(t^2-t_0^2)\right]$$

$$x = x_0 e^{B[(t-t_0) + \frac{1}{2}AB(t^2-t_0^2)]} \qquad (1a)$$

Also $v = Cy = dy/dt$, $\int_{t_0}^{t} C\,dt = \int_{y_0}^{y}\frac{dy}{y}$, $\therefore y = y_0 e^{C(t-t_0)}$ (1b)

The velocity vector is tangent to the streamline

$$\left.\frac{dy}{dx}\right)_{streamline} = \frac{v}{u} = \frac{Cy}{Bx(1+At)} \quad \text{and} \quad (1+At)\frac{dy}{y} = \frac{C}{B}\frac{dx}{x}$$

then

$$(1+At)\ln y = \frac{C}{B}\ln x + \ln c_1 \quad \text{and} \quad c_1 x^{C/B} = y^{(1+At)}$$

Streamline through point (1,1,0) gives $c_1 = 1$. Then on substituting for A, B, and C we obtain

$$x = y^{(1+0.5t)}$$

Streamline

At $t = 0$ $x = y$
$t = 1s$ $x = y^{1.5}$
$t = 2s$ $x = y^2$
} These streamlines through (1,1,0) are shown on the plot

Points on the streakline have coordinates given by Eqs 1a & 1b

$$x = x_0 e^{B[(t-t_0) + \frac{1}{2}AB(t^2-t_0^2)]} \qquad y = y_0 e^{C(t-t_0)}$$

Substituting for A, B, and C

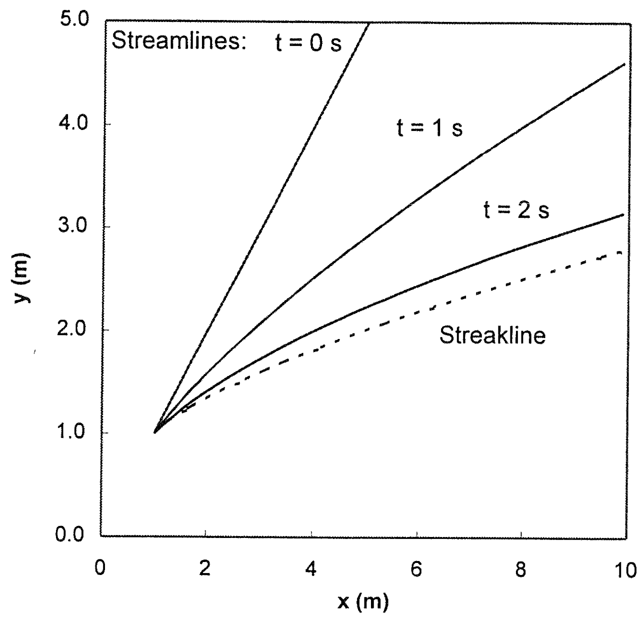
$$x = x_0 e^{[(t-t_0) + 0.25(t^2-t_0^2)]} \qquad y = y_0 e^{(t-t_0)}$$

The streakline through $(x_0, y_0) = (1,1)$ at time $t = 3s$ is obtained by substituting $x_0 = 1$, $y_0 = 1$, $t = 3s$ and varying t_0 in these equations.

Thus, $x = e^{[(3-t_0)+0.25(9-t_0^2)]}$

$y = e^{(3-t_0)}$

give points (obtained by varying t_0) on the streakline through (1,1,0) at t = 3 s



Given: Velocity field $\vec{V} = ax(1+bt)\hat{i} + cy\hat{j}$, where $a = c = 1\,s^{-1}$, $b = 0.2\,s^{-1}$, and coordinates are measured in meters.

Plot: the streakline that passes through the point $(x_0, y_0) = (1,1)$ during the interval $0 \le t \le 3\,s$.
Compare with the streamlines plotted through the same point at $t = 0$, 1, and $2\,s$

Solution:

Streakline at $t = 3\,s$ connects particles that passed through point (x_0, y_0) at earlier times $\tau = 0, 1, 2$, and $3\,s$.

For a particle, $u = dx/dt$ and $v = dy/dt$

Then

$$u = ax(1+bt) = \frac{dx}{dt} \quad \text{and} \quad \int_{x_0}^{x} \frac{dx}{x} = \int_{\tau}^{t} a(1+bt)\,dt$$

$$\ln \frac{x}{x_0} = a\left(t + \frac{b}{2}t^2\right)\Big]_{\tau}^{t} = a\left[(t-\tau) + \frac{b}{2}(t^2-\tau^2)\right]$$

$$x = x_0\, e^{a\left[(t-\tau) + \frac{b}{2}(t^2-\tau^2)\right]}$$

Also

$$v = \frac{dy}{dt} = cy, \quad \int_{y_0}^{y} \frac{dy}{y} = \int_{\tau}^{t} c\,dt, \quad \ln\frac{y}{y_0} = c(t-\tau), \quad y = y_0\, e^{c(t-\tau)}$$

Substituting for a, b, c, x_0, and y_0, gives

$$x = e^{[(t-\tau) + 0.1(t^2-\tau^2)]}, \quad y = e^{(t-\tau)}$$ ← (x,y) streakline

The streakline may be plotted by substituting values for τ in the range $0 \le \tau \le 3\,s$ as shown below.

The streamline is found (at given t) from $dy/dx)_s = \frac{v}{u}$

Thus

$$dy/dx = \frac{cy}{ax(1+bt)} \quad \text{and} \quad \int_{y_0}^{y} \frac{dy}{y} = \int_{x_0}^{x} \frac{c}{a(1+bt)} \frac{dx}{x}$$

$$\ln \frac{y}{y_0} = \frac{c}{a(1+bt)} \ln \frac{x}{x_0} \quad \text{or} \quad y = y_0 \left[\frac{x}{x_0}\right]^{c/a(1+bt)}$$

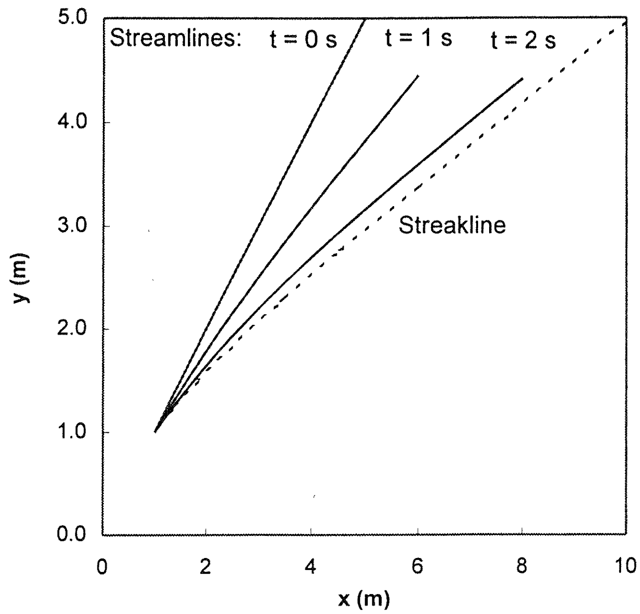
Substituting values for x_0, y_0, a, b, c, then

$$y = x^{1/(1+0.2t)} \quad \text{or} \quad x = y^{(1+0.2t)}$$ ← streamline

At $t = 0$, $x = y$

$t = 1\,s$, $x = y^{1.2}$

$t = 2\,s$, $x = y^{1.4}$


Streamlines: t = 0 s
t = 1 s
t = 2 s
Streakline
y (m)
x (m)
5.0
4.0
3.0
2.0
1.0
0.0
0
2
4
6
8
10

Problem 2.20 (In Excel)

Tiny hydrogen bubbles are being used as tracers to visualize a flow. All the bubbles are generated at the origin ($x = 0$, $y = 0$). The velocity field is unsteady and obeys the equations:

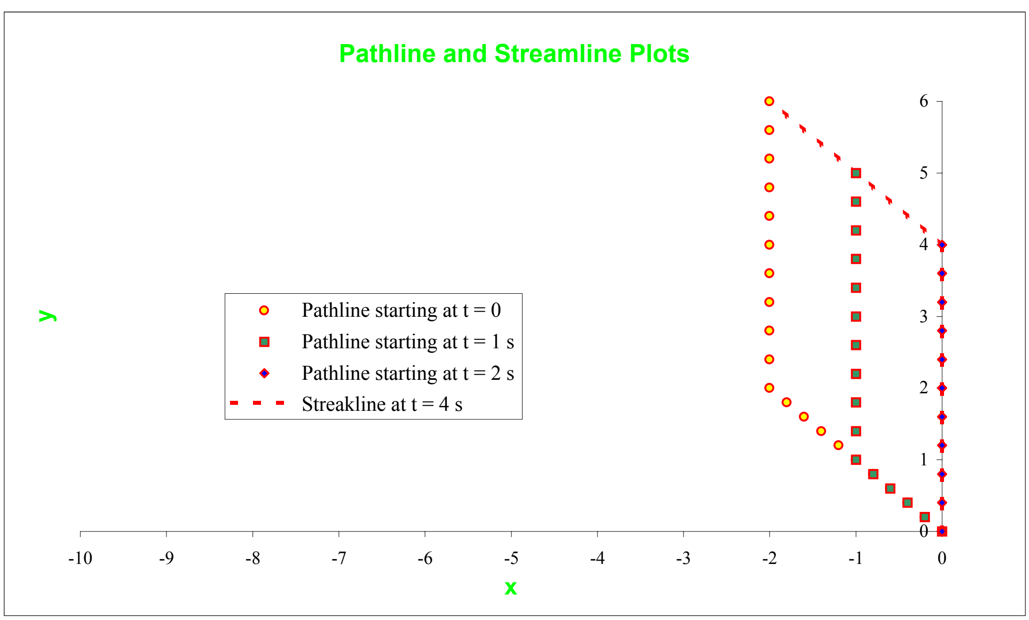
$$u = -1 \text{ m/s} \qquad v = 1 \text{ m/s} \qquad 0 \le t < 2 \text{ s}$$
$$u = 0 \qquad v = 2 \text{ m/s} \qquad 2 \le t \le 4 \text{ s}$$

Plot the pathlines of bubbles that leave the origin at $t = 0$, 1, 2, 3, and 4 s. Mark the locations of these five bubbles at $t = 4$ s. Use a dashed line to indicate the position of a streakline at $t = 4$ s.

Solution

Pathlines: | **Starting at t = 0** | **Starting at t = 1 s** | **Starting at t = 2 s** | **Streakline at t = 4 s**

Pathlines:	Starting at t = 0		Starting at t = 1 s		Starting at t = 2 s		Streakline at t = 4 s	
t	x	y	x	y	x	y	x	y
0.00	0.00	0.00					0.00	0.00
0.20	-0.20	0.20					0.00	0.40
0.40	-0.40	0.40					0.00	0.80
0.60	-0.60	0.60					0.00	1.20
0.80	-0.80	0.80					0.00	1.60
1.00	-1.00	1.00	0.00	0.00			0.00	2.00
1.20	-1.20	1.20	-0.20	0.20			0.00	2.40
1.40	-1.40	1.40	-0.40	0.40			0.00	2.80
1.60	-1.60	1.60	-0.60	0.60			0.00	3.20
1.80	-1.80	1.80	-0.80	0.80			0.00	3.60
2.00	-2.00	2.00	-1.00	1.00	0.00	0.00	0.00	4.00
2.20	-2.00	2.40	-1.00	1.40	0.00	0.40	-0.20	4.20
2.40	-2.00	2.80	-1.00	1.80	0.00	0.80	-0.40	4.40
2.60	-2.00	3.20	-1.00	2.20	0.00	1.20	-0.60	4.60
2.80	-2.00	3.60	-1.00	2.60	0.00	1.60	-0.80	4.80
3.00	-2.00	4.00	-1.00	3.00	0.00	2.00	-1.00	5.00
3.20	-2.00	4.40	-1.00	3.40	0.00	2.40	-1.20	5.20
3.40	-2.00	4.80	-1.00	3.80	0.00	2.80	-1.40	5.40
3.60	-2.00	5.20	-1.00	4.20	0.00	3.20	-1.60	5.60
3.80	-2.00	5.60	-1.00	4.60	0.00	3.60	-1.80	5.80
4.00	-2.00	6.00	-1.00	5.00	0.00	4.00	-2.00	6.00



Given: Velocity field $\vec{V} = axt\hat{i} + b\hat{j}$, where $a = 0.2\ s^{-1}$, $b = 1\ m/s$, and coordinates are in meters.

Plot: the pathline (during the interval $0 \le t \le 3s$) of the particle that passed through the point $(x_0, y_0) = (1, 2)$ at time $t = 0$.
Compare with the streakline through the same point at the instant $t = 3s$.

Solution:

The pathline and streakline are based on parametric equations for a particle.

For a particle $u = dx/dt$ and $v = dy/dt$

then $u = \frac{dx}{dt} = axt$, $\int_{x_0}^{x} \frac{dx}{x} = \int_{t_0}^{t} at\,dt$, $\ln \frac{x}{x_0} = \frac{1}{2}a(t^2 - t_0^2)$

$x = x_0 e^{\frac{1}{2}a(t^2 - t_0^2)}$

Also $v = dy/dt = b$, $\int_{y_0}^{y} dy = \int_{t_0}^{t} b\,dt$, $y = y_0 + b(t - t_0)$

In the above equations, x_0, y_0 are coordinates of particle at t_0

(a) The pathline is obtained by following the particle that passed through the point $(x_0, y_0) = (1, 2)$ at time $t_0 = 0$

Thus $x = x_0 e^{\frac{1}{2}at^2} = e^{0.1t^2}$

$y = y_0 + bt = 2 + t$ } ← (x,y) pathline

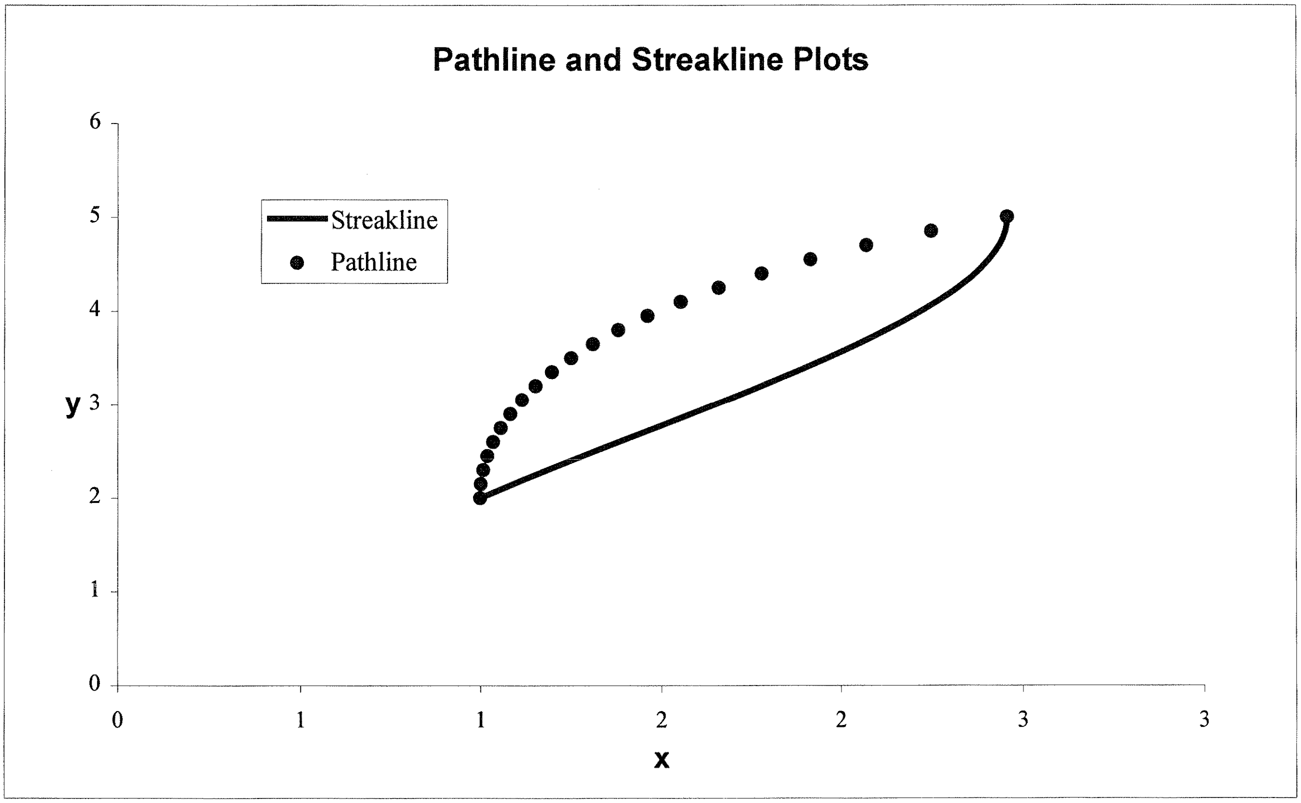
The pathline may be plotted by varying t $(0 \le t \le 3s)$ as shown below

(b) The streakline is obtained by locating (and connecting) at time $t = 3s$, all the particles that passed through the point $(x_0, y_0) = (1, 2)$ at some earlier time t_0.

Thus $x = x_0 e^{\frac{1}{2}a(9 - t_0^2)} = e^{0.1(9 - t_0^2)}$

$y = y_0 + b(t - t_0) = 2 + (3 - t_0) = 5 - t_0$ } ← (x,y) streakline

The streakline may be plotted by varying t_0 $(0 \le t_0 \le 3s)$ as shown below.


Pathline and Streakline Plots
Streakline
Pathline
y
x
0
1
2
3
4
5
6
0
1
1
2
2
3
3

Given: Velocity field in xy plane, $\vec{V} = a\hat{i} + bx\hat{j}$, where

$a = 2$ m/s and $b = 1\ s^{-1}$

Find: (a) Equation for streamline through $(x,y) = (2,5)$.

(b) At $t = 2$ s, coordinates of particle $(0,4)$ at $t = 0$.

(c) At $t = 3$ s, coordinates of particle $(1, 4.25)$ at $t = 1$ s.

(d) compare pathline, streamline, streakline.

Solution: For a streamline $\frac{dx}{u} = \frac{dy}{v}$

For $\vec{V} = a\hat{i} + bx\hat{j}$, $u = a$ and $v = bx$, so $\frac{dx}{a} = \frac{dy}{bx}$ or

$$x\,dx = \frac{a}{b}\,dy$$

Integrating

$$\frac{x^2}{2} = \frac{a}{b}y + c' \quad \text{or} \quad y = \frac{b}{2a}x^2 + c$$

Evaluating c at $(x,y) = (2,5)$,

$$c = y - \frac{b}{2a}x^2 = 5\,m - \frac{1}{2}\times\frac{1}{s}\times\frac{s}{2m}(2m)^2 = 4\ m$$

Streamline through $(x,y) = (2,5)$ is $y = \frac{x^2}{4} + 4$ (a)

To locate particles, derive parametric equations

$$u_p = \frac{dx}{dt} = a, \quad dx = a\,dt, \quad \text{and} \quad x - x_0 = a(t - t_0)$$

$$v_p = \frac{dy}{dt} = bx, \quad dy = bx\,dt = b(x_0 + at - at_0)$$

$$y - y_0 = bx_0(t - t_0) + \frac{a}{2}(t^2 - t_0^2) - at_0(t - t_0)$$

For the particle at $(x_0, y_0) = (0,4)$ at $t = 0$,

$x = 0 + at$ so at $t = 2$ s, $x = 2\frac{m}{s}\times 2\,s = 4\,m$

$y = 4 + \frac{at^2}{2}$ so at $t = 2$ s, $y = 4 + \frac{1}{2}\times 2\frac{m}{s}\times(2)^2 s^2$

$y = 8\,m$ (b)

For the particle at $(x,y) = (1, 4.25)$ at $t = 1\,s$,

$$x = x_0 + a(t - t_0) = 1 + a(t-1)$$

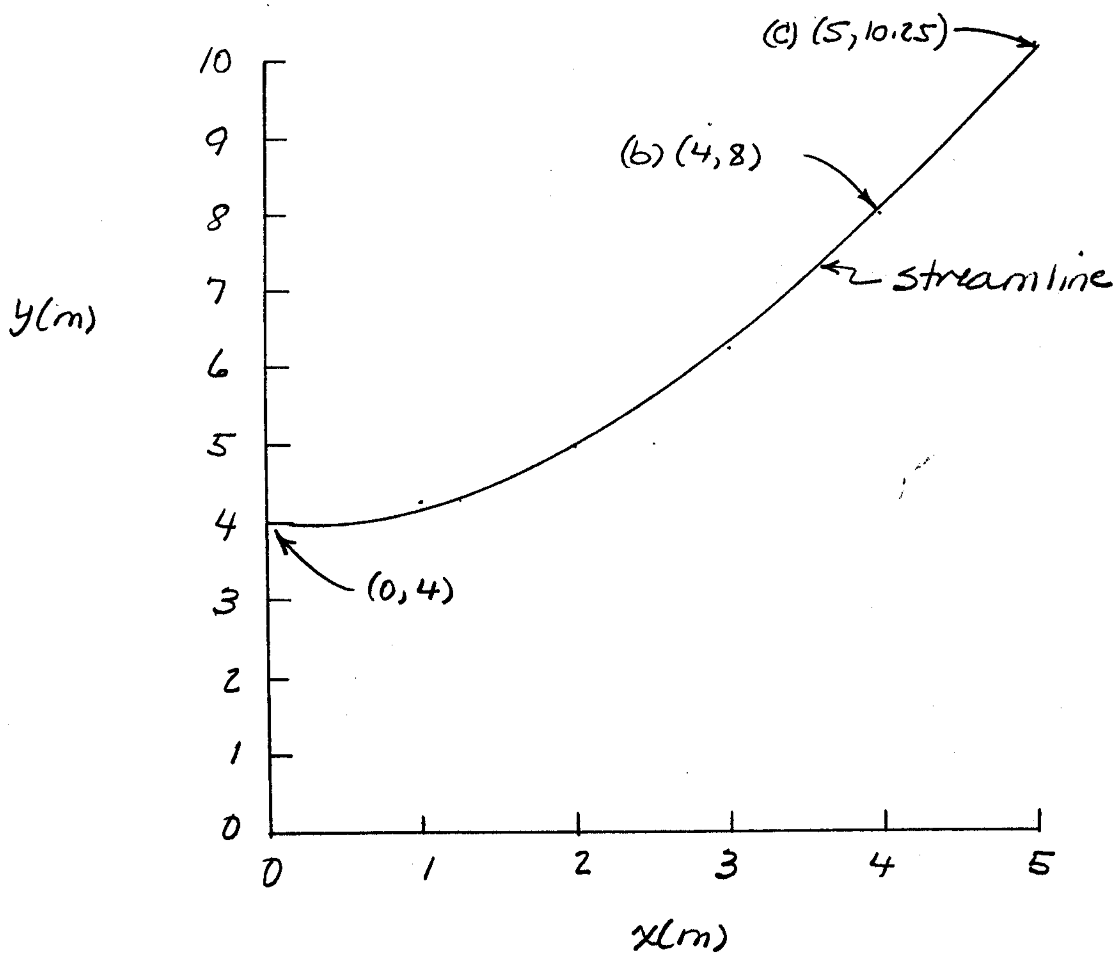
so at $t = 3\,s$, $x = 1 + 2\frac{m}{s}(3-1)s = 5\,m$

$$y = y_0 + bx_0(t - t_0) + \frac{a}{2}(t^2 - t_0^2) - at_0(t - t_0)$$

$$= 4.25 + \frac{1}{s} \times 1m \times (t-1) + \frac{1}{2} \times 2\frac{m}{s}(t^2 - 1) - 2\frac{m}{s} \times 1s\,(t-1)$$

so at $t = 3\,s$, $y = 4.25 + 2 + 8 - 4 = 10.25\,m$ (c)

All these points lie on the same streamline, as shown below:



For this steady flow, streamlines, pathlines, and streaklines coincide, as expected.

Problem 2.23

Given: Velocity field $\vec{V} = ay\hat{i} + b\hat{j}$, where $a = 1\ s^{-1}$, and $b = 2\ m/s$; coordinates are measured in meters.

Find: (a) Equation of streamline through $(x,y) = (6,6)$
(b) At $t = 1\ s$, coordinates of particle that passed through point $(x_0, y_0) = (1,4)$ at $t = 0$
(c) At $t = 3\ s$, coordinates of particle that passed through point $(x_0, y_0) = (-3,0)$ at $t_0 = 1\ s$.

Solution

The velocity vector is tangent to the streamlines

$$\left.\frac{dy}{dx}\right)_{streamline} = \frac{v}{u} = \frac{b}{ay} \quad \text{or} \quad \int_6^y ay\,dy = \int_6^x b\,dx$$

Then $\frac{1}{2}ay^2\Big]_6^y = bx\Big]_x^6$, $2b(x-6) = a(y^2 - 36)$

and $4(x-6) = y^2 - 36$ or $x = \frac{y^2}{4} - 3$ ← Streamline

(b) Follow particle that passed through (1,4) @ $t=0$

$u = \frac{dx}{dt} = ay \quad \therefore \int_{x_0}^x dx = \int_0^t ay\,dt \quad \{\text{need } y = y(t)\}$

$v = \frac{dy}{dt} = b \quad \therefore \int_{y_0}^y dy = \int_0^t b\,dt \quad \text{and } y = y_0 + bt \quad (1a)$

Then

$$x - x_0 = \int_{x_0}^x dx = \int_0^t a(y_0 + bt)\,dt = ay_0t + \frac{1}{2}bt^2$$

$$x = x_0 + ay_0t + \frac{1}{2}bt^2 \quad (1b)$$

Following particle through (1,4) at $t=0$, then at $t = 1\ s$

$x_p = 1 + (1)(4)(1) + \frac{1}{2}(2)(1)^2 = 6$ and $y_p = 4 + 2(1) = 6$ ← (x_p, y_p)

(c) Streakline. At $t = 3\ s$, locate position of particle that passed through $(x_0, y_0) = (-3,0)$ at earlier time $t_0 = 1\ s$.

For a particle

$v = \frac{dy}{dt} = b \quad \therefore \int_{y_0}^y dy = \int_{t_0}^t b\,dt \quad \text{and } y = y_0 + b(t - t_0) \quad (2a)$

$u = \frac{dx}{dt} = ay \quad \therefore \int_{x_0}^x dx = \int_{t_0}^t ay\,dt = \int_{t_0}^t a\left[y_0 + b(t-t_0)\right]dt$

and

$$x = x_0 + ay_0(t - t_0) + \frac{ab}{2}(t^2 - t_0^2) - abt_0(t - t_0) \quad \text{-----}(2b)$$

Then from Eqs 2a & 2b for $t = 3\ s$ and $t_0 = 1\ s$

$x = -3 + 0 + \frac{(1)2}{2}\left[(3)^2 - (1)^2\right] - (1)(2)(1)(3-1) = 1$

$y = 0 + 2(3-1) = 4$ $\qquad (x,y) = (1,4)$ ←

Since points (6,6), (1,4), and (-3,0) are all on the same streamline ($x = y^2/4 - 3$), pathlines, streaklines & streamlines coincide

Given: Velocity field $\vec{V} = at\hat{i} + b\hat{j}$, where $a = 0.4\ m/s^2$, $b = 2\ m/s$, and coordinates are measured in meters

Find: (a) At $t = 2s$, coordinates of particle that passed through $(x_0, y_0) = (2,1)$ at $t = 0$
(b) At $t = 3s$, coordinates of the particle that passed through (x_0, y_0) at $t = 2s$

Plot: the pathline and streakline through point (2,1); compare with the streamlines through the same point at $t = 0, 1, 2s$

Solution:

The pathline and streakline are based on parametric equations for a particle.

For a particle $u = dx/dt$ and $v = dy/dt$

Thus $u = \frac{dx}{dt} = at$, $\int_{x_0}^{x} dx = \int_{t_0}^{t} at\,dt$, $x = x_0 + \frac{1}{2}a(t^2 - t_0^2)$ (1a)

$v = \frac{dy}{dt} = b$, $\int_{y_0}^{y} dy = \int_{t_0}^{t} b\,dt$, $y = y_0 + b(t - t_0)$ (1b)

In the above equations, x_0, y_0 are coordinates of the particle at time t_0

(a) The pathline is obtained by following the particle that passed through the point $(x_0, y_0) = (2,1)$ at time $t_0 = 0$

Thus $x = x_0 + \frac{1}{2}at^2 = 2 + 0.2t^2$
$y = y_0 + bt = 1 + 2t$ } ← (x,y) pathline

At $t = 2s$, particle is at $(x,y) = (2.8, 5)\ m$ ← (a)

The pathline may be plotted by varying t $(0 \le t \le 3s)$ as shown below

(b) The streakline is obtained by locating (and connecting) at time $t = 3s$, all the particles that passed through the point $(x_0, y_0) = (2,1)$ at some earlier time t_0

Thus $x = x_0 + \frac{1}{2}a(9 - t_0^2) = 2 + 0.2(9 - t_0^2)$
$y = y_0 + b(t - t_0) = 1 + 2(3 - t_0)$ } ← (x,y) streakline

At $t = 2s$, particle is at $(x,y) = (3,3)$ ← (b)

The streakline may be plotted by varying t_0 $(0 \le t_0 \le 3s)$ as shown below.

The streamline is found (at given t) from $\left.\frac{dy}{dx}\right)_s = \frac{v}{u}$

Then, $dy/dx = \frac{b}{at}$, $\int_{y_0}^{y} dy = \int_{x_0}^{x} \frac{b}{at} dx$, $y - y_0 = \frac{b}{at}(x - x_0)$

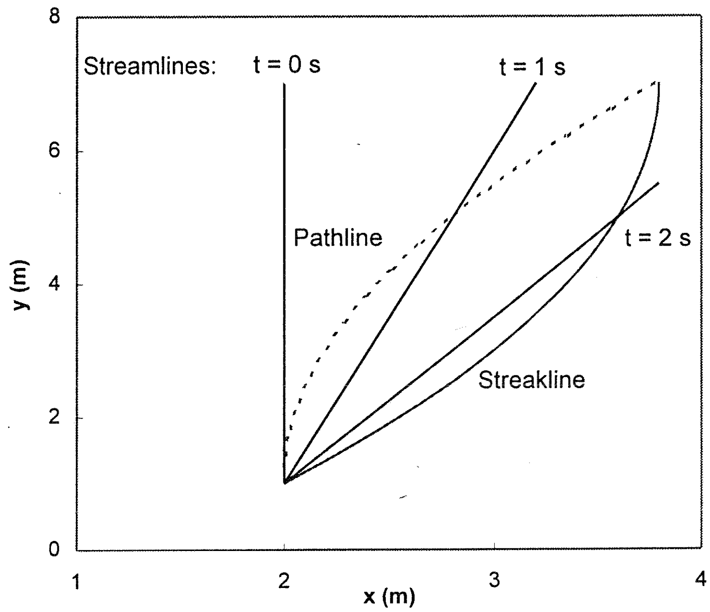
Streamline through point (2,1) gives $y - 1 = \frac{b}{at}(x-2)$

$y = 1 + \frac{5(x-2)}{t}$ or $x = 2 + \frac{t}{5}(y-1)$ ← streamline

At $t=0$, $x = 2$

$t = 1$, $y = 5x - 9$

$t = 2$, $y = 2.5x - 4$



Given: Velocity field $\vec{V} = ay\hat{i} + bt\hat{j}$, where $a = 1\,s^{-1}$; $b = 0.5\,m/s^2$; t in s.

Find: (a) At $t = 2\,s$, particle that passed $(1,2)$ at $t = 0\,s$

(b) At $t = 3\,s$, particle that passed $(1,2)$ at $t = 2\,s$

(c) Plot pathline and streakline through $(1,2)$; compare with streamlines at $t = 0, 1, 2\,s$.

Solution: Pathline and streakline are based on parametric equations for a particle. Thus

$$v = \frac{dy}{dt} = bt, \text{ so } dy = bt\,dt, \text{ and } y - y_0 = \frac{b}{2}(t^2 - t_0^2)$$

and $$u = \frac{dx}{dt} = ay = a\left[y_0 + \frac{b}{2}(t^2 - t_0^2)\right]$$

so $$x\Big]_{x_0}^{x} = a\left[y_0 t + \frac{b}{2}\left(\frac{t^3}{3} - t_0^2 t\right)\right]_{t_0}^{t} \; ; \; x = x_0 + a y_0 (t - t_0) + \frac{ab}{2}\left(\frac{t^3 - t_0^3}{3} + t_0^2(t_0 - t)\right)$$

where x_0, y_0 are coordinates of particle at t_0.

For (a), $t_0 = 0$, and $(x_0, y_0) = (1,2)$. Thus at $t = 2\,s$, $y = y_0 + \frac{bt^2}{2}$

$$y = 2\,m + \frac{1}{2} \times 0.5\frac{m}{s^2} \times (2)^2 s^2 = 3.00\,m$$

$$x = 1\,m + \frac{1}{s} \times 2\,m\,(2-0)\,s + \frac{1}{2} \times \frac{1}{s} \times 0.5\frac{m}{s^2}\left(\frac{(2)^3 - 0}{3} + 0\right)s^3 = 5.67\,m$$

$t_0 = 0$ (a)

At $t = 2\,s$, $(x, y) = (5.67, 3.00)\,m$

For (b), $t_0 = 2\,s$, and $(x_0, y_0) = (1,2)$. Thus at $t = 3\,s$, the particle is at

$$y(3) = 2\,m + \frac{1}{2} \times 0.5\frac{m}{s^2}\left[(3)^2 - (2)^2\right]s^2 = 3.25\,m$$

$$x(3) = 1\,m + \frac{1}{s} \times 2\,m\,(3-2)\,s + \frac{1}{2} \times \frac{1}{s} \times 0.5\frac{m}{s^2}\left(\frac{(3)^3 - (2)^3}{3} + (2)^2(2-3)\right)s^3 = 3.58\,m$$

At $t = 3\,s$, $t_0 = 2\,s$ (b)

$(x, y) = (3.58, 3.25)\,m$

For (c), the streakline may be plotted at any t by varying t_0, as shown on the next page.

The streamline is found (at given t) from $\frac{dx}{u} = \frac{dy}{v}$

Substituting $u = ay$ and $v = bt$, $dx = \frac{ay}{bt}dy$ or $y^2 = \frac{2bt}{a}x + c$

Thus $c = y_0^2 - \frac{2bt}{a}x_0$

For $t = 0$, $y^2 = c$; at $(x_0, y_0) = (1,2)$, then $c = 4$

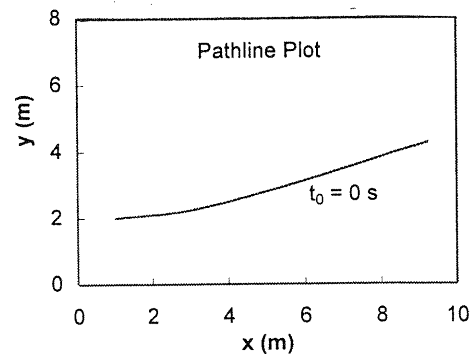
$t = 1$, $y^2 = \frac{2b}{a}x + c$; at $(x_0, y_0) = (1,2)$, then $c = 3$ (c)

$t = 2$, $y^2 = \frac{4b}{a}x + c$; at $(x, y) = (1,2)$, $c = 2$; for $t = 3\,s$, $c = 1$

Recall $\vec{V} = ay\hat{i} + bt\hat{j}$, where $a = 1\ s^{-1}$; $b = 0.5\ m/s^2$; $(x_0, y_0) = (1, 2)\ m$.

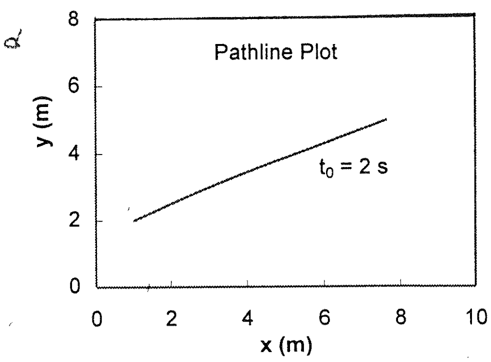
Part (a): Pathline of particle located at (x_0, y_0) at $t_0 = 0\ s$:

t_0 (s)	t (s)	x (m)	y (m)
0	0	1.00	2.00
0	1	3.08	2.25
0	2	5.67	3.00
0	3	9.25	4.25



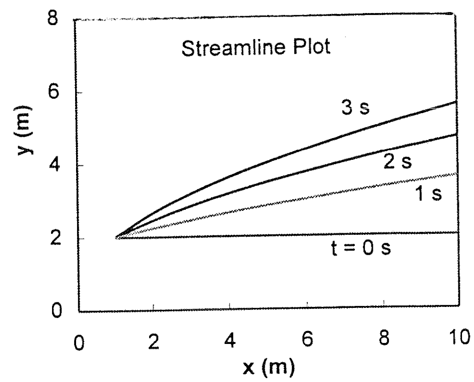
Part (b): Pathline of particle located at (x_0, y_0) at $t_0 = 2\ s$:

t_0 (s)	t (s)	x (m)	y (m)
2	2	1.00	2.00
2	3	3.58	3.25
2	4	7.67	5.00



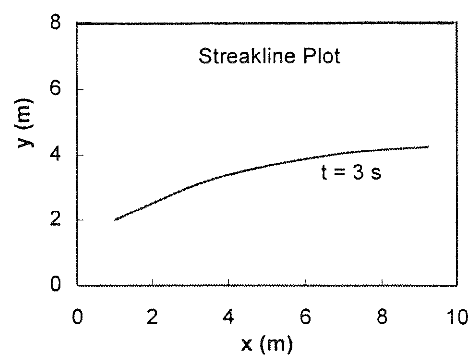
Part (c): Streamlines through point (x_0, y_0) at $t = 0, 1, 2$, and $3\ s$:

	t (s)	0	1	2	3
	c =	4.0	3.0	2.0	1.0
t_0 (s)	x (m)	y (m)	y (m)	y (m)	y (m)
0	1	2.00	2.00	2.00	2.00
0	2	2.00	2.24	2.45	2.65
0	3	2.00	2.45	2.83	3.16
0	4	2.00	2.65	3.16	3.61
0	5	2.00	2.83	3.46	4.00
0	6	2.00	3.00	3.74	4.36
0	7	2.00	3.16	4.00	4.69
0	8	2.00	3.32	4.24	5.00
0	9	2.00	3.46	4.47	5.29
0	10	2.00	3.61	4.69	5.57



Streakline at $t = 3\ s$ of particles that passed thru point (x_0, y_0):

t_0 (s)	t (s)	x (m)	y (m)
0	3	9.25	4.25
1	3	6.67	4.00
2	3	3.58	3.25
3	3	1.00	2.00



Problem 2.26

Given: Variation of air viscosity with temperature (absolute) is

$$\mu = \frac{b T^{1/2}}{1 + S/T}$$

where $b = 1.458 \times 10^{-6}\ kg/m \cdot s \cdot K^{1/2}$, $S = 110.4K$

Find: Equation for calculating air viscosity in British Gravitational units as a function of absolute temperature in degrees Rankine. Check result using data from Appendix A

Solution:

Convert constants.

$$b = 1.458 \times 10^{-6} \frac{kg}{m \cdot s \cdot K^{1/2}} \times \frac{lbm}{0.4536\,kg} \times \frac{slug}{32.17\,lbm} \times \frac{lb \cdot s^2}{slug \cdot ft} \times \frac{0.3048\,m}{ft} \times \left(\frac{5K}{9°R}\right)^{1/2}$$

$$b = 2.27 \times 10^{-8}\ lbf \cdot s/ft^2 \cdot °R^{1/2}$$

$$S = 110.4\,K \times \frac{9°R}{5K} = 198.7°R$$

Then in British Gravitational Units

$$\mu = \frac{2.27 \times 10^{-8}\ T^{1/2}}{1 + 198.7/T} \longleftarrow \mu$$

where units of T are °R; μ is in $lbf \cdot s/ft^2$

Evaluate at T = 80°F (539.7°R)

$$\mu = \frac{2.27 \times 10^{-8} \times (539.7)^{1/2}}{1 + 198.7/539.7} = 3.855 \times 10^{-7}\ lbf \cdot s/ft^2$$

From Table A.9 (Appendix A) at T = 80°F

$\mu = 3.86 \times 10^{-7}\ lbf \cdot s/ft^2$ ✓ check.

Problem 2.27

Given: Variation of air viscosity with temperature (absolute) is

$$\mu = \frac{bT^{1/2}}{1+S/T}$$

where $b = 1.458\times10^{-6}\ \frac{kg}{m \cdot s \cdot K^{1/2}}$

$S = 110.4\ K$

Find: Equation for kinematic viscosity of air (in SI units) as a function of temperature at atmospheric pressure. Assume ideal gas behavior. Check result using data from Appendix A.

Solution:

For an ideal gas, $P = \rho RT$. From Table A.6, $R = 286.9\ N\cdot m/kg\cdot K$

The kinematic viscosity, $\nu \equiv \mu/\rho$

$$\therefore \nu = \frac{\mu}{\rho} = \frac{\mu RT}{p} = \frac{RT}{p}\frac{bT^{1/2}}{1+S/T} = \frac{Rb}{p}\frac{T^{3/2}}{1+S/T} = \frac{b' T^{3/2}}{1+S/T}$$

where $b' = \frac{Rb}{p} = 286.9\ \frac{N\cdot m}{kg\cdot K} \times 1.458\times10^{-6}\ \frac{kg}{m\cdot s\cdot K^{1/2}} \times \frac{m^2}{101.3\times10^3\ N}$

$$b' = 4.129\times10^{-9}\ m^2/s\cdot K^{3/2}$$

$$\therefore \nu = \frac{b' T^{3/2}}{1+S/T}$$

where $b' = 4.129\times10^{-9}\ m^2/s\cdot K^{3/2}$, $S = 110.4\ K$
units of T are (K); ν is in m^2/s

Evaluate at $T = 20°C = 293.2\ K$

$$\nu = \frac{4.129\times10^{-9}\ (293.2)^{3/2}}{1 + 110.4/293.2} = 1.506\times10^{-5}\ m^2/s$$

From Table A.10 (Appendix A) at $T = 20°C$

$\nu = 1.51\times10^{-5}\ m^2/s$ ✓ Check.

Problem 2.28 (In Excel)

Some experimental data for the viscosity of helium at 1 atm are

T, °C	0	100	200	300	400
μ, N · s/m^2($\times 10^5$)	1.86	2.31	2.72	3.11	3.46

Using the approach described in Appendix A-3, correlate these data to the empirical Sutherland equation

$$\mu = \frac{bT^{1/2}}{1 + S/T}$$

(where T is in kelvin) and obtain values for constants b and S.

Solution

Pathlines: **Data:** **Using procedure of Appendix A.3:**

T (°C)	T (K)	μ(x10^5)
0	273	1.86E-05
100	373	2.31E-05
200	473	2.72E-05
300	573	3.11E-05
400	673	3.46E-05

T (K)	$T^{3/2}/\mu$
273	2.43E+08
373	3.12E+08
473	3.78E+08
573	4.41E+08
673	5.05E+08

The equation to solve for coefficients S and b is

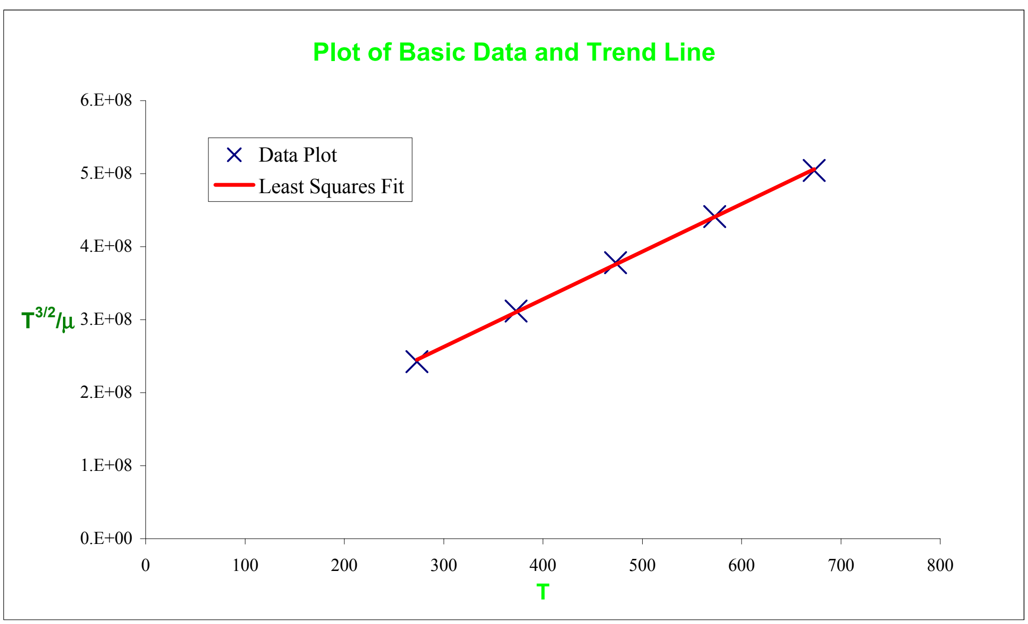
$$\frac{T^{3/2}}{\mu} = \left(\frac{1}{b}\right)T + \frac{S}{b}$$

From the built-in *Excel* *Linear Regression* functions:

Slope = 6.534E+05
Intercept = 6.660E+07
R^2 = 0.9996

Hence:

b = 1.53E-06 kg/m·s·K$^{1/2}$
S = 101.9 K

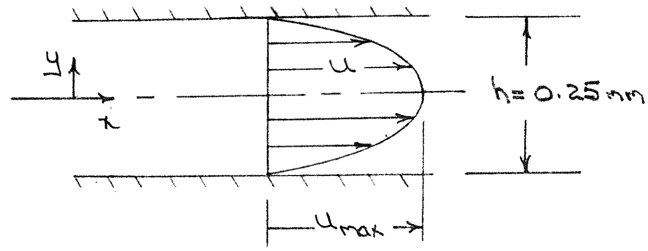


Problem 2.29

Given: Flow of water @ 15°C between parallel plates as shown.

$$\frac{u}{u_{max}} = \left[1 - \left(\frac{2y}{h}\right)^2\right]$$

$$u_{max} = 0.10 \text{ m/s}$$



Find: Shear stress on upper plate (indicate direction); sketch the variation of shear stress across the channel

Solution

Basic equation $\tau_{yx} = \mu \dfrac{du}{dy}$

$$\frac{du}{dy} = \frac{d}{dy}\left\{u_{max}\left[1 - \left(\frac{2y}{h}\right)^2\right]\right\}$$

$$\frac{du}{dy} = u_{max}\left(-\frac{4}{h^2}\right)2y = -\frac{8\,u_{max}\,y}{h^2}$$

At upper plate, $y = +\frac{h}{2}$, so

$$\tau_{yx}\left(@\, y = \frac{h}{2}\right) = \mu \left.\frac{du}{dy}\right)_{y=\frac{h}{2}} = -\frac{8\mu\, u_{max}}{h^2}\left(\frac{h}{2}\right) = -\frac{4\mu\, u_{max}}{h}$$

From Table A.8, for water @ 15°C, $\mu = 1.14 \times 10^{-3}$ N.s/m². Thus

$$\tau_{yx} = -\frac{4\mu\, u_{max}}{h} = -4 \times 1.14 \times 10^{-3}\,\frac{\text{N.s}}{\text{m}^2} \times 0.10\,\frac{\text{m}}{\text{s}} \times \frac{1}{2.5 \times 10^{-4}\,\text{m}}$$

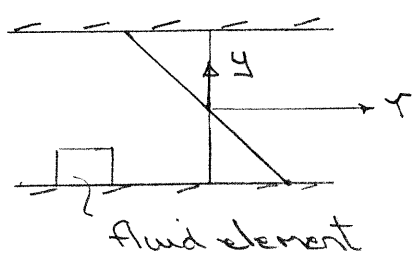
$$\tau_{yx} = -1.83 \text{ N/m}^2 \qquad \tau_{yx}$$

The upper plate is a minus y surface. Since $\tau_{yx} < 0$, the shear stress on the upper plate must act in the plus x direction.

The shear stress varies linearly with y

$$\tau = \mu \frac{du}{dy} = -\frac{8\,u_{max}}{h^2}\,y$$

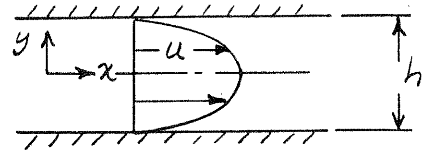
The shear stress on the surface of the fluid element shown (a positive y surface) is illustrated in the sketch.



Problem 2.30

Given: Laminar flow between parallel plates.

$$\frac{u}{u_{max}} = 1 - \left(\frac{2y}{h}\right)^2$$



$T = 15°C$, $u_{max} = 0.05$ m/s, $h = 1$ mm, water

Find: Force on $A = 0.1\ m^2$ section of lower plate.

Solution: Apply definitions of Newtonian fluid, shear stress.

Basic equations: $\tau = \frac{F}{A}$, $\tau_{yx} = \mu \frac{du}{dy}$

Assumptions: (1) Newtonian fluid

From the given profile, $u = u_{max}\left[1 - \left(\frac{2y}{h}\right)^2\right]$, so $\frac{du}{dy} = u_{max}(-2)\left(\frac{2y}{h}\right)\left(\frac{2}{h}\right)$

At lower surface, $y = -h/2$ $\qquad = -\frac{8 u_{max} y}{h^2}$

$$\tau_{yx}(\text{lower}) = \mu \left.\frac{du}{dy}\right)_{y=-h/2} = \mu\left[-\frac{8 u_{max}(-h/2)}{h^2}\right] = \frac{4 \mu u_{max}}{h}$$

$\tau_{yx} > 0$ and surface is positive, so to right.

$$F = \tau_{yx} A = \frac{4 \mu u_{max} A}{h}$$

From Appendix A, Table A.8, $\mu = 1.14 \times 10^{-3}$ N·s/m at 15°C, so

$$F = 4 \times 1.14 \times 10^{-3} \frac{N \cdot s}{m^2} \times 0.05 \frac{m}{s} \times 0.1\ m^2 \times \frac{1}{5\ mm} \times 10^3 \frac{mm}{m}$$

$F = 0.228$ N (to right) $\qquad$ F

Problem 2.31

Open-Ended Problem Statement: Explain how an ice skate interacts with the ice surface. What mechanism acts to reduce sliding friction between skate and ice?

Discussion: The normal freezing and melting temperature of ice is 0°C (32°F) at atmospheric pressure. The melting temperature of ice decreases as pressure is increased. Therefore ice can be caused to melt at a temperature below the normal melting temperature when the ice is subjected to increased pressure.

A skater is supported by relatively narrow blades with a short contact against the ice. The blade of a typical skate is less than 3 mm wide. The length of blade in contact with the ice may be just ten or so millimeters. With a 3 mm by 10 mm contact patch, a 75 kg skater is supported by a pressure between skate blade and ice on the order of tens of megaPascals (hundreds of atmospheres). Such a pressure is enough to cause ice to melt rapidly.

When pressure is applied to the ice surface by the skater, a thin surface layer of ice melts to become liquid water and the skate glides on this thin liquid film. Viscous friction is quite small, so the effective friction coefficient is much smaller than for sliding friction.

The magnitude of the viscous drag force acting on each skate blade depends on the speed of the skater, the area of contact, and the thickness of the water layer on top of the ice.

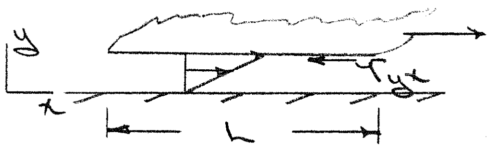
The phenomenon of static friction giving way to viscous friction is similar to the hydroplaning of a pneumatic tire caused by a layer of water on the road surface.

Given: Skater, of weight $W = 100$ lbf, glides on one skate at speed $V = 20$ ft/s. Skate blade, of length $L = 11.5$ in and width $w = 0.125$ in. glides on thin film of water of height $h = 5.75 \times 10^{-5}$ in.

Find: the deceleration of the skater due to viscous shear.

Solution:

Model flow as one-dimensional shear flow



Basic equation: $\tau_{yx} = \mu \dfrac{du}{dy}$

Assumptions:
1. Newtonian fluid
2. Linear velocity profile
3. Neglect end effects.

From Table A.7, Appendix A, at 32°F

$\mu = 3.66 \times 10^{-5}$ lbf·s/ft²

$$\tau_{yx} = \mu \frac{du}{dy} = \mu \frac{V}{h} = 3.66 \times 10^{-5} \frac{\text{lbf}\cdot\text{s}}{\text{ft}^2} \times 20 \frac{\text{ft}}{\text{s}} \times \frac{1}{5.75 \times 10^{-5}\ \text{in}} \times \frac{12\ \text{in}}{\text{ft}}$$

$$\tau_{yx} = 153 \ \text{lbf/ft}^2$$

$$\Sigma F_x = m a_x \qquad \therefore \ \tau_{yx} A = -\frac{W}{g} a_x$$

$$a_x = -\frac{\tau_{yx} A g}{W} = -\frac{\tau_{yx} L w g}{W}$$

$$= -153 \frac{\text{lbf}}{\text{ft}^2} \times 11.5\ \text{in} \times 0.125\ \text{in} \times 32.2 \frac{\text{ft}}{\text{s}^2} \times \frac{1}{100\ \text{lbf}} \times \frac{\text{ft}^2}{144\ \text{in}^2}$$

$$a_x = -0.491 \ \text{ft/s}^2 \quad \longleftarrow \quad a_x$$

Given: Thin film of crude oil ($SG = 0.85$, $\mu = 2.15\times10^{-3}\ \text{lbf}\cdot\text{s/ft}^2$) with thickness $h = 0.125$ in, flows down a 30° incline. The velocity profile is given by

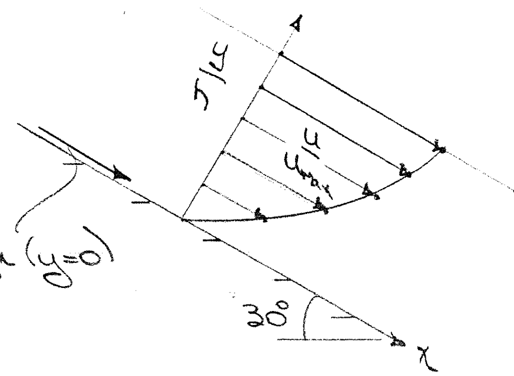
$$u = \frac{\rho g}{\mu}\left(hy - \frac{y^2}{2}\right)\sin\theta$$

Find: (a) the magnitude and direction of the shear stress acting on the surface
(b) Plot the velocity profile.

Solution:

To plot the profile, note that $u = u_{max}$ at $y = h$

$$u_{max} = \frac{\rho g}{\mu}\frac{h^2}{2}\sin\theta \qquad \therefore \frac{u}{u_{max}} = 2\left[\frac{y}{h} - \frac{1}{2}\left(\frac{y}{h}\right)^2\right]$$



The shear stress is given by $\tau_{yx} = \mu\frac{du}{dy}$

$$\therefore \tau_{yx} = \mu\frac{d}{dy}\left[\frac{\rho g}{\mu}\left(hy - \frac{y^2}{2}\right)\sin\theta\right] = \mu\frac{\rho g}{\mu}\sin\theta\,(h-y)$$

At the inclined surface, $y = 0$

$$\therefore \tau_{yx} = \rho g h\sin\theta = SG\,\rho_{H_2O}\,g h\sin\theta$$

$$\tau_{yx} = 0.85\times1.94\,\frac{\text{slug}}{\text{ft}^3}\times32.2\,\frac{\text{ft}}{\text{s}^2}\times0.125\,\text{in}\times\frac{\text{ft}}{12\,\text{in}}\times\sin30^\circ\times\frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}}$$

$$\tau_{yx} = 0.277\ \text{lbf/ft}^2 \qquad \tau_{yx}$$

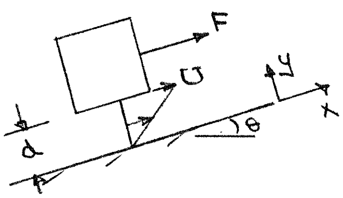
The surface is a positive y surface. Since $\tau_{yx} > 0$, the stress must act in the positive x direction as shown on the sketch above.

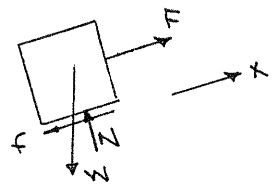
Given: Block of weight 10 lbf, 10 in. on each edge, is pulled up a plane, inclined at 25° to the horizontal, over a film of SAE 10W oil at 100°F. The speed of the block is constant at 2 ft/s and the oil film thickness is 0.001 in. Velocity profile in film is linear.

Find: Force required.

Solution:

Since the block is moving at constant velocity, U, then $\Sigma \vec{F}_{ext} = 0$. Consider the forces along the direction of motion and look at a free body diagram of the block.





Since $\Sigma F_x = 0$, then $F - f - W \sin\theta = 0$

Now the friction force, $f = \tau A$

where $\tau = \mu \frac{du}{dy}$

For small gap (linear velocity profile) $\tau = \mu \frac{U}{d}$

Hence

$$f = \tau A = \mu \frac{U}{d} A$$

and

$$F - \mu \frac{U}{d} A - W \sin\theta = 0$$

Thus

$$F = \mu \frac{U}{d} A + W \sin\theta$$

From Fig A.2, Appendix A, for SAE 10W oil @ 100°F (38°C), $\mu = 3.7 \times 10^{-2}$ N·s/m²

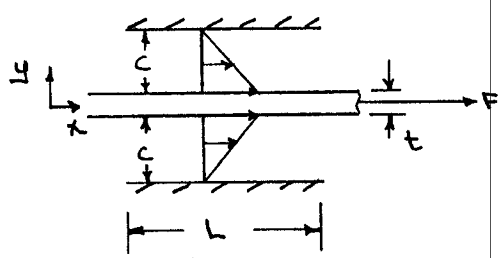
$$F = \mu \frac{U}{d} A + W \sin\theta$$

$$= 3.7 \times 10^{-2} \frac{N \cdot s}{m^2} \times 2.09 \times 10^{-2} \frac{lbf \cdot s}{ft^2} \cdot \frac{m^2}{N \cdot s} \times 2 \frac{ft}{s} \times (10)^2 in^2 \times \frac{1}{0.001 in} \times \frac{ft}{12 in} + 10 lbf \sin 25°$$

F = 17.1 lbf ← F

Problem 2.35

Given: Tape, of width $w = 1.00$ in is to be coated on both sides with lubricant by drawing in through narrow gap of length, L, as shown.



$c = 0.012$ in. $t = 0.015$ in, $L = 0.75$ in

Lubricant: $\mu = 0.021$ slug/ft·s completely fills gap, velocity distribution is linear

Maximum allowable force in tape is $F = 7.5$ lbf.

Find: Maximum allowable tape speed.

Solution:

$$\Sigma F_x = m a_x$$

Since V_{tape} = constant, then $\Sigma F_x = 0$ and driving force is balanced by friction force, F_f

$$F_f = \tau A \quad \text{where} \quad \tau = \mu \frac{du}{dy}$$

On top surface of tape, $\tau_t = \mu \frac{du}{dy} = \mu \frac{V_{c+\frac{t}{2}} - V_{t/2}}{(t/2 + c) - t/2} = -\mu \frac{V}{c}$

negative τ on positive surface means F_f acts to left

On bottom surface of tape, $\tau_b = \mu \frac{du}{dy} = \mu \frac{V_{-(c+t/2)} - V_{-t/2}}{(-t/2 - c) - (-t/2)} = \mu \frac{V}{c}$

positive τ on negative surface means F_f acts to left

Hence, $\Sigma F_x = 0 = F - F_{f_t} - F_{f_b}$

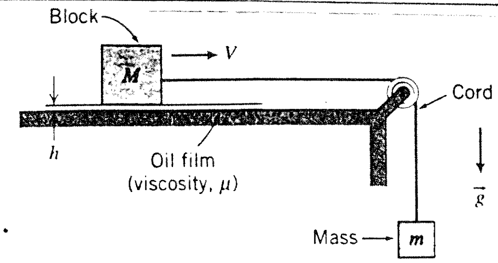
$$F = F_{f_t} + F_{f_b} = |\tau_t A| + |\tau_b A|$$

$$F = \mu \frac{V}{c} A + \mu \frac{V}{c} A = 2\mu \frac{V}{c} A$$

Solving for V,

$$V = \frac{Fc}{2\mu A} = 7.5\ \text{lbf} \times 0.012\ \text{in} \times \frac{1}{2} \times \frac{\text{ft} \cdot \text{s}}{0.021\ \text{slug}} \times \frac{1}{(1.00\ \text{in})(0.75\ \text{in})} \times \frac{\text{ft} \cdot \text{slug}}{\text{lbf} \cdot \text{s}^2} \times \frac{12\ \text{in}}{\text{ft}}$$

$$V = 34.3\ \text{ft/s}$$

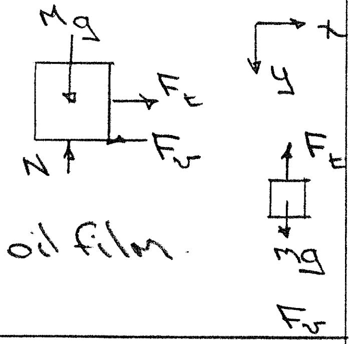


Given: Block of mass M slides on thin film of oil of thickness h. Contact area of block is A. At time t=0, mass m is released from rest.

M = 5 kg, m = 1 kg, A = 25 cm², h = 0.5 M

Find: (a) Expression for viscous force on block when moving at speed V
(b) Differential equation governing block speed as a function of time
(c) Expression for block speed V = V(t); plot
(d) If V = 1 m/s at t = 1 s, find μ

Solution:



Basic equations: $\tau_{yx} = \mu \frac{du}{dy}$ $\qquad \Sigma \vec{F} = m\vec{a}$

Assumptions: (1) Newtonian fluid
(2) Linear velocity profile in oil film

Then, $F_v = \tau A = \mu \frac{du}{dy} A = \mu \frac{\Delta u}{\Delta y} A = \mu \frac{V}{h} A$ ← F_v

For the block, $\Sigma F_x = F_t - F_v = M \frac{dV_b}{dt}$ (1)

For the falling mass $\Sigma F_y = mg - F_t = m \frac{dV_m}{dt}$, or

$F_t = mg - m\frac{dV_m}{dt}$ (2)

Since $V_b = V_m = V$, then substituting from Eq. (2) into (1) gives

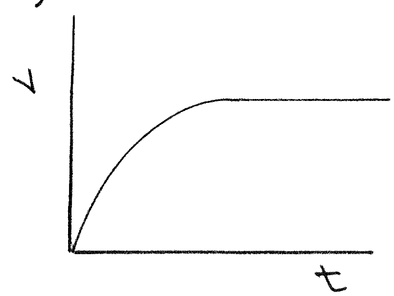
$$mg - m\frac{dV}{dt} - F_v = M\frac{dV}{dt} = mg - m\frac{dV}{dt} - \mu\frac{V}{h}A$$

Finally, $mg - \mu \frac{V}{h} A = (M+m)\frac{dV}{dt}$ ← Diff. Eq.

To solve we separate variables and integrate

$$t = \int_0^t dt = \int_0^V \frac{(M+m)}{mg - \mu\frac{V}{h}A} dV = -\frac{(M+m)h}{\mu A} \ln\left(mg - \frac{\mu V A}{h}\right)\Big]_0^V$$

$$t = -\frac{(M+m)h}{\mu A} \ln\left(1 - \frac{\mu V A}{mgh}\right)$$



Taking antilogarithms,

$$1 - \frac{\mu V A}{mgh} = e^{-\frac{\mu A t}{(M+m)h}}$$

Solving for V,

$$V = \frac{mgh}{\mu A}\left(1 - e^{-\frac{\mu A t}{(M+m)h}}\right)$$ ← V

The velocity increases exponentially to $V_{max} = \frac{mgh}{\mu A}$

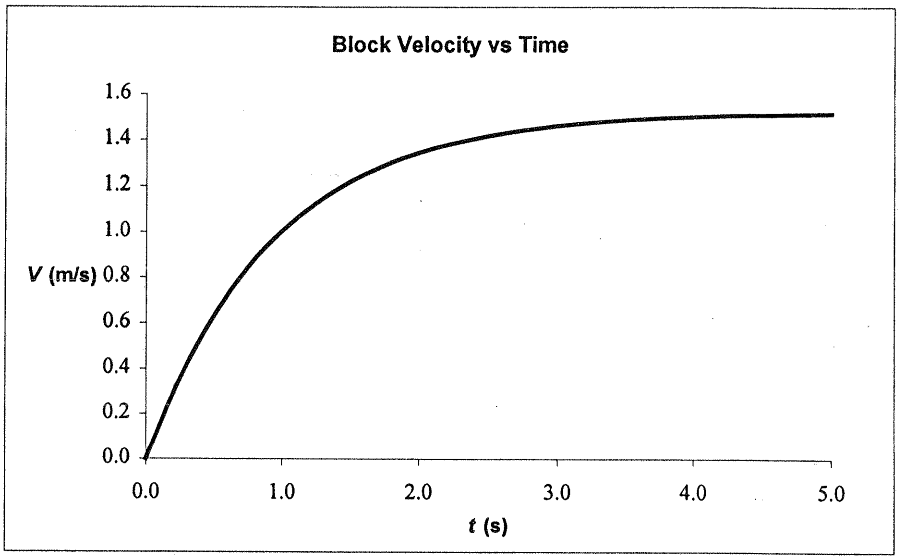
Using Excel's solver, with $V = 1$ m/s at $t = 1$ s, the oil viscosity is found to be $\mu = 1.29\ N.s/m^2$ ← μ

The $V(t)$ plot, with $M = 5$ kg, $m = 1$ kg, $A = 25\ cm^2$, $h = 0.5$ mm and $\mu = 1.29\ N.s/m^2$, is generated from

$$V = \frac{mgh}{\mu A}\left[1 - e^{-\frac{\mu A t}{(M+m)h}}\right]$$

t (s)	V (m/s)
0.00	0.00
0.25	0.36
0.50	0.63
0.75	0.84
1.00	1.00
1.25	1.12
1.50	1.22
1.75	1.29
2.00	1.34
2.25	1.39
2.50	1.42
2.75	1.44
3.00	1.46
3.25	1.47
3.50	1.49
3.75	1.49
4.00	1.50
4.25	1.51
4.50	1.51
4.75	1.51
5.00	1.51

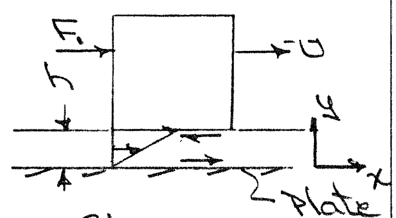
M = 5 kg
m = 1 kg
A = 25 cm^2
h = 0.5 mm
μ = **1.29** $N.s/m^2$ (From *Solver* or *Goal Seek*)



Given: Block of mass M moves at steady speed U under influence of constant force F_1 on a thin film of oil of thickness h and viscosity μ; block is square, a mm on a side.

Find: (a) Magnitude and direction of shear stress acting on bottom of block and supporting plate.
(b) Expression for time required to lose 95 % of its initial speed when force is suddenly removed.
(c) Expect shape of speed vs time curve.

Solution:



Basic equations: $\tau_{yx} = \mu \frac{du}{dy}$ $\quad \Sigma \vec{F} = m\vec{a}$

Assumptions: (1) Newtonian fluid
(2) Linear velocity profile in oil film

$$\tau_{yx} = \mu \frac{du}{dy} = \mu \frac{\Delta u}{\Delta y} = \mu \frac{U}{h} \qquad \tau_{yx}$$

Bottom of block is −y surface, so τ_{yx} acts to left
Plate surface is +y surface, so τ_{yx} acts to right

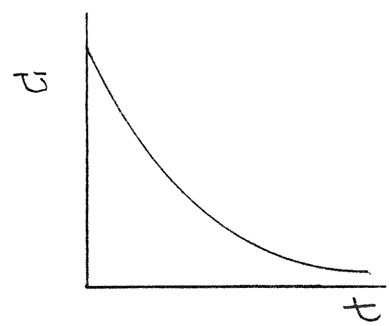
Viscous shear force on block is $F_v = \tau A = \tau a^2 = \frac{\mu U a^2}{h}$

When F_1 is removed, block slows under action of F_v

$$\Sigma F_x = m\frac{dU}{dt} = -F_v = -\frac{\mu U a^2}{h}$$

Separating variables and integrating we have

$$\int_{U_i}^{U} \frac{dU}{U} = -\int_0^t \frac{\mu a^2}{mh}\,dt$$



Then

$$\ln\frac{U}{U_i} = -\frac{\mu a^2}{mh}t \quad \text{....(1)}$$

and

$$t = -\frac{mh}{\mu a^2}\ln\frac{U}{U_i}$$

For $U/U_i = 0.05$

$$t = 3.0\frac{mh}{\mu a^2} \qquad t$$

From Eq.(1) we can write

$$U = U_i e^{-\frac{\mu a^2 t}{h}}$$

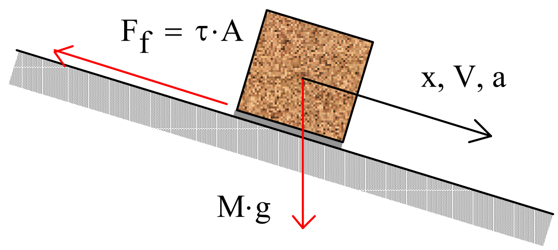
The speed thus decreases exponentially with time.

Problem 2.38

A block 0.2 m square, with 5 kg mass, slides down a smooth incline, 30° below the horizontal, a film of SAE 30 oil at 20°C that is 0.20 mm thick. If the block is released from rest at $t = 0$, wl is its initial acceleration? Derive an expression for the speed of the block as a function of time. the curve for V(t). Find the speed after 0.1 s. If we want the mass to instead reach a speed of 0. m/s at this time, find the viscosity μ of the oil we would have to use.

Given: Data on the block and incline

Find: Initial acceleration; formula for speed of block; plot; find speed after 0.1 s. Find oil viscosity if speed is 0.3 m/s after 0.1 s



Solution

Given data $M = 5 \cdot kg$ $A = (0.2 \cdot m)^2$ $d = 0.2 \cdot mm$ $\theta = 30 \cdot deg$

From Fig. A.2 $\mu = 0.4 \cdot \frac{N \cdot s}{m^2}$

Applying Newton's 2nd law to initial instant (no friction)

$$M \cdot a = M \cdot g \cdot \sin(\theta) - F_f = M \cdot g \cdot \sin(\theta)$$

so $$a_{init} = g \cdot \sin(\theta) = 9.81 \cdot \frac{m}{s^2} \times \sin(30)$$ $$a_{init} = 4.9 \frac{m}{s^2}$$

Applying Newton's 2nd law at any instant

$$M \cdot a = M \cdot g \cdot \sin(\theta) - F_f$$

and $$F_f = \tau \cdot A = \mu \cdot \frac{du}{dy} \cdot A = \mu \cdot \frac{V}{d} \cdot A$$

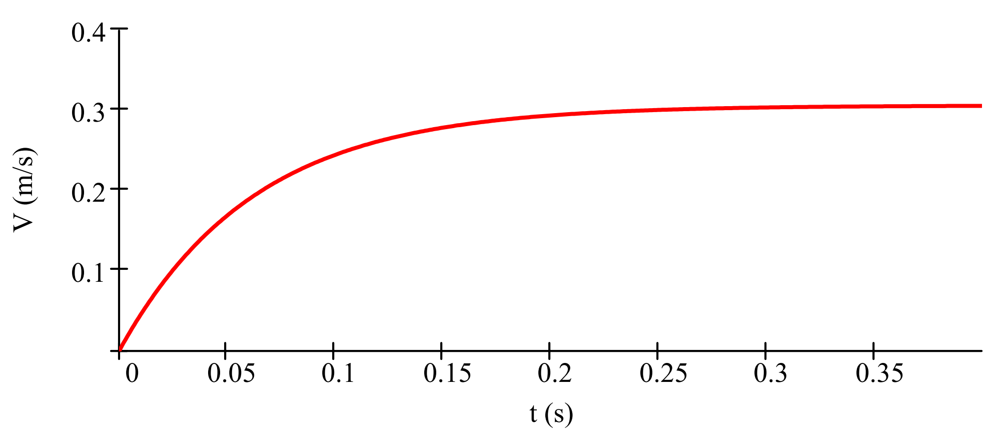
so $$M \cdot a = M \cdot \frac{dV}{dt} = M \cdot g \cdot \sin(\theta) - \frac{\mu \cdot A}{d} \cdot V$$

Separating variables $$\frac{dV}{g \cdot \sin(\theta) - \frac{\mu \cdot A}{M \cdot d} \cdot V} = dt$$

Integrating and using limits

$$-\frac{M \cdot d}{\mu \cdot A} \cdot \ln\left(1 - \frac{\mu \cdot A}{M \cdot g \cdot d \cdot \sin(\theta)} \cdot V\right) = t$$

or $$V(t) = \frac{M \cdot g \cdot d \cdot \sin(\theta)}{\mu \cdot A} \cdot \left(1 - e^{\frac{-\mu \cdot A}{M \cdot d} \cdot t}\right)$$



At t = 0.1 s

$$V = 5\cdot kg \times 9.81\cdot\frac{m}{s^2} \times 0.0002\cdot m\cdot \sin(30) \times \frac{m^2}{0.4\cdot N\cdot s\cdot(0.2\cdot m)^2} \times \frac{N\cdot s^2}{kg\cdot m} \times \left[1 - e^{-\left(\frac{0.4\cdot 0.04}{5\cdot 0.002}\cdot 0.1\right)}\right]$$

$$V = 0.245\frac{m}{s}$$

To find the viscosity for which V(0.1 s) = 0.3 m/s, we must solve

$$V(t = 0.1\cdot s) = \frac{M\cdot g\cdot d\cdot \sin(\theta)}{\mu\cdot A}\cdot\left[1 - e^{\frac{-\mu\cdot A}{M\cdot d}\cdot(t=0.1\cdot s)}\right]$$

The viscosity μ is implicit in this equation, so solution must be found by manual iteration, or by of a number of classic root-finding numerical methods, or by using *Excel*'s *Goal Seek*

From the *Excel* workbook for this problem the solution is

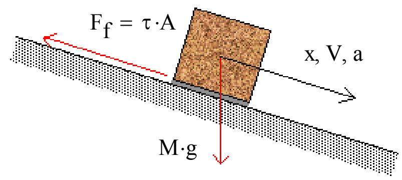
$$\mu = 0.27\frac{N\cdot s}{m^2}$$

Excel workbook

Problem 2.38 (In Excel)

A block 0.2 m square, with 5 kg mass, slides down a smooth incline, 30° below the horizontal, on a film of SAE 30 oil at 20°C that is 0.20 mm thick. If the block is released from rest at $t = 0$, what is its initial acceleration? Derive an expression for the speed of the block as a function of time. Plot the curve for $V(t)$. Find the speed after 0.1 s. If we want the mass to instead reach a speed of 0.3 m/s at this time, find the viscosity μ of the oil we would have to use.

Solution



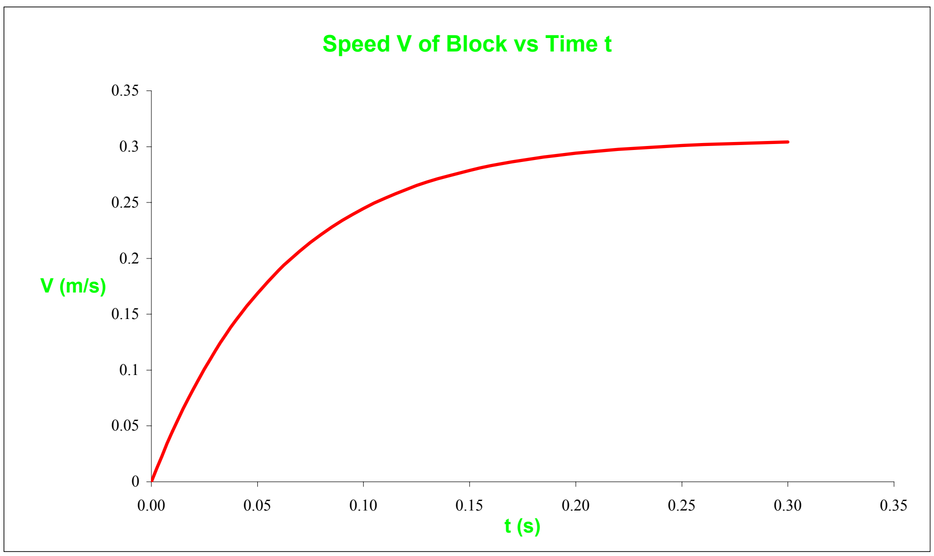
The solution is

$$V(t) = \frac{M \cdot g \cdot d \cdot \sin(\theta)}{\mu \cdot A} \cdot \left(1 - e^{\frac{-\mu \cdot A}{M \cdot d} \cdot t}\right)$$

The data is

M =	5.00	kg
θ =	30	deg
μ =	0.40	N.s/m^2
A =	0.04	m^2
d =	0.2	mm

t (s)	V (m/s)
0.00	0.000
0.01	0.045
0.02	0.084
0.03	0.117
0.04	0.145
0.05	0.169
0.06	0.189
0.07	0.207
0.08	0.221
0.09	0.234
0.10	0.245
0.11	0.254
0.12	0.262
0.13	0.268
0.14	0.274
0.15	0.279
0.16	0.283
0.17	0.286
0.18	0.289
0.19	0.292
0.20	0.294
0.21	0.296
0.22	0.297
0.23	0.299
0.24	0.300
0.25	0.301
0.26	0.302
0.27	0.302
0.28	0.303
0.29	0.304
0.30	0.304



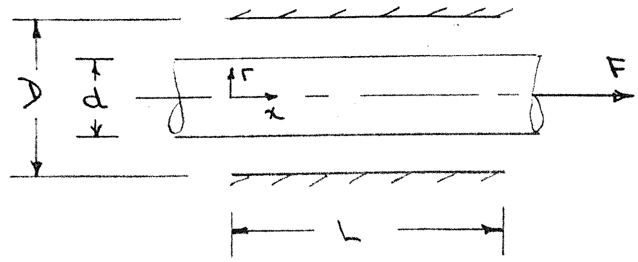
To find the viscosity for which the speed is 0.3 m/s after 0.1 s use *Goal Seek* with the velocity targeted to be 0.3 by varying the viscosity in the set of cell below:

t (s)	V (m/s)
0.10	0.300

for μ = 0.270 N.s/m^2

Problem 2.39

Given: Wire, of diameter d, is to be coated with varnish by drawing it through a circular die of diameter, D, and length, L.



$d = 0.9\text{ mm}$, $D = 1.0\text{ mm}$, $L = 50\text{ mm}$

Varnish, $\mu = 20$ centipoise fills the space between wire and die. Wire is drawn through at speed, $V = 50\text{ m/s}$

Find: Force required to pull the wire

<u>Solution</u>

$$\Sigma F_x = m a_x$$

Since V_{wire} = constant, applied force must be sufficient to balance friction force, F_f

$$F_f = \tau A \quad \text{where } \tau = \mu \frac{du}{dr} \quad \text{and } A = \pi d L$$

Assuming a linear velocity distribution in varnish

$$\tau_s = \mu \left.\frac{du}{dr}\right)_s = \mu \frac{V_{D/2} - V_{d/2}}{D/2 - d/2} = -\mu \frac{V}{(D-d)/2}$$

(negative stress on positive r surface must act in negative x direction)

$$F - F_f = 0$$

$$F = \tau A = \mu \frac{2V}{(D-d)} \times \pi d L$$

$$F = 20\text{ cp} \times \frac{\text{gm}}{100\text{ cm.s.cp}} \times 2\pi \times 50\frac{\text{m}}{\text{s}} \times 0.9\text{ mm} \times 50\text{ mm} \times \frac{1}{0.1\text{ mm}} \times \frac{\text{cm}}{10\text{ mm}} \times \frac{\text{kg}}{1000\text{ gm}} \times \frac{\text{N.s}^2}{\text{kg.m}}$$

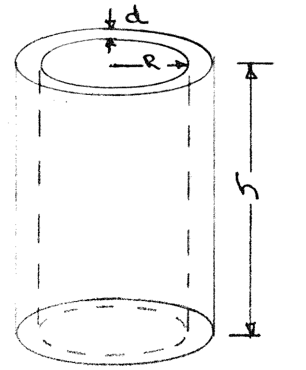
$F = 2.83\text{ N}$ ← F

Given: Concentric cylinder viscometer.

$R = 2.0$ in $d = 0.001$ in $h = 8$ in.

Inner cylinder rotates at 400 rpm

Gap filled with castor oil at 90°F.



Determine: Torque required to rotate the inner cylinder

Solution:

The required torque must balance the resisting torque of the shear force.

The shear force is given by $F = \tau A$ where $A = 2\pi R h$

For a Newtonian fluid $\tau = \mu \frac{du}{dy}$

For small gap (linear profile) $\tau = \mu \frac{V}{d}$

where V = tangential velocity of inner cylinder $= R\omega$

Hence

$$F = \tau A = \mu \frac{R\omega}{d} 2\pi R h = \frac{2\pi \mu R^2 \omega h}{d}$$

and the torque $T = RF = \dfrac{2\pi \mu R^3 \omega h}{d}$

From Fig A.2, for castor oil at 90°F (32°C), $\mu = 3.80 \times 10^{-1}$ N.s/m²

Substituting numerical values.

$$T = \frac{2\pi \mu R^3 \omega h}{d} = 2\pi \times 3.80\times10^{-1} \frac{N.s}{m^2} \times 2.09\times10^{-2} \frac{lbf.s.m^2}{ft^2.N.s} \times (2.0)^3 in^3 \times 400 \frac{rev}{min} \times 8\,in \times \frac{1}{10^{-3}\,in}$$

$$\times\, 2\pi \frac{rad}{rev} \times \frac{min}{60\,s} \times \frac{ft^3}{1728\,in^3}$$

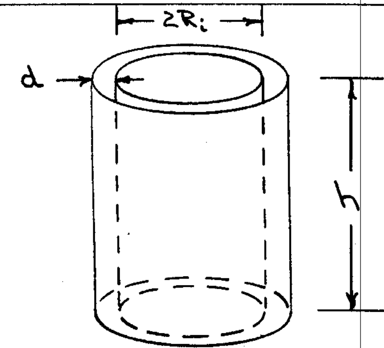
$T = 77.4$ ft·lbf ← Torque

Problem 2.41

Given: Concentric cylinder viscometer

$R_i = 37.5$ mm , $d = 0.02$ mm, $h = 150$ mm

Inner cylinder rotates at $\omega = 100$ rpm, under torque, $T = 0.021$ N.m



Find: Viscosity of liquid in clearance gap.

Solution

The imposed torque must balance the resisting torque of the shear force.

The shear force is given by $F = \tau A$ where $A = 2\pi R_i h$

For a Newtonian fluid $\tau = \mu \frac{du}{dy}$

Since the velocity profile is assumed to be linear, $\tau = \mu \frac{V}{d}$

where V is the tangential velocity of the inner cylinder, $V = R_i \omega$

Thus,

$$F = \tau A = \mu \frac{V}{d} 2\pi R_i h = \frac{2\pi \mu R_i^2 \omega h}{d}$$

and the torque $T = R_i F = \frac{2\pi \mu R_i^3 \omega h}{d}$

Solving for μ,

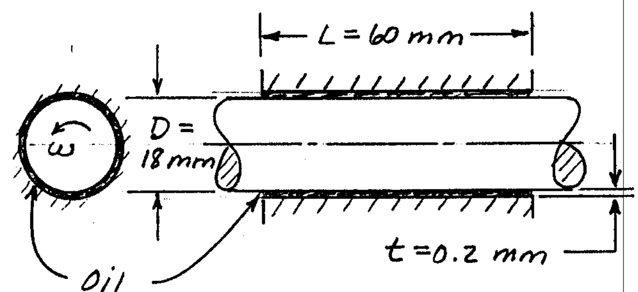
$$\mu = \frac{T d}{2\pi R_i^3 \omega h} = 0.021 \text{ N.m} \times 0.02 \text{ mm} \times \frac{1}{2\pi} \times \frac{1}{(37.5)^3 \text{ mm}^3} \times \frac{\text{min}}{100 \text{ rev}} \times \frac{1}{150 \text{ mm}} \times \frac{\text{rev}}{2\pi \text{ rad}} \times \frac{60 \text{ s}}{\text{min}} \times (1000)^3 \frac{\text{mm}^3}{\text{m}^3}$$

$$\mu = 8.07 \times 10^{-4} \text{ N.s/m}^2 \longleftarrow \mu$$

Problem 2.42

Given: Shaft turning inside stationary journal as shown, $N = 20$ rps.

Torque, $T = 0.0036\ \text{N}\cdot\text{m}$



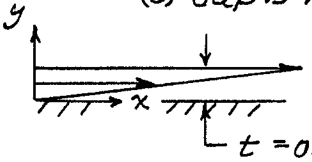
Find: Estimate viscosity of oil.

Solution: Basic equation $\tau_{yx} = \mu \dfrac{du}{dy}$

Assumptions: (1) Newtonian fluid

(2) Gap is narrow, so velocity profile is linear, $\dfrac{du}{dy} \approx \dfrac{\Delta u}{\Delta y}$

Then



Shear stress is

$$\tau_{yx} \approx \mu \frac{\Delta u}{\Delta y} = \mu \frac{U}{t} = \frac{\mu \omega D}{2t}$$

Neglecting end effects, torque is

$$T = FR = \tau_{yx} A R = \tau_{yx} (\pi D L) \frac{D}{2} = \frac{\mu \pi \omega D^3 L}{4t}$$

Solving for viscosity

$$\mu = \frac{4tT}{\pi \omega D^3 L}$$

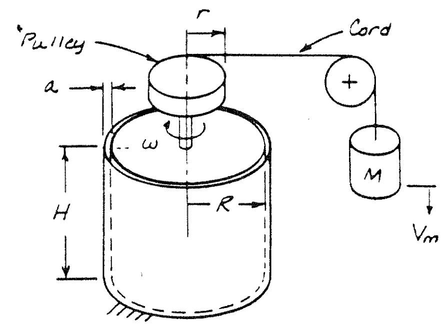
$$= \frac{4}{\pi} \times 0.2\ \text{mm} \times 0.0036\ \text{N}\cdot\text{m} \times \frac{\text{s}}{20\ \text{rev}} \times \frac{1}{(18)^3\ \text{mm}^3} \times \frac{1}{60\ \text{mm}} \times \frac{\text{rev}}{2\pi\ \text{rad}} \times (1000)^3 \frac{\text{mm}^3}{\text{m}^3}$$

$$\mu = 0.0208\ \text{N}\cdot\text{s}/\text{m}^2 \longleftarrow \mu$$

{ From Fig. A.2, this oil appears somewhat less viscous than SAE 10W, assuming the oil is at room temperature. }

Problem 2.43

Given: Concentric-cylinder viscometer, driven by falling mass.



$M = 0.10$ kg $r = 25$ mm
$R = 50$ mm $a = 0.20$ mm
$H = 80$ mm $V_m = 30$ mm/s

After starting transient, $V_m = $ const.

Find: (a) An algebraic expression for viscosity of the liquid, in terms of $M, g, V_m, r, R, a,$ and H.

(b) Evaluate using the data given.

Solution: Apply Newton's law of viscosity.

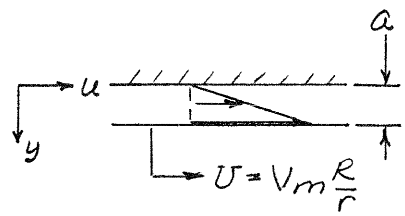
Basic equations: $\tau = \mu \dfrac{du}{dy}$ $\Sigma M = 0$ $T = \tau A R$

Assumptions: (1) Newtonian liquid
(2) Narrow gap, so linear velocity profile
(3) Steady angular speed

Summing torques on the rotor

$$\Sigma M = Mgr - \tau A R = I\alpha = 0 \,;\quad A = 2\pi R H \qquad (\alpha = 0 \text{ by } (3))$$

Because $a \ll R$, treat the gap as plane. Then



$$\tau = \mu \frac{du}{dy} = \mu \frac{\Delta u}{\Delta y} = \mu \frac{U - 0}{a - 0} = \mu \frac{U}{a} = \frac{\mu V_m R}{a r}$$

Substituting,

$$Mgr - \frac{\mu V_m R}{ar} 2\pi R H\, R = Mgr - \frac{2\pi \mu V_m R^3 H}{ar} = 0$$

so

$$\mu = \frac{M g r^2 a}{2\pi V_m R^3 H}$$ μ

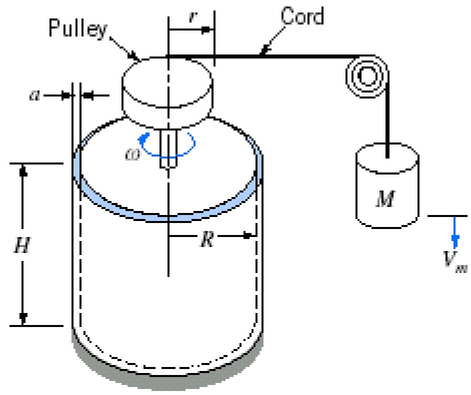
Evaluating for the given data

$$\mu = \frac{1}{2\pi} \times 0.10\,\text{kg} \times 9.81\,\frac{\text{m}}{\text{s}^2} \times (0.025)^2\,\text{m}^2 \times 0.0002\,\text{m} \times \frac{\text{s}}{0.030\,\text{m}} \times \frac{1}{(0.050)^3\,\text{m}^3} \times \frac{1}{0.080\,\text{m}} \times \frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}}$$

$$\mu = 0.0651\ \text{N}\cdot\text{s/m}^2 \quad (65.1\ \text{mPa}\cdot\text{s})$$ μ

Problem 2.44

The viscometer of Problem 2.43 is being used to verify that the viscosity of a particular fluid is μ = 0.1 N.s/m^2. Unfortunately the cord snaps during the experiment. How long will it take the cylinder to lose 99% of its speed? The moment of inertia o the cylinder/pulley system is 0.0273 kg.m^2.



Given: Data on the viscometer

Find: Time for viscometer to lose 99% of speed

Solution

The given data is

$R = 50 \cdot mm \quad H = 80 \cdot mm \quad a = 0.20 \cdot mm \quad I = 0.0273 \cdot kg \cdot m^2 \quad \mu = 0.1 \cdot \frac{N \cdot s}{m^2}$

The equation of motion for the slowing viscometer is

$$I \cdot \alpha = \text{Torque} = -\tau \cdot A \cdot R$$

where α is the angular acceleration and τ
viscometer

The stress is given by $\tau = \mu \cdot \frac{du}{dy} = \mu \cdot \frac{V - 0}{a} = \frac{\mu \cdot V}{a} = \frac{\mu \cdot R \cdot \omega}{a}$

where V and ω are the instantaneous linear and angular velocities.

Hence

$$I\cdot\alpha = I\cdot\frac{d\omega}{dt} = -\frac{\mu\cdot R\cdot\omega}{a}\cdot A\cdot R = \frac{\mu\cdot R^2\cdot A}{a}\cdot\omega$$

Separating variables

$$\frac{d\omega}{\omega} = -\frac{\mu\cdot R^2\cdot A}{a\cdot I}\cdot dt$$

Integrating and using IC $\omega = \omega_0$

$$\omega(t) = \omega_0\cdot e^{-\frac{\mu\cdot R^2\cdot A}{a\cdot I}\cdot t}$$

The time to slow down by 99% is obtained from solving

$$0.01\cdot\omega_0 = \omega_0\cdot e^{-\frac{\mu\cdot R^2\cdot A}{a\cdot I}\cdot t}$$

so $$t = -\frac{a\cdot I}{\mu\cdot R^2\cdot A}\cdot\ln(0.01)$$

Note that $$A = 2\cdot\pi\cdot R\cdot H$$

so $$t = -\frac{a\cdot I}{2\cdot\pi\cdot\mu\cdot R^3\cdot H}\cdot\ln(0.01)$$

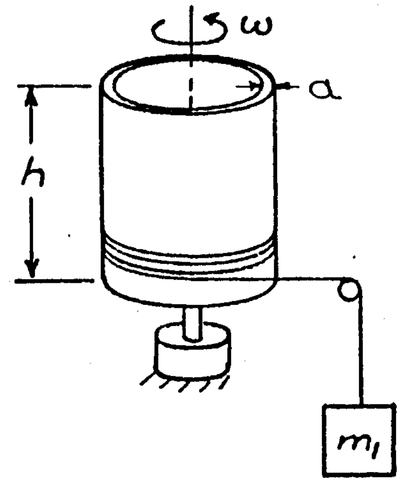
$$t = -\frac{0.0002\cdot m\cdot 0.0273\cdot kg\cdot m^2}{2\cdot\pi}\cdot\frac{m^2}{0.1\cdot N\cdot s}\cdot\frac{1}{(0.05\cdot m)^3}\cdot\frac{1}{0.08\cdot m}\cdot\frac{N\cdot s^2}{kg\cdot m}\cdot\ln(0.01)$$

$t = 4\,s$

Problem 2.45

Given: Thin outer cylinder (mass, M_2, and radius R) of a concentric-cylinder viscometer is driven by the falling mass, M_1. Clearance between outer cylinder and stationary inner cylinder is a. Bearing friction, air resistance and mass of liquid in the viscometer may be neglected

Find: (a) algebraic expression for the torque due to viscous shear acting on cylinder at angular speed ω.
(b) differential equation and solution for $\omega(t)$
(c) expression for ω_{max}



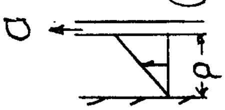
Solution:

Basic equations: $\tau = \mu \frac{du}{dy}$

$\Sigma F = ma$, $\Sigma M = I\alpha$

Assume: (1) Newtonian fluid
(2) linear velocity profile

In the gap,

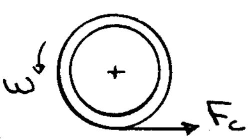


$$\tau = \mu \frac{du}{dy} = \mu \frac{U}{a} = \frac{\mu R \omega}{a}$$

$$T = \tau A R = \frac{\mu R \omega}{a}(2\pi R h)R$$

$$T = \frac{2\pi R^3 \mu h}{a}\omega \leftarrow T$$

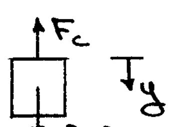
During acceleration, let the tension in the cord be F_c



For the cylinder $\Sigma M = F_c R - T = I\alpha = M_2 R^2 \frac{d\omega}{dt}$ ... (1)

For the mass $\Sigma F_y = M_1 g - F_c = ma = M_1 \frac{dV}{dt} = M_1 R \frac{d\omega}{dt}$ ... (2)

$\therefore F_c = M_1 g - M_1 R \frac{d\omega}{dt}$



Substituting into eq. (1)

$$M_1 g R - \frac{2\pi R^3 \mu h}{a}\omega = (M_1 + M_2)R^2 \frac{d\omega}{dt}$$

Let $M_1 g R = b$, $-2\pi R^3 \mu h / a = c$, $(M_1+M_2)R^2 = f$

Then, $b + c\omega = f\frac{d\omega}{dt}$ or $\int_0^t \frac{1}{f}\,dt = \int_0^\omega \frac{d\omega}{(b+c\omega)}$

Integrating, $\frac{1}{f}t = \frac{1}{c}\ln(b+c\omega)\Big]_0^\omega = \frac{1}{c}\ln\frac{(b+c\omega)}{b} = \frac{1}{c}\ln\left(1+\frac{c}{b}\omega\right)$.

$$\frac{c}{f}t = \ln\left(1+\frac{c}{b}\omega\right) \Rightarrow e^{\frac{c}{f}t} = \left(1+\frac{c}{b}\omega\right) \Rightarrow \omega = \frac{b}{c}\left(e^{\frac{c}{f}t}-1\right)$$

Substituting for b, c, and f

$$\omega = \frac{M_1 g R a}{2\pi R^3 \mu h}\left(1 - e^{\frac{-2\pi R^3 \mu h}{a(M_1+M_2)R^2}t}\right) = \frac{mga}{2\pi R^2 \mu h}\left[1 - e^{-\frac{2\pi R \mu h}{a(M_1+M_2)}t}\right] \leftarrow \omega(t)$$

Maximum ω occurs at $t \to \infty$

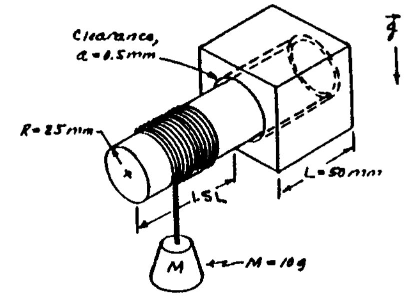
$$\omega_{max} = \frac{mga}{2\pi R^2 \mu h} \leftarrow \omega_{max}$$

Given: Circular aluminum shaft in journal.

Symmetric clearance gap

filled with SAE 10W-30 at 30°C.

Shaft turned by mass and cord.

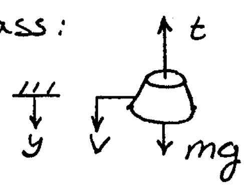


Find: (a) Develop and solve a differential equation for angular speed as a function of time.
(b) Calculate maximum angular speed.
(c) Estimate time needed to reach 95 percent of maximum speed.

Solution: Apply summation of torques and Newton's second law.

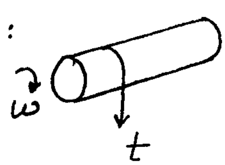
Basic equations: $\Sigma T = I\frac{d\omega}{dt}$ $\quad \Sigma F = m\frac{dV}{dt}$ $\quad V = R\omega$

For the mass:



$$\Sigma F_y = mg - t = m\frac{dV}{dt} = mR\frac{d\omega}{dt} \qquad (1)$$

For the shaft:



$$\Sigma T = tR - T_{viscous} = I\frac{d\omega}{dt} \qquad (2)$$

$$T_{viscous} = \tau RA = \mu\frac{V}{a}R\,2\pi RL = \frac{2\pi\mu\omega R^3L}{a}$$

Assume: (1) Newtonian liquid, (2) Small gap, (3) Linear profile

Then Eq. 2 becomes $\quad tR - \frac{2\pi\mu R^3L}{a}\omega = I\frac{d\omega}{dt}$; $I = \frac{1}{2}MR^2$ $\quad (3)$

Multiplying Eq. 1 by R and combining with Eq. 3 gives

$$mgR - mR^2\frac{d\omega}{dt} - \frac{2\pi\mu R^3L}{a}\omega = I\frac{d\omega}{dt} \quad \text{or} \quad mgR - \frac{2\pi\mu R^3L}{a}\omega = (I + mR^2)\frac{d\omega}{dt} \qquad (4)$$

This may be written $A - B\omega = C\frac{d\omega}{dt}$ where $A = mgR$, $B = \frac{2\pi\mu R^3L}{a}$, $C = I + mR^2$

Separating variables $\quad \frac{d\omega}{A - B\omega} = \frac{dt}{C}$

Integrating $\quad \int_0^\omega \frac{d\omega}{A - B\omega} = -\frac{1}{B}\ln(A - B\omega)\Big]_0^\omega = -\frac{1}{B}\ln\left(1 - \frac{B\omega}{A}\right) = \int_0^t \frac{dt}{C} = \frac{t}{C}$

Simplifying $\quad 1 - \frac{B\omega}{A} = e^{-Bt/C}$ or $\omega = \frac{A}{B}\left[1 - e^{-Bt/C}\right]$ $\quad (5)$ $\quad \omega(t)$

The maximum angular speed $(t \to \infty)$ is $\omega = A/B$.

$$A = mgR = 0.010\,kg \times 9.81\,\frac{m}{s^2} \times 0.025\,m \times \frac{N\cdot s^2}{kg\cdot m} = 2.45\times10^{-3}\,N\cdot m$$

$$B = \frac{2\pi\mu R^3L}{a} = 2\pi \times 0.095\,\frac{kg}{m\cdot s} \times (0.025)^3\,m^3 \times 0.050\,m \times \frac{1}{0.0005\,m} \times \frac{N\cdot s^2}{kg\cdot m} = 9.33\times10^{-4}\,N\cdot m\cdot s$$

Evaluating, $\omega_{max} = \frac{A}{B} = 2.45\times10^{-5}\,N\cdot m \times \frac{1}{9.33\times10^{-4}\,N\cdot m\cdot sec} = 2.63$ rad/s.

Thus

$$\omega_{max} = 2.63\,\frac{rad}{s}\times\frac{rev}{2\pi\,rad}\times\frac{60\,s}{min} = 25.1\text{ rpm}$$

ω_{max}

From Eq. 5, $\omega = 0.95\,\omega_{max}$ when $e^{-Bt/C} = 0.05$, or $Bt/C \simeq 3$; $t \simeq \frac{3C}{B}$

$$C = I + mR^2 = \frac{1}{2}MR^2 + mR^2 = \left(\frac{1}{2}M + m\right)R^2$$

$$M = \pi R^2(1.5L + L)\rho = 2.5\pi R^2 L\, SG\, \rho_w$$

$$M = 2.5\pi\times(0.025)^2 m^2 \times 0.050\,m \times (2.64)1000\,\frac{kg}{m^3} = 0.648\text{ kg}$$

$$C = \left(\frac{1}{2}\times0.648\text{ kg} + 0.010\text{ kg}\right)(0.025)^2 m^2 = 2.09\times10^{-4}\text{ kg}\cdot m^2$$

Thus

$$t_{95} = 3\times2.09\times10^{-4}\,kg\cdot m^2 \times \frac{1}{9.33\times10^{-4}\,N\cdot m\cdot s}\times\frac{N\cdot s^2}{kg\cdot m} = 0.671\text{ s}$$

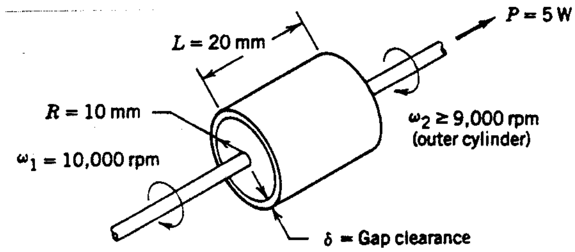
t_{95}

{The terminal speed could have been computed from Eq. 4 by setting $d\omega/dt \to 0$, without solving the differential equation.}

Problem 2.47

Given: Coupling, fabricated of concentric cylinders as shown, must transmit power $\mathcal{P} = 5\,W$. Minimum clearance gap, $\delta = 0.5\,mm$ is to be filled with fluid of viscosity μ. Other dimensions and properties are as indicated.

Find: viscosity of fluid.



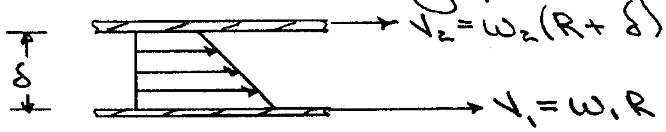
Solution:

Basic equations: $\tau_{r\theta} = \mu \dfrac{du}{dr}$

- shear force, $F = \tau A$
- torque, $T = FR$
- power, $\mathcal{P} = T\omega$

Assumptions:
(1) Newtonian fluid
(2) linear velocity profile in the gap.

Model flow in the gap



$$\tau_{r\theta} = \mu \frac{du}{dr} = \mu \frac{\Delta V}{\Delta r} = \mu \frac{[\omega_1 R - \omega_2(R+\delta)]}{\delta}$$

$$\tau_{r\theta} \approx \mu \frac{(\omega_1 - \omega_2)R}{\delta} \quad \{\delta \ll R\}$$

For the output

$$\mathcal{P} = T\omega_2 = \omega_2 F R = \omega_2 \tau A_2 R = \omega_2 \frac{\mu(\omega_1 - \omega_2)R}{\delta} \times 2\pi R L \times R$$

$$\mathcal{P} = \frac{2\pi \mu \omega_2 (\omega_1 - \omega_2) R^3 L}{\delta}$$

Solving for the viscosity,

$$\mu = \frac{\mathcal{P}\delta}{2\pi \omega_2 (\omega_1 - \omega_2) R^3 L}$$

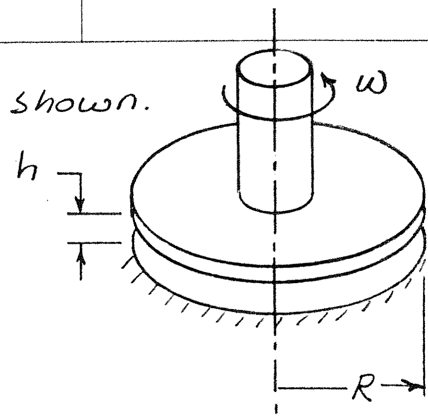
$$= \frac{5\,W \times 5\times10^{-4}\,m}{2\pi} \times \frac{min}{9000\,rev} \times \frac{min}{1000\,rev} \times \frac{1}{(0.01)^3\,m^3} \times \frac{1}{0.02\,m} \times \frac{N\cdot m}{s\cdot W} \times (2\pi)^2 \frac{rev^2}{rad^2} \times \frac{3600\,s^2}{min^2}$$

$$\mu = 0.202\ N\cdot s/m^2 \quad \mu$$

{This viscosity corresponds to SAE 30 oil at 30°C}

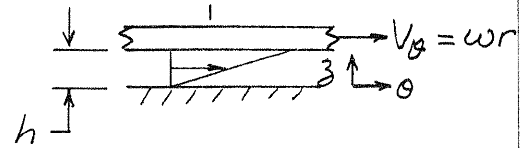
Problem 2.48

Given: Parallel-disk apparatus as shown.



Find: (a) Algebraic expression for shear stress at any radial location.

(b) Expression for the torque needed to turn the upper disk.



Solution: Use r, θ, z coordinates at right:

Basic equations: $\tau_{z\theta} = \mu \frac{dV_\theta}{dz}$

$$dT = r\,dF = r\,\tau_{z\theta}\,dA$$

Assumptions: (1) Newtonian fluid

(2) No-slip condition

(3) Linear velocity profile (in narrow gap)

The velocity at any radial location on the rotating disk is $V_\theta = \omega r$.

Since the velocity profile is linear, then

$$\tau_{z\theta} = \mu \frac{dV_\theta}{dz} = \mu \frac{\Delta V}{\Delta z} = \mu \frac{(\omega r - 0)}{(h - 0)} = \frac{\mu \omega r}{h}$$ $\tau_{z\theta}$

and

$$dT = r\,\tau_{z\theta}\,dA = r \mu \frac{\omega r}{h} 2\pi r\,dr = \frac{2\pi \mu \omega r^3}{h} dr$$

Integrating

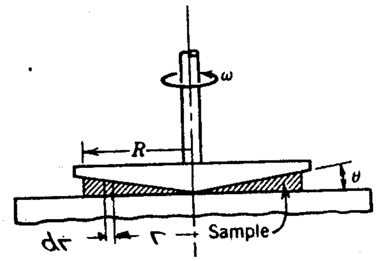
$$T = \int_A dT = \int_0^R \frac{2\pi \mu \omega r^3}{h} dr = \frac{\pi \mu \omega r^4}{2h}\Bigg]_0^R$$

$$T = \frac{\pi \mu \omega R^4}{2h}$$ T

The device could not be used to measure the viscosity of a non-Newtonian fluid because the applied shear stress is not uniform. It varies from zero at the center of the disks to $\mu\omega R/h$ at the edge

Problem 2.49

Given: Cone and plate viscometer shown. Apex of cone just touches the plate, θ is very small

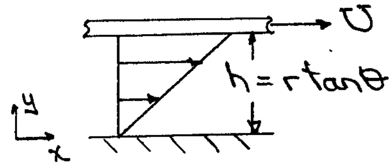


Find: (a) Derive an expression for the shear rate in the liquid that fills the gap
(b) Evaluate the torque on the driven cone in terms of the shear stress and geometry of the system.

Solution:

Since the angle θ is very small, the average gap width is also very small.

It is reasonable to assume a linear velocity profile across the gap and to neglect end effects.



The shear (deformation) rate is given by

$$\dot{\gamma} = \frac{du}{dy} = \frac{\Delta u}{\Delta y}$$

At any radius, r,

the velocity $U = \omega r$ and

the gap width $h = r \tan\theta$

$$\therefore \quad \dot{\gamma} = \frac{\omega r}{r \tan\theta} = \frac{\omega}{\tan\theta}$$

Since θ is very small, $\tan\theta \approx \theta$ and

$$\dot{\gamma} = \frac{\omega}{\theta} \qquad \dot{\gamma}$$

Note: The shear rate is independent of r. The entire sample is subjected to the same shear rate.

The torque on the driven cone is given by

$$T = \int r\,dF \qquad \text{where } dF = \tau_{yx}\,dA$$

Since $\dot{\gamma}$ is a constant (for a given ω) then τ_{yx} = constant

and

$$T = \int r\,dF = \int_A r\,\tau_{yx}\,dA = \tau_{yx}\int_0^R r\,2\pi r\,dr$$

$$T = \frac{2}{3}\pi R^3 \tau_{yx} \qquad T$$

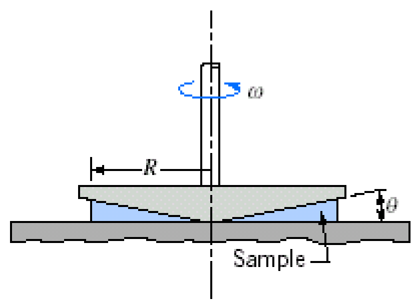
Problem 2.50

The viscometer of Problem 2.49 is used to measure the apparent viscosity of a fluid. The data below are obtained. What kind of non-Newtonian fluid is this? Find the values of k and n used in Eqs. 2.11 and 2.12 in defining the apparent viscosity of a fluid. (Assume θ is 0.5 degrees.) Predict the viscosity at 90 and 100 rpm.

Speed (rpm)	10	20	30	40	50	60	70	80
$\mu(\mathrm{N \cdot s/m^2})$	0.121	0.139	0.153	0.159	0.172	0.172	0.183	0.185

Given: Data from viscometer

Find: The values of coefficients k and n; determine the kind of non-Newtonial fluid it is; estimate viscosity at 90 and 100 rpm



Solution

The velocity gradient at any radius r is $$\frac{du}{dy} = \frac{r \cdot \omega}{r \cdot \tan(\theta)}$$

where ω (rad/s) is the angular velocity $$\omega = \frac{2 \cdot \pi \cdot N}{60}$$ where N is the speed in rpm

For small θ, tan(θ) can be replace with θ, so $$\frac{du}{dy} = \frac{\omega}{\theta}$$

From Eq 2.11. $$k \cdot \left(\left|\frac{du}{dy}\right|\right)^{n-1} \frac{du}{dy} = \eta \cdot \frac{du}{dy}$$

where η is the apparent viscosity. Hence $$\eta = k \cdot \left(\frac{du}{dy}\right)^{n-1} = k \cdot \left(\frac{\omega}{\theta}\right)^{n-1}$$

The data in the table conform to this equation. The corresponding *Excel* workbook shows how *Excel*'s *Trendline* analysis is used to fit the data.

From *Excel*

$k = 0.0449$ $n = 1.21$

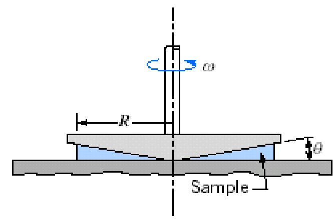
$\eta(90 \cdot rpm) = 0.191 \cdot \frac{N \cdot s}{m^2}$ $\eta(100 \cdot rpm) = 0.195 \cdot \frac{N \cdot s}{m^2}$

For $n > 1$ the fluid is dilatant

Problem 2.50 (In Excel)

The viscometer of Problem 2.49 is used to measure the apparent viscosity of a fluid. The data below are obtained. What kind of non-Newtonian fluid is this? Find the values of k and n used in Eqs. 2.11 and 2.12 in defining the apparent viscosity of a fluid. (Assume θ is 0.5 degrees.) Predict the viscosity at 90 and 100 rpm.

Speed (rpm)	10	20	30	40	50	60	70	80
$\mu(\mathrm{N \cdot s/m^2})$	0.121	0.139	0.153	0.159	0.172	0.172	0.183	0.185



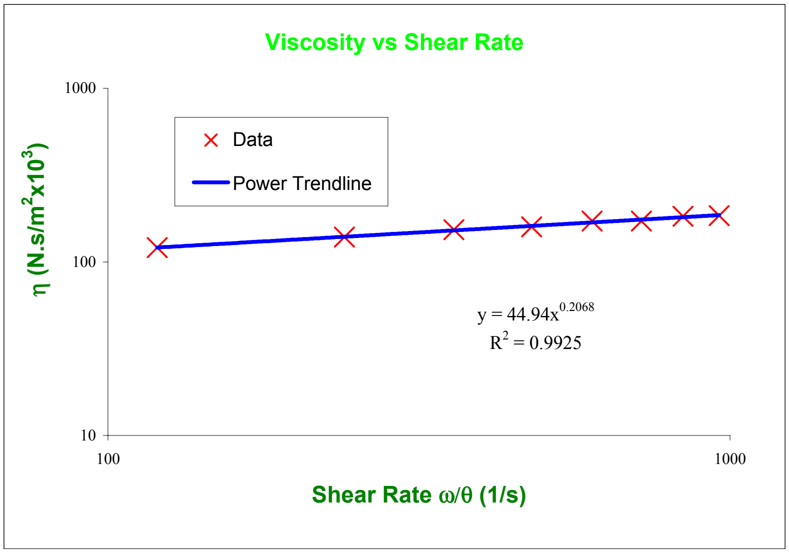
Solution

The data is

N (rpm)	μ (N.s/m^2)
10	0.121
20	0.139
30	0.153
40	0.159
50	0.172
60	0.172
70	0.183
80	0.185

The computed data is

ω (rad/s)	ω/θ (1/s)	η (N.s/m^{2}x10^3)
1.047	120	121
2.094	240	139
3.142	360	153
4.189	480	159
5.236	600	172
6.283	720	172
7.330	840	183
8.378	960	185



From the *Trendline* analysis

k = 0.0449

n - 1 = 0.2068

n = 1.21 The fluid is dilatant

The apparent viscosities at 90 and 100 rpm can now be computed

N (rpm)	ω (rad/s)	ω/θ (1/s)	η (N.s/m^{2}x10^3)
90	9.42	1080	191
100	10.47	1200	195

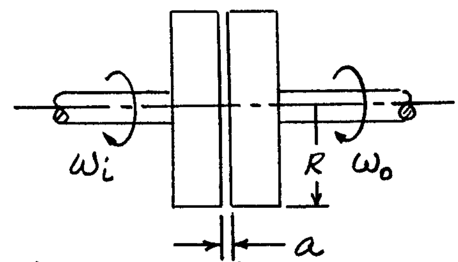
Problem 2.51

Given: Viscous clutch made from pair of closely spaced disks.

Input speed, ω_i

Output speed, ω_o

Viscous oil in gap, μ



Find algebraic expressions in terms of μ, R, a, ω_i, and ω_o for:

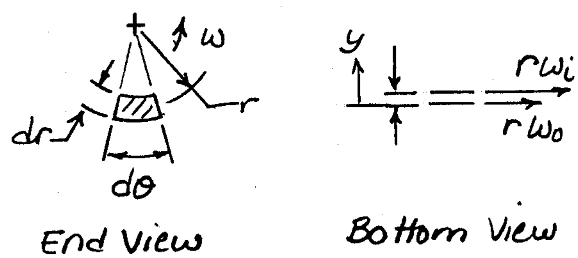
(a) Torque transmitted, T
(b) Power transmitted
(c) Slip ratio, $s = \Delta\omega/\omega_i$, in terms of T
(d) Efficiency, η, in terms of s, ω_i, and T

Solution: Apply Newton's law of viscosity

Basic equations: $\tau = \mu \frac{du}{dy}$ $\quad dF = \tau dA \quad dT = r dF$

Assumptions: (1) Newtonian liquid
(2) Narrow gap so velocity profile is linear

Consider a segment of plates:



$$\tau = \mu \frac{du}{dy} = \mu \frac{\Delta u}{\Delta y} = \mu \frac{r(\omega_i - \omega_o)}{a}$$

$$dA = r\,dr\,d\theta$$

$$dF = \tau dA = \frac{\mu r \Delta\omega}{a} r\,dr\,d\theta = \frac{\mu\Delta\omega}{a} r^2\,dr\,d\theta \; ; \; dT = r\,dF = \frac{\mu\Delta\omega}{a} r^3\,dr\,d\theta$$

Integrating

$$T = \int_0^{2\pi}\int_0^R dT = \frac{\mu\Delta\omega}{a}\int_0^{2\pi}\int_0^R r^3\,dr\,d\theta = \frac{2\pi\mu\Delta\omega}{a}\int_0^R r^3\,dr = \frac{\pi\mu\Delta\omega R^4}{2a}$$ T

$$P_o = T\omega_o = \frac{\pi\mu\omega_o\Delta\omega R^4}{2a} \quad \text{(power transmitted)}$$ P

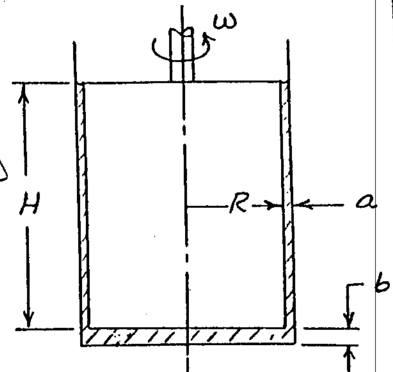
$$s = \frac{\Delta\omega}{\omega_i} = \frac{2aT}{\pi\mu R^4 \omega_i}$$ s

Efficiency is $\eta = \frac{\text{Power out}}{\text{Power in}} = \frac{T\omega_o}{T\omega_i} = \frac{\omega_o}{\omega_i}$. But $\omega_o = \omega_i - \Delta\omega$, so

$$\eta = \frac{\omega_i - \Delta\omega}{\omega_i} = 1 - \frac{\Delta\omega}{\omega_i} = 1 - s$$ η

Problem 2.52

Given: Concentric-cylinder viscometer shown. When inner cylinder rotates at angular speed ω viscous retarding torque arises around circumference of inner cylinder and on cylinder bottom.

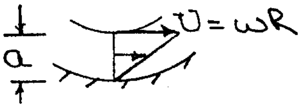


Find: (a) expression for viscous torque due to gap of width, a
(b) expression for viscous torque on bottom due to gap of width b
(c) For $T_{bottom}/T_{annulus} \le 0.01$, plot b/a vs geometric variables.
(d) What are design implications?
(e) What design modifications can you recommend?

Solution: Basic equation $\tau_{yx} = \mu \dfrac{du}{dy}$

Assumptions: (1) linear velocity profile (2) Newtonian liquid

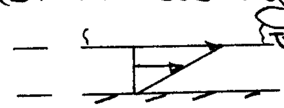
(a) in annular gap



$$\tau = \mu \frac{du}{dr} = \mu \frac{\Delta u}{\Delta r} = \mu \frac{U}{a} = \mu \frac{\omega R}{a}$$

$$\text{Torque} = RF_t = R\tau A = R\mu \frac{\omega R}{a}(2\pi R H) = \frac{2\pi \mu \omega R^3 H}{a} \quad \text{(a)}$$

(b) in bottom gap



$$\tau = \mu \frac{du}{dz} = \mu \frac{\Delta u}{\Delta z} = \mu \frac{U}{a} = \mu \frac{\omega r}{b} \quad \text{(varies with } r\text{)}$$

$$\text{Torque} = \int dT = \int r\, dF = \int r\tau\, dA = \int_0^R r \mu \frac{\omega r}{b} 2\pi r\, dr$$

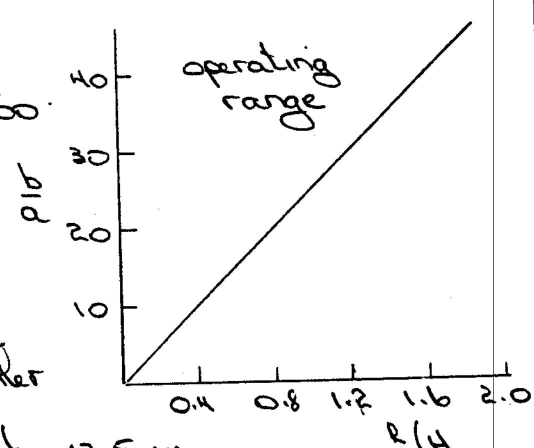
$$\text{Torque} = \frac{2\pi\mu\omega}{b}\int_0^R r^3\, dr = \frac{2\pi\mu\omega}{b}\left[\frac{r^4}{4}\right]_0^R = \frac{\pi\mu\omega}{2b}R^4 \quad \text{(b)}$$

(c) For $T_{bottom}/T_{annulus} \le \frac{1}{100}$, then

$$\frac{T_{bot}}{T_{an}} = \frac{\pi\mu\omega}{2b}R^4 \times \frac{a}{2\pi\mu\omega R^3 H} \le \frac{1}{100}$$

$$\frac{aR}{4bH} \le \frac{1}{100}$$

or $$\frac{b}{a} \ge 25\frac{R}{H}$$



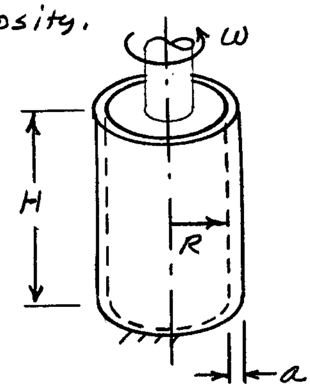
(d) The plot shows the operating range. Specific design would depend on other constraints.
For a = 1mm with R/H = 1/2 gives b = 12.5mm

(e) For a given value of R/H, the dimension b could be effectively increased by "hollowing out" the inner cylinder as shown by the dashed lines in the diagram above.

Problem 2.53

Given: Concentric-cylinder viscometer, liquid similar to water. Goal is to obtain ±1 percent accuracy in viscosity value.

Specify: Configuration and dimensions to achieve ±1% measurement. Parameter to be measured to compute viscosity.

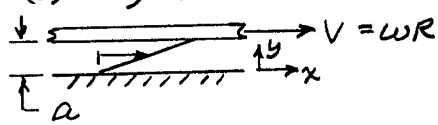


Solution: Apply definition of Newtonian fluid

Computing equation: $\tau = \mu \frac{du}{dy}$

Assumptions:
(1) Steady
(2) Newtonian liquid
(3) Narrow gap, so "unroll" it
(4) Linear velocity profile in gap
(5) Neglect end effects

Flow model:



$u = V\frac{y}{a} = \omega R \frac{y}{a}\ ;\ \frac{du}{dy} = \frac{\omega R}{a}$

Thus $\tau = \mu \frac{du}{dy} = \mu \frac{\omega R}{a}$ and torque on rotor is $T = R\tau A$, where $A = 2\pi R H$

Consequently $T = R\mu \frac{\omega R}{a} 2\pi R H = \frac{2\pi \mu \omega R^3 H}{a}$, or

$$\mu = \frac{Ta}{2\pi \omega R^3 H}$$ μ

From this equation the uncertainty in μ is (see Appendix F),

$$u_\mu = \pm\left[u_T^2 + u_a^2 + u_\omega^2 + (3u_R)^2 + u_H^2\right]^{\frac{1}{2}} = \pm\left[13\,u^2\right]^{\frac{1}{2}} = \pm 3.61\,u$$

if the uncertainty of each parameter equals u. Thus

$$u = \pm\frac{u_\mu}{3.61} = \pm\frac{1 \text{ percent}}{3.61} = \pm 0.277 \text{ percent}$$ u

Typical dimensions for a bench-top unit might be

$H = 200$ mm, $R = 75$ mm, $a = 0.02$ mm, and $\omega = 10.5$ rad/s (100 rpm)

From Appendix A, Table A.8, water has $\mu = 1.00\times10^{-3}$ N·s/m² at $T = 20°$C.

The corresponding torque would be

$$T = 2\pi \times 1.00\times10^{-3}\frac{\text{N}\cdot\text{s}}{\text{m}^2} \times \frac{10.5}{\text{s}} \times (0.075)^3\text{m}^3 \times 0.2\,\text{m} \times \frac{1}{0.00002\,\text{m}} = 0.278 \text{ N}\cdot\text{m}$$ T

It should be possible to measure this torque quite accurately.

{Many details would need to be considered (e.g. bearings, temperature rise, etc.) to produce a workable device.}

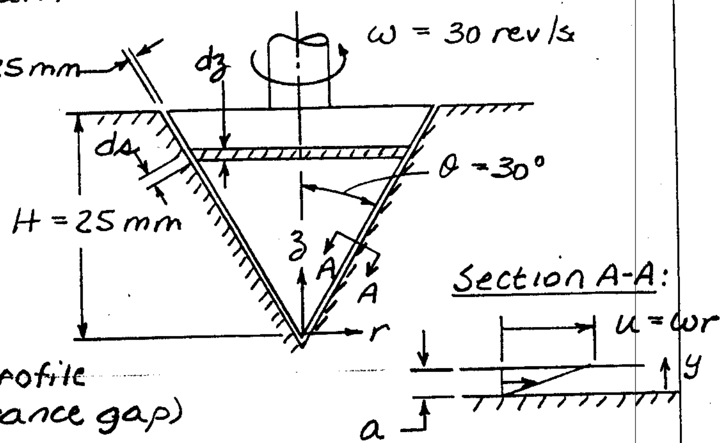
Problem 2.54

Given: Conical pointed shaft turning in conical bearing.

Lubricant is heavy oil with viscosity of SAE 30 at 30°C.

Find: (a) Algebraic expression for shear stress at height, z.
(b) Torque that acts on shaft.

Solution: Basic equations:



$$\tau = \mu \frac{du}{dy}$$

$$dT = r\tau dA$$

Assumptions: (1) Newtonian fluid
(2) No-slip condition
(3) Linear velocity profile
(in narrow clearance gap)

Along the conical surface, $\tan\theta = \frac{r}{z}$, so $r = z\tan\theta$

Then $\tau = \mu \frac{du}{dy} = \mu \frac{\Delta u}{\Delta y} = \mu \frac{(\omega r - 0)}{(a - 0)} = \frac{\mu \omega z \tan\theta}{a}$ τ

Consider the cross-hatched element of area: $dz = ds\cos\theta$

$$dA = 2\pi r\, ds = 2\pi r \frac{dz}{\cos\theta}$$

The viscous torque on the element of area is:

$$dT = r\tau dA = r\frac{\mu\omega z\tan\theta}{a} 2\pi r \frac{dz}{\cos\theta} \quad ; \; r = z\tan\theta$$

$$dT = \frac{2\pi\mu\omega z^3 \tan^3\theta}{a\cos\theta} dz$$

Integrating

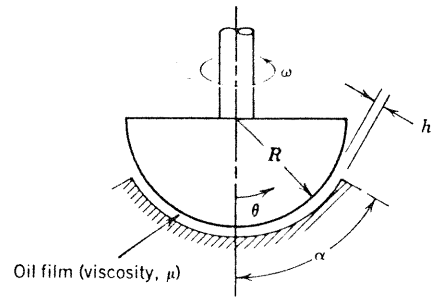
$$T = \int_A dT = \int_0^H \frac{2\pi\mu\omega\tan^3\theta}{a\cos\theta} z^3 dz = \frac{2\pi\mu\omega\tan^3\theta}{a\cos\theta} \frac{z^4}{4}\Big]_0^H$$

$$T = \frac{\pi\mu\omega\tan^3\theta\, H^4}{2a\cos\theta} \qquad (\mu \approx 0.2\ N\cdot s/m^2 \text{ from Fig. A.2})$$

$$= \frac{\pi}{2} \times 0.2 \frac{N\cdot s}{m^2} \times 30 \frac{rev}{s} \times \tan^3 30° \times (0.025)^4 m^4 \times \frac{1}{0.25\times10^{-3} m} \times \frac{1}{\cos 30°} \times 2\pi \frac{rad}{rev}$$

$$T = 0.0206\ N\cdot m$$

Given: Spherical thrust bearing shown:



Find: Obtain and plot an algebraic expression for the torque on the spherical member, as a function of α.

Solution: Apply definitions

Computing equations: $\tau = \mu \frac{du}{dy}$ $\quad T = \int_A r\tau \, dA$

Assumptions: (1) Newtonian fluid, (2) Narrow gap, (3) Laminar flow

From the figure, $r = R\sin\theta \quad u = \omega r = \omega R \sin\theta$

$$\tau = \mu\frac{du}{dy} = \mu\left(\frac{u-0}{h}\right) = \mu\frac{u}{h} = \mu\frac{\omega R\sin\theta}{h}$$

$$dA = 2\pi r R\,d\theta = 2\pi R^2 \sin\theta\, d\theta$$

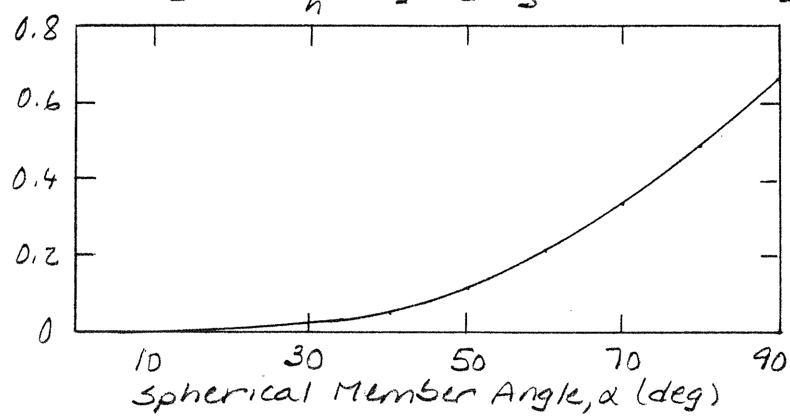
Thus

$$T = \int_0^\alpha R\sin\theta\left(\frac{\mu\omega R\sin\theta}{h}\right) 2\pi R^2 \sin\theta\, d\theta = \frac{2\pi\mu\omega R^4}{h}\int_0^\alpha \sin^3\theta\, d\theta$$

$$T = \frac{2\pi\mu\omega R^4}{h}\left[\frac{\cos^3\theta}{3} - \cos\theta\right]_0^\alpha = \frac{2\pi\mu\omega R^4}{h}\left[\frac{\cos^3\alpha}{3} - \cos\alpha + \frac{2}{3}\right]$$ T

To plot, normalize to $\left[T\Big/\frac{2\pi\mu\omega R^4}{h}\right] = \left[\frac{\cos^3\alpha}{3} - \cos\alpha + \frac{2}{3}\right]$

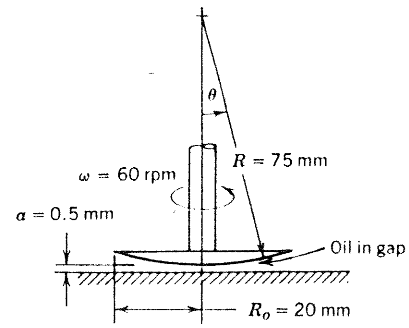
Plotting:



$\left\{\text{Check dimensions: } \left[\frac{\mu\omega R^4}{h}\right] = \frac{Ft}{L^2} \times \frac{1}{t} \times L^4 \times \frac{1}{L} = FL \ \checkmark\checkmark\right\}$

Given: Rotating bearing shown:

Narrow gap filled with viscous oil, $\mu = 1250$ cp.



Find: (a) Algebraic expression for shear stress on spherical member.
(b) Find maximum shear stress.
(c) Algebraic expression for viscous torque on spherical member.
(d) Evaluate torque.

Solution: Apply definitions

Computing equations: $\tau = \mu \frac{du}{dy}$ $\quad T = \int_A r \tau \, dA$

Assumptions: (1) Newtonian fluid, (2) Narrow gap, (3) Laminar motion

From the figure, $r = R\sin\theta$, $u = \omega r = \omega R \sin\theta$, $\frac{du}{dy} = \frac{u-0}{h} = \frac{u}{h}$

$$h = a + R(1-\cos\theta) \qquad dA = 2\pi r\,dr = 2\pi R\sin\theta R\cos\theta d\theta$$

Thus $\tau = \dfrac{\mu \omega R \sin\theta}{a + R(1-\cos\theta)}$ ← τ

From the table below, $\tau_{max} = 67.9 \text{ N/m}^2$ at $\theta = 6.5°$ (not at R_o) ← τ_{max}

Torque is $T = \displaystyle\int_0^{\theta_{max}} \frac{\mu \omega R^4 \sin^2\theta \cos\theta}{a + R(1-\cos\theta)} d\theta$ ← T

This must be evaluated numerically or graphically. From Appendix G, 1 Poise = 0.1 kg/m·s. Thus $\mu = 1.25$ kg/m·s $= 1.25$ N·s/m²

$$\tau = 1.25 \frac{\text{N·s}}{\text{m}^2} \times 2\pi \frac{\text{rad}}{\text{s}} \times 0.075 \text{ m} \times \sin 6.5° \times \frac{1}{0.0005 + 0.075(1-\cos 6.5°)\text{m}} = 67.9 \text{ N/m}^2$$

Tabulating results of similar calculations gives:

theta (deg)	tau (N/m^2)	function (---)	torque (N-m)	sumt (N-m)
0.5	10.2	0.15	4.13E-06	4.13E-06
1.5	29.3	1.30	3.55E-05	3.96E-05
2.5	45.0	3.33	9.07E-05	1.30E-04
3.5	56.2	5.81	1.58E-04	2.89E-04
4.5	63.2	8.39	2.29E-04	5.17E-04
5.5	66.8	10.82	2.95E-04	8.12E-04
6.5	67.9	12.96	3.53E-04	1.17E-03
7.5	67.3	14.80	4.03E-04	1.57E-03
8.5	65.8	16.32	4.45E-04	2.01E-03
9.5	63.6	17.58	4.79E-04	2.49E-03
10.5	61.1	18.60	5.07E-04	3.00E-03
11.5	58.5	19.42	5.29E-04	3.53E-03
12.5	56.0	20.08	5.47E-04	4.08E-03
13.5	53.5	20.60	5.61E-04	4.64E-03
14.5	51.0	21.01	5.72E-04	5.21E-03
15.5	48.8	21.32	5.81E-04	5.79E-03

Integrated torque = 5.50E-03 N-m ← T

Here "function" is $\dfrac{\sin^2\theta\cos\theta}{a + R(1-\cos\theta)}$

and $\Delta\theta = 1$ deg $= 0.0175$ rad

for the numerical integration.

$$\theta_{max} = \sin^{-1}\frac{R_o}{R} = \sin^{-1}\left(\frac{20}{75}\right) = 15.5°$$

Problem 2.57

Given: Small gas bubbles form in soda when opened; $D = 0.1\ mm$.

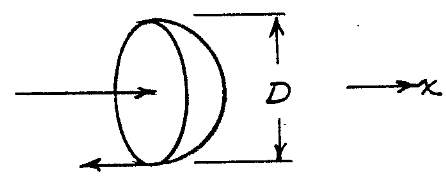
Find: Estimate pressure difference from inside to outside such a bubble.

Solution: Consider a free-body diagram of half a bubble:

Two forces act:

Pressure: $F_p = \Delta p \dfrac{\pi D^2}{4}$

Surface tension: $F_\sigma = \sigma \pi D$



Summing forces for equilibrium

$$\Sigma F_x = F_p - F_\sigma = \Delta p \frac{\pi D^2}{4} - \sigma \pi D = 0$$

so $\dfrac{\Delta p D}{4} - \sigma = 0$ or $\Delta p = \dfrac{4\sigma}{D}$

Assuming soda-gas interface is similar to water-air, then $\sigma = 72.8\ mN/m$, and

$$\Delta p = 4 \times 72.8 \times 10^{-3} \frac{N}{m} \times \frac{1}{0.1 \times 10^{-3}\ m} = 2.91 \times 10^{3} \frac{N}{m^2} = 2.41\ kPa$$ Δp

Problem 2.58

You intend to gently place several steel needles on the free surface of the water in a large tank. The needles come in two lengths: Some are 5 cm long, and some are 10 cm long. Needles of each length are available with diameters of 1 mm, 2.5 mm, and 5 mm. Make a prediction as to which needles, if any, will float.

Given: Data on size of various needles

Find: Which needles, if any, will float

Solution

For a steel needle of length L, diameter D, density ρ_s, to float in water with surface tension σ a contact angle θ, the vertical force due to surface tension must equal or exceed the weight

$$2 \cdot L \cdot \sigma \cdot \cos(\theta) \geq W = m \cdot g = \frac{\pi \cdot D^2}{4} \cdot \rho_s \cdot L \cdot g$$

or $$D \leq \sqrt{\frac{8 \cdot \sigma \cdot \cos(\theta)}{\pi \cdot \rho_s \cdot g}}$$

From Table A.4 $\sigma = 72.8 \cdot \frac{mN}{m}$ $\theta = 0 \cdot deg$ and for water $\rho = 999 \cdot \frac{kg}{m^3}$

From Table A.1, for steel $SG = 7.83$

Hence

$$\sqrt{\frac{8 \cdot \sigma \cdot \cos(\theta)}{\pi \cdot SG \cdot \rho \cdot g}} = \sqrt{\frac{8}{\pi \cdot 7.83} \times 72.8 \times 10^{-3} \cdot \frac{N}{m} \times \frac{m^3}{999 \cdot kg} \times \frac{s^2}{9.81 \cdot m} \times \frac{kg \cdot m}{N \cdot s^2}} = 1.55 \times 10^{-3} \cdot m$$

Hence $D < 1.55$ mm. Only the 1 mm needles float (needle length is irrelevant)

Problem 2.59

Open-Ended Problem Statement: Slowly fill a glass with water to the maximum possible level before it overflows. Observe the water level closely. Explain how it can be higher than the rim of the glass.

Discussion: Surface tension can cause the maximum water level in a glass to be higher than the rim of the glass. The same phenomenon causes an isolated drop of water to "bead up" on a smooth surface.

Surface tension between the water/air interface and the glass acts as an invisible membrane that allows trapped water to rise above the level of the rim of the glass. The mechanism can be envisioned as forces that act in the surface of the liquid above the rim of the glass. Thus the water appears to defy gravity by attaining a level higher than the rim of the glass.

To experimentally demonstrate that this phenomenon is the result of surface tension, set the liquid level nearly as far above the glass rim as you can get it, using plain water. Add a drop of liquid detergent (the detergent contains additives that reduce the surface tension of water). Watch as the excess water runs over the side of the glass.

Problem 2.60

Open-Ended Problem Statement: Plan an experiment to measure the surface tension of a liquid similar to water. If necessary, review the NCFMF video *Surface Tension* for ideas. Which method would be most suitable for use in an undergraduate laboratory? What experimental precision could be expected?

Discussion: Two basic kinds of experiment are possible for an undergraduate laboratory:

(1) Using a clear small-diameter tube, compare the capillary rise of the unknown liquid with that of a known liquid (compare with water, because it is similar to the unknown liquid).

This method would be simple to set up and should give fairly accurate results. A vertical traversing optical microscope could be used to increase the precision of measuring the liquid height in each tube.

A drawback to this method is that the specific gravity and contact angle of the two liquids must be the same to allow the capillary rises to be compared.

The capillary rise would be largest and therefore easiest to measure accurately in a tube with the smallest practical diameter. Tubes of several diameters could be used if desired.

(2) Dip an object into a pool of test liquid and measure the vertical force required to pull the object from the liquid surface.

The object might be made rectangular (e.g., a sheet of plastic material) or circular (e.g., a metal ring). The net force* needed to pull the same object from each liquid should be proportional to the surface tension of each liquid.

This method would be simple to set up. However, the force magnitudes to be measured would be quite small.

A drawback to this method is that the contact angles of the two liquids must be the same.

The first method is probably best for undergraduate laboratory use. A quantitative estimate of experimental measurement uncertainty is impossible without knowing details of the test setup. It might be reasonable to expect results accurate to within ±10% of the true surface tension.

---

* Net force is the total vertical force minus the weight of the object. A buoyancy correction would be necessary if part of the object were submerged in the test liquid.

Problem 2.61

Given: Water, with bulk modulus assumed constant.

Find: (a) Percent change in density at 100 atm

(b) Plot percent change vs. p/p_{atm} up to 50,000 psi.

(c) Comment on assumption of constant density.

<u>Solution</u>: By definition, $E_v = \dfrac{dp}{d\rho/\rho}$. Assume E_v = Constant. Then

$$\frac{d\rho}{\rho} = \frac{dp}{E_v}$$

Integrating, from ρ_0 to ρ gives $\ln\dfrac{\rho}{\rho_0} = \dfrac{p-p_0}{E_v} = \dfrac{\Delta p}{E_v}$, so $\dfrac{\rho}{\rho_0} = e^{\Delta p/E_v}$

The relative change in density is

$$\frac{\Delta\rho}{\rho_0} = \frac{\rho-\rho_0}{\rho_0} = \frac{\rho}{\rho_0} - 1 = e^{\Delta p/E_v} - 1$$

From Table A.2, $E_v = 2.24$ GPa for water at 20°C.

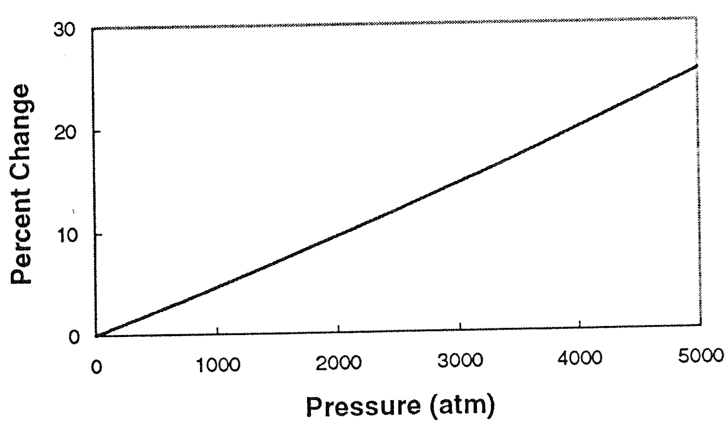
For $p = 100$ atm (gage), $\Delta p = 100$ atm, so

$$\frac{\Delta\rho}{\rho_0} = \exp\left(100\ \text{atm} \times \frac{1}{2.24\times10^9\ \text{Pa}} \times 101.325\times10^3\ \frac{\text{Pa}}{\text{atm}}\right) - 1 = 0.00453, \text{ or } 0.453\%$$

For $\Delta p = 50{,}000$ psi,

$$\frac{\Delta\rho}{\rho_0} = \exp\left(50{,}000\ \text{psi} \times \frac{1}{2.24\times10^9\ \text{Pa}} \times \frac{101.325\times10^3\ \text{Pa}}{14.696\ \text{psi}}\right) - 1 = 0.166 \text{ or } 16.6\%$$

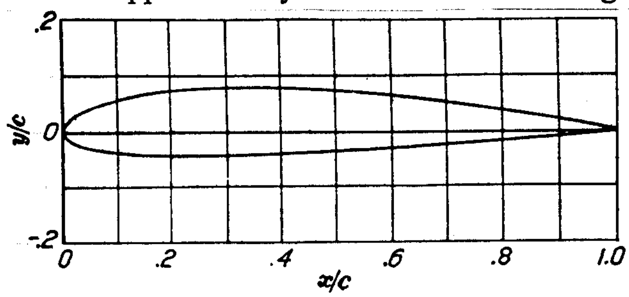
Thus constant density is not a reasonable assumption for a cutting jet operating at 50,000 psi. Constant density (5% change) would be reasonable up to $\Delta p \approx 16{,}000$ psi.



Problem 2.62

Open-Ended Problem Statement: How does an airplane wing develop lift?

Discussion: The sketch shows the cross-section of a typical airplane wing. The airfoil section is rounded at the front, curved across the top, reaches maximum thickness about a third of the way back, then tapers slowly to a fine trailing edge. The bottom of the airfoil section is relatively flat. (The discussion below also applies to a symmetric airfoil at an angle of incidence that produces lift.)



NACA 2412 Wing Section

It is both a popular expectation and an experimental fact that air flows more rapidly over the curved top surface of the airfoil section than along the relatively flat bottom. In the NCFMF video *Flow Visualization*, timelines placed in front of the airfoil indicate that fluid flows more rapidly along the top of the section than along the bottom.

In the absence of viscous effects (this is a valid assumption outside the boundary layers on the airfoil) pressure falls when flow speed increases. Thus the pressures on the top surface of the airfoil where flow speed is higher are lower than the pressures on the bottom surface where flow speed does not increase. (Actual pressure profiles measured for a lifting section are shown in the NCFMF video *Boundary Layer Control*.) The unbalanced pressures on the top and bottom surfaces of the airfoil section create a net force that tends to develop lift on the profile.

Problem 3.1

D = 0.75 m. The gas is at an absolute pressure of 25 MPa and a temperature of 25°C. What is the mass in the tank? If the maximum allowable wall stress in the tank is 210 MPa, find the minimum theoretical wall thickness of the tank.

Given: Data on nitrogen tank

Find: Mass of nitrogen; minimum required wall thickness

Solution

Assuming ideal gas behavior: $p \cdot V = M \cdot R \cdot T$

where, from Table A.6, for nitrogen $R = 297 \cdot \frac{J}{kg \cdot K}$

Then the mass of nitrogen is $M = \frac{p \cdot V}{R \cdot T} = \frac{p}{R \cdot T} \cdot \left(\frac{\pi \cdot D^3}{6}\right)$

$$M = \frac{25 \cdot 10^6 \cdot N}{m^2} \times \frac{kg \cdot K}{297 \cdot J} \times \frac{1}{298 \cdot K} \times \frac{J}{N \cdot m} \times \frac{\pi \cdot (0.75 \cdot m)^3}{6}$$

$$M = 62\,kg$$

To determine wall thickness, consider a free body diagram for one hemisphere:

$$\Sigma F = 0 = p \cdot \frac{\pi \cdot D^2}{4} - \sigma_c \cdot \pi \cdot D \cdot t$$

where σ_c is the circumferential stress in the container

Then

$$t = \frac{p \cdot \pi \cdot D^2}{4 \cdot \pi \cdot D \cdot \sigma_c} = \frac{p \cdot D}{4 \cdot \sigma_c}$$

$$t = 25 \cdot 10^6 \cdot \frac{N}{m^2} \times \frac{0.75 \cdot m}{4} \times \frac{1}{210 \cdot 10^6} \cdot \frac{m^2}{N}$$

$t = 0.0223\,m$ $t = 22.3\,mm$

Problem 3.2

Ear "popping" is an unpleasant phenomenon sometimes experienced when a change in pressure occurs, for example in a fast-moving elevator or in an airplane. If you are in a two-seater airplane at 3000 m and a descent of 100 m causes your ears to "pop," what is the pressure change that your ears "pop" at, in millimeters of mercury? If the airplane now rises to 8000 m and again begins descending, how far will the airplane descend before your ears "pop" again? Assume a U.S. Standard Atmosphere.

Given: Data on flight of airplane

Find: Pressure change in mm Hg for ears to "pop"; descent distance from 8000 m to cause ears to "pop."

Solution

Assume the air density is approximately constant constant from 3000 m to 2900 m.
From table A.3

$$\rho_{air} = 0.7423 \cdot \rho_{SL} = 0.7423 \times 1.225 \cdot \frac{kg}{m^3}$$

$$\rho_{air} = 0.909 \frac{kg}{m^3}$$

We also have from the manometer equation, Eq. 3.7

$$\Delta p = -\rho_{air} \cdot g \cdot \Delta z \qquad \text{and also} \qquad \Delta p = -\rho_{Hg} \cdot g \cdot \Delta h_{Hg}$$

Combining

$$\Delta h_{Hg} = \frac{\rho_{air}}{\rho_{Hg}} \cdot \Delta z = \frac{\rho_{air}}{SG_{Hg} \cdot \rho_{H2O}} \cdot \Delta z \qquad SG_{Hg} = 13.55 \text{ from Table A.2}$$

$$\Delta h_{Hg} = \frac{0.909}{13.55 \times 999} \times 100 \cdot m$$

$$\Delta h_{Hg} = 6.72\,mm$$

For the ear popping descending from 8000 m, again assume the air density is approximately con constant, this time at 8000 m.
From table A.3

$$\rho_{air} = 0.4292 \cdot \rho_{SL} = 0.4292 \times 1.225 \cdot \frac{kg}{m^3}$$

$$\rho_{air} = 0.526 \frac{kg}{m^3}$$

We also have from the manometer equation

$$\rho_{air8000} \cdot g \cdot \Delta z_{8000} = \rho_{air3000} \cdot g \cdot \Delta z_{3000}$$

where the numerical subscripts refer to conditions at 3000m and 8000m.
Hence

$$\Delta z_{8000} = \frac{\rho_{air3000} \cdot g}{\rho_{air8000} \cdot g} \cdot \Delta z_{3000} = \frac{\rho_{air3000}}{\rho_{air8000}} \cdot \Delta z_{3000}$$

$$\Delta z_{8000} = \frac{0.909}{0.526} \times 100 \cdot m$$

$$\Delta z_{8000} = 173\,m$$

Given: Pure water on a standard day

Find: Boiling temperature at (a) 1000 m, and (b) 2000 m. Compare with sea level value

Solution

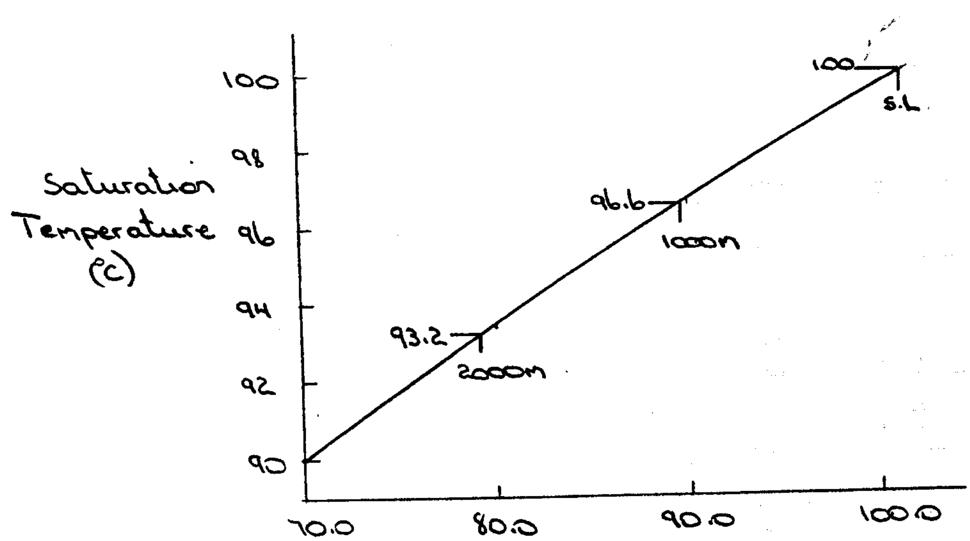
We can determine the atmospheric pressure at the given altitudes from table A.3, Appendix A

Elevation (m)	$\frac{P}{P_0}$	P (kPa)	T_{sat}^* (C)
0	1.000	101	100
1000	0.887	89.6	96.6
2000	0.785	79.3	93.2

* T_{sat} obtained from plot of T_{sat} vs P given below

Data from Steam Tables gives T_{sat}

P (kPa)	T_{sat} (C)
70	90.0
80	93.5
90	96.7
101.325	100.0



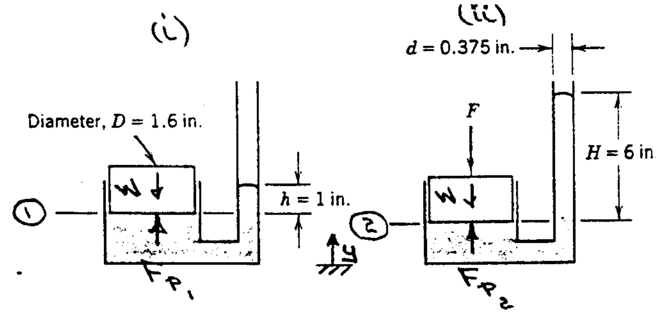
{ These data show that T_{sat} drops about 3.4°C / 1000m }

Problem 3.4

Given: The tube shown is filled with mercury at 20°C

Find: the force applied to the piston

Solution:



Basic equations: $\frac{dp}{dy} = -\rho g$

$\vec{F} = -\int p \, d\vec{A}$

For ρ = constant in a static fluid

$p = p_{atm} - \rho g (y - y_0)$

where $p = p_{atm}$ at $y = y_0$

Then

$p_1 = p_{atm} + \rho g h$ and $F_{p_1} = \rho g h A$ (gage).

For fbd (i) $\sum F_y = 0 = F_{p_1} - W = 0$ and $W = F_{p_1} = \rho g h A$

Also $p_2 = p_{atm}$ and $\rho g H$ and $F_2 = \rho g H A$ (gage).

For fbd (ii) $\sum F_y = 0 = F_{p_2} - W - F = 0$

$\therefore F = F_{p_2} - W = \rho g H A - \rho g h A = \rho g A (H - h)$

$F = \rho_{H_2O} SG \, g \frac{\pi D^2}{4} (H - h)$ From Fig. A.1, App. A, SG = 13.54

$F = 1000 \frac{kg}{m^3} \times 13.54 \times 9.81 \frac{m}{s^2} \times \frac{\pi}{4} (1.6)^2 \, in^2 \, (6-1) \, in \times (0.0254)^3 \frac{m^3}{in^3} \times \frac{N \cdot s^2}{kg \cdot m}$

$F = 21.9 \, N$ ← F

Problem 3.4 (In Excel)

When you are on a mountain face and boil water, you notice that the water temperature is 90°C. What is your approximate altitude? The next day, you are at a location where it boils at 85°C. How high did you climb between the two days? Assume a U.S. Standard Atmosphere.

Given: Boiling points of water at different elevations
Find: Change in elevation

Solution

From the steam tables, we have the following data for the boiling point (saturation temperature) of water

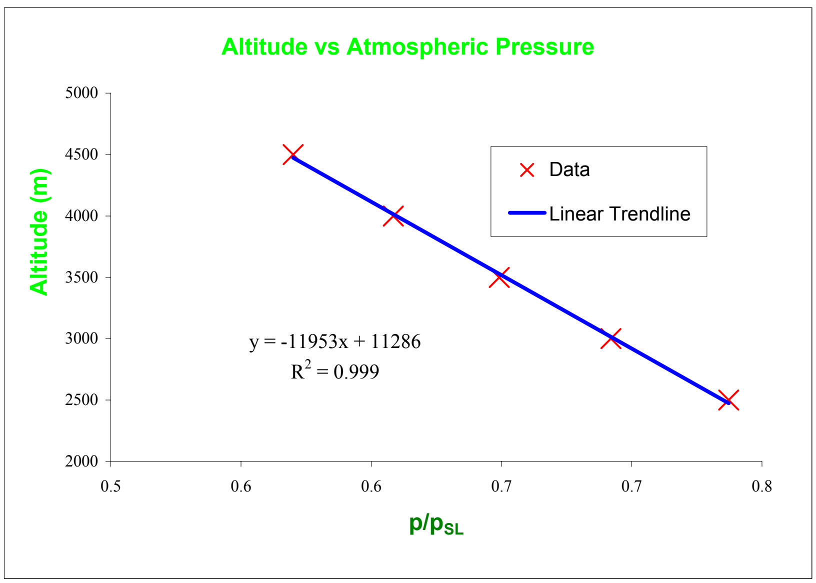
T_{sat} (°C)	p (kPa)
90	70.14
85	57.83

The sea level pressure, from Table A.3, is

p_{SL} = 101 kPa

Hence

T_{sat} (°C)	p/p_{SL}
90	0.694
85	0.573



From Table A.3

p/p_{SL}	Altitude (m)
0.7372	2500
0.6920	3000
0.6492	3500
0.6085	4000
0.5700	4500

Then, any one of a number of *Excel* functions can be used to interpolate
(Here we use *Excel*'s *Trendline* analysis)

p/p_{SL}	Altitude (m)
0.694	2985
0.573	4442

Current altitude is approximately 2980 m

The change in altitude is then 1457 m

Alternatively, we can interpolate for each altitude by using a linear regression between adjacent data points

For

p/p_{SL}	Altitude (m)
0.7372	2500
0.6920	3000

p/p_{SL}	Altitude (m)
0.6085	4000
0.5700	4500

Then

0.6940	2978

0.5730	4461

The change in altitude is then 1483 m

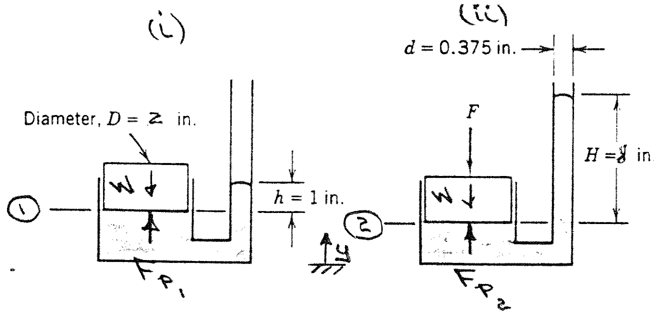
or approximately 1480 m

Problem 3.5

Given: The tube shown is filled with mercury at 20°C

Find: the force applied to the piston

Solution:



Basic equations: $\frac{dp}{dy} = -\rho g$

$\vec{F} = -\int p\,d\vec{A}$

For ρ = constant in a static fluid

$p = p_{atm} - \rho g(y - y_0)$

where $p = p_{atm}$ at $y = y_0$

Then

$p_1 = p_{atm} + \rho g h$ and $F_{p_1} = \rho g h A$ (gage).

For fbd (i) $\Sigma F_y = 0 = F_{p_1} - W = 0$ and $W = F_{p_1} = \rho g h A$

Also $p_2 = p_{atm}$ and $\rho g H$ and $F_2 = \rho g H A$ (gage).

For fbd (ii) $\Sigma F_y = 0 = F_{p_2} - W - F = 0$

$\therefore F = F_{p_2} - W = \rho g H A - \rho g h A = \rho g A (H - h)$

$F = \rho_{H_2O} SG\, g \frac{\pi D^2}{4}(H - h)$. From Fig. A.1, App. A, SG = 13.54

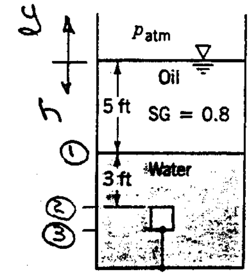
$F = 1000 \frac{kg}{m^3} \times 13.54 \times 9.81 \frac{m}{s^2} \times \frac{\pi}{4} (2)^2\, in^2 (8-1)\, in \times (0.0254)^3 \frac{m^3}{in^3} \times \frac{N \cdot s^2}{kg \cdot m}$

$F = 47.9\,N$ ← F

Problem 3.6

Given: Cube of solid oak, 1 ft on a side, is submerged by tether as shown.

Find: (a) the force of water on bottom surface
(b) the tension in the tether.



Solution:

Basic equations: $\frac{dP}{dh} = \rho g$, $\vec{F} = -\int P\, d\vec{A}$

Assumptions: (a) static fluid
(b) SG_{oil} = constant, ρ_{H_2O} = constant

Then $\int_{P_1}^{P_3} dP = \int_{h_0}^{h_3} \rho g\, dh = \int_{h_0}^{h_1} SG_{oil}\, \rho_{H_2O}\, g\, dh + \int_{h_1}^{h_3} \rho_{H_2O}\, g\, dh$

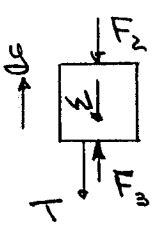
$P_3 - P_{atm} = SG_{oil}\, \rho_{H_2O}\, g\,(h_1 - h_0) + \rho_{H_2O}\, g\,(h_3 - h_1)$

$= 0.8 \times 1.94 \frac{slug}{ft^3} \times 32.2 \frac{ft}{s^2} \times 5 ft + 1.94 \frac{slug}{ft^3} \times 32.2 \frac{ft}{s^2} \times 4 ft$

$P_3 - P_{atm} = 500 \frac{slug}{ft \cdot s^2} \times \frac{lbf \cdot s^2}{ft \cdot slug} = 500\ lbf/ft^2$

Since the pressure over the bottom surface is uniform,

$\vec{F}_3 = -\int_A P\, d\vec{A} = -P\vec{A} = \left[P_{atm} + 500 \frac{lbf}{ft^2}\right] 1\ ft^2\, \hat{j} = 2620\, \hat{j}\ lbf$ ← $\vec{F}_3$



The force F_2 on the top of the cube is $F_2 = P_2 A$

The pressure on the top of the cube is

$P_2 - P_{atm} = SG_{oil}\, \rho_{H_2O}\, g\,(h_1 - h_0) + \rho_{H_2O}\, g\,(h_2 - h_1)$

The weight of the block is $W = \rho g \forall = SG_{oak}\, \rho_{H_2O}\, g \forall$
where $SG_{oak} = 0.77$ (Table A.1, Appendix A)

Then for the fbd of the block, $\Sigma F_y = 0 = F_3 - F_2 - W - T$

$T = F_3 - F_2 - W = \left[P_{atm} + SG_{oil}\, \rho_{H_2O}\, g\,(h_1 - h_0) + \rho_{H_2O}\, g\,(h_3 - h_1)\right] A$
$\quad - \left[P_{atm} + SG_{oil}\, \rho_{H_2O}\, g\,(h_1 - h_0) + \rho_{H_2O}\, g\,(h_2 - h_1)\right] A - SG_{oak}\, \rho_{H_2O}\, g \forall$

$T = \rho_{H_2O}\, g\,(h_3 - h_2) A - SG_{oak}\, \rho_{H_2O}\, g \forall$

$= 1.94 \frac{slug}{ft^3} \times 32.2 \frac{ft}{s^2} \times 1 ft \times 1 ft^2 - 0.77 \times 1.94 \frac{slug}{ft^3} \times 32.2 \frac{ft}{s^2} \times 1 ft^3$

$T = 14.4 \frac{slug \cdot ft}{s^2} \times \frac{lbf \cdot s^2}{ft \cdot slug} = 14.4\ lbf$ ← T

Problem 3.7

A cube with 6 in. sides is suspended in a fluid by a wire. The top of the cube is horizontal and 8 in. below the free surface. If the cube has a mass of 2 slugs and the tension in the wire is *T* = 50.7 lbf, compute the fluid specific gravity, and from this determine the fluid. What are the gage pressures on the upper and lower surfaces?

Given: Properties of a cube suspended by a wire in a fluid

Find: The fluid specific gravity; the gage pressures on the upper and lower surfaces

Solution

Consider a free body diagram of the cube: $\Sigma F = 0 = T + (p_L - p_U) \cdot d^2 - M \cdot g$

where M and d are the cube mass and size and p_L and p_U are the pressures on the lower and upp surfaces

For each pressure we can use Eq. 3.7 $\quad p = p_0 + \rho \cdot g \cdot h$

Hence $\quad p_L - p_U = \left[p_0 + \rho \cdot g \cdot (H + d) \right] - \left(p_0 + \rho \cdot g \cdot H \right) = \rho \cdot g \cdot d = SG \cdot \rho_{H2O} \cdot d$

where H is the depth of the upper surface

Hence the force balance gives $\quad SG = \dfrac{M \cdot g - T}{\rho_{H2O} \cdot g \cdot d^3}$

$$SG = \frac{2 \cdot slug \times 32.2 \cdot \frac{ft}{s^2} \times \frac{lbf \cdot s^2}{slug \cdot ft} - 50.7 \cdot lbf}{1.94 \cdot \frac{slug}{ft^3} \times 32.2 \cdot \frac{ft}{s^2} \times \frac{lbf \cdot s^2}{slug \cdot ft} \times (0.5 \cdot ft)^3}$$

$SG = 1.75$

From Table A.1, the fluid is Meriam blue.

The individual pressures are computed from Eq 3.7

$$p = p_0 + \rho \cdot g \cdot h$$

or

$$p_g = \rho \cdot g \cdot h = SG \cdot \rho_{H2O} \cdot h$$

For the upper surface $$p_g = 1.754 \times 1.94 \cdot \frac{slug}{ft^3} \times 32.2 \cdot \frac{ft}{s^2} \times \frac{2}{3} \cdot ft \times \frac{lbf \cdot s^2}{slug \cdot ft} \times \left(\frac{1 \cdot ft}{12 \cdot in}\right)^2$$

$$p_g = 0.507\,psi$$

For the lower surface $$p_g = 1.754 \times 1.94 \cdot \frac{slug}{ft^3} \times 32.2 \cdot \frac{ft}{s^2} \times \left(\frac{2}{3} + \frac{1}{2}\right) \cdot ft \times \frac{lbf \cdot s^2}{slug \cdot ft} \times \left(\frac{1 \cdot ft}{12 \cdot in}\right)^2$$

$$p_g = 0.89\,psi$$

Note that the SG calculation can also be performed using a buoyancy approach (discussed later in the chapter):

Consider a free body diagram of the cube: $\Sigma F = 0 = T + F_B - M \cdot g$

where M is the cube mass and F_B is the buoyancy force $F_B = SG \cdot \rho_{H2O} \cdot L^3 \cdot g$

Hence $T + SG \cdot \rho_{H2O} \cdot L^3 \cdot g - M \cdot g = 0$

or $SG = \frac{M \cdot g - T}{\rho_{H2O} \cdot g \cdot L^3}$ as before

$SG = 1.75$

Problem 3.8

A hollow metal cube with sides 100 mm floats at the interface between a layer of water and a la
of SAE 10W oil such that 10% of the cube is exposed to the oil. What is the pressure differenc
between the upper and lower horizontal surfaces? What is the average density of the cube?

Given: Properties of a cube floating at an interface

Find: The pressures difference between the upper and lower surfaces; average cube density

Solution

The pressure difference is obtained from two applications of Eq. 3.7

$$p_U = p_0 + \rho_{SAE10} \cdot g \cdot (H - 0.1 \cdot d)$$

$$p_L = p_0 + \rho_{SAE10} \cdot g \cdot H + \rho_{H2O} \cdot g \cdot 0.9 \cdot d$$

where p_U and p_L are the upper and lower pressures, p_0 is the oil free surface pressure, H is the depth of the interface, and d is the cube size

Hence the pressure difference is

$$\Delta p = p_L - p_U = \rho_{H2O} \cdot g \cdot 0.9 \cdot d + \rho_{SAE10} \cdot g \cdot 0.1 \cdot d$$

$$\Delta p = \rho_{H2O} \cdot g \cdot d \cdot \left(0.9 + SG_{SAE10} \cdot 0.1\right)$$

From Table A.2, for SAE 10W oil: $SG_{SAE10} = 0.92$

$$\Delta p = 999 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times 0.1 \cdot m \times (0.9 + 0.92 \times 0.1) \times \frac{N \cdot s^2}{kg \cdot m}$$

$\Delta p = 972\,\text{Pa}$

For the cube density, set up a free body force balance for the cube

$$\Sigma F = 0 = \Delta p \cdot A - W$$

Hence $\quad W = \Delta p \cdot A = \Delta p \cdot d^2$

$$\rho_{cube} = \frac{m}{d^3} = \frac{W}{d^3 \cdot g} = \frac{\Delta p \cdot d^2}{d^3 \cdot g} = \frac{\Delta p}{d \cdot g}$$

$$\rho_{cube} = 972 \cdot \frac{N}{m^2} \times \frac{1}{0.1 \cdot m} \times \frac{s^2}{9.81 \cdot m} \times \frac{kg \cdot m}{N \cdot s^2}$$

$$\rho_{cube} = 991 \frac{kg}{m^3}$$

Problem 3.9

Your pressure gage indicates that the pressure in your cold tires is 0.25 MPa (gage) on a mountain at an elevation of 3500 m. What is the absolute pressure? After you drive down to sea level, your tires have warmed to 25°C. What pressure does your gage now indicate?Assume a U.S. Standard Atmosphere.

Given: Data on tire at 3500 m and at sea level

Find: Absolute pressure at 3500 m; pressure at sea level

Solution

At an elevation of 3500 m, from Table A.3:

$$p_{atm} = 0.6492 \cdot p_{SL} = 0.6492 \times 101 \cdot kPa$$

$$p_{atm} = 65.6\,kPa$$

Then the absolute pressure is:

$$p_{abs} = p_{atm} + p_{gage} = 65.6 \cdot kPa + 250 \cdot kPa$$

$$p_{abs} = 316\,kPa$$

At sea level $p_{atm} = 101 \cdot kPa$

Meanwhile, the tire has warmed up, from the ambient temperature at 3500 m, to 25°C.

At an elevation of 3500 m, from Table A.3 $T_{cold} = 265.4 \cdot K$

Hence, assuming ideal gas behavior, $pV = mRT$
the absolute pressure of the hot tire is

$$p_{hot} = \frac{T_{hot}}{T_{cold}} \cdot p_{cold} = \frac{298 \cdot K}{265.4 \cdot K} \times 316 \cdot kPa$$

$$p_{hot} = 355\,kPa$$

Then the gage pressure is

$$p_{gage} = p_{hot} - p_{atm} = 355 \cdot kPa - 101 \cdot kPa$$

$$p_{gage} = 254\,kPa$$

Problem 3.10

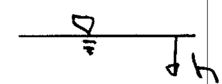
Given: Air bubble, $D = 10\text{ mm}$, released at depth $h = 30\text{ m}$ below surface of sea; water at $T = 30°C$.

Find: Estimate of bubble diameter as it reaches the water surface.

Solution:

Basic equations: $\frac{dp}{dh} = \rho g \qquad p = \rho R T \qquad \rho = \frac{m}{\forall}$

Assumptions: (1) T = constant = $30°C$
(2) air behaves as ideal gas
(3) ρ_{H_2O} = constant.



$h_1 = 30\text{ m}$
$h_2 = 0$

From ideal gas eq. $p = \rho R T = \frac{m}{\forall} R T$

Since m and T are constant, then $p_1 \forall_1 = p_2 \forall_2$ (1)

Also $\int_{p_1}^{p_2} dp = \int_{h_1}^{h_2} \rho g \, dh \qquad p_2 - p_1 = \rho g (h_2 - h_1)$

$$p_1 = p_2 + \rho g h_1 = p_{atm} + \rho g h_1$$

From Eq.(1)

$$\frac{\forall_2}{\forall_1} = \frac{p_1}{p_2} = \frac{p_{atm} + \rho g h_1}{p_{atm}} = 1 + \frac{\rho g h_1}{p_{atm}}$$

From Table A.8 (Appendix A) at $T = 30°C$, $\rho = 996\ kg/m^3$

From Table A.2, $SG_{sea\,water} = 1.025$

$$\therefore \frac{\forall_2}{\forall_1} = 1 + (996)(1.025)\frac{kg}{m^3} \times 9.81\frac{m}{s^2} \times 30\,m \times \frac{N \cdot s^2}{kg \cdot m} \times \frac{1}{1.01 \times 10^5}\frac{m^2}{N}$$

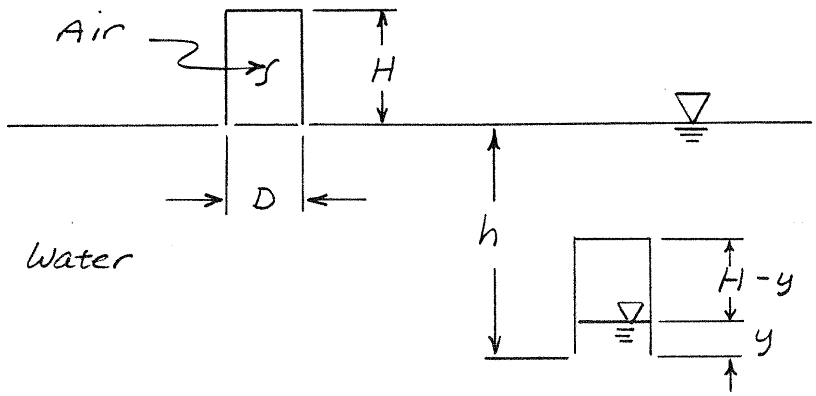
$$\frac{\forall_2}{\forall_1} = 3.975$$

Since $\forall \propto D^3$, $\left(\frac{D_2}{D_1}\right)^3 = \frac{\forall_2}{\forall_1}$

and $D_2 = D_1 \left(\frac{\forall_2}{\forall_1}\right)^{1/3} = 10\text{ mm}(3.975)^{1/3} = 15.8\text{ mm}$ ← D_2

Problem 3.11

Given: Cylindrical cup lowered slowly beneath pool surface.



Find: Expression for y in terms of h and H. Plot: y/H vs. h/H.

Solution: Apply ideal gas and hydrostatic equations.

Basic equations: $p\forall = mRT \qquad \frac{dp}{dh} = \rho g$

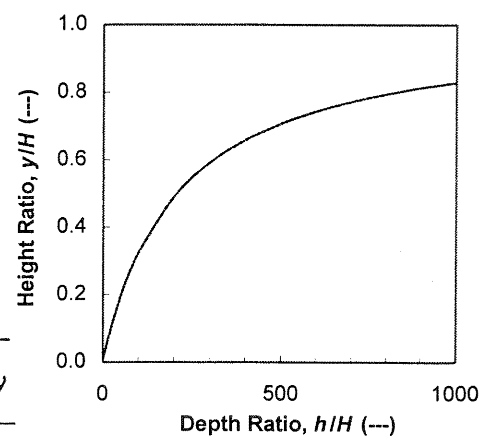
Assumptions: (1) T = constant
(2) Static liquid
(3) Incompressible liquid

Using (1), $p\forall = p_a \frac{\pi D^2}{4} H = p \frac{\pi D^2}{4}(H-y)$; or $\quad p_a H = p(H-y)$

Integrating $\frac{dp}{dh} = \rho g$ gives $p - p_a = \rho g(h-y)$ in container.

Thus

$$p_a H = [p_a + \rho g(h-y)](H-y) = p_a H - p_a y + \rho g(h-y)(H-y)$$



Expanding,

$$0 = \rho g h H - \rho g h y - \rho g y H + \rho g y^2 - p_a y$$

or

$$0 = hH - \left[(h+H) + \frac{p_a}{\rho g}\right] y + y^2$$

Using the quadratic equation

$$y = \frac{h + H + p_a/\rho g - \sqrt{\left[h + H + \frac{p_a}{\rho g}\right]^2 - 4hH}}{2} \qquad y$$

(Note $y \leq H$, so the minus sign must be used.) In terms of y/H, this becomes

$$\frac{y}{H} = \frac{\frac{h}{H} + 1 + \frac{p_a}{\rho g H} - \sqrt{\left[\frac{h}{H} + 1 + \frac{p_a}{\rho g H}\right]^2 - 4\frac{h}{H}}}{2} \qquad y/H$$

(See plot above.)

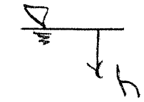
Given: Behavior of seawater to be modeled by assuming constant bulk modulus

Find: The percent deviations in (a) density, and (b) pressure, at depth $h = 10$ km, as compared to values obtained assuming constant density.

Plot: the results over range of $0 \le h \le 10$ km

Solution

Basic equation: $\frac{dp}{dh} = \rho g$ Definition: $E_v = \frac{dp}{d\rho/\rho}$



Then, $dp = \rho g\, dh = \frac{d\rho}{\rho} E_v$ and $\int_{\rho_0}^{\rho} \frac{d\rho}{\rho^2} = \int_0^h \frac{g\, dh}{E_v}$

We obtain

$$-\frac{1}{\rho}\Big]_{\rho_0}^{\rho} = -\frac{1}{\rho} + \frac{1}{\rho_0} = \frac{-\rho_0 + \rho}{\rho\rho_0} = \frac{gh}{E_v} \quad \text{or} \quad \rho - \rho_0 = \rho\rho_0 \frac{gh}{E_v}$$

Then

$$\rho\left(1 - \frac{\rho_0 g h}{E_v}\right) = \rho_0 \quad \text{and} \quad \frac{\rho}{\rho_0} = \frac{1}{\left(1 - \frac{\rho_0 g h}{E_v}\right)}$$

Finally,

$$\frac{\Delta\rho}{\rho_0} = \frac{\rho - \rho_0}{\rho_0} = \frac{\rho}{\rho_0} - 1 = \frac{\rho_0 g h / E_v}{(1 - \rho_0 g h / E_v)} \quad \text{---- (1)}$$

To determine an expression for the percent deviation in pressure we write $\int_{p_{atm}}^{p} dp = E_v \int_{\rho_0}^{\rho} \frac{d\rho}{\rho}$

Then $p - p_{atm} = E_v \ln \rho/\rho_0$

For ρ = constant, $\int_{p_{atm}}^{p} dp = \rho_0 g \int_0^h dh$ and $p_{\rho=c} - p_{atm} = \rho_0 g h$

Then

$$\frac{p - p_{\rho=c}}{p_{\rho=c}} = \frac{\Delta p}{p_{\rho=c}} = \frac{E_v \ln \rho/\rho_0 - \rho_0 g h}{\rho_0 g h} = \frac{E_v}{\rho_0 g h} \ln\frac{\rho}{\rho_0} - 1 \quad \text{---- (2)}$$

From Table A.2 for seawater SG = 1.025, $E_v = 2.42$ GN/m^2. Then

$$\frac{E_v}{\rho_0 g} = 2.42 \times 10^9 \frac{N}{m^2} \times \frac{1}{(1000)(1.025)} \frac{m^3}{kg} \times \frac{s^2}{9.81\, m} \times \frac{kg \cdot m}{N \cdot s^2} \times \frac{km}{10^3\, m} = 240.7 \text{ km}$$

Substituting into eqs (1) and (2)

$$\frac{\Delta\rho}{\rho_0} = \frac{4.155 \times 10^{-3} h}{1 - 4.155 \times 10^{-3} h} \quad \text{.....(1a)}$$

$$\frac{\Delta p}{p_0} = \frac{240.7}{h} \ln\left[\frac{1}{1 - 4.155 \times 10^{-3} h}\right] - 1 \quad \text{.... (2a)}$$

At $h = 10$ km, $\frac{\Delta\rho}{\rho_0} = 0.0434$ or 4.34% ______ $\frac{\Delta\rho}{\rho}\Big)_{h=10\,km}$

$\frac{\Delta p}{p_0} = 0.0215$ or $2.15\,\%$ ← $\left.\frac{\Delta p}{p_0}\right)_{h=10\,km}$

Both $\Delta \rho/\rho_0$ and $\Delta p/p_0$ are plotted as a function of depth h (in km) below.

The computing equations are

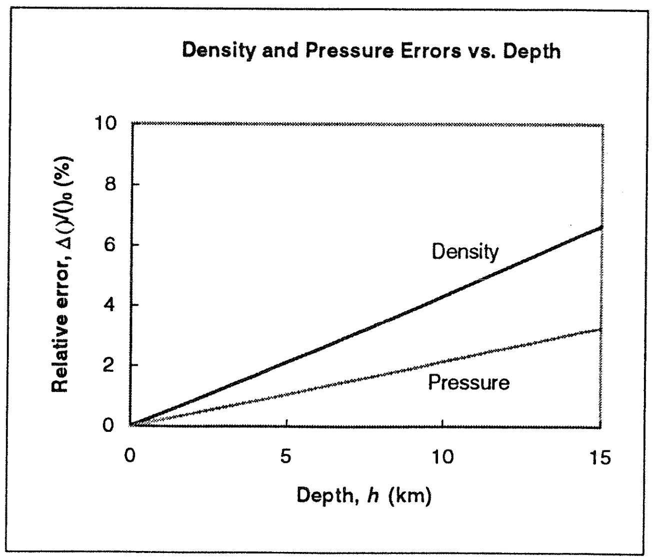
$$\Delta\rho/\rho_0 = \frac{\rho_0 g h/E_v}{(1-\rho_0 g h/E_v)}$$

$$\Delta p/p_0 = \frac{E_v}{\rho_0 g h}\ln\frac{\rho}{\rho_0} - 1$$

Density and pressure variation of seawater:

$E_v =$ 2.42 GN/m^2 Bulk modulus of seawater

Depth, h (km)	Density Error, $\Delta\rho/\rho_0$ (---)	Pressure Error, $\Delta p/p_0$ (---)
0	0	0
1	0.417	0.219
2	0.838	0.429
3	1.26	0.639
4	1.69	0.851
5	2.12	1.06
6	2.56	1.28
7	3.00	1.49
8	3.44	1.71
9	3.88	1.93
10	4.34	2.15
11	4.79	2.37
12	5.25	2.59
13	5.71	2.81
14	6.18	3.04
15	6.65	3.26

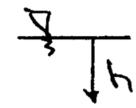


Problem 3.13

Given: Model behavior of seawater by assuming constant bulk modulus

Find: (a) expression density as a function of depth, h.
(b) Show that result may be written as $\rho \approx \rho_0 + bh$
(c) evaluate the constant b.
(d) use results of (b) to obtain equation for $p(h)$
(e) determine percent error in predicted pressure at h = 1000 m

Solution: From Table A.2, App. A, $SG)_0 = 1.025$, $E_v = 2.42\ \text{GN/m}^2$



Basic equation: $\frac{dp}{dh} = \rho g$ Definition: $E_v = \frac{dp}{d\rho/\rho}$

Then, $dp = \rho g\, dh = E_v \frac{d\rho}{\rho}$ and $\frac{d\rho}{\rho^2} = \frac{g}{E_v} dh$

Integrating, $\int_{\rho_0}^{\rho} \frac{d\rho}{\rho^2} = \int_0^h \frac{g}{E_v} dh$ and $-\frac{1}{\rho}\Big]_{\rho_0}^{\rho} = \frac{gh}{E_v}$

then, $\frac{gh}{E_v} = -\frac{1}{\rho} + \frac{1}{\rho_0} = \frac{-\rho_0 + \rho}{\rho\rho_0}$ or $\rho - \rho_0 = \rho\rho_0 \frac{gh}{E_v}$

$\therefore\ \rho\left(1 - \rho_0 \frac{gh}{E_v}\right) = \rho_0$ and $\frac{\rho}{\rho_0} = \frac{1}{\left\{1 - \frac{\rho_0 g h}{E_v}\right\}}$ ← $\rho(h)$

For $\frac{\rho_0 g}{E_v} \ll 1$, $\frac{\rho}{\rho_0} \approx 1 + \frac{\rho_0 g}{E_v} h$

Thus, $\rho \approx \rho_0 + \frac{\rho_0^2 g}{E_v} h = \rho_0 + bh$ where $b = \frac{\rho_0^2 g}{E_v}$ ← Q.E.D.

Since $dp = \rho g\, dh$, then an approximate expression for $p(h)$ is

$$p - p_{atm} = \int_{p_{atm}}^{p} dp = \int_0^h (\rho_0 + bh) g\, dh = \left(\rho_0 h + \frac{bh^2}{2}\right) g$$

$$p_{approx} = p_{atm} + \left(\rho_0 h + \frac{\rho_0^2 g h^2}{2E_v}\right) g = p_{atm} + \rho_0 h g\left[1 + \frac{\rho_0 g h}{2E_v}\right]$$ ← p_{approx}

The exact solution for $p(h)$ is obtained by utilizing the exact equation for $\rho(h)$. Thus.

$$p - p_{atm} = \int_{p_{atm}}^{p} dp = \int_{\rho_0}^{\rho} E_v \frac{d\rho}{\rho} = E_v \ln\frac{\rho}{\rho_0}$$

$$p = p_{atm} + E_v \ln\left\{1 - \frac{\rho_0 g h}{E_v}\right\}^{-1}$$ ← p_{exact}

$$\frac{\rho_0 g h}{E_v} = (1.025)1000\frac{\text{kg}}{\text{m}^3} \times 9.81\frac{\text{m}}{\text{s}^2} \times 10^3\,\text{m} \times \frac{1}{2.42\times10^9}\frac{\text{m}^2}{\text{N}} \times \frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}} = 4.16\times10^{-3}$$

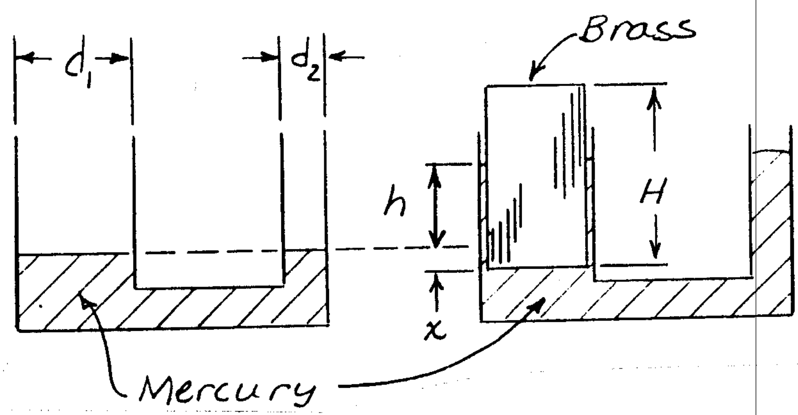
Substituting numerical values, $p_{approx} = p_{atm} + 9.851$ MPa
$p_{exact} = p_{atm} + 10.076$ MPa

$$\text{error} = \frac{p_{exact} - p_{app}}{p_{exact}} = \frac{10.076 - 9.851}{10.076} = 0.0224 = 2.24\%$$ ← error

Problem 3.14

Given: Container of mercury with vertical tubes $d_1 = 39.5$ mm and $d_2 = 12.7$ mm.

Brass cylinder with $D = 37.5$ mm and $H = 76.2$ mm is introduced into larger tube, where it floats.

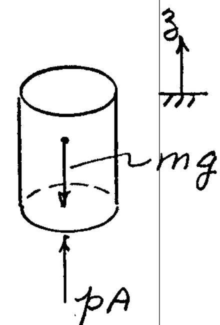


Find: (a) Pressure on bottom of cylinder.

(b) New equilibrium level, h, of mercury.

<u>Solution:</u> Analyze free-body diagram of cylinder, apply hydrostatics.

Computing equations: $\Sigma F_z = 0$; $\frac{dp}{dz} = -\rho g$; $\rho = SG\,\rho_{H_2O}$



Assumptions: (1) Static liquid

(2) Incompressible liquid

For the cylinder $\Sigma F_z = p\frac{\pi D^2}{4} - \rho_{brass}\, g\, \frac{\pi D^2}{4} H = 0$

Thus $p = \rho_{brass}\, g\, H = SG_{brass}\, \rho_{H_2O}\, g\, H$

From Table A.1, $SG_{brass} = 8.55$ at 20°C, so

$$p = 8.55 \times 1000\,\frac{kg}{m^3} \times 9.81\,\frac{m}{s^2} \times 0.0762\,m \times \frac{N \cdot s^2}{kg \cdot m} = 6.39\ kPa\ (gage)$$

This pressure must be produced by a column of mercury $h+x$ in height. Thus, using SG_{Hg} from Table A.1,

$$p = \rho_{Hg}\, g\,(h+x) = SG_{Hg}\, \rho_{H_2O}\, g\,(h+x) = SG_{brass}\, \rho_{H_2O}\, g\, H$$

Thus $h + x = \frac{SG_{brass}}{SG_{Hg}} H = \frac{8.55}{13.55} H = 0.631\ H$ (1)

But the volume of mercury must remain constant. Therefore

$$\frac{\pi D^2}{4} x = \frac{\pi(d_1^2 - D^2)}{4} h + \frac{\pi d_2^2}{4} h \quad \text{or} \quad x\left[\left(\frac{d_1}{D}\right)^2 - 1 + \left(\frac{d_2}{D}\right)^2\right] = 0.224\ h$$

Substituting into Eq. 1,

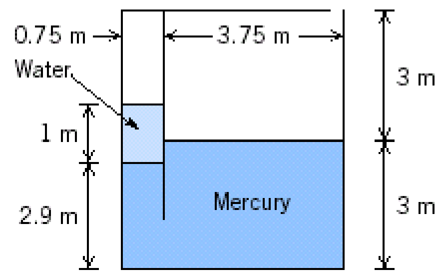
$$h + x = h + 0.224 h = 1.224 h = 0.631\ H \quad \text{or} \quad h = \frac{0.631}{1.224} H = 0.516\ H$$

Problem 3.15

A partitioned tank as shown contains water and mercury. What is the gage pressure in the air trapped in the left chamber? What pressure would the air on the left need to be pumped to in order to bring the water and mercury free surfaces level?

Given: Data on partitioned tank

Find: Gage pressure of trapped air; pressure to make water and mercury levels equal



Solution

The pressure difference is obtained from repeated application of Eq. 3.7, or in other words, from 3.8. Starting from the right air chamber

$$p_{gage} = SG_{Hg} \times \rho_{H2O} \times g \times (3 \cdot m - 2.9 \cdot m) - \rho_{H2O} \times g \times 1 \cdot m$$

$$p_{gage} = \rho_{H2O} \times g \times \left(SG_{Hg} \times 0.1 \cdot m - 1.0 \cdot m\right)$$

$$p_{gage} = 999 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times (13.55 \times 0.1 \cdot m - 1.0 \cdot m) \times \frac{N \cdot s^2}{kg \cdot m}$$

$$p_{gage} = 3.48\,kPa$$

If the left air pressure is now increased until the water and mercury levels are now equal, Eq. 3.8 leads to

$$p_{gage} = SG_{Hg} \times \rho_{H2O} \times g \times 1.0 \cdot m - \rho_{H2O} \times g \times 1.0 \cdot m$$

$$p_{gage} = \rho_{H2O} \times g \times \left(SG_{Hg} \times 1 \cdot m - 1.0 \cdot m\right)$$

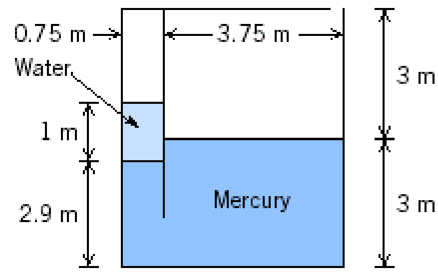
$$p_{gage} = 999 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times (13.55 \times 1 \cdot m - 1.0 \cdot m) \times \frac{N \cdot s^2}{kg \cdot m}$$

$$p_{gage} = 123\,kPa$$

Problem 3.16

In the tank of Problem 3.15, if the opening to atmosphere on the right chamber is first sealed, what pressure would the air on the left now need to be pumped to in order to bring the water and mercury free surfaces level? (Assume the air trapped in the right chamber behaves isothermally.)

Given: Data on partitioned tank



Find: Pressure of trapped air required to bring water and mercury levels equal if right air opening is sealed

Solution

First we need to determine how far each free surface moves.

In the tank of Problem 3.15, the ratio of cross section areas of the partitions is 0.75/3.75 or 1:5. Suppose the water surface (and therefore the mercury on the left) must move down distance x to bring the water and mercury levels equal. Then by mercury volume conservation, the mercury f surface (on the right) moves up $(0.75/3.75)x = x/5$. These two changes in level must cancel the original discrepancy in free surface levels, of (1m + 2.9m) - 3 m = 0.9 m. Hence $x + x/5 = 0.9$ m or $x = 0.75$ m. The mercury level thus moves up $x/5 = 0.15$ m.

Assuming the air (an ideal gas, $pV=RT$
will be

$$p_{right} = \frac{V_{rightold}}{V_{rightnew}} \cdot p_{atm} = \frac{A_{right} \cdot L_{rightold}}{A_{right} \cdot L_{rightnew}} \cdot p_{atm} = \frac{L_{rightold}}{L_{rightnew}} \cdot p_{atm}$$

where V, A and L
Hence

$$p_{right} = \frac{3}{3 - 0.15} \times 101 \cdot \text{kPa}$$

$$p_{right} = 106\,\text{kPa}$$

When the water and mercury levels are equal application of Eq. 3.8 gives:

$$p_{left} = p_{right} + SG_{Hg} \times \rho_{H2O} \times g \times 1.0 \cdot m - \rho_{H2O} \times g \times 1.0 \cdot m$$

$$p_{left} = p_{right} + \rho_{H2O} \times g \times \left(SG_{Hg} \times 1.0 \cdot m - 1.0 \cdot m\right)$$

$$p_{left} = 106 \cdot kPa + 999 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times (13.55 \cdot 1.0 \cdot m - 1.0 \cdot m) \times \frac{N \cdot s^2}{kg \cdot m}$$

$$p_{left} = 229\,kPa$$

$$p_{gage} = p_{left} - p_{atm} \quad p_{gage} = 229 \cdot kPa - 101 \cdot kPa$$

$$p_{gage} = 128\,kPa$$

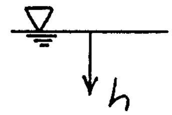
Problem 3.17

Given: U-tube manometer, partially filled with water, then $\forall_{oil} = 3.25\ cm^3$ of Meriam red oil is added to the left side.

Find: Equilibrium height, H, when both legs are open to atmosphere.

Solution: Apply basic pressure-height relation.

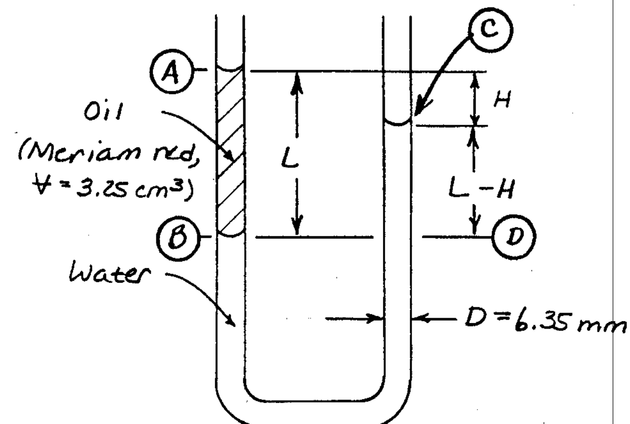
Basic equation: $\dfrac{dp}{dh} = +\rho g$



Assumptions: (1) Incompressible liquid
(2) h measured down

Integration gives

$$p_2 - p_1 = \rho g (h_2 - h_1)$$



Thus

$$p_B = p_A + \rho_{oil}\, g\, L$$

$$p_D = p_C + \rho_{water}\, g (L - H)$$

Since $p_A = p_C = p_{atm}$, then

$$\rho_{oil}\, g\, L = \rho_{water}\, g (L - H)$$

or

$$SG_{oil}\, L = L - H$$

Thus

$$H = L(1 - SG_{oil})$$

From the volume of oil, $\forall = \dfrac{\pi D^2}{4} L$, so

$$L = \frac{4\forall}{\pi D^2} = \frac{4}{\pi} \times 3.25\ cm^3 \times \frac{1}{(6.35)^2 mm^2} \times (10)^3 \frac{mm^3}{cm^3} = 103\ mm$$

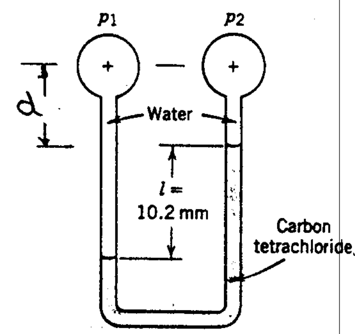
Finally, since $SG = 0.827$ (Table A.1, Appendix A), then

$$H = 103\ mm\,(1 - 0.827) = 17.8\ mm$$ H

Problem 3.18

Given: Two-fluid manometer shown.

Find: Pressure difference, $p_1 - p_2$



Solution:

Basic equation: $\frac{dp}{dh} = \rho g$

Assumptions: (1) static liquid
(2) incompressible
(3) g = constant

Then, $dp = \rho g\, dh$ and $\Delta p = \rho g \Delta h$

Starting at point ① and progressing to point ② we have

$$p_1 + \rho_{H_2O} g (d+l) - \rho_{c.t.} g l - \rho_{H_2O} g d = p_2$$

$$\therefore p_1 - p_2 = \rho_{ct} g l - \rho_{H_2O} g l = SG_{ct} \rho_{H_2O} g l - \rho_{H_2O} g l$$

$$p_1 - p_2 = \rho_{H_2O} g l (SG_{ct} - 1)$$

From Table A.2, Appendix A, $SG_{ct} = 1.595$

$$\therefore p_1 - p_2 = 1000 \frac{kg}{m^3} \times 9.81 \frac{m}{s^2} \times 10.2\,mm \times \frac{m}{10^3\,mm} (1.595 - 1) \frac{N \cdot s^2}{kg \cdot m}$$

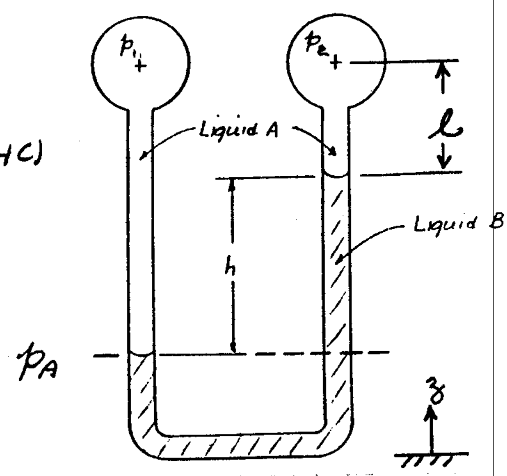
$$p_1 - p_2 = 59.5\ N/m^2 \qquad p_1 - p_2$$

Problem 3.19

Given: Manometer with two liquids as shown. $SG_A = 0.88$, $SG_B = 2.95$.

Find: Deflection, h, when $p_1 - p_2 = 870$ Pa.

Solution: Apply hydrostatics.



Basic equation: $\frac{dp}{dz} = -\rho g \qquad SG = \frac{\rho}{\rho_{H_2O}(4C)}$

Assumptions: (1) Static liquids
(2) Incompressible

Integrating, $p_A - p_B = -\rho(z_A - z_B)g$

$$p_A - p_B = \rho g(z_B - z_A)$$

or $\Delta p = \rho g \Delta h$

For left leg, $p_A = p_1 + (l + h)\rho_A g$

For right leg, $p_A = p_2 + l\rho_A g + h\rho_B g$

Subtracting, $p_1 - p_2 + hg(\rho_A - \rho_B) = 0$

$$p_1 - p_2 = hg(\rho_B - \rho_A)$$

Thus $$h = \frac{p_1 - p_2}{(\rho_B - \rho_A)g} = \frac{p_1 - p_2}{(SG_B - SG_A)\rho_{H_2O}\, g}$$

$$h = 870 \frac{N}{m^2} \times \frac{m^3}{(2.95 - 0.88)\, 1000\, kg} \times \frac{s^2}{9.81\, m} \times \frac{kg \cdot m}{N \cdot s^2} = 0.0428\ m$$

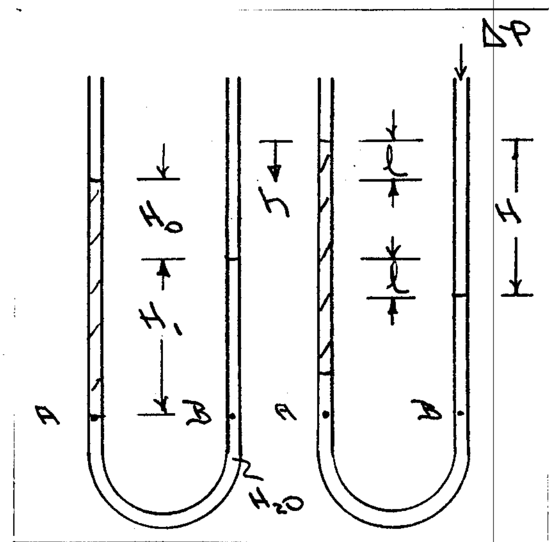
$h = 42.8$ mm

h

Problem 3.20

Given: Two fluid manometer contains water and kerosene. With both tubes open to atmosphere, the free surface elevations differ by $H_o = 20.0$ mm

Find: Elevation difference, H, between free-surface of fluids when a gage pressure of 98.0 Pa is applied to the right tube.



Solution:

Basic equation: $\frac{dp}{dh} = \rho g$; $\Delta p = \rho g \Delta h$

Assumptions: (1) static fluid
(2) gravity is the only body force

When the gage pressure $\Delta p = 98.0$ Pa is applied to the right tube, the water in the right tube is displaced downward a distance, l; the kerosene in the left tube is displaced upward the same distance, l.

Under the applied gage pressure, Δp, the elevation difference, H, is

$$H = H_o + 2l$$

Since points A & B are at the same elevation in the same fluid $p_A = p_B$.

Initially (left diagram), $p_A = \rho_k g(H_o + H_1)$, $p_B = \rho g H_1$ and hence

$$\rho_k g(H_o + H_1) = \rho g H_1$$

or

$$H_1 = \frac{\rho_k H_o}{\rho - \rho_k} = \frac{SG_k H_o}{(1 - SG_k)}$$. From table A.2, $SG_k = 0.82$

$$\therefore H_1 = \frac{0.82}{(1-0.82)} 20\text{mm} = 91.1 \text{ mm}$$

Under the applied pressure Δp (right diagram),

$$p_A = \rho_k g(H_o + H_1) + \rho g l \quad , \quad p_B = \Delta p + \rho g (H_1 - l).$$

$$\therefore SG_k(H_o + H_1) + l = \frac{\Delta p}{\rho g} + (H_1 - l)$$

Solving for l,

$$l = \frac{1}{2}\left[H_1 + \frac{\Delta p}{\rho g} - SG_k(H_o + H_1)\right]$$

$$= \frac{1}{2}\left[91.1\text{ mm} + \frac{98\text{N}}{\text{m}^2} \times \frac{\text{m}^3}{999\text{kg}} \times \frac{\text{s}^2}{9.81\text{m}} \times \frac{\text{kg}\cdot\text{m}}{\text{N}\cdot\text{s}^2} \times \frac{10^3\text{mm}}{\text{m}} - 0.82(20 + 91.1)\text{mm}\right]$$

$$l = 5\text{mm}$$

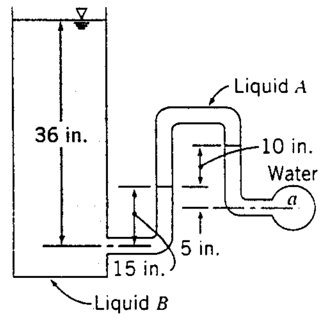
$$H = H_o + 2l = 30\text{mm}$$

Probelm 3.21

Given: Manometer system as shown

SG Liquid A = 0.75

SG Liquid B = 1.20



Find: Gage pressure at point "a"

Solution:

Basic equation: $\frac{dP}{dz} = -\gamma dz$

Assumptions: (1) static fluid

(2) gravity is only body force

(3) z axis direction vertically

(4) γ = constant

$dP = -\gamma dz$

For γ = constant, then $\Delta P = -\gamma \Delta z$, i.e. $P_j - P_i = -\gamma(z_j - z_i)$

$$P_2 - P_1 = -\gamma_B (z_2 - z_1)$$
$$P_3 - P_2 = -\gamma_B (z_3 - z_2)$$
$$P_4 - P_3 = -\gamma_A (z_4 - z_3)$$
$$P_5 - P_4 = -\gamma_{H_2O} (z_5 - z_4)$$

Summing these equations recognizing that $P_5 = P_a$ and $P_1 = P_{atm}$ then

$$P_a - P_{atm} = -\gamma_B (z_3 - z_1) - \gamma_A (z_4 - z_3) - \gamma_{H_2O} (z_5 - z_4)$$

$$= 1.20 \times 62.4 \frac{lbf}{ft^3} \times 21 in \times \frac{ft}{12 in} - 0.75 \times 62.4 \frac{lbf}{ft^3} \times \frac{10}{12} ft + 62.4 \frac{lbf}{ft^3} \times \frac{15}{12} ft$$

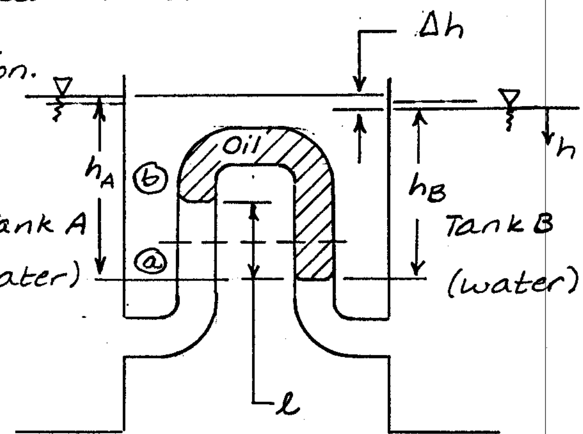
$$P_{a\,gage} = 170 \frac{lbf}{ft^2} \times \frac{ft^2}{144 in^2}$$

$$P_{a\,gage} = 1.18 \text{ psig}$$

P_a

Given: Two-fluid manometer; oil is second fluid.

Find: SG needed for 10 to 1 amplification.



Solution: Basic equation $\frac{dp}{dz} = -\rho g$

Assumptions: (1) Static liquid
(2) Incompressible

Then $dp = \rho g\,dh$

$$p = p_0 + \rho g h$$

For left leg, $p_a = p_{atm} + \rho_{H_2O}\, g\, h_A$

$$p_b = p_a - \rho_{H_2O}\, g\, \ell = p_{atm} + \rho_{H_2O}\, g\,(h_A - \ell) \qquad (1)$$

For right leg, $p_a = p_{atm} + \rho_{H_2O}\, g\, h_B$

$$p_b = p_a - SG_{oil}\,\rho_{H_2O}\, g\, \ell = p_{atm} + \rho_{H_2O}\, g\,(h_B - SG_{oil}\,\ell) \qquad (2)$$

Combining,

$$\cancel{p_{atm}} + \rho_{H_2O}\, g\,(h_A - \ell) = \cancel{p_{atm}} + \rho_{H_2O}\, g\,(h_B - SG_{oil}\,\ell)$$

or

$$h_A - \ell = h_B - SG_{oil}\,\ell \ ; \quad h_A - h_B = \Delta h = \ell\,(1 - SG_{oil})$$

Finally

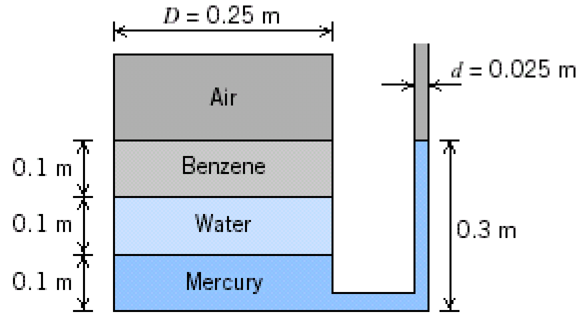
$$SG_{oil} = 1 - \frac{\Delta h}{\ell} = 1 - \frac{1}{10} = 0.900$$ ⟵ SG

Problem 3.23

Consider a tank containing mercury, water, benzene, and air as shown. Find the air pressure (gage). If an opening is made in the top of the tank, find the equilibrium level of the mercury in the manometer.

Given: Data on fluid levels in a tank

Find: Air pressure; new equilibrium level if opening appears



Solution

Using Eq. 3.8, starting from the open side and working in gage pressure

$$p_{air} = \rho_{H2O} \times g \times \left[SG_{Hg} \times (0.3 - 0.1) \cdot m - 0.1 \cdot m - SG_{Benzene} \times 0.1 \cdot m \right]$$

Using data from Table A.2

$$p_{air} = 999 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times (13.55 \times 0.2 \cdot m - 0.1 \cdot m - 0.879 \times 0.1 \cdot m) \times \frac{N \cdot s^2}{kg \cdot m}$$

$$p_{air} = 24.7\,kPa$$

To compute the new level of mercury in the manometer, assume the change in level from 0.3 m an increase of x. Then, because the volume of mercury is constant, the tank mercury level will fall by distance $(0.025/0.25)^2 x$

x

$$SG_{Hg} \times \rho_{H2O} \times g \times (0.3 \cdot m + x) = SG_{Hg} \times \rho_{H2O} \times g \times \left[0.1 \cdot m - x \cdot \left(\frac{0.025}{0.25} \right)^2 \right] \cdot m \; ...$$

$$+ \rho_{H2O} \times g \times 0.1 \cdot m + SG_{Benzene} \times \rho_{H2O} \times g \times 0.1 \cdot m$$

Hence $$x = \frac{[\, 0.1 \cdot m + 0.879 \times 0.1 \cdot m + 13.55 \times (0.1 - 0.3) \cdot m \,]}{\left[1 + \left(\frac{0.025}{0.25} \right)^2 \right] \times 13.55}$$

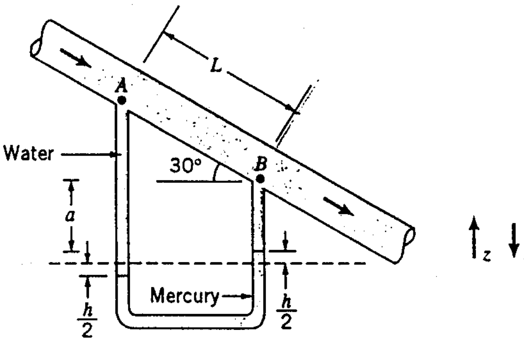
$x = -0.184\,m$ (The negative sign indicates the manometer level actually fell)

The new manometer height is $h = 0.3 \cdot m + x$

$h = 0.116\,m$

Problem 3.24

Given: Water flow in an inclined pipe as shown. Pressure difference, $p_A - p_B$, measured with two-fluid manometer
$L = 5\,ft$, $h = 6\,in.$



Find: Pressure difference, $p_A - p_B$.

Solution:

Basic equation: $\frac{dp}{dh} = \rho g$ where h is measured positive down

Assumptions: (1) static liquid
(2) incompressible
(3) g = constant

Then,
$$dp = \rho g\, dh \quad \text{and} \quad \Delta p = \rho g \Delta h$$

Start at p_A and progress through manometer to p_B

$$p_A + \rho_{H_2O} g L \sin 30° + \cancel{\rho_{H_2O} g a} + \rho_{H_2O} g h - \rho_{Hg} g h - \cancel{\rho_{H_2O} g a} = p_B$$

$$p_A - p_B = \rho_{Hg} g h - \rho_{H_2O} g h - \rho_{H_2O} g L \sin 30°$$

$$= SG_{Hg} \rho_{H_2O} g h - \rho_{H_2O} g h - \rho_{H_2O} g L \sin 30°$$

$$p_A - p_B = \rho_{H_2O}\, g \left[h(SG_{Hg} - 1) - L \sin 30° \right]$$

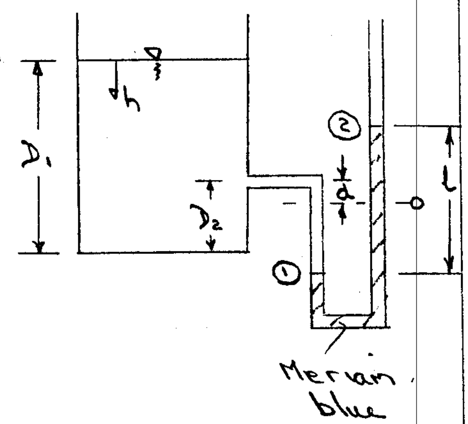
From Table A.2, $SG_{Hg} = 13.55$

Then,
$$p_A - p_B = 1.94 \frac{slug}{ft^3} \times 32.2 \frac{ft}{s^2} \left[0.5\,ft\,(13.55 - 1) - 5\,ft \sin 30° \right] \times \frac{lbf \cdot s^2}{ft \cdot slug}$$

$$p_A - p_B = 236\ lbf/ft^2 \quad (1.64\ psi)$$ ← $p_A - p_B$

Problem 3.25

Given: A U-tube manometer is connected to the open tank filled with water as shown (manometer fluid is Merian blue)



$$D_1 = 2.5\,m,\quad D_2 = 0.7\,m,\quad d = 0.2\,m$$

Find: The manometer deflection, l.

Solution:

Basic equation: $\frac{dP}{dh} = \rho g$

For γ = constant $\Delta P = \rho g \Delta h$

Then, beginning at the free surface and accounting for the changes in pressure with elevation,

$$P_{atm} + (P_1 - P_{atm}) + (P_2 - P_1) = P_2 = P_{atm}$$

$$\rho_{H_2O}\, g\left[(D_1 - D_2) + d + \frac{l}{2}\right] - \rho_{mb}\, g\, l = 0$$

$$(D_1 - D_2) + d + \frac{l}{2} = \frac{\rho_{mb}}{\rho_{H_2O}}\, l = (S.G)_{mb}\, l$$

and

$$l = \frac{(D_1 - D_2) + d}{\left[(S.G)_{mb} - \frac{1}{2}\right]}$$

(From Table A.1, Appendix A, SG = 1.75.)

$$l = \frac{(2.5 - 0.7)\,m + 0.2\,m}{(1.75 - 0.5)}$$

$$l = 1.6\,m$$ ← l

Problem 3.26

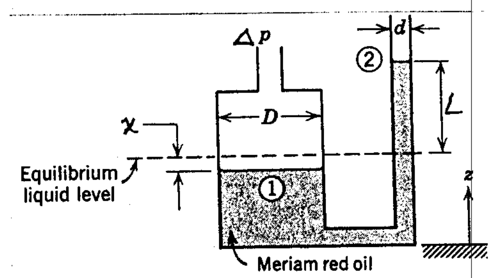
Given: Reservoir manometer with vertical tubes $D = 18$ mm and $d = 6$ mm diameter. Gage liquid is Meriam red oil.

Find: (a) Algebraic expression for deflection L in small tube when gage pressure Δp is applied to the reservoir.
(b) Evaluate L when Δp is equivalent to 25 mm H_2O (gage).

Solution: Use the diagram of Example Problem 3.2, apply hydrostatics.

Computing equations: $\frac{dp}{dh} = +\rho g$; $\Delta p = \rho g \Delta h$; $\rho = SG \rho_{H_2O}$

Assumptions: (1) Static liquid
(2) Incompressible liquid



Then $\Delta p = \rho_{oil}\, g\,(x + L)$

From conservation of volume,

$$\frac{\pi D^2}{4} x = \frac{\pi d^2}{4} L \; ; \; x = \left(\frac{d}{D}\right)^2 L$$

so

$$\Delta p = \rho_{water}\, g\, \Delta h = \rho_{oil}\, g\left[\left(\frac{d}{D}\right)^2 L + L\right] = \rho_{oil}\, g\, L\left[1 + \left(\frac{d}{D}\right)^2\right]$$

Solving for L,

$$L = \frac{\Delta p}{\rho_{oil}\, g\left[1 + (d/D)^2\right]}$$ L

Substituting $\Delta p = \rho_{water}\, g\, \Delta h$,

$$L = \frac{\rho_{water}\, g\, \Delta h}{SG_{oil}\, \rho_{water}\, g\left[1 + (d/D)^2\right]} = \frac{\Delta h}{SG_{oil}\left[1 + (d/D)^2\right]}$$

Evaluating, with $SG_{oil} = 0.827$ (Table A.1),

$$L = \frac{25.0 \text{ mm}}{0.827\left[1 + (6/18)^2\right]} = 27.2 \text{ mm}$$ L

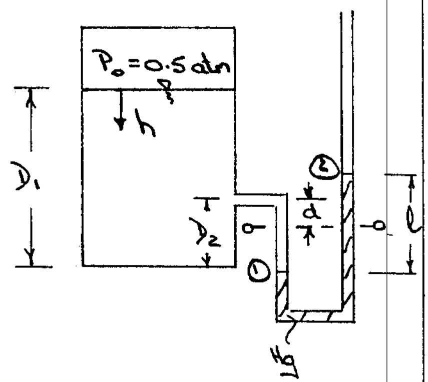
{Note: $s \equiv \frac{L}{\Delta h_e} = \frac{27.2 \text{ mm}}{25.0 \text{ mm}} = 1.09$ for this manometer.}

Problem 3.27

Given: A U-tube manometer is connected to a closed tank filled with water as shown. The manometer fluid is Hg.

$D_1 = 2.5\,m$, $D_2 = 0.7\,m$, $d = 0.2\,m$

At the water surface $P_o = 0.5$ atm (gage)



Find: The manometer deflection ℓ.

Solution

Basic equation $\frac{dP}{dh} = \rho g$

For γ = constant $\Delta P = \rho g \Delta h$

Then, beginning at the free surface and accounting for pressure changes with elevation,

$$P_o + (P_1 - P_o) + (P_2 - P_1) = P_2 = P_{atm}$$

$$P_o + \rho_{H_2O}\, g\left[(D_1 - D_2) + d + \frac{\ell}{2}\right] - \rho_{Hg}\, g\, \ell = P_{atm}$$

$$\frac{P_o - P_{atm}}{\rho_{H_2O}\, g} + (D_1 - D_2) + d + \frac{\ell}{2} = \frac{\rho_{Hg}\, g}{\rho_{H_2O}\, g}\ell = (S.G)_{Hg}\,\ell$$

and

$$\ell = \frac{(P_o - P_{atm})/\rho_{H_2O}\, g + (D_1 - D_2) + d}{(S.G)_{Hg} - 0.5}$$

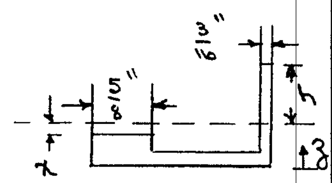
$$= \frac{0.5\,atm \times 1.01\times10^5 \frac{N}{m^2 \cdot atm} \times \frac{m^3}{999\,kg} \times \frac{s^2}{9.81\,m} \times \frac{kg \cdot m}{N \cdot s^2} + (2.5 - 0.7)m + 0.2\,m}{13.6 - 0.5}$$

$$\ell = 0.546\,m$$ ← ℓ

Problem 3.28

Given: Reservoir manometer with dimensions shown
Monometer fluid SG = 0.827

Find: required distance between marks on vertical scale for 1 in of water ΔP



Solution:

Basic equation: $\frac{dP}{dz} = -\gamma$

Assumptions: (1) static fluid
(2) gravity is only body force
(3) z axis directed vertically

$$dP = -\gamma\, dz$$

For constant γ, $\Delta P = P_1 - P_2 = -\gamma(z_1 - z_2)$

Under applied pressure $\Delta P = \gamma_{oil}(x+h)$

But conditions of problem require $\Delta P = \gamma_{H_2O}\, l$ where $l = 1$ in

$$\therefore \gamma_{oil}(x+h) = \gamma_{H_2O}\, l$$

Since the volume of the oil must remain constant

$$x A_{res} = h A_{tube}$$

$$\therefore x = h \frac{A_{tube}}{A_{res}}$$

and $\gamma_{oil}\left(h \frac{A_t}{A_r} + h\right) = \gamma_{H_2O}\, l$

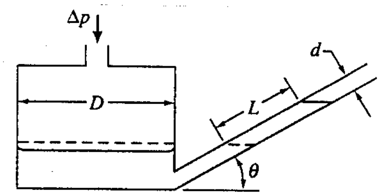
$$\therefore \frac{h}{l} = \frac{\gamma_{H_2O}}{\gamma_{oil}}\left(\frac{1}{\frac{A_t}{A_r}+1}\right) = \frac{1}{SG_{oil}\left[\left(\frac{D_t}{D_r}\right)^2+1\right]}$$

$$\frac{h}{l} = \frac{1}{0.827\left[\left(\frac{3}{16}\times\frac{8}{5}\right)^2+1\right]} = \frac{1}{0.827\left[(0.3)^2+1\right]}$$

$$\frac{h}{l} = 1.11$$

For $l = 1.0$ in as given, then $h = 1.11$ in.

Problem 3.29



Given: Inclined manometer as shown filled with oil, SG = 0.897

Find: Angle, θ, such that applied pressure of 1 in. H_2O gage gives 5" oil deflection along incline. Also determine sensitivity

Solution:

Basic equation: $\frac{dP}{dz} = -\gamma$

Assumptions: (1) static fluid
(2) gravity is only body force
(3) z axis directed vertically

$dP = -\gamma \, dz$

For constant γ, $\Delta P = P_1 - P_2 = -\gamma(z_1 - z_2)$

Under applied pressure $\Delta P = \gamma_{oil}(L\sin\theta + x)$

where $\Delta P = 1 \text{ in } H_2O = \gamma_{H_2O} h = 62.4 \frac{lbf}{ft^3} \times 1 \text{ in} \times \frac{ft}{12 \text{ in}} = 5.2 \frac{lbf}{ft^2}$.

Since the volume of the oil must remain constant

$x A_{res} = L A_{tube}$

$\therefore x = L \frac{A_{tube}}{A_{res}}$

and

$$\Delta P = \gamma_{oil}\left(L\sin\theta + L\frac{A_t}{A_r}\right) = \gamma_{oil}\left[L\sin\theta + L\left(\frac{d}{D}\right)^2\right]$$

Solving for $\sin\theta$,

$$\sin\theta = \frac{\Delta P}{\gamma_{oil} L} - \left(\frac{d}{D}\right)^2$$

$$= 5.2 \frac{lbf}{ft^2} \times \frac{ft^3}{0.897(62.4)\,lbf} \times \frac{1}{5 \text{ in}} \times \frac{12 \text{ in}}{ft} - \left(\frac{1}{4(3)}\right)^2$$

$\sin\theta = 0.2161$

$\theta = 12.5°$ ← θ

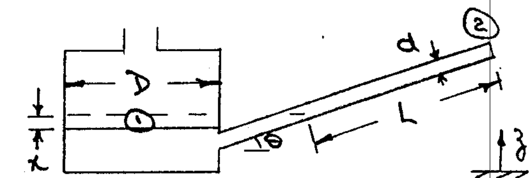
The manometer sensitivity, $s = \frac{L}{\Delta h_e} = \frac{5 \text{ in}}{1 \text{ in}} = 5$ ← s

Problem 3.30

Given: Inclined manometer as shown

$D = 96\,mm$, $d = 8\,mm$

Angle θ is such that liquid deflection is five times that of U-tube manometer under same applied pressure difference



Find: angle, θ and manometer sensitivity

Solution:

Basic equation $\dfrac{dP}{dz} = -\rho g$

Then $dP = -\rho g\, dz$ and for constant ρ

$$\Delta P = P_1 - P_2 = -\rho g (z_1 - z_2)$$

For the inclined manometer,

$$P_1 - P_{atm} = \rho g (L \sin\theta + x)$$

Since the volume of the oil must remain constant,

$$x A_{res} = L A_{tube}$$

$$x = L \frac{A_{tube}}{A_{res}} = L \left(\frac{d}{D}\right)^2$$

Then

$$P_1 - P_{atm} = \rho g (L\sin\theta + x) = \rho g \left(L\sin\theta + L\left(\frac{d}{D}\right)^2\right) = \rho g L \left(\sin\theta + \left(\frac{d}{D}\right)^2\right)$$

For a U-tube manometer

$$P_1 - P_{atm} = -\rho g (z_1 - z_2) = \rho g h$$

Hence,

$$\frac{(P_1 - P_{atm})_{incl}}{(P_1 - P_{atm})_{U\text{-}tube}} = \frac{\rho g L \left[\sin\theta + \left(\frac{d}{D}\right)^2\right]}{\rho g h}$$

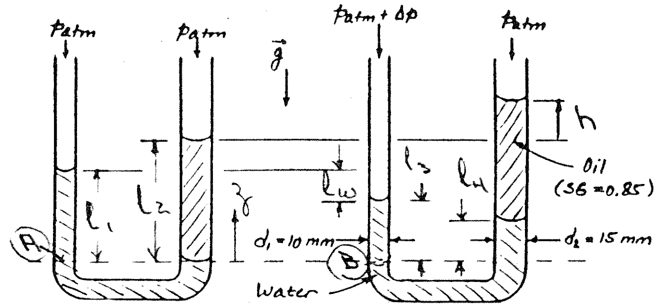
For same applied pressure and $L/h = 5$

$$1 = 5\left[\sin\theta + \left(\frac{d}{D}\right)^2\right]$$

$$\theta = \sin^{-1}\left[0.2 - \left(\frac{d}{D}\right)^2\right] = \sin^{-1}\left[0.2 - \left(\frac{8}{96}\right)^2\right] = 11.1^\circ \qquad \theta$$

The manometer sensitivity, $s = \dfrac{L}{\Delta h_e} = \dfrac{L}{SG\, h} = \dfrac{5}{SG}$

Given: U-tube manometer with tubes of different diameter and two liquids, as shown.



Find: (a) the deflection, h, for $\Delta p = 250\ N/m^2$.
(b) the sensitivity of the manometer.

Plot: the manometer sensitivity as a function of d_2/d_1

Solution:

Basic equation: $\frac{dp}{dz} = -\rho g$

Assumptions: (1) static liquid (2) incompressible

Integrating the basic equation from reference state at z_0 to general state at z gives

$$p - p_0 = -\rho g (z - z_0) = \rho g (z_0 - z)$$

From the left diagram: $p_A - p_{atm} = \rho_w g l_1 = \rho_o g l_2$ ......(1)

From the right diagram $p_B - (p_{atm} + \Delta p) = \rho_w g l_3$ ......(2)

$p_B - p_{atm} = \rho_w g l_4 + \rho_o g l_2$ ......(3)

Subtracting Eq. 2 from Eq. 3 and then employing Eq. 1 gives

$$\Delta p = \rho_w g (l_4 - l_3) + \rho_o g l_2 = \rho_w g (l_4 + l_1 - l_3)$$

Define $l_w = l_1 - l_3$. Note $l_4 = h$. Then $\Delta p = \rho_w g (h + l_w)$ ......(4)

We can relate l_w to h by recognizing the volume of water must be conserved

$$\therefore \pi \frac{d_1^2}{4} l_w = \pi \frac{d_2^2}{4} h \quad \text{and} \quad l_w = h \left(\frac{d_2}{d_1}\right)^2$$

Substituting into Eq. 4 gives

$$\Delta p = \rho_w g \left[h + h\left(\frac{d_2}{d_1}\right)^2\right] = \rho_w g h \left[1 + \left(\frac{d_2}{d_1}\right)^2\right]$$

Solving for h,

$$h = \frac{\Delta p}{\rho_w g [1 + (d_2/d_1)^2]} = 250 \frac{N}{m^2} \times \frac{m^3}{999\,kg} \times \frac{s^2}{9.81\,m} \times \frac{1}{[1 + (15/10)^2]} \times \frac{kg \cdot m}{N \cdot s^2} \times 10^3 \frac{mm}{m}$$

$h = 7.85$ mm ← h

(b) The sensitivity of the manometer is defined as

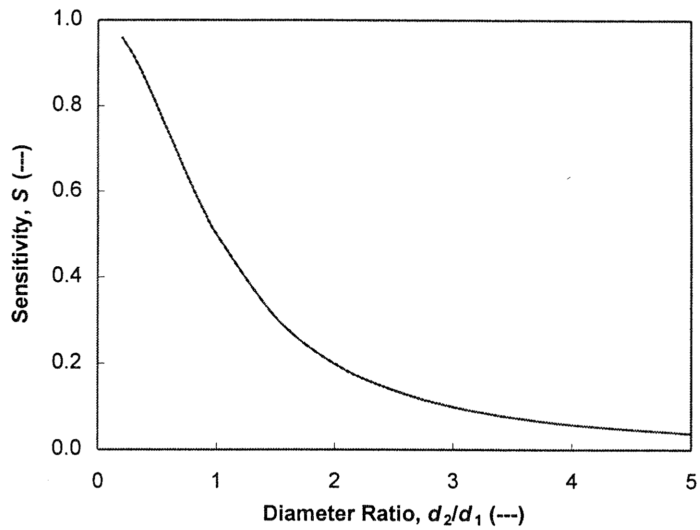
$$s = \frac{h}{\Delta h_e} = \frac{\text{actual deflection}}{\text{equivalent } \Delta h_{H_2O}} \quad \text{where } \Delta p = \rho_{H_2O} g \Delta h_e$$

$$\therefore s = \frac{h}{\Delta h_e} = \frac{1}{[1 + (d_2/d_1)^2]} = \frac{1}{[1 + (1.5)^2]} = 0.308$$ ← s

The design is a poor one. The sensitivity could be improved by interchanging d_2 and d_1, i.e. having $d_2/d_1 < 1.0$ as shown in the plot below

$$S = \frac{1}{[1 + (d_2/d_1)^2]}$$

The manometer sensitivity, as a function of diameter ratio d_2/d_1 is shown below.

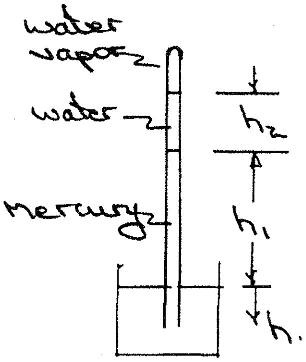


PROBLEM 3.32

Given: Barometer with 6.5 in. of water on top of the mercury column of height 28.35 in.; Temperature $T = 70°F$.

Find: (a) Barometric pressure in psia.
(b) Effect of increase in ambient temperature (to $T_a = 85°F$) on length of mercury column for same barometric pressure.

Solution:



Basic equation: $\frac{dp}{dh} = \rho g$

Assumptions: (1) static liquid
(2) incompressible
(3) $g =$ constant

Then, $dp = \rho g\, dh$ and $\Delta p = \rho g\, \Delta h$

Start at the free surface of the mercury ($p = p_{atm}$) and progress through the barometer to p_v (vapor pressure of the water)

$$p_{atm} - \rho_{Hg} g h_1 - \rho_{H_2O} g h_2 = p_v$$

$$p_{atm} = \rho_{Hg} g h_1 + \rho_{H_2O} g h_2 + p_v = \rho_{H_2O} SG_{Hg} g h_1 + \rho_{H_2O} g h_2 + p_v$$

$$p_{atm} = \rho_{H_2O}\, g\,[SG_{Hg} h_1 + h_2] + p_v$$

From Table A.2, $SG_{Hg} = 13.55$
Table A.7 $\rho_{H_2O} = 1.93\ slug/ft^3$, $p_v = 0.363$ psia.

Evaluating,

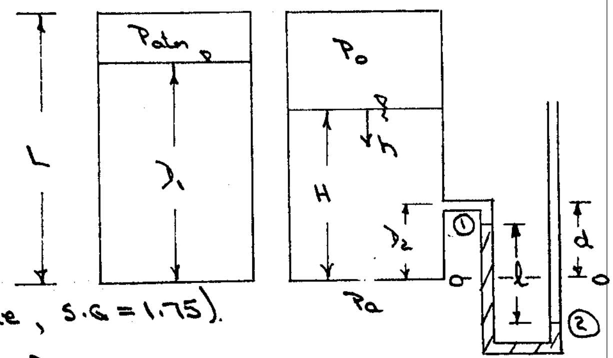
$$p_{atm} = 1.93\ \frac{slug}{ft^3} \times 32.2\ \frac{ft}{s^2}\left[13.55 \times 28.35\ in + 6.5\ in\right]\frac{ft}{12\ in} \times \frac{ft^2}{144\ in^2} \times \frac{lbf \cdot s^2}{ft \cdot slug} + 0.363\ psia$$

$$p_{atm} = 14.4\ psia \longleftarrow$$

At $T = 85°F$, the vapor pressure of water is estimated (from Table A.7) to be ≈ 0.60 psia. For the same barometric pressure the length of the mercury column would be shorter at the higher ambient temperature.

Problem 3.33

Given: Sealed tank of cross-section A and height, $L = 3.0\,m$, is filled with water to a depth $D_1 = 2.5\,m$.
Water drains slowly from the tank until system attains equilibrium.
U-tube manometer is connected to tank as shown.
(manometer fluid is merian blue, S.G = 1.75).



$D_1 = 2.5\,m$, $D_2 = 0.7\,m$, $d = 0.2\,m$

Find: The manometer deflection, l, under equilibrium conditions

Solution:

Basic equations: $\frac{dP}{dh} = \rho g \qquad P\forall = MRT$

For γ = constant $\quad \Delta P = \rho g \Delta h$

To determine the surface pressure P_o under equilibrium conditions treat air above water as an ideal gas

$$\frac{P_a \forall_a}{P_o \forall_o} = \frac{MRT_a}{MRT_o} \qquad \text{Assuming } T_a = T_o \text{, then}$$

$$P_o = \frac{\forall_a}{\forall_o} P_a = \frac{A(L-D_1)}{A(L-H)} P_a = \frac{(L-D_1)}{(L-H)} P_a$$

Under equilibrium conditions, $P_o + \rho_{H_2O} g H = P_a$

Hence, $\frac{(L-D_1)}{(L-H)} P_a + \rho_{H_2O} g H = P_a \quad$ or $\quad \rho_{H_2O} g H^2 - H(P_a + \rho_{H_2O} g L) + D_1 P_a = 0$

and

$$H = \frac{(P_a + \rho_{H_2O} g L) \pm \sqrt{(P_a + \rho_{H_2O} g L)^2 - 4 \rho_{H_2O} g D_1 P_a}}{2 \rho_{H_2O} g}$$

$$H = \frac{\left[1.01\times10^5 \frac{N}{m^2} + 999\frac{kg}{m^3} \times 9.81\frac{m}{sec^2} \times 3m \times \frac{N\cdot sec^2}{kg\cdot m}\right] \pm \sqrt{[\;]^2 - 4 \times 999\frac{kg}{m^3} \times 9.81\frac{m}{sec^2} \times 2.5m \times 1.01\times10^5 \frac{N}{m^2} \times \frac{N\cdot sec^2}{kg\cdot m}}}{2\times 999\frac{kg}{m^3} \times 9.81\frac{m}{sec^2} \times \frac{N\cdot sec^2}{kg\cdot m}}$$

$H = 10.9\,m$ or $2.36\,m$. From physical considerations $H = 2.36\,m$

$$P_o = \frac{(L-D_1)}{(L-H)} P_a = \frac{(3.0-2.5)}{(3.0-2.36)} \times 1.01\times10^5\,N/m^2 = 7.89\times10^4\,N/m^2$$

For the manometer, $P_o + (P_1 - P_o) + (P_2 - P_1) = P_2 = P_{atm}$

$$P_o + \rho_{H_2O} g \left(H - D_2 + d - \frac{l}{2}\right) + \rho_{mb} g l = P_{atm}$$

$$\frac{P_{atm} - P_o}{\rho_{H_2O} g} - H + D_2 - d = (S.G)_{mb} l - \frac{l}{2} = l\left[(S.G)_{mb} - 0.5\right]$$

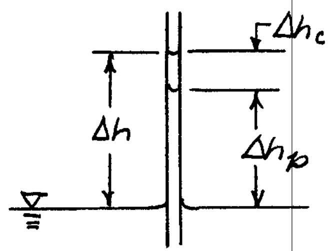
$$l = \frac{(P_{atm} - P_o)/\rho_{H_2O} g - H + D_2 - d}{(S.G)_{mb} - 0.5} = \frac{(10.1 - 7.89)\times10^4 \frac{N}{m^2} \times \frac{m^3}{999\,kg} \times \frac{s^2}{9.81\,m} \times \frac{kg\,m}{N\cdot s^2} - 2.36m + 0.7m - 0.2}{1.75 - 0.5}$$

$l = 0.316\,m$ ← $\quad l$

Problem 3.34

Given: Water column standing at $\Delta h = 50$ mm in $D = 2.5$ mm glass tube.

Find: (a) Column height if surface tension were zero.
(b) Column height in $D = 1$ mm tube.

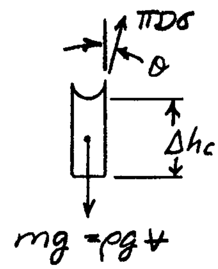


Solution: Assume column height is sum of capillary rise and rise caused by pressure difference,

$$\Delta h = \Delta h_c + \Delta h_p$$

Choose a free-body diagram of Δh_c for analysis:

$$\sum F_z = \pi D \sigma \cos\theta - \frac{\pi D^2}{4}\rho g \Delta h_c = 0$$



Assumptions: (1) Neglect volume under meniscus
(2) Δh_p remains constant

Then $\Delta h_c = \frac{4\sigma}{\rho g D}\cos\theta$

For water (Table A.4), $\sigma = 72.8$ mN/m and $\theta \approx 0$, so $\cos\theta = 1$, and

$$\Delta h_c = \frac{4\sigma}{\rho g D}$$

For the $D = 2.5$ mm tube,

$$\Delta h_c = 4 \times 72.8\times10^{-3}\frac{N}{m} \times \frac{m^3}{999\ kg} \times \frac{s^2}{9.81\ m} \times \frac{1}{0.0025\ m} \times \frac{kg\cdot m}{N\cdot s^2} = 0.0119\ m \text{ or } 11.9\ mm$$

Then

$$\Delta h_p = \Delta h - \Delta h_c = (50.0 - 11.9)\ mm = 38.1\ mm \quad (\sigma = 0)$$ ← Δh_p

For the $D = 1.0$ mm tube,

$$\Delta h_c = 4 \times 72.8\times10^{-3}\frac{N}{m} \times \frac{m^3}{999\ kg} \times \frac{s^2}{9.81\ m} \times \frac{1}{0.001\ m} \times \frac{kg\cdot m}{N\cdot s^2} = 0.0297\ m \text{ or } 29.7\ mm$$

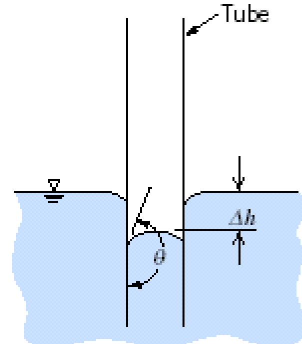
So

$$\Delta h = \Delta h_c + \Delta h_p = (29.7 + 38.1)\ mm = 67.8\ mm \quad (D = 1.0\ mm \text{ tube})$$ ← Δh

Problem 3.35

Consider a small diameter open-ended tube inserted at the interface between two immiscible fluids of different densities. Derive an expression for the height difference Δh between the interface level inside and outside the tube in terms of tube diameter D, the two fluid densities, ρ and ρ_2, and the surface tension σ and angle θ
water and mercury, find the tube diameter such that Δh < 10 mm.

Given: Two fluids inside and outside a tube



Find: An expression for height Δh; find diameter for Δh < 10 mm for water/mercury

Solution

A free-body vertical force analysis for the section of fluid 1 height Δh in the tube below the "free surface" of fluid 2 leads to

$$\sum F = 0 = \Delta p \cdot \frac{\pi \cdot D^2}{4} - \rho_1 \cdot g \cdot \Delta h \cdot \frac{\pi \cdot D^2}{4} + \pi \cdot D \cdot \sigma \cdot \cos(\theta)$$

where Δp $\qquad$ Δh $\Delta p = \rho_2 \cdot g \cdot \Delta h$

Assumption: Neglect meniscus curvature for column height and volume calculations

Hence

$$\Delta p \cdot \frac{\pi \cdot D^2}{4} - \rho_1 \cdot g \cdot \Delta h \cdot \frac{\pi \cdot D^2}{4} = \rho_2 \cdot g \cdot \Delta h \cdot \frac{\pi \cdot D^2}{4} - \rho_1 \cdot g \cdot \Delta h \cdot \frac{\pi \cdot D^2}{4} = -\pi \cdot D \cdot \sigma \cdot \cos(\theta)$$

Solving for Δh

$$\Delta h = -\frac{4 \cdot \sigma \cdot \cos(\theta)}{g \cdot D \cdot (\rho_2 - \rho_1)}$$

For fluids 1 and 2 being water and mercury (for mercury σ = 375 mN/m and θ = 140^o, from Table A.4), solving for D to make Dh = 10 mm

$$D = -\frac{4\cdot\sigma\cdot\cos(\theta)}{g\cdot\Delta h\cdot(\rho_2 - \rho_1)} = -\frac{4\cdot\sigma\cdot\cos(\theta)}{g\cdot\Delta h\cdot\rho_{H2O}\cdot(SG_{Hg} - 1)}$$

$$D = \frac{4\times 0.375\cdot\frac{N}{m}\times\cos(140^o)}{9.81\cdot\frac{m}{s^2}\times 0.01\cdot m\times 1000\cdot\frac{kg}{m^3}\times(13.6-1)}\times\frac{kg\cdot m}{N\cdot s^2}$$

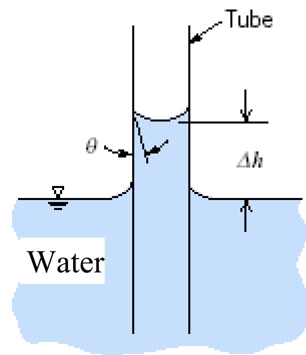
$$D = 9.3\times 10^{-4}\,m \qquad D \geq 9.3\cdot mm$$

Problem 3.36

Compare the height due to capillary action of water exposed to air in a circular tube of diameter D = 0.5 mm, and between two infinite vertical parallel plates of gap a = 0.5 mm.

Given: Water in a tube or between parallel plates

Find: Height Δh; for each system



Solution

a) Tube: A free-body vertical force analysis for the section of water height Δh above the "free surface" in the tube, as shown in the figure, leads to

$$\sum F = 0 = \pi \cdot D \cdot \sigma \cdot \cos(\theta) - \rho \cdot g \cdot \Delta h \cdot \frac{\pi \cdot D^2}{4}$$

Assumption: Neglect meniscus curvature for column height and volume calculations

Solving for Δh $$\Delta h = \frac{4 \cdot \sigma \cdot \cos(\theta)}{\rho \cdot g \cdot D}$$

b) Parallel Plates: A free-body vertical force analysis for the section of water height Δh above the "free surface" between plates arbitrary width w (similar to the figure above), leads to

$$\sum F = 0 = 2 \cdot w \cdot \sigma \cdot \cos(\theta) - \rho \cdot g \cdot \Delta h \cdot w \cdot a$$

Solving for Δh

$$\Delta h = \frac{2 \cdot \sigma \cdot \cos(\theta)}{\rho \cdot g \cdot a}$$

For water σ = 72.8 mN/m and $\theta = 0^o$ (Table A.4), so

a) Tube

$$\Delta h = \frac{4 \times 0.0728 \cdot \frac{N}{m}}{999 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times 0.005 \cdot m} \times \frac{kg \cdot m}{N \cdot s^2}$$

$$\Delta h = 5.94 \times 10^{-3} m \qquad \Delta h = 5.94 mm$$

b) Parallel Plates

$$\Delta h = \frac{2 \times 0.0728 \cdot \frac{N}{m}}{999 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times 0.005 \cdot m} \times \frac{kg \cdot m}{N \cdot s^2}$$

$$\Delta h = 2.97 \times 10^{-3} m \qquad \Delta h = 2.97 mm$$

Problem 3.37

Open-Ended Problem Statement: A hydropneumatic elevator consists of a piston-cylinder assembly to lift the elevator cab. Hydraulic oil, stored in an accumulator tank pressurized by air, is valved to the piston as needed to lift the elevator. When the elevator descends, oil is returned to the accumulator. Design the least expensive accumulator that can satisfy the system requirements. Assume the lift is 3 floors, the maximum load is 10 passengers, the maximum system pressure is 800 kPa (gage). For column bending strength, the piston diameter must be at least 150 mm. The elevator cab and piston have a combined mass of 3,000 kg, and are to be purchased. Perform the analysis needed to define, as a function of system operating pressure, the piston diameter, the accumulator volume and diameter, and the wall thickness. Discuss safety features that your company should specify for the complete elevator system. Would it be preferable to use a completely pneumatic design or a completely hydraulic design? Why?

Discussion: The design requirements are specified, except that a typical floor height is about 12 ft, making the total required lift about 36 ft.)

A spreadsheet was used to calculate the system properties for various pressures. Results are presented on the next page, followed by a sample calculation.

Total cost dropped quickly as system pressure was increased. A shallow minimum was reached in the 100-110 psig range.

The lowest-cost solution was obtained at a system pressure of about 100 psig. At this pressure, the reservoir of 140 gal required a 3.30 ft diameter pressure sphere with a 0.250 in. wall thickness. The welding cost was \$311 and the material cost \$433, for a total cost of \$744.

Accumulator wall thickness was constrained at 0.250 in. for pressures below 100 psi; it increased for higher pressures (this caused the discontinuity in slope of the curve at 100 psig). The mass of steel became constant above 110 psig.

No allowance was made for the extra volume needed to pressurize the accumulator.

Fail-safe design is essential for an elevator to be used by the public. The control circuitry should be redundant. Failures must be easy to spot. For this reason, hydraulic actuation is good: leaks will be readily apparent. The final design must be reviewed, approved, and stamped by a professional engineer since the design involves public safety.

The terminology used in the solution is defined in Table 1.

Table 1. Symbols, definitions, and units

Symbol	Definition	Units
p	system pressure	psig
A_p	area of lift piston	in.^{2}
$\mathcal{V}_{oil}$	volume of oil	gal
D_s	diameter of (spherical) accumulator	ft
t	wall thickness of spherical accumulator	in.
A_w	area of weld	in.^{2}
C_w	cost of weld	\$
M_s	mass of (steel) accumulator	lbm
C_s	cost of steel	\$
C_t	*total cost*	\$

Results of the system simulation and sample calculations are presented on the next page.

Table 2. Results of system simulation

Input Data:	Cab and piston weight:	W_{cab} =	6,000	lbf
	Passenger weight:	W_{pax} =	1,500	lbf
	Total weight:	W_{tot} =	7,500	lbf
	Allowable stress:	σ =	4,000	psi
	Minimum wall thickness:	t =	0.250	in.
	Welding cost factor:	cf_w =	5.00	\$/in.^{2}
	Steel cost factor:	cf_s =	1.25	\$/pound

Results:

p (psig)	A_p (in.^{2})	V_{oil} (gal)	D_s (ft)	t (in.)	A_w (in.^{2})	C_w (\$)	M_s (lbm)	C_s (\$)	C_t (\$)
20	375	701	5.64	0.250	106.2	\$531	1012	\$1,265	\$1,796
30	250	468	4.92	0.250	92.8	\$464	772	\$965	\$1,429
40	188	351	4.47	0.250	84.3	\$422	638	\$797	\$1,218
50	150	281	4.15	0.250	78.3	\$391	549	\$687	\$1,078
60	125	234	3.91	0.250	73.7	\$368	487	\$608	\$976
70	107	200	3.71	0.250	70.0	\$350	439	\$549	\$899
80	93.8	175	3.55	0.250	66.9	\$335	402	\$502	\$837
90	83.3	156	3.41	0.250	64.4	\$322	371	\$464	\$786
100	75.0	140	3.30	0.250	62.1	\$311	346	\$433	\$743
110	68.2	128	3.19	0.263	63.4	\$317	342	\$428	\$745
120	62.5	117	3.10	0.279	65.3	\$326	342	\$428	\$754
130	57.7	108	3.02	0.294	67.1	\$335	342	\$428	\$763

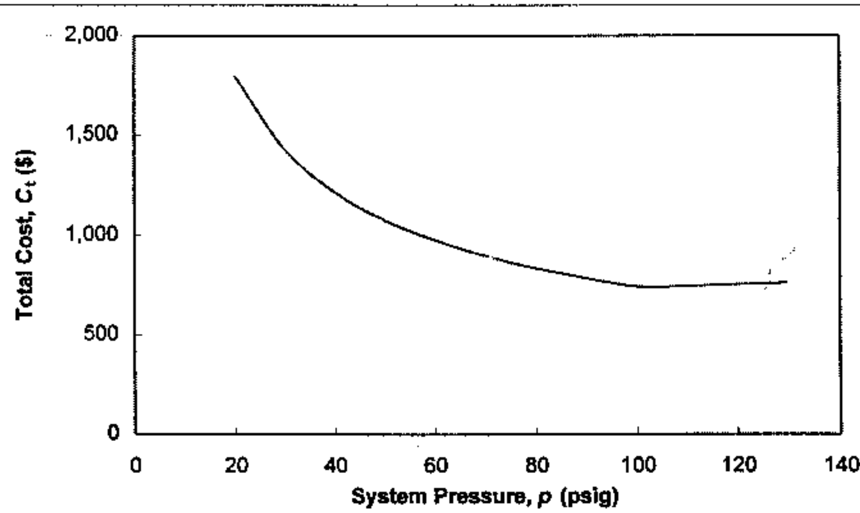


Fig. 1 Total cost versus system pressure

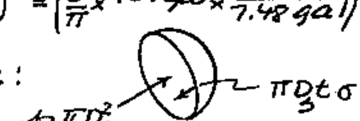
Sample Calculation (p = 20 psig):

$W_t = p A_p$; $A_p = \frac{W_t}{p} = 7500 \text{ lbf} \times \frac{\text{in.}^2}{20 \text{ lbf}} = 375 \text{ in.}^2$

$\forall_{oil} = A_p L = 375 \text{ in.}^2 \times \frac{1}{36 \text{ ft}} \times \frac{\text{ft}^2}{144 \text{ in.}^2} \times 7.48 \frac{\text{gal}}{\text{ft}^3} = 701 \text{ gal}$

$\forall_{oil} = \forall_s = \frac{4\pi R_s^3}{3} = \frac{\pi D_s^3}{6}$; $D_s = \left(\frac{6\forall_{oil}}{\pi}\right)^{1/3} = \left(\frac{6}{\pi} \times 701 \text{ gal} \times \frac{\text{ft}^3}{7.48 \text{ gal}}\right)^{1/3} = 5.64 \text{ ft}$

From a force balance on the sphere:



Thus $p\frac{\pi D_s^2}{4} = \pi D_s t \sigma$, so $t = \frac{p}{\sigma}\frac{D_s}{4} = \frac{1}{4} \times 20\,\frac{\text{lbf}}{\text{in.}^2} \times \frac{\text{in.}^2}{4000\,\text{lbf}} \times 5.64\,\text{ft} \times 12\,\frac{\text{in.}}{\text{ft}} = 0.0846$

Therefore $t = t_{min} = 0.250$ in.

$$A_w = \pi D_s t = \pi \times 5.64\,\text{ft} \times 0.25\,\text{in.} \times 12\,\frac{\text{in.}}{\text{ft}} = 106\ \text{in.}^2$$

$$C_w = \frac{\$5.00}{\text{in.}^2} \times 106\ \text{in.}^2 = \$531$$

$$M_s = 4\pi R_s^2 t \rho_s = \pi D_s^2 t\, SG_s\, \rho_{H_2O} = \pi \times (5.64)^2\,\text{ft}^2 \times 0.25\,\text{in.} \times 7.8 \times 62.4\,\frac{\text{lbm}}{\text{ft}^3} \times \frac{\text{ft}}{12\,\text{in.}} = 1012\,\text{lbm}$$

$$C_s = \frac{\$1.25}{\text{lbm}} \times 1012\ \text{lbm} = \$1265$$

and

$$C_t = C_w + C_s = \$531 + \$1265 = \$1,796$$

C_t

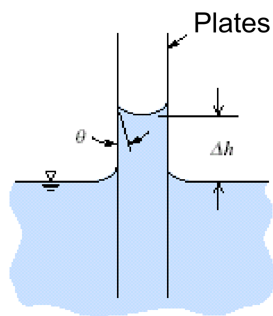
Problem 3.37 (In Excel)

Two vertical glass plates 300 mm x 300 mm are placed in an open tank containing water. At one end the gap between the plates is 0.1 mm, and at the other it is 2 mm. Plot the curve of water height between the plates from one end of the pair to the other.

Given: Geometry on vertical plates
Find: Curve of water height due to capillary action

Solution

A free-body vertical force analysis (see figure) for the section of water height Δh above the "free surface" between plates arbitrary separated by width a, (per infinitesimal length dx of the plates) leads to



$$\sum F = 0 = 2 \cdot dx \cdot \sigma \cdot \cos(\theta) - \rho \cdot g \cdot \Delta h \cdot dx \cdot a$$

Solving for Δh

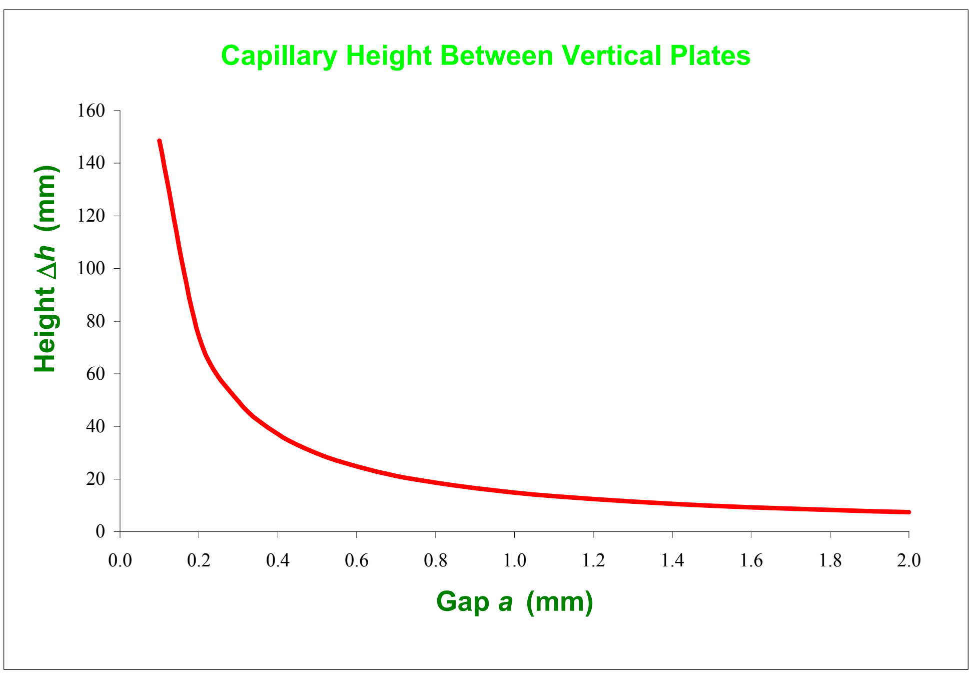
$$\Delta h = \frac{2 \cdot \sigma \cdot \cos(\theta)}{\rho \cdot g \cdot a}$$

For water $\sigma = 72.8$ mN/m and $\theta = 0^o$ (Table A.4)

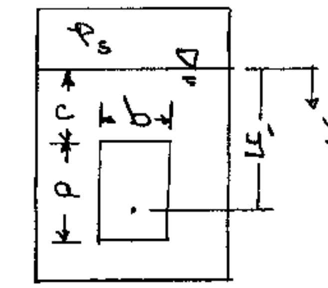
$\sigma =$ 72.8 mN/m
$\rho =$ 999 kg/m^3

Using the formula above

a (mm)	Δh (mm)
0.1	149
0.2	74.3
0.3	49.5
0.4	37.1
0.5	29.7
0.6	24.8
0.7	21.2
0.8	18.6
0.9	16.5
1.0	14.9
1.1	13.5
1.2	12.4
1.3	11.4
1.4	10.6
1.5	9.90
1.6	9.29
1.7	8.74
1.8	8.25
1.9	7.82
2.0	7.43



Given: Door located in plane vertical wall of water tank as shown
$a = 1.5\,m$, $b = 1\,m$, $c = 1\,m$.
Atmospheric pressure acts on outer surface of door.



Find: (a) For $p_s = p_{atm}$, resultant force on door and line of action of force
(b) Resultant force and line of action if $p_s = 0.3$ atm (gage)

Plot: F/F_o and y'/y_c over range of p_s/p_{atm}. (F_o is resultant force when $p_s = p_{atm}$; y_c is y coordinate of centroid).

Solution:

Basic equations: $\frac{dp}{dy} = \rho g$; $F_R = \int_A p\,dA$; $y' F_R = \int y\,p\,dA$

Assumptions: (1) static liquid
(2) incompressible liquid

Note: We will obtain a general expressions for F and y' (needed for the plot) and then simply for cases (a) & (b).

Since $dp = \rho g\,dy$ then $p = p_s + \rho g y$

Because p_{atm} acts on the outside of the door, then p_s is the surface gage pressure.

$$F_R = \int p\,dA = \int_c^{c+a} p\,b\,dy = \int_c^{c+a} (p_s + \rho g y)\,b\,dy = b\left[p_s y + \rho g \frac{y^2}{2}\right]_c^{c+a}$$

$$F_R = b\left[p_s a + \frac{\rho g}{2}\{(c+a)^2 - c^2\}\right] = b\left[p_s a + \frac{\rho g}{2}(a^2 + 2ac)\right] \quad \text{---- (1)}$$

$y' F_R = \int y\,p\,dA$ and $y' = \frac{1}{F_R}\int_c^{c+a} y(p_s + \rho g y)\,b\,dy$

$$y' = \frac{b}{F_R}\left[p_s \frac{y^2}{2} + \rho g \frac{y^3}{3}\right]_c^{c+a}$$

$$y' = \frac{b}{F_R}\left[\frac{p_s}{2}\{(c+a)^2 - c^2\} + \frac{\rho g}{3}\{(c+a)^3 - c^3\}\right] \quad \text{---- (2)}$$

(a) For $p_s = 0$ (gage) then

from Eq. 1 $F_R = \frac{\rho g b}{2}(a^2 + 2ac)$.

$$F_R = \frac{1}{2} \times 999\frac{kg}{m^3} \times 9.81\frac{m}{s^2} \times 1\,m\left[(1.5\,m)^2 + 2(1.5\,m)(1\,m)\right]\frac{N \cdot s^2}{kg \cdot m} = 25.7\,kN \quad F_R$$

From Eq. 2

$$y' = \frac{b}{F_{R_o}} \frac{\rho g}{3}\left[(c+a)^3 - c^3\right]$$

$$y' = \frac{1\,m}{25.7\,kN} \times \frac{999}{3}\frac{kg}{m^3} \times 9.81\frac{m}{s^2}\left[(2.5)^3 - 1\right]m^3 \frac{N \cdot s^2}{kg \cdot m} \times \frac{kN}{10^3\,N} = 1.86\,m \quad y'_c$$

(b) For $p_s = 0.3$ atm (gage) then

from Eq. 1 $\quad F_R = b\left[p_s a + \frac{\rho g}{2}(a^2 + 2ac)\right]$

$$F_R = 1\,m\left[0.3\,atm \times 1.01\times10^5 \frac{N}{m^2\cdot atm}(1.5\,m) + \frac{1}{2}\times 999\frac{kg}{m^3}\times 9.81\frac{m}{s^2}\left\{(1.5)^2 + 2(1.5)(1)\right\}m^2 \cdot \frac{N\cdot s^2}{kg\cdot m}\right]$$

$F_R = 71.2$ kN $\quad F_R$

$$y' = \frac{b}{F_R}\left[\frac{p_s}{2}\left\{(c+a)^2 - c^2\right\} + \frac{\rho g}{3}\left\{(c+a)^3 - c^3\right\}\right]$$

$$y' = \frac{1\,m}{71.2\,kN}\left[\frac{1}{2}\times 0.3\,atm \times 1.01\times10^5\frac{N}{m^2\cdot atm}\left\{(2.5)^2 - 1\right\} + \frac{1}{3}\times 999\frac{kg}{m^3}\times 9.81\frac{m}{s^2}\times\left\{(2.5)^3 - 1\right\}m^3 \times \frac{N\cdot s^2}{kg\cdot m}\right]\frac{kN}{10^3\,N}$$

$y' = 1.79$ m $\quad y'$

The value of F/F_o is obtained from Eq. 1 and $F_{R_o} = 25.7$ kN.

$$\frac{F}{F_o} = \frac{1}{25.7\,kN}\, b\left[p_s a + \frac{\rho g}{2}(a^2 + 2ac)\right] = 0.0389\left[151.5\,p_s + 25.7\right]$$

with p_s in atm. $\quad F/F_o$

For the gate $y_c = c + \frac{a}{2} = 1.75$ m. Then from Eq. 2

$$y'/y_c = \frac{b}{F_R(1.75)}\left[\frac{p_s}{2}\left\{(c+a)^2 - c^2\right\} + \frac{\rho g}{3}\left\{(c+a)^3 - c^3\right\}\right] = \frac{0.571}{F}\left[265\,p_s + 47.8\right]$$

with F in kN, p_s in atm $\quad y'/y_c$

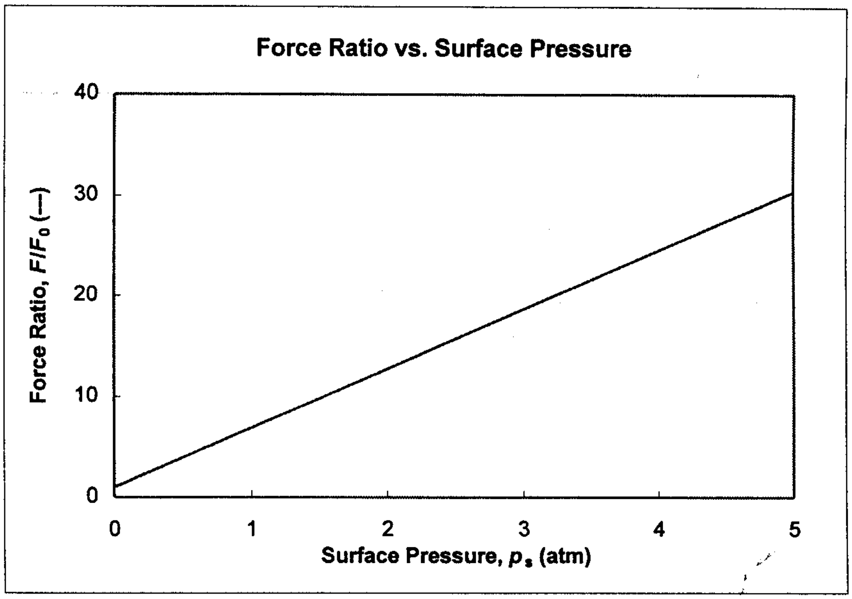
The plots are shown below

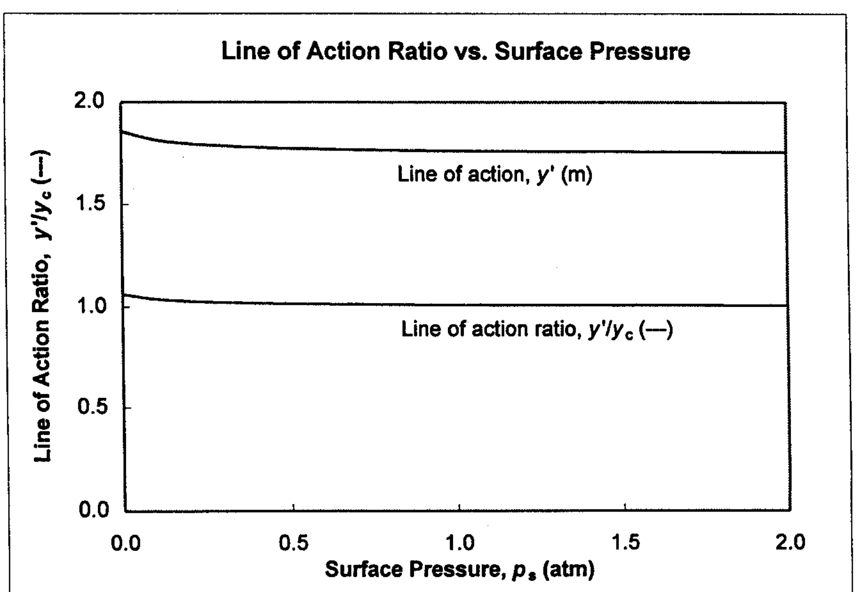
Note: The force on the gate increases linearly with increase in surface pressure.

The line of action of the resultant force is always below the centroid of the gate; y'/y_c approaches unity as the surface pressure is increased.

Force ratio and line of action ratio vs. surface pressure:

Surface Pressure, p_s (atm)	Force Ratio, F/F_0 (---)	Force, F_0 (kN)	Line of Action Ratio, y'/y_c (---)	Line of Action, y' (m)
0	1.00	25.7	1.0623	1.86
0.1	1.59	40.8	1.0388	1.82
0.2	2.18	56.0	1.0281	1.80
0.3	2.77	71.1	1.0219	1.79
0.5	3.95	101	1.0151	1.78
1.0	6.89	177	1.00822	1.76
2.0	12.8	329	1.00399	1.76
3.0	18.7	480		
4.0	24.6	632		
5.0	30.5	783		

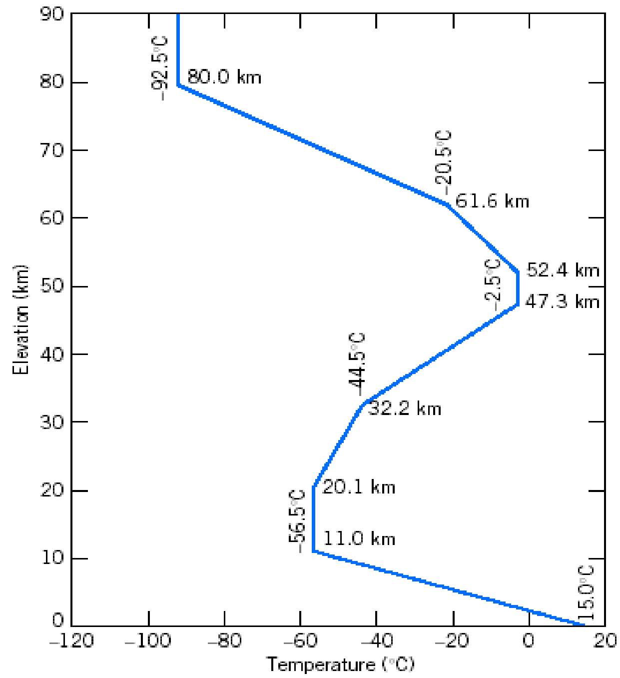




Problem 3.38 (In Excel)

Based on the atmospheric temperature data of the U.S. Standard Atmosphere of Fig. 3.3, compute and plot the pressure variation with altitude, and compare with the pressure data of Table A.3.

Given: Atmospheric temperature data
Find: Pressure variation; compare to Table A.3



Solution

From Section 3-3:

$$\frac{dp}{dz} = -\rho \cdot z \qquad \text{(Eq. 3.6)}$$

For linear temperature variation (m = - dT/dz) this leads to

$$p = p_0 \cdot \left(\frac{T}{T_0}\right)^{\frac{g}{m \cdot R}} \qquad \text{(Eq. 3.9)}$$

For isothermal conditions Eq. 3.6 leads to

$$p = p_0 \cdot e^{-\frac{g \cdot (z - z_0)}{R \cdot T}} \qquad \text{Example Problem 3.4}$$

In these equations p_0, T_0, and z_0 are reference conditions

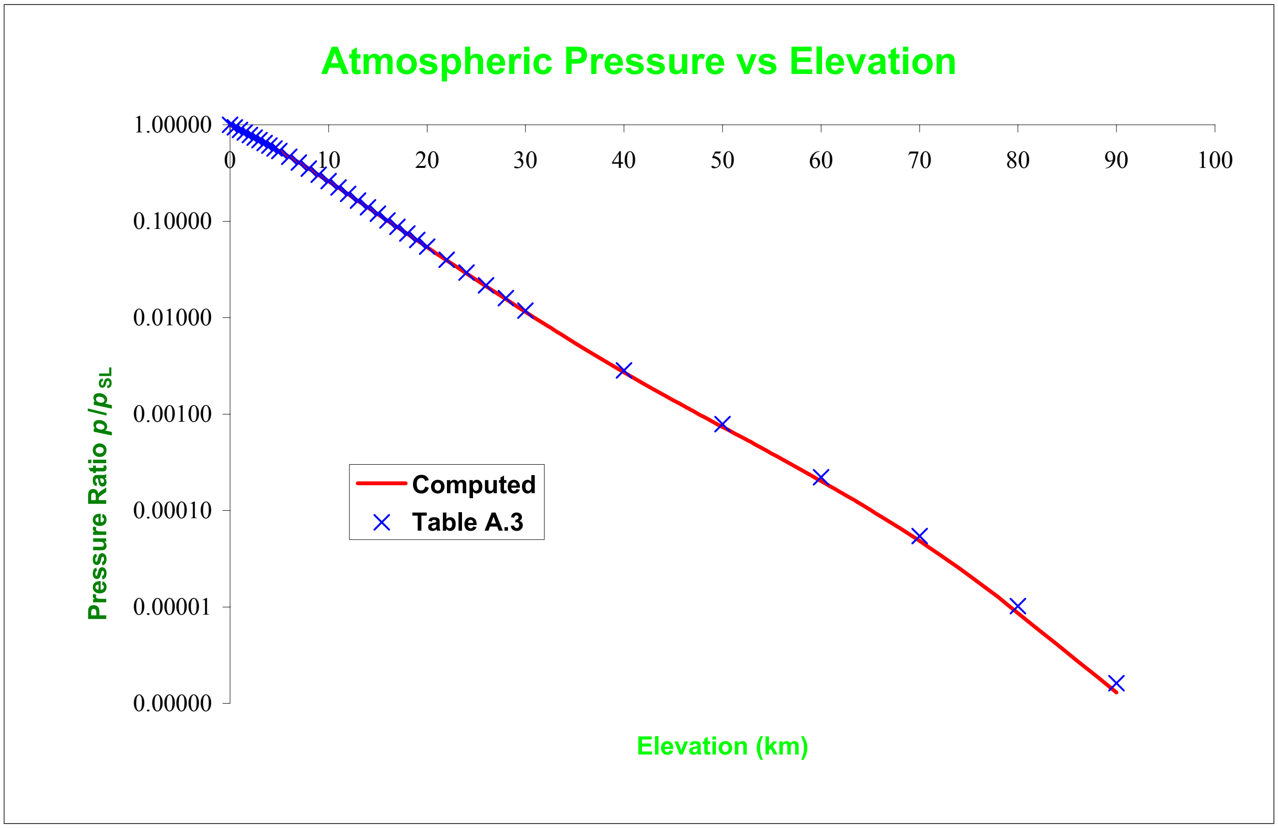
p_{SL} =	101	kPa
R =	286.9	J/kg.K
ρ =	999	kg/m^3

The temperature can be computed from the data in the figure
The pressures are then computed from the appropriate equation

z (km)	T (°C)	T (K)		p/p_{SL}
0.0	15.0	288.0	m =	1.000
2.0	2.0	275.00	0.0065	0.784
4.0	-11.0	262.0	(K/m)	0.608
6.0	-24.0	249.0		0.465
8.0	-37.0	236.0		0.351
11.0	-56.5	216.5		0.223
12.0	-56.5	216.5	T = const	0.190
14.0	-56.5	216.5		0.139
16.0	-56.5	216.5		0.101
18.0	-56.5	216.5		0.0738
20.1	-56.5	216.5		0.0530
22.0	-54.6	218.4	m =	0.0393
24.0	-52.6	220.4	-0.000991736	0.0288
26.0	-50.6	222.4	(K/m)	0.0211
28.0	-48.7	224.3		0.0155
30.0	-46.7	226.3		0.0115
32.2	-44.5	228.5		0.00824
34.0	-39.5	233.5	m =	0.00632
36.0	-33.9	239.1	-0.002781457	0.00473
38.0	-28.4	244.6	(K/m)	0.00356
40.0	-22.8	250.2		0.00270
42.0	-17.2	255.8		0.00206
44.0	-11.7	261.3		0.00158
46.0	-6.1	266.9		0.00122
47.3	-2.5	270.5		0.00104
50.0	-2.5	270.5	T = const	0.000736
52.4	-2.5	270.5		0.000544
54.0	-5.6	267.4	m =	0.000444
56.0	-9.5	263.5	0.001956522	0.000343
58.0	-13.5	259.5	(K/m)	0.000264
60.0	-17.4	255.6		0.000202
61.6	-20.5	252.5		0.000163
64.0	-29.9	243.1	m =	0.000117
66.0	-37.7	235.3	0.003913043	0.0000880
68.0	-45.5	227.5	(K/m)	0.0000655
70.0	-53.4	219.6		0.0000482
72.0	-61.2	211.8		0.0000351
74.0	-69.0	204.0		0.0000253
76.0	-76.8	196.2		0.0000180
78.0	-84.7	188.3		0.0000126
80.0	-92.5	180.5	T = const	0.00000861
82.0	-92.5	180.5		0.00000590
84.0	-92.5	180.5		0.00000404
86.0	-92.5	180.5		0.00000276
88.0	-92.5	180.5		0.00000189
90.0	-92.5	180.5		0.00000130

From Table A.3

z (km)	p/p_{SL}
0.0	1.000
0.5	0.942
1.0	0.887
1.5	0.835
2.0	0.785
2.5	0.737
3.0	0.692
3.5	0.649
4.0	0.609
4.5	0.570
5.0	0.533
6.0	0.466
7.0	0.406
8.0	0.352
9.0	0.304
10.0	0.262
11.0	0.224
12.0	0.192
13.0	0.164
14.0	0.140
15.0	0.120
16.0	0.102
17.0	0.0873
18.0	0.0747
19.0	0.0638
20.0	0.0546
22.0	0.0400
24.0	0.0293
26.0	0.0216
28.0	0.0160
30.0	0.0118
40.0	0.00283
50.0	0.000787
60.0	0.000222
70.0	0.0000545
80.0	0.0000102
90.0	0.00000162



Agreement between calculated and tabulated data is very good (as it should be, considering the table data is also computed!)

Given: Atmosphere in which T = constant = 30°C between sea level and 5 km altitude.

Find: (a) Elevation change, Δz, corresponding to a 1% reduction in air pressure.
(b) Elevation change, Δz, necessary to effect a 15% reduction in density

Plot: p_2/p_1 and ρ_2/ρ_1 vs. z.

Solution:

Basic equations: $\frac{dp}{dz} = -\rho g$, $p = \rho RT$

Assumptions: (1) static, isothermal fluid
(2) g = constant
(3) ideal gas behavior

Then, $\frac{dp}{dz} = -\rho g = -\frac{p}{RT}$ and $\frac{dp}{dz} = -\frac{pg}{RT}dz$

Separating variables and integrating,

$$\int_{p_1}^{p} \frac{dp}{p} = -\frac{g}{RT_0}\int_{z_1}^{z_2} dz \quad \text{and} \quad \ln \frac{p_2}{p_1} = -\frac{g}{RT}\Delta z$$

$$\Delta z = -\frac{RT_0}{g}\ln\frac{p_2}{p_1} \qquad (1)$$

For an ideal gas, $\frac{p_2}{p_1} = \frac{\rho_2 RT_0}{\rho_1 RT_0} = \frac{\rho_2}{\rho_1}$

Thus,

$$\Delta z = -\frac{RT_0}{g}\ln\frac{\rho_2}{\rho_1} \qquad (2)$$

From Table A.6 $R_{air} = 287 \ N\cdot m / kg\cdot K$

Evaluating, $\frac{RT_0}{g} = 287\frac{N\cdot m}{kg\cdot K} \times (273+30)K \times \frac{s^2}{9.81 m} \times \frac{kg\cdot m}{N\cdot s^2} = 8860 \ m$

For a one percent reduction in pressure, $p_2/p_1 = 0.99$. From (1)

$$\Delta z = -8860 \ m \ln(0.99) = 89.0 \ m \qquad (a)$$

For a 15% reduction in density, $\rho_2/\rho_1 = 0.85$. From (2)

$$\Delta z = -8860 \ m \ln(0.85) = 1440 \ m \qquad (b)$$

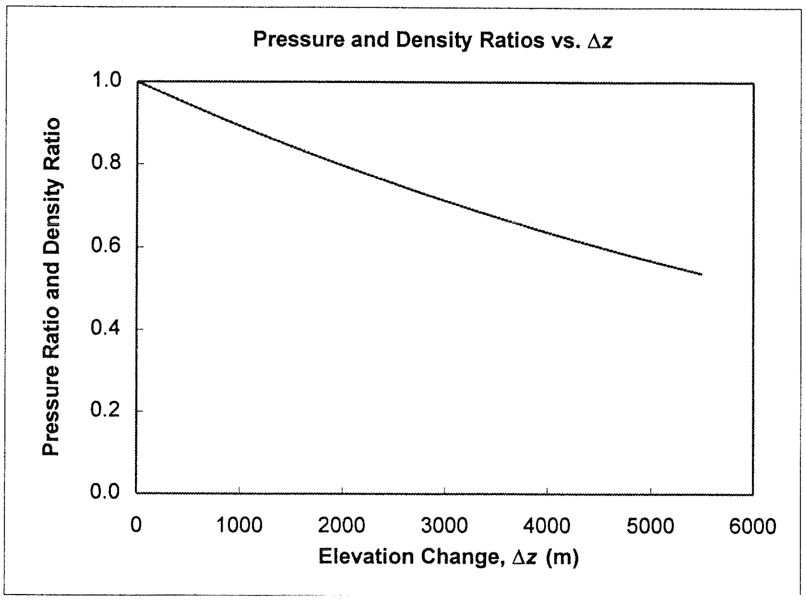
To plot p_2/p_1 and ρ_2/ρ_1, we rewrite eqs. (1) and (2) as

$$\frac{p_2}{p_1} = \frac{\rho_2}{\rho_1} = e^{-g\Delta z/RT_0}$$

The plot is presented below

Pressure and density ratio variation with altitude (T = 30°C):

Elevation Change Δz (m)	Pressure Ratio p_2/p_1 (---)	Density Ratio ρ_2/ρ_1 (---)
0	1.00	1.00
500	0.945	0.945
1000	0.893	0.893
1500	0.844	0.844
2000	0.798	0.798
2500	0.754	0.754
3000	0.713	0.713
3500	0.674	0.674
4000	0.637	0.637
4500	0.602	0.602
5000	0.569	0.569
5500	0.538	0.538



Note: Since T = constant, both ratios are the same!

Given: Martian atmosphere behaves as an ideal gas, T = constant
$M_m = 32.0$, $T = 200\ K$, $g = 3.92\ m/s^2$, $\rho_0 = 0.015\ kg/m^3$

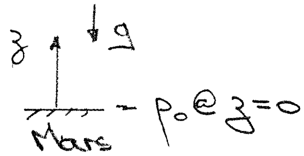
Find: Density at $z = 20\ km$.

Plot: the ratio ρ/ρ_0 (ratio of density to surface density) vs z; compare with earth's atmosphere

Solution:

Basic equations: $\frac{dp}{dz} = -\rho g$; $p = \rho R T$; $R = R_u / M_m$

Assumptions: (1) static fluid
(2) g constant
(3) ideal gas



Since T = constant, $dp = d(\rho RT) = RT\,d\rho$

$$\frac{dp}{dz} = RT\frac{d\rho}{dz} = -\rho g \quad \text{and} \quad \int_{\rho_0}^{\rho} \frac{d\rho}{\rho} = -\int_0^z \frac{g}{RT}\,dz$$

$$\ln\frac{\rho}{\rho_0} = -gz/RT \quad \text{and} \quad \frac{\rho}{\rho_0} = e^{-gz/RT} \quad \text{-----(1)}$$

Evaluating

$$R = \frac{R_u}{M_m} = 8314.3 \frac{N\cdot m}{kgmole\cdot K} \times \frac{kgmole}{32.0\,kg} = 260 \frac{N\cdot m}{kg\cdot K}$$

$$\rho = 0.015\frac{kg}{m^3} \times \exp\left[-3.92\frac{m}{s^2} \times 20\times10^3 m \times \frac{kg\cdot K}{260\,N\cdot m} \times \frac{1}{200\,K} \times \frac{N\cdot s^2}{kg\cdot m}\right]$$

$\rho = 0.00332\ kg/m^3$ ——— $\rho_{z=20\,km}$

For the Martian atmosphere, Eq. 1 gives $\rho/\rho_0 = e^{-0.0754\,z(km)}$

For the earth's atmosphere, ρ/ρ_0 is given in Table A.3

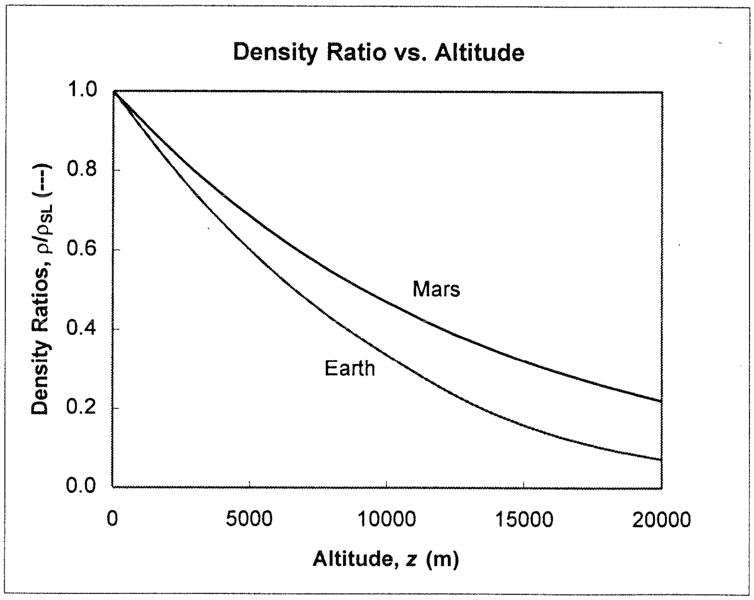
Both ρ/ρ_0 variations are plotted below

Note from the plot:
- on Mars $\rho/\rho_0 = 0.221$ at $z = 20\ km$, whereas
- on Earth, $\rho/\rho_0 = 0.073$ at $z = 20\ km$.

The difference is caused by (a) the larger gravity on Earth, and (b) temperature decrease with altitude in our atmosphere.

Density vs. elevation in Martian and Earth atmospheres:

Elevation Change Δz (m)	Density Ratio (Earth) ρ/ρ_{SL} (---)	Density Ratio (Mars) ρ/ρ_{SL} (---)
0	1.000	1.00
2000	0.8217	0.860
4000	0.6689	0.740
6000	0.5389	0.636
8000	0.4292	0.547
10000	0.3376	0.470
12000	0.2546	0.405
14000	0.1860	0.348
16000	0.1359	0.299
18000	0.09930	0.257
20000	0.07258	0.221



Given: Atmospheric conditions at ground level ($z=0$) in Denver, Colorado are $p_0 = 83.2$ kPa, $T_0 = 25°C$. Pike's peak is at elevation $z = 2690$ m

Find: Pressure on Pike's peak assuming (a) an incompressible, and (b) an adiabatic atmosphere.

Plot: p/p_0 vs z for both cases.

Solution:

Basic equations: $dp/dz = -\rho g$; $p = \rho RT$

Assumptions: (1) static fluid (2) g = constant (3) ideal gas behavior

(a) For an incompressible atmosphere $\int_{p_0}^{p} dp = -\int_0^z \rho g\, dz$

$$p - p_0 = \rho g z = \rho_0 g z = \frac{p_0}{RT_0} g z \quad \text{and} \quad p = p_0\left[1 - \frac{gz}{RT_0}\right] \text{------(1)}$$

At $z = 2690$ m

$$p = 83.2\text{ kPa}\left[1 - \frac{9.81\text{ m}}{\text{s}^2} \times 2690\text{ m} \times \frac{\text{kg}\cdot\text{K}}{287\text{ N}\cdot\text{m}} \times \frac{1}{298\text{ K}} \times \frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}}\right] = 57.5\text{ kPa} \quad p_{\rho=c}$$

(b) For an adiabatic atmosphere p/ρ^k = constant, $\rho = \rho_0\left(\frac{p}{p_0}\right)^{1/k}$

$$\frac{dp}{dz} = -\rho g = -g\rho_0\left(\frac{p}{p_0}\right)^{1/k} dz \quad \text{or} \quad \int_{p_0}^{p} \frac{dp}{p^{1/k}} = -\int_0^z \rho_0 p_0^{-1/k} g\, dz$$

Then

$$\frac{k}{(k-1)} p^{-\frac{1}{k}+1}\Big]_{p_0}^{p} = -\rho_0 p_0^{-1/k} g\, dz \quad \text{or} \quad \frac{k}{(k-1)}\left[p^{(k-1)/k} - p_0^{(k-1)/k}\right] = -\rho_0 p_0^{-1/k} g z$$

and

$$\left(\frac{k}{k-1}\right) p_0^{(k-1)/k}\left[\left(\frac{p}{p_0}\right)^{(k-1)/k} - 1\right] = -\rho_0 p_0^{-1/k} g z$$

$$\left(\frac{p}{p_0}\right)^{(k-1)/k} = 1 - \frac{(k-1)}{k} p_0^{-1/k-(k-1)/k} \rho_0 g z = 1 - \frac{(k-1)}{k} p_0^{-1} \rho_0 g z$$

and

$$\frac{p}{p_0} = \left[1 - \frac{(k-1)}{k}\frac{\rho_0}{p_0} g z\right]^{k/k-1} = \left[1 - \frac{(k-1)}{k}\frac{gz}{RT_0}\right]^{k/k-1} \text{------(2)}$$

Evaluating at $z = 2690$ m

$$p = 83.2\text{ kPa}\left[1 - \frac{0.4}{1.4} \times \frac{9.81\text{ m}}{\text{s}^2} \times 2690\text{ m} \times \frac{\text{kg}\cdot\text{K}}{287\text{ N}\cdot\text{m}} \times \frac{1}{298\text{ K}} \times \frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}}\right]^{\frac{1.4}{0.4}}$$

$$p = 60.2\text{ kPa} \quad p_{adiab.}$$

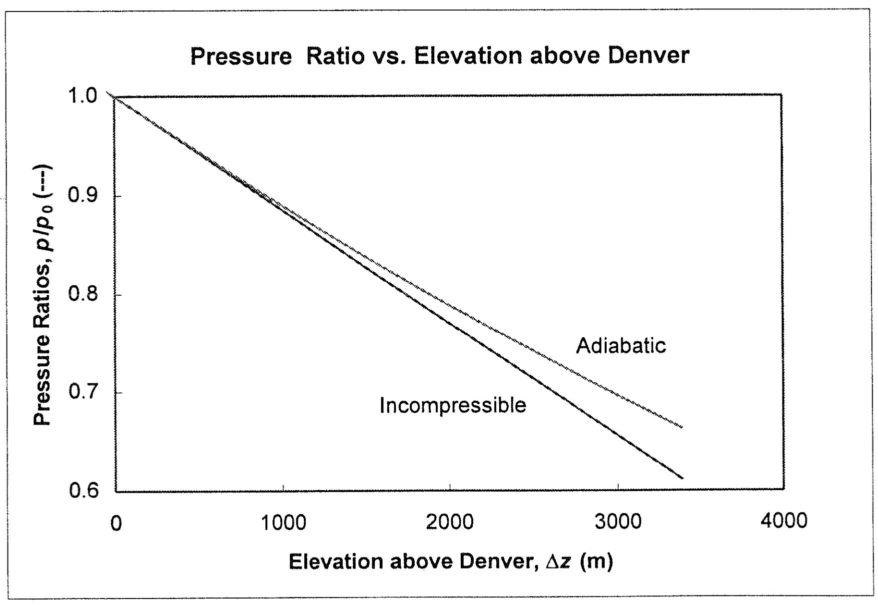
The pressure ratio p/p_0 vs z is plotted for an incompressible atmosphere (Eq. 1) and an adiabatic atmosphere (Eq. 2) below

Incompressible case $p/p_0 = [1 - 0.115 z]$ (z in km).

Adiabatic case $p/p_0 = [1 - 0.0328 z]^{3.5}$ (z in km)

Pressure ratio vs. elevation above Denver:

Elevation z (m)	Elevation above Denver z (m)	Pressure Ratio (T = C) p/p_0 (---)	Pressure Ratio (adiabatic) p/p_0 (---)
0	-1610	1.185	1.20
500	-1110	1.127	1.13
1000	-610	1.070	1.07
1500	-110	1.013	1.01
2000	390	0.955	0.956
2500	890	0.898	0.902
3000	1390	0.841	0.849
3500	1890	0.783	0.800
4000	2390	0.726	0.752
4300	2690	0.691	0.724
4500	2890	0.669	0.706
5000	3390	0.611	0.662



Problem 3.42

Open-Ended Problem Statement: A hydropneumatic elevator consists of a piston-cylinder assembly to lift the elevator cab. Hydraulic oil, stored in an accumulator tank pressurized by air, is valved to the piston as needed to lift the elevator. When the elevator descends, oil is returned to the accumulator. Design the least expensive accumulator that can satisfy the system requirements. Assume the lift is 3 floors, the maximum load is 10 passengers, the maximum system pressure is 800 kPa (gage). For column bending strength, the piston diameter must be at least 150 mm. The elevator cab and piston have a combined mass of 3,000 kg, and are to be purchased. Perform the analysis needed to define, as a function of system operating pressure, the piston diameter, the accumulator volume and diameter, and the wall thickness. Discuss safety features that your company should specify for the complete elevator system. Would it be preferable to use a completely pneumatic design or a completely hydraulic design? Why?

Discussion: The design requirements are specified, except that a typical floor height is about 12 ft, making the total required lift about 36 ft.)

A spreadsheet was used to calculate the system properties for various pressures. Results are presented on the next page, followed by a sample calculation.

Total cost dropped quickly as system pressure was increased. A shallow minimum was reached in the 100-110 psig range.

The lowest-cost solution was obtained at a system pressure of about 100 psig. At this pressure, the reservoir of 140 gal required a 3.30 ft diameter pressure sphere with a 0.250 in. wall thickness. The welding cost was $311 and the material cost $433, for a total cost of $744.

Accumulator wall thickness was constrained at 0.250 in. for pressures below 100 psi; it increased for higher pressures (this caused the discontinuity in slope of the curve at 100 psig). The mass of steel became constant above 110 psig.

No allowance was made for the extra volume needed to pressurize the accumulator.

Fail-safe design is essential for an elevator to be used by the public. The control circuitry should be redundant. Failures must be easy to spot. For this reason, hydraulic actuation is good: leaks will be readily apparent. The final design must be reviewed, approved, and stamped by a professional engineer since the design involves public safety.

The terminology used in the solution is defined in Table 1.

Table 1. Symbols, definitions, and units

Symbol	Definition	Units
p	system pressure	psig
A_p	area of lift piston	in.^{2}
$\forall_{oil}$	volume of oil	gal
D_s	diameter of (spherical) accumulator	ft
t	wall thickness of spherical accumulator	in.
A_w	area of weld	in.^{2}
C_w	cost of weld	$
M_s	mass of (steel) accumulator	lbm
C_s	cost of steel	$
C_t	total cost	$

Results of the system simulation and sample calculations are presented on the next page.

Table 2. Results of system simulation

Input Data:

Cab and piston weight:	W_{cab} =	6,000	lbf
Passenger weight:	W_{pax} =	1,500	lbf
Total weight:	W_{tot} =	7,500	lbf
Allowable stress:	σ =	4,000	psi
Minimum wall thickness:	t =	0.250	in.
Welding cost factor:	cf_w =	5.00	\$/in.^{2}
Steel cost factor:	cf_s =	1.25	\$/pound

Results:

p (psig)	A_p (in.^{2})	V_{oil} (gal)	D_s (ft)	t (in.)	A_w (in.^{2})	C_w (\$)	M_s (lbm)	C_s (\$)	C_t (\$)
20	375	701	5.64	0.250	106.2	\$531	1012	\$1,265	\$1,796
30	250	468	4.92	0.250	92.8	\$464	772	\$965	\$1,429
40	188	351	4.47	0.250	84.3	\$422	638	\$797	\$1,218
50	150	281	4.15	0.250	78.3	\$391	549	\$687	\$1,078
60	125	234	3.91	0.250	73.7	\$368	487	\$608	\$976
70	107	200	3.71	0.250	70.0	\$350	439	\$549	\$899
80	93.8	175	3.55	0.250	66.9	\$335	402	\$502	\$837
90	83.3	156	3.41	0.250	64.4	\$322	371	\$464	\$786
100	75.0	140	3.30	0.250	62.1	\$311	346	\$433	\$743
110	68.2	128	3.19	0.263	63.4	\$317	342	\$428	\$745
120	62.5	117	3.10	0.279	65.3	\$326	342	\$428	\$754
130	57.7	108	3.02	0.294	67.1	\$335	342	\$428	\$763

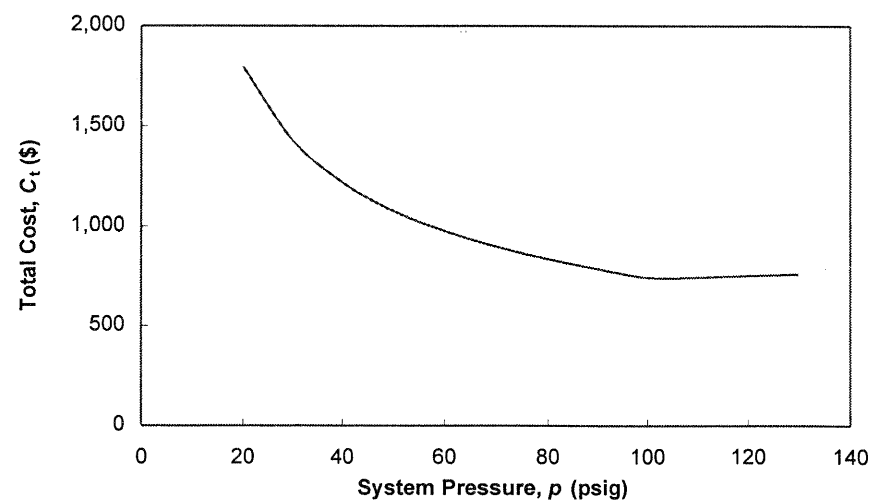


Fig. 1 Total cost versus system pressure

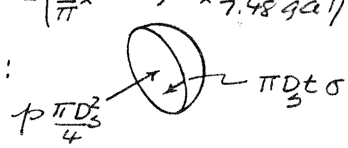
Sample Calculation (p = 20 psig):

$$W_t = p A_p \; ; \; A_p = \frac{W_t}{p} = 7500\ lbf \times \frac{in.^2}{20\ lbf} = 375\ in.^2$$

$$\forall_{oil} = A_p L = 375\ in.^2 \times \frac{1}{36\ ft} \times \frac{ft^2}{144\ in.^2} \times 7.48 \frac{gal}{ft^3} = 701\ gal$$

$$\forall_{oil} = \forall_s = \frac{4\pi R_s^3}{3} = \frac{\pi D_s^3}{6} \; ; \; D_s = \left(\frac{6\forall_{oil}}{\pi}\right)^{1/3} = \left(\frac{6}{\pi} \times 701\ gal \times \frac{ft^3}{7.48\ gal}\right)^{1/3} = 5.64\ ft$$

From a force balance on the sphere:



Thus $p\frac{\pi D_s^2}{4} = \pi D_s t\sigma$, so $t = \frac{p}{\sigma}\frac{D_s}{4} = \frac{1}{4} \times 20\,\frac{lbf}{in.^2} \times \frac{in.^2}{4000\,lbf} \times 5.64\,ft \times 12\,\frac{in.}{ft} = 0.0846\,in.$

Therefore $t = t_{min} = 0.250$ in.

$$A_W = \pi D_s t = \pi \times 5.64\,ft \times 0.25\,in. \times 12\,\frac{in.}{ft} = 106\ in.^2$$

$$C_W = \frac{\$5.00}{in.^2} \times 106\,in.^2 = \$531$$

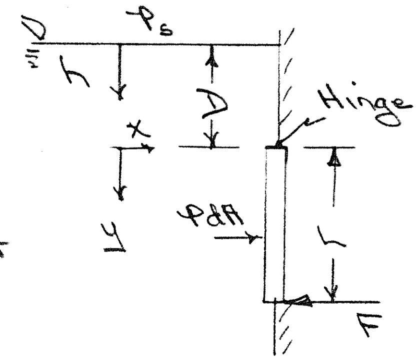
$$M_s = 4\pi R_s^2 t \rho_s = \pi D_s^2 t\, SG_s\, \rho_{H_2O} = \pi \times (5.64)^2 ft^2 \times 0.25\,in. \times 7.8 \times 62.4\,\frac{lbm}{ft^3} \times \frac{ft}{12\,in.} = 1012\,lbm$$

$$C_s = \frac{\$1.25}{lbm} \times 1012\,lbm = \$1265$$

and

$$C_t = C_W + C_s = \$531 + \$1265 = \$1,796$$

C_t



Given: Door, of width $b = 1\,m$, located in plane vertical wall of water tank is hinged along upper edge.
$D = 1\,m$, $L = 1.5\,m$
Atmospheric pressure acts on outer surface of door; force F is applied at lower applied at lower edge to keep door closed

Find: (a) Force F, if $p_s = p_{atm}$.
(b) Force F, if $p_s = 0.5$ atm.

Plot: F/F_0 over range of p_s/p_{atm}. (F_0 is force required when $p_s = p_{atm}$)

Solution:

Basic equations: $\frac{dp}{dh} = \rho g$; $F_R = \int p\,dA$; $\sum M_z = 0$

Assumptions: (1) static fluid (2) ρ = constant
(3) door is in equilibrium

Since $\sum M_z = 0$ for equilibrium, taking moments about the hinge

$$\sum M_z = 0 = FL - \int y\,p\,dA = F.L - \int_0^L y\,p\,b\,dy$$

and

$$F = \frac{1}{L}\int_0^L y\,p\,b\,dy \quad \text{------------ (1)}$$

Note: We will obtain a general expression for F (needed for the plot) and then simplify for cases (a) and (b)

Since $dp = \rho g\,dh$, then $p = p_s + \rho g h$
$h = D + y$ and hence $p = p_s + \rho g(D+y)$.

Because p_{atm} acts on the outside of the door, p_s is the surface gage pressure.

From Eq (1), $F = \frac{1}{L}\int_0^L y\left[p_{s_g} + \rho g(D+y)\right] b\,dy$

$$F = \frac{b}{L}\left[p_{s_g}\frac{y^2}{2} + \rho g\left(\frac{Dy^2}{2} + \frac{y^3}{3}\right)\right]_0^L$$

$$F = \frac{b}{L}\left[p_{s_g}\frac{L^2}{2} + \rho g\left(\frac{DL^2}{2} + \frac{L^3}{3}\right)\right] = b\left[p_{s_g}\frac{L}{2} + \rho g L\left(\frac{D}{2} + \frac{L}{3}\right)\right] \quad \text{---- (2)}$$

(a) For $p_s = p_{atm}$, $p_{s_g} = 0$

$$F_0 = \rho g b L\left(\frac{D}{2} + \frac{L}{3}\right) \quad \text{------------ (3)}$$

$$F_0 = 999\,\frac{kg}{m^3} \times 9.81\,\frac{m}{s^2} \times 1\,m \times 1.5\,m\left(\frac{1\,m}{2} + \frac{1.5\,m}{3}\right) \times \frac{N\,s^2}{kg\,m} \times \frac{kN}{10^3\,N} = 14.7\,kN \leftarrow F_0$$

(b) For $p_{sg} = 0.5$ atm (50.6 kPa), from Eq (2)

$$F = p_{sg}\frac{bL}{2} + \rho g L b\left(\frac{D}{2}+\frac{L}{3}\right)$$

$$F = 50.6\,\frac{kN}{m^2}\times 1\,m \times \frac{1.5\,m}{2} + 14.7\,kN = 52.7\,kN \quad \longleftarrow \quad F)_{p_s=0.5\,atm}$$

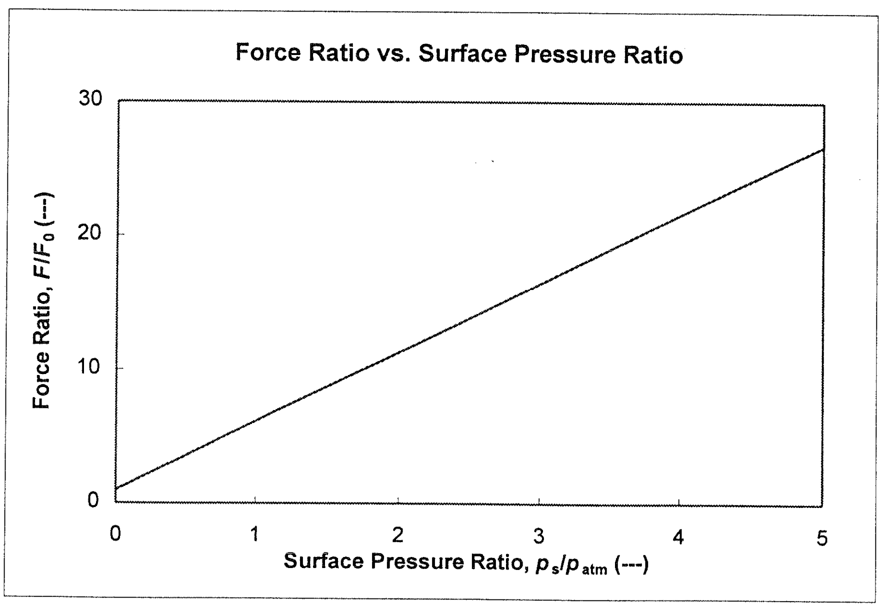
From Eqs (2) and (3) we can write

$$\frac{F}{F_0} = \frac{b\left[p_{sg}\frac{L}{2} + \rho g L\left(\frac{D}{2}+\frac{L}{3}\right)\right]}{\rho g b L\left(\frac{D}{2}+\frac{L}{3}\right)} = 1 + \frac{p_{sg}}{2\rho g\left(\frac{D}{2}+\frac{L}{3}\right)}$$

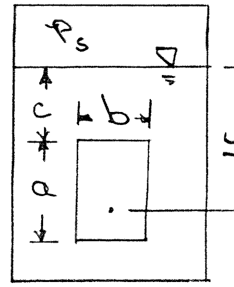
Substituting values

$$\frac{F}{F_0} = 1 + \frac{p_{sg}}{0.194} \quad \text{(with } p_{sg} \text{ in atmospheres)}. \qquad (4)$$

F/F_0 is plotted as a function of $p_{s\,gage}/p_{atm}$



Given: Door located in plane vertical wall of water tank as shown
$a = 1.5\,m$, $b = 1\,m$, $c = 1\,m$.
Atmospheric pressure acts on outer surface of door



Find: (a) For $p_s = p_{atm}$, resultant force on door and line of action of force
(b) Resultant force and line of action if $p_s = 0.3$ atm (gage)

Plot: F/F_0 and y'/y_c over range of p_s/p_{atm}. (F_0 is resultant force when $p_s = p_{atm}$; y_c is y coordinate of centroid)

Solution:

Basic equations: $\dfrac{dp}{dy} = \rho g$; $F_R = \int_A p\,dA$; $y' F_R = \int y p\,dA$

Assumptions: (1) static liquid
(2) incompressible liquid

Note: We will obtain a general expressions for F and y' (needed for the plot) and then simply for cases (a) & (b).

Since $dp = \rho g\,dy$ then $p = p_s + \rho g y$
Because p_{atm} acts on the outside of the door, then p_s is the surface gage pressure

$$F_R = \int p\,dA = \int_c^{c+a} p\,b\,dy = \int_c^{c+a} (p_s + \rho g y)\,b\,dy = b\left[p_s y + \rho g \frac{y^2}{2}\right]_c^{c+a}$$

$$F_R = b\left[p_s a + \frac{\rho g}{2}\{(c+a)^2 - c^2\}\right] = b\left[p_s a + \frac{\rho g}{2}(a^2 + 2ac)\right] \quad \text{---- (1)}$$

$$y' F_R = \int y p\,dA \quad \text{and} \quad y' = \frac{1}{F_R}\int_c^{c+a} y(p_s + \rho g y)\,b\,dy$$

$$y' = \frac{b}{F_R}\left[p_s \frac{y^2}{2} + \rho g \frac{y^3}{3}\right]_c^{c+a}$$

$$y' = \frac{b}{F_R}\left[\frac{p_s}{2}\{(c+a)^2 - c^2\} + \frac{\rho g}{3}\{(c+a)^3 - c^3\}\right] \quad \text{------ (2)}$$

(a) For $p_s = 0$ (gage) then
from Eq. 1 $\quad F_R = \dfrac{\rho g b}{2}(a^2 + 2ac)$.

$$F_R = \frac{1}{2} \times 999\,\frac{kg}{m^3} \times 9.81\,\frac{m}{s^2} \times 1\,m\,\left[(1.5\,m)^2 + 2(1.5\,m)(1\,m)\right]\frac{N \cdot s^2}{kg \cdot m} = 25.7\,kN \quad \leftarrow F_{R_0}$$

From Eq. 2

$$y' = \frac{b}{F_{R_0}}\frac{\rho g}{3}\left[(c+a)^3 - c^3\right]$$

$$y' = \frac{1\,m}{25.7\,kN} \times \frac{999}{3}\,\frac{kg}{m^3} \times 9.81\,\frac{m}{s^2}\left[(2.5)^3 - 1\right]m^3\,\frac{N \cdot s^2}{kg \cdot m} \times \frac{kN}{10^3\,N} = 1.86\,m \quad \leftarrow y'_0$$

(b) For $p_s = 0.3$ atm (gage) then

from Eq. 1 $\quad F_R = b\left[p_s a + \frac{\rho g}{2}(a^2 + 2ac)\right]$

$$F_R = 1\,m\left[0.3\,atm \times 1.01\times10^5 \frac{N}{m^2 \cdot atm}(1.5\,m) + \frac{1}{2}\times 999\frac{kg}{m^3}\times 9.81\frac{m}{s^2}\left\{(1.5)^2 + 2(1.5)(1)\right\}m^2 \cdot \frac{N \cdot s^2}{kg \cdot m}\right]$$

$F_R = 71.2\,kN$ ——— F_R

$$y' = \frac{b}{F_R}\left[\frac{p_s}{2}\left\{(c+a)^2 - c^2\right\} + \frac{\rho g}{3}\left\{(c+a)^3 - c^3\right\}\right]$$

$$y' = \frac{1\,m}{71.2\,kN}\left[\frac{1}{2}\times 0.3\,atm\times 1.01\times10^5\frac{N}{m^2\cdot atm}\left\{(2.5)^2 - 1\right\} + \frac{1}{3}\times 999\frac{kg}{m^3}\times 9.81\frac{m}{s^2}\times\left\{(2.5)^3 - 1\right\}m^3 \times \frac{N\cdot s^2}{kg\cdot m}\right]\frac{kN}{10^3\,N}$$

$y' = 1.79\,m$ ——— y'

The value of F/F_o is obtained from Eq. 1 and $F_{R_o} = 25.7\,kN$.

$$\frac{F}{F_o} = \frac{1}{25.7\,kN}\, b\left[p_s a + \frac{\rho g}{2}(a^2 + 2ac)\right] = 0.0389\left[151.5\, p_s + 25.7\right]$$

with p_s in atm. ——— F/F_o

For the gate $y_c = c + \frac{a}{2} = 1.75\,m$. Then from Eq. 2

$$y'/y_c = \frac{b}{F_R(1.75)}\left[\frac{p_s}{2}\left\{(c+a)^2 - c^2\right\} + \frac{\rho g}{3}\left\{(c+a)^3 - c^3\right\}\right] = \frac{0.571}{F}\left[265\, p_s + 47.8\right]$$

with F in kN, p_s in atm ——— y'/y_c

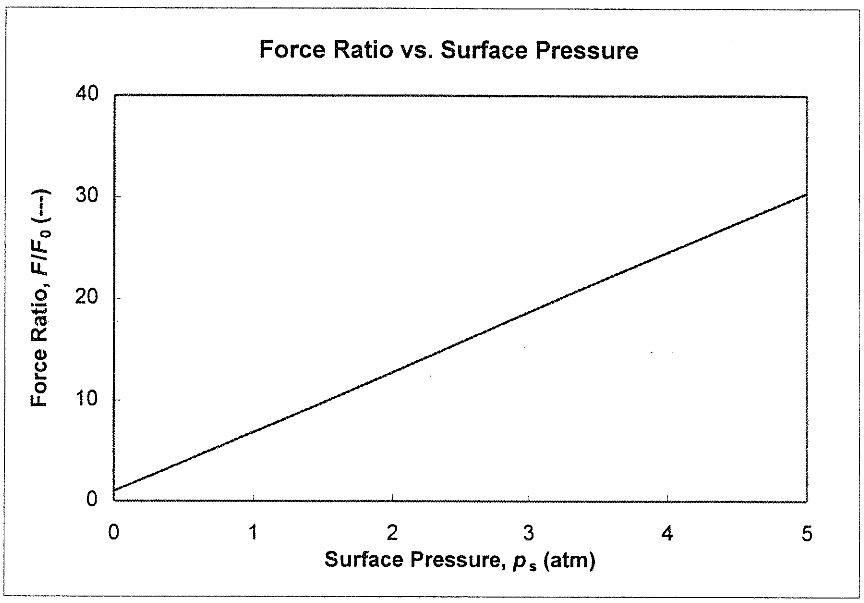
The plots are shown below

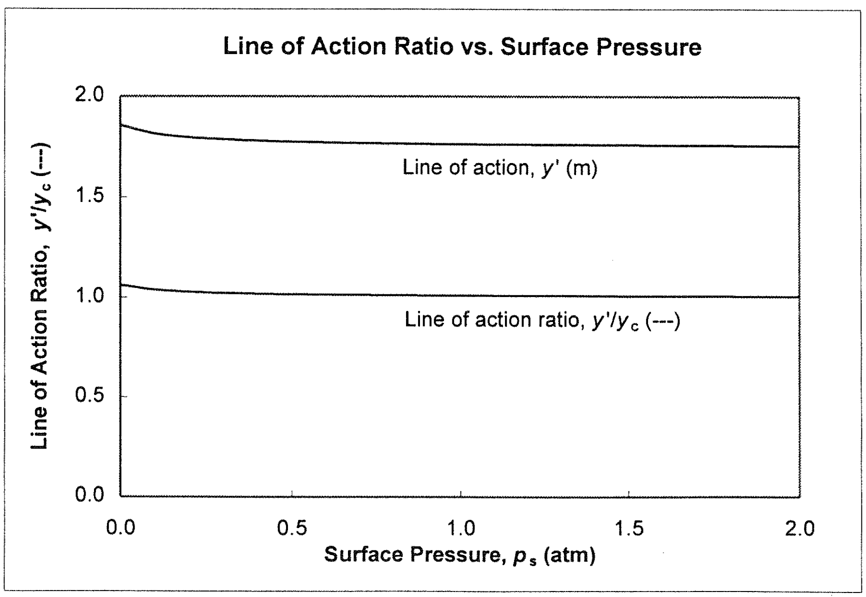
Note: The force on the gate increases linearly with increase in surface pressure.

The line of action of the resultant force is always below the centroid of the gate; y'/y_c approaches unity as the surface pressure is increased.

Force ratio and line of action ratio vs. surface pressure:

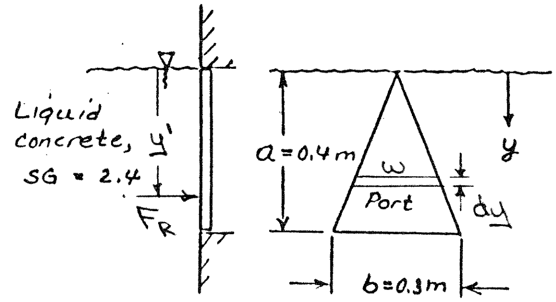
Surface Pressure, p_s (atm)	Force Ratio, F/F_0 (---)	Force, F_0 (kN)	Line of Action Ratio, y'/y_c (---)	Line of Action, y' (m)
0	1.00	25.7	1.0623	1.86
0.1	1.59	40.8	1.0388	1.82
0.2	2.18	56.0	1.0281	1.80
0.3	2.77	71.1	1.0219	1.79
0.5	3.95	101	1.0151	1.78
1.0	6.89	177	1.00822	1.76
2.0	12.8	329	1.00399	1.76
3.0	18.7	480		
4.0	24.6	632		
5.0	30.5	783		





Given: Triangular port in the side of a form containing liquid concrete, as shown.

Find: (a) the resultant force that acts on the port.
(b) point of application of the resultant force.



Solution:

Basic equations: $F_R = \int p\,dA \qquad \frac{dp}{dy} = \rho g \qquad \rho \equiv SG\,\rho_{H_2O}$

$$\Sigma M_s = y' F_R = \int y\, dF_R = \int y\, p\, dA$$

Assumptions: (1) static fluid (2) ρ = constant
(3) p_{atm} acts at surface & on outside of port.

Under these assumptions, the pressure at any point in the liquid is given by $p = \rho g y$.

Also $dA = w\,dy$ where $\frac{w}{b} = \frac{y}{a}$ or $w = \frac{b}{a}y$

Then
$$F_R = \int p\,dA = \int_0^a \rho g y\, w\, dy = \int_0^a \rho g y \frac{b}{a} y\, dy = \int_0^a \rho g \frac{b}{a} y^2\, dy$$

$$F_R = \rho g \frac{b}{a}\left[\frac{y^3}{3}\right]_0^a = \rho g \frac{ba^2}{3} = SG\,\rho_{H_2O}\, g\, \frac{ba^2}{3}$$

$$F_R = \frac{2.4}{3} \times 999\,\frac{kg}{m^3} \times 9.81\,\frac{m}{s^2} \times 0.3\,m \times (0.4)^2\,m^2 \times \frac{N\cdot s^2}{kg\cdot m} = 376\,N \qquad F_R$$

$$\Sigma M_{surf} = y' F_R = \int y\, p\, dA = \int_0^a y \rho g y \frac{b}{a} y\, dy = \int_0^a \rho g \frac{b}{a} y^3\, dy$$

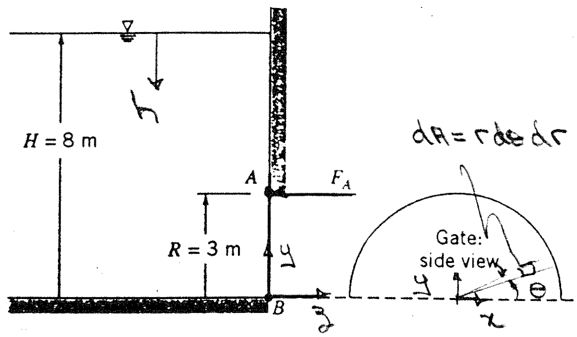
$$y' F_R = \rho g \frac{b}{a}\left[\frac{y^4}{4}\right]_0^a = \rho g \frac{b}{a}\frac{a^4}{4} = SG\,\rho_{H_2O}\, g\, \frac{ba^3}{4}$$

$$y' = \frac{M_{sur}}{F_R} = SG\,\rho_{H_2O}\, g\, \frac{ba^3}{4} \times \frac{3}{SG\,\rho_{H_2O}\, g\, b a^2} = \frac{3a}{4} = \frac{3}{4} \times 0.4\,m = 0.3\,m \qquad y'$$

Problem 3.46

Given: Semicircular plane gate AB is hinged along B and held in place by horizontal force F_A.

Find: Force F_A required to hold gate in place



Solution:

Basic equations: $\frac{dp}{dh} = \rho g$; $F_R = \int p\,dA$; $\sum M_z = 0$

Assumptions: (1) static fluid (2) ρ = constant
(3) door is in equilibrium.

Since $\sum M_z = 0$ for equilibrium, taking moments about the hinge B,

$$F_A = \frac{1}{R}\int y\,p\,dA$$

Since $dp = \rho g\,dh$ and $p = p_s + \rho g h$

Because the free surface is at atmospheric pressure, and atmospheric pressure acts on the outside of the gate, $p = \rho g h$

and $F_A = \frac{1}{R}\int y \rho g h\,dA$

For the circular gate, $dA = r\,dr\,d\theta$, $y = r\sin\theta$, $h = H - y$.

so $F_A = \frac{1}{R}\int_0^{\pi}\int_0^{R} r\sin\theta\,\rho g(H - r\sin\theta)\,r\,dr\,d\theta$

$$F_A = \frac{\rho g}{R}\int_0^{\pi}\int_0^{R}(Hr^2 - r^3\sin\theta)\sin\theta\,dr\,d\theta = \frac{\rho g}{R}\int_0^{\pi}\left[\frac{Hr^3}{3} - \frac{r^4}{4}\sin\theta\right]_0^R \sin\theta\,d\theta$$

$$F_A = \frac{\rho g}{R}\int_0^{\pi}\left[\frac{HR^3}{3} - \frac{R^4}{4}\sin\theta\right]\sin\theta\,d\theta = \frac{\rho g}{R}\int_0^{\pi}\left[\frac{HR^3}{3}\sin\theta - \frac{R^4}{4}\sin^2\theta\right]d\theta$$

$$= \frac{\rho g}{R}\left[\left(-\frac{HR^3}{3}\cos\theta\right)_0^{\pi} - \frac{R^4}{4}\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi}\right] = \frac{\rho g}{R}\left[\frac{2HR^3}{3} - \frac{\pi R^4}{8}\right]$$

$$F_A = \rho g\left[\frac{2HR^2}{3} - \frac{\pi R^3}{8}\right]$$

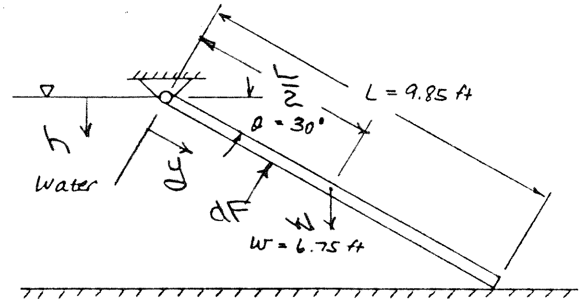
$$= 999\frac{kg}{m^3} \times 9.81\frac{m}{s^2}\left[\frac{2 \times 8m \times 9m^2}{3} - \frac{\pi}{8}27m^3\right] \times \frac{N\cdot s^2}{kg\cdot m} \times \frac{kN}{10^3 N}$$

$F_A = 366$ kN ← F_A

Problem 3.47

Given: Plane gate of uniform thickness and width $w = 6.75$ ft holds back a depth of water as shown.

Find: the minimum weight, W, of the gate needed to insure gate remains closed.



Solution:

Basic equations: $F = \int p\,dA \qquad \frac{dp}{dh} = \rho g$

$\Sigma M_o = 0 \qquad M = \int y\,dF$

Assumptions: (1) static fluid (2) ρ = constant
(3) p_{atm} acts at surface of water and along top surface of the gate.

Under these assumptions, the pressure at any point in the liquid is given by $p = \rho g h = \rho g y \sin\theta$

$$\Sigma M_o = 0 = \int y\,dF - W \frac{L}{2}\cos\theta \qquad \left\{ \begin{array}{l} \text{moment about axis} \\ \text{through O is + cc} \end{array} \right\}$$

then

$$W = \frac{2}{L\cos\theta}\int y\,dF = \frac{2}{L\cos\theta}\int y\,p\,dA = \frac{2}{L\cos\theta}\int_0^L y\,\rho g y \sin\theta\, w\,dy$$

$$W = \frac{2\rho g w \tan\theta}{L}\int_0^L y^2\,dy = \frac{2\rho g w\tan\theta}{L}\left[\frac{y^3}{3}\right]_0^L$$

$$W = \frac{2}{3}\rho g w L^2 \tan\theta$$

$$W = \frac{2}{3} \times 1.94\,\frac{\text{slug}}{\text{ft}^3} \times 32.2\,\frac{\text{ft}}{\text{s}^2} \times 6.75\,\text{ft} \times (9.85)^2\,\text{ft}^2 \tan 30^\circ \times \frac{\text{lbf}\cdot\text{s}^2}{\text{ft}\cdot\text{slug}}$$

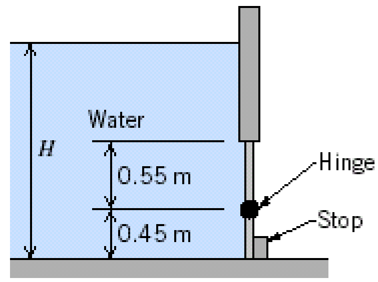
$W_{min} \approx 15{,}800$ lbf ← W_{min}

Problem 3.48

A rectangular gate (width w
what depth H will the gate tip?

Given: Gate geometry

Find: Depth H at which gate tips



Solution

This is a problem with atmospheric pressure on both sides of the plate, so we can first determine the location of the center of pressure with respect to the free surface, using Eq.3.11c (assuming depth H)

$$y' = y_c + \frac{I_{xx}}{A \cdot y_c} \quad \text{and} \quad I_{xx} = \frac{w \cdot L^3}{12} \quad \text{with} \quad y_c = H - \frac{L}{2}$$

where L = 1 m is the plate height and w is the plate width

Hence

$$y' = \left(H - \frac{L}{2}\right) + \frac{w \cdot L^3}{12 \cdot w \cdot L \cdot \left(H - \frac{L}{2}\right)} = \left(H - \frac{L}{2}\right) + \frac{L^2}{12 \cdot \left(H - \frac{L}{2}\right)}$$

But for equilibrium, the center of force must always be at or below the level of the hinge so tha
stop can hold the gate in place. Hence we must have

$$y' > H - 0.45 \cdot m$$

Combining the two equations

$$\left(H - \frac{L}{2}\right) + \frac{L^2}{12 \cdot \left(H - \frac{L}{2}\right)} \geq H - 0.45 \cdot m$$

Solving for *H*

$$H \leq \frac{L}{2} + \frac{L^2}{12 \cdot \left(\frac{L}{2} - 0.45 \cdot m\right)}$$

$$H \leq \frac{1 \cdot m}{2} + \frac{(1 \cdot m)^2}{12 \times \left(\frac{1 \cdot m}{2} - 0.45 \cdot m\right)}$$

$$H \leq 2.167 \cdot m$$

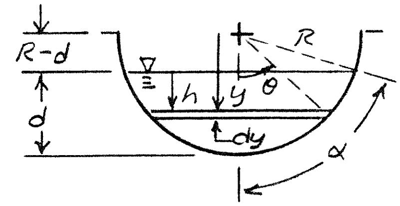
Given: Semi-cylindrical trough, partly filled with water to depth d.

Find: (a) General expressions for F_R and y' on end of trough, if open to atmosphere.
(b) Plots of results vs. d/R for $0 \le d/R \le 1$.

Solution: Apply basic equations for hydrostatics of incompressible liquid.

Computing equations: $p = \rho g h \qquad F_R = \int_A p\,dA \qquad y' F_R = \int_A y\,p\,dA$

Assumptions: (1) Static liquid
(2) ρ = constant



$$p = \rho g h = \rho g\left[y - (R-d)\right]$$

$$= \rho g R\left[\frac{y}{R} - \left(1 - \frac{d}{R}\right)\right] = \rho g R(\cos\theta - \cos\alpha)$$

$h = y - (R-d)$

$dA = w\,dy = 2R\sin\theta\,dy\,; \quad y = R\cos\theta$

$\cos\alpha = \frac{R-d}{R} = 1 - \frac{d}{R}$

$dy = -R\sin\theta\,d\theta$

$w = 2R\sin\theta$

$$F_R = \int_{R-d}^{R} p\,w\,dy = \int_{R-d}^{R} \rho g R(\cos\theta - \cos\alpha)\,2R\sin\theta\,(-R\sin\theta)\,d\theta$$

The new limits are $y = R \to \theta = 0$ and $y = R-d \to \theta = \alpha$, so

$$F_R = 2\rho g R^3 \int_\alpha^0 (-\sin^2\theta\cos\theta + \sin^2\theta\cos\alpha)\,d\theta = 2\rho g R^3 \int_0^\alpha (\sin^2\theta\cos\theta - \sin^2\theta\cos\alpha)\,d\theta$$

$$= 2\rho g R^3\left[\frac{\sin^3\theta}{3} - \cos\alpha\left(\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right)\right]_0^\alpha = 2\rho g R^3\left[\frac{\sin^3\theta}{3} - \cos\alpha\left(\frac{\theta}{2} - \frac{\sin\theta\cos\theta}{2}\right)\right]_0^\alpha$$

$$F_R = 2\rho g R^3\left[\frac{\sin^3\alpha}{3} - \cos\alpha\left(\frac{\alpha}{2} - \frac{\sin\alpha\cos\alpha}{2}\right)\right]$$ F_R

$$y' F_R = \int_{R-d}^{R} y\,p\,w\,dy = \int_{R-d}^{R} R\cos\theta\,\rho g R(\cos\theta - \cos\alpha)\,2R\sin\theta(-R\sin\theta)\,d\theta$$

$$= 2\rho g R^4 \int_0^\alpha \sin^2\theta\cos\theta(\cos\theta - \cos\alpha)\,d\theta = 2\rho g R^4 \int_0^\alpha (\sin^2\theta\cos^2\theta - \cos\alpha\sin^2\theta\cos\theta)\,d\theta$$

$$= 2\rho g R^4\left[\frac{1}{8}\left(\theta - \frac{\sin 4\theta}{4}\right) - \cos\alpha\,\frac{\sin^3\theta}{3}\right]_0^\alpha$$

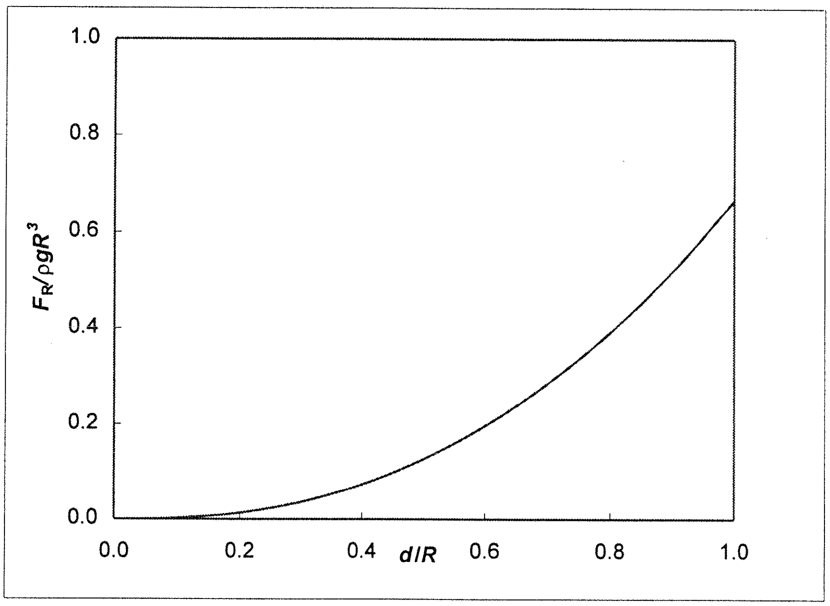
$$y' F_R = 2\rho g R^4\left[\frac{1}{8}\left(\alpha - \frac{\sin 4\alpha}{4}\right) - \cos\alpha\,\frac{\sin^3\alpha}{3}\right]$$ $y'F_R$

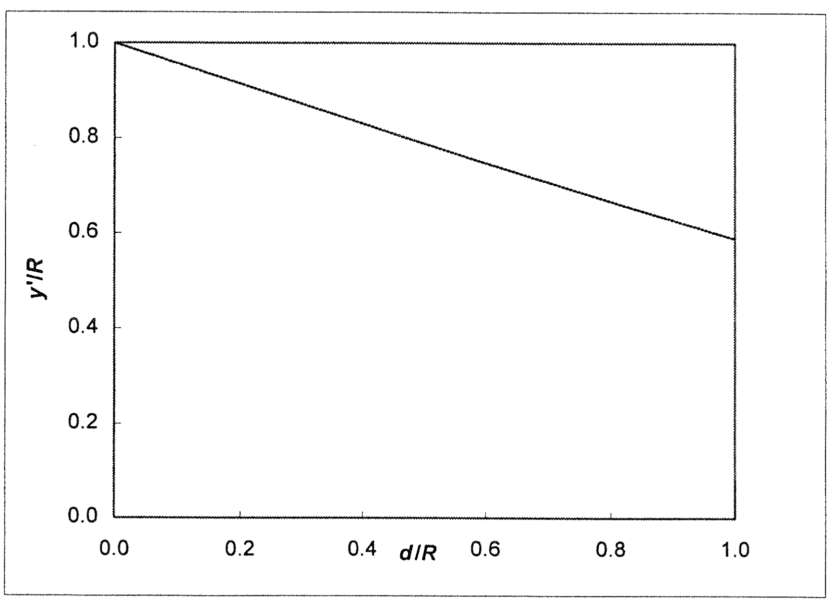
and

$$y' = \frac{y' F_R}{F_R} \quad \text{or} \quad y'/R = \frac{y' F_R}{R F_R}$$ y'

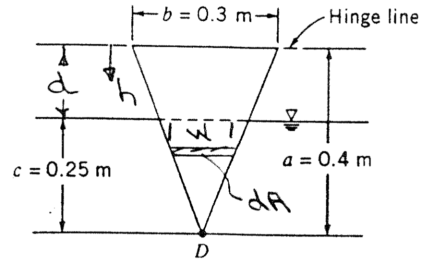
Resultant force and line of action on end of semi-cylindrical water trough:

d/R	α (rad)	α (deg)	$F_R/\rho g R^3$	$y'F_R/\rho g R^4$	y'/R
0	0.001	0.08	7.54E-16	7.54E-16	1.000
0.05	0.318	18.2	0.000419	0.000410	0.979
0.1	0.451	25.8	0.00236	0.00226	0.957
0.2	0.644	36.9	0.0132	0.0121	0.915
0.3	0.795	45.6	0.0360	0.0314	0.873
0.4	0.927	53.1	0.0730	0.0606	0.831
0.5	1.05	60.0	0.126	0.0994	0.790
0.6	1.16	66.4	0.196	0.147	0.749
0.7	1.27	72.5	0.285	0.202	0.708
0.8	1.37	78.5	0.392	0.262	0.668
0.9	1.47	84.3	0.520	0.326	0.628
1.0	1.57	90.0	0.667	0.393	0.589





Given: Window, in shape of isosceles triangle and hinged at the top is located in the vertical wall of a form that contains concrete.



Find: the minimum force applied at D needed to keep the window closed.

Plot: the results over the range of concrete depth $0 \le c \le a$

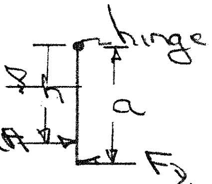
Solution:

Basic equations: $\frac{dp}{dh} = \rho g$, $F = \int p\,dA$, $\Sigma M = 0$

Assumptions: (1) static fluid (2) ρ = constant
(3) p_{atm} acts at the free surface and on the outside of the window.

Then $dp = \rho g\,dh$ gives $p = \rho g(h-d)$ for $h > d$
and $p = 0$ for $h < d$
where $d = a - c$

Summing moments about the hinge



$$F_D = \frac{1}{a}\int h\,P\,dA = \frac{1}{a}\int_d^a h\,\rho g(h-d)\,w\,dh$$

From the law of similar triangles $\frac{w}{b} = \frac{a-h}{a}$; $w = \frac{b}{a}(a-h)$

$$F_D = \frac{b}{a^2}\rho g\int_d^a h(h-d)(a-h)\,dh \qquad \{\rho = SG_{concrete}\,\rho_{H_2O}\}$$

$$F_D = \frac{b}{a^2}\rho g\int_d^a \left[-h^3 + h^2(a+d) - adh\right]dh$$

$$F_D = \frac{b}{a^2}\rho g\left[-\frac{h^4}{4} + \frac{h^3}{3}(a+d) - \frac{1}{2}adh^2\right]_d^a$$

$$F_D = \frac{b}{a^2}\rho g\left[-\frac{1}{4}(a^4-d^4) + \frac{1}{3}(a^3-d^3)(a+d) - \frac{1}{2}ad(a^2-d^2)\right]$$

$$F_D = b\rho g a^2\left[-\frac{1}{4}\left(1-\frac{d^4}{a^4}\right) + \frac{1}{3}\left(1-\frac{d^3}{a^3}\right)\left(1+\frac{d}{a}\right) - \frac{1}{2}\frac{d}{a}\left(1-\frac{d^2}{a^2}\right)\right] \quad (1)$$

Evaluating with $\rho = SG_{conc}\,\rho_{H_2O}$ (SG = 2.5 - Table A.1)

$$b\rho g a^2 = 0.3\,m \times 2.5\times10^3\,\frac{kg}{m^3} \times 9.81\,\frac{m}{s^2} \times (0.4)^2\,m^2 \times \frac{N\cdot s^2}{kg\cdot m} = 1177\,N$$

For $a = 0.4\,m$, $c = 0.25\,m$, $d = a - c = 0.15\,m$, $\frac{d}{a} = 0.375$

the term [ ] in Eq. 1 has a value of 0.0280

Then for the conditions given

$F_D = 1177N \times 0.0280 = 33.0N$

To plot F_D vs c/a for $0 \le c \le a$, recognize

. Since $d = a - c$, then $\frac{d}{a} = 1 - \frac{c}{a}$

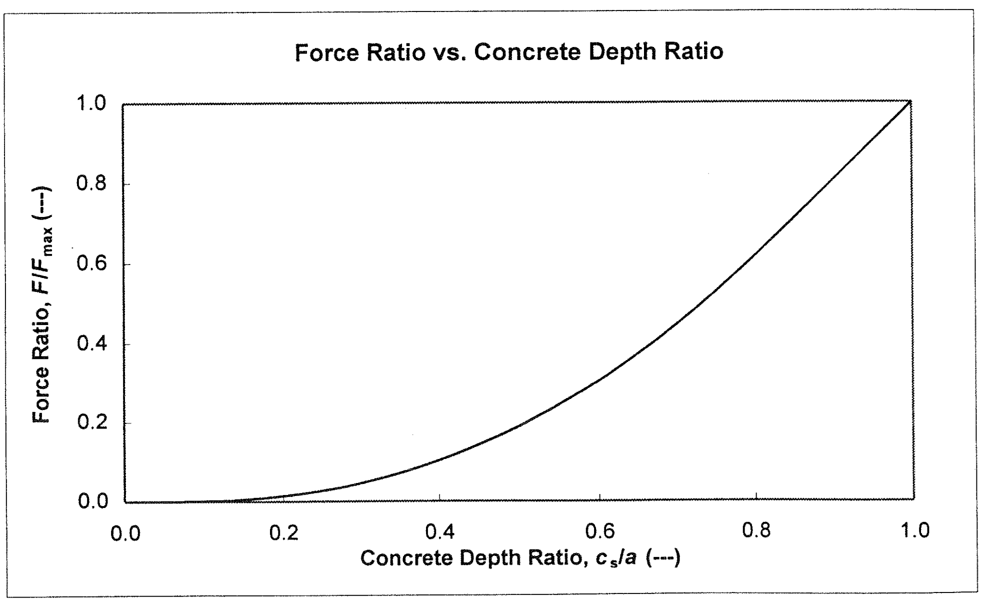
and

$$F_D = 1177N\left[-\frac{1}{4}\left\{1-\left(\frac{d}{a}\right)^4\right\} + \frac{1}{3}\left\{1-\left(\frac{d}{a}\right)^3\right\}\left(1+\frac{d}{a}\right) - \frac{1}{2}\frac{d}{a}\left\{1-\left(\frac{d}{a}\right)^2\right\}\right]$$

The results are plotted below

Hinge force vs. concrete depth ratio:

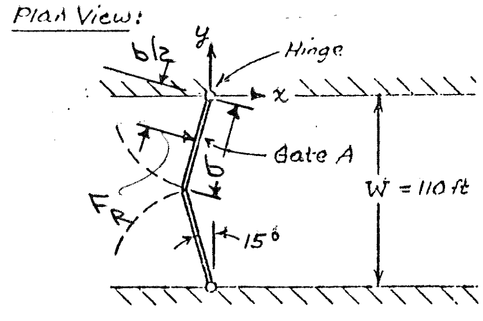
Depth Ratio, c/a (---)	Depth Ratio, d/a (---)	Force Ratio, F/F_{max} (---)
0	1.0	0.0000
0.1	0.9	0.0019
0.2	0.8	0.0144
0.3	0.7	0.0459
0.4	0.6	0.102
0.5	0.5	0.187
0.6	0.4	0.302
0.625	0.375	0.336
0.7	0.3	0.446
0.8	0.2	0.614
0.9	0.1	0.802
1.0	0.0	1.000



Problem 3.51

Given: Pair of plane gates close a channel of width, $W = 110\,ft$; each gate is hinged at channel wall. Gate edges are forced together at the channel center by water pressure. Water depth, $D = 32\,ft$. Neglect the weight of the gate.

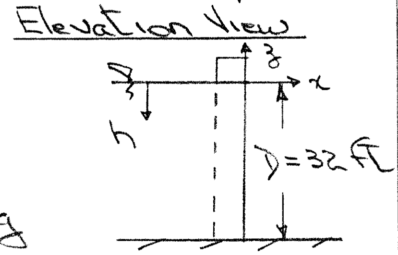
Find: (a) force exerted by water on gate A
(b) force components exerted by the gate on hinge A



Solution:

Basic equations: $\dfrac{dp}{dh} = \rho g$; $p = p_{atm} + \rho g h$

Assumptions: (1) static liquid
(2) gravity only body force
(3) h positive down from free surface.
(4) p_{atm} acts on both sides of gate



Then

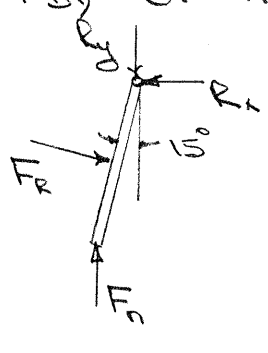
$$F_R = \int p\,dA = \int_0^D \rho g h b\,dh = \rho g \frac{W}{2\cos 15}\frac{h^2}{2}\Big]_0^D = \frac{\rho g W D^2}{4\cos 15°}$$

$$= \frac{1}{4\cos 15} \times 1.94\,\frac{slug}{ft^3} \times 32.2\,\frac{ft}{s^2} \times 110\,ft \times (32)^2\,ft^2 \times \frac{lbf \cdot s^2}{ft \cdot slug}$$

$F_R = 1.82 \times 10^6\ lbf$ ←

Since the gate width, $b = \dfrac{W}{2\cos 15} = 56.9\,ft$, is constant the line of action of F_R is located at $b/2$ from the hinge

To find the reaction forces at the hinge, consider a FBD of the gate.



Since we have neglect the weight of the gate, the reaction force at the hinge has only components R_x and R_y.

The contact force, F_n, between the pair of gates must act perpendicular to the channel walls (from symmetry conditions).

$$\sum M_o = 0 = F_R \frac{b}{2} - F_n b \sin 15°$$

$$\therefore F_n = \frac{F_R}{2\sin 15} = \frac{1.82 \times 10^6\ lbf}{2\sin 15} = 3.52 \times 10^6\ lbf$$

$\sum F_x = F_R \cos 15 - R_x = 0 \quad \therefore R_x = F_R \cos 15 = 1.82 \times 10^6\,lbf \cos 15 = 1.76 \times 10^6\,lbf$

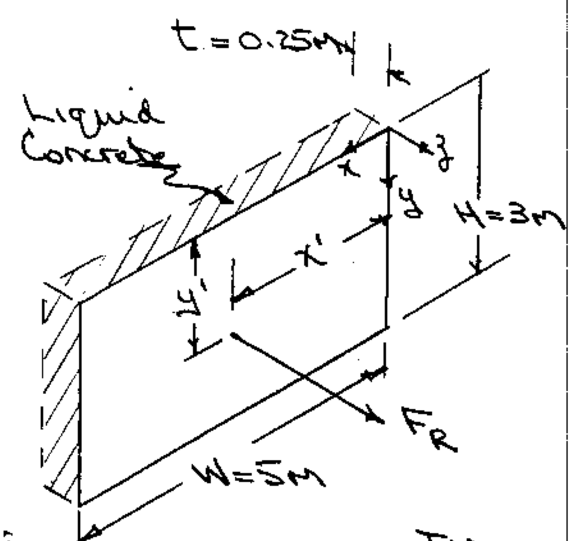
$\sum F_y = -R_y - F_R \sin 15 + F_n = 0 \quad \therefore R_y = F_n - F_R \sin 15 = 3.52 \times 10^6 - 1.82 \times 10^6 \sin 15$

$R_y = 3.04 \times 10^6\ lbf$

The force on the hinge (from the gate) is $\vec{F}_A = (1.76\hat{i} + 3.04\hat{j})10^6\ lbf$ ← $\vec{F}_A$

Problem 3.52

Given: Liquid concrete poured between vertical forms as shown



Find: (a) Resultant force on form
(b) Line of application

Solution:

Basic equation: $\frac{dp}{dy} = \rho g$

Computing equations:

$F_R = p_c A$ (3.14) ; $y' = y_c + \frac{I_{\hat{x}\hat{x}}}{A y_c}$ (3.15a); $x' = x_c + \frac{I_{\hat{x}\hat{y}}}{A y_c}$

For the rectangular plate: $x_c = 2.5\,m$, $y_c = 1.5\,m$.

$I_{\hat{x}\hat{x}} = \frac{1}{12} W H^3$, $I_{\hat{x}\hat{y}} = 0$

Assumptions: (1) static liquid (2) incompressible liquid
(3) p_{atm} acts at free surface and on the vertical form.

Then on integrating $dp = \rho g\, dy$, we obtain $p = \rho g y$

$F_R = p_c A = \rho g y_c A = \rho g y_c W H = SG_{conc}\, \rho_{H_2O}\, y_c W H$

$F_R = 2.5 \times 10^3 \frac{kg}{m^3} \times 9.81 \frac{m}{s^2} \times 1.5\,m \times 5\,m \times 3\,m \times \frac{N \cdot s^2}{kg \cdot m}$ {SG = 2.5, Table A.1}

$F_R = 552\,kN$ ← F_R

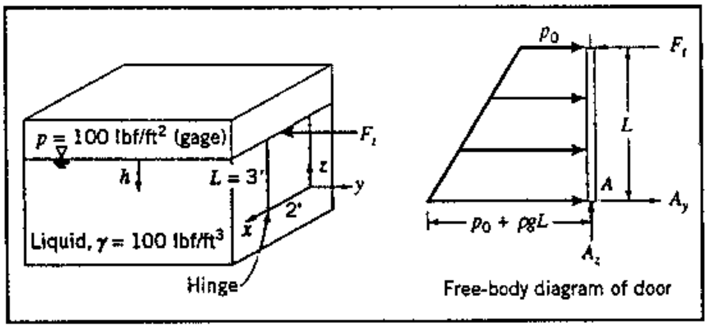
$y' = y_c + \frac{I_{\hat{x}\hat{x}}}{A y_c} = y_c + \frac{1}{12}\frac{W H^3}{W H y_c} = y_c + \frac{1}{12}\frac{H^2}{y_c} = 1.5\,m + \frac{1}{12}\frac{(3\,m)^2}{1.5\,m} = 2.0\,m$

$x' = x_c = 2.5\,m$

Line of application is through $(x', y') = (2.5, 2.0)\,m$ ← (x', y')

Problem 3.53

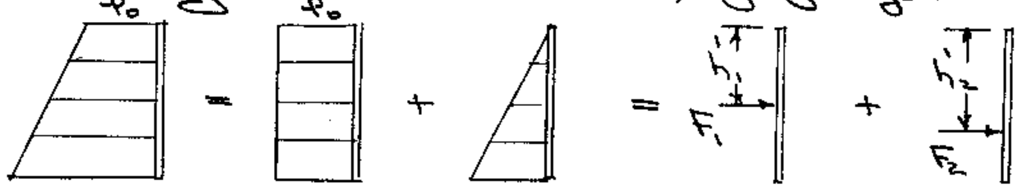
Given: Door as shown in the figure; x axis is along the hinge
From Ex. Prob 3.6, pressure in liquid is $p = p_{gage} + \rho g h$



Find: Force required to keep door shut (by considering the distributed force to be the sum of a force F_1 caused by uniform gage pressure, and force F_2 caused by the liquid)

Solution:

Computing equations: $F_R = p_c A$; $y' = y_c + \frac{I_{\hat{x}\hat{x}}}{y_c A}$; $I_{\hat{x}\hat{x}} = \frac{bL^3}{12}$



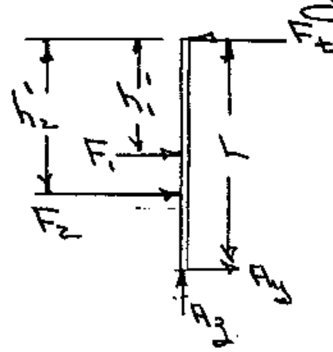
$F_1 = p_0 A = 100 \frac{lbf}{ft^2} \times 3ft \times 2ft = 600\ lb$ { applied at $(x', z') = (1.0, 1.5)$ ft

$F_2 = p_c A = \rho g h_c L b = \gamma h_c L b = 100 \frac{lb}{ft^3} \times 1.5ft \times 3ft \times 2ft = 900\ lbf$.

For the rectangular door $I_{\hat{x}\hat{x}} = \frac{1}{12} bL^3$

$h_2' = h_c + \frac{I_{\hat{x}\hat{x}}}{A h_c} = h_c + \frac{1}{12}\frac{bL^3}{bLh_c} = h_c + \frac{1}{12}\frac{L^2}{h_c} = 1.5m + \frac{1}{12}\frac{(3m)^2}{1.5m} = 2.0m$

The free-body diagram of the door is then



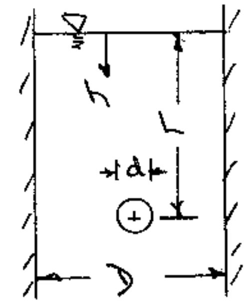
$\Sigma M_{Ax} = 0 = L F_t - F_1(L - h_1') - F_2(L - h_2')$.

$F_t = F_1\left(1 - \frac{h_1'}{L}\right) + F_2\left(1 - \frac{h_2'}{L}\right)$

$= 600\ lb\left(1 - \frac{1.5}{3.0}\right) + 900\ lb\left(1 - \frac{2}{3}\right)$

$F_t = 600\ lb$ ←——— F_t

Problem 3.54



Given: Circular access port, of diameter $d = 0.6\,m$, in side of water standpipe, of diameter $D = 7\,m$, is held in place by eight bolts evenly spaced around circumference of the port.
Center of the port is located at distance $L = 12\,m$ below the free surface of the water

Find: (a) Total force on the port
(b) Appropriate bolt diameter

Solution:

Basic equations: $\frac{dp}{dh} = \rho g$, $\sigma = \frac{F}{A}$

Computing equation: $F_R = p_c A$

Assumptions: (1) static fluid
(2) incompressible
(3) force distributed uniformly over the bolts
(4) appropriate working stress for steel bolts is $\sigma = 100$ MPa
(5) p_{atm} acts at free surface and on the outside of the port

Then on integrating $dp = \rho g dh$ we obtain $p = \rho g h$

$F_R = p_c A = \rho g h_c \pi R^2 = \rho g L \pi R^2$

$F_R = 999 \frac{kg}{m^3} \times 9.81 \frac{m}{s^2} \times 12\,m \times \pi \times (0.3\,m)^2 \times \frac{N \cdot s^2}{kg \cdot m} = 33.3\,kN$ $\quad F_R$

$\sigma = \frac{F}{A}$ where A (total area of bolts) $= 8 \times \frac{\pi d_b^2}{4}$

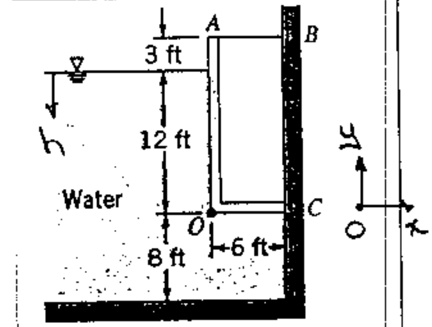
Then

$\sigma = \frac{F}{2\pi d_b^2}$

$d_b = \left[\frac{F}{2\pi\sigma}\right]^{1/2} = \left[\frac{33.3 \times 10^3\,N}{2\pi} \times \frac{m^2}{10^8\,N} \times 10^6 \frac{mm^2}{m^2}\right]^{1/2} = 7.28\,mm$ $\quad d_b$

Problem 3.55

Given: Gate AOC, hinged along O, has width $b = 6$ ft; weight of gate may be neglected. Gate is sealed at C.



Find: Force in bar AB.

Solution:

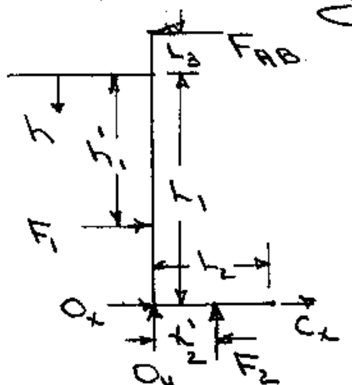
Basic equations: $\frac{dp}{dh} = \rho g$; $\sum M_z = 0$

Computing equations: $F_R = p_c A$; $y' = y_c + \frac{I_{\hat{x}\hat{x}}}{y_c A}$; $I_{\hat{x}\hat{x}} = \frac{bL^3}{12}$

Assumptions: (1) static liquid (2) ρ = constant
(3) p_{atm} acts at free surface and on outside of gate
(4) no resisting moment in hinge along O
(5) no vertical resisting force at C

Then on integrating $dp = \rho g\, dh$, we obtain $p = \rho g h$

The free body diagram of the gate is as shown.



F_1 is resultant of distributed force on L_1
F_2 " " " uniform force on L_2
F_{AB} is force of bar
C_x is force from seal at C

$$F_1 = p_c A_1 = \rho g h_{c_1} b L_1$$

$$F_1 = 1.94 \frac{\text{slug}}{\text{ft}^3} \times 32.2 \frac{\text{ft}}{\text{s}^2} \times 6\text{ft} \times 6\text{ft} \times 12\text{ft} \times \frac{\text{lbf}\cdot\text{s}^2}{\text{ft}\cdot\text{slug}} = 27.0 \times 10^3 \text{ lbf}$$

$$h_1' = h_{c_1} + \frac{bL_1^3}{12 h_{c_1} b L_1} = \frac{L_1}{2} + \frac{L_1^2}{12} \frac{2}{L_1} = \frac{L_1}{2} + \frac{L_1}{6} = \frac{2}{3}L_1 = \frac{2}{3} \times 12\text{ft} = 8\text{ft}$$

$$F_2 = p_c A_2 = \rho g h_c b L_2 = \rho g L_1 b L_2$$

$$F_2 = 1.94 \frac{\text{slug}}{\text{ft}^3} \times 32.2 \frac{\text{ft}}{\text{s}^2} \times 12\text{ft} \times 6\text{ft} \times 6\text{ft} = 27.0 \times 10^3 \text{ lbf}$$

Since the pressure is uniform over surface ②, the force F_2 acts at the centroid of the surface, i.e. $x_2' = L_2/2 = 3$ ft.

Then summing moments about O gives

$$\sum M_{O_z} = 0 = (L_1 + L_3) F_{AB} + x_2' F_2 - (L_1 - h_1') F_1$$

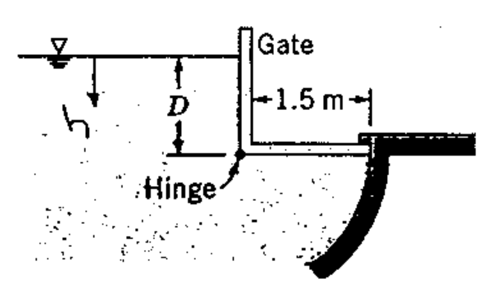
$$F_{AB} = \frac{1}{(L_1 + L_3)} \left[(L_1 - h_1') F_1 - x_2' F_2 \right] = \frac{1}{15\text{ft}} \left[(12-8)\text{ft} \times 27{,}000\text{lbf} - 3\text{ft} \times 27{,}000\text{lbf} \right]$$

$F_{AB} = 1800$ lbf ← F_{AB}

Thus bar AB is in compression

Problem 3.56

Given: Water rising on the left side of the gate causes it to open automatically. Neglect weight of gate.



Find: Depth, D, above the hinge at which the gate begins to open.

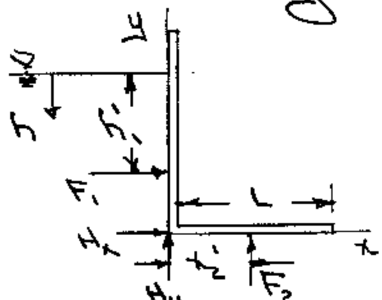
Solution:

Basic equations: $\frac{dp}{dh} = \rho g$; $\Sigma M_z = 0$

Computing equations: $F_R = p_c A$; $y' = y_c + \frac{I_{\hat{x}\hat{x}}}{y_c A}$; $I_{\hat{x}\hat{x}} = \frac{bD^3}{12}$

Assumptions: (1) static liquid (2) ρ = constant
(3) p_{atm} acts at free surface and on outside of gate
(4) no resisting moment in hinge.

Then on integrating $dp = \rho g\, dh$, we obtain $p = \rho g h$
The free body diagram of the gate is as shown.



F_1 is resultant of distributed force on vertical section
F_2 " " " uniform force on horizontal "

Let width of gate be b.

$$F_1 = p_c A_1 = \rho g h_c bD = \rho g \frac{D}{2} bD = \frac{1}{2}\rho g b D^2$$

$$h_1' = h_{c_1} + \frac{bD^3}{12 h_c bD} = \frac{D}{2} + \frac{D^2}{12\frac{D}{2}} = \left(\frac{1}{2}+\frac{1}{6}\right)D = \frac{2}{3}D$$

$$F_2 = p_c A_2 = \rho g h_{c_2} bL = \rho g D bL$$

Since the pressure is uniform over the horizontal surface, the force F_2 acts at the centroid of the surface, i.e. $x_2' = L/2$

Then summing moments about the hinge

$$\Sigma M_{z_H} = 0 = F_2 x_2' - F_1(D - h_1') = \rho g D b L \frac{L}{2} - \frac{1}{2}\rho g b D^2\left(D - \frac{2}{3}D\right)$$

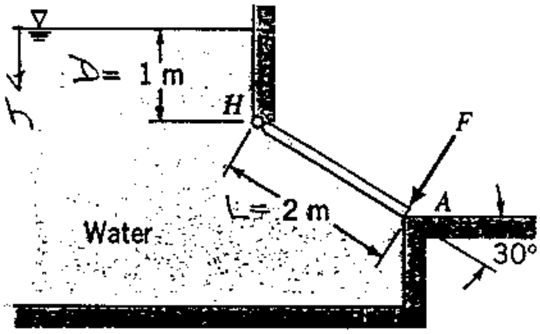
$$\therefore \quad L^2 - \frac{D^2}{3} = 0$$

$$D = \sqrt{3}L = \sqrt{3} \times 1.5\,\text{m} = 2.60\,\text{m} \qquad D$$

Problem 3.57

Given: Gate of width $b = 2\,m$, hinged at H.

Find: Force F_A required to hold gate closed.



Solution:

Basic equations: $\frac{dp}{dh} = \rho g$

$\Sigma M_z = 0$

Computing equations: $F_R = p_c A$; $y' = y_c + \frac{I_{\hat{x}\hat{x}}}{y_c A}$; $I_{\hat{x}\hat{x}} = \frac{bL^3}{12}$

Assumptions: (1) static liquid (2) ρ = constant (3) p_{atm} acts at free surface and on top of the gate.

Then on integrating $dp = \rho g\, dh$, we obtain $p = \rho g h$

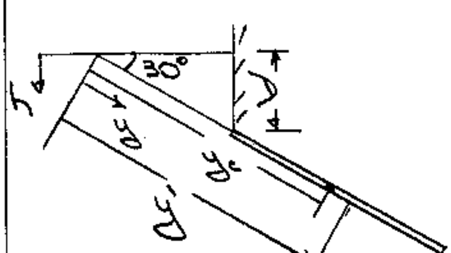
$F_R = p_c A = \rho g h_c A = \rho g h_c L b$ $\qquad h_c = D + \frac{L}{2}\sin 30° = 1m + \frac{2m}{2}\sin 30°$

$h_c = 1.5\,m$

$F_R = 999\,\frac{kg}{m^3} \times 9.81\,\frac{m}{s^2} \times 1.5m \times 2m \times 2m \times \frac{N \cdot s^2}{kg \cdot m}$

$F_R = 58.8\ kN$

When using the computing equation to find y', we must use coordinates, with origin at the location where $p_{gage} = 0$

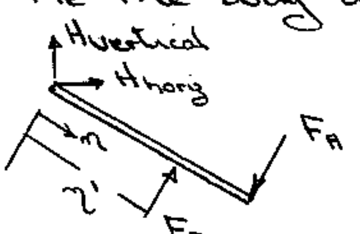


$y_c = \frac{D}{\sin 30} + \frac{L}{2} = \frac{1m}{\sin 30°} + \frac{2m}{2} = 3.0\,m$

$y' = y_c + \frac{I_{xx}}{A y_c} = y_c + \frac{bL^3}{12 y_c bL} = y_c + \frac{L^2}{12 y_c}$

$y' = 3.0m + \frac{(2m)^2}{(12)3.0m} = 3.111\,m$

The free body diagram of the gate is as shown.



Summing moments about H

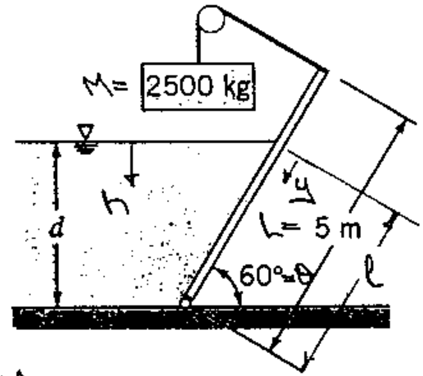
$\Sigma M_H = 0 = \eta' F_R - L F_A$

where $\eta' = y' - \frac{D}{\sin 30°} = 3.111m - \frac{D}{\sin 30°} = 1.11\,m$

$F_A = \frac{1}{L}\eta' F_R = \frac{1.11m}{2.0m} \times 58.8kN = 32.6\,kN$ $\qquad F_A$ ←

Problem 3.58

Given: Gate shown has width $b = 3\,m$; mass of gate is negligible.
Gate is in equilibrium



Find: Water depth, d

Solution:

Basic equation: $\frac{dp}{dh} = \rho g \qquad \Sigma M_z = 0$

Computing equations: $F_R = p_c A$; $y' = y_c + \frac{I_{\hat{x}\hat{x}}}{y_c A}$; $I_{\hat{x}\hat{x}} = \frac{bL^3}{12}$

Assumptions: (1) static liquid (2) ρ = constant
(3) p_{atm} acts at free surface and on underside of gate.

Then on integrating $dp = \rho g dh$, we obtain $p = \rho g h$

$F_R = p_c A = \rho g h_c A \qquad h_c = \frac{d}{2}, \quad A = b \times \frac{d}{\sin\theta}$

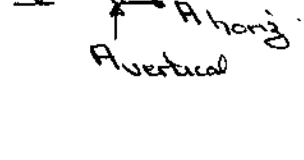
$F_R = \rho g \frac{d}{2} \frac{db}{\sin\theta} = \frac{\rho g b d^2}{2\sin\theta}$

$y' = y_c + \frac{I_{\hat{x}\hat{x}}}{y_c A} = y_c + \frac{1}{12}\frac{b\ell^3}{y_c \ell b}$ where ℓ is length of gate in contact with the water

$y' = y_c + \frac{\ell^2}{12 y_c} \qquad \ell = \frac{d}{\sin\theta}, \quad y_c = \frac{\ell}{2} = \frac{d}{2\sin\theta}$

$y' = \frac{d}{2\sin\theta} + \frac{1}{12}\left(\frac{d}{\sin\theta}\right)^2 \frac{2\sin\theta}{d} = \frac{d}{2\sin\theta} + \frac{d}{6\sin\theta} = \frac{2d}{3\sin\theta}$

The free body diagram of the gate is as shown.



Summing moments about A

$\Sigma M_z = 0 = TL - (\ell - y')F_R \qquad T = Mg$

$MgL = (\ell - y')F_R = \left(\frac{d}{\sin\theta} - \frac{2d}{3\sin\theta}\right)\frac{\rho g b d^2}{2\sin\theta}$

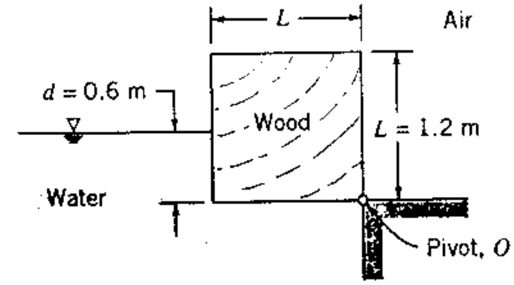
$MgL = \frac{1}{3}\frac{d}{\sin\theta} \times \frac{\rho g b d^2}{2\sin\theta} = \frac{\rho g b d^3}{6 \sin^2\theta}$

$d^3 = \frac{6\sin^2\theta\, ML}{\rho b}$

$d = \left[6 \times \sin^2 60° \times 2500\,kg \times 5\,m \times \frac{m^3}{999\,kg} \times \frac{1}{3\,m}\right]^{1/3} = 2.66\,m$ ← d

Problem 3.59

Given: Long, square wooden block, pivoted on one edge, in equilibrium in water as shown. Friction in pivot is negligible.



Find: Specific gravity of the wood.

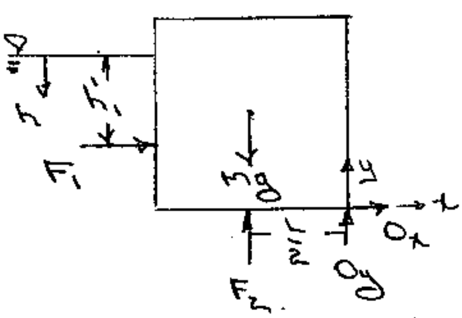
Solution:

Basic equations: $\frac{dp}{dh} = \rho g$, $\sum M_{z_O} = 0$

Computing equations: $F_R = p_c A$; $y' = y_c + \frac{I_{\hat{x}\hat{x}}}{y_c A}$; $I_{\hat{x}\hat{x}} = \frac{bd^3}{12}$

Assumptions: (1) static liquid (2) ρ = constant
(3) p_{atm} acts at free surface and on outside of the block.
(4) no resisting moment in hinge (given)

Then on integrating $dp = \rho g\, dh$, we obtain $p = \rho g h$
The free body diagram of the block is as shown.



F_1 is the resultant of distributed force on vertical face
F_2 is the resultant of the uniform force on the bottom face

$m = \text{mass} = \rho_w g \forall = SG\rho L^2 b$
where b is the length of the block

$$F_1 = p_{c_1} A_1 = \rho g h_c\, db = \rho g \frac{d}{2} db = \frac{1}{2}\rho g b d^2$$

$$h'_1 = h_{c_1} + \frac{bd^3}{12 h_c bd} = \frac{d}{2} + \frac{d^2}{12\frac{d}{2}} = d\left(\frac{1}{2} + \frac{1}{6}\right) = \frac{2}{3}d$$

$$F_2 = p_c A_2 = \rho g h_{c_2} bL = \rho g d b L$$

F_2 due to uniform pressure acts at centroid of surface

Then summing moments about the hinge gives

$$mg\frac{L}{2} - F_1 h'_1 - F_2\frac{L}{2} = 0$$

$$SG\,\rho L^2 b g \frac{L}{2} - \frac{1}{2}\rho g b d^2\left(d - \frac{2}{3}d\right) - \rho g d b L \cdot \frac{L}{2} = 0$$

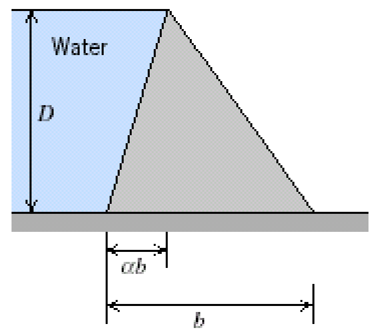
$$SG\frac{L^3}{2} - \frac{d^3}{6} - \frac{dL^2}{2} = 0$$

$$SG = \frac{1}{3}\left(\frac{d}{L}\right)^3 + \frac{d}{L} = \frac{1}{3}\left(\frac{0.6}{1.2}\right)^3 + \frac{0.6}{1.2} = 0.542$$ SG

Problem 3.60

A solid concrete dam is to be built to hold back a depth *D* of water. For ease of construction the walls of the dam must be planar. Your supervisor asks you to consider the following dam cross-sections: a rectangle, a right triangle with the hypotenuse in contact with the water, and a right triangle with the vertical in contact with the water. She wishes you to determine which of these would require the least amount of concrete. What will your report say? You decide to look at one more possibility: a nonright triangle, as shown. Develop and plot an expression for the cross-section area *A* as a function of α, and find the minimum cross-sectional area.

Given: Various dam cross-sections

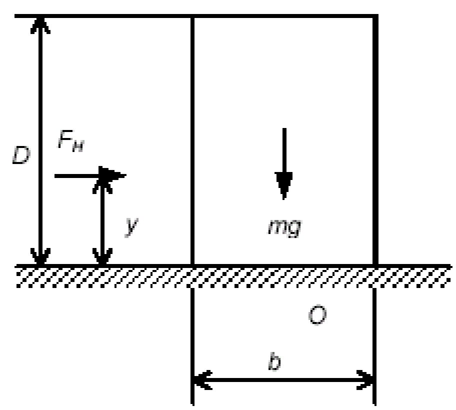


Find: Which requires the least concrete; plot cross-section area *A* as a function of α

Solution

For each case, the dam width *b*
enough moment to balance the moment due to fluid hydrostatic force(s). By doing a moment balance this value of *b* can be found

a) Rectangular dam



Straightforward application of the computing equations of Section 3-5 yields

$$F_H = p_c \cdot A = \rho \cdot g \cdot \frac{D}{2} \cdot w \cdot D = \frac{1}{2} \cdot \rho \cdot g \cdot D^2 \cdot w$$

$$y' = y_c + \frac{I_{xx}}{A \cdot y_c} = \frac{D}{2} + \frac{w \cdot D^3}{12 \cdot w \cdot D \cdot \frac{D}{2}} = \frac{2}{3} \cdot D$$

so $\quad y = D - y' = \frac{D}{3}$

Also $\quad m = \rho_{cement} \cdot g \cdot b \cdot D \cdot w = SG \cdot \rho \cdot g \cdot b \cdot D \cdot w$

Taking moments about *O*

$$\sum M_{0.} = 0 = -F_H \cdot y + \frac{b}{2} \cdot m \cdot g$$

so $$\left(\frac{1}{2} \cdot \rho \cdot g \cdot D^2 \cdot w\right) \cdot \frac{D}{3} = \frac{b}{2} \cdot (SG \cdot \rho \cdot g \cdot b \cdot D \cdot w)$$

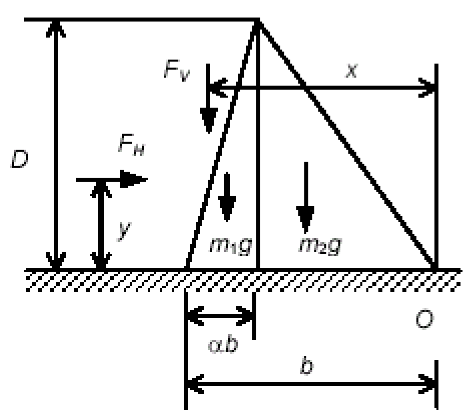
Solving for *b* $$b = \frac{D}{\sqrt{3 \cdot SG}}$$

The minimum rectangular cross-section area is $A = b \cdot D = \frac{D^2}{\sqrt{3 \cdot SG}}$

For concrete, from Table A.1, SG = 2.4, so $$A = \frac{D^2}{\sqrt{3 \cdot SG}} = \frac{D^2}{\sqrt{3 \times 2.4}}$$

$$A = 0.373 \cdot D^2$$

a) Triangular dams



made, at the end of which right triangles are analysed as special cases by setting α = 0 or 1.

Straightforward application of the computing equations of Section 3-5 yields

$$F_H = p_c \cdot A = \rho \cdot g \cdot \frac{D}{2} \cdot w \cdot D = \frac{1}{2} \cdot \rho \cdot g \cdot D^2 \cdot w$$

$$y' = y_c + \frac{I_{xx}}{A \cdot y_c} = \frac{D}{2} + \frac{w \cdot D^3}{12 \cdot w \cdot D \cdot \frac{D}{2}} = \frac{2}{3} \cdot D$$

so $$y = D - y' = \frac{D}{3}$$

Also $$F_V = \rho \cdot V \cdot g = \rho \cdot g \cdot \frac{\alpha \cdot b \cdot D}{2} \cdot w = \frac{1}{2} \cdot \rho \cdot g \cdot \alpha \cdot b \cdot D \cdot w$$

$$x = (b - \alpha \cdot b) + \frac{2}{3} \cdot \alpha \cdot b = b \cdot \left(1 - \frac{\alpha}{3}\right)$$

For the two triangular masses

$$m_1 = \frac{1}{2} \cdot SG \cdot \rho \cdot g \cdot \alpha \cdot b \cdot D \cdot w \qquad x_1 = (b - \alpha \cdot b) + \frac{1}{3} \cdot \alpha \cdot b = b \cdot \left(1 - \frac{2 \cdot \alpha}{3}\right)$$

$$m_2 = \frac{1}{2} \cdot SG \cdot \rho \cdot g \cdot (1 - \alpha) \cdot b \cdot D \cdot w \qquad x_2 = \frac{2}{3} \cdot b(1 - \alpha)$$

Taking moments about *O*

$$\sum M_{0.} = 0 = -F_H \cdot y + F_V \cdot x + m_1 \cdot g \cdot x_1 + m_2 \cdot g \cdot x_2$$

so $$-\left(\frac{1}{2} \cdot \rho \cdot g \cdot D^2 \cdot w\right) \cdot \frac{D}{3} + \left(\frac{1}{2} \cdot \rho \cdot g \cdot \alpha \cdot b \cdot D \cdot w\right) \cdot b \cdot \left(1 - \frac{\alpha}{3}\right) \ldots = 0$$
$$+ \left(\frac{1}{2} \cdot SG \cdot \rho \cdot g \cdot \alpha \cdot b \cdot D \cdot w\right) \cdot b \cdot \left(1 - \frac{2 \cdot \alpha}{3}\right) + \left[\frac{1}{2} \cdot SG \cdot \rho \cdot g \cdot (1 - \alpha) \cdot b \cdot D \cdot w\right] \cdot \frac{2}{3} \cdot b(1 - \alpha)$$

Solving for *b* $$b = \frac{D}{\sqrt{\left(3 \cdot \alpha - \alpha^2\right) + SG \cdot (2 - \alpha)}}$$

For a $\alpha = 1$, and

$$b = \frac{D}{\sqrt{3 - 1 + SG}} = \frac{D}{\sqrt{3 - 1 + 2.4}}$$

$$b = 0.477 \cdot D$$

The cross-section area is $A = \frac{b \cdot D}{2} = 0.238 \cdot D^2$

$$A = 0.238 \cdot D^2$$

For a $\alpha = 0$, and

$$b = \frac{D}{\sqrt{2 \cdot SG}} = \frac{D}{\sqrt{2 \cdot 2.4}}$$

$$b = 0.456 \cdot D$$

The cross-section area is $A = \frac{b \cdot D}{2} = 0.228 \cdot D^2$

$$A = 0.228 \cdot D^2$$

For a general triangle $A = \frac{b \cdot D}{2} = \frac{D^2}{2 \cdot \sqrt{\left(3 \cdot \alpha - \alpha^2\right) + SG \cdot (2 - \alpha)}}$

$$A = \frac{D^2}{2\cdot\sqrt{\left(3\cdot\alpha - \alpha^2\right) + 2.4\cdot(2-\alpha)}}$$

The final result is $$A = \frac{D^2}{2\cdot\sqrt{4.8 + 0.6\cdot\alpha - \alpha^2}}$$

From the corresponding Excel workbook, the minimum area occurs at $\alpha = 0.3$

$$A_{min} = \frac{D^2}{2\cdot\sqrt{4.8 + 0.6\times 0.3 - 0.3^2}}$$

$$A = 0.226\cdot D^2$$

The final results are that a triangular cross-section with $\alpha = 0.3$ uses the least concrete; the next best is a right triangle with the vertical in contact with the water; next is the right triangle with the hypotenuse in contact with the water; and the cross-section requiring the most concrete is the rectangular cross-section.

Problem 3.60 (In Excel)

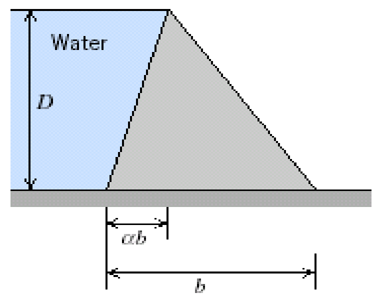
A solid concrete dam is to be built to hold back a depth D of water. For ease of construction the walls of the dam must be planar. Your supervisor asks you to consider the following dam cross-sections: a rectangle, a right triangle with the hypotenuse in contact with the water, and a right triangle with the vertical in contact with the water. She wishes you to determine which of these would require the least amount of concrete. What will your report say? You decide to look at one more possibility: a nonright triangle, as shown. Develop and plot an expression for the cross-section area A as a function of α, and find the minimum cross-sectional area.

Given: Various dam cross-sections

Find: Plot cross-section area as a function of α

Solution

The triangular cross-sections are considered in this workbook

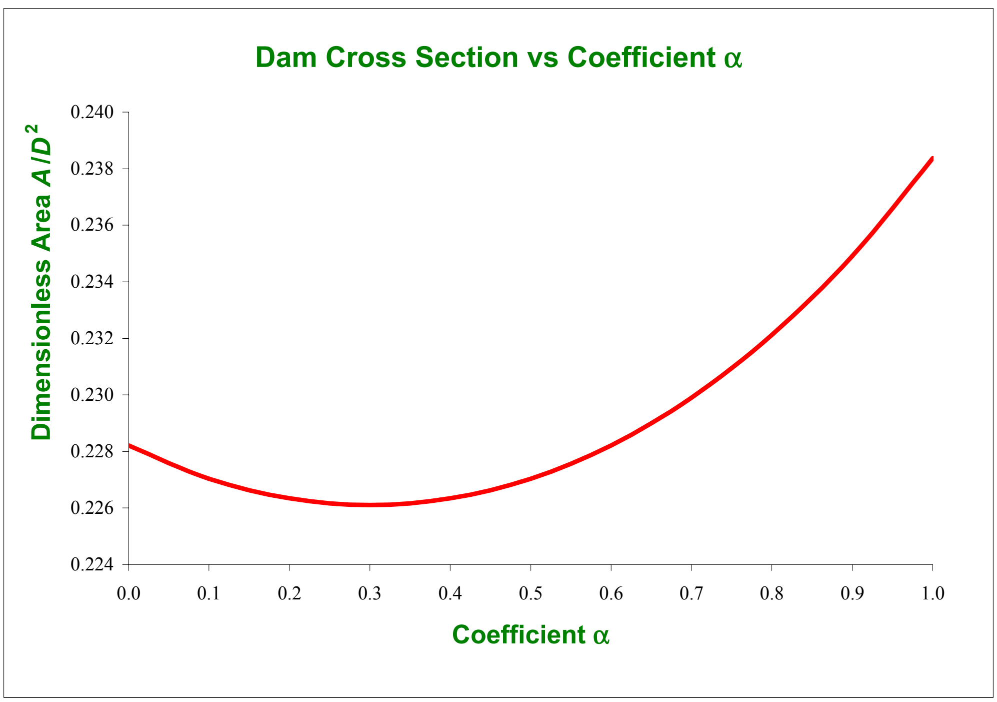


The final result is

$$A = \frac{D^2}{2\cdot\sqrt{4.8 + 0.6\cdot\alpha - \alpha^2}}$$

The dimensionless area, A/D^2, is plotted

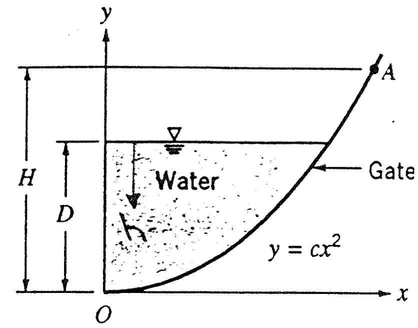
α	A/D^2
0.0	0.2282
0.1	0.2270
0.2	0.2263
0.3	0.2261
0.4	0.2263
0.5	0.2270
0.6	0.2282
0.7	0.2299
0.8	0.2321
0.9	0.2349
1.0	0.2384



Solver can be used to find the minimum area

α	A/D^2
0.30	0.2261

Given: Parabolic gate, hinged at O, has width $b = 2m$.
$c = 0.25 m^{-1}$, $D = 2m$, $H = 3m$



Find: (a) Magnitude and line of action of vertical force on gate due to water
(b) Horizontal force applied at A needed for equilibrium
(c) Vertical force applied at A needed for equilibrium

Solution:

Basic equations: $\frac{dp}{dh} = \rho g$, $\Sigma M_{O_z} = 0$, $F_V = \int p\, dA_y$, $x' F_V = \int x\, dF_V$

Computing equations $F_H = p_c A$, $h' = h_c + \frac{I_{xx}}{h_c A}$

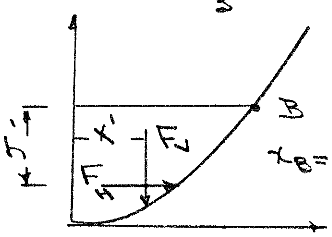
Assumptions: (1) static liquid (2) ρ = constant
(3) p_{atm} acts on the surface of the water and along the outside surface of the gate

Then on integrating $dp = \rho g\, dh$, we obtain $p = \rho g h$

(a) $F_V = \int p\, dA_y = \int_0^{\sqrt{D/c}} \rho g h\, b\, dx = \int_0^{\sqrt{D/c}} \rho g (D - y)\, b\, dx = \int_0^{\sqrt{D/c}} \rho g (D - cx^2)\, b\, dx$

$F_V = \rho g b \left[Dx - \frac{cx^3}{3} \right]_0^{\sqrt{D/c}} = \rho g b \left[\frac{D^{3/2}}{c^{1/2}} - \frac{c}{3}\left(\frac{D}{c}\right)^{3/2} \right] = \frac{2}{3} \frac{\rho g b}{c^{1/2}} D^{3/2}$ ——(1)

$F_V = \frac{2}{3} \times 999 \frac{kg}{m^3} \times 9.81 \frac{m}{s^2} \times 2m \times (2m)^{3/2} \left(\frac{m}{0.25}\right)^{0.5} \times \frac{N \cdot s^2}{kg \cdot m} = 73.9 \, kN$ ← F_V



$x' = \frac{1}{F_V} \int x\, dF_V = \frac{1}{F_V} \int x\, p\, dA_y = \frac{1}{F_V} \int_0^{\sqrt{D/c}} x\, \rho g h\, b\, dx$

$x' = \frac{1}{F_V} \int_0^{\sqrt{D/c}} x\, \rho g (D - cx^2)\, b\, dx$

$x' = \frac{b\rho g}{F_V} \left[\frac{Dx^2}{2} - \frac{cx^4}{4} \right]_0^{\sqrt{D/c}} = \frac{b\rho g}{F_V} \left[\frac{D}{2} \cdot \frac{D}{c} - \frac{c}{4} \frac{D^2}{c^2} \right] = \frac{b\rho g}{F_V} \frac{D^2}{4c}$

Substituting for F_V from Eq. 1

$x' = \frac{b\rho g D^2}{4c} \times \frac{3}{2} \frac{c^{1/2}}{\rho g b D^{3/2}} = \frac{3}{8} \left(\frac{D}{c}\right)^{1/2} = \frac{3}{8} \left[2m \times \frac{m}{0.25} \right]^{1/2} = 1.06 m$ ← x'

In order to sum moments about point O to find the required force at A required for equilibrium we need to find the horizontal force of the water on the gate and its line of action

$F_H = p_c A = \rho g h_c b D = \rho g b \frac{D^2}{2}$ $\quad \{h_c = D/2\}$

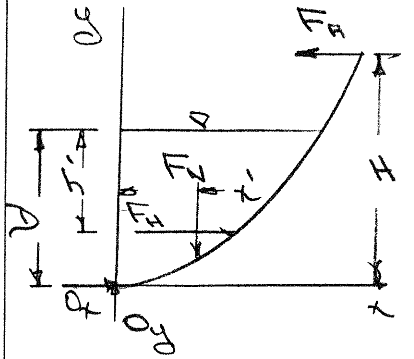
$F_H = 999 \frac{kg}{m^3} \times 9.81 \frac{m}{s^2} \times 2m \times \frac{(2m)^2}{2} \times \frac{N \cdot s^2}{kg \cdot m} = 39.2\ kN$

$h' = h_c + \frac{I_{\hat{x}\hat{x}}}{A h_c} = h_c + \frac{D^2}{12 h_c}$ $\quad \{ I_{\hat{x}\hat{x}} = \frac{bD^3}{12}$ and $A = bD \}$.

$h' = \frac{D}{2} + \frac{D^2}{12} \times \frac{2}{D}$ $\quad \{h_c = \frac{D}{2}\}$

$h' = \frac{2}{3} D = \frac{4}{3} m$

(b) Horizontal force applied at A for equilibrium



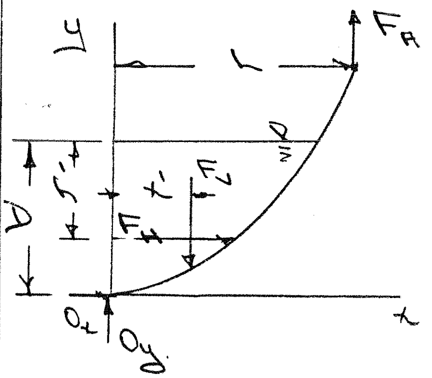
$\Sigma M_O = 0 = F_A H - F_V x' - F_H (D - h')$

$F_A = \frac{1}{H} \left[F_V x' + F_H (D - h') \right]$

$= \frac{1}{3m} \left[73.9\ kN \times 1.06m + 39.2\ kN \times \left(2 - \frac{4}{3}\right) m \right]$

$F_{A_H} = 34.8\ kN$ ← F_{AH}

(c) Vertical force applied at A for equilibrium



$\Sigma M_O = 0 = F_A L - F_V x' - F_H (D - h')$.

$F_A = \frac{1}{L} \left[F_V x' + F_H (D - h') \right]$

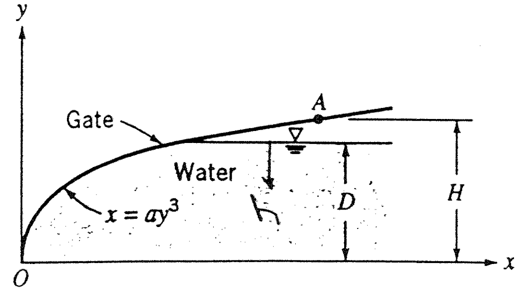
$L = x$ @ $y = H$. Since $y = cx^2$

$L = \sqrt{\frac{H}{c}} = \left[3m \times \frac{m}{0.25} \right]^{1/2} = 3.46m$

$F_A = \frac{1}{3.46m} \left[73.9\ kN \times 1.06m + 39.2\ kN \times \left(2 - \frac{4}{3}\right) m \right]$

$F_{A_V} = 30.2\ kN$ ← F_{AV}

Given: Gate, hinged at O, has width $b = 1.5$ m
$a = 1.0\ m^{-2}$, $D = 1.20$ m, $H = 1.40$ m



Find: (a) Magnitude and moment about O of vertical force on gate due to water
(b) Horizontal force applied at A needed for equilibrium

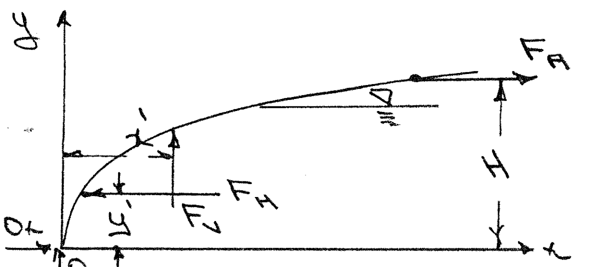
Solution

Basic equations: $\frac{dp}{dh} = \rho g$, $F_V = \int p\,dA_y$, $x' F_V = \int x\,dF_V$

$y' F_H = \int y\,dF_H$, $F_H = \int p\,dA_x$, $\sum M_{O_z} = 0$

Assumptions: (1) static liquid (2) ρ = constant
(3) p_{atm} acts on the surface of the water and along the top surface of the gate

Then on integrating $dp = \rho g\,dh$, we obtain $p = \rho g h$



$F_V = \int p\,dA_y = \int \rho g h\, b\,dx$

$h = D - y \qquad x = ay^3 \qquad dx = 3ay^2\,dy$

$F_V = \int_0^D \rho g (D-y)\, b\, 3ay^2\,dy$

$F_V = 3\rho g b a \left[D\frac{y^3}{3} - \frac{y^4}{4} \right]_0^D = 3\rho g b a \frac{D^4}{12} = \rho g b a \frac{D^4}{4}$

$F_V = 999\frac{kg}{m^3} \times 9.81\frac{m}{s^2} \times 1.5\,m \times \frac{1.0}{m^2} \times \frac{(1.20\,m)^4}{4} \times \frac{N\cdot s^2}{kg\cdot m} = 7.62\ kN$ ← F_V

The moment of F_V about O is given by

$x' F_V = \int x\,dF_V = \int x\,p\,dA_y = \int x\,\rho g h\, b\,dx$

$= \rho g b \int_0^D ay^3 (D-y)\, 3ay^2\,dy = 3\rho g b a^2 \int_0^D y^5 (D-y)\,dy$

$= 3\rho g b a^2 \left[D\frac{y^6}{6} - \frac{y^7}{7} \right]_0^D = \rho g b \frac{a^2 D^7}{14}$

$x' F_V = 999\frac{kg}{m^3} \times 9.81\frac{m}{s^2} \times 1.5\,m \times \frac{(1.0)^2}{m^4} \times \frac{(1.20\,m)^7}{14} \times \frac{N\cdot s^2}{kg\cdot m}$

$x' F_V = 3.76\ kN\cdot m$ ← (counterclockwise) $x' F_V$

From the free body diagram of the gate

$$\Sigma M_{oz} = x'F_V + y'F_H - H F_A$$

$$y'F_H = \int y\,dF_H = \int y\,p\,dA_x = \int y\,\rho g h\,b\,dy = \rho g b \int_0^D y(D-y)\,dy$$

$$= \rho g b \left[\frac{D y^2}{2} - \frac{y^3}{3}\right]_0^D = \rho g b \frac{D^3}{6}$$

$$y'F_H = \frac{1}{6} \times 999\,\frac{kg}{m^3} \times 9.81\,\frac{m}{s^2} \times 1.5\,m \times (1.20\,m)^3 \times \frac{N \cdot s^2}{kg \cdot m} = 4.23\ kN \cdot m$$

(counterclockwise)

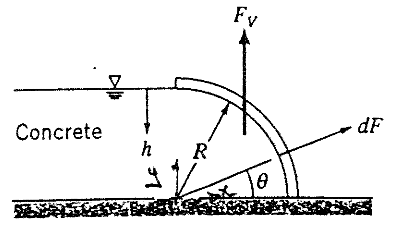
Then

$$F_A = \frac{1}{H}\left[x'F_V + y'F_H\right] = \frac{1}{1.40\,m}\left[3.76 + 4.23\right] kN \cdot m$$

$F_A = 5.71$ kN ◄ F_A

Problem 3.63

Given: Liquid concrete is poured into form shown; width $w = 4.25\,m$



Find: Magnitude and line of action of vertical force on form

Solution:

Basic equations: $\dfrac{dp}{dh} = \rho g$, $F_v = \int p\, dA_y$, $x' F_v = \int x\, dF_v$

Assumptions: (1) static liquid (2) ρ = constant
(3) p_{atm} acts on the liquid surface and along the outside of the form.

Then on integrating $dp = \rho g\, dh$, we obtain $p = \rho g h$

$$F_v = \int p\, dA_y = \int \rho g h\, dA \sin\theta \qquad dA = wR\,d\theta,\ h = R - y = R - R\sin\theta$$

$$F_v = \int_0^{\pi/2} \rho g R(1-\sin\theta)\sin\theta\, wR\, d\theta = \rho g R^2 w \int_0^{\pi/2} (\sin\theta - \sin^2\theta)\, d\theta$$

$$F_v = \rho g R^2 w \left[-\cos\theta - \frac{\theta}{2} + \frac{\sin 2\theta}{4} \right]_0^{\pi/2} = \rho g R^2 w \left[-0 + 1 - \frac{\pi}{4} + 0 + 0 - 0 \right]$$

$$F_v = \rho g R^2 w \left(1 - \frac{\pi}{4}\right) \qquad \{ \rho = SG\, \rho_{H_2O}\,;\ SG = 2.5 \text{ (Table A.1)}$$

$$F_v = 2.5 \times 1000 \frac{kg}{m^3} \times 9.81 \frac{m}{s^2} \times (0.313\,m)^2 \times 4.25\,m \left(1 - \frac{\pi}{4}\right) \times \frac{N \cdot s^2}{kg \cdot m}$$

$$F_v = 2.19\ kN \longleftarrow F_v$$

$$x' F_v = \rho g R^2 w \int_0^{\pi/2} x(\sin\theta - \sin^2\theta)\, d\theta = \rho g R^2 w \int_0^{\pi/2} R\cos\theta(\sin\theta - \sin^2\theta)\, d\theta$$

$$= \rho g R^3 w \int_0^{\pi/2} (\sin\theta\cos\theta - \sin^2\theta\cos\theta)\, d\theta = \rho g R^3 w \left[\frac{\sin^2\theta}{2} - \frac{\sin^3\theta}{3} \right]_0^{\pi/2}$$

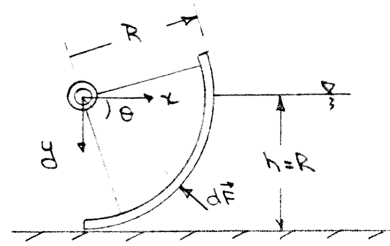
$$x' F_v = \rho g R^3 w \left[\frac{1}{2} - \frac{1}{3} \right] = \frac{\rho g R^3 w}{6}$$

$$x' = \frac{\rho g R^3 w}{6 F_v} = \frac{\rho g R^3 w}{6} \times \frac{1}{\rho g R^2 w \left(1 - \frac{\pi}{4}\right)} = \frac{R}{6\left(1 - \frac{\pi}{4}\right)} = \frac{0.313\,m}{6(1 - \pi/4)}$$

$$x' = 0.243\ m \longleftarrow x'$$

Given: Gate formed in the shape of a circular arc has width of w meters. Liquid is water; depth $h = R$

Find: (a) magnitude and direction of the net vertical force component due to fluids acting on the gate
(b) line of action of vertical component of the force.



Solution

Basic equations: $\vec{F}_R = -\int P\,d\vec{A}$ $\quad \frac{dP}{dy} = \rho g$ $\quad x' F_{R_y} = \int x\,dF$

Assumptions: (1) static fluid
(2) ρ = constant
(3) y is measured positive downward from free surface

$$F_{R_y} = \vec{F}_R \cdot \hat{j} = \int d\vec{F} \cdot \hat{j} = -\int P\,d\vec{A} \cdot \hat{j} = -\int P\,dA \sin\theta = -\int_0^{\pi/2} P \sin\theta \, wR\,d\theta$$

We can obtain an expression for P as a function of y

$$\frac{dP}{dy} = \rho g \qquad dP = \rho g\,dy \qquad \text{and} \quad P - P_0 = \int_{P_0}^{P} dP = \int_0^y \rho g\,dy = \rho g y$$

Since atmospheric pressure acts at the free surface and on the back surface of the gate, then the appropriate expression for P is $P = \rho g y$

Along the surface of the gate,

$$y = R \sin\theta \quad \text{and hence} \quad P = \rho g R \sin\theta$$

Thus,

$$F_{R_y} = -\int_0^{\pi/2} P \sin\theta \, wR\,d\theta = -\rho g w R^2 \int_0^{\pi/2} \sin^2\theta\,d\theta = -\rho g w R^2 \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi/2}$$

$$F_{R_y} = -\frac{\rho g w R^2 \pi}{4} \quad \{F_{R_y} \text{ acts upward}\}$$ F_{R_y}

For any element of surface area, $d\vec{A}$, the force, $d\vec{F}$, acts normal to the surface. Thus each $d\vec{F}$ has a line of action through the origin. Consequently, the line of action of $\vec{F}_R$ must also be through the origin.

We can find the line of action of F_{R_y} by recognizing that the moment of F_{R_y} about an axis through the origin must be equal to the sum of the moments of dF_y about the same axis.

$$x' F_{R_y} = \int x\,dF_y = \int x(-P\,dA \sin\theta) = -\int x P\,dA \sin\theta$$

$$x' F_{R_y} = -\int_0^{\pi/2} R\cos\theta \, \rho g R \sin\theta \, wR\,d\theta \sin\theta = -\rho g w R^3 \int_0^{\pi/2} \sin^2\theta \cos\theta\,d\theta$$

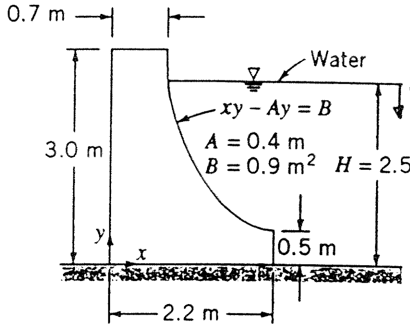
$$x' = \frac{-\rho g w R^3}{F_{R_y}} \int_0^{\pi/2} \sin^2\theta \cos\theta\,d\theta = \frac{-\rho g w R^3}{-\frac{\rho g w R^2 \pi}{4}} \left[\frac{1}{3}\sin^3\theta\right]_0^{\pi/2}$$

$$x' = \frac{4R}{3\pi}$$

Problem 3.65

Given: Dam with cross-section shown (width $b = 50\,m$)

Find: (a) Magnitude and line of action of vertical force on dam due to water.
(b) If it is possible for water force to overturn the dam



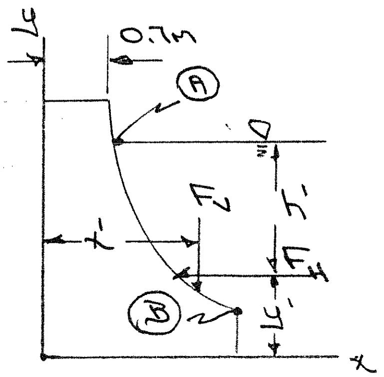
Solution:

Basic equations: $\frac{dP}{dh} = \rho g$, $F_V = \int -P\,dA_y$, $x' F_V = \int x\,dF_V$, $\sum M_z = 0$

Computing equations: $F_H = P_c A$, $h' = h_c + \frac{I_{\hat{x}\hat{x}}}{h_c A}$

Assumptions: (1) static fluid (2) ρ = constant
(3) P_{atm} acts on the surface of the water and on the back side of the dam.

Then on integrating $dP = \rho g\,dh$ we obtain $P = \rho g h$



$$F_V = \int -P\,dA_y = \int_{x_A}^{x_B} \rho g h b\,dx = \rho g b \int_{x_A}^{x_B} (H-y)\,dx$$

$$y(x-A) = B \quad \text{so} \quad y = \frac{B}{(x-A)}$$

$$F_V = \rho g b \int_{x_A}^{x_B} \left(H - \frac{B}{(x-A)} \right) dx$$

$$= \rho g b \left[Hx - B \ln(x-A) \right]_{x_A}^{x_B}$$

$$F_V = \rho g b \left[H(x_B - x_A) - B \ln \frac{(x_B - A)}{(x_A - A)} \right] \quad \text{--- (1)}$$

$$F_V = 999\,\frac{kg}{m^3} \times 9.81\,\frac{m}{s^2} \times 50\,m \left[2.5\,m(2.2-0.76)\,m - 0.9\,m^2 \ln \frac{(2.2-0.4)}{(0.76-0.4)} \right] \frac{N \cdot s^2}{kg \cdot m}$$

$$F_V = 1.05 \times 10^6\,N \qquad F_V$$

$$x' F_V = \int x\,dF_V = \int_{x_A}^{x_B} x \rho g b \left(H - \frac{B}{(x-A)} \right) dx = \rho g b \int_{x_A}^{x_B} \left[Hx - \frac{Bx}{(x-A)} \right] dx$$

$$x' F_V = \rho g b \left[H \frac{x^2}{2} - Bx - BA \ln(x-A) \right]_{x_A}^{x_B}$$

$$x' F_V = \rho g b \left[\frac{H}{2}(x_B^2 - x_A^2) - B(x_B - x_A) - BA \ln \frac{(x_B - A)}{(x_A - A)} \right] \quad \text{--- (2)}$$

$$x' = 999\,\frac{kg}{m^3} \times 9.81\,\frac{m}{s^2} \times 50\,m \left\{ \frac{2.5\,m}{2} \left[(2.2)^2 m^2 - (0.76)^2 m^2 \right] - 0.9\,m^2 (2.2-0.76)\,m - 0.9\,m^2 \times 0.4\,m \ln \frac{2.2-0.4}{0.76-0.4} \right\} \frac{N \cdot s^2}{kg \cdot m} \times \frac{1}{1.05 \times 10^6\,N}$$

$$x' = 1.61\,m \qquad x'$$

From the free-body diagram of the dam we see that it is the horizontal component of the resultant force of the water that tends to overturn the dam

Thus, neglecting the weight of the dam, the net moment tending to overturn the dam is

$$\sum M_{oz} = x' F_V - y' F_H$$

$$F_H = p_c A = \rho g h_c bH = \rho g \frac{H}{2} bH = \rho g b \frac{H^2}{2} \quad , \quad y' = H - h'$$

$$h' = h_c + \frac{I_{\bar{x}\bar{x}}}{h_c A} = \frac{H}{2} + \frac{bH^3}{12 \times \frac{H}{2} bH} = \frac{H}{2} + \frac{H}{6} = \frac{2}{3}H$$

$$\therefore \; y' F_H = \left(H - \frac{2}{3}H\right) \rho g b \frac{H^2}{2} = \rho g b \frac{H^3}{6} \qquad (3)$$

The tipping moment is a maximum at $H = 3.0\,m$. At $H = 3.0\,m$,

$$y' F_H = \rho g b \frac{(3m)^3}{6} = 4.50\, \rho g b$$

From Eq. (2), at these conditions

$$x' F_V = \rho g b \left\{ \frac{3.0}{2} m \left[(2.2m)^2 - (0.7m)^2 \right] - 0.9m^2 \left[(2.2 - 0.7)m \right] - 0.9m^2 \times 0.4m \times \ln \frac{2.2 - 0.4}{0.7 - 0.4} \right\}$$

$$x' F_V = 4.53\, \rho g b$$

Thus at $H = 3.0\,m$, $\sum M_{oz} = 4.50\,\rho g b - 4.53\,\rho g b = -0.03\,\rho g b$

The weight of the gate would produce a clockwise moment. Even neglecting this, the gate would not tip

Note: The maximum net tipping moment occurs at a water depth $H = 0.5\,m$
At this condition

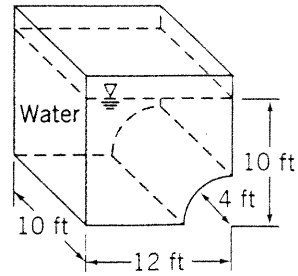
$$y' F_H = \rho g b \frac{H^3}{6} = 999 \frac{kg}{m^3} \times 9.81 \frac{m}{s^2} \times 50\,m \times \frac{(0.5m)^3}{6} \times \frac{N \cdot s^2}{kg \cdot m}$$

$$y' F_H = 10.2\ kN \cdot m$$

The moment from the weight of the gate would be sufficient to prevent tipping.

Problem 3.66

Given: Open tank as shown
width of curved surface b = 10 ft

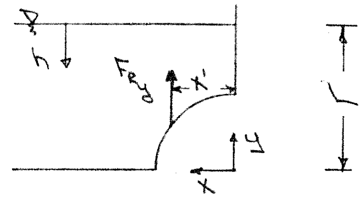


Find: (a) vertical force component, F_{R_y}, on curved surface
(b) line of action of F_{R_y}

Solution:

Basic equations: $\vec{F}_R = -\int P\,d\vec{A}$ $\quad \frac{dP}{dh} = \gamma \quad \vec{r}' \times \vec{F}_R = \int \vec{r} \times d\vec{F} = -\int \vec{r} \times P\,d\vec{A}$

Assumptions: (1) static fluid
(2) gravity is only body force
(3) γ = constant = 62.4 lbf/ft³
(4) h is measured positive downward from free surface



$F_{R_y} = \vec{F}_R \cdot \hat{j} = -\int P\,d\vec{A}\cdot\hat{j} = -\int P\,dA_y = -\int P b\,dx$

We can obtain an expression for P as a function of y

$\frac{dP}{dh} = \gamma \qquad dP = \gamma\,dh \qquad P - P_0 = \int_{P_0}^{P} dP = \int_0^h \gamma\,dh = \gamma h$

Since atmospheric pressure acts at the free surface and on the underside of the curved surface, then the appropriate expression for P is $P = \gamma h$

Now, $h = L - y \quad \therefore P = \gamma(L-y)$

$F_{R_y} = -\int P b\,dx = -\int \gamma(L-y) b\,dx$. Along the surface $y = (R^2 - x^2)^{1/2}$ and so

$F_{R_y} = -\gamma b \int_0^R \{L - (R^2-x^2)^{1/2}\}\,dx = -\gamma b\left[Lx - \frac{1}{2}\left(x\sqrt{R^2-x^2} + R^2 \arcsin\frac{x}{R}\right)\right]_0^R$

$= -\gamma b\left\{LR - \frac{1}{2}(R^2 \arcsin 1) + \frac{1}{2}R^2 \arcsin 0\right\} = \gamma b R\left\{L - \frac{R}{2}\arcsin 1\right\}$

$= -\gamma b R\left\{L - R\frac{\pi}{4}\right\}$

$F_{R_y} = -62.4\,\frac{\text{lbf}}{\text{ft}^3} \times 10\,\text{ft} \times 4\,\text{ft} \times \left\{10\,\text{ft} - 4\,\text{ft} \times \frac{\pi}{4}\right\} = -17{,}100\ \text{lbf}$ (ie acts downward) $\qquad F_{R_y}$

$x'\hat{i} \times F_{R_y}\hat{j} = \int x\hat{i} \times dF_{R_y}\hat{j} = \int x\hat{i} \times (-P\,dA_y\hat{j}) = -\int x\hat{i} \times P b\,dx\,\hat{j}$

$x' F_{R_y}\hat{k} = -\hat{k}\int x P b\,dx$

$x' = -\frac{1}{F_{R_y}}\int_0^R x P b\,dx = -\frac{1}{F_{R_y}}\int_0^R x\gamma(L-y) b\,dx = -\frac{\gamma b}{F_{R_y}}\int_0^R x\{L - (R^2-x^2)^{1/2}\}\,dx$

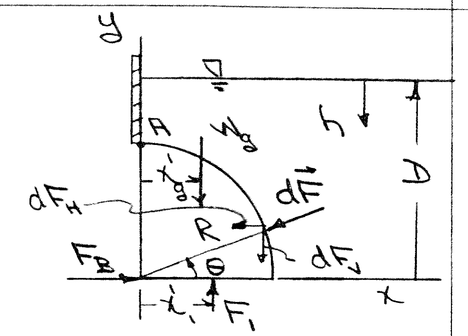
$= -\frac{\gamma b}{F_{R_y}}\left[L\frac{x^2}{2} + \frac{1}{3}\sqrt{(R^2-x^2)^3}\right]_0^R = -\frac{\gamma b}{F_{R_y}}\left[L\frac{R^2}{2} - \frac{1}{3}R^3\right] = -\frac{\gamma b R^2}{F_{R_y}}\left[\frac{L}{2} - \frac{R}{3}\right]$

$x' = -62.4\,\frac{\text{lbf}}{\text{ft}^3} \times 10\,\text{ft} \times (4)^2\,\text{ft}^2 \times \frac{1}{(-17{,}100)\,\text{lbf}}\left[\frac{10\,\text{ft}}{2} - \frac{4\,\text{ft}}{3}\right]$

$x' = 2.14\ \text{ft} \qquad x'$

Problem 3.67

Given: Concrete gate in the form of a quarter cylinder, hinged at A, has width $b = 2\,m$. Liquid is water. $R = 2\,m$, $D = 3\,m$



Find: Force on the stop at B.

Solution:

Basic equations: $\dfrac{dp}{dh} = \rho g$, $\vec{F}_R = -\int p\, d\vec{A}$, $\sum M_{A_z} = 0$

Assumptions: (1) static liquid (2) ρ = constant

$$\sum M_{A_z} = 0 = x_1' F_1 - W_g x_g' + R F_B - \int (R-y)\, dF_H - \int x\, dF_V$$

$$dF_V = dF \sin\theta = p\,dA \sin\theta \; ; \; dF_H = dF\cos\theta = p\,dA\cos\theta$$

$dp = \rho g\, dh$ and $p - p_0 = \rho g h = \rho g (D-y)$. Also $dA = bR\,d\theta$

Then
$$R F_B = W_g x_g' - F_1 x_1' + \int_0^{\pi/2} (R-y)\, p\, bR\cos\theta\, d\theta + \int_0^{\pi/2} x\, p\, bR \sin\theta\, d\theta$$

$$R F_B = W_g x_g' - F_1 x_1' + \int_0^{\pi/2} (R - R\sin\theta)\, \rho g (D-y)\, bR\cos\theta\, d\theta + \int_0^{\pi/2} R\cos\theta\, \rho g (D-y)\, bR\sin\theta\, d\theta$$

$$= W_g x_g' - F_1 x_1' + \rho g b R^2 \int_0^{\pi/2} (1-\sin\theta)(D - R\sin\theta)\cos\theta\, d\theta + \rho g b R^2 \int_0^{\pi/2} \sin\theta\cos\theta\,(D - R\sin\theta)\, d\theta$$

$$= W_g x_g' - F_1 x_1' + \rho g b R^2 \int_0^{\pi/2} \left[D\cos\theta - (D+R)\sin\theta\cos\theta + R\sin^2\theta\cos\theta \right] d\theta$$
$$+ \rho g b R^2 \int_0^{\pi/2} \left[D\sin\theta\cos\theta - R\sin^2\theta\cos\theta \right] d\theta$$

$$= W_g x_g' - F_1 x_1' + \rho g b R^2 \left[D\sin\theta - (D+R)\tfrac{1}{2}\sin^2\theta + R\frac{\sin^3\theta}{3} \right]_0^{\pi/2}$$
$$+ \rho g b R^2 \left[D\frac{\sin^2\theta}{2} - R\frac{\sin^3\theta}{3} \right]_0^{\pi/2}$$

$$= W_g x_g' - F_1 x_1' + \rho g b R^2 \left[D - \tfrac{1}{2}(D+R) + \tfrac{1}{2}D \right]$$

$$R F_B = W_g x_g' - F_1 x_1' + \rho g b R^2 \left[D - \frac{R}{2} \right]$$

$F_1 = p_1 A_1 = \rho g D\, bR$ also $x_1' = \dfrac{R}{2}$

$W_g = \rho_g g \forall_g = \rho_g g \dfrac{\pi R^2}{4} b = SG\,\rho g \dfrac{\pi R^2}{4} b$ {From Table A.1, SG = 2.4}

also $x_g' = \dfrac{4R}{3\pi}$

$$\therefore \; R F_B = SG\,\rho g \frac{\pi R^2}{4} b \times \frac{4R}{3\pi} - \rho g D bR \times \frac{R}{2} + \rho g b R^2 \left[D - \frac{R}{2} \right]$$

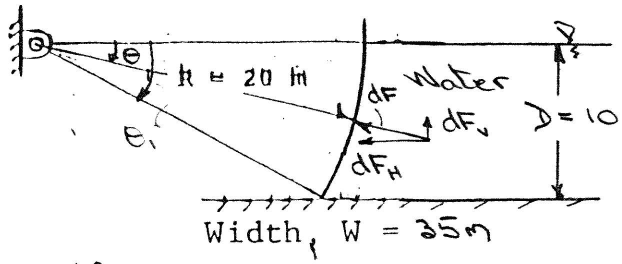
$$F_B = SG_{concrete}\, \rho g \frac{bR^2}{3} + \rho g b R \frac{(D-R)}{2} = \rho g b R \left[\frac{SG\,R}{3} + \frac{(D-R)}{2} \right]$$

$$F_B = 1000\,\frac{kg}{m^3} \times 9.81\,\frac{m}{s^2} \times 2\,m \times 2\,m \left[\frac{(2.4)(2)}{3} + \frac{1}{2} \right] m \times \frac{N\cdot s^2}{kg\cdot m}$$

$F_B = 82.4\ kN$ ← F_B

Given: Tainter gate as shown

Find: Force of the water acting on the gate



Solution:

Basic equations: $dF = p\,dA$; $\dfrac{dp}{dh} = \rho g$

Assumptions: (1) static fluid
(2) ρ = constant
(3) p_{atm} acts at free surface and on surface of gate

For ρ = const, $\int dp = \int \rho g\, dh$ yields $p - p_{atm} = \rho g h = \rho g R \sin\theta$ (3)

$dF_H = dF\cos\theta = p\,dA\cos\theta = \rho g R\sin\theta\, wR\,d\theta \cos\theta$ $\{dA = wR\,d\theta\}$

$F_H = \int dF_H = \int_0^{\theta_1} \rho g w R^2 \sin\theta\cos\theta\, d\theta$ where $\theta_1 = \sin^{-1}\frac{10}{20} = 30°$

$F_H = \rho g w R^2 \int_0^{30°} \sin\theta\cos\theta\, d\theta = \rho g w R^2 \left[\frac{\sin^2\theta}{2}\right]_0^{30°} = \frac{\rho g w R^2}{8}$

$F_H = \frac{1}{8} \times 999\,\frac{kg}{m^3} \times 9.81\,\frac{m}{sec^2} \times 35\,m \times (20\,m)^2 \times \frac{N\cdot sec^2}{kg\cdot m} = 1.72 \times 10^7\,N$ ←

$dF_V = dF\sin\theta = p\,dA\sin\theta = \rho g R\sin\theta\, wR\,d\theta\sin\theta$

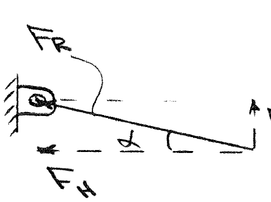
$F_V = \int dF_V = \rho g w R^2 \int_0^{30°} \sin^2\theta\, d\theta = \rho g w R^2 \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi/6}$

$F_V = \rho g w R^2 \left[\frac{\pi}{12} - \frac{0.866}{4}\right] = 0.0453\,\rho g w R^2$

$F_V = 0.0453 \times 999\,\frac{kg}{m^3} \times 9.81\,\frac{m}{s^2} \times 35\,m \times (20\,m)^2 \times \frac{N\cdot s^2}{kg\cdot m} = 6.22 \times 10^6\,N$ ←

Since the gate surface in contact with the water is a circular arc, all elements dF, of the force and hence the line of action of the resultant force must pass through the pivot. Thus

$F_R = \left[F_H^2 + F_V^2\right]^{1/2} = \left[(17.2\times10^6)^2 + (6.22\times10^6)^2\right]^{1/2} = 1.83 \times 10^7\,N$ ← F_R



$\alpha = \tan^{-1}\frac{F_V}{F_R} = \tan^{-1}\frac{6.22}{17.2}$

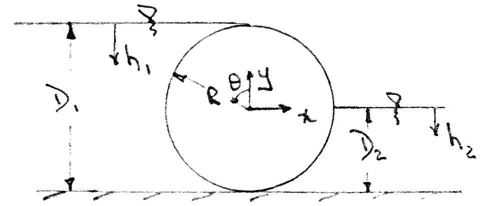
$\alpha = 19.9°$ ← α

F_R passes through pivot at angle α to the horizontal

Problem 3.69

Given: Cylindrical weir of radius, $R = 1.5\,m$ and length, $L = 6\,m$ as shown. Liquid is water.

$D_1 = 3\,m \quad D_2 = 1.5\,m$



Find: Magnitude and direction of resultant force of water on the weir.

Solution:

Basic equations: $\vec{F}_R = -\int P\,d\vec{A} \qquad \dfrac{dP}{dh} = \rho g$

Assumptions: (1) static fluid
(2) ρ = constant
(3) h is measured positive down from free surface

$F_{R_x} = \int dF_x = \vec{F}_R \cdot \hat{i} = \int d\vec{F}\cdot\hat{i} = -\int P\,d\vec{A}\cdot\hat{i} = -\int P\,dA\cos(90+\theta) = \int P\,dA\sin\theta$

$F_{R_y} = \int dF_y = \vec{F}_R\cdot\hat{j} = \int d\vec{F}\cdot\hat{j} = -\int P\,d\vec{A}\cdot\hat{j} = -\int P\,dA\cos\theta$

Since $dA = LR\,d\theta$,

$$F_{R_x} = \int_0^{3\pi/2} PLR\sin\theta\,d\theta \quad \text{and} \quad F_{R_y} = -\int_0^{3\pi/2} PLR\cos\theta\,d\theta$$

We can obtain an expression for P as a function of h,

$$\frac{dP}{dh} = \rho g \qquad dP = \rho g\,dh \quad \text{and} \quad P - P_0 = \int_{P_0}^{P} dP = \int_0^h \rho g\,dh = \rho g h$$

Since atmospheric pressure acts over the first quadrant of the cylinder and both free surfaces, the appropriate expression for P is $P = \rho g h$.

For

$0 \le \theta \le \pi$, $h_1 = R - R\cos\theta = R(1-\cos\theta)$ and hence $P_1 = \rho g R(1-\cos\theta)$

$\pi \le \theta \le \frac{3\pi}{2}$, $h_2 = -R\cos\theta$ and hence $P_2 = -\rho g R\cos\theta$

$$F_{R_x} = \int_0^{3\pi/2} PLR\sin\theta\,d\theta = \int_0^{\pi} \rho g R(1-\cos\theta)LR\sin\theta\,d\theta + \int_{\pi}^{3\pi/2}(-\rho g R\cos\theta)LR\sin\theta\,d\theta$$

$$= \rho g R^2 L\int_0^{\pi}(1-\cos\theta)\sin\theta\,d\theta - \rho g R^2 L\int_{\pi}^{3\pi/2}\cos\theta\sin\theta\,d\theta$$

$$= \rho g R^2 L\left[-\cos\theta - \tfrac{1}{2}\sin^2\theta\right]_0^{\pi} - \rho g R^2 L\left[\tfrac{1}{2}\sin^2\theta\right]_{\pi}^{3\pi/2} = \rho g R^2 L\left[2 - \tfrac{1}{2}\right] = \tfrac{3}{2}\rho g R^2 L$$

$$F_{R_x} = \frac{3}{2} \times 999\,\frac{kg}{m^3} \times 9.81\,\frac{m}{s^2} \times (1.5)^2\,m^2 \times 6\,m \times \frac{N\cdot s^2}{kg\cdot m} = 198\,kN$$

$$F_{R_y} = -\int_0^{3\pi/2} PLR\cos\theta = -\int_0^{\pi}\rho g R(1-\cos\theta)LR\cos\theta\,d\theta - \int_{\pi}^{3\pi/2}(-\rho g R\cos\theta)LR\cos\theta\,d\theta$$

$$= -\rho g R^2 L\int_0^{\pi}(1-\cos\theta)\cos\theta\,d\theta + \rho g R^2 L\int_{\pi}^{3\pi/2}\cos^2\theta\,d\theta$$

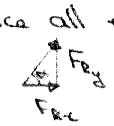
$$= -\rho g R^2 L\left[\sin\theta - \frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi} + \rho g R^2 L\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_{\pi}^{3\pi/2} = \rho g R^2 L\left[\frac{\pi}{2} + \frac{3\pi}{4} - \frac{\pi}{2}\right] = \frac{3\pi}{4}\rho g R^2 L$$

$$F_{R_y} = \frac{3\pi}{4} \times 999\,\frac{kg}{m^3} \times 9.81\,\frac{m}{s^2} \times (1.5)^2\,m^2 \times 6\,m \times \frac{N\cdot s^2}{kg\cdot m} = 312\,kN$$

$$\vec{F}_R = \hat{i}F_{R_x} + \hat{j}F_{R_y} = 198\hat{i} + 312\hat{j}\ kN$$

$$F_R = \sqrt{F_{R_x}^2 + F_{R_y}^2} = \left[(198)^2 + (312)^2\right]^{1/2}\,kN = 370\,kN \quad \longleftarrow \quad F_R$$

Since all elements of force $d\vec{F}$ are normal to the surface, the direction α,

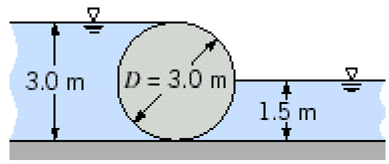


$$\alpha = \tan^{-1} F_{R_y}/F_{R_x} = \tan^{-1} 312/198 = 57.6^\circ \quad \longleftarrow \quad \alpha$$

Problem 3.70

Consider the cylindrical weir of diameter 3 m and length 6 m. If the fluid on the left has a specific gravity of 1.6, and on the right has a specific gravity of 0.8, find the magnitude and direction of the resultant force.

Given: Sphere with different fluids on each side



Find: Resultant force and direction

Solution

The horizontal and vertical forces due to each fluid are treated separately. For each, the horizon force is equivalent to that on a vertical flat plate; the vertical force is equivalent to the weight of "above".

For horizontal forces, the computing equation of Section 3-5 is $F_H = p_c \cdot A$ where A is the area of the equivalent vertical plate.

For vertical forces, the computing equation of Section 3-5 is $F_V = \rho \cdot g \cdot V$ where V is the volume of fluid above the curved surface.

The data are For water $\rho = 999 \cdot \frac{kg}{m^3}$

For the fluids $SG_1 = 1.6$ $SG_2 = 0.8$

For the weir $D = 3 \cdot m$ $L = 6 \cdot m$

(a) Horizontal Forces

For fluid 1 (on the left) $F_{H1} = p_c \cdot A = \left(\rho_1 \cdot g \cdot \frac{D}{2}\right) \cdot D \cdot L = \frac{1}{2} \cdot SG_1 \cdot \rho \cdot g \cdot D^2 \cdot L$

$$F_{H1} = \frac{1}{2}\cdot 1.6\cdot 999\cdot\frac{kg}{m^3}\cdot 9.81\cdot\frac{m}{s^2}\cdot(3\cdot m)^2\cdot 6\cdot m\cdot\frac{N\cdot s^2}{kg\cdot m}$$

$$F_{H1} = 423\,kN$$

For fluid 2 (on the right) $F_{H2} = p_c\cdot A = \left(\rho_2\cdot g\cdot\frac{D}{4}\right)\cdot\frac{D}{2}\cdot L = \frac{1}{8}\cdot SG_2\cdot\rho\cdot g\cdot D^2\cdot L$

$$F_{H2} = \frac{1}{8}\cdot 0.8\cdot 999\cdot\frac{kg}{m^3}\cdot 9.81\cdot\frac{m}{s^2}\cdot(3\cdot m)^2\cdot 6\cdot m\cdot\frac{N\cdot s^2}{kg\cdot m}$$

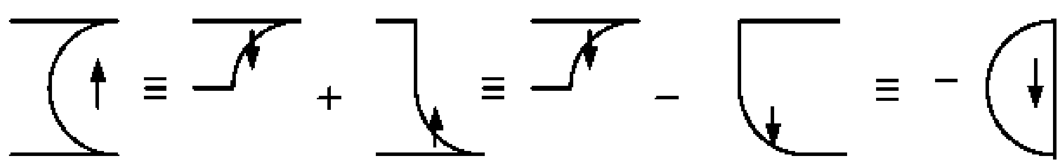
$$F_{H2} = 53\,kN$$

The resultant horizontal force is

$$F_H = F_{H1} - F_{H2} \qquad F_H = 370\,kN$$

(b) Vertical forces

For the left geometry, a "thought experiment" is needed to obtain surfaces with fluid "abov



Hence $$F_{V1} = SG_1\cdot\rho\cdot g\cdot\frac{\frac{\pi\cdot D^2}{4}}{2}\cdot L$$

$$F_{V1} = 1.6 \times 999 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times \frac{\pi \cdot (3 \cdot m)^2}{8} \times 6 \cdot m \times \frac{N \cdot s^2}{kg \cdot m}$$

$$F_{V1} = 332\,kN$$

(Note: Use of buoyancy leads to the same result!)

For the right side, using a similar logic

$$F_{V2} = SG_2 \cdot \rho \cdot g \cdot \frac{\frac{\pi \cdot D^2}{4}}{4} \cdot L$$

$$F_{V2} = 0.8 \times 999 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times \frac{\pi \cdot (3 \cdot m)^2}{16} \times 6 \cdot m \times \frac{N \cdot s^2}{kg \cdot m}$$

$$F_{V2} = 83\,kN$$

The resultant vertical force is

$$F_V = F_{V1} + F_{V2} \qquad F_V = 415\,kN$$

Finally the resultant force and direction can be computed

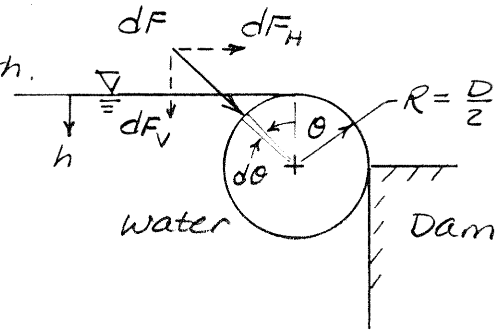
$$F = \sqrt{F_H^2 + F_V^2} \qquad F = 557\,kN$$

$$\alpha = \operatorname{atan}\left(\frac{F_V}{F_H}\right) \qquad \alpha = 48.3\,deg$$

Given: Cylindrical log floating against dam.

Find: (a) Mass per unit length

(b) Contact force per unit length.



Solution: Use hydrostatic equations

Basic equations: $\frac{dp}{dh} = \rho g \qquad dF = p\,dA$

Assumptions: (1) Static liquid

(2) Incompressible

(3) Neglect p_{atm} (it acts everywhere)

Then

$$p - \cancel{p_0}^{(3)} = \rho g h = \rho g R(1-\cos\theta)$$

$$dF = p\,dA = p w R\,d\theta, \quad dF_H = dF\sin\theta, \quad dF_V = dF\cos\theta$$

$$F_H = \int_0^{3\pi/2} \rho g R(1-\cos\theta)\, wR\sin\theta\, d\theta = \rho g w R^2\left[-\cos\theta - \frac{\sin^2\theta}{2}\right]_0^{3\pi/2} = \rho g w R^2\left[-\frac{(-1)^2}{2} - (-1)\right]$$

$$F_H = \frac{1}{2}\rho g w R^2 \qquad \frac{F_H}{w} = \frac{1}{2}\rho g R^2$$ $\frac{F_H}{w}$

$$F_V = \int_0^{3\pi/2} \rho g R(1-\cos\theta)\, wR\cos\theta\, d\theta = \int_0^{3\pi/2} \rho g w R^2\left(\cos\theta - \frac{1+\cos 2\theta}{2}\right)d\theta$$

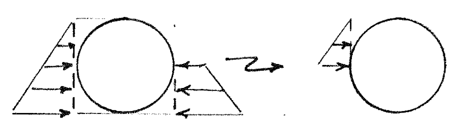
$$F_V = \rho g w R^2\left[\sin\theta - \frac{\theta + \frac{1}{2}\sin 2\theta}{2}\right]_0^{3\pi/2} = \rho g w R^2\left[-1 - \frac{3\pi}{4}\right] = -\rho g w R^2\left[1 + \frac{3\pi}{4}\right]$$

From a free-body diagram of the log

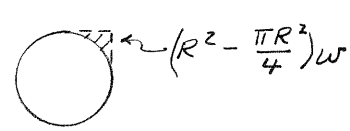
$$\Sigma F_y = -mg - F_V = 0 \qquad m = -\frac{F_V}{g} = \rho w R^2\left[1 + \frac{3\pi}{4}\right]$$

$$\frac{m}{w} = \rho R^2\left[1 + \frac{3\pi}{4}\right]$$ $\frac{m}{w}$

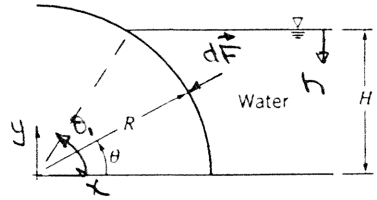
Check: $F_H = p_c A = \rho g \frac{R}{2} wR = \frac{1}{2}\rho g w R^2$ ✓



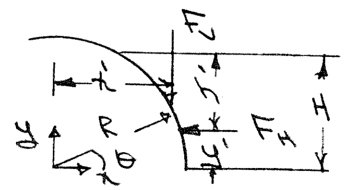
$$F_V = -\rho g \forall - \rho g\left[R^2 - \frac{\pi R^2}{4}\right]w = -\rho g w\left[-\pi R^2 - R^2 + \frac{\pi R^2}{4}\right] = -\rho g w R^2\left[1 + \frac{3\pi}{4}\right]$$ ✓



Given: Curved surface, in shape of quarter cylinder, with radius $R = 0.750$ m and width $w = 3.55$ m; water stands to depth $H = 0.650$ m



Find: Magnitude and line of action of:
(a) vertical force, and
(b) horizontal force
on the curved surface.



Solution:

Basic equations: $\frac{dp}{dh} = \rho g$, $F_V = \int p\, dA_y$, $x' F_V = \int x\, dF_V$

Computing equations: $F_H = p_c A$, $h' = h_c + \frac{I_{xx}}{h_c A}$

Assumptions: (1) static liquid (2) ρ = constant
(3) p_{atm} acts at free surface of the water

Then on integrating $dp = \rho g\, dh$, we obtain $p = \rho g h$.

From the geometry $h = H - R\sin\theta$, $y = R\sin\theta$, $x = R\cos\theta$
$\theta_1 = \sin^{-1} H/R$, $dA = wR\, d\theta$

$$F_V = \int p\, dA_y = \int \rho g h\, dA \sin\theta = \int_0^{\theta_1} \rho g (H - R\sin\theta)\sin\theta\, wR\, d\theta$$

$$F_V = \rho g w R \int_0^{\theta_1} (H\sin\theta - R\sin^2\theta)\, d\theta = \rho g w R \left[-H\cos\theta - R\left(\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right)\right]_0^{\theta_1}$$

$$F_V = \rho g w R \left[H(1 - \cos\theta_1) - R\left(\frac{\theta_1}{2} - \frac{\sin 2\theta_1}{4}\right)\right] \qquad (1)$$

Evaluating for $\theta_1 = \sin^{-1}\frac{H}{R} = \sin^{-1}\frac{0.650}{0.750} = 60°\ (\pi/3)$.

$$F_V = 999\frac{kg}{m^3} \times 9.81\frac{m}{s^2} \times 3.55\,m \times 0.75\,m \left[0.65\,m(1 - \cos 60°) - 0.75\,m\left(\frac{\pi}{6} - \frac{\sin 120°}{4}\right)\right]\frac{N\cdot s^2}{kg\cdot m}$$

$F_V = 2.47$ kN ← F_V

$$x' F_V = \rho g w R \int_0^{\theta_1} R\cos\theta (H\sin\theta - R\sin^2\theta)\, d\theta = \rho g w R^2 \int_0^{\theta_1} (H\sin\theta\cos\theta - R\sin^2\theta\cos\theta)\, d\theta$$

$$x' F_V = \rho g w R^2 \left[H\frac{\sin^2\theta}{2} - R\frac{\sin^3\theta}{3}\right]_0^{\theta_1}$$

$$x' = \frac{\rho g w R^2}{F_V}\left[\frac{H}{2}\sin^2\theta_1 - \frac{R}{3}\sin^3\theta_1\right] \qquad (2)$$

$$x' = 999\frac{kg}{m^3} \times 9.81\frac{m}{s^2} \times 3.55\,m \times (0.75\,m)^2 \times \frac{1}{2.47\times10^3\,N}\left[\frac{0.650\,m}{2}\sin^2 60° - \frac{0.750\,m}{3}\sin^3 60°\right]\frac{N\cdot s^2}{kg\cdot m}$$

$x' = 0.645$ m ← x'

$$F_H = p_c A = \rho g h_c H w = \rho g \frac{H}{2} H w = \frac{\rho g H^2 w}{2} \qquad (3)$$

$$F_H = \frac{1}{2} \times 999\frac{kg}{m^3} \times 9.81\frac{m}{s^2} \times (0.65\,m)^2 \times 3.55\,m \times \frac{N\cdot s^2}{kg\cdot m} = 7.35\ kN$$ ← F_H

$$h' = h_c + \frac{I_{xx}}{h_c A} = h_c + \frac{\frac{1}{12} W H^3}{h_c A} = \frac{H}{2} + \frac{1}{12}\frac{W H^3}{\frac{H}{2} H W} = \frac{H}{2} + \frac{H}{6} = \frac{2}{3} H$$

$$y' = H - h' = H - \frac{2}{3} H = \frac{1}{3} H \quad \text{———— (4)}$$

$$y' = \frac{1}{3} H = \frac{1}{3} \times 0.650\ \text{m} = 0.217\ \text{m} \quad \text{———— } y'$$

The computing equations for the plot are:

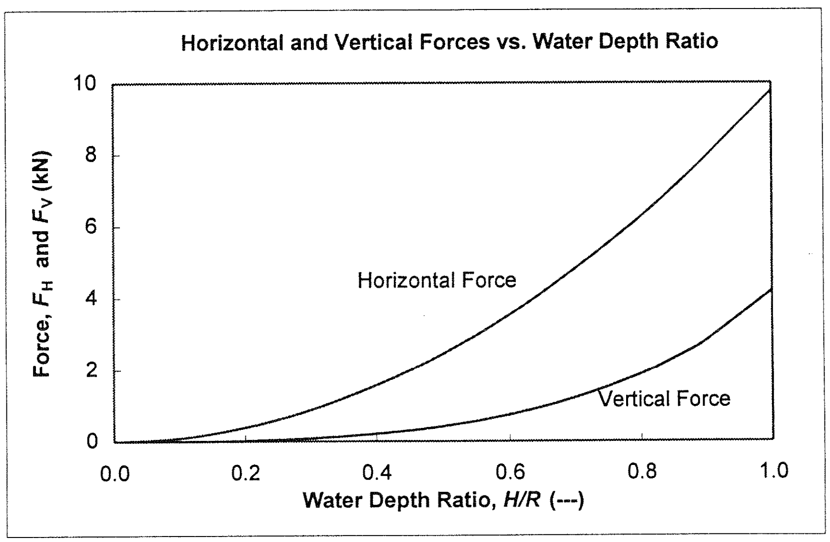
$$\theta_1 = \sin^{-1} H/R$$

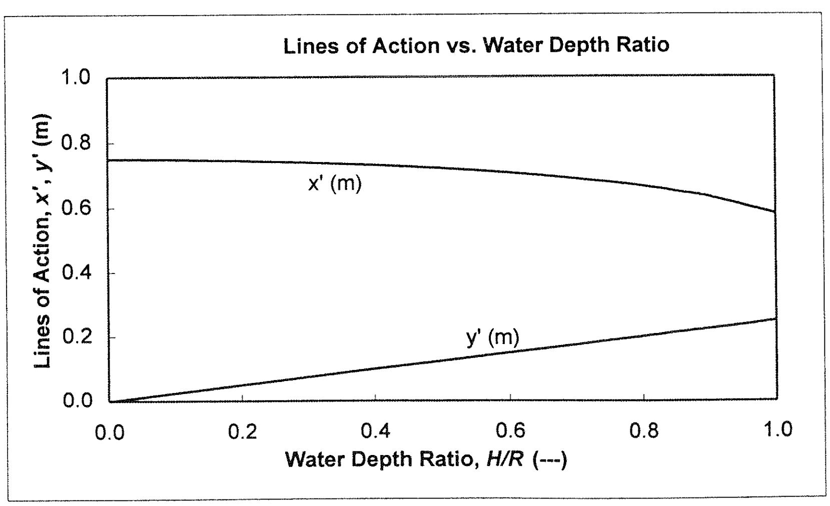
$$F_V = \rho g W R^2 \left[\frac{H}{R}(1 - \cos\theta_1) - \frac{\theta_1}{2} + \frac{\sin 2\theta_1}{4} \right]$$

$$x' = \frac{\rho g W R^3 \sin^2\theta_1}{F_V} \left[\frac{1}{2}\frac{H}{R} - \frac{1}{3}\sin\theta_1 \right]$$

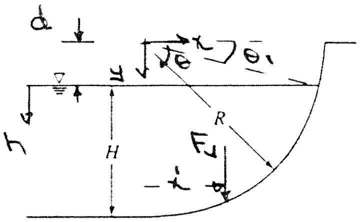
$$F_H = \frac{\rho g H^2 W}{2}$$

$$y' = \frac{H}{3}$$





Given: Curved surface, in shape of quarter cylinder, with radius $R = 0.3\,m$ and width $w = 1.25\,m$ is filled to depth $H = 0.24\,m$ with liquid concrete.



Find: (a) Magnitude, and (b) line of action, of the vertical force on the form from the concrete.

Plot: F_V and x' over the range of depth $0 \le H \le R$

Solution:

Basic equations: $\frac{dp}{dh} = \rho g$, $F_V = \int p\, dA_y$, $x' F_V = \int x\, dF_V$

Assumptions: (1) static liquid (2) ρ = constant
(3) p_{atm} acts at surface of concrete

Then on integrating $dp = \rho g\, dh$, we obtain $p = \rho g h$

$F_V = \int p\, dA_y = \int \rho g h\, dA \sin\theta \qquad dA = wR\,d\theta$

From the geometry: $y = R\sin\theta$, $h = y - d$, $d = R - H$

$F_V = \int_{\theta_1}^{\pi/2} \rho g (R\sin\theta - d)\sin\theta\, wR\, d\theta \qquad$ where $\theta_1 = \sin^{-1}\frac{d}{R}$

$F_V = \rho g R w \int_{\theta_1}^{\pi/2} (R\sin^2\theta - d\sin\theta)\, d\theta = \rho g R w \left[R\left(\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right) + d\cos\theta \right]_{\theta_1}^{\pi/2}$

$F_V = \rho g R w \left[R\left(\frac{\pi}{4} - \frac{\theta_1}{2} + \frac{\sin 2\theta_1}{4}\right) - d\cos\theta_1 \right] \quad (1)$

Evaluating, $\theta_1 = \sin^{-1}\frac{d}{R} = \sin^{-1}\frac{0.3 - 0.24}{0.30} = 11.5°$

$\rho = SG\, \rho_{H_2O} \qquad \{ SG = 2.50$, Table A.1$)$

$F_V = 1000\frac{kg}{m^3} \times 2.5 \times 9.81\frac{m}{s^2} \times 0.3m \times 1.25m \times \frac{N \cdot s^2}{kg \cdot m}\left[0.3m\left(\frac{\pi}{4} - 0.0639\frac{\pi}{2} + \frac{\sin 23°}{4}\right) - 0.06m \cos 11.5° \right]$

$F_V = 1.62\, kN \qquad F_V$

$x' F_V = \rho g R w \int_{\theta_1}^{\pi/2} x (R\sin^2\theta - d\sin\theta)\, d\theta = \rho g R^2 w \int_{\theta_1}^{\pi/2} (R\sin^2\theta\cos\theta - d\sin\theta\cos\theta)\, d\theta$

$= \rho g R^2 w \left[R\frac{\sin^3\theta}{3} + d\frac{\cos^2\theta}{2} \right]_{\theta_1}^{\pi/2}$

$x' F_V = \rho g R^2 w \left[\frac{R}{3}(1 - \sin^3\theta_1) - \frac{d}{2}\cos^2\theta_1 \right]$

$$x' = SG\rho_{H_2O}\frac{gR^2W}{F_V}\left[\frac{R}{3}(1-\sin^3\theta_1) - \frac{d}{2}\cos^2\theta_1\right] \quad (2)$$

$$x' = 2.5\times1000\frac{kg}{m^3}\times9.81\frac{m}{s^2}\times(0.3m)^2\times1.25m\times\frac{1}{1.62\times10^3 N}\times\frac{N\cdot s^2}{kg\cdot m}\times\left[\frac{0.3m}{3}(1-\sin^3 11.5^\circ) - \frac{0.06m}{2}\cos^2 11.5^\circ\right]$$

$$x' = 0.120\ m \qquad x'$$

The computing equations for the required plots are:

$$\theta_1 = \sin^{-1}\frac{R-H}{R} = \sin^{-1}\left(1-\frac{H}{R}\right) \quad (3)$$

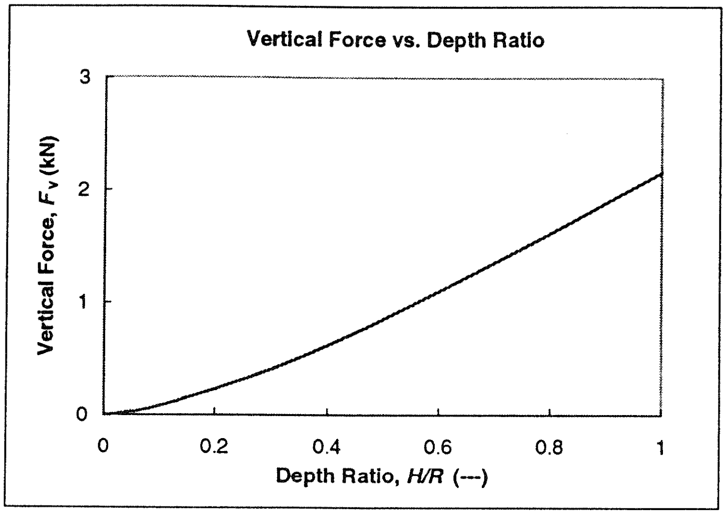
$$F_V = SG\rho_{H_2O}gR^2W\left[\frac{\pi}{4} - \frac{\theta_1}{2} + \frac{\sin 2\theta_1}{4} - \left(1-\frac{H}{R}\right)\cos\theta_1\right] \quad (1a)$$

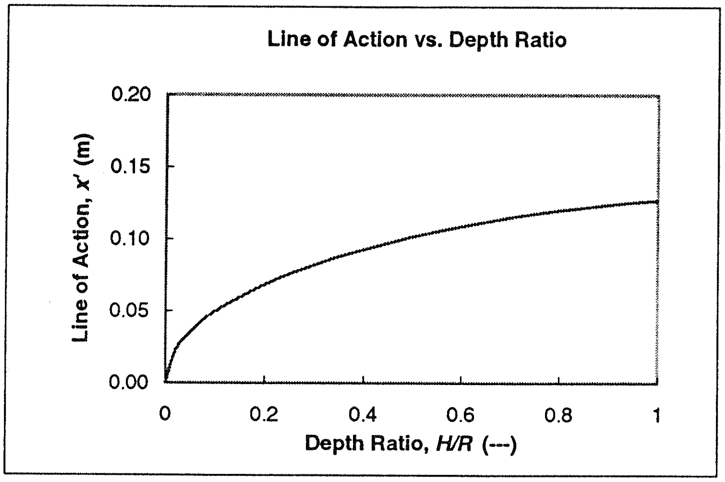
$$x' = SG\frac{\rho_{H_2O}gR^3W}{F_V}\left[\frac{1}{3}(1-\sin^3\theta_1) - \frac{1}{2}\left(1-\frac{H}{R}\right)\cos^2\theta_1\right] \quad (2a)$$

Force and line of action vs. liquid concrete depth:

Radius:	***R* =**	**0.3**	**m**
Specific gravity:	***SG* =**	**2.5**	**---**
Width:	***W* =**	**1.25**	**m**

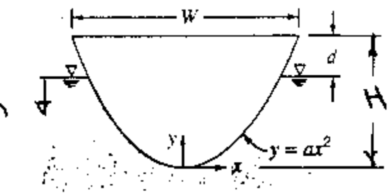
Depth Ratio, *H/R* (---)	Concrete Depth, *H* (m)	Angle, θ_1 (deg)	Vertical Force, F_V (kN)	Line of Action, *x'* (m)
0	0	90.0	0	0
0.02	0.006	78.5	0.00734	0.0224
0.05	0.015	71.8	0.0289	0.0352
0.1	0.03	64.2	0.0810	0.0494
0.2	0.06	53.1	0.226	0.0685
0.3	0.09	44.4	0.408	0.0822
0.4	0.12	36.9	0.617	0.0930
0.5	0.15	30.0	0.847	0.102
0.6	0.18	23.6	1.09	0.109
0.7	0.21	17.5	1.35	0.115
0.8	0.24	11.5	1.62	0.120
0.9	0.27	5.7	1.89	0.124
1.0	0.30	0.0	2.17	0.127


Vertical Force vs. Depth Ratio
Vertical Force, F_V (kN)
3
2
1
0
0
0.2
0.4
0.6
0.8
1
Depth Ratio, H/R (---)


Line of Action vs. Depth Ratio
Line of Action, x' (m)
0.20
0.15
0.10
0.05
0.00
0
0.2
0.4
0.6
0.8
1
Depth Ratio, H/R (---)

Problem 3.74

Given: Model cross section of canoe by $y = ax^2$, where $a = 3.89\ m^{-1}$; coordinates are in meters. Assume constant width $W = 0.6\ m$ over entire length $L = 5.25\ m$.



Find: Expression relating total mass of canoe and contents to distance d; determine maximum allowable total mass without swamping the canoe.

Solution:

At any value of d the weight of the canoe and its contents is balanced by the net vertical force of the water on the canoe.

Basic equations: $\frac{dp}{dh} = \rho g$, $F_v = \int p\, dA_y$

Assumptions: (1) static liquid (2) ρ = constant
(3) p_{atm} acts at free surface of the water and on inner surface of canoe.

Then on integrating $dp = \rho g\, dh$, we obtain $p = \rho g h$

$F_v = \int p\, dA_y = \int \rho g h L\, dx$ where $h = (H-d) - y$

$y = ax^2$, At surface $y = H - d$ $\therefore$ $x = \sqrt{\frac{H-d}{a}}$

$$F_v = 2\int_0^{\sqrt{\frac{H-d}{a}}} \rho g\left[(H-d) - ax^2\right] L\, dx = 2\rho g L\left[(H-d)x - a\frac{x^3}{3}\right]_0^{\sqrt{\frac{H-d}{a}}}$$

$$F_v = 2\rho g L\left[\frac{(H-d)^{3/2}}{\sqrt{a}} - \frac{a}{3}\frac{(H-d)^{3/2}}{a^{3/2}}\right] = \frac{2\rho g L (H-d)^{3/2}}{\sqrt{a}}\left[1 - \frac{1}{3}\right]$$

$$F_v = \frac{4}{3}\frac{\rho g L}{\sqrt{a}}(H-d)^{3/2} = Mg$$

$$\therefore M = \frac{4\rho L (H-d)^{3/2}}{3\sqrt{a}}$$ At $d = 0$, $x = W/2$, $y = H = 0.35\ m$

For $d = 0$, $M = \frac{4}{3} \times 999\frac{kg}{m^3} \times 5.25\ m \times (0.35\ m)^{3/2} \times \left(\frac{m}{3.89}\right)^{1/2} = 734\ kg$

This does not provide any cushion from swamping.

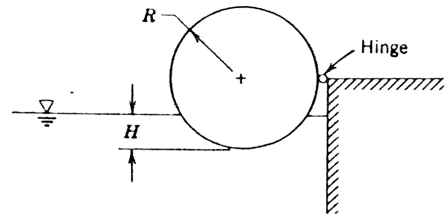
Set $d = 0.050\ m$

$M = \frac{4}{3} \times 999\frac{kg}{m^3} \times 5.25\ m \times (0.30\ m)^{3/2} \times \left(\frac{m}{3.89}\right)^{1/2} = 583\ kg$ ← M

The answer clearly depends on the allowed risk of swamping!

Given: Cylinder, of mass M, length L, and radius R, is hinged along it's length and immersed in an incompressible liquid to depth H.

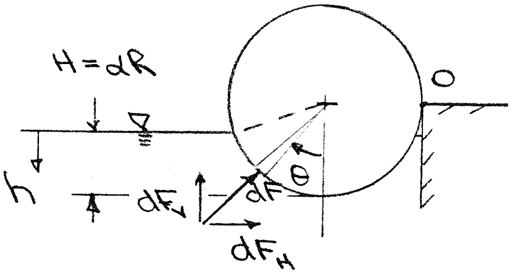
Find: a general expression for the cylinder specific gravity as a function of $\alpha = H/R$ needed to hold the cylinder in equilibrium for $0 \le \alpha \le 1$.



Solution: Apply fluid statics

Basic eqs.: $\frac{dp}{dh} = \rho g$, $F = \int p\,dA$, $\Sigma M = 0$

Assumptions: (1) static liquid
(2) ρ = constant
$\therefore p = \rho g h$



For $0 \le \alpha \le 1$, F_H causes no net moment about O.

$$dF_V = dF\cos\theta = p\,dA\cos\theta = \rho g h\, w R\, d\theta \cos\theta$$

$$h + R(1-\cos\theta) = H, \quad \therefore h = H - R(1-\cos\theta).$$

$$dF_V = \rho g[H - R(1-\cos\theta)]\, w R\cos\theta\, d\theta = \rho g w R^2\left[\frac{H}{R} - (1-\cos\theta)\right]\cos\theta\, d\theta$$

$$dF_V = \rho g w R^2[(\alpha-1)\cos\theta + \cos^2\theta]\,d\theta = \rho g w R^2\left[(\alpha-1)\cos\theta + \frac{1+\cos 2\theta}{2}\right]$$

For $\alpha \le 1$, $F_H = 0$, and

$$F_V = \int_{-\theta_{max}}^{\theta_{max}} dF_V = 2\int_0^{\theta_{max}} dF_V \quad \text{where } \cos\theta_{max} = \frac{R-H}{R} = 1-\alpha$$

$$\theta_{max} = \cos^{-1}(1-\alpha)$$

$$F_V = 2\rho g w R^2 \int_0^{\theta_{max}}\left[(\alpha-1)\cos\theta + \frac{1}{2} + \frac{1}{2}\cos 2\theta\right]d\theta$$

$$F_V = 2\rho g w R^2\left[(\alpha-1)\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\theta_{max}}$$

$$\sin\theta_{max} = \sqrt{1-\cos^2\theta_{max}} = \left[1-(1-\alpha)^2\right]^{1/2} = \left[1 - 1 + 2\alpha - \alpha^2\right]^{1/2} = \sqrt{\alpha(2-\alpha)}.$$

$$\sin 2\theta_{max} = 2\sin\theta_{max}\cos\theta_{max} = 2\sqrt{\alpha(2-\alpha)}\,(1-\alpha)$$

Then,

$$F_V = 2\rho g w R^2\left[(\alpha-1)\sqrt{\alpha(2-\alpha)} + \frac{1}{2}\cos^{-1}(1-\alpha) + \frac{1}{2}(1-\alpha)\sqrt{\alpha(2-\alpha)}\right]$$

$$F_V = 2\rho g w R^2\left[\frac{1}{2}\cos^{-1}(1-\alpha) - \frac{1}{2}(1-\alpha)\sqrt{\alpha(2-\alpha)}\right]$$

$$F_V = \rho g w R^2\left[\cos^{-1}(1-\alpha) - (1-\alpha)\sqrt{\alpha(2-\alpha)}\right]$$

The line of action of the vertical force due to the liquid is through the centroid of the displaced liquid, i.e. through the center of the cylinder

The weight of the cylinder is given by

$$W = mg = \rho_c \forall g = SG\,\rho\,\pi R^2 w\,g$$

where $SG = \rho_c/\rho$ and the gravity force acts through the center of the cylinder

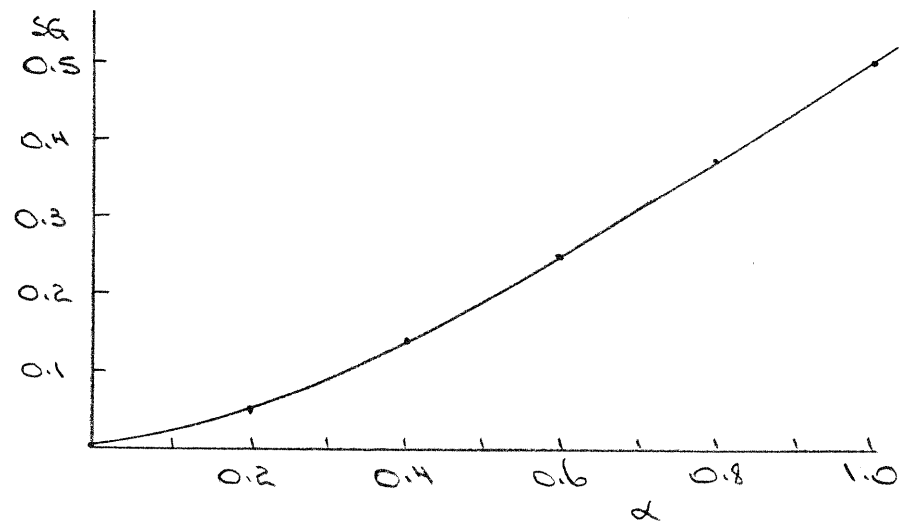
$$\Sigma M_o = WR - F_V R = 0 \quad \therefore \quad W = F_V \text{ and}$$

$$SG\,\rho\,\pi R^2 w\,g = \rho g w R^2 \left[\cos^{-1}(1-\alpha) - (1-\alpha)\sqrt{\alpha(2-\alpha)}\right]$$

$$SG = \frac{1}{\pi}\left[\cos^{-1}(1-\alpha) + (\alpha-1)\sqrt{\alpha(2-\alpha)}\right] \longleftarrow SG\,(0 \le \alpha \le 1)$$

Tabulating values.

α	SG
0	0
0.2	0.052
0.4	0.142
0.6	0.252
0.8	0.374
1.0	0.500



Given: Canoe, modelled as a right circular semi-cylindrical shell, floats in water of depth, d. The shell has outer radius, $R = 0.35$ m and length, $L = 5.25$ m.

Find: (a) a general algebraic expression for the maximum total mass that can be floated, as a function of depth and

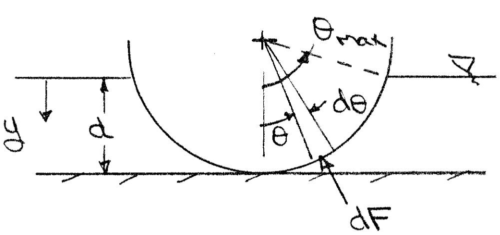
(b) evaluate for the given conditions with $d = 0.245$ m

Plot: the results over the range of water depth $0 \le d \le R$.

Solution:

Basic equations: $\frac{dp}{dy} = \rho g$; $p = p_{atm} + \rho g y$; $F_R = \int p\, dA$

End view of canoe



Assumptions: (1) static liquid
(2) p_{atm} acts on both inside & outside surfaces.

Geometry $y = y(\theta)$ for given d

$$y = d - (R - R\cos\theta) = d - R + R\cos\theta$$

$$\theta_{max} = \cos^{-1}\frac{R-d}{R}$$

A fbd of the canoe gives $\Sigma F_y = 0 = Mg - F_v$

where F_v is the vertical force of the water on the canoe

$$F_v = \int dF_v = \int dF\cos\theta = \int_A p\, dA\cos\theta = \int_{-\theta_{max}}^{\theta_{max}} \rho g y\, LR\, d\theta\cos\theta$$

$$F_v = 2\int_0^{\theta_{max}} \rho g LR\left[(d-R)\cos\theta + R\cos^2\theta\right] d\theta$$

$$F_v = 2\rho g LR\left[(d-R)\sin\theta + R\left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right)\right]_0^{\theta_{max}}$$

$$F_v = 2\rho g LR\left[(d-R)\sin\theta_{max} + R\left(\frac{\theta_{max}}{2} + \frac{\sin 2\theta_{max}}{4}\right)\right]$$

where $\theta_{max} = \cos^{-1}\frac{(R-d)}{R}$.

Since $M = F_v / g$

$$M = 2\rho LR\left[(d-R)\sin\theta_{max} + R\left(\frac{\theta_{max}}{2} + \frac{\sin 2\theta_{max}}{4}\right)\right] \qquad M(d)$$

For $R = 0.35$ m, $L = 5.25$ m and $d = 0.245$ m,

$$\theta_{max} = \cos^{-1}\frac{(R-d)}{R} = \cos^{-1}\frac{(0.35-0.245)}{0.35} = \cos^{-1}0.30 = 72.5°$$

$$\theta_{max} = 0.403\,\pi$$

$$M = 2 \times 999\,\frac{kg}{m^3} \times 5.25\,m \times 0.35\,m\left[(0.245-0.35)\sin 72.5 + 0.35\left(\frac{0.403\pi}{2} + \frac{1}{4}\sin 145\right)\right] m$$

$$M = 631\ kg \qquad M$$

The computing equations for the plot are.

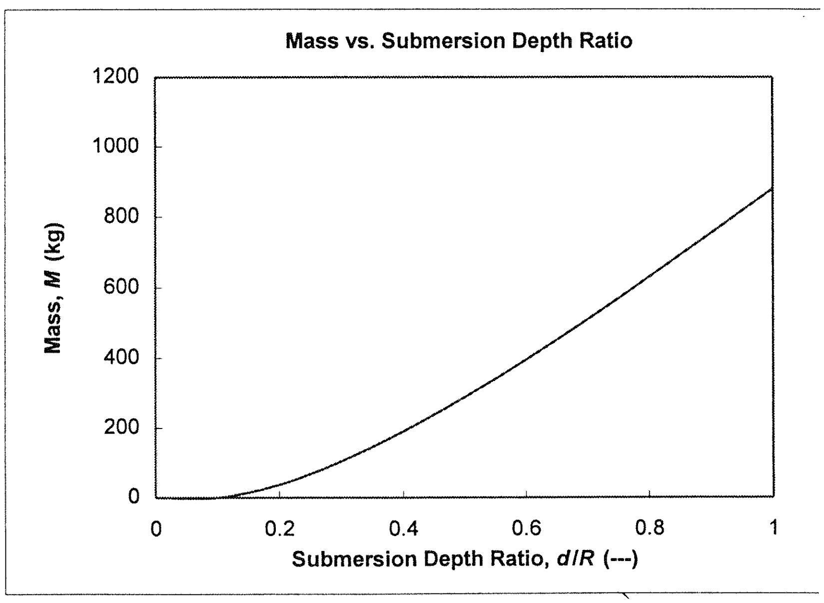
$$\theta_{max} = \cos^{-1}\left(1 - \frac{d}{R}\right)$$

$$M = 2\rho L R^2 \left[\frac{\theta_{max}}{2} + \frac{\sin 2\theta_{max}}{4} - \left(1 - \frac{d}{R}\right) \sin\theta_{max} \right]$$

Mass of canoe vs. depth of submersion ratio:

Density:	ρ =	999	kg/m^3
Length:	L =	5.25	m
Radius:	R =	0.35	m

d (m)	d/R (---)	θ_{max} (rad)	θ_{max} (deg)	Mass (kg)
0	0	0	0	0
0.035	0.10	0.45	25.8	37.7
0.070	0.20	0.64	36.9	105
0.105	0.30	0.80	45.6	190
0.140	0.40	0.93	53.1	287
0.175	0.50	1.05	60.0	395
0.210	0.60	1.16	66.4	509
0.245	0.70	1.27	72.5	630
0.280	0.80	1.37	78.5	754
0.315	0.90	1.47	84.3	881
0.350	1.00	1.57	90.0	1009



Problem 3.77

A glass observation room is to be installed at the corner of the bottom of an aquarium. The aquarium is filled with seawater to a depth of 10 m. The glass is a segment of a sphere, radius 1.5 m, mounted symmetrically in the corner. Compute the magnitude and direction of the net force on the glass structure.

Given: Geometry of glass observation room

Find: Resultant force and direction

Solution

The *x*, *y* and *z* components of force due to the fluid are treated separately. For the *x*, *y* components, the horizontal force is equivalent to that on a vertical flat plate; for the *z* componen (vertical force) the force is equivalent to the weight of fluid above.

For horizontal forces, the computing equation of Section 3-5 is $F_H = p_c \cdot A$ where A is the area of the equivalent vertical plate.

For the vertical force, the computing equation of Section 3-5 is $F_V = \rho \cdot g \cdot V$ where V is the volume of fluid above the curved surface.

The data are For water $\rho = 999 \cdot \frac{kg}{m^3}$

For the fluid (Table A.2) $SG = 1.025$

For the aquarium $R = 1.5 \cdot m$ $H = 10 \cdot m$

(a) Horizontal Forces

Consider the *x* component

The center of pressure of the glass is $y_c = H - \frac{4 \cdot R}{3 \cdot \pi}$ $y_c = 9.36\,m$

Hence $$F_{Hx} = p_c \cdot A = \left(SG \cdot \rho \cdot g \cdot y_c\right) \cdot \frac{\pi \cdot R^2}{4}$$

$$F_{Hx} = 1.025 \times 999 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times 9.36 \cdot m \times \frac{\pi \cdot (1.5 \cdot m)^2}{4} \times \frac{N \cdot s^2}{kg \cdot m}$$

$$F_{Hx} = 166\,kN$$

The *y* component is of the same magnitude as the *x* component

$$F_{Hy} = F_{Hx} \qquad F_{Hy} = 166\,kN$$

The resultant horizontal force (at 45º to the *x* and *y* axes) is

$$F_H = \sqrt{F_{Hx}^2 + F_{Hy}^2} \qquad F_H = 235\,kN$$

(b) Vertical forces

The vertical force is equal to the weight of fluid above (a volume defined by a rectangular column minus a segment of a sphere)

The volume is $$V = \frac{\pi \cdot R^2}{4} \cdot H - \frac{\frac{4 \cdot \pi \cdot R^3}{3}}{8} \qquad V = 15.9\,m^3$$

Then $$F_V = SG \cdot \rho \cdot g \cdot V = 1.025 \times 999 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times 15.9 \cdot m^3 \times \frac{N \cdot s^2}{kg \cdot m}$$

$$F_V = 160\,kN$$

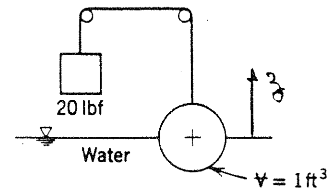
Finally the resultant force and direction can be computed

$$F = \sqrt{F_H^{\,2} + F_V^{\,2}} \qquad F = 284\,kN$$

$$\alpha = atan\left(\frac{F_V}{F_H}\right) \qquad \alpha = 34.2\,deg$$

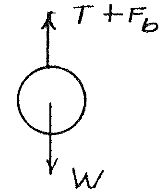
Note that α

Given: Sphere of $\forall = 1\ ft^3$ floating as shown.



Find: (a) Specific weight of sphere.
(b) New equilibrium position if 20-pound weight is removed.

Solution: Balance forces on sphere.



Basic equation: $\Sigma F_z = m a_z = 0$

Computing equation: $F_{buoyancy} = \rho_{H_2O}\, g\, \forall_{displaced} = \gamma \forall_{disp}$

Then $\Sigma F_z = T + F_b - W = 0$

$$\Sigma F_z = T + \gamma_{H_2O}\forall_{disp} - W = T + \gamma_{H_2O}\frac{\forall_s}{2} - \gamma_s \forall_s = 0$$

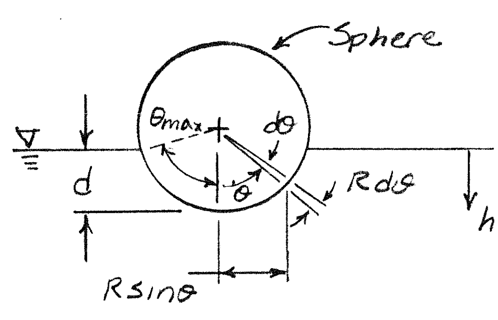
Thus $\gamma_s = \dfrac{T + \gamma_{H_2O}\frac{\forall_s}{2}}{\forall_s} = \dfrac{T}{\forall_s} + \dfrac{\gamma_{H_2O}}{2}$

$$\gamma_s = 20\ lbf \times \frac{1}{1\ ft^3} + \frac{1}{2} \times 62.4\ \frac{lbf}{ft^3} = 51.2\ lbf/ft^3$$ γ_s

To find new equilibrium position, evaluate force from water on sphere.

Basic equations: $\dfrac{dp}{dh} = \gamma_{H_2O}$

$$dF = p\,dA$$



Assumptions: (1) Static liquid
(2) Incompressible
(3) Neglect patm, because it acts everywhere

Then $dF_V = \cos\theta\, p\, dA$; $p = \gamma h$; $d = h + R(1-\cos\theta)$; $h = d - R(1-\cos\theta)$

$$dA = 2\pi R \sin\theta\, R\,d\theta = 2\pi R^2 \sin\theta\, d\theta$$

$$dF_V = \cos\theta\, \gamma\left[d - R(1-\cos\theta)\right] 2\pi R^2 \sin\theta\, d\theta = 2\pi R^3 \gamma\left[\frac{d}{R} - (1-\cos\theta)\right]\sin\theta\cos\theta\, d\theta$$

Now $F_V = \int_A dF_V = \int_0^{\theta_{max}} 2\pi R^3\gamma\left[\frac{d}{R} - (1-\cos\theta)\right]\sin\theta\cos\theta\,d\theta$

At θ_{max}, $\cos\theta_{max} = \dfrac{R-d}{R} = 1 - \dfrac{d}{R}$, so

$$F_V = 2\pi R^3\gamma\left\{\left(1-\frac{d}{R}\right)\left[\frac{1}{2}\left(1-\frac{d}{R}\right)^2 - \frac{1}{2}\right] - \frac{1}{3}\left(1-\frac{d}{R}\right)^3 + \frac{1}{3}\right\}$$

$$F_V = 2\pi R^3\gamma\left[\frac{1}{6}\left(1-\frac{d}{R}\right)^3 - \frac{1}{2}\left(1-\frac{d}{R}\right) + \frac{1}{3}\right]$$ F_V

But $\forall_s = \dfrac{4\pi R^3}{3}$, so

$F_V = F_{buoyancy} = \frac{3}{2}\left(\frac{4\pi R^3}{3}\right)\gamma_{H_2O}\, f\left(\frac{d}{R}\right)$; $f\left(\frac{d}{R}\right) = \frac{1}{6}\left(1-\frac{d}{R}\right)^3 - \frac{1}{2}\left(1-\frac{d}{R}\right) + \frac{1}{3}$

Because $T = 0$, $F_B = W = \gamma_S \forall_S = \frac{3}{2}\gamma_{H_2O}\forall_S\, f\left(\frac{d}{R}\right)$

Thus at equilibrium $f\left(\frac{d}{R}\right) = \frac{2}{3}\frac{\gamma_S}{\gamma_{H_2O}} = \frac{2}{3} SG_S = \frac{2}{3}\times\frac{51.2}{62.4} = 0.547$

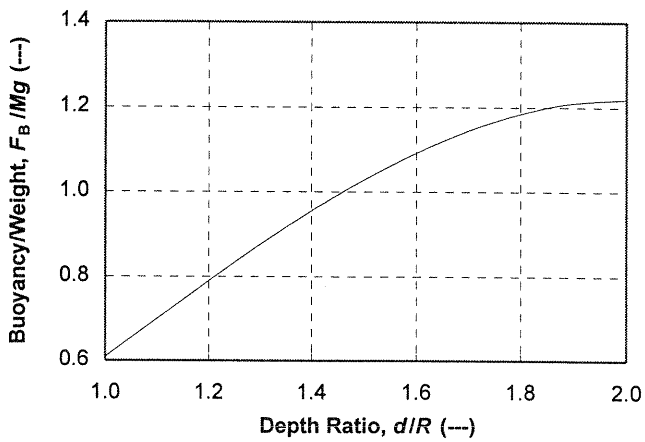
By iteration, using a spreadsheet, $f(d/R) = 0.547$ when $d/R = 1.46$ — d/R

For $\forall = 1\,ft^3$, $\forall = \frac{4\pi R^3}{3}$ and $R = \left(\frac{3\forall}{4\pi}\right)^{1/3} = 0.620\,ft$

Thus $d = 1.46R = 1.46 \times 0.62\,ft = 0.906\,ft$ — d

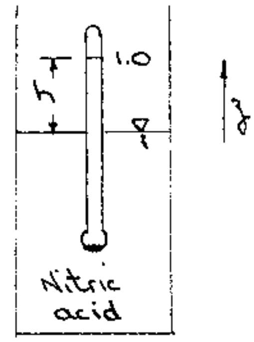
The spreadsheet results and plot are shown below.

d/R (---)	f(d/R)	Fv/Mg (---)
1	0.333	0.609
1.1	0.383	0.700
1.2	0.432	0.790
1.3	0.479	0.875
1.4	0.523	0.956
1.46	0.547	1.000
1.5	0.563	1.03
1.6	0.597	1.09
1.7	0.626	1.14
1.8	0.648	1.18
1.9	0.662	1.21
2	0.667	1.22



Problem *3.79

Given: Hydrometer, as shown, submerged in nitric acid, S.G = 1.5
When immersed in water, h = 0 and immersed volume is 15 cm³.
Stem diameter d = 6 mm.



Find: The distance, h

Solution:

Basic equation: $\Sigma \vec{F} = m\vec{a} = 0$

Computing equation: $\vec{F}_{buoyancy} = \rho g \forall_d \hat{k}$

Assumptions: (1) static conditions

(2) ρ = constant

$$\Sigma \vec{F} = 0 = M\vec{g} + \vec{F}_{buoyancy}$$

Using the data given for water, we can calculate M

$$-Mg + F_b = 0 \qquad M = \frac{F_b}{g} = \rho_{H_2O} \forall_{H_2O}$$

When immersed in nitric acid

$$M = \rho_{n.a} \forall_{n.a} \qquad \text{where } \forall_{n.a} = \forall_{H_2O} - \frac{\pi d^2 h}{4}$$

Since the mass is the same in both cases

$$M = \rho_{H_2O} \forall_{H_2O} = \rho_{n.a}\left(\forall_{H_2O} - \frac{\pi d^2 h}{4}\right)$$

$$\frac{\pi d^2 h}{4} = \forall_{H_2O} - \frac{\rho_{H_2O}}{\rho_{n.a}} \forall_{H_2O} = \forall_{H_2O}\left(1 - \frac{1}{S.G_{n.a}}\right)$$

$$h = \frac{4 \forall_{H_2O}}{\pi d^2}\left(1 - \frac{1}{S.G_{n.a}}\right)$$

$$h = \frac{4}{\pi} \times 15\,cm^3 \times \frac{1}{6^2\,mm^2}\left(1 - \frac{1}{1.5}\right) \times \frac{1000\,mm^3}{cm^3} = 177\,mm$$ h

Problem *3.80

Given: Experiment performed by Archimedes to identify the material content of King Hero's crown.

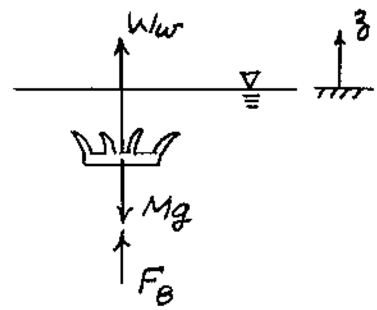
Measured weight of crown in air, W_a, and in water, W_w.

Find: Expression for specific gravity of crown as function of W_a and W_w.

Solution: Apply principle of bouyancy to free-body of crown:

Computing equation: $F_B = \rho_{H_2O} g \forall$

Assumptions: (1) Static liquid
(2) Incompressible liquid



Free-body diagram of crown in water:

$$\Sigma F_z = W_w - Mg + F_B = m a_z = 0$$

or

$$W_w - Mg + \rho_{H_2O} \forall g = 0$$

For the crown in air, $W_a = Mg$

Combining, $W_w - W_a + \rho_{H_2O} g \forall$, so $\forall = \dfrac{W_a - W_w}{\rho_{H_2O} g}$

The crown's density is $\rho_c = \dfrac{M}{\forall} = \dfrac{W_a}{g\forall} = \rho_{H_2O} \dfrac{W_a}{W_a - W_w}$

The crown's specific gravity is $SG = \dfrac{\rho_c}{\rho_{H_2O}} = \dfrac{W_a}{W_a - W_w}$ SG

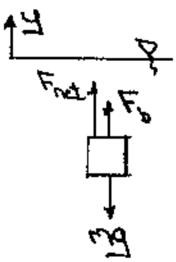
{Note: by definition, $SG = \rho / \rho_{H_2O}(4°C)$, so the measured temperature of water and data from Table A.7 or A.8 may be used to correct the density to 4°C.}

Problem *3.81

Given: Specific gravity of a person is to be determined from measurements of weight in air and the net weight when totally immersed in water.

Find: Expression for the specific gravity of a person from the measurements.

Solution:



For equilibrium $\Sigma F_y = 0$

$$F_{net} = mg - F_b$$

$$F_b = \rho_{H_2O}\, g\, V \qquad F_{air} = mg$$

$$\therefore F_{net} = F_{air} - \rho_{H_2O}\, g\, V \qquad \text{and } V = \frac{F_{air} - F_{net}}{\rho_{H_2O}\, g}$$

$$F_{air} = mg = \rho V g = \frac{\rho}{\rho_{H_2O}}(F_{air} - F_{net})$$

Let $\rho^* = \rho_{H_2O}$ at 4C. Then

$$F_{air} = \frac{\rho/\rho^*}{\rho_{H_2O}/\rho^*}(F_{air} - F_{net}) = \frac{SG}{SG_{H_2O}}(F_{air} - F_{net}).$$

Solving for SG,

$$SG = SG_{H_2O}\,\frac{F_{air}}{(F_{air} - F_{net})}$$

SG

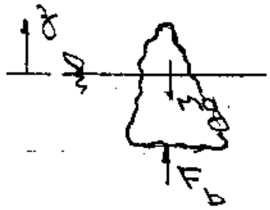
Problem *3.82

Given: Iceberg floating in sea water

Find: Quantify the statement "only the tip of an iceberg shows"

Solution:

A floating body is buoyed up by a force equal to the weight of the displaced liquid.



$$\Sigma F_z = 0 = F_b - mg$$

$$F_b = \rho_s \forall_{sub}\, g \qquad m = \rho \forall_{total}$$

$$\therefore \rho_s \forall_{sub}\, g = \rho \forall_{tot}\, g$$

$$\therefore \forall_{sub} = \forall_{tot} \frac{\rho}{\rho_{sw}} = \forall_{tot} \frac{\rho/\rho^*}{\rho_{sw}/\rho^*}$$

where $\rho^* = \rho_{H_2O}$ at 4C.

$$\forall_{sub} = \forall_{tot} \frac{SG_{ice}}{SG_{sw}}$$

$$\forall_{not\,sub} = \forall_{tot} - \forall_{sub} = \forall_{tot}\left(1 - \frac{SG_{ice}}{SG_{sw}}\right)$$ {Table A.1, $SG_{ice} = 0.917$; Table A.2, $SG_{sw} = 1.025$}

$$\therefore \frac{\forall_{not\,sub}}{\forall_{tot}} = 1 - \frac{SG_{ice}}{SG_s} = 1 - \frac{0.917}{1.025}$$

$$\frac{\forall_{not\,sub}}{\forall_{tot}} = 0.105$$ (10% shows)

Problem *3.83

An open tank is filled to the top with water. A steel cylindrical container, wall thickness $\delta = 1$ mm, outside diameter $D = 100$ mm, and height $H = 1$ m, with an open top, is gently placed in the water. What is the volume of water that overflows from the tank? How many 1 kg weights must be placed in the container to make it sink? Neglect surface tension effects.

Given: Geometry of steel cylinder

Find: Volume of water displaced; number of 1 kg wts to make it sink

Solution

The data are For water $\rho = 999 \cdot \frac{kg}{m^3}$

For steel (Table A.1) $SG = 7.83$

For the cylinder $D = 100 \cdot mm \quad H = 1 \cdot m \quad \delta = 1 \cdot mm$

The volume of the cylinder is $V_{steel} = \delta \cdot \left(\frac{\pi \cdot D^2}{4} + \pi \cdot D \cdot H \right) \quad V_{steel} = 3.22 \times 10^{-4} m^3$

The weight of the cylinder is $W = SG \cdot \rho \cdot g \cdot V_{steel}$

$$W = 7.83 \times 999 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times 3.22 \times 10^{-4} \cdot m^3 \times \frac{N \cdot s^2}{kg \cdot m}$$

$$W = 24.7 N$$

At equilibium, the weight of fluid displaced is equal to the weight of the cylinder

$$W_{displaced} = \rho \cdot g \cdot V_{displaced} = W$$

$$V_{displaced} = \frac{W}{\rho \cdot g} = 24.7 \cdot N \times \frac{m^3}{999 \cdot kg} \times \frac{s^2}{9.81 \cdot m} \times \frac{kg \cdot m}{N \cdot s^2}$$

$$V_{displaced} = 2.52 \times 10^{-3} m^3$$

To determine how many 1 kg wts will make it sink, we first need to find the extra volume that w
need to be dsiplaced

Distance cylinder sank $x_1 = \frac{V_{displaced}}{\left(\frac{\pi \cdot D^2}{4}\right)}$ $x_1 = 0.321\,m$

Hence, the cylinder must be made to sink an additional distanc $x_2 = H - x_1$ $x_2 = 0.679\,m$

We deed to add n weights so that $1 \cdot kg \cdot n \cdot g = \rho \cdot g \cdot \frac{\pi \cdot D^2}{4} \cdot x_2$

$$n = \frac{\rho \cdot \pi \cdot D^2 \cdot x_2}{4 \times 1 \cdot kg} = 999 \cdot \frac{kg}{m^3} \times \frac{\pi}{4} \times (0.1 \cdot m)^2 \times 0.679 \cdot m \times \frac{1}{1 \cdot kg} \times \frac{N \cdot s^2}{kg \cdot m}$$

$$n = 5.328$$

Hence we need $n = 6$ weights to sink the cylinder

Problem *3.84

Given: Hydrogen bubble, with diameter $d = 0.025$ mm, rise slowly when immersed in water.
The drag force on a bubble is given by $F_D = 3\pi\mu V d$, where V is bubble speed relative to the water.

Find: (a) the buoyancy force on a hydrogen bubble immersed in water.
(b) estimate of terminal speed of bubble rising in water

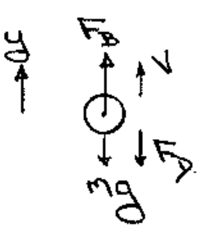
Solution:

Basic equations: $F_B = \rho g \forall$, $\Sigma \vec{F} = m\vec{a}$

For a sphere, $\forall = \frac{\pi d^3}{6}$

$$\therefore F_B = \rho g \forall = \frac{\rho g \pi d^3}{6} = \frac{\pi}{6} \times 999\frac{kg}{m^3} \times 9.81\frac{m}{s^2} \times (0.025\times10^{-3})^3 m^3 \times \frac{N\cdot s^2}{kg\cdot m}$$

$$F_B = 8.02 \times 10^{-11}\ N \quad \longleftarrow F_B$$



$$\Sigma F_y = F_B - mg - F_D = ma_y$$

At terminal speed, $a_y = 0$. Hence

$$F_D = 3\pi\mu V d = F_B - mg$$

and
$$V = \frac{F_B - mg}{3\pi\mu d}$$

At $T = 20°C$, from Table A.8 (Appendix A) $\mu = 1.0\times10^{-3}\ N\cdot s/m^2$

Treat hydrogen as an ideal gas. Assume $T = 20\,c$, $p = 1.1$ atm

$$mg = \rho \forall g = \frac{p}{RT}\forall g = \frac{p}{RT}\frac{\pi d^3}{6} g \qquad \left\{\text{From Table A.6, } R = 4124\frac{J}{kg\cdot K}\right\}$$

$$mg = 1.1\,atm \times 1.01\times10^{5}\frac{N}{m^2\cdot atm} \times \frac{kg\cdot K}{4124\,J} \times \frac{1}{293\,K} \times \frac{\pi}{6} \times (0.025\times10^{-3})^3 \times 9.81\frac{m}{s^2} \times \frac{J}{N\cdot m} \times \frac{N\cdot s^2}{kg\cdot m}$$

$$mg = 7.38\times10^{-15}\ N$$

$$\therefore V = \frac{(8.02\times10^{-11} - 7.38\times10^{-15})N}{3\pi} \times \frac{1}{1\times10^{-3}}\frac{m^2}{N\cdot s} \times \frac{1}{0.025\times10^{-3}\,m}$$

$$V = 3.40\times10^{-4}\ m/s \quad \text{or} \quad 0.341\ mm/s \quad \longleftarrow V$$

(As noted by Prof. Kline in the movie, "Flow Visualization," bubbles rise slowly!).

Problem *3.85

Open-Ended Problem Statement: Gas bubbles are released from the regulator of a submerged Scuba diver. What happens to the bubbles as they rise through the seawater?

Discussion: Air bubbles released by a submerged diver should be close to ambient pressure at the depth where the diver is swimming. The bubbles are small compared to the depth of submersion, so each bubble is exposed to essentially constant pressure. Therefore the released bubbles are nearly spherical in shape.

The air bubbles are buoyant in water, so they begin to rise toward the surface. The bubbles are quite light, so they reach terminal speed quickly. At low speeds the spherical shape should be maintained. At higher speeds the bubble shape may be distorted.

As the bubbles rise through the water toward the surface, the hydrostatic pressure decreases. Therefore the bubbles expand as they rise. As the bubbles grow larger, one would expect the tendency for distorted bubble shape to be exaggerated.

Problem *3.86

Given: Balloons with hot air, helium, and hydrogen. Claim lift per cubic foot of 0.018, 0.066, and 0.071 lbf/ft³ for respective gases, with air heated to 150°F over ambient.

Find: (a) Evaluate claims
(b) Compare air at 250°F above ambient.

Solution: Assume ambient conditions are STP, $p_{gas} = p_{air}$, and apply ideal gas equation of state.

(Use data from Table A.6.)

Basic equations: $\text{Lift} = \rho_{air} g \forall - \rho_{gas} g \forall$, $p = \rho R T$

Then $$\text{Lift}/\forall = g(\rho_a - \rho_g) = \rho_a g\left(1 - \frac{\rho_g}{\rho_a}\right) = \rho_a g\left(1 - \frac{R_a T_a}{R_g T_g}\right); \; \rho_a g = 0.0765 \frac{\text{lbf}}{\text{ft}^3}$$

For helium

$$\frac{L}{\forall} = 0.0765 \frac{\text{lbf}}{\text{ft}^3}\left[1 - 53.33 \frac{\text{ft}\cdot\text{lbf}}{\text{lbm}\cdot R} \times (460+59)R \times \frac{\text{lbm}\cdot R}{386.1\ \text{ft}\cdot\text{lbf}} \times \frac{1}{(460+59\,F)}\right]$$

$$\frac{L}{\forall} = 0.0659\ \text{lbf}/\text{ft}^3 \quad (\text{rounds to } 0.066)$$ He

For hydrogen

$$\frac{L}{\forall} = 0.0765 \frac{\text{lbf}}{\text{ft}^3}\left(1 - \frac{53.33}{766.5}\right) = 0.0712\ \text{lbf}/\text{ft}^3 \quad (\text{rounds to } 0.071)$$ H_2

For air at 150°F above ambient,

$$\frac{L}{\forall} = 0.0765 \frac{\text{lbf}}{\text{ft}^3}\left[1 - \frac{53.33\,(460+59)}{53.33\,(460+59+150)}\right] = 0.0172\ \text{lbf}/\text{ft}^3$$ Air $\Delta T = 150$

For air at 250°F above ambient,

$$\frac{L}{\forall} = 0.0765 \frac{\text{lbf}}{\text{ft}^3}\left[1 - \frac{53.33\,(460+59)}{53.33\,(460+59+250)}\right] = 0.0249\ \text{lbf}/\text{ft}^3$$ Air $\Delta T = 250$ F

Agreement with claims is good.

Air at $\Delta T = 250°F$ gives 45 percent more lift than at $\Delta T = 150°F$.

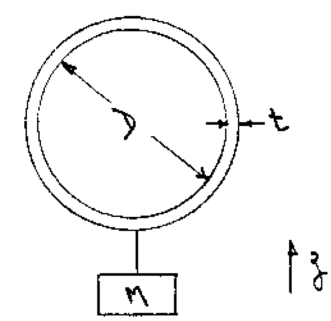
{ Hot air balloon needs 40.2 ft³/lbf of lift at $\Delta T = 250°F$! }

Problem *3.87

Given: Spherical balloon of diameter D, and skin thickness, $t = 0.013$ mm, filled with helium lifted a payload of mass $M = 230$ kg to an altitude of 49 km. At altitude,

$p = 0.95$ mbar and $T = -20°C$.

The helium temperature is $-10°C$. The specific gravity of the skin material is 1.28



Find: The diameter and mass of the balloon.

Solution: Basic equation $\Sigma \vec{F} = m\vec{a} = 0$

Assumptions: (1) static equilibrium at altitude of 49 km
(2) air and helium exhibit ideal gas behavior.

$$\Sigma F_z = 0 = F_{buoy} - M_{He}g - M_s g - Mg = \rho_{air} g \forall_b - \rho_{He} g \forall_b - \rho_s \forall_s - Mg$$

$$0 = \forall_b(\rho_{air} - \rho_{He}) - \rho_s A_s t - M = \frac{4}{3}\pi R^3(\rho_{air} - \rho_{He}) - \rho_s 4\pi R^2 t - M$$

$$0 = \frac{\pi D^3}{6}(\rho_{air} - \rho_{He}) - \rho_s \pi D^2 t - M$$

This is a cubic equation which requires an iterative solution

$$\pi D^2\left[\frac{D}{6}(\rho_{air} - \rho_{He}) - \rho_s t\right] - M = 0 \qquad \text{Solving for } D,$$

$$D = \frac{6}{(\rho_{air} - \rho_{He})}\left[\frac{M}{\pi D^2} + \rho_s t\right] = 6\left[\frac{M}{\pi D^2(\rho_{air} - \rho_{He})} + \frac{\rho_s t}{(\rho_{air} - \rho_{He})}\right]$$

From the ideal gas law,

$$\rho_{air} = \frac{p}{RT} = 0.95\times10^{-3}\,\text{bar} \times \frac{\text{kg}\cdot\text{K}}{287\,\text{J}} \times \frac{1}{253\,\text{K}} \times \frac{10^5\,\text{Pa}}{\text{bar}} \times \frac{\text{N}}{\text{Pa}\cdot\text{m}^2} \times \frac{\text{J}}{\text{N}\cdot\text{m}} = 1.31\times10^{-3}\,\frac{\text{kg}}{\text{m}^3}$$

$$\rho_{He} = \frac{p}{RT} = 0.95\times10^{-3}\,\text{bar} \times \frac{\text{kg}\cdot\text{K}}{2080} \times \frac{1}{263\,\text{K}} \times \frac{10^5\,\text{Pa}}{\text{bar}} \times \frac{\text{N}}{\text{Pa}\cdot\text{m}^2} \times \frac{\text{J}}{\text{N}\cdot\text{m}} = 1.74\times10^{-4}\,\frac{\text{kg}}{\text{m}^3}$$

Substituting into the expression for D

$$D = 6\left[\frac{1}{\pi D^2} \times 230\,\text{kg} \times \frac{\text{m}^3}{11.4\times10^{-4}\,\text{kg}} + (1.28)999\,\frac{\text{kg}}{\text{m}^3} \times 1.3\times10^{-5}\,\text{m} \times \frac{\text{m}^3}{11.4\times10^{-4}\,\text{kg}}\right]$$

$$D = \left[\frac{38.5\times10^4}{D^2} + 87.5\right] \qquad \text{where } D \text{ is in meters}$$

Organizing Calculations: Guess D (m) = 100 120 116
R.H.S = 126 114 116.1

$\therefore D = 116$ m ← D

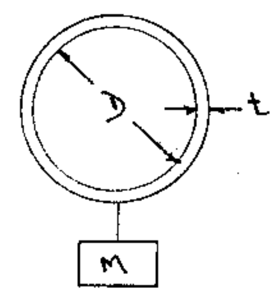
$$M_b = \rho_s \forall_s = \rho_s A_s t = \rho_s \pi D^2 t = 1.28 \times 999\,\frac{\text{kg}}{\text{m}^3} \times \pi(116)^2\,\text{m}^2 \times 1.3\times10^{-5}\,\text{m}$$

$M_b = 703$ kg ← M_b

Problem *3.88

Given: A pressurized helium balloon is to be designed to lift a payload of mass, M, to an altitude of 40 km, where
$P = 3.0$ mbar and $T = -25°C$.
The balloon skin has a specific gravity, S.G = 1.28 and thickness, $t = 0.015$ mm
The gage pressure of the helium is 0.45 mbar. The allowable tensile stress in the balloon skin is $\sigma = 62$ MN/m²

Find: (a) Maximum balloon diameter
(b) Payload, M

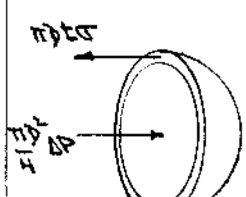


Solution:

Basic equation: $\Sigma \vec{F} = m\vec{a} = 0$

Assumptions: (1) static equilibrium at altitude
(2) air and helium exhibit ideal gas behavior

The balloon diameter is limited by tensile stress

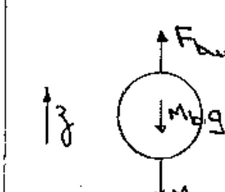


$$\Sigma F = 0 = \frac{\pi D^2}{4}\Delta P - \pi D t \sigma$$

$$D_{max} = \frac{4t\sigma}{\Delta P}$$

$$D_{max} = 4 \times 1.50 \times 10^{-5}\,\text{m} \times 62 \times 10^{6}\,\frac{\text{N}}{\text{m}^2} \times \frac{1}{0.45 \times 10^{-3}\,\text{bar}} \times \frac{\text{bar}\,\text{m}^2}{10^5\,\text{N}}$$

$D_{max} = 82.7$ m ← D_{max}



$$\Sigma F_z = 0 = F_{buoy} - M_{He}g - M_b g - Mg \qquad M_{He} = \rho_{He} \forall$$

$$F_{buoy} - M_{He}g = (\rho_{air} - \rho_{He})g\forall = (\rho_{air} - \rho_{He})g\frac{\pi D^3}{6}$$

$$M_b = \rho_s \forall_s = \rho_s A_s t_s = \rho_s \pi D^2 t$$

$$\therefore M = \frac{F_{buoy}}{g} - M_b = (\rho_{air} - \rho_{He})\frac{\pi D^3}{6} - \rho_s \pi D^2 t$$

$$M = \pi D^2\left[(\rho_{air} - \rho_{He})\frac{D}{6} - \rho_s t\right]$$

From ideal gas law

$$\rho_{air} = \frac{P}{RT} = 3.0 \times 10^{-3}\,\text{bar} \times \frac{\text{kg}\cdot\text{K}}{287\,\text{J}} \times \frac{1}{248\,\text{K}} \times \frac{10^5\,\text{Pa}}{\text{bar}} \times \frac{\text{N}}{\text{Pa}\,\text{m}^2} \times \frac{\text{J}}{\text{N}\cdot\text{m}} = 4.21 \times 10^{-3}\,\text{kg/m}^3$$

$$\rho_{He} = \frac{(P+\Delta P)}{RT} = 3.45 \times 10^{-3}\,\text{bar} \times \frac{\text{kg}\cdot\text{K}}{2080\,\text{J}} \times \frac{1}{248\,\text{K}} \times \frac{10^5\,\text{Pa}}{\text{bar}} \times \frac{\text{N}}{\text{Pa}\cdot\text{m}^2} \times \frac{\text{J}}{\text{N}\cdot\text{m}} = 6.69 \times 10^{-4}\,\text{kg/m}^3$$

Then,

$$M = \pi(82.7)^2\,\text{m}^2\left[(42.1 - 6.69)\times 10^{-4}\,\frac{\text{kg}}{\text{m}^3} \times \frac{82.7\,\text{m}}{6} - 1.28 \times 999\,\frac{\text{kg}}{\text{m}^3} \times 1.5 \times 10^{-5}\,\text{m}\right]$$

$M = 637$ kg ← M

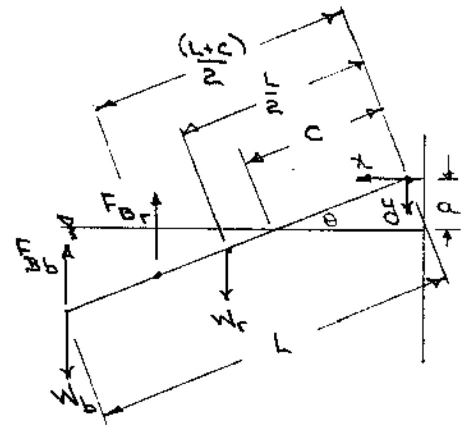
Problem *3.89

Given: Weight as shown in water on rod.

$L = 10\,ft$, $A = 3\,in^2$, $W_r = 3\,lbf$

$a = 1\,ft$, $W_b = 67\,lbf$.

Find: θ for equilibrium condition



Solution:

Basic equations: $\sum \vec{M} = 0$ for equilibrium

Moment of force $= \vec{r} \times \vec{F}$

Computing equation: $\vec{F}_B = -\gamma \forall_{displaced} \hat{k}$

Let $()_r$ refer to rod

$()_b$ refer to block

Summing moments about the hinge

$$(W_b - F_{B_b})\hat{j} \times L(\hat{i}\cos\theta + \hat{j}\sin\theta) + (-F_{B_r})\hat{j} \times \frac{(L+c)}{2}(\hat{i}\cos\theta + \hat{j}\sin\theta) + W_r\hat{j} \times \frac{L}{2}(\hat{i}\cos\theta + \hat{j}\sin\theta)$$

$$\left\{-W_b L\cos\theta + F_{B_b} L\cos\theta + F_{B_r}\frac{(L+c)}{2}\cos\theta - W_r\frac{L}{2}\cos\theta\right\}\hat{k} = 0$$

$$-W_b L + F_{B_b} L + F_{B_r}\frac{(L+c)}{2} - W_r\frac{L}{2} = 0$$

$$F_{B_r} = \gamma \forall_{dis} = \gamma A l = \gamma A (L-c)$$

$$\therefore -W_b L + F_{B_b} L + \gamma A (L-c)\frac{(L+c)}{2} - W_r\frac{L}{2} = 0$$

$$-2W_b L + 2F_{B_b} L + \gamma A (L^2 - c^2) - W_r L = 0$$

$$\therefore \gamma A (L^2 - c^2) = W_r L + 2W_b L - 2F_{B_b} L$$

and

$$c = \left[L^2 - \frac{1}{\gamma A}(W_r L + 2W_b L - 2F_{B_b} L)\right]^{1/2} = \frac{a}{\sin\theta}$$

$$c = \left[(10)^2\,ft^2 - \frac{ft^3}{62.4\,lbf} \times \frac{1}{3\,in^2}\left(3\,lbf \times 10\,ft + 2 \times 67\,lbf \times 10\,ft - 2 \times 62.4\frac{lbf}{ft^3} \times 1\,ft^3 \times 10\,ft\right) \times 144\frac{in^2}{ft^2}\right]^{1/2}$$

$$c = [6.18]^{1/2} = 2.48\,ft$$

$$\sin\theta = \frac{a}{c} = \frac{1.00\,ft}{2.48\,ft} = 0.403 \quad \therefore \theta = 23.8^\circ$$

Problem *3.90

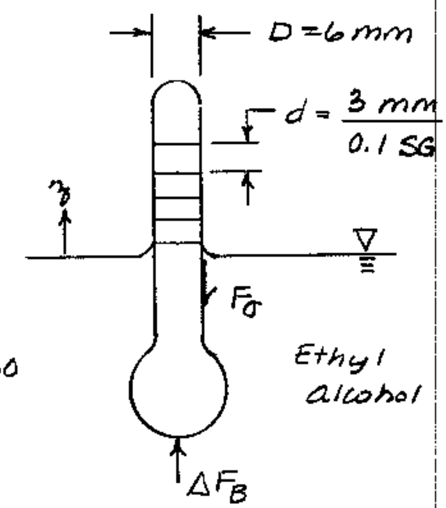
Given: Glass hydrometer used to measure SG of liquids.

Stem has $D = 6$ mm; distance between marks on stem is $d = 3$ mm per 0.1 SG

Hydrometer floats in ethyl alcohol (assume contact angle is 0°).

Find: Magnitude of error introduced by surface tension.

Solution: Consider a free-body diagram of the floating hydrometer



Surface tension will cause the hydrometer to sink Δh lower into the liquid. Thus for this change,

$$\Sigma F_z = \Delta F_B - F_\sigma = m a_z = 0$$

Computing equation: $\Delta F_B = \rho g \Delta \forall$

Assumptions: (1) Static liquid (2) Incompressible liquid (3) $\theta \approx 0$

Then $\Delta \forall = \frac{\pi D^2}{4} \Delta h$ and $\Delta F_B = \rho g \frac{\pi D^2}{4} \Delta h$

and $F_\sigma = \pi D \sigma \cos\theta = \pi D \sigma$

Combining $\rho g \frac{\pi D^2}{4} \Delta h = \pi D \sigma$ or $\Delta h = \frac{4\sigma}{\rho g D} = \frac{4\sigma}{SG \rho_{H_2O} g D}$

From Table A.2, SG = 0.789 and from Table A.4, $\sigma = 22.3$ mN/m for ethanol, so

$$\Delta h = \frac{4}{0.789} \times 22.3 \times 10^{-3} \frac{N}{m} \times \frac{m^3}{1000\,kg} \times \frac{s^2}{9.81\,m} \times \frac{1}{0.006\,m} \times \frac{kg \cdot m}{N \cdot s^2} = 1.92 \times 10^{-3}\,m$$

Thus the change in SG will be

$$\Delta SG = 1.92 \times 10^{-3}\,m \times \frac{0.1\,SG}{3\,mm} \times 1000 \frac{mm}{m} = 0.0640$$

ΔS

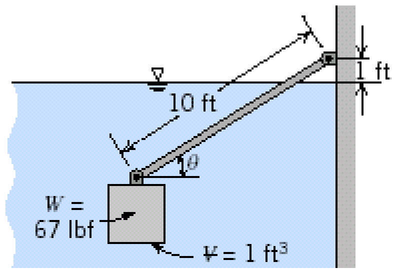
{ From the diagram, surface tension acts to cause the hydrometer to float lower in the liquid. Therefore surface tension results in an indicated SG <u>smaller</u> than the actual SG. }

Problem *3.91

If the weight *W* in Problem 3.89 is released from the rod, at equilibrium how much of the rod will remain submerged? What will be the minimum required upward force at the tip of the rod to just lift it out of the water?

Given: Data on rod

Find: How much is submerged if weight is removed; force required to lift out of water



Solution

The data are For water $\gamma = 62.4\cdot\frac{lbf}{ft^3}$

For the cylinder $L = 10\cdot ft$ $A = 3\cdot in^2$ $W = 3\cdot lbf$

The semi-floating rod will have zero net force and zero moment about the hinge

For the moment $$\sum M_{hinge} = 0 = W\cdot\frac{L}{2}\cdot\cos(\theta) - F_B\cdot\left[(L-x)+\frac{x}{2}\right]\cdot\cos(\theta)$$

where $F_B = \gamma\cdot A\cdot x$ is the buoyancy force *x* is the submerged length of rod

Hence $$\gamma\cdot A\cdot x\cdot\left(L-\frac{x}{2}\right) = \frac{W\cdot L}{2}$$

$$x = L-\sqrt{L^2-\frac{W\cdot L}{\gamma\cdot A}} = 10\cdot ft-\sqrt{(10\cdot ft)^2-3\cdot lbf\times 10\cdot ft\times\frac{ft^3}{62.4\cdot lbf}\times\frac{1}{3\cdot in^2}\times\frac{144\cdot in^2}{1\cdot ft^2}}$$

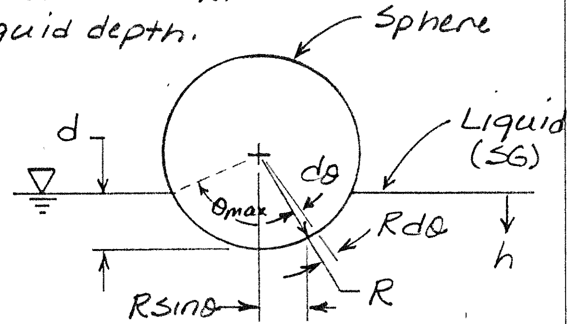
$x = 1.23\,ft$

gives a physically unrealistic value)

To just lift the rod out of the water requires $F = 1.5\cdot lbf$ (half of the rod weight)

Given: Sphere partially immersed in liquid of specific gravity, SG.

Find: (a) Formula, algebraic expression for buoyancy force, as a function of submersion depth, d, for $0 \le d \le R$.
(b) Plot of results over range of liquid depth.



Solution: Apply fluid statics

Basic equations: $\frac{dp}{dh} = \rho g$

$dF = p\,dA$

Assumptions: (1) Static liquid
(2) Incompressible, so $p = p_0 + \rho g h$
(3) Neglect p_{atm} since it acts everywhere

Then $dF_V = \cos\theta\, p\, dA$; $p = \rho g h$; $d = h + R(1-\cos\theta)$; $h = d - R(1-\cos\theta)$

$$dA = 2\pi(R\sin\theta)R\,d\theta = 2\pi R^2 \sin\theta\, d\theta$$

$$dF_V = \cos\theta\, \rho g[d - R(1-\cos\theta)]\,2\pi R^2 \sin\theta\, d\theta = 2\pi R^3\left[\frac{d}{R} - (1-\cos\theta)\right]\sin\theta\cos\theta\, d\theta\, \rho g$$

Now

$$F_V = \int_A dF_V = \int_0^{\theta_{max}} 2\pi R^3\left[\frac{d}{R} - (1-\cos\theta)\right]\sin\theta\cos\theta\, d\theta\,\rho g$$

$$F_V = 2\pi R^3\left[(1-d/R)\frac{\cos^2\theta}{2} - \frac{\cos^3\theta}{3}\right]_0^{\theta_{max}}\rho g \quad ; \rho = SG\,\rho_{H_2O}$$

At θ_{max}, $\cos\theta_{max} = \frac{R-d}{R} = 1 - d/R$, so

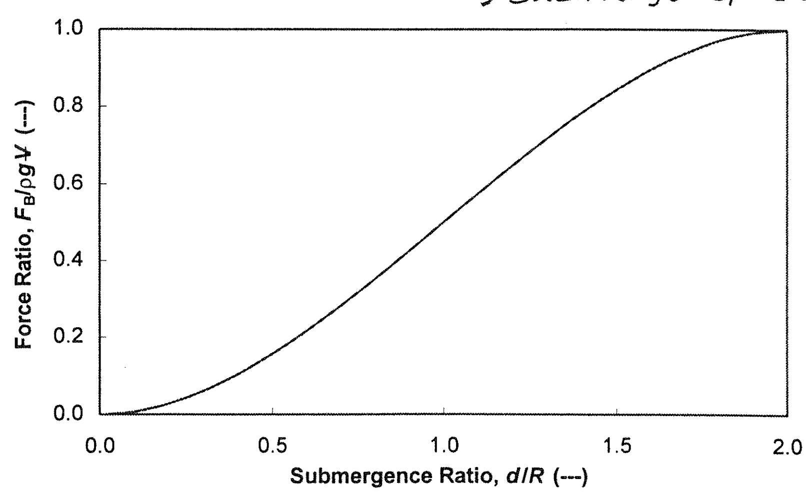
$$F_V = 2\pi\rho g R^3\left\{(1-d/R)\left[(1-d/R)^2/2 - 1/2\right] - \frac{(1-d/R)^3}{3} + \frac{1}{3}\right\}$$

$$F_V = 2\pi\rho g R^3\left[\frac{1}{6}\left(1-\frac{d}{R}\right)^3 - \frac{1}{2}\left(1-\frac{d}{R}\right) + \frac{1}{3}\right] \qquad F_V$$

Dividing both sides by the vertical force on a fully submerged sphere,

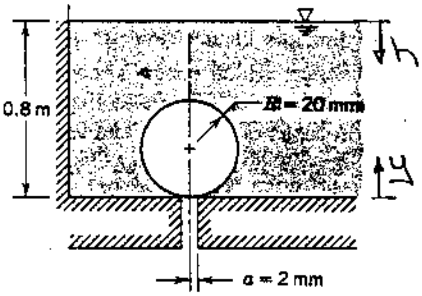
$$\frac{F_V}{\rho g \frac{4\pi R^3}{3}} = \frac{3}{2}\left[\frac{1}{6}()^3 - \frac{1}{2}() + \frac{1}{3}\right]$$

where $() = (1 - \frac{d}{R})$.



Problem *3.93

Given: Sphere, of radius R and specific gravity SG, is submerged in a tank of water. Sphere is placed over a hole, of radius a, in the tank bottom.



Find: (a) general expression for the range of SG for which sphere will float to the surface

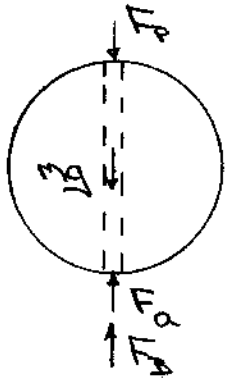
(b) minimum SG required for sphere to remain in the position shown

Solution:

Basic equations: $F_{buoy} = \rho g V$ $\quad \frac{dp}{dh} = \rho g \quad dF = p\,dA$

Assumptions: (1) static liquid
(2) incompressible, so $p = p_0 + \rho g h$
(3) $p = p_{atm}$ at free surface and at hole
(4) $a/R \ll 1$

Draw fbd of sphere. $\Sigma F_y = 0$



$\Sigma F_y = 0 = F_a - F_p + F_B - mg$

F_a = force of air on area of sphere of radius a. $F_a = p_{atm}\pi a^2$

F_p = total force on on area of sphere of radius a at depth $h = H - 2R$
$F_p = [p_{atm} + \rho g(H-2R)]\pi a^2$

F_B = net buoyant force on sphere excluding cylinder of radius a
$F_B = \rho_w g V_{net} = \rho_w g \left[\frac{4\pi R^3}{3} - \pi a^2 (2R)\right]$

$mg = \rho_w SG\, g \frac{4\pi R^3}{3}$

Substituting,

$0 = p_{atm}\pi a^2 - p_{atm}\pi a^2 - \rho g(H-2R)\pi a^2 + \rho_w g \frac{4\pi R^3}{3} - \rho_w g \pi a^2 2R - \rho_w SG\, g \frac{4\pi R^3}{3}$

$0 = -(H-2R)a^2 + \frac{4R^3}{3} - 2a^2 R - SG\frac{4R^3}{3}$

$0 = -\left(\frac{H}{R} - 2\right)\left(\frac{a}{R}\right)^2 + \frac{4}{3} - 2\left(\frac{a}{R}\right)^2 - \frac{4}{3}SG$

$SG = 1 - \frac{3}{4}\frac{H}{R}\left(\frac{a}{R}\right)^2$ ← SG

For dimensions given $\frac{a}{R} = \frac{2}{20} = 0.1$, $\frac{H}{R} = \frac{800}{20} = 40$

$\therefore SG = 1 - \frac{3}{4} \times 40 \times (0.1)^2 = 0.70$

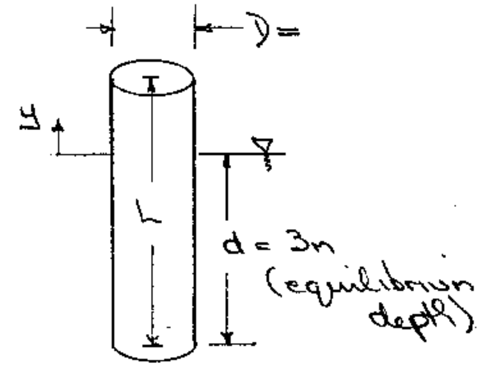
For $SG \geq 0.70$ sphere will stay in position shown ← SG_{min}

Problem *3.94

Given: Cylindrical timber, $D = 0.3$ m and $L = 4$ m is weighted on lower end so it floats vertically with 3 m submerged in sea water.
When displaced vertically from equilibrium position, the timber oscillates in a vertical direction upon release.

Find: Estimate frequency of oscillation. (Neglect any viscous effects or water motion)

Solution:



At equilibrium

$$\sum F_y = 0 = F_b - mg = \rho A d - mg$$

$$\therefore m = \frac{\rho A d}{g}$$

For displacement y

$$\sum F_y = m \frac{d^2 y}{dt^2} = m\ddot{y}$$

$$F_b - mg = m\ddot{y} \quad \text{where } F_b = \rho A(d - y)$$

$$\therefore \rho A(d-y) - mg = m\ddot{y}$$

$$\rho A d - \rho A y - \frac{\rho A d}{g} g = m\ddot{y}$$

or

$$m\ddot{y} + \rho A y = 0$$

$$\ddot{y} + \frac{\rho A}{m} y = 0 = \ddot{y} + \omega^2 y = 0$$

$$\text{where } \omega^2 = \frac{\rho A}{m} = \frac{\rho A\, g}{\rho A d} = \frac{g}{d}$$

$$\omega = \left(\frac{g}{d}\right)^{1/2} = \left[9.81 \frac{m}{s^2} \times \frac{1}{3\,m}\right]^{1/2} = 1.81 \text{ rad/s}$$

ω

$$f = \frac{\omega}{2\pi} = 1.81 \frac{rad}{s} \times \frac{cycle}{2\pi\, rad} = 0.288 \text{ cycle/s}$$

$$\tau = \frac{1}{f} = 3.47 \text{ s}$$

Problem *3.95

Open-Ended Problem Statement: A proposed ocean salvage scheme involves pumping air into "bags" placed within and around a wrecked vessel on the sea bottom. Comment on the practicality of this plan, supporting your conclusions with analyses.

Discussion: This plan has several problems that render it impractical. First, pressures at the sea bottom are very high. For example, *Titanic* was found in about 12,000 ft of seawater. The corresponding pressure is nearly 6,000 psi. Compressing air to this pressure is possible, but would require a multi-stage compressor and very high power.

Second, it would be necessary to manage the buoyancy force after the bag and object are broken loose from the sea bed and begin to rise toward the surface. Ambient pressure would decrease as the bag and artifact rise toward the surface. The air would tend to expand as the pressure decreases, thereby tending to increase the volume of the bag. The buoyancy force acting on the bag is directly proportional to the bag volume, so it would increase as the assembly rises. The bag and artifact thus would tend to accelerate as they approach the sea surface. The assembly could broach the water surface with the possibility of damaging the artifact or the assembly.

If the bag were of constant volume, the pressure inside the bag would remain essentially constant at the pressure of the sea floor, e.g., 6,000 psi for *Titanic*. As the ambient pressure decreases, the pressure differential from inside the bag to the surroundings would increase. Eventually the difference would equal sea floor pressure. This probably would cause the bag to rupture.

If the bag permitted some expansion, a control scheme would be needed to vent air from the bag during the trip to the surface to maintain a constant buoyancy force just slightly larger than the weight of the artifact in water. Then the trip to the surface could be completed at low speed without danger of broaching the surface or damaging the artifact.

Problem *3.96

Open-Ended Problem Statement: In the "Cartesian diver" child's toy, a miniature "diver" is immersed in a column of liquid. When a diaphragm at the top of the column is pushed down, the diver sinks to the bottom. When the diaphragm is released, the diver again rises. Explain how the toy might work.

Discussion: A possible scenario is for the toy to have a flexible bladder that contains air. Pushing down on the diaphragm at the top of the liquid column would increase the pressure at any point in the liquid. The air in the bladder would be compressed slightly as a result. The volume of the bladder, and therefore its buoyancy, would decrease, causing the diver to sink to the bottom of the liquid column.

Releasing the diaphragm would reduce the pressure in the water column. This would allow the bladder to expand again, increasing its volume and therefore the buoyancy of the diver. The increased buoyancy would permit the diver to rise to the top of the liquid column and float in a stable, partially submerged position, on the surface of the liquid.

Problem *3.97

Open-Ended Problem Statement: Consider a conical funnel held upside down and submerged slowly in a container of water. Discuss the force needed to submerge the funnel if the spout is open to the atmosphere. Compare with the force needed to submerge the funnel when the spout opening is blocked by a rubber stopper.

Discussion: Let the weight of the funnel in air be W_a. Assume the funnel is held with its spout vertical and the conical section down. Then W_a will also be vertical.

Two possible cases are with the funnel spout open to atmosphere or with the funnel spout sealed.

With the funnel spout open to atmosphere, the pressures inside and outside the funnel are equal, so no net pressure force acts on the funnel. The force needed to support the funnel will remain constant until it first contacts the water. Then a buoyancy force will act vertically upward on every element of volume located beneath the water surface.

The first contact of the funnel with the water will be at the widest part of the conical section. The buoyancy force will be caused by the volume formed by the funnel thickness and diameter as it begins to enter the water. The buoyancy force will reduce the force needed to support the funnel. The buoyancy force will increase as the depth of submergence of the funnel increases until the funnel is fully submerged. At that point the buoyancy force will be constant and equal to the weight of water displaced by the volume of the material from which the funnel is made.

If the funnel material is less dense than water, it would tend to float partially submerged in the water. The force needed to support the funnel would decrease to zero and then become negative (i.e., down) to fully submerge the funnel.

If the funnel material were more dense than water it would not tend to float even when fully submerged. The force needed to support the funnel would decrease to a minimum when the funnel became fully submerged, and then would remain constant at deeper submersion depths.

With the funnel spout sealed, air will be trapped inside the funnel. As the funnel is submerged gradually below the water surface, it will displace a volume equal to the volume of the funnel material plus the volume of trapped air. Thus its buoyancy force will be much larger than when the spout is open to atmosphere. Neglecting any change in air volume (pressures caused by submersion should be small compared to atmospheric pressure) the buoyancy force would be from the entire volume encompassed by the outside of the funnel. Finally, when fully submerged, the volume of the rubber stopper (although small) will also contribute to the total buoyancy force acting on the funnel.

Problem *3.98

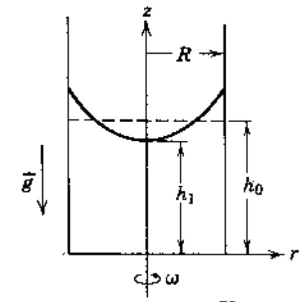
Given: Cylindrical container rotating as in Example Problem 3.9

$R = 0.5$ ft

$h_0 = 4$ in.

Determine: (a) value of ω such that $h_1 = 0$

(b) if solution is dependent on ρ



Solution:

In order to obtain the solution we need an expression for the shape of the free surface in terms of ω, r, and h_0

The required expression was derived in Example Problem 3.9. The equation is

$$z = h_0 - \frac{(\omega R)^2}{2g}\left[\frac{1}{2} - \left(\frac{r}{R}\right)^2\right]$$

Since $h_1 = 0$ corresponds to $z = 0$ and $r = 0$ we must determine ω such that

$$0 = h_0 - \frac{(\omega R)^2}{4g}$$

Solving for ω,

$$\omega = \frac{2}{R}\sqrt{g h_0}$$

$$= \frac{2}{0.5\ \text{ft}}\left(32.2\ \frac{\text{ft}}{\text{s}^2} \times 4\ \text{in} \times \frac{\text{ft}}{12\ \text{in}}\right)^{1/2}$$

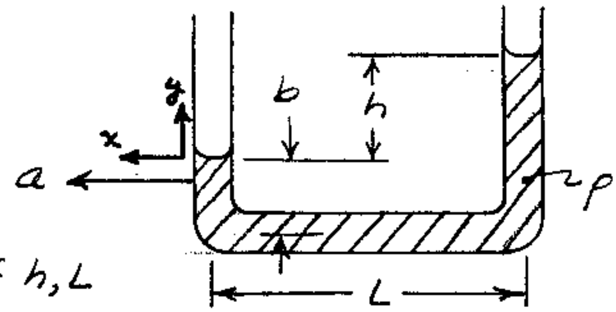
$$= 4 \times 3.28\ \frac{1}{\text{s}}$$

$$\omega = 13.1\ \text{rad/s}$$ ω

The solution is independent of ρ since the equation of the free surface is independent of ρ.

Problem *3.99

Given: U-tube accelerometer



Find: Acceleration in terms of h, L

Solution: Apply x, y components of hydrostatic equation.

Basic equations:

$$-\frac{\partial p}{\partial x} + \rho g_x = \rho a_x \qquad a_x = a \quad g_x = 0$$

$$-\frac{\partial p}{\partial y} + \rho g_y = \rho a_y \qquad a_y = 0 \quad g_y = -g$$

Assumptions: (1) Neglect sloshing
(2) Ignore corners

Then $\frac{\partial p}{\partial x} = -\rho a$, $\frac{\partial p}{\partial y} = -\rho g$. Evaluate Δp from left leg to right:

$$dp = \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy$$

$$\Delta p = \frac{\partial p}{\partial x} \Delta x + \frac{\partial p}{\partial y} \Delta y$$

$$= (-\rho g)(-b) + (-\rho a)(-L) + (-\rho g)(b+h)$$

$$\Delta p = \rho a L - \rho g h = 0$$

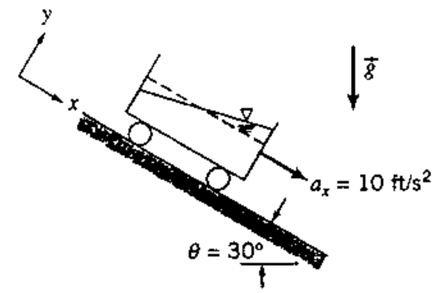
Solving,

$$a = g\left(\frac{h}{L}\right)$$

a

Problem *3.100

Given: Rectangular container of water undergoing constant acceleration as shown



Determine: The slope of the free surface

Solution:

Basic equation: $-\nabla P + \rho\vec{g} = \rho\vec{a}$

Writing the component equations

$$-\frac{\partial P}{\partial x} + \rho g_x = \rho a_x \qquad -\frac{\partial P}{\partial y} + \rho g_y = \rho a_y \qquad -\frac{\partial P}{\partial z} + \rho g_z = \rho a_z$$

For given coordinates

$$a_y = a_z = 0 \qquad g_y = -g\cos\theta \qquad g_x = g\sin\theta \qquad g_z = 0$$

$$\Rightarrow \quad \frac{\partial P}{\partial x} = \rho g \sin\theta - \rho a_x \qquad \frac{\partial P}{\partial y} = -\rho g\cos\theta \qquad \frac{\partial P}{\partial z} = 0$$

From the component equations we conclude that $P = P(x,y)$

then

$$dP = \frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy$$

Along the free surface P = constant and $dP = 0$. Hence

$$\left.\frac{dy}{dx}\right)_{surface} = -\frac{\partial P/\partial x}{\partial P/\partial y} = \frac{\rho g \sin\theta - \rho a_x}{\rho g\cos\theta}$$

$$= \frac{g\sin\theta - a_x}{g\cos\theta}$$

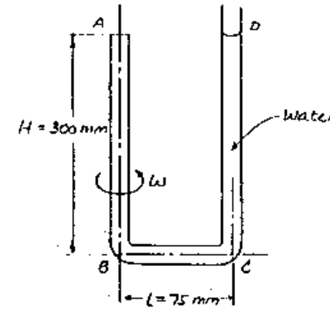
$$= \frac{32.2\,(0.5)\ \text{ft/s}^2 - 10\ \text{ft/s}^2}{32.2\,(0.866)\ \text{ft/s}^2}$$

$$\left.\frac{dy}{dx}\right)_{surface} = 0.22$$

$\left.\frac{dy}{dx}\right)_{sur}$

Problem *3.101

Given: U-tube, sealed at A and open to the atmosphere at D, is filled with water at T=20 C and rotated about vertical axis AB. Dimensions are shown on the digram



Find: the maximum allowable angular speed, ω, for no cavitation.

Solution:

Basic equation: $-\nabla p + \rho\vec{g} = \rho\vec{a}$

Assumptions: (1) incompressible fluid (2) solid body rotation

Component equations $-\frac{\partial p}{\partial r} = \rho a_r = -\rho\frac{V^2}{r} = -\rho\omega^2 r$

$$\frac{\partial p}{\partial z} = -\rho g$$

Between D and C, r = constant, so $\frac{dp}{dz} = -\rho g$ and $p_D - p_C = \rho g H$ .....(1)

Between B and A, r = constant, so $\frac{dp}{dz} = -\rho g$ and $p_A - p_B = -\rho g H$ ...(2)

Between B and C, z = constant, so $\frac{dp}{dr} = \rho\omega^2 r$. Hence

$$\int_{p_B}^{p_C} dp = \int_0^L \rho\omega^2 r\,dr = \rho\omega^2 \frac{r^2}{2}\Big]_0^L = \rho\omega^2\frac{L^2}{2}$$

$$p_C - p_B = \rho\omega^2\frac{L^2}{2} \quad ....(3)$$

Since $p_D = p_{atm}$, then from Eq(1) $p_C = p_{atm} + \rho g H$

From Eq(3) $p_B = p_C - \rho\omega^2\frac{L^2}{2}$ so $p_B = p_{atm} + \rho g H - \rho\omega^2\frac{L^2}{2}$

From Eq(2) $p_A = p_B - \rho g H$ so $p_A = p_{atm} - \rho\omega^2\frac{L^2}{2}$

Thus the minimum pressure occurs at point A

At T = 20C the vapor pressure of water is $p_v = 2.34\times10^3\ N/m^2$

Solving for ω with $p_A = p_v$, we obtain

$$\omega = \left[\frac{2(p_{atm} - p_v)}{\rho L^2}\right]^{1/2} = \left[2(101.33 - 2.34)\times10^3\ \frac{N}{m^2}\times\frac{m^3}{999\ kg}\times\frac{1}{(0.075)^2\ m^2}\times\frac{kg\cdot m}{N\cdot s^2}\right]^{1/2}$$

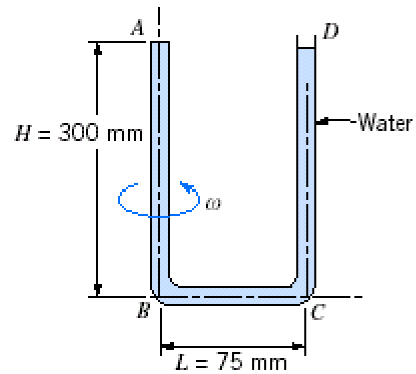
$\omega = 188$ rad/s ← ω

Problem *3.102

If the U-tube of Problem 3.101 is spun at 200 rpm, what will be the pressure at *A*? If a small leak appears at *A*, how much water will be lost at *D*?

Given: Data on U-tube

Find: Pressure at A at 200 rpm; water loss due to leak



Solution

For water $\rho = 999\cdot\frac{kg}{m^3}$

The speed of rotation is $\omega = 200\cdot rpm$ $\quad \omega = 20.9\frac{rad}{s}$

The pressure at *D* is $p_D = 0\cdot kPa$ (gage)

From the analysis of Example Problem 3.10, the pressure *p* at any point (*r*,*z*) in a continuous rotating fluid is given by

$$p = p_0 + \frac{\rho\cdot\omega^2}{2}\cdot\left(r^2 - r_0^2\right) - \rho\cdot g\cdot\left(z - z_0\right)$$

where p_0 is a reference pressure at point (r_0, z_0)

In this case $p = p_A$ $\quad p_0 = p_D$

$z = z_A = z_D = z_0 = H \quad r = 0 \quad r_0 = r_D = L$

Hence $$p_A = \frac{\rho\cdot\omega^2}{2}\cdot\left(-L^2\right) - \rho\cdot g\cdot(0) = -\frac{\rho\cdot\omega^2\cdot L^2}{2}$$

$$p_A = -\frac{1}{2} \times 999 \cdot \frac{kg}{m^3} \times \left(20.9 \cdot \frac{rad}{s}\right)^2 \times (0.075 \cdot m)^2 \times \frac{N \cdot s^2}{kg \cdot m}$$

$$p_A = -1.23\,kPa$$

When the leak appears,the water level at *A* will fall, forcing water out at point *D*. Once again, fr
the analysis of Example Problem 3.10, the pressure *p* at any point (r,z) in a continuous rotating f
is given by

$$p = p_0 + \frac{\rho \cdot \omega^2}{2} \cdot \left(r^2 - r_0^2\right) - \rho \cdot g \cdot \left(z - z_0\right)$$

where p_0 is a reference pressure at point (r_0,z_0)

In this case $p = p_A = 0 \qquad p_0 = p_D = 0$

$$z = z_A \qquad z_0 = z_D = H \qquad r = 0 \qquad r_0 = r_D = L$$

Hence $$0 = \frac{\rho \cdot \omega^2}{2} \cdot \left(-L^2\right) - \rho \cdot g \cdot \left(z_A - H\right)$$

$$z_A = H - \frac{\omega^2 \cdot L^2}{2 \cdot g}$$

$$z_A = 0.3 \cdot m - \frac{1}{2} \times \left(20.9 \cdot \frac{rad}{s}\right)^2 \times (0.075 \cdot m)^2 \times \frac{s^2}{9.81 \cdot m} \times \frac{N \cdot s^2}{kg \cdot m}$$

$$z_A = 0.175\,m$$

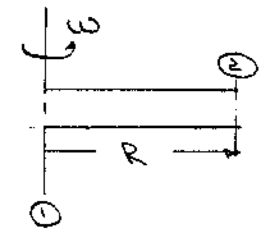
The amount of water lost is $\Delta h = H - z_A = 300 \cdot mm - 175 \cdot mm \qquad \Delta h = 125\,mm$

Problem *3.103

Given: Centrifugal micromanometer consists of pair of parallel disks that rotate to develop a radial pressure difference. There is no flow between the disks.

Find: (a) An expression for the pressure difference, ΔP, as a function of ω, R, and ρ
(b) Find ω if $\Delta P = 8\ \mu m\ H_2O$ and $R = 50\ mm$.

Solution:



Basic equation: $-\nabla P + \rho\vec{g} = \rho\vec{a}$

(r component) $-\frac{\partial P}{\partial r} + \rho g_r = \rho a_r$

Assumptions: (1) standard air between disks
(2) r horizontal, so $g_r = 0$
(3) rigid body motion, so $a_r = -\frac{V^2}{r} = -\frac{(r\omega)^2}{r} = -r\omega^2$

Then

$$\frac{\partial P}{\partial r} = \rho r \omega^2 \qquad (\rho \text{ is a constant})$$

Separating variables and integrating, we obtain

$$\int_P^{P+\Delta P} dP = \rho\omega^2 \int_0^R r\,dr$$

$$\Delta P = \frac{\rho\omega^2 R^2}{2} \qquad \Delta P$$

Then

$$\omega^2 = \frac{2\Delta P}{\rho R^2}$$

where $\Delta P = \rho_{H_2O}\, g\, \Delta h$ and $\Delta h = 8\times10^{-6}\ m$

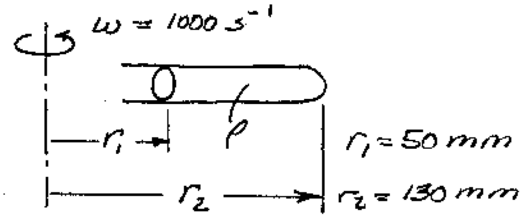
$$\omega^2 = \frac{2\rho_{H_2O}\, g\, \Delta h}{\rho R^2}$$

$$= 2 \times \frac{999\ kg/m^3}{1.225\ kg/m^3} \times 9.81\ \frac{m}{s^2} \times 8\times10^{-6}\ m \times \frac{1}{(0.05)^2\ m^2}$$

$$\omega^2 = 51.2\ s^{-2}$$

$$\omega = 7.16\ rad/s \qquad \omega$$

Problem *3.104



Given: Test tube with water

Find: (a) Radial acceleration
(b) Radial pressure gradient, $\partial p/\partial r$
(c) Maximum pressure on bottom.

Solution: Apply equation for rigid-body motion

Basic equation: $-\nabla p + \rho\vec{g} = \rho\vec{a}$

(r component) $-\frac{\partial p}{\partial r} + \rho g_r = \rho a_r$

Assumptions: (1) Rigid-body motion, so $a_r = -\frac{V^2}{r} = -\frac{(r\omega)^2}{r} = -r\omega^2$ ← a_r
(2) r horizontal, so $g_r = 0$

Then $\frac{\partial p}{\partial r} = -\rho a_r = -\rho(-r\omega^2) = \rho r\omega^2$ ← $\frac{\partial p}{\partial r}$

Integrating, $p_2 - p_1 = \int_1^2 \frac{\partial p}{\partial r} dr = \int_{r_1}^{r_2} \rho r\omega^2 = \rho \frac{r^2\omega^2}{2}\Big]_{r_1}^{r_2} = \frac{1}{2}\rho\omega^2(r_2^2 - r_1^2)$

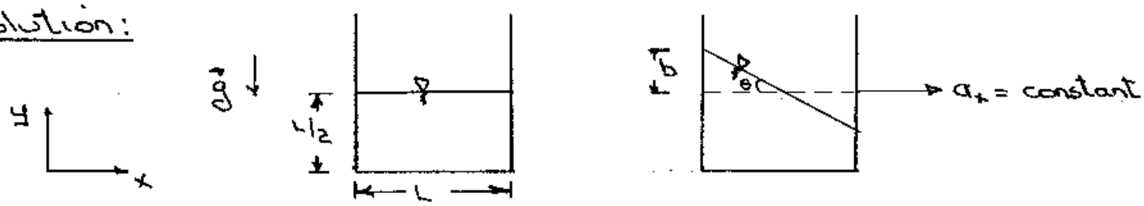
$p_{max} = p_2 - p_1 = \frac{1}{2} \times 999\frac{kg}{m^3} \times \frac{(1000)^2}{s^2} \times [(0.130)^2 - (0.050)^2]\,m^2 \times \frac{N\cdot s^2}{kg\cdot m} = 7.19\ MPa$ ← p_{ma}

Problem *3.105

Given: Box, 1m×1m×1m, half filled with oil (SG = 0.80), subjected to a constant horizontal acceleration of $0.2g$.

Determine: (a) slope of free surface
(b) pressure along bottom of box

Solution:



Basic equation: $-\nabla p + \rho \vec{g} = \rho \vec{a}$

Writing the component equations

$$-\frac{\partial p}{\partial x} + \rho g_x = \rho a_x \quad \Rightarrow \quad \frac{\partial p}{\partial x} = -\rho a_x$$

$$-\frac{\partial p}{\partial y} + \rho g_y = \rho a_y \quad \Rightarrow \quad \frac{\partial p}{\partial y} = \rho g_y = -\rho g$$

$$-\frac{\partial p}{\partial z} + \rho g_z = \rho a_z \quad \Rightarrow \quad \frac{\partial p}{\partial z} = 0$$

From the component equations we conclude that $p = p(x, y)$

Then

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy$$

Along the free surface p = constant and $dp = 0$. Hence

$$\left.\frac{dy}{dx}\right)_{surface} = -\frac{\partial p/\partial x}{\partial p/\partial y} = -\frac{a_x}{g} = -\frac{0.2g}{g} = -0.2$$ $\frac{dy}{dx}\Big)$

Since $p = p(x,y)$

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy$$

Substituting for the partial derivatives

$$dp = -\rho a_x dx - \rho g dy$$

Integrating for ρ = constant

$$p = -\rho a_x x - \rho g y + c$$

To evaluate the constant of integration note that

$p = p_{atm}$ at $x = 0$, $y = \frac{L}{2} + b$

Hence $p_{atm} = -\rho g\left(\frac{L}{2} + b\right) + c$ and $c = p_{atm} + \rho g\left(\frac{L}{2} + b\right)$

Thus

$$p = p_{atm} - \rho a_x x + \rho g\left(\frac{L}{2} + b - y\right)$$

where $b = \frac{L}{2}\tan\theta = \frac{L}{2}\left(-\frac{dy}{dx}\right)_{surf} = \frac{L}{2}\frac{a_x}{g}$ {Note $\theta > 0$ for $\frac{dy}{dx} < 0$}

$$\therefore \quad p = p_{atm} - \rho a_x x + \rho g\left(\frac{L}{2} + \frac{L}{2}\frac{a_x}{g} - y\right)$$ {Note: This gives $p = p_{atm}$ at $x = y = \frac{L}{2}$ as it should}

Along the bottom surface $y = 0$ and hence

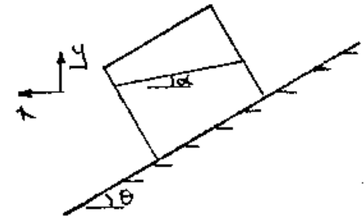
$$p(x,0) = p_{atm} - \rho a_x x + \rho g\left(\frac{L}{2} + \frac{L}{2}\frac{a_x}{g}\right)$$

$$p(x,0) = 101\,\text{kPa} - (0.8)\times 999\frac{kg}{m^3}\times 0.2\times 9.81\frac{m}{s^2}\times\frac{N\cdot s^2}{kg\cdot m}\,x + \left(0.8\times 999\frac{kg}{m^3}\right)9.81\frac{m}{s^2}\left(\frac{1}{2} + \frac{1}{2}\times 0.2\right)m\times\frac{N\cdot s^2}{kg\cdot m}$$

$$p(x,0) = 106 - 1.57x \text{ kPa} \quad (x \text{ in meters})$$ p(

Problem *3.106

Given: Rectangular container of base dimensions 0.4m × 0.2m and height 0.4m is filled with water to a depth, $d = 0.2m$
Mass of empty container is $M_c = 10$ kg
Container slides down an incline, $\theta = 30°$
Coefficient of sliding friction is 0.30



Find: The angle of the water surface relative to the horizontal

Solution:

Basic equations: $-\nabla P + \rho \vec{g} = M\vec{a}$ $\quad \Sigma \vec{F} = M\vec{a}$

Assumptions: (1) fluid moves as solid body, ie no sloshing

Writing component equations,

$$-\frac{\partial P}{\partial x} = \rho a_x \qquad \frac{\partial P}{\partial x} = -\rho a_x$$

$$-\frac{\partial P}{\partial y} - \rho g = \rho a_y \qquad \frac{\partial P}{\partial y} = -\rho(g + a_y)$$

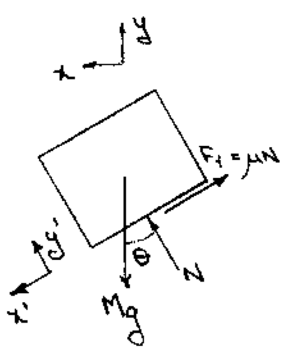
$P = P(x,y)$ $\quad dP = \frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy$. Along the water surface, $dP = 0$

$$\frac{dy}{dx} = -\frac{\partial P/\partial x}{\partial P/\partial y} = -\frac{a_x}{g + a_y}$$

To determine a_x and a_y consider the container and contents

$$M = M_c + M_{H_2O} = M_c + \rho V = 10\,kg + 999\frac{kg}{m^3} \times 0.4m \times 0.2m \times 0.2m$$

$$M = 26\ kg$$



$$\Sigma F_{y'} = 0 = N - Mg\cos\theta$$

$$N = Mg\cos\theta = 26\,kg \times 9.81\frac{m}{s^2} \times \cos 30° \times \frac{N \cdot s^2}{kg \cdot m} = 221\ N$$

$$\Sigma F_{x'} = Ma_{x'} = Mg\sin 30° - F_f = Mg\sin 30° - \mu N$$

$$a_{x'} = g\sin 30° - \mu\frac{N}{M} = 9.81\frac{m}{s^2}\sin 30° - 0.3 \times 221\,N \times \frac{1}{26\,kg} \times \frac{kg \cdot m}{N \cdot s^2}$$

$$a_{x'} = 2.36\ m/sec^2$$

Then

$$a_x = a_{x'}\cos\theta = 2.36\frac{m}{sec^2} \times \cos 30° = 2.04\ m/s^2$$

$$a_y = -a_{x'}\sin\theta = -2.36\frac{m}{sec^2} \times \sin 30° = -1.18\ m/s^2$$

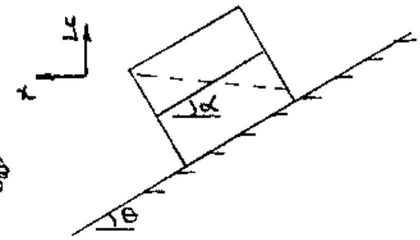
and

$$\frac{dy}{dx} = \frac{-a_x}{g + a_y} = -\frac{2.04}{9.81 - 1.18} = -0.236$$

$$\alpha = \tan^{-1} 0.236 = 13.3°$$ ← α

Problem *3.107

Given: Rectangular container of base dimensions 0.4m × 0.2m and height 0.4m is filled with water to a depth, d = 0.2m
Mass of empty container is $M_c = 10$ kg
Container slides down an incline, $\theta = 30°$ without friction



Find: (a) The angle of the water surface relative to the horizontal.
(b) Slope of the free surface for the same acceleration up the plane

Solution:

Basic equations: $-\nabla P + \rho\vec{g} = M\vec{a}$ $\qquad \Sigma\vec{F} = M\vec{a}$

Assumptions: (1) fluid moves as solid body, i.e. no sloshing

Writing component equations,

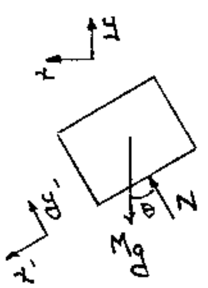
$$-\frac{\partial P}{\partial x} = \rho a_x \qquad \frac{\partial P}{\partial x} = -\rho a_x$$

$$-\frac{\partial P}{\partial y} - \rho g = \rho a_y \qquad \frac{\partial P}{\partial y} = -\rho(g + a_y)$$

$P = P(x,y)$ $\qquad dP = \frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy$ $\qquad$ Along the water surface, $dP = 0$

$$\frac{dy}{dx} = \frac{-\partial P/\partial x}{\partial P/\partial y} = -\frac{a_x}{(g + a_y)}$$

For motion without friction



$$\Sigma F_{x'} = Ma_{x'} = Mg\sin\theta \qquad \therefore a_{x'} = g\sin\theta$$

$$a_x = a_{x'}\cos\theta = g\sin\theta\cos\theta$$

$$a_y = -a_{x'}\sin\theta = -g\sin^2\theta$$

$$\frac{dy}{dx} = -\frac{a_x}{(g + a_y)} = -\frac{g\sin\theta\cos\theta}{(g - g\sin^2\theta)} = -\frac{\sin\theta}{\cos\theta} = -\tan\theta$$

$$\frac{dy}{dx} = -\tan 30° = -0.577$$

$$\alpha = \tan^{-1} 0.577 = 30°$$ α

For the same acceleration up the incline, $a_x = -g\sin\theta\cos\theta$ $\quad a_y = g\sin^2\theta$

$$\frac{dy}{dx} = \frac{-a_x}{(g + a_y)} = \frac{g\sin\theta\cos\theta}{(g + g\sin^2\theta)} = \frac{\sin\theta\cos\theta}{1 + \sin^2\theta} = \frac{\sin 30\cos 30}{1 + \sin^2 30}$$

$$\frac{dy}{dx} = 0.346$$

Problem *3.108

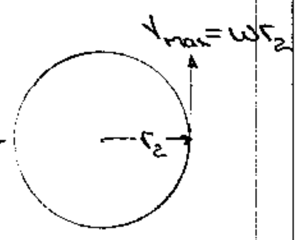
Given: Gas centrifuge, with maximum peripheral speed, $V_{max} = 300$ m/sec contains uranium hexafluoride gas ($M = 352$ kg/kgmol) at 325C.

Find: (a) Develop an expression for ratio of maximum pressure to pressure at centrifuge axis
(b) Evaluate for given conditions

Solution:

Basic equation: $-\nabla p + \rho\vec{g} = \rho\vec{a}$ $\qquad p = \rho RT$

(r component) $-\frac{\partial p}{\partial r} + \rho g_r = \rho a_r$



Assumptions: (1) ideal gas behavior, T = constant
(2) r horizontal, so $g_r = 0$
(3) rigid body motion, so
$$a_r = -\frac{V^2}{r} = -\frac{(r\omega)^2}{r} = -r\omega^2$$

Then $\frac{\partial p}{\partial r} = -\rho a_r = \rho r\omega^2 = \frac{p}{RT} r\omega^2$

Separating variables and integrating, we obtain

$$\int_{p_1}^{p_2} \frac{dp}{p} = \frac{\omega^2}{RT}\int_{r_1=0}^{r_2} r\,dr = \frac{\omega^2}{RT}\frac{r_2^2}{2} \qquad V_{max} = \omega r_2$$

$$\ln\frac{p_2}{p_1} = \frac{V_{max}^2}{2RT}$$

$$\frac{p_2}{p_1} = e^{\frac{V_{max}^2}{2RT}} \longleftarrow \frac{p_2}{p_1}$$

To evaluate, $R = \frac{R_u}{M} = 8314\,\frac{N\cdot m}{kgmol\cdot K} \times \frac{kgmol}{352\,kg} = 23.62\,\frac{N\cdot m}{kg\cdot K}$

$$\frac{V_{max}^2}{2RT} = \frac{(300)^2}{2}\frac{m^2}{s^2} \times \frac{kg\cdot K}{23.62\,N\cdot m} \times \frac{1}{598\,K} \times \frac{N\cdot s^2}{kg\cdot m} = 3.186$$

$$\therefore \frac{p_2}{p_1} = e^{3.186} = 24.2 \longleftarrow$$

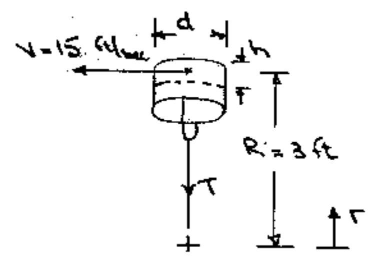
Problem *3.109

Given: Pail, 1 ft in diameter and 1 ft deep, weighs 3 lbf and contains 8 in. of water.
Pail is swung in a vertical circle of 3 ft radius and a speed of 15. ft/s.
Water moves as solid body
Point of interest is top of trajectory.

Determine: (a) tension in string
(b) pressure on pail bottom from water

Solution

Assumption: center of mass of bucket and of water are located at $r = 3$ ft where $V = r\omega = 15.$ ft/s



Summing forces in radial direction

$$\Sigma F_r \hat{e}_r = M_b a_{b_r} \hat{e}_r + M_w a_{w_r} \hat{e}_r$$

$$-T - (M_b + M_w) g = M_b a_{b_r} + M_w a_{w_r}$$

But $a_{b_r} = a_{w_r} = -\omega^2 r = -\frac{V^2}{r}$

$$\therefore T = \left(\frac{V^2}{r} - g\right)(M_b + M_w)$$

where

$$M_w = \rho_w \forall_w = \rho_w \frac{\pi d^2}{4} h = 1.94 \frac{\text{slug}}{\text{ft}^3} \pi \times \frac{1\,\text{ft}^2}{4} \times 8\,\text{in} \times \frac{\text{ft}}{12\,\text{in}} = 1.02\ \text{slug}$$

Then

$$T = \left((15)^2 \frac{\text{ft}^2}{\text{s}^2} \times \frac{1}{3\,\text{ft}} - 32.2 \frac{\text{ft}}{\text{s}^2}\right)\left(3\,\text{lbf} \cdot \frac{1}{32.2}\frac{\text{s}^2}{\text{ft}} \cdot \frac{\text{slug-ft}}{\text{lbf-s}^2} + 1.02\ \text{slug}\right) \frac{\text{lbf} \cdot \text{s}^2}{\text{ft} \cdot \text{slug}}$$

$$T = 47.6\ \text{lbf}$$ ← T

In the water $-\nabla P + \rho \vec{g} = \rho \vec{a}$

Writing the component in the r direction

$$-\frac{\partial P}{\partial r} - \rho g = \rho a_r = -\rho \frac{V^2}{r}$$

$$\frac{\partial P}{\partial r} = \rho\left(\frac{V^2}{r} - g\right) = 1.94 \frac{\text{slug}}{\text{ft}^3}\left((15)^2 \frac{\text{ft}^2}{\text{sec}^2} \times \frac{1}{3\,\text{ft}} - 32.2 \frac{\text{ft}}{\text{sec}^2}\right) \times \frac{\text{lbf} \cdot \text{sec}^2}{\text{ft} \cdot \text{slug}}$$

$$\frac{\partial P}{\partial r} = 83.0\ \text{lbf}/\text{ft}^3$$

Assuming that $\partial P/\partial r$ is constant throughout the water then

$$P_{bottom} \cong P_{surface} + \frac{\partial P}{\partial r} \Delta r$$

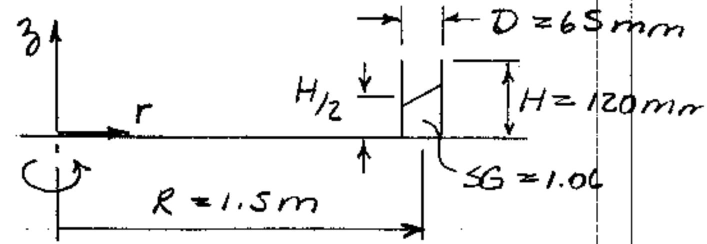
$$P_{bottom} = P_{atm} + 83.0 \frac{\text{lbf}}{\text{ft}^3} \times 8\,\text{in} \times \frac{\text{ft}}{12\,\text{in}} = P_{atm} + 55.3 \frac{\text{lbf}}{\text{ft}^2}$$

$$P_{bottom} - P_{atm} = 55.3\ \text{lbf}/\text{ft}^2\ (\text{gage})$$ ←

Problem *3.110

Given: Soft drink can at outer edge of merry-go-round.

$\omega = 0.3$ rev/sec



Find: (a) Slope of free surface
(b) Spin rate to spill
(c) Likelihood of spilling vs. slipping

Solution: Assume rigid-body motion

Basic equation: $-\nabla p + \rho \vec{g} = \rho \vec{a}_r \qquad a_r = -\frac{V^2}{r} = -\frac{(r\omega)^2}{r} = -r\omega^2$

$$\left. \begin{aligned} -\frac{\partial p}{\partial r} + \rho g_r &= \rho a_r \quad (g_r = 0\ (2)) \\ -\frac{\partial p}{\partial z} + \rho g_z &= \rho a_z \quad (a_z = 0\ (3)) \end{aligned} \right\} \qquad \begin{aligned} \frac{\partial p}{\partial r} &= -\rho a_r = \rho r \omega^2 \\ \frac{\partial p}{\partial z} &= +\rho g_z = -\rho g \end{aligned}$$

Assumptions: (1) Rigid-body motion, (2) $g_r = 0$, (3) $a_z = 0$, (4) $g_z = -g$

Then $p = p(r, z)$ so $dp = \frac{\partial p}{\partial r} dr + \frac{\partial p}{\partial z} dz$

$dp = 0$ along free surface, so $\frac{dz}{dr} = -\frac{\partial p / \partial r}{\partial p / \partial z} = -\frac{\rho r \omega^2}{-\rho g} = \frac{r\omega^2}{g}$

$$\omega = 0.3\ \frac{\text{rev}}{\text{sec}} \times 2\pi\ \frac{\text{rad}}{\text{rev}} = 1.88\ \text{rad/s}$$

$$\left.\frac{dz}{dr}\right)_{\text{surface}} = 1.5\ \text{m} \times (1.88)^2\ \frac{\text{rad}^2}{\text{s}^2} \times \frac{\text{s}^2}{9.81\ \text{m}} = 0.540$$ Slope

To spill, slope must be $H/D = 120/65 = 1.85$

$$\text{Thus} \quad \omega = \left[\frac{g}{r}\frac{dz}{dr}\right]^{1/2} = \left[9.81\ \frac{\text{m}}{\text{s}^2} \times 1.85 \times \frac{1}{1.5\ \text{m}}\right]^{1/2} = 3.48\ \text{rad/s}$$ Spill

This is nearly double the speed.

The coefficient of static friction between the can and surface is probably $\mu_s \le 0.5$.

Thus the can would likely not spill or tip: it would slide off!

Open-Ended Problem Statement: When a water polo ball is submerged below the surface in a swimming pool and released from rest, it is observed to pop out of the water. How would you expect the height to which it rises above the water to vary with depth of submersion below the surface? Would you expect the same results for a beach ball? For a table-tennis ball?

Discussion: Separate the problem into two parts: (1) motion of the ball in water below the pool surface, and (2) motion of the ball in air above the pool surface.

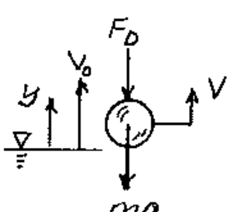
Below the pool water surface the motion of each ball is controlled by buoyancy force and inertia. For small depths of submersion ball speed upon reaching the pool surface will be small. As depth is increased, ball speed will increase until terminal speed in water is approached. For large depths, the actual depth will be irrelevant because the ball will reach terminal speed before reaching the pool water surface. All three balls are relatively light for their diameters, so terminal speed in water should be reached quickly. The depth of submersion needed to reach terminal speed should be fairly small, perhaps 1 meter or less.[1]

Buoyancy is proportional to volume and inertia is proportional to mass. The ball with the largest volume per unit mass should accelerate most quickly to terminal speed. This probably will be the beach ball, followed by the table-tennis ball and the water polo ball.

The ball with the largest diameter has the smallest frontal area per unit volume; the terminal speed should be highest for this ball. The beach ball should have the highest terminal speed, followed by the water polo ball and the table-tennis ball.

Above the pool water surface the motion of each ball is controlled by aerodynamic drag force, gravity force, and inertia (see equation below). Without aerodynamic drag, the height above the pool water surface reached by each ball would depend only its initial speed.[2] Aerodynamic drag reduces the height reached by each ball.

Aerodynamic drag force is proportional to frontal area. The heaviest ball per unit frontal area (probably the water polo ball) should reach the maximum height and the lightest ball per unit frontal area (probably the beach ball) should reach the minimum height.



$$\Sigma F_y = -F_D - mg = ma_y = m\frac{dV}{dt}$$

$$-C_D A \tfrac{1}{2}\rho V^2 - mg = m\frac{dV}{dt}, \quad \text{since } F_D = C_D A \tfrac{1}{2}\rho V^2$$

$$-\frac{C_D A \frac{1}{2}\rho V^2}{m} - g = \frac{dV}{dt} = V\frac{dV}{dy} \qquad (1)$$

Separating variables
$$\frac{V\,dV}{1 + \frac{C_D A \rho}{mg}\frac{V^2}{2}} = -g\,dy$$

Integrating,
$$\frac{mg}{\rho C_D A}\ln\left[1 + \frac{\rho C_D A}{mg}\frac{V^2}{2}\right]_{V_0}^{0} = -\frac{mg}{\rho C_D A}\ln\left[1 + \frac{\rho C_D A}{mg}\frac{V_0^2}{2}\right] = -g y_{max}$$

[1] The initial water depth required to reach terminal speed may be calculated using the methods of Chapter 9.

[2] The maximum height reached by a ball in air with aerodynamic drag may be calculated using the methods of Chapter 9.

Thus $$y_{max} = \frac{m}{\rho C_D A} \ln\left[1 + \frac{\rho C_D A}{mg}\frac{V_0^2}{2}\right] = \frac{m}{\rho C_D A}\ln\left[1 + \frac{F_{D_0}}{mg}\right] \qquad (2)$$

y_{max}

With no aerodynamic drag, Eq. 1 reduces to

$$-mg = mV\frac{dV}{dy} \quad \text{or} \quad V\,dV = -g\,dy$$

Integrating from V_0 to 0, $\left.\frac{V^2}{2}\right]_{V_0}^{0} = -g\,y_{max}$

$$y_{max} = \frac{V_0^2}{2g} \qquad (3)$$

y_{max} (C_D =

Check the limiting value predicted by Eq. 2 as $C_D \to 0$:

$$\lim_{C_D \to 0} y_{max} = \lim_{C_D \to 0} \frac{m}{\rho C_D A}\frac{\rho C_D A}{mg}\frac{V_0^2}{2} = \frac{V_0^2}{2g} \quad \checkmark\checkmark$$

Problem *3.112

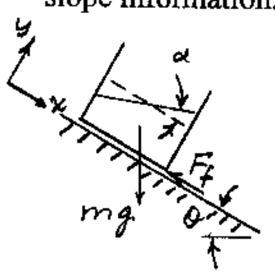
Open-Ended Problem Statement: The analysis of problem 3.98 suggests that it maybe possible to determine the coefficient of sliding friction between two surfaces by measuring the slope of the free surface in a liquid-filled container sliding down an inclined surface. Investigate the feasibility of this idea.

Discussion: A certain minimum angle of inclination would be needed to overcome static friction and start the container into motion down the incline. Once the container is in motion, the retarding force would be provided by sliding (dynamic) friction. The coefficient of dynamic friction usually is smaller than the static friction coefficient. Thus the container would continue to accelerate as it moved down the incline. This acceleration would provide a non-zero slope to the free surface of the liquid in the container.

In principle the slope could be measured and the coefficient of dynamic friction calculated. In practice several problems would arise.

To calculate dynamic friction coefficient one must assume the liquid moves as a solid body (i.e., that there is no sloshing). This condition could only be achieved if there were minimum initial disturbance and the sliding distance were long.

It would be difficult to measure the slope of the free surface of liquid in the moving container. Images made with a video camera or digital still camera might be processed to obtain the required slope information.



$$\Sigma F_y = N - mg\cos\theta \; ; \; N = mg\cos\theta$$

$$\Sigma F_x = mg\sin\theta - F_f = ma_x \; ; \; F_f = \mu_k N = \mu_k mg\cos\theta$$

$$a_x = g\sin\theta - \mu_k g\cos\theta = g(\sin\theta - \mu_k\cos\theta)$$

For static liquid $\quad -\nabla p + \rho\vec{g} = \rho\vec{a}$

$$-\frac{\partial p}{\partial x} + \rho g\sin\theta = \rho a_x = \rho g(\sin\theta - \mu_k\cos\theta) \; ; \; \frac{\partial p}{\partial x} = \rho g\mu_k\cos\theta$$

$$-\frac{\partial p}{\partial y} - \rho g\cos\theta = \rho a_y^{\,0} \; ; \; \frac{\partial p}{\partial y} = -\rho g\cos\theta$$

For the free surface, $dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy = 0$, so $\frac{dy}{dx} = -\frac{\partial p/\partial x}{\partial p/\partial y}$

Thus $\frac{dy}{dx} = -\frac{\rho g\mu_k\cos\theta}{-\rho g\cos\theta} = \mu_k \; ; \; \alpha = \tan^{-1}(\mu_k)$

Since it was necessary to make the container slip on the surface,

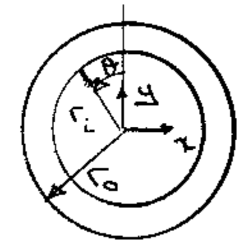
$$\theta > \tan^{-1}(\mu_s) > \tan^{-1}(\mu_k) = \alpha$$ α

Thus $\alpha < \theta$, as shown in the sketch above.

Problem *3.113

Given: A steel liner of length $L = 2\,m$, outer radius $r_o = 0.15\,m$, and inner radius $r_i = 0.10\,m$ is to be formed in a spinning horizontal mold. To insure uniform thickness the minimum radial acceleration should be $10\,g$. For steel, S.G = 7.8.

Find: (a) The required angular velocity
(b) the maximum and minimum pressures on the surface of the mold.



Solution:

Basic equation: $\nabla p + \rho \vec{g} = \rho \vec{a}$

Writing component equations,

$-\frac{\partial p}{\partial r} + \rho g_r = \rho a_r$ and $\frac{\partial p}{\partial r} = \rho g_r - \rho a_r = \rho(-g\cos\theta) - \rho(-r\omega^2) = \rho r\omega^2 - \rho g\cos\theta$

$-\frac{1}{r}\frac{\partial p}{\partial \theta} + \rho g_\theta = 0$ and $\frac{\partial p}{\partial \theta} = \rho g_\theta r = \rho g \sin\theta\, r$

Then, $dp = \frac{\partial p}{\partial r}dr + \frac{\partial p}{\partial \theta}d\theta = (\rho r\omega^2 - \rho g\cos\theta)dr + \rho g r\sin\theta\, d\theta$

$\left.\frac{\partial p}{\partial r}\right)_\theta = \text{const} = \rho r\omega^2 - \rho g\cos\theta$. Since $p = p_{atm}$ at $r = r_i$, then

$p - p_{atm} = \int_{r_i}^{r}(\rho r\omega^2 - \rho g\cos\theta)\,dr + f(\theta)$ where, $f(\theta)$ is an arbitrary function

$\therefore p = p_{atm} + \rho\omega^2\frac{(r^2 - r_i^2)}{2} - \rho g\cos\theta(r - r_i) + f(\theta)$. Then,

$\frac{\partial p}{\partial \theta} = \rho g\sin\theta(r - r_i) + \frac{df}{d\theta} = \rho g\sin\theta\, r$

Hence, $\frac{df}{d\theta} = \rho g\sin\theta\, r_i$ and $f = -\rho g r_i\cos\theta + c$

$\therefore p = p_{atm} + \rho\omega^2\frac{(r^2 - r_i^2)}{2} - \rho g\cos\theta(r - r_i) - \rho g r_i\cos\theta + c$

At $r = r_i$, $p = p_{atm}$ for any value of θ. Hence, $c = \rho g r_i\cos\theta$ and

$p = p_{atm} + \rho\omega^2\frac{(r^2 - r_i^2)}{2} - \rho g\cos\theta(r - r_i)$

Minimum value of $a_r = 10g = r\omega^2$ occurs at r_i for given ω. Hence,

$\omega_{min} = \left[\frac{10g}{r_i}\right]^{1/2} = \left[10 \times 9.81\frac{m}{s^2} \times \frac{1}{0.10\,m}\right]^{1/2} = 31.3\ \text{rad/s}$ ← ω

p_{max} on the surface of the mold ($r = r_o$) occurs at $\theta = \pi$

$p_{max} - p_{atm} = \frac{\rho\omega^2}{2}(r_o^2 - r_i^2) - \rho g\cos\theta(r - r_i)$.

$p_{max} - p_{atm} = \frac{1}{2} \times 7.8 \times 999\frac{kg}{m^3} \times \frac{(31.3)^2}{s^2} \times [(0.15)^2 - (0.10)^2]\,m^2 \times \frac{N\cdot s^2}{kg\cdot m} - 7.8 \times 999\frac{kg}{m^3} \times 9.81\frac{m}{s^2}(-1)[0.15 - 0.10]\frac{N\cdot s^2}{kg\cdot m}$

$p_{max} = 51.5\ kPa$ (gage) ← p_{max}

p_{min} on the surface of the mold ($r = r_o$) occurs at $\theta = 0$

$p_{min} - p_{atm} = \frac{\rho\omega^2}{2}(r_o^2 - r_i^2) - \rho g\cos\theta(r - r_i)$

$p_{min} - p_{atm} = \frac{1}{2} \times 7.8 \times 999\frac{kg}{m^3} \times \frac{(31.3)^2}{s^2}[(0.15)^2 - (0.10)^2]\,m^2 \times \frac{N\cdot s^2}{kg\cdot m} - 7.8 \times 999\frac{kg}{m^3} \times 9.81\frac{m}{s^2}(+1)[0.15 - 0.10]\frac{N\cdot s^2}{kg\cdot m}$

$p_{min} = 43.9\ kPa$ (gage) ← p_{min}

Problem 4.1

Given: Six-pack cooled from 25°C to 5°C in freezer.

Find: Change in specific entropy.

Solution: Apply the Tds equation.

Basic equation: $Tds = du + p\,dv$ ($dv \simeq 0$ (1))

Assumptions: (1) Neglect volume change
(2) Liquid properties are similar to water

Then

$$Tds = du = c_v\,dT$$

or

$$ds = c_v \frac{dT}{T}$$

Integrating,

$$s_2 - s_1 = c_v \ln\left(\frac{T_2}{T_1}\right)$$

$$= 1\,\frac{\text{kcal}}{\text{kg}\cdot\text{K}} \times \ln\left(\frac{273+5}{273+25}\right) \times 4190\,\frac{\text{J}}{\text{kcal}}$$

$$s_2 - s_1 = -0.291\ \text{kJ/kg}\cdot\text{K}$$

Δs

Problem 4.2

A mass of 3 kg falls freely a distance of 5 m before contacting a spring attached to the ground. If the spring stiffness is 400 N/m, what is the maximum spring compression?

Given: Data on mass and spring

Find: Maximum spring compression

Solution

The given data is $M = 3 \cdot kg \qquad h = 5 \cdot m \qquad k = 400 \cdot \frac{N}{m}$

Apply the First Law of Thermodynamics: for the system consisting of the mass and the spring (t spring has gravitional potential energy and the spring elastic potential energy)

Total mechanical energy at initial state $E_1 = M \cdot g \cdot h$

Total mechanical energy at instant of maximum compression x $E_2 = M \cdot g \cdot (-x) + \frac{1}{2} \cdot k \cdot x^2$

Note: The datum for zero potential is the top of the uncompressed spring

But $E_1 = E_2$

so $M \cdot g \cdot h = M \cdot g \cdot (-x) + \frac{1}{2} \cdot k \cdot x^2$

Solving for x $x^2 - \frac{2 \cdot M \cdot g}{k} \cdot x - \frac{2 \cdot M \cdot g \cdot h}{k} = 0$

$$x = \frac{M \cdot g}{k} + \sqrt{\left(\frac{M \cdot g}{k}\right)^2 + \frac{2 \cdot M \cdot g \cdot h}{k}}$$

$$x = 3\,kg \cdot 9.81\,\frac{m}{s^2} \cdot \frac{m}{400\,N} + \sqrt{\left(3\,kg \cdot 9.81\,\frac{m}{s^2} \cdot \frac{m}{400\,N}\right)^2 + 2 \cdot 3\,kg \cdot 9.81\,\frac{m}{s^2} \cdot 5\,m \cdot \frac{m}{400\,N}}$$

$$x = 0.934\,m$$

Note that ignoring the loss of potential of the mass due to spring compression x gives

$$x = \sqrt{\frac{2 \cdot M \cdot g \cdot h}{k}} \qquad x = 0.858\,m$$

Note that the deflection if the mass is dropped from immediately above the spring is

$$x = \frac{2 \cdot M \cdot g}{k} \qquad x = 0.147\,m$$

Given: Jet aircraft with $W = 715{,}000$ lbf.
Takeoff speed is $V_t = 140$ mph.
Twin engines develop 102,000 lbf thrust each.

Assume thrust is constant. Neglect resistance.

Estimate: (a) Runway length needed.
(b) Time to reach V_t.

Solution: Apply Newton's second law of motion.

$$\sum F_x = m\frac{dV}{dt} = mV\frac{dV}{ds} = F_t = \text{constant} \;;\; m = \frac{W}{g}$$

For distance calculation

$$mV\frac{dV}{ds} = \frac{W}{g}V\frac{dV}{ds} = F_t \;;\quad V\,dV = \frac{gF_t}{W}ds$$

Integrating,

$$\int V\,dV = \frac{V^2}{2} = \int \frac{gF_t}{W}ds = \frac{gF_t}{W}s$$

Thus

$$s = \frac{WV^2}{2gF_t}$$

$$= \frac{1}{2} \times 715{,}000\,\text{lbf}\left[140\frac{\text{mi}}{\text{hr}} \times 5280\frac{\text{ft}}{\text{mi}} \times \frac{\text{hr}}{3600\,\text{s}}\right]^2 \frac{\text{s}^2}{32.2\,\text{ft}} \times \frac{1}{2(102{,}000)\,\text{lbf}}$$

$$s = 2{,}290 \text{ ft}$$ s

For time calculation,

$$m\frac{dV}{dt} = \frac{W}{g}\frac{dV}{dt} = F_t \;;\quad dV = \frac{gF_t}{W}dt$$

Integrating,

$$\int dV = V = \int \frac{gF_t}{W}dt = \frac{gF_t}{W}t$$

Thus

$$t = \frac{WV}{gF_t}$$

$$= 715{,}000\,\text{lbf} \times 140\frac{\text{mi}}{\text{hr}} \times \frac{\text{s}^2}{32.2\,\text{ft}} \times \frac{1}{2(102{,}000)\,\text{lbf}} \times 5280\frac{\text{ft}}{\text{mi}} \times \frac{\text{hr}}{3600\,\text{s}}$$

$$t = 22.4 \text{ s}$$ t

{Aerodynamic and rolling resistance would cause these values to increase for an actual aircraft.}

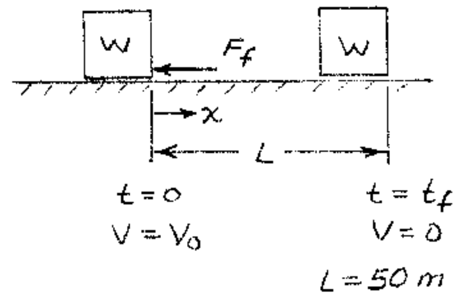
Problem 4.4

Given: Auto skids to stop in 50 meters on level road with $\mu = 0.6$.

Find: Initial speed.

Solution: Apply Newton's second law to a system (auto).

Basic equations: $\Sigma F_x = m a_x = \frac{W}{g}\frac{d^2x}{dt^2}$



Assumptions: (1) $F_f = \mu W$

(2) Neglect air resistance

Then $\Sigma F_x = -F_f = -\mu W = \frac{W}{g}\frac{d^2x}{dt^2}$

or $\frac{d^2x}{dt^2} = -\mu g$

Integrating,

$$\frac{dx}{dt} = -\mu g t + C_1 = -\mu g t + V_0 \qquad (1)$$

since $V = V_0$ at $t = 0$. Integrating again,

$$x = -\frac{1}{2}\mu g t^2 + V_0 t + c_2 = -\frac{1}{2}\mu g t^2 + V_0 t \qquad (2)$$

since $x = 0$ at $t = 0$.

Now at $x = L$, $\frac{dx}{dt} = 0$, and $t = t_f$. From Eq. 1,

$$0 = -\mu g t_f + V_0 \quad \text{or} \quad t_f = \frac{V_0}{\mu g}$$

Substituting into Eq. 2, evaluated at $t = t_f$,

$$L = -\frac{1}{2}\mu g t_f^2 + V_0 t_f = -\frac{1}{2}\mu g \frac{V_0^2}{(\mu g)^2} + V_0 \frac{V_0}{\mu g}$$

$$L = -\frac{1}{2}\frac{V_0^2}{\mu g} + \frac{V_0^2}{\mu g} = \frac{1}{2}\frac{V_0^2}{\mu g}$$

Solving, $V_0 = \sqrt{2\mu g L} = \sqrt{2(0.6)\, 9.81 \frac{m}{s^2} \times 50\, m} = 24.3\ m/s$

or

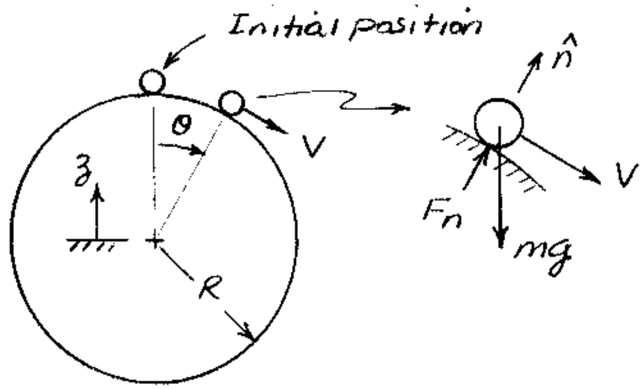
$$V_0 = 24.3 \frac{m}{s} \times \frac{km}{1000\, m} \times 3600 \frac{s}{hr} = 87.5\ km/hr$$

V_0

Given: Small steel ball of radius, r, atop large sphere of radius, R, begins to roll. Neglect rolling and air resistance.

Find: Location where ball loses contact and becomes a projectile.

Solution: Sum forces in n direction



$$\Sigma F_n = F_n - mg\cos\theta = ma_n$$

$$a_n = -\frac{V^2}{(R+r)}$$

Contact is lost when $F_n \to 0$, or

$$-mg\cos\theta = -m\frac{V^2}{(R+r)}$$

or

$$V^2 = (R+r)g\cos\theta \qquad (1)$$

Energy must be conserved if there is no resistance. Thus

$$E = mgz + m\frac{V^2}{2} = mg(R+r)\cos\theta + m\frac{V^2}{2} = E_0 = mg(R+r)$$

Thus from energy considerations

$$V^2 = 2g(R+r)(1-\cos\theta) \qquad (2)$$

Combining Eqs. 1 and 2,

$$V^2 = 2g(R+r)(1-\cos\theta) = (R+r)g\cos\theta$$

or

$$2(1-\cos\theta) = 2 - 2\cos\theta = \cos\theta$$

Thus $\cos\theta = \frac{2}{3}$ and $\theta = \cos^{-1}(\frac{2}{3}) = 48.2$ degrees ← θ

Given: Air at 20°C and 1 atm compressed adiabatically, without friction, to 3 atm (abs.).

Find: Change in internal energy, in J/kg.

Solution: Apply the first law of thermodynamics. Treat the air as a system.

Basic equation: $\delta Q - \delta W = dE$ ($\delta Q = 0$ (1))

Assumptions: (1) Adiabatic process, so $\delta Q = 0$
(2) Stationary system, $dE = dU$
(3) Frictionless process, $\delta W = p\,d\mathcal{V} = m\,p\,dv$
(4) Ideal gas, $pv = RT$

Then

$$\Delta U = \int dU = -\int \delta W = -\int m\,p\,dv$$

The problem is to relate p and v so that the integral may be evaluated. A frictionless adiabatic process is isentropic. Recall from thermodynamics that an ideal gas follows the isentropic process equation

$$pv^k = C \quad \text{where } k = c_p/c_v$$

Thus $v = C^{1/k}p^{-1/k}$ and $dv = C^{1/k}\frac{1}{k}p^{-1/k-1}dp$. Substituting,

$$\Delta u = \frac{\Delta U}{m} = -\int p\,\frac{C^{1/k}}{k}p^{-1/k-1}dp = -\frac{C^{1/k}}{k}\int_{p_1}^{p_2} p^{-1/k}\,dp$$

$$= -\frac{C^{1/k}}{k}\left[\frac{1}{-\frac{1}{k}+1}\,p^{-\frac{1}{k}+1}\right]_{p_1}^{p_2} = -\frac{C^{1/k}}{k}\left[-\frac{k}{k-1}\,p^{\left(\frac{k-1}{k}\right)}\right]_{p_1}^{p_2}$$

$$\Delta u = \frac{C^{1/k}}{k-1}p_1^{\left(\frac{k-1}{k}\right)}\left[\left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}} - 1\right]$$

But $C^{1/k}p^{\left(\frac{k-1}{k}\right)} = C^{1/k}p^{-1/k}p = pv = RT$. Thus

$$\Delta u = \frac{RT_1}{k-1}\left[\left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}} - 1\right]$$

From Table A.6, $R = 287$ J/(kg·K) and $k = 1.40$ for air. Substituting,

$$\Delta u = \frac{1}{0.40}\times 287\,\frac{\text{J}}{\text{kg}\cdot\text{K}}\times(273+20)\,\text{K}\left[\left(\frac{3}{1}\right)^{\frac{1.40-1}{1.40}} - 1\right] = 77.5\ \text{kJ/kg}$$

Δu

Problem 4.7

Given: Auditorium, with volume, $\forall \approx 1.2\times10^7$ ft³ contains 6000 people. Ventilation system fails. Average heat loss per person is 300 Btu/hr.

Find: (a) increase in internal energy of air in 15 min.
(b) change in internal energy for system of people and air; account for increase in air temperature
(c) estimate rate of temperature rise.

Solution
Apply the first law of thermodynamics for a system.

Basic equation: $Q - W = \Delta E$

Assumptions: (1) no work is done, so $W=0$
(2) stationary system, so $\Delta E = \Delta U$

(a) Consider the air in the auditorium to be the system

$$\Delta U_{air} = Q = \frac{300\ \text{Btu}}{\text{hr}\cdot\text{person}} \times 6000\ \text{persons} \times \frac{1}{4}\ \text{hr} = 4.50\times10^5\ \text{Btu}$$ ΔU_{air}

(b) Consider the air and people to be the system

$$\Delta U_{a\,\&\,p} = Q_{from\ surroundings} = 0$$ $\Delta U_{a\,\&\,p}$

The increase in internal energy of the air is equal and opposite to the change in internal energy of the people

(c) To estimate the rate of temperature rise we write the first law on a rate basis

$$\dot{Q} - \dot{W} = \frac{dE}{dt}$$

Taking the air in the auditorium to be the system, then

$$\dot{Q} = \frac{dU}{dt} = M_{air}\frac{du}{dt} = M_{air}c_v\frac{dT}{dt} = \rho_{air}\forall c_v\frac{dT}{dt}$$

Assumptions: (3) air behaves as an ideal gas
(4) initial $T_{aud} = 74°$F, P @ atm

$$\rho = \frac{p}{RT} = \frac{14.7\ \text{lbf}}{\text{in}^2}\times\frac{\text{lbm}\cdot°\text{R}}{53.3\ \text{ft}\cdot\text{lbf}}\times\frac{1}{534\ \text{R}}\times\frac{144\ \text{in}^2}{\text{ft}^2} = 0.0744\ \frac{\text{lbm}}{\text{ft}^3}$$

Then
$$\frac{dT}{dt} = \frac{\dot{Q}}{\rho\forall c_v} = \frac{300\ \text{Btu}}{\text{hr}\cdot\text{person}}\times 6000\ \text{persons}\times\frac{\text{ft}^3}{0.0744\ \text{lbm}}\times\frac{1}{1.2\times10^7\ \text{ft}^3}\times\frac{\text{lbm}\cdot°\text{R}}{0.171\ \text{Btu}}$$

$$\frac{dT}{dt} = 11.8\ °\text{R/hr}$$ $\frac{dT}{dt}$

Problem 4.8

In an experiment with a can of soda, it took 3 hr to cool from an initial temperature of 25°C to 10°C in a 5°C refrigerator. If the can is now taken from the refrigerator and placed in a room at 20°C, how long will the can take to reach 15°C? You may assume that for both processes the heat transfer is modeled by $\dot{Q} \approx -k(T - T_{amb})$, where T is the can temperature, T_{amb} is the ambient temperature, and k is a heat transfer coefficient.

Given: Data on cooling of a can of soda in a refrigerator

Find: How long it takes to warm up in a room

Solution

The First Law of Thermodynamics for the can (either warming or cooling) is

$$M \cdot c \cdot \frac{dT}{dt} = -k \cdot (T - T_{amb}) \quad \text{or} \quad \frac{dT}{dt} = -A \cdot (T - T_{amb}) \quad \text{where} \quad A = \frac{k}{M \cdot c}$$

where M is the can mass, c T is the temperature, and T_{amb} is the ambient temperature

Separating variables $$\frac{dT}{T - T_{amb}} = -A \cdot dt$$

Integrating $$T(t) = T_{amb} + (T_{init} - T_{amb}) \cdot e^{-A t}$$

where T_{init} is the initial temperature. The available data from the coolling can now be used to ob a value for constant A

Given data for cooling $T_{init} = (25 + 273) \cdot K$ $\quad T_{init} = 298\,K$

$T_{amb} = (5 + 273) \cdot K$ $\quad T_{amb} = 278\,K$

$$T = (10 + 273)\cdot K \qquad T = 283\,K \qquad \text{when} \qquad t = \tau = 10\cdot hr$$

Hence $$A = \frac{1}{\tau}\cdot\ln\left(\frac{T_{init} - T_{amb}}{T - T_{amb}}\right) = \frac{1}{3\cdot hr}\cdot\frac{1\cdot hr}{3600\cdot s}\cdot\ln\left(\frac{298 - 278}{283 - 278}\right)$$

$$A = 1.284\cdot 10^{-4}\,s^{-1}$$

Then, for the warming up process

$$T_{init} = (10 + 273)\cdot K \qquad T_{init} = 283\,K$$

$$T_{amb} = (20 + 273)\cdot K \qquad T_{amb} = 293\,K$$

$$T_{end} = (15 + 273)\cdot K \qquad T_{end} = 288\,K$$

with $$T_{end} = T_{amb} + \left(T_{init} - T_{amb}\right)\cdot e^{-A\,\tau}$$

Hence the time τ is $$\tau = \frac{1}{A}\cdot\ln\left(\frac{T_{init} - T_{amb}}{T_{end} - T_{amb}}\right) = \frac{s}{1.284\cdot 10^{-4}}\cdot\ln\left(\frac{283 - 293}{288 - 293}\right)$$

$$\tau = 5.398\cdot 10^{3}\,s \qquad \tau = 1.5\,hr$$

Given: Aluminum beverage can, $m_c = 20\,g$, $D = 65\,mm$, $H = 120\,mm$.

Maximum contents level is h_{max}, when $\forall_b = 354\,mL$ of beverage.

SG of beverage is 1.05.

Find: (a) Center of mass, y_c, vs. level, h.
(b) Level for least tendency to tip.
(c) Minimum coefficient of friction, μ_s, for full can to tip, not slide.
(d) Plot $\mu_s)_{minimum}$ for can to tip (not slide) as a function of beverage level in can.

Solution: $M_b = SG\,\rho\,\forall_b = 1.05 \times 1.0\,\frac{g}{cm^3} \times 354\,mL \times \frac{cm^3}{mL} = 372\,g$ (max)

$$h_{max} = \frac{\forall_b}{A} = \frac{4\forall_b}{\pi D^2} = \frac{4}{\pi} \times 354\,mL \times \frac{1}{(6.5)^2\,cm^2} \times \frac{cm^3}{mL} \times \frac{10\,mm}{cm} = 107\,mm$$

At any level, $m_b = \frac{h}{h_{max}} M_b$; $m_b(g) = \frac{h(mm)}{107\,mm} \times 372\,g = 3.47\,h(mm)$

From moment considerations,

$$y_c M = \frac{h}{2} m_b + \frac{H}{2} m_c = \frac{1}{2}\left[h(3.47h) + 120(20)\right] = \frac{1}{2}(3.47h^2 + 2400)$$

$$M = m_b + m_c = 3.47h + 20$$

$$y_c = \frac{3.47h^2 + 2400}{6.94h + 40} \quad (h \text{ in mm})$$

y_c

Tendency to tip will be least when y_c is a minimum. Thus

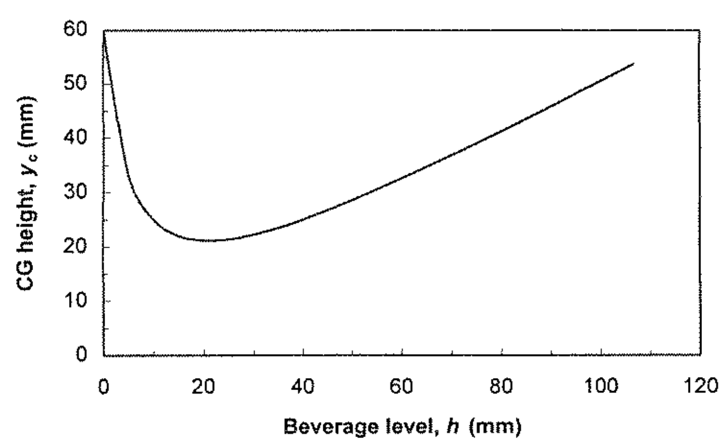
$$\frac{dy_c}{dh} = \frac{2(3.47h)}{6.94h + 40} + (-1)(6.95)\frac{3.47h^2 + 2400}{(6.94h + 40)^2} = \frac{24.1h^2 + 278h - 16{,}700}{(6.94h + 40)^2} = 0$$

Using the quadratic formula,

$$h\,(\text{at } y_c \text{ min}) = \frac{-278 \pm \sqrt{(278)^2 + 4(24.1)16{,}700}}{2(24.1)} = 21.2\,mm$$

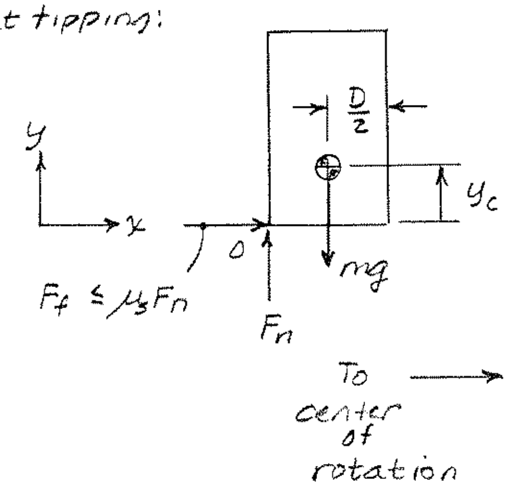
h (y_c min)

Plotting,



Next page →

Draw a free-body diagram of the can at tipping:



$$\sum F_x = F_f = ma_x$$

$$\sum F_y = F_n - mg = ma_y = 0$$

$$F_n = mg$$

Since $F_f \le \mu_s F_n$, then $\mu_s F_n \geq ma_x$

Summing moments about point O:

$$\sum \overset{\curvearrowleft +}{M_0} = y_c\, ma_x - \frac{D}{2} F_n = 0$$

or $\quad y_c\, ma_x = \frac{D}{2} F_n$

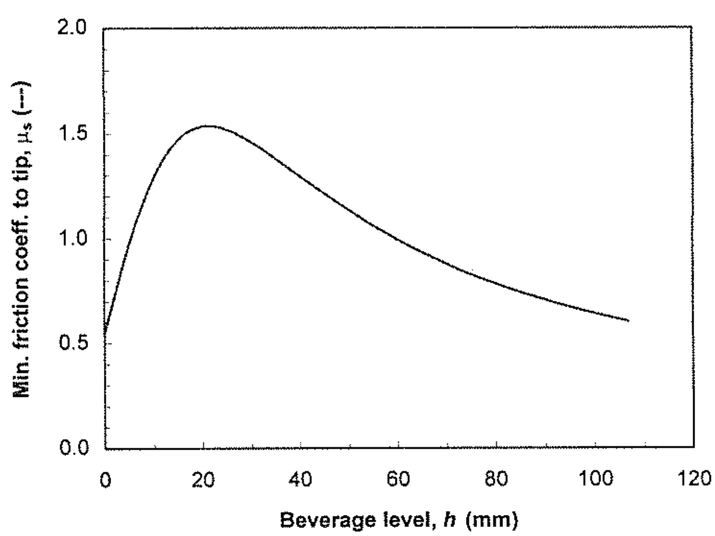
But $ma_x \le \mu_s F_n$, so

$$y_c \mu_s F_n \geq \frac{D}{2} F_n$$

Thus to tip

$$\mu_s \geq \frac{D}{2y_c}$$ μ_s

Plotting,



For the full can with $y_c = 53.8$ mm,

$$\mu_s \geq \frac{1}{2} \times 65\,\text{mm} \times \frac{1}{53.8\,\text{mm}} = 0.604$$ μ_s

This value is much higher than the can could develop. Therefore the can <u>will</u> <u>not</u> tip; it will slide.

The corresponding acceleration is $a_x = \mu_s g = 0.593\ \text{m/s}^2$

Problem 4.10

The velocity field in the region shown is given by $\vec{V} = az\hat{j} + b\hat{k}$, where $a = 10\ \text{s}^{-1}$ and $b = 5$ m/s. For the 1 m × 1 m triangular control volume (depth $w = 1$ m perpendicular to the diagram), an element of area ① may be represented by $w(-dz\,\hat{j} + dy\,\hat{k})$ and an element of area ② by $wdz\,\hat{j}$.

(a) Find an expression for $\vec{V} \cdot d\vec{A}_1$.

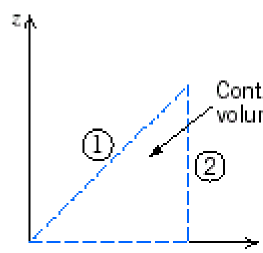
(b) Evaluate $\int_{A_1} \vec{V} \cdot d\vec{A}_1$.

(c) Find an expression for $\vec{V} \cdot d\vec{A}_2$.

(d) Find an expression for $\vec{V}(\vec{V} \cdot d\vec{A}_2)$.

(e) Evaluate $\int_{A_2} \vec{V}(\vec{V} \cdot d\vec{A}_2)$.

Given: Data on velocity field and control volume geometry



Find: Several surface integrals

Solution

$d\vec{A}_1 = -wdz\hat{j} + wdy\hat{k}$ $\qquad$ $d\vec{A}_1 = -dz\hat{j} + dy\hat{k}$

$d\vec{A}_2 = wdz\hat{j}$ $\qquad$ $d\vec{A}_2 = dz\hat{j}$

$\vec{V} = \left(az\hat{j} + b\hat{k}\right)$ $\qquad$ $\vec{V} = \left(10z\hat{j} + 5\hat{k}\right)$

(a) $$\vec{V} \cdot d\vec{A}_1 = \left(10z\hat{j} + 5\hat{k}\right)\cdot\left(-dz\hat{j} + dy\hat{k}\right) = -10zdz + 5dy$$

(b) $$\int_{A_1} \vec{V} \cdot d\vec{A}_1 = -\int_0^1 10zdz + \int_0^1 5dy = -5z^2\Big|_0^1 + 5y\Big|_0^1 = 0$$

(c) $$\vec{V} \cdot d\vec{A}_2 = \left(10z\hat{j} + 5\hat{k}\right)\cdot\left(dz\hat{j}\right) = 10zdz$$

(d) $$\vec{V}\left(\vec{V} \cdot d\vec{A}_2\right) = \left(10z\hat{j} + 5\hat{k}\right)10zdz$$

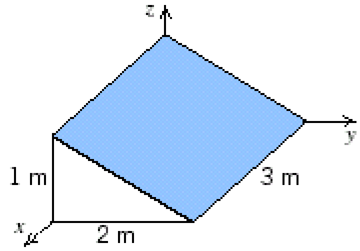
(e) $$\int_{A_2} \vec{V}\left(\vec{V} \cdot d\vec{A}_2\right) = \int_0^1 \left(10z\hat{j} + 5\hat{k}\right)10zdz = \frac{100}{3}z^3\hat{j}\Big|_0^1 + 25z^2\hat{k}\Big|_0^1 = 33.3\hat{j} + 25\hat{k}$$

Problem 4.11

The shaded area shown is in a flow where the velocity field is given by $\vec{V} = ax\hat{i} - by\hat{j}$; $a = b = 1\ \mathrm{s}^{-1}$, and the coordinates are measured in meters. Evaluate the volume flow rate and the momentum flux through the shaded area.

Given: Data on velocity field and control volume geometry

Find: Volume flow rate and momentum flux through shaded area



Solution

$$\vec{dA} \,|\, dxdz\hat{j}\ 2\ dxdy\hat{k}$$

$$\vec{V} \,|\, ax\hat{i}\ 4\ by\hat{j} \qquad \vec{V} \,|\, x\hat{i}\ 4\ y\hat{j}$$

(a) Volume flow rate

$$Q \,|\, \int_A \vec{V} \cdot \vec{dA} \,|\, \int_A \left(x\hat{i}\ 4\ y\hat{j}\right) \cdot \left(dxdz\hat{j}\ 2\ dxdy\hat{k}\right)$$

$$|\, \int_0^3 \int_0^1 4\ ydzdx \,|\, \int_0^1 4\ 3ydz \,|\, \int_0^1 4\ 3\left(2\ 4\ 2z\right)dz \,|\, \left.4\ 6z\ 2\ 3z^2\right|_0^1$$

$$Q \,|\, 43\frac{\mathrm{m}^3}{\mathrm{s}}$$

(b) Momentum flux

$$\psi \int_A \vec{V}\left(\vec{V} \cdot \vec{dA}\right) \,|\, \psi \int_A \left(x\hat{i}\ 4\ y\hat{j}\right)\left(4\ ydxdz\right)$$

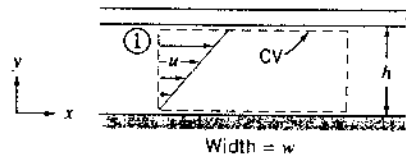
$$|\, \psi \int_0^3 \int_0^1 \left(4\ xy\right)dzdx\ 2\ \psi \int_0^1 3y^2dz$$

$$|\, \psi\left(4\ \left.\frac{x^2}{2}\right|_0^3\right)\left(2z\ 4\ \left.z^2\right|_0^1\right)2\ \psi\left(4\frac{4}{3}\left.\left(z\ 4\ 1^3\right)\right|_0^1\right) \,|\, \psi\left(4\frac{9}{2}\ 2\ \frac{4}{3}\right)$$

$$|\ 43.167\psi$$

Problem 4.12

Given: Control volume with linear velocity distribution across surface ① as shown; width = w.



Find: (a) Volume flow rate, and (b) Momentum flux, through surface ①

Solution:

The volume flow rate is $Q = \int \vec{V} \cdot d\vec{A}$

At surface ①, $\vec{V} = \frac{V}{h} y \hat{i}$ and $d\vec{A} = -w\,dy\,\hat{i}$

Thus

$$Q = \int_{y=0}^{h} \frac{V}{h} y \hat{i} \cdot (-w\,dy\,\hat{i}) = -\frac{Vw}{h} \int_{y=0}^{h} y\,dy = -\frac{Vw}{h} \frac{y^2}{2}\Big]_0^h$$

$$Q = -\frac{1}{2} Vhw \qquad \text{Volume flow rate}$$

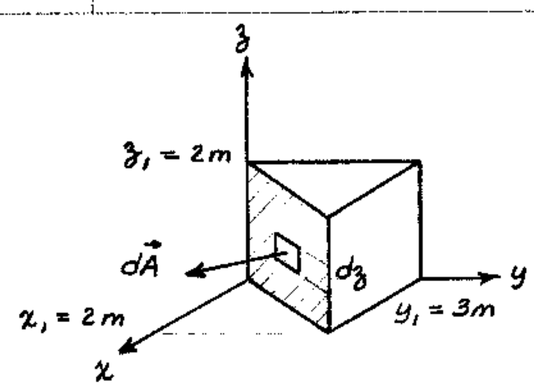
The momentum flux is given by $m.f. = \int \vec{V} (\rho \vec{V} \cdot d\vec{A})$

Thus,

$$m.f. = \int_0^h \frac{V}{h} y \hat{i} \left(-\rho \frac{Vw}{h} y\,dy\right) = -\rho \frac{V^2 w}{h^2} \hat{i} \int_0^h y^2\,dy = -\rho \frac{V^2 w}{h^2} \hat{i} \frac{y^3}{3}\Big]_0^h$$

$$m.f. = -\frac{1}{3} \rho V^2 w h \hat{i} \qquad \text{Momentum flux}$$

Problem 4.13



Given: The shaded area shown, with

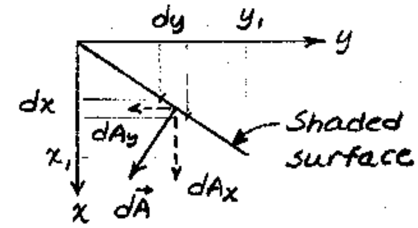
$$\vec{V} = -ax\hat{i} + by\hat{j} + c\hat{k}$$

where $a = b = 1\,sec^{-1}$ and $c = 1\,m/s$

Find: $d\vec{A}$, $\int_A \vec{V}\cdot d\vec{A}$, $\int_A \vec{V}(\vec{V}\cdot d\vec{A})$

Solution: From sketch at right,

$$d\vec{A} = dA_x\,\hat{i} + dA_y(-\hat{j})$$



where

$dA_x = dy\,dz$, the projection of $d\vec{A}$ on yz plane

$dA_y = dx\,dz$, the projection of $d\vec{A}$ on xz plane

Thus

$$d\vec{A} = dy\,dz\,\hat{i} - dx\,dz\,\hat{j}$$ $d\vec{A}$

Along the shaded surface, $\frac{x}{y} = \frac{x_1}{y_1} = \frac{2m}{3m}$, or $y = \frac{3}{2}x$ and $x = \frac{2}{3}y$.

$$\int_A \vec{V}\cdot d\vec{A} = \int_A (-ax\hat{i} + by\hat{j} + c\hat{k})\cdot(dy\,dz\,\hat{i} - dx\,dz\,\hat{j}) = \int_A -ax\,dy\,dz - by\,dx\,dz$$

$$= \int_A -\tfrac{2}{3}ay\,dy\,dz + \int_A -\tfrac{3}{2}bx\,dx\,dz = -\int_{z=0}^{z_1}\int_{y=0}^{y_1}\tfrac{2}{3}ay\,dy\,dz - \int_{z=0}^{z_1}\int_{x=0}^{x_1}\tfrac{3}{2}bx\,dx\,dz$$

$$= -\int_{z=0}^{z_1}\tfrac{1}{3}ay_1^2\,dz - \int_{z=0}^{z_1}\tfrac{3}{4}bx_1^2\,dz = -\tfrac{1}{3}ay_1^2 z_1 - \tfrac{3}{4}bx_1^2 z_1$$

$$\int_A \vec{V}\cdot d\vec{A} = -\tfrac{1}{3}\left(\tfrac{1}{s}\right)(3)^2 m^2 \times 2m - \tfrac{3}{4}\left(\tfrac{1}{s}\right)(2)^2 m^2 \times 2m = -12.0\,m^3/s$$ {a scalar} $\int\vec{V}\cdot d$

$$\int_A \vec{V}(\vec{V}\cdot d\vec{A}) = \int_A (-ax\hat{i} + by\hat{j} + c\hat{k})\left[(-ax\hat{i} + by\hat{j} + c\hat{k})\cdot(dy\,dz\,\hat{i} - dx\,dz\,\hat{j})\right]$$

$$= \int_A (-ax\hat{i} + by\hat{j} + c\hat{k})(-ax\,dy\,dz - by\,dx\,dz)$$

Substituting $y = \frac{3}{2}x$ and $dy = \frac{3}{2}dx$, and noting that $a = b$,

$$\int_A \vec{V}(\vec{V}\cdot d\vec{A}) = \int_A (-ax\hat{i} + \tfrac{3}{2}ax\hat{j} + c\hat{k})(-3ax\,dx\,dz)$$

$$= \int_{z=0}^{z_1}\int_{x=0}^{x_1}(3a^2x^2\hat{i} - \tfrac{9}{2}a^2x^2\hat{j} - 3acx\,\hat{k})\,dx\,dz$$

$$= \int_{z=0}^{z_1}(a^2x_1^3\hat{i} - \tfrac{3}{2}a^2x_1^3\hat{j} - \tfrac{3}{2}acx_1^2\,\hat{k})\,dz$$

$$= a^2x_1^3 z_1\hat{i} - \tfrac{3}{2}a^2x_1^3 z_1\hat{j} - \tfrac{3}{2}acx_1^2 z_1\hat{k}$$

$$= \left(\tfrac{1}{s}\right)^2(2)^3 m^3 \times 2m\,\hat{i} - \tfrac{3}{2}\left(\tfrac{1}{s}\right)^2(2)^3 m^3 \times 2m\,\hat{j} - \tfrac{3}{2}\left(\tfrac{1}{s}\right)\left(1\tfrac{m}{s}\right)(2)^2 m^2 \times 2m\,\hat{k}$$

$$\int_A \vec{V}(\vec{V}\cdot d\vec{A}) = 16\hat{i} - 24\hat{j} - 12\hat{k}\;\; m^4/s^2$$ {result is a vector} $\int_A \vec{V}(\vec{V}\cdot d$

Problem 4.14

Given: Velocity distribution for laminar flow in a long circular tube

$$\vec{V} = u\hat{i} = U_{max}\left[1 - \left(\frac{r}{R}\right)^2\right]\hat{i}$$

where R is the tube radius.

Evaluate: (a) The volume flow rate, and (b) the momentum flux, through a section normal to the pipe axis.

Solution: The volume flow rate is given by

$$\int_{A_{tube}} \vec{V}\cdot d\vec{A} = \int_0^R U_{max}\left[1 - \left(\frac{r}{R}\right)^2\right]\hat{i} \cdot 2\pi r\, dr\, \hat{i} \qquad \{A = \pi r^2,\ dA = 2\pi r\, dr\}$$

$$= U_{max}\, 2\pi \int_0^R \left[1 - \left(\frac{r}{R}\right)^2\right] r\, dr = U_{max}\, 2\pi \int_0^R \left[r - \frac{r^3}{R^2}\right] dr$$

$$= U_{max}\, 2\pi \left[\frac{r^2}{2} - \frac{r^4}{4R^2}\right]_0^R = U_{max}\, 2\pi \left[\frac{R^2}{2} - \frac{R^2}{4}\right]$$

$$\int_{A_{tube}} \vec{V}\cdot d\vec{A} = \frac{1}{2} U_{max}\, \pi R^2$$ ← volume flow rate

The momentum flux is given by

$$\int_{A_{tube}} \vec{V}(\vec{V}\cdot d\vec{A}) = \int_0^R U_{max}\left[1 - \left(\frac{r}{R}\right)^2\right]\hat{i} \left\{ U_{max}\left[1 - \left(\frac{r}{R}\right)^2\right]\hat{i} \cdot 2\pi r\, dr\, \hat{i} \right\}$$

$$= \int_0^R U_{max}\left[1 - \left(\frac{r}{R}\right)^2\right]\hat{i} \left\{ U_{max}\, 2\pi \left[r - \frac{r^3}{R^2}\right] dr \right\}$$

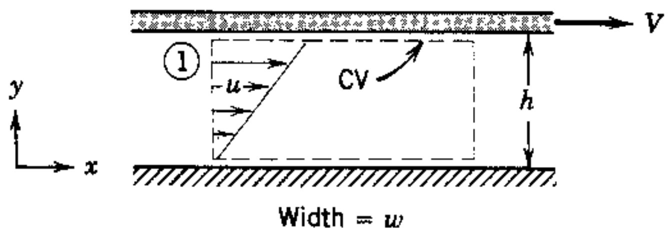
$$= U_{max}^2\, 2\pi \int_0^R \left(r - \frac{2r^3}{R^2} + \frac{r^5}{R^4}\right) dr\, \hat{i}$$

$$= U_{max}^2\, 2\pi \left[\frac{r^2}{2} - \frac{r^4}{2R^2} + \frac{r^6}{6R^4}\right]_0^R \hat{i}$$

$$= U_{max}^2\, 2\pi R^2 \left[\frac{1}{2} - \frac{1}{2} + \frac{1}{6}\right]\hat{i}$$

$$\int_{A_{tube}} \vec{V}(\vec{V}\cdot d\vec{A}) = \frac{1}{3} U_{max}^2\, \pi R^2\, \hat{i}$$ ← momentum flux

Given: Flow and CV of Problem 4.11, as shown



Find: Expression for kinetic energy flux through cross-section ① of CV.

Solution: Kinetic energy flux is defined as $kef = \int_A \frac{V^2}{2} \rho \vec{V} \cdot d\vec{A}$

Model the velocity profile as $u = V\frac{y}{h}$. Then

$$\vec{V} = u\hat{i} = V\frac{y}{h}\hat{i} \; ; \quad V^2 = V^2\left(\frac{y}{h}\right)^2$$

Since flow is into the CV, $\vec{V} \cdot d\vec{A} = -u\,dA = -V\frac{y}{h}w\,dy$

Substituting,

$$kef = \int_A \frac{V^2}{2}\left(\frac{y}{h}\right)^2\left\{-\rho V\frac{y}{h}w\,dy\right\} = -\frac{\rho V^3 w}{2h^3}\int_0^h y^3\,dy$$

$$= -\frac{\rho V^3 w}{2h^3}\left[\frac{y^4}{4}\right]_0^h$$

$$kef = -\frac{\rho V^3 w h}{8}$$ kef

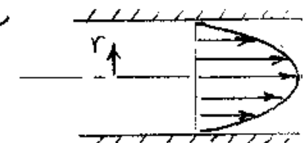
Check dimensions:

$$[kef] = \frac{M}{L^3}\left(\frac{L}{t}\right)^3 LL = \frac{ML^2}{t^3} \times \frac{Ft^2}{ML} = \frac{FL}{t} = \frac{\text{Energy}}{\text{Time}} \quad \checkmark\checkmark$$

Problem 4.16

Given: Velocity profile in a circular tube,

$$\vec{V} = u\hat{i} = u_{max}\left[1-\left(\frac{r}{R}\right)^2\right]\hat{i}$$

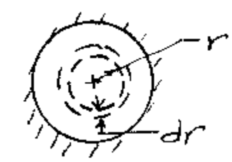


Find: Expression for kinetic energy flux, $kef = \int \frac{V^2}{2}\rho\vec{V}\cdot d\vec{A}$

Solution: $V^2 = \vec{V}\cdot\vec{V} = u_{max}^2\left[1-\left(\frac{r}{R}\right)^2\right]^2 = u_{max}^2\left[1-2\left(\frac{r}{R}\right)^2+\left(\frac{r}{R}\right)^4\right]$

$$d\vec{A} = 2\pi r\,dr\,\hat{i}$$

$$\vec{V}\cdot d\vec{A} = 2\pi r\,u_{max}\left[1-\left(\frac{r}{R}\right)^2\right]$$



Then

$$kef = \int_0^R \frac{u_{max}^2}{2}\left[1-2\left(\frac{r}{R}\right)^2+\left(\frac{r}{R}\right)^4\right]\rho\,2\pi r\,u_{max}\left[1-\left(\frac{r}{R}\right)^2\right]dr$$

$$= \pi\rho u_{max}^3\int_0^R\left[1-3\left(\frac{r}{R}\right)^2+3\left(\frac{r}{R}\right)^4-\left(\frac{r}{R}\right)^6\right]r\,dr$$

$$= \pi\rho u_{max}^3 R^2\int_0^1\left[1-3\left(\frac{r}{R}\right)^2+3\left(\frac{r}{R}\right)^4-\left(\frac{r}{R}\right)^6\right]\frac{r}{R}\,d\left(\frac{r}{R}\right)$$

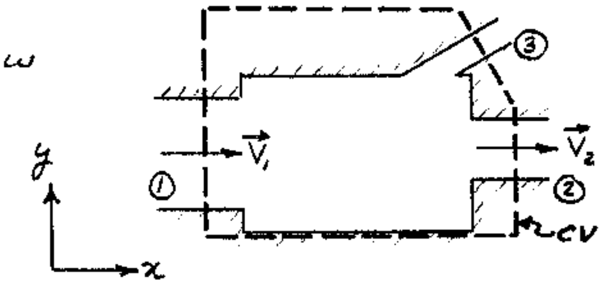
$$= \pi\rho u_{max}^3 R^2\left[\frac{1}{2}\left(\frac{r}{R}\right)^2-\frac{3}{4}\left(\frac{r}{R}\right)^4+\frac{1}{2}\left(\frac{r}{R}\right)^6-\frac{1}{8}\left(\frac{r}{R}\right)^8\right]_0^1$$

$$= \pi R^2\rho u_{max}^3\left[\frac{1}{2}-\frac{3}{4}+\frac{1}{2}-\frac{1}{8}\right]$$

$$kef = \frac{\pi R^2\rho u_{max}^3}{8}$$

kef

Problem 4.17



Given: Steady, incompressible flow through device shown.

$A_1 = 1\,ft^2;\ A_2 = 0.5\,ft^2;\ A_3 = 0.2\,ft^2$

$\vec{V}_1 = 10\hat{i}\ ft/s,\quad \vec{V}_2 = 30\hat{i}\ ft/s$

Find: Volume flow rate through port 3.

Solution: Apply conservation of mass to CV shown

Basic equation: $$0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A}$$ (first term $= 0$ (1))

Assumptions: (1) Steady flow
(2) Uniform flow at each section
(3) Incompressible flow, ρ = constant

Then $$0 = \vec{V}_1\cdot\vec{A}_1 + \vec{V}_2\cdot\vec{A}_2 + \vec{V}_3\cdot\vec{A}_3$$

or $$0 = -|V_1A_1| + |V_2A_2| + \vec{V}_3\cdot\vec{A}_3$$ (flow in at ①, out at ②)

Solving,

$$\vec{V}_3\cdot\vec{A}_3 = |V_1A_1| - |V_2A_2|$$

$$\vec{V}_3\cdot\vec{A}_3 = \left|10\frac{ft}{s}\times 1\,ft^2\right| - \left|30\frac{ft}{s}\times 0.5\,ft^2\right| = -5.00\frac{ft^3}{s}$$

Therefore

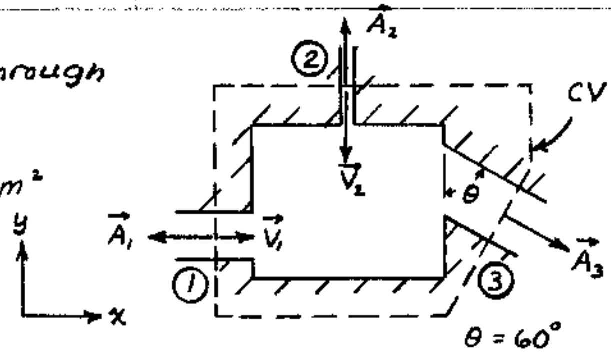
$$Q_3 = \vec{V}_3\cdot\vec{A}_3 = -5.00\,ft^3/s$$ (minus sign means into CV) Q_3

Problem 4.18

Given: Steady, incompressible flow through the device shown.

$A_1 = 0.05\,m^2$, $A_2 = 0.01\,m^2$, $A_3 = 0.06\,m^2$

$\vec{V}_1 = 4\hat{i}\ m/s$, $\vec{V}_2 = -8\hat{j}\ m/s$



Find: Velocity, $\vec{V}_3$

Solution: Apply conservation of mass, using CV shown.

Basic equation: $$0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A}$$ (first term $= 0$ (1))

Assumptions: (1) Steady flow
(2) Incompressible flow, ρ = constant
(3) Uniform flow at each section

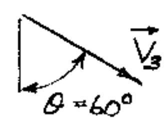
Then
$$\int_{CS} \vec{V}\cdot d\vec{A} = \vec{V}_1\cdot\vec{A}_1 + \vec{V}_2\cdot\vec{A}_2 + \vec{V}_3\cdot\vec{A}_3 = 0$$

or
$$\vec{V}_3\cdot\vec{A}_3 = -\vec{V}_1\cdot\vec{A}_1 - \vec{V}_2\cdot\vec{A}_2 = -4\hat{i}\,\frac{m}{s}\cdot 0.05(-\hat{i})\,m^2 - (-8\hat{j})\,\frac{m}{s}\times 0.01\hat{j}\,m^2$$
$$\vec{V}_3\cdot\vec{A}_3 = 0.28\ m^3/s$$

Since $\vec{V}_3\cdot\vec{A}_3 > 0$, flow at section ③ is out of CV. Thus $\vec{V}_3\cdot\vec{A}_3 = V_3 A_3$

$$V_3 = \frac{1}{A_3}\times 0.28\,\frac{m^3}{s} = \frac{1}{0.06\,m^2}\times 0.28\,\frac{m^3}{s} = 4.67\ m/s$$

Finally, from the geometry of the sketch,

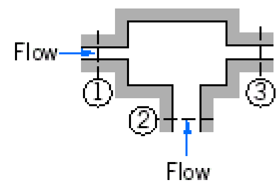


$$\vec{V}_3 = V_3\sin\theta\,\hat{i} - V_3\cos\theta\,\hat{j} = 4.67\,\frac{m}{s}\times\sin 60°\,\hat{i} - 4.67\,\frac{m}{s}\times\cos 60°\,\hat{j}$$
$$\vec{V}_3 = 4.04\hat{i} - 2.34\hat{j}\ \ m/s$$

Problem 4.19

In the incompressible flow through the device shown, velocities may be considered uniform over the inlet and outlet sections. The following conditions are known: $A_1 = 0.1\ \text{m}^2$, $A_2 = 0.2\ \text{m}^2$, $A_3 = 0.15\ \text{m}^2$, $V_1 = 10e^{-t/2}$ m/s, and $V_2 = 2\cos(2\pi t)$ m/s (t in seconds). Obtain an expression for the velocity at section ③, and plot V_3 as a function of time. At what instant does V_3 first become zero? What is the total mean volumetric flow at section ③?

Given: Data on flow through device



Find: Velocity V_3; plot V_3 against time; find when V_3 is zero; total mean flow

Solution

Governing equation: For incompressible flow (Eq. 4.13) and uniform flow

$$\int \vec{V} \cdot d\vec{A} = \sum_{CS} \vec{V} \cdot \vec{A} = 0$$

Applying to the device (assuming V_3 is out)

$$-V_1 \cdot A_1 - V_2 \cdot A_2 + V_3 \cdot A_3 = 0$$

$$V_3 = \frac{V_1 \cdot A_1 + V_2 \cdot A_2}{A_3} = \frac{10 \cdot e^{-\frac{t}{2}} \cdot \frac{\text{m}}{\text{s}} \times 0.1 \cdot \text{m}^2 + 2 \cdot \cos(2 \cdot \pi \cdot t) \cdot \frac{\text{m}}{\text{s}} \times 0.2 \cdot \text{m}^2}{0.15 \cdot \text{m}^2}$$

The velocity at A_3 is $\quad V_3 = 6.67 \cdot e^{-\frac{t}{2}} + 2.67 \cdot \cos(2 \cdot \pi \cdot t)$

The total mean volumetric flow at A_3 is

$$Q \mid \int_0^{\leftarrow} V_3 \cdot A_3 \, dt \mid \int_0^{\leftarrow} \left(6.67 \cdot e^{4\frac{t}{2}} \, 2 \; 2.67 \cdot \cos\left(2 \cdot \phi \cdot t \right) \right) \cdot 0.15 \, dt \cdot \frac{m}{s} \cdot m^2$$

$$Q \mid \lim_{t \Downarrow \leftarrow} 42 \cdot e^{4\frac{t}{2}} \, 2 \, \frac{1}{5 \cdot \phi} \cdot \sin\left(2 \cdot \phi \cdot t \right) \; 4 \, (42) \mid 2 \cdot m^3$$

$$Q \mid 2 \cdot m^3$$

The time at which V_3 first is zero, and the plot of V_3 is shown in the corresponding *Excel* workb

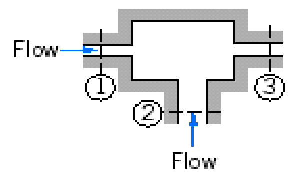
$$t \mid 2.39 \cdot s$$

Problem 4.19 (In Excel)

In the incompressible flow through the device shown, velocities may be considered uniform over the inlet and outlet sections. The following conditions are known: $A_1 = 0.1\ \text{m}^2$, $A_2 = 0.2\ \text{m}^2$, $A_3 = 0.15\ \text{m}^2$, $V_1 = 10e^{-t/2}$ m/s, and $V_2 = 2\cos(2\pi t)$ m/s (t in seconds). Obtain an expression for the velocity at section (3), and plot V_3 as a function of time. At what instant does V_3 first become zero? What is the total mean volumetric flow at section (3)?

Given: Data on flow rates and device geometry

Find: When V_3 is zero; plot V_3

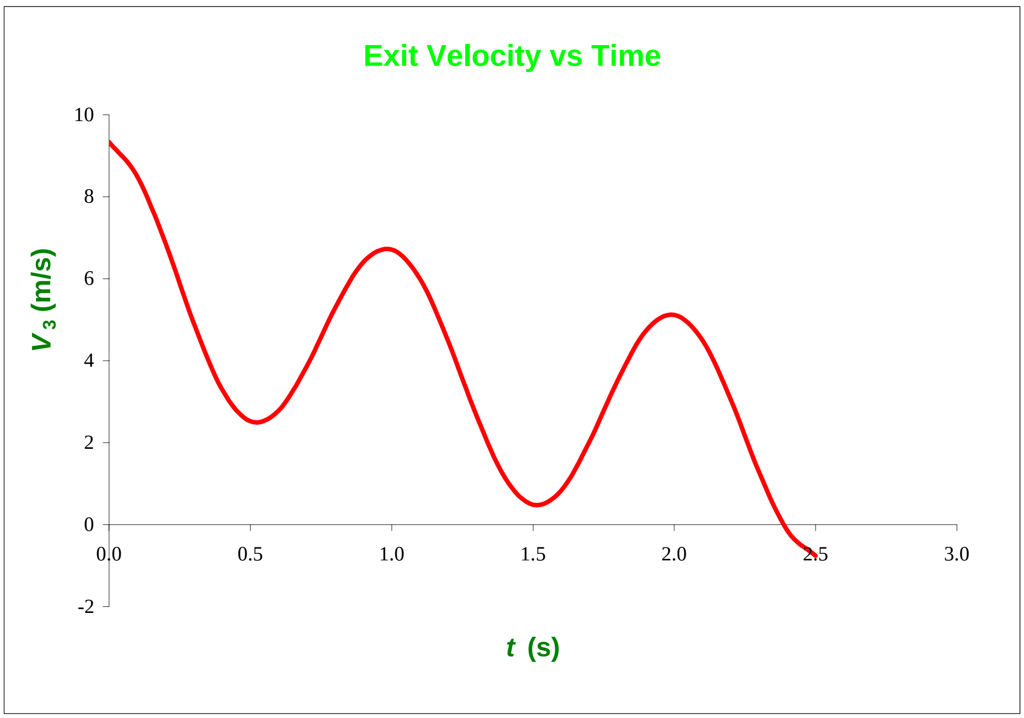


Solution

The velocity at A_3 is

$$V_3 = 6.67 e^{-\frac{t}{2}} + 2.67 \cos(2\pi t)$$

t (s)	V_3 (m/s)
0.00	9.33
0.10	8.50
0.20	6.86
0.30	4.91
0.40	3.30
0.50	2.53
0.60	2.78
0.70	3.87
0.80	5.29
0.90	6.41
1.00	6.71
1.10	6.00
1.20	4.48
1.30	2.66
1.40	1.15
1.50	0.48
1.60	0.84
1.70	2.03
1.80	3.53
1.90	4.74
2.00	5.12
2.10	4.49
2.20	3.04
2.30	1.29
2.40	-0.15
2.50	-0.76

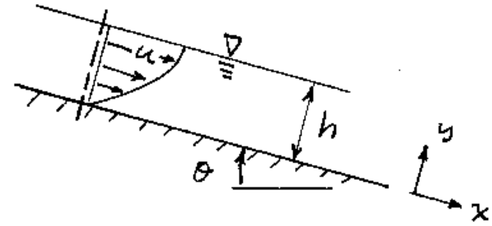


The time at which V_3 first becomes zero can be found using *Goal Seek*

t (s)	V_3 (m/s)
2.39	0.00

Given: Oil flow down inclined plane.

$$u = \frac{\rho g \sin\theta}{\mu}\left(hy - \frac{y^2}{2}\right)$$



Find: Mass flow rate per unit width.

Solution: At the dashed cross-section, $\dot{m} = \int \rho u \, dA$

$dA = w\,dy$, where w = width

$$\dot{m} = \int_0^h \rho \frac{\rho g \sin\theta}{\mu}\left(hy - \frac{y^2}{2}\right) w\,dy = \frac{\rho^2 g \sin\theta}{\mu}\int_0^h \left(hy - \frac{y^2}{2}\right) w\,dy$$

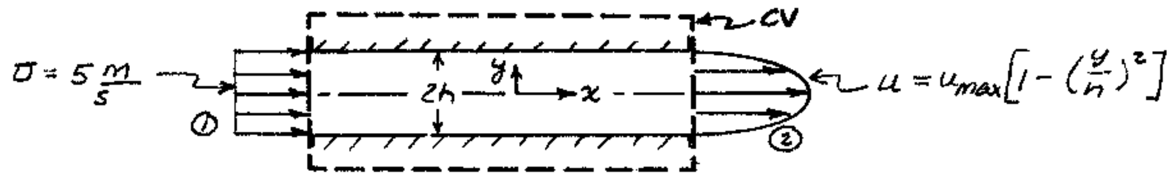
$$\dot{m} = \frac{\rho^2 g \sin\theta}{\mu} w\left[\frac{hy^2}{2} - \frac{y^3}{6}\right]_0^h = \frac{\rho^2 g \sin\theta\, w}{\mu}\,\frac{h^3}{3} = \frac{\rho^2 g \sin\theta\, w h^3}{3\mu}$$

Thus

$$\dot{m}/w = \frac{\rho^2 g \sin\theta\, h^3}{3\mu}$$

$\dot{m}/w$

Given: Water flow between parallel plates as shown.



Find: Exit centerline velocity, U_{max}.

Solution: Apply continuity using the CV shown.

Basic equation: $$0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A}$$ (first term $= 0(1)$)

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Uniform flow at inlet section

Then

$$0 = \vec{V}_1\cdot\vec{A}_1 + \int_{②} \vec{V}_2\cdot d\vec{A}_2 \;;\; \vec{V}_2 = u\hat{i},\; d\vec{A}_2 = w\,dy\,\hat{i}\;(w = \text{width})$$

$$0 = -U(2hw) + \int_{-h}^{h} u_{max}\left[1-\left(\frac{y}{h}\right)^2\right] w\,dy$$

or

$$U = \frac{1}{2h}\int_{-h}^{h} u_{max}\left[1-\left(\frac{y}{h}\right)^2\right]dy = \frac{u_{max}}{2}\int_{-1}^{1}\left[1-\left(\frac{y}{h}\right)^2\right]d\left(\frac{y}{h}\right)$$

$$U = u_{max}\int_0^1\left[1-\left(\frac{y}{h}\right)^2\right]d\left(\frac{y}{h}\right) = u_{max}\left[\left(\frac{y}{h}\right)-\frac{1}{3}\left(\frac{y}{h}\right)^3\right]_0^1 = \frac{2}{3}u_{max}$$

Thus

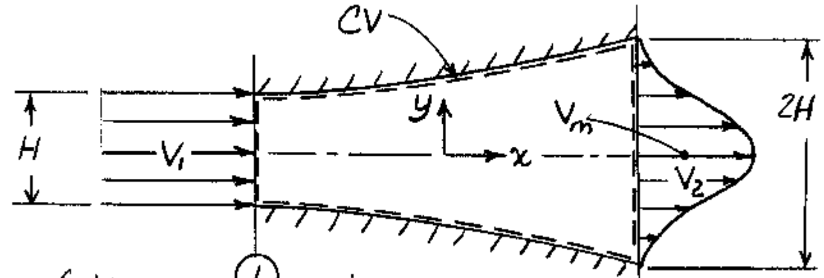
$$u_{max} = \frac{3}{2}U = \frac{3}{2}\times 5\frac{m}{s} = 7.50\ m/s$$ ← u_{max}

{ The maximum speed at the outlet section is 3/2 that of the uniform flow speed at the inlet section. }

Given: Incompressible flow in a diverging channel, as shown.

V_1 = constant

$V_2 = V_m \cos\left(\frac{\pi y}{2H}\right)$



Find: Express V_m in terms of V_1.

Solution: Apply conservation of mass using the CV shown.

Basic equation: $0 = \cancel{\frac{\partial}{\partial t}\int_{CV} \rho \, d\forall}^{=0(1)} + \int_{CS} \rho \vec{V} \cdot d\vec{A}$

Assumptions: (1) Steady flow
(2) Uniform flow at section 1
(3) Incompressible flow

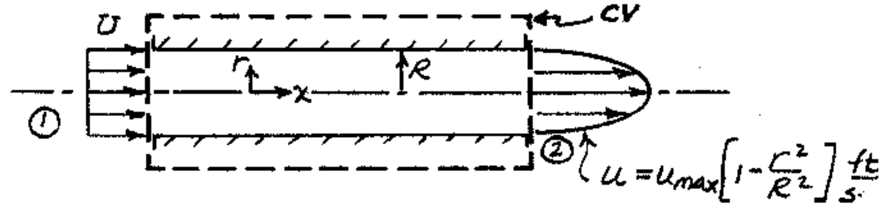
Then $0 = \left\{ -|\cancel{\rho} V_1 A_1| \right\} + \int_{-H}^{H} \cancel{\rho} V_2 w \, dy$

Since $A_1 = wH$, then $V_1 w H = \int_{-H}^{H} V_m \cos\left(\frac{\pi}{2}\frac{y}{H}\right) w \, dy = 2\int_0^H V_m \cos\left(\frac{\pi}{2}\frac{y}{H}\right) w \, dy$

So $V_1 H = 2V_m\left(\frac{2H}{\pi}\right)\int_0^H \cos\left(\frac{\pi}{2}\frac{y}{H}\right) d\left(\frac{\pi}{2}\frac{y}{H}\right) = \frac{4V_m H}{\pi}\left[\sin\left(\frac{\pi}{2}\frac{y}{H}\right)\right]_0^H = \frac{4V_m H}{\pi}$

Thus $V_m = \frac{\pi}{4} V_1$ $\qquad V_m$

Given: Water flow in a pipe as shown. $R = 3$ in. $u_{max} = 10$ ft/s



Find: Uniform inlet velocity, U.

Solution: Apply continuity using the CV shown.

Basic equation: $0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$ ($\frac{\partial}{\partial t}$ term $= 0(1)$)

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Uniform flow at inlet section

Then

$$0 = \vec{V}_1 \cdot \vec{A}_1 + \int_{②} \vec{V}_2 \cdot d\vec{A}_2 \quad ; \quad \vec{V}_2 = u\hat{i}\,,\ d\vec{A}_2 = 2\pi r\,dr\,\hat{i}$$

$$0 = -U\pi R^2 + \int_0^R u_{max}\left[1 - \frac{r^2}{R^2}\right] 2\pi r\,dr$$

or

$$U = \frac{1}{\pi R^2}\int_0^R u_{max}\left[1 - \frac{r^2}{R^2}\right] 2\pi r\,dr = 2u_{max}\int_0^1 \left[1 - \left(\frac{r}{R}\right)^2\right]\left(\frac{r}{R}\right) d\left(\frac{r}{R}\right)$$

$$U = 20\left[\frac{1}{2}\left(\frac{r}{R}\right)^2 - \frac{1}{4}\left(\frac{r}{R}\right)^4\right]_0^1 = 5.00 \text{ ft/s}$$

U

{ The speed of the uniform inlet flow is half the maximum speed at the outlet section. }

Problem 4.24

The velocity profile for laminar flow in an annulus is given by

$$u(r) = -\frac{\Delta p}{4\mu L}\left[R_o^2 - r^2 + \frac{R_o^2 - R_i^2}{\ln(R_i/R_o)}\ln\frac{R_o}{r}\right]$$

where $\Delta p/L = -10$ kPa/m is the pressure gradient, μ is the viscosity (SAE 10 oil at 20°C), and $R_o = 5$ mm and $R_i = 1$ mm are the outer and inner radii. Find the volume flow rate, the average velocity, and the maximum velocity. Plot the velocity distribution.

Given: Velocity distribution in annulus

Find: Volume flow rate; average velocity; maximum velocity; plot velocity distribution

Solution

Governing equation

$$Q = \int \vec{V} \cdot d\vec{A} \qquad V_{av} = \frac{Q}{A}$$

The given data is $R_o = 5\cdot mm$ $\qquad R_i = 1\cdot mm$ $\qquad \frac{\Delta p}{L} = -10\cdot\frac{kPa}{m}$

$$\mu = 0.1\cdot\frac{N\cdot s}{m^2} \qquad \text{(From Fig. A.2)}$$

$$u(r) = -\frac{\Delta p}{4\cdot\mu\cdot L}\cdot\left[R_o^2 - r^2 + \frac{R_o^2 - R_i^2}{\ln\left(\frac{R_i}{R_o}\right)}\cdot\ln\left(\frac{R_o}{r}\right)\right]$$

The flow rate is given by

$$Q = \int_{R_i}^{R_o} u(r)\cdot 2\cdot \pi \cdot r\, dr$$

Considerable mathematical manipulation leads to

$$Q = \frac{\Delta p\cdot \pi}{8\cdot \mu\cdot L}\cdot\left(R_o^{\,2} - R_i^{\,2}\right)\cdot\left[\frac{\left(R_o^{\,2} - R_i^{\,2}\right)}{\ln\left(\frac{R_o}{R_i}\right)} - \left(R_i^{\,2} + R_o^{\,2}\right)\right]$$

Substituting values

$$Q = \frac{\pi}{8}\cdot\left(-10\cdot 10^3\right)\cdot\frac{N}{m^2\cdot m}\cdot\frac{m^2}{0.1\cdot N\cdot s}\cdot\left(5^2 - 1^2\right)\cdot\left(\frac{m}{1000}\right)^2\cdot\left[\frac{\left(5^2 - 1^2\right)}{\ln\left(\frac{5}{1}\right)} - \left(5^2 + 1^2\right)\right]\cdot\left(\frac{m}{1000}\right)^2$$

$$Q = 1.045\times 10^{-5}\,\frac{m^3}{s} \qquad\qquad Q = 10.45\,\frac{mL}{s}$$

The average velocity is

$$V_{av} = \frac{Q}{A} = \frac{Q}{\pi\cdot\left(R_o^{\,2} - R_i^{\,2}\right)}$$

$$V_{av} = \frac{1}{\pi}\times 1.045\times 10^{-5}\,\frac{m^3}{s}\times\frac{1}{5^2 - 1^2}\cdot\left(\frac{1000}{m}\right)^2 \qquad\qquad V_{av} = 0.139\,\frac{m}{s}$$

The maximum velocity occurs when $\frac{du}{dr} = 0$

$$\frac{du}{dr} = \frac{d}{dx}\left[\frac{-\Delta p}{4 \cdot \mu \cdot L}\cdot\left(R_o^{\,2} - r^2 - \frac{R_o^{\,2} - R_i^{\,2}}{\ln\left(\frac{R_i}{R_o}\right)}\cdot\ln\left(\frac{R_o}{r}\right)\right)\right] = -\frac{\Delta p}{4 \cdot \mu \cdot L}\left[-2 \cdot r - \frac{R_o^{\,2} - R_i^{\,2}}{\ln\left(\frac{R_i}{R_o}\right)\cdot r}\right] = 0$$

$$r = \sqrt{\frac{R_i^{\,2} - R_o^{\,2}}{2 \cdot \ln\left(\frac{R_i}{R_o}\right)}} \qquad r = 2.73\,\text{mm}$$

Substituting in u(r) $\quad u_{max} = u(2.73 \cdot \text{mm}) = 0.213 \cdot \frac{m}{s}$

The maximum velocity, and the plot, are also shown in the corresponding *Excel* workbook

Problem 4.24 (In Excel)

The velocity profile for laminar flow in an annulus is given by

$$u(r) = -\frac{\Delta p}{4\mu L}\left[R_o^2 - r^2 + \frac{R_o^2 - R_i^2}{\ln(R_i/R_o)}\ln\frac{R_o}{r}\right]$$

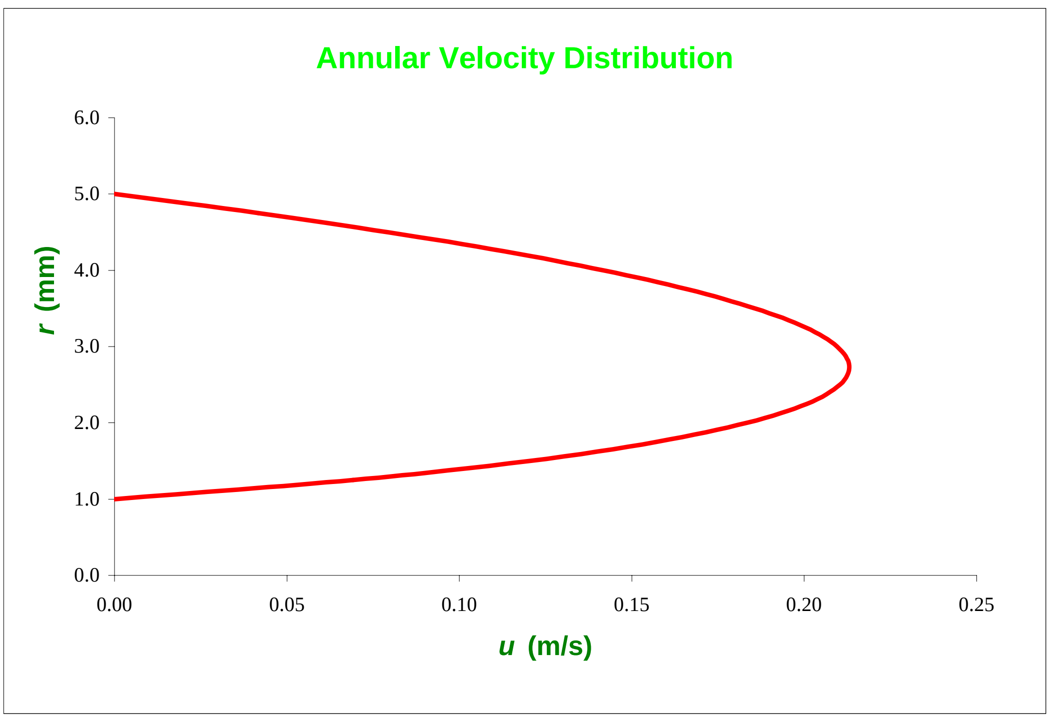
where $\Delta p/L = -10$ kPa/m is the pressure gradient, μ is the viscosity (SAE 10 oil at 20°C), and $R_o = 5$ mm and $R_i = 1$ mm are the outer and inner radii. Find the volume flow rate, the average velocity, and the maximum velocity. Plot the velocity distribution.

Given: Velocity distribution in annulus
Find: Maximum velocity; plot velocity distribution

Solution

R_o = 5 mm
R_i = 1 mm
$\Delta p/L$ = -10 kPa/m
μ = 0.1 $N.s/m^2$

r (mm)	*u* (m/s)
1.00	0.000
1.25	0.069
1.50	0.120
1.75	0.157
2.00	0.183
2.25	0.201
2.50	0.210
2.75	0.213
3.00	0.210
3.25	0.200
3.50	0.186
3.75	0.166
4.00	0.142
4.25	0.113
4.50	0.079
4.75	0.042
5.00	0.000



The maximum velocity can be found using *Solver*

r (mm)	*u* (m/s)
2.73	0.213

Problem 4.25

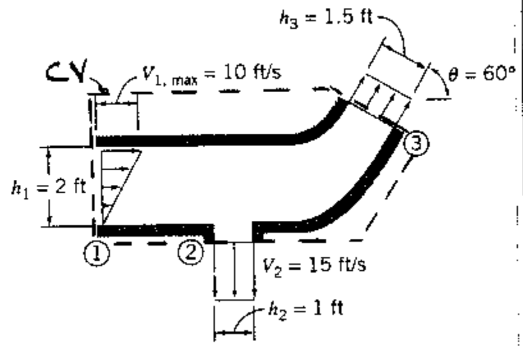
Given: Two-dimensional reducing bend as shown.

Find: Magnitude and direction of uniform velocity at section ③.

Solution: Apply conservation of mass using CV shown.

Basic equation:

$$0 = \cancelto{=0(1)}{\frac{\partial}{\partial t}}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$



Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Uniform flow at ② and ③

Then

$$0 = \int_{CS} \vec{V} \cdot d\vec{A} = \int_{A_1} \vec{V}_1 \cdot d\vec{A}_1 + \vec{V}_2 \cdot \vec{A}_2 + \vec{V}_3 \cdot \vec{A}_3$$

or

$$\vec{V}_3 \cdot \vec{A}_3 = -\int_{A_1} \vec{V}_1 \cdot d\vec{A}_1 - \vec{V}_2 \cdot \vec{A}_2 = +\int_0^{h_1} V_{1,max} \frac{y}{h_1} w \, dy - V_2 w h_2$$

$$\vec{V}_3 \cdot \vec{A}_3 = V_{1,max} w \left[\frac{y^2}{2h_1} \right]_0^{h_1} - V_2 w h_2 = \frac{V_{1,max} w h_1}{2} - V_2 w h_2$$

so

$$\frac{\vec{V}_3 \cdot \vec{A}_3}{w} = \frac{1}{2} \times 10 \frac{ft}{s} \times 2 ft - 15 \frac{ft}{s} \times 1 ft = -5 \, ft^2/s$$

Since $\vec{V}_3 \cdot \vec{A}_3 < 0$, flow at ③ is into the CV ← Direction

Thus

$$\frac{\vec{V}_3 \cdot \vec{A}_3}{w} = -\frac{V_3 A_3}{w} = -\frac{V_3 w h_3}{w} = -V_3 h_3 = -5 \, ft^2/s$$

$$V_3 = \frac{1}{h_3} \times 5 \frac{ft^2}{s} = \frac{1}{1.5 \, ft} \times 5 \frac{ft^2}{s} = 3.33 \, ft/s \quad \text{(into CV)}$$ ← V_3

Problem 4.26

Given: Water flow in the two-dimensional square channel shown.

$v_{max} = 2v_{min}$, $U = 7.5$ m/s , $h = 75.5$ mm

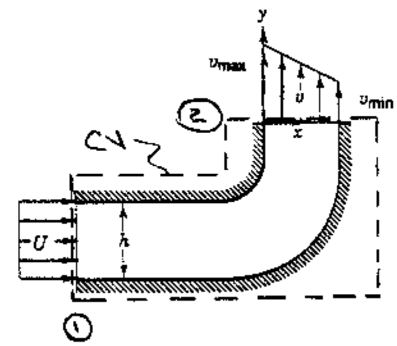
Find: v_{min}

Solution: Apply conservation of mass to the CV shown.

Basic equation:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$

(1) [term crossed out]



Assumptions: (1) steady flow
(2) incompressible flow
(3) uniform flow at section ①

Then

$$0 = \vec{V}_1 \cdot \vec{A}_1 + \int \vec{V}_2 \cdot d\vec{A}_2$$

$$0 = -Uwh + \int_0^h v \, w \, dx$$

The velocity distribution across the exit at ② is linear

$$v_2 = v_{max} - (v_{max} - v_{min})\frac{x}{h} = 2v_{min} - v_{min}\frac{x}{h} = v_{min}\left(2 - \frac{x}{h}\right)$$

$$\therefore Uwh = \int_0^h v_{min}\left(2 - \frac{x}{h}\right) w \, dx = v_{min} \, w \left[2x - \frac{x^2}{2h}\right]_0^h$$

$$U \not{w} \not{h} = v_{min} \, w \left[2h - \frac{h}{2}\right] = \frac{3}{2} v_{min} \not{w} \not{h}$$

$$\therefore v_{min} = \frac{2}{3} U = \frac{2}{3} \times 7.5 \frac{m}{s} = 5.0 \text{ m/s}$$

v_{min}

Problem 4.27

Given: Water flows in a porous round tube of diameter $D = 60$ mm. At the pipe inlet the flow is uniform with $V_1 = 7.0$ m/sec. Flow out through the porous wall is radial and axisymmetric with velocity distribution

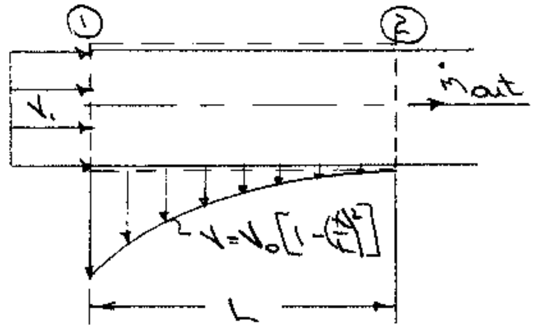
$$v = V_0\left[1 - \left(\frac{x}{L}\right)^2\right]$$

where $V_0 = 0.03$ m/s and $L = 0.950$ m.

Find: the mass flow rate, $\dot{m}_2$, inside the tube at $x = L$

Solution:

Basic equation: $0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$ (the $\frac{\partial}{\partial t}$ term is crossed out: = 0(1))



Assumptions: (1) steady flow
(2) ρ = constant

Then

$$0 = \int_{A_1} \rho \vec{V} \cdot d\vec{A} + \int_{A_2} \rho \vec{V} \cdot d\vec{A} + \int_{A_{circ}} \rho \vec{V} \cdot d\vec{A}$$

$$= -|\rho V_1 A_1| + \dot{m}_2 + \int_0^L \rho V_0\left[1 - \left(\frac{x}{L}\right)^2\right] 2\pi R \, dx$$

$$\dot{m}_2 = \rho V_1 A_1 - 2\pi R \rho V_0 \int_0^L \left[1 - \frac{x^2}{L^2}\right] dx$$

$$= \rho V_1 \frac{\pi D_1^2}{4} - 2\pi R \rho V_0 \left[x - \frac{x^3}{3L^2}\right]_0^L$$

$$= \frac{\pi}{4} \rho V_1 D_1^2 - \frac{4}{3}\pi R \rho V_0 L$$

$$\dot{m}_2 = \frac{\pi}{4} \times 999 \frac{kg}{m^3} \times 7.0 \frac{m}{s} \times (0.06)^2 m^2 - \frac{4}{3}\pi \times 0.03m \times 999 \frac{kg}{m^3} \times 0.03\frac{m}{s} \times 0.95m$$

$$\dot{m}_2 = 19.8 \frac{kg}{s} - 3.6 \frac{kg}{s} = 16.2 \text{ kg/s}$$ ← $\dot{m}_{out}$

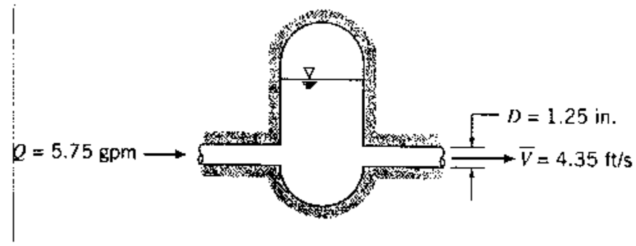
Problem 4.28

Given: A hydraulic accumulator, designed to reduce pressure pulsations in a hydraulic system, is operating under conditions shown, at a given instant.

Find: Rate at which accumulator gains or loses hydraulic oil.

Solution:

Use the control volume shown



Basic equation:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$

Assumptions: (1) uniform flow at section ②

(2) ρ = constant

Then,

$$0 = \frac{\partial}{\partial t}(M_{CV}) + \int_{A_1}\{-|\rho V_1 dA_1|\} + \int_{A_2}\{|\rho V_2 dA_2|\}$$

But $\int_{A_1} \rho V_1 dA_1 = \rho Q_1$ where Q = volume flowrate and $\rho = SG \, \rho_{H_2O}$

So $0 = \frac{\partial}{\partial t} M_{CV} - \rho Q_1 + \rho V_2 A_2$

or

$$\frac{\partial M_{CV}}{\partial t} = \rho (Q_1 - V_2 A_2)$$

$$= SG \, \rho_{H_2O} \left(Q_1 - V_2 \pi \frac{D_2^2}{4} \right) \quad \text{where } SG = 0.88 \text{ (Table A.2)}$$

$$= 0.88 \times 1.94 \frac{slug}{ft^3}\left[5.75 \frac{gal}{min} \times \frac{ft^3}{7.48 gal} \times \frac{min}{60 s} - 4.35 \frac{ft}{s} \times \frac{\pi}{4} \times (1.25)^2 in^2 \times \frac{ft^2}{144 in^2}\right]$$

$$\frac{\partial M_{CV}}{\partial t} = -4.14 \times 10^{-2} \frac{slug}{s} \quad \text{or} \quad -1.33 \ lbm/s \qquad \longleftarrow \frac{\partial M_{CV}}{\partial t}$$

(mass is decreasing in the CV)

Since $M_{CV} = \rho_{oil} \forall_{oil}$

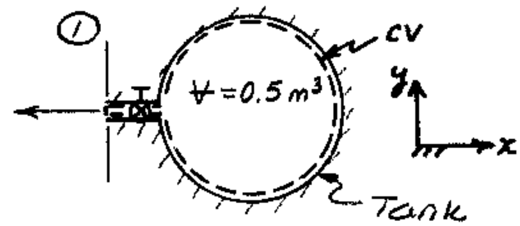
$$\frac{\partial M_{CV}}{\partial t} = \frac{\partial}{\partial t}(\rho_{oil} \forall_{oil}) = \rho_{oil} \frac{\partial \forall_{oil}}{\partial t} = SG_{oil} \, \rho_{H_2O} \frac{\partial \forall_{oil}}{\partial t}$$

$$\frac{\partial \forall_{oil}}{\partial t} = \frac{1}{SG_{oil} \, \rho_{H_2O}} \frac{\partial M_{CV}}{\partial t} = \frac{1}{0.88} \times \frac{ft^3}{1.94 \ slugs} \times (-4.14) \times 10^{-2} \frac{slug}{s}$$

$$\frac{\partial \forall_{oil}}{\partial t} = -2.43 \times 10^{-2} \frac{ft^3}{s} \quad \text{or} \quad 0.181 \ gal/s \qquad \longleftarrow \frac{\partial \forall_{oil}}{\partial t}$$

Problem 4.29

Given: Tank shown in sketch with air escaping.



$\vec{V}_1 = -300\,\hat{i}\ m/s$

$\vec{A}_1 = -130\,\hat{i}\ mm^2$

$T_1 = -15\,C$

$p_1 = 350\ kPa\ (abs)$

Find: Rate of change of density in tank.

Solution: Apply conservation of mass using CV shown

Basic equation: $$0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A}$$

Assume: (1) Density is uniform in tank, so $\frac{\partial}{\partial t}\int_{CV} \rho\, d\forall = \frac{\partial}{\partial t}(\rho_t \forall)$

(2) Exit flow is uniform, so

$$\int_{CS} \rho \vec{V}\cdot d\vec{A} = \rho_1 \vec{V}_1 \cdot \vec{A}_1 = |\rho_1 V_1 A_1|$$

(3) Air is an ideal gas, $p_1 = \rho_1 R T_1$

Then

$$\rho_1 = \frac{p_1}{RT_1} = 3.50\times 10^5\,\frac{N}{m^2}\times\frac{kg\cdot K}{287\,N\cdot m}\times\frac{1}{(273-15)\,K} = 4.73\ kg/m^3$$

$$\frac{\partial}{\partial t}(\rho_t \forall) = \rho_t \frac{\partial \forall}{\partial t}^{\nearrow 0} + \forall \frac{\partial \rho_t}{\partial t} = -|\rho_1 V_1 A_1|$$

$$\frac{\partial \rho_t}{\partial t} = \frac{-|\rho_1 V_1 A_1|}{\forall} = -4.73\,\frac{kg}{m^3}\times 300\,\frac{m}{s}\times 130\,mm^2\times\frac{1}{0.5\,m^3}\times\frac{m^2}{(1000)^2\,mm^2}$$

$$\frac{\partial \rho_t}{\partial t} = -0.369\ kg/m^3\cdot s$$

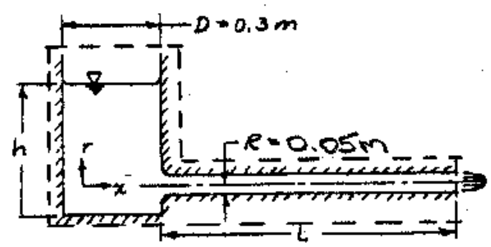
$\frac{\partial \rho_t}{\partial t}$

{ Note since $\frac{\partial \rho_t}{\partial t} < 0$, mass in tank decreases as expected. }

Problem 4.30

Given: Liquid drains from a tank through a long circular tube. Flow is laminar; velocity profile at tube discharge is given by

$$u = U_{max}\left[1 - \left(\frac{r}{R}\right)^2\right]$$



Find: (a) Show that $\bar{V} = 0.5\,U_{max}$ at any instant
(b) rate of change of liquid level in tank when $U_{max} = 0.155$ m/s

Solution:

(a) The average velocity $\bar{V}$ is defined as Q/A.

Since $Q = \int u\,dA$, $dA = 2\pi r\,dr$ and $A = \pi R^2$, then

$$\bar{V} = \frac{Q}{A} = \frac{1}{\pi R^2}\int_0^R U_{max}\left[1-\left(\frac{r}{R}\right)^2\right]2\pi r\,dr = \frac{2U_{max}}{R^2}\int_0^R\left[1-\left(\frac{r}{R}\right)^2\right]r\,dr$$

$$\bar{V} = \frac{2U_{max}}{R^2}R^2\int_0^1\left[1-\left(\frac{r}{R}\right)^2\right]\left(\frac{r}{R}\right)d\left(\frac{r}{R}\right) = 2U_{max}\left[\frac{1}{2}\left(\frac{r}{R}\right)^2 - \frac{1}{4}\left(\frac{r}{R}\right)^4\right]_0^1$$

$$\bar{V} = \frac{1}{2}U_{max} \quad \longleftarrow \bar{V}$$

(b) Apply conservation of mass to the CV shown

Basic equation: $0 = \frac{\partial}{\partial t}\int_{CV}\rho\,d\forall + \int_{CS}\rho\vec{V}\cdot d\vec{A}$

Assumptions: (1) neglect air entering the CV
(2) incompressible flow

Then

$$0 = \rho_L\frac{\partial}{\partial t}\forall_{CV} + \left\{\left|\rho_L\bar{V}A_e\right|\right\} = \rho_L\frac{\partial}{\partial t}\left[\frac{\pi D^2}{4}h + L\pi R^2\right] + \rho\bar{V}\pi R^2$$

$$0 = \frac{\pi D^2}{4}\frac{dh}{dt} + \bar{V}\pi R^2 \qquad \left(\text{note } \frac{dL}{dt} = 0\right)$$

$$\therefore \frac{dh}{dt} = -4\bar{V}\left(\frac{R}{D}\right)^2 \qquad \text{But } \bar{V} = \frac{1}{2}U_{max} \text{ and hence}$$

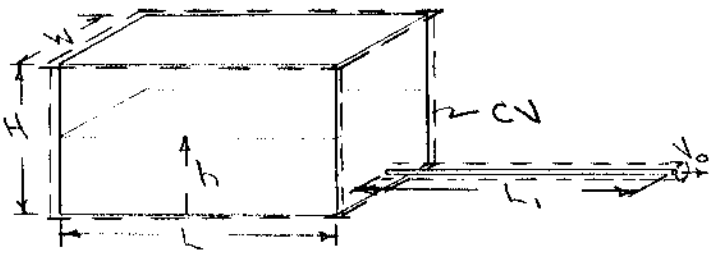
$$\frac{dh}{dt} = -2U_{max}\left(\frac{R}{D}\right)^2 = -2\times 0.155\frac{m}{s}\times\left(\frac{0.05\,m}{0.30\,m}\right)^2\times 1000\frac{mm}{m}$$

$$\frac{dh}{dt} = -8.61 \text{ mm/s} \quad (\text{level is falling}) \quad \longleftarrow \frac{dh}{dt}$$

Problem 4.31

Given: Rectangular tank with dimensions $H = 230$ mm, $W = 150$ mm, $L = 230$ mm, supplies water to an outlet tube of diameter, $D = 6.35$ mm. When the tank is half full the flow in the tube is at Reynolds number $Re = 2000$. At this instant there is no water flow into the tank.

Find: the rate of change of water level in the tank at this instant.



Solution:

Apply conservation of mass to CV which includes tank and tube.

Basic equation:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$

Definition: $Re = \frac{\rho D \bar{V}}{\mu} = \frac{D\bar{V}}{\nu}$

Assumptions: (1) uniform flow at exit of tube
(2) incompressible flow
(3) neglect air entering the control volume

Then,

$$0 = \frac{\partial}{\partial t}\left[\rho W L h + \rho \pi \frac{D^2}{4} L_1\right] + \left\{ +\left| \rho \bar{V}_o \frac{\pi D^2}{4} \right| \right\}$$

$$0 = W L \frac{dh}{dt} + \bar{V}_o \pi \frac{D^2}{4} \qquad (\text{note } L_1 = \text{constant})$$

$$\therefore \quad \frac{dh}{dt} = -\bar{V}_o \frac{\pi D^2}{4 W L}$$

To find $\bar{V}$ use the definition of Re

$$\bar{V}_o = \frac{Re\, \nu}{D}$$

For water at 20C $\nu = 1 \times 10^{-6}$ m²/sec (Table A.8)

$$V_o = 2000 \times 1 \times 10^{-6} \frac{m^2}{sec} \times \frac{1}{6.35 \times 10^{-3}\, m} = 0.315 \text{ m/sec}$$

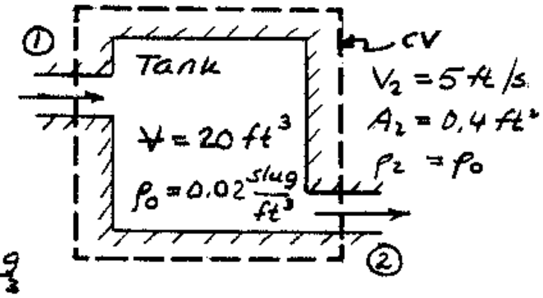
$$\frac{dh}{dt} = -\bar{V}_o \frac{\pi D^2}{4WL} = -\frac{0.315}{4} \frac{m}{sec} \times \frac{\pi (6.35)^2 \, mm^2}{150\, mm \times 230\, mm} \times 10^3 \frac{mm}{m}$$

$$\frac{dh}{dt} = -0.289 \text{ mm/sec} \quad (\text{falling})$$

$\frac{dh}{dt}$

Given: Air flow through tank with conditions shown at time, t_0.

$V_1 = 15$ ft/s
$A_1 = 0.2$ ft^2
$\rho_1 = 0.03 \frac{\text{slug}}{\text{ft}^3}$



Find: $\frac{\partial \rho}{\partial t}$ in tank at time, t_0.

Solution: Apply conservation of mass, using CV shown.

Basic equation: $0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$

Assumptions: (1) Density is uniform in tank, so $\frac{\partial}{\partial t}\int_{CV} \rho \, d\forall = \frac{\partial}{\partial t}(\rho_0 \forall)$

(2) Flow is uniform at inlet and outlet sections.

Then

$$0 = \frac{\partial}{\partial t}(\rho_0 \forall) + \rho_1 \vec{V}_1 \cdot \vec{A}_1 + \rho_0 \vec{V}_2 \cdot \vec{A}_2$$

$$0 = \rho_0 \underbrace{\frac{\partial \forall}{\partial t}}_{=0} + \forall \frac{\partial \rho_0}{\partial t} - |\rho_1 V_1 A_1| + |\rho_0 V_2 A_2|$$

or

$$\frac{\partial \rho_0}{\partial t} = \frac{|\rho_1 V_1 A_1| - |\rho_0 V_2 A_2|}{\forall}$$

Substituting magnitudes

$$\frac{\partial \rho_0}{\partial t} = \frac{1}{20 \text{ ft}^3}\left[\frac{0.03 \text{ slug}}{\text{ft}^3} \times \frac{15 \text{ ft}}{\text{s}} \times 0.2 \text{ ft}^2 - \frac{0.02 \text{ slug}}{\text{ft}^3} \times \frac{5 \text{ ft}}{\text{s}} \times 0.4 \text{ ft}^2\right]$$

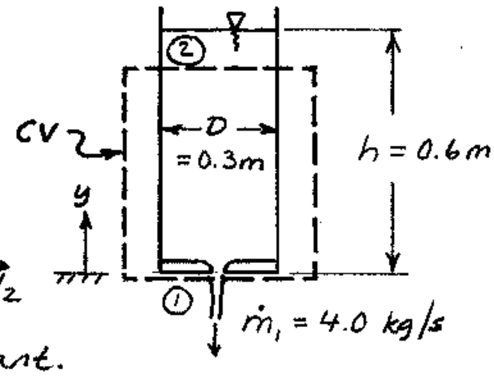
$$\frac{\partial \rho_0}{\partial t} = 2.50 \times 10^{-3} \text{ slug/(ft}^3 \cdot \text{s)}$$

$\frac{\partial \rho_0}{\partial t}$

{Note since $\frac{\partial \rho_0}{\partial t} > 0$, mass in tank increases.}

Problem 4.33

Given: Circular tank, with D = 1 ft draining through a hole in its bottom. Fluid is water



Find: Rate of change of water level at the instant shown.

Solution: Apply conservation of mass to CV shown. Note section ② cuts below free surface, so $\vec{V}_2$ corresponds to free surface velocity; volume of CV is constant.

Basic equation:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$

(unsteady term $= 0(1)$)

Assumptions: (1) Incompressible flow, so unsteady term is zero, since volume of CV is fixed
(2) Uniform flow at each section

Then

$$0 = \rho \vec{V}_1 \cdot \vec{A}_1 + \rho \vec{V}_2 \cdot \vec{A}_2 = \dot{m}_1 + \rho \vec{V}_2 \cdot \vec{A}_2$$

and

$$\vec{V}_2 \cdot \vec{A}_2 = -\frac{\dot{m}_1}{\rho} = -4.0 \frac{kg}{s} \times \frac{m^3}{999\ kg} = -0.004\ m^3/s$$

Since $\vec{V}_2 \cdot \vec{A}_2 < 0$, flow at section ② is <u>into</u> CV. Therefore

$$V_2 = \frac{|V_2 A_2|}{A_2} = 0.004 \frac{m^3}{s} \times \frac{4}{\pi} \times \frac{1}{(0.3)^2 m^2} = 0.0566\ m/s$$

The water level is <u>falling</u> at 56.6 mm/s.

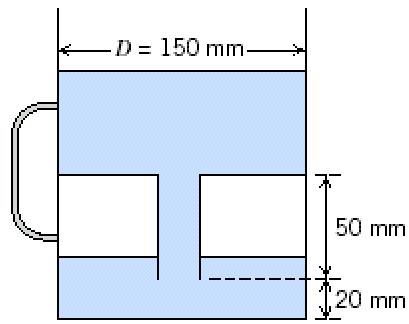
$$\vec{V}_s = -V_2 \hat{j} = -56.6 \hat{j}\ mm/s$$

$\vec{V}_s$

Problem 4.34

A home water filter container as shown is initially completely empty. The upper chamber is no
filled to a depth of 80 mm with water. How long will it take the lower chamber water level to ju
touch the bottom of the filter? How long will it take for the water level in the lower chamber to
reach 50 mm? Note that both water surfaces are at atmospheric pressure, and the filter material
itself can be assumed to take up none of the volume. Plot the lower chamber water level as a
Q = *kH* where *k* = 2x10^{-4} m^2/s and *H*
(m) is the net hydrostatic head across the filter.

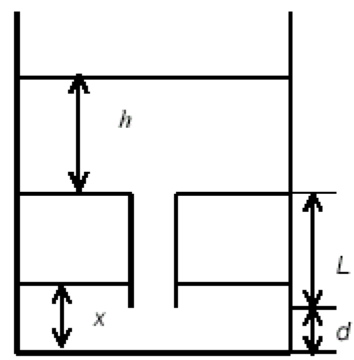
Given: Geometry on water filter



Solution

Given data $Q = k \cdot H$ where $k = 2 \cdot 10^{-4} \cdot \frac{m^2}{s}$

Let the instantaneous depth of water in the upper chamber be *h*; let the filter height be *L*; let the gap between the filter and the bottom be *d*; and let the level in the lower chamber be *x*.



Then $h(t = 0) = h_0$ $h_0 = 80 \cdot mm$ $x(t = 0) = 0$

$L = 50 \cdot mm$ $d = 20 \cdot mm$ $D = 150 \cdot mm$

Governing equation For the flow rate out of the upper chamber

$$Q = -A \cdot \frac{dh}{dt} = k \cdot H$$

where A is the cross-section area $A = \frac{\pi \cdot D^2}{4}$ $A = 0.0177\,m^2$

There are two flow regimes: before the lower chamber water level reaches the bottom of the filte
and after this point

(a) First Regime: water level in lower chamber not in contact with filter, $x < d$

The head H is given by $H = h + L$

Hence the governing equation becomes

$$-A \cdot \frac{dh}{dt} = H = h + L$$

Separating variables $\frac{dh}{h + L} = -\frac{dt}{A}$

Integrating and using the initial condition $h = h_0$

$$h = (h_0 + L) \cdot e^{-\frac{k}{A} \cdot t} - L$$

Note that the initial condition is satisfied, and that as time increases h approaches $-L$, that is, upper chamber AND filter completely drain

We must find the instant that the lower chamber level reaches the bottom of the filter

Note that the increase in lower chamber level is equal to the decrease in upper chamber level $A \cdot x = A \cdot (h_0 - h)$

so $$x = h_0 - h = h_0 - \left[(h_0 + L) \cdot e^{-\frac{k}{A} \cdot t} - L \right]$$

$$x = (h_0 + L) \cdot \left(1 - e^{-\frac{k}{A} \cdot t} \right)$$

Hence we need to find when $x = d$, or

$$d = (h_0 + L)\cdot\left(1 - e^{-\frac{k}{A}\cdot t}\right)$$

Solving for t

$$t = -\frac{A}{k}\cdot\ln\left(1 - \frac{d}{h_0 + L}\right)$$

$$t = -0.0177\cdot m^2 \times \frac{s}{2\cdot 10^{-4} \times m^2} \times \ln\left(1 - \frac{20}{80 + 50}\right)$$

$$t = 14.8\,s$$

(a) Second Regime:water level in lower chamber in contact with filter, $x > d$

The head H is now given by $H = h + L + d - x$

Note that the increase in lower chamber level is equal to the decrease in upper chamber level

$$A\cdot x = A\cdot(h_0 - h) \qquad \text{so} \qquad x = h_0 - h$$

Hence the governing equation becomes

$$-A\cdot\frac{dh}{dt} = H = h + L + d - x = 2\cdot h + L + d - h_0$$

Separating variables

$$\frac{dh}{2\cdot h + L + d - h_0} = -\frac{dt}{A}$$

Before integrating we need an initial condition for this regime

Let the time at which $x = d$ be $t_1 = 14.8$ s

Then the initial condition is $h = h_0 - x = h_0 - d$

Integrating and using this IC yields eventually

$$h = \frac{1}{2}\left(h_0 + L - d\right) e^{-\frac{2 \cdot k}{A} \cdot (t - t_1)} - \frac{1}{2} \cdot \left(L + d - h_0\right)$$

or

$$x = \frac{1}{2} \cdot \left(L + d + h_0\right) - \frac{1}{2} \cdot \left(h_0 + L - d\right) \cdot e^{-\frac{2 \cdot k}{A} \cdot (t - t_1)}$$

Note that the start of Regime 2 ($t = t_1$), $x = d$, which is correct.

We must find the instant that the lower chamber level reaches a level of 50 mm

Let this point be $x = x_{end} = 50 \cdot mm$

We must solve $x_{end} = \frac{1}{2} \cdot \left(L + d + h_0\right) - \frac{1}{2} \cdot \left(h_0 + L - d\right) \cdot e^{-\frac{2 \cdot k}{A} \cdot (t - t_1)}$

Solving for t $t = -\frac{A}{2 \cdot k} \cdot \ln\left(\frac{L + d + h_0 - 2 \cdot x_{end}}{h_0 + L - d}\right) + t_1$

$t = 49.6\,s$

The complete solution for the lower chamber water level is

$$x = \left(h_0 + L\right)\cdot\left(1 - e^{-\frac{k}{A}\cdot t}\right) \qquad x < d$$

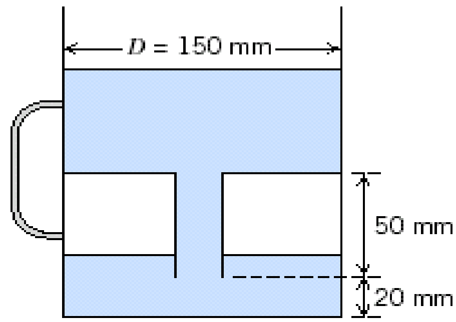
$$x = \frac{1}{2}\cdot\left(L + d + h_0\right) - \frac{1}{2}\cdot\left(h_0 + L - d\right)\cdot e^{-\frac{2\cdot k}{A}\cdot\left(t - t_1\right)} \qquad x > d$$

The solution is plotted in the corresponding *Excel* workbook; in addition, *Goal Seek* is used to find the two times asked for

Problem 4.34 (In Excel)

A home water filter container as shown is initially completely empty. The upper chamber is now filled to a depth of 80 mm with water. How long will it take the lower chamber water level to just touch the bottom of the filter? How long will it take for the water level in the lower chamber to reach 50 mm? Note that both water surfaces are at atmospheric pressure, and the filter material itself can be assumed to take up none of the volume. Plot the lower chamber water level as a function of time.

For the filter, the flow rate is given by $Q = kH$ where $k = 2\text{x}10^{-4}$ m^2/s and H (m) is the net hydrostatic head across the filter.



Given: Geometry of water filter

Find: Times to reach various levels; plot lower chamber level

Solution

The complete solution for the lower chamber water level is

$$x \mid \ / h_0 \ 2 \ L() \, f^{M} \ 4 e^{4 \frac{k}{A} f} \quad ® © \qquad x \ \Omega \ d$$

$$x \mid \ \frac{1}{2} \, f L \ 2 \ d \ 2 \ h_0 () \ 4 \ \frac{1}{2} \, f h_0 \ 2 \ L \ 4 \ d() \, e^{4 \frac{2 \, k}{A} f t 4 t_1 ()} \qquad x \ \} \ d$$

h_o =	80	mm
d =	20	mm
L =	50	mm
D =	150	mm
k =	2.00E-04	m^2/s
A =	0.0177	mm
t_1 =	14.8	s

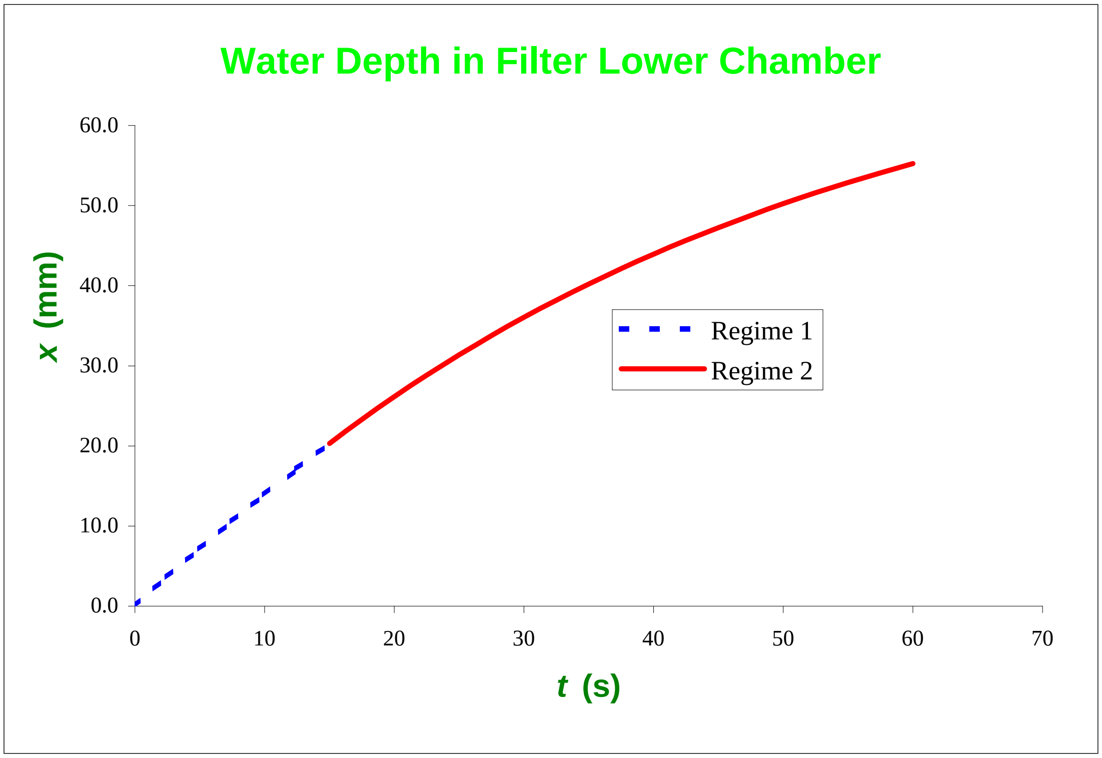
To find when $x = d$, use *Goal Seek*

t (s)	*x* (mm)
14.8	20.0

To find when x = 50 mm, use *Goal Seek*

t (s)	*x* (mm)
49.6	50

t (s)	*x* (mm)
0.0	0.0
2.5	3.6
5.0	7.2
7.5	10.6
10.0	13.9
12.5	17.1
15.0	20.3
17.5	23.3
20.0	26.2
22.5	28.8
25.0	31.4
27.5	33.8
30.0	36.0
32.5	38.2
35.0	40.2
37.5	42.1
40.0	43.9
42.5	45.6
45.0	47.3
47.5	48.8
50.0	50.2
52.5	51.6
55.0	52.9
57.5	54.1
60.0	55.2



Problem 4.35

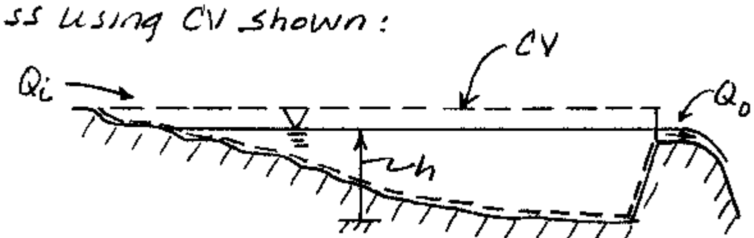
Given: Lake being drained at 2,000 cubic feet per second (cfs). Level falls at 1 ft per 8 hr. Normal flow rate is 290 cfs.

Find: (a) Actual flow rate during draining (gal/s).
(b) Estimate surface area of lake.

Solution: Convert units

$$Q = 2000\,\frac{ft^3}{s} = 2000\,\frac{ft^3}{s} \times 7.48\,\frac{gal}{ft^3} = 1.50 \times 10^4\ gal/s$$ Q

Apply conservation of mass using CV shown:



Basic equation: $0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$

Assumption: (1) ρ = constant

Then $\frac{d\forall}{dt} = A\frac{dh}{dt} = -\int_{CS} \vec{V} \cdot d\vec{A} = -Q_0 + Q_i$

$$A = -\frac{Q_0 - Q_i}{dh/dt} = -\frac{\Delta Q}{dh/dt}\ ;\ \Delta Q = Q_0 - Q_i$$

But $\Delta Q = 1{,}710\ ft^3/s$ and $dh/dt = -1\ ft/8\ hr$, since decreasing.

Thus

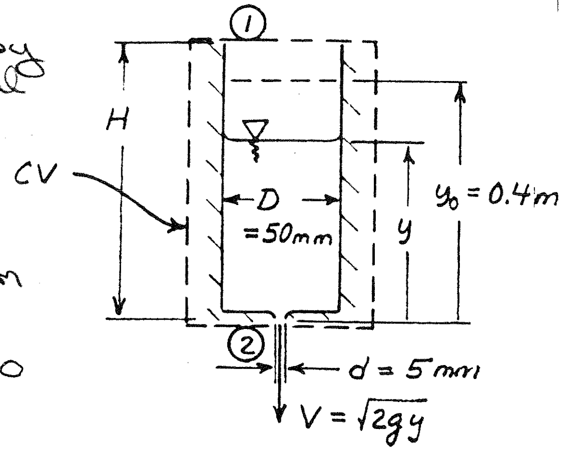
$$A = -1{,}710\,\frac{ft^3}{s} \times \frac{8\,hr}{-1\,ft} \times 3600\,\frac{s}{hr} = 4.92 \times 10^7\ ft^2$$ A

Since 1 acre = 43,600 ft^2,

$$A = 4.92 \times 10^7\ ft^2 \times \frac{acre}{43{,}600\ ft^2} \approx 1{,}130\ acres$$

Since 1 square mile = 640 acres, the lake surface area is slightly less than 2 square miles!

Given: Cylindrical tank, draining by gravity as shown; initial depth is y_0



Find: Water depth at $t = 12\,s$

Plot: (a) y/y_0 vs t for $0.1 \le y_0 \le 1m$ and $D/d = 10$

(b) y/y_0 vs t for $2 \le D/d \le 10$ and $y_0 = 0.4m$

Solution:

Apply conservation of mass using CV shown

Basic equation: $0 = \frac{\partial}{\partial t}\int_{CV} \rho\, dV + \int_{CS} \rho \vec{V} \cdot d\vec{A}$

Assumptions: (1) incompressible flow
(2) uniform flow at each section
(3) neglect ρ_{air} compared to ρ_{H_2O}

For the CV, $dV = A_t\, dy$, so

$$0 = \frac{\partial}{\partial t}\int_0^y \rho_{H_2O} A_t\, dy + \frac{\partial}{\partial t}\int_y^H \rho_{air} A_t\, dy + \{-|\rho_{air} V_1 A_t|\} + \{|\rho_{H_2O} V_2 A_2|\}$$

(terms with ρ_{air} ≈ 0 (3))

or

$$0 = \rho A_t \frac{dy}{dt} + \rho A_2 V_2 = A_t \frac{dy}{dt} + A_2 \sqrt{2gy}$$

Separating variables, $\frac{dy}{y^{1/2}} = -\sqrt{2g}\,\frac{A_2}{A_t}\, dt$

Integrating from y_0 at $t=0$ to y at t

$$\int_{y_0}^{y} y^{-1/2}\, dy = 2\left[y^{1/2} - y_0^{1/2}\right] = -\sqrt{2g}\,\frac{A_2}{A_t}\, t$$

$$\frac{y^{1/2}}{y_0^{1/2}} = 1 - \sqrt{\frac{g}{2y_0}}\,\frac{A_2}{A_t}\, t \quad \text{or} \quad y = y_0\left[1 - \sqrt{\frac{g}{2y_0}}\left(\frac{d}{D}\right)^2 t\right]^2 \quad \text{--- (1)}$$

At $t = 12$ sec

$$y = 0.4m\left[1 - \left(\frac{9.81}{2}\frac{m}{s^2} \times \frac{1}{0.4m}\right)^{1/2}\left(\frac{5mm}{50mm}\right)^2 12s\right]^2 = 0.134m \longleftarrow y_{t=12s}$$

For $D/d = 10$, Eq. 1 gives

$$\frac{y}{y_0} = \left[1 - 2.215 \times 10^{-2}\, y_0^{-1/2}\, t\right]^2$$

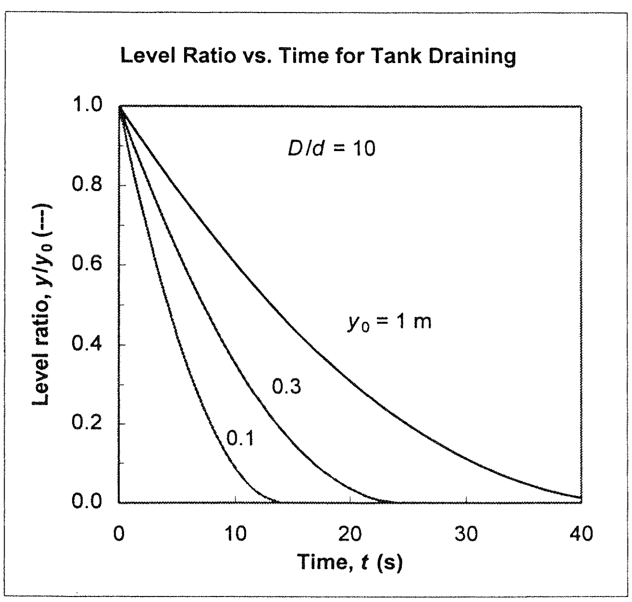
For $y_0 = 0.4\,\text{m}$, Eq. 1 gives

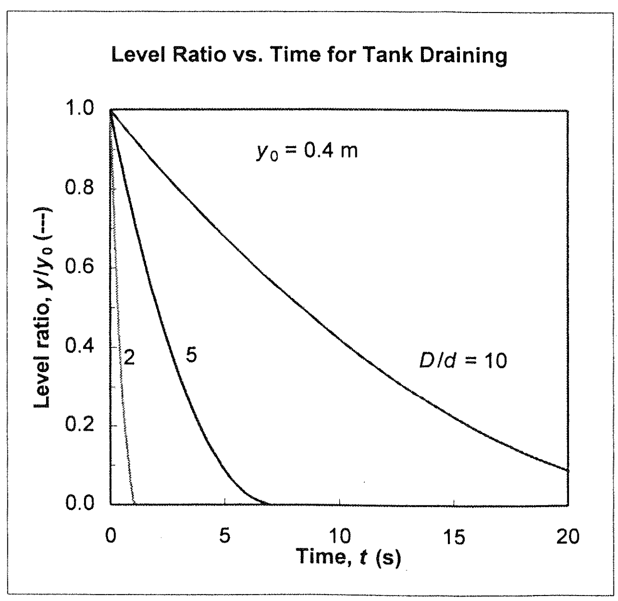
$$\frac{y}{y_0} = \left[1 - \frac{3.502}{(D/d)^2}\, t\right]^2$$

The variation of y/y_0 with t is plotted below for:

- $D/d = 10$ and $0.1 < y_0 \le 1.0\,\text{m}$
- $y_0 = 0.4\,\text{m}$ and $2 \le D/d \le 10$

y_0 (m) =	0.1	0.3	1	D/d (---) =	2	5	10
Time, t (s)	y/y_0 (---)	y/y_0 (---)	y/y_0 (---)	Time, t (s)	y/y_0 (---)	y/y_0 (---)	y/y_0 (---)
0	1.000	1.000	1.000	0	1.000	1.000	1.000
2	0.739	0.845	0.913	0.5	0.316	0.865	0.965
4	0.518	0.703	0.831	1	0.016	0.739	0.931
6	0.336	0.574	0.752	1.1	0.001	0.716	0.924
8	0.193	0.458	0.677	2		0.518	0.865
10	0.090	0.355	0.606	3		0.336	0.801
12	0.025	0.265	0.539	4		0.193	0.739
14	0.000	0.188	0.476	5		0.090	0.680
16		0.125	0.417	6		0.025	0.624
18		0.074	0.362	7		0.000	0.570
20		0.037	0.310	10			0.422
22		0.012	0.263	12			0.336
24		0.001	0.219	14			0.260
26			0.180	16			0.193
28			0.144	18			0.137
30			0.113	20			0.090
32			0.085	22			0.053
34			0.061	24			0.025
36			0.041	26			0.008
38			0.025	28			0.000
40			0.013				
45			0.000				

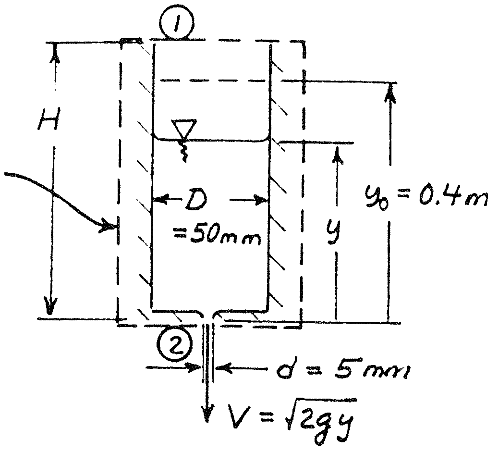




Given: Cylindrical tank, draining by gravity as shown; initial depth is y_0.

Find: Time to drain tank to depth $y = 20$ mm

Plot: Time t to drain the tank (to $y = 20$ mm) as a function of y/y_0 for $0.1 \le y_0 \le 1$ m with d/D as a parameter for $0.1 \le d/D \le 0.5$.



Solution:

Apply conservation of mass using CV shown.

Basic equation: $0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$

Assumptions: (1) incompressible flow
(2) uniform flow at each section.
(3) neglect ρ_{air} compared to ρ_{H_2O}

For the CV, $d\forall = A_t \, dy$, so

$$0 = \frac{\partial}{\partial t}\int_0^y \rho_{H_2O} A_t \, dy + \frac{\partial}{\partial t}\int_y^H \rho_{air} A_t \, dy + \{-|\rho_{air} V_1 A_t|\} + \{|\rho_{H_2O} V_2 A_2|\}$$

(the ρ_{air} terms ≈ 0 (3))

or

$$0 = \frac{\partial}{\partial t}\int_0^y \rho_{H_2O} A_t \, dy + \rho_{H_2O} V_2 A_2 = A_t \frac{dy}{dt} + A_2\sqrt{2gy}$$

Separating variables,

$$\frac{dy}{y^{1/2}} = -\sqrt{2g}\,\frac{A_2}{A_t}\,dt$$

Integrating from y_0 at $t=0$ to y at t

$$\int_{y_0}^{y} \frac{dy}{y^{1/2}} = 2\left[y^{1/2} - y_0^{1/2}\right] = -\sqrt{2g}\,\frac{A_2}{A_t}\,t$$

$$-\sqrt{2g}\,\frac{A_2}{A_t}\,t = 2y_0^{1/2}\left[\left(\frac{y}{y_0}\right)^{1/2} - 1\right] \quad \text{or} \quad t = \sqrt{\frac{2y_0}{g}}\left(\frac{D}{d}\right)^2\left[1 - \left(\frac{y}{y_0}\right)^{1/2}\right] \quad (1)$$

Evaluating at $y = 20$ mm

$$t = \left[2 \times 0.4\text{m} \times \frac{s^2}{9.81\text{m}}\right]^{1/2}\left[\frac{50\text{mm}}{5\text{mm}}\right]^2\left[1 - \left(\frac{0.02\text{m}}{0.40\text{m}}\right)^{1/2}\right] = 22.2\text{s} \quad \leftarrow t_{y=20mm}$$

Time t is plotted as a function of y/y_0 ($y = 20$ mm) with d/D as a parameter.

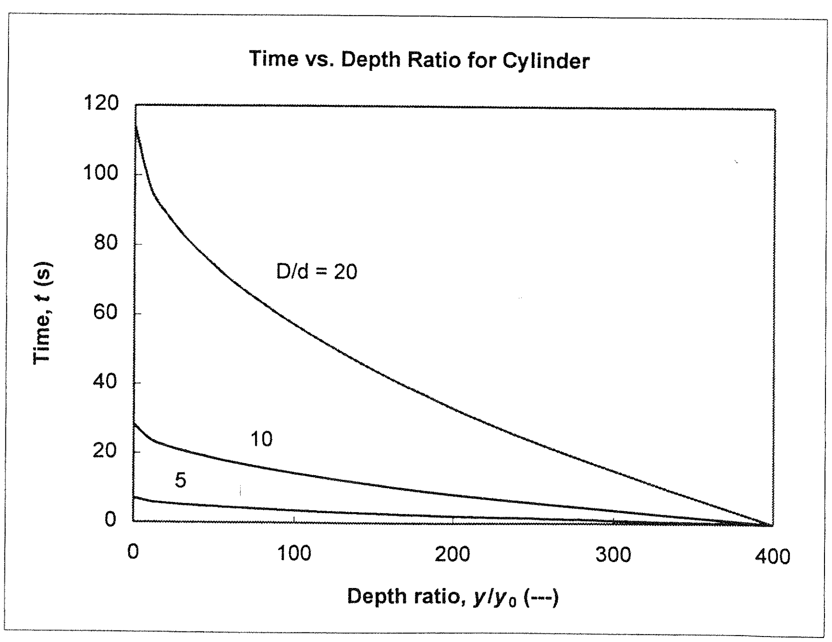
Draining of a cylindrical liquid tank:

Input Data:

Initial height:	y_0	0.4	m			
Diameter ratio:	D/d	20		10	5	---

Calculated Results:

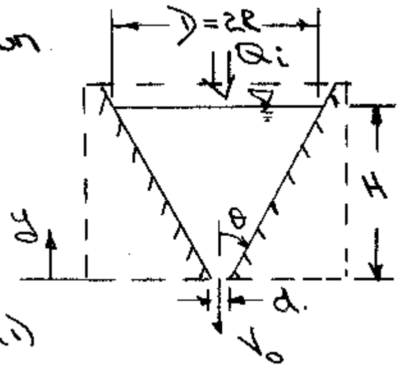
Level, y (mm)	D/d =	Time, t (s) 20	10	5
400		0	0	0
380		2.89	0.723	0.181
360		5.86	1.47	0.366
340		8.91	2.23	0.557
320		12.1	3.01	0.754
300		15.3	3.83	0.96
280		18.7	4.66	1.17
260		22.1	5.53	1.38
240		25.7	6.44	1.61
220		29.5	7.38	1.84
200		33.5	8.36	2.09
180		37.6	9.40	2.35
160		42.0	10.5	2.62
140		46.6	11.7	2.92
120		51.7	12.9	3.23
100		57.1	14.3	3.57
80		63.1	15.8	3.95
60		70.0	17.5	4.37
40		78.1	19.5	4.88
30		82.9	20.7	5.18
20		88.7	22.2	5.54
10		96.2	24.0	6.01
0		114	28.6	7.14



Problem 4.38

Given: Water flows into the top of a conical flask at a constant rate of $Q = 3.75 \times 10^{-7}\ m^3/hr$. Water drains out through the round opening of diameter $d = 7.35\ mm$ at the apex of the cone; the flow speed at the exit is $V = (2gy)^{1/2}$ where y is the water depth above the exit plane. At the instant of interest, the water depth $H = 36.8\ mm$ and the corresponding diameter $D = 29.4\ mm$

Find: At the instant of interest:
(a) find the volume flow rate from the bottom of the flask
(b) evaluate the direction and rate of change of water surface level



Solution: Apply continuity to the CV shown.

Basic eq.: $0 = \frac{\partial}{\partial t}\int_{cv} \rho\, d\forall + \int_{cs} \rho \vec{V} \cdot d\vec{A}$

Assumptions: (1) uniform flow at each section
(2) neglect mass of air.

Then

$$0 = \rho \left.\frac{d\forall}{dt}\right)_{water} + \rho Q_{out} - \rho Q_{in} \quad \text{----(1)}$$

$$Q_{out} = V_o A_o = (2gH)^{1/2}\ \frac{\pi d^2}{4}$$

$$Q_{out} = \left[2 \times 9.81\frac{m}{s^2} \times 0.0368\,m\right]^{1/2} \frac{\pi}{4} \times (0.00735)^2\ m^2$$

$$Q_{out} = 3.61 \times 10^{-5}\ m^3/s \quad (0.130\ m^3/hr) \longleftarrow Q_{out}$$

From eq. (1)

$$\left.\frac{d\forall}{dt}\right)_{water} = Q_{in} - Q_{out}$$

$$\forall = \frac{1}{3}\text{ area of base} \times \text{altitude} = \frac{1}{3}\pi R^2 y$$

Since $R = y\tan\theta$, $\forall = \frac{1}{3}\pi y^3 \tan^2\theta$

$$\frac{d\forall}{dt} = \frac{1}{3}\pi \tan^2\theta \times 3y^2\frac{dy}{dt} = \pi y^2 \tan^2\theta\frac{dy}{dt} = \pi R^2\frac{dy}{dt}$$

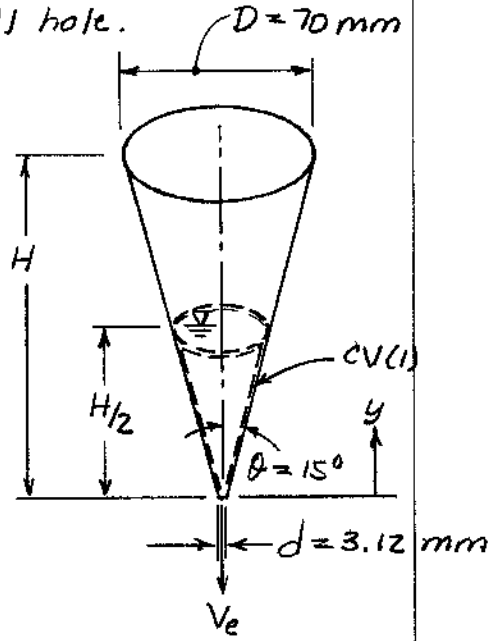
$$\therefore\ \frac{dy}{dt} = \frac{Q_{in} - Q_{out}}{\pi R^2} = \frac{4}{\pi D^2}(Q_{in} - Q_{out})$$

$$= \frac{4}{\pi} \times \frac{1}{(0.0294)^2\,m^2}\left(3.75 \times 10^{-7} - 0.130\right)\frac{m^3}{hr} \times \frac{hr}{3600\,s}$$

$$\frac{dy}{dt} = -0.0532\ m/s \quad \text{(surface moves downward)} \longleftarrow \frac{dy}{dt}$$

Given: Conical funnel draining through small hole.

$$V_e = \sqrt{2gy}$$



Find: Rate of change of surface level when $y = H/2$.

Solution: Apply conservation of mass.

(1) Choose CV with top just below surface level.

Basic equation: $0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$ ($\frac{\partial}{\partial t}$ term $= 0(1)$)

Assumptions: (1) ρ = constant, $\forall$ = const, so $\partial/\partial t = 0$
(2) Uniform flow at each section.

For CV(1): $0 = \{-|\rho V_s A_s|\} + \{+|\rho V_e A_e|\}$ or $V_s = V_e \frac{A_e}{A_s}$ (in; out)

Thus

$$V_s = V_e\left(\frac{d}{D/2}\right)^2 = \sqrt{2g\frac{H}{2}}\, 4\left(\frac{d}{D}\right)^2 = 4\sqrt{gH}\left(\frac{d}{D}\right)^2 = -\frac{dy}{dt} \text{ (since } y \text{ decreases)}$$

But $\tan\theta = \frac{D/2}{H}$ so $H = \frac{D}{2\tan\theta} = \frac{0.070\text{ m}}{2\tan 15°} = 0.131\text{ m}$

Substituting,

$$\frac{dy}{dt} = -4\sqrt{9.81\frac{\text{m}}{\text{s}^2} \times 0.131\text{ m}}\left(\frac{0.00312\text{ m}}{0.070\text{ m}}\right)^2 1000\frac{\text{mm}}{\text{m}} = -9.01\text{ mm/s}$$ $\frac{dy}{dt}$

Alternate solution: Choose CV(2) enclosing entire funnel.

Basic equation: $0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$

Assumptions: (1) ρ = constant, but $\forall$ changes (Note: $\forall = \frac{\pi}{3} r^2 h$ for a cone.)
(2) Neglect air
(3) Uniform flow at outlet section

Then

$$0 = \rho \frac{\partial}{\partial t}\forall_{H_2O} + \{+|\rho V_e A_e|\} \quad \text{or} \quad \frac{d\forall}{dt} = -V_e A_e$$ (out)

The volume of water is $\forall = \frac{\pi}{3} r^2 h = \frac{\pi}{3}(y\tan\theta)^2 y = \frac{\pi y^3 \tan^2\theta}{3}$

so $\frac{d\forall}{dt} = \pi y^2 \tan^2\theta \frac{dy}{dt} = \pi\left(\frac{D}{4}\right)^2 \frac{dy}{dt}$ and $\frac{\pi D^2}{16}\frac{dy}{dt} = -V_e A_e = -\sqrt{2gy}\,\frac{\pi d^2}{4}$

Finally, since $y = H/2$, $\frac{dy}{dt} = -4\sqrt{2gH}\left(\frac{d}{D}\right)^2$ as before.

{Note: Flow is not steady in either CV. The $\partial/\partial t$ term vanishes for CV(1) because there is no change in mass inside the CV.}

Problem 4.40

Given: Steady flow of water past a porous flat plate. Suction is constant. Velocity profile at section cd is

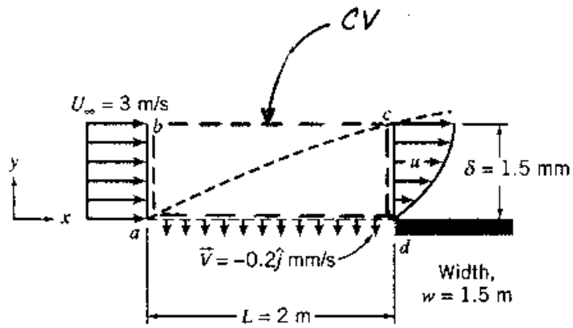
$$\frac{u}{U_\infty} = 3\left(\frac{y}{\delta}\right) - 2\left(\frac{y}{\delta}\right)^{1.5}$$

Find: Mass flow rate across section bc.

Solution: Apply conservation of mass using the CV shown.

Basic equation:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$



Assumptions: (1) Steady flow
(2) Incompressible flow
(3) $\vec{V} = -v_0 \hat{j}$ along da

Then

$$0 = \int_{CS} \rho \vec{V} \cdot d\vec{A} = \int_{ab} \rho \vec{V} \cdot d\vec{A} + \dot{m}_{bc} + \int_{cd} \rho \vec{V} \cdot d\vec{A} + \int_{da} \rho \vec{V} \cdot d\vec{A}$$

or

$$0 = -\rho U_\infty w \delta + \dot{m}_{bc} + \int_0^\delta \rho U_\infty \left[3\left(\frac{y}{\delta}\right) - 2\left(\frac{y}{\delta}\right)^{1.5}\right] w \, dy + \rho v_0 w L$$

Thus

$$\dot{m}_{bc} = \rho U_\infty w \delta - \rho U_\infty w \delta \int_0^1 \left[3\left(\frac{y}{\delta}\right) - 2\left(\frac{y}{\delta}\right)^{1.5}\right] d\left(\frac{y}{\delta}\right) - \rho v_0 w L$$

$$= \rho w \left\{ U_\infty \delta - U_\infty \delta \left[\frac{3}{2}\left(\frac{y}{\delta}\right)^2 - \frac{2}{2.5}\left(\frac{y}{\delta}\right)^{1.5}\right]_0^1 - v_0 L \right\}$$

$$= \rho w \left[U_\infty \delta - U_\infty \delta \left(\frac{3}{2} - \frac{2}{2.5}\right) - v_0 L \right] = \rho w \left(0.3 U_\infty \delta - v_0 L\right)$$

$$= 999 \frac{kg}{m^3} \times 1.5\, m \left(0.3 \times 3 \frac{m}{s} \times 0.0015\, m - 0.0002 \frac{m}{s} \times 2\, m\right)$$

$$\dot{m}_{bc} = 1.42 \text{ kg/s} \quad (\dot{m} > 0, \text{ so } \underline{\text{out}} \text{ of CV})$$

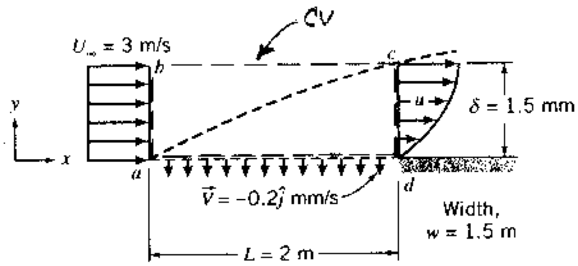
$\dot{m}_{bc}$

Problem 4.41

Given: Steady incompressible flow of air on porous surface shown in Fig. P4.38. Velocity profile at downstream end is parabolic. Uniform suction is applied along ad.

Find: (a) Volume flow rate across cd.
(b) Volume flow rate through porous surface (ad).
(c) Volume flow rate across bc.

Solution: Apply conservation of mass to CV shown.



Basic equation:

$$0 = \cancel{\frac{\partial}{\partial t}}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A} \quad (\text{first term} = 0 \ (1))$$

Assumptions: (1) Incompressible flow
(2) Parabolic profile at section cd: $\frac{u}{U_\infty} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$

Then $0 = \int_{CS} \vec{V}\cdot d\vec{A} = Q_{ab} + Q_{bc} + Q_{cd} + Q_{da}$ (1)

$$Q_{cd} = \int_{cd} \vec{V}\cdot d\vec{A} = \int_0^\delta u w \, dy = w U_\infty \delta \int_0^1 \frac{u}{U} d\left(\frac{y}{\delta}\right) = w U_\infty \delta \int_0^1 \left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right] d\left(\frac{y}{\delta}\right)$$

$$= w U_\infty \delta \left[\left(\frac{y}{\delta}\right)^2 - \frac{1}{3}\left(\frac{y}{\delta}\right)^3\right]_0^1 = \frac{2}{3} w \delta U_\infty$$

$Q_{cd} = \frac{2}{3} \times 1.5\,m \times 0.0015\,m \times 3\,\frac{m}{s} = 4.50\times10^{-3}\,m^3/s$ (out of CV) Q_{cd}

Flow across ad is uniform, so

$Q_{ad} = \vec{V}\cdot\vec{A} = v\hat{j}\cdot wL(-\hat{j}) = -vwL$

$Q_{ad} = -0.2\,\frac{mm}{s} \times 1.5\,m \times 2\,m \times \frac{m}{1000\,mm} = 6.00\times10^{-4}\,m/s$ (out of CV) Q_{ad}

Finally, from Eq. 1,

$Q_{bc} = -Q_{ab} - Q_{cd} - Q_{da}$ (2)

But $Q_{ab} = \vec{U}_\infty \cdot \vec{A}_{ab} = U_\infty \hat{i}\cdot w\delta(-\hat{i}) = -w\delta U_\infty$

$Q_{ab} = -1.5\,m \times 0.0015\,m \times 3\,\frac{m}{s} = -6.75\times10^{-3}\,m^3/s$ (into CV)

Substituting into Eq. 2,

$Q_{bc} = \left[-(-6.75\times10^{-3}) - 4.50\times10^{-3} - 6.00\times10^{-4}\right] m^3/s$

$Q_{bc} = 1.65\times10^{-3}\,m^3/s$ (out of CV) Q_{bc}

Problem 4.42

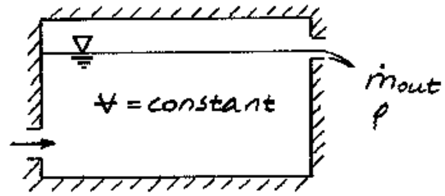
Given: Tank containing brine with steady inlet stream of water.
Initial density is $\rho_i > \rho_{H_2O}$.

Find: (a) Rate of change of liquid density in tank.
(b) Time required to reach density, ρ_f, where $\rho_i > \rho_f > \rho_{H_2O}$.

Solution: Apply conservation of mass using the CV shown.

Basic equation:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$



Assumptions: (1) $\forall_{tank} = \text{constant}$
(2) ρ uniform in tank
(3) Uniform flows at inlet and outlet sections

Then $V_1 A_1 = V_2 A_2$ since tank volume is constant, and

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \rho VA - \rho_{H_2O} VA = \frac{\partial}{\partial t}\rho\forall + (\rho - \rho_{H_2O})VA = \forall\frac{d\rho}{dt} + (\rho - \rho_{H_2O})VA$$

So that

$$\frac{d\rho}{dt} = -\frac{(\rho - \rho_{H_2O})VA}{\forall}$$

$\dfrac{d\rho}{dt}$

Separating variables,

$$\frac{d\rho}{\rho - \rho_{H_2O}} = -\frac{VA}{\forall} dt$$

Integrating from ρ_i at $t=0$ to ρ_f at t,

$$\int_{\rho_i}^{\rho_f} \frac{d\rho}{\rho - \rho_{H_2O}} = \ln(\rho - \rho_{H_2O})\Big]_{\rho_i}^{\rho_f} = \ln\left(\frac{\rho_f - \rho_{H_2O}}{\rho_i - \rho_{H_2O}}\right) = \int_0^t -\frac{VA}{\forall} dt = -\frac{VA}{\forall} t$$

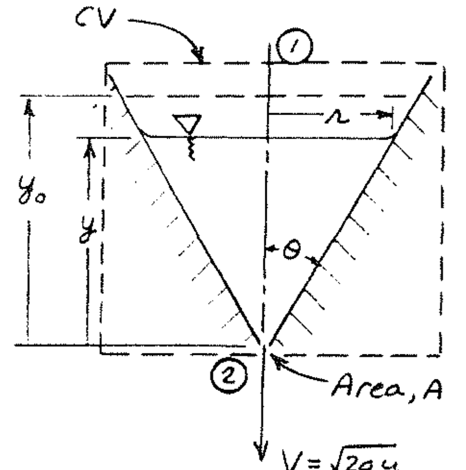
Finally,

$$t = -\frac{\forall}{VA}\ln\left(\frac{\rho_f - \rho_{H_2O}}{\rho_i - \rho_{H_2O}}\right)$$

t

{ Note that $\rho_f \to \rho_{H_2O}$ asymptotically as $t \to \infty$. }

Given: Funnel of liquid draining through a small hole of diameter $d = 5\text{ mm}$ (area A) as shown; y_0 is initial depth.



Find: (a) Expression for time to drain
(b) Expression for result in terms of
· initial volume $\forall_0$, and
· initial volume flow rate $Q_0 = AV_0 = A\sqrt{2gy_0}$

Plot: t as a function of y_0 ($0.1 \le y_0 \le 1\text{ m}$) with angle θ as a parameter for $15^\circ \le \theta \le 85^\circ$.

Solution

Apply conservation of mass using CV shown.

Basic equation: $0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$

Assumptions: (1) Incompressible flow
(2) Uniform flow at each section
(3) Neglect ρ_{air} compared to ρ_{H_2O}

Then,

$$0 = \frac{\partial}{\partial t}\int_{\forall_{air}} \rho_{air}\, d\forall + \frac{\partial}{\partial t}\int_{\forall_{H_2O}} \rho_{H_2O}\, d\forall + \{-|\rho_{air} V_1 A_1|\} + \{|\rho_{H_2O} V A|\}$$

(the air terms ≈ 0 (3))

For the CV,

$$d\forall = A_s\, dy = \pi r^2 dy = \pi (y \tan\theta)^2 dy \;;\; \forall = \pi \tan^2\theta \frac{y^3}{3}$$

Thus

$$0 = \rho_{H_2O} \frac{\partial}{\partial t}\left(\pi \tan^2\theta \frac{y^3}{3}\right) + \rho_{H_2O} A\sqrt{2gy}$$

$$0 = \pi \tan^2\theta\, y^2 \frac{dy}{dt} + A\sqrt{2g}\, y^{1/2}$$

Separating variables, $y^{3/2}\, dy = -\frac{\sqrt{2g}\,A}{\pi \tan^2\theta}\, dt$

Integrating from y_0 at $t = 0$ to 0 at t,

$$\int_{y_0}^{0} y^{3/2}\, dy = \frac{2}{5}\left(-y_0^{5/2}\right) = -\frac{\sqrt{2g}\,A}{\pi \tan^2\theta}\, t$$

or

$$t = \frac{2}{5}\, \frac{\pi \tan^2\theta\, y_0^{5/2}}{\sqrt{2g}\,A} \qquad t$$

But $\forall_o = \pi \tan^2\theta \, \frac{y_o^3}{3}$ and $Q_o = AV_o = A\sqrt{2gy_o}$, so

$$t = \frac{2}{5}\,\frac{\pi \tan^2\theta \, y_o^{5/2}}{\sqrt{2g}\;A} \times \frac{3}{3} \cdot \frac{y_o^{1/2}}{y_o^{1/2}} = \frac{6}{5}\,\frac{\forall_o}{Q_o} \longleftarrow t$$

Since $A = \frac{\pi d^2}{4}$, we can write

$$t = \frac{2}{5}\,\frac{\pi \tan^2\theta \, y_o^{5/2}}{\sqrt{2g}\;\pi \frac{d^2}{4}} = \frac{8}{5}\,\frac{\tan^2\theta \, y_o^{5/2}}{d^2\sqrt{2g}}$$

t is plotted as a function of y_o with θ as a parameter

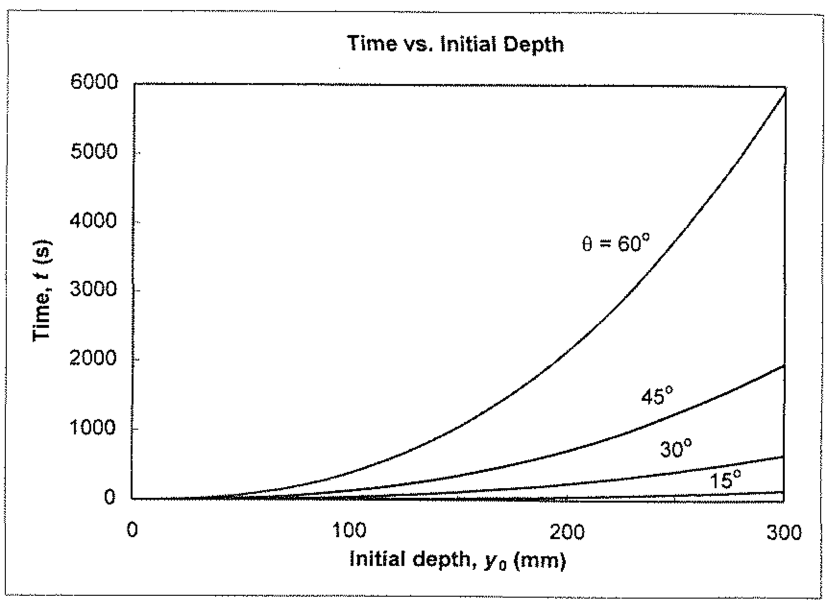
Draining of a conical liquid tank:

Input Data:

Orifice diameter: $d =$ 3 mm

Calculated Results:

Initial Height, y_0 (mm)	Cone Half Angle, θ (deg)	Drain Time, t (s): 60	45	30	15
300		5935	1978	659	142
275		4775	1592	531	114
250		3763	1254	418	90.0
225		2891	964	321	69.2
200		2154	718	239	51.5
175		1543	514	171	36.9
150		1049	350	117	25.1
125		665	222	74	15.9
100		381	127	42	9.11
75		185	62	21	4.44
50		67	22	7	1.61
25		12	4	1	0.285
0		0	0	0	0



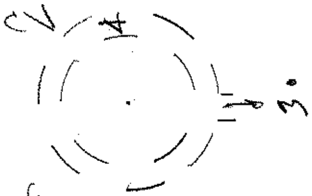
Given: The instantaneous leakage mass flow rate $\dot{m}$ from a bicycle tire is proportional to the air density ρ in the tire and to the gage pressure p_g in the tire.
Air in the tire is nearly isothermal (because the leakage rate is slow).
The initial air pressure is $p_0 = 0.60$ MPa (gage) and the initial rate of pressure loss is 1 psi/day

Find: (a) Pressure in the tire after 30 days
(b) Accuracy of rule of thumb which says a tire loses pressure at the rate of "a pound [1 psi] a day".

Plot: the pressure as a function of time over the 30 days; show rule of thumb results for comparison.

Solution:

Apply conservation of mass to tire as the CV



Basic equation: $0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$

Assumptions: (1) uniform properties in tire
(2) air inside CV behaves as ideal gas
(3) T = constant, $\forall$ = constant
(4) $\dot{m} = c(p - p_{atm})\rho$

Then we can write

$$0 = \forall \frac{\partial \rho}{\partial t} + \dot{m} = \forall \frac{\partial \rho}{\partial t} + c(p - p_{atm})\rho \quad \ldots\ldots\ldots (1)$$

But $\rho = p/RT$ and $\frac{\partial \rho}{\partial t} = \frac{1}{RT}\frac{dp}{dt}$, so

$$0 = \frac{\forall}{RT}\frac{dp}{dt} + \frac{cp}{RT}(p - p_{atm})$$

At $t = 0$, $p = p_0$ and $dp/dt = dp/dt)_0$. Thus

$$0 = \forall \left.\frac{dp}{dt}\right)_0 + c p_0 (p_0 - p_{atm}) \quad \text{and} \quad c = -\frac{\forall}{p_0(p_0 - p_{atm})}\left.\frac{dp}{dt}\right)_0$$

Substituting into Eq. 1 we obtain

$$0 = \frac{dp}{dt} - \frac{p(p - p_{atm})}{p_0(p_0 - p_{atm})}\left.\frac{dp}{dt}\right)_0$$

Separating variables and integrating

$$\int_{p_0}^{p} \frac{dp}{p(p - p_{atm})} = \frac{dp/dt)_0}{p_0(p_0 - p_{atm})}\int_0^t dt$$

$$\frac{1}{p_{atm}}\left[\ln \frac{p_0(p - p_{atm})}{p(p_0 - p_{atm})}\right] = \frac{dp/dt)_0}{p_0(p_0 - p_{atm})}\, t$$

$$\ln\left[\frac{1 - p_{atm}/p}{1 - p_{atm}/p_0}\right] = \frac{dp/dt)_0}{p_0(p_0/p_{atm} - 1)}\, t$$

Taking antilogs,

$$1-\frac{p_{atm}}{p} = \left(1-\frac{p_{atm}}{p_0}\right) e^{\left\{\frac{dp/dt)_0}{p_0(p_0/p_{atm}-1)} t\right\}} = \left(1-\frac{p_{atm}}{p_0}\right) e^{kt}$$

where

$$k = \frac{dp/dt)_0}{p_0(p_0/p_{atm}-1)} = -1\,psi \times \frac{6.895\,kPa}{psi} \times \frac{1}{701\,kPa}\,\frac{1}{(701/101-1)}$$

$$k = -0.00166\ day^{-1}$$

Then

$$\frac{p_{atm}}{p} = 1 - \left(\frac{p_0-p_{atm}}{p_0}\right) e^{kt}$$

and

$$p = \frac{p_{atm}}{1-\left(\frac{p_0-p_{atm}}{p_0}\right) e^{kt}} \qquad (2)$$

Evaluating at t = 30 days.

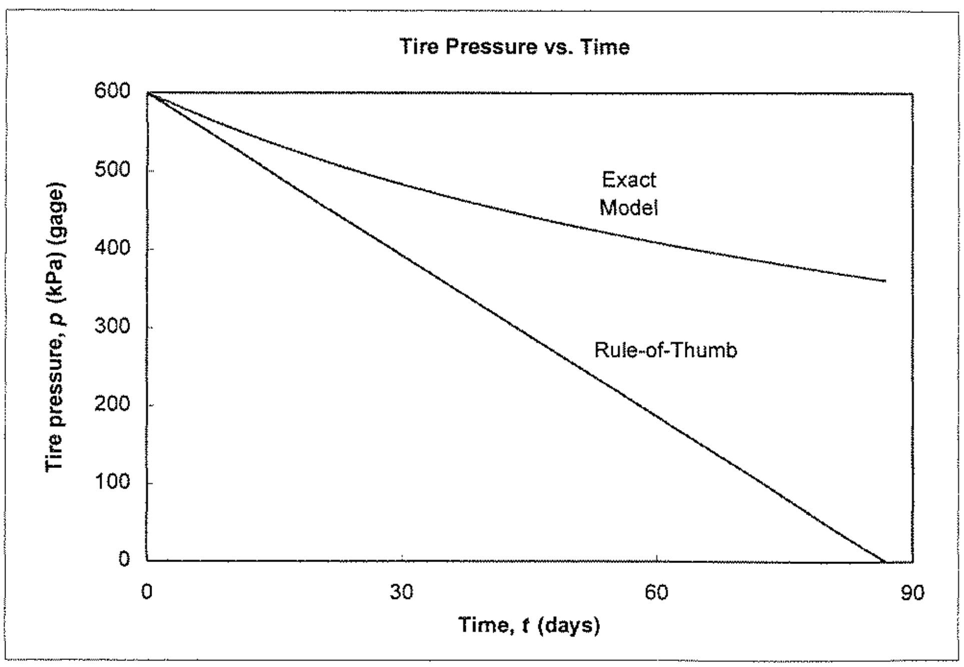
$$p = \frac{101\,kPa}{1-\frac{600}{701} e^{-30(0.00166)}} = 544\,kPa \quad p_{t=30\,days}$$

Rule of thumb gives $p = p_0 - 6.895 \frac{kPa}{day} t$ ...... (3)

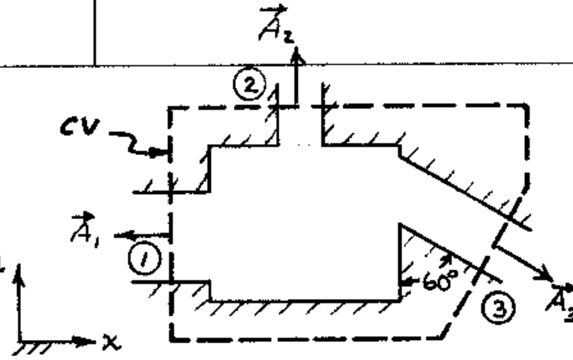
At t = 30 days $p = 600\,kPa - 207\,kPa = 493\,kPa$ p_{ROT}

The rule of thumb predicts a larger pressure loss

Results for both models are presented below



Problem 4.45



Given: Steady, incompressible flow ($\rho = 1050\ kg/m^3$) through rectangular box shown.

$A_1 = 0.05\ m^2,\ A_2 = 0.01\ m^2,\ A_3 = 0.06\ m^2$

$\vec{V}_1 = 4\ \hat{i}\ m/s,\quad \vec{V}_2 = -8\ \hat{j}\ m/s$

and, from Problem 4.19, $\vec{V}_3 = 4.04\ \hat{i} - 2.34\ \hat{j}\ m/s$; $V_3 = 4.67\ m/s$

Find: Net rate of efflux of momentum through CV.

Solution: The net rate of momentum efflux is given by the term

$$\int_{CS} \vec{V} \rho \vec{V} \cdot d\vec{A}$$

Assumption: (1) Flow is uniform at each section.

Then $\int_{CS} \vec{V} \rho \vec{V} \cdot d\vec{A} = \vec{V}_1 \rho \vec{V}_1 \cdot \vec{A}_1 + \vec{V}_2 \rho \vec{V}_2 \cdot \vec{A}_2 + \vec{V}_3 \rho \vec{V}_3 \cdot \vec{A}_3$

or in components, since $\vec{V} = u\hat{i} + v\hat{j}$,

$$\int_{CS} \vec{V} \rho \vec{V} \cdot d\vec{A} = (u_1 \rho \vec{V}_1 \cdot \vec{A}_1 + u_2 \rho \vec{V}_2 \cdot \vec{A}_2 + u_3 \rho \vec{V}_3 \cdot \vec{A}_3)\hat{i}$$
$$+ (v_1 \rho \vec{V}_1 \cdot \vec{A}_1 + v_2 \rho \vec{V}_2 \cdot \vec{A}_2 + v_3 \rho \vec{V}_3 \cdot \vec{A}_3)\hat{j}$$

$$\vec{mf} = \left[u_1 \{-|\rho V_1 A_1|\} + u_2 \{-|\rho V_2 A_2|\} + u_3 \{|\rho V_3 A_3|\} \right]\hat{i}$$

$u_1 = 4\ m/s \qquad u_2 = 0 \qquad u_3 = 4.04\ m/s$

$$+ \left[v_1 \{-|\rho V_1 A_1|\} + v_2 \{-|\rho V_2 A_2|\} + v_3 \{|\rho V_3 A_3|\} \right]\hat{j}$$

$v_1 = 0 \qquad v_2 = -8\ m/s \qquad v_3 = -2.34\ m/s$

$$= 1050 \frac{kg}{m^3} \left[4.0 \frac{m}{s} \left\{ -4.0 \frac{m}{s} \times 0.05 m^2 \right\} + 4.04 \frac{m}{s} \left\{ 4.67 \frac{m}{s} \times 0.06 m^2 \right\} \right] \hat{i}$$

$$+ 1050 \frac{kg}{m^3} \left[-8.0 \frac{m}{s} \left\{ -8.0 \frac{m}{s} \times 0.01 m^2 \right\} - 2.34 \frac{m}{s} \left\{ 4.67 \frac{m}{s} \times 0.06 m^2 \right\} \right] \hat{j}$$

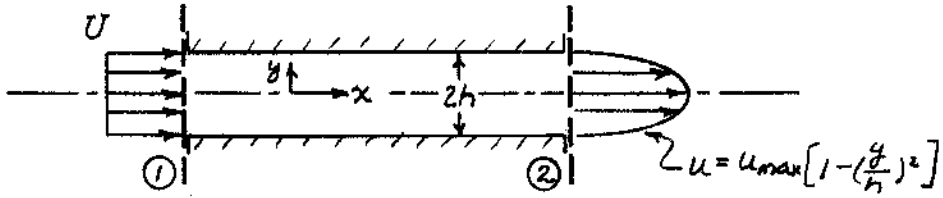
$$= (349\ \hat{i} - 13.5\ \hat{j}) \frac{kg \cdot m}{s^2} \times \frac{N \cdot s^2}{kg \cdot m}$$

$$\vec{mf} = 349\hat{i} - 16.5\hat{j}\ N$$

Ne
mf

Problem 4.46

Given: Water flow between parallel plates, as shown.



From Problem 4.21, $u_{max} = \frac{3}{2}U$ (from continuity).

Evaluate: Ratio of x direction momentum flux at outlet to that at inlet.

Solution: The x direction momentum flux at a section is given by

$$mf_x = \int_A u \rho V dA$$

Assumptions: (1) Uniform flow at section ①
(2) Incompressible flow

Then at ① $\quad mf_x)_1 = \int_{A_1} U \rho U dA = \rho U^2 2hw \quad$ (w = channel depth)

At section ②, the velocity varies and we must integrate. Using $dA = w\,dy$,

$$mf_x)_2 = \int_{-h}^{h} u \rho u\, w\,dy = \rho w u_{max}^2 \int_{-h}^{h} \left[1-\left(\frac{y}{h}\right)^2\right]^2 dy$$

$$= \rho w h u_{max}^2 \int_{-1}^{1} \left[1-\left(\frac{y}{h}\right)^2\right]^2 d\left(\frac{y}{h}\right) = 2\rho w h u_{max}^2 \int_0^1 \left[1-\left(\frac{y}{h}\right)^2\right]^2 d\left(\frac{y}{h}\right)$$

$$= 2\rho w h u_{max}^2 \int_0^1 \left[1 - 2\left(\frac{y}{h}\right)^2 + \left(\frac{y}{h}\right)^4\right] d\left(\frac{y}{h}\right)$$

$$= 2\rho w h u_{max}^2 \left[\left(\frac{y}{h}\right) - \frac{2}{3}\left(\frac{y}{h}\right)^3 + \frac{1}{5}\left(\frac{y}{h}\right)^5\right]_0^1$$

$$mf_x)_2 = \frac{16}{15}\rho w h u_{max}^2$$

The ratio of x direction momentum fluxes is

$$\frac{mf_x)_2}{mf_x)_1} = \frac{\frac{16}{15}\rho w h u_{max}^2}{2\rho w h U^2} = \frac{8}{15}\left(\frac{u_{max}}{U}\right)^2$$

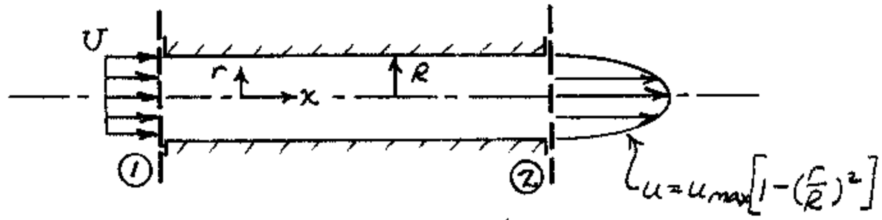
But from Problem 4.21, $u_{max} = \frac{3}{2}U$, so

$$\frac{mf_x)_2}{mf_x)_1} = \frac{8}{15}\left(\frac{3}{2}\right)^2 = \frac{72}{60} = \frac{6}{5} = 1.2$$

Rat.

Problem 4.47

Given: Water flow through a circular pipe as shown.



From Problem 4.23, $u_{max} = 2U$ (from continuity).

Evaluate: Ratio of x direction momentum flux at outlet to that at inlet.

Solution: The x direction momentum flux at a section is given by

$$mf_x = \int_A u\rho V dA$$

Assumptions: (1) Uniform flow at section ①
(2) Incompressible flow

Then at ① $\quad mf_x)_1 = \int_{A_1} U\rho U dA = \rho U^2 \pi R^2$

At section ② the velocity varies and we must integrate.
Using $dA = 2\pi r dr$,

$$mf_x)_2 = \int_0^R u\rho u \, 2\pi r dr = 2\pi\rho u_{max}^2 \int_0^R \left[1-(\tfrac{r}{R})^2\right]^2 r dr$$

$$= \rho u_{max}^2 2\pi R^2 \int_0^1 \left[1-(\tfrac{r}{R})^2\right]^2 (\tfrac{r}{R}) d(\tfrac{r}{R})$$

$$= \rho u_{max}^2 2\pi R^2 \int_0^1 \left[(\tfrac{r}{R}) - 2(\tfrac{r}{R})^3 + (\tfrac{r}{R})^5\right] d(\tfrac{r}{R})$$

$$= \rho u_{max}^2 2\pi R^2 \left[\tfrac{1}{2}(\tfrac{r}{R})^2 - \tfrac{1}{2}(\tfrac{r}{R})^4 + \tfrac{1}{6}(\tfrac{r}{R})^6\right]_0^1$$

or

$$mf_x)_2 = \rho u_{max}^2 \pi R^2 (\tfrac{1}{3})$$

The ratio of x direction momentum fluxes is

$$\frac{mf_x)_2}{mf_x)_1} = \frac{\rho u_{max}^2 \pi R^2 (\frac{1}{3})}{\rho U^2 \pi R^2} = \frac{1}{3}\left(\frac{u_{max}}{U}\right)^2$$

But from Problem 4.23, $u_{max} = 2U$, so

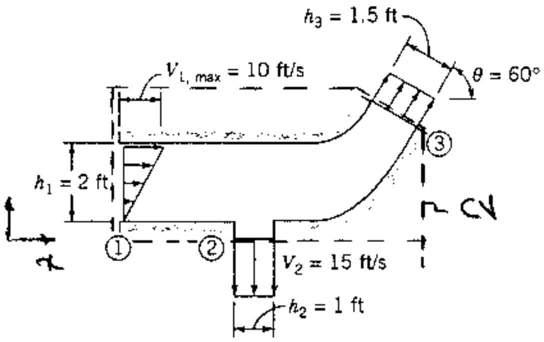
$$\frac{mf_x)_2}{mf_x)_1} = \frac{1}{3}(2)^2 = \frac{4}{3}$$

Rati

Problem 4.48

Given: Two-dimensional reducing bend shown has width $w = 3$ ft.

$V_3 = 3.33$ ft/s into CV (from Problem 4.24)



Find: Momentum flux through the bend.

Solution:

The momentum flux is defined as $m.f = \int \vec{V}(\rho\vec{V}\cdot d\vec{A})$

The net momentum flux through the CV is

$$m.f = \int_{A_1} \vec{V}(\rho\vec{V}\cdot d\vec{A}) + \int_{A_2} \vec{V}(\rho\vec{V}\cdot d\vec{A}) + \int_{A_3} \vec{V}(\rho\vec{V}\cdot d\vec{A})$$

where $\vec{V}_1 = V_{1\max}\frac{y}{h_1}\hat{i}$, $\vec{V}_2 = -V_2\hat{j}$, $\vec{V}_3 = -V_3(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$

$V_{1\max} = 10$ ft/s, $V_2 = 15$ ft/s, $V_3 = 3.33$ ft/s

Assumptions: (1) incompressible flow
(2) fluid is water
(3) uniform flow at ② and ③ (given)

$$\int_{A_1} \vec{V}(\rho\vec{V}\cdot d\vec{A}) = \int_0^{h_1} V_{1\max}\frac{y}{h_1}\hat{i}\,\rho\left\{-V_{1\max}\frac{y}{h_1}\right\}w\,dy = -\hat{i}\rho V_{1\max}^2\frac{w}{h_1^2}\int_0^{h_1} y^2\,dy$$

$$\int_{A_1} \vec{V}(\rho\vec{V}\cdot d\vec{A}) = -\hat{i}\rho V_{1\max}^2\frac{w}{h_1^2}\left.\frac{y^3}{3}\right]_0^{h_1} = -\hat{i}\rho V_{1\max}^2\frac{wh_1}{3} \quad \text{----(1)}$$

$$\int_{A_2} \vec{V}(\rho\vec{V}\cdot d\vec{A}) = \vec{V}_2|\rho V_2 h_2 w| = -V_2\hat{j}|\rho V_2 h_2 w| = -\hat{j}\rho V_2^2 h_2 w \quad \text{----(2)}$$

$$\int_{A_3} \vec{V}(\rho\vec{V}\cdot d\vec{A}) = \vec{V}_3(-|\rho V_3 h_3 w|) = -V_3(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})(-|\rho V_3 h_3 w|)$$

$$\int_{A_3} \vec{V}(\rho\vec{V}\cdot d\vec{A}) = \rho V_3^2 h_3 w(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}) \quad \text{---------(3)}$$

$$m.f = \hat{i}\left[\rho V_3^2 h_3 w\cos\theta - \rho V_{1\max}^2\frac{wh_1}{3}\right] + \hat{j}\left[\rho V_3^2 h_3 w\sin\theta - \rho V_2^2 h_2 w\right]$$

$$m.f = \rho w\left\{\left[V_3^2 h_3\cos\theta - V_{1\max}^2\frac{h_1}{3}\right]\hat{i} + \left[V_3^2 h_3\sin\theta - V_2^2 h_2\right]\hat{j}\right\}$$

Evaluating

$$m.f = 1.94\,\frac{\text{slug}}{\text{ft}^3} \times 3\,\text{ft} \times \frac{\text{lbf}\cdot\text{s}^2}{\text{ft}\cdot\text{slug}}\left\{\left[(3.33)^2\frac{\text{ft}^2}{\text{s}^2}\times 1.5\,\text{ft}\times\cos 60° - (10)^2\frac{\text{ft}^2}{\text{s}^2}\times\frac{2\,\text{ft}}{3}\right]\hat{i} + \left[(3.33)^2\frac{\text{ft}^2}{\text{s}^2}\times 1.5\,\text{ft}\times\sin 60° - (15)^2\frac{\text{ft}^2}{\text{s}^2}\times 1\,\text{ft}\right]\hat{j}\right\}$$

$$m.f = -340\,\hat{i} - 1230\,\hat{j}\ \text{lbf}$$ ← m.f.

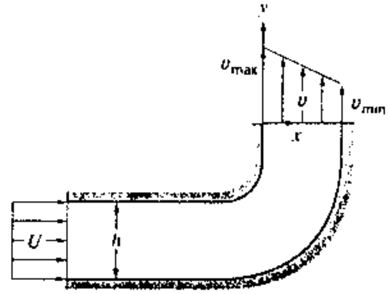
Problem 4.49

Given: Water flow in the two-dimensional square channel shown.

$\bar{U} = 7.5$ m/s, $h = w = 75.5$ mm

$v_{max} = 2\, v_{min}$

$v_{min} = 5.0$ m/s (from Problem 4.25)



Find: Momentum flux through the channel; comment on expected outlet pressure (relative to pressure at the inlet.

Solution:

The momentum flux is defined as $m.f. = \int \vec{V}(\rho \vec{V} \cdot d\vec{A})$

The net momentum flux through the CV is

$$m.f. = \int_{A_1} \vec{V}(\rho \vec{V} \cdot d\vec{A}) + \int_{A_2} \vec{V}(\rho \vec{V} \cdot d\vec{A})$$

where $\vec{V}_1 = U\hat{i}$, $\vec{V}_2 = \left\{ v_{max} - (v_{max} - v_{min})\frac{x}{h} \right\}\hat{j}$

$$\vec{V}_2 = \left\{ 2v_{min} - v_{min}\frac{x}{h} \right\}\hat{j} = v_{min}\left(2 - \frac{x}{h}\right)\hat{j}$$

Assumptions: (1) incompressible flow
(2) uniform flow at ① (given).

$$\int_{A_1} \vec{V}(\rho \vec{V} \cdot d\vec{A}) = \vec{V}_1\{-|\rho V_1 A_1|\} = -\rho U^2 h^2 \hat{i} \quad \text{.......(1)}$$

$$\int_{A_2} \vec{V}(\rho \vec{V} \cdot d\vec{A}) = \int_0^h v_{min}\left(2 - \frac{x}{h}\right)\hat{j}\, \rho v_{min}\left(2 - \frac{x}{h}\right) h\, dx$$

$$= \hat{j}\rho v_{min}^2 h \int_0^h \left(4 - 4\frac{x}{h} + \frac{x^2}{h^2}\right) dx$$

$$= \hat{j}\rho v_{min}^2 h \left[4x - 2\frac{x^2}{h} + \frac{x^3}{3h^2}\right]_0^h = \hat{j}\rho v_{min}^2 h \left[4h - 2h + \frac{h}{3}\right]$$

$$= \hat{j}\,\frac{7}{3}\rho v_{min}^2 h^2$$

$$\therefore\ m.f. = -\rho U^2 h^2 \hat{i} + \frac{7}{3}\rho v_{min}^2 h^2 \hat{j} = \rho h^2\left[-U^2\hat{i} + \frac{7}{3}v_{min}^2\hat{j}\right]$$

Evaluating

$$m.f. = 999\,\frac{kg}{m^3} \times (0.0755)^2 m^2 \left[-(7.5)^2\frac{m^2}{s^2}\hat{i} + \frac{7}{3}(5)^2\frac{m^2}{s^2}\hat{j}\right] \times \frac{N \cdot s^2}{kg \cdot m}$$

$$m.f. = -320\hat{i} + 332\hat{j}\ N$$ m.f.

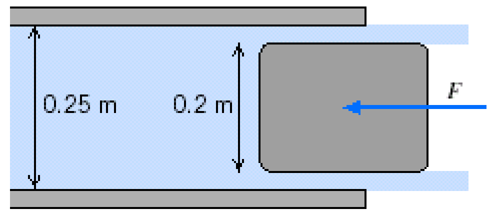
For viscous (real) flow friction causes a pressure drop in the direction of flow (Chapter 8)
For flow in a bend streamline curvature results in a pressure gradient normal to the flow (Chapter 6)

Problem 4.50

Find the force required to hold the plug in place at the exit of the water pipe. The flow rate is 1 m3/s, and the upstream pressure is 3.5 MPa.

Given: Data on flow and system geometry

Find: Force required to hold plug



Solution

The given data are

$D_1 \,|\, 0.25 \cdot m$ $\quad D_2 \,|\, 0.2 \cdot m$ $\quad Q \,|\, 1.5 \cdot \frac{m^3}{s}$ $\quad p_1 \,|\, 3500 \cdot kPa$ $\quad \psi \,|\, 999 \cdot \frac{kg}{m^3}$

Then $\quad A_1 \,|\, \frac{\phi \cdot D_1^2}{4} \qquad A_1 \,|\, 0.0491\,m^2$

$A_2 \,|\, \frac{\phi}{4} \cdot \left(D_1^2 \; 4 \; D_2^2\right) \qquad A_2 \,|\, 0.0177\,m^2$

$V_1 \,|\, \frac{Q}{A_1} \qquad V_1 \,|\, 30.6\,\frac{m}{s}$

$V_2 \,|\, \frac{Q}{A_2} \qquad V_2 \,|\, 84.9\,\frac{m}{s}$

Governing equation:

Momentum

$$\vec{F} = \vec{F}_S + \vec{F}_B = \frac{\partial}{\partial t}\int_{CV} \vec{V}\,\rho\, d\forall + \int_{CS} \vec{V}\,\rho\vec{V}\cdot d\vec{A} \qquad (4.17)$$

Applying this to the current system

$$-F + p_1 \cdot A_2 - p_2 \cdot A_2 = 0 + V_1 \cdot \left(-\rho \cdot V_1 \cdot A_1\right) + V_2 \cdot \left(\rho \cdot V_2 \cdot A_2\right)$$

and $\quad p_2 = 0 \;\; \text{(gage)}$

Hence $\quad F = p_1 \cdot A_1 + \rho \cdot \left(V_1^{\,2} \cdot A_1 - V_2^{\,2} \cdot A_2\right)$

$$F = 3500 \cdot \frac{\text{kN}}{\text{m}^2} \cdot 0.0491 \cdot \text{m}^2 \ldots$$

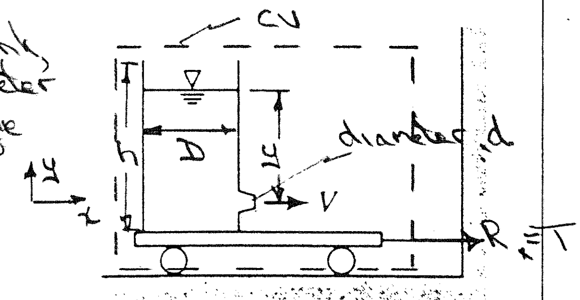
$$+ 999 \cdot \frac{\text{kg}}{\text{m}^3} \cdot \left[\left(30.6 \cdot \frac{\text{m}}{\text{s}}\right)^2 \cdot 0.0491 \cdot \text{m}^2 - \left(84.9 \cdot \frac{\text{m}}{\text{s}}\right)^2 \cdot 0.0177 \cdot \text{m}^2\right]$$

$$F = 90.4\,\text{kN}$$

Problem 4.51

Given: Water discharges from tank of height $h = 1\,m$ and diameter $D = 0.6\,m$ through a nozzle of diameter $d = 10\,mm$.

$V_{jet} = \sqrt{2gy}$ where y is height of free surface above the nozzle.



Find: Tension in wire holding the cart when $y = 0.8\,m$.

Plot: tension in wire as a function of water depth for $0 \le y \le 0.8\,m$.

Solution:

Apply the x component of the momentum equation, using the inertial CV shown.

Basic equation: $F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u \rho \, d\forall + \int_{CS} u \rho \vec{V} \cdot d\vec{A}$ (with $F_{B_x} = 0$ (2), $\frac{\partial}{\partial t} = 0$ (3))

Assumptions:
(1) There are no net pressure forces
(2) $F_{B_x} = 0$
(3) Steady flow
(4) Uniform flow across the jet

Then,

$$R_x = T = u\{|\rho V_j A_j|\} = \rho V_j^2 A = \rho\, 2gy\, \pi \frac{d^2}{4}$$

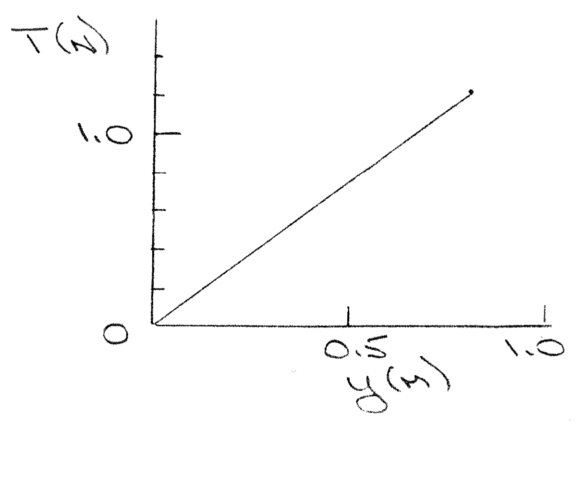
$$T = \rho g y \pi \frac{d^2}{2} \qquad (1)$$

Evaluating for $y = 0.8\,m$

$$T = 999\frac{kg}{m^3} \times 9.81\frac{m}{s^2} \times 0.8\,m \times \frac{\pi}{2} \times (0.010)^2 m^2 \times \frac{N \cdot s^2}{kg \cdot m}$$

$$T = 1.23\,N \qquad T$$

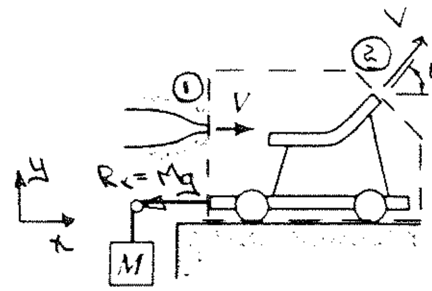
From Eq. 1 we see that T varies linearly with y



Problem 4.52

Given: Cart with vane, struck by water jet.

$V_j = 15\ \text{m/s} \qquad A_j = 0.05\ \text{m}^2$



Find: Mass needed to hold cart stationary for $\theta = 50^\circ$

Plot: mass needed to hold cart stationary for $0 \le \theta \le 180$ degrees.

Solution:

Apply the x component of the momentum equation to the inertial CV shown.

Basic equation: $F_{S_x} + \cancel{F_{B_x}}^{=0(2)} = \cancel{\frac{\partial}{\partial t}\int_{CV} u \rho\, d\forall}^{=0(3)} + \int_{CS} u \rho \vec{V} \cdot d\vec{A}$

Assumptions:
(1) atmospheric pressure surrounds CV
(2) $F_{B_x} = 0$
(3) steady flow
(4) jet velocity (and area) remain constant on vane
(5) uniform flow at each section
(6) incompressible flow

Then

$$-Mg = u_1\{-|\rho V_1 A_1|\} + u_2\{|\rho V_2 A_2|\} \qquad \begin{cases} u_1 = V\ ;\ u_2 = V\cos\theta \\ V_1 = V_2 = V\ ;\ A_1 = A_2 = A \end{cases}$$

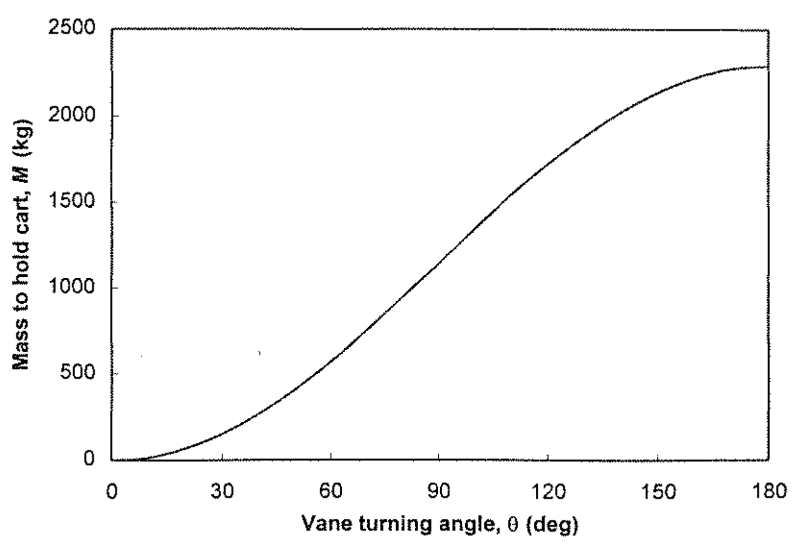
$$-Mg = V(-\rho V A) + V\cos\theta(\rho V A) = \rho V^2 A(\cos\theta - 1)$$

$$M = \frac{\rho V^2 A}{g}(1 - \cos\theta) \qquad (1)$$

Evaluating for $\theta = 50^\circ$

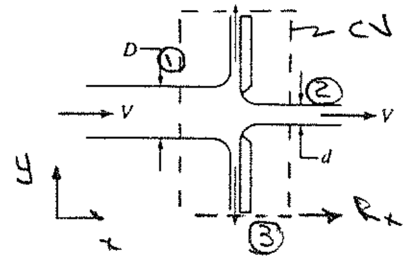
$$M = 999\ \frac{\text{kg}}{\text{m}^3} \times (15)^2\ \frac{\text{m}^2}{\text{s}^2} \times (0.05\ \text{m}^2) \times \frac{\text{s}^2}{9.81\ \text{m}}(1 - \cos 50^\circ) = 409\ \text{kg} \leftarrow M$$

M is plotted as a function of θ



Given: Plate with orifice struck concentrically by water jet as shown.

Find: (a) Expression for force needed to hold the plate.
(b) Value of force for $V = 5$ m/s, $D = 100$ mm, and $d = 25$ mm



Plot: required force as a function of diameter ratio d/D

Solution:

Apply the x component of the momentum equation to the inertial CV shown.

Basic equation: $$F_{S_x} + \cancel{F_{B_x}}^{=0(2)} = \cancel{\frac{\partial}{\partial t}\int_{CV} u \rho \, d\forall}^{=0(3)} + \int_{CS} u \rho \vec{V} \cdot d\vec{A}$$

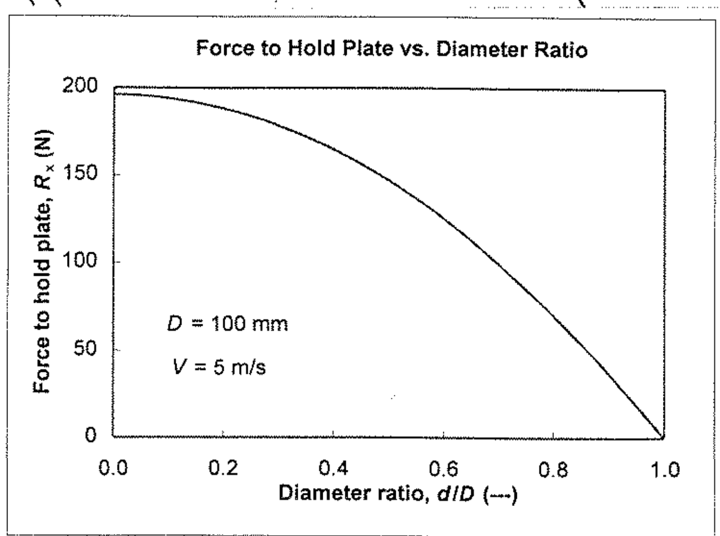
Assumptions: (1) atmospheric pressure surrounds CV
(2) $F_{B_x} = 0$
(3) steady flow
(4) uniform flow at each section
(5) incompressible flow

Then,
$$R_x = u_1\{-|\rho V_1 A_1|\} + u_2\{|\rho V_2 A_2|\} + u_3\{|\rho V_3 A_3|\}$$
$$u_1 = V, \quad A_1 = \frac{\pi D^2}{4} \qquad u_2 = V, \quad A_2 = \frac{\pi d^2}{4} \qquad u_3 = 0$$

and
$$R_x = -\rho V^2 A_1 + \rho V^2 A_2 = \rho V^2 (A_2 - A_1) = \rho V^2 \frac{\pi}{4}(d^2 - D^2)$$
$$R_x = -\rho V^2 \frac{\pi}{4} D^2 \left[1 - \left(\frac{d}{D}\right)^2\right] \qquad R_x$$

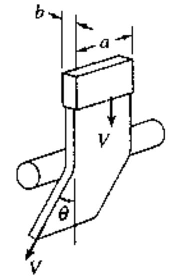
Evaluating for $d = 25$ mm
$$R_x = -\frac{\pi}{4} \times 999 \frac{\text{kg}}{\text{m}^3} (5)^2 \frac{\text{m}^2}{\text{s}^2} \times (0.10)^2 \text{m}^2 \left[1 - \left(\frac{25 \text{ mm}}{100 \text{ mm}}\right)^2\right] \frac{\text{N} \cdot \text{s}^2}{\text{kg} \cdot \text{m}} = -184 \text{ N} \qquad R_x$$

Since $R_x < 0$, it must be applied to the left. R_x is plotted as a function of d/D.



Problem 4.54

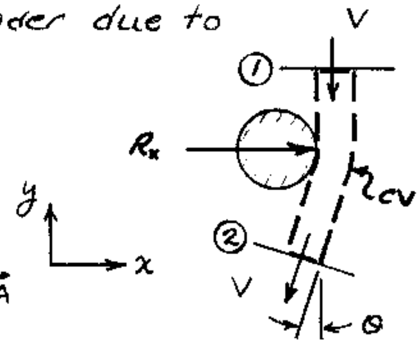
Given: Circular cylinder deflecting flat jet of water as shown.



$a = 0.5\,in.$, $b = 0.1\,in.$

$V = 10\,\frac{ft}{s}$, $\theta = 20°$

Find: Horizontal component of force on cylinder due to flowing water.



Solution: Apply x component of momentum equation to inertial CV shown.

Basic equation: $F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$ (F_{B_x} = 0(2), $\frac{\partial}{\partial t}$ = 0(3))

Assumptions: (1) No pressure forces on jet
(2) $F_{Bx} = 0$
(3) Steady flow
(4) Incompressible flow
(5) Uniform flow at each section
(6) Jet speed and area remain constant

Then

$$R_x = u_1\{-|\rho V_1 A_1|\} + u_2\{|\rho V_2 A_2|\}$$

$u_1 = 0$ $\quad$ $u_2 = -V\sin\theta$, $V_2 = V$, $A_2 = ab$

$$R_x = -\rho V^2 ab \sin\theta$$

$$= -1.94\,\frac{slug}{ft^3} \times (10)^2 \frac{ft^2}{s^2} \times 0.5\,in. \times 0.1\,in. \times \frac{ft^2}{144\,in.^2} \times \sin 20° \times \frac{lbf\cdot s^2}{slug\cdot ft}$$

$$R_x = -0.0230\ lbf$$

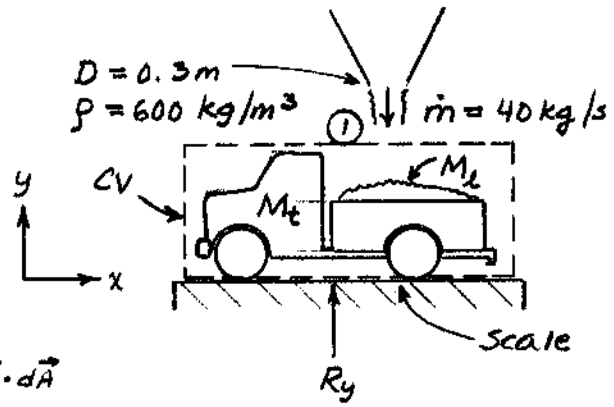
But R_x is force of cylinder on CV. Force of CV on cylinder is

$K_x = -R_x = 0.0230\ lbf$ (to the right) $\quad K_x$

Problem 4.55

Given: Farmer purchases 675 kg of bulk grain. The grain is loaded into a pickup truck from a hopper as shown. Grain flow is terminated when the scale reading reaches the desired gross value.

Find: The true payload.



Solution: Apply the y component of momentum equation using CV shown.

Basic equation:

$$F_{S_y} + F_{B_y} = \overset{\approx 0(2)}{\frac{\partial}{\partial t}\int_{CV} v\,\rho\,d\forall} + \int_{CS} v\,\rho\,\vec{V}\cdot d\vec{A}$$

Assumptions: (1) No net pressure force; $F_{S_y} = R_y$

(2) Neglect v inside CV

Then (3) Uniform flow of grain at inlet section ①

$$R_y - (M_t + M_\ell)g = v_1\{-|\dot{m}|\}$$

$$v_1 = -V_1 = -\frac{\dot{m}}{\rho A}$$

or

$$R_y = (M_t + M_\ell)g + \frac{\dot{m}^2}{\rho A} \quad \text{(indicated during grain flow)}$$

Loading is terminated when

$$\frac{R_y}{g} - M_t = M_\ell + \frac{\dot{m}^2}{\rho g A} = 675\text{ kg}$$

Thus

$$M_\ell = 675\text{ kg} - \frac{\dot{m}^2}{\rho g A}$$

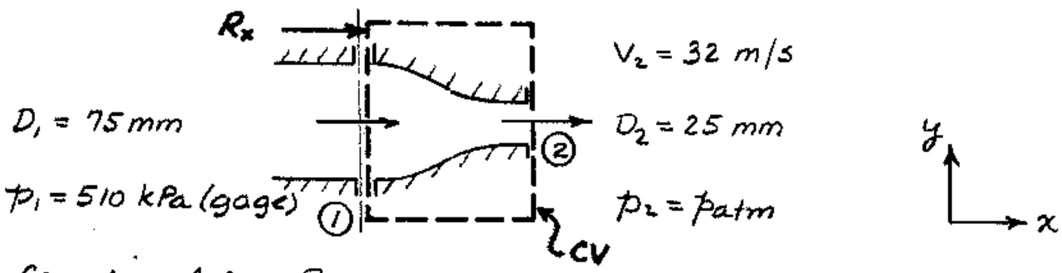
$$= 675\text{ kg} - (40)^2\frac{\text{kg}^2}{\text{s}^2}\times\frac{\text{m}^3}{600\text{ kg}}\times\frac{\text{s}^2}{9.81\text{ m}}\times\frac{4}{\pi}\frac{1}{(0.3)^2\text{ m}^2}$$

$$M_\ell = 671\text{ kg}$$

M_ℓ

Problem 4.56

Given: Water flow through a fire hose and nozzle.



Find: (a) Coupling force, R_x
(b) Indicate if in tension or compression.

Solution: Apply continuity and x component of momentum equation to inertial CV shown; use gage pressures to cancel p_{atm}.

Basic equations:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho\, dV + \int_{CS} \rho \vec{V}\cdot d\vec{A} \quad (=0(1))$$

$$F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u \rho\, dV + \int_{CS} u \rho \vec{V}\cdot d\vec{A} \quad (F_{B_x}=0(4),\ \tfrac{\partial}{\partial t}=0(1))$$

Assumptions: (1) Steady flow
(2) Uniform flow at each section
(3) Incompressible flow
(4) $F_{B_x} = 0$

Then

$$0 = \{-|\rho V_1 A_1|\} + \{|\rho V_2 A_2|\} = -\rho V_1 A_1 + \rho V_2 A_2$$

$$V_1 = V_2 \frac{A_2}{A_1} = V_2\left(\frac{D_2}{D_1}\right)^2 = 32\,\frac{m}{s} \times \left(\frac{25\,mm}{75\,mm}\right)^2 = 3.56\ m/s$$

and

$$R_x + p_{1g} A_1 = u_1\{-|\rho V_1 A_1|\} + u_2\{|\rho V_2 A_2|\}$$

$$u_1 = V_1 \qquad u_2 = V_2$$

$$R_x = -p_{1g}A_1 - V_1 \rho V_1 A_1 + V_2 \rho V_2 A_2 = -p_{1g} A_1 + \rho V_2 A_2 (V_2 - V_1)$$

$$= -510\times 10^3\,\frac{N}{m^2}\times\frac{\pi}{4}(0.075)^2 m^2 + 999\,\frac{kg}{m^3}\times 32\,\frac{m}{s}\times\frac{\pi}{4}(0.025)^2 m^2 (32.0-3.56)\frac{m}{s}\times\frac{N\cdot s^2}{kg\cdot m}$$

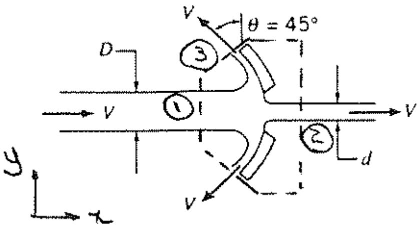
$R_x = -1.81$ kN (i.e. force on CV is to the left) R_x

Thus the coupling must be in tension.

Problem 4.57

Given: Circular dish with central orifice struck concentrically by water jet as shown

Find: (a) Expression for force needed to hold the dish in place
(b) Value of force for $V = 5$ m/s, $D = 100$ mm, and $d = 20$ mm



Plot: required force as a function of θ ($0 \le \theta \le 90°$) with d/D as a parameter.

Solution:

Apply the x component of the momentum equation to the inertial CV shown.

Basic equation: $F_{S_x} + \cancel{F_{B_x}} = \cancel{\frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall} + \int_{CS} u(\rho \vec{V}\cdot d\vec{A})$ $\quad$ $=0(2)$, $=0(3)$

Assumptions: (1) atmospheric pressure acts on all CV surfaces
(2) $F_{B_x} = 0$
(3) steady flow
(4) uniform flow at each section
(5) incompressible flow
(6) no change in jet speed on dish: $V_1 = V_2 = V_3 = V$

Then,

$$R_x = u_1\{-|\rho V_1 A_1|\} + u_2\{|\rho V_2 A_2|\} + u_3\{|\rho V_3 A_3|\}$$

$u_1 = V \quad A_1 = \frac{\pi D^2}{4} \quad u_2 = V \quad A_2 = \frac{\pi d^2}{4} \quad u_3 = -V\sin\theta \quad A_3 = A_1 - A_2$

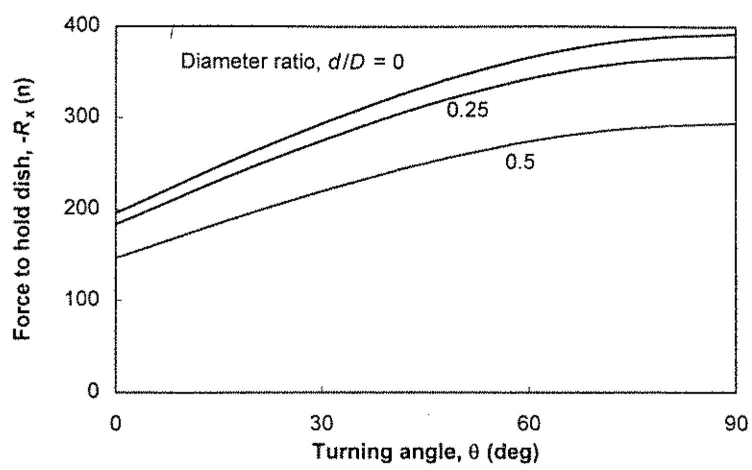
$$R_x = -\rho V^2 \frac{\pi D^2}{4} + \rho V^2 \frac{\pi d^2}{4} - \rho V^2 \sin\theta \frac{\pi}{4}(D^2 - d^2) = \rho V^2 \frac{\pi}{4}(1+\sin\theta)(d^2 - D^2)$$

$$R_x = -\rho V^2 \frac{\pi D^2}{4}(1+\sin\theta)\left[1 - \left(\frac{d}{D}\right)^2\right] \longleftarrow R_x$$

Evaluating for $d = 25$ mm

$$R_x = -\frac{\pi}{4} \times 999\frac{\text{kg}}{\text{m}^3} \times (5)^2\frac{\text{m}^2}{\text{s}^2} \times (0.10)^2\text{m}^2\,(1+\sin 45°)\left[1 - \left(\frac{25}{100}\right)^2\right]\frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}} = -314\text{ N} \longleftarrow R_x$$

Since $R_x < 0$, it must be applied to the left. R_x is plotted as a function of θ for different values of d/D

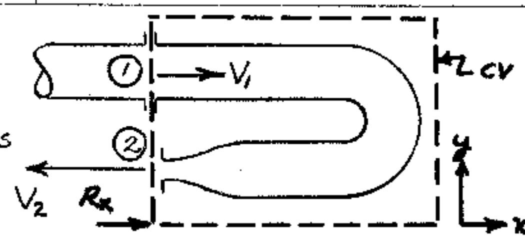


Problem 4.58

Given: Elbow assembly shown, water flow.

$p_1 = 96$ kPa (gage), $V_1 = 3.05$ m/s

$A_1 = 2600\ mm^2$, $A_2 = 650\ mm^2$



Find: Horizontal force required to hold in place.

Solution: Use CV shown, apply x component of momentum eq.

Basic equation: $$F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$$
($F_{B_x} = 0(1)$, $\frac{\partial}{\partial t} = 0(2)$)

Assumptions: (1) $F_{Bx} = 0$
(2) Steady flow
(3) Incompressible flow
(4) Uniform flow at each section

Then
$$R_x + p_{1g}A_1 = u_1\{-|\rho V_1 A_1|\} + u_2\{|\rho V_2 A_2|\}$$
$$u_1 = V_1 \qquad u_2 = -V_2$$

$$R_x = -p_{1g}A_1 + \dot{m}(-V_1 - V_2)$$

From continuity, $\dot{m} = \rho V_1 A_1 = \rho V_2 A_2$; $V_2 = V_1 \frac{A_1}{A_2} = 3.05\frac{m}{s} \times \frac{2600}{650} = 12.2$ m/s

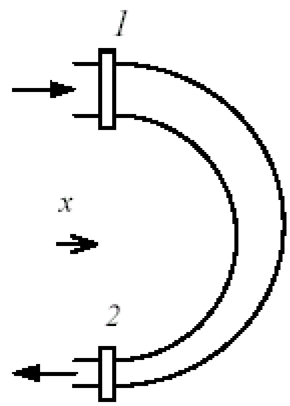
$$\dot{m} = 999\frac{kg}{m^3} \times 3.05\frac{m}{s} \times 2600\ mm^2 \times \frac{m^2}{10^6 mm^2} = 7.92\ kg/s$$

Thus
$$R_x = -96\times 10^3\frac{N}{m^2} \times 2600\ mm^2 \times \frac{m^2}{10^6 mm^2} + 7.92\frac{kg}{s}(-3.05 - 12.2)\frac{m}{s} \times \frac{N\cdot s^2}{kg\cdot m}$$

$$R_x = -370\ N \quad \text{(to the left)}$$ R_x

Problem 4.59

A 180° elbow takes in water at an average velocity of 1 m/s and a pressure of 400 kPa (gage) at the inlet, where the diameter is 0.25 m. The exit pressure is 50 kPa, and the diameter is 0.05 m. What is the force required to hold the elbow in place?



Given: Data on flow and system geometry

Find: Force required to hold elbow in place

Solution

The given data are

$\rho = 999 \cdot \frac{kg}{m^3}$ $\quad D_1 = 0.25 \cdot m$ $\quad D_2 = 0.05 \cdot m$ $\quad p_1 = 400 \cdot kPa$ $\quad p_2 = 50 \cdot kPa$

$V_1 = 1 \cdot \frac{m}{s}$

Then $\quad A_1 = \frac{\pi \cdot D_1^{\,2}}{4} \qquad A_1 = 0.0491\,m^2$

$A_2 = \frac{\pi}{4} \cdot D_2^{\,2} \qquad A_2 = 0.00196\,m^2$

$Q = V_1 \cdot A_1 \qquad Q = 0.0491\,\frac{m^3}{s}$

$V_2 = \frac{Q}{A_2} \qquad V_2 = 25\,\frac{m}{s}$

Governing equation:

Momentum

$$\vec{F} = \vec{F}_S + \vec{F}_B = \frac{\partial}{\partial t}\int_{\mathrm{CV}} \vec{V}\,\rho\, d\mathcal{V} + \int_{\mathrm{CS}} \vec{V}\,\rho\vec{V}\cdot d\vec{A} \qquad (4.17)$$

Applying this to the current system

$$4F\ 2\ p_1\ A_2\ 2\ p_2\ A_2\ |\ \ 0\ 2\ V_1\ (4\psi\ V_1\ A_1)\ 4\ V_2\ (\psi\ V_2\ A_2)$$

Hence

$$F\ |\ \ p_1\ A_1\ 2\ p_2\ A_2\ 2\ \psi\ (V_1^{\ 2}\ A_1\ 2\ V_2^{\ 2}\ A_2)$$

$$F\ |\ \ 400\ \frac{\mathrm{kN}}{\mathrm{m}^2}\ 0.0491\ \mathrm{m}^2\ 2\ 50\ \frac{\mathrm{kN}}{\mathrm{m}^2}\ 0.00196\ \mathrm{m}^2\ 55$$

$$2\ 999\ \frac{\mathrm{kg}}{\mathrm{m}^3}\left(\left(1\ \frac{\mathrm{m}}{\mathrm{s}}\right)^2 0.0491\ \mathrm{m}^2\ 2\ \left(25\ \frac{\mathrm{m}}{\mathrm{s}}\right)^2 0.00196\ \mathrm{m}^2\right)$$

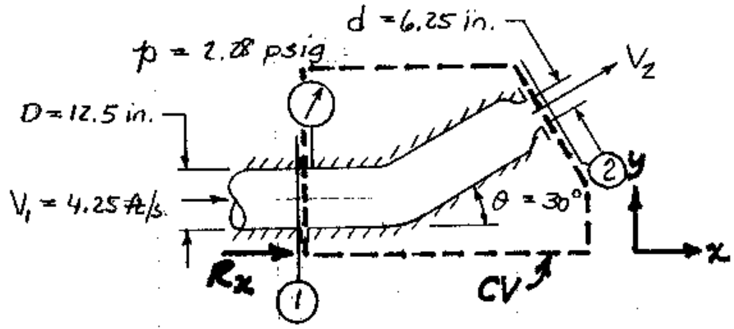
$$F\ |\ \ 21\,\mathrm{kN}$$

Problem 4.60

Given: Water flow through nozzle shown, discharging to p_{atm}.

Find: (a) Horizontal force component in the joint.

(b) Indicate whether joint is in tension or compression.



Solution: Apply continuity & x momentum using CV & CS shown.

Basic equation:

$$F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u \rho \, d\forall + \int_{CS} u \rho \vec{V} \cdot d\vec{A}$$

(F_{B_x} = 0(1), $\frac{\partial}{\partial t}$ term = 0(2))

Assumptions: (1) $F_{Bx} = 0$ (2) Steady flow (3) Incompressible (4) Uniform flow at each section (5) Use gage pressures

Then

$$p_{1g} A_1 + R_x = u_1 \{-|\rho V_1 A_1|\} + u_2 \{+|\rho V_2 A_2|\}$$

$$u_1 = V_1 \qquad u_2 = V_2 \cos\theta$$

So $R_x = V_1(-\rho V_1 A_1) + V_2 \cos\theta (+\rho V_2 A_2) - p_{1g} A_1$

From continuity, $\rho V_1 A_1 = \rho V_2 A_2$ and $V_2 = V_1 A_1 / A_2 = V_1 \left(\frac{D_1}{D_2}\right)^2$

$$V_2 = 4.25 \frac{ft}{s} \left(\frac{12.5\ in.}{6.25\ in.}\right)^2 = 17.0\ ft/s$$

$$R_x = \rho V_1 A_1 (V_2 \cos\theta - V_1) - p_{1g} A_1$$

$$= 1.94 \frac{slug}{ft^3} \times 4.25 \frac{ft}{s} \times \frac{\pi}{4}\left(\frac{12.5}{12}\right)^2 ft^2 \left[17.0 \frac{ft}{s} \times \cos 30° - 4.25 \frac{ft}{s}\right] \frac{lbf \cdot s^2}{slug \cdot ft}$$

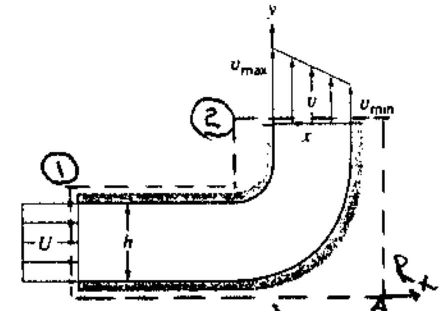
$$- 2.28 \frac{lbf}{in.^2} \times \frac{\pi}{4} (12.5)^2\ in.^2$$

$R_x = -206$ lbf (on CV; to left, since < 0) R_x

Thus $K_x = -R_x = +206$ lbf (from CV on joint)

∴ Joint is in tension

Problem 4.61



Given: Two-dimensional square bend shown is a segment of a larger channel, lies in horizontal plane.

$U = 7.5$ m/s, $h = w = 75.5$ mm

$p_1 = 170$ kPa(abs), $p_2 = 130$ kPa(abs)

$v_{max} = 2 v_{min}$; $v_{min} = 5.0$ m/s (from Problem 4.25)

Find: Force required to hold the bend in place.

Solution:

Basic equation: $\vec{F}_S + \vec{F}_B = \frac{\partial}{\partial t}\int_{CV} \vec{V}\rho\, d\forall + \int_{CS} \vec{V}(\rho\vec{V}\cdot d\vec{A})$ (the $\frac{\partial}{\partial t}$ term $= 0(1)$)

Assumptions: (1) steady flow
(2) $F_{B_x} = F_{B_y} = 0$
(3) incompressible flow
(4) atmospheric pressure acts on outside surfaces.

The x-momentum equation becomes

$$R_x + p_1 A_1 + F_{B_x} = \int_{CS} u(\rho\vec{V}\cdot d\vec{A}) = U\{-|\rho U A_1|\}$$ ($F_{B_x} = 0(2)$)

$$R_x = -p_1 A_1 - \rho U^2 A_1 = -h^2(p_1 + \rho U^2)$$

$$R_x = -(0.0755)^2 m^2 \left[(170-101)10^3 \frac{N}{m^2} + 999 \frac{kg}{m^3} \times (7.5)^2 \frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m}\right] = -714 \text{ N}$$

The y-momentum equation becomes

$$R_y - p_2 A_2 + F_{B_y} = \int_{CS} v(\rho\vec{V}\cdot d\vec{A})$$ ($F_{B_y} = 0(2)$)

$$v_2 = V_2 = v_{max} - (v_{max} - v_{min})\frac{x}{h} = 2v_{min} - v_{min}\frac{x}{h} = v_{min}\left(2 - \frac{x}{h}\right)$$

$$R_y - p_2 A_2 = \int_0^h v_{min}\left(2-\frac{x}{h}\right)\rho v_{min}\left(2-\frac{x}{h}\right) h\, dx$$

$$R_y = p_2 A_2 + \rho v_{min}^2 h \int_0^h \left(4 - 4\frac{x}{h} + \frac{x^2}{h^2}\right) dx$$

$$= p_2 A_2 + \rho v_{min}^2 h \left[4x - 2\frac{x^2}{h} + \frac{x^3}{3h^2}\right]_0^h$$

$$R_y = p_2 A_2 + \rho v_{min}^2 h \left[4h - 2h + \frac{h}{3}\right] = p_2 A_2 + \frac{7}{3}\rho v_{min}^2 h^2$$

$$R_y = h^2\left(p_2 + \frac{7}{3}\rho v_{min}^2\right)$$

$$= (0.0755)^2 m^2 \left[(130-101)10^3 \frac{N}{m^2} + \frac{7}{3} \times 999 \frac{kg}{m^3} \times (5.0)^2 \frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m}\right]$$

$$R_y = 498 \text{ N}$$

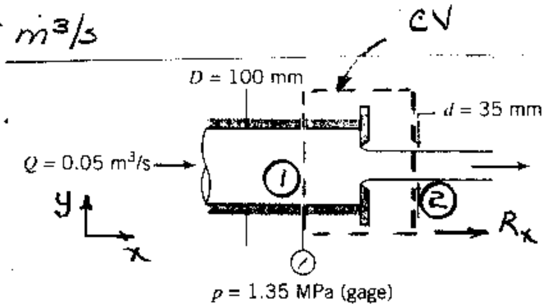
$$\therefore \vec{R} = -714\hat{i} + 498\hat{j} \text{ N}$$ ($\vec{R}$)

Problem 4.62

Given: Flat plate orifice at end of pipe, as shown.

$D = 100\ mm,\ d = 35\ mm,\ Q = 0.05\ m^3/s$

Neglect friction on pipe wall.



Find: Force to hold orifice plate.

Solution: Apply the x component of the momentum equation. The CV and CS are shown.

Basic equation:

$$F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$$

($F_{B_x} = 0$ (1), $\frac{\partial}{\partial t} = 0$ (2))

Assumptions:
(1) $F_{Bx} = 0$
(2) Steady flow
(3) Uniform flow at each section
(4) Use gage pressures to cancel p_{atm}
(5) Incompressible flow

Then

$$Q = V_1 A_1 = V_2 A_2;\quad V_1 = \frac{Q}{A_1} = \frac{4Q}{\pi D_1^2} = \frac{4}{\pi} \times 0.05\frac{m^3}{s} \times \frac{1}{(0.1)^2 m^2} = 6.37\ m/s$$

$$V_2 = V_1\left(\frac{D_1}{D_2}\right)^2 = 6.37\frac{m}{s}\left(\frac{100}{35}\right)^2 = 52.0\ m/s$$

From momentum,

$$R_x + p_{1g}A_1 = u_1\{-\rho Q\} + u_2\{+\rho Q\} = (V_2 - V_1)\rho Q$$

$$u_1 = V_1 \qquad u_2 = V_2$$

$$R_x = -p_{1g}A_1 + (V_2 - V_1)\rho Q$$

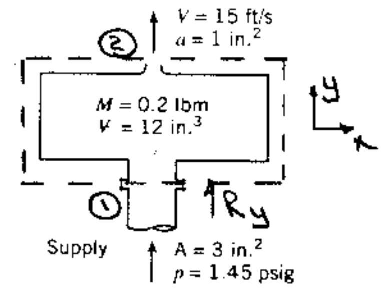
$$= -1.35\times10^6\frac{N}{m^2}\times\frac{\pi}{4}(0.1)^2 m^2 + (52.0 - 6.37)\frac{m}{s}\times 999\frac{kg}{m^3}\times 0.05\frac{m^3}{s}\times\frac{N\cdot s^2}{kg\cdot m}$$

$R_x = -8.32\ kN$ (to left) $\qquad R_x$

Problem 4.63

Given: Spray system, of mass $M = 0.200$ lbm and internal volume $\forall = 12\ \text{in}^3$ operates under steady state conditions shown.

Find: the vertical force exerted on the supply pipe by the spray system



Solution:

Apply the y component of the momentum equation to the fixed control volume shown.

Basic Equation:

$$F_{S_y} + F_{B_y} = \cancelto{0(1)}{\frac{\partial}{\partial t}\int_{CV} v\,\rho\, d\forall} + \int_{CS} v\,\rho\vec{V}\cdot d\vec{A} \quad ---- (1)$$

Assumptions:
(1) steady flow
(2) incompressible flow
(3) uniform flow at each section
(4) calculation of surface forces is simplified through use of gage pressures

From continuity, $0 = \cancel{\frac{\partial}{\partial t}\int \rho\, d\forall} + \int_{CS} \rho\vec{V}\cdot d\vec{A}$, for given conditions

$$0 = -|\rho V_1 A_1| + |\rho V_2 A_2| \quad \text{and} \quad V_1 = V_2\frac{A_2}{A_1} = V\frac{a}{A}$$

The momentum flux is

$$\int_{CS} v\,\rho\vec{V}\cdot d\vec{A} = v_1\{-|\rho V_1 A_1|\} + v_2\{|\rho V_2 A_2|\} = V_1(-\rho V_1 A_1) + V(\rho V a)$$

$$= V\frac{a}{A}(-\rho V a) + V(\rho V a) = \rho V^2 a\left(1 - \frac{a}{A}\right)$$

Then from eq (1) we can write

$$R_y + p_{1g}A - \rho\forall g - Mg = \rho V^2 a\left(1 - \frac{a}{A}\right)$$ Solving for R_y,

$$R_y = -p_{1g}A + \rho\forall g + Mg + \rho V^2 a\left(1 - \frac{a}{A}\right)$$

$$= -1.45\,\frac{\text{lbf}}{\text{in}^2}\times 3\,\text{in}^2 + 1.94\,\frac{\text{slug}}{\text{ft}^3}\times 12\,\text{in}^3\times 32.2\,\frac{\text{ft}}{\text{s}^2}\times\frac{\text{ft}^3}{1728\,\text{in}^3}\times\frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}}$$

$$+ 0.2\,\text{lbm}\times 32.2\,\frac{\text{ft}}{\text{s}^2}\times\frac{\text{slug}}{32.2\,\text{lbm}}\times\frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}}$$

$$+ 1.94\,\frac{\text{slug}}{\text{ft}^3}\times(15)^2\,\frac{\text{ft}^2}{\text{s}^2}\times 1\,\text{in}^2\times\frac{\text{ft}^2}{144\,\text{in}^2}\times\frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}}\left(1 - \frac{1\,\text{in}^2}{3\,\text{in}^2}\right)$$

$$R_y = -1.70\ \text{lbf}$$

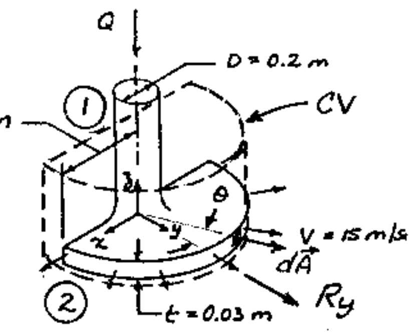
The force of the spray system on the supply pipe is

$$K_y = -R_y = 1.70\ \text{lbf (upward)} \leftarrow K_y$$

Problem 4.64

Given: Flow through semi-circular nozzle, as shown.

Find: (a) Volume flow rate
(b) y-component of force required to hold in place



Solution: Choose CV and coordinates shown. Apply continuity and momentum equation in y-direction.

Basic equations: $Q = \int_A \vec{V} \cdot d\vec{A}$

$$F_{S_y} + F_{B_y} = \frac{\partial}{\partial t}\int_{CV} v \rho \, dV + \int_{CS} v \rho \vec{V} \cdot d\vec{A}$$

($F_{B_y} = 0$ (2), $\frac{\partial}{\partial t}\int_{CV} = 0$ (3))

Assumptions: (1) Flow uniform across exit section
(2) $F_{B_y} = 0$
(3) Steady flow

At section (2), $\vec{V} \cdot d\vec{A} = VRt\,d\theta$, since flow out of CV. Then

$$Q = \int_{-\pi/2}^{\pi/2} VRt\,d\theta = VRt\left[\theta\right]_{-\pi/2}^{\pi/2} = VRt\pi$$

$$Q = 15\frac{m}{s} \times 0.3\,m \times 0.03\,m \times \pi = 0.424\ m^3/s$$ Q

From momentum

$$R_y = \int_{CS} v \rho \vec{V} \cdot d\vec{A} = \int_{A_1} v_1 \left\{-|\rho V_1 dA_1|\right\} + \int_{A_2} v_2 \left\{+|\rho V_2 dA_2|\right\}$$

with $v_1 = 0$ $\qquad v_2 = V\cos\theta$

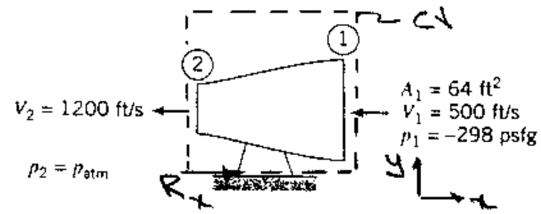
$$R_y = \int_{-\pi/2}^{\pi/2} V\cos\theta\, \rho V R t\, d\theta = \rho V^2 R t\left[\sin\theta\right]_{-\pi/2}^{\pi/2} = 2\rho V^2 R t$$

$$R_y = 2 \times 999\frac{kg}{m^3} \times (15)^2\frac{m^2}{s^2} \times 0.3\,m \times 0.03\,m \times \frac{N \cdot s^2}{kg \cdot m} = 4.05\ kN$$ Ry

Problem 4.65

Given: Jet engine on test stand.
Fuel enters vertically at rate

$\dot{m}_{fuel} = 0.02\,\dot{m}_{air}$



Find: (a) Air flow rate
(b) Estimate of engine thrust.

Solution:

Apply x-component of the momentum equation to CV shown

Basic equations: $F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho\vec{V}\cdot d\vec{A}$ (with $F_{B_x} = 0$ (1), $\frac{\partial}{\partial t} = 0$ (2))

$\dot{m}_{air} = \rho_1 V_1 A_1 \quad , \quad \rho = p/RT$

Assumptions: (1) $F_{B_x} = 0$
(2) steady flow
(3) uniform flow at inlet and outlet sections.
(4) air behaves as ideal gas; $T_1 = 70°F$
(5) fuel enters vertically (given).

$$\rho_1 = \frac{p_1}{RT_1} = \left(14.7\frac{lbf}{in^2}\times 144\frac{in^2}{ft^2} - 298\frac{lbf}{ft^2}\right)\times\frac{lbm\,°R}{53.3\,ft\,lbf}\times\frac{1}{530°R} = 0.0644\,\frac{lbm}{ft^3}$$

$$\dot{m}_{air} = \rho_1 V_1 A_1 = 0.0644\,\frac{lbm}{ft^3}\times 500\frac{ft}{s}\times 64\,ft^2 = 2060\ lbm/s \quad \dot{m}$$

From the x-momentum equation

$$R_{x} - p_{1g}A_1 + p_{2g}A_2 = u_1\{-\dot{m}_1\} + u_2\{\dot{m}_2\} + u_f\{-\dot{m}_f\}$$

(with $p_{2g} = 0$ and $u_f = 0$ (5))

$u_1 = -V_1 \quad , u_2 = -V_2 \quad , \quad \dot{m}_2 = \dot{m}_1 + \dot{m}_f$

Also thrust $T = K_x$ (force of engine on surroundings) $= -R_x$

So

$$-T - p_{1g}A_1 = \dot{m}_1 V_1 - \dot{m}_2 V_2 = \dot{m}_1 V_1 - (1.02\dot{m}_1)V_2$$

$$T = \dot{m}_1(1.02V_2 - V_1) - p_{1g}A_1$$

$$T = 2060\,\frac{lbm}{s}\left[1.02\times 1200\frac{ft}{s} - 500\frac{ft}{s}\right]\times\frac{slug}{32.2\,lbm}\times\frac{lbf\,s^2}{ft\,slug} - \left(-298\frac{lbf}{ft^2}\right)64\,ft^2$$

$$T = 65{,}400\ lbf \quad T$$

Problem 4.66

Given: Liquid-fueled rocket motor consumes 180 lbm/s of nitric acid as oxidizer and 70 lbm/s of analine as fuel. Flow leaves axially at $V = 6000$ ft/s relative to nozzle and $p = 16.5$ psia. Nozzle exit diameter $D = 2$ ft. Motor run on test stand at standard sea-level.

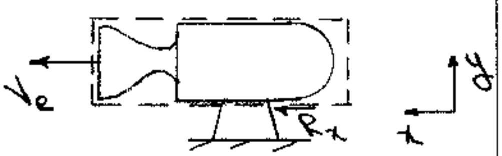
Find: Thrust produced by the motor on test stand.

Solution: Apply x-component of momentum equation to CV shown.

Basic eq.:

$$F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u \rho \, dV + \int_{CS} u \rho \vec{V} \cdot d\vec{A}$$

($F_{B_x} = 0$ (1); $\frac{\partial}{\partial t} \approx 0$ (2))



Assumptions:
(1) $F_{B_x} = 0$
(2) Neglect $\partial/\partial t$ of x momentum inside CV.
(3) uniform flow at nozzle exit.

Then

$$R_x - p_{eg} A_e = u_e \dot{m}$$

where $\dot{m} = \dot{m}_{n.a} + \dot{m}_a = (180 + 70)\,\text{lbm/s} = 250\,\text{lbm/s}$

R_x is force from test stand on CV

$$\therefore R_x = p_{eg} A_e + V_e \dot{m} = p_{eg} \frac{\pi D_e^2}{4} + V_e \dot{m}$$

$$= (16.5 - 14.7)\frac{\text{lbf}}{\text{in}^2} \times \frac{\pi}{4}(2)^2\,\text{ft}^2 \times 144\frac{\text{in}^2}{\text{ft}^2} + 6000\frac{\text{ft}}{\text{s}} \times 250\frac{\text{lbm}}{\text{s}} \times \frac{\text{slug}}{32.2\,\text{lbm}} \times \frac{\text{lbf}\cdot\text{s}^2}{\text{ft}\cdot\text{slug}}$$

$$R_x = 814\,\text{lbf} + 46{,}600\,\text{lbf} = 47{,}400\,\text{lbf}$$

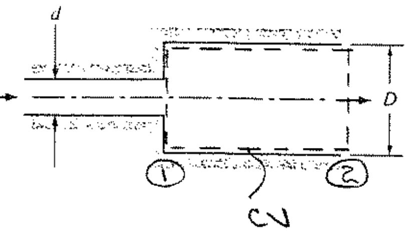
The thrust of the motor, $T = -R_x$

$$\vec{T} = -47{,}400\,\hat{i}\ \text{lbf} \quad (\text{to the right})$$

Given: Incompressible, frictionless flow through a sudden expansion as shown.

Show: Pressure rise, $\Delta p = p_2 - p_1$, is given by

$$\frac{\Delta p}{\frac{1}{2}\rho \bar{V}_1^2} = 2\left(\frac{d}{D}\right)^2\left[1-\left(\frac{d}{D}\right)^2\right]$$



Plot: the nondimensional pressure rise vs d/D to determine the optimum d/D and corresponding nondimensional pressure rise

Solution:

Apply x component of momentum equation, using fixed CV shown

Basic equation: $F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u(\rho \vec{V}\cdot d\vec{A})$ [with $F_{B_x} = 0$ (2), $\frac{\partial}{\partial t} = 0$ (3)]

Assumptions: (1) no friction, so surface force due to pressure only
(2) $F_{B_x} = 0$
(3) steady flow (4) incompressible flow (given)
(5) uniform flow at sections ① and ②
(6) uniform pressure p_1 on vertical surface of expansion

Then,

$$p_1 A_2 - p_2 A_2 = u_1\{-|\rho \bar{V}_1 A_1|\} + u_2\{|\rho \bar{V}_2 A_2|\} \qquad u_1 = \bar{V}_1, \ u_2 = \bar{V}_2$$

From continuity for uniform flow, $\dot{m} = \rho A_1 \bar{V}_1 = \rho A_2 \bar{V}_2$; $\bar{V}_2 = \bar{V}_1 \frac{A_1}{A_2}$

Thus,

$$p_2 - p_1 = \rho \bar{V}_1 \frac{A_1}{A_2} V_1 - \rho \bar{V}_1 \frac{A_1}{A_2} V_2 = \rho \bar{V}_1 \frac{A_1}{A_2}(\bar{V}_1 - \bar{V}_2)$$

$$p_2 - p_1 = \rho \bar{V}_1^2 \frac{A_1}{A_2}\left(1 - \frac{\bar{V}_2}{\bar{V}_1}\right) = \rho \bar{V}_1^2 \frac{A_1}{A_2}\left(1 - \frac{A_1}{A_2}\right)$$

and

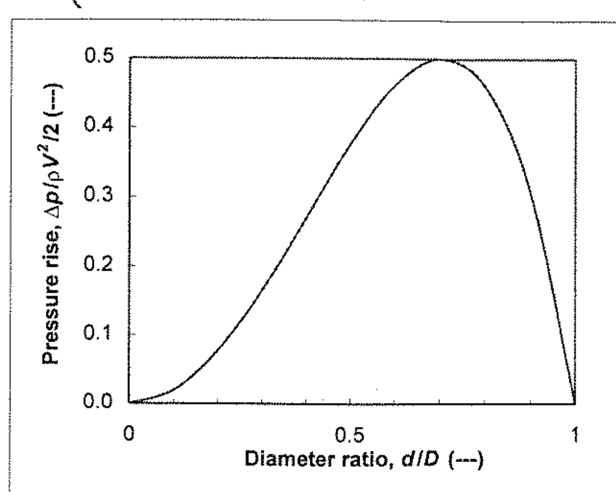
$$\frac{p_2 - p_1}{\frac{1}{2}\rho \bar{V}_1^2} = 2\frac{A_1}{A_2}\left(1 - \frac{A_1}{A_2}\right) = 2\left(\frac{d}{D}\right)^2\left[1-\left(\frac{d}{D}\right)^2\right]$$

Q.E.D.

From the plot below we see that $\frac{\Delta p}{\frac{1}{2}\rho \bar{V}_1^2}$ has an optimum value of ≈ 0.5 at $d/D = 0.70$

Note: As expected
- for $d = D$, $\Delta p = 0$ for straight pipe
- for $\frac{d}{D} \to 0$, $\Delta p = 0$ for free jet

Also note that the location of section ② would have to be chosen with care to make assumption (5) reasonable.

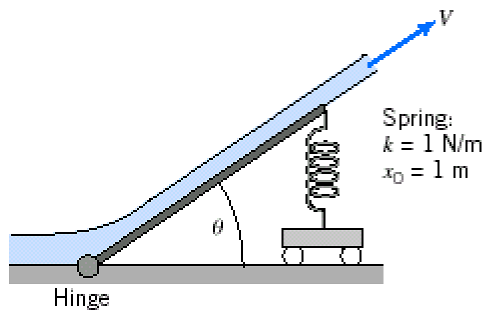


Problem 4.68

^{2} is deflected by a hinged plate of length 2 m supported by a spring with spring constant k = 1 N/m and uncompressed length x_0 = 1 m. Find and plot the deflection angle θ as a function of jet speed V. What jet speed has a deflection of 10°?

Given: Data on flow and system geometry

Find: Deflection angle as a function of speed; jet spee for 10° deflection



Solution

The given data are

$\psi \mid 999 \cdot \frac{kg}{m^3}$ $\quad A \mid 0.005 \cdot m^2$ $\quad L \mid 2 \cdot m$ $\quad k \mid 1 \cdot \frac{N}{m}$ $\quad x_0 \mid 1 \cdot m$

Governing equation:

Momentum

$$\vec{F} = \vec{F}_S + \vec{F}_B = \frac{\partial}{\partial t}\int_{CV} \vec{V}\,\rho\, d\Psi + \int_{CS} \vec{V}\,\rho\vec{V}\cdot d\vec{A} \qquad (4.17)$$

$$F_{spring} \mid V \cdot \sin/\chi 0 \cdot \psi \cdot V \cdot A 0$$

But $\quad F_{spring} \mid k \cdot x \mid k \cdot (x_0 \, 4 \, L \cdot \sin/\chi 0)$

Hence $\quad k \cdot (x_0 \, 4 \, L \cdot \sin/\chi 0) \mid \psi \cdot V^2 \cdot A \cdot \sin/\chi 0$

Solving for θ

$$\theta = \operatorname{asin}\left(\frac{k \cdot x_0}{k \cdot L - 2 \cdot \rho \cdot A \cdot V^2}\right)$$

For the speed at which θ = 10º, solve

$$V = \sqrt{\frac{k \cdot (x_0 - 4 \cdot L \cdot \sin(\theta))}{\rho \cdot A \cdot \sin(\theta)}}$$

$$V = \sqrt{\frac{1 \cdot \frac{N}{m} \cdot (1 - 4 \cdot 2 \cdot \sin(10)) \cdot m}{999 \cdot \frac{kg}{m^3} \cdot (0.005 \cdot m)^2 \cdot \sin(10)} \cdot \frac{kg \cdot m}{N \cdot s^2}}$$

$$V = 0.867\,\frac{m}{s}$$

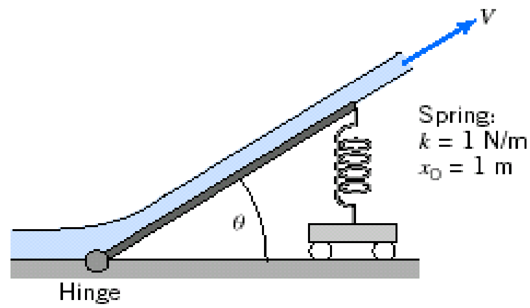
The deflection is plotted in the corresponding *Excel* workbook, where the above velocity is obtained using *Goal Seek*

Problem 4.68 (In Excel)

A free jet of water with constant cross-section area 0.005 m^2 is deflected by a hinged plate of length 2 m supported by a spring with spring constant k = 1 N/m and uncompressed length x_0 = 1 m. Find and plot the deflection angle θ as a function of jet speed V. What jet speed has a deflection of 10°?

Given: Geometry of system

Find: Speed for angle to be 10^o; plot angle versus speed



Solution

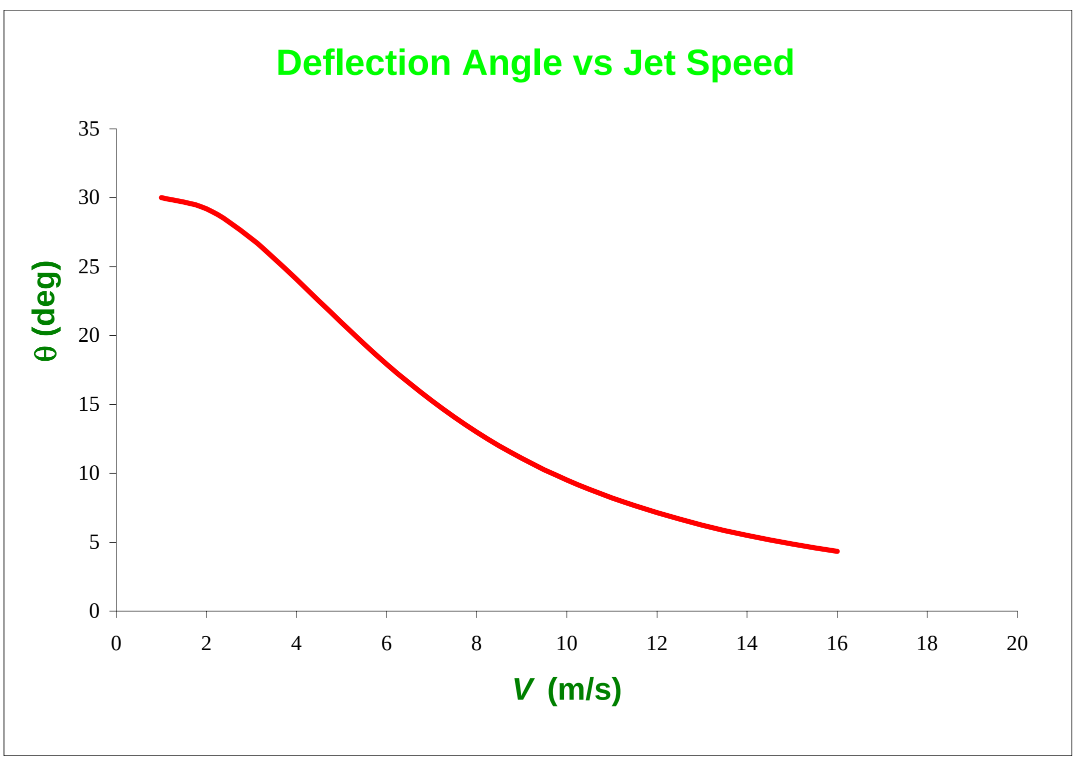
The equation for θ is $$\theta = \mathrm{asin}\left(\frac{k\,x_0}{k\,L + 2\,\rho\,A\,V^2}\right)$$

ρ =	999	kg/m^3
x_o =	1	m
L =	2	m
k =	1	N/m
A =	0.005	m^2

To find when θ = 10^o, use *Goal Seek*

V (m/s)	θ (°)
0.867	10

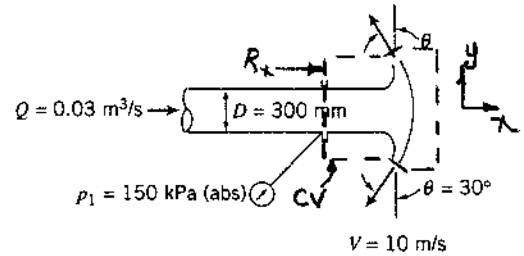
V (m/s)	θ (°)
0.0	30.0
0.1	29.2
0.2	27.0
0.3	24.1
0.4	20.9
0.5	17.9
0.6	15.3
0.7	13.0
0.8	11.1
0.9	9.52
1.0	8.22
1.1	7.14
1.2	6.25
1.3	5.50
1.4	4.87
1.5	4.33



Problem 4.69

Given: Conical spray head discharging water, as shown.

Find: (a) Thickness of spray sheet at $R = 400$ mm radius.
(b) Axial force exerted on supply pipe.



Solution: Apply continuity and the x component of the momentum equation, using the CV, CS shown.

Basic equation:

$$F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$$

($F_{B_x} = 0(1)$, $\frac{\partial}{\partial t} = 0(2)$)

Assumptions: (1) $F_{B_x} = 0$
(2) Steady flow.
(3) Incompressible flow
(4) Uniform flow at each section
(5) Use gage pressure to cancel p_{atm}

From continuity,

$$V_1 = \frac{Q}{A_1} = \frac{4Q}{\pi D_1^2} = \frac{4}{\pi} \times 0.03 \frac{m^3}{sec} \times \frac{1}{(0.3)^2\, m^2} = 0.424\ m/s$$

Assume velocity in jet sheet is constant at $V = 10$ m/s. Then

$$Q = 2\pi R t V;\quad t = \frac{Q}{2\pi R V} = \frac{1}{2\pi} \times 0.03 \frac{m^3}{s} \times \frac{1}{0.4\, m} \times \frac{s}{10\, m} \times 1000 \frac{mm}{m} = 1.19\ mm$$ t

From momentum,

$$R_x + p_{1g} A_1 = u_1\{-\rho Q\} + u_2\{+\rho Q\}$$

$$u_1 = V_1 \qquad u_2 = -V\sin\theta$$

$$R_x + p_{1g} A_1 = -(V_1 + V\sin\theta)\rho Q$$

or

$$R_x = -p_{1g} A_1 - (V_1 + V\sin\theta)\rho Q$$

$$= -(150-101)10^3 \frac{N}{m^2} \times \frac{\pi}{4}(0.3)^2 m^2 - (0.424 + 10\sin 30°)\frac{m}{s} \times 999 \frac{kg}{m^3} \times 0.03 \frac{m^3}{s} \times \frac{N\cdot s^2}{kg\cdot m}$$

$$R_x = -3.63\ kN$$

But R_x is force on CV; force on supply pipe is K_x,

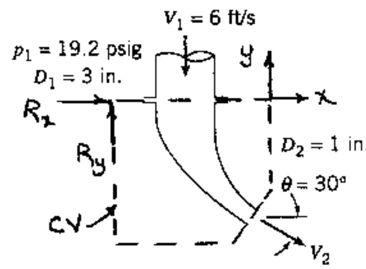
$$K_x = -R_x = 3.63\ kN \text{ (to the right)}$$ K_x

Problem 4.70

Given: Curved nozzle assembly, as shown.

$W = 10\ \text{lbf} \qquad \forall = 150\ \text{in.}^3$

Fluid is water.



Find: Force of nozzle assembly on inlet pipe.

Solution: Apply the x and y components of the momentum equation, using the CV and coordinates shown. Use gage pressures to cancel p_{atm}.

Basic equations:

$$\overset{=0(1)}{F_{Sx}} + \overset{=0(2)}{F_{Bx}} = \overset{=0(3)}{\frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall} + \int_{CS} u\rho\vec{V}\cdot d\vec{A}$$

$$F_{Sy} + F_{By} = \overset{=0(3)}{\frac{\partial}{\partial t}\int_{CV} v\rho\, d\forall} + \int_{CS} v\rho\vec{V}\cdot d\vec{A}$$

Assumptions:
(1) $F_{Sx} = 0$
(2) $F_{Bx} = 0$
(3) Steady flow
(4) Uniform flow at each section
(5) Incompressible flow

From continuity,

$$V_1 A_1 = V_2 A_2\ ;\ V_2 = V_1 \frac{A_1}{A_2} = V_1\left(\frac{D_1}{D_2}\right)^2 = 6\,\frac{\text{ft}}{\text{s}}\left(\frac{3\ \text{in.}}{1\ \text{in.}}\right)^2 = 54\ \text{ft/s}$$

From x component of momentum,

$$R_x = \overset{=0}{u_1}\{-|\rho V_1 A_1|\} + u_2\{+|\rho V_2 A_2|\} = \rho V_2^2 A_2 \cos\theta$$

$$R_x = 1.94\,\frac{\text{slug}}{\text{ft}^3}\times(54)^2\frac{\text{ft}^2}{\text{s}^2}\ \frac{\pi}{4}(1)^2\ \text{in.}^2\times\cos 30^\circ\times\frac{\text{ft}^2}{144\ \text{in.}^2}\times\frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}} = 26.7\ \text{lbf}$$

$K_x = -R_x = -26.7\ \text{lbf}$ (to left on pipe) K_x

From y component of momentum,

$$R_y - p_{1g}A_1 - W - \rho g\forall = v_1\{-|\rho V_1 A_1|\} + v_2\{+|\rho V_2 A_2|\} = (V_1 - V_2\sin\theta)\rho V_1 A_1$$

$$v_1 = -V_1 \qquad v_2 = -V_2\sin\theta$$

$$R_y = p_{1g}A_1 + W + \rho g\forall + (V_1 - V_2\sin\theta)\rho V_1 A_1$$

$$= 19.2\,\frac{\text{lbf}}{\text{in.}^2}\times\frac{\pi}{4}(3)^2\text{in.}^2 + 10\ \text{lbf} + 1.94\,\frac{\text{slug}}{\text{ft}^3}\times 32.2\,\frac{\text{ft}}{\text{s}^2}\times 150\ \text{in.}^3\times\frac{\text{ft}^3}{1728\ \text{in.}^3}\times\frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}}$$

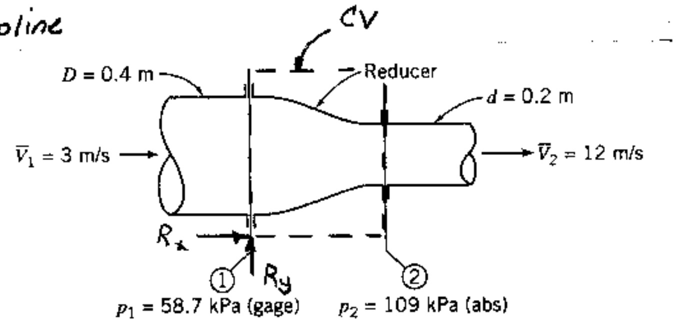
$$+\left[6 - 54\left(\tfrac{1}{2}\right)\right]\frac{\text{ft}}{\text{s}}\times 1.94\,\frac{\text{slug}}{\text{ft}^3}\times 6\,\frac{\text{ft}}{\text{s}}\times\frac{\pi}{4}(3)^2\text{in.}^2\times\frac{\text{ft}^2}{144\ \text{in.}^2}\times\frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}}$$

$R_y = 139\ \text{lbf}\ ;\ K_y = -R_x = -139\ \text{lbf}$ (down on pipe) K_y

Problem 4.71

Given: Flow through reducer in gasoline piping system, as shown.

$M = 25\ kg \quad V = 0.2\ m^3$



Find: Force needed to hold reducer in place.

Solution: Apply the x and y components of the momentum equation, using the CV and coordinates shown. Use gage pressures to cancel patm.

Basic equations:

$$F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\rho\, dV + \int_{CS} u\rho \vec{V}\cdot d\vec{A} \qquad (F_{Bx}=0(1),\ \tfrac{\partial}{\partial t}=0(2))$$

$$F_{Sy} + F_{By} = \frac{\partial}{\partial t}\int_{CV} v\rho\, dV + \int_{CS} v\rho \vec{V}\cdot d\vec{A} \qquad (\tfrac{\partial}{\partial t}=0(2))$$

Assumptions: (1) $F_{Bx} = 0$
(2) Steady flow
(3) Uniform flow at each section
(4) Incompressible flow, $SG = 0.72$ {Table A.2, Appendix A}

From the x component of momentum,

$$R_x + p_{1g}A_1 - p_{2g}A_2 = u_1\{-|\rho V_1 A_1|\} + u_2\{+|\rho V_2 A_2|\} = (V_2 - V_1)\rho V_1 A_1$$

$$u_1 = V_1 \qquad u_2 = V_2$$

$$R_x = p_{2g}A_2 - p_{1g}A_1 + (\bar{V}_2 - \bar{V}_1)\rho \bar{V}_1 A_1 \qquad \text{Note } \rho = SG\,\rho_{H_2O}$$

$$= (109-101)10^3 \frac{N}{m^2} \times \frac{\pi}{4}(0.2)^2 m^2 - 58.7\times10^3 \frac{N}{m^2} \times \frac{\pi}{4}(0.4)^2 m^2$$
$$+ (12-3)\frac{m}{s} \times (0.72)1000\frac{kg}{m^3} \times 3\frac{m}{s} \times \frac{\pi}{4}(0.4)^2 m^2 \times \frac{N\cdot s^2}{kg\cdot m}$$

$$R_x = -4.68\ kN \quad \text{(force must be applied to \underline{left})}$$ R_x

From the y component of momentum,

$$R_y - Mg - \rho g V = v_1\{-|\rho V_1 A_1|\} + v_2\{+|\rho V_2 A_2|\} \qquad (v_1 = 0,\ v_2 = 0)$$

$$R_y = Mg + \rho g V$$

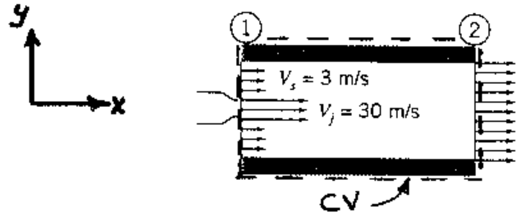
$$= 25\ kg \times 9.81\frac{m}{s^2} \times \frac{N\cdot s^2}{kg\cdot m} + (0.72)1000\frac{kg}{m^3} \times 9.81\frac{m}{s^2} \times 0.2\ m^3 \times \frac{N\cdot s^2}{kg\cdot m}$$

$$R_y = 1.66\ kN \quad \text{(force must be applied \underline{up})}$$ R_y

Problem 4.72

Given: Water jet pump as shown in the sketch.

$A_j = 0.01\,m^2$



$A_2 = 0.075\,m^2$

The two streams are thoroughly mixed at section ②, and the inlet pressures are the same.

Find: (a) The velocity at the pump exit
(b) The pressure rise, $p_2 - p_1$

Solution: Apply continuity and the x component of momentum to the inertial CV shown.

Basic equations:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A} \quad \left(\frac{\partial}{\partial t} = 0\,(1)\right)$$

$$F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho \vec{V}\cdot d\vec{A} \quad \left(F_{Bx} = 0\,(5),\ \frac{\partial}{\partial t} = 0\,(1)\right)$$

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Uniform flow at each section
(4) No viscous forces act on CV
(5) $F_{Bx} = 0$

Then from continuity

$$0 = \{-|\rho V_s A_s|\} + \{-|\rho V_j A_j|\} + \{|\rho V_2 A_2|\} = -\rho V_s A_s - \rho V_j A_j + \rho V_2 A_2$$

$$V_2 = \frac{1}{A_2}(V_s A_s + V_j A_j) \quad ; \quad A_s = A_2 - A_j = (0.075 - 0.01)\,m^2 = 0.065\,m^2$$

$$V_2 = \frac{1}{0.075\,m^2}\left(3\frac{m}{s} \times 0.065\,m^2 + 30\frac{m}{s} \times 0.01\,m^2\right) = 6.60\,\frac{m}{s}$$

V_2

and

$$p_1 A_2 - p_2 A_2 = u_s\{-|\rho V_s A_s|\} + u_j\{-|\rho V_j A_j|\} + u_2\{|\rho V_2 A_2|\}$$

$$u_s = V_s \qquad u_j = V_j \qquad u_2 = V_2$$

$$\Delta p = p_2 - p_1 = \frac{1}{A_2}\left(+\rho V_s^2 A_s + \rho V_j^2 A_j - \rho V_2^2 A_2\right) = \frac{\rho}{A_2}\left(+V_s^2 A_s + V_j^2 A_j - V_2^2 A_2\right)$$

$$= 999\frac{kg}{m^3} \times \frac{1}{0.075\,m^2}\left[(3.0)^2(0.065) + (30)^2(0.01) - (6.6)^2(0.075)\right]\frac{m^2}{s^2} \times m^2 \times \frac{N\cdot s^2}{kg\cdot m}$$

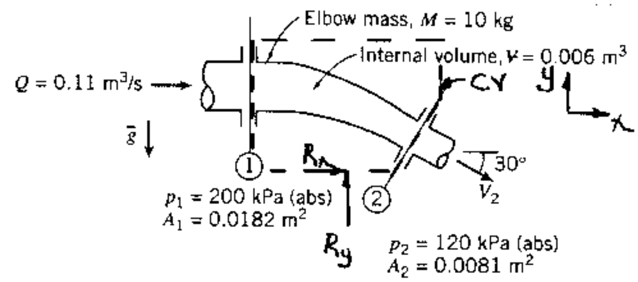
$$p_2 - p_1 = 84.2\ kPa$$

$p_2 - p$

Problem 4.73

Given: Reducing elbow shown.

Fluid is water.



Find: Force components needed to keep elbow from moving.

Solution: Apply the x and y components of the momentum equation using the cs and CV shown.

Basic equations:
$$F_{S_x} + \overset{=0(4)}{F_{B_x}} = \overset{=0(1)}{\frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall} + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$$
$$F_{S_y} + F_{B_y} = \overset{=0(1)}{\frac{\partial}{\partial t}\int_{CV} v\rho\, d\forall} + \int_{CS} v\rho \vec{V}\cdot d\vec{A}$$

Assumptions: (1) Steady flow (2) Uniform flow at each section (3) Use gage pressures (4) x horizontal

x comp: $R_x + p_{1g}A_1 - p_{2g}A_2\cos\theta = u_1\{-|\rho Q|\} + u_2\{+|\rho Q|\}$

$u_1 = V_1 \qquad u_2 = V_2\cos\theta$

$R_x = (-V_1 + V_2\cos\theta)\rho Q - p_{1g}A_1 + p_{2g}A_2\cos\theta$

$V_1 = \frac{Q}{A_1} = 0.11\,\frac{m^3}{s} \times \frac{1}{0.0182\,m^2} = 6.04\,\frac{m}{s}$

$V_2 = \frac{Q}{A_2} = 0.11\,\frac{m^3}{s} \times \frac{1}{0.0081\,m^2} = 13.6\,\frac{m}{s}$

$= \left(-6.04\,\frac{m}{s} + 13.6\,\frac{m}{s} \times \cos 30°\right) 999\,\frac{kg}{m^3} \times 0.11\,\frac{m^3}{s} \times \frac{N\cdot s^2}{kg\cdot m} - (200-101)10^3\,\frac{N}{m^2} \times 0.0182\,m^2 + (120-101)10^3\,\frac{N}{m^2} \times 0.0081\,m^2 \times \cos 30°$

$R_x = +631 - 1800 + 133\,N = -1040\,N$ $\qquad R_x$

y comp: $R_y + p_{2g}A_2\sin\theta - Mg - \rho\forall g = v_1\{-|\rho Q|\} + v_2\{+|\rho Q|\}$

$v_1 = 0 \qquad v_2 = -V_2\sin\theta$

$R_y = -V_2\sin\theta\,\rho Q + Mg + \rho\forall g - p_{2g}A_2\sin\theta$

$= -13.6\,\frac{m}{s} \times \sin 30° \times 999\,\frac{kg}{m^3} \times 0.11\,\frac{m^3}{s} \times \frac{N\cdot s^2}{kg\cdot m} + 10\,kg \times 9.81\,\frac{m}{s^2} \times \frac{N\cdot s^2}{kg\cdot m}$

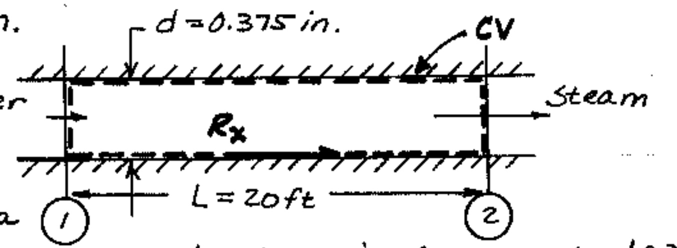
$+ 999\,\frac{kg}{m^3} \times 0.006\,m^3 \times 9.81\,\frac{m}{s^2} \times \frac{N\cdot s^2}{kg\cdot m} - (120-101)10^3\,\frac{N}{m^2} \times 0.0081\,m^2 \times \sin 30°$

$R_y = -747 + 98.1 + 58.8 - 77 = -667\,N$ $\qquad R_y$

{ R_x and R_y are the horizontal and vertical components of force that must be supplied by the adjacent pipes to keep the elbow (the control volume) from moving. }

Problem 4.74

Given: Monotube boiler, as shown.



$\dot{m} = 0.3\ lbm/s \qquad p_1 = 500\ psia$

$p_2 = 400\ psig, \ \rho_2 = 0.024\ slug/ft^3$

Find: Magnitude and direction of force exerted by fluid on tube.

Solution: Apply the x component of the momentum equation, using the CV and coordinates shown.

Basic equation:

$$F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u \rho \, d\forall + \int_{CS} u \rho \vec{V} \cdot d\vec{A}$$

($F_{B_x} = 0(1)$, $\frac{\partial}{\partial t} = 0(2)$)

Assumptions: (1) $F_{B_x} = 0$
(2) Steady flow
(3) Uniform flow at each section
(4) Use gage pressures to cancel patm

From continuity,

$\dot{m} = \rho_1 V_1 A_1 = \rho_2 V_2 A_2$; A = constant, so $\rho_1 V_1 = \rho_2 V_2$. Thus

$$V_1 = \frac{\dot{m}}{\rho_1 A} = 0.3\,\frac{lbm}{s} \times \frac{ft^3}{1.94\,slug} \times \frac{4}{\pi}\frac{1}{(0.375)^2\,in.^2} \times \frac{slug}{32.2\,lbm} \times 144\,\frac{in.^2}{ft^2} = 6.26\ ft/s$$

and

$$V_2 = V_1 \frac{\rho_1}{\rho_2} = 6.26\,\frac{ft}{s} \times 1.94\,\frac{slug}{ft^3} \times \frac{ft^3}{0.024\,slug} = 506\ ft/s$$

From momentum,

$$R_x + p_{1g} A_1 - p_{2g} A_2 = u_1\{-\dot{m}\} + u_2\{+\dot{m}\} = (V_2 - V_1)\dot{m}$$

$$u_1 = V_1 \qquad u_2 = V_2$$

$$R_x = (p_{2g} - p_{1g})A + (V_2 - V_1)\dot{m}$$

$$= [400 - (500 - 14.7)]\,\frac{lbf}{in.^2} \times \frac{\pi}{4}(0.375)^2\,in.^2 + (506 - 6.26)\frac{ft}{s} \times 0.3\,\frac{lbm}{s} \times \frac{slug}{32.2\,lbm} \times \frac{lbf \cdot s^2}{slug \cdot ft}$$

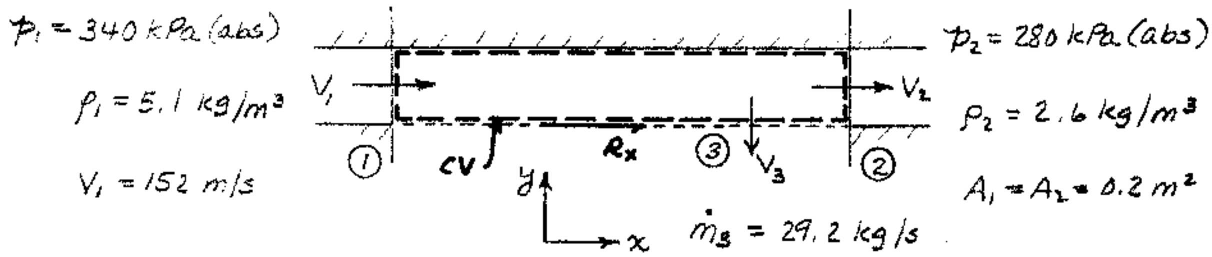
$$R_x = -4.77\ lbf$$

But R_x is force on CV; force on pipe is K_x,

$K_x = -R_x = 4.77\ lbf$ (to right) K_x

Problem 4.75

Given: Gas flow through a porous pipe of constant area.



V_3 is uniform over surface ③ and normal to pipe wall.

Find: Axial force of fluid on pipe.

Solution: Apply continuity and x component of momentum equation using inertial CV shown.

Basic equations: $$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$ (first term $=0$ (1))

$$F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u \rho \, d\forall + \int_{CS} u \rho \vec{V} \cdot d\vec{A}$$ ($F_{B_x} = 0$ (3), time derivative $=0$ (1))

Assumptions: (1) Steady flow
(2) Flow uniform at each section
(3) $F_{B_x} = 0$
(4) Flow at section ③ normal to wall; $u_3 = 0$

Then

$$0 = \{-|\rho_1 V_1 A|\} + \{|\rho_2 V_2 A|\} + \dot{m}_3 = -\rho_1 V_1 A + \rho_2 V_2 A + \dot{m}_3$$

$$V_2 = \frac{1}{\rho_2 A}\left[\rho_1 V_1 A - \dot{m}_3\right] = V_1 \frac{\rho_1}{\rho_2} - \frac{\dot{m}_3}{\rho_2 A}$$

$$V_2 = 152 \frac{m}{s} \times 5.1 \frac{kg}{m^3} \times \frac{m^3}{2.6\,kg} - 29.2 \frac{kg}{s} \times \frac{m^3}{2.6\,kg} \times \frac{1}{0.2\,m^2} = 242 \; m/s$$

and

$$R_x + p_1 A - p_2 A = u_1\{-|\rho_1 V_1 A|\} + u_2\{|\rho_2 V_2 A|\} + u_3\{|\dot{m}_3|\}$$ ($u_3 = 0$ (4))

$$u_1 = V_1 \qquad u_2 = V_2$$

$$R_x = (p_2 - p_1 + \rho_2 V_2^2 - \rho_1 V_1^2) A$$

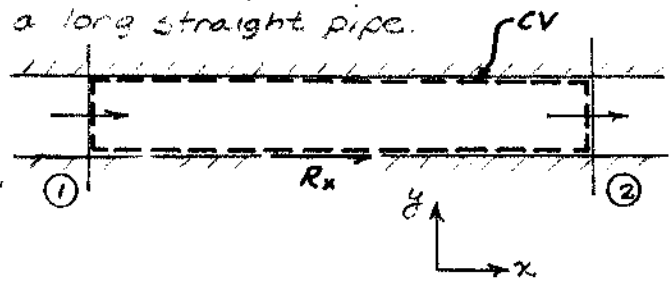
$$= \left[(280-340)10^3 \frac{N}{m^2} + \left(2.6 \frac{kg}{m^3} \times (242)^2 \frac{m^2}{s^2} - 5.1 \frac{kg}{m^3} \times (152)^2 \frac{m^2}{s^2}\right) \frac{N \cdot s^2}{kg \cdot m}\right] 0.2\,m^2$$

$R_x = -5.11$ kN (this is force of the duct wall on the gas)

The force of the gas on the duct wall is $K_x = -R_x = 5.11$ kN (acting to the right) K_x

Problem 4.76

Given: Air flow in a long straight pipe.



$V_1 = 500\ \frac{ft}{s}$ $\qquad$ $A_1 = A_2 = 0.5\ ft^2$

$p_1 = 30$ psia $\qquad$ $V_2 = 985\ \frac{ft}{s}$

$T_1 = 140°F$ $\qquad$ $p_2 = 11.3$ psia

Find: Axial force of the air on the pipe.

Solution: Apply the x component of the momentum equation to the inertial CV shown. Also use continuity and ideal gas.

Basic equations:

$$F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$$

($F_{B_x} = 0(1)$, $\frac{\partial}{\partial t} = 0(2)$)

$$p = \rho R T$$

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A}$$

($\frac{\partial}{\partial t} = 0(2)$)

Assumptions:
(1) $F_{B_x} = 0$
(2) Steady flow
(3) Uniform flow at each section
(4) Air behaves as an ideal gas

Then

$$R_x + p_1 A_1 - p_2 A_2 = u_1\{-|\rho_1 V_1 A_1|\} + u_2\{|\rho_2 V_2 A_2|\}$$

But from continuity $\quad 0 = \{-|\rho_1 V_1 A_1|\} + \{|\rho_2 V_2 A_2|\}$, so

$$R_x = p_2 A_2 - p_1 A_1 + u_1\{-|\rho_1 V_1 A_1|\} + u_2\{|\rho_1 V_1 A_1|\} \quad ; A_1 = A_2 = A$$

$$u_1 = V_1 \qquad\qquad u_2 = V_2$$

$$R_x = (p_2 - p_1)A + (V_2 - V_1)\rho_1 V_1 A$$

From the ideal gas equation of state

$$\rho_1 = \frac{p_1}{RT_1} = 30\,\frac{lbf}{in.^2} \times 144\,\frac{in.^2}{ft^2} \times \frac{lbm\cdot°R}{53.3\,ft\cdot lbf} \times \frac{slug}{32.2\,lbm} \times \frac{1}{600°R} = 0.00420\ \frac{slug}{ft^3}$$

and

$$R_x = (11.3 - 30)\frac{lbf}{in.^2} \times 0.5\,ft^2 \times 144\,\frac{in.^2}{ft^2}$$
$$+ (985 - 500)\frac{ft}{s} \times 0.0042\,\frac{slug}{ft^3} \times 500\,\frac{ft}{s} \times 0.5\,ft^2 \times \frac{lbf\cdot s^2}{slug\cdot ft}$$

$R_x = -837$ lbf (this is force of the pipe wall on the CV)

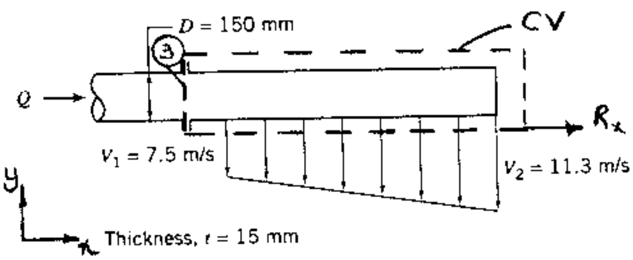
The force of the gas on the pipe is then

$K_x = -R_x = 837$ lbf (to the right) $\qquad K_x$

Problem 4.77

Given: Water flow discharging nonuniformly from slot, as shown.

$p_{1g} = 30 \text{ kPa}$



Find: (a) Volume flow rate.
(b) Forces to hold pipe.

Solution: Apply x, y components of momentum, using the CV, CS shown.

Basic equations:

$$F_{S_x} + \cancel{F_{B_x}}^{=0(1)} = \cancel{\frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall}^{=0(2)} + \int_{CS} u\rho \vec{V}\cdot d\vec{A}\;;\quad F_{S_y} + \cancel{F_{B_y}}^{=0(1)} = \cancel{\frac{\partial}{\partial t}\int_{CV} v\rho\, d\forall}^{=0(2)} + \int_{CS} v\rho\vec{V}\cdot d\vec{A}$$

Assumptions: (1) $F_{B_x} = F_{B_y} = 0$
(2) Steady flow
(3) Uniform flow at inlet section
(4) Use gage pressures to cancel p_{atm}

From continuity,

$$Q = \bar{V}A = \frac{1}{2}(V_1 + V_2)Lt = \frac{1}{2}(7.5 + 11.3)\frac{m}{s} \times 1\,m \times 0.015\,m = 0.141\ m^3/s$$ Q

$$V_3 = \frac{Q}{A_3} = 0.141\frac{m^3}{s} \times \frac{4}{\pi}\frac{1}{(0.15)^2 m^2} = 7.98\ m/s$$

From x momentum, since flow leaves slot vertically ($u = 0$),

$$R_x + p_{3g}A_3 = u_3\{-\rho Q\} = -V_3\rho Q\;;\quad R_x = -p_{3g}A_3 - V_3\rho Q$$

$$R_x = -30\times10^3\frac{N}{m^2} \times \frac{\pi}{4}(0.15)^2 m^2 - 7.98\frac{m}{s} \times 999\frac{kg}{m^3} \times 0.141\frac{m^3}{s} \times \frac{N\cdot s^2}{kg\cdot m}$$

$R_x = -1.65$ kN (to left) R_x

From y momentum, since $v_3 = 0$,

$$R_y = \cancel{v_3}^{=0}\{-\rho Q\} + \int_0^L v\rho V t\,dx = -\rho t\int_0^L \left(V_1 + \frac{V_2 - V_1}{L}x\right)^2 dx$$

$$= -\rho t\left[V_1^2 x + 2V_1\left(\frac{V_2 - V_1}{L}\right)\frac{x^2}{2} + \left(\frac{V_2 - V_1}{L}\right)^2\frac{x^3}{3}\right]_0^L$$

$$= -999\frac{kg}{m^3} \times 0.015\,m\left[(7.5)^2\frac{m^2}{s^2} + 7.5\frac{m}{s} \times (11.3 - 7.5)\frac{m}{s} \times \frac{1}{1m} \times (1)^2 m^2 + (11.3 - 7.5)^2\frac{m^2}{s^2} \times \frac{1}{(1)^2 m^2} \times \frac{(1)^3 m^3}{3}\right]$$

$R_y = -1.34$ kN (down) R_y

{A moment also would be required at the coupling.}

Problem 4.78

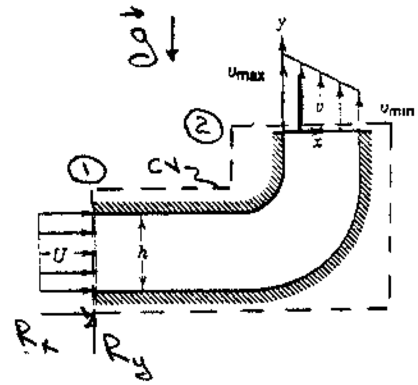
Given: Steady flow of water through square channel shown

$V_{max} = 2V_{min}$, $U = 7.5$ m/s , $p_1 = 185$ kPa (gage) , $p_2 = p_{atm}$

$M_c = 2.05$ kg , $\forall_c = 0.00355$ m^3 , $h = 75.5$ mm $= w$

Find: Force exerted by channel assembly on the supply duct.

Solution: Apply conservation of mass + momentum equations to the CV shown.



Basic equations:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A} \quad (1)$$

$$F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u \rho \, d\forall + \int_{CS} u \rho \vec{V} \cdot d\vec{A} \quad (2)$$

$$F_{S_y} + F_{B_y} = \frac{\partial}{\partial t}\int_{CV} v \rho \, d\forall + \int_{CS} v \rho \vec{V} \cdot d\vec{A} \quad (3)$$

(unsteady terms = 0 (1))

Assumptions:
(1) steady flow (2) incompressible flow
(3) uniform flow at inlet
(4) use gage pressures.

From continuity, $0 = \vec{V}_1 \cdot \vec{A}_1 + \int \vec{V}_2 \cdot d\vec{A}_2 = -Uwh + \int_0^h v w \, dx$

$\therefore Uh = \int_0^h v \, dx = \int_0^h v_{min}\left(2 - \frac{x}{h}\right) dx = v_{min}\left[2x - \frac{x^2}{2h}\right]_0^h = \frac{3}{2} v_{min} h$

and $v_{min} = \frac{2}{3} U = \frac{2}{3} \times 7.5 \frac{m}{s} = 5.0$ m/s

From Eq. 2,

$R_x + p_{1g} A_1 = u_1\{-\rho U A_1\} + \int_0^h u_2 \rho v_{min}\left(2 - \frac{x}{h}\right) w \, dx = -\rho U^2 A_1$ ($u_2 = 0$)

$R_x = -p_{1g} A_1 - \rho U^2 A_1 = -(185-101)10^3 \frac{N}{m^2} \times (0.0755)^2 m^2 - 999 \frac{kg}{m^3} \times (7.5)^2 \frac{m^2}{s^2} \times (0.0755)^2 m^2$

$R_x = -479 \, N - 320 \frac{kg \cdot m}{s^2} \cdot \frac{N \cdot s^2}{kg \cdot m} = -479 N - 320 N = -799$ N

$K_x = -R_x = 799$ N (on supply duct to the right) ← K_x

From Eq. 3,

$R_y - M_c g - \rho \forall g = v_1\{-\rho U A_1\} + \int_0^h v_2 \{\rho v_2 w \, dx\}$ ($v_1 = 0$)

$R_y - M_c g - \rho \forall g = \int_0^h v_{min}\left(2 - \frac{x}{h}\right) \rho v_{min}\left(2 - \frac{x}{h}\right) w \, dx$

$= \rho v_{min}^2 w \int_0^h \left(4 - 4\frac{x}{h} + \frac{x^2}{h^2}\right) dx$

$= \rho v_{min}^2 w \left[4x - 2\frac{x^2}{h} + \frac{x^3}{3h^2}\right]_0^h = \rho v_{min}^2 w h \frac{7}{3}$

$\therefore R_y = \left[2.05 \, kg \times 9.81 \frac{m}{s^2} + 999 \frac{kg}{m^3} \times (0.00355 \, m^3) 9.8 \frac{m}{s^2} + \frac{7}{3} \times 999 \frac{kg}{m^3} \times (5.0)^2 \frac{m^2}{s^2} \times (0.0755)^2 m^2\right] \frac{N \cdot s^2}{kg \cdot m}$

$R_y = (20.1 + 34.8 + 332)$ N $= 387$ N (on CV)

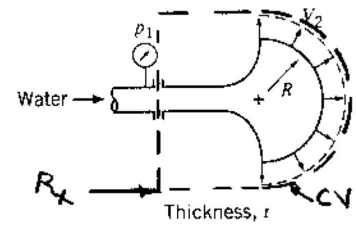
$K_y = -R_y = -387$ N (on supply duct, down) ← K_y

Problem 4.79

Given: Nozzle discharging flat, radial sheet of water, as shown.

Find: Axial force of nozzle on coupling.

$D_1 = 35\ mm$



Solution: Apply the x component of momentum, using CV and coordinates shown.

Basic equation:

$$F_{S_x} + \cancel{F_{B_x}}^{=0(1)} = \cancel{\frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall}^{=0(2)} + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$$

Assumptions: (1) $F_{B_x} = 0$
(2) Steady flow
(3) Uniform flow at each section
(4) Use gage pressure to cancel p_{atm}

From continuity

$$Q = V_1 A_1 = V_2 A_2 = V_2 \pi R t = \pi \times 10\frac{m}{sec} \times 0.05\, m \times 0.0015\, m = 0.00236\ m^3/s$$

$$V_1 = \frac{Q}{A_1} = \frac{4Q}{\pi D_1^2} = \frac{4}{\pi} \times 0.00236\frac{m^3}{sec} \times \frac{1}{(0.035)^2 m^2} = 2.45\ m/s$$

From momentum $\left\{ \text{Note } A_1 = \frac{\pi D_1^2}{4} = 0.000962\ m^2 \right\}$

$$R_x + p_{1g} A_1 = u_1\{-\rho Q\} + \int_{A_2} u_2 \rho V_2\, dA_2$$

$$u_1 = V_1 \qquad u_2 = V_2 \cos\theta\ ;\ dA_2 = Rt\, d\theta$$

$$\int_{A_2} = \int_{-\pi/2}^{\pi/2} V_2 \cos\theta\, \rho V_2 R t\, d\theta = 2\rho V_2^2 R t \int_0^{\pi/2} \cos\theta\, d\theta = 2\rho V_2^2 R t$$

Thus

$$R_x = -p_{1g} A_1 - V_1 \rho Q + 2\rho V_2^2 R t$$

$$= -(150-101)10^3\frac{N}{m^2} \times 0.000962\, m^2 - 2.45\frac{m}{sec} \times 999\frac{kg}{m^3} \times 0.00236\frac{m^3}{sec} \times \frac{N\cdot sec^2}{kg\cdot m}$$

$$+ 2 \times 999\frac{kg}{m^3} \times (10)^2\frac{m^2}{sec^2} \times 0.05\, m \times 0.0015\, m \times \frac{N\cdot sec^2}{kg\cdot m}$$

$$R_x = -37.9\ N$$

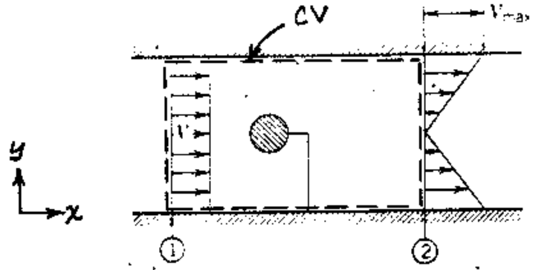
But R_x is force on CV; force on coupling is K_x,

$$K_x = -R_x = 37.9\ N \text{ (to right)}$$ K_x

Problem 4.80

Given: Small round object tested in wind tunnel. Neglect friction.

Find: (a) Mass flow rate
(b) $V_{2,max}$
(c) Drag of object



$p_1 = 20$ mm H_2O (gage) $\quad p_2 = 10$ mm H_2O (gage)

$V_1 = 10$ m/s

Solution:

Basic equations:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A} \qquad \left(\frac{\partial}{\partial t} = 0 \ (1)\right)$$

$$F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u \rho \, d\forall + \int_{CS} u \rho \vec{V} \cdot d\vec{A} \qquad \left(F_{B_x} = 0 \ (4), \ \frac{\partial}{\partial t} = 0 \ (1)\right)$$

$$p_1 = \rho g h_1 = 999 \frac{kg}{m^3} \times 9.81 \frac{m}{sec^2} \times 0.02\, m = 196 \text{ Pa (gage)}$$

$$p_2 = 98.0 \text{ Pa (gage)}$$

Assumptions: (1) Steady flow
(2) Density uniform at each section
(3) Uniform flow at Section ①, so $\dot{m} = \rho V_1 A$
(4) Horizontal flow; $F_{B_x} = 0$

Then

$$\dot{m} = \rho_1 V_1 A = 1.23 \frac{kg}{m^3} \times 10 \frac{m}{s} \times \frac{\pi}{4}(1)^2 m^2 = 9.67 \text{ kg/s} \qquad \dot{m}$$

From continuity,

$$\dot{m} = \int_{A_2} \rho_2 u_2 \, dA_2 = \rho_2 \int_0^R V_{2,max} \frac{r}{R} 2\pi r \, dr = 2\pi \rho_2 V_{2,max} R^2 \int_0^1 \left(\frac{r}{R}\right)^2 d\left(\frac{r}{R}\right) = \frac{2\pi}{3} \rho_2 V_{2,max} R^2$$

$$V_{2,max} = \frac{3\dot{m}}{2\pi \rho_2 R^2} = \frac{3}{2\pi} \times 9.67 \frac{kg}{s} \times \frac{m^3}{1.23\, kg} \times \frac{1}{(0.5)^2 m^2} = 15.0 \text{ m/s} \qquad V_{2,max}$$

From the momentum equation,

$$R_x + p_1 A - p_2 A = u_1\{-\dot{m}\} + \int_{A_2} u \rho_2 V_2 \, dA_2 = -V_1 \dot{m} + 2\pi \rho_2 V_{2,max}^2 R^2 \int_0^1 \left(\frac{r}{R}\right)^3 d\left(\frac{r}{R}\right)$$

$$u_1 = V_1 \qquad u_2 = V_{2,max} \frac{r}{R}$$

$$R_x = (p_2 - p_1)A - V_1 \dot{m} + 2\pi \rho_2 V_{2,max}^2 R^2 \left(\frac{1}{4}\right)$$

$$= (98.0 - 196) \frac{N}{m^2} \times \frac{\pi}{4}(1)^2 m^2 + \left[-10 \frac{m}{s} \times 9.67 \frac{kg}{s} + \frac{\pi}{2} \times 1.23 \frac{kg}{m^3} \times (15)^2 \frac{m^2}{s^2} \times (0.5)^2 m^2\right] \frac{N \cdot s^2}{kg \cdot m}$$

$$R_x = -65.0 \text{ N}$$

R_x is force to hold CV in place. CV cuts strut, so R_x is force needed to hold object. Drag of object and strut is

$$F_D = |R_x| = 65.0 \text{ N} \qquad F_D$$

Problem 4.81

The horizontal velocity in the wake behind an object in an air stream of velocity *U* is given by

$$u(r) = U\left(1 - \cos^2\left(\frac{\pi r}{2}\right)\right) \qquad |r| \le 1$$

$$u(r) = U \qquad |r| > 1$$

where *r* is the non-dimensional radial coordinate, measured perpendicular to the flow. Find an expression for the drag on the object.

Given: Data on wake behind object

Find: An expression for the drag

Solution

Governing equation:

Momentum

$$\vec{F} = \vec{F}_S + \vec{F}_B = \frac{\partial}{\partial t}\int_{CV} \vec{V}\,\rho\, dV + \int_{CS} \vec{V}\,\rho\vec{V}\cdot d\vec{A} \qquad (4.17)$$

Applying this to the horizontal motion

$$-F = U\left(-\rho\,\pi\,1^2\,U\right) + \int_0^1 u(r)\,\rho\,2\,\pi\,r\,u(r)\,dr$$

$$F = \pi\,\rho\left(U^2 - 2\int_0^1 r\,u(r)^2\,dr\right)$$

$$F | \phi \psi U^2 \left(1 \; 4 \; 2 \int_0^1 r \left(1 \; 4 \cos\left(\frac{\phi r}{2}\right)\right)^2 dr \right)$$

$$F | \phi \psi U^2 \left(1 \; 4 \; 2 \int_0^1 r \; 4 \; 2 \, r \cos\left(\frac{\phi r}{2}\right)^2 \; 2 \, r \cos\left(\frac{\phi r}{2}\right)^4 dr \right)$$

Integrating and using the limits

$$F | \phi \psi U^2 \left(1 \; 4 \; \frac{3}{8} \; 2 \; \frac{2}{\phi^2} \right)$$

$$F | \left(\frac{5 \phi}{8} \; 4 \; \frac{2}{\phi} \right) \psi U^2$$

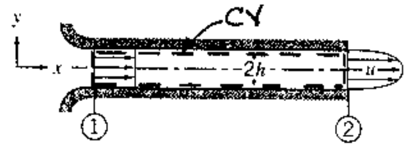
Problem 4.82

Given: Incompressible flow in entrance region of two-dimensional channel.

$u_2 = u_{max}\left[1-\left(\frac{y}{h}\right)^2\right]$

Find: (a) Maximum velocity at section ②.
(b) Pressure drop if viscous friction could be neglected.

Solution: Apply continuity and the x momentum equations.
Use the CV and CS shown.



$U_1 = 20$ ft/s
$\rho = 0.00238$ slug/ft³

Basic equations:

$$0 = \frac{\partial}{\partial t}\int_{CV}\rho\, d\forall + \int_{CS}\rho\vec{V}\cdot d\vec{A} \quad (=0(1))$$

$$F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho\vec{V}\cdot d\vec{A} \quad (F_{Bx}=0(4),\ \tfrac{\partial}{\partial t}=0(1))$$

Assumptions: (1) Steady flow
(2) Uniform flow at section ①
(3) $F_{Bx} = 0$
(4) Neglect friction at duct wall
(5) Incompressible flow

Then

$$0 = \{-|\rho U_1 2hw|\} + \int_{-h}^{h}\rho u_{max}\left[1-\left(\frac{y}{h}\right)^2\right] w\,dy$$

or

$$2U_1 hw = 2u_{max}wh\int_0^1\left[1-\left(\frac{y}{h}\right)^2\right]d\left(\frac{y}{h}\right) = 2u_{max}wh\left[\frac{y}{h}-\frac{1}{3}\left(\frac{y}{h}\right)^3\right]_0^1$$

Thus

$$u_{max} = \frac{3}{2}U_1 = \frac{3}{2}\times 20\,\frac{\text{ft}}{\text{s}} = 30\text{ ft/s}$$

← u_{max}

From the momentum equation,

$$p_1 2hw - p_2 2hw = u_1\{-\rho U_1 2hw\} + \int_{-h}^{h} u_2\rho u_2\,dA_2 \rightarrow 2\int_0^h \rho u_{max}^2\left[1-\left(\frac{y}{h}\right)^2\right]^2 w\,dy$$

$u_1 = U_1 \qquad u_2 = u_{max}\left[1-\left(\frac{y}{h}\right)^2\right]$

or

$$p_1 - p_2 = -\rho U_1^2 + \rho u_{max}^2\int_0^1(1-\eta^2)^2 d\eta \;;\; \eta = \frac{y}{h}$$

But

$$\int_0^1(1-\eta^2)^2 d\eta = \int_0^1(1-2\eta^2+\eta^4)d\eta = \eta - \frac{2}{3}\eta^3 + \frac{1}{5}\eta^5\Big]_0^1 = \frac{15-10+3}{15} = \frac{8}{15}$$

and $u_{max}^2 = \left(\frac{3}{2}U_1\right)^2 = \frac{9}{4}U_1^2$, so

$$p_1 - p_2 = -\rho U_1^2 + \frac{9}{4}\rho U_1^2\left(\frac{8}{15}\right) = \rho U_1^2\left(\frac{6}{5}-1\right) = \frac{1}{5}\rho U_1^2$$

$$= \frac{1}{5}\times 0.00238\,\frac{\text{slug}}{\text{ft}^3}\times(20)^2\frac{\text{ft}^2}{\text{s}^2}\times\frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}}$$

$$p_1 - p_2 = 0.190\text{ lbf/ft}^2$$

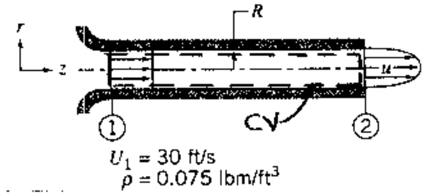
← $p_1 - p_2$

Problem 4.83

Given: Incompressible flow in entrance region of circular tube of radius, R.

$$u_2 = u_{max}\left[1-\left(\frac{r}{R}\right)^2\right]$$

Find: (a) Maximum velocity at Section ②.
(b) Pressure drop if viscous friction could be neglected.



Solution: Apply continuity and the x direction momentum equations. Use the CV and CS shown.

Basic equations:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho\, dV + \int_{CS} \rho \vec{V}\cdot d\vec{A} \qquad (=0(1))$$

$$F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\rho\, dV + \int_{CS} u\rho\vec{V}\cdot d\vec{A} \qquad (F_{B_x}=0(3),\ \tfrac{\partial}{\partial t}=0(1))$$

Assumptions: (1) Steady flow
(2) Uniform flow at Section ①
(3) $F_{Bx} = 0$
(4) Neglect friction at duct wall
(5) Incompressible flow

Then

$$0 = \{-|\rho U_1 \pi R^2|\} + \int_0^R \rho u_{max}\left[1-\left(\frac{r}{R}\right)^2\right]2\pi r\, dr$$

or

$$\pi\rho U_1 R^2 = 2\pi\rho u_{max} R^2\int_0^1\left[1-\left(\frac{r}{R}\right)^2\right]\left(\frac{r}{R}\right)d\left(\frac{r}{R}\right) = 2\pi\rho u_{max}R^2\left[\frac{1}{2}\left(\frac{r}{R}\right)^2-\frac{1}{4}\left(\frac{r}{R}\right)^4\right]_0^1$$

Thus $u_{max} = 2U_1 = 2\times 30\,\frac{ft}{s} = 60\ ft/s$ $\qquad u_{ma}$

From the momentum equation,

$$p_1\pi R^2 - p_2\pi R^2 = u_1\{-\rho U_1\pi R^2\} + \int_0^R u_2\rho u_2\, dA_2 = -\rho U_1\pi R^2 + \rho u_{max}^2 2\pi R^2\int_0^1\left[1-\left(\frac{r}{R}\right)^2\right]^2\left(\frac{r}{R}\right)d\left(\frac{r}{R}\right)$$

$$u_1 = U_1 \qquad u_2 = u_{max}\left[1-\left(\frac{r}{R}\right)^2\right]$$

or

$$p_1 - p_2 = -\rho U_1^2 + 2\rho u_{max}^2\int_0^1 (1-\eta^2)^2\eta\, d\eta\ ;\ \eta = \frac{r}{R}$$

But $\int_0^1 (1-\eta^2)\eta\, d\eta = \int_0^1 (1-2\eta^2+\eta^4)\eta\, d\eta = \frac{1}{2}\eta^2 - \frac{1}{2}\eta^4 + \frac{1}{6}\eta^6\Big]_0^1 = \frac{1}{6}$

and $u_{max}^2 = (2U_1)^2 = 4U_1^2$, so

$$p_1 - p_2 = -\rho U_1^2 + \frac{8}{6}\rho U_1^2 = \rho U_1^2\left(\frac{4}{3}-1\right) = \frac{1}{3}\rho U_1^2$$

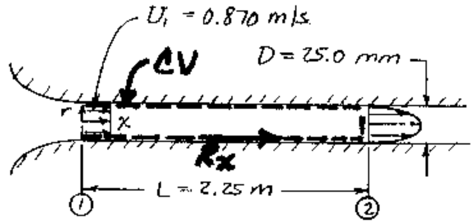
$$= \frac{1}{3}\times 0.075\,\frac{lbm}{ft^3}\times(30)^2\frac{ft^2}{s^2}\times\frac{slug}{32.2\ lbm}\times\frac{lbf\cdot s^2}{slug\cdot ft}$$

$$p_1 - p_2 = 0.699\ lbf/ft^2 \qquad p_1 - p_2$$

Problem 4.84

Given: Uniform flow into, fully developed flow from duct shown.

Air



$$\frac{u(r)}{U_c} = 1 - \left(\frac{r}{R}\right)^2 \quad \text{at } ②$$

$$p_1 - p_2 = 1.92 \text{ N/m}^2$$

Find: Total force exerted by tube on the flowing air.

Solution: Apply continuity and momentum to CV, CS shown.

Basic equations:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A} \qquad (\tfrac{\partial}{\partial t} = 0(1))$$

$$F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u \rho \, d\forall + \int_{CS} u \rho \vec{V} \cdot d\vec{A} \qquad (F_{B_x} = 0(4),\ \tfrac{\partial}{\partial t} = 0(1))$$

Assumptions: (1) Steady flow (2) Incompressible flow (3) Uniform flow at inlet (4) $F_{B_x} = 0$

Then

$$0 = \{-|\rho U_1 A_1|\} + \int_{②} \rho u \, dA = -\rho U_1 \pi R^2 + \int_0^R \rho U_c \left[1 - \left(\tfrac{r}{R}\right)^2\right] 2\pi r \, dr$$

$$0 = -\rho U_1 \pi R^2 + 2\rho \pi R^2 U_c \int_0^1 (1-\lambda^2)\lambda \, d\lambda \quad \text{or} \quad 0 = -U_1 + 2U_c\left[\frac{\lambda^2}{2} - \frac{\lambda^4}{4}\right]_0^1$$

Thus $0 = -U_1 + \frac{1}{2}U_c$ or $U_c = 2U_1$ $\qquad (\lambda = r/R)$

From momentum $R_x + p_1 A_1 - p_2 A_2 = u_1\{-|\rho U_1 A_1|\} + \int_{②} u_2 \{+\rho u_2 \, dA_2\}$

$$u_1 = U_1 \qquad u_2 = U_c\left[1 - \left(\tfrac{r}{R}\right)^2\right]$$

so

$$\int_{②} = \int_0^R U_c\left[1-\left(\tfrac{r}{R}\right)^2\right]\rho U_c\left[1-\left(\tfrac{r}{R}\right)^2\right] 2\pi r \, dr = 2\pi \rho U_c^2 R^2 \int_0^1 (1-\lambda^2)(1-\lambda^2)\lambda \, d\lambda$$

$$= 2\pi \rho U_c^2 R^2 \int_0^1 (1 - 2\lambda^2 + \lambda^4)\lambda \, d\lambda = 2\pi\rho U_c^2 R^2\left[\frac{\lambda^2}{2} - \frac{\lambda^4}{2} + \frac{\lambda^6}{6}\right]_0^1 = \frac{1}{3}\pi\rho U_c^2 R^2$$

Substituting,

$$R_x + (p_1 - p_2)\pi R^2 = -\pi\rho U_1^2 R^2 + \frac{1}{3}\pi\rho U_c^2 R^2 = -\pi\rho U_1^2 R^2 + \frac{1}{3}\pi\rho(2U_1)^2 R^2$$

$$R_x = -(p_1 - p_2)\frac{\pi D^2}{4} + \frac{1}{3}\rho U_1^2 \frac{\pi D^2}{4}$$

$$= -1.92\,\frac{\text{N}}{\text{m}^2} \times \frac{\pi}{4}(0.025)^2 \text{m}^2 + \frac{1}{3} \times 1.23\,\frac{\text{kg}}{\text{m}^3} \times (0.870)^2\frac{\text{m}^2}{\text{s}^2} \times \frac{\pi}{4}(0.025)^2\text{m}^2 \times \frac{\text{N} \cdot \text{s}^2}{\text{kg} \cdot \text{m}}$$

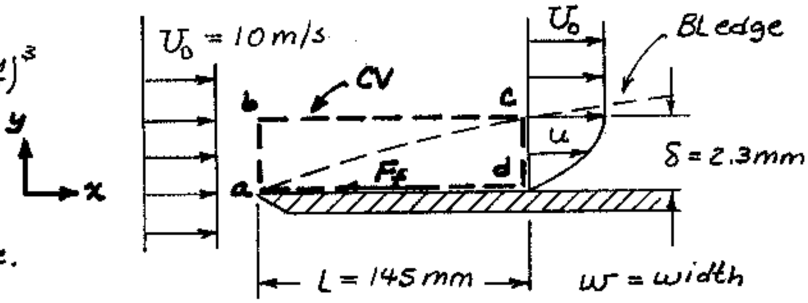
$$R_x = -7.90 \times 10^{-4} \text{ N} \quad (\text{to left } \underline{\text{on}} \text{ CV, since} < 0) \qquad R_x$$

Problem 4.85

Given: Incompressible flow in boundary layer, as shown.

In BL: $\frac{u}{U_0} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$

Standard air



Find: Horizontal force per unit width to hold plate.

Solution: Apply continuity and x component momentum. Use CV, CS shown.

Basic equations: $0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$ (=0(1))

$$F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u \rho\, d\forall + \int_{CS} u \rho \vec{V} \cdot d\vec{A}$$

(=0(3), =0(1))

Assumptions: (1) Steady flow
(2) No net pressure force; $F_{S_x} = -F_f$
(3) $F_{B_x} = 0$
(4) Uniform flow at section ab
(5) Incompressible flow

Then from continuity,

$$0 = \{-|\rho U_0 w \delta|\} + \dot{m}_{bc} + \left\{\int_0^\delta \rho u w\, dy\right\}; \quad \delta = \int_0^\delta dy; \quad \dot{m}_{bc} = \int_0^\delta \rho (U_0 - u) w\, dy$$

From momentum equation,

$$-F_f = U_0\{-|\rho U_0 w \delta|\} + U_0 \dot{m}_{bc} + \left\{\int_0^\delta u \rho u w\, dy\right\} = \int_0^\delta \left[-U_0^2 + u^2 + U_0(U_0 - u)\right] w\, dy$$

$$\text{Drag force} = F_f = \int_0^\delta \rho u (U_0 - u) w\, dy = \int_0^\delta \rho U_0^2 \frac{u}{U_0}\left(1 - \frac{u}{U_0}\right) w\, dy$$

At section cd, $\frac{u}{U_0} = \frac{3}{2}\eta - \frac{1}{2}\eta^3$; $dy = \delta\, d\eta$

$$\frac{F_f}{w} = \int_0^{\eta=1} \rho U_0^2 \delta \frac{u}{U_0}\left(1 - \frac{u}{U_0}\right) d\eta = \rho U_0^2 \delta \int_0^1 \left(\frac{3}{2}\eta - \frac{1}{2}\eta^3\right)\left(1 - \frac{3}{2}\eta + \frac{1}{2}\eta^3\right) d\eta$$

$$= \rho U_0^2 \delta \int_0^1 \left(\frac{3}{2}\eta - \frac{9}{4}\eta^2 - \frac{1}{2}\eta^3 + \frac{3}{2}\eta^4 - \frac{1}{4}\eta^6\right) d\eta$$

$$= \rho U_0^2 \delta \left[\frac{3}{4}\eta^2 - \frac{3}{4}\eta^3 - \frac{1}{8}\eta^4 + \frac{3}{10}\eta^5 - \frac{1}{28}\eta^7\right]_0^1 = \rho U_0^2 \delta (0.139)$$

$$= 0.139 \times 1.23 \frac{kg}{m^3} \times (10)^2 \frac{m^2}{s^2} \times 0.0023\ m \times \frac{N \cdot s^2}{kg \cdot m}$$

$$\frac{F_f}{w} = 0.0393\ N/m \quad \text{(to right)}$$ $\frac{F_f}{w}$

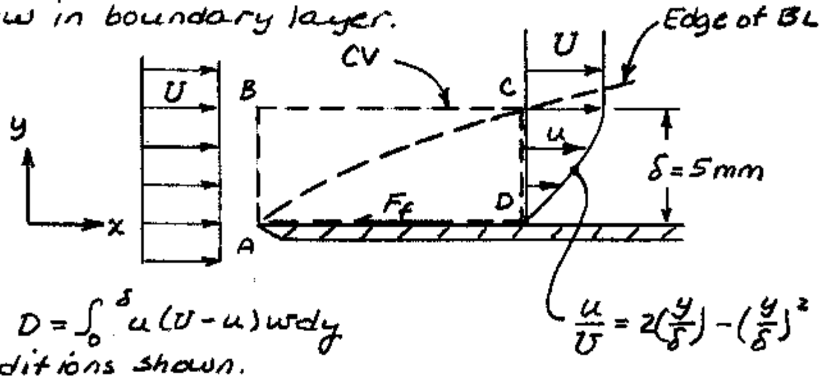
Problem 4.86

Given: Incompressible flow in boundary layer.

$w = 0.6\ m$

$U = 30\ m/s$

$\rho = 1.24\ kg/m^3$



Find: (a) Show that drag, $D = \int_0^\delta u(U-u)w\,dy$
(b) Evaluate for conditions shown.

Solution: Apply continuity and x component of momentum using CV shown.

Basic equations: $$0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A} \quad \left(\frac{\partial}{\partial t} = 0(1)\right)$$

$$F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho\vec{V}\cdot d\vec{A} \quad \left(F_{B_x} = 0(3),\ \frac{\partial}{\partial t} = 0(1)\right)$$

Assumptions: (1) Steady flow
(2) No net pressure force; $F_{S_x} = -F_f$
(3) $F_{B_x} = 0$
(4) Uniform flow at section (AB)
(5) Incompressible flow

Then from continuity

$$0 = \{-|\rho U w \delta|\} + \left\{\left|\int_0^\delta \rho u w\, dy\right|\right\} + \dot{m}_{BC}\ ;\ \delta = \int_0^\delta dy\ ;\ \dot{m}_{BC} = \rho\int_0^\delta (U-u)w\,dy$$

From momentum

$$-F_f = U\{-|\rho U w\delta|\} + \left\{\int_0^\delta \rho u^2 w\,dy\,|\right\} + U\dot{m}_{BC} = \rho\int_0^\delta\left[-U^2 + u^2 + U(U-u)\right]w\,dy$$

$$\text{Drag} = F_f = \int_0^\delta \rho u(U-u)w\,dy$$ Drag

At CD, $\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2 = 2\eta - \eta^2$; $dy = \delta\, d\left(\frac{y}{\delta}\right) = \delta\, d\eta$

$$\text{Drag} = \int_0^\delta \rho U\left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right]\left(U - U\left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right]\right)w\,dy = \rho U^2 w\delta\int_0^1 (2\eta-\eta^2)(1-2\eta+\eta^2)\,d\eta$$

$$= \rho U^2 w\delta\int_0^1 (2\eta - 5\eta^2 + 4\eta^3 - \eta^4)\,d\eta = \rho U^2 w\delta\left[\eta^2 - \frac{5}{3}\eta^3 + \eta^4 - \frac{1}{5}\eta^5\right]_0^1$$

$$= \frac{2}{15}\rho U^2 w\delta$$

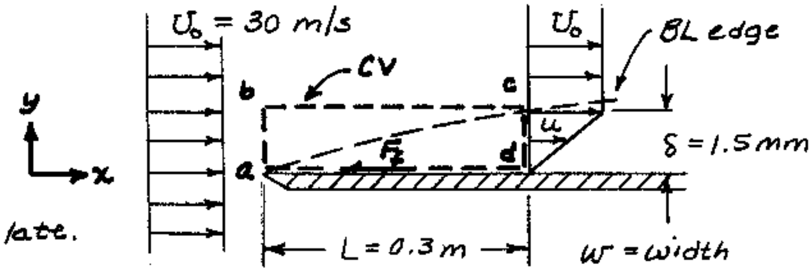
$$\text{Drag} = \frac{2}{15}\times 1.24\,\frac{kg}{m^3}\times(30)^2\frac{m^2}{s^2}\times 0.6\,m\times 0.005\,m\times\frac{N\cdot s^2}{kg\cdot m} = 0.446\ N$$ Drag

Problem 4.87

Given: Incompressible flow in boundary layer, as shown.

In BL: $\frac{u}{U_0} = \frac{y}{\delta}$

standard air



Find: Horizontal force per unit width to hold plate.

Solution: Apply continuity and x component momentum. Use CV, CS shown.

Basic equations: $0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A}$ (first term $= 0(1)$)

$$F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho\vec{V}\cdot d\vec{A}$$ ($F_{Bx} = 0(3)$, $\frac{\partial}{\partial t}$ term $= 0(1)$)

Assumptions: (1) Steady flow
(2) No net pressure force; $F_{Sx} = -F_f$
(3) $F_{Bx} = 0$
(4) Uniform flow at section ab
(5) Incompressible flow

Then from continuity,

$$0 = \{-|\rho U_0 w\delta|\} + \dot{m}_{bc} + \left\{\int_0^\delta \rho u w\, dy\,|\right\}; \quad \delta = \int_0^\delta dy; \quad \dot{m}_{bc} = \int_0^\delta \rho(U_0 - u)w\, dy$$

From momentum equation,

$$-F_f = U_0\{-|\rho U_0 w\delta|\} + U_0\dot{m}_{bc} + \left\{\int_0^\delta u\rho u w\, dy\right\} = \int_0^\delta \rho\left[-U_0^2 + u^2 + U_0(U_0 - u)\right]w\, dy$$

$$\text{Drag force} = F_f = \int_0^\delta \rho u (U_0 - u) w\, dy$$

At cd, $\frac{u}{U_0} = \frac{y}{\delta} = \eta$; $dy = \delta\, d(\frac{y}{\delta}) = \delta\, d\eta$

$$\frac{F_f}{w} = \int_0^{y/\delta = 1} \rho U_0 \frac{y}{\delta}\left(U_0 - U_0\frac{y}{\delta}\right)\delta\, d\left(\frac{y}{\delta}\right) = \rho U_0^2 \delta \int_0^1 \eta(1-\eta)\, d\eta = \rho U_0^2\delta\left[\frac{\eta^2}{2} - \frac{\eta^3}{3}\right]_0^1$$

$$= \frac{\rho U_0^2 \delta}{6} = \frac{1}{6} \times 1.23\frac{\text{kg}}{\text{m}^3} \times (30)^2\frac{\text{m}^2}{\text{s}^2} \times 0.0015\,\text{m} \times \frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}}$$

$$\frac{F_f}{w} = 0.277\ \text{N/m (to left)}$$ $\frac{F_f}{w}$

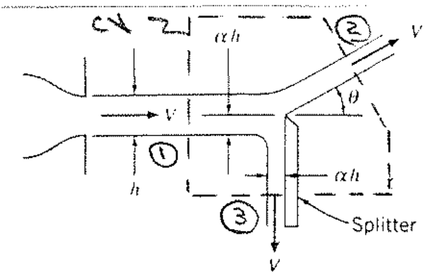
Given: Flow of flat jet over sharp-edged splitter plate, as shown. Neglect friction force between water and plate; $0 \le \alpha \le 0.5$.

Find: (a) Expression for angle θ as a function of α
(b) Expression for force R_x needed to hold splitter plate in place.

Plot: both θ and R_x as functions of θ.

Solution

Apply the x and y components of the momentum equation to the CV shown.



Basic equations:

$$F_{S_x} + \cancel{F_{B_x}}^{=0(3)} = \cancel{\frac{\partial}{\partial t}\int_{CV} u \rho \, d\forall}^{=0(4)} + \int_{CS} u(\rho \vec{V} \cdot d\vec{A})$$

$$\cancel{F_{S_y}}^{=0(2)} + \cancel{F_{B_y}}^{=0(3)} = \cancel{\frac{\partial}{\partial t}\int_{CV} v \rho \, d\forall}^{=0(4)} + \int_{CS} v(\rho \vec{V} \cdot d\vec{A})$$

Assumptions: (1) no net pressure forces on CV
(2) no friction in y direction, so $F_{S_y} = 0$
(3) neglect body forces
(4) steady flow
(5) no change in jet speed: $V_1 = V_2 = V_3 = V$
(6) uniform flow at each section

Then from the y equation

$$0 = v_1\{-|\rho V_1 A_1|\} + v_2\{|\rho V_2 A_2|\} + v_3\{|\rho V_3 A_3|\}$$

$v_1 = 0 \qquad v_2 = V\sin\theta \qquad v_3 = -V$

{w is depth} $\quad A_1 = wh \qquad A_2 = w(1-\alpha)h \qquad A_3 = w\alpha h$

$$0 = 0 + \rho V^2 \sin\theta \, w(1-\alpha)h - \rho V^2 w\alpha h$$

Thus

$$\sin\theta = \frac{\rho V^2 w \alpha h}{\rho V^2 w(1-\alpha)h} = \frac{\alpha}{(1-\alpha)} \; ; \quad \theta = \sin^{-1}\left(\frac{\alpha}{1-\alpha}\right) \qquad \theta(\alpha)$$

From the x equation

$$R_x = u_1\{-|\rho_1 V_1 A_1|\} + u_2\{|\rho_2 V_2 A_2|\} + u_3\{|\rho_3 V_3 A_3|\}$$

$u_1 = V \qquad u_2 = V\cos\theta \qquad u_3 = 0$

$$R_x = -\rho V^2 wh + \rho V^2 \cos\theta \, w(1-\alpha)h = \rho V^2 wh\left[\cos\theta(1-\alpha) - 1\right]$$

But $\cos\theta = (1-\sin^2\theta)^{1/2} = \left(1 - \frac{\alpha^2}{(1-\alpha)^2}\right)^{1/2} = \frac{(1-2\alpha)^{1/2}}{(1-\alpha)}$

$$\therefore R_x = -\rho V^2 wh\left[1 - (1-2\alpha)^{1/2}\right] \qquad (R_x < 0; \text{ so to left}) \qquad R_x$$

{Check: $\alpha = 0$, $R_x = 0$ ✓ ; $\alpha = \frac{1}{2}$, $R_x = -\rho V^2 wh$ ✓}

Plots of: $\theta = \sin^{-1}\left(\frac{\alpha}{1-\alpha}\right)$ and

$$\frac{R_x}{R_{x\,\alpha=0.5}} = 1 - \sqrt{1-2\alpha}$$

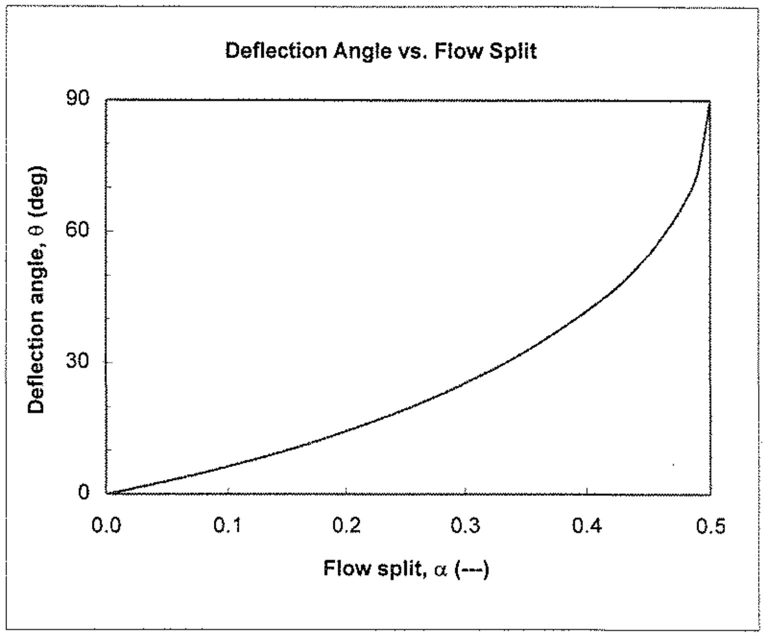
are presented below

Flow deflection by sharp-edged splitter:

α = fraction of jet intercepted by splitter

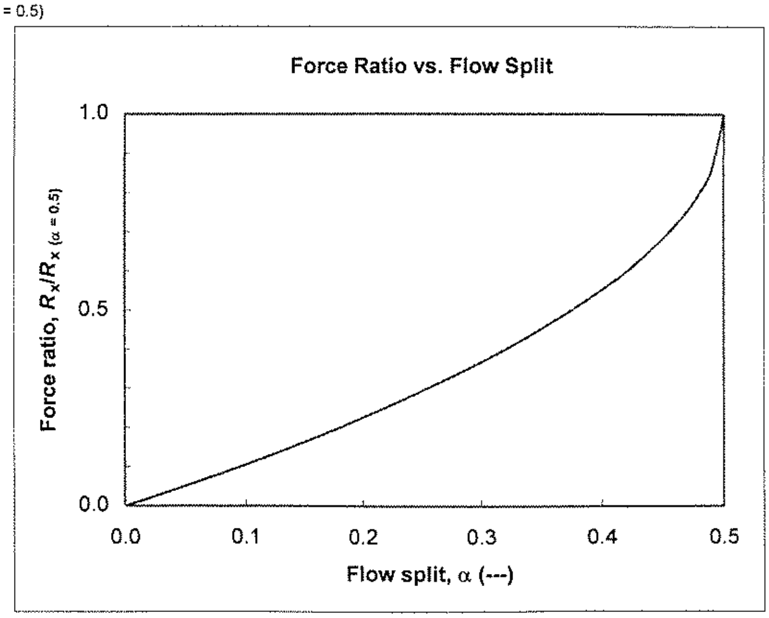
Calculated Results: Deflection angle

α (---)	θ (deg)
0	0
0.05	3.02
0.10	6.38
0.15	10.2
0.20	14.5
0.25	19.5
0.30	25.4
0.35	32.6
0.40	41.8
0.425	47.7
0.45	54.9
0.470	62.5
0.480	67.4
0.490	73.9
0.50	90.0



Calculated Results: Force over maximum force

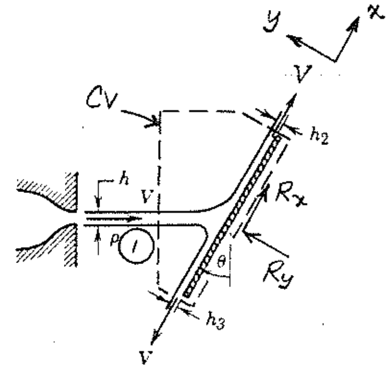
α (---)	$R_x/R_{x\,(\alpha=0.5)}$
0	0
0.05	0.0513
0.10	0.106
0.15	0.163
0.20	0.225
0.25	0.293
0.30	0.368
0.35	0.452
0.40	0.553
0.425	0.613
0.45	0.684
0.470	0.755
0.480	0.800
0.490	0.859
0.50	1.00



Given: Plane jet striking inclined plate, as shown. No frictional force along plate surface.

Find: (a) Expression for h_2/h as a function of θ.
(b) Plot of results.
(c) Comment on limiting cases, $\theta = 0$ and $\theta = 90°$

Solution: Apply the x component of the momentum equation using the CV and coordinates shown.



Basic equation:

$$\overset{=0(1)}{F_{S_x}} + \overset{=0(2)}{F_{B_x}} = \overset{=0(3)}{\frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall} + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$$

Assumptions: (1) No surface force on CV
(2) Neglect body forces
(3) Steady flow
(4) No change in jet speed: $V_1 = V_2 = V_3 = V$
(5) Uniform flow at each section

From continuity for uniform incompressible flow $0 = -\rho V w h + \rho V w h_2 + \rho V w h_3$

or

$$h = h_2 + h_3 = h_1 \quad \text{or} \quad h_3 = h_1 - h_2$$

From momentum

$$0 = u_1\{-|\rho V w h_1|\} + u_2\{+|\rho V w h_2|\} + u_3\{+|\rho V w h_3|\}$$

$$u_1 = V\sin\theta \qquad u_2 = V \qquad u_3 = -V$$

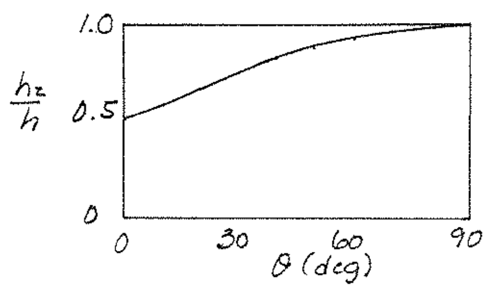
$$0 = -\rho V^2 \sin\theta\, w h_1 + \rho V^2 w h_2 - \rho V^2 w h_3$$

Substituting from continuity and simplifying

$$0 = -\sin\theta\, h_1 + h_2 - (h_1 - h_2) \quad \text{so} \quad \frac{h_2}{h} = \frac{h_2}{h_1} = \frac{1 + \sin\theta}{2}$$

$\frac{h_2}{h}$

Plot:



At $\theta = 0$, $\frac{h_2}{h} = 0.5$; flow is equally split when plate is ⊥ to jet.

At $\theta = 90°$, $\frac{h_2}{h} = 1.0$; plate has no effect on flow.

Given: Model gas flow in a propulsion nozzle as a spherical source; V_e = constant

Find: (a) Expression for axial thrust, T_a, and compare to the 1-D approximation, $T = \dot{m} V_e$
(b) Percent error for $\alpha = 15°$.

Plot: the percent error vs α for $0 \le \alpha \le 22.5°$.

Solution:

Apply definitions $\dot{m} = \int_A \rho V dA$, $T_a = \int_A u \rho V dA$. Use spherically symmetric flow.

The mass flow rate is [assuming $\rho_e \neq \rho_e(\theta)$]

$$\dot{m} = \int_A \rho V dA = \int_0^\alpha \rho_e V_e (2\pi R \sin\theta) R d\theta = 2\pi \rho_e V_e R^2 \left[-\cos\theta\right]_0^\alpha = 2\pi \rho_e V_e R^2 (1-\cos\alpha)$$

The one-dimensional approximation for thrust is then

$$T = \dot{m} V_e = 2\pi \rho_e V_e^2 R^2 (1-\cos\alpha) \qquad T_{1-D}$$

The axial thrust is given by

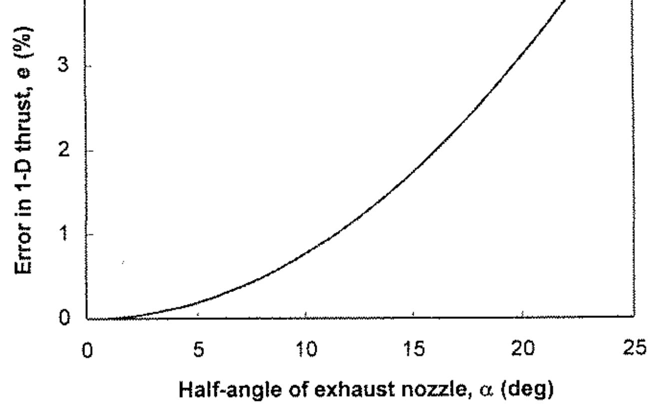
$$T_a = \int u \rho V dA = \int_0^\alpha V_e \cos\theta \, \rho_e V_e (2\pi R \sin\theta) R d\theta = 2\pi \rho_e V_e^2 R^2 \int_0^\alpha \sin\theta \cos\theta \, d\theta$$

$$T_a = 2\pi \rho_e V_e^2 R^2 \left[\frac{\sin^2\theta}{2}\right]_0^\alpha = \pi \rho_e V_e^2 R^2 \sin^2\alpha \qquad T_a$$

The error in the one-dimensional approximation is

$$e = \frac{T_{1-D} - T_a}{T_a} = \frac{T_{1-D}}{T_a} - 1 = \frac{2\pi \rho_e V_e^2 R^2 (1-\cos\alpha)}{\pi \rho_e V_e^2 R^2 \sin^2\alpha} - 1 = \frac{2(1-\cos\alpha)}{\sin^2\alpha} - 1 \quad \text{---(1)}$$

The percent error is plotted as a function of α



For $\alpha = 15°$

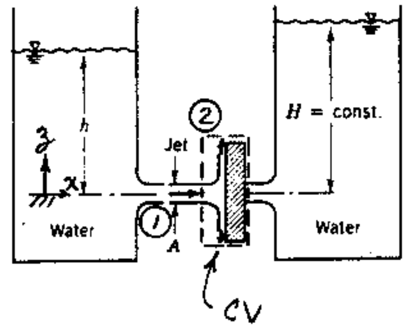
$$e_{15} = \frac{2(1-\cos 15°)}{\sin^2 15°} - 1$$

$$e_{15} = 0.0173 \text{ or } 1.73\% \qquad e_{15}$$

Problem *4.91

Given: Tanks and flat plate shown.

Find: Minimum height h needed to keep plate in place.



Solution: Apply Bernoulli and momentum equations. Use CV enclosing plate, as shown.

Basic equations: $\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$

$$F_{S_x} + \cancel{F_{B_x}}^{=0(5)} = \cancel{\frac{\partial}{\partial t}\int_{CV} u \rho \, d\forall}^{=0(1)} + \int_{CS} u \rho \vec{V} \cdot d\vec{A}$$

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Flow along a streamline
(4) No friction
(5) $F_{Bx} = 0$

Apply Bernoulli from water surface to jet

$$\frac{\cancel{p}}{\cancel{\rho}} + \cancel{\frac{V_s^2}{2}}^{\ll V^2} + gh = \frac{\cancel{p}}{\cancel{\rho}} + \frac{V^2}{2} + g(0) \quad \text{so that} \quad V^2 = 2gh \quad \text{or} \quad V = \sqrt{2gh}$$

From fluid statics, $p_{3g} = \rho g H$

From momentum

$$-p_{3g}A = -\rho g H A = u_1\{-\rho V A\} + u_2\{+\rho V A\} = -\rho V^2 A$$

$$u_1 = V \qquad u_2 = 0$$

Thus, using Bernoulli,

$$\rho g H A = \rho V^2 A = \rho(2gh)A = 2\rho g h A$$

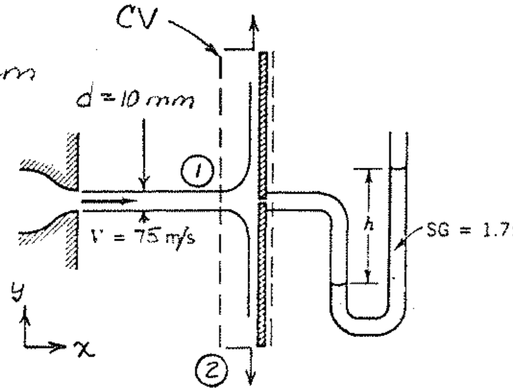
and

$$h = \frac{H}{2}$$ h

Problem *4.92

Given: Air jet striking disk of diameter, D = 200 mm, as shown.

Find: (a) Manometer deflection.
(b) Force to hold disk.



Solution: Apply Bernoulli and momentum equations. Use CV shown.

Basic equations: $\frac{p}{\rho} + \frac{V^2}{2} + g\not{z} = \text{constant}$ (5)

$$F_{S_x} + \not{F}_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$$

(with $F_{B_x} = 0$ (5) and $\frac{\partial}{\partial t}\int_{CV} = 0$ (1))

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Flow along a streamline
(4) No friction
(5) $F_{B_x} = 0$; horizontal flow
(6) Uniform flow in jet

Apply Bernoulli between jet exit and stagnation point

$$\frac{p}{\rho} + \frac{V^2}{2} = \frac{p_0}{\rho} + 0 \; ; \; p_0 - p = \frac{1}{2}\rho V^2$$

From hydrostatics, $p_0 - p = SG\, \rho_{H_2O}\, g\, \Delta h$

Thus $\Delta h = \frac{\frac{1}{2}\rho V^2}{SG\, \rho_{H_2O}\, g} = \frac{\rho V^2}{2 SG\, \rho_{H_2O}\, g}$

$$\Delta h = 1.23\,\frac{kg}{m^3} \times (75)^2\frac{m^2}{s^2} \times \frac{1}{2(1.75)} \times \frac{m^3}{999\,kg} \times \frac{s^2}{9.81\,m} = 0.202 \text{ m or } 202 \text{ mm}$$ Δh

From momentum,

$$R_x = u_1\{-\rho V A\} + u_2\{\rho V A\} = -\rho V^2 A$$

$$u_1 = V \qquad u_2 = 0$$

$$R_x = -1.23\,\frac{kg}{m^3} \times (75)^2\frac{m^2}{s^2} \times \frac{\pi}{4}(0.01)^2 m^2 \times \frac{N\cdot s^2}{kg\cdot m} = -0.543 \text{ N (to } \underline{\text{left}})$$ R_x

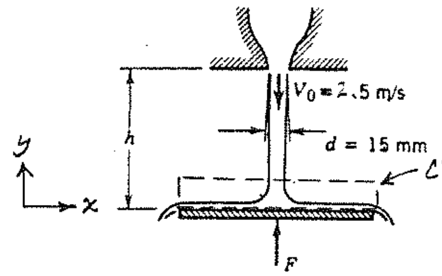
This is the force needed to hold the plate.

The "force" of the jet on the plate is

$$K_x = -R_x = 0.543 \text{ N (to } \underline{\text{right}})$$

Given: Jet flowing downward, striking horizontal disk, as shown.

Find: (a) Velocity in jet at h.
(b) Expression for force to hold disk.
(c) Evaluate for $h = 3.0\,m$.



Solution: Apply Bernoulli and momentum equations. Use CV shown.

Basic equations: $\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$ (5)

$F_{S_z} + F_{B_z} = \frac{\partial}{\partial t}\int_{CV} w\rho\, d\forall + \int_{CS} w\rho \vec{V}\cdot d\vec{A}$ (F_{B_z} = 0 (6), $\frac{\partial}{\partial t}$ = 0 (1))

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Flow along a streamline
(4) Frictionless flow
(5) Atmospheric pressure along jet
(6) Neglect water on plate; $F_{B_z} = 0$
(7) Uniform flow at each section

The Bernoulli equation becomes

$$\frac{V_0^2}{2} + gh = \frac{V^2}{2} + g(0) \quad \text{or} \quad V^2 = V_0^2 + 2gh\,; \quad V = \sqrt{V_0^2 + 2gh}$$ V

From the momentum equation

$$R_z = w_1\{-\rho V A\} + w_2\{+\rho V_0 A_0\} = +\rho V^2 A$$

$$w_1 = -V \qquad w_2 = 0$$

But from continuity, $\dot{m} = \rho V_0 A_0 = \rho V A$. Thus $VA = V_0 A_0$, and

$$R_z = \rho V_0 A_0 V = \rho V_0 A_0 \sqrt{V_0^2 + 2gh}$$ R_z

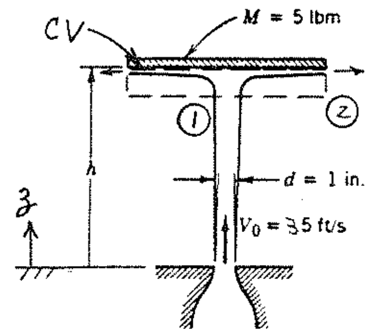
At $h = 3.0\,m$,

$$R_z = 999\frac{kg}{m^3} \times 2.5\frac{m}{s} \times \frac{\pi}{4}(0.015)^2 m^2 \left[(2.5)^2\frac{m^2}{s^2} + 2 \times 9.81\frac{m}{s^2} \times 3.0\,m\right]^{1/2} \frac{N\cdot s^2}{kg\cdot m}$$

$R_z = 3.56\ N$ (upward force) R_z

Problem *4.44

Given: Horizontal disk above jet, as shown. No external force applied to disk.



Find: (a) Expression for jet speed, V(h).
(b) Equilibrium height for disk.

Solution: Apply Bernoulli and momentum equations. Use CS, CV shown.

Basic equations: $\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$ (5)

$$F_{S_z} + F_{B_z} = \frac{\partial}{\partial t}\int_{CV} w\rho\, d\forall + \int_{CS} w\rho \vec{V}\cdot d\vec{A}$$
(F_{S_z} crossed out; $\frac{\partial}{\partial t}$ term $= 0(1)$)

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Flow along a streamline
(4) Frictionless flow
(5) Atmospheric pressure along jet
(6) Neglect mass of water in CV
(7) Uniform flow at each section

The Bernoulli equation becomes

$$\frac{V_0^2}{2} + g(0) = \frac{V^2}{2} + gh \quad \text{or} \quad V^2 = V_0^2 - 2gh \;;\; V = \sqrt{V_0^2 - 2gh}$$ V

From the momentum equation,

$$-Mg = w_1\{-\rho VA\} + w_2\{+\rho VA\} = -\rho V^2 A$$

$$w_1 = V \qquad w_2 = 0$$

But from continuity, $\dot{m} = \rho V_0 A_0 = \rho VA$. Thus $VA = V_0 A_0$, and

$$Mg = \rho V_0 A_0 V = \rho V_0 A_0 \sqrt{V_0^2 - 2gh}$$

Solving for h,

$$h = \frac{1}{2g}\left[V_0^2 - \left(\frac{Mg}{\rho V_0 A_0}\right)^2\right]$$

$$= \frac{1}{2} \times \frac{s^2}{32.2\text{ ft}}\left[(35)^2\frac{ft^2}{s^2} - \left(5\text{ lbm} \times 32.2\frac{ft}{s^2} \times \frac{ft^3}{62.4\text{ lbm}} \times \frac{s}{25\text{ ft}} \times \frac{4}{\pi}\frac{1}{(1)^2\text{ in.}^2} \times \frac{144\text{ in.}^2}{ft^2}\right)^2\right]$$

$h = 16.2$ ft h

Problem *4.95

Given: Stream of air at standard conditions strikes a curved vane. Stagnation tube with water-filled manometer in exit plane.

Find: (a) Speed of air leaving nozzle.
(b) Horizontal component of force exerted on vane by jet.
(c) Comment on each assumption used to solve this problem.

Solution: Apply the definition of stagnation pressure and the x component of the momentum equation.

By definition $p_0 = p + \frac{1}{2}\rho_{air} V^2$

From fluid statics, $p_0 - p = \rho_{water}\, g\, \Delta h$

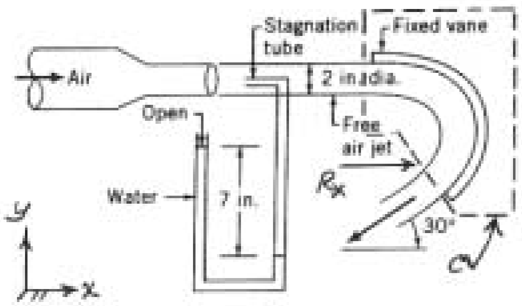
Combining, $\rho_{water}\, g\, \Delta h = \frac{1}{2}\rho_{air} V^2$ or $V = \sqrt{\dfrac{2\rho_{water}\, g\, \Delta h}{\rho_{air}}}$

$$V = \left[2 \times 1.94\,\frac{slug}{ft^3} \times 32.2\,\frac{ft}{s^2} \times 7\,in. \times \frac{ft^3}{0.00238\,slug} \times \frac{ft}{12\,in.}\right]^{\frac{1}{2}} = 175\ ft/s$$ V

The momentum equation is

$$F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\, d\forall + \int_{CS} u\,\rho\,\vec{V}\cdot d\vec{A}$$

($F_{B_x} = 0(2)$, $\frac{\partial}{\partial t} = 0(3)$)



Assumptions: (1) No net pressure force
(2) $F_{B_x} = 0$
(3) Steady flow
(4) Uniform flow
(5) Constant speed on vane

Then

$$R_x = u_1\{-\rho V A\} + u_2\{\rho V A\} = -\rho V^2 A(1+\cos\theta)$$

$$u_1 = V \qquad u_2 = -V\cos\theta$$

$$R_x = -0.00238\,\frac{slug}{ft^3} \times (175)^2\frac{ft^2}{s^2} \times \frac{\pi}{4}\left(\frac{2}{12}\right)^2 ft^2 \cdot (1+\cos 30°) = -2.97\ lbf$$

Force of air on vane is $K_x = -R_x = +2.97$ lbf (to right) K_x

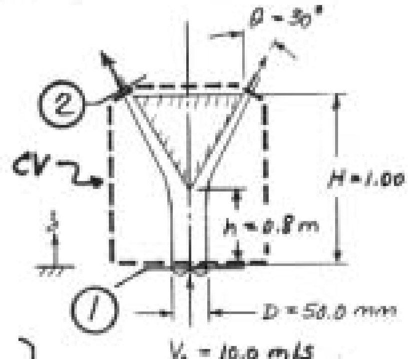
Comments on each assumption used to solve this problem:

- Frictionless flow in the nozzle is a good assumption.
- Incompressible flow is a good assumption for this low-speed flow.
- No horizontal component of body force is exact.
- No net pressure force on the control volume is exact.
- Frictionless flow along the vane is not realistic; air flow along the vane would be slowed by friction, reducing the momentum flux at the exit.

Given: Water jet supporting conical object, as shown.

Find: (a) Combined mass of cone and water, M, supported.
(b) Estimate mass of water in CV.

Solution: Apply continuity, Bernoulli, and momentum equations using CV shown.



Basic equations: $0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A}$ (=0(1) on the first term)

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + g z_2$$ (pressure terms =0 (6))

$$F_{S_z} + F_{B_z} = \frac{\partial}{\partial t}\int_{CV} w \rho\, d\forall + \int_{CS} w \rho \vec{V}\cdot d\vec{A}$$ (F_{S_z} = 0 (6); time term = 0 (1))

Assumptions: (1) Steady flow
(2) No friction
(3) Flow along a streamline
(4) Incompressible flow
} required for Bernoulli
(5) Uniform flow at each cross-section
(6) $F_{S_z} = 0$ since p_{atm} acts everywhere

Then

$$0 = \{-|\rho V_1 A_1|\} + \{+|\rho V_2 A_2|\} \quad \text{so} \quad V_1 A_1 = V_2 A_2$$

From Bernoulli

$$\frac{V_1^2}{2} + g z_1 = \frac{V_2^2}{2} + g z_2 = \frac{V_0^2}{2} = \frac{V_2^2}{2} + gH \; ; \quad V_2^2 = V_0^2 - 2gH$$

From momentum

$$F_{B_z} = \int_{CS} w \rho \vec{V}\cdot d\vec{A} = -Mg = w_1\{-|\rho V_1 A_1|\} + w_2\{+|\rho V_2 A_2|\}$$

$$w_1 = V_0 \qquad w_2 = V_2 \cos\theta$$

or

$$-Mg = -V_0 \rho V_1 A_1 + V_2 \cos\theta\, \rho V_2 A_2 = \rho V_0 A_1 (V_2 \cos\theta - V_0)$$

so

$$M = \frac{(V_0 - V_2 \cos\theta)\rho V_0 A_1}{g}$$

From Bernoulli

$$V_2 = (V_0^2 - 2gH)^{1/2} = \left[(10)^2 \frac{m^2}{s^2} - 2 \times 9.81 \frac{m}{s^2} \times 1\, m\right]^{1/2} = 8.97 \text{ m/s}$$

Substituting

$$M = \left(10.0 \frac{m}{s} - 8.97 \frac{m}{s} \times \cos 30°\right) 999 \frac{kg}{m^3} \times 10 \frac{m}{s} \times \frac{\pi}{4}(0.050)^2 m^2 \times \frac{s^2}{9.81\, m}$$

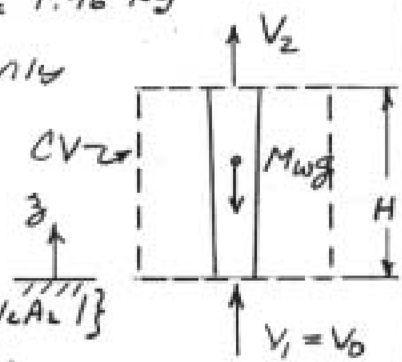
$M = 4.46$ kg ← (total mass in CV: water + object)   M

To find mass of water in CV, we have 3 options:

(1) assume area of jet is constant

$$M = \rho \forall \simeq \rho A_1 H = 999 \frac{kg}{m^3} \times \frac{\pi}{4}(0.05)^2 m^2 \times 1 m = 1.96 \text{ kg}$$

(2) Use a CV that encloses the free jet only



Continuity $V_1 A_1 = V_2 A_2$

Bernoulli $V_2 = (V_1^2 - 2gH)^{1/2}$

Momentum $-M_w g = w_1\{-|\rho V_1 A_1|\} + w_2\{+|\rho V_2 A_2|\}$

$$w_1 = V_1 = V_0 \qquad w_2 = V_2$$

Substituting in momentum

$$-M_w g = V_0(-\rho V_0 A_1) + V_2(+\rho V_0 A_1) = \rho V_0 A_1 (V_2 - V_0)$$

$$M_w = \frac{\rho V_0 A_1 (V_0 - V_2)}{g}$$

$$= 999 \frac{kg}{m^3} \times 10 \frac{m}{s} \times \frac{\pi}{4}(0.05)^2 m^2 (10 - 8.97)\frac{m}{s} \times \frac{s^2}{9.81 m}$$

$$M_w = 2.06 \text{ kg}$$ M_w

(3) Evaluate the area at each cross-section using Bernoulli and continuity, then integrate to find $\forall$.

$$VA = V_0 A_1 = (V_0^2 - 2gz)^{1/2} A = V_0 A_1 \quad \text{so} \quad A = \frac{V_0 A_1}{(V_0^2 - 2gz)^{1/2}}$$

$$\forall = \int_0^H A\,dz = \int_0^H \frac{V_0 A_1}{(V_0^2 - 2gz)^{1/2}} dz = A_1 \int_0^H \frac{V_0^2}{2g} \frac{1}{(1 - \frac{2gz}{V_0^2})^{1/2}} d\left(\frac{2gz}{V_0^2}\right)$$

This can be integrated. Let $u = 1 - 2gz/V_0^2$, so $\int = \int \frac{-du}{u^{1/2}}$

Then $$\forall = A_1 \frac{V_0^2}{2g}\left[-2\left(1 - \frac{2gz}{V_0^2}\right)^{1/2}\right]_{z=0}^{z=h} = \frac{A_1}{g}\left[V_0^2 - V_0(V_0^2 - 2gh)^{1/2}\right]$$

and $$M_w = \rho \forall = \frac{\rho A_1 V_0 (V_0 - V_2)}{g} = 2.06 \text{ kg (same as (2) above)}$$

Thus the mass of the cone is $M_c = M - M_w = 2.40$ kg. M_c

{Note: If V_0 were smaller or H larger, V_2 would differ more from V_0 and the jet area would increase significantly. Option (2) would still give the correct result with little effort.}

Problem *4.97

A venturi meter installed along a water pipe consists of a convergent section, a constant-area throat, and a divergent section. The pipe diameter is *D* = 100 mm and the throat diameter is *d* = 40 mm. Find the net fluid force acting on the convergent section if the water pressure in the pipe is 600 kPa (gage) and the average velocity is 5 m/s. For this analysis neglect viscous effects.

Given: Data on flow and venturi geometry

Find: Force on convergent section

Solution

The given data are

$\rho = 999 \cdot \frac{kg}{m^3}$ $\quad D = 0.1 \cdot m$ $\quad d = 0.04 \cdot m$ $\quad p_1 = 600 \cdot kPa$ $\quad V_1 = 5 \cdot \frac{m}{s}$

Then

$A_1 = \frac{\pi \cdot D^2}{4}$ $\qquad A_1 = 0.00785\,m^2$

$A_2 = \frac{\pi}{4} \cdot d^2$ $\qquad A_2 = 0.00126\,m^2$

$Q = V_1 \cdot A_1$ $\qquad Q = 0.0393\,\frac{m^3}{s}$

$V_2 = \frac{Q}{A_2}$ $\qquad V_2 = 31.3\,\frac{m}{s}$

Governing equations:

Bernoulli equation $\frac{p}{\rho} + \frac{V^2}{2} + g \cdot z = \text{const}$ (4.24)

Momentum $\vec{F} = \vec{F}_S + \vec{F}_B = \frac{\partial}{\partial t}\int_{CV} \vec{V} \rho \, dV + \int_{CS} \vec{V} \rho \vec{V} \cdot d\vec{A}$ (4.17)

Applying Bernoulli between inlet and throat

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} = \frac{p_2}{\rho} + \frac{V_2^2}{2}$$

Solving for p_2

$$p_2 = p_1 + \frac{\rho}{2} \cdot \left(V_1^2 - V_2^2\right)$$

$$p_2 = 600 \cdot kPa + 999 \cdot \frac{kg}{m^3} \times \left(5^2 - 31.3^2\right) \cdot \frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m} \times \frac{kN}{1000 \cdot N}$$

$$p_2 = 125\,kPa$$

Applying the horizontal component of momentum

$$-F + p_1 \cdot A_2 - p_2 \cdot A_2 = V_1 \cdot \left(-\rho \cdot V_1 \cdot A_1\right) + V_2 \cdot \left(\rho \cdot V_2 \cdot A_2\right)$$

Hence $F = p_1 \cdot A_1 - p_2 \cdot A_2 + \rho \cdot \left(V_1^2 \cdot A_1 - V_2^2 \cdot A_2\right)$

$$F = 600\left(\frac{kN}{m^2}\right)\cdot 0.00785\left(m^2\right) - 125\left(\frac{kN}{m^2}\right)\cdot 0.00126\left(m^2\right)$$

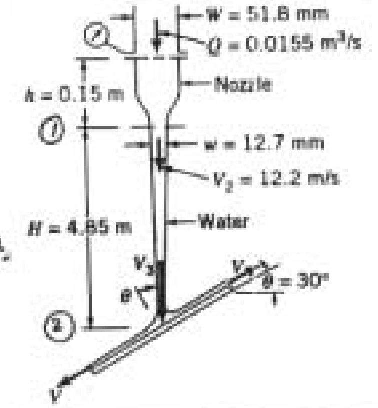
$$+ 999\left(\frac{kg}{m^3}\right)\cdot\left(\left(5\left(\frac{m}{s}\right)\right)^2\left(0.00785\left(m^2\right)\right) - \left(31.3\left(\frac{m}{s}\right)\right)^2\left(0.00126\left(m^2\right)\right)\right)\left(\frac{N\,s^2}{kg\cdot m}\right)$$

$$F = 3.52\,kN$$

Problem *4.98

Given: Plane nozzle discharging water steadily, striking an inclined plate.

Neglect friction in nozzle and along plate surface.

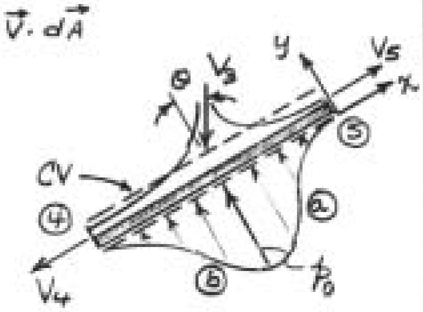


Find: (a) Minimum gage pressure at nozzle inlet.
(b) Magnitude and direction of force exerted by water stream on plate.
(c) Sketch pressure distribution on plate. Explain why the pressure distribution is shaped as you show it.

<u>Solution</u>: Apply continuity, Bernoulli, and momentum equations using the CV and coordinates shown.

Basic equations: $V_1 A_1 = V_2 A_2$ $\quad \frac{p_1}{\rho} + \frac{V_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + g z_2$

$$R_y + \cancel{F_{B_y}}^{=0(6)} = \cancel{\frac{\partial}{\partial t}\int_{CV} v \rho \, d\forall}^{=0(3)} + \int_{CS} v \rho \vec{V} \cdot d\vec{A}$$



Assumptions: (1) Frictionless flow
(2) Incompressible flow
(3) Steady flow
(4) Flow along a streamline
(5) Uniform flow at each section

Then from continuity $V_1 = \frac{A_2}{A_1} V_2 = \frac{w}{W} V_2 = \frac{12.7 \text{ mm}}{51.8 \text{ mm}} \times 12.2 \frac{m}{s} = 2.99 \text{ m/s}$

From Bernoulli $p_{1g} = \frac{\rho}{2}(V_2^2 - V_1^2) - \rho g (z_1 - z_2) + \cancel{p_{2g}}^{=0}$; $z_1 - z_2 = h$

$$p_{1g} = \left[\frac{1}{2} \times 999 \frac{kg}{m^3} \times [(12.2)^2 - (2.99)^2] \frac{m^2}{s^2} - 999 \frac{kg}{m^3} \times 9.81 \frac{m}{s^2} \times 0.15 m\right] \frac{N \cdot s^2}{kg \cdot m} = 68.4 \text{ kPa (gage)}$$ p_{1g}

Calculate V_3 in the absense of the plate using Bernoulli ($p_2 = p_3$)

$$V_3 = \sqrt{V_2^2 + 2gH} = \left[(12.2)^2 \frac{m^2}{s^2} + 2 \times 9.81 \frac{m}{s^2} \times 4.85 m\right]^{\frac{1}{2}} = 15.6 \text{ m/s}$$

From momentum: $R_x = 0$ since there is no friction on the plate surface.

Assumptions: (6) Neglect mass of plate and of water on plate.
(7) Atmospheric pressure acts on entire CV; $F_{s_y} = R_y$

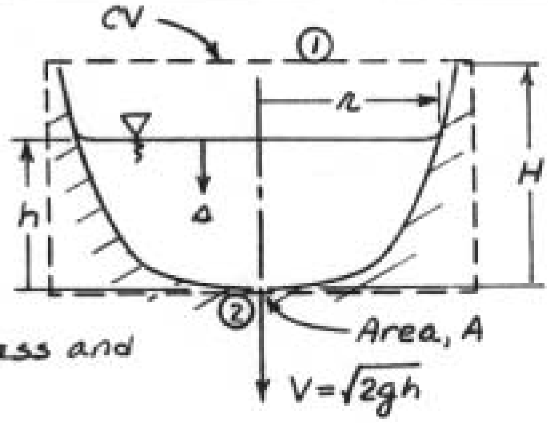
Then $R_y = v_3 \{-\dot{m}_3\} + \cancel{v_4}^{0}\{+\dot{m}_4\} + \cancel{v_5}^{0}\{+\dot{m}_5\} = V_3 \cos\theta \, \rho Q$, since $v_3 = -V_3 \cos\theta$

$$R_y = 15.6 \frac{m}{s} \times \cos 30° \times 999 \frac{kg}{m^3} \times 0.0155 \frac{m^3}{s} \times \frac{N \cdot s^2}{kg \cdot m} = 209 \text{ N}; \; K_y = -R_y = -209 \text{ N}$$ K_y

Pressure is maximum at stagnation, minimum (p_{atm}) at ④ and ⑤.
Pressure at ⓐ is higher than at ⓑ because of streamline curvature.

Problem *4.99

Given: Egyptian water clock. Surface level drops at rate, s = constant.



Find: (a) Expression for $r(h)$.
(b) Volume needed for n hours' operation.

Solution: Apply conservation of mass and the Bernoulli equation.

Basic equations: $0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A}$

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$$

Assumptions: (1) Quasi-steady flow; $\frac{\partial}{\partial t}$ small
(2) Incompressible flow
(3) Uniform flow at each cross section
(4) Flow along a streamline
(5) No friction
(6) $\rho_{air} \ll \rho_{H_2O}$

Writing Bernoulli from the liquid surface to the jet exit,

$$\frac{p_{atm}}{\rho} + \frac{s^2}{2} + gh = \frac{p_{atm}}{\rho} + \frac{V^2}{2} + g(0);$$

For $s \ll V$, then $V = \sqrt{2gh}$.

For the CV,

$$0 = \frac{\partial}{\partial t}\int_{\forall_{air}} \rho_{air}\, d\forall + \frac{\partial}{\partial t}\int_{\forall_{H_2O}} \rho_{H_2O}\, d\forall + \left\{-|\rho_{air} V_1 A_1|\right\} + \left\{|\rho_{H_2O} V A|\right\}$$

($\rho_{air} \ll \rho_{H_2O}$ (6), for both air terms)

or

$$0 = \rho \frac{d\forall}{dt} + \rho V A = \rho \pi r^2 \frac{dh}{dt} + \rho\sqrt{2gh}\, A = 0$$

But h decreases, so $\frac{dh}{dt} = -s$. Thus

$$\pi r^2 s = \sqrt{2gh}\, A \quad \text{or} \quad r = \sqrt[4]{2g}\sqrt{\frac{A}{\pi s}}\, h^{1/4}$$ ← r

For n hours' operation, $H = ns$, and

$$\forall = \int_0^H \pi r^2 \, dh = \int_0^{ns} \sqrt{2gh}\,\frac{A}{s}\, dh = \frac{2A}{3s}\sqrt{2g}\,(ns)^{3/2}$$

or

$$\forall = \frac{2A\sqrt{2g}\, n^{3/2} s^{1/2}}{3}$$ ← ∀

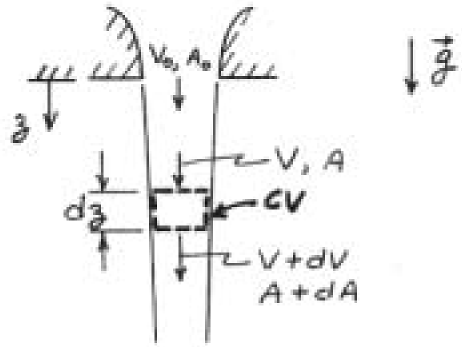
Check dimensions:

$$[\forall] = L^3 = \left[A\sqrt{g}\, n^{3/2} s^{1/2}\right] = L^2 \frac{L^{1/2}}{t}\, t^{3/2} \frac{L^{1/2}}{t^{1/2}} = L^3 \checkmark$$

Problem *4.100

Given: Low-speed jet of incompressible liquid moving downward from nozzle.

Find: Expressions for $V(z)$, $A(z)$.
Location where $A = A_0/2$.



Solution: Apply continuity and momentum equations using CV shown.

Basic equations:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A} \qquad (=0(1))$$

$$F_{S_z} + F_{B_z} = \frac{\partial}{\partial t}\int_{CV} w\rho \, d\forall + \int_{CS} w\rho \vec{V} \cdot d\vec{A} \qquad (F_{S_z}=0(4,5);\ \partial/\partial t=0(1))$$

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Uniform flow at each section
(4) p_{atm} acts everywhere } $F_{S_z} = 0$
(5) No friction

Then

$$0 = \int_{CS} \vec{V} \cdot d\vec{A} = \{-VA\} + \{+(V+dV)(A+dA)\}; \quad VA = V_0 A_0 = \text{constant}$$

From momentum,

$$\rho g \left(A + \frac{dA}{2}\right)dz = V\{-\rho VA\} + (V+dV)\{-\rho(V+dV)(A+dA)\} = \rho V A\, dV$$

since $dV dA \ll dA$. Also, since $dA\,dz \ll dz$, the left side is $\rho g A\, dz$.
Thus

$$\rho g A\, dz = \rho V A\, dV \quad \text{or} \quad V dV = g\, dz$$

Integrating from V_0 at $z_0 = 0$ to V at z,

$$\int_{V_0}^{V} V dV = \frac{V^2}{2}\bigg]_{V_0}^{V} = \frac{V^2}{2} - \frac{V_0^2}{2} = \int_{z_0}^{z} g\, dz = g(z - z_0) = gz$$

Thus

$$V^2 = V_0^2 + 2gz \quad \text{or} \quad V(z) = \sqrt{V_0^2 + 2gz}$$ $V(z)$

Since $VA = V_0 A_0$, $A = A_0 \dfrac{V_0}{V}$

$$A(z) = A_0 \frac{V_0}{\sqrt{V_0^2 + 2gz}} = \frac{A_0}{\sqrt{1 + 2gz/V_0^2}}$$ $A(z)$

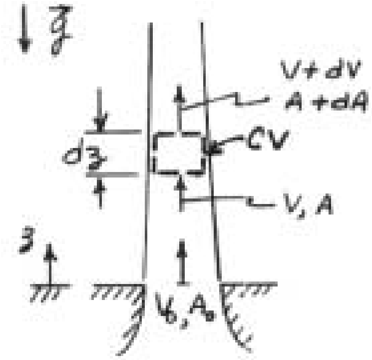
Solving for z,

$$z = \frac{V_0^2}{2g}\left[\left(\frac{A_0}{A}\right)^2 - 1\right]; \text{ for } \frac{A}{A_0} = \frac{1}{2},\ \frac{A_0}{A} = 2, \text{ and } z_{1/2} = \frac{3V_0^2}{2g}$$ $z_{1/2}$

Problem *4.101

Given: Low-speed jet of incompressible liquid moving upward from nozzle.

Find: Expressions for $V(z)$, $A(z)$. Location where $V = 0$.



Solution: Apply continuity and momentum equation using CV shown.

Basic equations:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A} \qquad \left(\frac{\partial}{\partial t} = 0(1)\right)$$

$$F_{S_z} + F_{B_z} = \frac{\partial}{\partial t}\int_{CV} w \rho \, d\forall + \int_{CS} w \rho \vec{V} \cdot d\vec{A} \qquad \left(F_{S_z} = 0(4,5),\ \frac{\partial}{\partial t} = 0(1)\right)$$

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Uniform flow at each section
(4) p_{atm} acts everywhere
(5) No friction
} $F_{S_z} = 0$

Then

$$0 = \int_{CS} \vec{V} \cdot d\vec{A} = \{-VA\} + \{+(V+dV)(A+dA)\}; \quad VA = V_0 A_0 = \text{constant}$$

From momentum,

$$-\rho g \left(A + \frac{dA}{2}\right) dz = V\{-\rho VA\} + (V+dV)\{+\rho(V+dV)(A+dA)\} = \rho VA\, dV$$

since $dV dA \ll dA$. Also, since $dA\,dz \ll dz$, the left side is $-\rho g A\,dz$. Thus

$$-\rho g A\,dz = \rho V A\,dV \quad \text{or} \quad V dV = -g\,dz$$

Integrating from V_0 at $z_0 = 0$ to V at z,

$$\int_{V_0}^{V} V dV = \frac{V^2}{2}\bigg]_{V_0}^{V} = \frac{V^2}{2} - \frac{V_0^2}{2} = \int_{z_0}^{z} -g\,dz = -g(z - z_0) = -gz$$

Thus $V^2 = V_0^2 - 2gz$ or $V(z) = \sqrt{V_0^2 - 2gz}$ ← $V(z)$

Since $VA = V_0 A_0$, then $A = A_0 \frac{V_0}{V}$

$$A(z) = A_0 \frac{V_0}{\sqrt{V_0^2 - 2gz}} = \frac{A_0}{\sqrt{1 - 2gz/V_0^2}}$$ ← $A(z)$

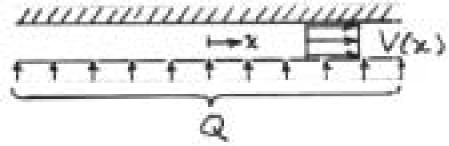
Solving for z at $V = 0$,

$$z = \frac{V_0^2}{2g}$$ ← z

Problem *4.102

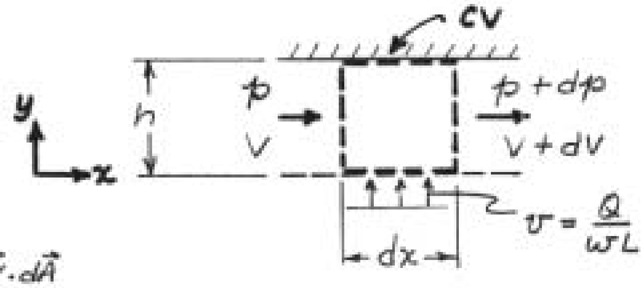
Given: Uniform flow in narrow gap between parallel plates, as shown.

Fluid in gap has only horizontal motion.



Find: Expression for $p(x)$.

Solution: Apply continuity and x component of momentum equation.



Basic equations:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A} \qquad \left(\frac{\partial}{\partial t} = 0\,(1)\right)$$

$$F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u \rho \, d\forall + \int_{CS} u \rho \vec{V}\cdot d\vec{A} \qquad \left(F_{B_x} = 0\,(5),\ \frac{\partial}{\partial t} = 0\,(1)\right)$$

Assumptions:
(1) Steady flow
(2) Incompressible flow
(3) Uniform flow at each section
(4) Neglect friction
(5) $F_{B_x} = 0$

Then

$$0 = \int_{CS} \vec{V}\cdot d\vec{A} = \{-Vwh\} + \left\{-\frac{Q}{wL} w\,dx\right\} + \{(V+dV)wh\}; \quad wh\,dV = \frac{Q}{L}dx$$

$$V = \frac{Q}{wh}\frac{x}{L} + c; \quad c = 0 \text{ since } V(0) = 0; \quad V(x) = \frac{Q}{wh}\frac{x}{L}$$

From momentum,

$$p\,wh - (p+dp)wh = u_x\{-\rho Vwh\} + u_{dx}\left\{-\rho\frac{Q}{wh}w\,dx\right\} + u_{x+dx}\{+\rho(V+dV)wh\}$$

$$u_x = V \qquad u_{dx} = 0 \qquad u_{x+dx} = V + dV$$

From continuity, $(V+dV)wh = Vwh + Q\frac{dx}{L}$, so

$$-dp\,wh = -\rho V^2 wh + 0 + (V+dV)\left(Vwh + Q\frac{dx}{L}\right)\rho$$

$$= -\rho V^2 wh + \rho V^2 wh + \rho Vwh\,dV + V\rho Q\frac{dx}{L} + \rho Q\,dV\frac{dx}{L}$$

Neglecting products of differentials ($dVdx \ll dx$), and with $dV = \frac{Q}{wh}\frac{dx}{L}$

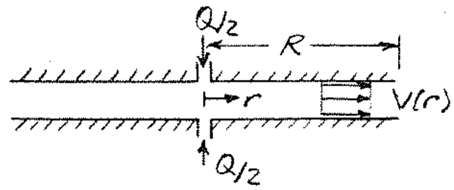
$$-dp = \rho V dV + \frac{V\rho Q}{wh}\frac{dx}{L} = \rho V\frac{Q}{wh}\frac{dx}{L} + \frac{V\rho Q}{wh}\frac{dx}{L} = 2\rho\frac{Q}{wh}\frac{x}{L}\frac{Q}{wh}\frac{dx}{L}$$

$$-dp = 2\rho\left(\frac{Q}{whL}\right)^2 x\,dx \qquad p(x) = -\rho\left(\frac{Q}{whL}\right)^2 x^2 + c$$

If $p(0) = p_0$, then $$p(x) = p_0 - \rho\left(\frac{Q}{wh}\right)^2\left(\frac{x}{L}\right)^2$$ $p(x)$

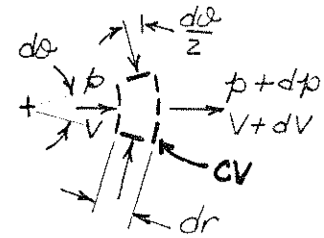
Given: Uniform flow in narrow gap between parallel disks, as shown.

Liquid in gap has only radial motion.



Find: Expression for $p(r)$; plot

<u>Solution</u>: Apply continuity and momentum equations to the differential CV shown.



Basic equations:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A} \quad (\frac{\partial}{\partial t} \text{ term} = 0(1))$$

$$F_{S_r} + F_{B_r} = \frac{\partial}{\partial t}\int_{CV} V_r \rho \, d\forall + \int_{CS} V_r \rho \vec{V}\cdot d\vec{A} \quad (F_{B_r} = 0(5), \ \frac{\partial}{\partial t} \text{ term} = 0(1))$$

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Uniform flow at each section
(4) Neglect friction
(5) $F_{B_r} = 0$
(6) No flow in θ direction
(7) $\sin\frac{d\theta}{2} = \frac{d\theta}{2}$

Then

$$0 = \int_{CS} \vec{V}\cdot d\vec{A} = \{-\rho V h r d\theta\} + \{\rho(V+dV)h(r+dr)d\theta\} \ ; \ Vr = \text{constant}$$

For $r = R$, $Q = V_R 2\pi r h$, so $V_R = Q/2\pi R h$

From momentum,

$$p h r d\theta + 2(p + \tfrac{dp}{2}) h dr \sin\frac{d\theta}{2} - (p+dp)h(r+dr)d\theta$$

$$= V\{-\rho V h r d\theta\} + (V+dV)\{\rho(V+dV)h(r+dr)d\theta\}$$

$$\cancel{phr}d\theta + \cancel{ph}dr\,d\theta + \tfrac{1}{2}dp\,h\,dr\,d\theta - (\cancel{pr} + \cancel{pdr} + r dp + dr\,dp)h\,d\theta$$

$$= dV(\rho V h r d\theta) \quad \{\text{Note terms in braces are equal.}\}$$

Assuming products of differentials are much smaller than single differentials,

$$-r\,dp\,h\,d\theta = dV(\rho V h r d\theta) \quad \text{or} \quad dp = -\rho V dV$$

Integrating, $p(r) - p(R) = -\rho\frac{V^2}{2} + \rho\frac{V_R^2}{2}$ or $p(r) - p_{atm} = \frac{\rho}{2}(V_R^2 - V^2)$
$= \frac{\rho V_R^2}{2}\left[1 - \left(\frac{V}{V_R}\right)^2\right]$

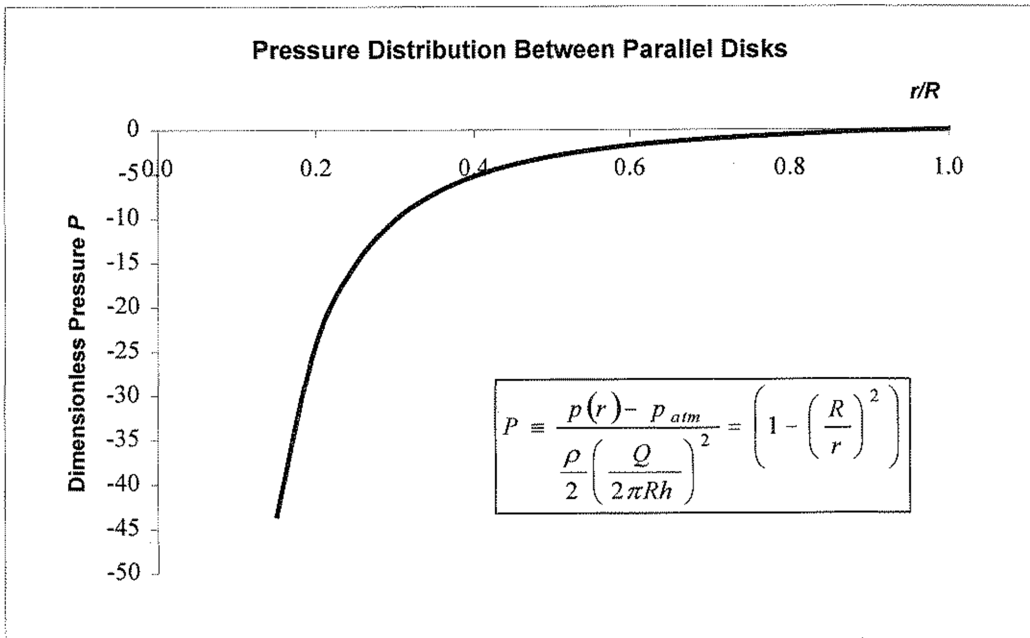
Since $V_R = \frac{Q}{2\pi R h}$, and Vr = constant, $\frac{V}{V_R} = \frac{R}{r}$, so

$$p(r) - p_{atm} = \frac{\rho}{2}\left(\frac{Q}{2\pi R h}\right)^2\left[1 - \left(\frac{R}{r}\right)^2\right] \qquad p(r)$$

Note since $r < R$, that $p(r) < p_{atm}$ between the disks.

The pressure distribution is computed and plotted in *Excel*:

r/R	*P*
0.15	-43.4
0.20	-24.0
0.25	-15.0
0.30	-10.1
0.35	-7.16
0.40	-5.25
0.45	-3.94
0.50	-3.00
0.55	-2.31
0.60	-1.78
0.65	-1.37
0.70	-1.04
0.75	-0.78
0.80	-0.563
0.85	-0.384
0.90	-0.235
0.95	-0.108
1.00	0.000



Given: Liquid falling vertically into short, horizontal, rectangular open channel. Neglect viscous effects.

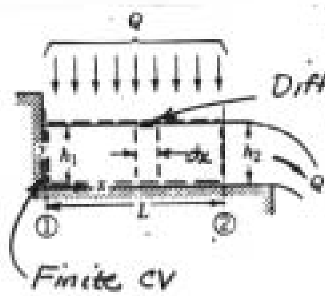
Find: (a) Expression for h_1 in terms of h_2, Q, and b.
(b) Sketch surface profile, $h(x)$.

Solution: Apply continuity and momentum equations to (i) finite CV, and (ii) differential CV, as shown.

Basic equations:

$$0 = \cancel{\frac{\partial}{\partial t}}^{=0(1)} \int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$

$$F_{S_x} + \cancel{F_{B_x}}^{=0(6)} = \cancel{\frac{\partial}{\partial t}}^{=0(1)} \int_{CV} u \rho \, d\forall + \int_{CS} u \rho \vec{V} \cdot d\vec{A}$$



Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Uniform flow at each section
(4) Hydrostatic pressure distribution; $F_p(h) = \rho g b \frac{h^2}{2}$
(5) No friction on bed
(6) Horizontal bed; $F_{B_x} = 0$

Then for finite CV shown,

$$0 = \int_{CS} \vec{V} \cdot d\vec{A} = -Q + V_2 b h_2 \; ; \; V_2 = \frac{Q}{b h_2}$$

From momentum

$$\rho g b \frac{h_1^2}{2} - \rho g b \frac{h_2^2}{2} = u_1\{0\} + u_2\{+\rho Q\} + u_3\{-\rho Q\}$$

$$u_2 = V_2 \qquad u_3 = 0$$

$$\frac{\rho g b}{2}(h_1^2 - h_2^2) = V_2 \rho Q = \frac{Q}{b h_2} \rho Q = \frac{\rho Q^2}{b h_2} \; ; \; h_1 = \sqrt{h_2^2 + \frac{2Q^2}{g b^2 h_2}} \longleftarrow h_1$$

For differential CV shown,

$$0 = \int_{CS} \vec{V} \cdot d\vec{A} = \{-Vbh\} + \{-\frac{Q}{bL} b\,dx\} + \{+(V+dV) b (h+dh)\}$$

$$0 = -\frac{Q}{L} dx + b(h\,dV + V\,dh) = -\frac{Q}{L} dx + b\, d(hV) \; ; \; \frac{d(hV)}{dx} = \frac{Q}{L}$$

From momentum,

$$\rho g b \frac{h^2}{2} - \rho g b \frac{(h+dh)^2}{2} = V\{-\rho V b h\} + 0\{-\rho \frac{Q}{L} dx\} + (V+dV)\{+\rho (V+dV) b (h+dh)\}$$

Using continuity,

$$\frac{\rho g b}{2}(-2h\,dh + \cancel{dh\,dh}^{\ll dh}) = -\rho V^2 b h + (V+dV)\{\rho V b h + \rho \frac{Q}{L} dx\}$$

Cont'd. →

$$-\rho g b h dh = -\rho V^2 bh + \rho V^2 bh + \rho V bh dV + \frac{\rho Q}{L} V dx + \frac{\rho Q}{L} dV dx$$ ($dV\,dx \ll dx$)

or

$$-g h dh = V h dV + \frac{Q}{bL} V dx$$

From continuity, $VhdV = -V^2 dh + \frac{Q}{bL} V dx$, so

$$-g h dh = -V^2 dh + \frac{2Q}{bL} V dx$$

Solving,

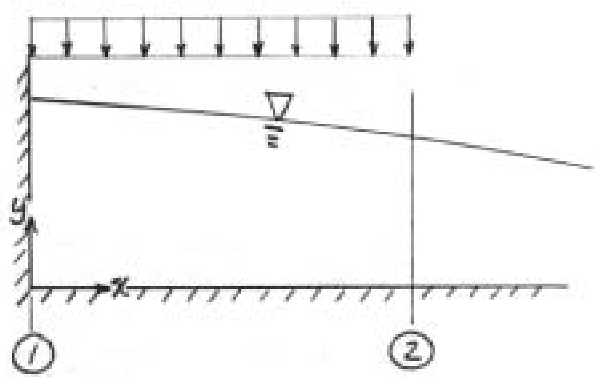
$$\frac{dh}{dx}(V^2 - gh) = \frac{2Q}{bL} V$$

$$\frac{dh}{dx} = \frac{2QV}{bL(V^2 - gh)} = \frac{2QV}{bLgh(V^2/gh - 1)}$$

From finite CV analysis, $h_1 > h_2$, so $\frac{dh}{dx} < 0$. Thus $V^2/gh < 1$. As x increases, $V \uparrow$ and $h \downarrow$. Therefore

$$\frac{V^2}{gh} \uparrow, \quad \frac{V}{h} \uparrow, \text{ and } \left|\frac{dh}{dx}\right| \uparrow.$$

Sketch:



Problem *4.105

Given: Narrow gap between parallel disks filled with liquid.

At $t = 0^+$, upper disk begins to move downward at V_0.

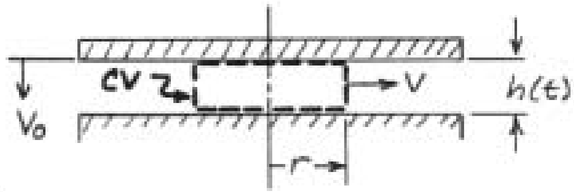
Neglect viscous effects; flow uniform in horizontal direction.

Find: Expression for velocity field, $V(r)$. Note flow is not steady.

Solution: Apply continuity, using the deformable CV shown.

Basic equation:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$



Assumptions: (1) Incompressible flow
(2) Uniform flow at each cross section

Then

$$0 = \frac{\partial}{\partial t}\int_{CV} d\forall + \int_{CS} \vec{V} \cdot d\vec{A} = \frac{\partial}{\partial t}\int_{CV} d\forall + V \, 2\pi r h$$

But

$$\int_{CV} d\forall = \pi r^2 h, \text{ so } \frac{\partial}{\partial t}\int_{CV} d\forall = \frac{\partial}{\partial t}(\pi r^2 h) = \pi r^2 \frac{dh}{dt}$$

Thus

$$0 = \pi r^2 \frac{dh}{dt} + V 2\pi r h = \pi r^2(-V_0) + V \, 2\pi r h$$

so

$$V(r) = V_0 \frac{r}{2h}$$ $V(r)$

If V_0 is constant, so $h = h_0 - V_0 t$, and

$$V(r,t) = \frac{V_0 r}{2(h_0 - V_0 t)} \quad \text{for } t < \frac{h_0}{V_0}$$ $V(r,t)$

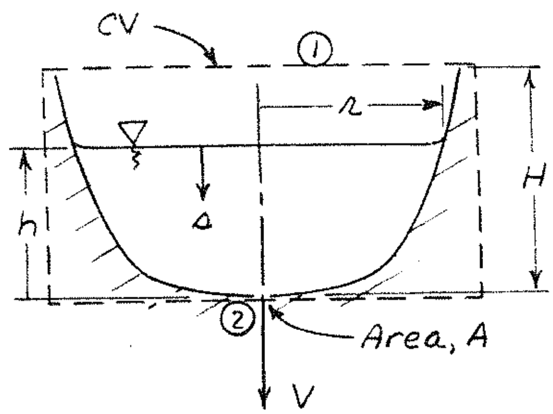
Problem *4.106

Open-Ended Problem Statement: Design a clepsydra (Egyptian water clock) — a vessel from which water drains by gravity through a hole in the bottom and time is indicated by the level of the remaining water. Specify the dimensions of the vessel and the size of the drain hole; indicate the amount of water needed to fill the vessel, and at what interval it must be filled. Plot the vessel shape. (This is an open-ended problem when choosing dimensions for a specific application.)

Discussion: The original Egyptian water clock was an open water-filled vessel with an orifice in the bottom. The vessel shape was designed so that the water level dropped at a constant rate during use.

Water leaves the orifice at higher speed when the water level within the vessel is high, and at lower speed when the water level within the vessel is low. The size of the orifice is constant. Thus the instantaneous volume flow rate depends on the water level in the vessel.

The rate at which the water level falls in the vessel depends on the volume flow rate and the area of the water surface. The surface area at each water level must be chosen so that the water level within the vessel decreases at a constant rate. The continuity and Bernoulli equations can be applied to determine the required vessel shape so that the water surface level drops at a constant rate.



Use the CV and notation shown (Problem 4.97):

Solution: Basic equations are

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$$

Assumptions: (1) Quasi-steady flow
(2) Incompressible flow
(3) Uniform flow at each cross-section
(4) Flow along a streamline
(5) No friction
(6) $\rho_{air} \ll \rho_{H_2O}$

Writing Bernoulli from the liquid surface to the jet exit,

$$\frac{p_{atm}}{\rho} + \frac{\dot{s}^2}{2} + gh = \frac{p_{atm}}{\rho} + \frac{V^2}{2} + g(0)$$

For $\dot{s} \ll V$, then $V = \sqrt{2gh}$

For the CV,

$$0 = \frac{\partial}{\partial t}\int_{\forall_{air}} \rho_{air} \, d\forall + \frac{\partial}{\partial t}\int_{\forall_{H_2O}} \rho_{H_2O} \, d\forall + \{-|\rho_{air} V_1 A_1|\} + \{|\rho_{H_2O} V A|\}$$

(The ρ_{air} terms are marked "$\ll \rho_{H_2O}$ (6)" and struck out.)

or $0 = \rho \frac{d\forall}{dt} + \rho V A = \rho \pi r^2 \frac{dh}{dt} + \rho \sqrt{2gh}\, A$

But h decreases, so $\frac{dh}{dt} = -s$. Thus

$$\pi r^2 s = \sqrt{2gh}\, A \quad \text{or} \quad r = \sqrt[4]{2g} \sqrt{\frac{A}{\pi s}}\, h^{1/4}$$ r

For n hours operation, $H = ns$, and

$$\forall = \int_0^H \pi r^2 dh = \int_0^{ns} \sqrt{2gh}\, \frac{A}{s}\, dh = \frac{2A}{3s} \sqrt{2g}\, (ns)^{3/2}$$

or

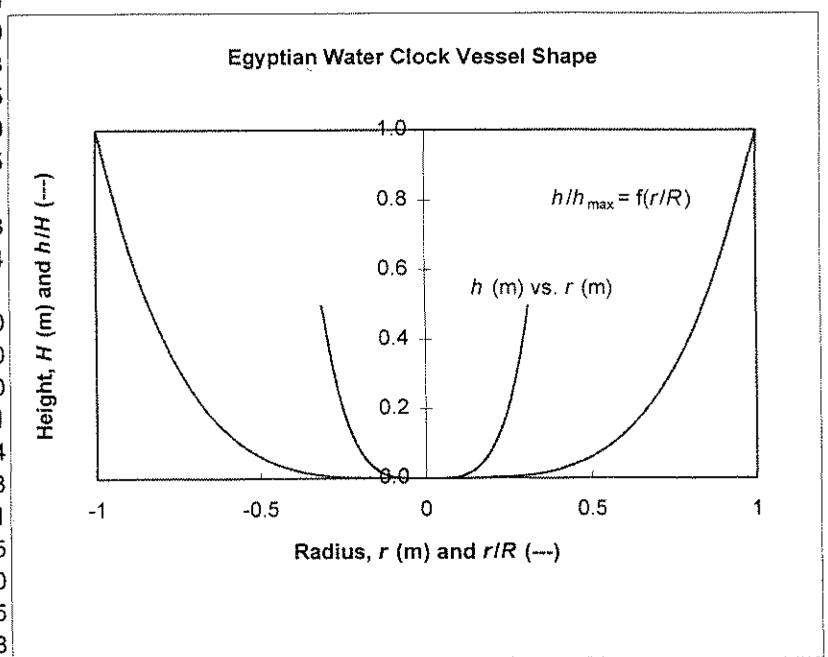
$$\forall = \frac{2A\sqrt{2g}\, n^{3/2}}{3} s^{1/2}$$ ∀

Evaluating and plotting:

Input Parameters:

Maximum water height:	*H* =	0.5	m
Number of hours' duration:	*n* =	24	hr

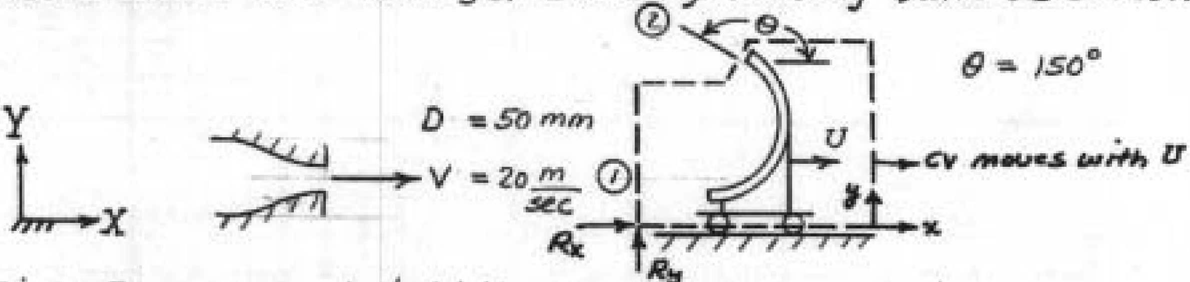
Dimensionless Shape		Actual Shape	
r/R	***h/H***	***r* (m)**	***h* (m)**
-1	1.00	-0.309	0.500
-0.9	0.656	-0.278	0.328
-0.8	0.410	-0.247	0.205
-0.7	0.240	-0.216	0.120
-0.6	0.130	-0.185	0.065
-0.5	0.063	-0.155	0.031
-0.4	0.026	-0.124	0.013
-0.3	0.008	-0.093	0.004
-0.2	0.002	-0.062	0.001
-0.1	0.000	-0.031	0.000
0	0	0	0
0.1	0.000	0.031	0.000
0.2	0.002	0.062	0.001
0.3	0.008	0.093	0.004
0.4	0.026	0.124	0.013
0.5	0.063	0.155	0.031
0.6	0.130	0.185	0.065
0.7	0.240	0.216	0.120
0.8	0.410	0.247	0.205
0.9	0.656	0.278	0.328
1	1.000	0.309	0.500



Problem 4.107

Given: Water flow from jet striking moving vane as shown.

$\theta = 150°$



Find: Force needed to hold the vane speed at $U = 5$ m/s.

Solution: Apply momentum equation to moving CV shown.

B.E.:
$$F_{S_x} + \cancel{F_{B_x}}^{=0(2)} = \cancel{\frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\, dV}^{=0(3)} + \int_{CS} u_{xyz}\,\rho\, \vec{V}_{xyz}\cdot d\vec{A}$$

$$F_{S_y} + \cancel{F_{B_y}}^{\approx 0(2)} = \cancel{\frac{\partial}{\partial t}\int_{CV} v_{xyz}\,\rho\, dV}^{=0(3)} + \int_{CS} v_{xyz}\,\rho\, \vec{V}_{xyz}\cdot d\vec{A}$$

Assume: (1) No pressure forces or friction, so $F_{S_x} = R_x$, $F_{S_y} = R_y$
(2) $F_{B_x} = 0$, neglect F_{B_y} since not given
(3) Steady flow
(4) Uniform flow at each section
(5) Relative velocity constant for jet stream crossing vane

Then

$$R_x = u_1\{-|\rho(V-U)A|\} + u_2\{|\rho(V-U)A|\} \quad ; A = \frac{\pi}{4}(0.05)^2\,m^2 = 1.96\times10^{-3}\,m^2$$

$$u_1 = V - U \qquad u_2 = (V-U)\cos\theta$$

$$R_x = \rho(V-U)^2 A(\cos\theta - 1)$$

$$R_x = 999\,\frac{kg}{m^3}(20-5)^2\frac{m^2}{s^2}\times 1.96\times10^{-3}\,m^2(\cos 150° - 1)\frac{N\cdot s^2}{kg\cdot m} = -822\ N$$

R_x ←

$$R_y = v_1\{-|\rho(V-U)A|\} + v_2\{|\rho(V-U)A|\}$$

$$v_1 = 0 \qquad v_2 = (V-U)\sin\theta$$

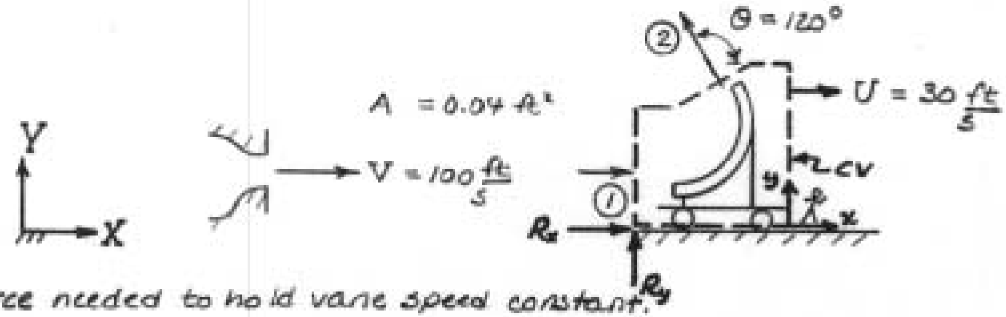
$$R_y = \rho(V-U)^2 A\sin\theta = 999\,\frac{kg}{m^3}(20-5)^2\frac{m^2}{s^2}\times 1.96\times10^{-3}\,m^2\times\sin 150°\times\frac{N\cdot s^2}{kg\cdot m} = 220\ N$$

R_y ←

{Thus a force of 822 N to the left and 220 N upward must be applied to the vane to maintain its motion at $U = 5$ m/s.}

Problem 4.108

Given: Jet of water striking a moving vane as shown.



Find: Force needed to hold vane speed constant.

Solution: Apply momentum equation using moving CV shown.

Basic equations:

$$F_{S_x} + \underbrace{F_{B_x}}_{=0(2)} = \underbrace{\frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\, d\forall}_{=0(3)} + \int_{CS} u_{xyz}\,\rho \vec{V}_{xyz}\cdot d\vec{A}$$

$$F_{S_y} + \underbrace{F_{B_y}}_{=0(2)} = \underbrace{\frac{\partial}{\partial t}\int_{CV} v_{xyz}\,\rho\, d\forall}_{=0(3)} + \int_{CS} v_{xyz}\,\rho \vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: (1) No pressure forces on CV; $F_{S_x} = R_x$, $F_{S_y} = R_y$
(2) $F_{B_x} = 0$; neglect F_{B_y}
(3) Steady flow relative to vane
(4) Flow uniform at each section
(5) Jet area and speed relative to vane are constant

The subscript xyz is a reminder that all velocities must be evaluated relative to the CV. Then

$$R_x = u_1\{-|\rho(V-U)A|\} + u_2\{|\rho(V-U)A|\}$$

$$u_1 = V - U \qquad\qquad u_2 = (V-U)\cos\theta$$

and

$$R_x = \rho(V-U)^2 A\,(\cos\theta - 1)$$

$$= 1.94\,\frac{\text{slug}}{\text{ft}^3} \times (100-30)^2\,\frac{\text{ft}^2}{\text{s}^2} \times 0.04\,\text{ft}^2 \times (\cos 120° - 1) \times \frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}}$$

$R_x = -570$ lbf (to the left) $\qquad R_x$

Also

$$R_y = v_1\{-|\rho(V-U)A|\} + v_2\{|\rho(V-U)A|\}$$

$$v_1 = 0 \qquad\qquad v_2 = (V-U)\sin\theta$$

$$R_y = \rho(V-U)^2 A\,\sin\theta$$

$$= 1.94\,\frac{\text{slug}}{\text{ft}^3} \times (100-30)^2\,\frac{\text{ft}^2}{\text{s}^2} \times 0.04\,\text{ft}^2 \times \sin 120° \times \frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}}$$

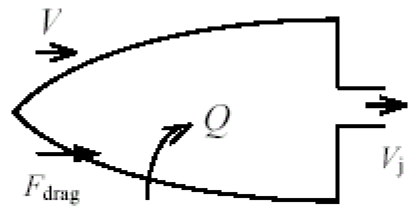
$R_y = 329$ lbf (force is up) $\qquad R_y$

Problem 4.109

A jet boat takes in water at a constant volumetric rate Q through side vents and ejects it at a high jet speed V_j at the rear. A variable-area exit orifice controls the jet speed. The drag on the boat is given by $F_{drag} = kV^2$, where V
speed V. If a jet speed $V_j = 25$ m/s produces a boat speed of 10 m/s, what jet speed will be required to double the boat speed?

Given: Data on jet boat

Find: Formula for boat speed; jet speed to double boat speed



Solution

CV in boat coordinates

Governing equation:

Momentum
$$\vec{F} = \vec{F}_S + \vec{F}_B = \frac{\partial}{\partial t}\int_{CV} \vec{V}_{xyz}\,\rho\, dV\!\!\!\!- + \int_{CS} \vec{V}_{xyz}\,\rho \vec{V}_{xyz}\cdot d\vec{A} \qquad (4.26)$$

Applying the horizontal component of momentum

$$F_{drag} = V\cdot(-\rho\cdot Q) + V_j\cdot(\rho\cdot Q)$$

Hence
$$k\cdot V^2 = \rho\cdot Q\cdot V_j - \rho\cdot Q\cdot V$$

$$k\cdot V^2 + \rho\cdot Q\cdot V - \rho\cdot Q\cdot V_j = 0$$

Solving for V
$$V = -\frac{\rho\cdot Q}{2\cdot k} + \sqrt{\left(\frac{\rho\cdot Q}{2\cdot k}\right)^2 + \frac{\rho\cdot Q\cdot V_j}{k}}$$

Let $\alpha = \frac{\rho \cdot Q}{2 \cdot k}$

$$V = -\alpha + \sqrt{\alpha^2 + 2 \cdot \alpha \cdot V_j}$$

We can use given data at V = 10 m/s to find α $\quad V = 10 \cdot \frac{m}{s} \quad V_j = 25 \cdot \frac{m}{s}$

$$10 \cdot \frac{m}{s} = -\alpha + \sqrt{\alpha^2 + 2 \cdot 25 \cdot \frac{m}{s} \cdot \alpha}$$

$$\alpha^2 + 50 \cdot \alpha = (10 + \alpha)^2 = 100 + 20 \cdot \alpha + \alpha^2$$

$$\alpha = \frac{10}{3} \cdot \frac{m}{s}$$

Hence $V = -\frac{10}{3} + \sqrt{\frac{100}{9} + \frac{20}{3} \cdot V_j}$

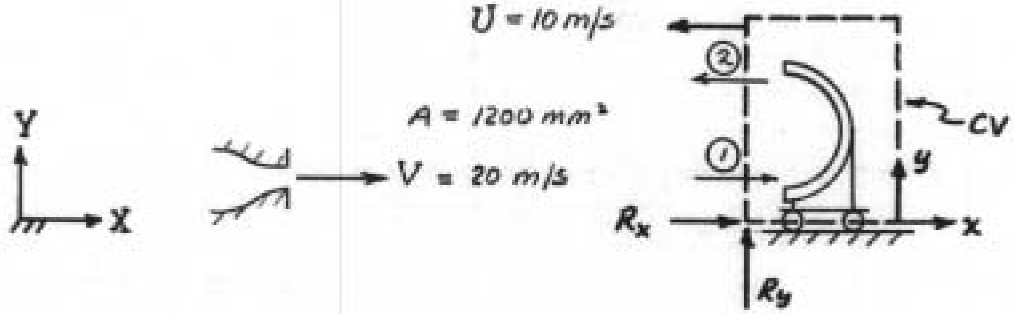
For V = 20 m/s $\quad 20 = -\frac{10}{3} + \sqrt{\frac{100}{9} + \frac{20}{3} \cdot V_j}$

$$\frac{100}{9} + \frac{20}{3} \cdot V_j = \frac{70}{3}$$

$$V_j = 80 \cdot \frac{m}{s}$$

Problem 4.110

Given: Jet of oil (SG = 0.8) striking moving vane.



Find: Force needed to maintain vane speed constant.

Solution: Apply x component of momentum equation to moving CV shown.

Basic equation: $F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\,d\forall + \int_{CS} u_{xyz}\,\rho\,\vec{V}_{xyz}\cdot d\vec{A}$ (F_{B_x} = 0(2), $\frac{\partial}{\partial t}$ = 0(3))

Assumptions: (1) No net pressure force on CV; $F_{S_x} = R_x$
(2) $F_{B_x} = 0$
(3) Steady flow
(4) Flow uniform at each section
(5) Jet area and speed relative to vane are constant

The subscript xyz is a reminder that all velocities must be evaluated relative to the CV. Then

$$R_x = u_1\{-|\rho(V+U)A|\} + u_2\{|\rho(V+U)A|\}$$

$$u_1 = V + U \qquad\qquad u_2 = -(V+U)$$

and $R_x = -\rho(V+U)^2A - \rho(V+U)^2A = -2\rho(V+U)^2A = -2SG\,\rho_{H_2O}(V+U)^2A$

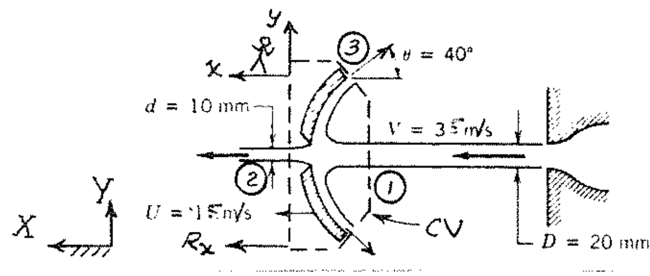
$$R_x = -2(0.8)\,999\,\frac{kg}{m^3}\,(20+10)^2\frac{m^2}{s^2}\times 1200\,mm^2\times\frac{m^2}{10^6\,mm^2}\times\frac{N\cdot s^2}{kg\cdot m} = -1.73\text{ kN}$$

R_x

This force must be applied to the left on the vane.

{Note $R_y = mg$, since there are no vertical components of velocity.}

Given: Circular dish and jet moving as shown.



Find: Force required to maintain dish motion.

Solution: Apply continuity and x momentum equation to CV moving with dish as shown.

Basic equations:

$$0 = \cancel{\frac{\partial}{\partial t}\int_{CV} \rho\, d\forall}^{=0(1)} + \int_{CS} \rho \vec{V}_{xyz} \cdot d\vec{A}$$

$$F_{S_x} + \cancel{F_{B_x}}^{=0(3)} = \cancel{\frac{\partial}{\partial t}\int_{CV} u_{xyz}\, \rho\, d\forall}^{=0(1)} + \int_{CS} u_{xyz}\, \rho \vec{V} \cdot d\vec{A}$$

Assumptions: (1) Steady flow w.r.t. CV
(2) No pressure forces on CV
(3) Horizontal; $F_{B_X} = 0$
(4) Uniform flow at each section
(5) No change in speed of jet relative to vane
(6) Incompressible flow

Then

$$0 = \int_{CS} \vec{V}_{xyz} \cdot d\vec{A} = (V-U)\left(-\frac{\pi D^2}{4} + \frac{\pi d^2}{4} + A_{3,4}\right)$$

$$A_{3,4} = \frac{\pi}{4}(D^2 - d^2) = \frac{\pi}{4}\left[(0.020)^2 - (0.010)^2\right] m^2 = 2.36 \times 10^{-4}\, m^2$$

From the momentum equation

$$R_x = u_1\left\{-\rho(V-U)\frac{\pi D^2}{4}\right\} + u_2\left\{+\rho(V-U)\frac{\pi d^2}{4}\right\} + u_3\left\{+\rho(V-U)A_{3,4}\right\}$$

$$u_1 = V - U \qquad u_2 = V - U \qquad u_3 = -(V-U)\cos 40^\circ$$

$$R_x = -\rho(V-U)^2\frac{\pi D^2}{4} + \rho(V-U)^2\frac{\pi d^2}{4} - \rho(V-U)^2\frac{\pi}{4}(D^2 - d^2)\cos 40^\circ$$

$$= -\rho(V-U)^2\frac{\pi}{4}(D^2 - d^2)(1 + \cos 40^\circ)$$

$$= -999\,\frac{kg}{m^3} \times (35-15)^2\,\frac{m^2}{s^2} \times 2.36 \times 10^{-4}\, m^2\,(1 + \cos 40^\circ) \times \frac{N \cdot s^2}{kg \cdot m}$$

$R_x = -167$ N (force must be applied to right) R_x

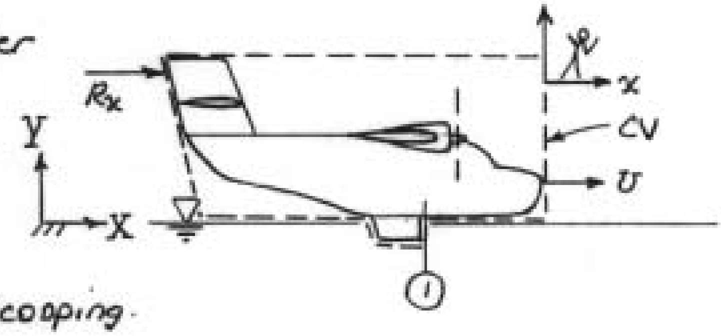
{ Note: $R_y = Mg$, since there is no net momentum flux in the y-direction. }

Problem 4.112

Given: Aircraft scooping water from lake:

1620 gal in 12 sec

Find: Added thrust needed to maintain steady aircraft speed during scooping.



Solution: Use CV moving with aircraft, as shown. Apply momentum.

Basic equation:

$$F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\, d\forall + \int_{CS} u_{xyz}\,\rho \vec{V}_{xyz}\cdot d\vec{A}$$

($F_{B_x} = 0(1)$, $\frac{\partial}{\partial t}\int_{CV} \approx 0(2)$)

Assumptions:
(1) Horizontal motion, so $F_{B_x} = 0$
(2) Neglect u_{xyz} within the CV
(3) Uniform flow at inlet cross-section
(4) Neglect hydrostatic pressure

Then

$$R_x = u_1\{-|\rho Q_1|\} = -U(-\rho Q) = +U\rho Q$$

$$u_1 = -U$$

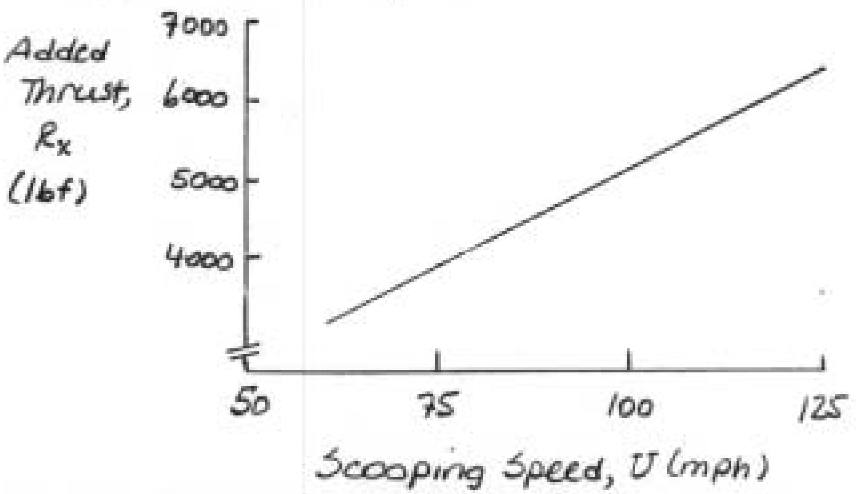
From data given

$$Q = \frac{\Delta\forall}{\Delta t} = \frac{1620\ \text{gal}}{12\ \text{sec}} \times \frac{\text{ft}^3}{7.48\ \text{gal}} = 18.0\ \text{ft}^3/\text{s}$$

For an aircraft speed of $U = 75$ mph (110 ft/s)

$$R_x = 110\,\frac{\text{ft}}{\text{s}} \times 1.94\,\frac{\text{slug}}{\text{ft}^3} \times 18.0\,\frac{\text{ft}^3}{\text{s}} \times \frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}} = 3{,}840\ \text{lbf}$$

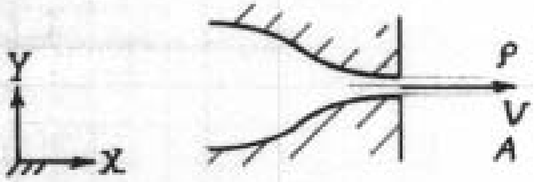
For a range of aircraft speeds:

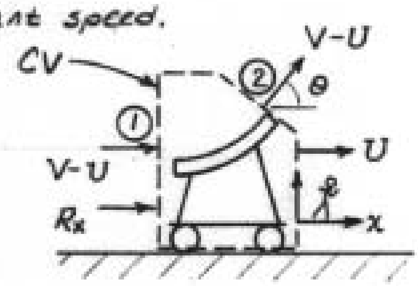


{Thus at 60 mph the added thrust is 3,070 lbf, while at 125 mph the added thrust is 6,400 lbf.}

Problem 4.113

Given: Turning vane moving at constant speed.





Find: (a) Force exerted by vane.
(b) Power produced by vane.
(c) Show power maximized when $U = V/3$.

Solution: Apply x component of momentum equation to moving CV.

$$\text{Basic equation:}\quad F_{S_x} + \cancel{F_{B_x}}^{=0(2)} = \cancel{\frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\, dV}^{=0(3)} + \int_{CS} u_{xyz}\,\rho\,\vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: (1) No net pressure force; $F_{S_x} = R_x$
(2) $F_{B_x} = 0$
(3) Steady flow with respect to observer on CV
(4) Uniform flow at each section
(5) No change in jet speed (relative to vane) on vane
(6) Incompressible flow, so A = constant

Then

$$R_x = u_{xyz_1}\{-|\rho V_{xyz_1} A_1|\} + u_{xyz_2}\{|\rho V_{xyz_2} A_2|\}$$

$$u_{xyz_1} = V-U \qquad u_{xyz_2} = (V-U)\cos\theta$$
$$V_{xyz_1} = V-U \qquad V_{xyz_2} = V-U$$

So

$$R_x = (V-U)[-\rho(V-U)A] + (V-U)\cos\theta[\rho(V-U)A] = \rho(V-U)^2A(\cos\theta - 1)$$

This is force exerted on vane. The force exerted by vane is

$$K_x = -R_x = \rho(V-U)^2A(1-\cos\theta)$$ K_x

The power produced by the vane is

$$\dot{W}_{out} = K_x U = \rho(V-U)^2 U A(1-\cos\theta)$$ $\dot{W}_{out}$

To maximize, set $d\dot{W}_{out}/dU = 0$

$$\frac{d\dot{W}}{dU} = \rho(V-U)^2A(1-\cos\theta) + (2)(-1)\rho(V-U)UA(1-\cos\theta) = 0$$

or

$$V-U-2U = V-3U = 0$$

so

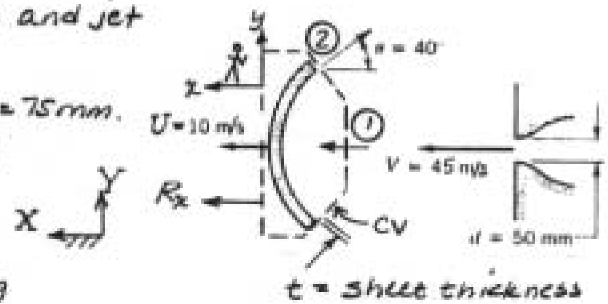
$$U = \frac{V}{3} \qquad \text{(vane speed for maximum power output)}$$ U

{ Note: $K_y = -R_y = -Mg - \rho(V-U)^2A\sin\theta$, but this force does not produce power. }

Problem 4.114

Given: Circular dish with $D = 0.15$ m and jet as shown.

Find: (a) Thickness of jet sheet at $R = 75$ mm.
(b) Horizontal force required to maintain dish motion.



Solution: Apply the momentum equation to a CV moving with the dish, as shown.

Basic equation:

$$F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\, d\forall + \int_{CS} u_{xyz}\,\rho\,\vec{V}_{xyz}\cdot d\vec{A}$$

($F_{B_x} = 0(2)$, $\frac{\partial}{\partial t} = 0(3)$)

Assumptions: (1) No pressure forces
(2) Horizontal; $F_{B_x} = 0$
(3) Steady flow w.r.t. CV
(4) Uniform flow at each section
(5) Use relative velocities
(6) No change in relative velocity on the dish

Then

$$R_x = u_1\{-\rho(V-U)A\} + u_2\{+\rho(V-U)A\}$$

$$u_1 = V-U \qquad u_2 = -(V-U)\cos\theta$$

$$R_x = -\rho(V-U)^2A - \rho(V-U)^2A\cos\theta = -\rho(V-U)^2A(1+\cos\theta)$$

$$= -999\frac{kg}{m^3}\times(45-10)^2\frac{m^2}{s^2}\times\frac{\pi}{4}(0.050)^2m^2(1+\cos 40°)\frac{N\cdot s^2}{kg\cdot m}$$

$R_x = -4.24$ kN (force must act to right) $\quad R_x$

Apply conservation of mass to determine the jet sheet thickness:

Basic equation: $0 = \frac{\partial}{\partial t}\int_{CV}\rho\, d\forall + \int_{CS}\rho\vec{V}\cdot d\vec{A}$

Using the above assumptions, then

$$0 = -\rho V_1A_1 + \rho V_2A_2$$

$$V_1 = V-U \; ; \; V_2 = V-U \; ; \; A_1 = \frac{\pi d^2}{4} \; ; \; A_2 = 2\pi Rt$$

Therefore $A_1 = A_2 = \frac{\pi d^2}{4} = 2\pi Rt$, and $t = \frac{d^2}{8R}$

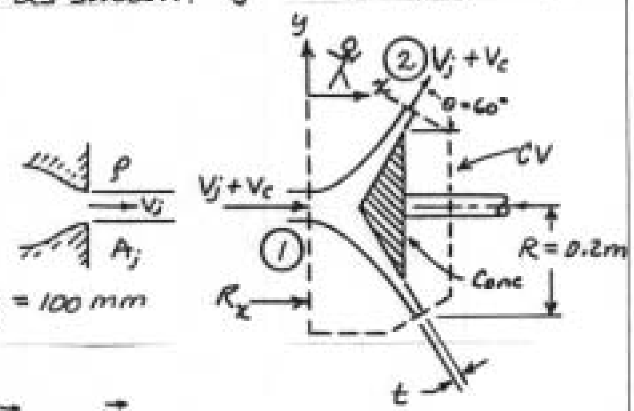
$t = \frac{1}{8}\times(0.050)^2m^2\times\frac{1}{0.075\,m} = 4.17\times10^{-3}$ m or 4.17 mm $\quad t$

Problem 4.115

Given: Water jet deflected by cone as shown. $V_j = 30$ m/sec, $V_c = 15$ m/sec

Find: (a) Thickness of jet sheet at $R = 200$ mm

(b) Force needed to move cone.



Solution: Apply continuity and x component of momentum.

Use moving coordinate system and control volume shown.

Basic equations:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}_{xyz} \cdot d\vec{A} \qquad \left(\frac{\partial}{\partial t} = 0\ (1)\right)$$

$$F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\, \rho\, d\forall + \int_{CS} u_{xyz}\, \rho \vec{V}_{xyz} \cdot d\vec{A} \qquad \left(F_{Bx} = 0\ (4),\ \frac{\partial}{\partial t} = 0\ (1)\right)$$

Assumptions: (1) Steady flow (in moving reference frame)

(2) Uniform flow at each section

(3) $V_2 = V_1$ (no change in relative velocity)

(4) $F_{Bx} = 0$; Neglect F_{By}

(5) No pressure forces

Then

$$0 = \int_{A_1} \{-\rho V_1 dA_1\} + \int_{A_2} \{+\rho V_2 dA_2\} = -\rho(V_j + V_c)\frac{\pi D_j^2}{4} + \rho(V_j + V_c)\, 2\pi R t$$

and

$$t = \frac{D_j^2}{8R} = \frac{1}{8} \times (0.1)^2\, m^2 \times \frac{1}{0.2\, m} \times 1000 \frac{mm}{m} = 6.25\ mm$$ t

Using relative velocities, momentum becomes

$$R_x = \int_{A_1} u_1 \{-\rho V_1 dA_1\} + \int_{A_2} u_2 \{+\rho V_2 dA_2\}$$

$$u_1 = V_j + V_c \qquad u_2 = (V_j + V_c)\cos\theta$$

$$V_1 = V_j + V_c \qquad V_2 = V_j + V_c$$

$$R_x = -\rho(V_j + V_c)^2 A_j + \rho(V_j + V_c)^2 A_j \cos\theta = (\cos\theta - 1)\rho(V_j + V_c)^2 A_j$$

$$= (\cos 60° - 1)\, 999 \frac{kg}{m^3} \times (30 + 15)^2 \frac{m^2}{s^2} \times \frac{\pi}{4}(0.1)^2\, m^2 \times \frac{N \cdot s^2}{kg \cdot m}$$

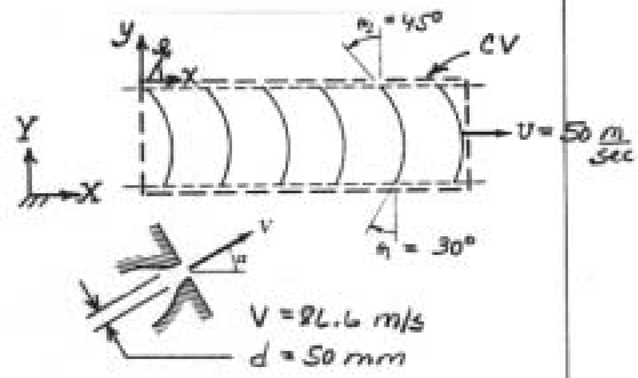
$R_x = -7.94$ kN (force must be applied to left) R_x

{Note: $R_y = Mg$ since there is no net momentum flux in the y-direction.}

Problem 4.116

Given: Series of vanes struck by continuous jet, as shown.

Find: (a) Nozzle angle, α.
(b) Force to hold vane speed constant.



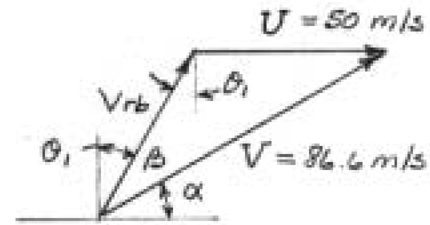
Solution: Apply momentum equation using CV moving with vanes, as shown.

Basic equation:

$$F_{S_x} + \cancelto{=0(2)}{F_{B_x}} = \cancelto{=0(3)}{\frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\, d\forall} + \int_{CS} u_{xyz}\,\rho\,\vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: (1) No pressure forces
(2) Horizontal; $F_{B_x} = 0$
(3) Steady flow w.r.t. CV
(4) Uniform flow at each section
(5) No change in relative velocity on vane
(6) Flow enters and leaves tangent to vanes

The nozzle angle may be obtained from trigonometry. The inlet velocity relationship is shown in the sketch:



From the law of sines,

$$\frac{\sin\alpha}{V_{rb}} = \frac{\sin(90+\theta_1)}{V} = \frac{\sin\beta}{U}$$

$$\beta = \sin^{-1}\left[\frac{U}{V}\sin(90+\theta_1)\right] = \sin^{-1}\left[\frac{50}{86.6}\sin(120°)\right] = 30°$$

From the sketch, $90° = \alpha + \beta + \theta_1$, so $\alpha = 90° - \beta - \theta_1 = 90° - 30° - 30° = 30°$ α

Also $V_{rb}\cos\theta_1 = V\sin\alpha$; $V_{rb} = V\frac{\sin\alpha}{\cos\theta_1} = 86.6\frac{m}{s}\times\frac{\sin 30°}{\cos 30°} = 50.0$ m/s

From momentum equation (note all of $\dot{m}$ flows across vanes)

$$R_x = u_1\{-\dot{m}\} + u_2\{\dot{m}\} = V_{rb}\sin\theta_1(-\dot{m}) - V_{rb}\sin\theta_2(\dot{m}) = V_{rb}\dot{m}(-\sin\theta_1 - \sin\theta_2)$$

$$u_1 = V_{rb}\sin\theta_1 \quad u_2 = -V_{rb}\sin\theta_2 ; \qquad R_y = \dot{m}V_{rb}(-\cos\theta_1 + \cos\theta_2)$$

Thus, since $\dot{m} = \rho Q$,

$$R_x = V_{rb}\rho Q(-\sin\theta_1 - \sin\theta_2)$$

$$= 50\frac{m}{s}\times 999\frac{kg}{m^3}\times 0.170\frac{m^3}{s}(-\sin 30° - \sin 45°)\times\frac{N\cdot s^2}{kg\cdot m}$$

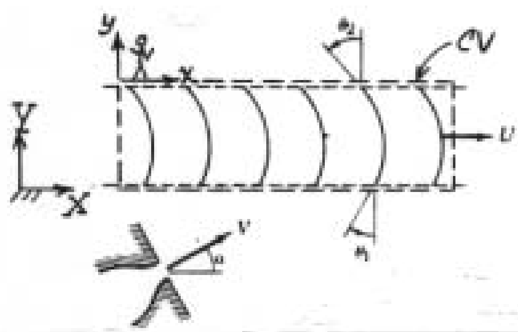
$R_x = -10.3$ kN (to left) R_x

{Note: The net force on the CV in the y-direction is $R_y = -1.35$ kN.}

Problem 4.117

Given: Series of vanes struck by continuous jet, as shown.

Find: For $\alpha \approx 0$ ($\theta_1 \approx 90°$), vane speed, U, to maximize power produced by vane.



Solution: Apply momentum equation using CV moving with vanes, as shown.

Basic equation:

$$F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\,d\forall + \int_{CS} u_{xyz}\,\rho\,\vec{V}_{xyz}\cdot d\vec{A}$$

($F_{B_x} = 0(2)$, $\frac{\partial}{\partial t} = 0(3)$)

Assumptions: (1) No pressure forces
(2) Horizontal; $F_{B_x} = 0$
(3) Steady flow w.r.t. CV
(4) Uniform flow at each section
(5) No change in relative velocity on vane
(6) Flow enters and leaves tangent to vanes

For $\alpha \approx 0$, $V_{rb} \approx V - U$; the momentum equation becomes

$$R_x = u_1\{-\dot{m}\} + u_2\{+\dot{m}\} = -\dot{m}(V-U) - \dot{m}(V-U)\sin\theta_2 = -\dot{m}(V-U)(1+\sin\theta_2)$$

$$u_1 \approx V_{rb} \approx V - U\,;\; u_2 \approx -V_{rb}\sin\theta_2 \approx -(V-U)\sin\theta_2$$

The vane system produces force, $K_x = -R_x$, and power $\mathcal{P} = K_x U$. Thus

$$\mathcal{P} = K_x U = -R_x U = \dot{m}(V-U)U(1+\sin\theta_2) \qquad (1)$$

To find maximum power, set $\frac{d\mathcal{P}}{dU} = 0$

$$\frac{d\mathcal{P}}{dU} = \dot{m}(-1)U(1+\sin\theta_2) + \dot{m}(V-U)(1)(1+\sin\theta_2) = \dot{m}(V-2U)(1+\sin\theta_2)$$

Thus power is maximized when $V - 2U = 0$, or $U = \frac{V}{2}$ (for $\mathcal{P}_{max}$) $\quad U$

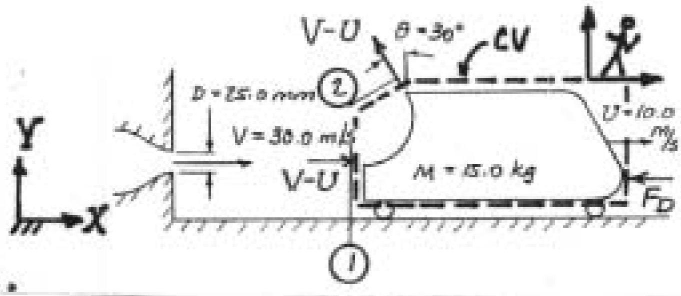
{Note from Eq. 1 that $\theta_2 \rightarrow 90°$ increases power also.}

{Note also that $K_y = -R_y = -\dot{m}V_{rb}\cos\theta_2$ but this force does not produce power.}

Given: Cart propelled by steady water jet, as shown. Total resistance to motion is

$$F_D = kU^2$$

Where $k = 0.92 \frac{N \cdot s^2}{m^2}$



Find: Acceleration of cart at instant when $U = 10$ m/s.

Solution: Apply the momentum equation using CV and CS shown.

Basic equation: $F_{Sx} + F_{Bx} - \int_{CV} a_{rfx} \rho \, d\forall = \frac{\partial}{\partial t}\int_{CV} u_{xyz} \rho \, d\forall + \int_{CS} u_{xyz} \rho \vec{V} \cdot d\vec{A}$ ($F_{Bx} = 0(2)$; $\frac{\partial}{\partial t} \approx 0(3)$)

Assumptions: (1) Only resistance is F_D; $F_{Sx} = -F_D = -kU^2$
(2) Horizontal; $F_{Bx} = 0$
(3) Neglect $\partial/\partial t$ of mass of water in CV
(4) No change in speed w.r.to vane
(5) Uniform flow at each cross-section

Then

$$-kU^2 - a_{rfx} M_{cv} = u_1\{-\rho(V-U)A\} + u_2\{+\rho(V-U)A\}$$

Measure u w.r.to CV: $u_1 = V-U$ $\qquad u_2 = -(V-U)\sin\theta$

$$-kU^2 - a_{rfx} M_{cv} = -\rho(V-U)^2 A - \rho(V-U)^2 A \sin\theta = -\rho(V-U)^2 A(1+\sin\theta)$$

so

$$a_{rfx} = \frac{1}{M}\left[\rho(V-U)^2 A(1+\sin\theta) - kU^2\right]$$

$$= \frac{1}{15\,kg}\left[999\frac{kg}{m^3}(30-10)^2\frac{m^2}{s^2}\frac{\pi}{4}(0.025)^2 m^2(1+\sin 30°) - 0.92\frac{N \cdot s^2}{m^2} \times (10)^2\frac{m^2}{sec^2} \times \frac{kg \cdot m}{N \cdot s^2}\right]$$

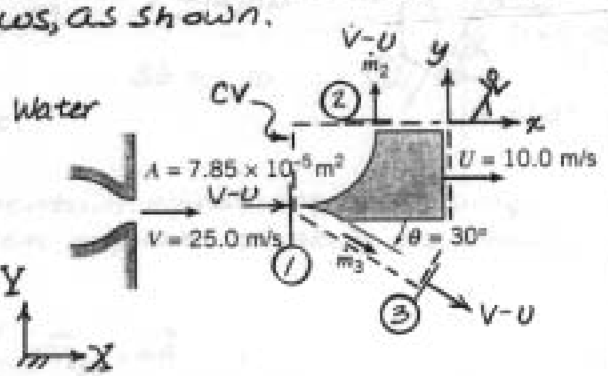
$a_{rfx} = 13.5$ m/s² (to right) $\qquad a_{rfx}$

Problem 4.119

Given: Splitter dividing flow into two flat streams, as shown.

Find: (a) Mass flow rate ratio, $\dot{m}_2/\dot{m}_3$, so net vertical force is zero.
(b) Horizontal force need to maintain constant speed.

Solution: Apply x and y components of momentum to CV drawn with boundaries ⊥ to flows, as shown.



Basic equations:

$$F_{S_y} + F_{B_y} = \frac{\partial}{\partial t}\int_{CV} v_{xyz}\rho\, dV + \int_{CS} v_{xyz}\rho \vec{V}\cdot d\vec{A}$$ (F_{S_y} =0(1), F_{B_y} =0(2), unsteady term =0(3))

$$F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\rho\, dV + \int_{CS} u_{xyz}\rho \vec{V}\cdot d\vec{A}$$ (F_{B_x} =0, unsteady term =0(3))

Assumptions: (1) No pressure forces
(2) Neglect mass of water on vane
(3) Steady flow w.r.to vane
(4) Uniform flow at each section
(5) No change in speed w.r. to vane

Then

$$0 = \int_{CS} v\rho\vec{V}\cdot d\vec{A} = v_1\{-\dot{m}_1\} + v_2\{+\dot{m}_2\} + v_3\{+\dot{m}_3\}$$

Measure w.r. to CV: $v_1 = 0$ $\quad v_2 = V-U \quad v_3 = -(V-U)\sin\theta$

so $\quad 0 = (V-U)\dot{m}_2 - (V-U)\sin\theta\,\dot{m}_3\ ; \quad \frac{\dot{m}_2}{\dot{m}_3} = \sin\theta = \frac{1}{2}$ $\qquad \dot{m}_2/\dot{m}_3$

and

$$F_{S_x} = \int_{CS} u\rho\vec{V}\cdot d\vec{A} = R_x = u_1\{-\dot{m}_1\} + u_2\{+\dot{m}_2\} + u_3\{+\dot{m}_3\}$$

Measure w.r. to CV: $\quad u_1 = V-U \quad u_2 = 0 \quad u_3 = (V-U)\cos\theta$

$$R_x = (V-U)(-\dot{m}_1) + (V-U)\cos\theta(\dot{m}_3) = (V-U)(\dot{m}_3\cos\theta - \dot{m}_1)$$

From continuity $\quad 0 = -\dot{m}_1 + \dot{m}_2 + \dot{m}_3 = -\dot{m}_1 + \frac{\dot{m}_3}{2} + \dot{m}_3\ ; \quad \dot{m}_3 = \frac{2}{3}\dot{m}_1$

$$R_x = (V-U)\left(\frac{2}{3}\dot{m}_1\cos\theta - \dot{m}_1\right) = (V-U)\dot{m}_1\left(\frac{2\cos\theta}{3} - 1\right)$$

$$R_x = (25-10)\frac{m}{s} \times 999\frac{kg}{m^3} \times (25-10)\frac{m}{s} \times 7.85\times10^{-5}\,m^2\left(\frac{2}{3}\cos 30^\circ - 1\right)\frac{N\cdot s^2}{kg\cdot m}$$

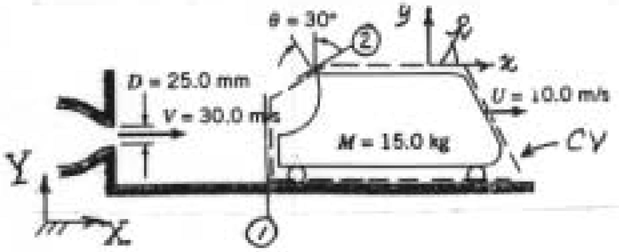
$$R_x = -7.46\ N\ \text{(to left)}$$ $\qquad R_x$

{ Force must be applied to <u>left</u> to maintain vane speed constant; if R_x were zero, vane would accelerate. }

Problem 4.120

Given: Hydraulic catapult of Problem 4.118, rolling on level track with negligible resistance, speed U.

Find: Time required to accelerate from rest to $U = V/2$.



Solution: Apply x component of momentum equation to accelerating CV.

Basic equation:

$$\overset{=0(1)}{F_{S_x}} + \overset{=0(2)}{F_{B_x}} - \int_{CV} a_{rf_x} \rho \, d\forall = \overset{\approx 0(3)}{\frac{\partial}{\partial t}} \int_{CV} u_{xyz} \rho \, d\forall + \int_{CS} u_{xyz} \rho \vec{V}_{xyz} \cdot d\vec{A}$$

Assumptions: (1) $F_{S_x} = 0$, since no pressure forces, no resistance
(2) $F_{B_x} = 0$, since horizontal
(3) Neglect mass of water on vane
(4) Uniform flow in jet
(5) No change in relative velocity on vane

Then

$$-a_{rf_x} M_{cv} = u_1\{-\rho(V-U)A\} + u_2\{+\rho(V-U)A\} = -(1+\sin\theta)\rho(V-U)^2 A$$

$$u_1 = V-U \qquad u_2 = -(V-U)\sin\theta$$

so

$$\frac{dU}{dt} = \frac{\rho A(1+\sin\theta)}{M}(V-U)^2$$

To integrate, note since V = constant, $d(V-U) = -dU$, so

$$-\int_0^{V/2} \frac{d(V-U)}{(V-U)^2} = \int_0^t \frac{\rho A(1+\sin\theta)}{M} dt$$

or

$$\frac{1}{V-U}\Bigg]_{U=0}^{U=V/2} = \frac{2}{V} - \frac{1}{V} = \frac{1}{V} = \frac{\rho A(1+\sin\theta)}{M} t$$

Thus

$$t = \frac{M}{\rho V A(1+\sin\theta)}$$

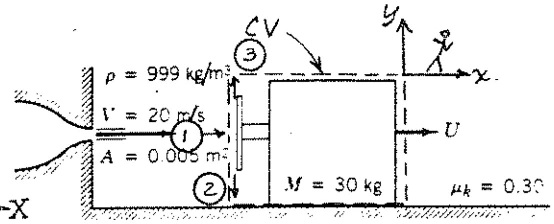
$$= 15.0\,\text{kg} \times \frac{\text{m}^3}{999\,\text{kg}} \times \frac{\text{s}}{30.0\,\text{m}} \times \frac{4}{\pi(0.025)^2\,\text{m}^2} \times \frac{1}{(1+\sin 30°)}$$

$$t = 0.680\ \text{s}$$

t

Problem 4.121

Given: Vane/slider assembly moving under influence of jet.



Find: Terminal speed.

Solution: Apply x momentum equation to linearly accelerating CV.

Basic equation:

$$F_{S_x} + \cancelto{=0(1)}{F_{B_x}} - \int_{CV} a_{rf_x} \rho \, dV = \cancelto{\approx 0(2)}{\frac{\partial}{\partial t}} \int_{CV} u_{xyz} \rho \, dV + \int_{CS} u_{xyz} \rho \vec{V}_{xyz} \cdot d\vec{A}$$

Assumptions: (1) Horizontal motion, so $F_{B_x} = 0$
(2) Neglect mass of liquid on vane, $u \approx 0$ on vane
(3) Uniform flow at each section
(4) Measure velocities relative to CV

Then

$$-Mg\mu_k - a_{rf_x} M = u_1\{-|\rho(V-U)A|\} + u_2\{+\dot{m}_2\} + u_3\{+\dot{m}_3\}$$

$$u_1 = V - U \qquad u_2 = 0 \qquad u_3 = 0$$

$$-Mg\mu_k - M\frac{dU}{dt} = -\rho(V-U)^2 A$$

or

$$\frac{dU}{dt} = \frac{\rho(V-U)^2 A}{M} - g\mu_k$$

At terminal speed, $dU/dt = 0$ and $U = U_t$, so

$$0 = \frac{\rho(V-U_t)^2 A}{M} - g\mu_k \quad \text{or} \quad V - U_t = \sqrt{\frac{Mg\mu_k}{\rho A}}$$

and

$$U_t = V - \sqrt{\frac{Mg\mu_k}{\rho A}}$$

$$= 20\,\frac{m}{s} - \left[30\,kg \times 9.81\,\frac{m}{s^2} \times 0.3 \times \frac{m^3}{999\,kg} \times \frac{1}{0.005\,m^2}\right]^{1/2}$$

$$U_t = 15.8\ m/s$$

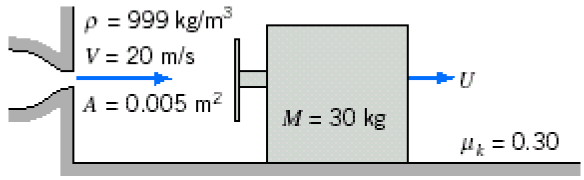
U_t

Problem 4.122

For the vane/slider problem of Problem 4.121, find and plot expressions for the acceleration, speed, and position of the slider as a function of time.

Given: Data on vane/slider

Find: Formula for acceleration, speed, and position; plot



Solution

The given data is

$$\psi \,|\, 999 \cdot \frac{kg}{m^3} \qquad M \,|\, 30 \cdot kg \qquad A \,|\, 0.005 \cdot m^2 \qquad V \,|\, 20 \cdot \frac{m}{s} \qquad \sigma_k \,|\, 0.3$$

The equation of motion, from Problem 4.121, is

$$\frac{dU}{dt} \,|\, \frac{\psi \cdot (V\ 4\ U)^2 \cdot A}{M}\ 4\ g \cdot \sigma_k$$

(The acceleration is)

$$a \,|\, \frac{\psi \cdot (V\ 4\ U)^2 \cdot A}{M}\ 4\ g \cdot \sigma_k$$

Separating variables

$$\frac{dU}{\frac{\psi \cdot (V\ 4\ U)^2 \cdot A}{M}\ 4\ g \cdot \sigma_k} \,|\, dt$$

Substitute $u \,|\, V\ 4\ U$ $\quad dU \,|\, 4du$

$$\frac{du}{\frac{\psi \hat{A}\, u^2}{M}\ 4\ g\, \sigma_k} \,|\, 4dt$$

$$\int \frac{1}{\left(\frac{\psi \hat{A}\, u^2}{M}\ 4\ g\, \sigma_k\right)} du \,|\, 4\sqrt{\frac{M}{g\, \sigma_k\, \psi\, \hat{A}}}\ \mathrm{atanh}\left(\sqrt{\frac{\psi\, \hat{A}}{g\, \sigma_k\, M}}\ u\right)$$

and u = V - U so

$$4\sqrt{\frac{M}{g\, \sigma_k\, \psi\, \hat{A}}}\ \mathrm{atanh}\left(\sqrt{\frac{\psi\, \hat{A}}{g\, \sigma_k\, M}}\ u\right) \,|\, 4\sqrt{\frac{M}{g\, \sigma_k\, \psi\, \hat{A}}}\ \mathrm{atanh}\left(\sqrt{\frac{\psi\, \hat{A}}{g\, \sigma_k\, M}}\ (V\ 4\ U)\right)$$

Using initial conditions

$$4\sqrt{\frac{M}{g\, \sigma_k\, \psi\, \hat{A}}}\ \mathrm{atanh}\left(\sqrt{\frac{\psi\, \hat{A}}{g\, \sigma_k\, M}}\ (V\ 4\ U)\right) 2 \sqrt{\frac{M}{g\, \sigma_k\, \psi\, \hat{A}}}\ \mathrm{atanh}\left(\sqrt{\frac{\psi\, \hat{A}}{g\, \sigma_k\, M}}\ V\right) \,|\, 4t$$

$$V\ 4\ U \,|\, \sqrt{\frac{g\, \sigma_k\, M}{\psi\, \hat{A}}}\ \tanh\left(\sqrt{\frac{g\, \sigma_k\, \psi\, \hat{A}}{M}}\ t\ 2\ \mathrm{atanh}\left(\sqrt{\frac{\psi\, \hat{A}}{g\, \sigma_k\, M}}\ V\right)\right)$$

$$U = V\ 4\ \sqrt{\frac{g\,\sigma_k\,M}{\psi\,A}}\ \tanh\left(\sqrt{\frac{g\,\sigma_k\,\psi\,A}{M}}\ t\ 2\ \operatorname{atanh}\left(\sqrt{\frac{\psi\,A}{g\,\sigma_k\,M}}\ V\right)\right)$$

Note that $\operatorname{atanh}\left(\sqrt{\frac{\psi\,A}{g\,\sigma_k\,M}}\ V\right) = 0.213\ 4\ \frac{\phi}{2}\,i$

which is complex and difficult to handle in *Excel*, so we use the identity

$$\operatorname{atanh}(x) = \operatorname{atanh}\left(\frac{1}{x}\right)\ 4\ \frac{\phi}{2}\,i \qquad \text{for } x > 1$$

so $$U = V\ 4\ \sqrt{\frac{g\,\sigma_k\,M}{\psi\,A}}\ \tanh\left(\sqrt{\frac{g\,\sigma_k\,\psi\,A}{M}}\ t\ 2\ \operatorname{atanh}\left(\frac{1}{\sqrt{\frac{\psi\,A}{g\,\sigma_k\,M}}\ V}\right)\ 4\ \frac{\phi}{2}\,i\right)$$

and finally the identity

$$\tanh\left(x\ 4\ \frac{\phi}{2}\,i\right) = \frac{1}{\tanh(x)}$$

to obtain

$$U = V\ 4\ \frac{\sqrt{\frac{g\,\sigma_k\,M}{\psi\,A}}}{\tanh\left(\sqrt{\frac{g\,\sigma_k\,\psi\,A}{M}}\ t\ 2\ \operatorname{atanh}\left(\sqrt{\frac{g\,\sigma_k\,M}{\psi\,A}}\ \frac{1}{V}\right)\right)}$$

For the position x

$$\frac{dx}{dt} = V - \frac{\sqrt{\dfrac{g \cdot \sigma_k \cdot M}{\psi \cdot A}}}{\tanh\left(\sqrt{\dfrac{g \cdot \sigma_k \cdot \psi \cdot A}{M}} \cdot t + \operatorname{atanh}\left(\sqrt{\dfrac{g \cdot \sigma_k \cdot M}{\psi \cdot A}} \cdot \dfrac{1}{V}\right)\right)}$$

This can be solved analytically, but is quite messy. Instead, in the corresponding *Excel* workboo it is solved numerically using a simple Euler method. The complete set of equations is

$$U = V - \frac{\sqrt{\dfrac{g \cdot \sigma_k \cdot M}{\psi \cdot A}}}{\tanh\left(\sqrt{\dfrac{g \cdot \sigma_k \cdot \psi \cdot A}{M}} \cdot t + \operatorname{atanh}\left(\sqrt{\dfrac{g \cdot \sigma_k \cdot M}{\psi \cdot A}} \cdot \dfrac{1}{V}\right)\right)}$$

$$a = \frac{\psi \cdot (V - U)^2 \cdot A}{M} - g \cdot \sigma_k$$

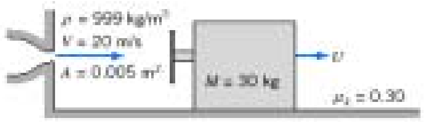
$$x(n+1) = x(n) + \left(V - \frac{\sqrt{\dfrac{g \cdot \sigma_k \cdot M}{\psi \cdot A}}}{\tanh\left(\sqrt{\dfrac{g \cdot \sigma_k \cdot \psi \cdot A}{M}} \cdot t + \operatorname{atanh}\left(\sqrt{\dfrac{g \cdot \sigma_k \cdot M}{\psi \cdot A}} \cdot \dfrac{1}{V}\right)\right)}\right) \cdot \Delta t$$

The plots are presented in the *Excel* workbook

Problem 4.122 (In Excel)

For the vane/slider problem of Problem 4.121, find and plot expressions for the acceleration, speed, and position of the slider as a function of time.

Given: Data on vane/slider
Find: Plot acceleration, speed and position



Solution

The solutions are

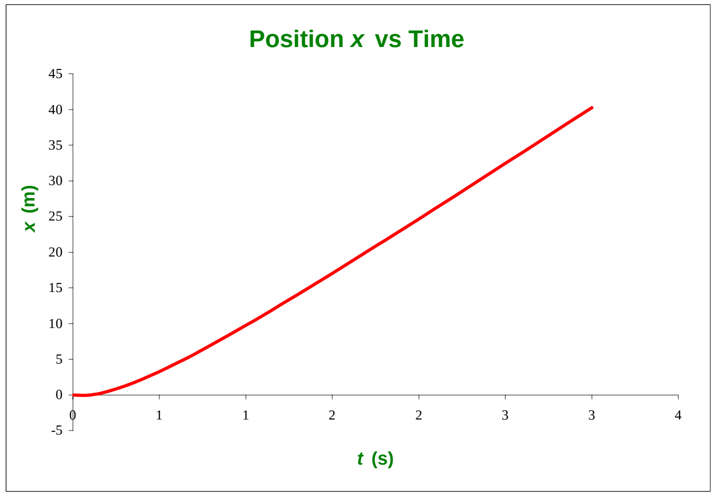
$$U = V - \frac{\sqrt{\frac{g\,\mu_k\,M}{\rho\,A}}}{\tanh\left(\sqrt{\frac{g\,\mu_k\,\rho\,A}{M}}\cdot t + \operatorname{atanh}\left(\sqrt{\frac{g\,\mu_k\,M}{\rho\,A}}\cdot\frac{1}{V}\right)\right)}$$

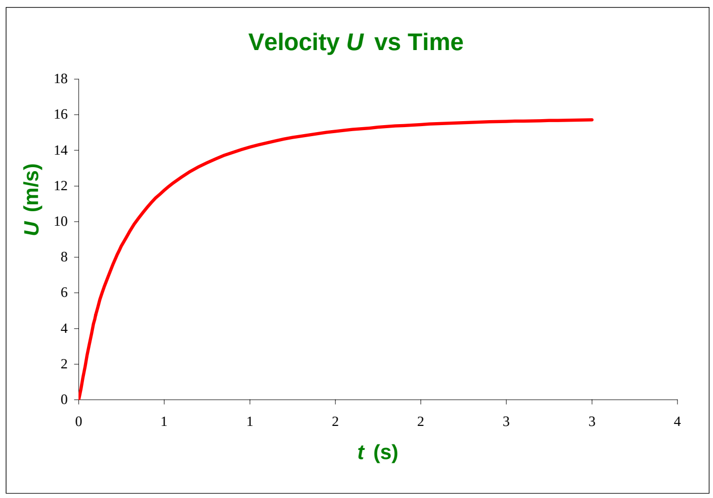
$$a = \frac{\rho\,(V-U)^2\,A}{M} - g\,\mu_k$$

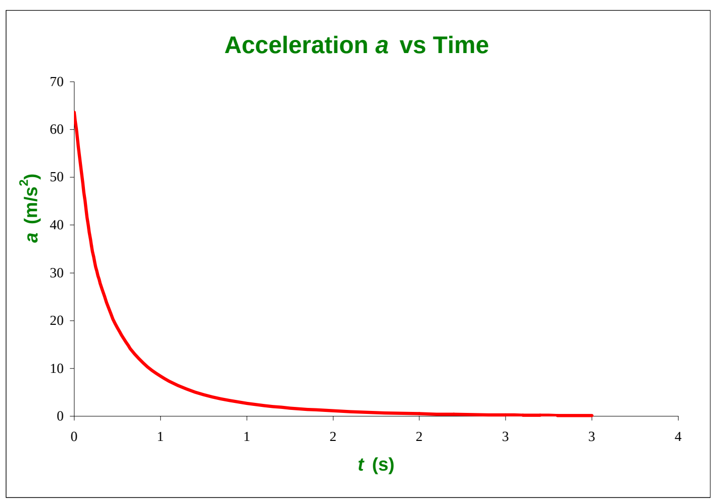
$$x(n+1) = x(n) + \left(V - \frac{\sqrt{\frac{g\,\mu_k\,M}{\rho\,A}}}{\tanh\left(\sqrt{\frac{g\,\mu_k\,\rho\,A}{M}}\cdot t + \operatorname{atanh}\left(\sqrt{\frac{g\,\mu_k\,M}{\rho\,A}}\cdot\frac{1}{V}\right)\right)}\right)\cdot\Delta t$$

ρ = 999 kg/m^3
μ_k = 0.3
A = 0.005 m^2
V = 20 m/s
M = 30 kg
Δt = 0.1 s

t (s)	*x* (m)	*U* (m/s)	*a* (m/s^2)
0.0	0.0	0.0	63.7
0.1	0.0	4.8	35.7
0.2	0.5	7.6	22.6
0.3	1.2	9.5	15.5
0.4	2.2	10.8	11.2
0.5	3.3	11.8	8.4
0.6	4.4	12.5	6.4
0.7	5.7	13.1	5.1
0.8	7.0	13.5	4.0
0.9	8.4	13.9	3.3
1.0	9.7	14.2	2.7
1.1	11.2	14.4	2.2
1.2	12.6	14.6	1.9
1.3	14.1	14.8	1.6
1.4	15.5	14.9	1.3
1.5	17.0	15.1	1.1
1.6	18.5	15.2	0.9
1.7	20.1	15.3	0.8
1.8	21.6	15.3	0.7
1.9	23.1	15.4	0.6
2.0	24.7	15.4	0.5
2.1	26.2	15.5	0.4
2.2	27.8	15.5	0.4
2.3	29.3	15.6	0.3
2.4	30.9	15.6	0.3
2.5	32.4	15.6	0.2
2.6	34.0	15.6	0.2
2.7	35.6	15.7	0.2
2.8	37.1	15.7	0.2
2.9	38.7	15.7	0.1
3.0	40.3	15.7	0.1

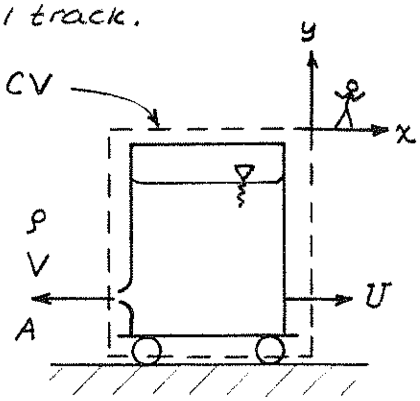




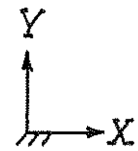


Given: Cart propelled by a horizontal liquid jet of constant speed. Neglect resistance along horizontal track.

Initial mass is M_0.



Find: (a) A general expression for speed, U, as cart accelerates from rest.
(b) V for $U = 1.5$ m/s @ $t = 30$ s



Solution:

a) Apply x component of momentum equation using linearly accelerating CV shown.

Basic equation:
$$\overset{=0(1)}{F_{S_x}} + \overset{=0(2)}{F_{B_x}} - \int_{CV} a_{rf_x}\rho\, d\forall = \overset{\approx 0(3)}{\frac{\partial}{\partial t}\int_{CV} u_{xyz}\rho\, d\forall} + \int_{CS} u_{xyz}\rho \vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: (1) No resistance
(2) $F_{B_x} = 0$ since track is horizontal
(3) Neglect u_{xyz} within CV
(4) Uniform flow at jet exit

Then
$$-a_{rf_x} M = u\{|\rho V A|\} = -\rho V^2 A$$
$$u = -V$$

From continuity, $M = M_0 - \dot{m}t = M_0 - \rho V A t$. Using $a_{rf_x} = \frac{dU}{dt}$,
$$\frac{dU}{dt} = \frac{\rho V^2 A}{M_0 - \rho V A t}$$

Separating variables and integrating,
$$\int_0^U dU = U = \int_0^t \frac{\rho V^2 A\, dt}{M_0 - \rho V A t} = -V \ln(M_0 - \rho V A t)\Big|_0^t = V \ln\left(\frac{M_0}{M_0 - \rho V A t}\right)$$

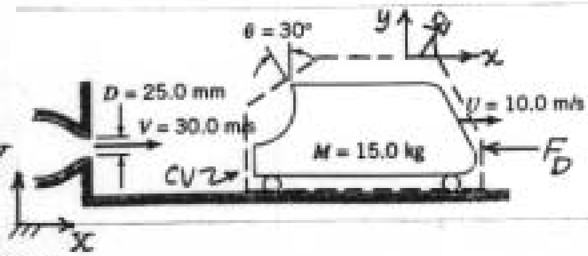
or
$$\frac{U}{V} = \ln\left(\frac{M_0}{M_0 - \rho V A t}\right) \qquad \frac{U}{V}$$

Check dimensions: $[\rho V A t] = \frac{M}{L^3}\frac{L}{t}L^2 t = M$ ✓

b) Using the given data in Excel (with Solver) the jet speed for $U = 1.5$ m/s @ $t = 30$ s is $V = 0.61$ m/s

Problem 1.24

Given: Hydraulic catapult of Problem 4.118, rolling on level track with resistance, $F_D = kU^2$, speed U, starting from rest at $t=0$.



Find: (a) When acceleration is maximum
(b) Sketch of acceleration vs. time
(c) Value of θ to maximize acceleration, why?
(d) If U will ever reach V; explanation

Solution: Apply x component of momentum equation to accelerating CV

Basic equation:

$$F_{S_x} + F_{B_x} - \int_{CV} a_{rf_x} \rho \, dV = \frac{\partial}{\partial t}\int_{CV} u_{xyz} \rho \, dV + \int_{CS} u_{xyz} \rho \vec{V}_{xyz} \cdot d\vec{A}$$

(with $F_{B_x} = 0$ (2), $\frac{\partial}{\partial t} \approx 0$ (3))

Assumptions: (1) $F_{S_x} = -F_D = -kU^2$, where $k = 0.92 \ N \cdot s^2/m^2$
(2) $F_{B_x} = 0$, since horizontal
(3) Neglect mass of water on vane
(4) Uniform flow in jet
(5) No change in relative velocity on vane

Then

$$-kU^2 - a_{rf_x} M_{CV} = u_1\{-\rho(V-U)A\} + u_2\{+\rho(V-U)A\} = -(1+\sin\theta)\rho(V-U)^2 A$$

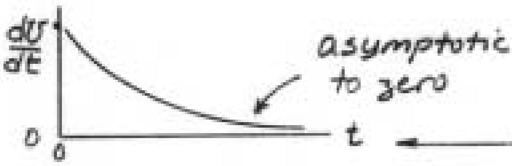
$$u_1 = V - U \qquad u_2 = -(V-U)\sin\theta$$

so

$$\frac{dU}{dt} = \frac{\rho A(1+\sin\theta)}{M}(V-U)^2 - kU^2/M \qquad (1)$$

(a) Acceleration is maximum at $t=0$, when $U=0$

(b) Acceleration vs. time will be



(c) From Eq. 1, dU/dt is maximum when $\theta = \pi/2$ and $\sin\theta = 1$

(d) From Eq. 1, $\frac{dU}{dt}$ will go to zero when $U < V$; this will be the terminal speed for the cart, U_t. From Eq. 1, $\frac{dU}{dt} = 0$ when

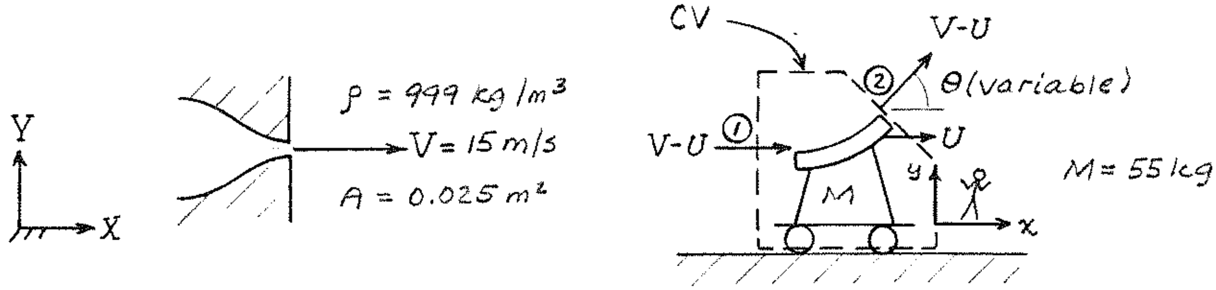
$$\rho A(1+\sin\theta)(V-U)^2 = kU^2$$

or

$$U = \frac{\left[\frac{\rho A(1+\sin\theta)}{k}\right]^{1/2}}{1+\left[\frac{\rho A(1+\sin\theta)}{k}\right]^{1/2}} V = 0.472V$$

U will be asymptotic to V.

Given: Vane/cart assembly driven by liquid jet. Motion to be controlled so that $a_{rfx} = 1.5\ m/s^2$ by varying turning angle, θ. Neglect resistance.



Find: θ at $t = 5\,s$. Plot: $\theta(t)$ over a suitable range.

<u>Solution</u>: Apply x component of momentum equation, using linearly accelerating CV shown above.

Basic equation:
$$\overset{=0(1)}{F_{S_x}} + \overset{=0(2)}{F_{B_x}} - \int_{CV} a_{rf_x}\,\rho\, d\forall = \overset{\approx 0(3)}{\frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\, d\forall} + \int_{CS} u_{xyz}\,\rho\, \vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: (1) $F_{S_x} = 0$
(2) $F_{B_x} = 0$
(3) Neglect u and rate of change of u within CV
(4) Uniform flow at each section
(5) Jet area and speed relative to vane are constant

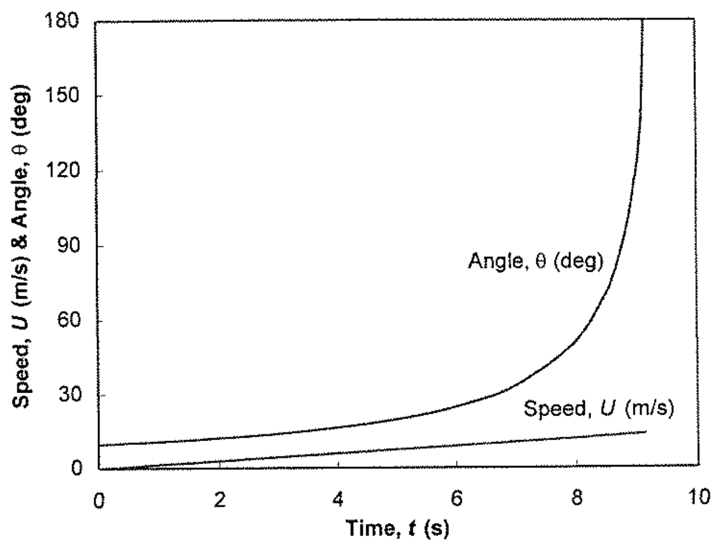
Then
$$-Ma_{rf_x} = u_1\{-|\rho(V-U)A|\} + u_2\{|\rho(V-U)A|\}$$
$$u_1 = V - U \qquad u_2 = (V-U)\cos\theta$$
$$-Ma_{rf_x} = -\rho(V-U)^2 A + \rho(V-U)^2 A\cos\theta = \rho(V-U)^2 A(\cos\theta - 1)$$
or
$$\cos\theta = 1 - \frac{Ma_{rf_x}}{\rho(V-U)^2 A}$$
Since a_{rf_x} = constant, $U = a_{rf_x} t$
$$\cos\theta = 1 - \frac{Ma_{rf_x}}{\rho(V - a_{rf_x} t)^2 A} \qquad (1)$$
and
$$\theta = \cos^{-1}\left[1 - \frac{Ma_{rf_x}}{\rho(V - a_{rf_x} t)^2 A}\right]$$
$$= \cos^{-1}\left\{1 - 55\,kg \times 1.5\frac{m}{s} \times \frac{m^3}{999\,kg}\left[\frac{1}{15\frac{m}{s} - 1.5\frac{m}{s^2}\times 5s}\right]^2 \frac{1}{0.025\,m^2}\right\}$$

$\theta = 19.7°$ (at $t = 5s$) θ

Equation 1 is only valid for $\theta \le 180°$. This occurs at $t \approx 9.14\ s$.

The plot is on the next page.

The plot is



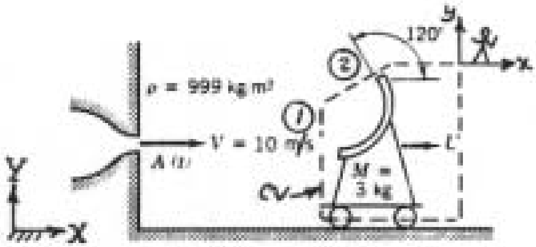
(Constant acceleration cannot be maintained after $t = 9.14\ s$.)

Problem 4.126

Given: Vaned cart rolling with negligible resistance.

$a_{rf_x} = 2\ m/s^2 =$ constant

Jet area is $A(t)$, programmed.



Find: (a) Expression for $A(t)$ at cart.
(b) Sketch for $t \leq 4s$.
(c) Evaluate at $t = 2s$.

Solution: Apply x momentum to CV with linear acceleration.

Basic equation:

$$\overset{=0(1)}{F_{S_x}} + \overset{=0(2)}{F_{B_x}} - \int_{CV} a_{rf_x} \rho\, dV = \overset{\approx 0(3)}{\frac{\partial}{\partial t}\int_{CV} u_{xyz} \rho\, dV} + \int_{CS} u_{xyz} \rho \vec{V}_{xyz} \cdot d\vec{A}$$

Assumptions: (1) No resistance to motion
(2) Horizontal motion, so $F_{B_x} = 0$
(3) Neglect mass of liquid in CV
(4) Uniform flow at each section
(5) All velocities measured relative to CV
(6) No change in stream area or speed on vane

Then (with $a_{rf_x} = a$)

$$-aM = u_1\{-|\rho(V-U)A|\} + u_2\{+|\rho(V-U)A|\} = -\frac{3}{2}\rho(V-U)^2 A$$

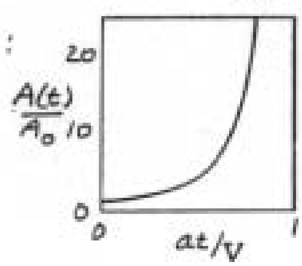
$$u_1 = V-U \qquad u_2 = (V-U)\cos 120° = -\frac{1}{2}(V-U)$$

Since $a =$ constant, $U = at$, and

$$A = A(t) = \frac{2aM}{3\rho(V-at)^2}$$ A(t)

At $t=0$, $A(0) = A_0 = \frac{2aM}{3\rho V^2}$. Thus $\frac{A}{A_0} = \frac{1}{(1-at/V)^2}$.

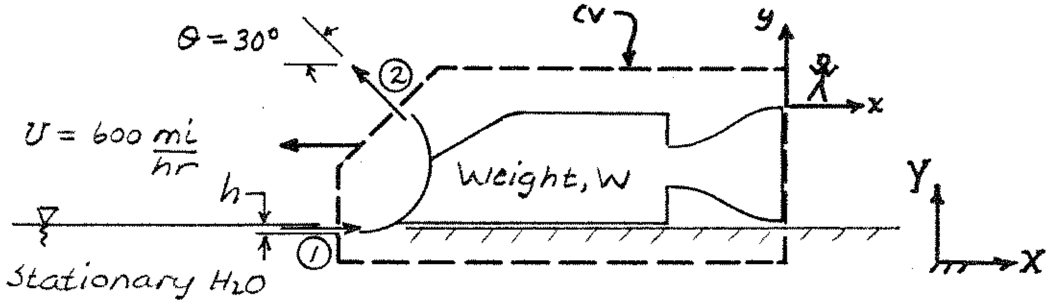
Sketch:



At $t = 2$ sec,

$$A = \frac{2}{3} \times 2\frac{m}{s^2} \times 3\ kg \times \frac{m^3}{999\ kg} \frac{1}{\left[10\frac{m}{s} - 2\frac{m}{s^2} \times 2s\right]^2} \times 10^6 \frac{mm^2}{m^2} = 111\ mm^2$$ A(2)

Given: Rocket sled with water scoop brake. $W = 10{,}000$ lbf



Scoop immersed in trough is $w = 6$ in. wide, $h = 3$ in. deep.

Find: Time needed to decelerate to 20 mph. Plot: Speed vs. time.

Solution: Apply x component of momentum equation to linearly accelerating CV. Basic equation is

$$\overset{=0(1)}{F_{S_x}} + \overset{=0(2)}{F_{B_x}} - \int_{CV} a_{rf_x}\rho\, d\forall = \overset{\approx 0(3)}{\frac{\partial}{\partial t}}\int_{CV} u_{xyz}\rho\, d\forall + \int_{CS} u_{xyz}\rho \vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: (1) $F_{S_x} = 0$
(2) $F_{B_x} = 0$
(3) Neglect u_{xyz} and its rate of change in CV
(4) Uniform flow at each section
(5) Speed of water relative to sled is constant

Then

$$-a_{rf_x} M = u_1\{-|\rho U w h|\} + u_2\{|\rho U w h|\}\ ;\ u_1 = U,\ u_2 = -U\cos\theta$$

$$-a_{rf_x}\frac{W}{g} = -\rho U^2 w h(1+\cos\theta), \text{ or } \quad a_{rf_x} = \frac{\rho g U^2 w h(1+\cos\theta)}{W}$$

Now $a_{rf_x} = -\frac{dU}{dt}$, because of coordinate choice. Thus

$$\frac{dU}{U^2} = -\frac{\gamma w h}{W}(1+\cos\theta)\, dt$$

and

$$\int_{U_i}^{U}\frac{dU}{U^2} = -\frac{1}{U} + \frac{1}{U_i} = -\frac{\gamma w h}{W}(1+\cos\theta)\, t \qquad (1)$$

Solving for t,

$$t = \left[\frac{1}{U} - \frac{1}{U_i}\right]\frac{W}{\gamma w h(1+\cos\theta)}$$

$$= \left[\frac{1}{20} - \frac{1}{600}\right]\frac{hr}{mi}\times\frac{mi}{5280\,ft}\times\frac{3600\,s}{hr}\times\frac{ft^3}{62.4\,lbf}\times\frac{1}{6\,in.}\times\frac{1}{3\,in.}\times\frac{144\,in.^2}{ft^2}\times\frac{10{,}000\,lbf}{1+\cos 30^\circ}$$

$t = 22.6$ s ← t

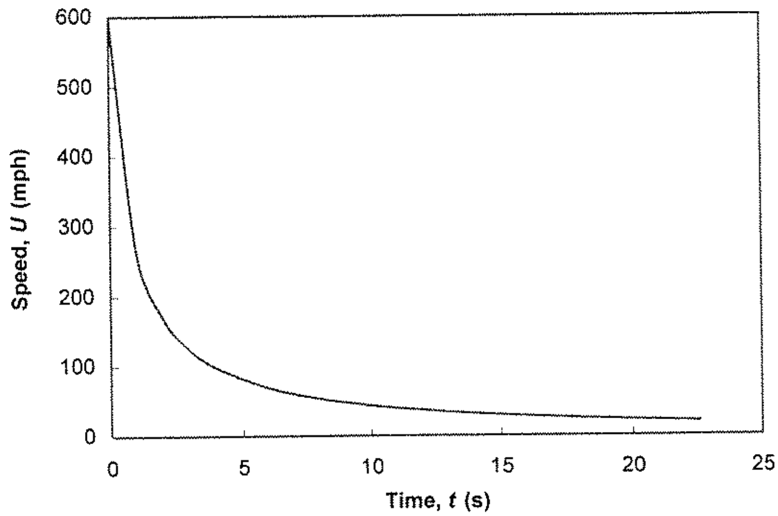
The plot is presented on the next page.

Solving Eq. 1 for U,

$$\frac{1}{U} = \frac{1}{U_i} + \frac{\gamma w h}{W}(1+\cos\theta)t = \frac{W + \gamma w h U_i (1+\cos\theta)t}{W U_i}$$

or $$U = \frac{W U_i}{W + \gamma w h U_i (1+\cos\theta)t} \qquad (2)$$

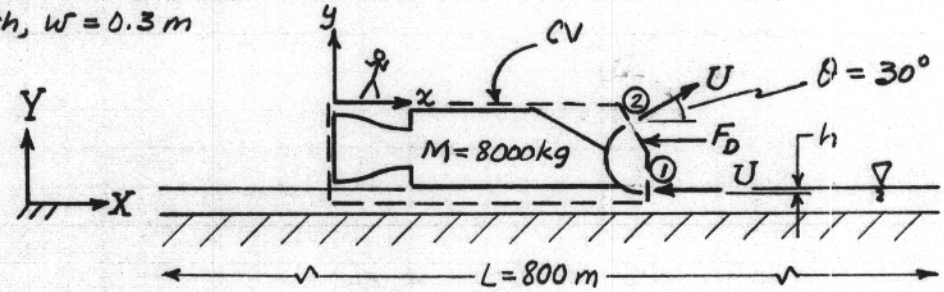
Plotting,



Problem 4.128

Given: Rocket sled slowed by scoop in water trough.

Aerodynamic drag proportional to U^2. At $U_0 = 300\ m/s$, $F_D = 90\ kN$.
Scoop width, $w = 0.3\ m$



Find: Depth of scoop immersion to slow to 100 m/s in trough length, L.

Solution: Apply x component of momentum equation using linearly accelerating CV shown.

Basic equation: $$F_{S_x} + F_{B_x} - \int_{CV} a_{rf_x}\rho\, dV = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\rho\, dV + \int_{CS} u_{xyz}\rho \vec{V}_{xyz}\cdot d\vec{A}$$
($F_{B_x} = 0$ (1); $\frac{\partial}{\partial t}\int_{CV} \approx 0$ (2))

Assumptions: (1) $F_{B_x} = 0$
(2) Neglect rate of change of u in CV
(3) Uniform flow at each section
(4) No change in relative speed of liquid crossing scoop

Then

$$-F_D - Ma_{rf_x} = u_1\{-|\rho U w h|\} + u_2\{|\rho U w h|\}\ ;\ h = \text{scoop immersion}$$

$$u_1 = -U \qquad u_2 = U\cos\theta$$

But $F_D = kU^2$; $$k = \frac{F_{D_0}}{U_0^2} = 90\ kN \times \frac{s^2}{(300)^2 m^2} \times \frac{10^3 N}{kN} \times \frac{kg\cdot m}{N\cdot s^2} = 1.00\ kg/m$$

$$-kU^2 - M\frac{dU}{dt} = \rho U^2 w h(1+\cos\theta)\ ,\ \text{since } a_{rf_x} = dU/dt,$$

Thus $$-M\frac{dU}{dt} = [k + \rho w h(1+\cos\theta)]U^2 = -MU\frac{dU}{dX}$$

or $$\frac{dU}{U} = -c\,dX\ ,\ \text{where } c = \frac{k+\rho w h(1+\cos\theta)}{M}$$

Integrating, $$\ln\frac{U}{U_0} = -cX\ ,\ \text{so } c = -\frac{1}{X}\ln\frac{U}{U_0}$$

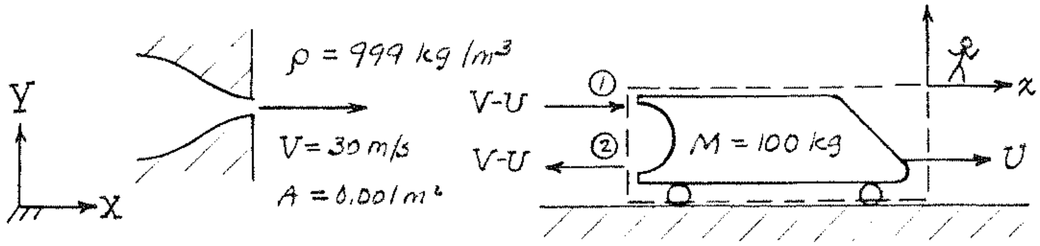
$$c = -\frac{1}{800\ m}\ln\left(\frac{100}{300}\right) = 1.37\times10^{-3}\ m^{-1}$$

Solving for h, $$h = \frac{Mc - k}{\rho w(1+\cos\theta)}$$

$$h = \left[8000\ kg \times \frac{1.37\times10^{-3}}{m} - 1.00\ \frac{kg}{m}\right]\frac{m^3}{999\ kg}\times\frac{1}{0.3\ m}\times\frac{1}{(1+\cos 30°)} = 0.0179\ m$$

$h = 17.9\ mm$ ← h

Given: Vehicle accelerated from rest by a hydraulic catapult. Neglect resistance.



Find: Vehicle speed at $t = 5$ sec. Plot: Vehicle speed vs. time.

Solution: Apply x component of momentum equation using the linearly accelerating CV shown above.

Basic equation: $$F_{S_x} + F_{B_x} - \int_{CV} a_{rf_x} \rho\, d\forall = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\rho\, d\forall + \int_{CS} u_{xyz}\rho \vec{V}_{xyz}\cdot d\vec{A}$$
($F_{S_x} = 0(1)$, $F_{B_x} = 0(2)$, $\frac{\partial}{\partial t}\int_{CV} \approx 0(3)$)

Assumptions: (1) $F_{S_x} = 0$
(2) $F_{B_x} = 0$
(3) Neglect mass of liquid and rate of change of u in CV
(4) Uniform flow at each section
(5) Jet area and speed with respect to vehicle are constant

Then
$$-Ma_{rf_x} = -M\frac{dU}{dt} = u_1\{-|\rho(V-U)A|\} + u_2\{|\rho(V-U)A|\}$$
$$u_1 = V-U \qquad u_2 = -(V-U)$$

or
$$\frac{dU}{dt} = \frac{2\rho(V-U)^2 A}{M}$$

Note that $dU = -d(V-U)$, and separate variables to obtain
$$-\frac{d(V-U)}{(V-U)^2} = \frac{2\rho A}{M}dt$$

Integrate from $U=0$ at $t=0$ to U at t,
$$\int_{V-U=V}^{V-U} -\frac{d(V-U)}{(V-U)^2} = \frac{1}{V-U}\Bigg]_V^{V-U} = \frac{1}{V-U} - \frac{1}{V} = \frac{V-(V-U)}{V(V-U)} = \frac{U}{V(V-U)} = \frac{2\rho A}{M}t$$

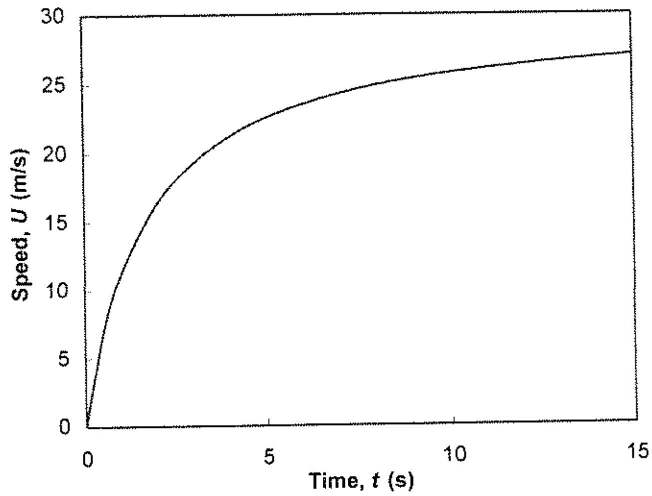
Solving,
$$U = (V-U)\frac{2\rho VA}{M}t \quad \text{or} \quad U = V\left[\frac{\frac{2\rho VA}{M}t}{1+\frac{2\rho VA}{M}t}\right] \qquad (1)$$

For the given conditions at $t = 5$ s,
$$\frac{2\rho VA}{M}t = 2\times 999\frac{kg}{m^3}\times 30\frac{m}{s}\times 0.001\,m^2\times 5s \times\frac{1}{100\,kg} = 3.00$$

$$U = 30\frac{m}{s}\left[\frac{3.00}{1+3.00}\right] = 22.5\ m/s$$ U

The plot is on the next page.

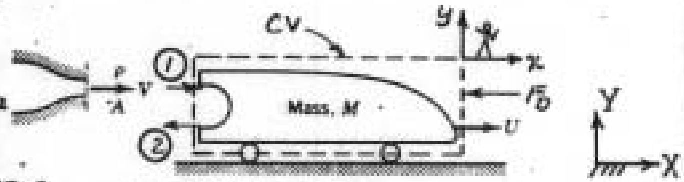
The speed vs. time plot is



Problem 4.130

Given: Cart accelerated from rest by hydraulic catapult.

$F_D = kU^2$; $k = 2.0\ N\cdot s^2/m^2$



Find: (a) Expression for acceleration in terms of speed, U.
(b) Evaluate at $U = 10$ m/s.
(c) Fraction of U_t.

$\rho = 999\ kg/m^3$
$V = 30$ m/s
$A = 0.001\ m^2$
$M = 100$ kg

Solution: Apply x momentum for CV with linear acceleration.

Basic equation:

$$F_{S_x} + F_{B_x} - \int_{CV} a_{rf_x}\rho\, d\forall = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\rho\, d\forall + \int_{CS} u_{xyz}\rho \vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: (1) Horizontal, $F_{B_x} = 0$
(2) Neglect mass of liquid in CV (components of u cancel)
(3) Uniform flow at each section
(4) Measure all velocities relative to the CV
(5) No change in stream area or speed on vane

Then

$$-kU^2 - a_{rf_x}M = u_1\{-|\rho(V-U)A|\} + u_2\{+|\rho(V-U)A|\} = -2\rho(V-U)^2A$$

$$u_1 = V-U \qquad u_2 = -(V-U)$$

or

$$a_{rf_x} = \frac{dU}{dt} = \frac{2\rho(V-U)^2A - kU^2}{M}$$ a(U)

At $U = 10$ m/sec

$$a_{rf_x} = \frac{2\times 999\frac{kg}{m^3}(30-10)^2\frac{m^2}{s^2}\times 0.001\,m^2 - 2.0\frac{N\cdot s^2}{m^2}\times(10)^2\frac{m^2}{s^2}\times\frac{kg\cdot m}{N\cdot s^2}}{100\,kg} = 5.99\frac{m}{s^2}$$ a_{rf_x}

At terminal speed, $a_{rf_x} = 0$. Then $2\rho(V-U_t)^2A = kU_t^2$, or

$$V - U_t = U_t\sqrt{\frac{k}{2\rho A}}$$

Solving, $$U_t = \frac{V}{1+\sqrt{k/2\rho A}}$$

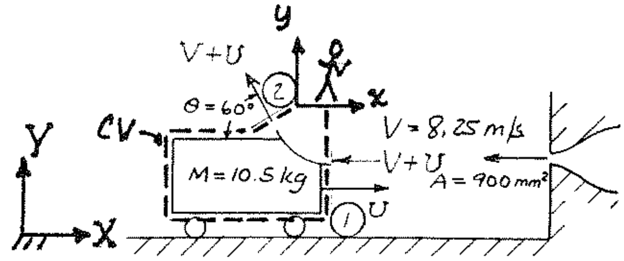
$$U_t = 30\frac{m}{s}\times\frac{1}{1+\left[\frac{1}{2}\times 2.0\frac{N\cdot s^2}{m^2}\times\frac{m^3}{999\,kg}\times\frac{1}{0.001\,m^2}\times\frac{kg\cdot m}{N\cdot s^2}\right]^{1/2}} = 15.0\ m/s$$

Finally,

$$\text{Fraction} = \frac{U}{U_t} = 10.0\frac{m}{s}\times\frac{s}{15.0\,m} = 0.667$$ Fraction

Given: Small vaned cart rolling on level track, struck by a water jet, as shown. At $t=0$, $U_0 = 12.5$ m/sec. Neglect air resistance and rolling resistance.

Find: (a) Time and (b) distance needed to bring cart to rest, and (c) Plot of $U(t)$, $x(t)$.



Solution: Apply x component of momentum using CS and CV shown.

Basic equation:
$$\overset{=0(1)}{F_{S_x}} + \overset{=0(2)}{F_{B_x}} - \int_{CV} a_{rf_x}\rho\, d\forall = \overset{\approx 0(3)}{\frac{\partial}{\partial t}\int_{CV} u_{xyz}\rho\, d\forall} + \int_{CS} u_{xyz}\rho \vec{V}\cdot d\vec{A}$$

Assumptions: (1) No resistance; $F_{S_x}=0$
(2) Horizontal; $F_{B_x}=0$
(3) Neglect mass of water on vane; $\partial/\partial t \approx 0$
(4) No change in speed w.r.to vane
(5) Uniform flow at each cross-section

Then
$$-a_{rf_x} M_{CV} = u_1\{-|\rho(V+U)A|\} + u_2\{+|\rho(V+U)A|\}$$

$$a_{rf_x} = \frac{dU}{dt} \qquad u_1 = -(V+U) \qquad u_2 = -(V+U)\cos\theta \quad \text{(w.r.to CV)}$$

So
$$-\frac{dU}{dt}M = \rho(V+U)^2A - \rho(V+U)^2A\cos\theta = \rho(V+U)^2A(1-\cos\theta) \tag{1}$$

Note V = constant, so $dU = d(V+U)$. Substituting

$$-\frac{d(V+U)}{(V+U)^2} = \frac{\rho A(1-\cos\theta)}{M}dt \tag{2}$$

Integrate from U_0 at $t=0$ to stop, when $U=0$

$$\left.\frac{1}{V+U}\right]_{U=U_0}^{U=0} = \frac{1}{V} - \frac{1}{V+U_0} = \frac{V+U_0-V}{V(V+U_0)} = \frac{U_0}{V(V+U_0)} = \frac{\rho A(1-\cos\theta)t}{M}$$

Thus
$$t = \frac{U_0 M}{\rho(V+U_0)VA(1-\cos\theta)}$$

$$= 12.5\frac{m}{sec}\times 10.5\,kg \times \frac{m^3}{999\,kg}\times\frac{sec}{(12.5+8.25)m}\times\frac{sec}{8.25\,m}\times\frac{1}{900\times10^{-6}m^2}\times\frac{1}{(1-\cos 60°)}$$

$t = 1.71$ sec (to stop) t

To find distance note $\frac{dU}{dt} = \frac{dU}{ds}\frac{ds}{dt} = \frac{dU}{ds}U = U\frac{dU}{ds}$; so from Eq. 1

$$-U\frac{dU}{ds}M = \rho(V+U)^2A(1-\cos\theta)$$

Separating variables
$$\frac{U\,dU}{(V+U)^2} = -\frac{\rho A(1-\cos\theta)}{M}ds \tag{3}$$

Equation 3 may be integrated. Using tables, and integrating from U_0 at $t=0$ to stop (when $U=0$),

$$\int_{U_0}^{0} \frac{U\,dU}{(V+U)^2} = \left[\ln(V+U) + \frac{V}{V+U}\right]_{U_0}^{0} = \ln\left(\frac{V}{V+U_0}\right) + \frac{V}{V} - \frac{V}{V+U_0} = -\frac{\rho A(1-\cos\theta)}{M}s$$

Simplifying and solving for s,

$$s = -\frac{M}{\rho A(1-\cos\theta)}\left(\ln\left(\frac{V}{V+U_0}\right) + 1 - \frac{V}{V+U_0}\right)$$

$$= -10.5\,\text{kg} \times \frac{\text{m}^3}{999\,\text{kg}} \times \frac{1}{900\times10^{-6}\,\text{m}^2} \times \frac{1}{(1-\cos 60°)}\left[\ln\left(\frac{8.25}{8.25+12.5}\right) + 1 - \frac{8.25}{8.25+12.5}\right]$$

$s = 7.47$ m (to stop) — s

From Eq. 2 the general solution is

$$\int_{U_0}^{U} -\frac{d(V+U)}{(V+U)^2} = \left.\frac{1}{V+U}\right]_{U_0}^{U} = \frac{1}{V+U} - \frac{1}{V+U_0} = \frac{(V+U_0)-(V+U)}{(V+U)(V+U_0)} = \frac{\rho A(1-\cos\theta)t}{M} = at$$

Thus $U_0 - U = a(V+U)(V+U_0)t = aV(V+U_0)t + aU(V+U_0)t$ {Let $b = V+U_0$}

Simplifying, $$U = \frac{U_0 - aVbt}{1+abt} \qquad (4)$$ — $U(t)$

Acceleration is found from Eq. 1

$$a_x = \frac{dU}{dt} = \frac{\rho A(1-\cos\theta)(V+U)^2}{M} = a(V+U)^2$$ — $a_x(U)$

Integrate Eq. 4 to get $X(t)$:

$$U = \frac{dX}{dt} = \frac{U_0 - abVt}{1+abt}$$

$$dX = \frac{U_0}{1+abt}dt - \frac{abVt}{1+abt}dt$$

Integrating

$$X = \frac{U_0}{ab}\ln(1+abt)\Big]_0^t - \frac{V}{ab}\int_0^t \frac{x}{1+x}dx = \left[\frac{U_0}{ab}\ln(1+abt) - \frac{V}{ab}\left(1+abt - \ln(1+abt)\right)\right]_0^t$$

$$X = \frac{U_0}{ab}\ln(1+abt) - \frac{V}{ab}\left[abt - \ln(1+abt)\right]$$ — $X(t)$

Numerical values and plots are on the next page.

Problem 4.131 (cont'd.)

Acceleration, Velocity, and Position of Cart vs. Time:

Input Parameters:

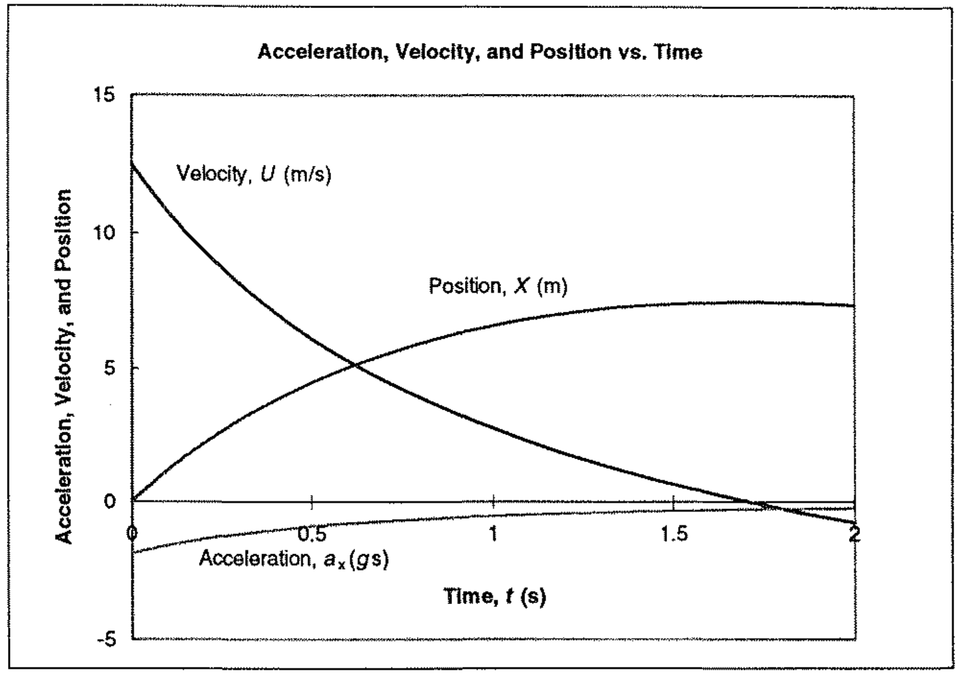
A =	900	mm^2	9.00E-04	m^2
M =	10.5	kg		
U_0 =	12.5	m/s		
V =	8.25	m/s		
θ =	60	degrees	1.047	rad
ρ =	999	kg/m^3		

Calculated Parameters:

a =	0.0428	m^{-1}
b =	20.75	m/s

Calculated Results:

Time, t (s)	Velocity, U (m/s)	Accel., a_x (m/s)	Accel., a_x (gs)	Position, X (m)
0	12.5	-18.4	-1.88	0.00
0.1	10.8	-15.5	-1.58	1.16
0.2	9.37	-13.3	-1.35	2.17
0.3	8.13	-11.5	-1.17	3.04
0.4	7.06	-10.0	-1.02	3.80
0.5	6.12	-8.84	-0.901	4.46
0.6	5.29	-7.84	-0.800	5.03
0.7	4.54	-7.01	-0.714	5.52
0.8	3.88	-6.30	-0.642	5.94
0.9	3.28	-5.69	-0.580	6.30
1.0	2.74	-5.17	-0.527	6.60
1.1	2.24	-4.72	-0.481	6.85
1.2	1.79	-4.32	-0.440	7.05
1.3	1.38	-3.97	-0.405	7.21
1.4	0.998	-3.66	-0.373	7.33
1.5	0.646	-3.39	-0.345	7.41
1.6	0.319	-3.14	-0.320	7.46
1.7	0.0160	-2.93	-0.298	7.47
1.705	0.00000	-2.91	-0.297	7.47
1.8	-0.267	-2.73	-0.278	7.46
1.9	-0.530	-2.55	-0.260	7.42
2.0	-0.777	-2.39	-0.244	7.35

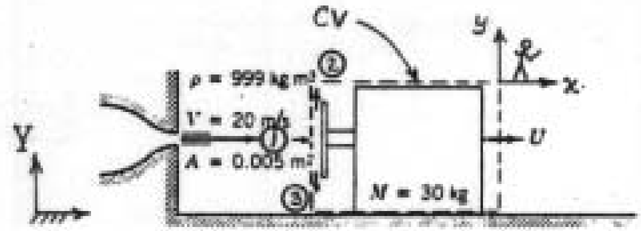


3/3

Problem 4.132

Given: Vane/slider assembly moving under influence of jet.

$F_R = kU$; $k = 7.5\ N\cdot s/m$



Find: (a) Acceleration at instant when $U = 10$ m/s.
(b) Terminal speed of slider.

Solution: Apply x momentum equation to linearly accelerating CV.

Basic equation:

$$F_{S_x} + F_{B_x} - \int_{CV} a_{rf_x}\rho\, dV = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\rho\, dV + \int_{CS} u_{xyz}\rho \vec{V}_{xyz}\cdot d\vec{A}$$

($F_{B_x} = 0(1)$; $\frac{\partial}{\partial t} \approx 0(2)$)

Assumptions: (1) Horizontal, so $F_{B_x} = 0$
(2) Neglect mass of liquid on vane, $u \approx 0$ on vane
(3) Uniform flow at each section
(4) Measure velocities relative to CV

Then

$$-kU - a_{rf_x}M = u_1\{-|\rho(V-U)A|\} + u_2\{+\dot{m}_2\} + u_3\{+\dot{m}_3\}$$

$$u_1 = V-U \qquad u_2 = 0 \qquad u_3 = 0$$

$$-kU - M\frac{dU}{dt} = -\rho(V-U)^2A$$

or

$$\frac{dU}{dt} = \frac{\rho(V-U)^2A}{M} - \frac{kU}{M}$$

$$= 999\frac{kg}{m^3}(20-10)^2\frac{m^2}{s^2}\times 0.005\,m^2\times\frac{1}{30\,kg} - 7.5\frac{N\cdot s}{m}\times 10\frac{m}{s}\times\frac{1}{30\,kg}\times\frac{kg\cdot m}{N\cdot s^2}$$

$$\frac{dU}{dt} = 14.2\ m/s^2 \quad (\text{at } U = 10\,m/s)$$ $\frac{dU}{dt}$

At terminal speed, $U = U_t$ and $dU/dt = 0$, so

$$0 = \frac{\rho(V-U)^2A}{M} - \frac{kU}{M} \quad \text{or} \quad V^2 - 2UV + U^2 - \frac{k}{\rho A}U = 0$$

$$U^2 - \left(2V + \frac{k}{\rho A}\right)U + V^2 = 0$$

$$U = \frac{2V + k/\rho A \pm\sqrt{(2V + k/\rho A)^2 - 4V^2}}{2} = V\left\{\left(1 + \frac{k}{2\rho VA}\right) \pm\sqrt{\left(1+\frac{k}{2\rho VA}\right)^2 - 1}\right\}$$

$$1 + \frac{k}{2\rho VA} = 1 + \frac{1}{2}\times 7.5\frac{N\cdot s}{m}\times\frac{m^3}{999\,kg}\times\frac{s}{20\,m}\times\frac{1}{0.005\,m^2}\times\frac{kg\cdot m}{N\cdot s^2} = 1.0375$$

$$U = V\left\{1.0375 \pm\sqrt{(1.0375)^2 - 1}\right\} = 0.761\,V = 0.761\times 20\frac{m}{s} = 15.2\,m/s$$ U_t

{The negative root was chosen so $U_t < V$, as required.}

Problem 4.133

For the vane/slider problem of Problem 4.132, find and plot expressions for the acceleration, speed, and position of the slider as functions of time. (Consider numerical integration.)

Given: Data on vane/slider

Find: Formula for acceleration, speed, and position; plot

Solution

The given data is

$$\rho = 999 \cdot \frac{kg}{m^3} \qquad M = 30 \cdot kg \qquad A = 0.005 \cdot m^2 \qquad V = 20 \cdot \frac{m}{s} \qquad k = 7.5 \cdot \frac{N \cdot s}{m}$$

The equation of motion, from Problem 4.132, is

$$\frac{dU}{dt} = \frac{\rho \cdot (V - U)^2 \cdot A}{M} - \frac{k \cdot U}{M}$$

(The acceleration is)

$$a = \frac{\rho \cdot (V - U)^2 \cdot A}{M} - \frac{k \cdot U}{M}$$

The differential equation for *U* can be solved analytically, but is quite messy. Instead we use a simple numerical method - Euler's method

$$U(n+1) = U(n) + \left(\frac{\psi\cdot(V-U)^2\cdot A}{M} - \frac{k\cdot U}{M}\right)\cdot\Delta t$$

where Δt is the time step

Finally, for the position x $\frac{dx}{dt} = U$

so $$x(n+1) = x(n) + U\cdot\Delta t$$

The final set of equations is

$$U(n+1) = U(n) + \left(\frac{\psi\cdot(V-U)^2\cdot A}{M} - \frac{k\cdot U}{M}\right)\cdot\Delta t$$

$$a = \frac{\psi\cdot(V-U)^2\cdot A}{M} - \frac{k\cdot U}{M}$$

$$x(n+1) = x(n) + U\cdot\Delta t$$

The results are plotted in the corresponding *Excel* workbook

Problem 4.133 (In Excel)

For the vane/slider problem of Problem 4.132, find and plot expressions for the acceleration, speed, and position of the slider as functions of time. (Consider numerical integration.)

Given: Data on vane/slider
Find: Plot acceleration, speed and position

Solution

The solutions are

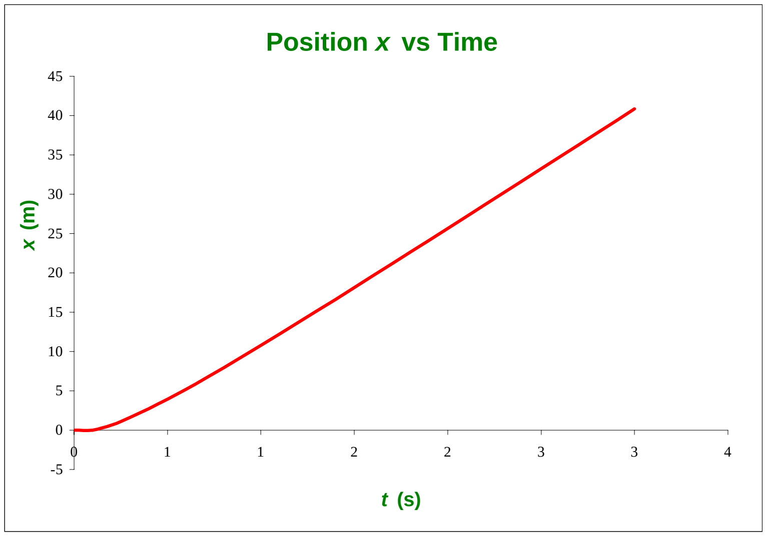
$$U(n+1) = U(n) + \left(\frac{\rho (V-U)^2 A}{M} - \frac{k U}{M}\right)\Delta t$$

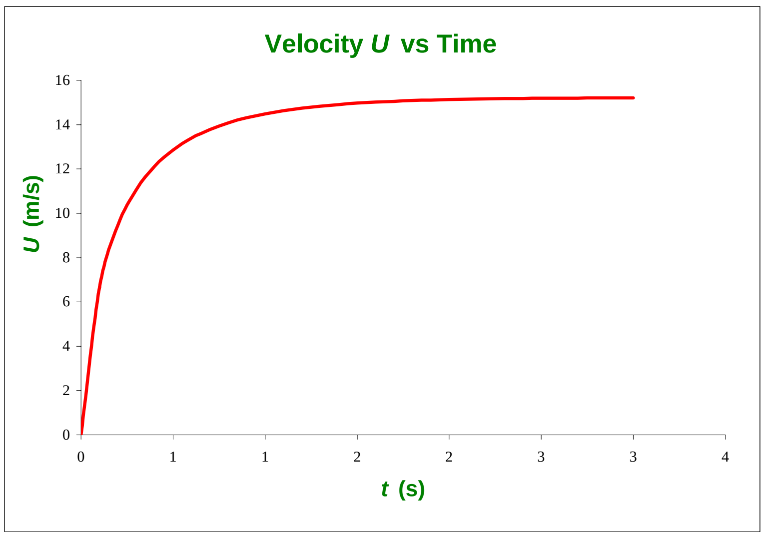
$$a = \frac{\rho (V-U)^2 A}{M} - \frac{k U}{M}$$

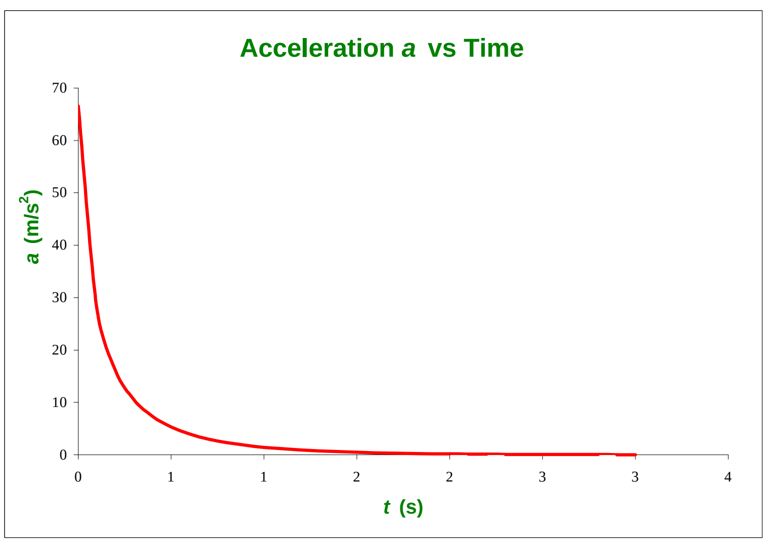
$$x(n+1) = x(n) + U\Delta t$$

ρ = 999 kg/m^3
k = 7.5 N.s/m
A = 0.005 m^2
V = 20 m/s
M = 30 kg
Δt = 0.1 s

t (s)	*x* (m)	*U* (m/s)	*a* (m/s^2)
0.0	0.0	0.0	66.6
0.1	0.0	6.7	28.0
0.2	0.7	9.5	16.1
0.3	1.6	11.1	10.5
0.4	2.7	12.1	7.30
0.5	3.9	12.9	5.29
0.6	5.2	13.4	3.95
0.7	6.6	13.8	3.01
0.8	7.9	14.1	2.32
0.9	9.3	14.3	1.82
1.0	10.8	14.5	1.43
1.1	12.2	14.6	1.14
1.2	13.7	14.7	0.907
1.3	15.2	14.8	0.727
1.4	16.6	14.9	0.585
1.5	18.1	15.0	0.472
1.6	19.6	15.0	0.381
1.7	21.1	15.1	0.309
1.8	22.6	15.1	0.250
1.9	24.1	15.1	0.203
2.0	25.7	15.1	0.165
2.1	27.2	15.1	0.134
2.2	28.7	15.2	0.109
2.3	30.2	15.2	0.0889
2.4	31.7	15.2	0.0724
2.5	33.2	15.2	0.0590
2.6	34.8	15.2	0.0481
2.7	36.3	15.2	0.0392
2.8	37.8	15.2	0.0319
2.9	39.3	15.2	0.0260
3.0	40.8	15.2	0.0212





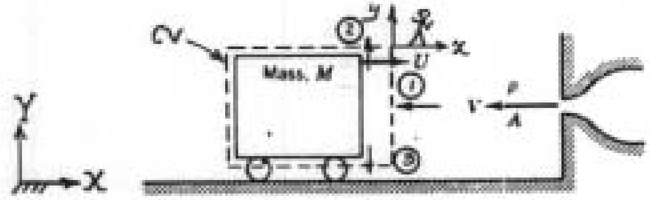


Problem 4.134

Given: Block and jet as shown.

Jet strikes block at $t > 0$.

Find: (a) Expression for acceleration.
(b) Time at which $U = 0$.



Solution: Apply x momentum equation to linearly accelerating CV.

Basic equation:

$$\overset{=0(1)}{F_{S_x}} + \overset{=0(2)}{F_{B_x}} - \int_{CV} a_{rf_x} \rho \, d\forall = \overset{\approx 0(3)}{\frac{\partial}{\partial t}} \int_{CV} u_{xyz} \rho \, d\forall + \int_{CS} u_{xyz} \rho \vec{V}_{xyz} \cdot d\vec{A}$$

Assumptions: (1) No pressure or friction forces, so $F_{S_x} = 0$
(2) Horizontal, so $F_{B_x} = 0$
(3) Neglect mass of liquid in CV, $u \approx 0$ in CV
(4) Uniform flow at each section
(5) Measure velocities relative to CV

Then

$$-M a_{rf_x} = -M \frac{dU}{dt} = u_1 \{ -|\rho(V+U)A| \} + u_2 \{ +\dot{m}_2 \} + u_3 \{ +\dot{m}_3 \}$$

$$u_1 = -(V+U) \qquad u_2 = 0 \qquad u_3 = 0$$

or

$$\frac{dU}{dt} = -\frac{\rho (V+U)^2 A}{M}$$ $\frac{dU}{dt}$

But, since V = constant, $dU = d(V+U)$, so

$$\frac{d(V+U)}{(V+U)^2} = -\frac{\rho A}{M} dt$$

Integrating from U_0 at $t = 0$ to $U = 0$ at t

$$\int_{V+U_0}^{V} \frac{d(V+U)}{(V+U)^2} = -\frac{1}{(V+U)}\Big]_{V+U_0}^{V} = -\frac{1}{V} + \frac{1}{V+U_0} = \frac{-U_0}{V(V+U_0)} = -\frac{\rho A t}{M}$$

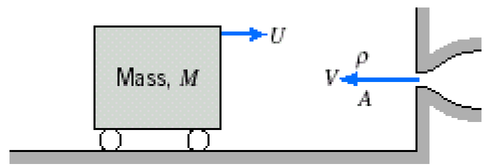
Solving, $$t = \frac{M U_0}{\rho V A (V+U_0)} = \frac{M}{\rho V A (1 + V/U_0)}$$ t

Problem 4.135

If M = 100 kg, ρ = 999 kg/m^3, and A = 0.01 m^2, find the jet speed V required for the cart to be brought to rest after one second if the initial speed of the cart is U_0 = 5 m/s. For this condition, plot the speed U and position x of the cart as functions of time. What is the maximum value of x, and how long does the cart take to return to its initial position?

Given: Data on system

Find: Jet speed to stop cart after 1 s; plot speed & position; maximum x; time to return to origin



Solution

The given data is $\rho = 999 \cdot \frac{kg}{m^3}$ $\quad M = 100 \cdot kg$ $\quad A = 0.01 \cdot m^2$ $\quad U_0 = 5 \cdot \frac{m}{s}$

The equation of motion, from Problem 4.134, is

$$\frac{dU}{dt} = -\frac{\rho \cdot (V + U)^2 \cdot A}{M}$$

which leads to

$$\frac{d(V + U)}{(V + U)^2} = -\left(\frac{\rho \cdot A}{M}\right) \cdot dt$$

Integrating and using the IC $U = U_0$ at $t = 0$

$$U = -V + \frac{V + U_0}{1 + \frac{\rho \cdot A \cdot (V + U_0)}{M} \cdot t}$$

To find the jet speed *V* *V*, with *U* = 0 and *t* = 1 s. (The equation becomes a quadratic in *V*). Instead we use *Excel*'s *Goal Seek* in the associated workbook

From *Excel* $V \mid 5 \cdot \frac{m}{s}$

For the position *x* we need to integrate

$$\frac{dx}{dt} \mid U \mid 4V\,2\,\frac{V\,2\,U_0}{1\,2\,\frac{\psi \cdot \hat{A} \cdot (V\,2\,U_0)}{M} \cdot t}$$

The result is

$$x \mid 4V \cdot t\,2\,\frac{M}{\psi \cdot \hat{A}} \cdot \ln\left(1\,2\,\frac{\psi \cdot \hat{A} \cdot (V\,2\,U_0)}{M} \cdot t\right)$$

This equation (or the one for *U* with *U* *x* b differentiating, as well as the time for *x* to be zero again. Instead we use *Excel*'s *Goal Seek* and *Solver* in the associated workbook

From *Excel* $x_{max} \mid 1.93 \cdot m$ $t(x \mid 0) \mid 2.51 \cdot s$

The complete set of equations is

$$U \mid 4V\,2\,\frac{V\,2\,U_0}{1\,2\,\frac{\psi \cdot \hat{A} \cdot (V\,2\,U_0)}{M} \cdot t}$$

$$x \mid 4V \cdot t\,2\,\frac{M}{\psi \cdot \hat{A}} \cdot \ln\left(1\,2\,\frac{\psi \cdot \hat{A} \cdot (V\,2\,U_0)}{M} \cdot t\right)$$

The plots are presented in the *Excel* workbook

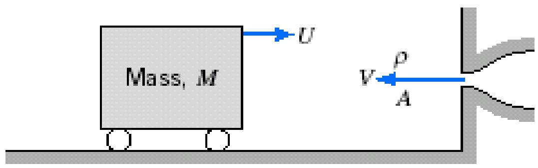
Problem 4.135 (In Excel)

If M = 100 kg, ρ = 999 kg/m^3, and A = 0.01 m^2, find the jet speed V required for the cart to be brought to rest after one second if the initial speed of the cart is U_0 = 5 m/s. For this condition, plot the speed U and position x of the cart as functions of time.
What is the maximum value of x, and how long does the cart take to return to its initial position?

Given: Data on system
Find: Jet speed to stop cart after 1 s; plot speed & position; maximum x; time to return to origin

Solution

M = 100 kg
ρ = 999 kg/m^3
A = 0.01 m^2
U_o = 5 m/s



$$U = -V + \frac{V + U_0}{1 + \frac{\rho A (V + U_0)}{M} t}$$

$$x = -V t + \frac{M}{\rho A} \ln\left(1 + \frac{\rho A (V + U_0)}{M} t\right)$$

t (s)	*x* (m)	*U* (m/s)
0.0	0.00	5.00
0.2	0.82	3.33
0.4	1.36	2.14
0.6	1.70	1.25
0.8	1.88	0.56
1.0	1.93	0.00
1.2	1.88	-0.45
1.4	1.75	-0.83
1.6	1.56	-1.15
1.8	1.30	-1.43
2.0	0.99	-1.67
2.2	0.63	-1.88
2.4	0.24	-2.06
2.6	-0.19	-2.22
2.8	-0.65	-2.37
3.0	-1.14	-2.50

To find V for U = 0 in 1 s, use *Goal Seek*

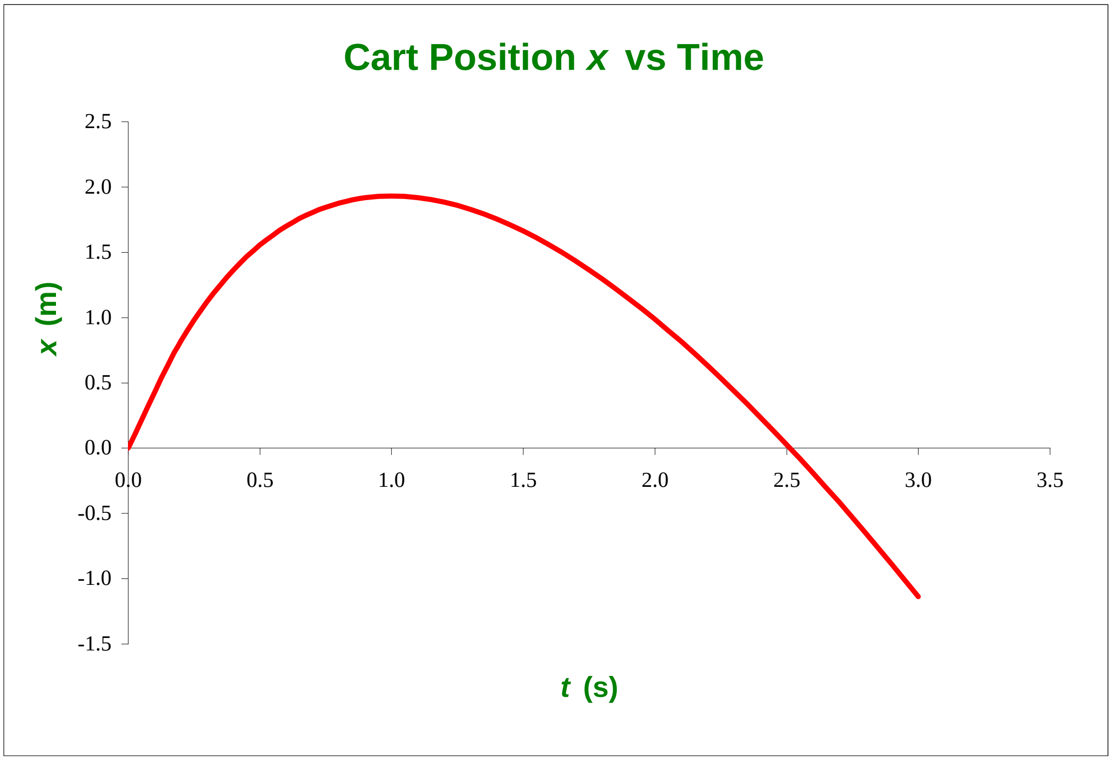
t (s)	*U* (m/s)	*V* (m/s)
1.0	0.00	5.00

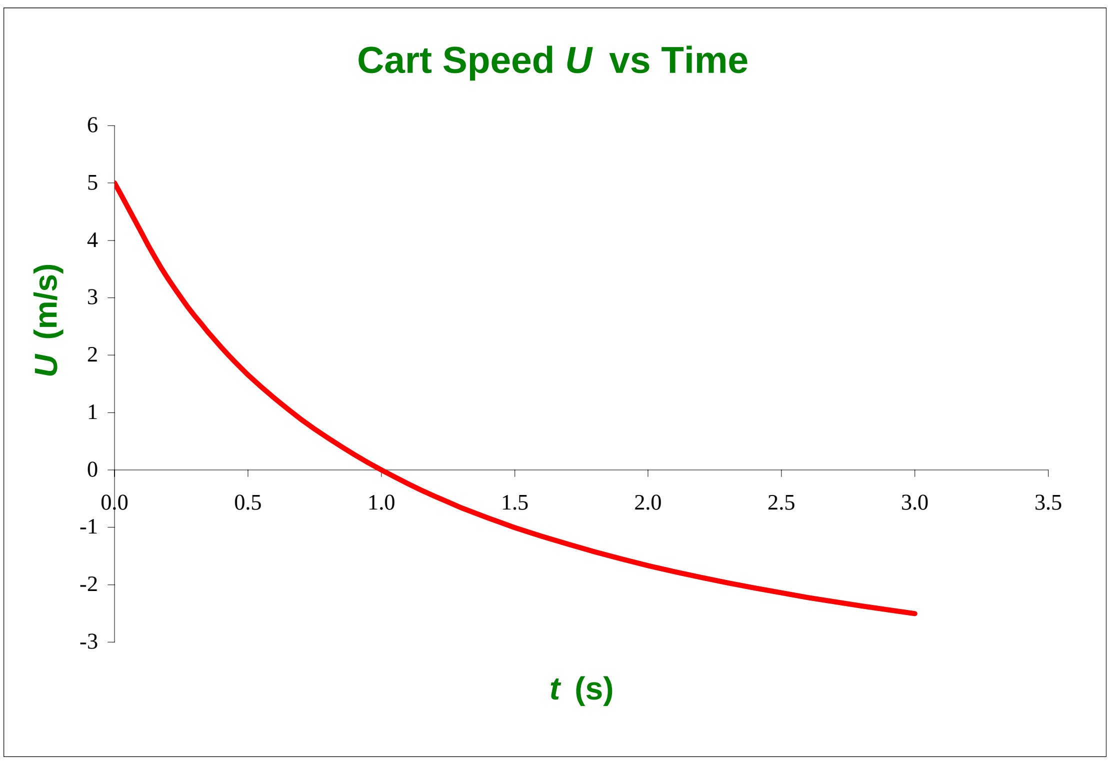
To find the maximum x, use *Solver*

t (s)	*x* (m)
1.0	1.93

To find the time at which x = 0 use *Goal Seek*

t (s)	*x* (m)
2.51	0.00


Cart Position x vs Time
x (m)
2.5
2.0
1.5
1.0
0.5
0.0
-0.5
-1.0
-1.5
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
t (s)

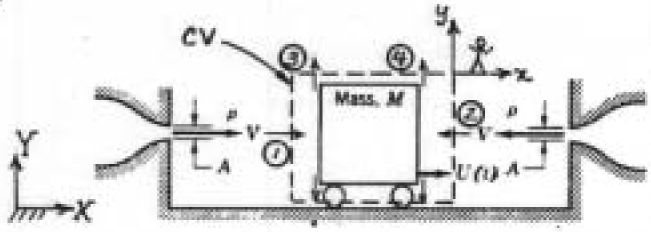

Cart Speed U vs Time
U (m/s)
6
5
4
3
2
1
0
-1
-2
-3
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
t (s)

Problem 4.136

Given: Block rolling between opposing jets, as shown.

Speed is U_0 at $t=0$.

There is no resistance for $t>0$.



Find: (a) Expression for acceleration, $a(t)$.
(b) Expression for speed, $U(t)$.

Solution: Apply x momentum to linearly accelerating CV.

Basic equation:

$$\overset{=0(1)}{F_{S_x}} + \overset{=0(2)}{F_{B_x}} - \int_{CV} a_{rf_x}\,\rho\,d\forall = \overset{\approx 0(3)}{\frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\,d\forall} + \int_{CS} u_{xyz}\,\rho\,\vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: (1) No pressure or friction forces, so $F_{S_x}=0$
(2) Horizontal, so $F_{B_x}=0$
(3) Neglect mass of liquid in CV; $u\approx 0$ in CV
(4) Uniform flow at each section
(5) Measure velocities relative to CV

Then

$$-a_{rf_x}M = -M\frac{dU}{dt} = u_1\{-|\rho(V-U)A|\} + u_2\{-|\rho(V+U)A|\} + u_3\{\dot m_3\} + u_4\{\dot m_4\}$$

$$u_1 = V-U \qquad u_2 = -(V+U) \qquad u_3=0 \qquad u_4=0$$

or

$$-M\frac{dU}{dt} = \rho A\left[-(V-U)^2+(V+U)^2\right] = \rho A\left[4UV\right] = 4\rho VAU$$

Thus

$$\frac{dU}{U} = -\frac{4\rho VA}{M}\,dt$$

Integrating

$$\int_{U_0}^{U}\frac{dU}{U} = \ln U\Big]_{U_0}^{U} = \ln\frac{U}{U_0} = -\frac{4\rho VA}{M}t$$

or

$$U(t) = U_0\,e^{-\frac{4\rho VA}{M}t}$$

U(t)

Also

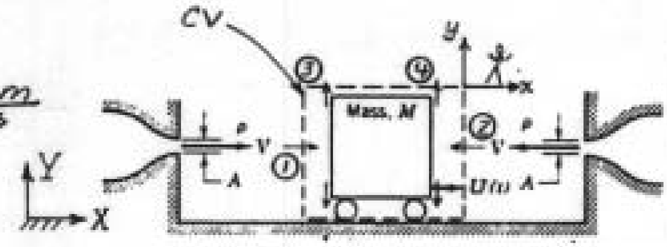
$$a(t) = \frac{dU}{dt} = -\frac{4\rho VA}{M}\,U_0\,e^{-\frac{4\rho VA}{M}t}$$

a(t)

Problem 4.137

Given: Block rolling between opposing jets, as shown.

At $t = 0$, block moves at $U_0 = 10 \frac{m}{s}$ starting from $X = 0$.



Find: (a) Time to reduce speed to $U = 0.5$ m/s.
(b) Position at that instant.

Solution: Apply x momentum equation to linearly accelerating CV.

Basic equation:

$$\overset{=0(1)}{F_{S_x}} + \overset{=0(2)}{F_{B_x}} - \int_{CV} a_{rf_x} \rho \, d\forall = \overset{\approx 0(3)}{\frac{\partial}{\partial t}\int_{CV} u_{xyz} \rho \, d\forall} + \int_{CS} u_{xyz} \rho \vec{V}_{xyz} \cdot d\vec{A}$$

Assumptions: (1) No pressure or friction forces, so $F_{S_x} = 0$
(2) Horizontal, so $F_{B_x} = 0$
(3) Neglect mass of liquid in CV; $u \approx 0$ in CV
(4) Uniform flow at each section
(5) Measure velocities relative to CV

Then

$$-a_{rf_x} M = -M\frac{dU}{dt} = u_1\{-|\rho(V-U)A|\} + u_2\{-|\rho(V+U)A|\} + u_3\{\dot{m}_3\} + u_4\{\dot{m}_4\}$$

$$u_1 = V - U \qquad u_2 = -(V+U) \qquad u_3 = 0 \qquad u_4 = 0$$

or

$$-M\frac{dU}{dt} = \rho A\left[-(V-U)^2 + (V+U)^2\right] = \rho A\left[4UV\right] = 4\rho V A U$$

Thus

$$\frac{dU}{U} = -\frac{4\rho VA}{M}\,dt$$

Integrating, $$\int_{U_0}^{U}\frac{dU}{U} = \ln U\Big]_{U_0}^{U} = \ln\frac{U}{U_0} = -\frac{4\rho VA}{M}\,t \qquad (1)$$

Thus $$t = -\frac{M}{4\rho VA}\ln\frac{U}{U_0} = -\frac{1}{4}\cdot\frac{M}{\rho VA}\ln\frac{0.5}{10} = 0.750\frac{M}{\rho VA} \qquad t$$

From Eq. 1, $$U(t) = \frac{dX}{dt} = U_0 e^{-\frac{4\rho VA}{M}t}$$

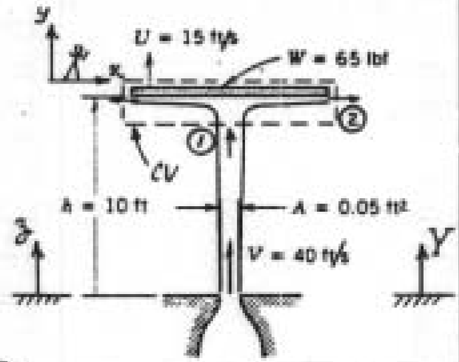
Integrating, $$X = \int_0^X dX = \int_0^t U_0 e^{-\frac{4\rho VA}{M}t}\,dt = -\frac{MU_0}{4\rho VA}\,e^{-\frac{4\rho VA}{M}t}\Big]_0^t$$

$$X = \frac{MU_0}{4\rho VA}\left[1 - e^{-\frac{4\rho VA}{M}t}\right] = \frac{0.95}{4}\frac{MU_0}{\rho VA} = 0.238\frac{MU_0}{\rho VA} \qquad X$$

Problem *4.138

Given: Vertical jet impinging on disk.

Find: Vertical acceleration of disk at the instant shown.



Solution: Apply Bernoulli equation to jet, then y momentum equation to a CV with linear acceleration.

Basic equations:

$$\frac{p_0}{\rho} + \frac{V_0^2}{2} + gz_0 = \frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1$$

$$F_{S_y} + F_{B_y} - \int_{CV} a_{rf_y} \rho \, d\forall = \frac{\partial}{\partial t}\int_{CV} v_{xyz} \rho \, d\forall + \int_{CS} v_{xyz} \rho \vec{V}_{xyz} \cdot d\vec{A}$$

$= 0(6)$, $\approx 0(7)$

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) No friction
(4) Flow along streamline
} in jet
(5) $p_0 = p_1 = p_{atm}$

From Bernoulli,

$$V_1 = \sqrt{V_0^2 + 2g(z_0 - z_1)} = \left[40^2 \frac{ft^2}{s^2} + 2 \times 32.2 \frac{ft}{s^2}(0-10)ft\right]^{1/2} = 30.9 \text{ ft/s}$$

(6) No pressure force on CV; $F_{S_y} = 0$
(7) Neglect mass of liquid in CV; $v \approx 0$ in CV
(8) Uniform flow at each section
(9) Measure velocities relative to CV

Then

$$-W - Ma_{rf_y} = v_1\{-|\rho(V_1 - U)A_1|\} + v_2\{\dot{m}_2\} = -\rho(V_1 - U)^2 A_1$$

$$v_1 = V_1 - U \qquad v_2 = 0$$

or

$$a_{rf_y} = \frac{\rho(V_1 - U)^2 A_1 - W}{M}$$

But from continuity, $V_0 A_0 = V_1 A_1$; $A_1 = A_0 \frac{V_0}{V_1}$. Thus, since $M = W/g$,

$$a_{rf_y} = \frac{\rho(V_1 - U)^2 \frac{V_0}{V_1} A_0 - W}{W/g} = \left[\frac{\rho(V_1 - U)^2 V_0 A_0}{W V_1} - 1\right] g$$

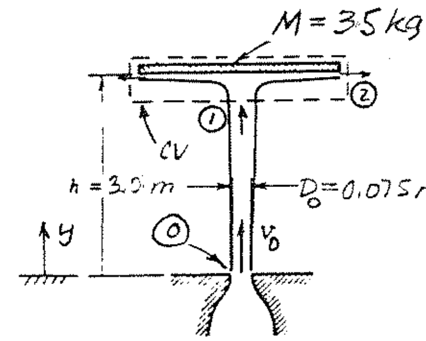
$$= \left[1.94 \frac{slug}{ft^3}(30.9 - 15)^2 \frac{ft^2}{s^2} \times 40 \frac{ft}{s} \times 0.05 \, ft^2 \times \frac{1}{65 \, lbf} \times \frac{s}{30.9 \, ft} \times \frac{lbf \cdot s^2}{slug \cdot ft} - 1\right] 32.2 \frac{ft}{s^2}$$

$$a_{rf_y} = -16.5 \text{ ft/s}^2 \quad \text{(down)}$$

a_{rf_y}

Problem *4.139

Given: Flow system of Problem 4.138, with dimensions shown at right.



Find: Plot disk mass vs. flow rate to determine the flow rate required to make h = 3.5 m.

Solution: Apply continuity, momentum, and Bernoulli equations using CV shown.

Basic equations: $0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A}$ (first term $=0(1)$)

$$F_{Sy} + F_{By} = \frac{\partial}{\partial t}\int_{CV} v\rho\, d\forall + \int_{CS} v\rho\vec{V}\cdot d\vec{A}$$ (first term $=0(1)$)

$$\frac{p_0}{\rho} + \frac{V_0^2}{2} + g y_0 = \frac{p_1}{\rho} + \frac{V_1^2}{2} + g y_1$$ ($y_0 = 0$)

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) No friction
(4) Flow along a streamline
} in jet
(5) $p_0 = p_1 = p_{atm}$
(6) No pressure force on CV, so $F_{Sy} = 0$
(7) Uniform flow at each cross-section

From momentum

$$-Mg = \int_{CS} v\rho\vec{V}\cdot d\vec{A} = v_1\{-\rho V_1 A_1\} + v_2\{+\rho V_2 A_2\} = -V_1 \rho V_1 A_1$$

$$v_1 = V_1 \qquad v_2 = 0$$

$$M = \frac{V_1 \rho V_1 A_1}{g}$$

From continuity $V_1 A_1 = V_0 A_0$

From Bernoulli $V_1 = \left[V_0^2 - 2gh\right]^{\frac{1}{2}}$

Substituting

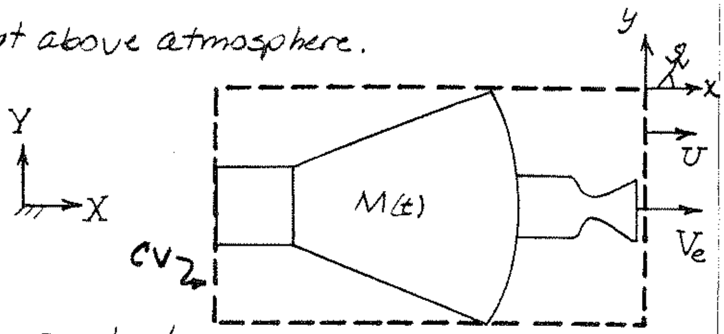
$$M = \frac{\rho}{g}\left[V_0^2 - 2gh\right]^{\frac{1}{2}} V_0 \frac{\pi D_0^2}{4}$$ M

This equation cannot be solved for V_0 directly, but it can be plotted for various values of V_0. Alternatively, using Excel's solver we obtain $Q = 0.0469 \text{ m}^3/\text{s}$ for h = 3 m.

Given: Space capsule in level flight above atmosphere.

$U_0 = 8.0$ km/s
$M_0 = 1600$ kg

$\dot{m} = 8$ kg/s
$V_e = 3000$ m/s



Find: Time to reduce speed to $U = 5.00$ km/s.

Solution: Apply x component of momentum to CV with linear acceleration.

Basic equation:

$$\overset{=0(1)}{F_{S_x}} + \overset{=0(2)}{F_{B_x}} - \int_{CV} a_{rf_x} \rho \, d\forall = \frac{\partial}{\partial t} \int_{CV} \overset{\approx 0(4)}{u_{xyz}} \rho \, d\forall + \int_{CS} u_{xyz} \rho \vec{V}_{xyz} \cdot d\vec{A}$$

Assumptions: (1) No resistance; $F_{S_x} = 0$
(2) Horizontal; $F_{B_x} = 0$
(3) Use velocities measured relative to CV
(4) Neglect velocity within CV
(5) Uniform flow at exit plane with negligible p_e (given)

From continuity,

$$\frac{dM}{dt} = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall = -\int_{CS} \rho \vec{V}_{xyz} \cdot d\vec{A} = -\dot{m}; \quad M(t) = M_0 - \dot{m}t$$

From momentum,

$$-a_{rf_x} M = -\frac{dU}{dt}(M_0 - \dot{m}t) = u_e\{+\dot{m}\} = V_e \dot{m}$$

$u_e = V_e$

Thus

$$\frac{dU}{dt} = -\frac{V_e \dot{m}}{M_0 - \dot{m}t}$$

Integrating, $$U - U_0 = V_e \int_0^t \frac{-\dot{m}\,dt}{M_0 - \dot{m}t} = V_e \ln(M_0 - \dot{m}t)\Big]_0^t = V_e \ln\left(\frac{M_0 - \dot{m}t}{M_0}\right)$$

Solving for t,

$$\frac{M_0 - \dot{m}t}{M_0} = e^{\frac{U-U_0}{V_e}}; \quad M_0 - \dot{m}t = M_0 e^{U-U_0/V_e}$$

$$t = \frac{M_0}{\dot{m}}\left(1 - e^{U-U_0/V_e}\right)$$

$$= 1600 \text{ kg} \times \frac{\text{s}}{8 \text{ kg}}\left\{1 - e^{\left[(5.00 - 8.0)\frac{\text{km}}{\text{s}} \times \frac{\text{s}}{3000 \text{ m}} \times 1000 \frac{\text{m}}{\text{km}}\right]}\right\}$$

$t = 126$ s

t

Problem 4.141

Given: Rocket sled on horizontal track, slowed by retro-rocket.

Initial mass $M_0 = 1500\ kg$ Initial speed $U_0 = 500\ m/s$

Mass flow rate $\dot{m} = 7.75\ kg/s$ Exhaust speed $V_e = 2500\ m/s$

Firing time $t_{bo} = 20.0\ s$

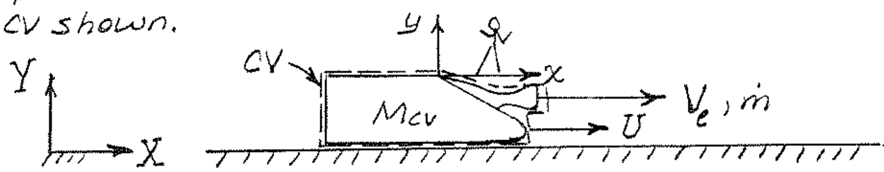
Neglect aerodynamic drag and rolling resistance.

Find: (a) Algebraic expression for sled speed U as a function of t.

(b) Speed at end of retro-rocket firing.

Solution: Apply x-component of momentum equation to the linearly accelerating CV shown.

From continuity,



$M_{cv} = M_0 - \dot{m}t, \quad t < t_{bo}$

Basic equation:

$$F_{S_x} + F_{B_x} - \int_{CV} a_{rf_x} \rho\, d\forall = \frac{\partial}{\partial t}\int_{CV} u_{xyz} \rho\, d\forall + \int_{CS} u_{xyz}\, \rho \vec{V}_{xyz} \cdot d\vec{A}$$

($F_{S_x} = 0$ (1), $F_{B_x} = 0$ (2), $\frac{\partial}{\partial t}\int_{CV} \approx 0$ (3))

Assumptions: (1) No pressure, drag, or rolling resistance, so $F_{S_x} = 0$

(2) Horizontal motion, so $F_{B_x} = 0$

(3) Neglect unsteady effects within CV

(4) Uniform flow at nozzle exit plane

(5) $p_e = p_{atm}$

Then $-a_{rf_x} M_{cv} = u_e \{+\dot{m}\} = +V_e \dot{m}$ or $\frac{dU}{dt} = -\frac{V_e \dot{m}}{M_{cv}} = -\frac{V_e \dot{m}}{M_0 - \dot{m}t}$

$u_e = V_e$

Thus $dU = V_e \left(\frac{-\dot{m}\,dt}{M_0 - \dot{m}t}\right)$ and $U - U_0 = V_e \ln(M_0 - \dot{m}t)\Big|_0^t = V_e \ln\left(1 - \frac{\dot{m}t}{M_0}\right)$

$$U(t) = U_0 + V_e \ln\left(1 - \frac{\dot{m}t}{M_0}\right); \quad t < t_{bo}$$ U(t)

At t_{bo}, $U(t_{bo}) = 500\ \frac{m}{s} + 2500\ \frac{m}{s} \times \ln\left(1 - 7.75\ \frac{kg}{s} \times 20.0\ s \times \frac{1}{1500\ kg}\right)$

$$U(t_{bo}) = 227\ m/s$$ $U(t_{bo})$

Problem 4.142

Given: Rocket sled accelerates from rest on a level track. Initial mass, $M_0 = 600$ kg, includes fuel - $M_f = 150$ kg. The rocket motor burns fuel at rate $\dot{m} = 15$ kg/s. Exhaust gases leave nozzle uniformly and axially at atmospheric pressure with $V_e = 2900$ m/s relative to the nozzle. Neglect air and rolling resistance.

Find: (a) Maximum speed reached by the sled.
(b) Maximum acceleration of sled during the run.

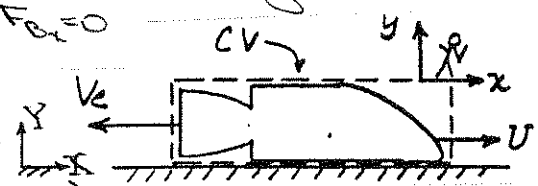
Plot: the sled speed and acceleration as functions of time.

Solution:

Apply the momentum equation to linearly accelerating CV shown.

Basic equation: $\cancel{F_{S_x}}^{=0(1)} + \cancel{F_{B_x}}^{=0(2)} - \int_{CV} a_{rf}\,\rho\, dV = \cancel{\frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\, dV}^{=0(3)} + \int_{CS} u_{xyz}\,\rho\,\vec{V}_{xyz}\cdot d\vec{A}$

Assumptions: (1) no net pressure forces ($p_e = p_{atm}$, given)
(2) horizontal motion, $F_{B_x} = 0$
(3) neglect $\partial/\partial t$ in CV
(4) uniform axial jet



From continuity, $M = M_0 - \dot{m}t$. Then

$$-a_{rf_x} M = -\frac{dU}{dt}(M_0 - \dot{m}t) = u_e\{\dot{m}\} = -V_e\dot{m} \quad \text{....(1)}$$

Separating variables,

$$dU = V_e \frac{\dot{m}\,dt}{M_0 - \dot{m}t}$$

Integrating from $U = 0$ at $t = 0$ to U at t gives

$$U = -V_e \ln(M_0 - \dot{m}t)\Big]_0^t = -V_e \ln\frac{(M_0 - \dot{m}t)}{M_0} = V_e \ln\frac{M_0}{(M_0 - \dot{m}t)} \quad \text{---(2)}$$

The speed is a maximum at burnout. At burnout $M_f = 0$ and $M = M_0 - \dot{m}t = 450$ kg

At burnout, $t = \frac{M_f)_{initial}}{\dot{m}_{fuel}} = 150\,\text{kg} \times \frac{s}{15\,\text{kg}} = 10\,s$

Then from Eq. 2

$$U_{max} = 2900\frac{m}{s} \ln\frac{600\,\text{kg}}{450\,\text{kg}} = 834\ m/s \longleftarrow U_{max}$$

From Eq. 1 the acceleration is $\frac{dU}{dt} = \frac{\dot{m}V_e}{M_0 - \dot{m}t}$

The maximum acceleration occurs at the instant prior to burn out

$$\left.\frac{dU}{dt}\right)_{max} = 15\frac{kg}{s} \times 2900\frac{m}{s} \times \frac{1}{450\,kg} = 96.7\ m/s^2 \longleftarrow \left.\frac{dU}{dt}\right)_{max}$$

The sled speed as a function of time is

$$U = V_e \ln \frac{M_0}{(M_0 - \dot{m}t)} \quad \text{for } 0 \le t \le 10\,s$$

$$U = \text{constant} = 834\ m/s \quad \text{for } t > 10 \quad \text{(neglecting resistance)}$$

The sled acceleration is given by

$$\frac{dU}{dt} = \frac{\dot{m} V_e}{(M_0 - \dot{m}t)} \quad \text{for } 0 \le t \le 10\,s$$

$$\frac{dU}{dt} = 0 \quad \text{for } t \ge 10\,s.$$

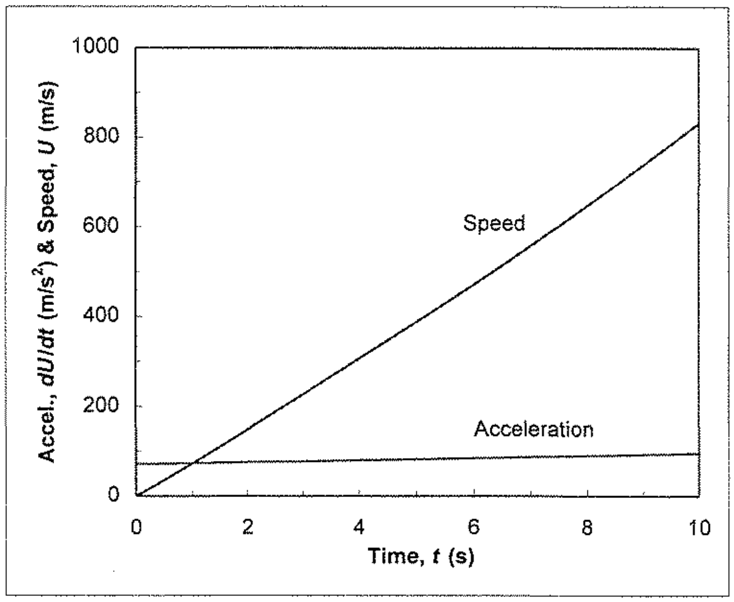
Acceleration and Velocity vs. Time for Rocket Sled:

Input Data:

M_0 =	600	kg
m(dot) =	15	kg/s
V_e =	2900	m/s

Calculated Results:

Time, t (s)	Acceleration, dU/dt (m/s^2)	Velocity, U (m/s)
0	72.5	0
1	74.4	73.4
2	76.3	149
3	78.4	226
4	80.6	306
5	82.9	387
6	85.3	471
7	87.9	558
8	90.6	647
9	93.5	739
10	96.7	834



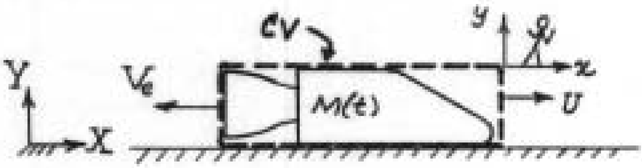
Problem 4.143

Given: Rocket sled moving on level track with no resistance.

$M_0 = 2000$ lbm

$\dot{m} = 30$ lbm/s

$V_e = 9000$ ft/s ; $p_e = p_{atm}$



Find: Minimum mass of fuel needed to accelerate sled to $U = 600$ mph.

Solution: Apply x component of momentum to CV accelerating linearly.

Basic equation:

$$F_{S_x} + F_{B_x} - \int_{CV} a_{rf_x} \rho \, dV = \frac{\partial}{\partial t}\int_{CV} u_{xyz} \rho \, dV + \int_{CS} u_{xyz} \rho \vec{V}_{xyz} \cdot d\vec{A}$$

($F_{S_x} = 0(1,2)$, $F_{B_x} = 0(3)$, $\frac{\partial}{\partial t}\int_{CV} \approx 0(5)$)

Assumptions: (1) No resistance
(2) $p_e = p_{atm}$ (given) } $F_{Sx} = 0$
(3) Horizontal track; $F_{Bx} = 0$
(4) Use relative velocities
(5) Neglect u and $\partial/\partial t$ within CV
(6) Uniform flow in exit plane

From continuity,

$$\frac{dM}{dt} = \frac{\partial}{\partial t}\int_{CV} \rho \, dV = -\int_{CS} \rho \vec{V}_{xyz} \cdot d\vec{A} = -\dot{m} \; ; \; M = M_0 - \dot{m}t$$

From momentum,

$$-a_{rf_x} M = -\frac{dU}{dt}(M_0 - \dot{m}t) = u_e\{+\dot{m}\} = -V_e \dot{m}$$

$$u_e = -V_e$$

Separating variables

$$dU = \frac{V_e \dot{m} \, dt}{M_0 - \dot{m}t}$$

Integrating, $U = -V_e \ln(M_0 - \dot{m}t)\Big|_0^t = V_e \ln\left(\frac{M_0}{M_0 - \dot{m}t}\right) = -V_e \ln\left(1 - \frac{\dot{m}t}{M_0}\right)$

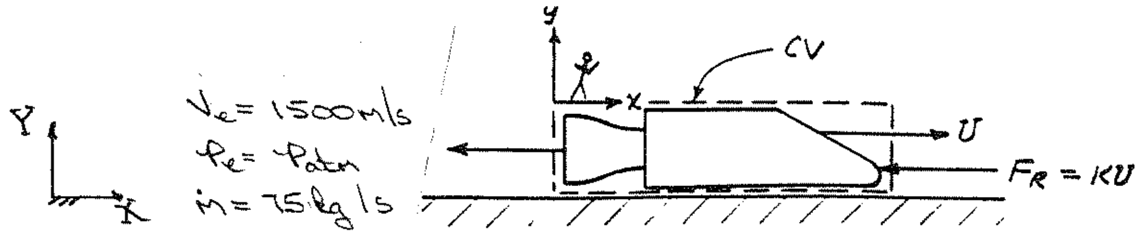
The mass of fuel consumed is $m_f = \dot{m}t$. From the above

$$m_f = \dot{m}t = M_0\left(1 - e^{-U/V_e}\right)$$

$$= 2000 \text{ lbm}\left[1 - e^{\left(-600\frac{mi}{hr} \times \frac{s}{9000 ft} \times \frac{5280 ft}{mi} \times \frac{hr}{3600 s}\right)}\right]$$

$m_f = 186$ lbm ← m_f

Given: Rocket sled with initial mass of 4 metric tons, including 1 ton of fuel. Motion resistance is given by kU where $k = 75\ N/m/s$.



Find: Sled speed 10 s after starting from rest, & U_{max}

Plot: sled speed and acceleration as functions of time.

Solution:

Apply the x component of the momentum equation to linearly accelerating CV shown

Basic equation: $F_{S_x} + F_{B_x} - \int_{CV} a_{rf_x}\,\rho\,d\forall = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\,d\forall + \int_{CS} u_{xyz}\,(\rho \vec{V}_{xyz}\cdot d\vec{A})$ (F_{B_x} = 0 (2); $\frac{\partial}{\partial t}$ ≈ 0 (3))

Assumptions: (1) $p_e = p_{atm}$ (given) so $F_{S_x} = -F_R$
(2) $F_{B_x} = 0$
(3) neglect unsteady effects within CV
(4) uniform flow at exit plane

Then,

$$-F_R - a_{rf_x} M = u_e\{+|\dot{m}|\} = -V_e \dot{m} \qquad \{F_R = kU,\ u_e = -V_e\}$$

From continuity, $M = M_o - \dot{m}t$. Substituting with $a_{rf_x} = \frac{dU}{dt}$

$$-kU - (M_o - \dot{m}t)\frac{dU}{dt} = -V_e\dot{m}$$

$$\frac{dU}{dt} = \frac{V_e\dot{m} - kU}{M_o - \dot{m}t} \quad \text{or} \quad \frac{dU}{V_e\dot{m} - kU} = \frac{dt}{M_o - \dot{m}t}$$

Integrating, $\frac{1}{k}\ln(V_e\dot{m} - kU)\Big]_0^U = \frac{1}{\dot{m}}\ln(M_o - \dot{m}t)\Big]_0^t$

and $\ln\frac{(V_e\dot{m} - kU)}{V_e\dot{m}} = \ln\left(1 - \frac{kU}{V_e\dot{m}}\right) = \frac{k}{\dot{m}}\ln\frac{(M_o - \dot{m}t)}{M_o} = \frac{k}{\dot{m}}\ln\left(1 - \frac{\dot{m}t}{M_o}\right)$

Then $1 - \frac{kU}{V_e\dot{m}} = \left(1 - \frac{\dot{m}t}{M_o}\right)^{k/\dot{m}}$ and

$$U = \frac{V_e\dot{m}}{k}\left[1 - \left(1 - \frac{\dot{m}t}{M_o}\right)^{k/\dot{m}}\right] \qquad (1)$$

At t = 10 s

$$U = 1500\frac{m}{s} \times 75\frac{kg}{s} \times \frac{m}{75\ N\cdot s} \times \frac{N\cdot s^2}{kg\cdot m}\left[1 - \left(1 - 75\frac{kg}{s} \times 10\,s \times \frac{1}{4000\,kg}\right)^{75\frac{N\cdot s}{m} \times \frac{s}{75\,kg} \times \frac{kg\cdot m}{N\cdot s^2}}\right]$$

$U = 281\ m/s$ ← U

Note that all fuel would be expended at $t_{bo} = \frac{M_f}{\dot{M}} = 1000\,kg \times \frac{s}{75\,kg}$

i.e. at $t = 13.3\,s$

The sled speed as a function of time is then

$$U = \frac{V_e \dot{M}}{k}\left[1 - \left(1 - \frac{\dot{M}t}{M_0}\right)^{k/\dot{M}}\right] \quad \text{for } 0 \le t \le 13.3\,s$$

The speed reaches a maximum at $t = 13.3\,s$ and decays with time due to the motion resistance. $U_{max} = 375\,m/s$

The sled acceleration is given by

$$\frac{dU}{dt} = \frac{V_e \dot{M} - kU}{M_0 - \dot{M}t} \quad \text{for } 0 \le t \le 11.1\,s$$

At $t \ge 13.3\,s$ $V_e = 0$ and

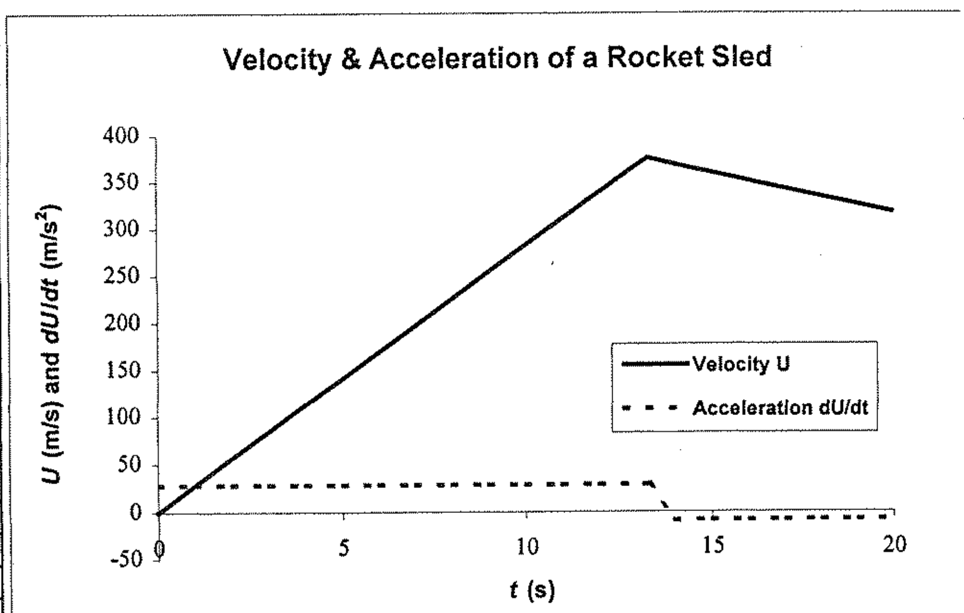
$$\frac{dU}{dt} = \frac{-kU}{M_0 - M_{fuel}}$$

Note that for $t > t_{bo} = 13.3\,s$, $\frac{dU}{dt} = -\frac{kU}{M_{bo}}$ and

$$\frac{dU}{U} = -\frac{k}{M_{bo}}dt \quad , \quad \ln\frac{U}{U_{bo}} = -\frac{k(t - t_{bo})}{M_{bo}}$$

and $U = U_{bo}\, e^{-k(t-t_{bo})/M_{bo}}$

t (s)	*U* (m/s)	*dU/dt* (m/s²)
0.0	0.0	28.1
1.0	28.1	28.1
2.0	56.3	28.1
3.0	84.4	28.1
4.0	113	28.1
5.0	141	28.1
6.0	169	28.1
7.0	197	28.1
8.0	225	28.1
9.0	253	28.1
10.0	281	28.1
11.0	309	28.1
12.0	338	28.1
13.2	371	28.1
13.3	375	28.1
14.0	369	-9.22
15.0	360	-8.99
16.0	351	-8.77
17.0	342	-8.55
18.0	334	-8.34
19.0	325	-8.14
20.0	317	-7.94



Problem 4.145

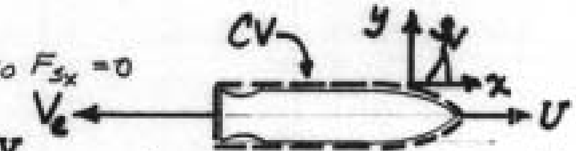
Given: Rocket launched from aircraft flying horizontally at $U_0 = 300$ m/s. Rocket accelerates to $U_f = 1.8$ km/s. Exhaust stream leaves nozzle at $V_e = 3000$ m/sec (relative to rocket) at atmospheric pressure. Neglect air resistance.

Find: (a) Algebraic expression for speed reached in horizontal flight.
(b) Minimum mass fraction needed to reach $U_f = 1.8$ km/s.

Solution: Apply x component of momentum using CV & CS shown.

Basic equation: $$F_{S_x} + F_{B_x} - \int_{CV} a_{rf_x}\rho\, d\forall = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\rho\, d\forall + \int_{CS} u_{xyz}\rho \vec{V}\cdot d\vec{A}$$

Assumptions: (1) No drag; $p_e = p_{atm}$ so $F_{S_x} = 0$
(2) Horiz; $F_{B_x} = 0$
(3) Neglect $\partial/\partial t$ in CV
(4) Constant mass flow rate; $\dot{m} = $ const; $M(t) = M_0 - \dot{m}t$
(5) Uniform, axial flow at nozzle exit



Then
$$-\int a_{rf_x}\rho\, d\forall = -\frac{dU}{dt}M(t) = u_e\{+\dot{m}\} = -V_e\dot{m}$$
so
$$u_e = -V_e$$
$$\frac{dU}{dt} = \frac{V_e\dot{m}}{M(t)}; \quad dU = \frac{V_e\dot{m}}{M_0 - \dot{m}t}dt = -V_e\frac{-\dot{m}dt}{M_0-\dot{m}t} = -V_e\frac{d(M_0-\dot{m}t)}{M_0-\dot{m}t}$$

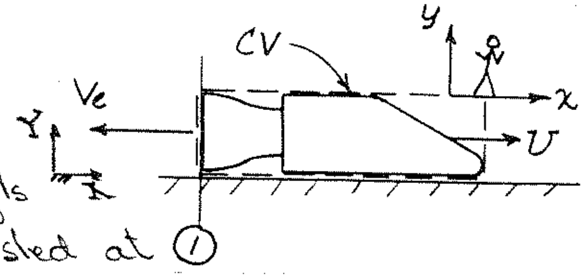
Integrating from U_0 at $t=0$ to U at t,
$$U - U_0 = -V_e \ln(M_0 - \dot{m}t)\Big]_0^t = -V_e\left[\ln(M_0-\dot{m}t) - \ln(M_0)\right] = -V_e\ln\left(\frac{M_0-\dot{m}t}{M_0}\right)$$
or
$$U = U_0 + V_e\ln\left(\frac{M_0}{M_0-\dot{m}t}\right)$$ U(t)

Solving,
$$\frac{M_0-\dot{m}t}{M_0} = e^{-\frac{U-U_0}{V_e}} = 1 - \frac{\dot{m}t}{M_0}; \quad \frac{\dot{m}t}{M_0} = 1 - e^{-\frac{U-U_0}{V_e}} = \text{mass fraction consumed}$$

Substituting,
$$\frac{\dot{m}t}{M_0} = 1 - e^{-(1800-300)\frac{m}{s}\times\frac{s}{3000m}} = 1 - e^{-0.5} = 0.393$$ Mass Fraction

{The mass fraction calculated here is a minimum because neither air resistance nor drag due to lift were included.}

Given: Rocket sled moving on level track without resistance
Inital mass, $M_0 = 3000$ kg
(includes $M_{fuel} = 1000$ kg)
$V_e = 2500$ m/s ; $p_e = p_{atm}$
Fuel consumption, $\dot{m} = 75$ kg/s



Find: Acceleration and speed of sled at $t = 10$ s.

Plot: sled speed and acceleration as functions of time.

Solution:

Apply x component of momentum to linearly accelerating CV; use continuity to find $M(t)$

Basic equations: $0 = \frac{\partial}{\partial t}\int_{CV} \rho \, dV + \int_{CS} \rho \vec{V}_{xyz} \cdot d\vec{A}$ ($\approx 0(3)$)

$F_{S_x} + F_{B_x} - \int_{CV} a_{rf} \, \rho \, dV = \frac{\partial}{\partial t}\int_{CV} u_{xyz} \, \rho \, dV + \int_{CS} u_{xyz} \, (\rho \vec{V}_{xyz} \cdot d\vec{A})$ ($=0(1)$, $=0(2)$, $\approx 0(3)$)

Assumptions: (1) $F_{S_x} = 0$, no resistance (given)
(2) $F_{B_x} = 0$, horizontal
(3) neglect $\partial/\partial t$ inside CV
(4) uniform flow at nozzle exit
(5) $p_e = p_{atm}$ (given)

From continuity, $0 = \frac{\partial M}{\partial t} + \{+|\dot{m}|\} = \frac{dM}{dt} + \dot{m}$ or $dM = -\dot{m}\,dt$

Integrating, $\int_{M_0}^{M} dM = M - M_0 = \int_0^t -\dot{m}\,dt = -\dot{m}t$ or $M = M_0 - \dot{m}t$

From the momentum equation

$$-a_{rf_x} M = -a_{rf_x}(M_0 - \dot{m}t) = u_1\{+|\dot{m}|\} = -V_e \dot{m} \qquad \{u_1 = -V_e\}$$

Thus

$$a_{rf_x} = \frac{dU}{dt} = \frac{V_e \dot{m}}{(M_0 - \dot{m}t)} \qquad (1)$$

At $t = 10$ s

$$\frac{dU}{dt} = 2500\frac{m}{s} \times 75\frac{kg}{s} \times \frac{1}{3000\,kg - 75\frac{kg}{s} \times 10\,s} = 83.3\ m/s^2 \longleftarrow a_{rf_x}$$

From Eq. 1, $dU = V_e \frac{\dot{m}\,dt}{M_0 - \dot{m}t}$

Integrating from $U = 0$ at $t = 0$ to U at t gives

$$U = -V_e \ln(M_0 - \dot{m}t)\Big]_0^t = -V_e \ln\frac{(M_0 - \dot{m}t)}{M_0}$$

$$U = V_e \ln\frac{M_0}{(M_0 - \dot{m}t)} \qquad (2)$$

At $t = 10\,s$

$$U = 2500\frac{m}{s}\ln\frac{3000\,kg}{3000\,kg - 75\frac{kg}{s}\times 10\,s} = 719\ m/s \qquad U.$$

Note that all fuel would be expended at $t_{bo} = \frac{M_f}{\dot{m}} = 1000\,kg \cdot \frac{s}{75\,kg}$

i.e. at $t_{b,o} = 13.3\,s$.

The sled speed as a function of time is then

$$U = V_e \ln\frac{M_0}{(M_0 - \dot{m}t)} \quad \text{for } t \le 13.3\,s$$

$$U = U_{max} = 1010\ m/s \quad \text{for } t \ge 13.3\,s$$

The sled acceleration is given by

$$\frac{dU}{dt} = \frac{\dot{m}V_e}{(M_0 - \dot{m}t)} \quad \text{for } 0 \le t \le 13.3\,s$$

$$\frac{dU}{dt} = 0 \quad \text{for } t \ge 13.3\,s$$

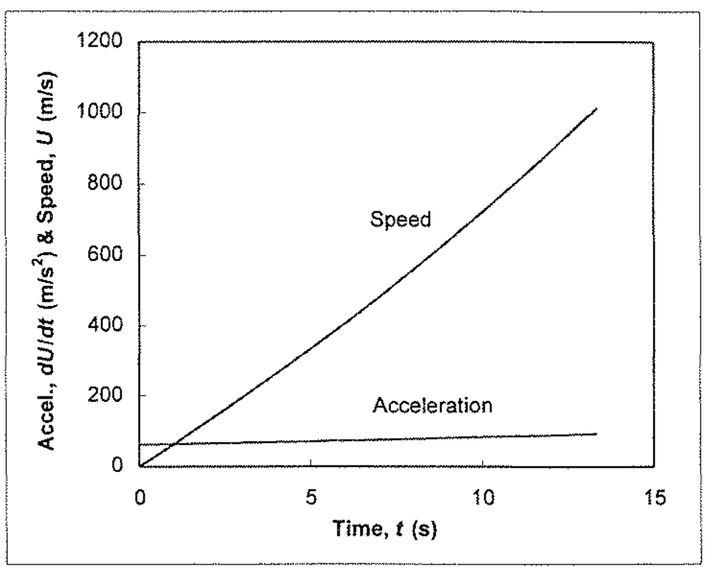
Acceleration and Speed vs. Time for Rocket Sled:

Input Data:

M_0 =	3000	kg
m(dot) =	75	kg/s
V_e =	2500	m/s

Calculated Results:

Time, t (s)	Acceleration, dU/dt (m/s^2)	Speed, U (m/s)
0	62.5	0
1	64.1	63.3
2	65.8	128
3	67.6	195
4	69.4	263
5	71.4	334
6	73.5	406
7	75.8	481
8	78.1	558
9	80.6	637
10	83.3	719
11	86.2	804
12	89.3	892
13	92.6	983
13.33	93.8	1014



Problem 4.147

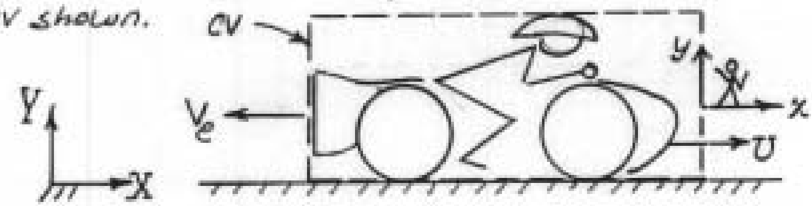
Given: Rocket-propelled motorcycle, to jump, standing start, level.

Speed needed $U_j = 875$ km/hr Rocket exhaust speed $V_e = 2510$ m/s

Total mass $M_B = 375$ kg (without fuel)

Find: Minimum fuel mass needed to reach V_j.

<u>Solution</u>: Apply x-component of momentum equation to linearly accelerating CV shown.



From continuity,

$$M_{cv} = M_0 - \dot m t$$

Basic equation:

$$\overset{\approx 0(1)}{F_{S_x}} + \overset{=0(2)}{F_{B_x}} - \int_{CV} a_{rf_x}\rho\, d\forall = \overset{\approx 0(3)}{\frac{\partial}{\partial t}}\int_{CV} u_{xyz}\rho\, d\forall + \int_{CS} u_{xyz}\rho \vec V_{xyz}\cdot d\vec A$$

Assumptions: (1) Neglect air and rolling resistance

(2) Level track, so $F_{B_x} = 0$

(3) Neglect unsteady effects within CV

(4) Uniform flow at nozzle exit plane

(5) $p_e = p_{atm}$

Then

$$-a_{rf_x} M_{cv} = u_e\{+\dot m\} = -V_e \dot m \quad \text{or} \quad \frac{dU}{dt} = \frac{V_e \dot m}{M_{cv}} = \frac{V_e \dot m}{M_0 - \dot m t}$$

$$u_e = -V_e$$

Separating variables and integrating,

$$dU = -V_e\left(\frac{-\dot m dt}{M_0 - \dot m t}\right) \quad \text{or} \quad U_j = -V_e \ln(M_0 - \dot m t)\Big|_0^t = V_e \ln\left(\frac{M_0}{M_0 - \dot m t}\right)$$

But $M_0 = M_B + M_F$ and $M_F = \dot m t$, so

$$\frac{U_j}{V_e} = \ln\left(\frac{M_B + M_F}{M_B}\right) = \ln\left(1 + \frac{M_F}{M_B}\right); \quad 1 + \frac{M_F}{M_B} = e^{U_j/V_e}; \quad \frac{M_F}{M_B} = e^{U_j/V_e} - 1$$

Finally, $M_F = M_B\left(e^{U_j/V_e} - 1\right)$

$$M_F = 375\ \text{kg} \times \exp\left[875\,\frac{\text{km}}{\text{hr}} \times \frac{\text{s}}{2510\ \text{m}} \times 1000\,\frac{\text{m}}{\text{km}} \times \frac{\text{hr}}{3600\ \text{s}} - 1\right]$$

$$M_F = 38.1\ \text{kg}$$ M_F

The fuel mass required is about 10 percent of the mass of the motorcycle and rider.

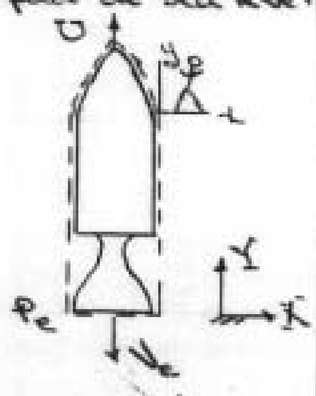
Problem 4.148

Given: Liquid-fueled rocket launched from pad at sea level

$M_0 = 30{,}000\ \text{kg}$ $\quad \dot{m} = 2450\ \text{kg/s}$

$V_e = 2270\ \text{m/s}$ $\quad p_e = 166\ \text{kPa (abs)}$

Exit plane diameter, $D_e = 2.6\ \text{m}$



Find: acceleration at lift-off.
expression for rocket speed, $U(t)$

Solution: Apply y component of momentum equation to CV with linear acceleration

Basic equation:
$$F_{S_y} + F_{B_y} - \int_{CV} a_{rf_y}\,\rho\, dV = \frac{\partial}{\partial t}\int_{CV} v_{xyz}\,\rho\, dV + \int_{CS} v_{xyz}\,\rho \vec{V}_{xyz}\cdot d\vec{A}$$
(term $\frac{\partial}{\partial t}\int_{CV}$ crossed out, $=0(2)$)

Assumptions: (1) F_{S_y} due to pressure, p_{atm} assumed constant, neglect air resistance
(2) neglect rate of change of momentum inside cv
(3) uniform flow at exit

Then,
$$(-p_e - p_{atm})A_e - Mg - a_{rf_y} M = v_e\{+\dot{m}\} = -\dot{m}V_e$$

Solving for a_{rf_y},
$$a_{rf_y} = \frac{dU}{dt} = \frac{1}{M}\left[\dot{m}V_e + (p_e - p_{atm})A_e\right] - g \quad \ldots(1)$$

$M = M(t)$. From conservation of mass $\frac{\partial}{\partial t}\int_{CV}\rho\, dV + \int_{CS}\rho\vec{V}\cdot d\vec{A} = 0$

Then $\frac{\partial}{\partial t}\int_{CV}\rho\, dV = \frac{dM}{dt} = -\int_{CS}\rho\vec{V}\cdot d\vec{A} = -\dot{m}_e$ (constant)

Hence $M(t) = M_0 - \dot{m}t$, and

$$a_{rf_y} = \frac{dU}{dt} = \frac{\dot{m}V_e}{M_0 - \dot{m}t} + \frac{(p_e - p_{atm})A_e}{M_0 - \dot{m}t} - g$$

$$U = \int_0^U dU = \int_0^t \frac{\dot{m}V_e}{M_0 - \dot{m}t}\,dt + \int_0^t \frac{(p_e - p_{atm})A_e}{M_0 - \dot{m}t}\,dt - \int_0^t g\,dt$$

$$U = -V_e \ln\left[\frac{M_0 - \dot{m}t}{M_0}\right] - \frac{(p_e - p_{atm})A_e}{\dot{m}}\ln\left[\frac{M_0 - \dot{m}t}{M_0}\right] - gt$$

$$U = -\left[V_e + \frac{(p_e - p_{atm})A_e}{\dot{m}}\right]\ln\left[\frac{M_0 - \dot{m}t}{M_0}\right] - gt \longleftarrow U(t)$$

At lift-off, $t = 0$, $M = M_0$

$$a_{rf_y} = \frac{1}{M}\left[\dot{m}V_e + (p_e - p_{atm})A_e\right] - g$$

$$= \frac{1}{3\times10^4\ \text{kg}}\left[2450\frac{\text{kg}}{\text{s}}\times 2270\frac{\text{m}}{\text{s}} + (166-101)10^3\frac{\text{N}}{\text{m}^2}\times\frac{\pi}{4}(2.6)^2\text{m}^2\times\frac{\text{kg}\cdot\text{m}}{\text{N}\cdot\text{s}^2}\right] - 9.81\frac{\text{m}}{\text{s}^2}$$

$$a_{rf_y} = 169\ \text{m/s}^2 \longleftarrow a_{rf_y}$$

Given: Home made rocket launched vertically from rest.

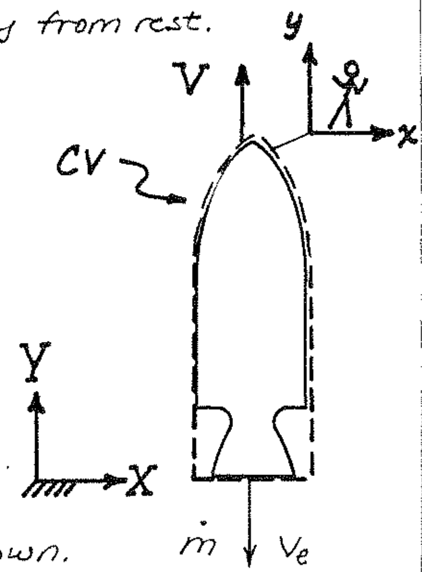
$M_0 = 20$ lbm, of which 15 lbm is fuel

$\dot{m} = 0.5$ lbm/s

$V_e = 6500$ ft/s (relative to rocket)

$p_e = p_{atm}$

Neglect aerodynamic drag.



Find: (a) Speed at $t = 20$ s. (b) Height at $t = 20$ s.

Plot: Speed and height as functions of time.

<u>Solution:</u> Apply y-component of momentum equation to accelerating CV using CS shown.

Basic equation:

$$\cancel{F_{S_y}}^{=0(1)} + F_{B_y} - \int_{CV} a_{rf_y}\rho\, d\forall = \cancel{\frac{\partial}{\partial t}}^{\approx 0(2)}\int_{CV} v_{xyz}\rho\, d\forall + \int_{CS} v_{xyz}\rho \vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: (1) Neglect air resistance; $p_e = p_{atm}$ (given)
(2) Neglect v_{xyz} and $\partial/\partial t$ within CV
(3) Uniform flow at nozzle exit section

Then

$$F_{B_y} - a_{rf_y}M = -Mg - Ma_{rf_y} = v_e\{+\dot{m}\} = -V_e\dot{m}$$

and $\qquad v_e = -V_e$

$$a_{rf_y} = \frac{dV}{dt} = \frac{V_e\dot{m}}{M} - g$$

Introducing $M = M_0 - \dot{m}t$ and separating variables,

$$dV = \left(\frac{V_e\dot{m}}{M_0 - \dot{m}t} - g\right)dt$$

Integrating from rest at $t = 0$

$$V = \int_0^t \left(\frac{V_e\dot{m}}{M_0 - \dot{m}t} - g\right)dt = -V_e \ln(M_0 - \dot{m}t)\Big]_0^t - gt$$

or

$$V = V_e \ln\left(\frac{M_0}{M_0 - \dot{m}t}\right) - gt \qquad (1)$$

At $t = 20$ sec,

$$V = 6500\,\frac{\text{ft}}{\text{s}} \ln\left(\frac{20 \text{ lbm}}{20 \text{ lbm} - 0.5\,\frac{\text{lbm}}{\text{s}} \times 20\text{ s}}\right) - 32.2\,\frac{\text{ft}}{\text{s}^2} \times 20\text{ s}$$

$$V(20\text{ s}) = 3{,}860 \text{ ft/s}$$ V_{20}

To find height, note $V = \frac{dY}{dt}$. Substitute into Eq. 1 to obtain

$$\frac{dY}{dt} = V_e \ln\left(\frac{M_0}{M_0 - \dot{m}t}\right) - gt = -V_e \ln\left(1 - \frac{\dot{m}t}{M_0}\right) - gt$$

Let $r = 1 - \frac{\dot{m}t}{M_0}$, and $dr = -\frac{\dot{m}}{M_0}dt$, then

$$dY = -V_e \ln r\, dt - gt\, dt = +\frac{V_e M_0}{\dot{m}} \ln r\, dr - gt\, dt$$

Integrating from $Y=0$ at $t=0$,

$$Y = \int_0^t \frac{V_e M_0}{\dot{m}} \ln r\, dr - \frac{1}{2}gt^2 = \frac{V_e M_0}{\dot{m}}\Big[r \ln r - r\Big]_0^t - \frac{1}{2}gt^2$$

$$= \frac{V_e M_0}{\dot{m}}\left\{\left(1 - \frac{\dot{m}t}{M_0}\right)\left[\ln\left(1 - \frac{\dot{m}t}{M_0}\right) - 1\right]\right\}\Bigg|_0^t - \frac{1}{2}gt^2$$

$$Y = \frac{V_e M_0}{\dot{m}}\left\{\left(1 - \frac{\dot{m}t}{M_0}\right)\left[\ln\left(1 - \frac{\dot{m}t}{M_0}\right) - 1\right] + 1\right\} - \frac{1}{2}gt^2$$

At $t = 20$ s,

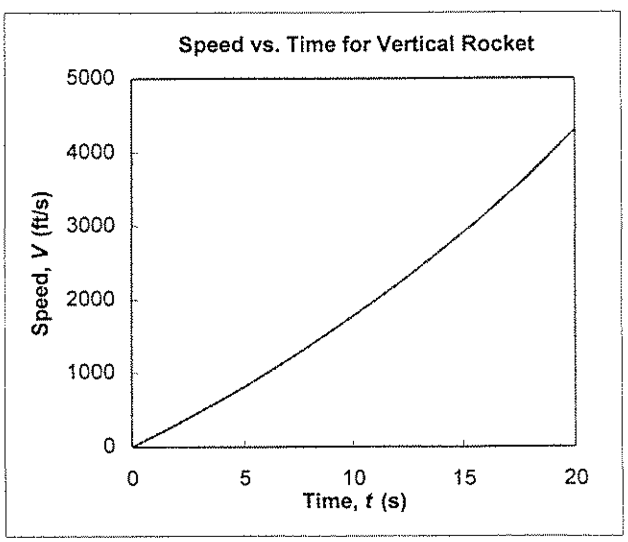
$$1 - \frac{\dot{m}t}{M_0} = 1 - 0.5\,\frac{\text{lbm}}{\text{s}} \times 20\,\text{s} \times \frac{1}{20\,\text{lbm}} = \frac{1}{2}$$

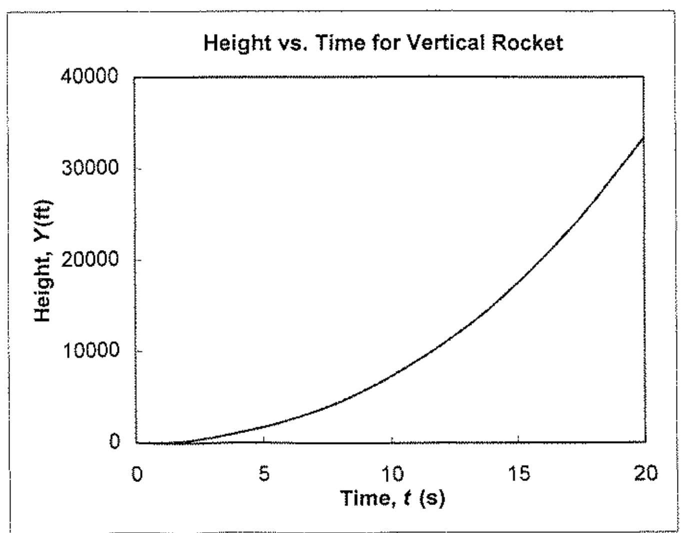
so

$$Y = 6500\,\frac{\text{ft}}{\text{s}} \times 20\,\text{lbm} \times \frac{\text{s}}{0.5\,\text{lbm}}\left\{\left(\frac{1}{2}\right)\left[\ln\left(\frac{1}{2}\right) - 1\right] + 1\right\} - \frac{1}{2} \times 32.2\,\frac{\text{ft}}{\text{s}^2}\,(20)^2\text{s}^2$$

$$Y = 33{,}500\ \text{ft}$$

Y



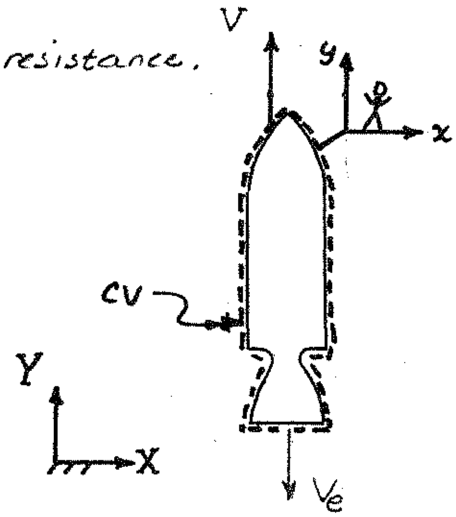


Given: Rocket fired vertically from rest, no resistance.

$M_0 = 200\ kg$, $\dot{m} = 10\ kg/sec$
$V_e = 2900\ m/s$, $p_e = p_{atm}$

Find: Speed at $t = 10\ s$.

Plot: Rocket speed as a function of time.



Solution: Apply y component momentum equation to accelerating CV.

Basic equation:

$$\overset{=0(1),(2)}{F_{S_y}} + F_{B_y} - \int_{CV} a_{rf_y} \rho \, d\forall = \overset{\simeq 0(3)}{\frac{\partial}{\partial t}} \int_{CV} v_{xyz} \rho \, d\forall + \int_{CS} v_{xyz} \rho \vec{V}_{xyz} \cdot d\vec{A}$$

Assumptions:
(1) Neglect air resistance
(2) Exit pressure is atmospheric (given)
(3) Neglect rate of change of v_{xyz} within CV
(4) Flow is uniform at exit section
(5) All velocities are relative to CV
(6) $M = M_0 - \dot{m}t$

Then

$$F_{B_y} - a_{rf_y} M = -Mg - a_{rf_y} M = v_e \{|\dot{m}|\} = -V_e \dot{m}, \text{ since } v_e = -V_e.$$

or
$$a_{rf_y} = \frac{dV}{dt} = \frac{V_e \dot{m}}{M} - g \qquad (1)$$

Introducing $M = M_0 - \dot{m}t$ and separating variables

$$dV = \left(\frac{V_e \dot{m}}{M_0 - \dot{m}t} - g\right) dt$$

Integrating,

$$\int_0^V dV = V = \int_0^t \left(\frac{V_e \dot{m}}{M_0 - \dot{m}t} - g\right) dt = -V_e \ln(M_0 - \dot{m}t)\Big]_0^t - gt$$

or
$$V = V_e \ln\left(\frac{M_0}{M_0 - \dot{m}t}\right) - gt \qquad (2)$$

Substituting at $t = 10\ s$,

$$V = 2900 \frac{m}{s} \times \ln\left(\frac{200\ kg}{200\ kg - 10 \frac{kg}{s} \times 10\ s}\right) - 9.81 \frac{m}{s^2} \times 10\ s$$

$$V = 1910\ m/s$$ V

Speed as a function of time is given by Eq. 2; acceleration is given by Eq. 1.

The results are tabulated and plotted on the next page.

over →

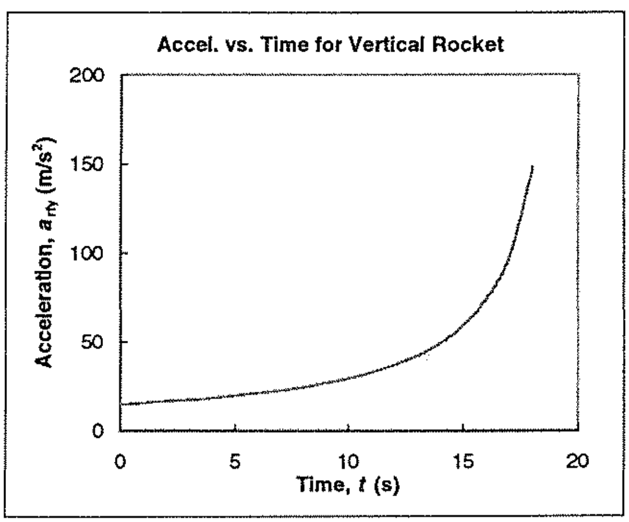
Acceleration and Speed as Functions of Time for Vertical Rocket:

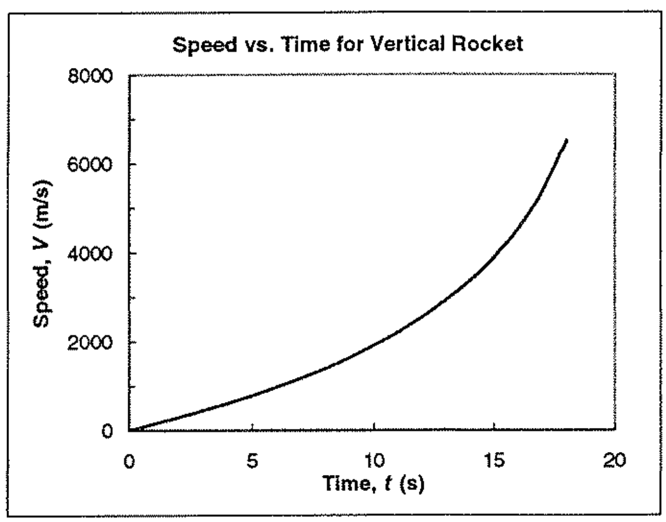
Input Data:

m(dot) =	10	kg/s	Mass flow rate
M_0 =	200	kg	Initial mass
V_e =	2900	m/s	Exhaust gas speed

Calculated Results:

Time, t (s)	Mass, M (kg)	Mass Ratio, M/M_0 (---)	Acceleration, a_{rfy} (m/s²)	Acceleration, a_{rfy} (gs)	Speed, U (m/s)
0	200	1	145	14.8	0.0
1	190	0.950	153	15.6	139
2	180	0.900	161	16.4	286
3	170	0.850	171	17.4	442
4	160	0.800	181	18.5	608
5	150	0.750	193	19.7	785
6	140	0.700	207	21.1	975
7	130	0.650	223	22.7	1181
8	120	0.600	242	24.6	1403
9	110	0.550	264	26.9	1645
10	100	0.500	290	29.6	1912
11	90	0.450	322	32.8	2208
12	80	0.400	363	37.0	2540
13	70	0.350	414	42.2	2917
14	60	0.300	483	49.3	3354
15	50	0.250	580	59.1	3873
16	40	0.200	725	73.9	4510
17	30	0.150	967	98.5	5335
18	20	0.100	1450	148	6501





Problem 4.151

Open-Ended Problem Statement: Inflate a toy balloon with air and release it. Watch as the balloon darts about the room. Explain what causes the phenomena you see.

Discussion: Air blown into a balloon to inflate it must be compressed to overcome the skin's resistance to stretching. (Remember how hard it is to create enough pressure to "start" the inflation process!) After decreasing briefly, the required pressure seems to increase as inflation of the balloon continues.

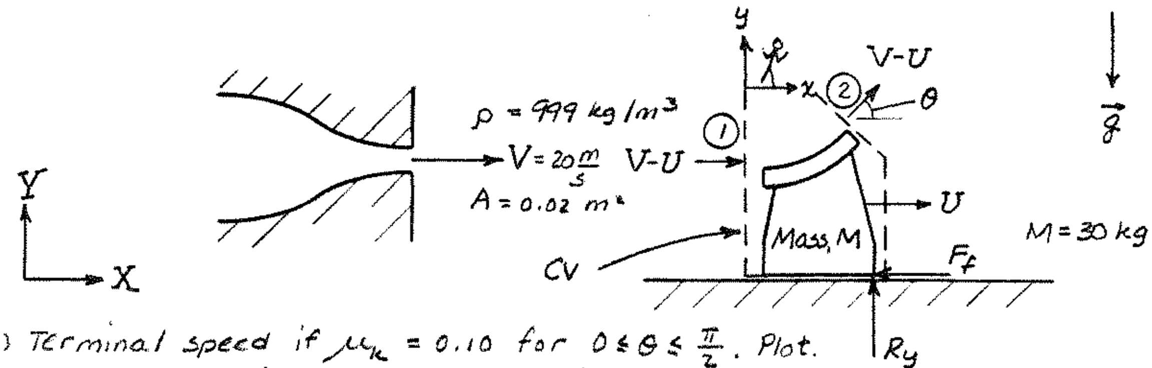
As the balloon is inflated, the skin stretches and stores energy. When the inflated balloon is released, the stored energy in the skin forces the compressed air out the open mouth of the balloon. The expansion of the compressed air to the lower surrounding atmospheric pressure creates a high-speed jet of air, which propels the relatively light balloon initially at a high speed.

The moving balloon is unstable because it has a poor aerodynamic shape. Therefore it darts about in a random pattern. The balloon keeps moving as long as it contains pressurized air to act as a propulsion jet. However, it is not long before the energy stored in the skin is exhausted and the air in the balloon is reduced to atmospheric pressure.

When the balloon reaches atmospheric pressure it is slowed by aerodynamic drag. Finally the empty, wrinkled balloon simply falls to the floor.

Some toys that use a balloon for propulsion are available. Most have stabilizing surfaces. It is instructive to study these toys carefully to understand how each works, and why each toy is shaped the way it is.

Given: Vane/cart assembly moving with friction under the influence of a jet, as shown.



Find: (a) Terminal speed if $\mu_k = 0.10$ for $0 \le \theta \le \frac{\pi}{2}$. Plot.
(b) Angle at which motion begins if $\mu_s = 0.15$.

Solution: From dynamics, $F_f \le \mu R_y$, so apply two components of momentum equation. Use the linearly accelerating CV shown.

Basic equations:

$$F_{S_x} + \cancelto{=0(3)}{F_{B_x}} - \int_{CV} a_{rf_x}\,\rho\, d\forall = \cancelto{\approx 0(2)}{\frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\, d\forall} + \int_{CS} u_{xyz}\,\rho \vec{V}_{xyz}\cdot d\vec{A}$$

$$F_{S_y} + F_{B_y} - \int_{CV} \cancelto{=0(3)}{a_{rf_y}}\,\rho\, d\forall = \cancelto{\approx 0(2)}{\frac{\partial}{\partial t}\int_{CV} v_{xyz}\,\rho\, d\forall} + \int_{CS} v_{xyz}\,\rho \vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: (1) No net pressure forces on assembly; $F_{S_x} = -F_f$, $F_{S_y} = R_y$
(2) Neglect mass of liquid on vane
(3) $a_{rf_y} = 0$; $F_{B_x} = 0$
(4) Uniform flow at each section
(5) No change in jet area or speed relative to vane
(6) Incompressible flow

The subscript xyz reminds us to use relative velocities. Then

$$-F_f - M a_{rf_x} = u_1\{-|\rho(V-U)A|\} + u_2\{|\rho(V-U)A|\}$$

$$u_1 = V-U \qquad u_2 = (V-U)\cos\theta$$

$$a_{rf_x} = \frac{\rho(V-U)^2A(1-\cos\theta) - F_f}{M} \qquad (1)$$

and

$$R_y - Mg = v_1\{-|\rho(V-U)A|\} + v_2\{|\rho(V-U)A|\}$$

$$v_1 = 0 \qquad v_2 = (V-U)\sin\theta$$

$$R_y = Mg + \rho(V-U)^2A\sin\theta \qquad (2)$$

At terminal speed, $a_{rf_x} = 0$ and $F_f = \mu_k R_y$. Substituting into Eq. 1,

$$0 = \frac{\rho(V-U_t)^2A(1-\cos\theta) - \mu_k[Mg + \rho(V-U_t)^2A\sin\theta]}{M} = \frac{\rho(V-U_t)^2A(1-\cos\theta - \mu_k\sin\theta)}{M} - \mu_k g$$

or

$$V - U_t = \left[\frac{\mu_k M g}{\rho A(1-\cos\theta - \mu_k\sin\theta)}\right]^{\frac{1}{2}}$$

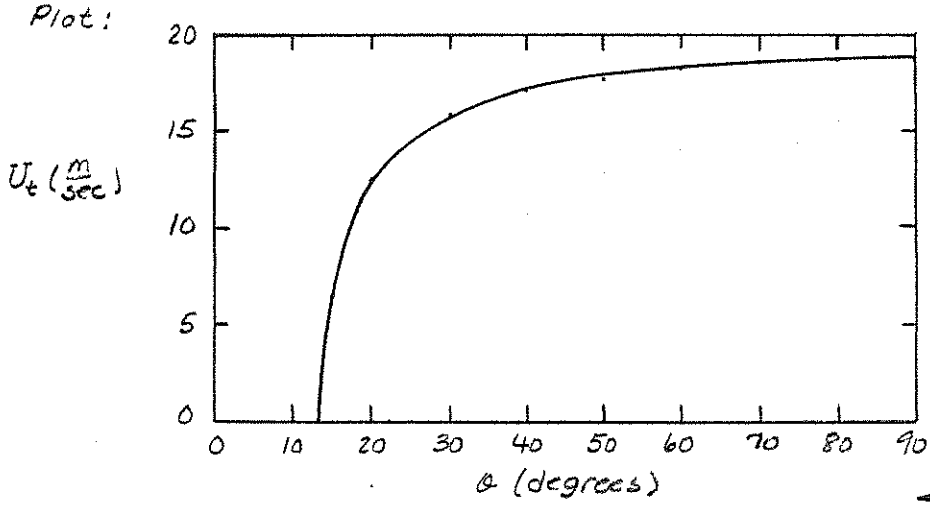
or $U_t = V - \left[\frac{\mu_k M g}{\rho A(1-\cos\theta - \mu_k \sin\theta)}\right]^{2}$; $\frac{U_t}{V} = 1 - \left[\frac{\mu_k M g}{\rho V^2 A(1-\cos\theta - \mu_k \sin\theta)}\right]^{\frac{1}{2}}$ $\frac{U_t}{V}$

Substituting values,

$$U_t \left\{1 - \left[(0.10)30\,kg \times 9.81\frac{m}{s^2} \times \frac{m^3}{999\,kg} \times \frac{s^2}{(20)^2 m^2} \times \frac{1}{0.02\,m^2} \frac{1}{(1-\cos\theta - 0.1\sin\theta)}\right]^{\frac{1}{2}}\right\} 20\frac{m}{s}$$

$$U_t = 20 - 1.21\left[\frac{1}{1-\cos\theta - 0.1\sin\theta}\right]^{\frac{1}{2}} \frac{m}{s}$$ U_t

Plot:



Plot

For the static case, $F_f \le \mu_s R_y$. Substituting into Eq. 1, with $U = 0$

$$a_{rf_x} \ge \frac{\rho V^2 A(1-\cos\theta) - \mu_s[\rho V^2 A \sin\theta + Mg]}{M} = \frac{\rho V^2 A(1-\cos\theta - \mu_s \sin\theta)}{M} - g$$

When the assembly is about to move, $F_f = \mu_s R_y$ and $a_{rf_x} = 0$. Thus

$$a_{rf_x} = 0 = \frac{\rho V^2 A(1-\cos\theta - \mu_s \sin\theta)}{M} - \mu_s g$$

or

$$\cos\theta + \mu_s \sin\theta = 1 - \frac{\mu_s M g}{\rho V^2 A} \; ; \; \cos\theta[1 + \mu_s \tan\theta] = 1 - \frac{\mu_s M g}{\rho V^2 A}$$

But $\frac{\mu_s M g}{\rho V^2 A} = (0.15)30\,kg \times 9.81\frac{m}{s^2} \times \frac{m^3}{999\,kg} \times \frac{s^2}{(20)^2 m^2} \times \frac{1}{0.02\,m^2} = 0.00552$

Thus

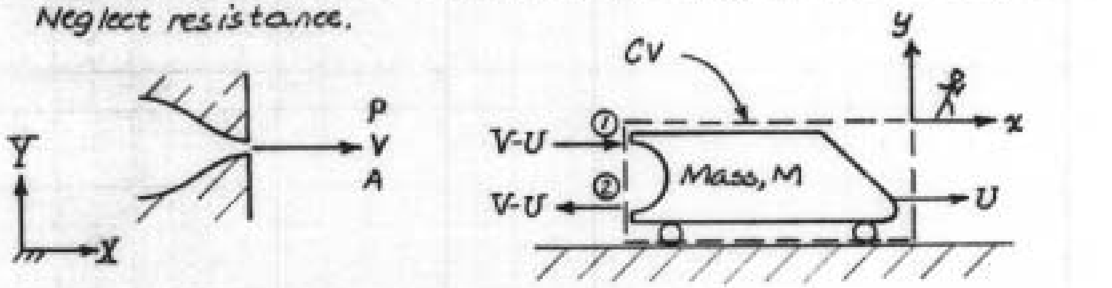
$$\cos\theta[1 + \mu_s \tan\theta] = 1 - 0.00552$$

Solving by iteration, $\theta \simeq 18.9°$ (to start motion) θ

{ Thus the assembly would have hysteresis. If θ were varied, it would start moving at $\theta \approx 18.9°$. Once motion began, it would continue, as θ decreased, to $\theta \approx 13°$. The reason is because $\mu_k < \mu_s$. }

Problem 4.153

Given: Vehicle accelerated from rest by a hydraulic catapult. Neglect resistance.



Find: (a) Expression for acceleration at any time, t.
(b) Time required to reach $U = V/2$.

Solution: Apply x component of momentum equation using linearly accelerating CV shown above.

Basic equation: $F_{S_x} + F_{B_x} - \int_{CV} a_{rf_x}\,\rho\, d\forall = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\, d\forall + \int_{CS} u_{xyz}\,\rho \vec{V}_{xyz}\cdot d\vec{A}$

($=0(1)$, $=0(2)$, $\approx 0(3)$)

Assumptions: (1) $F_{S_x} = 0$
(2) $F_{B_x} = 0$
(3) Neglect mass of liquid and rate of change of u in CV
(4) Uniform flow at each section
(5) Jet area and speed with respect to vehicle are constant

Then

$$-Ma_{rf_x} = -M\frac{dU}{dt} = u_1\{-|\rho(V-U)A|\} + u_2\{|\rho(V-U)A|\}$$

$$u_1 = V-U \qquad u_2 = -(V-U)$$

or

$$a_{rf_x} = \frac{dU}{dt} = \frac{2\rho(V-U)^2A}{M} \quad ; \quad \frac{dU}{(V-U)^2} = \frac{2\rho A}{M}dt \quad ; -\frac{d(V-U)}{(V-U)^2} = \frac{2\rho A}{M}dt$$

To obtain $a_{rf_x}(t)$, we must first find $U(t)$. Integrating from $U=0$ at $t=0$ to U at t,

$$\int_{V-U=V}^{V-U} -\frac{d(V-U)}{(V-U)^2} = \left.\frac{1}{V-U}\right]_V^{V-U} = \frac{1}{V-U} - \frac{1}{V} = \frac{V-(V-U)}{V(V-U)} = \frac{2\rho A}{M}t \;;\; \frac{U}{V-U} = \frac{2\rho VA}{M}t$$

Solving,

$$U = (V-U)\frac{2\rho VA}{M}t \;,\quad U = V\frac{\frac{2\rho VA}{M}t}{1+\frac{2\rho VA}{M}t} \quad \text{and} \quad V-U = V\left[1-\frac{\frac{2\rho VA}{M}t}{1+\frac{2\rho VA}{M}t}\right]$$

Substituting,

$$a_{rf_x} = \frac{2\rho V^2A}{M}\left[1-\frac{\frac{2\rho VA}{M}t}{1+\frac{2\rho VA}{M}t}\right]^2 = \frac{2\rho V^2A}{M}\left[\frac{1}{1+\frac{2\rho VA}{M}t}\right]^2$$ $a_{rf_x}(t)$

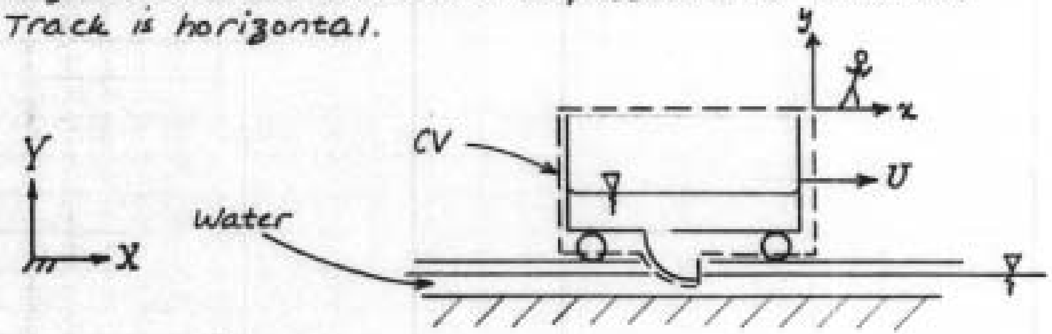
The time to reach $U = V/2$ is

$$\frac{U}{V} = \frac{1}{2} = \frac{\frac{2\rho VA}{M}t}{1+\frac{2\rho VA}{M}t} \quad \text{or} \quad t = \frac{M}{2\rho VA}$$ $t(\frac{V}{2})$

Check: $\left[\frac{M}{\rho VA}\right] = M\frac{L^3}{M}\frac{t}{L}\frac{1}{L^2} = t$ ✓ ; $\left[\frac{\rho V^2A}{M}\right] = \frac{M}{L^3}\frac{L^2}{t^2}L^2\frac{1}{M} = \frac{L}{t^2}$ ✓

Problem 4.154

Given: Moving tank slowed by lowering scoop into water trough. Initial mass and speed are M_0 and U_0, respectively. Neglect external forces due to pressure or friction. Track is horizontal.



Find: (a) Apply continuity and momentum to show $U = U_0 M_0 / M$.
(b) Obtain a general expression for $U(t)$.

Solution: Apply continuity and momentum equations to linearly accelerating CV shown.

Basic equations: $0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}_{xyz} \cdot d\vec{A}$

$$F_{S_x} + F_{B_x} - \int_{CV} a_{rf_x} \rho\, d\forall = \frac{\partial}{\partial t}\int_{CV} u_{xyz} \rho\, d\forall + \int_{CS} u_{xyz} \rho \vec{V}_{xyz} \cdot d\vec{A}$$

($F_{S_x} = 0(1)$, $F_{B_x} = 0(2)$, $\frac{\partial}{\partial t}\int_{CV} u_{xyz}\rho\, d\forall \approx 0(3)$)

Assumptions: (1) $F_{S_x} = 0$
(2) $F_{B_x} = 0$
(3) Neglect u within CV
(4) Uniform flow across inlet section

From continuity

$$0 = \frac{\partial}{\partial t} M_{cv} + \{-|\rho U A|\} \quad \text{or} \quad \frac{dM}{dt} = \rho U A$$

From momentum

$$-a_{rf_x} M = -\frac{dU}{dt} M = u\{-|\rho U A|\} = U \rho U A, \text{ since } u = -U$$

But from continuity, $\rho U A = \frac{dM}{dt}$, so

$$M\frac{dU}{dt} + U\frac{dM}{dt} = 0 \quad \text{or} \quad UM = \text{constant} = U_0 M_0 \; ; \; U = U_0 M_0 / M$$ U

Substituting $M = M_0 U_0 / U$ into momentum, $-\frac{dU}{dt}\frac{M_0 U_0}{U} = \rho U^2 A$, or

$$\frac{dU}{U^3} = -\frac{\rho A}{U_0 M_0} dt$$

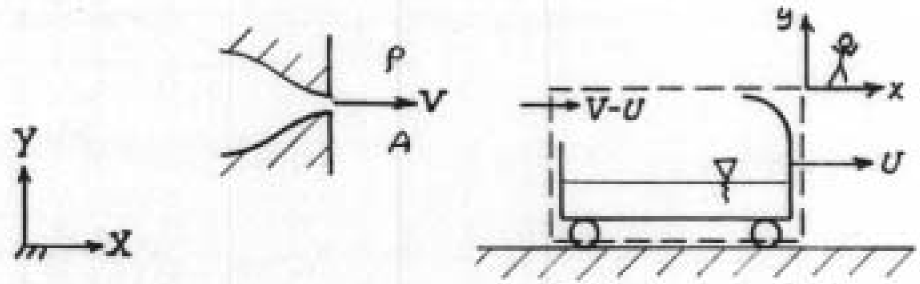
Integrating, $\int_{U_0}^{U} \frac{dU}{U^3} = -\frac{1}{2}\frac{1}{U^2}\Big]_{U_0}^{U} = -\frac{1}{2}\left(\frac{1}{U^2} - \frac{1}{U_0^2}\right) = -\int_0^t \frac{\rho A}{U_0 M_0} dt = -\frac{\rho A}{U_0 M_0} t$

Solving for U,

$$U = \frac{U_0}{\left[1 + \frac{2\rho U_0 A}{M_0} t\right]^{\frac{1}{2}}}$$ U(t)

Problem 4.155

Given: Tank driven by jet along horizontal track. Neglect resistance. Acceleration is from rest. Initial mass is M_0. Track horizontal.



Find: (a) Apply continuity and momentum to show $M = M_0V/(V-U)$
(b) General expression for U/V as a function of time.

<u>Solution</u>: Apply continuity and x component of momentum equation to linearly accelerating CV shown.

Basic equations: $0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V}_{xyz} \cdot d\vec{A}$

$$\overset{=0(1)}{F_{S_x}} + \overset{=0(2)}{F_{B_x}} - \int_{CV} a_{rf_x} \rho \, d\forall = \frac{\partial}{\partial t}\int_{CV} \overset{\simeq 0(3)}{u_{xyz}} \rho \, d\forall + \int_{CS} u_{xyz} \rho \vec{V}_{xyz} \cdot d\vec{A}$$

Assumptions: (1) $F_{S_x} = 0$
(2) $F_{B_x} = 0$
(3) Neglect u within CV
(4) Uniform flow in jet

From continuity

$$0 = \frac{\partial}{\partial t} M_{cv} + \{-|\rho(V-U)A|\} \quad \text{or} \quad \frac{dM}{dt} = \rho(V-U)A$$

From momentum

$$-a_{rf_x} M = -\frac{dU}{dt} M = u\{-|\rho(V-U)A|\} = (V-U)[-\rho(V-U)A]; \; u = V-U$$

But from continuity, $\rho(V-U)A = \frac{dM}{dt}$, and $dU = -d(V-U)$, so

$$-\frac{dU}{dt} M = \frac{d(V-U)}{dt} M = -(V-U)\frac{dM}{dt} \quad \text{or} \quad M(V-U) = \text{constant} = M_0 V$$

Thus $M = M_0V/(V-U)$ ← M

Substituting into momentum, $-\frac{dU}{dt} M = \frac{d(V-U)}{dt}\frac{M_0 V}{(V-U)} = -\rho(V-U)^2 A$, or

$$\frac{d(V-U)}{(V-U)^3} = -\frac{\rho A}{V M_0} dt$$

Integrating, $\int_V^{V-U} \frac{d(V-U)}{(V-U)^3} = -\frac{1}{2}\left[\frac{1}{(V-U)^2} - \frac{1}{V^2}\right] = -\int_0^t \frac{\rho A}{V M_0} dt = -\frac{\rho A}{V M_0} t$

Solving,

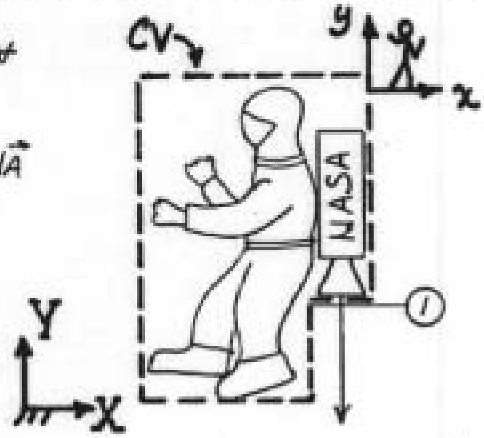
$$\frac{U}{V} = \left\{1 - \frac{1}{\left[1 + \frac{2\rho V A}{M_0} t\right]^{1/2}}\right\}$$ ← $\frac{U}{V}$

Problem 4.156

Given: Small rocket "jet pack" used to lift astronaut above Earth. Exhaust jet speed is constant but mass flow rate varies.

Find: (a) Algebraic expression for mass flow rate needed to hover.
(b) Maximum hover time.

Solution: Apply continuity and momentum using CV & CS shown.



Basic equation: $F_{S_y} + F_{B_y} - \int_{CV} a_{rf_y} \rho \, d\forall = \frac{\partial}{\partial t}\int_{CV} v_{xyz} \rho \, d\forall + \int_{CS} v_{xyz} \rho \vec{V} \cdot d\vec{A}$

($F_{S_y} = 0$ (1), $a_{rf_y} = 0$ (2), $\frac{\partial}{\partial t} \approx 0$ (3))

Assumptions: (1) Hover; $F_{S_y} = 0$
(2) $a_{rf_y} = 0$
(3) Neglect $\partial/\partial t$ in CV
(4) Uniform flow exhaust

$V_e = 2940$ m/s
$M_0 = 130$ kg
$M_f = 40$ kg

Then

$$-Mg = v_1 \{+\dot{m}\}$$

$$v_1 = -V_e$$

$$-Mg = -V_e \dot{m}$$

so

$$\dot{m} = \frac{Mg}{V_e}$$ $\dot{m}(t)$

From conservation of mass, $0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A} = \frac{dM}{dt} + \dot{m}$

so $\frac{dM}{dt} = -\dot{m} = -\frac{Mg}{V_e}$ or $\frac{dM}{M} = -\frac{g}{V_e} dt$

Integrating from M_0 at $t = 0$ to $M_0 - M_f$ at t,

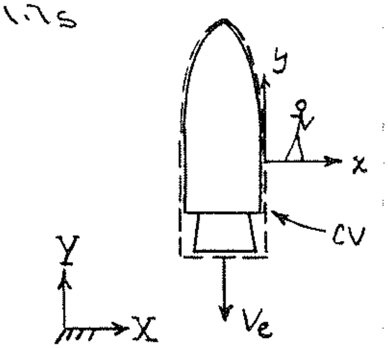
$$\int_{M_0}^{M_0 - M_f} \frac{dM}{M} = \ln M \Big]_{M_0}^{M_0 - M_f} = \ln\left(\frac{M_0 - M_f}{M_0}\right) = \ln\left(1 - \frac{M_f}{M_0}\right) = -\frac{gt}{V_e}$$

Solving for t,

$$t = -\frac{V_e}{g} \ln\left(1 - \frac{M_f}{M_0}\right) = -2940 \frac{m}{s} \times \frac{s^2}{9.81 \, m} \ln\left(1 - \frac{40 \, kg}{130 \, kg}\right)$$

$t = 110$ s (hover time) t

Given: Model solid propellant rocket: $M_0 = 69.6\,g$, $M_f = 12.5\,g$
Thrust, $F_t = 1.3\,lbf$; burn time, $t_b = 1.7\,s$



Find: Maximum speed and height (neglecting drag)

Plot: speed and distance traveled as functions of time

Solution

Apply the y momentum equation to analyze motion

Basic equation: $F_{S_y} + F_{B_y} - \int_{CV} a_{rf_y}\,\rho\, d\forall = \frac{\partial}{\partial t}\int_{CV} v_{xyz}\,\rho\, d\forall + \int_{CS} v_{xyz}\,\rho \vec{V}_{xyz}\cdot d\vec{A}$ (the $\frac{\partial}{\partial t}$ term is crossed out, marked ≈ 0 (2))

Assumptions: (1) neglect pressure forces and aerodynamic drag
(2) neglect rate of change of momentum inside CV
(3) uniform flow from CV

Thrust is produced by momentum flux from CV

$R_y = -F_t = v_e \dot{m}$, Since $v_e = -V_e$, then $-F_t = -V_e \dot{m}$

$$V_e = \frac{F_t}{\dot{m}} = 1.3\,lbf \times \frac{1.7\,s}{12.5\,g} \times \frac{4.448\,N}{lbf} \times \frac{kg\cdot m}{N\cdot s^2} \times \frac{10^3\,g}{kg} = 786\,m/s$$

From conservation of mass, $M = M_0 - \dot{m}t$. Then from momentum

$$-Mg - M a_{rf_y} = -V_e \dot{m} \quad \text{or} \quad a_{rf_y} = \frac{dV}{dt} = \frac{V_e \dot{m}}{M} - g$$

$$\frac{dV}{dt} = \frac{V_e \dot{m}}{M_0 - \dot{m}t} - g \quad \text{or} \quad dV = \left(\frac{V_e \dot{m}}{M_0 - \dot{m}t} - g\right)$$

At $t = t_b$, $\dot{m} = 0$. Max velocity occurs at $t = t_b = 1.7\,s$

Integrating between $V = 0$ at $t = 0$ and V at $t \le t_b$

$$V = V_e \ln\left[M_0/(M_0 - \dot{m}t)\right] - gt \quad \text{for } 0 \le t \le t_b \quad \text{....(1)}$$

Evaluating at $t = t_b = 1.7\,s$ with $\dot{m} = M_f/t_b = 7.35 \times 10^{-3}\,kg/s$

$$V_{max} = 786\frac{m}{s} \ln\left[\frac{69.6\,g}{(69.6 - 12.5)\,g}\right] - 9.81\frac{m}{s} \times 1.7\,s = 139\,m/s \quad \leftarrow V_{max}$$

To obtain $Y = Y(t)$, set $V = \frac{dY}{dt}$ in Eq. 1, separate variables and integrate from $Y = 0$ at $t = 0$ to Y at $t \le t_b$.

$$Y = \frac{V_e M_0}{\dot{m}}\left\{\left(1 - \frac{\dot{m}t}{M_0}\right)\left[\ln\left(1 - \frac{\dot{m}t}{M_0}\right) - 1\right] + 1\right\} - \frac{1}{2}gt^2 \quad \text{for } 0 \le t \le t_b \quad \text{....(2)}$$

Evaluating at $t = t_b = 1.7\,s$

$$Y = 786\frac{m}{s} \times 69.6\,g \times \frac{s}{7.35\,g}\left\{\left(1 - \frac{12.5}{69.6}\right)\left[\ln\left(1 - \frac{12.5}{69.6}\right) - 1\right] + 1\right\} - \frac{1}{2} \times 9.81\frac{m}{s^2} \times (1.7)^2 s^2$$

$$Y_{t = t_b} = 114\,m$$

After burnout the rocket will "coast" until its kinetic energy is converted to potential energy. In the absence of aerodynamic drag

$$Mg Y_b + \frac{1}{2} M V_b^2 = Mg Y_{max} \quad \therefore \quad Y_{max} = Y_b + \frac{V_{max}^2}{2g}$$

$$Y_{max} = 114\,m + \frac{1}{2} \times (139)^2 \frac{m^2}{s^2} \times \frac{s^2}{9.81\,m} = 1100\,m \longleftarrow Y_{max}$$

After burnout the rocket travels at $a_y = \text{const} = -g$

$$\therefore \int_{V_{max}}^{V} dV = -\int_{t_b}^{t} g\,dt \qquad V = V_{max} - g(t - t_b)$$

$$V = 0 \ @ \ t = \frac{V_{max}}{g} + t_b = 15.9\,s$$

Since

$$V = \frac{dY}{dt} = V_{max} - g(t - t_b), \text{ then } \int_{Y_b}^{Y} dY = \int_{t_b}^{t} \{V_{max} - g(t - t_b)\}\, dt$$

$$Y = Y_b + V_{max}(t - t_b) - \frac{1}{2} g (t - t_b)^2 \qquad \text{for } 1.7 \le t \le 15.9\,s$$

Summarizing

$$V = V_e \ln\left[\frac{M_0}{M_0 - \dot{m}t}\right] - gt \qquad \text{for } 0 \le t \le t_b = 1.7\,s$$

$$V = V_{max} - g(t - t_b) \qquad \text{for } t_b \le t \le 15.9\,s$$

$$Y = \frac{V_e M_0}{\dot{m}} \left\{ \left(1 - \frac{\dot{m}t}{M_0}\right)\left[\ln\left(1 - \frac{\dot{m}t}{M_0}\right) - 1\right] + 1 \right\} - \frac{1}{2} g t^2 \qquad \text{for } 0 \le t \le t_b = 1.7\,s$$

$$Y = Y_b + V_{max}(t - t_b) - \frac{1}{2} g (t - t_b)^2 \qquad \text{for } t_b \le t \le 15.9\,s.$$

Acceleration, Speed, and Height for Model Solid-Propellant Rocket:

Input Parameters:

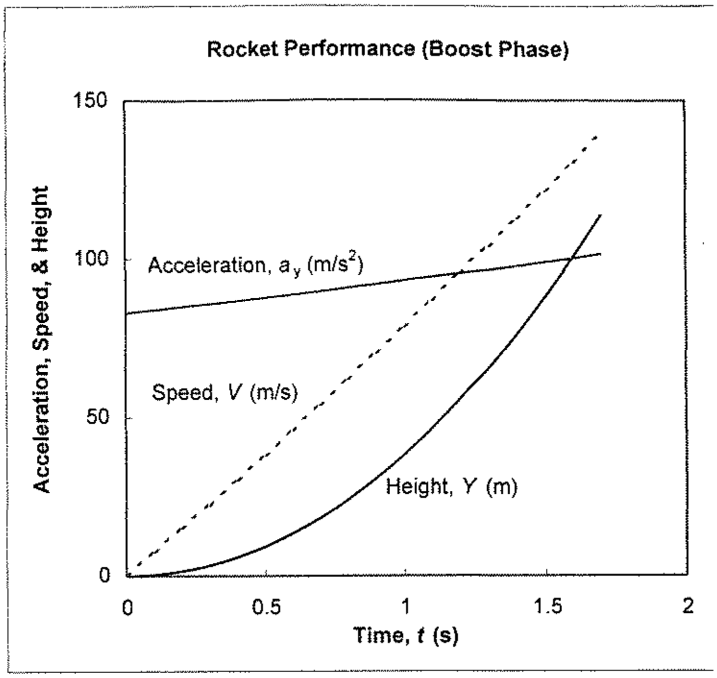
F_t =	1.3	lbf	Thrust
g =	9.81	m/s^2	
m_f =	12.5	g	Fuel mass
M_0 =	69.6	g	Initial mass
t_b =	1.7	s	Burn time

Calculated Values:

m(dot) =	0.00735	kg/s	Mass flow rate
V_e =	787	m/s	Exhaust gas velocity

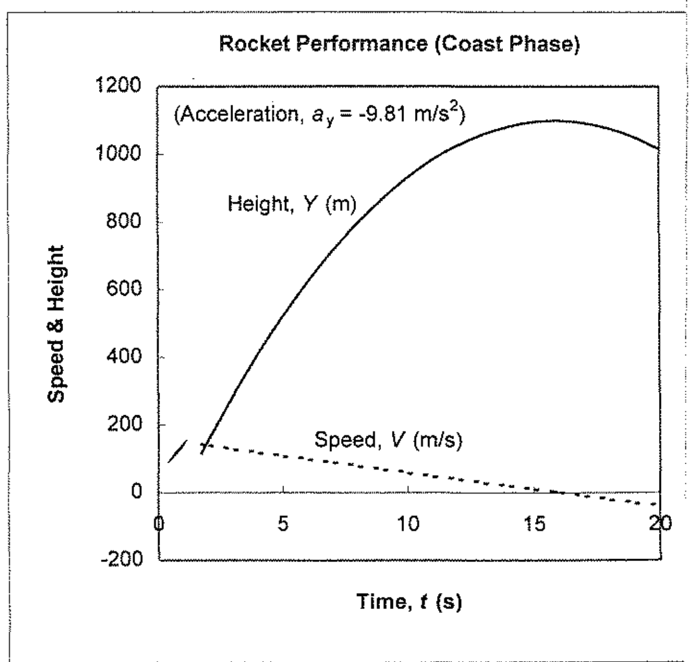
Calculated Results (Boost Phase):

Time, t (s)	[ ]	Accel, a_y (m/s^2)	Speed, V (m/s)	Height, Y (m)
0.0	1.00	83.1	0	0
0.1	0.99	84.0	7.37	0.368
0.2	0.98	84.9	14.8	1.48
0.3	0.97	85.8	22.4	3.34
0.4	0.96	86.8	30.0	5.96
0.5	0.95	87.7	37.8	9.35
0.6	0.94	88.7	45.6	13.5
0.7	0.93	89.8	53.6	18.5
0.8	0.92	90.8	61.6	24.2
0.9	0.90	91.8	69.8	30.8
1.0	0.89	92.9	78.0	38.2
1.1	0.88	94.0	86.4	46.4
1.2	0.87	95.2	94.9	55.5
1.3	0.86	96.3	103	65.4
1.4	0.85	97.5	112	76.2
1.5	0.84	98.8	121	87.8
1.6	0.83	100	130	100
1.7	0.82	101	139	114



Calculated Results (Coast Phase):

Time, t (s)	Accel, a_y (m/s^2)	Speed, V (m/s)	Height, Y (m)
1.7	-9.81	139	114
2	-9.81	136	155
3	-9.81	126	286
4	-9.81	117	408
5	-9.81	107	519
6	-9.81	97	621
7	-9.81	87	713
8	-9.81	77	795
9	-9.81	67	868
10	-9.81	58	930
11	-9.81	48	983
12	-9.81	38	1026
13	-9.81	28	1059
14	-9.81	18	1082
15	-9.81	9	1096
15.88	-9.81	0.000	1100
16	-9.81	-1	1099
17	-9.81	-11	1093
18	-9.81	-21	1077
19	-9.81	-31	1052
20	-9.81	-40	1016



Open-Ended Problem Statement: Several toy manufacturers sell water "rockets" that consist of plastic tanks to be partially filled with water and then pressurized with air. Upon release, the compressed air forces water out the nozzle rapidly, propelling the rocket. You are asked to help specify optimum conditions for this water-jet propulsion system. To simplify the analysis, consider horizontal motion only. Perform the analysis and design needed to define the acceleration performance of the compressed air/water-propelled rocket. Identify the fraction of tank volume that initially should be filled with compressed air to achieve optimum performance (i.e., maximum speed from the water charge). Describe the effect of varying the initial air pressure in the tank.

Discussion: The process may be modeled as a polytropic expansion of the trapped air which forces water out the jet nozzle, causing the "rocket" to accelerate. The polytropic exponent may be varied to model anything from an isothermal expansion process ($n = 1$) to an adiabatic expansion process ($n = k$), which is more likely to be an accurate model for the sudden expansion of the air.

Speed of the water jet leaving the "rocket" is proportional to the square root of the pressure difference between the tank and atmosphere.

Qualitatively it is apparent that the smaller the initial volume fraction of trapped air, the larger will be the expansion ratio of the air, and the more rapid will be the pressure reduction as the air expands. This will cause the water jet speed to drop rapidly. The combination of low water jet speed and relatively large mass of water will produce sluggish acceleration.

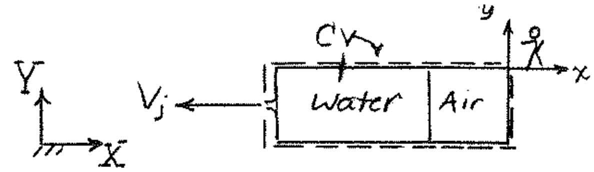
Increasing the initial volume fraction of air will reduce the expansion ratio, so higher pressure will be maintained longer in the tank and the water jet will maintain higher speed longer. This combined with the relatively small mass of water in the tank will produce rapid acceleration.

If the initial volume fraction of air is too large, all water will be expended before the air pressure is reduced significantly. In this situation, some of the stored energy of the air will be dissipated in a relatively ineffective air jet. Consequently, for any initial pressure in the tank, there is an optimum initial air fraction.

This problem cannot be solved in closed form because of the varying air pressure, mass flow rate, and mass of water in the tank;. it can only be solved numerically. One possible integration scheme is to increment time and solve for all properties of the system at each instant. The drawback to this scheme is that the water is unlikely to be exhausted at an even increment of time. A second scheme is to increment the volume of water remaining and solve for properties using the average flow rate during the interval. This scheme is outlined below.

Model the air/water jet-propelled "rocket" using the CV and coordinates shown.

First choose dimensions and mass of "rocket" to be simulated:



Input Data:

Jet diameter:	D_j =	0.003	m
Tank diameter:	D_t =	0.035	m
Tank length:	L =	0.1	m
Tank mass:	M_t =	0.01	kg
Polytropic exponent:	n =	1.4	---

Next choose initial conditions for the simulation (see sample calculations below):

Initial Conditions:

Air fraction in tank:	$\alpha =$	0.5	---
Tank pressure:	$p_0 =$	200	kPa (gage)
Volume increment:	$\Delta\alpha =$	0.02	---

Compute reference parameters:

Calculated Parameters:

Jet area:	$A_j =$	7.07E-06	m^2
Tank volume:	$V_t =$	9.62E-05	m^3
Initial air volume:	$V_0 =$	4.81E-05	m^3
Initial water mass:	$M_0 =$	0.0481	kg

(These are used in the spreadsheet below.)

Then decrease the water fraction in the tank by $\Delta\alpha$:

Calculated Results:

Water Fraction, V_w/V_t (---)	Gage Pressure, p (kPa)	Water Mass, M_w (kg)	Jet Speed, V_j (m/s)	Flow Rate, dm/dt (kg/s)	Time Interval, Δt (s)	Current Time, t (s)	"Rocket" Accel., a (m/s^2)	"Rocket" Speed, U (m/s)
0.50	200	0.0481	20.0	0.141	0	0	48.7	0
0.48	184	0.0461	19.2	0.135	0.0139	0.0139	47.5	0.668

The computation is made as follows:

(1) Decrease α by $\Delta\alpha$

(2) Compute p from $p = p_0\left(\frac{V_0}{V}\right)^n$

$$p = (200+101.325)\,kPa\left(\frac{0.50}{0.52}\right)^{1.4} - 101.325 = 183.9\ kPa\ (gage)$$

(3) Use Bernoulli to calculate jet speed

$$V_j = \sqrt{\frac{2\Delta p}{\rho}} = \left[2 \times 183.9\times10^3 \frac{N}{m^2} \times \frac{m^3}{999\ kg} \times \frac{kg \cdot m}{N \cdot s^2}\right]^{\frac{1}{2}} = 19.10\ m/s^*$$

(4) Calculate water mass using α.

(5) Use conservation of mass to compute mass flow rate

$$\dot{m} = \rho V_j A_j = 999\frac{kg}{m^3} \times 19.10\frac{m}{s} \times 7.07\times10^{-6} m^2 = 0.1349\ kg/s$$

(6) Use the average mass flow rate during the interval to approximate Δt:

$$\Delta t = \frac{\Delta m}{dm/dt} = \frac{\Delta m}{\bar{\dot{m}}} = (0.0481-0.0461)\,kg \times \frac{s}{0.138\ kg} = 0.01449\ s^*$$

(7) Use momentum to compute acceleration (note $M = M_w + M_t$):

$$a_{rf_x} = \frac{\dot{m}V_j}{M} = 0.135\frac{kg}{s} \times 19.2\ \frac{m}{s} \times \frac{1}{0.0461+0.0100\ kg} = 46.2\ m/s^{2*}$$

(8) Finally: use average acceleration to get speed

$$U = U_0 + \bar{a}\Delta t = 0 + 48.1\frac{m}{s^2} \times 0.0139\ s = 0.669\ m/s^*$$

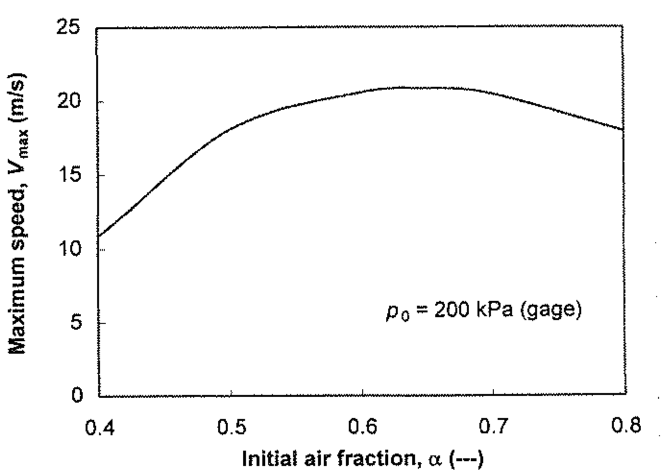
*Note effect of roundoff error.

Repeat these calculations until water is depleted or air pressure falls to zero, as shown below:

Water Fraction, V_w/V_t (---)	Gage Pressure, p (kPa)	Water Mass, M_w (kg)	Jet Speed, V_j (m/s)	Flow Rate, dm/dt (kg/s)	Time Interval, Δt (s)	Current Time, t (s)	"Rocket" Accel., a (m/s^2)	"Rocket" Speed, U (m/s)
0.50	200	0.0481	20.0	0.141	0	0	48.7	0
0.48	184	0.0461	19.2	0.135	0.0139	0.0139	47.5	0.668
0.46	169	0.0442	18.4	0.130	0.0145	0.0284	45.2	1.34
0.44	156	0.0423	17.7	0.125	0.0151	0.0435	43.1	2.01
0.42	143	0.0404	16.9	0.120	0.0157	0.0592	41.2	2.67
0.40	132	0.0384	16.3	0.115	0.0164	0.0756	39.4	3.33
0.38	122	0.0365	15.6	0.110	0.0171	0.0927	37.8	3.99
0.36	112	0.0346	15.0	0.106	0.0178	0.110	36.2	4.65
0.34	103	0.0327	14.4	0.101	0.0186	0.129	34.8	5.31
0.32	94.6	0.0308	13.8	0.0972	0.0194	0.148	33.5	5.97
0.30	86.8	0.0288	13.2	0.0931	0.0202	0.169	32.2	6.63
0.28	79.5	0.0269	12.6	0.0891	0.0211	0.190	31.0	7.30
0.26	72.7	0.0250	12.1	0.0852	0.0221	0.212	29.9	7.97
0.24	66.3	0.0231	11.5	0.0814	0.0231	0.235	28.9	8.65
0.22	60.4	0.0211	11.0	0.0776	0.0242	0.259	27.9	9.34
0.20	54.7	0.0192	10.5	0.0739	0.0254	0.284	26.9	10.0
0.18	49.4	0.0173	9.95	0.0702	0.0267	0.311	26.0	10.7
0.16	44.4	0.0154	9.43	0.0666	0.0281	0.339	25.2	11.5
0.14	39.7	0.0135	8.92	0.0630	0.0297	0.369	24.3	12.2
0.12	35.2	0.0115	8.40	0.0593	0.0314	0.400	23.5	12.9
0.10	31.0	0.00961	7.88	0.0556	0.0334	0.434	22.7	13.7
0.08	27.0	0.00769	7.35	0.0519	0.0357	0.469	22.0	14.5
0.06	23.2	0.00577	6.81	0.0481	0.0384	0.508	21.2	15.3
0.04	19.6	0.00384	6.26	0.0442	0.0416	0.550	20.4	16.2
0.02	16.1	0.00192	5.68	0.0401	0.0456	0.595	19.5	17.1
0.00	12.9	0.0000	5.07	0.0358	0.0506	0.646	18.6	18.1

In this simulation, the water is depleted when $t \approx 0.65$ s; $V_{max} = 18.1$ m/s.

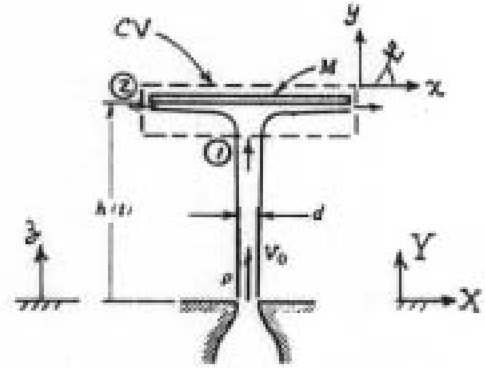
Varying the initial air fraction produces the following:



For this combination of parameters, a peak speed of about 20.8 m/s is attained with an initial air fraction of about 0.66.

Given: Vertical jet impinging on disk.

Disk is unconstrained vertically.



Find: (a) Differential equation for $h(t)$, if disk released from $H > h_0$, where h_0 is equilibrium height.
(b) Sketch $h(t)$ and explain.

Solution: Apply Bernoulli equation to jet, then y momentum equation to CV with linear acceleration.

Basic equations:

$$\frac{p_0}{\rho} + \frac{V_0^2}{2} + g z_0 = \frac{p_1}{\rho} + \frac{V_1^2}{2} + g z_1 \qquad (z_0 = 0)$$

$$F_{S_y} + F_{B_y} - \int_{CV} a_{rf_y} \rho \, d\forall = \frac{\partial}{\partial t}\int_{CV} v_{xyz} \rho \, d\forall + \int_{CS} v_{xyz} \rho \vec{V}_{xyz} \cdot d\vec{A}$$

($F_{S_y} = 0$ (6); $\frac{\partial}{\partial t}\int_{CV} \approx 0$ (7))

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) No friction
(4) Flow along a streamline
} in jet
(5) $p_1 = p_0 = p_{atm}$

(6) No pressure force on CV, so $F_{S_y} = 0$
(7) Neglect mass of liquid in CV and $v \approx 0$ in CV
(8) Uniform flow at each section
(9) Measure velocities relative to CV

From momentum

$$-(M + m_{cv})g - a_{rf_y}(M + m_{cv}) = v_1\{-|\rho(V_1 - U)A_1|\} + v_2\{\dot{m}_2\}$$

($m_{cv} \approx 0$ (7))

$$v_1 = V_1 - U \qquad v_2 \approx 0$$

With $a_{rf_y} = \frac{d^2h}{dt^2}$, $U = \frac{dh}{dt}$, then

$$-Mg - M\frac{d^2h}{dt^2} = -\rho\left(V_1 - \frac{dh}{dt}\right)^2 A_1$$

But from Bernoulli, $\frac{V_1^2}{2} = \frac{V_0^2}{2} - g z_1$, so $V_1 = \sqrt{V_0^2 - 2gh}$, since $z_1 = h(t)$

Also from continuity, $V_1 A_1 = V_0 A_0$, so $A_1 = A_0 V_0 / V_1$. Substituting

$$\frac{d^2h}{dt^2} = \rho\left(\sqrt{V_0^2 - 2gh} - \frac{dh}{dt}\right)^2 \frac{A_0 V_0}{M\sqrt{V_0^2 - 2gh}} - g$$

$\ddot{h}(t)$

At equilibrium height, $h = h_0$, $\frac{dh}{dt} = 0$, and $\frac{d^2h}{dt^2} = 0$. Then

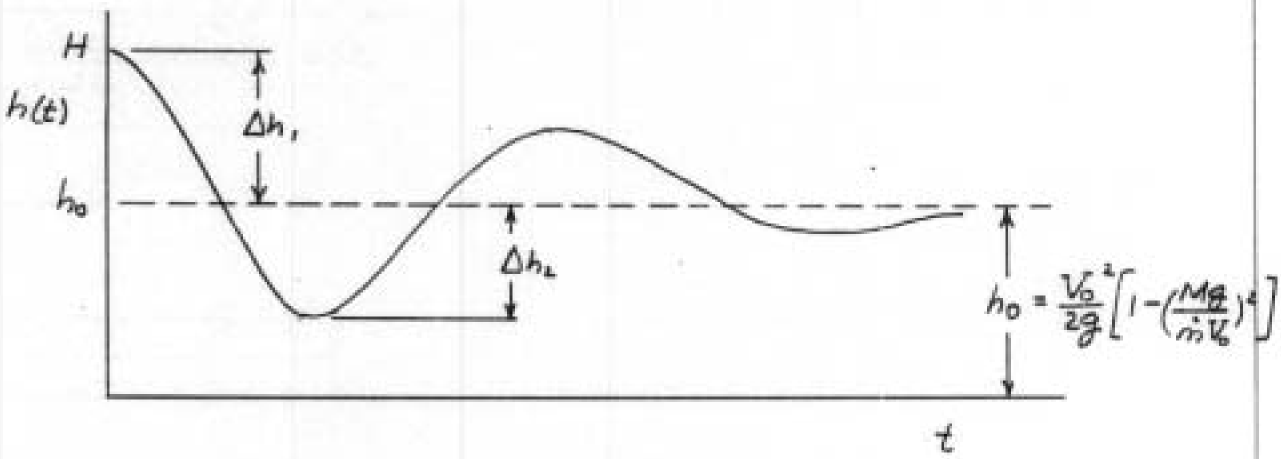
$$\rho\sqrt{V_0^2 - 2gh_0}\, A_0 V_0 - Mg = 0$$

Thus $$V_0^2 - 2gh_0 = \left(\frac{Mg}{\rho V_0 A_0}\right)^2$$

This may be solved to obtain

$$h_0 = \frac{V_0^2}{2g}\left[1-\left(\frac{Mg}{\rho V_0^2 A_0}\right)^2\right] = \frac{V_0^2}{2g}\left[1-\left(\frac{Mg}{\dot{m}V_0}\right)^2\right]$$

When released, $H > h_0$, and $dh/dt = 0$. Because the equation for d^2h/dt^2 is nonlinear, oscillations will occur. The expected behavior is sketched below:

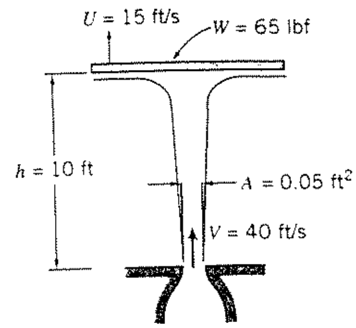


Notes: (1) Expect oscillations

(2) $\Delta h_3 < \Delta h_2 < \Delta h_1$, due to nonlinear equation

Given: Configuration of vertical jet striking horizontal disc shown in P*4.136.

Assume disk released from rest at $h = 10$ ft above nozzle exit plane.



Find: (a) Solve for subsequent motion of disc.
(b) Steady-state height.

Solution: As in Problem 4.159, apply Bernoulli to jet, then y-momentum to CV with linear acceleration.

Basic equations:
$$\frac{p_0}{\rho} + \frac{V_0^2}{2} + g z_0 = \frac{p_1}{\rho} + \frac{V_1^2}{2} + g z_1 \qquad (z_0 \approx 0)$$
$$F_{S_y} + F_{B_y} - \int_{CV} a_{rf_y} \rho \, d\forall = \frac{\partial}{\partial t}\int_{CV} v_{xyz} \rho \, d\forall + \int_{CS} v_{xyz} \rho \vec{V}_{xyz} \cdot d\vec{A}$$
($F_{S_y} = 0$ (6); $\frac{\partial}{\partial t}\int_{CV} \approx 0$ (7))

Assumptions: (1) Steady
(2) Incompressible flow
(3) No friction
(4) Flow along a streamline
} in jet
(5) $p_1 = p_0 = p_{atm}$
(6) No pressure force on CV, so $F_{S_y} = 0$
(7) Neglect mass of water and $v \approx 0$ in CV
(8) Uniform flow at each cross-section
(9) Measure velocities relative to CV

From momentum,
$$-(M + m_w)g - a_{rf_y}(M + m_w) = v_1\{-\rho(V_1 - U)A_1\} + v_2\{\dot m_2\} \qquad (m_w \approx 0\ (7))$$
$v_1 = V_1 - U \qquad v_2 = 0$

With $a_{rf_y} = \frac{d^2h}{dt^2}$ and $U = \frac{dh}{dt}$, then
$$-Mg - M\frac{d^2h}{dt^2} = -\rho\left(V_1 - \frac{dh}{dt}\right)^2 A_1$$

But from Bernoulli, $\frac{V_1^2}{2} = \frac{V_0^2}{2} - g z_1$, so $V_1 = \sqrt{V_0^2 - 2gh}$, since $z_1 = h(t)$

Also from continuity $V_1 A_1 = V_0 A_0$ so $A_1 = A_0 \frac{V_0}{V_1}$. Substituting
$$\frac{d^2h}{dt^2} = \rho\left(\sqrt{V_0^2 - 2gh} - \frac{dh}{dt}\right)^2 \frac{A_0 V_0}{M\sqrt{V_0^2 - 2gh}} - g \qquad (1)$$
$\ddot h(t)$

This may be solved numerically (next page).

At equilibrium height, $h = h_0(ss)$, $\dot h(t) = 0$, and $\ddot h(t) = 0$. Then
$$\rho\sqrt{V_0^2 - 2gh_0}\, A_0 V_0 - Mg = 0 \quad \text{or} \quad V_0^2 - 2gh_0 = \left(\frac{Mg}{\rho V_0 A_0}\right)^2$$

so
$$h_0 = \frac{V_0^2}{2g}\left[1 - \left(\frac{Mg}{\rho V_0 A_0}\right)^2\right]$$
$h_0(ss)$

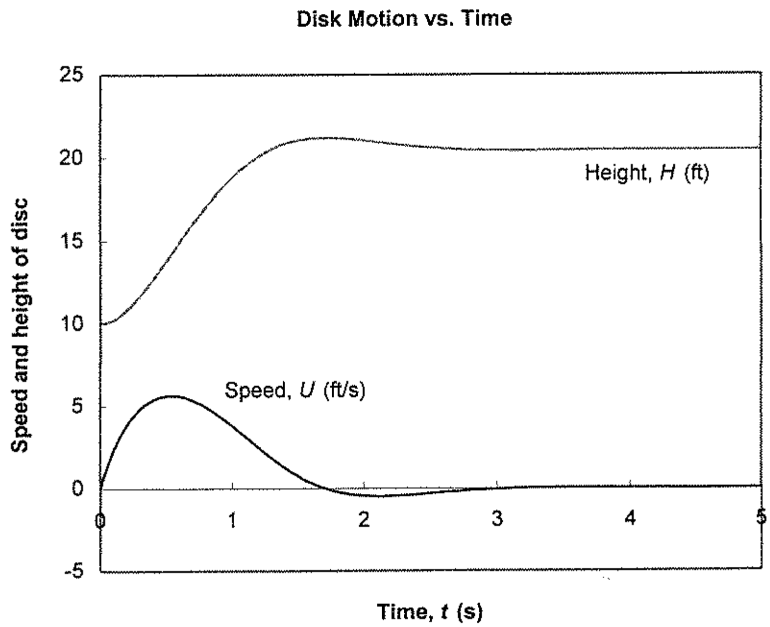
From the given data

$$h_0(ss) = \frac{1}{2} \times (40)^2 \frac{ft^2}{s^2} \times \frac{s^2}{32.2\,ft}\left[1 - \left(65\,lbf \times \frac{ft^3}{1.94\,slug} \times \frac{s^2}{(40)^2 ft^2} \times \frac{1}{0.05\,ft^2} \times \frac{slug \cdot ft}{lbf \cdot s^2}\right)^2\right] = 20.49\,ft$$

$h_0(ss)$

Several methods may be used to integrate Eq. 1 numerically. The 4th-order Runge-Kutta method works well with a spreadsheet (next page).

The results are plotted in the figure below:



The analysis shows the motion overshoots, then the oscillation damps out and the disc settles at its steady-state height of 20.49 ft.

Problem *4.160 (cont'd.)

Analysis of disc motion (using 4th-order Runge-Kutta numerical integration):

Input Data:

A_0 =	0.050	ft^2	Nozzle area
g =	32.2	ft/s^2	Acceleration of gravity
V_0 =	40.0	ft/s	Jet speed at nozzle
W =	65.0	lbf	Weight of disc
Δt =	0.10	s	Time step size
ρ =	1.94	$slug/ft^3$	Density of water in jet

Initial Conditions:

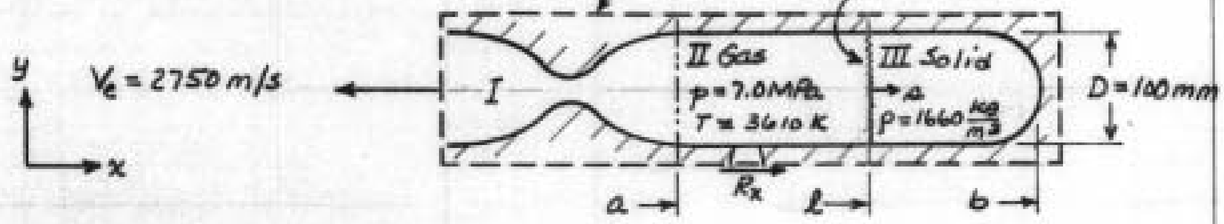
H_0 =	10.0	ft	Initial height of disc
U =	0.0	ft/s	Initial speed of disc

Calculated Values:

$H_{0\,(ss)}$ =	20.49	ft	Steady-state height
V_1 =	30.9	ft/s	Initial speed of jet reaching disc
$V_{1\,(ss)}$ =	16.75	ft/s	Steady-state jet speed
$z_{1\,(max)}$ =	24.8	ft	Maximum height reached by jet

Time Step, n (---)	Time, t (s)	Initial Speed, U (ft/s)	Initial Height, H (ft)	Jet Speed, V_1 (ft/s)	Accel, a_y (ft/s^2)	ΔU_1 (ft/s)	ΔU_2 (ft/s)	ΔU_3 (ft/s)	ΔU_4 (ft/s)	Final Speed, U (ft/s)	ΔH (ft)	Final Height, H (ft)
Initial Conditions:										0.0		10.0
0	0.0	0.00	10.00	30.92	27.23	2.72	2.21	2.31	1.87	2.27	0.23	10.23
1	0.1	2.27	10.23	30.68	18.37	1.84	1.52	1.57	1.29	3.82	0.61	10.84
2	0.2	3.82	10.84	30.04	11.78	1.18	0.98	1.01	0.84	4.82	0.86	11.70
3	0.3	4.82	11.70	29.09	6.72	0.67	0.56	0.58	0.49	5.40	1.02	12.72
4	0.4	5.40	12.72	27.94	2.75	0.28	0.23	0.24	0.20	5.64	1.10	13.83
5	0.5	5.64	13.83	26.64	-0.37	-0.04	-0.03	-0.03	-0.03	5.60	1.12	14.95
6	0.6	5.60	14.95	25.24	-2.83	-0.28	-0.24	-0.25	-0.21	5.36	1.10	16.05
7	0.7	5.36	16.05	23.80	-4.73	-0.47	-0.40	-0.41	-0.35	4.95	1.03	17.08
8	0.8	4.95	17.08	22.37	-6.14	-0.61	-0.52	-0.54	-0.45	4.42	0.94	18.01
9	0.9	4.42	18.01	20.97	-7.09	-0.71	-0.60	-0.62	-0.52	3.81	0.82	18.84
10	1.0	3.81	18.84	19.67	-7.62	-0.76	-0.64	-0.66	-0.55	3.16	0.70	19.53
11	1.1	3.16	19.53	18.49	-7.75	-0.78	-0.65	-0.67	-0.56	2.49	0.57	20.10
12	1.2	2.49	20.10	17.48	-7.50	-0.75	-0.63	-0.65	-0.53	1.86	0.44	20.53
13	1.3	1.86	20.53	16.66	-6.92	-0.69	-0.57	-0.59	-0.48	1.27	0.31	20.85
14	1.4	1.27	20.85	16.04	-6.06	-0.61	-0.50	-0.52	-0.42	0.76	0.20	21.05
15	1.5	0.76	21.05	15.63	-5.02	-0.50	-0.41	-0.43	-0.34	0.34	0.11	21.16
16	1.6	0.34	21.16	15.40	-3.90	-0.39	-0.32	-0.33	-0.26	0.02	0.04	21.20
17	1.7	0.02	21.20	15.33	-2.82	-0.28	-0.23	-0.24	-0.19	-0.21	-0.02	21.18
18	1.8	-0.21	21.18	15.37	-1.83	-0.18	-0.15	-0.15	-0.12	-0.37	-0.06	21.12
19	1.9	-0.37	21.12	15.49	-1.01	-0.10	-0.08	-0.08	-0.07	-0.45	-0.08	21.04
20	2.0	-0.45	21.04	15.66	-0.35	-0.04	-0.03	-0.03	-0.02	-0.48	-0.09	20.95
21	2.1	-0.48	20.95	15.85	0.13	0.01	0.01	0.01	0.01	-0.47	-0.09	20.85
22	2.2	-0.47	20.85	16.04	0.45	0.05	0.04	0.04	0.03	-0.43	-0.09	20.76
23	2.3	-0.43	20.76	16.22	0.65	0.06	0.05	0.05	0.04	-0.38	-0.08	20.68
24	2.4	-0.38	20.68	16.38	0.74	0.07	0.06	0.06	0.05	-0.32	-0.07	20.61
25	2.5	-0.32	20.61	16.51	0.76	0.08	0.06	0.06	0.05	-0.25	-0.06	20.55
26	2.6	-0.25	20.55	16.62	0.73	0.07	0.06	0.06	0.05	-0.19	-0.04	20.51
27	2.7	-0.19	20.51	16.71	0.66	0.07	0.05	0.06	0.04	-0.14	-0.03	20.48
28	2.8	-0.14	20.48	16.77	0.57	0.06	0.05	0.05	0.04	-0.09	-0.02	20.45
29	2.9	-0.09	20.45	16.82	0.47	0.05	0.04	0.04	0.03	-0.05	-0.01	20.44
30	3.0	-0.05	20.44	16.84	0.37	0.04	0.03	0.03	0.03	-0.02	-0.01	20.43
31	3.1	-0.02	20.43	16.86	0.28	0.03	0.02	0.02	0.02	0.00	0.00	20.43
32	3.2	0.00	20.43	16.86	0.20	0.02	0.02	0.02	0.01	0.02	0.00	20.43
33	3.3	0.02	20.43	16.86	0.13	0.01	0.01	0.01	0.01	0.03	0.00	20.44
34	3.4	0.03	20.44	16.85	0.07	0.01	0.01	0.01	0.00	0.04	0.01	20.44
35	3.5	0.04	20.44	16.83	0.02	0.00	0.00	0.00	0.00	0.04	0.01	20.45
36	3.6	0.04	20.45	16.82	-0.01	0.00	0.00	0.00	0.00	0.04	0.01	20.46
37	3.7	0.04	20.46	16.81	-0.03	0.00	0.00	0.00	0.00	0.03	0.01	20.47
38	3.8	0.03	20.47	16.79	-0.05	0.00	0.00	0.00	0.00	0.03	0.01	20.47
39	3.9	0.03	20.47	16.78	-0.06	-0.01	0.00	0.00	0.00	0.02	0.01	20.48
40	4.0	0.02	20.48	16.77	-0.06	-0.01	0.00	0.00	0.00	0.02	0.00	20.48
41	4.1	0.02	20.48	16.76	-0.06	-0.01	0.00	0.00	0.00	0.01	0.00	20.48
42	4.2	0.01	20.48	16.76	-0.05	-0.01	0.00	0.00	0.00	0.01	0.00	20.49
43	4.3	0.01	20.49	16.75	-0.04	0.00	0.00	0.00	0.00	0.01	0.00	20.49
44	4.4	0.01	20.49	16.75	-0.04	0.00	0.00	0.00	0.00	0.00	0.00	20.49
45	4.5	0.00	20.49	16.75	-0.03	0.00	0.00	0.00	0.00	0.00	0.00	20.49

Given: Small solid fuel rocket motor on test stand. The fuel burns uniformly at $\dot{s} = 12.7$ mm/s. Exhaust gases leave at ambient pressure.



Treat combustion products as ideal gas with molecular mass, $M_m = 25.8$.

Find: (a) Evaluate rate of change of mass and of linear momentum within rocket motor.
(b) Express rate of change of momentum as a percentage of thrust.

Solution: Apply continuity and x component of momentum equations using fixed CV shown.

Basic equations:
$$0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A}$$

$$F_{S_x} + \cancelto{=0(2)}{F_{B_x}} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$$

Assumptions: (1) No net pressure force; $F_{S_x} = R_x$
(2) $F_{B_x} = 0$
(3) All properties constant at each point, except at surface where combustion takes place
(4) Uniform flow at exit section

The continuity equation becomes

$$0 = \cancelto{=0(3)}{\frac{\partial}{\partial t}\int_I \rho\, d\forall} + \frac{\partial}{\partial t}\int_a^{\ell} \rho_g A\, dx + \frac{\partial}{\partial t}\int_{\ell}^{b} \rho_f A\, dx + \{|\rho_e V_e A_e|\}$$

$$0 = \frac{\partial}{\partial t}[\rho_g A(\ell - a)] + \frac{\partial}{\partial t}[\rho_f A(b-\ell)] + \dot{m}_e = (\rho_g - \rho_f)A\frac{d\ell}{dt} + \dot{m}_e$$

or

$$\dot{m}_e = (\rho_f - \rho_g)A\frac{d\ell}{dt} = (\rho_f - \rho_g)A\dot{s}$$

For an ideal gas,

$$\rho_g = \frac{p_g}{RT_g} = \frac{p_g M_m}{R_u T_g} = 7.0\times10^6\frac{N}{m^2}\times\frac{25.8\,kg}{mol}\times\frac{mol\cdot K}{8314\,N\cdot m}\times\frac{1}{3610\,K} = 6.02\ kg/m^3$$

so

$$\dot{m}_e = (1660-6)\frac{kg}{m^3}\times\frac{\pi}{4}(0.1)^2 m^2\times 0.0127\frac{m}{s} = 0.165\ kg/s$$

Mass flow is out, so $\frac{\partial M_{cv}}{\partial t} = -0.165$ kg/s $\qquad \frac{\partial M_{cv}}{\partial t}$

From the momentum equation,

$$R_x = \frac{\partial}{\partial t}\int_I^{=0(3)} u\rho\, d\forall + \frac{\partial}{\partial t}\int_a^{\ell} u_g \rho_g A\, dx + \frac{\partial}{\partial t}\int_\ell^{b} u_f \rho_f A\, dx^{=0} + u_e\{|\rho_e V_e A_e|\}$$

$$= \frac{\partial}{\partial t}\left[u_g \rho_g A(\ell - a)\right] + u_e \dot{m}_e \quad ; \quad u_g = -V_g \text{ and } u_e = -V_e$$

$$R_x = -\rho_g V_g A \frac{d\ell}{dt} - V_e \dot{m}_e = -\rho_g V_g A s - V_e \dot{m}_e$$

But from continuity, $\rho_g V_g A = \dot{m}_e$, since no mass accumulates in region I of the CV. Thus

$$R_x = -\dot{m}_e (V_e + s)$$

R_x is the force <u>on</u> the CV. The thrust is

$$K_x = \text{Thrust} = -R_x = \dot{m}_e(V_e + s)$$

$$K_x = 0.165\,\frac{kg}{s}\,(2750 + 0.0127)\frac{m}{s} \times \frac{N \cdot s^2}{kg \cdot m} = 454 \text{ N}$$

The rate of change of linear momentum within the CV is

$$\frac{\partial P_{x_{cv}}}{\partial t} = -\dot{m}_e s = -0.165\,\frac{kg}{s} \times 0.0127\,\frac{m}{s} \times \frac{N \cdot s^2}{kg \cdot m} = -2.10 \text{ mN}$$

$\frac{\partial P_{x_{cv}}}{\partial t}$

The ratio of rate of change of linear momentum to thrust is

$$\frac{\frac{\partial P_{x_{cv}}}{\partial t}}{K_x} = \frac{-\dot{m}_e s}{\dot{m}_e(V_e + s)} = -\frac{s}{(V_e + s)} = -\frac{0.0127\,\frac{m}{s}}{(2750 + 0.0127)\frac{m}{s}} = -4.62 \times 10^{-6}$$

or

$$\frac{\frac{\partial P_{x_{cv}}}{\partial t}}{K_x} = -4.62 \times 10^{-4} \text{ percent}$$

Ratio

{ Neglecting the unsteady momentum term in the analysis of this rocket motor would cause an error of approximately 1 part in 217,000. The assumption that $\partial P_{x_{cv}}/\partial t \simeq 0$ is certainly justified for engineering work. }

Open-Ended Problem Statement: A classroom demonstration of linear momentum is planned, using a water-jet propulsion system for a cart traveling on a horizontal linear air track. The track is 5 m long, and the cart mass is 155 g. The objective of the design is to obtain the best performance for the cart, using 1 L of water contained in an open cylindrical tank made from plastic sheet with density of 0.0819 g/cm^2. For stability, the maximum height of the water tank cannot exceed 0.5 m. The diameter of the smoothly rounded water jet may not exceed 10 percent of the tank diameter. Determine the best dimensions for the tank and the water jet by modeling the system performance. Plot acceleration, velocity, and distance as functions of time. Find the optimum dimensions of the water tank and jet opening from the tank. Discuss the limitations on your analysis. Discuss how the assumptions affect the predicted performance of the cart. Would the actual performance of the cart be better or worse than predicted? Why? What factors account for the difference(s)?

Discussion: This solution is an extension of Problem *4.162. The analyses for tank level, acceleration, and velocity are identical; please refer to the solution for Problem *4.162 for equations describing each of these variables as functions of time.

One new feature of this problem is computation of distance traveled. Equation 7 of Problem *4.162 could be integrated in closed form to provide an equation for distance traveled as a function of time. However, the integral would be messy, and it would provide little insight into the dependence on key parameters. Consequently, a numerical analysis has been chosen in this problem. The results are presented in the plots and spreadsheet on the next page.

We have chosen to define velocity as the output to be maximized.

A second new feature of this problem is the geometric constraints: the maximum track length is 5 m. Intuitively jet diameter should be chosen as the largest possible fraction of tank diameter for optimum performance. Using the spreadsheet to vary $\beta = d/D$ verifies that this is the case. Therefore we have used the maximum allowable ratio, $\beta = 0.1$, for all computations.

Tank height should be a factor in performance. Intuition suggests that increasing tank height should improve performance. Using the spreadsheet shows a very weak dependence on tank height. Performance is best at smaller tank heights, corresponding to the minimum tank mass.

As tank height is decreased, diameter increases because tank volume is held constant. Since diameter ratio is constant, then jet diameter increases with decreasing tank height. This effect almost overshadows the effect of tank height.

The principal limitations on the analysis are the assumptions of negligible motion resistance and no slope to the free surface of water in the tank. Actual performance of the cart would likely be less than predicted because of motion resistance.

Distance is modeled as

$$x_{i+1} = x_i + U_i \Delta t + \frac{1}{2} a_{x,i} \Delta t^2$$

The accuracy of this model for position is consistent with the accuracy of modeling the water-jet propulsion system.

Analysis of Cart Propelled by Gravity-Driven Water Jet:

Input Data:

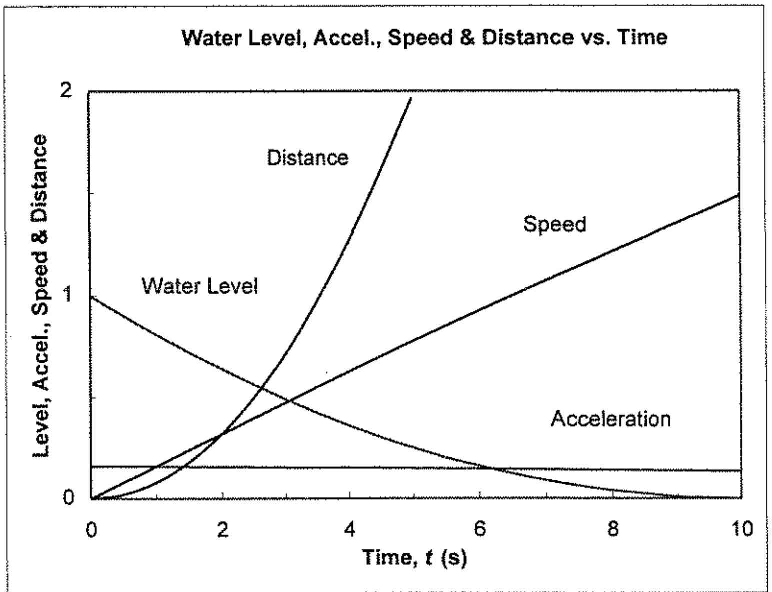
g =	9.81	m/s^2	Acceleration of gravity
H =	500	mm	Height of tank
M_c =	0.155	kg	Mass of cart
V =	1.00	L	Tank volume
β =	0.100	(---)	Ratio of jet diameter to tank diameter
ρ =	999	kg/m^3	Density of water
ρ'' =	0.819	kg/m^2	(Area) density of tank material

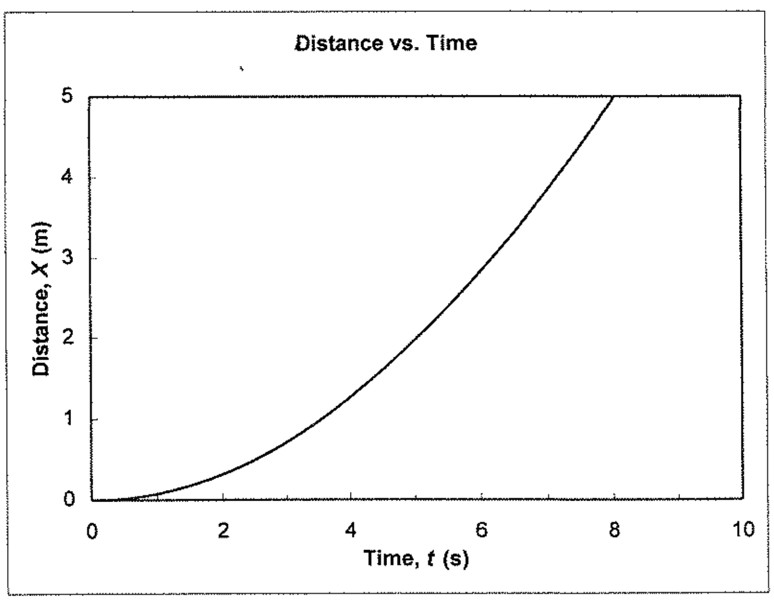
Calculated Parameters:

a =	0.471	(---)	(a^2 =) Ratio of mass of tank to initial mass of water
b =	0.0313	s^{-1}	Geometric parameter of solution
d =	5.05	mm	Diameter of water jet
D =	50.5	mm	Diameter of tank
M_0 =	1.00	kg	Initial mass of water in tank
M_p =	0.0666	kg	Mass of plastic in tank
M_t =	0.222	kg	Mass of plastic tank plus cart

Calculated Results:

Time, t (s)	Level, y/H (---)	Accel., a_x (m/s^2)	Velocity, U (m/s)	Position, X (m)
0	1	0.161	0	0
0.5	0.903	0.160	0.080	0.0201
1.0	0.810	0.159	0.160	0.080
1.5	0.723	0.158	0.239	0.180
2.0	0.640	0.157	0.317	0.319
2.5	0.563	0.156	0.395	0.497
3.0	0.490	0.154	0.473	0.714
3.5	0.423	0.153	0.550	0.97
4.0	0.360	0.152	0.626	1.26
4.5	0.303	0.151	0.702	1.60
5.0	0.250	0.150	0.777	1.97
5.5	0.203	0.148	0.852	2.37
6.0	0.160	0.147	0.925	2.82
6.5	0.123	0.145	1.00	3.30
7.0	0.0900	0.144	1.07	3.82
7.5	0.0625	0.142	1.14	4.37
8.0	0.0400	0.141	1.21	4.96
8.03	0.0388	0.141	1.22	5.00
9.0	0.0100	0.137	1.35	
9.5	0.0025	0.135	1.42	
10.0	0.0000	0.133	1.49	





Problem *4.163

Open-Ended Problem Statement: The capability of the Aircraft Landing Loads and Traction Facility at NASA's Langley Research Center is to be upgraded. The facility consists of a rail-mounted carriage propelled by a water jet issuing from a pressurized tank. (The setup is identical in concept to the hydraulic catapult of Problem 4.118.) The 49,000 kg carriage must accelerate to 220 knots in 122 m. (The vane turning angle is 170°.) Identify a range of water jet sizes and speeds needed to accomplish this performance. Specify the recommended operating pressure for the water jet system and determine the shape and estimated size of tankage to contain the pressurized water.

Discussion: The analysis of Example Problem 4.11 forms the basis for the solution outlined below. Use a control volume attached to and moving with the carriage to analyze the motion. Neglect aerodynamic and rolling resistance to obtain a best-case solution. Solve the resulting differential equation of motion for carriage speed and position as functions of time, and for speed as a function of position along the rails.

Computing equations are summarized and results tabulated below. As shown in Example Problem 4.11, analysis of the carriage motion results in the differential equation

$$\frac{dU}{dt} = \frac{\rho (V_j - U)^2 (1-\cos\theta)}{M} \qquad (1)$$

Integrating with respect to time gives carriage speed versus time

$$U = V_j \frac{bt}{1+bt} \qquad (2)$$

where parameter b is

$$b = \frac{\rho V_j A_j (1-\cos\theta)}{M} \qquad (3)$$

Equation 2 is integrated to obtain carriage position versus time

$$x = V_j \left[t - \frac{\ln(1+bt)}{b} \right] \qquad (4)$$

Substitute $dU/dt = UdU/dx$ and integrate Eq. 1 for distance traveled versus carriage speed

$$x = \frac{V_j}{b} \left[\ln(1 - U/V_j) + \frac{1}{1 - U/V_j} - 1 \right] \qquad (5)$$

Relate jet speed to water tank pressure using the Bernoulli equation

$$V_j = \sqrt{2\Delta p/\rho} \qquad (6)$$

The required volume of water is computed as follows:

1. Assume a range of tank pressures.
2. Compute the jet speed corresponding to each tank pressure from Eq. 6.
3. Solve for parameter b from Eq. 5 using the known maximum speed and specified distance.
4. Obtain jet area from Eq. 3.
5. Compute the time required to accelerate the carriage from Eq. 2.
6. Calculate jet diameter from jet area.
7. Compute the required volume of water from the product of mass flow rate and acceleration time.

The optimum operating pressure requires the least costly tankage. (Assume the most efficient spherical shape for pressurized tankage and constant tank pressure during acceleration.) Tankage calculations are organized as follows:

1. Obtain tank diameter from tank volume.
2. Calculate wall thickness from a force balance on the thin wall of the tank.
3. Calculate steel volume from tank surface area and wall thickness.
4. Assume steel cost is proportional to steel volume.

Sample calculation: assume $p = 6000$ psig

$$V_j = \left[2 \times 6000 \frac{lbf}{in.^2} \times \frac{ft^3}{1.94\,slug} \times 144 \frac{in.^2}{ft^2} \times \frac{slug \cdot ft}{lbf \cdot s^2}\right]^{\frac{1}{2}} = 944\ ft/s\ ;\ \frac{U}{V_j} = \frac{371}{944} = 0.393$$

$$b = 944 \frac{ft}{s} \times \frac{1}{400\,ft}\left[\ln(1-0.393) + \frac{1}{1-0.393} - 1\right] = 0.350\ s^{-1}$$

$$A_j = \frac{bM}{\rho V_j (1-\cos\theta)} = \frac{0.350}{s} \times 3350\,slug \times \frac{ft^3}{1.94\,slug} \times \frac{s}{944\,ft} \times \frac{1}{(1-\cos 170°)} = 0.323\ ft^2$$

$$D = \sqrt{\frac{4A}{\pi}} = \left[\frac{4}{\pi} \times 0.323\,ft^2 \times 144 \frac{in.^2}{ft^2}\right]^{\frac{1}{2}} = 7.69\ in.$$

$$t = \frac{1}{b}\left(\frac{U/V_j}{1-U/V_j}\right) = \frac{s}{0.350} \times \frac{0.393}{1-0.393} = 1.85\ s$$

$$Q = V_j A = 944 \frac{ft}{s} \times 0.323\,ft^2 \times 7.48 \frac{gal}{ft^3} = 2280\ gal/s$$

$$\forall = Qt = 2280 \frac{gal}{s} \times 1.85\,s = 4220\ gal$$

$$D = (6\forall/\pi)^{1/3} = \left(\frac{6}{\pi} \times 4220\,gal \times \frac{ft^3}{7.48\,gal}\right)^{1/3} = 10.3\ ft$$

$$\Delta p \frac{\pi D^2}{4} = \pi D t\ ;\ t = \frac{pD}{4\sigma} = \frac{1}{4} \times 6000 \frac{lbf}{in.^2} \times 10.3\,ft \times \frac{in.^2}{40{,}000\,lbf} \times 12 \frac{in.}{ft} = 4.64\ in.$$

$$\forall_{steel} = \pi D^2 t = \pi \times (10.3)^2 ft^2 \times 4.64\,in. \times \frac{ft}{12\,in.} = 129\ ft^3$$

Discussion: The results show the steel volume plummets as tank pressure is raised, with a broad minimum between 3,000 and 4000 psig.

Problem *4.163 (cont'd.)

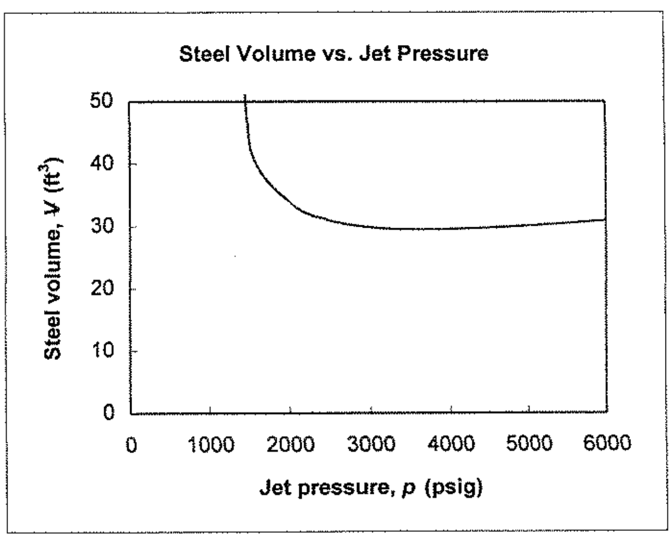
Input Data:

M =	49000	kg	3355	slug
U =	220	kt	371.3	ft/s
X =	122	m	400.3	ft
θ =	170	degrees		

Calculated Results:

Jet Pressure (psig)	Jet Speed (ft/s)	Parameter b (s^{-1})	Jet Area (ft^2)	Jet Diameter (in.)	Flow Rate (gal/s)	Flow Time (s)	Water Volume (gal)
6000	944	0.351	0.324	7.70	2285	1.85	4227
5500	904	0.380	0.367	8.20	2477	1.84	4546
5000	862	0.417	0.421	8.79	2715	1.82	4936
4500	817	0.463	0.494	9.51	3019	1.80	5426
4000	771	0.525	0.593	10.4	3419	1.77	6061
3500	721	0.610	0.737	11.6	3973	1.74	6924
3000	667	0.736	0.961	13.3	4797	1.70	8174
2500	609	0.944	1.35	15.7	6155	1.65	10172
2000	545	1.35	2.17	19.9	8830	1.58	13942
1500	472	2.53	4.67	29.3	16490	1.46	24061
1000	385	22.4	50.6	96.3	145835	1.19	173113

Jet Pressure (psig)	Water Volume (gal)	Tank Diameter (ft)	Wall Thickness (in.)	Steel Volume (ft^3)	Steel Mass (ton)
6000	4227	10.3	4.6	127.2	30.9
5500	4546	10.5	4.3	125.4	30.5
5000	4936	10.8	4.1	123.7	30.1
4500	5426	11.1	3.8	122.4	29.8
4000	6061	11.6	3.5	121.5	29.6
3500	6924	12.1	3.2	121.5	29.6
3000	8174	12.8	2.9	122.9	29.9
2500	10172	13.7	2.6	127.5	31.0
2000	13942	15.3	2.3	139.8	34.0
1500	24061	18.3	2.1	180.9	44.0
1000	173113	35.4	2.7	867.9	211.2



Open-Ended Problem Statement: Analyze the design and optimize the performance of a cart propelled along a horizontal track by a water jet that issues under gravity from an open cylindrical tank carried on board the cart. (A water-jet-propelled cart is shown in the diagram for Problem 4.121.) Neglect any change in slope of the liquid free surface in the tank during acceleration. Analyze the motion of the cart along a horizontal track, assuming it starts from rest and begins to accelerate when water starts to flow from the jet. Derive algebraic equations or solve numerically for the acceleration and speed of the cart as functions of time. Present results as plots of acceleration and speed versus time, neglecting the mass of the tank. Determine the dimensions of a tank of minimum mass required to accelerate the cart from rest along a horizontal track to a specified speed in a specified time interval.

Discussion: This problem solution consists of two parts. The first is to analyze the acceleration and velocity of a cart propelled by a gravity-driven water jet. The second is to optimize the dimensions of the cart and jet to accelerate to a specified speed in a specified time interval.

To analyze the problem, apply conservation of mass and the Bernoulli equation to the draining of the tank, then apply the x component of the momentum equation for a control volume to analyze the resulting linear acceleration. A representative plot of the results is presented below.

To optimize the performance of the water-jet-propelled cart, manipulate the solution dimensions until the best performance is attained.

Input Data:

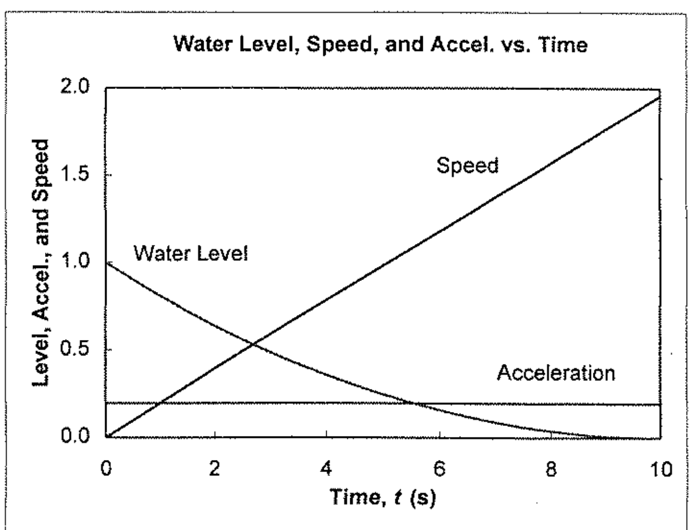
d =	10	mm	Diameter of water jet
D =	100	mm	Diameter of tank
g =	9.81	ft/s^2	Acceleration of gravity
H =	150	mm	Height of tank
M_t =	0.001	kg	Mass of tank
ρ =	999	kg/m^3	Density of water

Calculated Parameters:

a =	0.029	(---)	(a^2 =) Ratio of mass of tank to initial mass of water
b =	0.0572	s^{-1}	Geometric parameter of solution
M_0 =	1.18	kg	Initial mass of water in tank
β =	0.1	(---)	Ratio of jet diameter to tank diameter

Calculated Results:

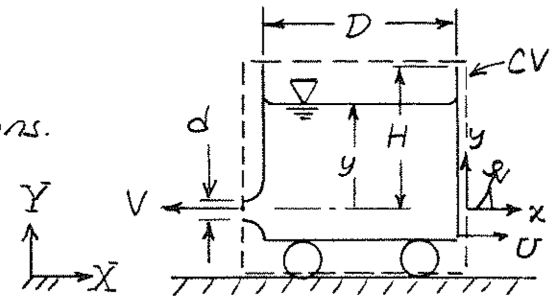
Time, t (s)	Level Ratio, y/H (---)	Accel., a_x (m/s^2)	Velocity, U (m/s)
0	1	0.196	0
1	0.810	0.196	0.196
2	0.640	0.196	0.392
3	0.490	0.196	0.588
4	0.360	0.196	0.784
5	0.250	0.196	0.980
6	0.160	0.196	1.176
7	0.0900	0.196	1.37
8	0.0400	0.196	1.57
9	0.0100	0.196	1.76
10	0	0.195	1.96



Given: Cart, propelled by water jet, accelerates along horizontal track.

Find: (a) Analyze motion, derive algebraic equations for acceleration and speed of cart as functions of time
(b) Plot acceleration and speed vs. time.

<u>Solution</u>: Apply conservation of mass, Bernoulli, and momentum equations.



Basic equations:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A}$$

$$\frac{p_j}{\rho} + \frac{V_j^2}{2} + g y_j = \frac{p}{\rho} + \frac{V^2}{2} + g y$$

(with p_j/ρ and p/ρ struck out, marked $=0(8)$; y_j struck out)

M_t = mass of tank, cart

$\beta = \frac{d}{D}$

$$F_{S_x} + F_{B_x} - \int_{CV} a_{rf_x} \rho \, d\forall = \frac{\partial}{\partial t}\int_{CV} u \rho \, d\forall + \int_{CS} u \rho \vec{V}\cdot d\vec{A}$$

(with $F_{S_x} = 0(9)$, $F_{B_x} = 0(10)$, $\frac{\partial}{\partial t} \approx 0(11)$)

Assumptions: (1) Uniform flow from exit jet (2) Neglect air in CV

$$0 = \frac{\partial}{\partial t}(\rho A_t y) + \{+|\rho V_j A_j|\} = \rho A_t \frac{dy}{dt} + \rho V_j A_j = -\rho A_t V + \rho V_j A_j \quad (1)$$

Thus $V = V_j \frac{A_j}{A_t} = V_j\left(\frac{d}{D}\right)^2 = \beta^2 V_j$ (2)

(3) No slope to free surface (given)
(4) Quasi-steady flow
(5) Frictionless flow
(6) Incompressible flow
(7) Flow along a streamline
(8) $p = p_j = p_{atm}$
(9) $y_j = 0$

From Bernoulli, $\frac{V_j^2}{2} = \frac{V^2}{2} + g y$ or $V_j^2 - V^2 = 2gy$

Substituting from (2), $V_j^2 - \beta^4 V_j^2 = V_j^2(1-\beta^4) = 2gy$; $V_j^2 = \frac{2gy}{(1-\beta^4)}$ (3)

Substituting into (1), $\frac{dy}{dt} = -\beta^2 V_j = -\beta^2 \frac{\sqrt{2gy}}{(1-\beta^4)}$ or $\frac{dy}{y^{1/2}} = -\frac{\beta^2\sqrt{2g}}{1-\beta^4} dt$

Integrating, $2y^{1/2}\Big]_{y_0}^{y} = -\frac{\beta^2\sqrt{2g}}{(1-\beta^4)} t$ or $y^{1/2} - y_0^{1/2} = -\frac{\beta^2\sqrt{2g}}{2(1-\beta^4)} t$

Thus $\left(\frac{y}{y_0}\right)^{1/2} = 1 - \left[\frac{g\beta^4}{2y_0(1-\beta^4)}\right]^{1/2} t = 1 - bt$; $b = \left[\frac{g\beta^4}{2y_0(1-\beta^4)}\right]^{1/2}$ (4)

From momentum (10) $F_{S_x} = 0$; no resistance
(11) $F_{B_x} = 0$; horizontal motion
(12) $u \approx 0$ in CV, so $\partial/\partial t \approx 0$

Then

$$-a_{rf_x} M(t) = u_j \{+|\rho V_j A_j|\} = -\rho V_j^2 A_j \qquad (5)$$

$$a_{rf_x} = \frac{dU}{dt} \qquad u_j = -V_j$$

But from (4), $M(t) = M_t + \rho A_t y = M_t + \rho A_t y_0 (1-bt)^2$

From (3), $V_j^2 = \frac{2gy}{1-\beta^4} = \frac{2g}{1-\beta^4} y_0 (1-bt)^2$

Substituting into (5)

$$\frac{dU}{dt}\left[M_t + \rho A_t y_0 (1-bt)^2\right] = \rho A_j \frac{2g}{1-\beta^4} y_0 (1-bt)^2 = \rho A_t y_0 \frac{2g\beta^2}{1-\beta^4}(1-bt)^2$$

Define M_0 = initial mass of water $= \rho A_t y_0$. Then

$$\frac{dU}{dt}\left[M_t + M_0 (1-bt)^2\right] = M_0 \frac{2g\beta^2}{1-\beta^4}(1-bt)^2$$

or

$$\frac{dU}{dt} = \frac{2g\beta^2}{1-\beta^4} \frac{M_0(1-bt)^2}{M_t + M_0(1-bt)^2} \qquad (6) \qquad \frac{dU}{dt}(t)$$

To integrate, let $r = 1-bt$, $dr = -b\,dt$, and $a^2 = M_t/M_0$. Then

$$U = \int_0^U dU = \frac{2g\beta^2}{1-\beta^4}\left(-\frac{1}{b}\right)\int_0^t \frac{r^2}{a^2+r^2}\,dr = -\frac{2g\beta^2}{1-\beta^4}\frac{1}{b}\left[r - a\tan^{-1}\left(\frac{r}{a}\right)\right]_0^t$$

$$= -\frac{2g\beta^2}{1-\beta^4}\frac{1}{b}\left[(1-bt) - a\tan^{-1}\left(\frac{1-bt}{a}\right)\right]_0^t$$

$$U = -\frac{2g\beta^2}{1-\beta^4}\frac{1}{b}\left[(1-bt) - a\tan^{-1}\left(\frac{1-bt}{a}\right) - 1 + a\tan^{-1}\left(\frac{1}{a}\right)\right]$$

Simplifying, then

$$U = \frac{2g\beta^2}{1-\beta^4}\left\{t + \frac{a}{b}\left[\tan^{-1}\left(\frac{1-bt}{a}\right) - \tan^{-1}\left(\frac{1}{a}\right)\right]\right\} \qquad (7) \qquad U(t)$$

$$a^2 = \frac{M_t}{M_0}\,;\quad b = \left[\frac{g\beta^4}{2y_0(1-\beta^4)}\right]^{1/2}$$

Given: Cart, propelled by water jet, accelerating on horizontal track.

$$\frac{dU}{dt} = \frac{2g\beta^2}{1-\beta^4}\frac{(1-bt)^2}{a^2+(1-bt)^2} \qquad (1)$$

$$U(t) = \frac{2g\beta^2}{1-\beta^4}\left\{t + \frac{a}{b}\left[\tan^{-1}\left(\frac{1-bt}{a}\right) - \tan^{-1}\left(\frac{1}{a}\right)\right]\right\} \qquad (2)$$

$$\beta = \frac{d}{D}, \quad a^2 = \frac{M_t}{M_0}, \quad b = \left[\frac{g\beta^4}{2y_0(1-\beta^4)}\right]^{1/2}$$

Find: (a) Shape for tank of minimum mass for given volume.
(b) Minimum water volume to reach $U = 2.5$ m/sec in $t = 25$ sec.

Solution: Mass of tank is $M = \rho_t A_s t$, where t = thickness of wall

$$A_s = A_{bottom} + A_{cylinder} = \pi\frac{D^2}{4} + \pi DH$$

Since volume is $\forall = \frac{\pi D^2}{4}H$, then $H = \frac{4\forall}{\pi D^2}$, and

$$A_s = \frac{\pi D^2}{4} + \pi D\left(\frac{4\forall}{\pi D^2}\right) = \frac{\pi D^2}{4} + \frac{4\forall}{D}$$

To minimize, set $dA_s/dD = 0$

$$\frac{dA_s}{dD} = \frac{\pi D}{2} + (-1)\frac{4\forall}{D^2} = 0 \quad \text{so} \quad D^3 = \frac{8\forall}{\pi} \quad \text{or} \quad D = \left(\frac{8\forall}{\pi}\right)^{1/3} \qquad (3) \quad D_{opt}$$

$$\text{Then} \quad \forall = \frac{\pi D^2 H}{4} = \frac{\pi D^3}{8} \quad \text{so} \quad \frac{H}{D} = \frac{1}{2} \qquad (4) \quad \left.\frac{H}{D}\right)_{opt}$$

The tank mass per volume for optimum H/D is

$$m = \frac{M}{\forall} = \frac{\rho_t\left(\frac{\pi D^2}{4} + \pi DH\right)t}{\frac{\pi D^2}{4}H} = \rho_t\left(\frac{t}{H} + \frac{4t}{D}\right) = \rho_t\frac{t}{H}\left(1 + 4\frac{H}{D}\right) = 3\rho_t\frac{t}{H}$$

Therefore mass depends on $\rho_t t$ for a given volume. The minimum mass is achieved for the smallest combination of ρ_t and t.

$$a^2 = \frac{M_t}{M_0} = \frac{M_t}{\rho\forall} = \frac{3\rho_t}{\rho}\frac{t}{H} = 3\,SG\left(\frac{t}{H}\right) \qquad (5)$$

which still depends on Volume, since it contains H.

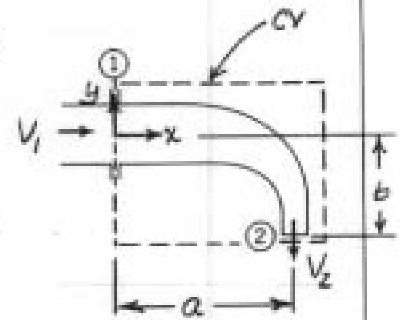
The best solution strategy seems to be: pick $\forall$, calculate H, D, β, a, and b, then plot $U(t)$.

Problem *4.165

Given: The 90° reducing elbow of Example Problem 4.7 discharges to atmosphere. Section ② is located 0.3 m to the right of Section ①.

Find: Estimate the moment exerted by the flange on the elbow.

Solution: Apply moment of momentum, using the CV and CS shown.



From Example Problem 4.7, $\vec{V}_2 = -16\hat{j}$ m/s, $A_1 = 0.01\ m^2$

Steady flow, $A_2 = 0.0025\ m^2$

Basic equation (fixed CV):

$$\vec{r}\times\vec{F}_s + \int_{CV}\vec{r}\times\vec{g}\rho\, d\forall + \vec{T}_{shaft} = \frac{\partial}{\partial t}\int_{CV}\vec{r}\times\vec{V}\rho\, d\forall + \int_{CS}\vec{r}\times\vec{V}\rho\vec{V}\cdot d\vec{A}$$

(terms marked $=0(1)$, $=0(2)$, $=0(3)$)

Assumptions: (1) Neglect body forces
(2) No shafts, so $\vec{T}_{shaft} = 0$
(3) Steady flow (given)
(4) Uniform flow at each cross section
(5) Incompressible flow

Then

$$\vec{M}_{flange} = \vec{r}\times\vec{F}_s\Big)_{flange} = \vec{r}_1\times\vec{V}_1\{-\rho V_1 A_1\} + \vec{r}_2\times\vec{V}_2\{+\rho V_2 A_2\} \qquad (1)$$

$$\vec{r}_1 = 0 \qquad \left.\begin{array}{l}\vec{r}_2 = a\hat{i} - b\hat{j} \\ \vec{V}_2 = -V_2\hat{j}\end{array}\right\} \vec{r}_2\times\vec{V}_2 = -aV_2\hat{k} + 0$$

Substituting into Eq. 1,

$$\vec{M}_{flange} = -aV_2\hat{k}\{+\rho V_2 A_2\} = -a\rho V_2^2 A_2\hat{k}$$

$$= 0.3\ m \times 999\ \frac{kg}{m^3}\times(16)^2\frac{m^2}{s^2}\times 0.0025\ m^2 \times \frac{N\cdot s^2}{kg\cdot m}(-\hat{k})$$

$$\vec{M}_{flange} = -192\ \hat{k}\ N\cdot m$$

$\vec{M}_{flange}$

This is the torque that must be exerted <u>on</u> the CV <u>by</u> the flange.

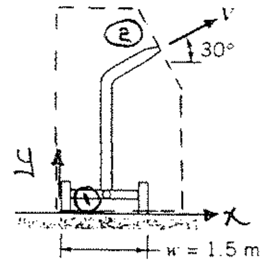
{ Since $\vec{M}_{flange}$ is in the $-\hat{k}$ direction, it must act cw in the xy-plane. }

Given: Irrigation sprinkler mounted on cart

$V = 40\,m/s$ $\theta = 30°$

$D = 50\,mm$ Flow is water

$h = 3\,m$ $M = 350\,kg$



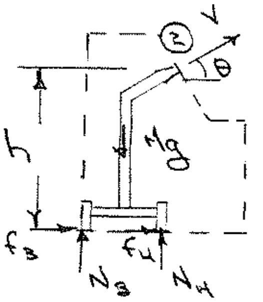
Find: (a) Magnitude of moment which tends to overturn the cart

(b) Value of V to cause impending motion; nature of impending motion.

(c) Effect on jet inclination on results

Plot: Jet velocity as a function of θ for the case of impending motion.

Solution:



Apply moment of momentum equation, using fixed CV shown at left. Origin of coordinates is on ground at left wheel of cart. With this coordinate system counterclockwise moments are positive (about the z axis).

Basic equation:

$$\vec{r}\times\vec{F}_s + \int_{CV}\vec{r}\times\vec{g}\,\rho\,dV + \vec{T}_{shaft} = \frac{\partial}{\partial t}\int_{CV}\vec{r}\times\vec{V}\rho\,dV + \int_{CS}\vec{r}\times\vec{V}(\rho\vec{V}\cdot d\vec{A})$$

($\vec{T}_{shaft} = 0$ (1); $\frac{\partial}{\partial t} = 0$ (2))

Assumptions: (1) $\vec{T}_{shaft} = 0$

(2) steady flow

(3) uniform flow at nozzle outlet

(4) neglect $\vec{r}\times\vec{V}$ of inlet flow

(5) center of mass located at $x = w/2$

(6) nozzle length is short; coordinates of nozzle exit at $(x_2, y_2) = (w/2, h)$

Then

$$\vec{r}\times\vec{F}_s + \vec{r}\times M\vec{g} = \vec{r}_1\times\vec{V}_1\{-|\rho V_1 A_1|\} + \vec{r}_2\times\vec{V}_2\{|\rho_2 V_2 A_2|\}$$

(first term on right $= 0$ (4))

$$r_2 = \frac{w}{2}\hat{i} + h\hat{j} \qquad V_2 = V(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}).$$

and

$$wN_4\hat{k} - \frac{w}{2}Mg\hat{k} = \frac{w}{2}V\sin\theta\,\dot{m}_2\hat{k} - hV\cos\theta\,\dot{m}\,\hat{k}$$

$$wN_4 - \frac{w}{2}Mg = \dot{m}_2V\left[\frac{w}{2}\sin\theta - h\cos\theta\right] \quad ------(1)$$

Rewriting Eq. 1 in the form $\Sigma M_z = 0$ {for static equilibrium}

$$wN_4 - \frac{w}{2}Mg + \dot{m}_2V\left[h\cos\theta - \frac{w}{2}\sin\theta\right] = 0 \quad ---(2)$$

The last term in Eq 2 is the moment (due to the jet) which tends to overturn the cart.

Evaluating, $\dot{m}_2 = \rho A_2 V_2 = \rho \frac{\pi D^2}{4} V_2$

$$\dot{m}_2 = 999 \frac{kg}{m^3} \times \frac{\pi}{4} (0.05)^2 m^2 \times 40 \frac{m}{s} = 78.5 \ kg/s$$

Then with $V_2 = 40 m/s$

$$\text{Moment from jet} = 78.5 \frac{kg}{s} \times 40 \frac{m}{s} \times \frac{N \cdot s^2}{kg \cdot m} \left[3m \cos 30° - \frac{1.5m}{2} \sin 30° \right]$$

$$\text{Moment}_{jet} = 6.98 \ kN \cdot m$$ ← Moment$_{jet}$

For the case of impending tipping (about point 3)

$N_4 \to 0$ and from Eq. 2

$$-\frac{W}{2} Mg + \dot{m}_2 V \left[h\cos\theta - \frac{W}{2} \sin\theta \right] = 0$$

To solve for V_2, write $\dot{m} = \rho A_2 V_2$

$$V_2^2 = \frac{W M g}{2 \rho A_2 \left[h\cos\theta - \frac{W}{2}\sin\theta \right]} \quad (3)$$

$$V_2^2 = \frac{1.5m}{2} \times 350kg \times 9.81 \frac{m}{s^2} \times \frac{m^3}{999 kg} \times \frac{1}{1.96 \times 10^{-3} m^2} \times \frac{1}{(3\cos 30° - 0.75 \sin 30°) m}$$

$$V_2^2 = 592 \ m^2 s^2 \qquad \therefore V_2 = 24.3 \ m/s$$ ← V_2

Thus, the maximum speed allowable without tipping is less than the value suggested.

The impending motion will be tipping since $f_3 < \mu N_3$

From the x momentum equation $f_3 = \dot{m} V_2 \cos\theta$

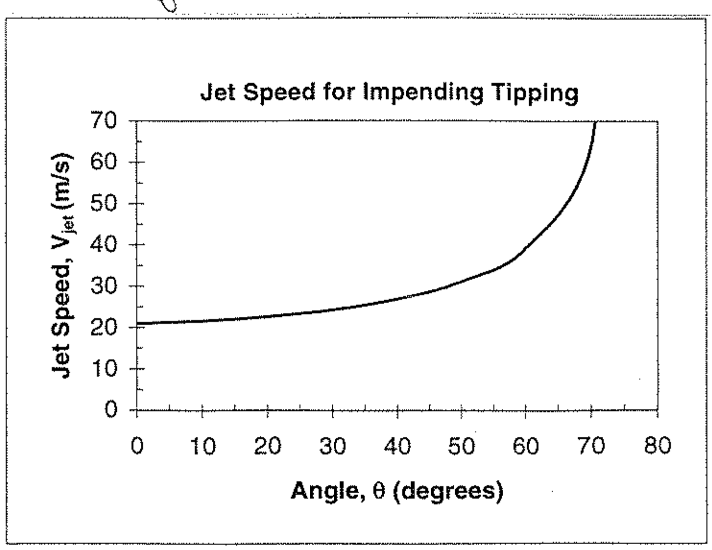
From the y momentum equation $N_3 = Mg + \dot{m} V_2 \sin\theta$

For tipping $\mu > 0.377$

From Eq. 2 we see that as θ increases the tendency to tip decreases

For impending motion from Eq. 3.

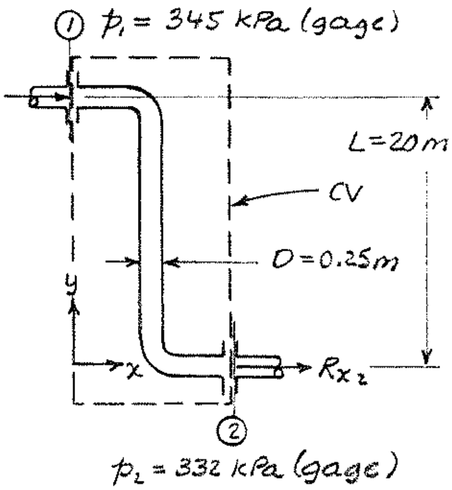
$$V = \left\{ \frac{W M g}{2\rho A_2 \left[h\cos\theta - \frac{W}{2}\sin\theta \right]} \right\}^{1/2}$$



Problem 4.167

Given: Crude oil ($SG = 0.95$) flow through a pipe assembly in the horizontal configuration shown.

$Q = 0.58\,m^3/s$



Find: Force and torque exerted by assembly on its supports.

Solution: No momentum components exist in the y direction. Apply x component of linear momentum and the moment of momentum equations using the CV shown. Location of coordinates is arbitrary; for simplicity, choose as shown.

Basic equations:

$$F_{S_x} + \underbrace{F_{B_x}}_{=0(1)} = \underbrace{\frac{\partial}{\partial t}\int_{CV} u\rho\,d\forall}_{=0(2)} + \int_{CS} u\rho\vec{V}\cdot d\vec{A}$$

$$\vec{r}\times\vec{F}_s + \underbrace{\int_{CV}\vec{r}\times\vec{g}\,\rho\,d\forall}_{=0(4)} + \underbrace{\vec{T}_{shaft}}_{=0(5)} = \underbrace{\frac{\partial}{\partial t}\int_{CV}\vec{r}\times\vec{V}\rho\,d\forall}_{=0(2)} + \int_{CS}\vec{r}\times\vec{V}\,\rho\vec{V}\cdot d\vec{A}$$

Assumptions:
(1) $F_{B_x} = 0$; $\vec{g}$ acts in z direction
(2) Steady flow
(3) Uniform flow at each section
(4) No z component of $\vec{r}\times\vec{g}$
(5) $\vec{T}_{shaft} = 0$

$$A = \frac{\pi D^2}{4} = \frac{\pi}{4}(0.25)^2\,m^2 = 0.049\ m^2$$

From momentum equation,

$$R_{x_1} + R_{x_2} + p_1A - p_2A = u_1\{-\dot{m}\} + u_2\{\dot{m}\} = 0\ ;\quad R_{x_1} + R_{x_2} = (p_2 - p_1)A$$

From moment of momentum,

$$\vec{r}_1\times(R_{x_1} + p_1A)\hat{i} + \underbrace{\vec{r}_2}_{=0}\times(R_{x_2} - p_2A)\hat{i} = \vec{r}_1\times V_1\hat{i}\{-\dot{m}\} + \underbrace{\vec{r}_2}_{=0}\times V_2\hat{i}\{\dot{m}\}\ ;\quad \vec{r}_1 = L\hat{j},\ \vec{r}_1\times\hat{i} = -L\hat{k}$$

$$-L(R_{x_1} + p_1A)\hat{k} = -LV_1(-\dot{m})\hat{k} = LV_1\dot{m}\,\hat{k} = L\frac{Q}{A}(\rho Q)\hat{k} = L\frac{\rho Q^2}{A}\hat{k}$$

$$R_{x_1} = -\frac{\rho Q^2}{A} - p_1A = -0.95\times 999\frac{kg}{m^3}\times(0.58)^2\frac{m^6}{s^2}\times\frac{1}{0.049\,m^2}\times\frac{N\cdot s^2}{kg\cdot m} - 3.45\times10^5\frac{N}{m^2}\times 0.049\,m^2 = -23.4\ kN$$

$$R_{x_2} = (p_2 - p_1)A - R_{x_1} = p_2A - p_1A + \frac{\rho Q^2}{A} + p_1A = p_2A + \frac{\rho Q^2}{A}$$

$$= 3.32\times10^5\frac{N}{m^2}\times 0.049\,m^2 + 0.95\times 999\frac{kg}{m^3}\times(0.58)^2\frac{m^6}{s^2}\times\frac{1}{0.049\,m^2}\times\frac{N\cdot s^2}{kg\cdot m} = 22.8\ kN$$

$$\vec{r}\times\vec{F}_s = \vec{r}_1\times R_{x_1}\hat{i} = L\hat{j}\times R_{x_1}\hat{i} = -LR_{x_1}\hat{k} = -20\,m\times(-46.0)\,kN\,\hat{k} = 468\,\hat{k}\ kN\cdot m$$

These are forces and torque on CV. The corresponding reactions are:

$$K_{x_1} = -R_{x_1} = 23.4\ kN,\quad K_{x_2} = -R_{x_2} = -22.8\ kN$$ ← Force

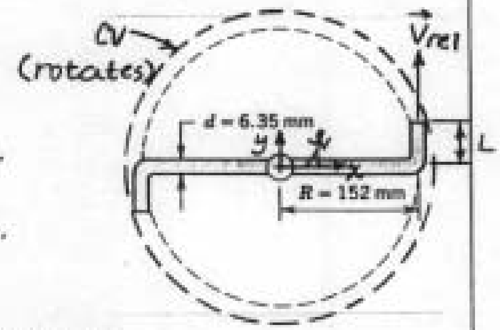
$$\vec{M} = -\vec{r}\times\vec{F}_s = -468\,\hat{k}\ kN\cdot m \quad (i.e.\ clockwise)$$ ← Torque

Problem *4.168

Given: Simplified lawn sprinkler rotating in horizontal plane, $Q = 4.5$ gal/min.

Water discharges horizontally from jets.

Neglect pivot friction, inertia of sprinkler.



Find: (a) Torque needed to hold at $\omega = 0$.
(b) Angular acceleration when torque is removed.

Solution: Choose rotating CV. Apply angular momentum principle, Eq. 4.53.

Basic equation:
$$\vec{r}\times\vec{F}_s \,(=0(1)) + \int_{CV} \vec{r}\times\vec{g}\,\rho\, d\forall \,(=0(2)) + \vec{T}_{shaft} - \int_{CV} \vec{r}\times[2\vec{\omega}\times\vec{V}_{xyz} \,(=0(3)) + \vec{\omega}\times(\vec{\omega}\times\vec{r}) \,(=0(3)) + \dot{\vec{\omega}}\times\vec{r}]\,\rho\, d\forall$$
$$= \frac{\partial}{\partial t}\int_{CV} \vec{r}\times\vec{V}_{xyz}\,\rho\, d\forall \,(=0(4)) + \int_{CS} \vec{r}\times\vec{V}_{xyz}\,\rho\vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: (1) No surface forces
(2) Body torques cancel
(3) Sprinkler stationary, $\vec{\omega} = 0$
(4) Steady flow
(5) Uniform flow at each section
(6) $L \ll R$

Analyze right arm of sprinkler. From geometry $\vec{r} = r\hat{i}$ in cv, $\vec{r} = R\hat{i}$ at jet. Then

$$T\hat{k} - \int_{CV} \vec{r}\times(\dot{\vec{\omega}}\times\vec{r})\rho\, d\forall = R\hat{i}\times V\hat{j}\,\rho\frac{Q}{2} = \rho\frac{QRV}{2}\hat{k} = \frac{\dot{m}RV}{2}\hat{k}$$
$$r\hat{i}\times(\dot{\omega}\hat{k}\times r\hat{i}) = r\hat{i}\times\dot{\omega}r\hat{j} = \dot{\omega}r^2\hat{k};\quad \int_{CV} = \dot{\omega}\frac{R^3}{3}\rho A\,\hat{k}$$

Dropping $\hat{k}$, $T - \frac{\dot{\omega}\rho A R^3}{3} = \frac{\dot{m}RV}{2}$. When arm is stationary, $\dot{\omega} = 0$, and

$$T = \frac{\dot{m}RV}{2} \qquad \dot{m} = \rho Q = 999\frac{kg}{m^3}\times 4.5\frac{gal}{min}\times 231\frac{in.^3}{gal}\times(0.0254)^3\frac{m^3}{in.^3}\times\frac{min}{60\,s} = 0.284\ \frac{kg}{s}$$
$$V = \frac{Q}{2A} = \frac{2Q}{\pi d^2} = \frac{2}{\pi}\times 2.84\times10^{-4}\frac{m^3}{s}\times\frac{1}{(0.00635)^2 m^2} = 4.48\ m/s$$
$$T = \frac{1}{2}\times 0.284\frac{kg}{sec}\times 0.152\,m\times 4.48\frac{m}{sec} = 0.0967\ N\cdot m\ (\text{per arm})$$

For two arms, $T_2 = 2T = 2\times 0.0967\ N\cdot m = 0.193\ N\cdot m$ $\qquad T_2$

When torque is removed, angular acceleration would be the same for each arm. Thus

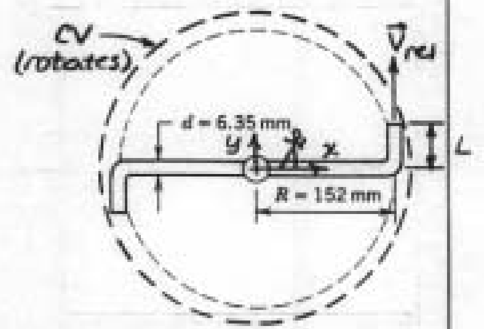
$$\dot{\omega} = \frac{\dot{m}RV}{2}\times\frac{3}{\rho A R^3} = \frac{3\dot{m}V}{2\rho A R^2}$$
$$\dot{\omega} = \frac{3}{2}\times 0.284\frac{kg}{s}\times 4.48\frac{m}{s}\times\frac{m^3}{999\,kg}\times\frac{4}{\pi(0.00635)^2 m^2}\times\frac{1}{(0.152)^2 m^2} = 2610\ rad/s^2$$ $\qquad \dot{\omega}$

Problem *4.169

Given: Simplified lawn sprinkler rotating in horizontal plane, Q = 4.5 gal/min.

Water discharges horizontally from jets.

Neglect pivot friction, inertia of sprinkler.



Find: (a) Derive a differential equation for angular speed as a function of time.
(b) Evaluate steady-state angular speed.

Solution: Choose rotating CV. Apply angular momentum principle, Eq. 4.53.

Basic equation: $\vec{r}\times\vec{F}_s + \int_{CV}\vec{r}\times\vec{g}\,\rho\,d\forall + \vec{T}_{shaft}$ (=0(1), =0(2), =0(3))

$$-\int_{CV}\vec{r}\times[2\vec{\omega}\times\vec{V}_{xyz}+\vec{\omega}\times(\vec{\omega}\times\vec{r})+\dot{\vec{\omega}}\times\vec{r}]\rho\,d\forall \quad (=0(4))$$

$$=\frac{\partial}{\partial t}\int_{CV}\vec{r}\times\vec{V}_{xyz}\rho\,d\forall + \int_{CS}\vec{r}\times\vec{V}_{xyz}\,\rho\vec{V}_{xyz}\cdot d\vec{A} \quad (=0(5))$$

Assumptions: (1) $\vec{F}_s = 0$, (2) Body torques cancel, (3) $\vec{T}_{shaft} = 0$, (4) No $\hat{k}$ component of centripetal acceleration, (5) Steady flow, (6) $L \ll R$

Analyze right arm of sprinkler. From geometry, $\vec{r} = r\hat{i}$ in CV, $\vec{r} = R\hat{i}$ at jet. Then

$$-\int_{CV} r\hat{i}\times[2\omega\hat{k}\times V\hat{i}+\dot{\omega}\hat{k}\times r\hat{i}]\rho A\,dr = R\hat{i}\times V\hat{j}\,\rho\frac{Q}{2} = \rho\frac{Q}{2}RV\hat{k}$$

$$r\hat{i}\times[2\omega V(+\hat{j})+\dot{\omega}r(+\hat{j})] = (2\omega rV+\dot{\omega}r^2)(+\hat{k});\ -\int_{CV} = -(\omega R^2V+\dot{\omega}\frac{R^3}{3})\rho A$$

Dropping $\hat{k}$,

$$-\omega\rho VAR^2 - \frac{\dot{\omega}\rho AR^3}{3} = \frac{\rho QRV}{2} \quad\text{or}\quad \dot{\omega} = \frac{3}{\rho AR^3}\left[-\omega\rho VAR^2 - \frac{\rho QRV}{2}\right]$$ O.D.E.

Thus $\frac{d\omega}{dt} = -a - b\omega$, where $a = \frac{3}{\rho AR^3}\frac{\rho QRV}{2} = \frac{3}{2}\frac{QV}{AR^2} = \frac{3V^2}{R^2}$, $b = \frac{3\rho VAR^2}{\rho AR^3} = \frac{3V}{R}$

$\frac{d\omega}{dt} = 0$ when $-a - b\omega_{max} = 0$, i.e., when $\omega_{max} = -a/b$. ↑Note $V = \frac{Q}{2A}$ (one arm)

$$Q = 4.5\,\frac{gal}{min}\times 231\,\frac{in^3}{gal}\times(0.0254)^3\,\frac{m^3}{in^3}\times\frac{min}{60\,s} = 2.84\times10^{-4}\ m^3/s$$

$$\omega_{max} = -\frac{a}{b} = -\frac{3V^2}{R^2}\times\frac{R}{3V} = -\frac{V}{R} = -4.48\,\frac{m}{s}\times\frac{1}{0.152\,m} = -29.5\ rad/s \quad (-281\ rpm)$$ ω_{max}

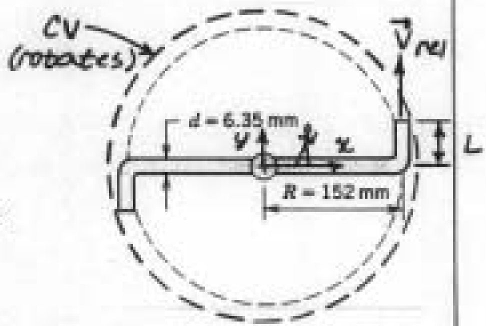
{Note it is not necessary to solve the differential equation to find ω_{max}.}

Problem *4.170

Given: Simplified lawn sprinkler rotating in horizontal plane, $Q = 4.5$ gal/min.

Water discharges horizontally from jets.

Neglect inertia of sprinkler; $T_f = 0.045$ ft·lbf



Find: (a) Derive a differential equation for angular speed as a function of time.
(b) Evaluate steady-state angular speed.

Solution: Choose rotating CV. Apply angular momentum principle, Eq. 4.53.

Basic equation:
$$\vec{r}\times\vec{F}_s^{\,=0(1)} + \int_{CV}\vec{r}\times\vec{g}^{\,=0(2)}\rho\,d\forall + \vec{T}_{shaft}$$
$$-\int_{CV}\vec{r}\times\left[2\vec{\omega}\times\vec{V}_{xyz} + \vec{\omega}\times(\vec{\omega}\times\vec{r})^{=0(4)} + \dot{\vec{\omega}}\times\vec{r}\right]\rho\,d\forall$$
$$= \frac{\partial}{\partial t}^{=0(5)}\int_{CV}\vec{r}\times\vec{V}_{xyz}\rho\,d\forall + \int_{CS}\vec{r}\times\vec{V}_{xyz}\,\rho\vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: (1) $\vec{F}_s = 0$, (2) Body torques cancel, (3) $\vec{T}_{shaft} = 0.045$ ft·lbf, (4) No $\hat{k}$ component of centripetal acceleration, (5) Steady flow, (6) $L \ll R$.

Analyze right arm of sprinkler. From geometry, $\vec{r} = r\hat{i}$ in CV, $\vec{r} = R\hat{i}$ at jet. Then
$$-\int_{CV} r\hat{i}\times\left[2\omega\hat{k}\times V\hat{i} + \dot{\omega}\hat{k}\times r\hat{i}\right]\rho A\,dr = R\hat{i}\times V\hat{j}\,\rho\frac{Q}{2} = \rho\frac{QRV}{2}\hat{k}$$
$$r\hat{i}\times\left[2\omega V\hat{j} + \dot{\omega}r\hat{j}\right] = (2\omega V r + \dot{\omega}r^2)\hat{k}\,;\quad -\int_{CV} = -\left(\omega V R^2 + \frac{\dot{\omega}R^3}{3}\right)\rho A\,\hat{k}$$

For both arms, dropping $\hat{k}$, $\{T = 0.045 \text{ ft·lbf} = 0.0610 \text{ N·m}\}$

$$T - 2\omega\rho VAR^2 - \frac{2\dot{\omega}\rho AR^3}{3} = \rho QRV \quad\text{or}\quad \dot{\omega} = \frac{3}{2\rho AR^3}\left[T - \rho QRV - 2\omega\rho VAR^2\right]$$ O.D.E.

Thus $\frac{d\omega}{dt} = a - b\omega$, where $a = \frac{3}{2\rho AR^3}(T - \rho QRV)$, $b = \frac{3}{2\rho AR^3}\times 2\rho VAR^2 = 3\frac{V}{R}$

The steady-state speed occurs when $\frac{d\omega}{dt} = 0$, i.e. when $\omega_{max} = \frac{a}{b}$

$$Q = 4.5\frac{\text{gal}}{\text{min}}\times 231\frac{\text{in}^3}{\text{gal}}\times(0.0254)^3\frac{\text{m}^3}{\text{in}^3}\times\frac{\text{min}}{60\text{ s}} = 2.84\times10^{-4}\text{ m}^3/\text{s}\,;\quad A = \frac{\pi d^2}{4} = 3.17\times10^{-5}\text{ m}^2$$

From the O.D.E., $\omega_{max} = \frac{T - \rho QRV}{2\rho VAR^2}$

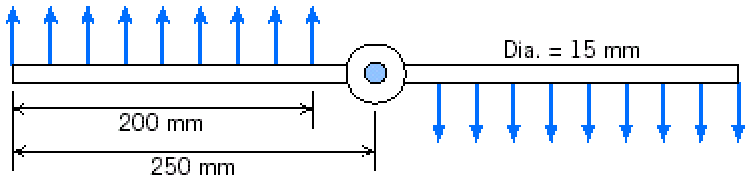
$$\omega_{max} = \frac{1}{2}\left[0.0610\text{ N·m}\times\frac{\text{kg·m}}{\text{N·s}^2} - 999\frac{\text{kg}}{\text{m}^3}\times 2.84\times10^{-4}\frac{\text{m}^3}{\text{s}}\times 0.152\text{ m}\times 4.48\frac{\text{m}}{\text{s}}\right]\frac{\text{m}^3}{999\text{ kg}}\times\frac{\text{s}}{4.48\text{ m}}\times\frac{1}{3.17\times10^{-5}\text{ m}^2}\times\frac{1}{(0.152)^2\text{ m}^2}$$

$\omega_{max} = -20.2$ rad/s $(-193$ rpm$)$ ω_{max}

Problem *4.171

Water flows in a uniform flow out of the 5 mm slots of the rotating spray system as shown. The flow rate is 15 kg/s. Find the torque required to hold the system stationary, and the steady-state speed of rotation after it is released.

Given: Data on rotating spray system



Solution

The given data is $\psi \mid 999 \cdot \frac{kg}{m^3}$ $\quad m_{flow} \mid 15 \cdot \frac{kg}{s}$

$D \mid 0.015 \cdot m \quad r_o \mid 0.25 \cdot m \quad r_i \mid 0.05 \cdot m \quad \iota \mid 0.005 \cdot m$

Governing equation: Rotating CV

$$\vec{r} \times \vec{F}_s + \int_{CV} \vec{r} \times \vec{g} \, \rho \, d\mathcal{V} + \vec{T}_{shaft} - \int_{CV} \vec{r} \times [2\vec{\omega} \times \vec{V}_{xyz} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \dot{\vec{\omega}} \times \vec{r}] \, \rho \, d\mathcal{V} = \frac{\partial}{\partial t} \int_{CV} \vec{r} \times \vec{V}_{xyz} \, \rho \, d\mathcal{V} + \int_{CS} \vec{r} \times \vec{V}_{xyz} \, \rho \vec{V}_{xyz} \cdot d\vec{A} \quad (4.52)$$

For no rotation (ω = 0) this equation reduces to a single scalar equation

$$T_{shaft} \mid \int r \Delta V_{xyz} \cdot \psi \cdot V_{xyz} \, dA$$

or $$T_{shaft} = 2\cdot\pi\cdot\int_{r_i}^{r_o} r\cdot V\cdot\rho\cdot V\,dr = 2\cdot\rho\cdot V^2\cdot\pi\cdot\int_{r_i}^{r_o} r\,dr = \rho\cdot V^2\cdot\pi\cdot\left(r_o^{\,2} - r_i^{\,2}\right)$$

where *V* is the exit velocity with respect to the CV

$$V = \frac{\frac{m_{flow}}{\rho}}{2\cdot\pi\cdot\left(r_o - r_i\right)}$$

Hence $$T_{shaft} = \rho\cdot\left[\frac{\frac{m_{flow}}{\rho}}{2\cdot\pi\cdot\left(r_o - r_i\right)}\right]^2\cdot\pi\cdot\left(r_o^{\,2} - r_i^{\,2}\right)$$

$$T_{shaft} = \frac{m_{flow}^{\,2}}{4\cdot\rho\cdot\pi}\cdot\frac{\left(r_o + r_i\right)}{\left(r_o - r_i\right)}$$

$$T_{shaft} = \frac{1}{4}\times\left(15\cdot\frac{kg}{s}\right)^2\times\frac{m^3}{999\cdot kg}\times\frac{1}{0.005\cdot m}\times\frac{(0.25+0.05)}{(0.25-0.05)}$$

$$T_{shaft} = 16.9\,N\cdot m$$

For the steady rotation speed the equation becomes

$$-\int \vec{r}\times\left(2\cdot\vec{\omega}\times\vec{V}_{xyz}\right)\cdot\rho\,dV = \int \vec{r}\times\vec{V}_{xyz}\cdot\rho\cdot\vec{V}_{xyz}\cdot d\vec{A}$$

The volume integral term $-\int_{CV} \vec{r} \times \left(2\vec{\omega} \times \vec{V}_{xyz}\right)\rho\, dV$ must be evaluated for the CV.

The velocity in the CV varies with *r*. This variation can be found from mass conservation

For an infinitesmal CV of length *dr* and cross-section *A* at radial position *r*, if the flow in is *Q*, the flow out is *Q* + *dQ*, and the loss through the slot is *V*δ*dr*. Hence mass conservation leads to

$$(Q + dQ) + V \cdot \delta \cdot dr - Q = 0$$

$$dQ = -V \cdot \delta \cdot dr$$

$$Q(r) = -V \cdot \delta \cdot r + \text{const}$$

At the inlet ($r = r_i$) $\quad Q = Q_i = \dfrac{m_{flow}}{2 \cdot \rho}$

Hence $\quad Q = Q_i + V \cdot \delta \cdot (r_i - r) = \dfrac{m_{flow}}{2 \cdot \rho} + \dfrac{m_{flow}}{2 \cdot \rho \cdot \delta \cdot (r_o - r_i)} \cdot \delta \cdot (r_i - r)$

$$Q = \frac{m_{flow}}{2 \cdot \rho} \cdot \left(1 + \frac{r_i - r}{r_o - r_i}\right) = \frac{m_{flow}}{2 \cdot \rho} \cdot \left(\frac{r_o - r}{r_o - r_i}\right)$$

and along each rotor the water speed is $v(r) = \dfrac{Q}{A} = \dfrac{m_{flow}}{2 \cdot \rho \cdot A} \cdot \left(\dfrac{r_o - r}{r_o - r_i}\right)$

Hence the term $-\int \mathrm{r}\,\Delta\, 2\,\varpi\,\Delta\,\mathrm{V}_{xyz}\,\psi\, dV$ becomes

$$4\int \mathrm{r}\,\Delta\, 2\,\varpi\,\Delta\,\mathrm{V}_{xyz}\,\psi\, dV \;|\; 4\,\psi\,A\,\varpi\int_{r_i}^{r_o} r\,V(r)\,dr \;|\; 4\,\psi\,\varpi\int_{r_i}^{r_o} r\left(\frac{m_{flow}}{2\,\psi}\,\frac{r_o\,4\,r}{r_o\,4\,r_i}\right)dr$$

or

$$4\int \mathrm{r}\,\Delta\, 2\,\varpi\,\Delta\,\mathrm{V}_{xyz}\,\psi\, dV \;|\; 2\,m_{flow}\,\varpi\int_{r_i}^{r_o} r\left(\frac{r_o\,4\,r}{r_o\,4\,r_i}\right)dr \;|\; m_{flow}\,\varpi\left(\frac{r_o^{\,3}\,2\,r_i^{\,2}\,(2\,r_i\,4\,3\,r_o)}{3\,(r_o\,4\,r_i)}\right)$$

Recall that $\int \mathrm{r}\,\Delta\,\mathrm{V}_{xyz}\,\psi\,\mathrm{V}_{xyz}\,dA \;|\; \psi\,V^2\,(r_o^{\,2}\,4\,r_i^{\,2})$

Hence equation $4\int \mathrm{r}\,\Delta\, 2\,\varpi\,\Delta\,\mathrm{V}_{xyz}\,\psi\, dV \;|\; \int \mathrm{r}\,\Delta\,\mathrm{V}_{xyz}\,\psi\,\mathrm{V}_{xyz}\,dA$ becomes

$$m_{flow}\,\varpi\left(\frac{r_o^{\,3}\,2\,r_i^{\,2}\,(2\,r_i\,4\,3\,r_o)}{3\,(r_o\,4\,r_i)}\right) \;|\; \psi\,V^2\,(r_o^{\,2}\,4\,r_i^{\,2})$$

Solving for ω $\quad \varpi \;|\; \dfrac{3\,(r_o\,4\,r_i)\,\psi\,V^2\,(r_o^{\,2}\,4\,r_i^{\,2})}{m_{flow}\,(r_o^{\,3}\,2\,r_i^{\,2}\,(2\,r_i\,4\,3\,r_o))}$ $\qquad \varpi \;|\; 461\,\text{rpm}$

Problem *4.172

If the same flow rate in the rotating spray system of Problem 4.171 is not uniform but instead varies linearly from a maximum at the outer radius to zero at a point 50 mm from the axis, find the torque required to hold it stationary, and the steady-state speed of rotation.

Given: Data on rotating spray system

Solution

The given data is $\psi \mid 999 \cdot \frac{kg}{m^3}$ $\quad m_{flow} \mid 15 \cdot \frac{kg}{s}$

$D \mid 0.015 \cdot m \qquad r_o \mid 0.25 \cdot m \qquad r_i \mid 0.05 \cdot m \qquad \iota \mid 0.005 \cdot m$

Governing equation: Rotating CV

$$\vec{r} \times \vec{F}_s + \int_{CV} \vec{r} \times \vec{g}\,\rho\, dV + \vec{T}_{shaft} - \int_{CV} \vec{r} \times [2\vec{\omega} \times \vec{V}_{xyz} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \dot{\vec{\omega}} \times \vec{r}]\,\rho\, dV = \frac{\partial}{\partial t} \int_{CV} \vec{r} \times \vec{V}_{xyz}\,\rho\, dV + \int_{CS} \vec{r} \times \vec{V}_{xyz}\,\rho \vec{V}_{xyz} \cdot d\vec{A} \qquad (4.52)$$

For no rotation (ω = 0) this equation reduces to a single scalar equation

$$T_{shaft} \mid \int r \,\Delta\, V_{xyz} \cdot \psi \cdot V_{xyz}\, dA$$

or $$T_{shaft} \mid 2 \cdot \iota \cdot \int_{r_i}^{r_o} r \cdot V \cdot \psi \cdot V\, dr$$

where *V* is the exit velocity with respect to the CV. We need to find *V*(*r*). To do this we use ma conservation, and the fact that the distribution is linear

$$V(r) = V_{max} \cdot \frac{(r - r_i)}{(r_o - r_i)}$$

and $$2 \cdot \frac{1}{2} \cdot V_{max} \cdot (r_o - r_i) \cdot \delta = \frac{m_{flow}}{\rho}$$

so $$V(r) = \frac{m_{flow}}{\rho \cdot \delta} \cdot \frac{(r - r_i)}{(r_o - r_i)^2}$$

Hence $$T_{shaft} = 2 \cdot \rho \cdot \delta \cdot \int_{r_i}^{r_o} r \cdot V^2 \, dr = 2 \cdot \frac{m_{flow}^2}{\rho \cdot \delta} \cdot \int_{r_i}^{r_o} r \cdot \left[\frac{(r - r_i)}{(r_o - r_i)^2} \right]^2 dr$$

$$T_{shaft} = \frac{m_{flow}^2 \cdot (r_i + 3 \cdot r_o)}{6 \cdot \rho \cdot \delta \cdot (r_o - r_i)}$$

$$T_{shaft} = \frac{1}{6} \times \left(15 \cdot \frac{kg}{s}\right)^2 \times \frac{m^3}{999 \cdot kg} \times \frac{1}{0.005 \cdot m} \times \frac{(0.05 + 3 \cdot 0.25)}{(0.25 - 0.05)}$$

$$T_{shaft} = 30 \, N \cdot m$$

For the steady rotation speed the equation becomes

$$-\int \vec{r} \times \left(2\vec{\omega} \times \vec{V}_{xyz}\right)\rho\, dV = \int \vec{r} \times \vec{V}_{xyz}\, \rho \vec{V}_{xyz} \cdot d\vec{A}$$

The volume integral term $-\int \vec{r} \times \left(2\vec{\omega} \times \vec{V}_{xyz}\right)\rho\, dV$ must be evaluated for the CV.

The velocity in the CV varies with *r*. This variation can be found from mass conservation

For an infinitesmal CV of length *dr* and cross-section *A* at radial position *r*, if the flow in is *Q*, the flow out is *Q* + *dQ*, and the loss through the slot is *Vδdr*. Hence mass conservation leads to

$$(Q + dQ) + V \cdot \delta \cdot dr - Q = 0$$

$$dQ = -V \cdot \delta \cdot dr$$

$$Q(r) = Q_i - \delta \int_{r_i}^{r} \frac{m_{flow}}{\rho \cdot \delta} \cdot \frac{(r - r_i)}{(r_o - r_i)^2}\, dr = Q_i - \int_{r_i}^{r} \frac{m_{flow}}{\rho} \cdot \frac{(r - r_i)}{(r_o - r_i)^2}\, dr$$

At the inlet ($r = r_i$) $\quad Q = Q_i = \dfrac{m_{flow}}{2 \cdot \rho}$

Hence $\quad Q(r) = \dfrac{m_{flow}}{2 \cdot \rho}\left[1 - \dfrac{(r - r_i)^2}{(r_o - r_i)^2}\right]$

and along each rotor the water speed is $v(r) = \frac{Q}{A} = \frac{m_{flow}}{2 \cdot \rho \cdot A} \cdot \left[1 - \frac{(r - r_i)^2}{(r_o - r_i)^2}\right]$

Hence the term $-\int r \times (2 \cdot \omega \times V_{xyz}) \cdot \rho \, dV$ becomes

$$-\rho \cdot A \cdot \omega \cdot \int_{r_i}^{r_o} r \cdot v(r) \, dr = -\rho \cdot \omega \cdot \int_{r_i}^{r_o} \frac{m_{flow}}{2 \cdot \rho} \cdot r \cdot \left[1 - \frac{(r - r_i)^2}{(r_o - r_i)^2}\right] dr$$

or

$$2 \cdot m_{flow} \cdot \omega \cdot \int_{r_i}^{r_o} r \cdot \left[1 - \frac{(r_o - r)^2}{(r_o - r_i)^2}\right] dr = m_{flow} \cdot \omega \cdot \left(\frac{1}{6} \cdot r_o^2 + \frac{1}{3} \cdot r_i \cdot r_o - \frac{1}{2} \cdot r_i^2\right)$$

Recall that $\int r \times V_{xyz} \, \rho \cdot V_{xyz} \cdot dA = \frac{m_{flow}^2 \cdot (r_i + 3 \cdot r_o)}{6 \cdot (r_o - r_i) \cdot \rho \cdot \iota}$

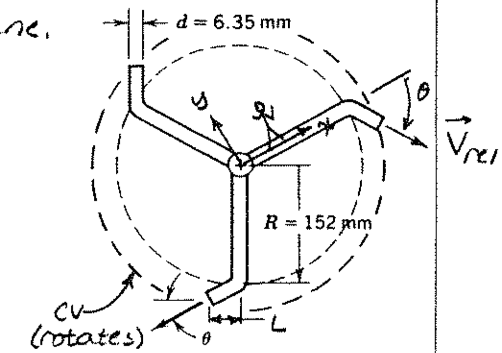
Hence equation $-\int r \times (2 \cdot \omega \times V_{xyz}) \cdot \rho \, dV = \int r \times V_{xyz} \, \rho \cdot V_{xyz} \cdot dA$ becomes

$$m_{flow} = \omega \cdot \rho \cdot \left(\frac{1}{6} \cdot r_o^2 - \frac{1}{3} \cdot r_i \cdot r_o + \frac{1}{2} \cdot r_i^2\right) = \frac{m_{flow}^2 \cdot (r_i - 3 \cdot r_o)}{6 \cdot (r_o + r_i) \cdot \ldots}$$

Solving for ω

$$\omega = \frac{m_{flow} \cdot (r_i - 3 \cdot r_o)}{\left(\rho_o^2 - 2 \cdot r_i \cdot r_o + 3 \cdot r_i^2\right) \cdot (r_o + r_i) \cdot \ldots}$$

$\omega = 1434\,\text{rpm}$



Given: Lawn sprinkler rotating in horizontal plane.

Neglect friction. $Q = 68\ L/min$

Find: Steady-state angular speed for $\theta = 30°$.

Plot: Steady-state angular speed for $0 \le \theta \le 90°$.

Solution: Choose rotating CV. Apply angular momentum principle, Eq. 4.53.

Basic equation: $\vec{r}\times\vec{F}_s + \int_{CV}\vec{r}\times\vec{g}\,\rho\,d\forall + \vec{T}_{shaft}$ (terms $=0(1)$, $=0(2)$, $=0(3)$)

$$-\int_{CV}\vec{r}\times\left[2\vec{\omega}\times\vec{V}_{xyz} + \vec{\omega}\times(\vec{\omega}\times\vec{r}) + \dot{\vec{\omega}}\times\vec{r}\right]\rho\,d\forall$$ (terms $=0(5)$, $=0(6)$)

$$= \frac{\partial}{\partial t}\int_{CV}\vec{r}\times\vec{V}_{xyz}\,\rho\,d\forall + \int_{CS}\vec{r}\times\vec{V}_{xyz}\,\rho\vec{V}_{xyz}\cdot d\vec{A}$$ (first term $=0(6)$)

Assumptions: (1) $\vec{F}_s = 0$, (2) Body torques cancel, (3) $\vec{T}_{shaft} = 0$, (4) Neglect aerodynamic drag, (5) No $\hat{k}$ component of centripetal acceleration, (6) Steady flow, (7) $L \ll R$

Analyze one arm of sprinkler. From geometry, $\vec{r} = r\hat{i}$ in CV, $\vec{r} = R\hat{i}$ at jet. Then

$$-\int_{CV}\vec{r}\times\left[2\vec{\omega}\times\vec{V}_{xyz}\right]\rho\,d\forall = R\hat{i}\times(-V\sin\theta\,\hat{j})\,\rho\frac{Q}{3} = -\rho\frac{QRV}{3}\sin\theta\,\hat{k}$$

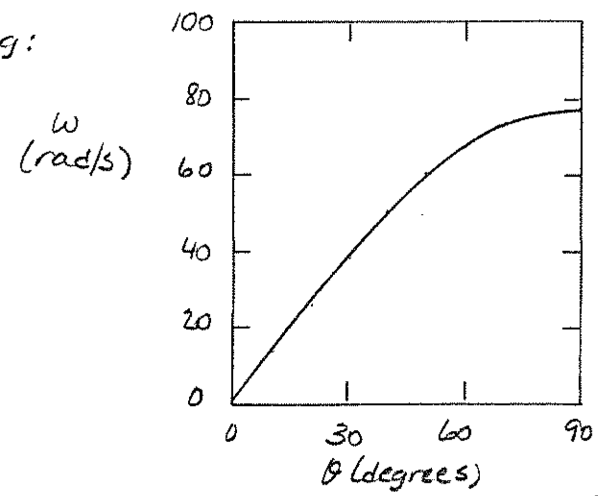
$$r\hat{i}\times(2\omega\hat{k}\times V\hat{i}) = 2\omega Vr\,\hat{k}\ ;\ -\int_{CV} = -\omega VR^2\rho A\,\hat{k}$$

Dropping $\hat{k}$, $\quad -\omega VR^2\rho A = -\rho\frac{QRV}{3}\sin\theta$, so with $VA = Q/3$,

$$\omega = \frac{V}{R}\sin\theta\ ;\ V = \frac{Q}{3A} = \frac{4Q}{3\pi d^2} = \frac{4}{3\pi}\times 68\times10^{-3}\frac{m^3}{min}\times\frac{1}{(0.00635)^2 m^2}\times\frac{min}{60\ s} = 11.9\ m/s$$

$$\omega = 11.9\frac{m}{s}\times\frac{1}{0.152\ m}\times\sin\theta = 78.3\sin\theta\ rad/s$$ ← ω

Plotting:



For $\theta = 30°$,

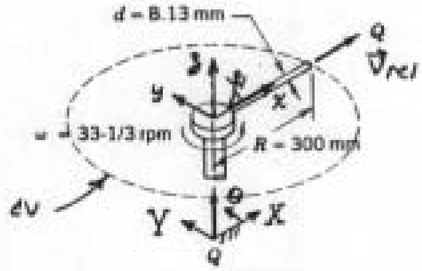
$$\omega = 78.3\sin 30°$$

$$\omega = 39.1\ rad/s$$ ← $\omega\ (\theta = 30°)$

← Plot

Given: Single rotating tube with water.

$Q = 13.8$ L/min



Find: Torque that must be applied to maintain steady rotation using:
(a) Rotating control volume.
(b) Fixed control volume.

Solution: Apply angular momentum principle. $\left\{\omega = 33\tfrac{1}{3}\,\frac{\text{rev}}{\text{min}} = 3.49\ \text{rad/s}\right\}$

(a) Rotating CV: use relative velocities, Eq. 4.53:

Basic equation:
$$\vec{r}\times\vec{F}_s^{\,=0(1)} + \int_{CV}\vec{r}\times\vec{g}^{\,=0(2)}\rho\,d\forall + \vec{T}_{shaft} - \int_{CV}\vec{r}\times\left[2\vec{\omega}\times\vec{V}_{xyz} + \vec{\omega}\times(\vec{\omega}\times\vec{r})^{=0(3)} + \dot{\vec{\omega}}^{=0(4)}\times\vec{r}\right]\rho\,d\forall$$
$$= \frac{\partial}{\partial t}\int_{CV}^{=0(5)}\vec{r}\times\vec{V}_{xyz}\rho\,d\forall + \int_{CS}\vec{r}\times\vec{V}_{xyz}^{\,=0(6)}\,\rho\vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: (1) $\vec{F}_s = 0$, (2) Body torques cancel, (3) No $\hat{k}$ in centripetal accel, (4) $\dot{\vec{\omega}} = 0$, (5) steady flow, (6) $\vec{r}\times\vec{V} = 0$

Then
$$T_{shaft}\,\hat{k} = \int_{CV}\vec{r}\times(2\vec{\omega}\times\vec{V})\rho\,d\forall = \int_0^R r\hat{i}\times(2\omega\hat{k}\times V\hat{i})\rho A\,dr = \omega\rho V A R^2\hat{k} = \omega\rho Q R^2\hat{k}$$

$$T_{shaft} = 3.49\,\frac{\text{rad}}{\text{s}}\times 999\,\frac{\text{kg}}{\text{m}^3}\times 13.8\times10^{-3}\,\frac{\text{m}^3}{\text{min}}\times(0.3)^2\,\text{m}^2\times\frac{\text{min}}{60\,\text{s}}\times\frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}} = 0.0722\ \text{N}\cdot\text{m}$$ T

(b) Fixed control volume: use absolute velocities, Eq. 4.47:

Basic equation:
$$\vec{r}\times\vec{F}_s^{\,=0(1)} + \int_{CV}\vec{r}\times\vec{g}^{\,=0(2)}\rho\,d\forall + \vec{T}_{shaft} = \frac{\partial}{\partial t}\int_{CV}\vec{r}\times\vec{V}\rho\,d\forall + \int_{CS}\vec{r}\times\vec{V}\rho\vec{V}_{xyz}\cdot d\vec{A}$$

Relative to fixed coordinates XY, $\vec{r} = r(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$

$$\vec{V} = V(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}) + r\omega(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})$$

$$\vec{r}\times\vec{V} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ r\cos\theta & r\sin\theta & 0\\ V\cos\theta - r\omega\sin\theta & V\sin\theta + r\omega\cos\theta & 0\end{vmatrix} = \hat{k}\,(rV\sin\theta\cos\theta + \omega r^2\cos^2\theta - rV\sin\theta\cos\theta + \omega r^2\sin^2\theta) = \omega r^2\hat{k}$$

Thus $\partial/\partial t = 0$ and $\int_{CS}\vec{r}\times\vec{V}\rho\vec{V}_{xyz}\cdot d\vec{A} = \omega R^2\hat{k}\{+\rho Q\} = \omega\rho Q R^2\hat{k}$ and

$T_{shaft}\,\hat{k} = \omega\rho Q R^2\hat{k}$ (as before); $T = 0.0722\ \text{N}\cdot\text{m}$ T

{Note that when applied correctly, either choice of CV produces the same result.}

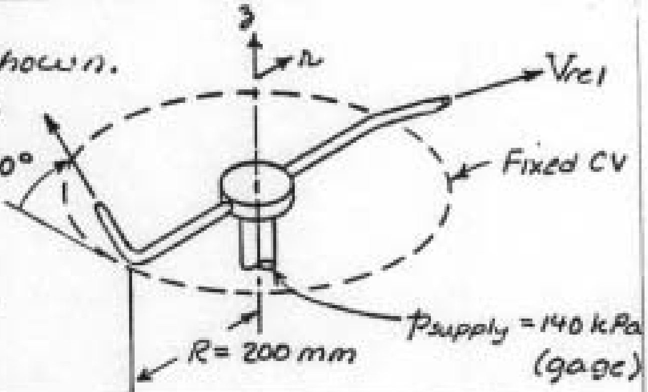
Problem *4.175

Given: Small lawn sprinkler as shown.

$V_{rel} = 17\ m/s$

Friction torque at pivot is $T_f = 0.18\ N\cdot m$.

Flowrate is $Q = 4.0$ liter/min.



Find: Torque to hold stationary.

Solution: Apply moment of momentum using fixed CV enclosing sprinkler arms.

Basic equation:

$$\vec{r}\times\vec{F}_s + \int_{CV} \vec{r}\times\vec{g}\,\rho\, d\forall + \vec{T}_{shaft} = \frac{\partial}{\partial t}\int_{CV} \vec{r}\times\vec{V}\,\rho\, d\forall + \int_{CS} \vec{r}\times\vec{V}\,\rho\,\vec{V}\cdot d\vec{A}$$

(first term $=0(1)$, second term $=0(2)$, time derivative term $=0(3)$)

Assumptions: (1) Neglect torque due to surface forces

(2) Torques due to body forces cancel by symmetry

(3) Steady flow

(4) Uniform flow leaving each jet

Then

$$-T_f\,\hat{k} = (\vec{r}\times\vec{V})_{in}\{-\rho Q\} + 2(\vec{r}\times\vec{V})_{jet}\left\{\tfrac{1}{2}\rho Q\right\}$$

$$(\vec{r}\times\vec{V})_{in} \simeq 0 \qquad \vec{r} = R\hat{e}_r$$

$$\vec{V} = (R\omega - V_{rel}\cos\alpha)\hat{e}_\theta + V_{rel}\sin\alpha\,\hat{e}_z$$

The absolute velocity of the jet leaving sprinkler is $\vec{V} = V_{rel}[\cos\alpha(-\hat{e}_\theta) + \sin\alpha(\hat{e}_z)]$

Then $(\vec{r}\times\vec{V})_z = \{R\hat{e}_r \times V_{rel}[\cos\alpha(-\hat{e}_\theta) + \sin\alpha(\hat{e}_z)]\}_z = \{RV_{rel}\cos\alpha(-\hat{e}_z) + RV_{rel}\sin\alpha(-\hat{e}_\theta)\}_z$

$$(\vec{r}\times\vec{V})_z = -RV_{rel}\cos\alpha$$

Substituting, $T_{shaft} = T_{ext} - T_f = 2(-RV_{rel}\cos\alpha)(\tfrac{1}{2}\rho Q)$

Thus $T_{ext} = T_f - \rho Q R V_{rel}\cos\alpha$

$$= 0.18\ N\cdot m - 999\frac{kg}{m^3}\times 4\frac{L}{min}\times 0.2\,m\times 17\frac{m}{s}\times 0.866\times\frac{m^3}{1000\,L}\times\frac{min}{60\,s}\times\frac{N\cdot s^2}{kg\cdot m}$$

$$T_{ext} = -0.0161\ N\cdot m \quad \text{(to hold sprinkler stationary)}$$

T_{ext}

{Since $T_{ext} < 0$, it must be applied in the minus z direction to oppose motion.}

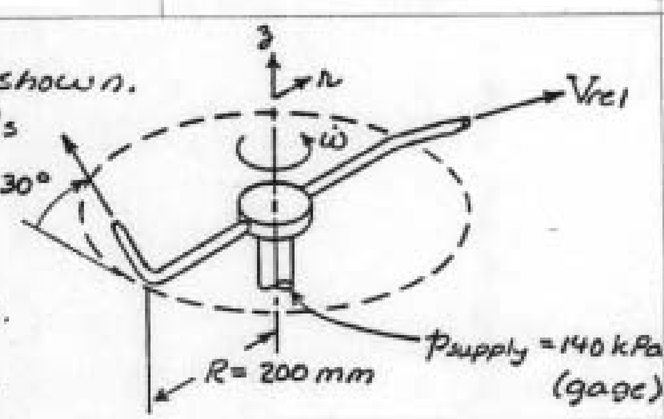
Problem *4.176

Given: Small lawn sprinkler as shown.

$V_{rel} = 17\ m/s$

Friction torque at pivot is zero. $I = 0.1\ kg \cdot m^2$

Flowrate is $Q = 4.0$ liter/min.



Find: Initial angular acceleration from rest.

Solution: Apply moment of momentum using fixed CV enclosing sprinkler arms.

Basic equation:

$$\vec{r} \times \vec{F}_s + \int_{CV} \vec{r} \times \vec{g}\, \rho\, d\forall + \vec{T}_{shaft} = \frac{\partial}{\partial t}\int_{CV} \vec{r} \times \vec{V}\, \rho\, d\forall + \int_{CS} \vec{r} \times \vec{V}\, \rho\, \vec{V} \cdot d\vec{A}$$

(the first term $=0(1)$, the second $=0(2)$, the time derivative term $=0(3)$)

Assumptions: (1) Neglect torque due to surface forces
(2) Torques due to body forces cancel by symmetry
(3) Steady flow
(4) Uniform flow leaving each jet

Then

$$-T_f\,\hat{k} = (\vec{r} \times \vec{V})_{in}\{-\rho Q\} + 2(\vec{r} \times \vec{V})_{jet}\{\tfrac{1}{2}\rho Q\}$$

$(\vec{r} \times \vec{V})_{in} \simeq 0$ $\qquad \vec{r} = R\hat{e}_r$

$\vec{V} = (R\omega - V_{rel}\cos\alpha)\hat{e}_\theta + V_{rel}\sin\alpha\,\hat{e}_z$

The jet leaves the sprinkler at $\vec{V}(abs) = V_{rel}[\cos\alpha(-\hat{e}_\theta) + \sin\alpha(\hat{e}_z)]$

Then $\vec{r} \times \vec{V} = R\hat{e}_r \times V_{rel}[\cos\alpha(-\hat{e}_\theta) + \sin\alpha(\hat{e}_z)] = RV_{rel}[\cos\alpha(-\hat{e}_z) + \sin\alpha(-\hat{e}_\theta)]$

Summing moments on the rotor, $\Sigma \vec{M} = I\dot{\vec{\omega}}$. Thus

$$\dot{\omega} = \frac{\Sigma T}{I} = \frac{\rho Q R V_{rel}\cos\alpha - T_f}{I}$$

$$= \left[\frac{999\ kg}{m^3} \times \frac{4\ L}{min} \times 0.2\ m \times \frac{17\ m}{s} \times 0.866 \times \frac{m^3}{1000\ L} \times \frac{min}{60\ s} - 0.18\ N\cdot m \times \frac{kg \cdot m}{N \cdot s^2}\right]\frac{1}{0.1\ kg \cdot m^2}$$

$\dot{\omega} = 0.161\ rad/s^2$ $\qquad \dot{\omega}$

{ It is not necessary to use a rotating CV, because at the instant considered, $\vec{\omega} = 0$ and I is known. }

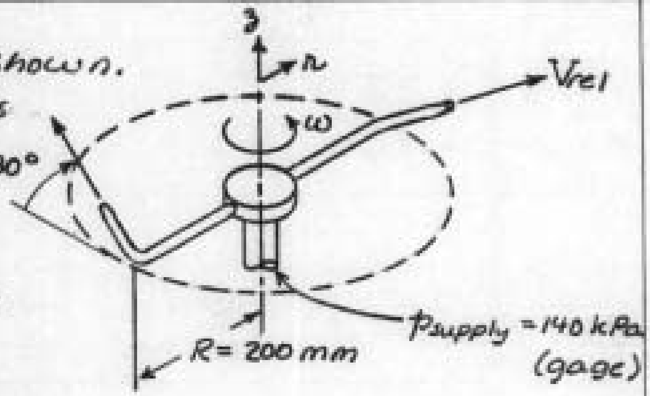
Problem #4.177

Given: Small lawn sprinkler as shown.

$V_{rel} = 17\ m/s$

Friction torque at pivot is $T_f = 0.18\ N\cdot m$.

Flowrate is $Q = 4.0$ liter/min.



Find: (a) Steady speed of rotation.
(b) Area covered by spray.

Solution: Apply moment of momentum using fixed CV enclosing sprinkler arms.

Basic equation:

$$\vec{r}\times\vec{F}_s^{\,=0(1)} + \int_{CV} \vec{r}\times\vec{g}^{\,=0(2)}\,\rho\,dV + \vec{T}_{shaft} = \frac{\partial}{\partial t}\int_{CV}^{=0(3)} \vec{r}\times\vec{V}\,\rho\,dV + \int_{CS} \vec{r}\times\vec{V}\,\rho\,\vec{V}\cdot d\vec{A}$$

Assumptions: (1) Neglect torque due to surface forces
(2) Torques due to body forces cancel by symmetry
(3) Steady flow
(4) Uniform flow leaving each jet

Then

$$-T_f\hat{k} = (\vec{r}\times\vec{V})_{in}\{-\rho Q\} + 2(\vec{r}\times\vec{V})_{jet}\{\tfrac{1}{2}\rho Q\}$$

$(\vec{r}\times\vec{V})_{in} \approx 0$

$\vec{r} = R\hat{e}_r$

$\vec{V} = (R\omega - V_{rel}\cos\alpha)\hat{e}_\theta + V_{rel}\sin\alpha\,\hat{e}_z$

$(\vec{r}\times\vec{V})_z = R(R\omega - V_{rel}\cos\alpha)$

or

$$-T_f = R(R\omega - V_{rel}\cos\alpha)\rho Q$$

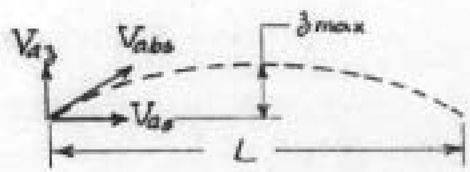
Thus

$$\omega = \frac{V_{rel}\cos\alpha}{R} - \frac{T_f}{\rho Q R^2}$$

$$= 17\frac{m}{s}\times\frac{\cos 30°}{0.2\,m} - 0.18\,N\cdot m\times\frac{m^3}{999\,kg}\times\frac{min}{4.0\,L}\times\frac{1}{(0.2)^2 m^2}\times\frac{60\,s}{min}\times\frac{10^3 L}{m^3}\times\frac{kg\cdot m}{N\cdot s^2}$$

$\omega = 6.04\ \frac{rad}{s}$ or 57.7 rpm ← ω

Treat the spray outside each nozzle as moving without air resistance:



For each particle, $\frac{dV_{az}}{dt} = -g$, so $V_z = V_{z_0} - gt$

At z_{max}, $V_z = 0$, so $t = \frac{V_{z_0}}{g}$; flight time is $2t$.

$$L = 2t\,V_{a\theta} = \frac{2V_{az}}{g}V_{a\theta} = \frac{2V_{rel}\sin\alpha(V_{rel}\cos\alpha - R\omega)}{g}$$

$$L = 2\times 17\frac{m}{s}\times\sin 30°\left(17\frac{m}{s}\times\cos 30° - 0.2\,m\times 6.04\frac{rad}{s}\right)\frac{s^2}{9.81\,m} = 23.4\ m$$

$R_{spray} = \sqrt{R^2 + L^2} = 23.4\ m$; $A_{spray} = \pi R_{spray}^2 = \pi\times(23.4)^2 m^2 = 1720\ m^2$ ← A_{spray}

Problem *4.178

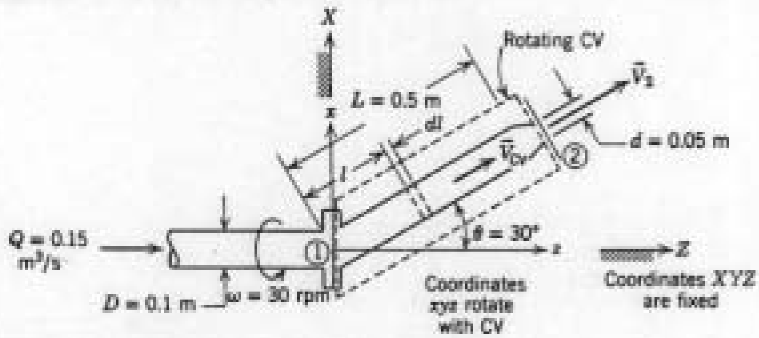
Open-Ended Problem Statement: When a garden hose is used to fill a bucket, water in the bucket may develop a swirling motion. Why does this happen? How could the amount of swirl be calculated approximately?

Discussion: Frequently when filling a bucket the hose is held so that the water stream entering the bucket is not vertical. If, in addition, the water stream is off-center in the bucket, then flow entering the bucket has a tangential component of velocity, a swirl component.

The tangential component of the water velocity entering the bucket has a moment-of-momentum (swirl) with respect to a control volume drawn around the stationary bucket. This entering swirl can only be reduced by a torque acting to oppose it. Viscous forces among fluid layers will tend to transfer swirl to other layers so that eventually all of the water in the bucket has a swirling motion.

Swirl in the bucket may be influenced by viscosity. The swirl may tend to nearly a rigid-body motion to minimize viscous forces between annular layers of water in the bucket. The rigid-body motion assumption may be a reasonable model to calculate the total angular momentum (moment-of-momentum) of the water in the bucket.

Given: Nozzle assembly rotating steadily, as shown in the sketch.



Find: (a) Torque required to drive the nozzle assembly
(b) Reaction torques at the flange.

Solution: Apply the moment of momentum equation to the rotating CV shown.

Basic equation:

$$\vec{r}\times\vec{F}_s + \int_{CV}\vec{r}\times\vec{g}\,\rho\,d\forall + \vec{T}_{shaft} \quad (\approx 0(2))$$

$$-\int_{CV}\vec{r}\times\left[2\vec{\omega}\times\vec{V}_{xyz} + \vec{\omega}\times(\vec{\omega}\times\vec{r}) + \dot{\vec{\omega}}\times\vec{r}\right]\rho\,d\forall = \frac{\partial}{\partial t}\int_{CV}\vec{r}\times\vec{V}_{xyz}\,\rho\,d\forall + \int_{CS}\vec{r}\times\vec{V}_{xyz}\,\rho\vec{V}_{xyz}\cdot d\vec{A}$$

($=0(3)$, $=0(5)$, $=0(7)$)

Assumptions: (1) Let $\vec{T}_{cv}$ represent all torques acting on the CV
(2) Neglect torque due to body force
(3) Constant angular speed
(4) Neglect mass of arm compared to water inside
(5) Steady flow in CV
(6) Neglect nozzle length compared to L
(7) $\vec{r}$ colinear with $\vec{V}$, so $\vec{r}\times\vec{V}_{xyz} = 0$

Then

$$\vec{T}_{cv} = \int_{CV}\vec{r}\times\left[2\vec{\omega}\times\vec{V}_{xyz} + \vec{\omega}\times(\vec{\omega}\times\vec{r})\right]\rho\,d\forall$$

Since $\vec{\omega} = \omega\hat{k}$ and $\vec{r} = l(\sin\theta\,\hat{i} + \cos\theta\,\hat{k})$, then

$$\vec{\omega}\times\vec{r} = \omega l\sin\theta\,\hat{j}$$

$$\vec{\omega}\times(\vec{\omega}\times\vec{r}) = \omega\hat{k}\times\omega l\sin\theta\,\hat{j} = \omega^2 l\sin\theta\,(-\hat{i})$$

and $\vec{r}\times[\vec{\omega}\times(\vec{\omega}\times\vec{r})] = l(\sin\theta\,\hat{i} + \cos\theta\,\hat{k})\times\omega^2 l\sin\theta\,(-\hat{i}) = \omega^2 l^2\sin\theta\cos\theta\,(-\hat{j})$

Since $\vec{V}_{xyz} = V_{cv}(\sin\theta\,\hat{i} + \cos\theta\,\hat{k})$, then

$$2\vec{\omega}\times\vec{V}_{xyz} = 2\omega\hat{k}\times V_{cv}(\sin\theta\,\hat{i} + \cos\theta\,\hat{k}) = 2\omega V_{cv}\sin\theta\,\hat{j}$$

and $\vec{r}\times[2\vec{\omega}\times\vec{V}_{xyz}] = l(\sin\theta\,\hat{i} + \cos\theta\,\hat{k})\times 2\omega V_{cv}\sin\theta\,\hat{j} = 2\omega l V_{cv}\sin^2\theta\,\hat{k}$

$$+\,2\omega l V_{cv}\sin\theta\cos\theta\,(-\hat{i})$$

Substituting and introducing $d\forall = A\,d\ell$,

$$\vec{T}_{cv} = \int_0^L \left(-2\omega\ell V_{cv}\sin\theta\cos\theta\,\hat{i} - \omega^2\ell^2\sin\theta\cos\theta\,\hat{j} + 2\omega\ell V_{cv}\sin^2\theta\,\hat{k}\right)\rho A\,d\ell$$

$$\vec{T}_{cv} = \left[-\omega L^2 V_{cv}\sin\theta\cos\theta\,\hat{i} - \frac{\omega^2 L^3}{3}\sin\theta\cos\theta\,\hat{j} + \omega L^2 V_{cv}\sin^2\theta\,\hat{k}\right]\rho A$$

The shaft torque needed to maintain steady rotation of the assembly is

$$T_{shaft} = T_{cv_z} = \omega L^2 V_{cv}\sin^2\theta\,\rho A = \omega L^2 \frac{Q}{A}\sin^2\theta\,\rho A = \rho Q\omega L^2\sin^2\theta$$

$$= 999\,\frac{kg}{m^3} \times 0.15\,\frac{m^3}{s} \times 30\,\frac{rev}{min} \times (0.5)^2 m^2 \times (0.5)^2 \times 2\pi\,\frac{rad}{rev} \times \frac{min}{60\,s} \times \frac{N\cdot s^2}{kg\cdot m}$$

$T_{shaft} = 29.4\ N\cdot m$ ← T_{shaft}

The reaction moments acting on the flange are

$$M_x = -T_{cv_x} = \omega L^2 V_{cv}\sin\theta\cos\theta\,\rho A = \rho Q\omega L^2\sin\theta\cos\theta$$

$$= 999\,\frac{kg}{m^3} \times 0.15\,\frac{m^3}{s} \times 30\,\frac{rev}{min} \times (0.5)^2 m^2 \times (0.5)(0.866)\,2\pi\,\frac{rad}{rev} \times \frac{min}{60\,s} \times \frac{N\cdot s^2}{kg\cdot m}$$

$M_x = 51.0\ N\cdot m$ (applied to flange by CV) ← M_x

$$M_y = -T_{cv_y} = \frac{1}{3}\rho\omega^2 L^3 A\sin\theta\cos\theta$$

$$= \frac{1}{3} \times 999\,\frac{kg}{m^3}\left[30\,\frac{rev}{min} \times 2\pi\,\frac{rad}{rev} \times \frac{min}{60\,s}\right]^2 (0.5)^3 m^3 \times \frac{\pi}{4}(0.1)^2 m^2 \times (0.5)(0.866) \times \frac{N\cdot s^2}{kg\cdot m}$$

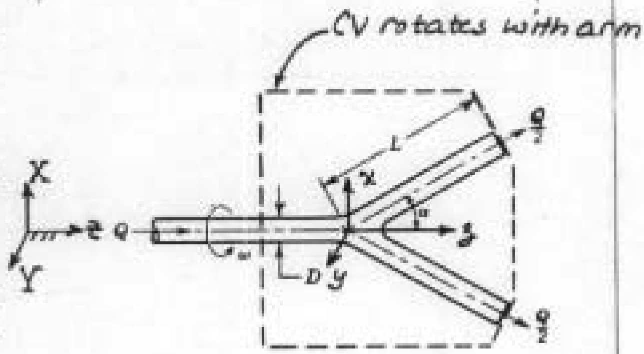
$M_y = 1.40\ N\cdot m$ (applied to flange by CV) ← M_y

{ Torques due to the masses of water, tube, and nozzle must be considered in the overall design. }

Given: Branched pipe with symmetrical legs as shown.

Angular momentum zero at inlet, relative to nonrotating frame.

Find: (a) External torque expression
(b) Additional torque to produce angular acceleration of $\dot{\omega}$.



Solution: Apply moment of momentum equation using rotating CV.

Basic equation:

$$\vec{r}\times\vec{F}_s + \int_{CV}\vec{r}\times\vec{g}\,\rho\,d\forall + \vec{T}_{shaft} \quad (=0(1),\ =0(2))$$

$$-\int_{CV}\vec{r}\times\left[2\vec{\omega}\times\vec{V}_{xyz}+\vec{\omega}\times(\vec{\omega}\times\vec{r})+\dot{\vec{\omega}}\times\vec{r}\right]\rho\,d\forall = \frac{\partial}{\partial t}\int_{CV}\vec{r}\times\vec{V}_{xyz}\,\rho\,d\forall + \int_{CS}\vec{r}\times\vec{V}_{xyz}\,\rho\vec{V}_{xyz}\cdot d\vec{A} \quad (=0(3),\ =0(4))$$

Assumptions: (1) No surface forces
(2) Body-forces produce no torque about axis (symmetry)
(3) Flow steady in rotating frame
(4) $\vec{r}$ and $\vec{V}_{xyz}$ are colinear: $\vec{r}\times\vec{V}_{xyz}=0$

Then

$$\vec{T}_{shaft} = \int_{CV}\vec{r}\times\left[2\vec{\omega}\times\vec{V}_{xyz}+\vec{\omega}\times(\vec{\omega}\times\vec{r})+\dot{\vec{\omega}}\times\vec{r}\right]\rho\,d\forall$$

Using the coordinates above, $\vec{\omega}=\omega\hat{k}$ $\qquad \dot{\vec{\omega}}=\dot{\omega}\hat{k}$

$\vec{r} = r(\cos\alpha\,\hat{k}+\sin\alpha\,\hat{i})$ (upper tube)

$\vec{V}_{xyz} = \frac{Q}{2A}(\cos\alpha\,\hat{k}+\sin\alpha\,\hat{i})$ (upper tube); $A=\frac{\pi D^2}{4}$

and

$$\dot{\vec{\omega}}\times\vec{r} = \dot{\omega}r\sin\alpha\,\hat{j}$$

$$\vec{\omega}\times(\vec{\omega}\times\vec{r}) = \omega\hat{k}\times\omega r\sin\alpha\,\hat{j} = -\omega^2 r\sin\alpha\,\hat{i}$$

$$2\vec{\omega}\times\vec{V}_{xyz} = 2\omega\frac{Q}{2A}\sin\alpha\,\hat{j} = \frac{\omega Q}{A}\sin\alpha\,\hat{j}$$

Thus for the upper tube,

$$\vec{T}_{shaft} = \int_0^L\left\{r(\cos\alpha\,\hat{k}+\sin\alpha\,\hat{i})\times\left[\left(\frac{\omega Q}{A}+\dot{\omega}r\right)\sin\alpha\,\hat{j}-\omega^2 r\sin\alpha\,\hat{i}\right]\right\}\rho A\,dr$$

$$=\int_0^L\left[\left(\frac{r\omega Q}{A}+\dot{\omega}r^2\right)(\sin\alpha\cos\alpha)\,\hat{i}+\left(\frac{r\omega Q}{A}+\dot{\omega}r^2\right)\sin^2\alpha\,\hat{k}+\omega^2r^2\sin\alpha\cos\alpha(-\hat{j})\right]\rho A\,dr$$

$$\vec{T}_{shaft}(\text{upper}) = \left[\left(\frac{L^2\omega Q}{2A}+\frac{\dot{\omega}L^3}{3}\right)\sin\alpha\cos\alpha\,\hat{i}+\left(\frac{L^2\omega Q}{2A}+\frac{\dot{\omega}L^3}{3}\right)\sin^2\alpha\,\hat{k}+\frac{\omega^2L^3}{3}\sin\alpha\cos\alpha(-\hat{j})\right]\rho A$$

For the lower tube, $\vec{\omega} = \omega\hat{k}$ $\qquad \dot{\vec{\omega}} = \dot{\omega}\hat{k}$

$$\vec{r} = r(\cos\alpha\,\hat{k} - \sin\alpha\,\hat{i}) \quad \text{(lower tube)}$$

$$\vec{V}_{xyz} = \frac{Q}{2A}(\cos\alpha\,\hat{k} - \sin\alpha\,\hat{i}) \quad \text{(lower tube)}$$

and

$$\dot{\vec{\omega}} \times \vec{r} = -r\dot{\omega}\sin\alpha\,\hat{j}$$

$$\vec{\omega} \times (\vec{\omega} \times \vec{r}) = \omega\hat{k} \times (-r\omega\sin\alpha\,\hat{j}) = r\omega^2\sin\alpha\,\hat{i}$$

$$2\vec{\omega} \times \vec{V}_{xyz} = 2\omega\frac{Q}{2A}(-\sin\alpha)(\hat{j}) = -\frac{\omega Q}{A}\sin\alpha\,\hat{j}$$

Thus for the lower tube,

$$\vec{T}_{shaft} = \int_0^L \left\{ r(\cos\alpha\,\hat{k} - \sin\alpha\,\hat{i}) \times \left[\left(\frac{\omega Q}{A} + r\dot{\omega}\right)\sin\alpha(-\hat{j}) + r\omega^2\sin\alpha \right] \right\} \rho A\,dr$$

$$= \int_0^L \left[\left(\frac{r\omega Q}{A} + r^2\dot{\omega}\right)\sin\alpha\cos\alpha(-\hat{i}) + \left(\frac{r\omega Q}{A} + r^2\dot{\omega}\right)\sin^2\alpha\,\hat{k} + r^2\omega^2\sin\alpha\cos\alpha\,\hat{j} \right] \rho A\,dr$$

$$\vec{T}_{shaft}(\text{lower}) = \left[-\left(\frac{L^2\omega Q}{2A} + \frac{L^3\dot{\omega}}{3}\right)\sin\alpha\cos\alpha\,\hat{i} + \left(\frac{L^2\omega Q}{2A} + \frac{L^3\dot{\omega}}{3}\right)\sin^2\alpha\,\hat{k} + \frac{L^3\omega^2}{3}\sin\alpha\cos\alpha\,\hat{j} \right] \rho A$$

Summing these expressions gives

$$\vec{T}_{shaft}(\text{total}) = \left(\frac{L^2\omega Q}{A} + \frac{2L^3\dot{\omega}}{3}\right)\sin^2\alpha\,\rho A\,\hat{k}$$

Thus the steady-state portion of the torque is

$$\vec{T}_{shaft}(\text{steady state}) = \left(\frac{L^2\omega Q}{A}\right)\sin^2\alpha\,\rho A\,\hat{k} = L^2\rho\omega Q\sin^2\alpha\,\hat{k}$$

Stea

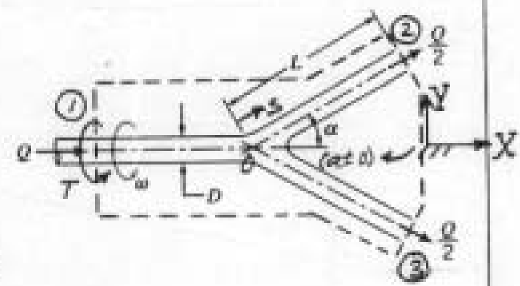
The additional torque needed to provide angular acceleration, $\dot{\omega}$, is

$$\vec{T}_{shaft}(\text{acceleration}) = \frac{2L^3\rho\dot{\omega}A}{3}\sin^2\alpha\,\hat{k}$$

Accel

{ Torques of individual tubes about the x and y axes are reacted internally; they must be considered in design of the tube. }

(b) Using fixed CV:



Basic equation: $\vec{r}\times\vec{F}_s + \int_{CV}\vec{r}\times\vec{g}\rho\, d\mathcal{V} + \vec{T}_{shaft} = \frac{\partial}{\partial t}\int_{CV}\vec{r}\times\vec{V}\rho\, d\mathcal{V} + \int_{CS}\vec{r}\times\vec{V}\rho\vec{V}\cdot d\vec{A}$

(first term $=0(1)$; second term $=0(2)$; first term on right $=0(3)$)

Assumptions: (1) No surface forces
(2) Body forces symmetric (no moment about X axis)
(3) No change in angular momentum within CV w.r.to time
(4) Symmetry in two branches
(5) Uniform flow at each cross-section

Then $\vec{T}_s = T\hat{I} = \vec{r}_1\times\vec{V}_1\{-\rho Q\} + \vec{r}_2\times\vec{V}_2\{+\rho\frac{Q}{2}\} + \vec{r}_3\times\vec{V}_3\{+\rho\frac{Q}{2}\} = 2\vec{r}_2\times\vec{V}_2\{\rho\frac{Q}{2}\}$

$\vec{r}_1 = 0 \qquad \vec{r}_2 = L\sin\alpha\,\hat{J}\ ;\ \vec{V}_2 = \omega r_2\,\hat{K}\ ;\ \vec{r}_2\times\vec{V}_2 = \omega L^2\sin^2\alpha\,\hat{I}$

or

$T_{ss} = \rho\omega Q L^2\sin^2\alpha$ (steady-state torque) $\qquad T_{ss}$

The torque required for acceleration is $T_{acc} = I\dot{\omega}$, where $I = \int r^2 dm$

For one leg of the branch, $I = \int r^2 dm = \int_0^L (s\sin\alpha)^2\rho A\, ds = \frac{\rho A L^3}{3}\sin^2\alpha$

(6) Neglect mass of pipe

For both sides, $I = \frac{2\rho A L^3}{3}\sin^2\alpha$.

Thus

$T_{acc} = \frac{2\rho\dot{\omega}AL^3}{3}\sin^2\alpha$ (torque required for angular acceleration) $\qquad T_{acc}$

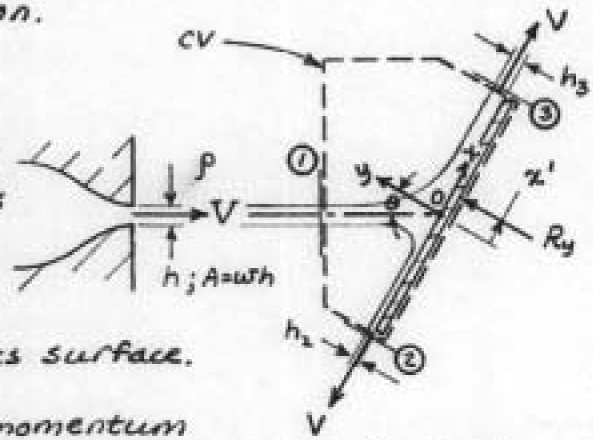
The total torque that must be applied is

$T = T_{ss} + T_{acc} = \rho\omega Q L^2\sin^2\alpha + \frac{2\rho\dot{\omega}AL^3}{3}\sin^2\alpha \qquad T_{total}$

Given: Thin sheet of liquid, of width, w, and thickness, h, striking inclined flat plate, as shown.

Neglect any viscous effects.

Find: (a) Magnitude and line of action of resultant force as functions of θ.
(b) Equilibrium angle of plate if force is applied at point O, where jet centerline intersects surface.



__Solution:__ Apply continuity, linear momentum and moment of momentum using CV and coordinates shown.

Basic equations:

$$0 = \overset{=0(1)}{\frac{\partial}{\partial t}\int_{CV} \rho\, d\forall} + \int_{CS} \rho \vec{V}\cdot d\vec{A}$$

$$\overset{=0(4)}{F_{Sx}} + \overset{=0(5)}{F_{Bx}} = \overset{=0(1)}{\frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall} + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$$

$$F_{Sy} + \overset{=0(5)}{F_{By}} = \overset{=0(1)}{\frac{\partial}{\partial t}\int_{CV} v\rho\, d\forall} + \int_{CS} v\rho \vec{V}\cdot d\vec{A}$$

$$\vec{r}\times\vec{F}_S + \overset{=0(5)}{\int_{CV} \vec{r}\times\vec{g}\,\rho\, d\forall} + \overset{=0(6)}{\vec{T}_{shaft}} = \overset{=0(1)}{\frac{\partial}{\partial t}\int_{CV} \vec{r}\times\vec{V}\rho\, d\forall} + \int_{CS} \vec{r}\times\vec{V}\rho\vec{V}\cdot d\vec{A}$$

Assumptions:
(1) Steady flow
(2) Uniform flow at each section
(3) No net pressure forces; $F_{Sx} = R_x$, $F_{Sy} = R_y$
(4) No viscous effects; $R_x = 0$ and $V_1 = V_2 = V_3 = V$
(5) Neglect body forces and torques
(6) $\vec{T}_{shaft} = 0$
(7) Incompressible flow, ρ = constant

Then from continuity,

$$0 = \{-|\rho V w h_1|\} + \{|\rho V w h_2|\} + \{|\rho V w h_3|\} \quad \text{or} \quad h_1 = h_2 + h_3 = h \qquad (1)$$

From x momentum

$$0 = u_1\{-|\rho V w h_1|\} + u_2\{|\rho V w h_2|\} + u_3\{|\rho V w h_3|\}$$

$$u_1 = V\sin\theta \qquad u_2 = -V \qquad u_3 = V$$

$$0 = \rho V^2 w\,(-h_1\sin\theta - h_2 + h_3) \quad \text{or} \quad h_3 - h_2 = h_1\sin\theta = h\sin\theta \qquad (2)$$

Combining Eqs. 1 and 2,

$$h_2 = h\left(\frac{1-\sin\theta}{2}\right) \qquad (3)$$

$$h_3 = h\left(\frac{1+\sin\theta}{2}\right) \qquad (4)$$

From y momentum, $R_y = v_1\{-|\rho V w h_1|\} + v_2\{|\rho V w h_2|\} + v_3\{|\rho V w h_3|\}$

$$v_1 = -V\cos\theta \qquad v_2 = 0 \qquad v_3 = 0$$

$$R_y = \rho V^2 w h \cos\theta \qquad (5) \quad R_y$$

From moment of momentum,

$$\vec{r}' \times \vec{F}_s = \vec{r}_1 \times \vec{V}_1\{-|\rho V w h_1|\} + \vec{r}_2 \times \vec{V}_2\{|\rho V w h_2|\} + \vec{r}_3 \times \vec{V}_3\{|\rho V w h_3|\}$$

$\vec{r}' = x'\hat{i}$ $\quad \vec{r}_1 \times \vec{V}_1 = 0$ $\quad \vec{r}_2 = \frac{h_2}{2}\hat{j}$ $\quad \vec{r}_3 = \frac{h_3}{2}\hat{j}$

$\vec{F}_s = R_y\hat{j}$ $\quad \vec{V}_2 = -V\hat{i}$ $\quad \vec{V}_3 = V\hat{i}$

$\vec{r}' \times \vec{F}_s = x' R_y \hat{k}$ $\quad \vec{r}_2 \times \vec{V}_2 = \frac{h_2 V}{2}\hat{k}$ $\quad \vec{r}_3 \times \vec{V}_3 = -\frac{h_3 V}{2}\hat{k}$

Combining and dropping $\hat{k}$,

$$x' R_y = \frac{1}{2}\rho V^2 w h_2^2 - \frac{1}{2}\rho V^2 w h_3^2 = \frac{1}{2}\rho V^2 w (h_2^2 - h_3^2)$$

or

$$x' = \frac{\rho V^2 w (h_2^2 - h_3^2)}{2R_y} = \frac{\rho V^2 w (h_2 + h_3)(h_2 - h_3)}{2R_y}$$

Substituting from Eqs. 3, 4 and 5,

$$x' = \frac{\rho V^2 w h^2 \left(\frac{1-\sin\theta}{2} + \frac{1+\sin\theta}{2}\right)\left(\frac{1-\sin\theta}{2} - \frac{1+\sin\theta}{2}\right)}{2\rho V^2 w h \cos\theta} = \frac{h(-\sin\theta)}{2\cos\theta}$$

or

$$x' = -\frac{h}{2}\tan\theta \qquad (6) \quad x'$$

Note that $x' < 0$. This means that R_y must be applied <u>below</u> point 0.

If R_y is applied <u>at</u> point 0, then $x' = 0$. For equilibrium, from Eq. 6, $\theta = 0$. Thus if force is applied at point 0, plate will be in equilibrium when <u>perpendicular</u> to jet.

Given: The rotating lawn sprinkler of Example Problem 4.14.

Find: (a) Jet angle α for maximum speed of rotation.
(b) What jet angle will provide the maximum area of coverage by the spray?
(c) Draw a velocity diagram to show the absolute velocity of the water jet leaving the nozzle.
(d) What governs the steady rotational speed of the sprinkler?
(e) Does the rotational speed of the sprinkler affect the area covered by the spray?
(f) How would you estimate the area of coverage?
(g) For fixed α, what might be done to increase or reduce the area covered by the spray?

Solution: The results of Example Problem 4.14 were computed assuming steady flow of water and constant frictional retarding torque at the sprinkler pivot.

$$T_f = R(V_{rel}\cos\alpha - \omega R)\rho Q$$

From these results,

$$\omega = \frac{V_{rel}\cos\alpha}{R} - \frac{T_f}{\rho Q R^2}$$

Thus rotational speed of the sprinkler increases as $\cos\alpha$ increases, i.e., as α decreases. The maximum rotational speed occurs when $\alpha = 0$. Then $\cos\alpha = 1$ and the rotational speed is

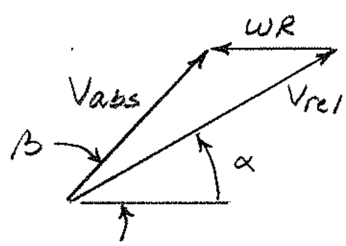
$$\omega = \frac{V_{rel}}{R} - \frac{T_f}{\rho Q R^2}$$

For the conditions of Example Problem 4.14 the maximum rotational speed is

$$\omega = 4.97\,\frac{m}{s} \times \frac{1}{0.150\,m} - 0.0718\,N\cdot m \times \frac{m^3}{999\,kg} \times \frac{min}{7.5\,L} \times \frac{1}{(0.150)^2 m^2} \times \frac{1000\,L}{m^3} \times \frac{60\,s}{min} = 7.58\ rad/s$$ ← ω_{max}

The steady rotation speed ω of the sprinkler is governed by torque T_f and angle α.

Maximum coverage by the spray occurs when the "carry" of each jet stream is the longest. When aerodynamic drag on the stream is neglected, maximum carry occurs when the absolute velocity of the stream leaves the sprinkler at $\beta = 45°$, as shown in the velocity diagram below.



Note $\vec{V}_{abs} = \vec{V}_{rel} - \omega R \hat{e}_\theta$

Both the magnitude and direction of $\vec{V}_{abs}$ vary with ω!

For $\omega = 0$, the relative velocity angle α and absolute velocity angle β are equal. Therefore maximum carry occurs when $\alpha = 45°$ (see graph on next page).

Any rotation rate ω reduces the magnitude V_{abs} and increases the angle β of the absolute velocity leaving the sprinkler jet. When $\omega > 0$, then $\beta > \alpha$, so for maximum carry α must be less than 45°. Consequently rotation reduces the carry of the stream and the area of coverage; at specified α the area of coverage decreases with increasing ω.

For the conditions of Example Problem 4.14 ($\omega = 30$ rpm), optimum carry occurs at $\alpha \approx 42°$, and the coverage area is reduced from approximately 20 m^2 with a fixed sprinkler to 15 m^2 with 30 rpm rotation. If the rotation speed is increased (by decreasing pivot friction or decreasing nozzle angle α), coverage area may be reduced still further, to 9 m^2 or less.

$$A \approx \pi (x_{max})^2$$

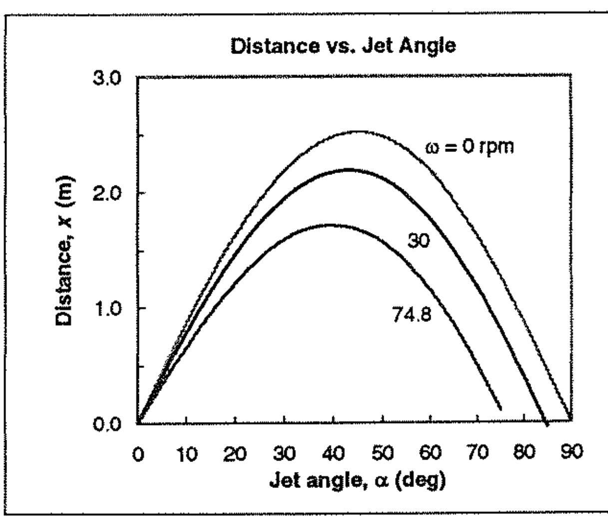
Analysis of Ground Area Covered by Rotating Lawn Sprinkler:

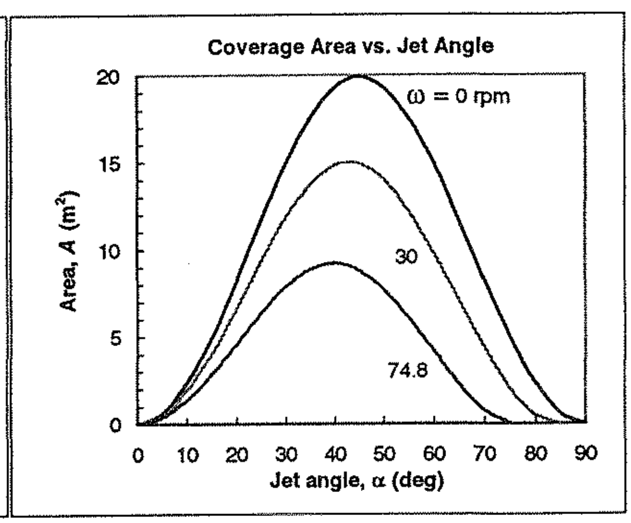
Variables:
- A = ground area covered by spray stream
- x = ground distance reached by spray stream
- α = angle of jet above ground plane
- β = angle of absolute velocity above ground plane

Input Data:
- R = 0.150 m
- V_{rel} = 4.97 m/s (Q = 7.5 L/min)

Results:

	ω (rpm) =	0		30		74.8
	ωR (m/s) =	0		0.471		1.17
α (deg)	x_{max} (m)	A (m²)	x_{max} (m)	A (m²)	x_{max} (m)	A (m²)
0	0.00	0.00	0.00	0.00	0.00	0.00
5	0.437	0.601	0.396	0.492	0.333	0.349
10	0.861	2.33	0.778	1.90	0.654	1.35
15	1.26	4.98	1.14	4.05	0.951	2.84
20	1.62	8.23	1.46	6.65	1.21	4.61
25	1.93	11.7	1.73	9.37	1.43	6.39
30	2.18	14.9	1.94	11.8	1.59	7.90
35	2.37	17.6	2.09	13.8	1.68	8.90
40	2.48	19.3	2.17	14.8	1.71	9.23
45	2.52	19.9	2.18	14.9	1.68	8.83
50	2.48	19.3	2.11	14.0	1.57	7.72
55	2.37	17.6	1.97	12.3	1.39	6.08
60	2.18	14.9	1.77	9.81	1.15	4.15
65	1.93	11.7	1.50	7.03	0.850	2.269
70	1.62	8.23	1.17	4.30	0.500	0.785
75	1.26	4.98	0.798	2.00	0.109	0.037
78	1.02	3.30	0.557	0.975		
80	0.861	2.33	0.391	0.480		
85	0.437	0.601	-0.04	0.00		
90	0.00	0.00				





Problem 4.183

Given: Compressor, $\dot{m} = 1.0$ kg/s

$p_1 = 101$ kPa (abs)

$T_1 = 288$ K

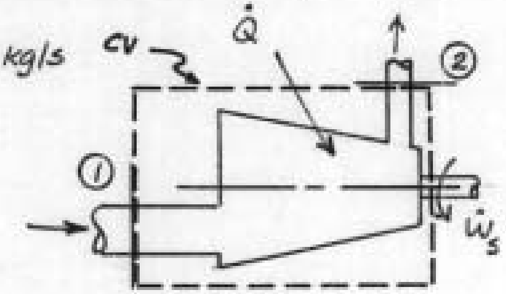
$V_1 = 75$ m/s

$p_2 = 200$ kPa (abs)

$T_2 = 345$ K

$V_2 = 125$ m/s

$\frac{dQ}{dm} = -18$ kJ/kg



Find: Power required.

Solution: Apply first law of thermodynamics, using CV shown.

B.E.

$$\dot{Q} - \dot{W}_s - \dot{W}_{shear} = \frac{\partial}{\partial t}\int_{CV} e\rho\, d\forall + \int_{CS} (e + pv)\rho \vec{V}\cdot d\vec{A}$$

($\dot{W}_{shear} = 0(1)$, $\frac{\partial}{\partial t} = 0(2)$)

Assume:
(1) $\dot{W}_{shear} = 0$
(2) Steady flow
(3) Uniform flow at each section
(4) Neglect Δz
(5) Ideal gas, $p = \rho RT$, $\Delta h = c_p \Delta T$; $c_p = 1.00$ kJ/kg·K
(6) From continuity, $\dot{m}_1 = \dot{m}_2 = \dot{m}$

Then

$$\dot{Q} - \dot{W}_s = \left(u_2 + \frac{V_2^2}{2} + gz_2 + p_2 v_2\right)\{|\dot{m}|\} + \left(u_1 + \frac{V_1^2}{2} + gz_1 + p_1 v_1\right)\{-|\dot{m}|\}$$

($gz_2 \approx 0(4)$, $gz_1 \approx 0(4)$)

Note that $h = u + pv$, and $\dot{Q} = \dot{m}\frac{dQ}{dm}$, so

$$\dot{W}_{in} = -\dot{W}_s = \dot{m}\left(\frac{V_2^2 - V_1^2}{2} + h_2 - h_1 - \frac{dQ}{dm}\right) = \dot{m}\left[\frac{V_2^2 - V_1^2}{2} + c_p(T_2 - T_1) - \frac{dQ}{dm}\right]$$

or

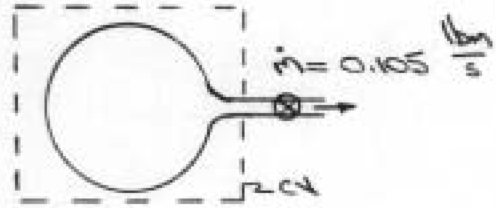
$$\dot{W}_{in} = 1.0\,\frac{kg}{s}\left\{\frac{1}{2}\left[(125)^2 - (75)^2\right]\frac{m^2}{s^2} \times \frac{N\cdot s^2}{kg\cdot m} \times \frac{kJ}{1000\,N\cdot m} + 1.00\,\frac{kJ}{kg\cdot K}(345 - 288)\,K - \left(-18\,\frac{kJ}{kg}\right)\right\}\frac{kW\cdot s}{kJ}$$

$$\dot{W}_{in} = 80.0\ kW$$

$\dot{W}_{in}$

Problem 4.184

Given: Pressure bottle, $\forall = 10\,ft^3$ contains compressed air at $p = 3000$ psia, $T = 140°F$. At $t=0$, $\dot{m} = 0.105$ lbm/s



Find: $\partial T/\partial t$ at $t=0$

Solution: Use CV shown

Basic equations:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A}$$

$$\dot{Q} - \dot{W}_s - \dot{W}_{shear} - \dot{W}_{other} = \frac{\partial}{\partial t}\int_{CV} e\rho\, d\forall + \int_{CS}\left(u + pv + \frac{V^2}{2} + gz\right)\rho\vec{V}\cdot d\vec{A}$$

$= 0(1)$, $= 0(2)$, $= 0(3)$, $= 0(3)$, $\approx 0(4)$, $\approx 0(5)$

$$e = u + \frac{V^2}{2} + gz$$

Assumptions:
(1) $\dot{Q} = 0$ (insulated)
(2) $\dot{W}_s = 0$
(3) $\dot{W}_{shear} = \dot{W}_{other} = 0$
(4) neglect V^2
(5) neglect z
(6) perfect gas, $u = c_v T$
(7) properties uniform in bottle and at exit

From continuity, $0 = \frac{\partial M_{cv}}{\partial t} + \dot{m}$ $\quad\therefore \frac{\partial M_{cv}}{\partial t} = -\dot{m}$

From the first law,

$$0 = \frac{\partial}{\partial t}\int u\rho\, d\forall + \left(u + \frac{p}{\rho}\right)\dot{m}$$

$$= u\frac{\partial M}{\partial t} + M\frac{\partial u}{\partial t} + \left(u + \frac{p}{\rho}\right)\dot{m}$$

$$0 = u(-\dot{m}) + Mc_v\frac{\partial T}{\partial t} + \left(u + \frac{p}{\rho}\right)\dot{m}$$

Thus,

$$\frac{\partial T}{\partial t} = -\frac{\dot{m}\, p/\rho}{Mc_v} = \frac{-\dot{m}\,p}{\rho\forall c_v \rho} = -\frac{\dot{m}\,p}{\forall c_v \rho^2}$$

where

$$\rho = \frac{p}{RT} = 3000\frac{lbf}{in^2}\times 144\frac{in^2}{ft^2}\times\frac{lbm\cdot°R}{53.3\,ft\cdot lbf}\times\frac{1}{600°R} = 13.5\frac{lbm}{ft^3}$$

$$\therefore \frac{\partial T}{\partial t} = -0.1\frac{lbm}{s}\times 3000\frac{lbf}{in^2}\times 144\frac{in^2}{ft^2}\times\frac{1}{10\,ft^3}\times\frac{lbm\cdot°R}{0.171\,Btu}\times\frac{Btu}{778\,ft\cdot lbf}\times\frac{1}{(13.5)^2}\frac{ft^6}{lbm^2}$$

$$\frac{\partial T}{\partial t} = -0.178\,°R/s$$

$\frac{\partial T}{\partial t}$

Problem 4.185

Given: Centrifugal water pump operating under conditions as follows:

$D_1 = D_2 = 4$ in. $\quad Q = 300$ gpm

$p_1 = 8$ in Hg (vacuum), $p_2 = 35$ psig $\quad z_1 = z_2$

$\mathcal{P}_{input} = 9.1$ hp

Find: pump efficiency.

Solution: Apply the energy equation to the CV shown. Neglect all losses to find the energy added to the fluid.

Basic equations: $\eta = \dfrac{\dot{W}_s}{\mathcal{P}_{in}}$ where $\dot{W}_s$ = power into fluid

$$\dot{Q} - \dot{W}_s - \dot{W}_{shear} - \dot{W}_{other} = \frac{\partial}{\partial t}\int_{CV} e\rho\, d\forall + \int_{CS}\left(u + pv + \frac{V^2}{2} + gz\right)\rho\vec{V}\cdot d\vec{A}$$

(with $\dot{Q}$ =0(1), $\dot{W}_{shear}$ =0(2), $\dot{W}_{other}$ =0(2), $\frac{\partial}{\partial t}$ =0(3), u =0(4), gz =0(5))

Assumptions: (1) $\dot{Q} = 0$
(2) $\dot{W}_{shear} = 0$ (by choice of CV); $\dot{W}_{other} = 0$
(3) steady flow
(4) neglect Δu
(5) $\Delta z = 0$
(6) incompressible flow
(7) uniform flow at inlet and outlet

Then
$$-\dot{W}_s = \left(p_1 v_1 + \frac{V_1^2}{2}\right)\{-\dot{m}\} + \left(p_2 v_2 + \frac{V_2^2}{2}\right)\{\dot{m}\}$$

Since $\dot{m} = \rho Q$ and $V_1 = V_2$ (from continuity)

$$-\dot{W}_s = \rho Q(p_2 v_2 - p_1 v_1) = Q(p_2 - p_1)$$

$$p_1 = \rho g h = SG\,\rho_{H_2O}\, g h$$

$$p_1 = 13.6 \times 1.94\,\frac{\text{slug}}{\text{ft}^3} \times 32.2\,\frac{\text{ft}}{\text{s}^2} \times (-8\text{ in}) \times \frac{\text{ft}}{12\text{ in}} \times \frac{\text{lbf}\cdot\text{s}^2}{\text{ft}\cdot\text{slug}} \times \frac{\text{ft}^2}{144\text{ in}^2} = -3.93 \text{ psig}$$

$$\therefore -\dot{W}_s = 300\,\frac{\text{gal}}{\text{min}} \times \frac{\text{ft}^3}{7.48\text{ gal}} \times \frac{\text{min}}{60\text{ s}} \times [35 - (-3.93)]\frac{\text{lbf}}{\text{in}^2} \times \frac{144\text{ in}^2}{\text{ft}^2} \times \frac{\text{hp}\cdot\text{s}}{550\text{ ft}\cdot\text{lbf}}$$

$\dot{W}_s = -6.81$ hp (negative sign indicates energy added).

Then
$$\eta = \frac{\dot{W}_s}{\mathcal{P}_{in}} = \frac{6.81}{9.1} = 0.748 \text{ or } 74.8 \text{ percent} \longleftarrow \eta$$

Problem 4.186

Given: Compressor operating at conditions shown

$\dot{m} = 20\ \frac{lbm}{s}$

$p_1 = 14$ psia

$T_1 = 80°F$

$V_1 \simeq 0$

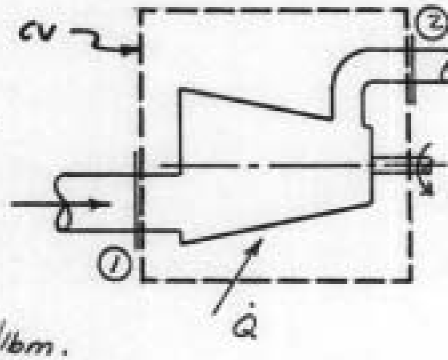
$p_2 = 70$ psia

$T_2 = 500°F$

$V_2 = 500\ \frac{ft}{s}$

$\dot{W}_{in} = 3200$ hp

Fluid is air.



Find: Heat transfer, in Btu/lbm.

Solution: Apply energy equation to CV shown.

Basic equations: $p = \rho RT$, $\Delta h = c_p \Delta T$

$$\dot{Q} - \dot{W}_s - \dot{W}_{shear} - \dot{W}_{other} = \frac{\partial}{\partial t}\int_{CV} e\rho\, dV + \int_{CS} \left(u + pv + \frac{V^2}{2} + gz\right)\rho \vec{V}\cdot d\vec{A}$$

($\dot{W}_{shear} = 0(2)$, $\dot{W}_{other} = 0(2)$, $\frac{\partial}{\partial t} = 0(3)$, $gz = 0(5)$)

Assumptions: (1) Ideal gas, constant specific heat
(2) $\dot{W}_{shear} = 0$ by choice of CV; $\dot{W}_{other} = 0$
(3) Steady flow
(4) Uniform flow at each section
(5) Neglect Δz
(6) $V_1 \approx 0$

By definition $h \equiv u + pv$, so

$$\dot{Q} - \dot{W}_s = \left(h_1 + \frac{V_1^2}{2}\right)\{-|\dot{m}|\} + \left(h_2 + \frac{V_2^2}{2}\right)\{|\dot{m}|\} = \dot{m}\left[\frac{V_2^2}{2} + c_p(T_2 - T_1)\right]$$

($V_1^2/2 \approx 0(6)$)

or

$$\frac{\delta Q}{dm} = \frac{\dot{Q}}{\dot{m}} = \frac{\dot{W}_s}{\dot{m}} + \frac{V_2^2}{2} + c_p(T_2 - T_1)$$

Noting $\dot{W}_s = -3200$ hp, so

$$\frac{\delta Q}{dm} = -3200\ hp \times \frac{2545\ Btu}{hp\cdot hr} \times \frac{s}{20\ lbm} \times \frac{hr}{3600\ s} + 0.240\ \frac{Btu}{lbm\cdot °F} \times (500-80)°F$$

$$+ \frac{(500)^2}{2}\frac{ft^2}{s^2} \times \frac{lbf\cdot s^2}{slug\cdot ft} \times \frac{slug}{32.2\ lbm} \times \frac{Btu}{778\ ft\cdot lbf}$$

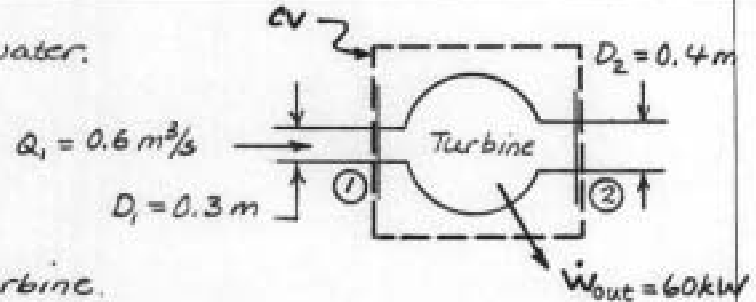
$$\frac{\delta Q}{dm} = -7.32\ Btu/lbm$$ $\frac{\delta Q}{dm}$

Therefore heat transfer is out of CV, since $\delta Q/dm < 0$. The rate of heat transfer is

$$\dot{Q} = -7.32\ \frac{Btu}{lbm} \times 20\ \frac{lbm}{s} = -146\ Btu/s$$ $\dot{Q}$

Problem 4.187

Given: Turbine operating on water.



Find: Pressure drop across turbine.

Solution: Apply continuity, energy equations, using CV shown.

Basic equations:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A} \quad (=0(1))$$

$$\dot{Q} - \dot{W}_s - \dot{W}_{shear} - \dot{W}_{other} = \frac{\partial}{\partial t}\int_{CV} e\rho\, d\forall + \int_{CS} \left(u + \frac{V^2}{2} + gz + pv\right)\rho \vec{V}\cdot d\vec{A}$$

$(\dot{Q} = 0(4),\ \dot{W}_{shear} = 0(5),\ \dot{W}_{other} = 0(5),\ \frac{\partial}{\partial t} = 0(1),\ u = 0(6),\ gz = 0(7))$

Assumptions:
(1) Steady flow
(2) Uniform flow at each section
(3) Incompressible flow
(4) $\dot{Q} = 0$
(5) $\dot{W}_{shear} = 0$ by choice of CV; $\dot{W}_{other} = 0$
(6) Neglect Δu
(7) Neglect Δz

Then

$$0 = \{-|\rho V_1 A_1|\} + \{|\rho V_2 A_2|\} \quad \text{or} \quad V_2 = V_1 \frac{A_1}{A_2} = V_1\left(\frac{D_1}{D_2}\right)^2$$

and

$$-\dot{W}_s = \left(\frac{V_1^2}{2} + p_1 v\right)\{-|\rho V_1 A_1|\} + \left(\frac{V_2^2}{2} + p_2 v\right)\{|\rho V_2 A_2|\}$$

$$-\dot{W}_s = -\left[\frac{V_1^2 - V_2^2}{2} + (p_1 - p_2)v\right]\rho Q = -\left\{\frac{V_1^2}{2}\left[1 - \left(\frac{D_1}{D_2}\right)^4\right] + (p_1 - p_2)v\right\}\rho Q$$

or

$$p_1 - p_2 = \frac{1}{v}\left\{\frac{\dot{W}_s}{\rho Q} - \frac{V_1^2}{2}\left[1 - \left(\frac{D_1}{D_2}\right)^4\right]\right\} = \frac{\dot{W}_s}{Q} - \frac{\rho V_1^2}{2}\left[1 - \left(\frac{D_1}{D_2}\right)^4\right]$$

But $V_1 = \frac{Q}{A_1} = 0.6\frac{m^3}{s} \times \frac{4}{\pi}\frac{1}{(0.3)^2 m^2} = 8.49\ m/s$, and $\dot{W}_s = \dot{W}_{out} = 60\ kW$, so

$$p_1 - p_2 = (60\ kW)\,10^3\frac{N\cdot m}{kW\cdot s} \times \frac{s}{0.6\,m^3} - \frac{1}{2} \times 999\frac{kg}{m^3} \times (8.49)^2\frac{m^2}{s^2}\left[1 - \left(\frac{0.3}{0.4}\right)^4\right]\frac{N\cdot s^2}{kg\cdot m}$$

$$p_1 - p_2 = 75.4\ kPa$$

$p_1 - p_2$

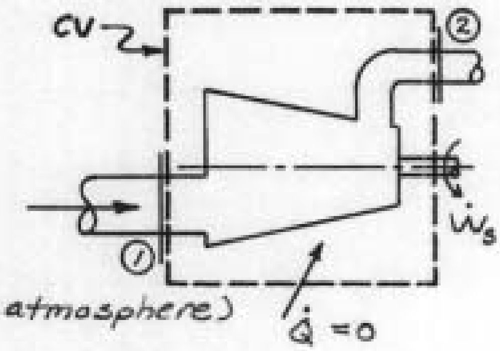
Problem 4.188

Given: Flow through turbomachine shown. Fluid is air.

$\dot{m} = 0.8$ kg/s

$T_1 = 288$ K

$p_1 = 101$ kPa (abs)



$T_2 = 130°$C

$p_2 = 500$ kPa (gage)

$V_2 = 100$ m/s

$V_1 \approx 0$ (from atmosphere)

Find: Shaft work interaction with surroundings.

Solution: Apply energy equation, using CV shown.

Basic equations: $p = \rho RT$, $\Delta h = c_p \Delta T$

$$\overset{=0(7)}{\dot{Q}} - \dot{W}_s - \overset{=0(2)}{\dot{W}_{shear}} - \overset{=0(2)}{\dot{W}_{other}} = \overset{=0(3)}{\frac{\partial}{\partial t}\int_{CV} e\rho\, d\forall} + \int_{CS}\left(u + pv + \frac{V^2}{2} + \overset{\approx 0(5)}{gz}\right)\rho \vec{V}\cdot d\vec{A}$$

Assumptions: (1) Ideal gas, constant specific heat
(2) $\dot{W}_{shear} = 0$ by choice of CV; $\dot{W}_{other} = 0$
(3) Steady flow
(4) Uniform flow at each section
(5) Neglect Δz
(6) $V_1 \approx 0$
(7) $\dot{Q} = 0$

By definition, $h \equiv u + pv$, so

$$-\dot{W}_s = \left(h_1 + \overset{\approx 0(6)}{\frac{V_1^2}{2}}\right)\{-|\dot{m}|\} + \left(h_2 + \frac{V_2^2}{2}\right)\{|\dot{m}|\} = \dot{m}\left(h_2 - h_1 + \frac{V_2^2}{2}\right)$$

or

$$-\dot{W}_s = \dot{m}\left(h_2 - h_1 + \frac{V_2^2}{2}\right) = \dot{m}\left[c_p(T_2 - T_1) + \frac{V_2^2}{2}\right]$$

$$= 0.8\,\frac{\text{kg}}{\text{s}}\left[1.00\,\frac{\text{kJ}}{\text{kg}\cdot\text{K}}(403 - 288)\,\text{K} + \frac{(100)^2}{2}\,\frac{\text{m}^2}{\text{s}^2} \times \frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}} \times \frac{\text{kJ}}{10^3\,\text{N}\cdot\text{m}}\right]\frac{\text{kW}\cdot\text{s}}{\text{kJ}}$$

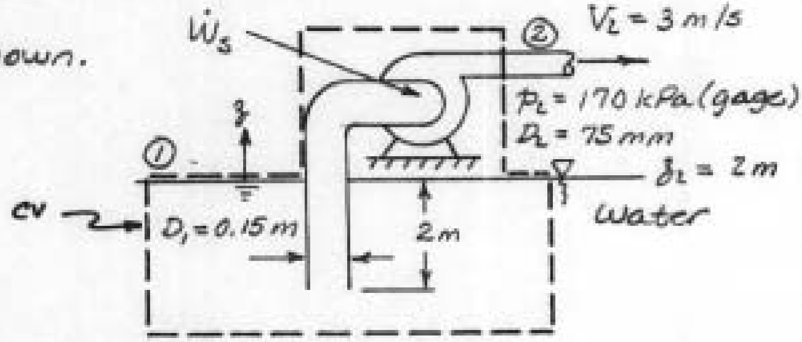
$-\dot{W}_s = 96.0$ kW or $\dot{W}_s = -96.0$ kW $\longleftarrow$ $\dot{W}_s$

{Power is into CV because $\dot{W}_s < 0$.}

Problem 4.189

Given: Pump system as shown.

$\eta_{pump} = 0.75$



Find: Power required.

Solution: Apply first law to CV shown, noting that flow enters with negligible velocity at section ①.

Basic equation:

$$\dot{Q} - \dot{W}_{shaft} - \dot{W}_{shear}^{=0(1)} - \dot{W}_{other}^{=0(1)} = \frac{\partial}{\partial t}\int_{CV} e\rho\, d\forall^{=0(2)} + \int_{CS}\left(e + \frac{p}{\rho}\right)\rho\vec{V}\cdot d\vec{A}$$

$$e = u + \frac{V^2}{2} + gz$$

Assumptions:
(1) $\dot{W}_{shear} = \dot{W}_{other} = 0$
(2) Steady flow
(3) $V_1 \simeq 0$
(4) $z_1 = 0$
(5) $p_1 = 0$ (gage)
(6) Uniform flow at each section
(7) Incompressible flow; $V_1 A_1 = V_2 A_2$

Then

$$\dot{Q} - \dot{W}_s = \left(u_1 + \frac{V_1^2}{2}^{\simeq 0(3)} + gz_1^{=0(4)} + \frac{p_1}{\rho}^{=0(5)}\right)\{-\dot{m}\} + \left(u_2 + \frac{V_2^2}{2} + gz_2 + \frac{p_2}{\rho}\right)\{\dot{m}\}$$

or

$$-\dot{W}_s = \dot{m}\left[\frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2 + \left(u_2 - u_1 - \frac{\delta Q}{dm}\right)\right]$$

Obtain the ideal or minimum power input by neglecting thermal effects. Thus

$$-\dot{W}_{s,ideal} = \dot{m}\left[\frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2\right]$$

For the system,

$$\dot{m} = \rho V_2 A_2 = 999\,\frac{kg}{m^3} \times 3\,\frac{m}{s} \times \frac{\pi}{4}(0.075)^2\,m^2 = 13.2\ kg/s$$

and

$$-\dot{W}_{s,ideal} = 13.2\,\frac{kg}{s}\left[1.70\times10^5\,\frac{N}{m^2}\times\frac{m^3}{999\,kg} + \frac{1}{2}(3)^2\frac{m^2}{s^2}\times\frac{N\cdot s^2}{kg\cdot m} + 9.81\,\frac{m}{s^2}\times 2\,m \times \frac{N\cdot s^2}{kg\cdot m}\right]$$

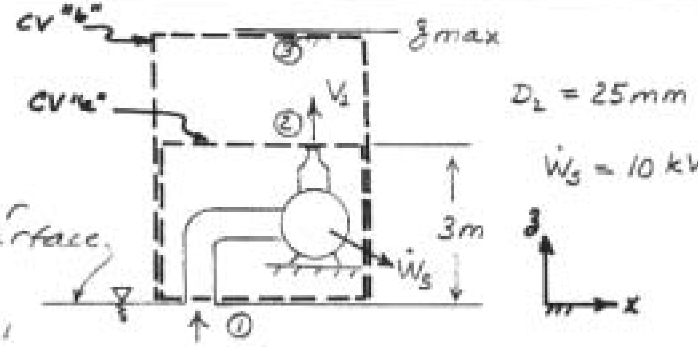
$$\dot{W}_{s,ideal} = -2560\,\frac{N\cdot m}{s}\times\frac{kW\cdot s}{10^3\,N\cdot m} = -2.56\ kW$$

Finally

$$\dot{W}_{s,actual} = \frac{\dot{W}_{s,ideal}}{\eta} = \frac{-2.56\,kW}{0.75} = -3.41\ kW$$

$\dot{W}_{s,actual}$

Given: Fire boat



Find: (a) Q_2
(b) z_{max}
(c) Force if horizontal

Solution: Apply first law to CV "a" shown above

Basic equation: $\dot{Q} - \dot{W}_s - \dot{W}_{shear} - \dot{W}_{other} = \frac{\partial}{\partial t}\int_{CV} e\rho\, dV + \int_{CS} (e + pv)\rho \vec{V}\cdot d\vec{A}$ (1) =0(2) =0(2) =0(3)

Assume: (1) Neglect losses, i.e. $u_2 - u_1 - \frac{dQ}{dm} \approx 0$ $\quad e = u + \frac{V^2}{2} + gz$
(2) $\dot{W}_{shear} = \dot{W}_{other} = 0$
(3) Steady flow
(4) Uniform flow at each section
(5) Neglect V_1
(6) $z_1 = 0$
(7) Incompressible flow, $v_2 = v_1 = v$, $\dot{m}_1 = \dot{m}_2 = \dot{m}_3$
(8) $p_2 = p_1 = p_{atm} = 0$ gage

Then

$$\dot{Q} - \dot{W}_s = \left(u_1 + \frac{V_1^2}{2} + gz_1 + p_1 v_1\right)(-\dot{m}) + \left(u_2 + \frac{V_2^2}{2} + gz_2 + p_2 v_2\right)(\dot{m})$$

($\approx 0(5)$, $=0(6)$, $=0(8)$)

or

$$-\dot{W}_s = \left(\frac{V_2^2}{2} + gz_2\right)\dot{m} \quad ; \dot{m} = \rho V_2 A_2$$

Note that this equation contains V_2 to the third power, so that it cannot be solved directly. As a first approximation, neglect z_2:

$$-\dot{W}_{in} \simeq \left(\frac{V_2^2}{2}\right)\rho V_2 A_2 = \frac{1}{2}\rho V_2^3 A_2 ; \quad A_2 = \frac{\pi}{4}D_2^2 = \frac{\pi}{4}(0.025)^2 m^2 = 4.91\times 10^{-4} m^2$$

$$V_2 \simeq \left[\frac{-2\dot{W}_s}{\rho A_2}\right]^{\frac{1}{3}} = \left[2\times 10\,kW \times \frac{m^3}{999\,kg} \times \frac{1}{4.91\times 10^{-4} m^2} \times \frac{10^3\,N\cdot m}{kW\cdot s} \times \frac{kg\cdot m}{N\cdot s^2}\right]^{\frac{1}{3}} = 34.4\,m/s$$

Comparing terms,

$$\frac{V_2^2}{2} \approx \frac{(34.4)^2}{2} = 592\,m^2/s^2 ; \quad gz_2 = 9.81\frac{m}{s^2} \cdot 3\,m = 29.4\,m^2/s^2, \text{ about 5 percent}$$

Therefore this value of V_2 is about $\frac{5}{3}$ percent too large. Assume $V_2 \simeq 33.8$ m/sec, and $Q = V_2 A_2 = 33.8\frac{m}{s} \cdot 4.91\times 10^{-4} m^2 = 0.0166\,m^3/s$ Q

To compute z_{max}, apply first law to CV "b" using above assumptions, plus

(9) $V_3 = 0$

Then $-\dot{W}_s = \left(\frac{V_2^2}{2} + g z_{max}\right)\dot{m}$; $\dot{m} = \rho Q = 999 \frac{kg}{m^3} \times 0.0166 \frac{m^3}{s} = 16.6\ kg/s$ (with $V_2^2/2 = 0$ (9))

or

$$z_{max} = \frac{-\dot{W}_s}{g\dot{m}} = 10\,kW \times \frac{s^2}{9.81\,m} \times \frac{s}{16.6\,kg} \times \frac{10^3\,N\cdot m}{kW\cdot s} \times \frac{kg\cdot m}{N\cdot s^2} = 61.4\ m$$ z_{max}

To find horizontal force, apply x component of momentum equation using CV "a", with flow at ② leaving horizontally.

Basic equation:

$$F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$$

($F_{B_x} = 0$ (11); $\frac{\partial}{\partial t} = 0$ (3))

Assumptions: (10) No net pressure force on CV; $F_{S_x} = R_x$
(11) $F_{B_x} = 0$

Then

$$R_x = u_1\{-\dot{m}_1\} + u_2\{\dot{m}\} = \dot{m}V_2\ ,\ \text{since } u_1 = 0$$

or

$$K_x = -\dot{m}V_2 = -16.6\frac{kg}{s} \times 33.8\frac{m}{s} \times \frac{N\cdot s^2}{kg\cdot m} = -561\ N$$ K_x

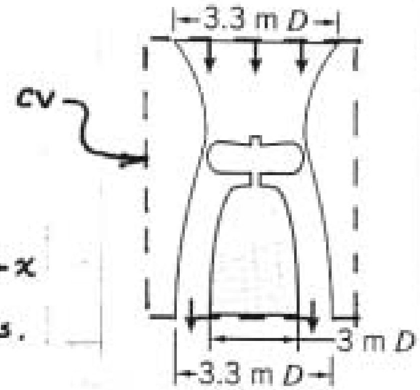
{Minus sign indicates reaction on boat is opposite from stream direction.}

Given: Helicopter-type craft hovering.

Mass, $M = 1500\,kg$

Assume atmospheric pressure at outlet, and treat as steady, uniform, incompressible flow.

Assume air is at standard conditions.



Find: (a) Speed of air leaving craft.
(b) Minimum power required.

Solution: Use inertial CV and coordinates shown. Apply continuity and momentum to determine V_2, then apply energy to find power.

Basic equations: $p = \rho RT$; $\Delta h = c_p \Delta T$; $\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A} \quad (=0(2))$$

$$F_{S_z} + F_{B_z} = \frac{\partial}{\partial t}\int_{CV} w \rho\, d\forall + \int_{CS} w \rho \vec{V} \cdot d\vec{A} \quad (=0(2))$$

Assumptions: (1) Air is an ideal gas, c_p = constant
(2) Steady flow
(3) Incompressible flow
(4) Uniform flow at each section
(5) Uniform pressure at inlet; $F_{S_z} = (p_{atm} - p_1)A_1 = -p_{1g}A_1$

Then

$$\rho = \frac{p}{RT} = 1.01 \times 10^5 \frac{N}{m^2} \times \frac{kg \cdot K}{287\, N \cdot m} \times \frac{1}{288\, K} = 1.22\, kg/m^3$$

and from continuity

$$0 = \{-|\rho V_1 A_1|\} + \{|\rho V_2 A_2|\} = \rho(V_2 A_2 - V_1 A_1) \quad \text{or} \quad V_1 = V_2\left(\frac{A_2}{A_1}\right)$$

Now $A_1 = \frac{\pi}{4} D_o^2 = \frac{\pi}{4}(3.3)^2 m^2 = 8.55\, m^2$

$$A_2 = \frac{\pi}{4}(D_o^2 - D_i^2) = \frac{\pi}{4}\left[(3.3)^2 - (3.0)^2\right] m^2 = 1.48\, m^2$$

From momentum

$$-p_{1g}A_1 - Mg = w_1\{-|\rho V_1 A_1|\} + w_2\{|\rho V_2 A_2|\}$$

$$w_1 = -V_1 \qquad w_2 = -V_2 \qquad \text{and} \quad \rho V_1 A_1 = \rho V_2 A_2$$

$$-p_{1g}A_1 - Mg = V_1 \rho V_2 A_2 - V_2 \rho V_2 A_2 = -\rho V_2 A_2 (V_2 - V_1)$$

For steady, incompressible flow without friction, along a streamline from atmosphere to ①, Bernoulli gives, neglecting Δz,

$$p_{atm} + \frac{1}{2}\rho V_0^2 + g z_0 = p_1 + \frac{1}{2}\rho V_1^2 + g z_1 \quad \text{so} \quad p_{1g} = -\frac{1}{2}\rho V_1^2$$

($V_0 \approx 0$)

Using continuity, $p_{1g} A_1 = -\frac{1}{2}\rho V_1^2 A_1 = -\frac{1}{2}\rho V_2 A_2 V_1 = -\frac{1}{2}\rho V_2^2 A_2 \frac{A_2}{A_1}$

Substituting into the momentum equation and using continuity,

$$\frac{1}{2}\rho V_2^2 A_2 \frac{A_2}{A_1} - Mg = -\rho V_2^2 A_2\left(1 - \frac{V_1}{V_2}\right) = -\rho V_2^2 A_2\left(1 - \frac{A_2}{A_1}\right) \quad \text{or} \quad Mg = \rho V_2^2 A_2\left(1 - \frac{1}{2}\frac{A_2}{A_1}\right)$$

Thus

$$V_2 = \sqrt{\frac{Mg}{\rho A_2\left(1 - \frac{1}{2}\frac{A_2}{A_1}\right)}} = \left[1500\,\text{kg} \times 9.81\frac{\text{m}}{\text{s}^2} \times \frac{\text{m}^3}{1.22\,\text{kg}} \times \frac{1}{1.48\,\text{m}^2} \times \frac{1}{\left(1 - \frac{1}{2}\frac{1.48}{8.55}\right)}\right]^{\frac{1}{2}} = 94.5\ \text{m/s}$$

V_2

Basic equation:

$$\dot{Q} - \dot{W}_s - \dot{W}_{shear} - \dot{W}_{other} = \frac{\partial}{\partial t}\int_{CV} e\rho\, d\forall + \int_{CS}\left(u + pv + \frac{V^2}{2} + gz\right)\rho \vec{V}\cdot d\vec{A}$$

($\dot{W}_{shear} = 0(6)$, $\dot{W}_{other} = 0(6)$, $\frac{\partial}{\partial t} = 0(2)$, $pv = 0(7)$, $gz \approx 0(8)$)

Additional assumptions: (6) $\dot{W}_{shear} = \dot{W}_{other} = 0$
(7) pv = constant
(8) Neglect Δz

Then

$$-\dot{W}_s = \left(u_1 + \frac{V_1^2}{2}\right)\{-|\dot{m}|\} + \left(u_2 + \frac{V_2^2}{2}\right)\{|\dot{m}|\} - \dot{Q}$$

$$-\dot{W}_s = \dot{m}\left(\frac{V_2^2 - V_1^2}{2}\right) + \dot{m}\left(u_2 - u_1 - \frac{dQ}{dm}\right)$$

The term $\left(u_2 - u_1 - \frac{dQ}{dm}\right)$ represents nonmechanical energy. The minimum possible work would be attained when the nonmechanical energy is zero. Thus

$$-\dot{W}_s\Big)_{min} = \dot{m}\left(\frac{V_2^2 - V_1^2}{2}\right) = \dot{m}\frac{V_2^2}{2}\left[1 - \left(\frac{V_1}{V_2}\right)^2\right] = \frac{\rho A_2 V_2^3}{2}\left[1 - \left(\frac{A_2}{A_1}\right)^2\right]$$

$$-\dot{W}_s = \frac{1}{2} \times 1.22\frac{\text{kg}}{\text{m}^3} \times 1.48\,\text{m}^2 \times \frac{(94.5)^3\text{m}^3}{\text{s}^3}\left[1 - \left(\frac{1.48}{8.55}\right)^2\right]\frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}} \times \frac{\text{kW}\cdot\text{s}}{10^3\,\text{N}\cdot\text{m}}$$

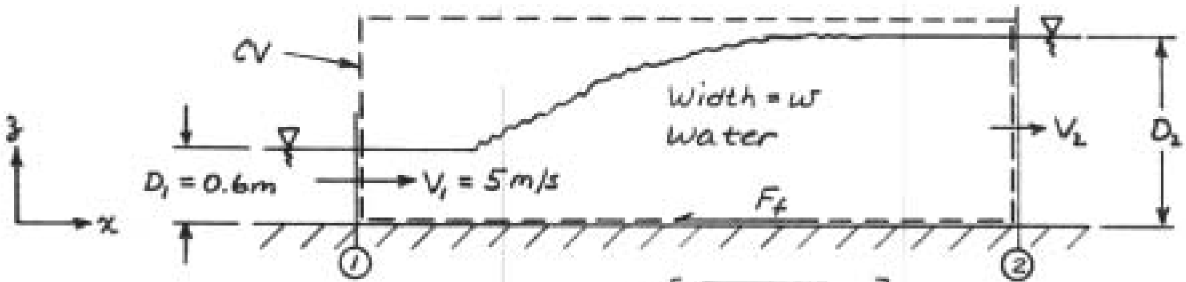
$$\dot{W}_s\Big)_{min} = -739\ \text{kW}\quad(\text{input})$$

$\dot{W}_s$

{The power required for hovering in a real craft would be greater due to flow losses, nonuniformities, etc.}

Problem 4.192

Given: Liquid flow in a wide, horizontal open channel, as shown.



Find: (a) Show that in general, $D_2 = \frac{D_1}{2}\left[\sqrt{1+\frac{8V_1^2}{gD_1}} - 1\right]$

(b) Change in mechanical energy across hydraulic jump.
(c) Temperature rise if no heat transfer.

Solution: Apply continuity, x component of momentum, and energy equations using CV shown.

Basic equations:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A} \qquad \left(\frac{\partial}{\partial t} = 0\,(1)\right)$$

$$F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} V_x \rho\, d\forall + \int_{CS} V_x \rho \vec{V}\cdot d\vec{A} \qquad \left(F_{B_x} = 0\,(8),\ \frac{\partial}{\partial t} = 0\,(1)\right)$$

$$\dot{Q} - \dot{W}_s - \dot{W}_{shear} - \dot{W}_{other} = \frac{\partial}{\partial t}\int_{CV} e\rho\, d\forall + \int_{CS}(e + pv)\rho\vec{V}\cdot d\vec{A}; \qquad e = u + \frac{V^2}{2} + gz$$

($\dot{Q} = 0\,(6)$, $\dot{W}_s = 0\,(7)$, $\dot{W}_{shear} = 0\,(7)$, $\dot{W}_{other} = 0\,(7)$, $\frac{\partial}{\partial t} = 0\,(1)$)

Assumptions:
(1) Steady flow
(2) Incompressible flow
(3) Uniform flow at each section
(4) Hydrostatic pressure distribution at sections ①, ②, so $p = \rho g(D - z)$
(5) Neglect friction force, F_f, on CV
(6) $\dot{Q} = 0$
(7) $\dot{W}_s = \dot{W}_{shear} = \dot{W}_{other} = 0$
(8) $F_{B_x} = 0$, since channel is horizontal

From continuity,

$$0 = \{-|\rho V_1 A_1|\} + \{|\rho V_2 A_2|\} = -\rho V_1 w D_1 + \rho V_2 w D_2\ ; \quad V_1 D_1 = V_2 D_2$$

From momentum,

$$F_{S_x} = \underbrace{\rho g \frac{D_1}{2} w D_1 - \rho g \frac{D_2}{2} w D_2}_{\text{hydrostatic forces}} = V_{x_1}\{-|\rho V_1 w D_1|\} + V_{x_2}\{|\rho V_2 w D_2|\}$$

$$V_{x_1} = V_1 \qquad V_{x_2} = V_2$$

or

$$\frac{g}{2}(D_1^2 - D_2^2) = V_1 D_1 (V_2 - V_1) = V_1^2 D_1\left(\frac{V_2}{V_1} - 1\right) = V_1^2 D_1\left(\frac{D_1}{D_2} - 1\right)$$

or

$$\frac{g}{2}(D_1 + D_2)\cancel{(D_1 - D_2)} = V_1^2 \frac{D_1}{D_2}\cancel{(D_1 - D_2)}$$

Thus $\frac{gD_1}{2}\left(1+\frac{D_2}{D_1}\right) = V_1^2\frac{D_1}{D_2}$ or $\frac{D_2}{D_1}\left(1+\frac{D_2}{D_1}\right) = \frac{2V_1^2}{gD_1}$ or $\left(\frac{D_2}{D_1}\right)^2 + \frac{D_2}{D_1} - \frac{2V_1^2}{gD_1} = 0$

Using the quadratic equation,

$$\frac{D_2}{D_1} = \frac{1}{2}\left[-1 \pm \sqrt{1+\frac{8V_1^2}{gD_1}}\right] \quad \text{or} \quad D_2 = \frac{D_1}{2}\left[\sqrt{1+\frac{8V_1^2}{gD_1}} - 1\right]$$

D_2

Solving for D_2

$$D_2 = \frac{1}{2}\times 0.6m\left[\sqrt{1+8\times(5)^2\frac{m^2}{s^2}\times\frac{s^2}{9.81m}\times\frac{1}{0.6m}} - 1\right] = 1.47\,m$$

$$V_2 = \frac{D_1}{D_2}V_1 = \frac{0.6}{1.47}\times 5\frac{m}{s} = 2.04\,m/s$$

From the energy equation, with $e_{mech} = \frac{V^2}{2} + gz + \frac{p}{\rho}$, and $dA = w\,dz$, the mechanical energy fluxes are

$$mef_1 = \int_0^{D_1}\left[\frac{V_1^2}{2} + gz + \frac{1}{\rho}\rho g(D-z)\right]\rho V_1 w\,dz = \left(\frac{V_1^2}{2} + gD_1\right)\rho V_1 w D_1$$

$$mef_2 = \int_0^{D_2}\left[\frac{V_2^2}{2} + gz + \frac{1}{\rho}\rho g(D-z)\right]\rho V_2 w\,dz = \left(\frac{V_2^2}{2} + gD_2\right)\rho V_2 w D_2$$

and

$$\Delta mef = mef_2 - mef_1 = \left[\frac{V_2^2 - V_1^2}{2} + g(D_2 - D_1)\right]\rho V_1 w D_1\text{, since } V_1D_1 = V_2D_2$$

Thus $\frac{\Delta mef}{\dot{m}} = \frac{1}{2}\left[V_2^2 - V_1^2 + 2g(D_2 - D_1)\right]$

$$\frac{\Delta mef}{\dot{m}} = \frac{1}{2}\left[(2.04)^2\frac{m^2}{s^2} - (5)^2\frac{m^2}{s^2} + 2\times 9.81\frac{m}{s^2}(1.47-0.6)m\right]\frac{N\cdot s^2}{kg\cdot m} = -1.88\,N\cdot m/kg$$

$\frac{\Delta mef}{\dot{m}}$

From the energy equation,

$$0 = \left[u_1 + \frac{V_1^2}{2} + gz + \frac{1}{\rho}\rho g(D-z)\right]\{-|\rho V_1 w D_1|\}$$
$$+\left[u_2 + \frac{V_2^2}{2} + gz + \frac{1}{\rho}\rho g(D-z)\right]\{|\rho V_2 w D_2|\}$$

or

$$0 = (u_2 - u_1)\dot{m} + \Delta mef$$

Thus

$$u_2 - u_1 = c_v(T_2 - T_1) = -\frac{\Delta mef}{\dot{m}}$$

$$\Delta T = T_2 - T_1 = -\frac{\Delta mef}{\dot{m}c_v} = -\left(-1.88\frac{N\cdot m}{kg}\right)\frac{kg\cdot K}{1\,kcal}\times\frac{kcal}{4187\,J} = 4.49\times 10^{-4}K$$

ΔT

{This small temperature change would be almost impossible to measure.}

Given: Velocity fields listed below

Find: Which are possible two-dimensional, incompressible flow cases?

<u>Solution:</u> Apply the continuity equation in differential form.

Basic equation: $\frac{\partial}{\partial x}\rho u + \frac{\partial}{\partial y}\rho v + \cancel{\frac{\partial}{\partial z}\rho w} + \cancel{\frac{\partial \rho}{\partial t}} = 0$ (the crossed-out terms are marked $= 0(1)$ and $= 0(2)$)

Assumptions: (1) Two-dimensional flow, $\vec{V} = \vec{V}(x,y)$, so $\frac{\partial}{\partial z} = 0$

(2) Incompressible flow

$\rho =$ constant, so $\frac{\partial \rho}{\partial t} = 0$, $\frac{\partial \rho}{\partial(\text{distance})} = 0$

Then, $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ is criterion.

(a) $u = 2x^2 + y^2 - x^2 y$, $v = x^3 + x(y^2 - 2y)$

$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = (4x - 2xy) + x(2y - 2)$

$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 4x - 2xy + 2xy - 2x \neq 0$

so $\rho \neq$ constant

(b) $u = 2xy - x^2 + y$, $v = 2xy - y^2 + x^2$

$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = (2y - 2x) + (2x - 2y) = 0$

so <u>possible</u>

(c) $u = xt + 2y$, $v = xt^2 - yt$

$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = t - t = 0$, so <u>possible</u>

(d) $u = (x + 2y)xt$, $v = -(2x + y)yt$

$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = (2xt + 2yt) + (-2xt - 2yt) = 0$

so <u>possible</u>

Given: Velocity fields listed below

Find: Which are possible two-dimensional, incompressible flow cases?

Solution: Apply the continuity equation in differential form

Basic equation: $\frac{\partial}{\partial x}\rho u + \frac{\partial}{\partial y}\rho v + \frac{\partial}{\partial z}\rho w + \frac{\partial \rho}{\partial t} = 0$ (the $\frac{\partial}{\partial z}\rho w$ term $= 0(1)$; the $\frac{\partial \rho}{\partial t}$ term $= 0(2)$)

Assumptions: (1) Two-dimensional flow, $\vec{V} = \vec{V}(x,y)$, so $\frac{\partial}{\partial z} = 0$

(2) Incompressible flow

ρ = constant, so $\frac{\partial \rho}{\partial t} = 0$, $\frac{\partial \rho}{\partial(\text{distance})} = 0$

Then

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \text{is the criterion}$$

(a) $u = -x + y$, $v = x - y^2$ $\quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -1 - 2y \neq 0$, so $\rho \neq$ constant

(b) $u = x + 2y$, $v = x^2 - y$ $\quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 1 - 1 = 0$, so possible

(c) $u = 4x^2 - y$, $v = x - y^2$ $\quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 8x - 2y \neq 0$, so $\rho \neq$ constant

(d) $u = xt + 2y$, $v = x^2 - yt$ $\quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = t - t = 0$, so possible

(e) $u = xt^2$, $v = xyt + y^2$ $\quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = t^2 + xt + 2y \neq 0$, so $\rho \neq$ constant

Problem 5.3

Given: Velocity field $u = Ax + By + Cz$

$v = Dx + Ey + Fz$

$w = Gx + Hy + Jz$

Find: The relationship among coefficients A thru J for this to be an incompressible flow field.

Solution: Flow must satisfy differential form of continuity.

Basic equation: $\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} + \frac{\partial \rho}{\partial t} = 0$ (with $\frac{\partial \rho}{\partial t} = 0$)

Assumption: Incompressible flow, so $\frac{\partial \rho}{\partial t} = \frac{\partial \rho}{\partial \rho} = 0$

Then $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$

For the given flow field, $\frac{\partial u}{\partial x} = A$, $\frac{\partial v}{\partial y} = E$, $\frac{\partial w}{\partial z} = J$. Thus

$A + E + J = 0$, and

B, C, D, F, G, H are arbitrary ←

Problem 5.4

Given: Velocity profiles listed below.

Find: Which are possible three-dimensional, incompressible cases?

Solution: Apply the continuity equation in differential form.

Basic equation: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$

Assumption: Incompressible flow

Field	Terms	Sum	Possible
(a) $u = x + y + z^2$	$\frac{\partial u}{\partial x} = 1$		
$v = x - y + z$	$\frac{\partial v}{\partial y} = -1$	0	Yes
$w = 2xy + y^2 + 4$	$\frac{\partial w}{\partial z} = 0$		
(b) $u = xyzt$	$\frac{\partial u}{\partial x} = yzt$		
$v = -xyzt^2$	$\frac{\partial v}{\partial y} = -xzt^2$	0	Yes
$w = \frac{z^2}{2}(xt^2 - yt)$	$\frac{\partial w}{\partial z} = xzt^2 - yzt$		
(c) $u = y^2 + 2xz$	$\frac{\partial u}{\partial x} = 2z$		
$v = -2yz + x^2yz$	$\frac{\partial v}{\partial y} = -2z + x^2z$	$\neq 0$	No
$w = \frac{x^2z^2}{2} + x^3y^4$	$\frac{\partial w}{\partial z} = x^2z$		

Problem 5.5

Given: Flow in xy plane, $u = Ax(y-B)$, where $A = 3\ m^{-1}s^{-1}$, $B = 2\ m$, and coordinates are measured in meters

Find: (a) Possible y component for steady, incompressible flow.
(b) If result is valid for unsteady, incompressible flow.
(c) Number of possible y components

Solution:

Basic equation: $\nabla \cdot \rho\vec{V} + \frac{\partial \rho}{\partial t} = 0 = \frac{\partial}{\partial x}\rho u + \frac{\partial}{\partial y}\rho v + \frac{\partial}{\partial z}\rho w + \frac{\partial \rho}{\partial t}$ (with $\frac{\partial}{\partial z}\rho w = 0$ (1), $\frac{\partial \rho}{\partial t} = 0$ (2))

Assumptions: (1) Flow in xy plane (given), $\frac{\partial}{\partial z} = 0$
(2) ρ = constant (given)

Then $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ or $\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}$

and $\frac{\partial v}{\partial y} = -\frac{\partial}{\partial x} Ax(y-B) = -A(y-B)$

Integrating

$$v = \int \frac{\partial v}{\partial y}\,dy = -A\int (y-B)\,dy = -A\left(\frac{y^2}{2} - By\right) + f(x)$$ ← v

The basic equation reduces to the same form for unsteady flow (as with steady flow). Hence the result is also valid for unsteady flow ← (b)

There are an infinite number of possible y components, since $f(x)$ is arbitrary. The simplest is obtained with $f(x) = 0$. ← (c)

Then,

$$v = -3\left(\frac{y^2}{2} - 2y\right)$$

Problem 5.6

Given: Flow in xy plane, $v = y^2 - 2x + 2y$, steady.

Find: (a) Possible x component for ρ = constant.
(b) Is it also valid for unsteady flow with $\rho = c$?
(c) Number of possible x components.

Solution:

Basic equation: $\nabla \cdot \rho \vec{V} + \frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x}\rho u + \frac{\partial}{\partial y}\rho v + \frac{\partial}{\partial z}\rho w + \frac{\partial \rho}{\partial t} = 0$ (where $\frac{\partial}{\partial z}\rho w = 0$ (1) and $\frac{\partial \rho}{\partial t} = 0$ (2))

Assume: (1) Flow in xy plane, $\frac{\partial}{\partial z} = 0$
(2) ρ = constant

Then $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ or $\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$

$$-\frac{\partial v}{\partial y} = -\frac{\partial}{\partial y}(y^2 - 2x + 2y) = -(2y + 2) = -2y - 2$$

Integrating,

$$u = \int \frac{\partial u}{\partial x}\,dx = \int -\frac{\partial v}{\partial y}\,dx = \int (-2y - 2)\,dx = -2yx - 2x + f(y)$$

u

The basic equation reduces to the same form for unsteady flow with ρ = constant. Therefore it is also valid for unsteady flow.

There are an infinite number of possible x components, since $f(y)$ is arbitrary. The simplest would be to choose $f(y) = 0$.

Problem 5.7

Given: Steady, incompressible flow field in the xy plane has an x component of velocity given by
$u = \frac{A}{x}$, where $A = 2\ m^2/s$ and x is in meters.

Find: the simplest y component of velocity for this flow field.

Solution:

Apply the continuity equation for the conditions given

Basic equation: $\nabla \cdot \rho \vec{V} + \frac{\partial \rho}{\partial t} = 0$

For steady flow $\frac{\partial \rho}{\partial t} = 0$ and for two-dimensional flow in the xy plane, $\frac{\partial}{\partial z}() = 0$. Thus the basic equation reduces to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Then

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -\frac{\partial}{\partial x}\left(\frac{A}{x}\right) = \frac{A}{x^2}$$

and

$$v = \int \frac{\partial v}{\partial y}\,dy + f(x) = \int \frac{A}{x^2}\,dy + f(x) = \frac{Ay}{x^2} + f(x)$$

The simplest y component of velocity is obtained with $f(x) = 0$

$$\therefore\ v = \frac{Ay}{x^2} \longleftarrow v$$

Problem 5.8

Given: The $\underline{y}$ component of velocity for a steady, incompressible flow in the $\underline{xy}$ plane is

$v = Ay^2/x^2$, where $A = 2$ m/s , x & y in m

Find: simplest x component.

Solution: Apply differential form of conservation of mass

For two-dimensional, incompressible flow,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \text{ . Thus } \frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} = -A\frac{y^2}{x^2}$$

Integrating,

$$u = \frac{2Ay}{x} + f(y)$$. The simplest form is for $f(y)=0$

Thus,

$$u = 2\frac{Ay}{x} = 4\frac{y}{x}$$ ← u

and

$$\vec{V} = 2A\frac{y}{x}\hat{i} + A\frac{y^2}{x^2}\hat{j} = 4\frac{y}{x}\hat{i} + 2\frac{y^2}{x^2}\hat{j}$$ ← $\vec{V}$

Problem 5.9

The x component of velocity in a steady incompressible flow field in the *xy* plane is $u = Ax/(x^2 + y^2)$, where $A = 10\ \text{m}^2/\text{s}$, and x and y are measured in meters. Find the simplest y component of velocity for this flow field.

Given: x component of velocity of incompressible flow

Find: y component of velocity

Solution

$$u(x,y) = \frac{A\cdot x}{x^2 + y^2}$$

For incompressible flow $$\frac{du}{dx} + \frac{dv}{dy} = 0$$

Hence $$v(x,y) = -\int \frac{d}{dx}u(x,y)\,dy$$

$$\frac{du}{dx} = \frac{A\cdot\left(y^2 - x^2\right)}{\left(x^2 + y^2\right)^2}$$

so $$v(x,y) = \int \frac{A\cdot\left(x^2 - y^2\right)}{\left(x^2 + y^2\right)^2}\,dy$$ $$v(x,y) = \frac{A\cdot y}{x^2 + y^2}$$

Problem 5.10

Given: Approximate profile for laminar boundary layer

$$u = cU\frac{y}{x^{1/2}}$$

Find: (a) Show simplest v is $v = \frac{U}{4}\frac{y}{x}$

(b) Evaluate maximum value of v/U where $\delta = 5\,mm$, $x = 0.5\,m$.

<u>Solution:</u> Apply continuity for incompressible flow

Basic equation: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$ (2-D flow: $\frac{\partial w}{\partial z}$ crossed out)

Thus

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -\left(-\frac{1}{2}\right)cU\frac{y}{x^{3/2}}$$

$$v = \int \frac{\partial v}{\partial y}dy + f(x) = \int \frac{1}{2}cU\frac{y}{x^{3/2}}dy + f(x) = \frac{1}{4}cU\frac{y^2}{x^{3/2}} + f(x)$$

or

$$v = \frac{U}{4}\frac{y}{x} \qquad [f(x) = 0 \text{ since } v = 0 \text{ along } y = 0]$$

v

From

$$\frac{v}{U} = \frac{1}{4}\frac{y}{x}$$

maximum value occurs at $y = \delta$. At the location given,

$$\left.\frac{v}{U}\right)_{max} = \frac{1}{4}\frac{\delta}{x} = \frac{1}{4}\frac{0.005\,m}{0.5\,m} = 0.0025$$

$\left.\frac{v}{U}\right)_{m}$

Problem 5.11

Given: Approximation for x component of velocity in laminar boundary layer

$$u = U \sin\left(\frac{\pi}{2}\frac{y}{\delta}\right) \qquad \text{where } \delta = c x^{1/2}$$

Show: $\frac{v}{U} = \frac{\delta}{\pi x}\left[\cos\left(\frac{\pi}{2}\frac{y}{\delta}\right) + \frac{\pi}{2}\frac{y}{\delta}\sin\left(\frac{\pi}{2}\frac{y}{\delta}\right) - 1\right]$ for incompressible flow.

Plot: $\frac{x}{\delta}\frac{v}{U}$ vs. y/δ to locate maximum value of v/U; evaluate at location where $x = 0.5\,m$ and $\delta = 5\,mm$.

Solution: Apply differential continuity for incompressible flow.

Basic equation: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$ ($\frac{\partial w}{\partial z} = 0$ (2-D flow))

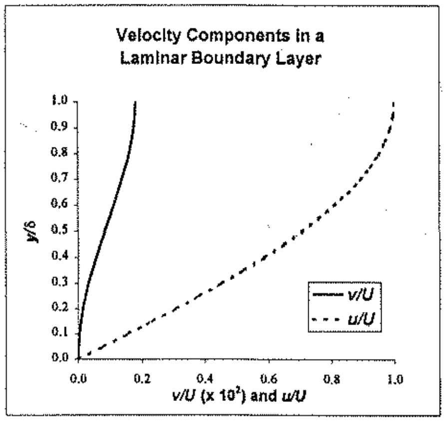
Thus $\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial \delta}\frac{d\delta}{dx} = -\left(\frac{\pi y}{2}\right)\left(-\frac{1}{\delta^2}\right)\cos\left(\frac{\pi y}{2\delta}\right)\frac{U}{2}c x^{-1/2} = \frac{U}{2x}\left(\frac{\pi y}{2\delta}\right)\cos\left(\frac{\pi y}{2\delta}\right)$

Integrating, $v = \int_0^y \frac{\partial v}{\partial y}dy + f(x) = \int_0^y \frac{U}{2x}\left(\frac{\pi}{2}\frac{y}{\delta}\right)\cos\left(\frac{\pi}{2}\frac{y}{\delta}\right)dy + f(x)$

$$v = \frac{2\delta}{\pi}\frac{U}{2x}\int_0^{\frac{\pi}{2}\frac{y}{\delta}} u\cos u\,du + f(x) = \frac{\delta}{\pi}\frac{U}{x}\Big[\cos u + u\sin u\Big]_0^{\frac{\pi}{2}\frac{y}{\delta}} + f(x)$$

($f(x) = 0$)

$$\frac{v}{U} = \frac{1}{\pi}\frac{\delta}{x}\left[\cos\left(\frac{\pi}{2}\frac{y}{\delta}\right) + \left(\frac{\pi}{2}\frac{y}{\delta}\right)\sin\left(\frac{\pi}{2}\frac{y}{\delta}\right) - 1\right]$$



This expression is a maximum at $y = \delta$, where

$$\frac{v}{U} = \frac{1}{\pi}\frac{\delta}{x}\left[\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right) - 1\right] = \frac{\delta}{\pi x}\left(\frac{\pi}{2} - 1\right)$$

and

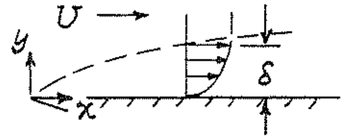
$$\left.\frac{v}{U}\right)_{max} = 0.182\,\frac{\delta}{x} \qquad \longleftarrow \left.\frac{v}{U}\right)_{max}$$

At the location given

$$\left.\frac{v}{U}\right)_{max} = 0.182 \times 0.005\,m \times \frac{1}{0.5\,m} = 0.00182 \text{ or } 0.182 \text{ percent} \qquad \longleftarrow \left.\frac{v}{U}\right)_{max}$$

Given: Laminar boundary layer, parabolic approximate profile.

$$\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2 \qquad \delta = c x^{1/2}$$



Find: Show $\frac{v}{U} = \frac{\delta}{x}\left[\frac{1}{2}\left(\frac{y}{\delta}\right)^2 - \frac{1}{3}\left(\frac{y}{\delta}\right)^3\right]$ for incompressible flow.

Plot: $\frac{v}{U}$ vs. $\frac{y}{\delta}$, evaluate max. at $x = 0.5$ m, if $\delta = 5$ mm.

Solution: Apply conservation of mass for incompressible flow.

Basic equation: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$ (with $\frac{\partial w}{\partial z}$ struck out, $= 0(2)$)

Assumptions: (1) Incompressible flow (ρ = const)
(2) $w = 0$

Then $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$; $\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}$; $v = \int_0^y -\frac{\partial u}{\partial x}\,dy + f(x)$ (with $f(x)$ struck out, $= 0$ (simplest))

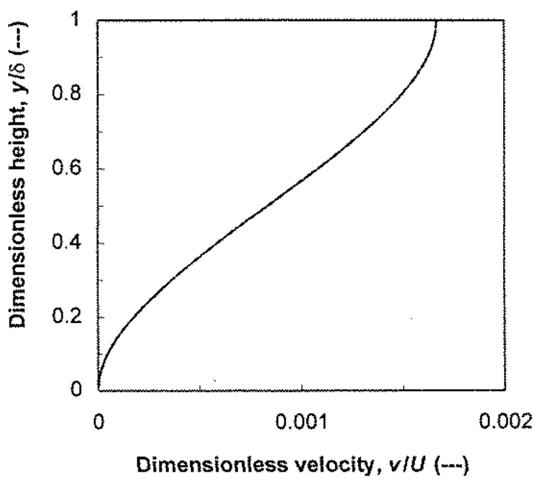
From the given profile

$$\frac{\partial u}{\partial x} = 2Uy(-1)\frac{1}{\delta^2}\frac{d\delta}{dx} - Uy^2(-2)\frac{1}{\delta^3}\frac{d\delta}{dx} = 2U\frac{d\delta}{dx}\left(\frac{y^2}{\delta^3} - \frac{y}{\delta^2}\right)$$

Since $\delta = cx^{1/2}$, $\frac{d\delta}{dx} = \frac{1}{2}cx^{-1/2} = \frac{cx^{1/2}}{2x} = \frac{\delta}{2x}$, so $\frac{\partial u}{\partial x} = \frac{U\delta}{x}\left(\frac{y^2}{\delta^3} - \frac{y}{\delta^2}\right)$

Integrating, $\frac{v}{U} = \frac{\delta}{x}\int_0^y\left(\frac{y}{\delta^2} - \frac{y^2}{\delta^3}\right)dy = \frac{\delta}{x}\left[\frac{1}{2}\left(\frac{y}{\delta}\right)^2 - \frac{1}{3}\left(\frac{y}{\delta}\right)^3\right]$ ← $\frac{v}{U}$

Plotting shows:



Maximum occurs at $\left(\frac{y}{\delta}\right) = 1$

$$\left.\frac{v}{U}\right)_{max} = \left.\frac{v}{U}\right)_{\frac{y}{\delta}=1} = \frac{\delta}{x}\left[\frac{1}{2}(1)^2 - \frac{1}{3}(1)^2\right] = \frac{\delta}{6x}$$ ← $\left.\frac{v}{U}\right)_{max}$

Evaluating, $\left.\frac{v}{U}\right)_{max} = \frac{1}{6} \times 0.005\,\text{m} \times \frac{1}{0.5\,\text{m}} = 0.00167$ or 0.167 percent ← $\left.\frac{v}{U}\right)_{max}$

Problem 5.13

A useful approximation for the *x*
layer is a cubic variation from $u = 0$ at the surface ($y = 0$) to the freestream velocity, *U*, at the edge of the boundary layer ($y = \delta$). The equation for the profile is $u/U = 3/2(y/\delta) - 1/2(y/\delta)^3$, where $\delta = cx^{1/2}$ and *c* is a constant. Derive the simplest expression for v/U, the *y* component of velocity ratio. Plot u/U and v/U versus y/δ, and find the location of the maximum value of the ratio v/U. Evaluate the ratio where $\delta = 5$ mm and $x = 0.5$ m.

Given: Data on boundary layer

Find: *y* component of velocity ratio; location of maximum value; plot velocity profiles; evaluate at particular point

Solution

$$u(x,y) = U\cdot\left[\frac{3}{2}\cdot\left(\frac{y}{\delta(x)}\right) - \frac{1}{2}\cdot\left(\frac{y}{\delta(x)}\right)^3\right]$$

and $$\delta(x) = c\cdot\sqrt{x}$$

so $$u(x,y) = U\cdot\left[\frac{3}{2}\cdot\left(\frac{y}{c\cdot\sqrt{x}}\right) - \frac{1}{2}\cdot\left(\frac{y}{c\cdot\sqrt{x}}\right)^3\right]$$

For incompressible flow $$\frac{du}{dx} + \frac{dv}{dy} = 0$$

Hence $$v(x,y) = -\int \frac{d}{dx}u(x,y)\,dy$$

$$\frac{du}{dx} = \frac{3}{4}\cdot U\cdot\left(\frac{y^3}{c^3\cdot x^{\frac{5}{2}}} - \frac{y}{c\cdot x^{\frac{3}{2}}}\right)$$

so $$v(x,y) = -\int \frac{3}{4}\cdot U\cdot\left(\frac{y^3}{c^3}\cdot\frac{x^5}{2} - \frac{y}{c}\cdot\frac{x^3}{2}\right) dy$$

$$v(x,y) = \frac{3}{8}\cdot U\cdot\left(\frac{y^2}{c\cdot x^{\frac{3}{2}}} - \frac{y^4}{2\cdot c^3\cdot x^{\frac{5}{2}}}\right)$$

$$v(x,y) = \frac{3}{8}\cdot U\cdot\frac{\delta}{x}\cdot\left[\left(\frac{y}{\delta}\right)^2 - \frac{1}{2}\cdot\left(\frac{y}{\delta}\right)^4\right]$$

The maximum occurs at $y = \delta$ as seen in the corresponding *Excel* workbook

$$v_{max} = \frac{3}{8}\cdot U\cdot\frac{\delta}{x}\cdot\left(1 - \frac{1}{2}\cdot 1\right)$$

At $\delta = 5\cdot mm$ and $x = 0.5\cdot m$, the maximum vertical velocity is

$$\frac{v_{max}}{U} = 0.00188$$

Problem 5.13 (In Excel)

A useful approximation for the x component of velocity in an incompressible laminar boundary layer is a cubic variation from $u = 0$ at the surface ($y = 0$) to the freestream velocity, U, at the edge of the boundary layer ($y = d$). The equation for the profile is $u/U = 3/2(y/d) - 1/2(y/d)^3$, where $d = cx^{1/2}$ and c is a constant. Derive the simplest expression for v/U, the y component of velocity ratio. Plot u/U and v/U versus y/d, and find the location of the maximum value of the ratio v/U. Evaluate the ratio where $d = 5$ mm and $x = 0.5$ m.

Given: Data on boundary layer

Find: y component of velocity ratio; location of maximum value;
plot velocity profiles; evaluate at particular point

Solution

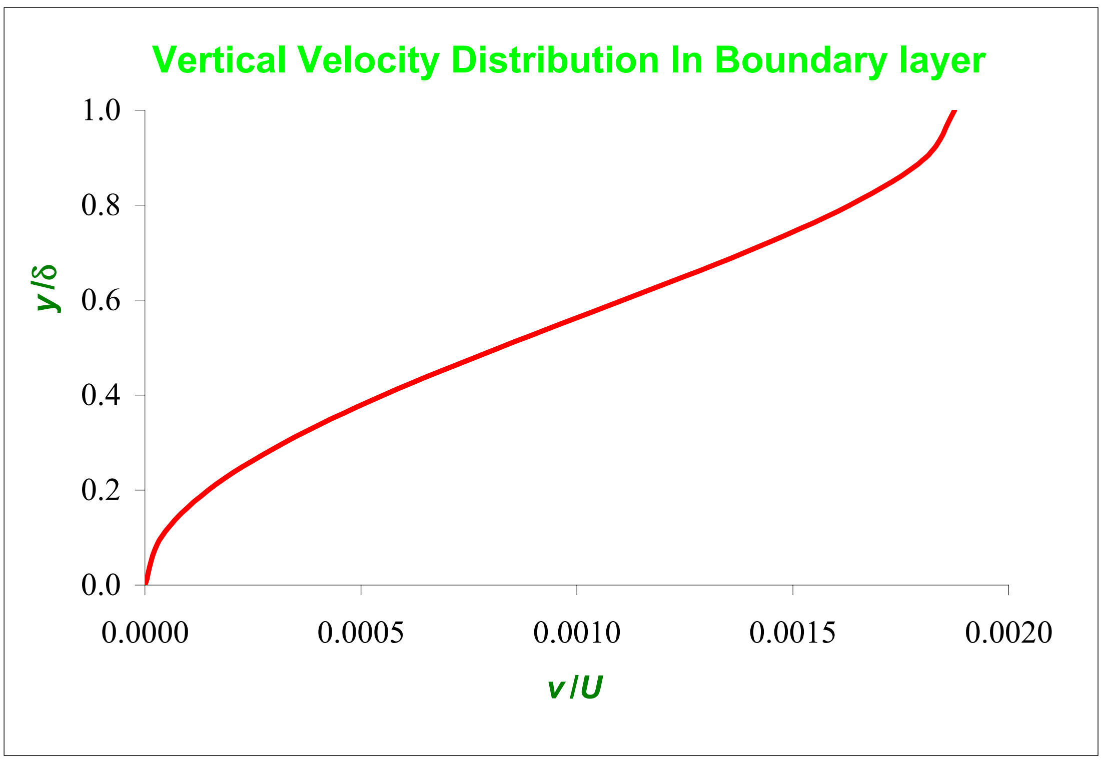
The solution is

$$\frac{v}{U} = \frac{3}{8}\cdot\frac{\delta}{x}\cdot\left[\left(\frac{y}{\delta}\right)^2 - \frac{1}{2}\cdot\left(\frac{y}{\delta}\right)^4\right]$$

To find when v/U is maximum, use *Solver*

v/U	y/δ
0.00188	1.0

v/U	y/δ
0.000000	0.0
0.000037	0.1
0.000147	0.2
0.000322	0.3
0.000552	0.4
0.00082	0.5
0.00111	0.6
0.00139	0.7
0.00163	0.8
0.00181	0.9
0.00188	1.0



Given: Flow in xy plane, $v = -Bxy^3$ where $B = 0.2\ m^{-3}\cdot s^{-1}$ and coordinates are measured in meters; steady, $\rho = c$.

Find: (a) Simplest x component of velocity.
(b) Equation of streamlines.

Plot: streamlines through points (1,4) and (2,4).

Solution:

Basic equation: $\nabla\cdot\rho\vec{V} + \frac{\partial\rho}{\partial t} = \frac{\partial}{\partial x}\rho u + \frac{\partial}{\partial y}\rho v + \frac{\partial}{\partial z}\rho w + \frac{\partial\rho}{\partial t}$ (the $\frac{\partial}{\partial z}$ term $= 0$ (1); the $\frac{\partial\rho}{\partial t}$ term $= 0$ (2))

Assumptions: (1) flow in the xy plane (given), $\frac{\partial}{\partial z} = 0$
(2) ρ = constant (given).

Then, $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ or $\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$.

and $\frac{\partial u}{\partial x} = -\frac{\partial}{\partial y}(-Bxy^3) = 3Bxy^2$

Integrating,

$$u = \int \frac{\partial u}{\partial x}dx = \int 3Bxy^2 = \frac{3}{2}Bx^2y^2 + f(y).$$

The simplest expression is obtained with $f(y) = 0$

$$\therefore\ u = \frac{3}{2}Bx^2y^2 \longleftarrow u$$

The equation of the streamlines is

$$\left.\frac{dy}{dx}\right)_{sl} = \frac{v}{u} = \frac{-Bxy^3}{\frac{3}{2}Bx^2y^2} = \frac{-2y}{3x}$$

Separating variables & integrating

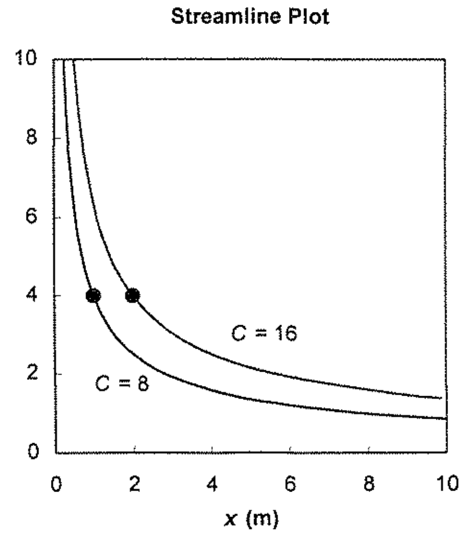
$$\frac{3}{2}\frac{dy}{y} + \frac{dx}{x} = 0$$

$$\frac{3}{2}\ln y + \ln x = \ln c$$

$$xy^{3/2} = C \quad \underline{\text{Streamline}}$$

pt (1,4) $\quad xy^{3/2} = 8$

pt (2,4) $\quad xy^{3/2} = 16$



Given: Flow in xy plane, $u = Ax^2y^2$ where $A = 0.3\ m^{-3}\cdot s^{-1}$, and coordinates are measured in meters

Find: (a) Possible y component for steady, incompressible flow.
(b) If result is valid for unsteady, incompressible flow
(c) Number of possible y components.
(d) Equation of streamlines for simplest value of v.

Plot: streamlines through points (1,4) and (2,4)

Solution:

Basic equation: $\nabla \cdot \rho\vec{V} + \frac{\partial \rho}{\partial t} = 0 = \frac{\partial}{\partial x}\rho u + \frac{\partial}{\partial y}\rho v + \frac{\partial}{\partial z}\rho w + \frac{\partial \rho}{\partial t}$ (with $\frac{\partial}{\partial z}\rho w = 0$ (1) and $\frac{\partial \rho}{\partial t} = 0$ (2))

Assumptions: (1) flow in xy plane (given), $\frac{\partial}{\partial z} = 0$
(2) ρ = constant (given)

Then,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \text{or} \quad \frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -\frac{\partial}{\partial x}(Ax^2y^2) = -2Axy^2$$

Integrating

$$v = \int \frac{\partial v}{\partial y}\,dy = -\int 2Axy^2 = -\frac{2}{3}Axy^3 + f(x)$$ ← v

The basic equation reduces to the same form for unsteady flow. Hence the result is also valid for unsteady flow. ← v (b)

There are an infinite number of possible y components, since $f(x)$ is arbitrary. The simplest is obtained with $f(x) = 0$. ← (c)

The equation of the streamline is

$$\left.\frac{dy}{dx}\right)_{s.l.} = \frac{v}{u} = -\frac{2}{3}\frac{Axy^3}{Ax^2y^2} = -\frac{2y}{3x}$$

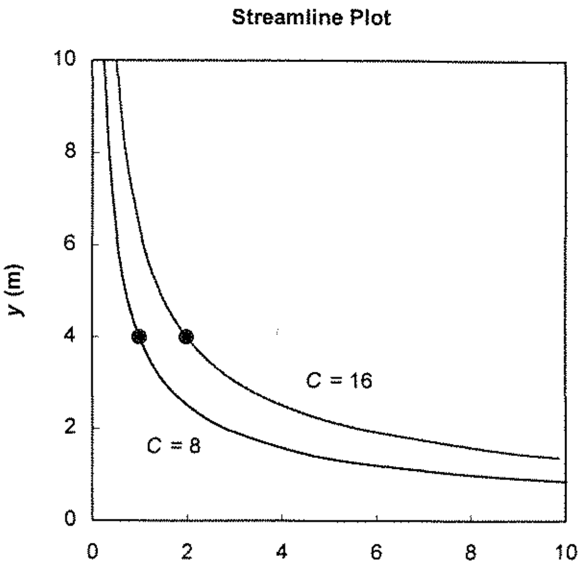
Separating variables & integrating

$$\frac{3}{2}\frac{dy}{y} + \frac{dx}{dx} = 0$$

$$\ln y^{3/2} + \ln x = \ln c$$

$$xy^{3/2} = c$$ ← Streamline

pt (1,4) $xy^{3/2} = 8$
(2,4) $xy^{3/2} = 16$



Problem 5.16

Given: Conservation of mass.

Find: Identical result to Eq. 5.1a by expanding products of density and velocity in Taylor series.

Solution: Use diagram of Fig. 5.1:

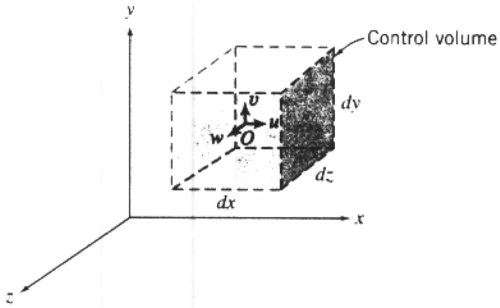


Fig. 5.1 Differential control volume in rectangular coordinates

Apply conservation of mass, using a Taylor series expansion of products. Evaluate derivatives at O.

For the x direction the mass flux is

$$\dot{m}_x = \rho u \, dA = \rho u \, dx \, dy$$

At the right face

$$\dot{m}_{x+dx/2} = \rho u \, dy \, dz + \frac{\partial}{\partial x}\rho u \frac{dx}{2} dy \, dz \quad \text{(out of CV)}$$

At the left face

$$\dot{m}_{x-dx/2} = \rho u \, dy \, dz + \frac{\partial}{\partial x}\rho u \left(-\frac{dx}{2}\right) dy \, dz \quad \text{(into CV)}$$

The net mass flux is "out" minus "in," so

$$\dot{m}_x(\text{net}) = \dot{m}_{x+dx/2} - \dot{m}_{x-dx/2} = \frac{\partial}{\partial x}\rho u \, dx \, dy \, dz$$

Summing terms for x, y, and z, and including $\frac{\partial \rho}{\partial t} dx\,dy\,dz$, we get

$$0 = \frac{\partial}{\partial x}\rho u + \frac{\partial}{\partial y}\rho v + \frac{\partial}{\partial z}\rho w + \frac{\partial \rho}{\partial t}$$

Problem 5.17

Open-Ended Problem Statement: Consider a water stream from a jet of an oscillating lawn sprinkler. Describe the corresponding pathline and streakline.

Discussion: Because the sprinkler jets oscillate, this is an unsteady flow. Therefore pathlines and streaklines need not coincide.

A *pathline* is a line tracing the path of an individual fluid particle. The path of each fluid particle is determined by the jet angle and the speed at which the particle leaves the jet.

Once a particle leaves the jet it is subject to gravity and drag forces. If aerodynamic drag were negligible, the path of each water particle would be parabolic. The horizontal speed of the particle would remain constant throughout its trajectory. The vertical speed would be slowed by gravity until reaching peak height, then it would become increasingly negative until the particle strikes the ground. The effect of aerodynamic drag is to reduce the particle speed. With drag the particle will not rise as far vertically nor travel as far horizontally. At each instant the particle trajectory will be lower and closer to the jet compared to the no-friction case. The trajectory after the particle reaches its peak height will be steeper than in the no-friction case.

A *streamline* is a line drawn in the flow that is tangent everywhere to the velocity vectors of the fluid motion. It is difficult to visualize the streamlines for an unsteady flow field because they may move laterally. However, the streamline pattern may be drawn at any instant.

A *streakline* is the locus of the present locations of fluid particles that passed a reference point at previous times. As an example, choose the exit of a jet as the reference point. Imagine marking particles that pass the jet exit at a given instant and at uniform time intervals later. The first particle will travel farthest from the jet exit and on the lowest trajectory; the last particle will be located right at the jet exit. The curve joining the present positions of the particles will resemble a spiral whose radius increases with distance from the jet opening.

Problem 5.18

Open-Ended Problem Statement: Consider a water stream from a nozzle attached to a rotating lawn sprinkler. Describe the corresponding pathline, streamline, and streakline.

Discussion: The rotating motion of the sprinkler jets makes this an unsteady flow. Therefore pathlines, streamlines, and streaklines need not coincide.

A *pathline* is a line tracing the path of an individual fluid particle. The trajectory of each particle depends on the absolute velocity with which it leaves the jet. Thus the path of each fluid particle is determined by the jet angle, the speed at which the particle leaves the jet, and the speed with which the sprinkler is rotating.

Once a particle leaves the jet it is subject to gravity and drag forces. The path of each water particle would be parabolic if aerodynamic drag were negligible. The absolute horizontal speed of the particle would remain constant throughout its trajectory. The particle would be slowed by gravity until reaching peak height, then its vertical speed would become increasingly negative until the particle strikes the ground. Aerodynamic drag reduces the particle speed. With drag the particle will not rise as far vertically nor travel as far horizontally. At each instant the particle trajectory will be lower and closer to the jet and the trajectory after the particle reaches its peak height will be steeper compared to the no-friction case.

A *streamline* is a line drawn in the flow that is tangent everywhere to the velocity vectors of the fluid motion. When unsteady effects are negligible, the streamline on which a given fluid particle lies is coincident with the pathline for the same particle. Flow unsteadiness creates different pathlines for particles that leave the sprinkler nozzle at different instants. It is difficult to visualize streamlines for an unsteady flow field because they may move laterally. The term “streamline” has little meaning for a rotating sprinkler with discrete jets.

A *streakline* is the locus of the present positions of fluid particles that passed a reference point at previous times. As an example, choose the exit of a jet as the reference point. Imagine marking particles that pass the jet exit at a given instant and at uniform time intervals later. The first particle will travel farthest from the jet exit where it was emitted; the last particle will be located right at the exit. In plan view the curve joining the positions of several particles will resemble a spiral with tighter radius close to the present position of the jet.

Problem 5.19

Given: Velocity fields listed below.

Find: Which are possible incompressible flow cases?

Solution: Apply the continuity equation in differential form.

Basic equation: $$\frac{1}{r}\frac{\partial r\rho V_r}{\partial r} + \frac{1}{r}\frac{\partial \rho V_\theta}{\partial \theta} + \frac{\partial \rho V_z}{\partial z} + \frac{\partial \rho}{\partial t} = 0$$
(where $\frac{\partial \rho V_z}{\partial z} = 0(1)$ and $\frac{\partial \rho}{\partial t} = 0(2)$)

Assumptions: (1) Two-dimensional flow, so $\frac{\partial}{\partial z} = 0$
(2) Incompressible flow

$$\rho = \text{constant}, \text{ so } \frac{\partial \rho}{\partial t} = \frac{\partial \rho}{\partial(\text{distance})} = 0$$

Then
$$\frac{1}{r}\frac{\partial r V_r}{\partial r} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} = 0$$

or
$$\frac{\partial r V_r}{\partial r} + \frac{\partial V_\theta}{\partial \theta} = 0 \quad \text{is the criterion.}$$

Field	V_r	V_θ	$\frac{\partial r V_r}{\partial r}$	$\frac{\partial V_\theta}{\partial \theta}$	$\frac{\partial r V_r}{\partial r} + \frac{\partial V_\theta}{\partial \theta}$	Possible?
(a)	$U\cos\theta$	$-U\sin\theta$	$U\cos\theta$	$-U\cos\theta$	0	Yes
(b)	$-\frac{q}{2\pi r}$	$\frac{K}{2\pi r}$	0	0	0	Yes
(c)	$U\cos\theta\left[1-\left(\frac{a}{r}\right)^2\right]$	$-U\sin\theta\left[1+\left(\frac{a}{r}\right)^2\right]$	$U\cos\theta\left[1+\left(\frac{a}{r}\right)^2\right]$*	$-U\cos\theta\left[1+\left(\frac{a}{r}\right)^2\right]$	0	Yes

* Note if $V_r = U\cos\theta\left[1-\left(\frac{a}{r}\right)^2\right]$, then $rV_r = U\cos\theta\left[r - \frac{a^2}{r}\right]$

and $\frac{\partial r V_r}{\partial r} = U\cos\theta\left[1 + \frac{a^2}{r^2}\right] = U\cos\theta\left[1 + \left(\frac{a}{r}\right)^2\right]$

Given: Incompressible flow in $r\theta$ plane with $V_\theta = -\frac{\Lambda \sin\theta}{r^2}$

Find: (a) A possible component, V_r
(b) How many possible r components are there?

<u>Solution:</u> Velocity field must satisfy the differential continuity equation

Basic equation: $\frac{1}{r}\frac{\partial r\rho V_r}{\partial r} + \frac{1}{r}\frac{\partial \rho V_\theta}{\partial \theta} + \frac{\partial \rho V_z}{\partial z} + \frac{\partial \rho}{\partial t} = 0$ (the $\frac{\partial \rho V_z}{\partial z}$ term is crossed out, marked $=0(1)$; the $\frac{\partial \rho}{\partial t}$ term is crossed out, marked $=0(2)$)

Assumptions: (1) Flow in $r\theta$ plane, so $\partial/\partial z = 0$
(2) Incompressible flow
$\rho =$ constant, so $\frac{\partial \rho}{\partial t} = \frac{\partial \rho}{\partial \text{distance}} = 0$

Then $\frac{1}{r}\frac{\partial r V_r}{\partial r} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} = 0$ or $\frac{\partial V_\theta}{\partial \theta} = -\frac{\partial r V_r}{\partial r}$

Solving for V_r,

$$V_r = -\frac{1}{r}\int \frac{\partial V_\theta}{\partial \theta}\,dr + f(\theta)$$

Since $V_\theta = -\frac{\Lambda \sin\theta}{r^2}$, $\frac{\partial V_\theta}{\partial \theta} = -\frac{\Lambda \cos\theta}{r^2}$

Thus

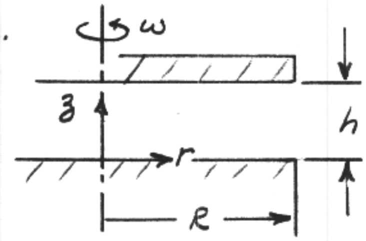
$$V_r = -\frac{1}{r}\int -\frac{\Lambda\cos\theta}{r^2}\,dr + f(\theta) = -\frac{1}{r}\frac{\Lambda\cos\theta}{r} + f(\theta)$$

$$V_r = -\frac{\Lambda\cos\theta}{r^2} + f(\theta) \quad \longleftarrow \quad V_r$$

There are an infinite number of solutions for V_r, one for each choice of $f(\theta)$

Problem 5.21

Given: Flow between parallel disks as shown.



Velocity is purely tangential.
No-slip condition is satisfied, so velocity varies linearly with z.

Find: Expression for velocity field.

Solution: A general velocity field would be

$$\vec{V} = V_r \hat{e}_r + V_\theta \hat{e}_\theta + V_z \hat{k}$$

but velocity is purely tangential, so $V_r = V_z = 0$. Then we seek

$$V_\theta = V_\theta(r, \theta, z)$$

By symmetry, $\dfrac{\partial V_\theta}{\partial \theta} = 0$, so

$$V_\theta = V_\theta(r, z)$$

Since the variation with z is linear, $V_\theta = z f(r) + c$ at most, that is

$$\frac{\partial V_\theta}{\partial z} = f(r)$$

at most.

Along the surface $z = 0$, $V_\theta = 0$, so $C = 0$.

Along the surface $z = h$, $V_\theta = \omega r$, so

$$V_\theta(z = h) = \omega r = h f(r)$$

or

$$f(r) = \frac{\omega r}{h}$$

and

$$V_\theta = \omega r \frac{z}{h}$$

Thus

$$\vec{V} = \omega r \frac{z}{h} \hat{e}_\theta$$

$\vec{V}$

Problem 5.22

A velocity field in cylindrical coordinates is given as $\vec{V} = \hat{e}_r A/r + \hat{e}_\theta B/r$, where A and B are constants with dimensions of m^2/s. Does this represent a possible incompressible flow? Sketch the streamline that passes through the point $r_0 = 1$ m, $\theta = 90°$ if $A = B = 1\ m^2/s$, if $A = 1\ m^2/s$ and $B = 0$, and if $B = 1\ m^2/s$ and $A = 0$.

Given: The velocity field

Find: Whether or not it is a incompressible flow; sketch various streamlines

Solution

$$V_r = \frac{A}{r} \qquad V_\theta = \frac{B}{r}$$

For incompressible flow

$$\frac{1}{r}\cdot\frac{d}{dr}\left(r\cdot V_r\right) + \frac{1}{r}\cdot\frac{d}{d\theta}V_\theta = 0$$

$$\frac{1}{r}\cdot\frac{d}{dr}\left(r\cdot V_r\right) = 0 \qquad \frac{1}{r}\cdot\frac{d}{d\theta}V_\theta = 0$$

Hence

$$\frac{1}{r}\cdot\frac{d}{dr}\left(r\cdot V_r\right) + \frac{1}{r}\cdot\frac{d}{d\theta}V_\theta = 0$$

Flow is incompressible

For the streamlines

$$\frac{dr}{V_r} = \frac{r\cdot d\theta}{V_\theta} \qquad \frac{r\cdot dr}{A} = \frac{r^2\cdot d\theta}{B}$$

so

$$\int \frac{1}{r}\,dr = \int \frac{A}{B}\,d\theta$$

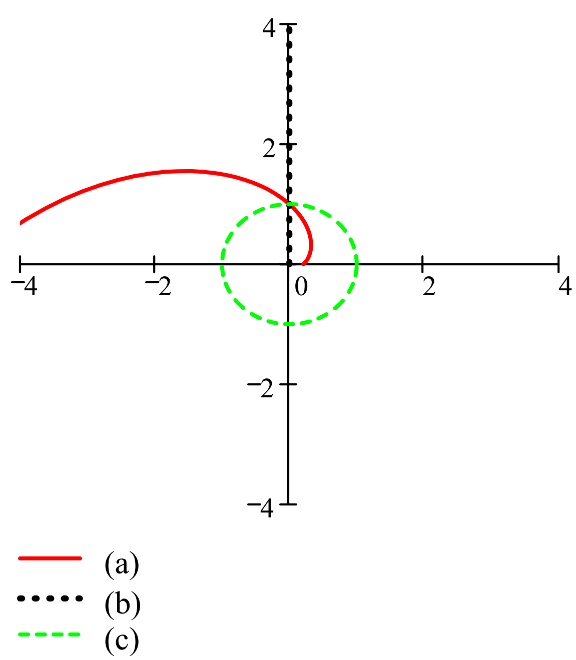
Integrating $\ln(r) = \frac{A}{B}\cdot\theta + \text{const}$

Equation of streamlines is $r = C\cdot e^{\frac{A}{B}\cdot\theta}$

(a) For $A = B = 1$ m²/s, passing through point (1m, π/2) $r = e^{\theta - \frac{\pi}{2}}$

(b) For $A = 1$ m²/s, $B = 0$ m²/s, passing through point (1m, π/2) $\theta = \frac{\pi}{2}$

(c) For $A = 0$ m²/s, $B = 1$ m²/s, passing through point (1m, π/2) $r = 1\cdot m$



Problem 5.23

Given: Definition of ∇ in cylindrical coordinates.

Obtain: $\nabla \cdot \rho\vec{V}$ in cylindrical coordinates (use hint on page 202).

Show result is identical to Eq. 5.2.

Solution: The definition of ∇ in cylindrical coordinates is

$$\nabla = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{k}\frac{\partial}{\partial z} \qquad (3.21)$$

Note $\rho\vec{V} = \rho(\hat{e}_r V_r + \hat{e}_\theta V_\theta + \hat{k} V_z)$

Hint: $\frac{\partial \hat{e}_r}{\partial \theta} = \hat{e}_\theta$, and $\frac{\partial \hat{e}_\theta}{\partial \theta} = -\hat{e}_r$ (p. 202)

Substituting $\nabla \cdot \rho\vec{V} = \left(\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{\partial}{\partial \theta} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \rho(\hat{e}_r V_r + \hat{e}_\theta V_\theta + \hat{k} V_z)$

$$\nabla \cdot \rho\vec{V} = \hat{e}_r \cdot \frac{\partial}{\partial r}\rho(\hat{e}_r V_r + \hat{e}_\theta V_\theta + \hat{k} V_z)$$
$$+ \hat{e}_\theta \cdot \frac{\partial}{\partial \theta}\rho(\hat{e}_r V_r + \hat{e}_\theta V_\theta + \hat{k} V_z)$$
$$+ \hat{k} \cdot \frac{\partial}{\partial z}\rho(\hat{e}_r V_r + \hat{e}_\theta V_\theta + \hat{k} V_z)$$

$$= \hat{e}_r \cdot \hat{e}_r \frac{\partial}{\partial r}\rho V_r + \hat{e}_\theta \cdot \underbrace{\frac{1}{r}\frac{\partial \hat{e}_r}{\partial \theta}}_{\hat{e}_\theta}\rho V_r + \underbrace{\hat{e}_\theta \cdot \hat{e}_r}_{0}\frac{\partial}{\partial \theta}\rho V_r$$
$$+ \hat{e}_\theta \cdot \underbrace{\frac{\partial \hat{e}_\theta}{\partial \theta}}_{-\hat{e}_r}\rho V_\theta + \hat{e}_\theta \cdot \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta}\rho V_\theta + \hat{k} \cdot \hat{k}\frac{\partial}{\partial z}\rho V_z$$

$$\nabla \cdot \rho\vec{V} = \frac{\partial}{\partial r}\rho V_r + \rho\frac{V_r}{r} + \frac{1}{r}\frac{\partial}{\partial \theta}\rho V_\theta + \frac{\partial}{\partial z}\rho V_z$$

Combining the first two terms, $\frac{\partial}{\partial r}\rho V_r + \rho\frac{V_r}{r} = \frac{1}{r}\frac{\partial}{\partial r} r\rho V_r$, as may be verified by differentiation. Substituting

$$\nabla \cdot \rho\vec{V} = \frac{1}{r}\frac{\partial}{\partial r}(r\rho V_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho V_\theta) + \frac{\partial}{\partial z}(\rho V_z) \longleftarrow$$

This result is identical to the corresponding terms in Eq. 5.2.

Problem *5.24

Given: Velocity field for viscometric flow of Example Problem 5.7

$$\vec{V} = U\frac{y}{h}\hat{i}$$

Find: (a) Stream function

(b) Locate streamline that divides flow rate equally.

Solution: Flow is incompressible, so stream function can be derived.

$$\frac{\partial\psi}{\partial y} = u = U\frac{y}{h}, \text{ so } \psi = \int \frac{\partial\psi}{\partial y}dy + f(x) = \int \frac{Uy}{h}dy + f(x) = \frac{Uy^2}{2h} + f(x)$$

Let $\psi = 0$ at $y = 0$, so $f(x) = 0$

$$\psi = \frac{Uy^2}{2h}$$

ψ

Stream function is maximum at $y = h$.

$$\psi_{max} = \frac{Uh^2}{2h} = \frac{Uh}{2}\ ;\ Q/w = \psi_{max} - \psi_{min} = \frac{Uh}{2} - 0 = \frac{Uh}{2}$$

$$\psi_{Q/2} = \frac{1}{2}\psi_{max} = \frac{Uh}{4} = \frac{Uy^2}{2h}$$

Thus

$$y^2 = \frac{2h}{U}\frac{Uh}{4} = \frac{h^2}{2} \quad \text{so} \quad y = \frac{h}{\sqrt{2}}$$

$\psi_{Q/2}$

Given: Velocity field $\vec{V} = (x+2y)\hat{i} + (x^2-y)\hat{j}$

Find: Corresponding family of stream functions.

Solution: ψ may be defined only if flow is incompressible

Basic equations: $\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \cancel{\frac{\partial \rho w}{\partial z}}^{=0(1)} + \cancel{\frac{\partial \rho}{\partial t}}^{=0(2)} = 0$

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$$

Assumptions: (1) $\vec{V} = \vec{V}(x,y)$, so $\partial/\partial z = 0$

(2) ρ = constant, so $\frac{\partial \rho}{\partial t} = \frac{\partial \rho}{\partial \text{distance}} = 0$

Then,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 1 - 1 = 0, \text{ so flow is incompressible}$$

Thus

$$u = x + 2y = \frac{\partial \psi}{\partial y}; \quad \psi = \int u\,dy + f(x) = xy + y^2 + f(x)$$

$$v = x^2 - y = -\frac{\partial \psi}{\partial x}; \quad \psi = \int -v\,dx + g(y) = -\frac{x^3}{3} + xy + g(y)$$

Comparing these two expressions for ψ, we see that

$$f(x) = -\frac{x^3}{3} \quad \text{and} \quad g(y) = y^2$$

so

$$\psi = -\frac{x^3}{3} + xy + y^2 \qquad \psi$$

Problem *5.26

Does the velocity field of Problem 5.22 represent a possible incompressible flow case? If so, evaluate and sketch the stream function for the flow. If not, evaluate the rate of change of density in the flow field.

Given: The velocity field

Find: Whether or not it is a incompressible flow; sketch stream function

Solution

$$V_r = \frac{A}{r} \qquad V_\theta = \frac{B}{r}$$

For incompressible flow $\quad \frac{1}{r}\cdot\frac{d}{dr}\left(r\cdot V_r\right) + \frac{1}{r}\cdot\frac{d}{d\theta}V_\theta = 0$

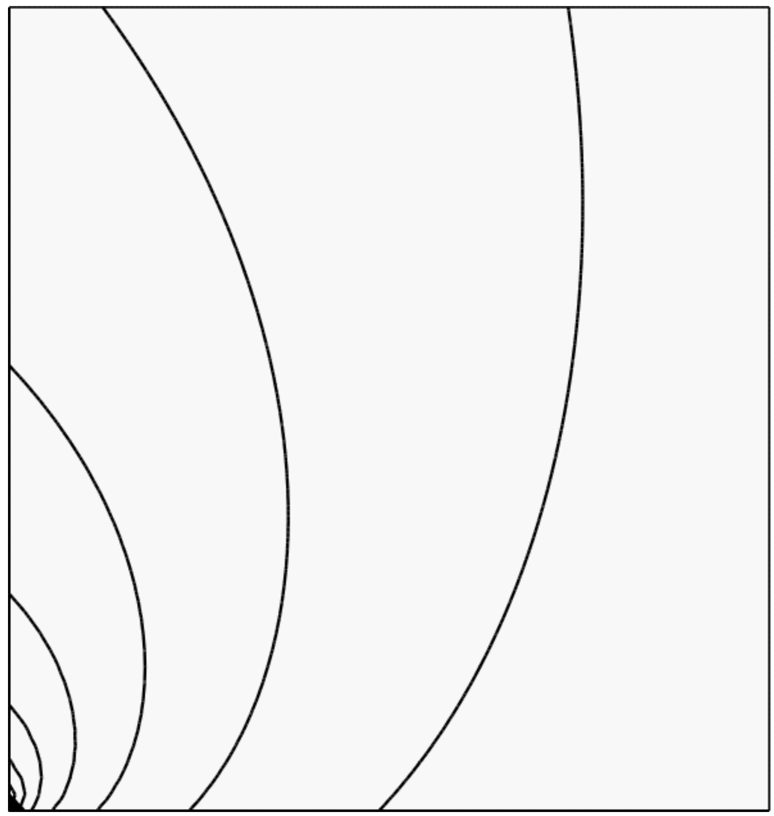
$$\frac{1}{r}\cdot\frac{d}{dr}\left(r\cdot V_r\right) = 0 \qquad \frac{1}{r}\cdot\frac{d}{d\theta}V_\theta = 0$$

Hence $\quad \frac{1}{r}\cdot\frac{d}{dr}\left(r\cdot V_r\right) + \frac{1}{r}\cdot\frac{d}{d\theta}V_\theta = 0 \qquad$ Flow is incompressible

For the stream function $\quad \frac{\partial}{\partial\theta}\psi = r\cdot V_r = A \qquad \psi = A\cdot\theta + f(r)$

Integrating $\quad \frac{\partial}{\partial r}\psi = -V_\theta = -\frac{B}{r} \qquad \psi = -B\cdot\ln(r) + g(\theta)$

Comparing, stream function is $\psi = A \cdot \theta - B \cdot \ln(r)$



ψ

Problem *5.27

Given: Stream function for an incompressible flow field,

$$\psi = -Ur\sin\theta + \frac{q}{2\pi}\theta$$

Find: (a) An expression for the velocity field.
(b) Points where $|\vec{V}| = 0$.
(c) Show $\psi = 0$ where $|\vec{V}| = 0$.

Solution: The velocity components are given by

$$V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = -U\cos\theta + \frac{q}{2\pi r}$$

$$V_\theta = -\frac{\partial \psi}{\partial r} = U\sin\theta$$

so $\vec{V} = V_r \hat{i}_r + V_\theta \hat{i}_\theta = \left(-U\cos\theta + \frac{q}{2\pi r}\right)\hat{i}_r + U\sin\theta\,\hat{i}_\theta$ ← $\vec{V}$

Now $|\vec{V}| = (V_r^2 + V_\theta^2)^{1/2} = 0$ only when both V_r and V_θ are zero.

From the component equations, $V_\theta = 0$ for $\theta = 0, \pi$. When $V_r = 0$,

$$r = \frac{q}{2\pi U\cos\theta}$$

For $r > 0$, then $V_r = 0$ for $\theta = 0$, and $r = \frac{q}{2\pi U}$.

Stagnation point ($|\vec{V}| = 0$) occurs at $(r, \theta) = \left(\frac{q}{2\pi U}, 0\right)$ ← $|\vec{V}| =$

Substituting, $\psi_{stagnation} = -Ur\sin\theta + \frac{q}{2\pi}\theta\Big]_{r = \frac{q}{2\pi U},\ \theta = 0}$

or $\psi_{stagnation} = 0$ ← ψ_{st}

Problem *5.28

Given: Flow with velocity components

$$u = 0, \quad v = -y^3 - 4z, \quad w = 3y^2 z$$

Find: (a) Is this one-, two- or three-dimensional?
(b) Incompressible?
(c) Stream function, if possible

Solution: $\vec{V} = u\hat{i} + v\hat{j} + w\hat{k} = \vec{V}(y,z)$

Velocity field is a function of two space coordinates. Therefore flow is two-dimensional. ← 2-D

If incompressible, it must satisfy differential continuity equation.

Basic equation: $\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} + \frac{\partial \rho}{\partial t} = 0$ (first term $= 0(1)$, last term $= 0(2)$)

Assumptions: (1) Two-dimensional flow, so $\frac{\partial}{\partial x} = 0$
(2) Incompressible flow

$$\rho = \text{constant}, \text{ so } \frac{\partial \rho}{\partial t} = \frac{\partial \rho}{\partial \ell} = 0$$

Then

$$\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = -3y^2 + 3y^2 = 0 \quad \therefore \text{Flow is incompressible}$$ ← $\rho = C$

For incompressible flow in yz plane, $\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$ will be satisfied identically if

$$v = \frac{\partial \psi}{\partial z} \quad \text{and} \quad w = -\frac{\partial \psi}{\partial y}$$

(Then continuity becomes $\frac{\partial^2 \psi}{\partial z \partial y} - \frac{\partial^2 \psi}{\partial y \partial z} = 0$.)

Thus $\psi = \int v \, dz + f(y) = -y^3 z - 2z^2 + f(y)$

and $\psi = \int -w \, dy + g(z) = -y^3 z + g(z)$

Comparing these two expressions, we see $f(y) = 0$ and $g(z) = -2z^2$.

$$\psi = -y^3 z - 2z^2$$ ← ψ

Given: An incompressible, frictionless flow specified by

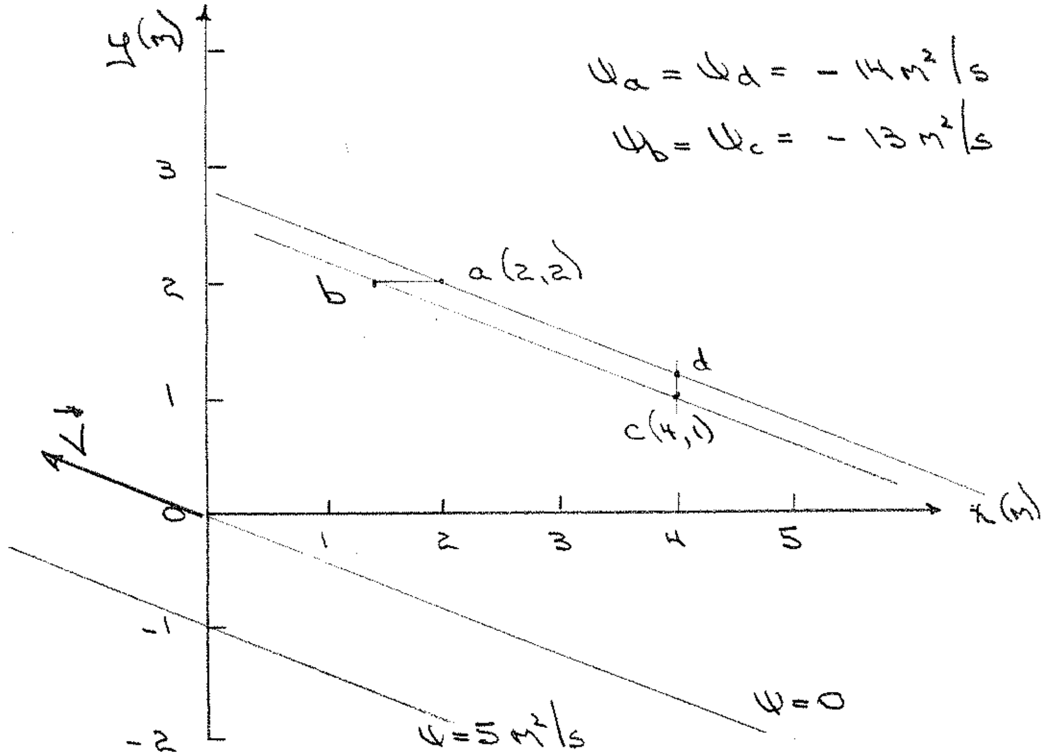
$\psi = -2Ax - 5Ay$; x, y in meters, A = 1 m/s

Find: (a) Sketch streamlines $\psi = 0$ and $\psi = 5\ m^2/s$
(b) Velocity vector at (0,0)
(c) Flow rate between streamlines passing through points (2,2) and (4,1)

Solution: Streamlines are lines ψ = constant

For $\psi = 0$, $0 = -2Ax - 5Ay$ or $y = -\frac{2}{5}x$

For $\psi = 5$, $5 = -2Ax - 5Ay$ or $y = -\frac{2}{5}x - \frac{1}{5} \times 5\frac{m^2}{s} \times \frac{s}{1m} = -\frac{2}{5}x - 1m$



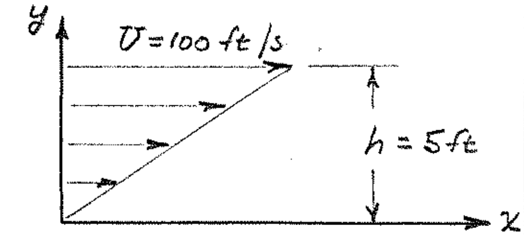
$u = \frac{\partial \psi}{\partial y} = -5A$; $v = -\frac{\partial \psi}{\partial x} = 2A$, so $\vec{V} = -5\hat{i} + 2\hat{j}$ m/s ← $\vec{V}$

$Q = \int_{x=b}^{x=a} v\,dx = \int_{x=b}^{x=a} -\frac{\partial \psi}{\partial x}\,dx = \int_{\psi_b}^{\psi_a} -d\psi = \psi_b - \psi_a = 1\ m^2/s$, i.e ↑

$Q = \int_{y=c}^{y=d} u\,dy = \int_{y=c}^{y=d} \frac{\partial \psi}{\partial y}\,dy = \int_{\psi_c}^{\psi_d} d\psi = \psi_d - \psi_c = -1\ m^2/s$, i.e ←

Thus $Q = 1\ m^3/s$ per meter of depth. ← Q

Given: Parallel one-dimensional flow in x direction with linear variation in velocity.



Find: (a) An expression for ψ.
(b) y coordinate below which half of flow passes.

Solution: Represent the velocity profile by $u = U\left(\frac{y}{h}\right)$,

where $U = 100 \frac{ft}{s}$, $h = 5$ ft.

Note that $u = \frac{\partial \psi}{\partial y}$, so

$$\psi = \int u\,dy + f(x) = \frac{Uy^2}{2h} + f(x)$$

Also $v = -\frac{\partial \psi}{\partial x}$, but $v = 0$, so

$$\psi = \int -v\,dx + g(y) = g(y)$$

Comparing these expressions, we find $f(x) = 0$ and $g(y) = \frac{Uy^2}{2h}$, so

$$\psi = \frac{Uy^2}{2h}$$ ψ

For the whole profile, $0 < y < h$, the flowrate is

$$Q = \int_0^h u\,dy = h\int_0^1 u\,d\left(\frac{y}{h}\right) = hU\int_0^1 \left(\frac{y}{h}\right) d\left(\frac{y}{h}\right) = \frac{hU}{2}$$

For half the flowrate, up to y^*

$$\frac{Q}{2} = \int_0^{y^*} u\,dy = \frac{U}{h}\int_0^{y^*} y\,dy = \frac{Uy^{*2}}{2h} = \frac{hU}{4} \quad \text{or} \quad y^{*2} = \frac{h^2}{2}$$

so

$$y^* = \frac{h}{\sqrt{2}} = 3.54 \text{ ft}$$ y^*

Problem *5.31

Given: Linear approximation to boundary layer velocity profile

$$u = U \frac{y}{\delta}$$

Find: (a) stream function for the flow field
(b) location of streamlines at one-quarter and one-half the total flow rate in the boundary layer.

Solution: For 2-D incompressible flow, ψ satisfies

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u = \frac{\partial \psi}{\partial y} = U \frac{y}{\delta} \qquad \therefore \psi = \int \frac{\partial \psi}{\partial y} dy + f(x) = \int U \frac{y}{\delta} dy + f(x)$$

Thus $\quad \psi = \frac{U y^2}{2\delta} + f(x)$

Let $\psi = 0$ along $y = 0$, so $f(x) = 0$ and $\psi = \frac{U}{2\delta} y^2$ ← ψ

The total flow rate within the boundary layer is

$$\frac{Q}{w} = \psi(\delta) - \psi(0) = \frac{1}{2} U \delta$$

At $\frac{1}{4}$ of total, $\psi - \psi_0 = \frac{U}{2\delta} y^2 = \frac{1}{4}\left(\frac{1}{2} U \delta\right)$

$$\therefore \left(\frac{y}{\delta}\right)^2 = \frac{1}{4} \quad \text{and} \quad \frac{y}{\delta} = \frac{1}{2} \quad \leftarrow \frac{1}{4}\frac{Q}{w}$$

At $\frac{1}{2}$ of total, $\psi - \psi_0 = \frac{U}{2\delta} y^2 = \frac{1}{2}\left(\frac{1}{2} U \delta\right)$

$$\therefore \left(\frac{y}{\delta}\right)^2 = \frac{1}{2} \quad \text{and} \quad \frac{y}{\delta} = \sqrt{\frac{1}{2}} = 0.707 \quad \leftarrow \frac{1}{2}\frac{Q}{w}$$

Problem *5.32

Given: Sinusoidal approximation to boundary layer velocity profile

$$u = U \sin\left(\frac{\pi}{2}\frac{y}{\delta}\right)$$

Find: Locate streamlines at quarter and half total flow rate.

Solution: Flow is incompressible so ψ may be derived.

$$u = \frac{\partial \psi}{\partial y} = U \sin\left(\frac{\pi}{2}\frac{y}{\delta}\right) \; ; \; \psi = \int \frac{\partial \psi}{\partial y} dy + f(x) = \int U \sin\left(\frac{\pi}{2}\frac{y}{\delta}\right) dy + f(x)$$

Thus $\psi = -\frac{2\delta U}{\pi}\cos\left(\frac{\pi}{2}\frac{y}{\delta}\right) + f(x)$

Let $\psi = 0$ along $y = 0$, so $f(x) = 0$ $\qquad \psi = -\frac{2\delta U}{\pi}\cos\left(\frac{\pi}{2}\frac{y}{\delta}\right)$

The total flow rate is $\frac{Q}{w} = \psi(\delta) - \psi(0) = -\frac{2\delta U}{\pi}\cos\left(\frac{\pi}{2}\right) + \frac{2\delta U}{\pi}\cos(0) = \frac{2\delta U}{\pi}$

At 1/4 of total, $\psi - \psi_0 = \frac{2\delta U}{\pi}\left[1 - \cos\left(\frac{\pi}{2}\frac{y}{\delta}\right)\right] = \frac{1}{4}\frac{2\delta U}{\pi} = \frac{\delta U}{2\pi}$

$$1 - \cos\left(\frac{\pi}{2}\frac{y}{\delta}\right) = \frac{\pi}{2\delta U}\frac{\delta U}{2\pi} = \frac{1}{4} \; ; \; \cos\left(\frac{\pi}{2}\frac{y}{\delta}\right) = \frac{3}{4} \; ; \; \frac{y}{\delta} = 0.460$$

At 1/2 of total, $\psi - \psi_0 = \frac{2\delta U}{\pi}\left[1 - \cos\left(\frac{\pi}{2}\frac{y}{\delta}\right)\right] = \frac{1}{2}\frac{2\delta U}{\pi} = \frac{\delta U}{\pi}$

$$1 - \cos\left(\frac{\pi}{2}\frac{y}{\delta}\right) = \frac{\pi}{2\delta U}\frac{\delta U}{\pi} = \frac{1}{2} \; ; \; \cos\left(\frac{\pi}{2}\frac{y}{\delta}\right) = \frac{1}{2} \; ; \; \frac{y}{\delta} = 0.667$$

Problem *5.33

Given: Parabolic approximation to boundary layer velocity profile

$$u = U\left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right]$$

Find: (a) stream function for the flow field
(b) location of streamlines at one-quarter and one-half the total flow rate in the boundary layer.

Solution: For 2-D incompressible flow, ψ satisfies

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u = \frac{\partial \psi}{\partial y} = U\left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right]$$

$$\therefore \quad \psi = \int \frac{\partial \psi}{\partial y}\,dy + f(x) = U\int\left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right]dy + f(x).$$

$$\psi = U\left[\frac{y^2}{\delta} - \frac{y^3}{3\delta^2}\right] + f(x)$$

Let $\psi = 0$ along $y = 0$, so $f(x) = 0$ and. $\psi = U\delta\left[\left(\frac{y}{\delta}\right)^2 - \frac{1}{3}\left(\frac{y}{\delta}\right)^3\right]$

The total flow rate within the boundary layer is

$$\frac{Q}{w} = \psi(\delta) - \psi(0) = U\delta\left[1 - \frac{1}{3}\right] = \frac{2}{3}U\delta$$

At $\frac{1}{4}$ of total, $\psi - \psi_0 = U\delta\left[\left(\frac{y}{\delta}\right)^2 - \frac{1}{3}\left(\frac{y}{\delta}\right)^3\right] = \frac{1}{4}\left(\frac{2}{3}U\delta\right)$

$$\therefore \left(\frac{y}{\delta}\right)^2 - \frac{1}{3}\left(\frac{y}{\delta}\right)^3 = \frac{1}{6} = 0.167$$

Trial and error solution gives $\frac{y}{\delta} = 0.442$ ← $\frac{1}{4}\frac{Q}{w}$

At $\frac{1}{2}$ of total, $\psi - \psi_0 = U\delta\left[\left(\frac{y}{\delta}\right)^2 - \frac{1}{3}\left(\frac{y}{\delta}\right)^3\right] = \frac{1}{2}\left(\frac{2}{3}U\delta\right)$

$$\therefore \left(\frac{y}{\delta}\right)^2 - \frac{1}{3}\left(\frac{y}{\delta}\right)^3 = \frac{1}{3} = 0.333$$

Trial and error solution gives $\frac{y}{\delta} = 0.652$ ← $\frac{1}{2}\frac{Q}{w}$

Problem *5.34

A cubic velocity profile was used to model flow in a laminar incompressible boundary layer in Problem 5.13. Derive the stream function for this flow field. Locate streamlines at one-quarter

Given: Data on boundary layer

Find: Stream function; locate streamlines at 1/4 and 1/2 of total flow rate

Solution

$$u(x,y) = U\cdot\left[\frac{3}{2}\cdot\left(\frac{y}{\delta}\right) - \frac{1}{2}\cdot\left(\frac{y}{\delta}\right)^3\right]$$

and $$\delta(x) = c\cdot\sqrt{x}$$

For the stream function $$u = \frac{\partial}{\partial y}\psi = U\cdot\left[\frac{3}{2}\cdot\left(\frac{y}{\delta}\right) - \frac{1}{2}\cdot\left(\frac{y}{\delta}\right)^3\right]$$

Hence $$\psi = \int U\cdot\left[\frac{3}{2}\cdot\left(\frac{y}{\delta}\right) - \frac{1}{2}\cdot\left(\frac{y}{\delta}\right)^3\right] dy$$

$$\psi = U\cdot\left(\frac{3}{4}\cdot\frac{y^2}{\delta} - \frac{1}{8}\cdot\frac{y^4}{\delta^3}\right) + f(x)$$

Let $\psi = 0$ along $y = 0$, so $f(x) = 0$

so $$\psi = U\cdot\delta\cdot\left[\frac{3}{4}\cdot\left(\frac{y}{\delta}\right)^2 - \frac{1}{8}\cdot\left(\frac{y}{\delta}\right)^4\right]$$

The total flow rate in the boundary layer is

$$\frac{Q}{W} = \psi(\delta) - \psi(0) = U\cdot\delta\cdot\left(\frac{3}{4} - \frac{1}{8}\right) = \frac{5}{8}\cdot U\cdot\delta$$

At 1/4 of the total $$\psi - \psi_0 = U\cdot\delta\cdot\left[\frac{3}{4}\cdot\left(\frac{y}{\delta}\right)^2 - \frac{1}{8}\cdot\left(\frac{y}{\delta}\right)^4\right] = \frac{1}{4}\cdot\left(\frac{5}{8}\cdot U\cdot\delta\right)$$

$$24\cdot\left(\frac{y}{\delta}\right)^2 - 4\cdot\left(\frac{y}{\delta}\right)^4 = 5$$

Trial and error (or use of *Excel*'s *Goal Seek*) leads to $$\frac{y}{\delta} = 0.465$$

At 1/2 of the total flow $$\psi - \psi_0 = U\cdot\delta\cdot\left[\frac{3}{4}\cdot\left(\frac{y}{\delta}\right)^2 - \frac{1}{8}\cdot\left(\frac{y}{\delta}\right)^4\right] = \frac{1}{2}\cdot\left(\frac{5}{8}\cdot U\cdot\delta\right)$$

$$12\cdot\left(\frac{y}{\delta}\right)^2 - 2\cdot\left(\frac{y}{\delta}\right)^4 = 5$$

Trial and error (or use of *Excel*'s *Goal Seek*) leads to $$\frac{y}{\delta} = 0.671$$

Problem *5.35

Given: Velocity field for a free vortex from Example Problem 5.6:

$$\vec{V} = \frac{c}{r}\hat{e}_\theta \qquad c = 0.5\ m^2/sec$$

Find: (a) Obtain the stream function for this flow.
(b) Evaluate the volume flow rate per unit depth between $r_1 = 0.10$ m and $r_2 = 0.12$ m.
(c) Sketch the velocity profile along a line of constant θ.
(d) Check the volume flow rate calculated from the stream function by integrating the velocity profile along this line.

Solution: From the definition of ψ, $\frac{\partial \psi}{\partial r} = -V_\theta = -\frac{c}{r}$

Thus $\psi = \int \frac{\partial \psi}{\partial r} dr + f(\theta) = \int -\frac{c}{r} dr + f(\theta) = -c \ln r + f(\theta)$

But $V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{1}{r} f'(\theta) = 0$. Therefore $f(\theta) = \text{constant} = c_1$, and

$$\psi = -c \ln r + c_1$$ ψ

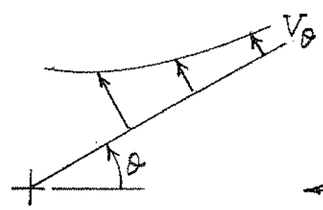
The volume flow rate per unit depth is

$$\frac{Q}{b} = \psi(r_2) - \psi(r_1) = -c\ln r_2 + c_1 - [-c \ln r_1 + c_1] = c(\ln r_1 - \ln r_2) = c \ln\left(\frac{r_1}{r_2}\right)$$

$$\frac{Q}{b} = 0.5 \frac{m^2}{s} \times \ln\left(\frac{0.10\ m}{0.12\ m}\right) = -0.0912\ m^3/s\ /m$$ Q/b

Because $Q/b < 0$, flow is in the direction of $\hat{e}_\theta$.

Along θ = constant, V_θ varies inversely with r:



Plot

From the expression for $\vec{V}$, $V_\theta = \frac{c}{r}$. Thus

$$\frac{Q}{b} = \int_{r_1}^{r_2} V_\theta\, dr = \int_{r_1}^{r_2} \frac{c}{r} dr = c \ln\left(\frac{r_2}{r_1}\right)$$ Q/b

From the sketch, this flow is in the direction of $\hat{e}_\theta$.

Comparing shows that the expressions for Q/b are the same except for sign.

Problem *5.36

Given: Rigid-body motion in Example Problem 5.6

$$\vec{V} = r\omega\,\hat{e}_\theta \qquad \omega = 0.5 \text{ rad/s}$$

Find: (a) Obtain the stream function for this flow.
(b) Evaluate the volume flow rate per unit depth between $r_1 = 0.10$ m and $r_2 = 0.12$ m.
(c) Sketch the velocity profile along a line of constant θ.
(d) Check the volume flow rate calculated from the stream function by integrating the velocity profile along this line.

Solution: From the definition of ψ, $\dfrac{\partial\psi}{\partial r} = -V_\theta = -r\omega$

Thus $\psi = \int \dfrac{\partial\psi}{\partial r}dr + f(\theta) = \int -r\omega\,dr + f(\theta) = -\frac{1}{2}r^2\omega + f(\theta)$

But $V_r = \dfrac{1}{r}\dfrac{\partial\psi}{\partial\theta} = \dfrac{1}{r}f'(\theta) = 0 \quad \therefore f(\theta) = c$

and $\psi = -\frac{1}{2}r^2\omega + c$ ← ψ

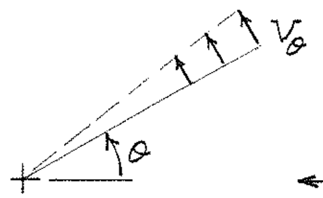
The volume flow rate per unit depth is

$$\frac{Q}{b} = \psi(r_2) - \psi(r_1) = -\frac{1}{2}r_2^2\omega + c - \left[-\frac{1}{2}r_1^2\omega + c\right] = \frac{\omega}{2}(r_1^2 - r_2^2)$$

$$\frac{Q}{b} = \frac{1}{2} \times 0.5\,\frac{\text{rad}}{\text{s}}\left[(0.10)^2 - (0.12)^2\right]\text{m}^2 = -0.0011 \quad \text{m}^3/\text{s}\,/\,\text{m}$$ ← Q/b

Because $Q/b < 0$, flow is in the direction of $\hat{e}_\theta$.

Along θ = constant, V_θ varies linearly:



← Plot

From the linear velocity variation, $V_\theta = \omega r$

Thus $\dfrac{Q}{b} = \int_{r_1}^{r_2} V_\theta\,dr = \int_{r_1}^{r_2} r\omega\,dr = \frac{1}{2}r^2\omega\Big]_{r_1}^{r_2} = \frac{\omega}{2}(r_2^2 - r_1^2)$ ← Q/b

From the sketch, this flow is in the direction of $\hat{e}_\theta$.

Comparing the expressions for Q/b shows they are the same except for sign.

Problem 5.37

Consider the velocity field $\vec{V} = A(x^2 + 2xy)\hat{i} - A(2xy + y^2)\hat{j}$ in the xy plane, where $A = 0.25\ \text{m}^{-1} \cdot \text{s}^{-1}$, and the coordinates are measured in meters. Is this a possible incompressible flow field? Calculate the acceleration of a fluid particle at point $(x,y) = (2, 1)$.

Given: Velocity field

Solution

The given data is $A = 0.25 \cdot \text{m}^{-1} \cdot \text{s}^{-1}$ $\quad x = 2 \cdot \text{m}$ $\quad y = 1 \cdot \text{m}$

$$u(x,y) = A \cdot \left(x^2 + 2 \cdot x \cdot y\right)$$

$$v(x,y) = -A \cdot \left(2 \cdot x \cdot y + y^2\right)$$

For incompressible flow $\quad \dfrac{du}{dx} + \dfrac{dv}{dy} = 0$

Hence $\quad \dfrac{du}{dx} + \dfrac{dv}{dy} = 2 \cdot A \cdot (x + y) - 2 \cdot A \cdot (x + y) = 0$

Incompressible flow

The acceleration is given by

$$\vec{a}_p = \underbrace{\frac{D\vec{V}}{Dt}}_{\substack{\text{total}\\ \text{acceleration}\\ \text{of a particle}}} = \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\substack{\text{convective}\\ \text{acceleration}}} + \underbrace{\frac{\partial \vec{V}}{\partial t}}_{\substack{\text{local}\\ \text{acceleration}}}$$

For the present steady, 2D flow

$$a_x = u \cdot \frac{du}{dx} + v \cdot \frac{du}{dy} = A \cdot \left(x^2 + 2 \cdot x \cdot y\right) \cdot 2 \cdot A \cdot (x + y) - A \cdot \left(2 \cdot x \cdot y + y^2\right) 2 \cdot A \cdot x$$

$$a_x = 2 \cdot A^2 \cdot x \cdot \left(x^2 + x \cdot y + y^2\right)$$

$$a_y = u \cdot \frac{dv}{dx} + v \cdot \frac{dv}{dy} = A \cdot \left(x^2 + 2 \cdot x \cdot y\right) \cdot (-2 \cdot A \cdot y) - A \cdot \left(2 \cdot x \cdot y + y^2\right) [-2 \cdot A \cdot (x + y)]$$

$$a_y = 2 \cdot A^2 \cdot y \cdot \left(x^2 + x \cdot y + y^2\right)$$

At point (2,1) the acceleration is

$$a_x = 2 \cdot A^2 \cdot x \cdot \left(x^2 + x \cdot y + y^2\right) \qquad a_x = 1.75 \frac{m}{s^2}$$

$$a_y = 2 \cdot A^2 \cdot y \cdot \left(x^2 + x \cdot y + y^2\right) \qquad a_y = 0.875 \frac{m}{s^2}$$

Problem 5.38

Given: Flow field $\vec{V} = xy^2\hat{i} - \frac{1}{3}y^3\hat{j} + xy\hat{k}$

Find: (a) Dimensions.
(b) If possible incompressible flow.
(c) Acceleration of particle at point $(x,y,z) = (1,2,3)$.

Solution: Apply continuity, use substantial derivative.

Basic equations: $\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} + \frac{\partial \rho}{\partial t} = 0$ (with $\frac{\partial \rho w}{\partial z} = 0$ (1), $\frac{\partial \rho}{\partial t} = 0$ (2))

$$\vec{a}_p = \frac{D\vec{V}}{Dt} = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} + \frac{\partial \vec{V}}{\partial t}$$

(with $w\frac{\partial \vec{V}}{\partial z} = 0$ (1), $\frac{\partial \vec{V}}{\partial t} = 0$ (2))

Assumptions: (1) Two-dimensional flow, $\vec{V} = \vec{V}(x,y)$, so $\partial/\partial z = 0$
(2) Incompressible flow
(3) Steady flow, $\vec{V} \neq \vec{V}(t)$

Then $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = y^2 - y^2 = 0$ Flow is a possible incompressible case. $\rho =$

$$\vec{a}_p = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} \;;\quad \frac{\partial \vec{V}}{\partial x} = y^2\hat{i} + y\hat{k} \;;\quad \frac{\partial \vec{V}}{\partial y} = 2xy\hat{i} - y^2\hat{j} + x\hat{k}$$

$$= (xy^2)(y^2\hat{i} + y\hat{k}) + (-\tfrac{1}{3}y^3)(2xy\hat{i} - y^2\hat{j} + x\hat{k})$$

$$= \hat{i}(xy^4 - \tfrac{2}{3}xy^4) + \hat{j}(\tfrac{1}{3}y^5) + \hat{k}(xy^3 - \tfrac{1}{3}xy^3)$$

$$\vec{a}_p = \hat{i}(\tfrac{1}{3}xy^4) + \hat{j}(\tfrac{1}{3}y^5) + \hat{k}(\tfrac{2}{3}xy^3)$$

At $(x,y,z) = (1,2,3)$

$$\vec{a}_p = \hat{i}\left[\tfrac{1}{3}(1)(16)\right] + \hat{j}\left[\tfrac{1}{3}(32)\right] + \hat{k}\left[\tfrac{2}{3}(1)(8)\right] = \tfrac{16}{3}\hat{i} + \tfrac{32}{3}\hat{j} + \tfrac{16}{3}\hat{k}$$

$\vec{a}_p$

($\vec{a}_p$ will be in m/s^2)

Problem 5.39

Given: Flow field $\vec{V} = ax^2y\hat{i} - by\hat{j} + cz^2\hat{k}$; $a = 1/m^2 \cdot s$, $b = 3/s$, $c = 2/m \cdot s$

Find: (a) Dimensions of flow field.
(b) If possible incompressible flow.
(c) Acceleration of a particle at $(x, y, z) = (3, 1, 2)$.

Solution: Apply continuity, use substantial derivative.

Basic equations:

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} + \frac{\partial \rho}{\partial t} = 0 \quad \left(\frac{\partial \rho}{\partial t} = 0\right)$$

$$\vec{a}_p = \frac{D\vec{V}}{Dt} = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} + \frac{\partial \vec{V}}{\partial t} \quad \left(\frac{\partial \vec{V}}{\partial t} = 0\right)$$

Assumption: Incompressible flow, ρ = constant

Then $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$ is criterion.

Note $\vec{V} = \vec{V}(x, y, z)$, so flow is three-dimensional, and ← 3-D

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 2xy - 3 + 4z \neq 0$$

Flow cannot be incompressible. ← $\rho \neq$

$$\vec{a}_p = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} \; ; \; \frac{\partial \vec{V}}{\partial x} = 2axy\hat{i}, \; \frac{\partial \vec{V}}{\partial y} = ax^2\hat{i} - b\hat{j}, \; \frac{\partial \vec{V}}{\partial z} = 2cz\hat{k}$$

$$= (ax^2y)(2axy\hat{i}) + (-by)(ax^2\hat{i} - b\hat{j}) + (cz^2)(2cz\hat{k})$$

$$\vec{a}_p = \hat{i}(2a^2x^3y^2 - abx^2y) + \hat{j}(b^2y) + \hat{k}(2c^2z^3)$$

At $(x, y, z) = (3, 1, 2)$,

$$\vec{a}_p = \hat{i}\left[2 \times \frac{(1)^2}{m^4 \cdot s^2} \times (3)^3 m^3 \times (1)^2 m^2 - \frac{1}{m^2 \cdot s} \times \frac{3}{s} \times (3)^2 m^2 \times 1\,m\right] + \hat{j}\left[\frac{(3)^2}{s^2} \times 1\,m\right] + \hat{k}\left[2 \times \frac{(2)^2}{m^2 \cdot s^2} \times (2)^3 m^3\right]$$

$$\vec{a}_p = 27\hat{i} + 9\hat{j} + 64\hat{k} \; \frac{m}{s^2}$$

← $\vec{a}_p$

Given: Velocity field (within a laminar boundary layer) is given by $\vec{V} = A\frac{Uy}{x^{1/2}}\left(\hat{i} + \frac{y}{4x}\hat{j}\right)$

where $A = 141\ m^{-1/2}$

$U = 0.240\ m/s$

Find: (a) Show that this velocity field represents a possible incompressible flow

(b) Calculate $\vec{a}$ of particle at $(x,y) = (0.5m, 5mm)$

(c) Slope of streamline through point $(0.5m, 5mm)$

Solution:

From given velocity field $\vec{V} = \vec{V}(x,y)$, $w=0$, flow is steady

(a) Check conservation of mass for ρ = constant

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}^{\,0} = 0$$

$$u = A\frac{Uy}{x^{1/2}} \qquad \frac{\partial u}{\partial x} = -\frac{1}{2}AU\frac{y}{x^{3/2}}$$

$$v = AU\frac{y^2}{4x^{3/2}} \qquad \frac{\partial v}{\partial y} = \frac{1}{2}AU\frac{y}{x^{3/2}}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$\therefore$ incompressible (Q.E.D.)

(b) $$\vec{a} = \frac{D\vec{V}}{Dt} = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}^{\,0} + \frac{\partial \vec{V}}{\partial t}^{\,0}$$

$$a_{p_x} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \quad ; \quad \frac{\partial u}{\partial y} = AU\frac{1}{x^{1/2}}$$

$$a_{p_x} = A\frac{Uy}{x^{1/2}}\left(-\frac{1}{2}AU\frac{y}{x^{3/2}}\right) + AU\frac{y^2}{4x^{3/2}}\left(AU\frac{1}{x^{1/2}}\right)$$

$$a_{p_x} = -\frac{1}{2}A^2U^2\frac{y^2}{x^2} + A^2U^2\frac{y^2}{4x^2} = -\frac{1}{4}\left(\frac{AUy}{x}\right)^2$$

$$a_{p_x} = -\frac{1}{4}\left[\frac{141}{m^{1/2}} \times 0.240\frac{m}{s} \times \frac{0.005m}{0.5m}\right]^2 = -0.0286\ m/s^2$$

$$a_{p_y} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} \quad ; \quad \frac{\partial v}{\partial x} = -\frac{3}{8}\frac{AUy^2}{x^{5/2}}$$

$$= AU\frac{y}{x^{1/2}}\left(-\frac{3}{8}AU\frac{y^2}{x^{5/2}}\right) + AU\frac{y^2}{4x^{3/2}}\left(\frac{1}{2}AU\frac{y}{x^{3/2}}\right)$$

$$= -\frac{3}{8}A^2U^2\frac{y^3}{x^3} + \frac{1}{8}A^2U^2\frac{y^3}{x^3} = -\frac{1}{4}A^2U^2\frac{y^3}{x^3}$$

$$a_{p_y} = -\frac{1}{4}\left(\frac{141}{m^{1/2}} \times 0.240\frac{m}{s}\right)^2\left(\frac{0.005m}{0.5m}\right)^3 = -2.86\times10^{-4}\ m/s^2$$

$$\therefore \vec{a}_p = -2.86\left(10^{-2}\hat{i} + 10^{-4}\hat{j}\right)m/s^2 \qquad \vec{a}_p$$

The slope of the streamline is given by

$$\left.\frac{dy}{dx}\right)_s = \frac{v}{u} = \frac{y}{4x} = \frac{5\times10^{-3}m}{4\times0.5m} = 0.0025 \qquad \left.\frac{dy}{dx}\right)_s$$

Problem 5.41

The x component of velocity in a steady, incompressible flow field in the xy plane is $u = A/x^2$, where $A = 2\ m^3/s$ and x is measured in meters. Find the simplest y component of velocity for this flow field. Evaluate the acceleration of a fluid particle at point $(x, y) = (1, 3)$.

Given: x component of incompressible flow field

Find: y component of velocity; find acceleration at a point

Solution

The given data is $A = 2\cdot\frac{m^3}{s}$ $\quad x = 1\cdot m \quad y = 3\cdot m$

$$u(x,y) = \frac{A}{x^2}$$

For incompressible flow $\frac{du}{dx} + \frac{dv}{dy} = 0$

Hence $v = -\int \frac{du}{dx}\,dy = \int \frac{2\cdot A}{x^3}\,dy$

$$v = \frac{2\cdot A\cdot y}{x^3}$$

The acceleration is given by

$$\vec{a}_p = \underbrace{\frac{D\vec{V}}{Dt}}_{\text{total acceleration of a particle}} = \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\text{convective acceleration}} + \underbrace{\frac{\partial \vec{V}}{\partial t}}_{\text{local acceleration}}$$

For the present steady, 2D flow

$$a_x = u\cdot\frac{du}{dx} + v\cdot\frac{du}{dy} = \frac{A}{x^2}\cdot\left(-\frac{2\cdot A}{x^3}\right) + \frac{A\cdot y}{x^2}\cdot 0 \qquad a_x = -\frac{2\cdot A^2}{x^5}$$

$$a_y = u\cdot\frac{dv}{dx} + v\cdot\frac{dv}{dy} = \frac{A}{x^2}\cdot\left(-\frac{6\cdot A\cdot y}{x^4}\right) + \frac{2\cdot A\cdot y}{x^3}\cdot\left(\frac{2\cdot A}{x^3}\right) \qquad a_y = -\frac{2\cdot A^2\cdot y}{x^6}$$

At point (1,3) the acceleration is

$$a_x = -\frac{2\cdot A^2}{x^5} \qquad a_x = -8\,\frac{m}{s^2}$$

$$a_y = -\frac{2\cdot A^2\cdot y}{x^6} \qquad a_y = -24\,\frac{m}{s^2}$$

Problem 5.42

Given: Incompressible, two-dimensional flow field with $w=0$, has a y component of velocity given by

$$v = -Axy$$

where units of v are m/s; x and y are in meters and A is a dimensional constant

Find: (a) the dimensions of the constant A
(b) the simplest x component of velocity for this flow field
(c) the acceleration of a fluid particle at the point $(x,y)=(1,2)$

Solution:

(a) Since $v = -Axy$, then the dimensions of A, [A], are given by

$$[A] = \left[\frac{v}{xy}\right] = \frac{L}{t}\cdot\frac{1}{L}\cdot\frac{1}{L} = \frac{1}{Lt}$$ [A]

(b) Apply the continuity equation for the conditions given

Basic equation: $\nabla\cdot\rho\vec{V} + \frac{\partial\rho}{\partial t} = 0$

For incompressible flow, $\frac{\partial\rho}{\partial t}=0$. Thus with $w=0$, the basic equation reduces to $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

Then,
$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} = -\frac{\partial}{\partial y}(-Axy) = Ax$$

and
$$u = \int \frac{\partial u}{\partial x}\,dx + f(y) = \int Ax\,dx + f(y) = \frac{1}{2}Ax^2 + f(y).$$

The simplest x component of velocity is obtained with $f(y)=0$

$$\therefore\ u = \frac{1}{2}Ax^2$$ u

(c) The acceleration of a fluid particle is given by

$$\vec{a}_p = \frac{D\vec{V}}{Dt} = u\frac{\partial\vec{V}}{\partial x} + v\frac{\partial\vec{V}}{\partial y} + w\frac{\partial\vec{V}}{\partial z}^{\,\approx 0} + \frac{\partial\vec{V}}{\partial t}^{\,\approx 0}$$

$$\vec{a}_p = \frac{1}{2}Ax^2\frac{\partial}{\partial x}\left[\frac{1}{2}Ax^2\hat{i} - Axy\hat{j}\right] - Axy\frac{\partial}{\partial y}\left[\frac{1}{2}Ax^2\hat{i} - Axy\hat{j}\right]$$

$$\vec{a}_p = \frac{1}{2}Ax^2\left[Ax\hat{i} - Ay\hat{j}\right] - Axy\left[-Ax\hat{j}\right] = \frac{1}{2}A^2x^3\hat{i} + \frac{1}{2}A^2x^2y\hat{j}$$

At the point $(x,y)=(1,2)$

$$\vec{a}_p = \frac{1}{2}A^2(1)^3\hat{i} + \frac{1}{2}A^2(1)^2(2)\hat{j} = A^2\left[\frac{1}{2}\hat{i} + \hat{j}\right]$$ $\vec{a}_p$

Problem 5.43

Consider the velocity field $\vec{V} = Ax/(x^2 + y^2)\hat{i} + Ay/(x^2 + y^2)\hat{j}$ in the xy plane, where $A = 10\ m^2/s$, and x and y are measured in meters. Is this an incompressible flow field? Derive an expression for the fluid acceleration. Evaluate the velocity and acceleration along the x axis, the y axis, and along a line defined by $y = x$. What can you conclude about this flow field?

Given: Velocity field

Find: Whether flow is incompressible; expression for acceleration; evaluate acceleration along axes and along $y = x$

Solution

The given data is $$A = 10 \cdot \frac{m^2}{s}$$

$$u(x,y) = \frac{A \cdot x}{x^2 + y^2}$$

$$v(x,y) = \frac{A \cdot y}{x^2 + y^2}$$

For incompressible flow $$\frac{du}{dx} + \frac{dv}{dy} = 0$$

Hence $$\frac{du}{dx} + \frac{dv}{dy} = -A \cdot \frac{\left(x^2 - y^2\right)}{\left(x^2 + y^2\right)^2} + A \cdot \frac{\left(x^2 - y^2\right)}{\left(x^2 + y^2\right)^2} = 0$$

Incompressible flow

The acceleration is given by

$$\vec{a}_p = \underset{\substack{\text{total}\\ \text{acceleration}\\ \text{of a particle}}}{\frac{D\vec{V}}{Dt}} = \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\substack{\text{convective}\\ \text{acceleration}}} + \underset{\substack{\text{local}\\ \text{acceleration}}}{\frac{\partial \vec{V}}{\partial t}}$$

For the present steady, 2D flow

$$a_x = u\cdot\frac{du}{dx} + v\cdot\frac{du}{dy} = \frac{A\cdot x}{x^2+y^2}\cdot\left[-\frac{A\cdot\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2}\right] + \frac{A\cdot y}{x^2+y^2}\cdot\left[-\frac{2\cdot A\cdot x\cdot y}{\left(x^2+y^2\right)^2}\right]$$

$$a_x = -\frac{A^2\cdot x}{\left(x^2+y^2\right)^2}$$

$$a_y = u\cdot\frac{dv}{dx} + v\cdot\frac{dv}{dy} = \frac{A\cdot x}{x^2+y^2}\cdot\left[-\frac{2\cdot A\cdot x\cdot y}{\left(x^2+y^2\right)^2}\right] + \frac{A\cdot y}{x^2+y^2}\cdot\left[\frac{A\cdot\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2}\right]$$

$$a_y = -\frac{A^2\cdot y}{\left(x^2+y^2\right)^2}$$

Along the *x* axis $\quad a_x = -\frac{A^2}{x^3} = -\frac{100}{x^3} \qquad a_y = 0$

Along the *y* axis $\quad a_x = 0 \qquad a_y = -\frac{A^2}{y^3} = -\frac{100}{y^3}$

Along the line $x = y$

$$a_x = -\frac{A^2 \cdot x}{r^4} = -\frac{100 \cdot x}{r^4}$$

$$a_y = -\frac{A^2 \cdot y}{r^4} = -\frac{100 \cdot y}{r^4}$$

where

$$r = \sqrt{x^2 + y^2}$$

For this last case the acceleration along the line $x = y$ is

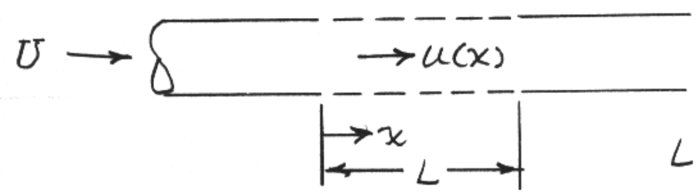
$$a = \sqrt{a_x^2 + a_y^2} = -\frac{A^2}{r^4} \cdot \sqrt{x^2 + y^2} = -\frac{A^2}{r^3} = -\frac{100}{r^3}$$

$$a = -\frac{A^2}{r^3} = -\frac{100}{r^3}$$

In each case the acceleration vector points towards the origin, so the flow field is a radial decelerating flow

Problem 5.44

Given: Duct flow with inviscid liquid, ρ = constant.



$L = 0.3\ m$

$$u(x) = U(1 - x/2L) \qquad U = 5\ m/s$$

Find: Expression for acceleration along ℄.

Solution: Computing equation

$$a_{p_x} = u\frac{\partial u}{\partial x} + \cancel{v}\frac{\partial u}{\partial y} + \cancel{w}\frac{\partial u}{\partial z} + \cancel{\frac{\partial u}{\partial t}}$$

($v = 0$ (1), $w = 0$ (1), $\frac{\partial u}{\partial t} = 0$ (2))

Assumptions: (1) Along ℄ $v = w = 0$

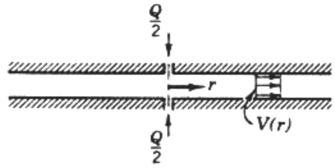
(2) Steady flow

Then

$$a_{p_x} = u\frac{\partial u}{\partial x} = U\left(1 - \frac{x}{2L}\right)U\left(-\frac{1}{2L}\right) = -\frac{U^2}{2L}\left(1 - \frac{x}{2L}\right) \qquad a_{p_x}$$

Problem 5.45

Given: Incompressible flow between parallel plates as shown.



Find: (a) Show $V_r = \dfrac{Q}{2\pi r h}$

(b) Acceleration in gap.

Solution: Apply conservation of mass

Basic equation: $\dfrac{1}{r}\dfrac{\partial}{\partial r}(rV_r) + \dfrac{1}{r}\dfrac{\partial}{\partial \theta}(V_\theta) + \dfrac{\partial}{\partial z}V_z = 0$ (second term $= 0(1)$, third term $= 0(2)$)

Assumptions: (1) $V_\theta = 0$

(2) $V_z = 0$

Then

$$\frac{1}{r}\frac{\partial}{\partial r}(rV_r) = 0 \quad \text{or} \quad rV_r = C \quad \text{or} \quad V_r = \frac{C}{r} \text{ is form of solution.}$$

The volume flow rate is $Q = 2\pi r h V_r$, so $V_r = \dfrac{Q}{2\pi r h}$ $\longleftarrow$ V_r

Because $V_\theta = 0$, $a_\theta = 0$. The radial acceleration is

$$a_r = V_r \frac{\partial V_r}{\partial r} = \frac{Q}{2\pi r h}\left[(-1)\frac{Q}{2\pi r^2 h}\right] = -\left(\frac{Q}{2\pi h}\right)^2 \frac{1}{r^3}$$

Thus

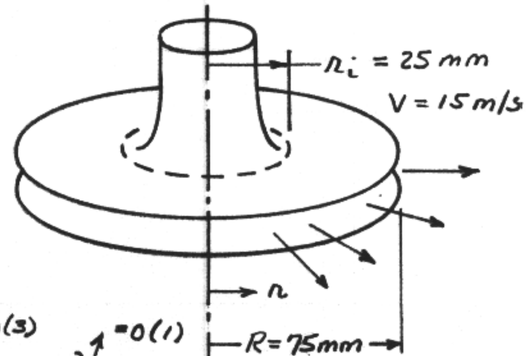
$$\vec{a}_p = -\left(\frac{Q}{2\pi h}\right)^2 \frac{1}{r^3}\hat{e}_r$$ $\longleftarrow$ $\vec{a}_p$

The above expressions are valid only for $r > 0$.

Problem 5.46

Given: Incompressible, inviscid flow of air between parallel disks.

Find: (a) Simplify continuity.
(b) Show $\vec{V} = V(R/r)\hat{e}_r$, $r_i < r < R$
(c) Calculate acceleration of a particle at $r = r_i, R$.



Solution: Apply continuity equation and substantial derivative

Basic equations: $\frac{1}{r}\frac{\partial}{\partial r}(r\rho V_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho V_\theta) + \frac{\partial}{\partial z}(\rho V_z) + \frac{\partial \rho}{\partial t} = 0$ (with $\frac{1}{r}\frac{\partial}{\partial \theta}(\rho V_\theta) = 0(2)$, $\frac{\partial}{\partial z}(\rho V_z) = 0(3)$, $\frac{\partial \rho}{\partial t} = 0(1)$)

$$a_r = V_r \frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} + \frac{\partial V_r}{\partial z} - \frac{V_\theta}{r} + \frac{\partial V_r}{\partial t}$$

(with $\frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} = 0(2)$, $\frac{\partial V_r}{\partial z} = 0(3)$, $\frac{V_\theta}{r} = 0(2)$, $\frac{\partial V_r}{\partial t} = 0(4)$)

Assumptions: (1) Incompressible flow, ρ = constant
(2) Radial flow, $V_\theta = 0$
(3) Uniform flow at each radial location, $\partial/\partial z = 0$
(4) Steady flow

Then

$$\frac{1}{r}\frac{\partial}{\partial r}(rV_r) = 0 \quad \text{or} \quad rV_r = \text{constant} = RV \; ; \; V_r = V\frac{R}{r}$$

So that $\vec{V} = V\frac{R}{r}\hat{e}_r$ ← $\vec{V}$

The radial acceleration of a fluid particle is

$$a_r = V_r\frac{\partial V_r}{\partial r} = V\frac{R}{r}(VR)\left(-\frac{1}{r^2}\right) = -\frac{V^2R^2}{r^3} = -\frac{V^2}{R}\left(\frac{R}{r}\right)^3$$

At $r = r_i = 25\,mm$,

$$a_r = -(15)^2\frac{m^2}{s^2} \times \frac{1}{0.075\,m}\left(\frac{75}{25}\right)^3 = -81.0\,\frac{km}{s^2}$$ ← $a_r(r_i)$

At $r = R = 75\,mm$

$$a_r = -(15)^2\frac{m^2}{s^2} \times \frac{1}{0.075\,m}\left(\frac{75}{75}\right)^3 = -3.00\,\frac{km}{s^2}$$ ← $a_r(R)$

Problem 5.47

Given: Temperature variation, $T = T_0 - \alpha e^{-x/L} \sin\left(\frac{2\pi t}{\tau}\right)$

Particle moves at speed U = constant

Find: Rate of change of T experienced by particle.

Solution: $T = T(x, t)$

$$dT = \frac{\partial T}{\partial x} dx + \frac{\partial T}{\partial t} dt$$

$$\left.\frac{dT}{dt}\right)_{particle} = \frac{\partial T}{\partial x} \left.\frac{dx}{dt}\right)_{particle} + \frac{\partial T}{\partial t}$$

or

$$\frac{DT}{Dt} = \left.\frac{dT}{dt}\right)_{particle} = u\frac{\partial T}{\partial x} + \frac{\partial T}{\partial t}$$

For the given data, $u = U$ = constant

$$\frac{\partial T}{\partial x} = \frac{\partial}{\partial x}\left[T_0 - \alpha e^{-x/L} \sin\left(\frac{2\pi t}{\tau}\right)\right] = \frac{\alpha}{L} e^{-x/L} \sin\left(\frac{2\pi t}{\tau}\right)$$

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial t}\left[T_0 - \alpha e^{-x/L} \sin\left(\frac{2\pi t}{\tau}\right)\right] = -\frac{2\pi}{\tau}\alpha e^{-x/L} \cos\left(\frac{2\pi t}{\tau}\right)$$

substituting,

$$\frac{DT}{Dt} = \left[\frac{U}{L} \sin\left(\frac{2\pi t}{\tau}\right) - \frac{2\pi}{\tau} \cos\left(\frac{2\pi t}{\tau}\right)\right] \alpha e^{-x/L} \quad \text{deg/s}$$

$\frac{DT}{Dt}$

Problem 5.48

Given: Instruments on board an aircraft flying through a cold front give the following information:
- rate of change of temperature is −0.5F/min
- air speed = 300 knots
- rate of climb = 3500 ft/min

Front is stationary and vertically uniform.

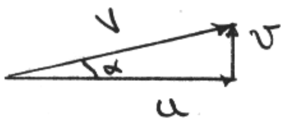
Find: rate of change of temperature with respect to horizontal distance through the cold front

Solution: Apply the substantial derivative concept.

Basic equation: $\frac{DT}{Dt} = u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + \frac{\partial T}{\partial t}$ ($\frac{\partial T}{\partial y} \to 0$ vertically uniform; $\frac{\partial T}{\partial t} \to 0$ (stationary front))

$\frac{DT}{Dt} = -0.5\,F/min$. Need to find $\frac{\partial T}{\partial x}$

Velocity picture.



$$V = 300\,\frac{nm}{hr} \times 6080\,\frac{ft}{nm} \times \frac{hr}{3600\,s} = 507\,\frac{ft}{s}$$

$$v = 3500\,\frac{ft}{min} \times \frac{min}{60\,s} = 58.3\,ft/s$$

Then $\alpha = \sin^{-1}\frac{v}{V} = \sin^{-1}\frac{58.3}{507} = 6.60°$

and $u = V\cos\alpha = 507\,\frac{ft}{sec}\cos 6.60° = 504\,ft/s$

$$\therefore \frac{\partial T}{\partial x} = \frac{1}{u}\frac{DT}{Dt} = \frac{s}{504\,ft} \times \frac{-0.5°F}{min} \times \frac{min}{60\,s} \times \frac{5280\,ft}{mi}$$

$$\frac{\partial T}{\partial x} = -0.0873°F/mile \qquad \frac{\partial T}{\partial x}$$

Problem 5.49

Given: Aircraft flying north with velocity component $u = 300$ mph is climbing at rate, $v = 3000$ ft/min. The rate of temperature change with vertical distance y is $\partial T/\partial y = -3°F/1000$ ft. The variation of temperature with position x is

$\partial T/\partial x = -1°F/\text{mile}$

Find: the rate of temperature change shown by a recorder on board the aircraft.

Solution: Apply the substantial derivative concept

Basic equation:

$$\frac{DT}{Dt} = u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + \cancel{\frac{\partial T}{\partial t}}^{\,0}$$

Substituting numerical values,

$$\frac{DT}{Dt} = 300\,\frac{\text{mile}}{\text{hr}} \times -\frac{1°F}{\text{mile}} \times \frac{\text{hr}}{60\,\text{min}} + 3000\,\frac{\text{ft}}{\text{min}} \times -\frac{3°F}{1000\,\text{ft}}$$

$$\frac{DT}{Dt} = (-5-9)°F/\text{min} = -14°F/\text{min} \longleftarrow \frac{DT}{Dt}$$

Problem 5.50

Given: Sediment concentration rates in a river after a rainfall are:

$$\frac{\partial c}{\partial t} = 100 \frac{ppm}{hr} \quad , \quad \frac{\partial c}{\partial x} = 50 \frac{ppm}{mi} \quad \text{(downstream)}$$

Stream speed is $u_s = 0.5$ mph, where a boat is used to survey concentration.

Boat speed is $V_b = 2.5$ mph.

Find: (a) Calculate rates of change of sediment concentration observed when boat travels upstream, drifts with the current, or travels downstream.

(b) Explain physically why the observed rates differ.

Solution: Apply substantial derivative concept

Basic equation: $$\frac{Dc}{Dt} = u\frac{\partial c}{\partial x} + \frac{\partial c}{\partial t}$$

To obtain rate of change seen from boat, set $u = u_B$.

(i) For travel upstream, $u_B = u_s - V_b = 0.5 - 2.5 = -2.0$ mph

$$\frac{Dc}{Dt}(up) = -2.0 \frac{mi}{hr} \times 50 \frac{ppm}{mi} + 100 \frac{ppm}{hr} = 0.00 \text{ ppm/hr}$$ up

(ii) For drifting, $u_B = u_s + 0 = 0.5$ mph

$$\frac{Dc}{Dt}(drift) = 0.5 \frac{mi}{hr} \times 50 \frac{ppm}{mi} + 100 \frac{ppm}{hr} = 125 \text{ ppm/hr}$$ drift

(iii) For travel downstream, $u_B = u_s + V_b = 0.5 + 2.5 = 3.0$ mph

$$\frac{Dc}{Dt}(down) = 3.0 \frac{mi}{hr} \times 50 \frac{ppm}{mi} + 100 \frac{ppm}{hr} = 250 \text{ ppm/hr}$$ down

Physically the observed rates of change differ because the observer is convected through the flow. The convective change may add to or subtract from the local rate of change.

Problem 5.51

Expand $(\vec{V}\cdot\nabla)\vec{V}$ in rectangular coordinates to obtain the convective acceleration of a fluid particle. Verify the results given in Eqs 5.11

Solution:

In rectangular coordinates $\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$, $\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$

$$(\vec{V}\cdot\nabla)\vec{V} = \left[(u\hat{i} + v\hat{j} + w\hat{k})\cdot\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\right] u\hat{i} + v\hat{j} + w\hat{k}$$

$$= \left[u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right] u\hat{i} + v\hat{j} + w\hat{k}$$

$$(\vec{V}\cdot\nabla)\vec{V} = \overset{①}{\left\{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right\}}\hat{i} + \overset{②}{\left\{u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right\}}\hat{j} + \overset{③}{\left\{u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right\}}\hat{k}$$

Term ① is x component of convective acceleration

Eq. 5.11a $\quad a_{x_p} = \left\{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right\} + \frac{\partial u}{\partial t}$

Term ② is the y component of convective acceleration

Eq. 5.11b $\quad a_{y_p} = \left\{u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right\} + \frac{\partial v}{\partial t}$

Term ③ is the z component of convective acceleration

Eq. 5.11c $\quad a_{z_p} = \left\{u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right\} + \frac{\partial w}{\partial t}$.

Given: Steady, two-dimensional velocity field, $\vec{V} = Ax\hat{i} - Ay\hat{j}$; $A = 1\,s^{-1}$, coordinates measured in meters.

Show: that streamlines are hyperbolas, $xy = C$

Find: (a) Expression for acceleration.
(b) Particle acceleration at $(x,y) = (\tfrac{1}{2},2), (1,1)$ and $(2,\tfrac{1}{2})$.

Plot: streamlines corresponding to $C = 0, 1,$ and $2\,m^2$; show acceleration vectors on the plot.

Solution:

Along a streamline, $\frac{dy}{dx} = \frac{v}{u} = \frac{-y}{x}$ or $\frac{dy}{y} + \frac{dx}{x} = 0$

Integrating we obtain $\ln y + \ln x = \ln C$ and $xy = C$ ← Streamline

The acceleration of a particle is

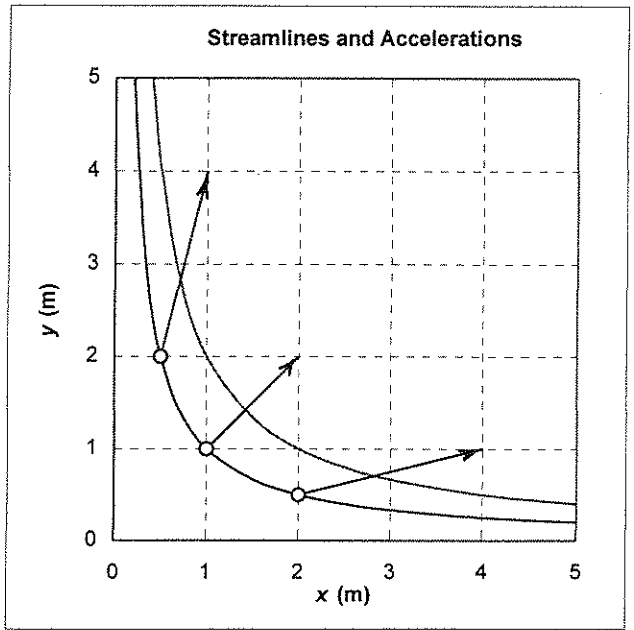
$$\vec{a}_p = \frac{D\vec{V}}{Dt} = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + \cancel{w\frac{\partial \vec{V}}{\partial z}} + \cancel{\frac{\partial \vec{V}}{\partial t}} \quad \{w=0 \text{ and steady flow}\}$$

$$\vec{a}_p = Ax(A\hat{i}) - (Ay)(-A\hat{j}) = A^2(x\hat{i} + y\hat{j}) \quad \leftarrow \vec{a}_p$$

$\vec{a}_p)_{1/2,2} = \frac{1}{2}\hat{i} + 2\hat{j}\ m/s^2$ $\quad \vec{a}_p)_{1,1} = \hat{i} + \hat{j}\ m/s^2$

$\vec{a}_p)_{2,1/2} = 2\hat{i} + \frac{1}{2}\hat{j}\ m/s^2$ $\quad \Big\} \leftarrow \vec{a}_p$

Plot:



Given: Velocity field represented by

$$\vec{V} = (Ax - B)\hat{i} + Cy\hat{j} + Dt\,\hat{k} \qquad (x, y \text{ in m})$$

where $A = 2\,s^{-1}$, $B = 4\,m/s$, and $D = 5\,m/s^2$

Find: (a) Proper value of C for incompressible flow.
(b) Acceleration of particle at $(x,y) = (3,2)$.
(c) Sketch streamlines in xy plane.

Solution: For incompressible flow, $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$. Since $w = Dt$, $\partial w/\partial z = 0$, and

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial v}{\partial y} = C = -\frac{\partial u}{\partial x} = -A = -2\,s^{-1}$$ C

$$\vec{a}_p = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} + \frac{\partial \vec{V}}{\partial t}$$

$$\vec{a}_p = (Ax - B)(A\hat{i}) + (Cy)(C\hat{j}) + (Dt)(0) + D\hat{k}$$

$$\vec{a}_p(3,2) = \left(\frac{2}{s} \times 3m - \frac{4m}{s}\right)\left(\frac{2}{s}\right)\hat{i} + \left(\frac{-2}{s} \times 2m\right)\left(-\frac{2}{s}\right)\hat{j} + 5\frac{m}{s^2}\,\hat{k}$$

$$\vec{a}_p(3,2) = 4\hat{i} + 8\hat{j} + 5\hat{k} \;\; m/s^2$$ $\vec{a}_p(3,2)$

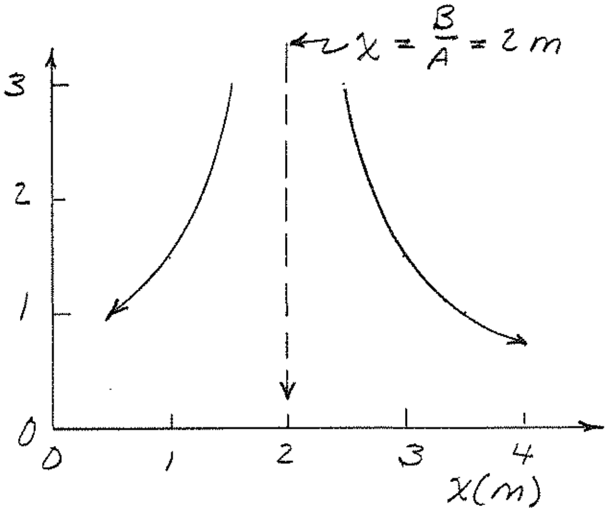
In the xy plane, streamlines are $\frac{dy}{dx} = \frac{v}{u} = \frac{Cy}{Ax - B}$. Thus

$$\frac{dx}{Ax - B} = \frac{dy}{Cy} \quad \text{or} \quad \frac{dx}{Ax - B} = -\frac{dy}{Ay} \quad \text{or} \quad \frac{dx}{x - B/A} + \frac{dy}{y} = 0$$

Integrating

$$\ln(x - B/A) + \ln y = \ln c_0$$

$$\left(x - \frac{B}{A}\right)y = \text{const}$$



Problem 5.54

Given: Velocity field $\vec{V} = (Ax - B)\hat{i} - Ay\hat{j}$; $A = 0.2\ s^{-1}$, $B = 0.6\ s^{-1}$, x in m.

Find: (a) General expression for acceleration of a fluid particle.
(b) Acceleration at $(x,y) = (0, 4/3), (1,2)$, and $(2,4)$.
(c) Plot of streamlines.
(d) Acceleration vectors on plot.

Solution: Note $w = 0$ and flow is steady. Then

$$\vec{a}_p = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} = (Ax - B)A\hat{i} + (-Ay)(-A)\hat{j} = (A^2x - AB)\hat{i} + A^2y\hat{j}$$ $\vec{a}_p$

At $(x,y) = (0, 4/3)$, $\vec{a}_p = -0.12\hat{i} + 0.0533\hat{j}\ m/s^2$

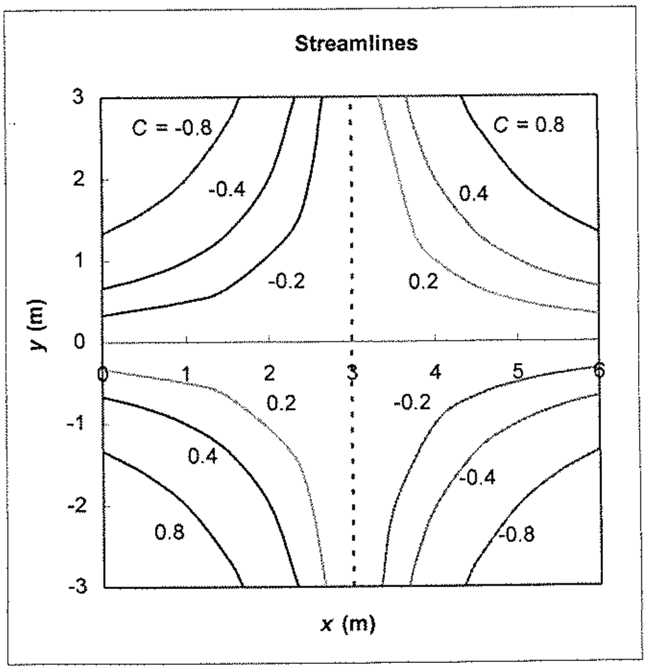
$(1,2)$, $\vec{a}_p = -0.08\hat{i} + 0.0800\hat{j}\ m/s^2$

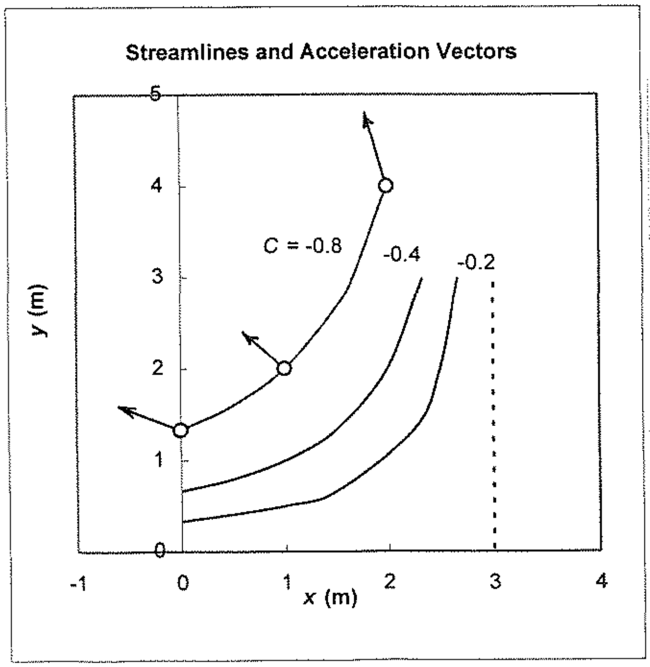
$(2,4)$, $\vec{a}_p = -0.04\hat{i} + 0.160\hat{j}\ m/s^2$ $\vec{a}_p$

Streamlines are $\frac{dx}{u} = \frac{dy}{v} = \frac{dx}{Ax - B} = \frac{dy}{-Ay}$. Integrating,

$$\frac{1}{A}\ln(Ax - B) + \frac{1}{A}\ln y = \frac{1}{A}\ln c \quad \text{or} \quad (Ax - B)y = c$$

The plots are:





Problem 5.55

Given: Air flowing downward toward infinite horizontal flat plate. Velocity field is

$$\vec{V} = (ax\hat{i} - ay\hat{j})(2+\cos\omega t)\,;\ a = 3\ s^{-1},\ \omega = \pi\ s^{-1}$$

Find: (a) Expression for streamline at $t = 1.5$ s.
(b) Plot of streamline through $(x,y) = (2,4)$ at this instant.
(c) Velocity vector
(d) Vectors representing local, convective, and total acceleration.

Solution: Streamline is $\frac{dx}{u} = \frac{dy}{v}$, or $\frac{dx}{x} + \frac{dy}{y} = 0$ or $xy = c$

At point $(x,y) = (2,4)$, $c = 2m \times 4m = 8m^2$; $xy = 8\,m^2$ ← Streamline

The plot is shown below. Note $u = ax\hat{i}[2+\cos\omega t]$, $v = -ay\hat{j}[2+\cos\omega t]$

At $(x,y,t) = (2m, 4m, 1.5s)$, $\vec{V} = (6\hat{i} - 12\hat{j})(2+0) = 12\hat{i} - 24\hat{j}$ ← $\vec{V}$

The local acceleration components at $(x,y,t) = (2m, 4m, 1.5s)$ are

$$a_{x,local} = \frac{\partial u}{\partial t} = ax\hat{i}(-\omega\sin\omega t) = \frac{3}{s} \times 2m \times \left(-\frac{\pi}{s}\right) \times \sin\left(\frac{3\pi}{2}\right) = 6\pi\hat{i}\ m/s^2$$

$$a_{y,local} = \frac{\partial v}{\partial t} = -ay\hat{j}(-\omega\sin\omega t) = \frac{3}{s} \times 4m \times \left(\frac{-\pi}{s}\right) \times \sin\left(\frac{3\pi}{2}\right) = -12\pi\hat{j}\ m/s^2$$ ← Local

The convective acceleration components at $(x,y,t) = (2m, 4m, 1.5s)$ are

$$a_{x,conv} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = ax(a\hat{i})\left[2+\cos\frac{3\pi}{2}\right]^2 = (3)(2)(3)[2]^2\hat{i} = 72\hat{i}$$

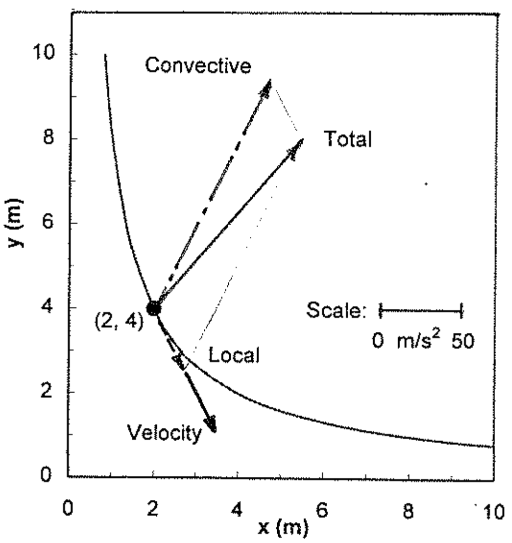
$$a_{y,conv} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = (-ay)(-a\hat{j})\left[2+\cos\frac{3\pi}{2}\right]^2 = 4a^2y\hat{j} = 4(3)^2 4\hat{j} = 144\hat{j}$$ ← Convective

The total acceleration is the sum of the convective and local values:

$$a_{x,total} = a_{x,conv} + a_{x,local} = (72 + 6\pi)\hat{i} = 90.8\hat{i}\ m/s^2$$

$$a_{y,total} = a_{y,conv} + a_{y,local} = (144 - 12\pi)\hat{j} = 106\ \hat{j}\ m/s^2$$ ← Total

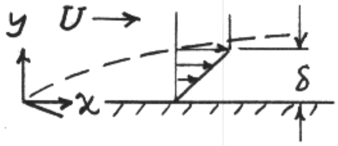
The plot is



Problem 5.56

Given: Laminar boundary layer, linear approximate profile.

$$\frac{u}{U} = \frac{y}{\delta} \qquad \delta = cx^{1/2}$$



From Problem 5.7, $v = \frac{uy}{4x} = U\frac{y^2}{4x\delta}$

Find: (a) x and y components of acceleration of a fluid particle.
(b) Locate maximum values.
(c) Ratio, $a_{x,max}/a_{y,max}$.

Solution: Basic equations:

$$a_{p_x} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} \qquad (w = 0\ (1),\ \partial/\partial t = 0\ (2))$$

$$a_{p_y} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t} \qquad (w = 0\ (1),\ \partial/\partial t = 0\ (2))$$

Assumptions: (1) w and $\partial/\partial z$ zero, (2) steady flow. $\frac{d\delta}{dx} = \frac{1}{2}cx^{-1/2} = \frac{\delta}{2x}$

$$u = U\frac{y}{\delta}\ ; \quad \frac{\partial u}{\partial x} = Uy\left(-\frac{1}{\delta^2}\right)\frac{d\delta}{dx} = -Uy\frac{1}{\delta^2}\frac{\delta}{2x} = -\frac{Uy}{2x\delta}\ ; \quad \frac{\partial u}{\partial y} = \frac{U}{\delta}$$

$$v = U\frac{y^2}{4x\delta}\ ; \quad \frac{\partial v}{\partial x} = \frac{Uy^2}{4}\left(-\frac{1}{x^2\delta} - \frac{1}{x\delta^2}\frac{d\delta}{dx}\right) = -\frac{3Uy^2}{8x^2\delta}\ ; \quad \frac{\partial v}{\partial y} = \frac{Uy}{2x\delta}$$

Thus

$$a_{p_x} = \left(U\frac{y}{\delta}\right)\left(-\frac{Uy}{2x\delta}\right) + \left(U\frac{y^2}{4x\delta}\right)\left(\frac{U}{\delta}\right) = -\frac{U^2}{2x}\left(\frac{y}{\delta}\right)^2 + \frac{U^2}{4x}\left(\frac{y}{\delta}\right)^2 = -\frac{U^2}{4x}\left(\frac{y}{\delta}\right)^2$$ a_{p_x}

$$a_{p_y} = \left(U\frac{y}{\delta}\right)\left(-\frac{3Uy^2}{8x^2\delta}\right) + \left(U\frac{y^2}{4x\delta}\right)\left(U\frac{y}{2x\delta}\right) = -\frac{3U^2}{8x}\left(\frac{y}{x}\right)\left(\frac{y}{\delta}\right)^2 + \frac{U^2}{8x}\left(\frac{y}{x}\right)\left(\frac{y}{\delta}\right)^2$$

$$a_{p_y} = -\frac{U^2}{4x}\left(\frac{y}{x}\right)\left(\frac{y}{\delta}\right)^2$$ a_{p_y}

Maximum values are at $y = \delta$

$$a_{p_x,max} = -\frac{U^2}{4x}$$ (max) a_{p_x}

$$a_{p_y,max} = -\frac{U^2}{4x}\left(\frac{\delta}{x}\right)$$ (max) a_{p_y}

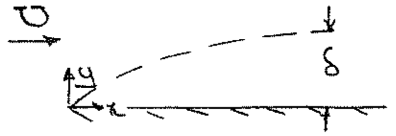
Thus $\frac{a_{p_x,max}}{a_{p_y,max}} = \frac{x}{\delta}$

At $x = 0.5\ m$, $\delta = 5\ mm$, $\frac{a_{p_x,max}}{a_{p_y,max}} = \frac{0.5\ m}{0.005\ m} = 100$ Ratio

Given: Laminar boundary layer on a flat plate. (Problem 5.11)

$$\frac{u}{U} = \sin\frac{\pi y}{2\delta}\ , \quad \delta = c x^{1/2}$$

$$\frac{v}{U} = \frac{1}{\pi}\frac{\delta}{x}\left[\cos\left(\frac{\pi}{2}\frac{y}{\delta}\right) + \left(\frac{\pi}{2}\frac{y}{\delta}\right)\sin\left(\frac{\pi}{2}\frac{y}{\delta}\right) - 1\right]$$



Find: Expression for a_{xp} and a_{yp}

Plot: a_x and a_y as functions of y/δ for $U = 5$ m/s, $x = 1$ m, $\delta = 1$ mm; determine maximum values at locations at which maxima occur.

Solution:

Basic equations: $$a_{px} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} \quad \text{---- (1)}$$

$$a_{py} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t} \quad \text{---- (2)}$$

(the w terms crossed out by (2), the $\partial/\partial t$ terms by (1))

Assumptions: (1) steady flow
(2) w and $\partial/\partial z = 0$

Let $\eta = \frac{\pi}{2}\frac{y}{\delta}$; $\eta = \eta(x, y)$ $\quad \frac{\partial\eta}{\partial y} = \frac{\pi}{2\delta}$; $\delta = c x^{1/2}$, $\frac{d\delta}{dx} = \frac{1}{2}c x^{-1/2} = \frac{\delta}{2x}$

$$\frac{\partial\eta}{\partial x} = \frac{\partial\eta}{\partial\delta}\frac{d\delta}{dx} = \frac{\pi y}{2}\left(-\frac{1}{\delta^2}\right)\frac{\delta}{2x} = -\frac{\pi}{4x}\left(\frac{y}{\delta}\right)$$

$u = U\sin\eta$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial\eta}\frac{\partial\eta}{\partial x} = U\cos\eta\left(-\frac{\pi}{4x}\frac{y}{\delta}\right) = -\frac{U}{2x}\left(\frac{\pi}{2}\frac{y}{\delta}\right)\cos\eta = -\frac{U}{2x}\eta\cos\eta \quad \text{---- (3)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial\eta}\frac{\partial\eta}{\partial y} = U\cos\eta\frac{\pi}{2\delta} = \frac{U\pi}{2\delta}\cos\eta \quad \text{---- (4)}$$

$v = U\frac{1}{\pi}\frac{\delta}{x}(\cos\eta + \eta\sin\eta - 1)$. Differentiating using product rule

$$\frac{\partial v}{\partial x} = \frac{U}{\pi}\left(\frac{1}{x}\frac{d\delta}{dx} - \frac{\delta}{x^2}\right)(\cos\eta + \eta\sin\eta - 1) + \frac{U}{\pi}\frac{\delta}{x}(-\sin\eta + \eta\cos\eta + \sin\eta)\frac{\partial\eta}{\partial x}$$

$$= \frac{U}{\pi}\left(\frac{1}{x}\frac{\delta}{2x} - \frac{\delta}{x^2}\right)(\cos\eta + \eta\sin\eta - 1) + \frac{U}{\pi}\frac{\delta}{x}\eta\cos\eta\left(-\frac{\pi}{4x}\right)\frac{y}{\delta}$$

$$\frac{\partial v}{\partial x} = -\frac{U}{\pi}\frac{\delta}{2x^2}(\cos\eta + \eta\sin\eta - 1) - \frac{U\delta}{4x^2}\left(\frac{y}{\delta}\right)\eta\cos\eta \quad \text{---- (5)}$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial\eta}\frac{\partial\eta}{\partial y} = \frac{U}{\pi}\frac{\delta}{x}(-\sin\eta + \eta\cos\eta + \sin\eta)\frac{\pi}{2\delta} = \frac{U}{2x}\eta\cos\eta \quad \text{---- (6)}$$

Substituting into Eq. 1,

$$a_x = U\sin\eta\left(-\frac{U}{2x}\eta\cos\eta\right) + \frac{U}{\pi}\frac{\delta}{x}(\cos\eta - \eta\sin\eta - 1)\frac{U\pi}{2\delta}\cos\eta$$

$$a_x = -\frac{U^2}{2x}\eta\sin\eta\cos\eta + \frac{U^2}{2x}(\cos\eta - \eta\sin\eta - 1)\cos\eta$$

$$a_x = \frac{U^2}{2x}\cos\eta\left[\cos\eta - \eta\sin\eta - 1 - \eta\sin\eta\right]$$

$$a_x = \frac{U^2}{2x}\cos\eta\,(\cos\eta - 1) = -\frac{U^2}{2x}\cos\eta\,(1-\cos\eta) \qquad a_x$$

Substituting into Eq. 2

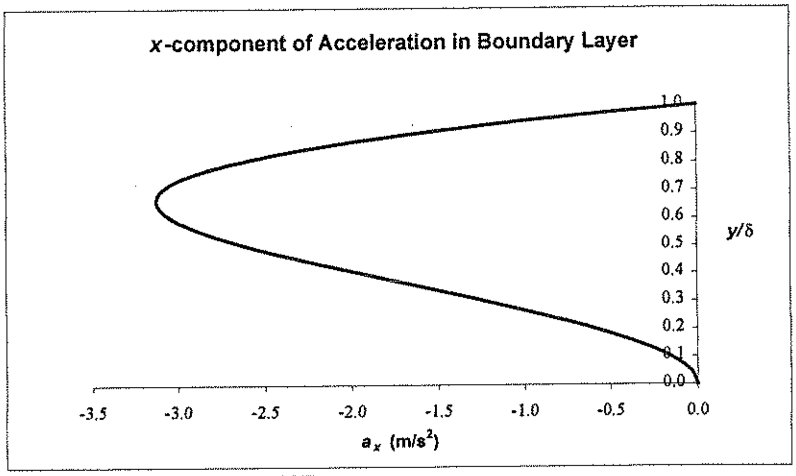
$$a_y = U\sin\eta\left[-\frac{U}{\pi}\frac{\delta}{2x^2}(\cos\eta+\eta\sin\eta-1) - \frac{U\delta}{4x^2}\left(\frac{y}{\delta}\right)\eta\cos\eta\right] + \frac{U}{\pi}\frac{\delta}{x}(\cos\eta+\eta\sin\eta-1)\frac{U}{2x}\eta\cos\eta$$

$$a_y = \frac{U^2\delta}{\pi 2x^2}\left\{\left[-\sin\eta(\cos\eta+\eta\sin\eta-1) - \frac{\pi}{2}\left(\frac{y}{\delta}\right)\eta\cos\eta\sin\eta\right] + \eta\cos\eta(\cos\eta+\eta\sin\eta-1)\right\}$$

$$a_y = \frac{U^2\delta}{\pi 2x^2}\left\{-\sin\eta(\cos\eta+\eta\sin\eta-1) - \frac{\pi}{2}\left(\frac{y}{\delta}\right)\eta\cos\eta\sin\eta + \eta\cos\eta(\cos\eta+\eta\sin\eta-1)\right\} \qquad a_y$$

x component

y/δ	η	a_x (m/s²)
0.00	0.000	0.000
0.05	0.0785	-0.0384
0.10	0.157	-0.152
0.15	0.236	-0.336
0.20	0.314	-0.582
0.25	0.393	-0.879
0.30	0.471	-1.21
0.35	0.550	-1.57
0.40	0.628	-1.93
0.45	0.707	-2.28
0.50	0.785	-2.59
0.55	0.864	-2.85
0.60	0.942	-3.03
0.65	1.02	-3.12
0.70	1.10	-3.10
0.75	1.18	-2.95
0.80	1.26	-2.67
0.85	1.34	-2.24
0.90	1.41	-1.65
0.95	1.49	-0.904
1.00	1.57	0.000

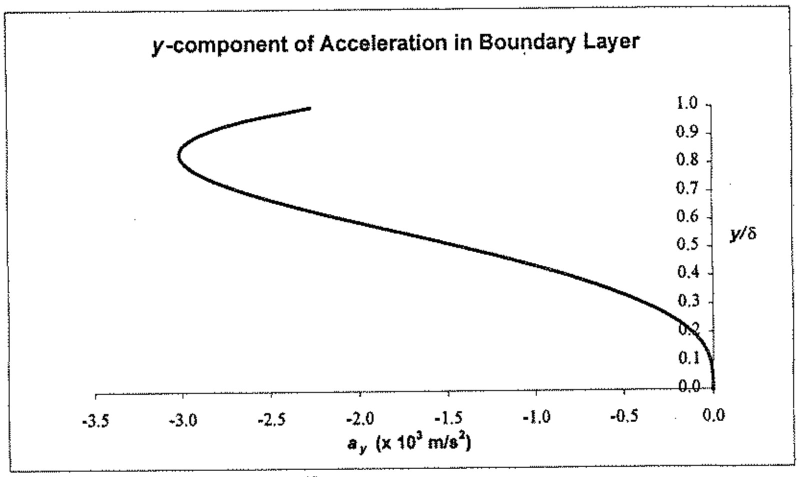


y/δ	η	a_x (m/s²)
0.667	1.05	-3.12

(Maximum absolute value using *Solver*)

y component

y/δ	η	a_y (x 10³ m/s²)
0.00	0.000	0.0000
0.05	0.0785	-0.00192
0.10	0.157	-0.0152
0.15	0.236	-0.0506
0.20	0.314	-0.117
0.25	0.393	-0.223
0.30	0.471	-0.372
0.35	0.550	-0.566
0.40	0.628	-0.803
0.45	0.707	-1.08
0.50	0.785	-1.39
0.55	0.864	-1.71
0.60	0.942	-2.04
0.65	1.02	-2.35
0.70	1.10	-2.62
0.75	1.18	-2.84
0.80	1.26	-2.98
0.85	1.34	-3.01
0.90	1.41	-2.91
0.95	1.49	-2.67
1.00	1.57	-2.27



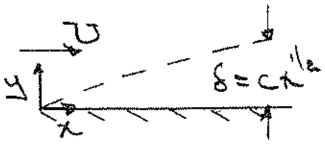
y/δ	η	a_y (x 10³ m/s²)
0.839	1.32	-3.01

(Maximum absolute value using *Solver*)

Note: a_y is normalized with x^2/δ and a_x is normalized with x. Thus $a_y = O\left(\frac{\delta}{x}a_x\right) \approx 0.001\, a_x$.

Given: Laminar boundary layer on a flat plate. (Problem 5.12)

$$\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2 , \quad \frac{v}{U} = \frac{\delta}{x}\left[\frac{1}{2}\left(\frac{y}{\delta}\right)^2 - \frac{1}{3}\left(\frac{y}{\delta}\right)^3\right]$$



Find: (a) Expression for a_x

(b) Plot a_x versus y/δ at location $x = 1\,m$, where $\delta = 1\,mm$, for a flow with $U = 5\,m/s$.

(c) Maximum value of a_x at this location.

Solution:

$$a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \qquad u = U\left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right] = U\left[2\eta - \eta^2\right] \text{ where } \eta = y/\delta$$

$$\frac{\partial u}{\partial x} = \frac{du}{d\eta}\frac{d\eta}{dx} = U\left[2 - 2\eta\right]\left(-\frac{y}{\delta^2}\right)\frac{d\delta}{dx} \qquad \frac{d\delta}{dx} = \frac{1}{2}c x^{-1/2}$$

$$\frac{\partial u}{\partial x} = U\left[2 - 2\eta\right]\left(-\frac{\eta}{\delta}\right)\frac{1}{2}c x^{-1/2} = U\left[2 - 2\eta\right]\left(-\frac{\eta}{c x^{1/2}}\right)\frac{1}{2}c x^{-1/2}$$

$$\frac{\partial u}{\partial x} = -U\left[2 - 2\eta\right]\frac{\eta}{2x} = -\frac{U(\eta - \eta^2)}{x} \longleftarrow$$

$$\frac{\partial u}{\partial y} = U\left[\frac{2}{\delta} - \frac{2y}{\delta^2}\right] = \frac{2U}{y}\left[\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2\right] = \frac{2U}{y}(\eta - \eta^2) \longleftarrow$$

Substituting into the expression for $a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}$

$$a_x = U\left[2\eta - \eta^2\right]U\frac{(\eta^2 - \eta)}{x} + \frac{U\delta}{x}\left[\frac{1}{2}\eta^2 - \frac{1}{3}\eta^3\right]\frac{2U}{y}(\eta - \eta^2)$$

$$= \frac{U^2}{x}\left(-2\eta^2 + 3\eta^3 - \eta^4\right) + \frac{U^2\delta}{xy}\left(\eta^3 - \frac{5}{3}\eta^4 + \frac{2}{3}\eta^5\right)$$

$$= \frac{U^2}{x}\left(-2\eta^2 + 3\eta^3 - \eta^4\right) + \frac{U^2}{x}\left(\eta^2 - \frac{5}{3}\eta^3 + \frac{2}{3}\eta^4\right)$$

$$a_x = \frac{U^2}{x}\left(-\eta^2 + \frac{4}{3}\eta^3 - \frac{1}{3}\eta^4\right) = -\frac{U^2}{x}\left[\left(\frac{y}{\delta}\right)^2 - \frac{4}{3}\left(\frac{y}{\delta}\right)^3 + \frac{1}{3}\left(\frac{y}{\delta}\right)^4\right] \longleftarrow a_x$$

To find value of $\eta (= y/\delta)$ for which a_x is a maximum, set

$$\frac{da_x}{d\eta} = 0 = \frac{U^2}{x}\left(-2\eta + 4\eta^2 - \frac{4}{3}\eta^3\right) = \frac{U^2}{x}\eta\left(-2 + 4\eta - \frac{4}{3}\eta^2\right)$$

At $\eta = 0$, $y/\delta = 0$ and $a_x = 0$

For $\left(-2 + 4\eta - \frac{4}{3}\eta^2\right) = 0$ or $\eta^2 - 3\eta + \frac{3}{2} = 0$

$$\eta = \frac{\pm 3 + \sqrt{(3)^2 - 4(1)3/2}}{2} = \frac{3 \pm \sqrt{3}}{2}$$

Choose $0 < \eta < 1$ (within $0 \le y \le \delta$) $\therefore \eta = 0.634 \longleftarrow y/\delta$

At $\eta = 0.634$

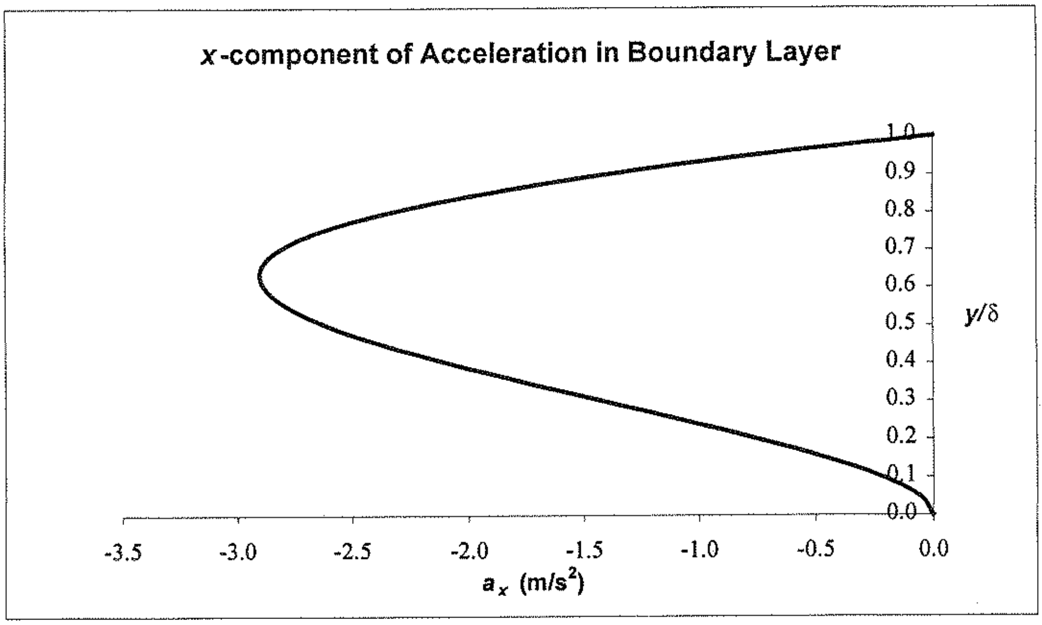
$$a_x = (5.0)^2\frac{m^2}{s^2} \times \frac{1}{1m}\left[-(0.634)^2 + \frac{4}{3}(0.634)^3 - \frac{1}{3}(0.634)^4\right] = -2.90\ m/s^2 \longleftarrow a_x$$

$$a_x = -\frac{U^2}{\delta}\left[\left(\frac{y}{\delta}\right)^2 - \frac{4}{3}\left(\frac{y}{\delta}\right)^3 + \frac{1}{3}\left(\frac{y}{\delta}\right)^4\right]$$

$$a_x = -25\left[\eta^2 - \frac{4}{3}\eta^3 + \frac{1}{3}\eta^4\right] \text{ m/s}^2 \quad \text{where } \eta = y/\delta$$

x component a_x

y/δ	a_x (m/s^2)
0.00	0.000
0.05	-0.058
0.10	-0.218
0.15	-0.454
0.20	-0.747
0.25	-1.074
0.30	-1.418
0.35	-1.758
0.40	-2.080
0.45	-2.367
0.50	-2.604
0.55	-2.779
0.60	-2.880
0.65	-2.896
0.70	-2.818
0.75	-2.637
0.80	-2.347
0.85	-1.942
0.90	-1.418
0.95	-0.771
1.00	0.000

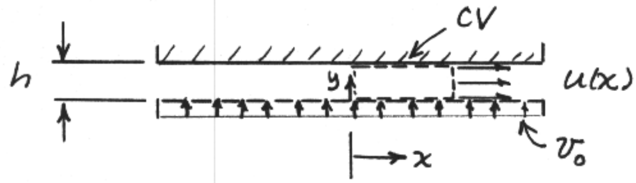


y/δ	a_x (m/s^2)
0.634	-2.90

(Maximum absolute value using *Solver*)

Problem 5.59

Given: Air flow through porous surface into narrow gap.



Find: (a) Show $u(x) = v_0 x/h$
(b) Component, v
(c) Acceleration of a fluid particle in the gap.

Solution: Apply conservation of mass to CV shown.

Basic equation: $0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$ (first term $= 0$ (1))

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Uniform flow at each section

Then

$$0 = \{-x w v_0\} + \{h w u(x)\} \quad \text{or} \quad u(x) = v_0 \frac{x}{h}$$ u(x)

Apply differential form to find v:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \frac{\partial u}{\partial x} = \frac{v_0}{h}$$

$$v - v_0 = \int_0^y \frac{\partial v}{\partial y} dy + f(x) = \int_0^y -\frac{v_0}{h} dy + f(x) = -\frac{v_0 y}{h} + f(x)$$

or

$$v = v_0\left(1 - \frac{y}{h}\right) \quad [f(x) = 0 \text{ since } v = v_0 = \text{const along } y = 0]$$ v(y)

$$a_{px} = \frac{Du}{Dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$$
($v\frac{\partial u}{\partial y} = 0$ (3), $w = 0$, $\frac{\partial u}{\partial t} = 0$ (1))

$$a_{px} = \left(v_0 \frac{x}{h}\right)\left(\frac{v_0}{h}\right) = \frac{v_0^2 x}{h^2}$$ a_{px}

$$a_{py} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t}$$

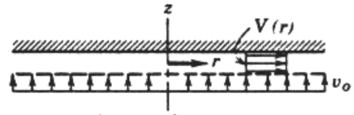
$$a_{py} = v_0\left(1 - \frac{y}{h}\right)\left(-\frac{v_0}{h}\right) = \frac{v_0^2}{h}\left(\frac{y}{h} - 1\right)$$ a_{py}

Thus

$$\vec{a}_p = a_{px}\hat{i} + a_{py}\hat{j} = \frac{v_0^2 x}{h^2}\hat{i} + \frac{v_0^2}{h}\left(\frac{y}{h} - 1\right)\hat{j}$$ $\vec{a}_p$

Problem 5.60

Given: Flow between parallel disks through porous surface.

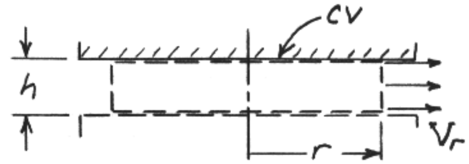


Find: (a) Show $V_r = v_0 r/2h$
(b) V_z, if $v_0 \ll V_r$
(c) Components of acceleration for a fluid particle in the gap.

Solution: Apply CV form of continuity to finite CV shown.

Basic equation:

$$0 = \cancelto{=0(1)}{\frac{\partial}{\partial t}\int_{CV} \rho \, d\forall} + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$



Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Uniform flow at each section

Then

$$0 = \{-|\rho v_0 \pi r^2|\} + \{+|\rho V_r 2\pi r h|\} \quad \text{or} \quad V_r = \frac{v_0 r}{2h}$$ V_r

Apply differential form of conservation of mass for incompressible flow.

Basic equation:

$$\frac{1}{r}\frac{\partial}{\partial r}(r V_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\cancelto{=0(1)}{V_\theta}) + \frac{\partial}{\partial z} V_z = 0$$

Assumptions: (4) $V_\theta = 0$ by symmetry
(5) $V_r = v_0 r/2h$ from above

Then

$$\frac{\partial V_z}{\partial z} = -\frac{1}{r}\frac{\partial}{\partial r}(r V_r) = -\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{v_0 r^2}{2h}\right) = -\frac{1}{r}\left(\frac{v_0 r}{h}\right) = -\frac{v_0}{h}$$

Integrating,

$$V_z = -\frac{v_0 z}{h} + f(r)$$

Boundary conditions are $V_z = v_0$ at $z = 0$, $V_z = 0$ at $z = h$

Thus from first BC, $f(r) = v_0$ = constant, so

$$V_z = v_0\left(1 - \frac{z}{h}\right)$$ V_z

The r component of acceleration is

$$a_r = V_r \frac{\partial V_r}{\partial r} + \cancel{\frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta}} + \cancel{V_z \frac{\partial V_r}{\partial z}} + \cancel{\frac{\partial V_r}{\partial t}} = \left(\frac{v_0 r}{2h}\right)\left(\frac{v_0}{2h}\right) = \left(\frac{v_0}{2h}\right)^2 r$$ a_r

The z component is

$$a_z = \cancel{V_r \frac{\partial V_z}{\partial r}} + \cancel{\frac{V_\theta}{r}\frac{\partial V_z}{\partial \theta}} + V_z \frac{\partial V_z}{\partial z} + \cancel{\frac{\partial V_z}{\partial t}} = v_0\left(1 - \frac{z}{h}\right)\left(-\frac{v_0}{h}\right) = \frac{v_0^2}{h}\left(\frac{z}{h} - 1\right)$$ a_z

Given: Steady, inviscid flow over a circular cylinder of radius R.

$$\vec{V} = U\cos\theta\left[1-\left(\frac{R}{r}\right)^2\right]\hat{e}_r - U\sin\theta\left[1+\left(\frac{R}{r}\right)^2\right]\hat{e}_\theta$$

Find: (a) Expression for acceleration of particle moving along $\theta=\pi$
(b) Expression for acceleration of particle moving along $r=R$
(c) Locations at which accelerations a_r and a_θ reach maximum and minimum values.

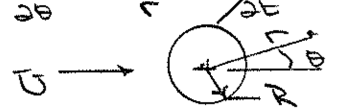
Plot: a_r as a function of R/r for $\theta=\pi$ and as a function of θ for $r=R$; plot a_θ as a function of θ for $r=R$

Solution:

Basic equations: $a_r = V_r\frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} - \frac{V_\theta^2}{r} + \cancel{\frac{\partial V_r}{\partial t}}$ $=0(1)$

$a_\theta = V_r\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{V_r V_\theta}{r} + \cancel{\frac{\partial V_\theta}{\partial t}}$ $=0(1)$

Assumptions: (1) steady flow



Along $\theta=\pi$

· $\cos\theta=-1$, $\sin\theta=0$, so $V_\theta=0$ and $V_r=-U\left[1-\left(\frac{R}{r}\right)^2\right]$

Then

$$a_r = V_r\frac{\partial V_r}{\partial r} = -U\left[1-\left(\frac{R}{r}\right)^2\right](-U)(-2)\left(-\frac{R^2}{r^3}\right) = \frac{2U^2}{R}\left[1-\left(\frac{R}{r}\right)^2\right]\left(\frac{R}{r}\right)^3 \quad a_r$$

$$a_\theta = 0 \quad a_\theta$$

$\}\ \theta=\pi$

To determine location of maximum a_r, let $\frac{R}{r}=\eta$ and evaluate $\frac{da_r}{d\eta}$

$$a_r = \frac{2U^2}{R}\left[1-\eta^2\right]\eta^3 = \frac{2U^2}{R}\left[\eta^3-\eta^5\right]$$

$\frac{da_r}{d\eta} = \frac{2U^2}{R}\left[3\eta^2-5\eta^4\right]$. Thus $\frac{da_r}{d\eta}=0$ at $\eta^2=\frac{3}{5}$ or $\eta=0.775$

Thus, $a_{r\,max}$ occurs at $r = R/0.775 = 1.29R$ $\quad r_{a_{max}}$

$$a_{r\,max} = \frac{2U^2}{R}(0.775)^3\left[1-(0.775)^2\right] = 0.372\,\frac{U^2}{R} \text{ @ } r=1.29R.$$

Since $a_\theta=0$, $\vec{a}_{max} = a_{r\,max}\hat{e}_r = 0.372\frac{U^2}{R}\hat{e}_r$ @ $r=1.29R$

Along $r=R$

· $r=R$, $V_r=0$ and $V_\theta=-2U\sin\theta$

$$a_r = -\frac{V_\theta^2}{r} = -\frac{(-2U\sin\theta)^2}{R} = -\frac{4U^2}{R}\sin^2\theta \quad a_r$$

$$a_\theta = \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} = \left(-\frac{2U\sin\theta}{R}\right)\left(-\frac{2U\cos\theta}{R}\right) = \frac{4U^2}{R^2}\sin\theta\cos\theta \quad a_\theta$$

$\}\ r=R$

a_r has maximum negative value at $\theta=\pm\pi/2$
has minimum value (of zero) at $\theta=0,\pi$

a_θ has maximum values at $\theta=\pm\pi/4,\ 3\pi/4$
has minimum values at $\theta=0,\ \pm\pi/2,\ \pi$

The acceleration magnitude is

$$|\vec{a}| = \left[a_r^2 + a_\theta^2\right]^{1/2} = \left[\left(\frac{-4U^2}{R}\right)^2 \sin^4\theta + \left(\frac{4U^2}{R}\right)^2 \sin^2\theta\cos^2\theta\right]^{1/2} = \frac{4U^2}{R}\sin\theta$$

. This is a maximum at $\theta = \pm \pi/2$.

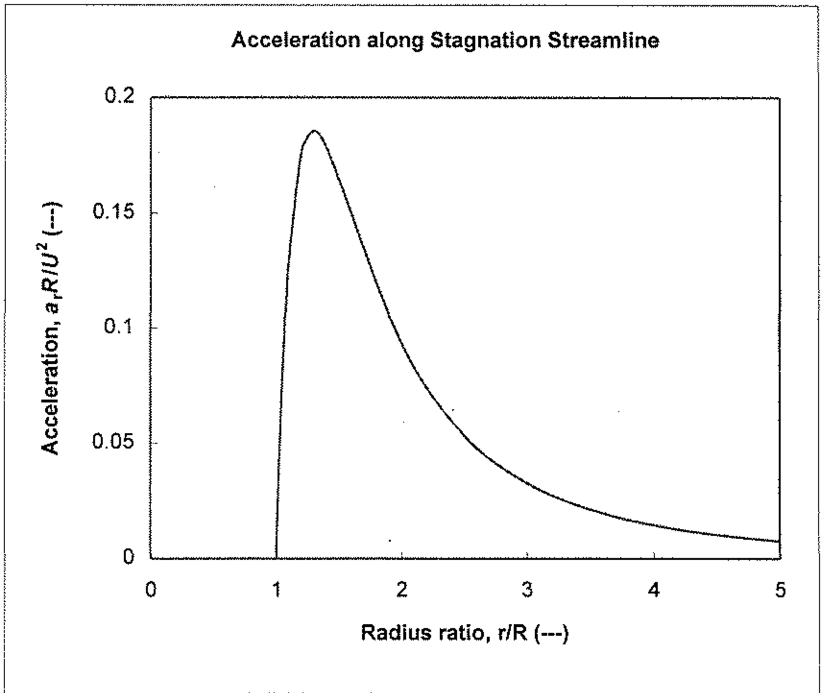
Thus $\vec{a}_{max} = \pm 4\frac{U^2}{R}$ at $\theta = \pm \pi/2$.

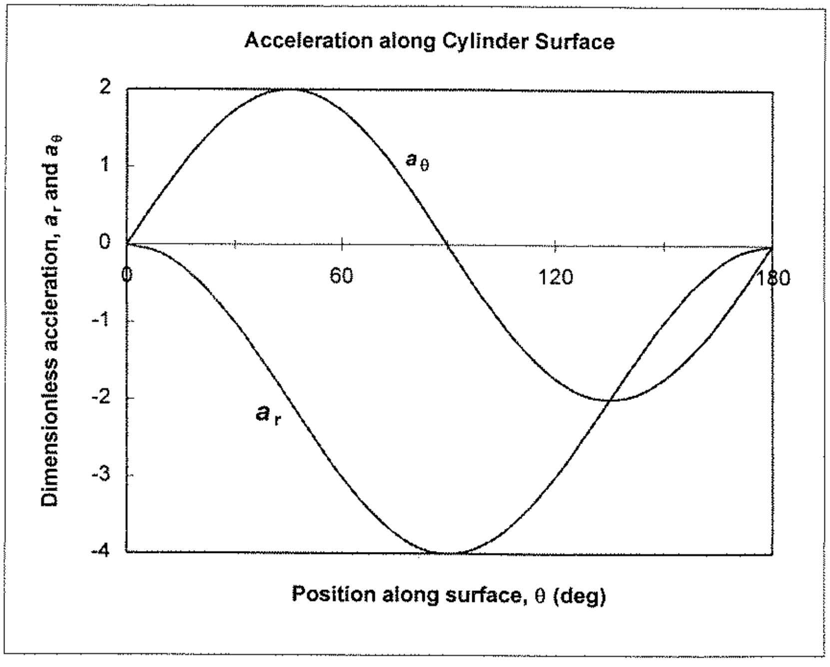
Plots:

(1) $\theta = \pi$ $\quad a_r = \frac{2U^2}{R}\left(\frac{R}{r}\right)^3\left[1-\left(\frac{R}{r}\right)^2\right]$; $\quad \frac{a_r}{U^2/R} = \left(\frac{R}{r}\right)^3\left[1-\left(\frac{R}{r}\right)^2\right]$

(2) $r = R$. $\quad a_r = -4\frac{U^2}{R}\sin^2\theta$; $\quad \frac{a_r}{U^2/R} = -4\sin^2\theta$

$a_\theta = 4\frac{U^2}{R}\sin\theta\cos\theta$; $\quad \frac{a_\theta}{U^2/R} = 4\sin\theta\cos\theta$



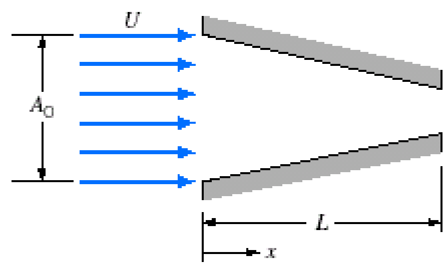


Problem 5.62

Consider the incompressible flow of a fluid through a nozzle as shown. The area of the nozzle is given by $A = A_0(1 - bx)$ and the inlet velocity varies according to $U = U_0(1 - e^{-\lambda t})$, where $A_0 = 0.5\ \text{m}^2$, $L = 5\ \text{m}$, $b = 0.1\ \text{m}^{-1}$, $\lambda = 0.2\ \text{s}^{-1}$, and $U_0 = 5\ \text{m/s}$. Find and plot the acceleration on the centerline, with time as a parameter.

Given: Velocity field and nozzle geometry

Find: Acceleration along centerline; plot



Solution

The given data is $A_0 = 0.5\cdot\text{m}^2$ $\quad L = 5\cdot\text{m}$ $\quad b = 0.1\cdot\text{m}^{-1}$ $\quad \lambda = 0.2\cdot\text{s}^{-1}$ $\quad U_0 = 5\cdot\frac{\text{m}}{\text{s}}$

$$A(x) = A_0\cdot(1 - b\cdot x)$$

The velocity on the centerline is obtained from continuity

$$u(x)\cdot A(x) = U_0\cdot A_o$$

so $$u(x,t) = \frac{A_0}{A(x)}\cdot U_0\cdot\left(1 - e^{-\lambda\cdot t}\right) = \frac{U_0}{(1 - b\cdot x)}\cdot\left(1 - e^{-\lambda\cdot t}\right)$$

The acceleration is given by

$$\vec{a}_p = \underbrace{\frac{D\vec{V}}{Dt}}_{\text{total acceleration of a particle}} = \underbrace{u\frac{\partial\vec{V}}{\partial x} + v\frac{\partial\vec{V}}{\partial y} + w\frac{\partial\vec{V}}{\partial z}}_{\text{convective acceleration}} + \underbrace{\frac{\partial\vec{V}}{\partial t}}_{\text{local acceleration}}$$

For the present 1D flow

$$a_x = \frac{\partial}{\partial t}u + u\cdot\frac{\partial}{\partial x}u = \frac{\lambda\cdot U_0}{(1-b\cdot x)}\cdot e^{-\lambda\cdot t} + \frac{U_0}{(1-b\cdot x)}\cdot\left(1\cdot -e^{-\lambda\cdot t}\right)\cdot\left[\frac{b\cdot U_0}{(1-b\cdot x)^2}\cdot\left(1\cdot -e^{-\lambda\cdot t}\right)\right]$$

$$a_x = \frac{U_0}{(1-b\cdot x)}\cdot\left[\lambda\cdot e^{-\lambda\cdot t} + \frac{b\cdot U_0}{(1-b\cdot x)^2}\cdot\left(1-e^{-\lambda\cdot t}\right)^2\right]$$

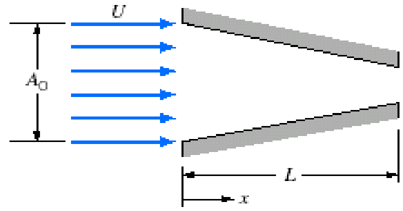
The plot is shown in the corresponding *Excel* workbook

Problem 5.62 (In Excel)

Consider the incompressible flow of a fluid through a nozzle as shown. The area of the nozzle is given by $A = A_0(1 - bx)$ and the inlet velocity varies according to $U = U_0(1 - e^{-\lambda t})$, where $A_0 = 0.5\ \text{m}^2$, $L = 5$ m, $b = 0.1\ \text{m}^{-1}$, $\lambda = 0.2\ \text{s}^{-1}$, and $U_0 = 5$ m/s. Find and plot the acceleration on the centerline, with time as a parameter.

Given: Velocity field and nozzle geometry

Find: Acceleration along centerline; plot



Given data:

$A_0 =$ 0.5 m^2

$L =$ 5 m

$b =$ 0.1 m^{-1}

$\lambda =$ 0.2 s^{-1}

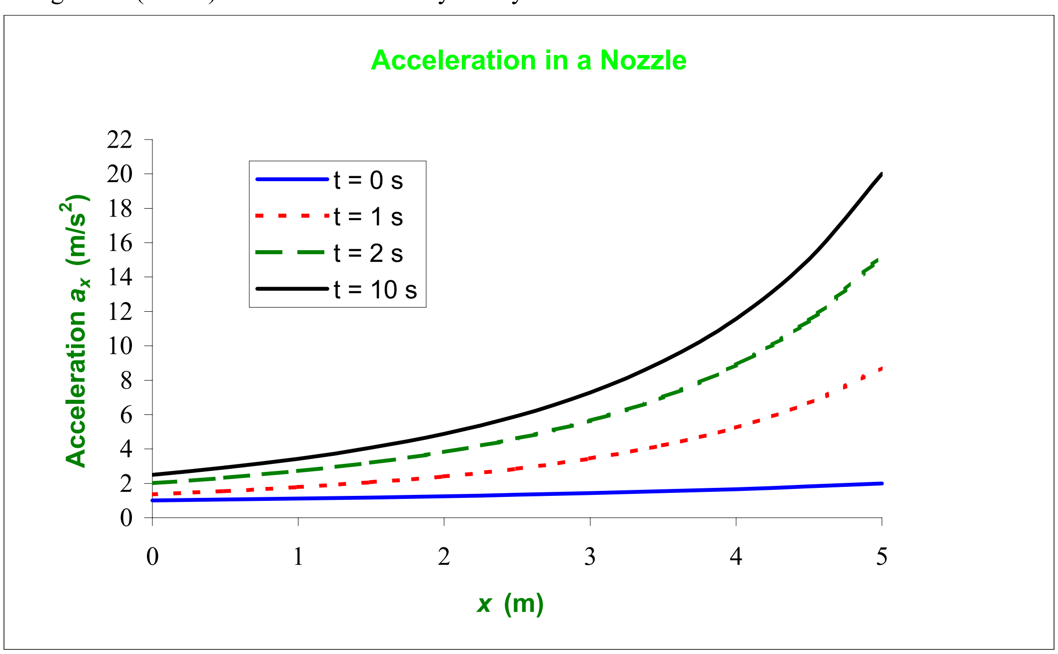
$U_0 =$ 5 m/s

The acceleration is

$$a_x = \frac{U_0}{(1 - b \cdot x)} \cdot \left[\lambda \cdot e^{-\lambda \cdot t} + \frac{b \cdot U_0}{(1 - b \cdot x)^2} \cdot \left(1 - e^{-\lambda \cdot t}\right)^2 \right]$$

***t* =**	**0**	**5**	**10**	**60**
***x* (m)**	***a*$_x$ (m/s^2)**	***a*$_x$ (m/s^2)**	***a*$_x$ (m/s^2)**	***a*$_x$ (m/s^2)**
0.0	1.00	1.367	2.004	2.50
0.5	1.05	1.552	2.32	2.92
1.0	1.11	1.78	2.71	3.43
1.5	1.18	2.06	3.20	4.07
2.0	1.25	2.41	3.82	4.88
2.5	1.33	2.86	4.61	5.93
3.0	1.43	3.44	5.64	7.29
3.5	1.54	4.20	7.01	9.10
4.0	1.67	5.24	8.88	11.57
4.5	1.82	6.67	11.48	15.03
5.0	2.00	8.73	15.22	20.00

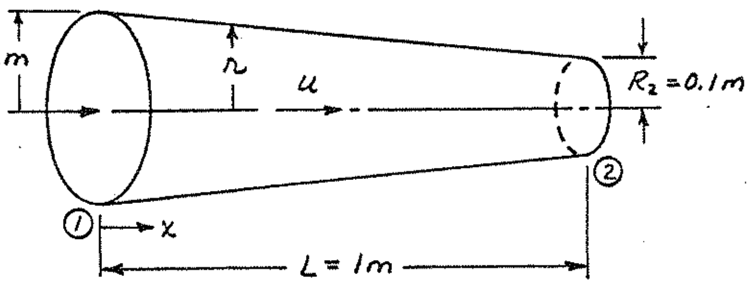
For large time (> 30 s) the flow is essentially steady-state



Given: One-dimensional, incompressible flow through circular channel.

$U = U_0 + U_1 \sin(\omega t)$

$U_0 = 20\,m/s$
$U_1 = 2\,m/s$
$\omega = 0.3\,rad/s$



Find: (a) The acceleration of a particle at the channel exit.
(b) Plot as a function of time for a complete cycle.
(c) On same plot, show acceleration if channel is constant area; explain

Solution: The acceleration of a particle in one-dimensional flow is

$$a_x = u\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t}$$

From continuity, $u = U\frac{A_1}{A} = U\,\frac{R_1^2}{r^2}$

From geometry, $r = R_1 - (R_1 - R_2)\frac{x}{L} = R_1 - \Delta R\frac{x}{L}$, so

$$u = U\frac{R_1^2}{(R_1 - \Delta R\frac{x}{L})^2} = \left[U_0 + U_1\sin(\omega t)\right]\frac{1}{\left[1 - \frac{\Delta R}{R_1}(\frac{x}{L})\right]^2}$$

Thus

$$a_x = \left[U_0 + U_1\sin(\omega t)\right]\frac{1}{\left[1-\frac{\Delta R}{R_1}(\frac{x}{L})\right]^2}\left[U_0 + U_1\sin(\omega t)\right](-2)(-\frac{\Delta R}{R_1 L})\frac{1}{\left[1-\frac{\Delta R}{R_1}(\frac{x}{L})\right]^3}$$

$$+ \frac{\omega U_1\cos(\omega t)}{\left[1-\frac{\Delta R}{R_1}(\frac{x}{L})\right]^2}$$

$$a_x = \frac{2\Delta R}{R_1 L}\frac{\left[U_0 + U_1\sin(\omega t)\right]^2}{\left[1-\frac{\Delta R}{R_1}(\frac{x}{L})\right]^5} + \frac{\omega U_1\cos(\omega t)}{\left[1-\frac{\Delta R}{R_1}(\frac{x}{L})\right]^2}$$

At $x/L = 1$, $\left[1 - \frac{\Delta R}{R_1}(\frac{x}{L})\right] = 1 - \frac{0.1\,m}{0.2\,m} = 0.5$, so

$$a_x = 2 \times 0.1\,m \times \frac{1}{0.2\,m}\times\frac{1}{1\,m}\left[20 + 2\sin(\omega t)\right]^2\frac{m^2}{s^2}\times\frac{1}{(0.5)^5} + 0.3\frac{rad}{s}\times\frac{2\,m}{s}\times\cos(\omega t)\times\frac{1}{(0.5)^2}$$

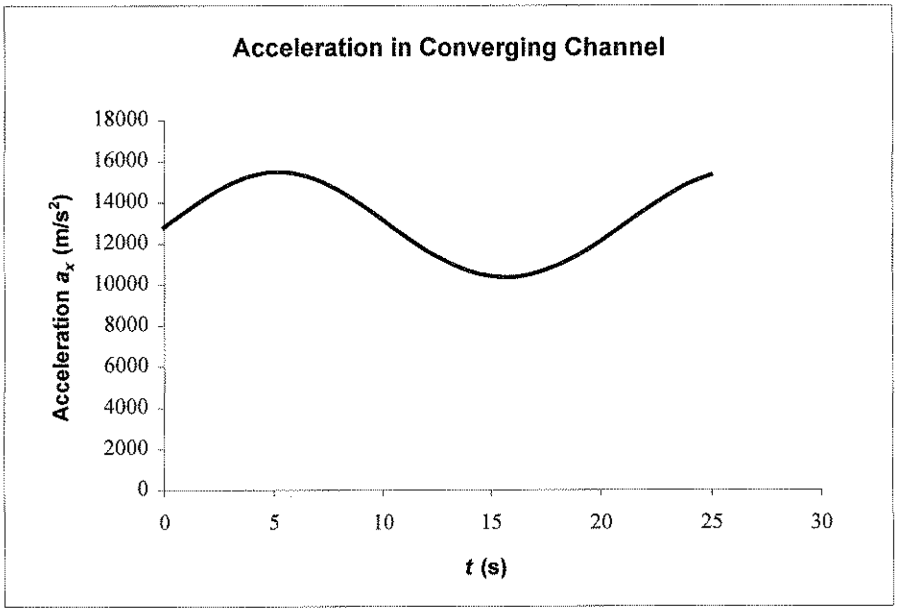
or

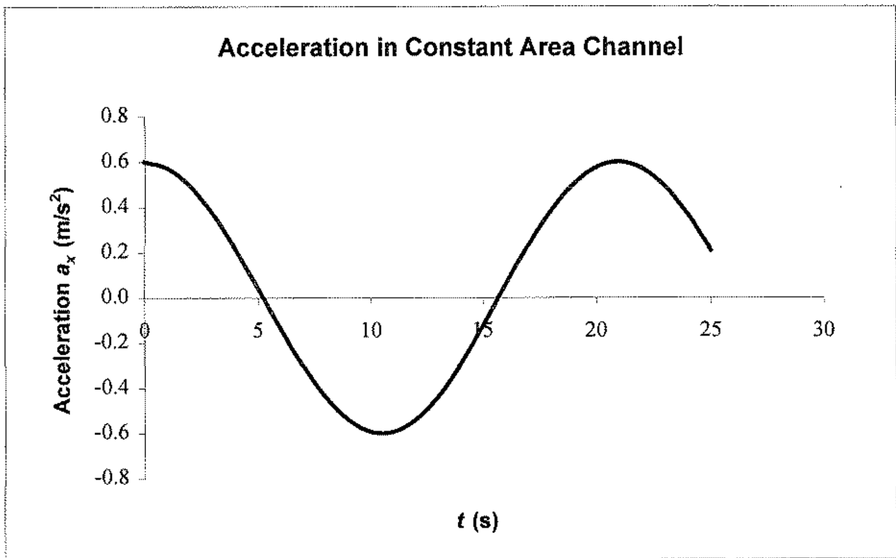
$$a_x\,(m/sec^2) = 32\left[20 + 2\sin(\omega t)\right]^2 + 2.4\cos(\omega t) \quad (\text{at } x = L)$$ a_x

(See next page for plots)

The acceleration in the channel and in a constant area are calculated and plotted below

t (s)	a_x (m/s^2) (Convergent)	a_x (m/s^2) (A = const.)
0	12802	0.600
1	13570	0.573
2	14288	0.495
3	14885	0.373
4	15298	0.217
5	15481	0.042
6	15414	-0.136
7	15104	-0.303
8	14586	-0.442
9	13915	-0.542
10	13161	-0.594
11	12397	-0.592
12	11690	-0.538
13	11098	-0.436
14	10665	-0.294
15	10419	-0.126
16	10377	0.052
17	10541	0.227
18	10900	0.381
19	11431	0.501
20	12097	0.576
21	12845	0.600
22	13612	0.570
23	14326	0.489
24	14914	0.365
25	15315	0.208





The acceleration in the convergent channel is massively larger than that in the constant area channel because very large convective acceleration is generated by the convergence (the constant area channel only has local acceleration)

Problem 5.64

Given: Steady, two-dimensional velocity field of Problem 5.47,

$$\vec{V} = Ax\hat{i} - Ay\hat{j} \quad ; \quad A = 1\,s^{-1}$$

Find: (a) Expressions for particle coordinates, $x_p = f_1(t)$ and $y_p = f_2(t)$.
(b) Time required for particle to travel from $(x_0, y_0) = (\frac{1}{2}, 2)$ to $(x, y) = (1, 1)$ and $(2, \frac{1}{2})$.
(c) Compare acceleration determined from $f_1(t)$ and $f_2(t)$ with those found in Problem 5.49.

Solution: For the given flow, $u = Ax$ and $v = -Ay$. Thus

$$u_p = \frac{df_1}{dt} = Ax_p = Af_1, \text{ or } \frac{df_1}{f_1} = A\,dt$$

Integrating from x_0 to f_1,

$$\int_{x_0}^{f_1} \frac{df_1}{f_1} = \ln f_1 \Big]_{x_0}^{f_1} = \ln\left(\frac{f_1}{x_0}\right) = At, \text{ or } f_1 = x_0 e^{At}$$

$f_1(t)$

Likewise $$v_p = \frac{df_2}{dt} = -Ay_p = -Af_2, \text{ or } \frac{df_2}{f_2} = -A\,dt$$

Integrating from y_0 to f_2,

$$\int_{y_0}^{f_2} \frac{df_2}{f_2} = \ln f_2 \Big]_{y_0}^{f_2} = \ln\left(\frac{f_2}{y_0}\right) = -At \text{ or } f_2 = y_0 e^{-At}$$

$f_2(t)$

For a particle initially at $(\frac{1}{2}, 2)$, $x_0 = \frac{1}{2}$ and $y_0 = 2$

To reach the point $(x,y) = (1,1)$, $e^{At} = \frac{x}{x_0} = 2$, so $t = \frac{\ln 2}{A} = 0.693$ sec

$e^{-At} = \frac{y}{y_0} = \frac{1}{2}$, so $t = \frac{-\ln\frac{1}{2}}{A} = 0.693$ sec

$t(1,1)$

To reach the point $(x,y) = (2, \frac{1}{2})$, $e^{At} = \frac{x}{x_0} = 4$, so $t = \frac{\ln 4}{A} = 1.39$ sec

$e^{-At} = \frac{y}{y_0} = \frac{1}{4}$, so $t = \frac{-\ln\frac{1}{4}}{A} = 1.39$ sec

$t(2, \frac{1}{2})$

The acceleration components are

$$a_{p_x} = \frac{d^2 f_1}{dt^2} = x_0 A^2 e^{At} = x_0 A^2 \frac{f_1}{x_0} = A^2 f_1$$

$$a_{p_y} = \frac{d^2 f_2}{dt^2} = y_0 A^2 e^{-At} = y_0 A^2 \frac{f_2}{y_0} = A^2 f_2$$

At $(x,y) = (1,1)$

$$\vec{a}_p = a_{p_x}\hat{i} + a_{p_y}\hat{j} = \frac{(1)^2}{s^2} \times 1\,m\,\hat{i} + \frac{(1)^2}{s^2} \times 1\,m\,\hat{j} = (\hat{i} + \hat{j})\frac{m}{s^2}$$

$\vec{a}_p(1,1)$

At $(x,y) = (2, \frac{1}{2})$

$$\vec{a}_p = \frac{(1)^2}{s^2} \times 2m\,\hat{i} + \frac{(1)^2}{s^2} \times \frac{1}{2}m\,\hat{j} = (2\hat{i} + \frac{1}{2}\hat{j})\frac{m}{s^2}$$

$\vec{a}_p(2, \frac{1}{2})$

These are identical to the accelerations found in Problem 5.49.

Problem 5.65

Expand $(\vec{V}\cdot\nabla)\vec{V}$ in cylindrical coordinates to obtain the convective acceleration of a fluid particle. Verify the results given in Eqs. 5.12.

Recall $\left.\dfrac{\partial \hat{e}_r}{\partial\theta}\right. = \hat{e}_\theta$ and $\left.\dfrac{\partial \hat{e}_\theta}{\partial\theta}\right. = -\hat{e}_r$

Solution:

In cylindrical coordinates $\nabla = \hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{k}\frac{\partial}{\partial z}$

$\vec{V} = V_r\hat{e}_r + V_\theta\hat{e}_\theta + V_z\hat{k}$

$$(\vec{V}\cdot\nabla)\vec{V} = \left[V_r\hat{e}_r + V_\theta\hat{e}_\theta + V_z\hat{k}\right]\cdot\left[\hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{k}\frac{\partial}{\partial z}\right](V_r\hat{e}_r + V_\theta\hat{e}_\theta + V_z\hat{k})$$

$$= \left[V_r\frac{\partial}{\partial r} + \frac{V_\theta}{r}\frac{\partial}{\partial\theta} + V_z\frac{\partial}{\partial z}\right](V_r\hat{e}_r + V_\theta\hat{e}_\theta + V_z\hat{k})$$

$$= V_r\frac{\partial}{\partial r}V_r\hat{e}_r + \frac{V_\theta}{r}\frac{\partial}{\partial\theta}V_r\hat{e}_r + V_z\frac{\partial}{\partial z}(V_r\hat{e}_r)$$
$$+ V_r\frac{\partial}{\partial r}V_\theta\hat{e}_\theta + \frac{V_\theta}{r}\frac{\partial}{\partial\theta}V_\theta\hat{e}_\theta + V_z\frac{\partial}{\partial z}V_\theta\hat{e}_\theta$$
$$+ V_r\frac{\partial}{\partial r}V_z\hat{k} + \frac{V_\theta}{r}\frac{\partial}{\partial\theta}V_z\hat{k} + V_z\frac{\partial}{\partial z}V_z\hat{k}$$

$$= \hat{e}_r\left\{V_r\frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial\theta} + V_z\frac{\partial V_r}{\partial z}\right\}① + \frac{V_\theta V_r}{r}\left(\frac{\partial\hat{e}_r}{\partial\theta}\right) \Rightarrow \hat{e}_\theta$$
$$+ \hat{e}_\theta\left\{V_r\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial\theta} + V_z\frac{\partial V_\theta}{\partial z}\right\}② + \frac{V_\theta^2}{r}\left(\frac{\partial\hat{e}_\theta}{\partial\theta}\right) = -\hat{e}_r$$
$$+ \hat{k}\left\{V_r\frac{\partial V_z}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_z}{\partial\theta} + V_z\frac{\partial V_z}{\partial z}\right\}③$$

$$(\vec{V}\cdot\nabla)\vec{V} = \hat{e}_r\left\{V_r\frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial\theta} - \frac{V_\theta^2}{r} + V_z\frac{\partial V_r}{\partial z}\right\}$$
$$+ \hat{e}_\theta\left\{V_r\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial\theta} + \frac{V_\theta V_r}{r} + V_z\frac{\partial V_\theta}{\partial z}\right\}$$
$$+ \hat{k}\left\{V_r\frac{\partial V_z}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_z}{\partial\theta} + V_z\frac{\partial V_z}{\partial z}\right\}$$

Term ① is the r component of convective acceleration

Eq. 5.12a $\quad a_{r_p} = \left\{V_r\frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial\theta} - \frac{V_\theta^2}{r} + V_z\frac{\partial V_r}{\partial z}\right\} + \frac{\partial V_r}{\partial t}$

Term ② is the θ component of convective acceleration

Eq. 5.12b $\quad a_{\theta_p} = \left\{V_r\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial\theta} + \frac{V_r V_\theta}{r} + V_z\frac{\partial V_\theta}{\partial z}\right\} + \frac{\partial V_\theta}{\partial t}$

Term ③ is the z component of convective acceleration

Eq. 5.12c $\quad a_{z_p} = \left\{V_r\frac{\partial V_z}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_z}{\partial\theta} + V_z\frac{\partial V_z}{\partial z}\right\} + \frac{\partial V_z}{\partial t}$

Problem 5.66

Given: Velocity field $\vec{V} = 10x\hat{i} - 10y\hat{j} + 30\hat{k}$

Determine if the field is: (a) Incompressible.
(b) Irrotational.

Solution: Apply continuity and irrotationality condition.

Basic equations: $\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} + \frac{\partial \rho}{\partial t} = 0$ (third term $= 0(1)$, fourth term $= 0(2)$)

$$\nabla \times \vec{V} = 0 \quad (\text{if irrotational})$$

Assumptions: (1) $\vec{V} = \vec{V}(x, y)$, so $\frac{\partial}{\partial z} = 0$

(2) Incompressible flow, so $\rho = \text{constant}$

Then

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 10 - 10 = 0$$

Flow is a possible incompressible flow. ←

$$\nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u & v & w \end{vmatrix} = \hat{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \hat{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \hat{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$\nabla \times \vec{V} = \hat{i}(0 - 0) + \hat{j}(0 - 0) + \hat{k}(0 - 0) = 0$$

Flow is irrotational. ←

Problem 5.67

Which, if any, of the flow fields of Problem 5.2 are irrotational?

Given: Velocity components

(a) $u = -x + y; v = x - y^2$ (b) $u = x + 2y; v = x^2 - y$
(c) $u = 4x^2 - y; v = x - y^2$ (d) $u = xt + 2y; v = x^2 - yt$
(e) $u = xt^2; v = xyt + y^2$

Find: Which flow fields are irrotational

Solution

For a 2D field, the irrotionality the test is $\frac{dv}{dx} - \frac{du}{dy} = 0$

(a) $\frac{dv}{dx} - \frac{du}{dy} = (1) - (1) = 0$ Irrotional

(b) $\frac{dv}{dx} - \frac{du}{dy} = (2 \cdot x) - (2) = 2 \cdot x - 2 \neq 0$ Not irrotional

(c) $\frac{dv}{dx} - \frac{du}{dy} = (1) - (-1) = 2 \neq 0$ Not irrotional

(d) $\frac{dv}{dx} - \frac{du}{dy} = (2 \cdot x) - (2) = 2 \cdot x - 2 \neq 0$ Not irrotional

(e) $\frac{dv}{dx} - \frac{du}{dy} = (y \cdot t) - (0) = y \cdot t \neq 0$ Not irrotional

Problem 5.68

Given: Sinusoidal approximation to boundary-layer velocity profile,

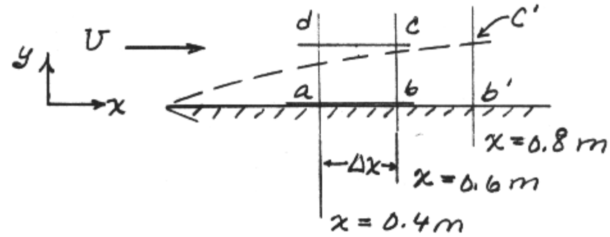
$$u = U \sin\left(\frac{\pi}{2}\frac{y}{\delta}\right) \quad \text{where } \delta = 5\text{ mm at } x = 0.5\text{ m (Problem 5.11)}$$

Neglect vertical component of velocity. $U = 0.5$ m/s.

Find: (a) Circulation about contour bounded by $x = 0.4$ m, $x = 0.6$ m, $y = 0$, and $y = 8$ mm.

(b) Result if evaluated $\Delta x = 0.2$ m further downstream?

Solution: Evaluate circulation



Defining equation:

$$\Gamma = \oint \vec{V} \cdot d\vec{s}$$

From the definition

$$\Gamma = \int_{ab} \underset{u=0}{\vec{V} \cdot d\vec{s}} + \int_{bc} \underset{u \perp dy}{\vec{V} \cdot d\vec{s}} + \int_{cd} \vec{V} \cdot d\vec{s} + \int_{da} \underset{u \perp dy}{\vec{V} \cdot d\vec{s}} = \int_0^{\Delta x} U\hat{i} \cdot dx(-\hat{i})$$

$$\Gamma = -U\Delta x = -5\frac{\text{m}}{\text{sec}} \times 0.2\text{ m} = -0.100\text{ m}^2/\text{sec}$$

Γ

At the downstream location, since $\delta = cx^{1/2}$

$$\delta' = \delta\left(\frac{x}{x'}\right)^{1/2} = 5\text{ mm}\left(\frac{0.8}{0.5}\right)^{1/2} = 6.32\text{ mm}$$

Point c' is also outside the boundary layer. Consequently the integral along c'c will be the same as along cd. Thus

$$\Gamma_{bb'c'c} = \Gamma_{abcd}$$

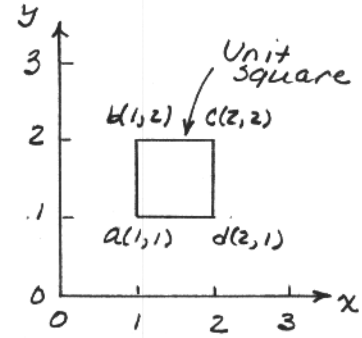
Γ

Problem 5.69

Given: Velocity field for flow in a rectangular "Corner,"

$$\vec{V} = Ax\hat{\imath} - Ay\hat{\jmath} \quad \text{with } A = 0.3\,s^{-1}$$

as in Example Problem 5.8.



Find: Circulation about unit square shown.

Solution: Evaluate circulation

Defining equation:

$$\Gamma = \oint \vec{V} \cdot d\vec{s}$$

The dot product is $\vec{V} \cdot d\vec{s} = (Ax\hat{\imath} - Ay\hat{\jmath}) \cdot (dx\hat{\imath} + dy\hat{\jmath}) = Ax\,dx - Ay\,dy$.

For the contour shown, $dy = 0$ along ad and cb, and $dx = 0$ along ba and dc. Thus

$$\Gamma = \int_a^d Ax\,dx + \int_d^c -Ay\,dy + \int_c^b Ax\,dx + \int_b^a -Ay\,dy$$

$$= \frac{Ax^2}{2}\Big]_{x_a}^{x_d} - \frac{Ay^2}{2}\Big]_{y_d}^{y_c} + \frac{Ax^2}{2}\Big]_{x_c}^{x_b} - \frac{Ay^2}{2}\Big]_{y_b}^{y_a}$$

$$= \frac{A}{2}(x_d^2 - x_a^2 + x_b^2 - x_c^2) - \frac{A}{2}(y_c^2 - y_d^2 + y_a^2 - y_b^2)$$

$\Gamma = 0$ (since $x_a = x_b$ and $x_c = x_d$

$y_a = y_d$ and $y_b = y_c$) Γ

This result is to be expected, since flow is irrotational ($\nabla \times \vec{V} = 0$). From Stokes' Theorem (Eq. 5.18),

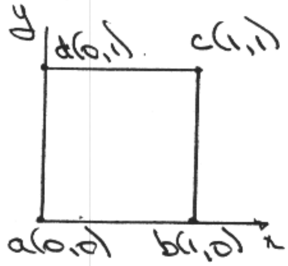
$$\Gamma = \int_A (\nabla \times \vec{V})_z \, dA = 0$$

Problem 5.70

Given: Two dimensional flow field $\vec{V} = Axy\hat{i} + By^2\hat{j}$, where $A = 1\,m^{-1}\cdot s^{-1}$, $B = -\frac{1}{2}\,m^{-1}\cdot s^{-1}$ and coordinates are measured in meters

Show: velocity field represents a possible incompressible flow

Find: (a) Rotation at point $(x,y) = (1,1)$
(b) Circulation about curve shown



Solution:

For incompressible flow $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

For given flow field

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial}{\partial x}(Axy) + \frac{\partial}{\partial y}(By^2) = Ay + 2By = (1)y + 2\left(-\tfrac{1}{2}\right)y = 0$$ ✓

The fluid rotation is defined as $\vec{\omega} = \frac{1}{2}\nabla\times\vec{V}$

$$\vec{\omega} = \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Axy & By^2 & 0 \end{vmatrix} = -\frac{1}{2}Ax\hat{k}$$

$$\vec{\omega}_{1,1} = -\frac{1}{2}\times\frac{1}{m\cdot s}\times 1\,m\,\hat{k} = -0.5\,\hat{k}\ \text{rad/s}$$ $\vec{\omega}_{1,1}$

The circulation is defined as $\Gamma = \oint \vec{V}\cdot d\vec{s}$

For the contour shown with $\vec{V} = Axy\hat{i} + By^2\hat{j}$

$$\Gamma = \int_a^b u\,dx + \int_b^c v\,dy + \int_c^d u(-dx) + \int_d^a v(-dy)$$

(the first term crossed out: $u = 0$ along ab.)

$$\Gamma = \int_0^1 By^2\,dy + \int_1^0 Axy\,dx + \int_1^0 By^2\,dy \qquad \{y = 1 \text{ along } cd\}$$

$$\Gamma = \frac{1}{3}By^3\Big]_0^1 + \frac{1}{2}Ax^2\Big]_1^0 + \frac{1}{3}By^3\Big]_1^0$$

$$\Gamma = \frac{1}{3}B - \frac{1}{2}A - \frac{1}{3}B = -\frac{1}{2}A = -\frac{1}{2}\,m^2/s$$ Γ

Problem *5.71

Consider the flow field represented by the stream function $\psi = (q/2\pi)\tan^{-1}(y/x)$, where q = constant. Is this a possible two-dimensional, incompressible flow? Is the flow irrotational?

Given: The stream function

Find: Whether or not the flow is incompressible; whether or not the flow is irrotational

Solution

The stream function is

$$\psi = \frac{q}{2\cdot\pi}\cdot\text{atan}\left(\frac{y}{x}\right)$$

The velocity components are

$$u = \frac{d\psi}{dy} = \frac{q\cdot x}{2\cdot\pi\cdot\left(x^2+y^2\right)}$$

$$v = -\frac{d\psi}{dx} = \frac{q\cdot y}{2\cdot\pi\cdot\left(x^2+y^2\right)}$$

Because a stream function exists, the flow is incompressible

Alternatively, we can check with

$$\frac{du}{dx} + \frac{dv}{dy} = 0$$

$$\frac{du}{dx} + \frac{dv}{dy} = -\frac{q\cdot\left(x^2-y^2\right)}{2\cdot\pi\cdot\left(x^2+y^2\right)^2} + \frac{q\cdot\left(x^2-y^2\right)}{2\cdot\pi\cdot\left(x^2+y^2\right)^2} = 0$$

Incompressible

For a 2D field, the irrotionality the test is $\frac{dv}{dx} - \frac{du}{dy} = 0$

$$\frac{dv}{dx} - \frac{du}{dy} = -\frac{q \cdot x \cdot y}{\pi \cdot \left(x^2 + y^2\right)^2} - \left[-\frac{q \cdot x \cdot y}{\pi \cdot \left(x^2 + y^2\right)^2}\right] = 0$$

Irrotational

Problem *5.72

Consider the flow field represented by the stream function $\psi = -A/2\pi(x^2 + y^2)$, where A = constant. Is this a possible two-dimensional, incompressible flow? Is the flow irrotational?

Given: The stream function

Find: Whether or not the flow is incompressible; whether or not the flow is irrotational

Solution

The stream function is $$\psi = -\frac{A}{2\cdot\pi\left(x^2 + y^2\right)}$$

The velocity components are $$u = \frac{d\psi}{dy} = \frac{A\cdot y}{\pi\left(x^2 + y^2\right)^2}$$

$$v = -\frac{d\psi}{dx} = -\frac{A\cdot x}{\pi\left(x^2 + y^2\right)^2}$$

Because a stream function exists, the flow is incompressible

Alternatively, we can check with $$\frac{du}{dx} + \frac{dv}{dy} = 0$$

$$\frac{du}{dx} + \frac{dv}{dy} = -\frac{4\cdot A\cdot x\cdot y}{\pi\left(x^2 + y^2\right)^3} + \frac{4\cdot A\cdot x\cdot y}{\pi\left(x^2 + y^2\right)^3} = 0$$

Incompressible

For a 2D field, the irrotionality the test is $\frac{dv}{dx} - \frac{du}{dy} = 0$

$$\frac{dv}{dx} - \frac{du}{dy} = \frac{A\cdot\left(x^2 - 3\cdot y^2\right)}{\pi\cdot\left(x^2 + y^2\right)^3} - \frac{A\cdot\left(3\cdot x^2 - y^2\right)}{\pi\cdot\left(x^2 + y^2\right)^3} = -\frac{2\cdot A}{\pi\cdot\left(x^2 + y^2\right)^2} \neq 0$$

Not irrotational

Problem *5.73

Given: Velocity field for motion in x direction with constant shear. The shear rate is

$$\frac{\partial u}{\partial y} = A \quad \text{where } A = 0.1\, s^{-1}$$

Find: (a) Expression for $\vec{V}$
(b) Rate of rotation
(c) Stream function.

Solution: The velocity field is

$$\vec{V} = u\hat{i} = \left[\int \frac{\partial u}{\partial y}\,dy + f(x)\right]\hat{i} = \left[Ay + f(x)\right]\hat{i}$$ $\vec{V}$

Fluid rotation is given by

$$\vec{\omega} = \frac{1}{2}\nabla\times\vec{V} = \frac{1}{2}\left(\frac{\partial v}{\partial x}^{\nearrow 0} - \frac{\partial u}{\partial y}\right)\hat{k} = -\frac{1}{2}\frac{\partial u}{\partial y}\hat{k} = -\frac{A}{2}\hat{k} = -0.05\, s^{-1}\ \hat{k}$$ $\vec{\omega}$

From the definition of the stream function,

$$u = \frac{\partial \psi}{\partial y} \quad \text{so} \quad \frac{\partial \psi}{\partial y} = Ay + f(x) \quad \text{and} \quad \psi = \frac{1}{2}Ay^2 + f(x)y + g(x)$$

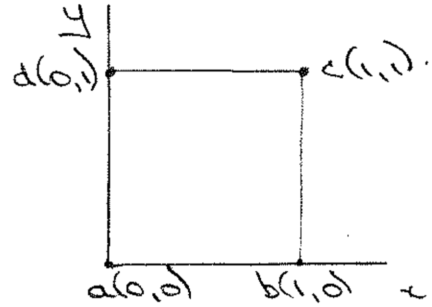
$$v = -\frac{\partial \psi}{\partial x} = f'(x)y + g'(x) = 0$$

Thus $f'(x) = 0$ and $g'(x) = 0$, and

$$\psi = \frac{1}{2}Ay^2 + c$$ c

Problem* 5.74

Given: Velocity field $\vec{V} = Axy\hat{i} + By^2\hat{j}$, where $A = 4\ m^{-1}s^{-1}$, $B = -2\ m^{-1}s^{-1}$ and coordinates are in meters.



Find: (a) Fluid rotation
(b) Circulation about "curve" shown
(c) Stream function

Plot: several streamlines in first quadrant.

Solution:

(a) The fluid rotation is given by

$$\vec{\omega} = \frac{1}{2}\nabla\times\vec{V} = \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Axy & By^2 & 0 \end{vmatrix} = \hat{k}\,\frac{1}{2}(-Ax) = -\frac{1}{2}\times\frac{4}{m\cdot s}\,x\,\hat{k} = -2x(m)\hat{k}\ \frac{rad}{s}$$ $\vec{\omega}$

(b) The circulation is defined as $\Gamma = \oint \vec{V}\cdot d\vec{s}$

For the contour shown with $\vec{V} = Axy\hat{i} + By^2\hat{j}$

$$\Gamma = \int_a^b Axy^{\,\nearrow y=0}\,dx + \int_b^c By^2\,dy + \int_c^d Axy^{\,\nearrow y=1}\,dx + \int_d^a By^2\,dy$$

$$\Gamma = \int_0^1 By^2\,dy + \int_1^0 Ax\,dx + \int_1^0 By^2\,dy = B\frac{y^3}{3}\Big]_0^1 + A\frac{x^2}{2}\Big]_1^0 + B\frac{y^3}{3}\Big]_1^0$$

$$\Gamma = \frac{1}{3}B - \frac{1}{2}A - \frac{1}{3}B = -\frac{1}{2}A = -2\ m^2/s$$ Γ

(c) For incompressible flow $u = \frac{\partial\psi}{\partial y}$, $v = -\frac{\partial\psi}{\partial x}$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = Ay + 2By = 4y + 2(-2)y = 0\ \checkmark \quad \therefore \text{incompressible}$$

Thus $u = Axy = \frac{\partial\psi}{\partial y}$ and

$$\psi = \int Axy\,dy + f(x)$$

$$\psi = \frac{1}{2}Axy^2 + f(x)$$

Then,

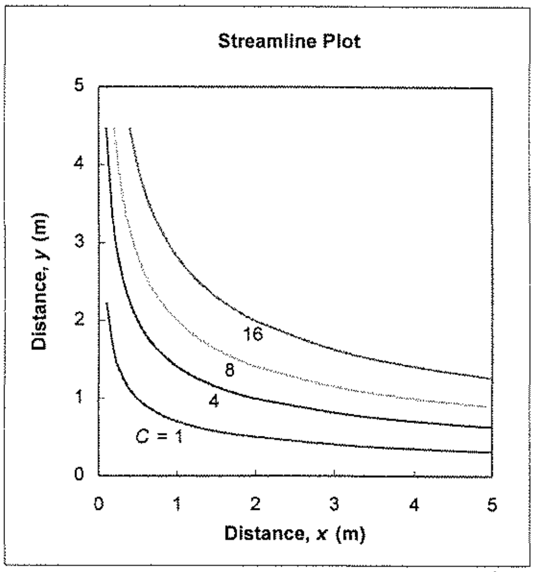
$$v = -\frac{\partial\psi}{\partial x} = -\frac{1}{2}Ay^2 - \frac{df}{dx} = By^2$$

$$\therefore \frac{df}{dx} = -\frac{1}{2}Ay^2 - By^2 = -2y^2 + 2y^2 = 0$$

Hence f = constant.

Taking $f = 0$ gives

$$\psi = \frac{1}{2}Axy^2 = 2xy^2$$ ψ



Given: Flow field represented by $\psi = x^2 - y^2$

Find: corresponding velocity field

Show: that flow field is irrotational

Plot: several streamlines and illustrate the velocity field

Solution:

Apply definition of ψ and irrotationality condition:

Computing equations: $u = \frac{\partial \psi}{\partial y}$ $\quad v = -\frac{\partial \psi}{\partial x}$

$\vec{\omega} = \frac{1}{2} \nabla \times \vec{V} = 0$

From the given $\psi = x^2 - y^2$

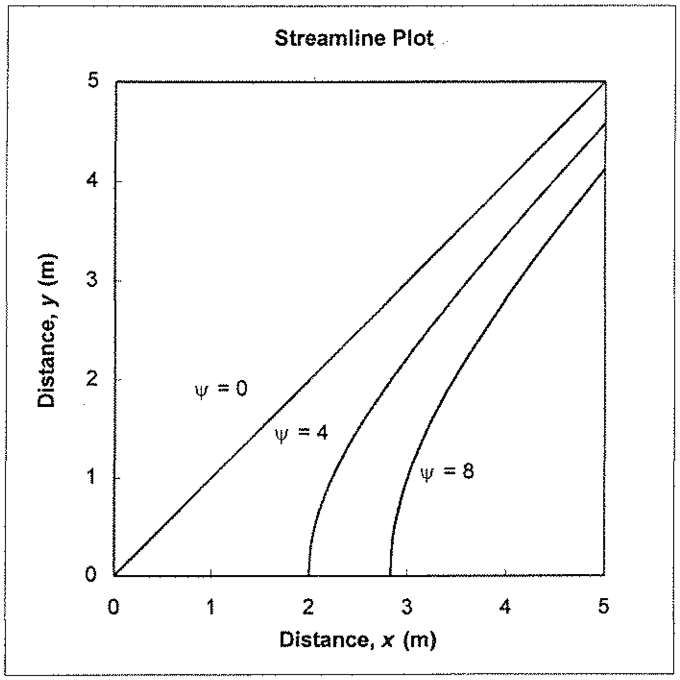
$u = \frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y}(x^2 - y^2) = -2y$

$v = -\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x}(x^2 - y^2) = -2x$

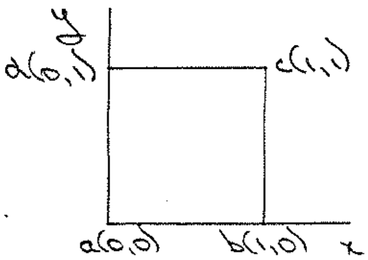
$\} \ \vec{V} = u\hat{i} + v\hat{j} = -2y\hat{i} - 2x\hat{j}$ ← $\vec{V}$

$$\nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -2y & -2x & 0 \end{vmatrix} = \hat{k}\left(-2 - (-2)\right) = 0$$

Since $\vec{\omega} = \frac{1}{2} \nabla \times \vec{V} = 0$ flow is irrotational ← $\vec{\omega} = 0$



Given: Velocity field, $\vec{V} = (Ay+B)\hat{i} + Ax\hat{j}$, where $A = 6\,s^{-1}$, $B = 3\,m\,s^{-1}$ and coordinates are in meters.



Find: (a) An expression for the stream function.
(b) Circulation about "curve" shown.

Plot: several streamlines (including stagnation streamline) in the first quadrant.

Solution

For incompressible flow $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$, $u = \frac{\partial \psi}{\partial y}$, $v = -\frac{\partial \psi}{\partial x}$

$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial}{\partial x}(Ay+B) + \frac{\partial}{\partial y}(Ax) = 0 + 0 = 0$ ∴ incompressible.

Then

$u = Ay + B = \frac{\partial \psi}{\partial y}$ and $\psi = \int (Ay+B)\,dy + f(x) = \frac{1}{2}Ay^2 + By + f(x)$

and

$v = -\frac{\partial \psi}{\partial x} = -\frac{df}{dx} = Ax$ and $f(x) = -\frac{1}{2}Ax^2 + \text{~~constant~~}$ set = 0

∴ $\psi = \frac{1}{2}A(y^2 - x^2) + By$ ← ψ

Several streamlines are plotted below. The stagnation point (where $\vec{V} = 0$) is at $x = 0$, $y = -B/A = -0.5\,m$.

The circulation is defined as $\Gamma = \oint \vec{V} \cdot d\vec{s}$

For the contour shown with $\vec{V} = (Ay+B)\hat{i} + Ax\hat{j}$

$\Gamma = \int_a^b u\,dx + \int_b^c v\,dy + \int_c^d u\,dx + \int_d^a v\,dy$ (last term $= 0$)

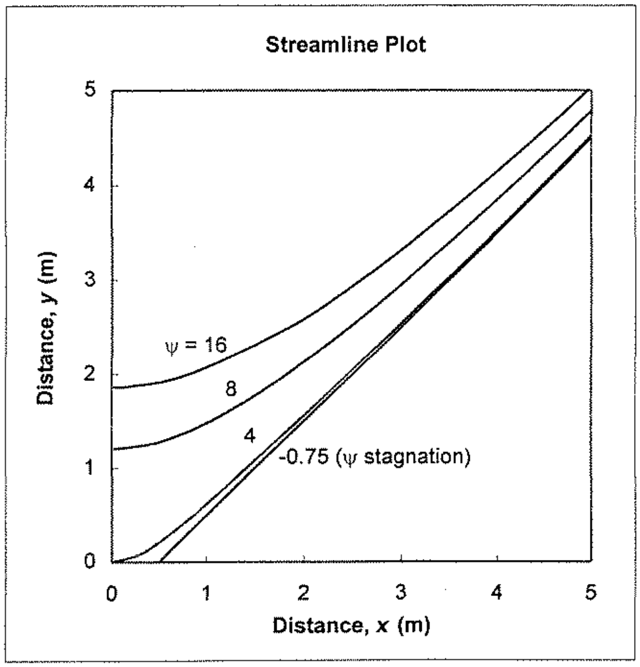
$\Gamma = \int_0^1 B\,dx + \int_0^1 A\,dy + \int_1^0 (A+B)\,dx$

{ $x = 1$ from b to c; $y = 1$ " c to d }

$\Gamma = Bx\Big]_0^1 + Ay\Big]_0^1 + (A+B)x\Big]_1^0$

$\Gamma = B + A - (A+B)$

$\Gamma = 0$ ← Γ



Note: The flow is irrotational, i.e. $\vec{\omega} = \frac{1}{2}\nabla \times \vec{V} = 0$ and hence we would expect $\Gamma = 0$

At stagnation, $\psi(x,y) = \psi(0,-0.5)$

$\psi(x,y) = 3[(-0.5)^2 - 0] + 3(-0.5) = -3/4$

Problem *5.77

Given: Flow field represented by $\psi = Axy + Ay^2$; $A = 1\ s^{-1}$

Find: (a) Show that this represents a possible incompressible flow field.
(b) Evaluate the rotation of the flow.
(c) Plot a few streamlines in the upper half plane.

Solution: For incompressible flow, $\nabla\cdot\vec{V} = 0$

The velocity field is determined from the stream function

$$u = \partial\psi/\partial y = Ax + 2Ay, \quad v = -\partial\psi/\partial x = -Ay \quad \Big\} \quad \vec{V} = A\{(x+2y)\hat{i} - y\hat{j}\}$$

Then

$$\nabla\cdot\vec{V} = \frac{\partial}{\partial x}A(x+2y) - \frac{\partial}{\partial y}(Ay) = A - A = 0 \qquad \text{Q.E.D.}$$

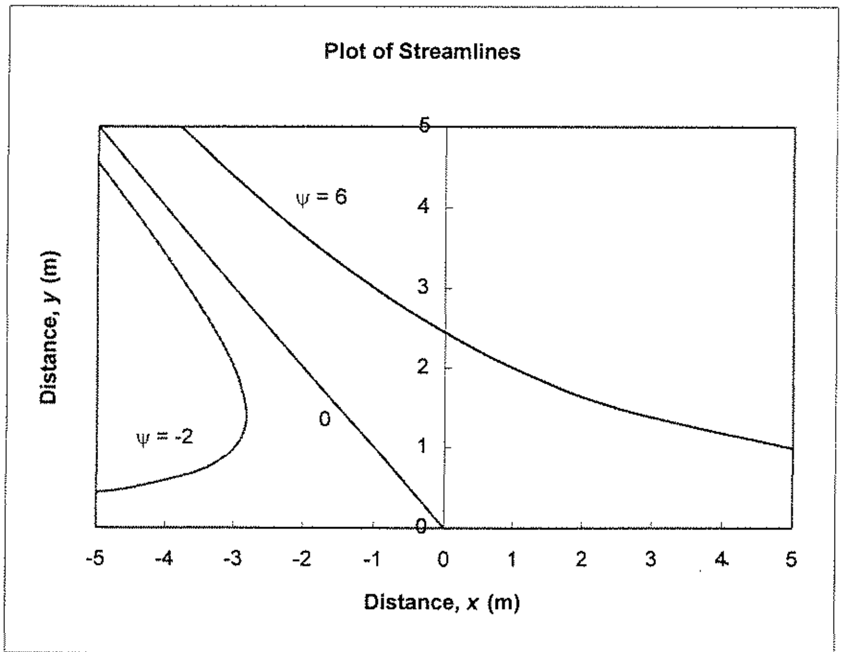
The rotation is given by $\vec{\omega} = \frac{1}{2}\nabla\times\vec{V} = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k}$

$$\vec{\omega} = \frac{1}{2}\left[\frac{\partial}{\partial x}(-Ay) - \frac{\partial}{\partial y}A(x+2y)\right]\hat{k} = \frac{1}{2}\left[0 - 2A\right]\hat{k} = -A\hat{k}$$

$$\vec{\omega} = -\hat{k}\ \text{rad/s} \qquad \vec{\omega}$$

To plot a few streamlines, $\psi = Axy + Ay^2$, note that for a given streamline

$$x = \frac{\psi}{y} - y$$

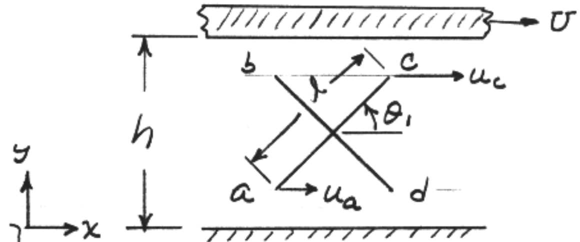


Problem 5.78

Given: Viscometric flow of Example Problem 5.7, $\vec{V} = U(y/h)\hat{i}$, where $U = 4$ mm/s and $h = 4$ mm.

Find: (a) Average rate of rotation of two line segments at $\pm 45°$
(b) Show that this is the same as in the Example.

<u>Solution</u>: Consider lines shown:

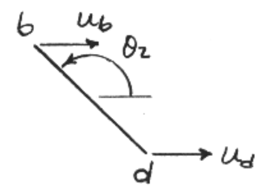


$$u_c = u_a + \frac{\partial u}{\partial y}(l\sin\theta_1)$$

$$-\omega_{ac} = \frac{(u_c - u_a)\sin\theta_1}{l} \quad \left\{\begin{array}{l}\text{Component} \perp \\ \text{to } l \text{ is } u\sin\theta_1.\end{array}\right\}$$

$$-\omega_{ac} = \frac{\frac{\partial u}{\partial y}(l\sin\theta_1)\sin\theta_1}{l} = \frac{\partial u}{\partial y}\sin^2\theta_1 = \frac{U}{h}\sin^2\theta_1$$

Sketch showing θ_2:



$$u_b = u_d + \frac{\partial u}{\partial y}(l\sin\theta_2)$$

$$-\omega_{bd} = \frac{(u_b - u_d)\sin\theta_2}{l} \quad \left\{\begin{array}{l}\text{Component} \perp \\ \text{to } l \text{ is } u\sin\theta_2.\end{array}\right\}$$

$$-\omega_{bd} = \frac{\frac{\partial u}{\partial y}(l\sin\theta_2)\sin\theta_2}{l} = \frac{\partial u}{\partial y}\sin^2\theta_2 = \frac{U}{h}\sin^2\theta_2$$

$$\omega\,(+\circlearrowleft) = \frac{1}{2}(\omega_{ac} + \omega_{bd}) = -\frac{1}{2}\frac{U}{h}(\sin^2\theta_1 + \sin^2\theta_2) = -\frac{1}{2}\frac{U}{h}(\sin^2 45° + \sin^2 135°)$$

$$= -\frac{1}{2}\frac{U}{h}\left[\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2\right] = -\frac{1}{2}\frac{U}{h}$$ ← ω

$$\omega = -\frac{1}{2} \times \frac{4\text{ mm}}{\text{sec}} \times \frac{1}{4\text{ mm}} = -0.5\ \text{s}^{-1}$$ ← ω

Problem *5.79

Given: Velocity field $\vec{V} = -\frac{q}{2\pi r}\hat{e}_r + \frac{K}{2\pi r}\hat{e}_\theta$ approximates a tornado.

Is it irrotational? Obtain the stream function.

Solution: Apply irrotationality condition.

Basic equation: $\nabla \times \vec{V} = 0$ (if irrotational)

It makes sense to work in cylindrical coordinates, where

$$\nabla = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{k}\frac{\partial}{\partial z}$$

But flow is in the $r\theta$ plane, so $\frac{\partial}{\partial z} = 0$. Then

$$\nabla \times \vec{V} = \left(\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta}\right) \times \left(V_r \hat{e}_r + V_\theta \hat{e}_\theta\right)$$

$$= \hat{e}_r \times \left(\frac{\partial V_r}{\partial r}\hat{e}_r + \frac{\partial V_\theta}{\partial r}\hat{e}_\theta\right)$$

$$+ \hat{e}_\theta \frac{1}{r} \times \left(\frac{\partial V_r}{\partial \theta}\hat{e}_r + V_r \frac{\partial \hat{e}_r}{\partial \theta} + \frac{\partial V_\theta}{\partial \theta}\hat{e}_\theta + V_\theta \frac{\partial \hat{e}_\theta}{\partial \theta}\right)$$

(where $\frac{\partial \hat{e}_r}{\partial \theta} = \hat{e}_\theta$ and $\frac{\partial \hat{e}_\theta}{\partial \theta} = -\hat{e}_r$)

$$\nabla \times \vec{V} = \hat{k}\left(\frac{\partial V_\theta}{\partial r} - \frac{1}{r}\frac{\partial V_r}{\partial \theta} + \frac{V_\theta}{r}\right) = \hat{k}\frac{1}{r}\left(\frac{\partial r V_\theta}{\partial r} - \frac{\partial V_r}{\partial \theta}\right)$$

For the given flow field, $\vec{V} = \vec{V}(r)$, so

$$\nabla \times \vec{V} = \hat{k}\frac{1}{r}\frac{\partial r V_\theta}{\partial r} = \hat{k}\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{k}{2\pi}\right) \equiv 0$$

Flow is irrotational.

$V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$; $\frac{1}{r}\frac{\partial \psi}{\partial \theta} = -\frac{q}{2\pi r}$; $\frac{\partial \psi}{\partial \theta} = \frac{-q}{2\pi}$; $\psi = \frac{-q}{2\pi}\theta + f(r)$

$V_\theta = -\frac{\partial \psi}{\partial r}$; $\frac{\partial \psi}{\partial r} = -\frac{K}{2\pi r}$; $\psi = -\frac{K}{2\pi}\ln r + g(\theta)$

Comparing,

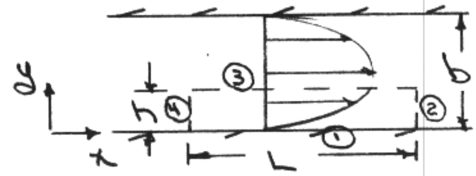
$$\psi = \frac{-q}{2\pi}\theta - \frac{K}{2\pi}\ln r$$

ψ

Problem 5.80

Given: Flow between parallel plates. Velocity field given by

$$u = U\left(\frac{y}{b}\right)\left[1-\frac{y}{b}\right]$$



Find: (a) expression for circulation about a closed contour of height h and length L.
(b) evaluate for $h = b/2$ and $h = b$.
(c) show that same result is obtained from area integral of Stokes Theorem (Eq. 5.18).

Solution:

Basic equations: $\Gamma = \oint \vec{V}\cdot d\vec{s} = \int_A (\nabla\times\vec{V})_z\, dA$

Then,

$$\Gamma = \int_1 \vec{V}\cdot d\vec{s} + \int_2 \vec{V}\cdot d\vec{s} + \int_3 \vec{V}\cdot d\vec{s} + \int_4 \vec{V}\cdot d\vec{s}$$

(the integrals over 1, 2 and 4 are crossed out and marked 0)

$$= \int_L^0 U\frac{y}{b}\left(1-\frac{y}{b}\right)dx$$

$$\Gamma = -UL\frac{h}{b}\left(1-\frac{h}{b}\right) \qquad \Gamma$$

For $h = y = \frac{b}{2}$, $\Gamma = -\frac{UL}{4}$

$h = y = b$, $\Gamma = 0$

From Stokes Theorem,

$$\Gamma = \int_A (\vec{\nabla}\times\vec{V})_z\, dA = \int_A \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dA = \int_A -U\left(\frac{1}{b}-\frac{2y}{b^2}\right)dA$$

$$\Gamma = -U\int\left(\frac{1}{b}-\frac{2y}{b^2}\right)L\,dy = -UL\left[\frac{y}{b}-\frac{y^2}{b^2}\right]_0^h$$

$$\Gamma = -UL\left[\frac{h}{b}-\frac{h^2}{b^2}\right] = -UL\frac{h}{b}\left(1-\frac{h}{b}\right) \qquad \Gamma$$

Problem 5.81

Given: Velocity profile for fully developed flow in a circular tube is

$$V_z = V_{max}\left[1 - \left(\frac{r}{R}\right)^2\right]$$

Find: (a) rates of linear and angular deformation for this flow.

(b) expression for the vorticity vector, $\vec{\xi}$

Solution:

Computing equations: B.1 and B.2 of Appendix B

Volume dilation rate $= \nabla \cdot \vec{V} = \frac{1}{r}\frac{\partial}{\partial r}(r V_r) + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial}{\partial z} V_z = 0$

Rates of linear deformation in each of the three coordinate directions r, θ, z are zero. ← Linear def

Angular deformation in the:

$r\theta$ plane is $r\frac{\partial}{\partial r}\left(\frac{V_\theta}{r}\right) + \frac{1}{r}\frac{\partial V_r}{\partial \theta} = 0$

θz plane is $\frac{\partial V_\theta}{\partial z} + \frac{1}{r}\frac{\partial V_z}{\partial \theta} = 0$

zr plane is $\frac{\partial V_r}{\partial z} + \frac{\partial V_z}{\partial r} = -V_{max}\frac{2r}{R^2}$ ← Angular def.

The vorticity vector is given by $\vec{\xi} = \nabla \times \vec{V}$

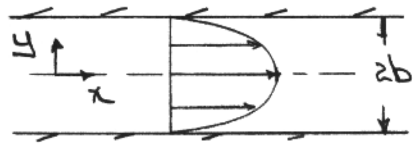
In cylindrical coordinates,

$$\nabla \times \vec{V} = \hat{e}_r\left(\frac{1}{r}\frac{\partial V_z}{\partial \theta} - \frac{\partial V_\theta}{\partial z}\right) + \hat{e}_\theta\left(\frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r}\right) + \hat{k}\left(\frac{1}{r}\frac{\partial r V_\theta}{\partial r} - \frac{1}{r}\frac{\partial V_r}{\partial \theta}\right) \quad (5.16)$$

$$\vec{\xi} = \nabla \times \vec{V} = \hat{e}_\theta V_{max}\frac{2r}{R^2}$$ ← $\vec{\xi}$

Given: Flow between parallel plates. Velocity field given by

$$u = U_{max}\left[1 - \left(\frac{y}{b}\right)^2\right]$$



Find: (a) rates of linear and angular deformation
(b) expression for the vorticity vector, $\vec{\zeta}$
(c) location of maximum vorticity

Solution:

The rate of linear deformation is zero since $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial w}{\partial z} = 0$

The rate of angular deformation in the xy plane is

$$\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = -\frac{2y\,U_{max}}{b^2}$$

The vorticity vector is given by $\vec{\zeta} = \nabla \times \vec{V}$

$$\vec{\zeta} = \hat{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \hat{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \hat{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

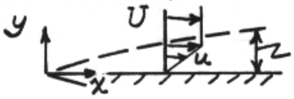
$$\vec{\zeta} = -\frac{\partial u}{\partial y}\hat{k} = \frac{2y\,U_{max}}{b^2}\hat{k}$$

$\vec{\zeta}$

The vorticity is a maximum at $y = \pm b$

Problem 5.83

Given: Linear approximate velocity profile in boundary layer.



$\delta = cx^{1/2}$; $u = U\frac{y}{\delta} = \frac{Uy}{cx^{1/2}}$; $v = \frac{uy}{4x} = \frac{Uy^2}{4cx^{3/2}}$

Find: (a) Express rotation, find maximum.
(b) Express angular deformation, locate maximum.
(c) Express linear deformation, locate maximum.
(d) Express shear force per unit volume, locate maximum.

Solution: Work in xy plane.

Computing equations: $\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$ $\qquad -\frac{d\gamma}{dt} = \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$

Linear def: $\frac{\partial u}{\partial x}$, $\frac{\partial v}{\partial y}$

Evaluating partial derivatives,

$$\frac{\partial u}{\partial x} = -\frac{1}{2}\frac{Uy}{cx^{3/2}} \qquad \frac{\partial u}{\partial y} = \frac{U}{cx^{1/2}} \qquad \frac{\partial v}{\partial x} = -\frac{3}{8}\frac{Uy^2}{cx^{5/2}} \qquad \frac{\partial v}{\partial y} = \frac{1}{2}\frac{Uy}{cx^{3/2}}$$

Then

$$\omega_z = \frac{1}{2}\left[-\frac{3}{8}\frac{Uy^2}{cx^{5/2}} - \frac{U}{cx^{1/2}}\right] = -\frac{U}{2cx^{1/2}}\left[1 + \frac{3}{8}\left(\frac{y}{x}\right)^2\right] \quad (\text{max at } y = \delta)$$ ω_z

$$-\frac{d\gamma}{dt} = -\frac{3}{8}\frac{Uy^2}{cx^{5/2}} + \frac{U}{cx^{1/2}} = \frac{U}{cx^{1/2}}\left[1 - \frac{3}{8}\left(\frac{y}{x}\right)^2\right] \quad (\text{max at } y = 0)$$ $-\frac{d\gamma}{dt}$

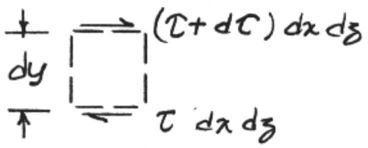
$$\frac{\partial u}{\partial x} = -\frac{1}{2}\frac{Uy}{cx^{3/2}} = -\frac{U}{2cx^{1/2}}\left(\frac{y}{x}\right) \quad (\text{max at } y = \delta)$$

$$\frac{\partial v}{\partial y} = +\frac{1}{2}\frac{Uy}{cx^{3/2}} = +\frac{U}{2cx^{1/2}}\left(\frac{y}{x}\right) \quad (\text{max at } y = \delta)$$

} sum = 0 $\qquad$ Lin Def

Shear stress is $\tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \mu\left(\frac{U}{cx^{1/2}} - \frac{3}{8}\frac{Uy^2}{cx^{5/2}}\right) = \frac{\mu U}{cx^{1/2}}\left[1 - \frac{3}{8}\left(\frac{y}{x}\right)^2\right]$

Net shear force on a fluid element is $d\tau\, dx\, dz$



$$d\tau = \frac{\partial \tau}{\partial y}dy = \frac{\mu U}{cx^{1/2}}\left(-\frac{3}{8}\frac{2y}{x^2}\right)dy = -\frac{3\mu}{4}\frac{Uy}{cx^{5/2}}dy$$

Shear stress per volume is $\frac{dF}{d\forall} = -\frac{3\mu U}{4cx^{3/2}}\left(\frac{y}{x}\right)$ (max at $y = \delta$) $\qquad \frac{dF}{d\forall}$

Problem 5.84

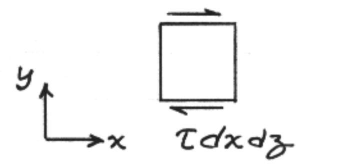
Given: x component of velocity in laminar boundary layer in water

$$u = U\sin\left(\frac{\pi}{2}\frac{y}{\delta}\right) \qquad U = 3\ m/s, \quad \delta = 2\ mm$$

y component is much smaller than u.

Find: (a) Expression for net shear force per unit volume in x direction.
(b) Maximum value for this flow

Solution: Consider a small element of fluid



Then

$$dF_{shear,x} = (\tau + d\tau)dxdz - \tau dxdz$$

$$= d\tau\, dxdz = \frac{d\tau}{dy}dxdydz$$

and

$$\frac{dF_{s,x}}{d\forall} = \frac{d\tau}{dy} = \frac{d}{dy}\left(\mu\frac{du}{dy}\right) = \mu\frac{d^2u}{dy^2}$$

From the given profile,

$$\frac{du}{dy} = \frac{\pi U}{2\delta}\cos\left(\frac{\pi}{2}\frac{y}{\delta}\right)$$

and

$$\frac{d^2u}{dy^2} = U\left(\frac{\pi}{2\delta}\right)^2\left(-\sin\left(\frac{\pi}{2}\frac{y}{\delta}\right)\right)$$

The maximum value occurs when $y = \delta$, when

$$\frac{dF_{sx,max}}{d\forall} = -\mu U\left(\frac{\pi}{2\delta}\right)^2$$

$\frac{dF_{sx}}{d\forall}$

$$= -1\times10^{-3}\frac{N\cdot sec}{m^2}\times 3\frac{m}{sec}\left(\frac{\pi}{2}\frac{1}{0.002\ m}\right)^2 = -1.85\times10^{3}\ N/m^3$$

$$\frac{dF_{sx,max}}{d\forall} = -1.85\ kN/m^3$$

$\frac{dF_{sx}}{d\forall}$

Problem 5.85

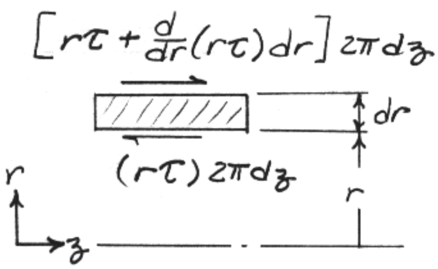
Given: Velocity profile for fully developed laminar flow in a tube

$$\frac{u}{u_{max}} = 1 - \left(\frac{r}{R}\right)^2$$

where $u_{max} = 10$ ft/s, $R = 3$ in., fluid is water.

Find: (a) Expression for shear force per unit volume in z direction.
(b) Maximum value for these conditions.

Solution: Consider a differential element:



Then

$$dF_{shear,z} = \left[r\tau + \frac{d}{dr}(r\tau)dr\right]2\pi dz - r\tau\, 2\pi dz$$

$$= \frac{d}{dr}(r\tau)\, 2\pi dr dz$$

Since $d\forall = 2\pi r dr dz$, then

$$\frac{dF_{sz}}{d\forall} = \frac{1}{2\pi r dr dz}\frac{d}{dr}(r\tau)\, 2\pi dr dz = \frac{1}{r}\frac{d}{dr}(r\tau)$$

In cylindrical coordinates, $\tau_{rz} = \mu\frac{du}{dr}$. For the given profile

$$\tau = \tau_{rz} = \mu\frac{du}{dr} = -\mu u_{max}\frac{2r}{R^2}$$

Substituting

$$\frac{dF_{sz}}{d\forall} = \frac{1}{r}\frac{d}{dr}\left[r\left(-\frac{2\mu u_{max} r}{R^2}\right)\right] = \frac{1}{r}\frac{d}{dr}\left[-\frac{2\mu u_{max} r^2}{R^2}\right] = \frac{1}{r}\left[\frac{-4\mu u_{max} r}{R^2}\right]$$

$$\frac{dF_{sz}}{d\forall} = -\frac{4\mu u_{max}}{R^2} = \text{constant} \longleftarrow \quad \frac{dF_{sz}}{d\forall}$$

Evaluating,

$$\frac{dF_{sz}}{d\forall} = -4\times 10^{-3}\frac{N\cdot s}{m^2}\times 10\frac{ft}{s}\times\frac{1}{(0.25)^2 ft^2}\times(0.3048)^2\frac{m^2}{ft^2}\times\frac{lbf}{4.448\,N}$$

$$\frac{dF_{sz}}{d\forall} = -0.0134\ lbf/ft^3 \longleftarrow$$

Given: Flow field $\vec{V} = Axy\hat{i} - By^2\hat{j}$, where $A = 10\ ft^{-1}\cdot s^{-1}$, $B = 1\ ft^{-1}\cdot s^{-1}$ and coordinates are measured in ft; $\rho = 2\ slug/ft^3$, gravity acts in negative y direction.

Find: (a) Acceleration of fluid particle at $(x,y) = 1,1$.
(b) Pressure gradient at $(1,1)$.

Solution.

Basic equations: $\vec{a}_p = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}^{\,=0} + \frac{\partial \vec{V}}{\partial t}^{\,=0}$

$$\rho\vec{g} - \nabla p = \rho\frac{D\vec{V}}{Dt}$$

Assumptions: (1) frictionless flow.

$$\vec{a}_p = \frac{D\vec{V}}{Dt} = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} = Axy\frac{\partial}{\partial x}(Axy\hat{i} - By^2\hat{j}) - By^2\frac{\partial}{\partial y}(Axy\hat{i} - By^2\hat{j})$$

$$\vec{a}_p = Axy(Ay\hat{i}) - By^2(Ax\hat{i} - 2By\hat{j})$$

$$\vec{a}_p = \hat{i}(A^2xy^2 - ABxy^2) + \hat{j}\,2B^2y^3 = Axy^2(A-B)\hat{i} + 2B^2y^3\hat{j}$$

At location (1,1)

$$\vec{a}_p = \frac{10}{ft\cdot s}\times 1ft\times 1ft^2\,\frac{(10-1)}{ft\cdot s}\,\hat{i} + 2\times\frac{1}{ft^2\cdot s^2}\times 1ft^3\,\hat{j} = 90\hat{i} + 2\hat{j}\ ft/s^2 \qquad \vec{a}_p$$

$$\nabla p = \rho\vec{g} - \rho\vec{a}_p = -\rho g\hat{j} - \rho\vec{a}_p = -\rho(g\hat{j} + \vec{a}_p)$$

$$= -2\frac{slug}{ft^3}(32.2\hat{j} + 90\hat{i} + 2\hat{j})\frac{ft}{s^2}\times\frac{lbf\cdot s^2}{ft\cdot slug}$$

$$\nabla p = -180\hat{i} - 68.4\hat{j}\ lbf/ft^2/ft \qquad \nabla p$$

Problem 6.2

Given: Incompressible flow field, $\vec{V} = (Ax - By)\hat{i} - Ay\hat{j}$

where: $A = 2\ s^{-1}$

$B = 1\ s^{-1}$

coordinates x, y, are in meters

Find: (a) Magnitude of $\vec{a}_p$ at location (1,1)

(b) Direction of $\vec{a}_p$ at location (1,1)

(c) Pressure gradient at (1,2) if $\vec{g} = -g\hat{j}$

Solution:

Basic equation: $\frac{D\vec{V}}{Dt} = \vec{a}_p = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}$

Substituting the given velocity field into the equation for $\vec{a}_p$

$$\vec{a}_p = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} = (Ax - By)\frac{\partial}{\partial x}\left[(Ax - By)\hat{i} - Ay\hat{j}\right] - Ay\frac{\partial}{\partial y}\left[(Ax - By)\hat{i} - Ay\hat{j}\right]$$

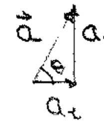
$$= (Ax - By)A\hat{i} - Ay\left[-B\hat{i} - A\hat{j}\right]$$

$$\vec{a}_p = A^2x\hat{i} + A^2y\hat{j} = A^2\left[x\hat{i} + y\hat{j}\right]$$

At location (1,1)

$$\vec{a}_p = \frac{(2)^2}{s^2}\times\left[\hat{i} + \hat{j}\right]m = 4\hat{i} + 4\hat{j}\ m/s^2$$

$$|\vec{a}_p| = \sqrt{a_x^2 + a_y^2} = \sqrt{(4)^2 + (4)^2}\ m/s^2 = 5.66\ m/s^2 \longleftarrow |\vec{a}_p|_{(1,1)}$$



$$\theta = \tan^{-1}\frac{a_y}{a_x} = \tan^{-1}1 = 45^\circ \longleftarrow \theta_{(1,1)}$$

Assume frictionless flow

$$\nabla p = -\rho g\hat{j} - \rho\vec{a}_p = -\rho(g\hat{j} + \vec{a}_p)$$

$$= -999\ \frac{kg}{m^3}\times(9.81\hat{j} + 4\hat{i} + 4\hat{j})\ \frac{m}{s^2}\times\frac{N\cdot s^2}{kg\cdot m}$$

$$\nabla p = -4.0\hat{i} - 13.8\hat{j}\ kN/m^2/m \longleftarrow \nabla p_{(1,2)}$$

Note: $\nabla\cdot\vec{V} = 0$ as required for incompressible flow

Given: Horizontal flow of water described by the velocity field

$$\vec{V} = (Ax + Bt)\hat{i} + (-Ay + Bt)\hat{j}$$

where: $A = 5\,s^{-1}$, $B = 10\,ft \cdot s^{-2}$, coordinates x, y in ft, t in s.

Find: (a) Expressions for (i) local, (ii) convective, (iii) total, acceleration
(b) Evaluate at point (2, 2) for t = 5 s
(c) Evaluate ∇p at same point and time

Solution:

Basic equations: $$\frac{D\vec{V}}{Dt} = \vec{a}_p = \underbrace{\frac{\partial \vec{V}}{\partial t}}_{\text{local}} + \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\text{convective}} \, ; \quad \rho\vec{g} - \nabla p = \rho\frac{D\vec{V}}{Dt}$$

(with $w\frac{\partial \vec{V}}{\partial z} \to 0$)

Assumptions: (1) frictionless flow
(2) ρ = constant = 1.94 slug/ft^3

$$\frac{\partial \vec{V}}{\partial t} = \frac{\partial}{\partial t}\left[(Ax + Bt)\hat{i} + (-Ay + Bt)\hat{j}\right] = B\hat{i} + B\hat{j} = 10(\hat{i} + \hat{j})\ ft/s^2 \quad \leftarrow \vec{a}_{local}$$

$$u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} = (Ax + Bt)\frac{\partial}{\partial x}\left[(Ax + Bt)\hat{i} + (-Ay + Bt)\hat{j}\right] + (-Ay + Bt)\frac{\partial}{\partial y}\left[(Ax + Bt)\hat{i} + (-Ay + Bt)\hat{j}\right]$$

$$= (Ax + Bt)\left[A\hat{i}\right] + (-Ay + Bt)\left[-A\hat{j}\right]$$

$$u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} = A(Ax + Bt)\hat{i} - A(-Ay + Bt)\hat{j} \quad \leftarrow$$

$$= \frac{5}{s}\left(\frac{5}{s} \times 2\,ft + 10\frac{ft}{s^2} \times 5\,s\right)\hat{i} - \frac{5}{s}\left(-\frac{5}{s} \times 2\,ft + 10\frac{ft}{s^2} \times 5\,s\right)\hat{j} = 300\hat{i} - 200\hat{j}\ \frac{ft}{s^2} \quad \leftarrow \vec{a}_{conv}$$

$$\vec{a} = \vec{a}_{local} + \vec{a}_{conv} = \left[B + A(Ax + Bt)\right]\hat{i} + \left[B - A(-Ay + Bt)\right]\hat{j} = 310\hat{i} - 190\hat{j}\ \frac{ft}{s^2} \quad \leftarrow \vec{a}$$

From Euler's equation,

$$\nabla p = \rho\vec{g} - \rho\frac{D\vec{V}}{Dt} = 1.94\frac{slug}{ft^3}\left[-32.2\hat{k} - (310\hat{i} - 190\hat{j})\right]\frac{ft}{s^2} \times \frac{lbf \cdot s^2}{ft \cdot slug}$$

$$\nabla p = -601\hat{i} + 367\hat{j} - 62\hat{k}\ \frac{lbf/ft^2}{ft} = -4.17\hat{i} + 2.56\hat{j} - 0.43\hat{k}\ psi/ft$$

Note: $\nabla \cdot \vec{V} = 0$ as required for incompressible flow

Problem 6.4

Given: Velocity field, $\vec{V} = (Axy - Bx^2)\hat{i} + (Axy - By^2)\hat{j}$

where: $A = 2\ ft^{-1}\cdot s^{-1}$

$B = 1\ ft^{-1}\cdot s^{-1}$

coordinates x, y are in ft

Fluid density, $\rho = 2\ slug/ft^3$. Body force $\vec{g} = -g\hat{j}$

Find: (a) Acceleration of fluid particle at (1,1)

(b) Pressure gradient at (1,1)

Solution:

Basic equations: $\rho\vec{g} - \nabla P = \rho\frac{D\vec{V}}{Dt}$

$$\frac{D\vec{V}}{Dt} = \vec{a}_p = \frac{\partial\vec{V}}{\partial t} + u\frac{\partial\vec{V}}{\partial x} + v\frac{\partial\vec{V}}{\partial y} + w\frac{\partial\vec{V}}{\partial z}$$

Assumptions: (1) frictionless flow

$$\vec{a}_p = u\frac{\partial\vec{V}}{\partial x} + v\frac{\partial\vec{V}}{\partial y} = (Axy - Bx^2)\frac{\partial}{\partial x}\left[(Axy - Bx^2)\hat{i} + (Axy - By^2)\hat{j}\right] + (Axy - By^2)\frac{\partial}{\partial y}\left[\quad\right]$$

$$= (Axy - Bx^2)\left[(Ay - 2Bx)\hat{i} + Ay\hat{j}\right] + (Axy - By^2)\left[Ax\hat{i} + (Ax - 2By)\hat{j}\right]$$

$$\vec{a}_p = \hat{i}\left[(Axy - Bx^2)(Ay - 2Bx) + Ax(Axy - By^2)\right] + \hat{j}\left[(Axy - Bx^2)Ay + (Axy - By^2)(Ax - 2By)\right]$$

At location (1,1)

$$\vec{a}_p = \hat{i}\left[(2-1)\frac{ft}{s}\times(2-2)\frac{1}{s} + \frac{2}{s}(2-1)\frac{ft}{s}\right] + \hat{j}\left[(2-1)\frac{ft}{s}\times\frac{2}{s} + (2-1)\frac{ft}{s}(2-2)\frac{1}{s}\right]$$

$$\vec{a}_p = 2\hat{i} + 2\hat{j}\ \ ft/s^2 \longleftarrow \vec{a}_p(1,1)$$

$$\rho\vec{B} - \nabla P = \rho\frac{D\vec{V}}{Dt} = \rho\vec{a}_p$$

$$\nabla P = \rho\vec{B} - \rho\vec{a}_p = \rho(\vec{B} - \vec{a}_p) = \rho(-g\hat{j} - \vec{a}_p) = -\rho(g\hat{j} + \vec{a}_p)$$

At location (1,1)

$$\nabla P = -2\frac{slug}{ft^3}\left[32.2\hat{j} + 2\hat{i} + 2\hat{j}\right]\frac{ft}{s^2}\cdot\frac{lbf\cdot s^2}{ft\cdot slug} = -\left[4\hat{i} + 68.4\hat{j}\right]\frac{lbf/ft^2}{ft} \longleftarrow \nabla P(1,1)$$

Note: For incompressible flow, $\nabla\cdot\vec{V} = 0$

$$\nabla\cdot\vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = Ay - 2Bx + Ax - 2By$$

$$\nabla\cdot\vec{V} = (A - 2B)(x + y) = 0$$

Hence given velocity field represents a possible incompressible flow

Problem 6.5

Given: Velocity field, $\vec{V} = (Ax - By)t\hat{i} - (Ay + Bx)t\hat{j}$

where $A = 1\ s^{-2}$

$B = 2\ s^{-2}$

coordinates x, y are in meters.

Fluid density is $\rho = 1500\ kg/m^3$. Body forces are negligible

Find: ∇P at location $(1,2)$ at $t = 1\ s$.

Solution:

Basic equations: $\rho\vec{g} - \nabla P = \rho \frac{D\vec{V}}{Dt}$ (with $\rho\vec{g}$ crossed out, $= 0$)

$$\frac{D\vec{V}}{Dt} = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}$$

Assumptions: (1) frictionless flow

Substituting for the velocity field in the equation for $\frac{D\vec{V}}{Dt}$,

$$\frac{D\vec{V}}{Dt} = \frac{\partial}{\partial t}\left[(Ax-By)t\hat{i} - (Ay+Bx)t\hat{j}\right] + (Ax-By)t\frac{\partial}{\partial x}\left[(Ax-By)t\hat{i} - (Ay+Bx)t\hat{j}\right]$$
$$- (Ay+Bx)t\frac{\partial}{\partial y}\left[(Ax-By)t\hat{i} - (Ay+Bx)t\hat{j}\right]$$
$$= \left[(Ax-By)\hat{i} - (Ay+Bx)\hat{j}\right] + (Ax-By)t\left[At\hat{i} - Bt\hat{j}\right] - (Ay+Bx)t\left[-Bt\hat{i} - At\hat{j}\right]$$
$$= \hat{i}\left\{Ax - By + A^2xt^2 - ABy t^2 + AByt^2 + B^2xt^2\right\} + \hat{j}\left\{-Ay - Bx - ABxt^2 + B^2yt^2 + A^2yt^2 + ABxt^2\right\}$$
$$\frac{D\vec{V}}{Dt} = \hat{i}\left\{Ax - By + xt^2(A^2+B^2)\right\} + \hat{j}\left\{-Ay - Bx + yt^2(A^2+B^2)\right\}$$

Then,

$$\nabla P = -\rho\frac{D\vec{V}}{Dt} = -\rho\left[\hat{i}\left\{Ax - By + xt^2(A^2+B^2)\right\} + \hat{j}\left\{-Ay - Bx + yt^2(A^2+B^2)\right\}\right]$$

At location $(1,2)$ at $t = 1\ s$

$$\nabla P = -1500\frac{kg}{m^3}\left[\hat{i}\left\{\frac{1}{s^2}\times 1m - \frac{2}{s^2}\times 2 + 1m\times 1s^2\left(\frac{(1)^2+(2)^2}{s^4}\right)\right\}\right.$$
$$\left. + \hat{j}\left\{-\frac{1}{s^2}\times 2m - \frac{2}{s^2}\times 1m + 2m\times 1s^2\times\left(\frac{(1)^2+(2)^2}{s^4}\right)\right\}\right]\frac{N\cdot s^2}{kg\cdot m}$$

$$\nabla P = -(3.0\hat{i} + 9.0\hat{j})\ \frac{kN/m^2}{m} \qquad \nabla P$$

Note: $\nabla\cdot\vec{V} = 0$ as required for incompressible flow

Problem 6.6

Consider the flow field with velocity given by $\vec{V} = Ax\sin(2\pi\omega t)\hat{i} - Ay\sin(2\pi\omega t)\hat{j}$, where $A = 2\ \mathrm{s}^{-1}$ and $\omega = 1\ \mathrm{s}^{-1}$. The fluid density is 2 kg/m^3. Find expressions for the local acceleration, the convective acceleration, and the total acceleration. Evaluate these at point (1, 1) at $t = 0$, 0.5 and 1 seconds. Evaluate ∇p at the same point and times.

Given: Velocity field

Find: Expressions for local, convective and total acceleration; evaluate at several points; evaluat pressure gradient

Solution

The given data is $A = 2\cdot\frac{1}{s}$ $\omega = 1\cdot\frac{1}{s}$ $\rho = 2\cdot\frac{kg}{m^3}$

$$u = A\cdot x\cdot \sin(2\cdot\pi\cdot\omega\cdot t) \qquad v = -A\cdot y\cdot \sin(2\cdot\pi\cdot\omega\cdot t)$$

Check for incompressible flow $\frac{du}{dx} + \frac{dv}{dy} = 0$

Hence $\frac{du}{dx} + \frac{dv}{dy} = A\cdot\sin(2\cdot\pi\cdot\omega\cdot t) - A\cdot\sin(2\cdot\pi\cdot\omega\cdot t) = 0$

Incompressible flow

The governing equation for acceleration is

$$\vec{a}_p = \underset{\substack{\text{total}\\\text{acceleration}\\\text{of a particle}}}{\frac{D\vec{V}}{Dt}} = \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\substack{\text{convective}\\\text{acceleration}}} + \underset{\substack{\text{local}\\\text{acceleration}}}{\frac{\partial \vec{V}}{\partial t}}$$

The local acceleration is then

x - component

$$\frac{\partial}{\partial t}u = 2\cdot\pi\cdot A\cdot\omega\cdot x\cdot\cos(2\cdot\pi\cdot\omega\cdot t)$$

y - component

$$\frac{\partial}{\partial t}v = -2\cdot\pi\cdot A\cdot\omega\cdot y\cdot\cos(2\cdot\pi\cdot\omega\cdot t)$$

For the present steady, 2D flow, the convective acceleration is

x - component

$$u\cdot\frac{du}{dx} + v\cdot\frac{du}{dy} = A\cdot x\cdot\sin(2\cdot\pi\cdot\omega\cdot t)\cdot(A\cdot\sin(2\cdot\pi\cdot\omega\cdot t)) \ \ldots$$
$$+(-A\cdot y\cdot\sin(2\cdot\pi\cdot\omega\cdot t))\cdot 0$$

$$u\cdot\frac{du}{dx} + v\cdot\frac{du}{dy} = A^2\cdot x\cdot\sin(2\cdot\pi\cdot\omega\cdot t)^2$$

y - component

$$u\cdot\frac{dv}{dx} + v\cdot\frac{dv}{dy} = A\cdot x\cdot\sin(2\cdot\pi\cdot\omega\cdot t)\cdot 0 + (-A\cdot y\cdot\sin(2\cdot\pi\cdot\omega\cdot t))\cdot(-A\cdot\sin(2\cdot\pi\cdot\omega\cdot$$

$$u\cdot\frac{dv}{dx} + v\cdot\frac{dv}{dy} = A^2\cdot y\cdot\sin(2\cdot\pi\cdot\omega\cdot t)^2$$

The total acceleration is then

x - component

$$\frac{\partial}{\partial t}u + u\cdot\frac{du}{dx} + v\cdot\frac{du}{dy} = 2\cdot\pi\cdot A\cdot\omega\cdot x\cdot\cos(2\cdot\pi\cdot\omega\cdot t) + A^2\cdot x\cdot\sin(2\cdot\pi\cdot\omega\cdot t)^2$$

y - component

$$\frac{\partial}{\partial t}v + u\cdot\frac{dv}{dx} + v\cdot\frac{dv}{dy} = -2\cdot\pi\cdot A\cdot\omega\cdot y\cdot\cos(2\cdot\pi\cdot\omega\cdot t) + A^2\cdot y\cdot\sin(2\cdot\pi\cdot\omega\cdot t)^2$$

Evaluating at point (1,1) at

$t = 0\cdot s$	Local	$12.6\cdot\frac{m}{s^2}$	and	$-12.6\cdot\frac{m}{s^2}$
	Convective	$0\cdot\frac{m}{s^2}$	and	$0\cdot\frac{m}{s^2}$
	Total	$12.6\cdot\frac{m}{s^2}$		$-12.6\cdot\frac{m}{s^2}$
$t = 0.5\cdot s$	Local	$-12.6\cdot\frac{m}{s^2}$	and	$12.6\cdot\frac{m}{s^2}$

	Convective	$0 \cdot \frac{m}{s^2}$	and	$0 \cdot \frac{m}{s^2}$
	Total	$-12.6 \cdot \frac{m}{s^2}$		$12.6 \cdot \frac{m}{s^2}$
$t = 1 \cdot s$	Local	$12.6 \cdot \frac{m}{s^2}$	and	$-12.6 \cdot \frac{m}{s^2}$
	Convective	$0 \cdot \frac{m}{s^2}$	and	$0 \cdot \frac{m}{s^2}$
	Total	$12.6 \cdot \frac{m}{s^2}$		$-12.6 \cdot \frac{m}{s^2}$

The governing equation (assuming inviscid flow) for computing the pressure gradient is

$$\rho \frac{D\vec{V}}{Dt} = \rho \vec{g} - \nabla p \qquad (6.1)$$

Hence, the components of pressure gradient (neglecting gravity) are

$$\frac{\partial}{\partial x} p = -\rho \cdot \frac{Du}{dt} \qquad \frac{\partial}{\partial x} p = -\rho \cdot \left(2 \cdot \pi \cdot A \cdot \omega \cdot x \cdot \cos(2 \cdot \pi \cdot \omega \cdot t) + A^2 \cdot x \cdot \sin(2 \cdot \pi \cdot \omega \cdot t)^2\right)$$

$$\frac{\partial}{\partial y}p = -\rho\cdot\frac{Dv}{dt} \qquad \frac{\partial}{\partial x}p = -\rho\cdot\left(-2\cdot\pi\cdot A\cdot\omega\cdot y\cdot\cos(2\cdot\pi\cdot\omega\cdot t) + A^2\cdot y\cdot\sin(2\cdot\pi\cdot\omega\cdot t)^2\right)$$

Evaluated at (1,1) and time $t = 0\cdot s$

x comp. $-25.1\cdot\frac{Pa}{m}$ *y* comp. $25.1\cdot\frac{Pa}{m}$

$t = 0.5\cdot s$

x comp. $25.1\cdot\frac{Pa}{m}$ *y* comp. $-25.1\cdot\frac{Pa}{m}$

$t = 1\cdot s$

x comp. $-25.1\cdot\frac{Pa}{m}$ *y* comp. $25.1\cdot\frac{Pa}{m}$

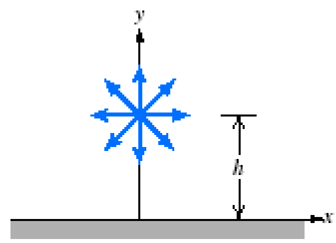
Problem 6.7

The velocity field for a plane source located distance $h = 1$ m above an infinite wall aligned along the x axis is given by

$$\vec{V} = \frac{q}{2\pi\left[x^2 + (y-h)^2\right]}\left[x\hat{i} + (y-h)\hat{j}\right] + \frac{q}{2\pi\left[x^2 + (y+h)^2\right]}\left[x\hat{i} + (y+h)\hat{j}\right]$$

where $q = 2$ m^3/s/m. The fluid density is 1000 kg/m^3 and body forces are negligible. Derive expressions for the velocity and acceleration of a fluid particle that moves along the wall, and plot from $x = 0$ to $x = +10h$. Verify that the velocity and acceleration normal to the wall are zero. Plot the pressure gradient $\partial p/\partial x$ along the wall. Is the pressure gradient along the wall adverse (does it oppose fluid motion) or not?

Given: Velocity field



Find: Expressions for velocity and acceleration along wall; plot; verify vertical components are zero; plot pressure gradient

Solution

The given data is $q = 2\cdot\frac{\frac{m^3}{s}}{m}$ $\qquad h = 1\cdot m \qquad \rho = 1000\cdot\frac{kg}{m^3}$

$$u = \frac{q\cdot x}{2\cdot\pi\left[x^2 + (y-h)^2\right]} + \frac{q\cdot x}{2\cdot\pi\left[x^2 + (y+h)^2\right]}$$

$$v = \frac{q\cdot(y-h)}{2\cdot\pi\left[x^2 + (y-h)^2\right]} + \frac{q\cdot(y+h)}{2\cdot\pi\left[x^2 + (y+h)^2\right]}$$

The governing equation for acceleration is

$$\vec{a}_p = \underbrace{\frac{D\vec{V}}{Dt}}_{\substack{\text{total}\\ \text{acceleration}\\ \text{of a particle}}} = \underbrace{u\frac{\partial\vec{V}}{\partial x} + v\frac{\partial\vec{V}}{\partial y} + w\frac{\partial\vec{V}}{\partial z}}_{\substack{\text{convective}\\ \text{acceleration}}} + \underbrace{\frac{\partial\vec{V}}{\partial t}}_{\substack{\text{local}\\ \text{acceleration}}}$$

x - component

$$u \cdot \frac{du}{dx} + v \cdot \frac{du}{dy} = -\frac{q^2 \cdot x \cdot \left[\left(x^2 + y^2\right)^2 - h^2 \cdot \left(h^2 - 4 \cdot y^2\right) \right]}{\left[x^2 + (y+h)^2 \right]^2 \cdot \left[x^2 + (y-h)^2 \right]^2 \cdot \pi^2}$$

$$a_x = -\frac{q^2 \cdot x \cdot \left[\left(x^2 + y^2\right)^2 - h^2 \cdot \left(h^2 - 4 \cdot y^2\right) \right]}{\pi^2 \cdot \left[x^2 + (y+h)^2 \right]^2 \cdot \left[x^2 + (y-h)^2 \right]^2}$$

y - component

$$u \cdot \frac{dv}{dx} + v \cdot \frac{dv}{dy} = -\frac{q^2 \cdot y \cdot \left[\left(x^2 + y^2\right)^2 - h^2 \cdot \left(h^2 + 4 \cdot x^2\right) \right]}{\pi^2 \cdot \left[x^2 + (y+h)^2 \right]^2 \cdot \left[x^2 + (y-h)^2 \right]^2}$$

$$a_y = -\frac{q^2 \cdot y \cdot \left[\left(x^2 + y^2\right)^2 - h^2 \cdot \left(h^2 + 4 \cdot x^2\right) \right]}{\pi^2 \cdot \left[x^2 + (y+h)^2 \right]^2 \cdot \left[x^2 + (y-h)^2 \right]^2}$$

For motion along the wall $y = 0 \cdot m$

$$u = \frac{q \cdot x}{\pi \cdot \left(x^2 + h^2\right)}$$ $v = 0$ (No normal velocity)

$$a_x = -\frac{q^2 \cdot x \cdot \left(x^2 - h^2\right)}{\pi^2 \cdot \left(x^2 + h^2\right)^3}$$ $a_y = 0$ (No normal acceleration)

The governing equation (assuming inviscid flow) for computing the pressure gradient is

$$\rho \frac{D\vec{V}}{Dt} = \rho\vec{g} - \nabla p \qquad (6.1)$$

Hence, the component of pressure gradient (neglecting gravity) along the wall is

$$\frac{\partial}{\partial x}p = -\rho\cdot\frac{Du}{dt} \qquad\qquad \frac{\partial}{\partial x}p = \frac{\rho\cdot q^2\cdot x\cdot\left(x^2 - h^2\right)}{\pi^2\cdot\left(x^2 + h^2\right)^3}$$

The plots of velocity, acceleration, and pressure gradient are shown in the associated *Excel* workbook. From the plots it is clear that the fluid experiences an adverse pressure gradient from the origin to $x = 1$ m, then a negative one promoting fluid acceleration. If flow separates, it will likely be in the region $x = 0$ to $x = h$.

Problem 6.7 (In Excel)

The velocity field for a plane source located distance $h = 1$ m above an infinite wall aligned along the x axis is given by

$$\vec{V} = \frac{q}{2\pi\left[x^2 + (y-h)^2\right]}\left[x\hat{i} + (y-h)\hat{j}\right] + \frac{q}{2\pi\left[x^2 + (y+h)^2\right]}\left[x\hat{i} + (y+h)\hat{j}\right]$$

where $q = 2$ m³/s/m. The fluid density is 1000 kg/m³ and body forces are negligible. Derive expressions for the velocity and acceleration of a fluid particle that moves along the wall, and plot from $x = 0$ to $x = +10h$. Verify that the velocity and acceleration normal to the wall are zero. Plot the pressure gradient $\partial p/\partial x$ along the wall. Is the pressure gradient along the wall adverse (does it oppose fluid motion) or not?

Given: Velocity field

Find: Plots of velocity, acceleration and pressure gradient along wall

Solution

The velocity, acceleration and pressure gradient are given by

$q =$ 2 $m^3/s/m$

$h =$ 1 m

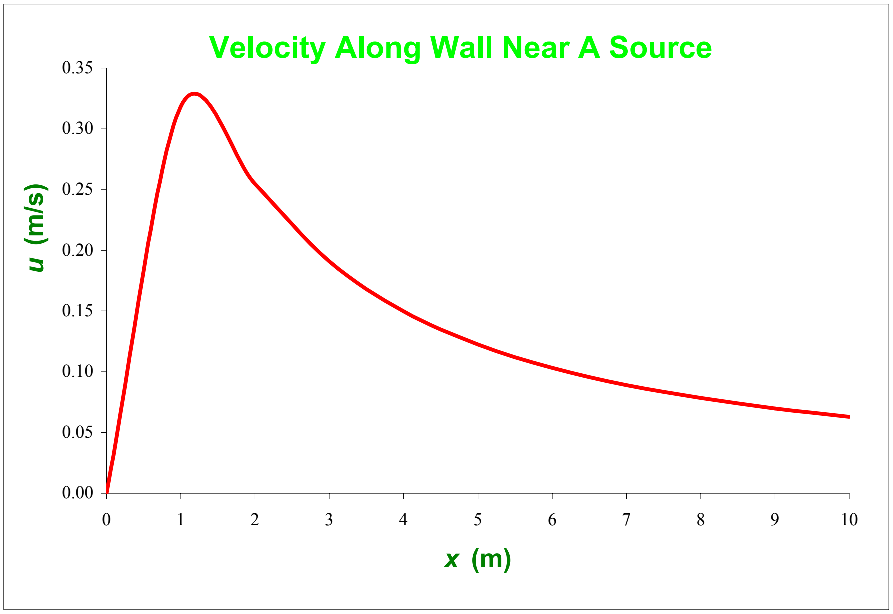
$\rho =$ 1000 kg/m^3

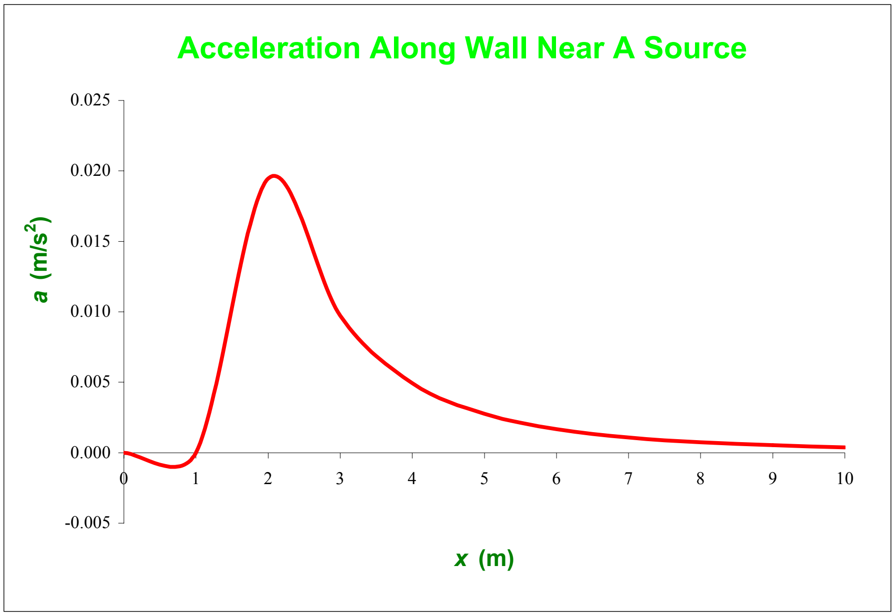
$$u = \frac{q \cdot x}{\pi \cdot \left(x^2 + h^2\right)}$$

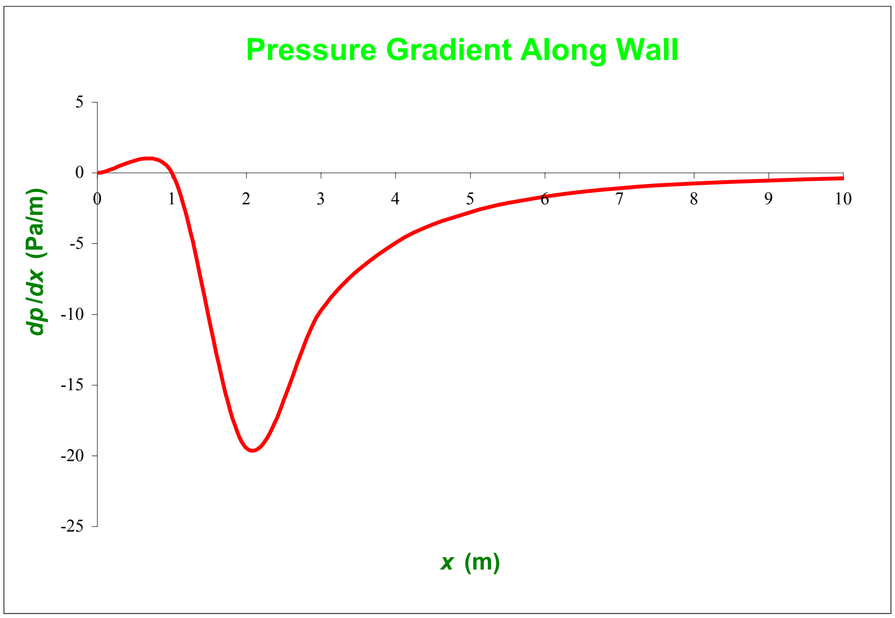
$$a_x = -\frac{q^2 \cdot x \cdot \left(x^2 - h^2\right)}{\pi^2 \cdot \left(x^2 + h^2\right)^3}$$

$$\frac{\partial}{\partial x}p = \frac{\rho \cdot q^2 \cdot x \cdot \left(x^2 - h^2\right)}{\pi^2 \cdot \left(x^2 + h^2\right)^3}$$

x (m)	*u* (m/s)	*a* (m/s²)	*dp*/*dx* (Pa/m)
0.0	0.00	0.00000	0.00
1.0	0.32	0.00000	0.00
2.0	0.25	0.01945	-19.45
3.0	0.19	0.00973	-9.73
4.0	0.15	0.00495	-4.95
5.0	0.12	0.00277	-2.77
6.0	0.10	0.00168	-1.68
7.0	0.09	0.00109	-1.09
8.0	0.08	0.00074	-0.74
9.0	0.07	0.00053	-0.53
10.0	0.06	0.00039	-0.39




Acceleration Along Wall Near A Source
0.025
0.020
0.015
0.010
0.005
0.000
-0.005
a (m/s2)
0
1
2
3
4
5
6
7
8
9
10
x (m)


Pressure Gradient Along Wall
5
0
-5
-10
-15
-20
-25
dp/dx (Pa/m)
0
1
2
3
4
5
6
7
8
9
10
x (m)

Given: y component of velocity for incompressible flow in the xy plane is

$v = Ay$ where $A = 2\ s^{-1}$ and x in m

Pressure is $p_0 = 190$ kPa(gage) at $(x,y) = (0,0)$.
Density is $\rho = 1.50\ kg/m^3$; z is vertical; neglect viscosity.

Find: (a) Simplest x component of velocity.
(b) Acceleration at point $(x,y) = (2,1)$.
(c) Pressure gradient at same point.
(d) Pressure distribution along x axis.

Solution: For 2-D incompressible flow, $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$, so $\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}$

$$u = -\int \frac{\partial v}{\partial y} dx + f(y) = \int -A\,dx + f(y) = -Ay + f(y)$$

For simplest case, $f(y) = 0$, and $u = -Ax$ — u

Acceleration is $\vec{a}_p = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y}$; $\vec{V} = Ax\hat{i} - Ay\hat{j}$

$$\vec{a}_p = (-Ax)(-A)\hat{i} + Ay(A)\hat{j} = A^2x\hat{i} + A^2y\hat{j}$$

At (1,2), $\vec{a}_p(1,2) = \frac{2^2}{s^2} \times 2m\,\hat{i} + \frac{2^2}{s^2} \times 1m\,\hat{j} = 8\hat{i} + 4\hat{j}\ m/s^2$ — $\vec{a}_p(2,1)$

To find pressure gradient, apply Euler's equation ($\mu = 0$):

BE $-\nabla p + \rho\vec{g} = \rho\vec{a}_p$

$$\nabla p = \rho\vec{g} - \rho\vec{a}_p = \rho(-g\hat{k}) - \rho(8\hat{i} + 4\hat{j}) = -\rho(8\hat{i} + 4\hat{j} + g\hat{k})$$

$$\nabla p = -1.50\frac{kg}{m^3}(8\hat{i} + 4\hat{j} + 9.81\hat{k})\frac{m}{s^2} \times \frac{N \cdot s^2}{kg \cdot m}$$

$\nabla p = -12\hat{i} - 6\hat{j} - 14.7\hat{k}\ N/m^3$ — ∇p

Along the x axis, $y = 0$, and $\vec{a}_p = A^2x\hat{i}$. Thus

$$\nabla p = \rho\vec{g} - \rho\vec{a}_p = \rho(-g\hat{k}) - \rho A^2x\hat{i} \quad \text{so} \quad \frac{\partial p}{\partial x} = -\rho A^2 x$$

Thus along the x axis $dp = \frac{\partial p}{\partial x}dx$. Integrating,

$$p(x) - p_0 = \int_0^x dp = \int_0^x -\rho A^2 x\,dx = -\rho A^2\frac{x^2}{2}\Big]_0^x = -\frac{\rho A^2 x^2}{2}$$

Finally

$$p(x) = p_0 - \frac{\rho A^2 x^2}{2} = 190\frac{N}{m^2} - \frac{1}{2} \times 1.50\frac{kg}{m^3} \times \frac{(2)^2}{s^2} \times (x)^2 m^2 \times \frac{N \cdot s^2}{kg \cdot m}$$

$p(x) = 190 - 3x^2$ Pa(gage) (x in m) — $p(x)$

Problem 6.9

Given: The velocity distribution in a steady (2-D) flow field in the xy plane is given by $\vec{V} = (Ax - B)\hat{i} + (C - Ay)\hat{j}$, where $A = 2\ s^{-1}$, $B = 5\ m \cdot s^{-1}$, $C = 3\ m \cdot s^{-1}$, and the body force distribution is $\vec{g} = -g\hat{k}$

Find: (a) Does the velocity field represent the flow of an incompressible fluid?
(b) Find the stagnation point of the flow field.
(c) Obtain an expression for the pressure gradient.
(d) Evaluate Δp between origin and point (1,3) if $\rho = 1.2\ kg/m^3$

Solution:

(a) Apply the continuity equation, $\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{V} = 0$, for the given conditions. If ρ = constant, then

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 = \frac{\partial}{\partial x}(2x - 5) + \frac{\partial}{\partial y}(3 - 2y) = 2 - 2 = 0 \checkmark$$

$\therefore$ velocity field represents an incompressible flow

(b) At the stagnation point, $\vec{V} = 0$. For $\vec{V} = 0$, then

$$u = 2x - 5 = 0 \quad \text{and} \quad v = (3 - 2y) = 0$$

Thus stagnation point is at $(x, y) = \left(\frac{5}{2}, \frac{3}{2}\right)$

(c) Euler's equation, $\rho\vec{g} - \nabla p = \rho \frac{D\vec{V}}{Dt}$, can be used to obtain an expression for the pressure gradient

$$\nabla p = \rho\vec{g} - \rho\frac{D\vec{V}}{Dt} = \rho\vec{g} - \rho\left[\frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}\right]$$

(steady; $w = 0$)

$$\nabla p = \rho\left[\vec{g} - u\frac{\partial \vec{V}}{\partial x} - v\frac{\partial \vec{V}}{\partial y}\right] = \rho\left[-g\hat{k} - (2x - 5)2\hat{i} - (3 - 2y)(-2\hat{j})\right]$$

$$\nabla p = -\rho\left[(4x - 10)\hat{i} + (4y - 6)\hat{j} + g\hat{k}\right]$$ ∇p

(d) Since $p = p(x, y, z)$ we can write

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz = -\rho(4x - 10)dx - \rho(4y - 6)dy - \rho g\,dz$$

We can integrate to obtain Δp between any two points in the field if, and only if, the integral of the right hand side is independent of the path of integration. This is true for the present case.

$$\therefore p_{1,3} - p_{0,0} = -\rho\left\{\int_0^1 (4x - 10)dx + \int_0^3 (4y - 6)dy\right\} = -\rho\left\{\left[2x^2 - 10x\right]_0^1 + \left[2y^2 - 6y\right]_0^3\right\}$$

$$= -\rho\{-8 - 0\} = 8\rho$$

$$p_{1,3} - p_{0,0} = 8\ \frac{m^2}{s^2} \times 1.2\ \frac{kg}{m^3} \cdot \frac{N \cdot s^2}{kg \cdot m} = 9.6\ N/m^2$$ Δp

Problem 6.10

Given: Frictionless, incompressible flow field with

$$\vec{V} = Ax\hat{i} - Ay\hat{j}$$

$$\vec{g} = -g\hat{k}$$

At $(0,0,0)$ $p = p_0$

Find: Expression for the pressure field $p(x,y,z)$

Solution:

Basic equations: $\rho\vec{g} - \nabla p = \rho\frac{D\vec{V}}{Dt}$

$$\frac{D\vec{V}}{Dt} = \frac{\partial\vec{V}}{\partial t} + u\frac{\partial\vec{V}}{\partial x} + v\frac{\partial\vec{V}}{\partial y} + w\frac{\partial\vec{V}}{\partial z}$$

$$\nabla p = \rho\left(\vec{g} - \frac{D\vec{V}}{Dt}\right) = \rho\left(-g\hat{k} - u\frac{\partial\vec{V}}{\partial x} - v\frac{\partial\vec{V}}{\partial y}\right)$$

$$= -\rho\left[g\hat{k} + Ax(A\hat{i}) - Ay(-A\hat{j})\right]$$

$$\nabla p = -\rho\left[A^2x\hat{i} + A^2y\hat{j} + g\hat{k}\right]$$

$$\hat{i}\frac{\partial p}{\partial x} + \hat{j}\frac{\partial p}{\partial y} + \hat{k}\frac{\partial p}{\partial z} = -\rho\left[A^2x\hat{i} + A^2y\hat{j} + g\hat{k}\right]$$

$$\frac{\partial p}{\partial x} = -\rho A^2 x \qquad \frac{\partial p}{\partial y} = -\rho A^2 y \qquad \frac{\partial p}{\partial z} = -\rho g$$

$$p = p(x,y,z)$$

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz = -\rho A^2 x\,dx - \rho A^2 y\,dy - \rho g\,dz$$

* $$p - p_0 = \int_{p_0}^{p} dp = -\int_0^x \rho A^2 x\,dx - \int_0^y \rho A^2 y\,dy - \int_0^z \rho g\,dz$$

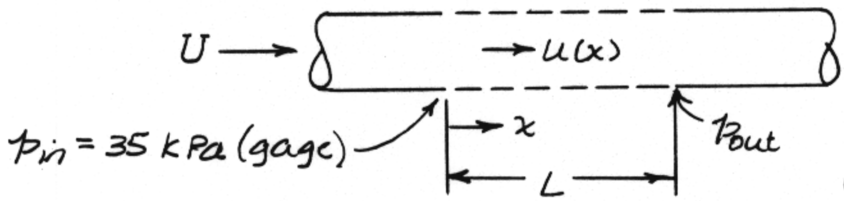
$$p - p_0 = -\rho\left[\frac{A^2x^2}{2} + \frac{A^2y^2}{2} + gz\right]$$

$$p = p_0 - \rho\left[\frac{A^2x^2}{2} + \frac{A^2y^2}{2} + gz\right] \qquad p$$

* We can integrate to obtain Δp between any two points in the flow field if, and only if, the integral of the right hand side is independent of the path of integration. This is true for the present case.

Problem 6.11

Given: Porous pipe with liquid ($\mu = 0$, $\rho = 900\ kg/m^3$)



$U = 5\ m/s$

$L = 0.3\ m$

$u(x) = U(1 - x/2L)$

Find: (a) Expression for acceleration along ℄.

(b) Expression for pressure gradient along ℄.

(c) Evaluate p_{out}

<u>Solution</u>: Computing equations (acceleration and Euler in x-direction)

$$a_{p_x} = u\frac{\partial u}{\partial x} + \overset{=0(1)}{v\frac{\partial u}{\partial y}} + \overset{=0(1)}{w\frac{\partial u}{\partial z}} + \overset{=0(2)}{\frac{\partial u}{\partial t}}; \quad \overset{=0(3)}{\rho g_x} - \frac{\partial p}{\partial x} = \rho a_{p_x}$$

Assumptions: (1) $v = w = 0$ along ℄

(2) steady flow

(3) $g_x = 0$

Then

$$a_{p_x} = u\frac{\partial u}{\partial x} = U\left(1 - \frac{x}{2L}\right)U\left(-\frac{1}{2L}\right) = -\frac{U^2}{2L}\left(1 - \frac{x}{2L}\right)$$ a_{p_x}

From Euler

$$\frac{\partial p}{\partial x} = \frac{dp}{dx} = -\rho a_{p_x} = \rho\frac{U^2}{2L}\left(1 - \frac{x}{2L}\right)$$ $\frac{dp}{dx}$

Integrating,

$$p_{out} - p_{in} = \int_0^L \frac{dp}{dx}dx = \rho\frac{U^2}{2L}\int_0^L\left(1 - \frac{x}{2L}\right)dx = \rho\frac{U^2}{2L}\left(x - \frac{x^2}{4L}\right)\Big]_0^L$$

or

$$p_{out} = p_{in} + \frac{\rho U^2}{2L}\left(\frac{3}{4}L\right) = p_{in} + \frac{3}{8}\rho U^2$$

$$= 35\ kPa + \frac{3}{8} \times 900\frac{kg}{m^3} \times (5)^2\frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m}$$

$p_{out} = 43.4\ kPa\ (gage)$ p_{out}

Problem 6.12

Given: Liquid, ρ = constant and negligible viscosity, is pumped at total volume flow rate, Q, through two small holes into the narrow gap between closely space parallel plates. The liquid flowing away from the holes has only radial motion. Flow may be assumed uniform at any section.

(a) Show that $V_r = Q/2\pi r h$, where h is the spacing between the plates.

(b) Obtain an expression for a_r and $\partial p/\partial r$

Solution:

Apply the conservation of mass to a CV with outer edge at r.

Basic equation: $0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A}$ — $\frac{\partial}{\partial t}$ term = 0(1)

Assumptions: (1) steady flow
(2) incompressible flow
(3) uniform flow at each section

Then

$$0 = \int_{CS} \vec{V}\cdot d\vec{A} = -2\times\frac{Q}{2} + V_r\, 2\pi r h$$

and $V_r = \dfrac{Q}{2\pi r h}$ ← V_r

From Eq. 6.4a

$$g_r - \frac{1}{\rho}\frac{\partial p}{\partial r} = a_r = \frac{\partial V_r}{\partial t} + V_r\frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} + V_z\frac{\partial V_r}{\partial z} - \frac{V_\theta^2}{r}$$

Since $V_r = V_r(r)$ and $V_\theta = 0$, then

$$a_r = V_r\frac{\partial V_r}{\partial r} = \frac{Q}{2\pi r h}\left[\frac{Q}{2\pi h}\left(-\frac{1}{r^2}\right)\right] = -\left(\frac{Q}{2\pi r h}\right)^2\frac{1}{r}$$

$a_r = -\dfrac{V_r^2}{r}$ ← a_r

Since $g_r = 0$, then

$$-\frac{1}{\rho}\frac{\partial p}{\partial r} = a_r$$

$\dfrac{\partial p}{\partial r} = -\rho a_r = \rho\dfrac{V_r^2}{r}$ ← $\dfrac{\partial p}{\partial r}$

Problem 6.13

The velocity field for a plane vortex sink is given by $\vec{V} = -\frac{q}{2\pi r}\hat{e}_r + \frac{K}{2\pi r}\hat{e}_\theta$, where $q = 2\ \text{m}^3/\text{s/m}$ and $K = 1\ \text{m}^3/\text{s/m}$. The fluid density is 1000 kg/m³. Find the acceleration at (1, 0), (1, π/2) and (2, 0). Evaluate ∇p under the same conditions.

Given: Velocity field

Find: The acceleration at several points; evaluate pressure gradient

Solution

The given data is $q = 2\cdot\frac{\frac{m^3}{s}}{m}$ $\quad K = 1\cdot\frac{\frac{m^3}{s}}{m}$ $\quad \rho = 1000\cdot\frac{kg}{m^3}$

$$V_r = -\frac{q}{2\cdot\pi\cdot r} \qquad V_\theta = \frac{K}{2\cdot\pi\cdot r}$$

The governing equations for this 2D flow are

$$\rho a_r = \rho\left(\frac{\partial V_r}{\partial t} + V_r\frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} + V_z\frac{\partial V_r}{\partial z} - \frac{V_\theta^2}{r}\right) = \rho g_r - \frac{\partial p}{\partial r} \qquad (6.3a)$$

$$\rho a_\theta = \rho\left(\frac{\partial V_\theta}{\partial t} + V_r\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + V_z\frac{\partial V_\theta}{\partial z} + \frac{V_r V_\theta}{r}\right) = \rho g_\theta - \frac{1}{r}\frac{\partial p}{\partial \theta} \qquad (6.3b)$$

The total acceleration for this steady flow is then

r - component

$$a_r = V_r\cdot\frac{\partial}{\partial r}V_r + \frac{V_\theta}{r}\cdot\frac{\partial}{\partial \theta}V_r \qquad a_r = -\frac{q^2}{4\cdot\pi^2\cdot r^3}$$

θ - component

$$a_\theta = V_r \cdot \frac{\partial}{\partial r} V_\theta + \frac{V_\theta}{r} \cdot \frac{\partial}{\partial \theta} V_\theta \qquad a_\theta = \frac{q \cdot K}{4 \cdot \pi^2 \cdot r^3}$$

Evaluating at point (1,0) $a_r = -0.101 \frac{m}{s^2}$ $\quad a_\theta = 0.051 \frac{m}{s^2}$

Evaluating at point (1,π/2) $a_r = -0.101 \frac{m}{s^2}$ $\quad a_\theta = 0.051 \frac{m}{s^2}$

Evaluating at point (2,0) $a_r = -0.0127 \frac{m}{s^2}$ $\quad a_\theta = 0.00633 \frac{m}{s^2}$

From Eq. 6.3, pressure gradient is

$$\frac{\partial}{\partial r} p = -\rho \cdot a_r \qquad \frac{\partial}{\partial r} p = \frac{\rho \cdot q^2}{4 \cdot \pi^2 \cdot r^3}$$

$$\frac{1}{r} \cdot \frac{\partial}{\partial \theta} p = -\rho \cdot a_\theta \qquad \frac{1}{r} \cdot \frac{\partial}{\partial \theta} p = -\frac{\rho \cdot q \cdot K}{4 \cdot \pi^2 \cdot r^3}$$

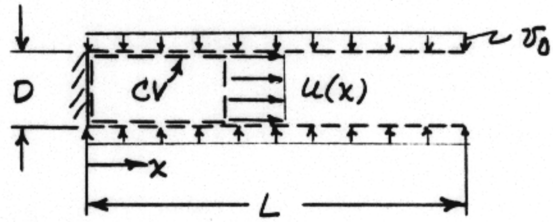
Evaluating at point (1,0) $\frac{\partial}{\partial r} p = 101 \cdot \frac{Pa}{m}$ $\quad \frac{1}{r} \cdot \frac{\partial}{\partial \theta} p = -50.5 \cdot \frac{Pa}{m}$

Evaluating at point (1,π/2) $\frac{\partial}{\partial r} p = 101 \cdot \frac{Pa}{m}$ $\quad \frac{1}{r} \cdot \frac{\partial}{\partial \theta} p = -50.5 \cdot \frac{Pa}{m}$

Evaluating at point (2,0) $\frac{\partial}{\partial r} p = 12.7 \cdot \frac{Pa}{m}$ $\quad \frac{1}{r} \cdot \frac{\partial}{\partial \theta} p = -6.33 \cdot \frac{Pa}{m}$

Problem 6.14

Given: Circular tube with porous wall; incompressible flow, uniform in x direction.



Find: (a) Algebraic expression for a_{p_x} at x.
(b) Pressure gradient at x.
(c) Integrate to obtain p at $x = 0$.

Solution: Apply conservation of mass using the CV shown.

Basic equations:
$$0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A} \qquad \left(\frac{\partial}{\partial t} = 0\ (1)\right)$$

$$a_{p_x} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} \;;\quad -\frac{\partial p}{\partial x} + \rho g_x = \rho a_{p_x}$$
($v \approx 0$ (5), $w \approx 0$ (5), $\frac{\partial u}{\partial t} = 0$ (1))

Assumptions: (1) Steady flow (2) Incompressible flow (3) Uniform flow at each cross-section (4) Horizontal; $g_x = 0$ (5) $v \approx 0$ in channel ($w \approx 0$ too) (6) Inviscid flow

Then
$$\int \vec{V}\cdot d\vec{A} = \left\{-|v_0 \pi D x|\right\} + \left\{+\left|u\frac{\pi D^2}{4}\right|\right\} = 0 \quad \text{or} \quad u(x) = 4 v_0 \frac{x}{D}$$

and
$$a_{p_x} = 4v_0\frac{x}{D}\left(4v_0\frac{1}{D}\right) = 16 v_0^2 \frac{x}{D^2} \qquad a_{p_x}$$

From the Euler equation,
$$-\frac{\partial p}{\partial x} = \rho a_{p_x} \quad \text{so} \quad \frac{\partial p}{\partial x} = -\rho a_{p_x} = -16\rho v_0^2 \frac{x}{D^2} \qquad \frac{\partial p}{\partial x}$$

Since $v \approx w \approx 0$, then $p(x)$ and $dp = \frac{\partial p}{\partial x}dx$. Integrating
$$\int_0^L dp = p_L - p(0) = \int_0^L -16\rho v_0^2 \frac{x}{D^2}\,dx = -\frac{16\rho v_0^2}{D^2}\left.\frac{x^2}{2}\right]_0^L = -\frac{8\rho v_0^2 L^2}{D^2}$$

Thus, since $p_L = p_{atm}$, the gage pressure at $x = 0$ is
$$p(0) = 8\rho v_0^2\left(\frac{L}{D}\right)^2 \qquad p(0)$$

Problem 6.15

ρ = 1000 kg/m^3 consists of a diverging section of pipe. At the inlet the diameter is D_i = 0.25 m, and at the outlet the diameter is D_o = 0.75 m. The diffuser length is L = 1 m, and the diameter increases linearly with distance x along the diffuser. Derive and plot the acceleration of a fluid particle, assuming uniform flow at each section, if the speed at the inlet is V_i = 5 m/s. Plot the pressure gradient through the diffuser, and find its maximum value. If the pressure gradient must be no greater than 25 kPa/m, how long would the diffuser have to be?

Given: Diffuser geometry

Find: Acceleration of a fluid particle; plot it; plot pressure gradient; find L such that pressure gradient is less than 25 kPa/m

Solution

The given data is $D_i = 0.25 \cdot m$ $\qquad D_o = 0.75 \cdot m$ $\qquad L = 1 \cdot m$

$V_i = 5 \cdot \frac{m}{s}$ $\qquad \rho = 1000 \cdot \frac{kg}{m^3}$

For a linear increase in diameter $\qquad D(x) = D_i + \frac{D_o - D_i}{L} \cdot x$

From continuity $\qquad Q = V \cdot A = V \cdot \frac{\pi}{4} \cdot D^2 = V_i \cdot \frac{\pi}{4} \cdot D_i^2$ $\qquad Q = 0.245 \frac{m^3}{s}$

Hence $\qquad V(x) \cdot \frac{\pi}{4} \cdot D(x)^2 = Q$ $\qquad V(x) = \frac{4 \cdot Q}{\pi \cdot \left(D_i + \frac{D_o - D_i}{L} \cdot x \right)^2}$

or $$V(x) = \frac{V_i}{\left(1 + \frac{D_o - D_i}{L \cdot D_i} \cdot x\right)^2}$$

The governing equation for this flow is

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho g_x - \frac{\partial p}{\partial x} \qquad (6.2a)$$

or, for steady 1D flow, in the notation of the problem

$$a_x = V \cdot \frac{d}{dx} V = \frac{V_i}{\left(1 + \frac{D_o - D_i}{L \cdot D_i} \cdot x\right)^2} \cdot \frac{d}{dx} \frac{V_i}{\left(1 + \frac{D_o - D_i}{L \cdot D_i} \cdot x\right)^2}$$

$$a_x(x) = -\frac{2 \cdot V_i^2 \cdot \left(D_o - D_i\right)}{D_i \cdot L \cdot \left[1 + \frac{\left(D_o - D_i\right)}{D_i \cdot L} \cdot x\right]^5}$$

This is plotted in the associated *Excel* workbook

From Eq. 6.2a, pressure gradient is

$$\frac{\partial}{\partial x} p = -\rho \cdot a_x \qquad \frac{\partial}{\partial x} p = \frac{2 \cdot \rho \cdot V_i^2 \cdot \left(D_o - D_i\right)}{D_i \cdot L \cdot \left[1 + \frac{\left(D_o - D_i\right)}{D_i \cdot L} \cdot x\right]^5}$$

This is also plotted in the associated *Excel* workbook. Note that the pressure gradient is adverse: separation is likely to occur in the diffuser, and occur near the entrance

At the inlet $\frac{\partial}{\partial x}p = 100 \cdot \frac{kPa}{m}$ At the exit $\frac{\partial}{\partial x}p = 412 \cdot \frac{Pa}{m}$

To find the length *L* for which the pressure gradient is no more than 25 kPa/m, we need to solve

$$\frac{\partial}{\partial x}p \le 25 \cdot \frac{kPa}{m} = \frac{2 \cdot \rho \cdot V_i^2 \cdot \left(D_o - D_i\right)}{D_i \cdot L \cdot \left[1 + \frac{\left(D_o - D_i\right)}{D_i \cdot L} \cdot x\right]^5}$$

with *x* = 0 m (the largest pressure gradient is at the inlet)

Hence $$L \ge \frac{2 \cdot \rho \cdot V_i^2 \cdot \left(D_o - D_i\right)}{D_i \cdot \frac{\partial}{\partial x}p}$$ $L \ge 4 \cdot m$

This result is also obtained using *Goal Seek* in the *Excel* workbook

Problem 6.15 (In Excel)

A diffuser for an incompressible, inviscid fluid of density $\rho = 1000\ kg/m^3$ consists of a diverging section of pipe. At the inlet the diameter is $D_i = 0.25$ m, and at the outlet the diameter is $D_o = 0.75$ m. The diffuser length is $L = 1$ m, and the diameter increases linearly with distance x along the diffuser. Derive and plot the acceleration of a fluid particle, assuming uniform flow at each section, if the speed at the inlet is $V_i = 5$ m/s. Plot the pressure gradient through the diffuser, and find its maximum value. If the pressure gradient must be no greater than 25 kPa/m, how long would the diffuser have to be?

Given: Diffuser geometry

Find: Acceleration of a fluid particle; plot it; plot pressure gradient; find L such that pressure gradient is less than 25 kPa/m

Solution

The acceleration and pressure gradient are given by

$D_i = 0.25$ m

$D_o = 0.75$ m

$L = 1$ m

$V_i = 5$ m/s

$\rho = 1000\ kg/m^3$

$$a_x(x) = -\frac{2\cdot V_i^2\cdot(D_o - D_i)}{D_i\cdot L\cdot\left[1 + \frac{(D_o - D_i)}{D_i\cdot L}\cdot x\right]^5}$$

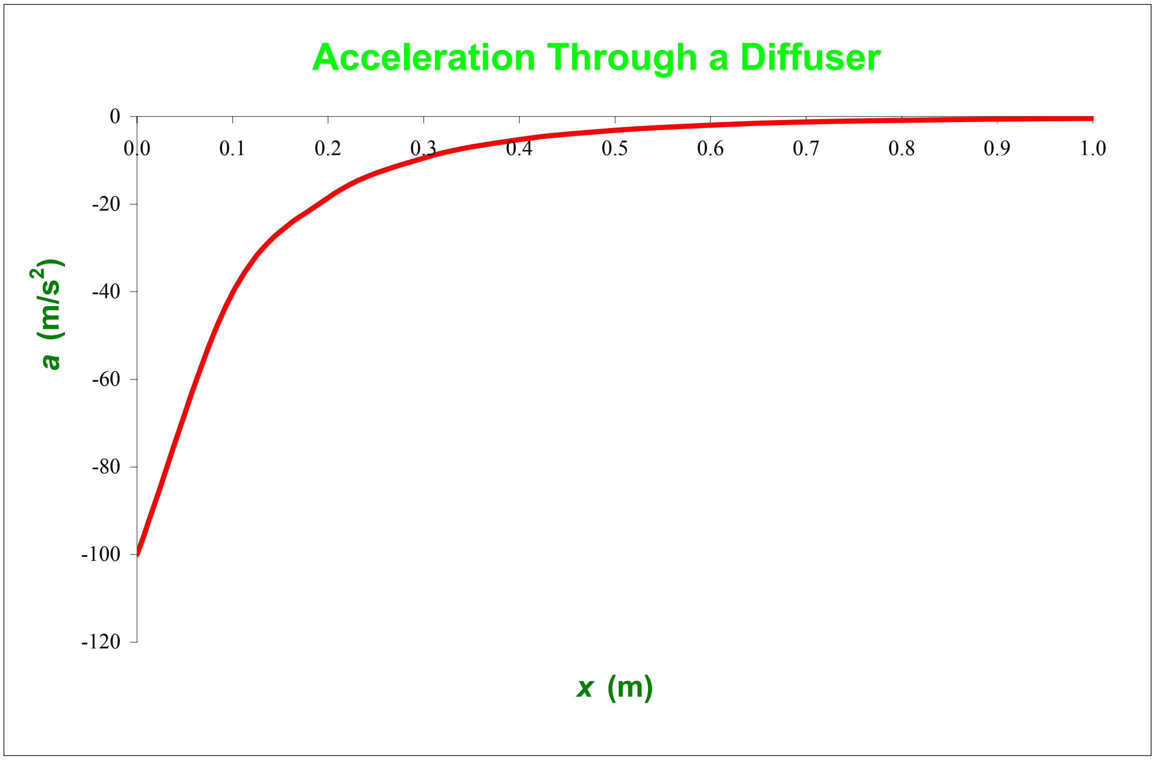
$$\frac{\partial}{\partial x}p = \frac{2\cdot\rho\cdot V_i^2\cdot(D_o - D_i)}{D_i\cdot L\cdot\left[1 + \frac{(D_o - D_i)}{D_i\cdot L}\cdot x\right]^5}$$

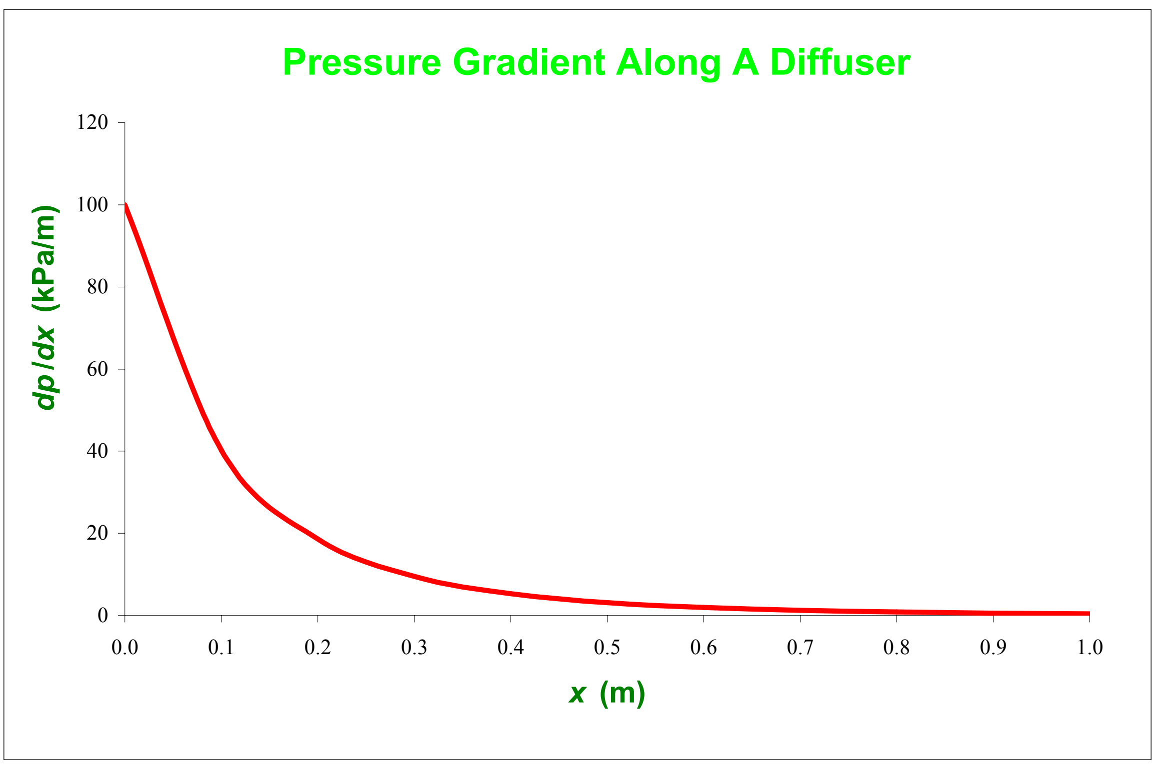
x (m)	a (m/s^2)	dp/dx (kPa/m)
0.0	-100	100
0.1	-40.2	40.2
0.2	-18.6	18.6
0.3	-9.5	9.54
0.4	-5.29	5.29
0.5	-3.13	3.13
0.6	-1.94	1.94
0.7	-1.26	1.26
0.8	-0.842	0.842
0.9	-0.581	0.581
1.0	-0.412	0.412

For the length L required for the pressure gradient to be less than 25 kPa/m use *Goal Seek*

$L = 4.00$ m

x (m)	dp/dx (kPa/m)
0.0	25.0


Acceleration Through a Diffuser
0
-20
-40
-60
-80
-100
-120
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
a (m/s2)
x (m)


Pressure Gradient Along A Diffuser
120
100
80
60
40
20
0
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
dp/dx (kPa/m)
x (m)

Problem 6.16

A nozzle for an incompressible, inviscid fluid of density ρ = 1000 kg/m³ consists of a converging section of pipe. At the inlet the diameter is D_i = 100 mm, and at the outlet the diameter is D_o = 20 mm. The nozzle length is L = 500 mm, and the diameter decreases linearly with distance x along the nozzle. Derive and plot the acceleration of a fluid particle, assuming uniform flow at each section, if the speed at the inlet is V_i = 1 m/s. Plot the pressure gradient through the nozzle, and find its maximum absolute value. If the pressure gradient mus be no greater than 5 MPa/m in absolute value, how long would the nozzle have to be?

Given: Nozzle geometry

Find: Acceleration of a fluid particle; plot it; plot pressure gradient; find L such that pressure gradient is less than 5 MPa/m in absolute value

Solution

The given data is $D_i = 0.1 \cdot m$ $\quad D_o = 0.02 \cdot m$ $\quad L = 0.5 \cdot m$

$V_i = 1 \cdot \frac{m}{s}$ $\quad \rho = 1000 \cdot \frac{kg}{m^3}$

For a linear decrease in diameter $\quad D(x) = D_i + \frac{D_o - D_i}{L} \cdot x$

From continuity $\quad Q = V \cdot A = V \cdot \frac{\pi}{4} \cdot D^2 = V_i \cdot \frac{\pi}{4} \cdot D_i^2$ $\quad Q = 0.00785 \frac{m^3}{s}$

Hence $\quad V(x) \cdot \frac{\pi}{4} \cdot D(x)^2 = Q$ $\quad V(x) = \frac{4 \cdot Q}{\pi \cdot \left(D_i + \frac{D_o - D_i}{L} \cdot x \right)^2}$

or $$V(x) = \frac{V_i}{\left(1 + \frac{D_o - D_i}{L \cdot D_i} \cdot x\right)^2}$$

The governing equation for this flow is

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho g_x - \frac{\partial p}{\partial x} \qquad (6.2a)$$

or, for steady 1D flow, in the notation of the problem

$$a_x = V \cdot \frac{d}{dx}V = \frac{V_i}{\left(1 + \frac{D_o - D_i}{L \cdot D_i} \cdot x\right)^2} \cdot \frac{d}{dx}\frac{V_i}{\left(1 + \frac{D_o - D_i}{L \cdot D_i} \cdot x\right)^2}$$

$$a_x(x) = -\frac{2 \cdot V_i^2 \cdot (D_o - D_i)}{D_i \cdot L \cdot \left[1 + \frac{(D_o - D_i)}{D_i \cdot L} \cdot x\right]^5}$$

This is plotted in the associated *Excel* workbook

From Eq. 6.2a, pressure gradient is

$$\frac{\partial}{\partial x}p = -\rho \cdot a_x \qquad \frac{\partial}{\partial x}p = \frac{2 \cdot \rho \cdot V_i^2 \cdot (D_o - D_i)}{D_i \cdot L \cdot \left[1 + \frac{(D_o - D_i)}{D_i \cdot L} \cdot x\right]^5}$$

This is also plotted in the associated *Excel* workbook. Note that the pressure gradient is

At the inlet $\frac{\partial}{\partial x}p = -3.2\cdot\frac{kPa}{m}$ At the exit $\frac{\partial}{\partial x}p = -10\cdot\frac{MPa}{m}$

To find the length *L* for which the absolute pressure gradient is no more than 5 MPa/m, we need solve

$$\left|\frac{\partial}{\partial x}p\right| \le 5\cdot\frac{MPa}{m} = \frac{2\cdot\rho\cdot V_i^2\cdot\left(D_o - D_i\right)}{D_i\cdot L\cdot\left[1 + \frac{\left(D_o - D_i\right)}{D_i\cdot L}\cdot x\right]^5}$$

with $x = L$ m (the largest pressure gradient is at the outlet)

Hence $$L \ge \frac{2\cdot\rho\cdot V_i^2\cdot\left(D_o - D_i\right)}{D_i\cdot\left(\frac{D_o}{D_i}\right)^5\cdot\left|\frac{\partial}{\partial x}p\right|}$$ $L \ge 1\cdot m$

This result is also obtained using *Goal Seek* in the *Excel* workbook

Problem 6.16 (In Excel)

A nozzle for an incompressible, inviscid fluid of density ρ = 1000 kg/m^3 consists of a converging section of pipe. At the inlet the diameter is D_i = 100 mm, and at the outlet the diameter is D_o = 20 mm. The nozzle length is L = 500 mm, and the diameter decreases linearly with distance x along the nozzle. Derive and plot the acceleration of a fluid particle, assuming uniform flow at each section, if the speed at the inlet is V_i = 5 m/s. Plot the pressure gradient through the nozzle, and find its maximum absolute value. If the pressure gradient must be no greater than 5 MPa/m in absolute value, how long would the nozzle have to be?

Given: Nozzle geometry

Find: Acceleration of a fluid particle; plot it; plot pressure gradient; find L such that the absolute pressure gradient is less than 5 MPa/m

Solution

The acceleration and pressure gradient are given by

D_i =	0.1	m
D_o =	0.02	m
L =	0.5	m
V_i =	1	m/s
ρ =	1000	kg/m^3

$$a_x(x) = -\frac{2\cdot V_i^2\cdot(D_o - D_i)}{D_i\cdot L\cdot\left[1 + \frac{(D_o - D_i)}{D_i\cdot L}\cdot x\right]^5}$$

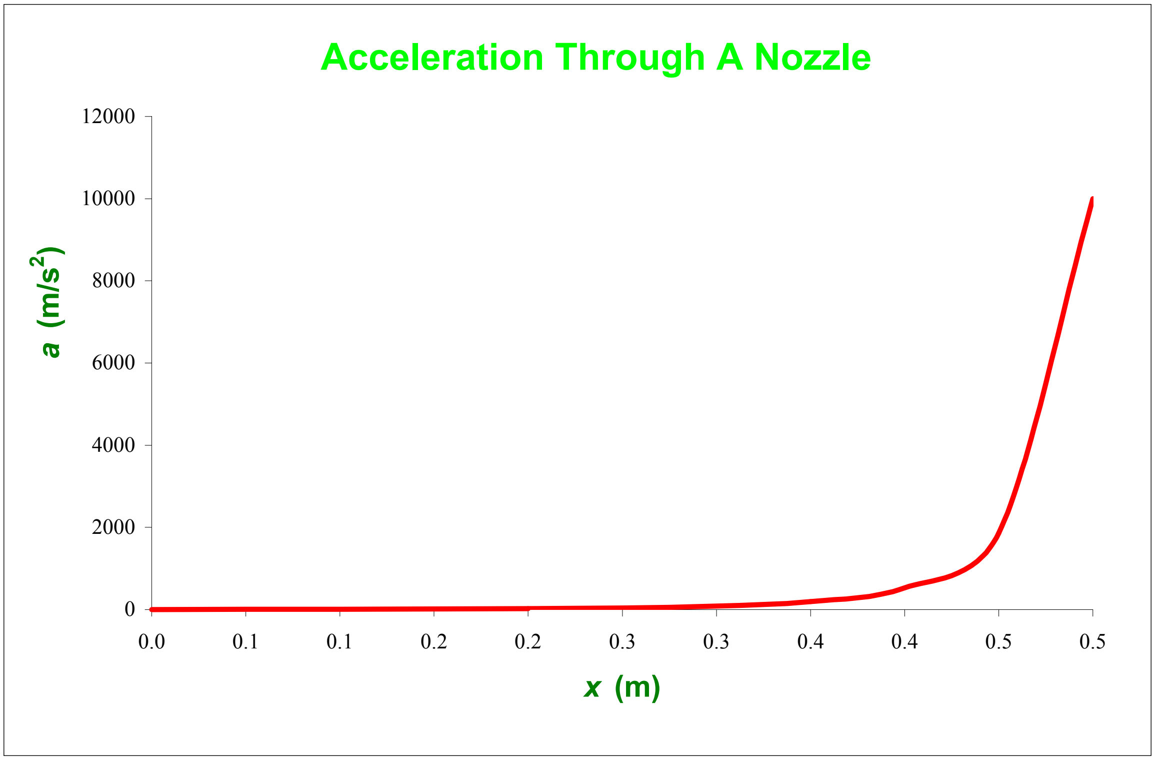
$$\frac{\partial}{\partial x}p = \frac{2\cdot\rho\cdot V_i^2\cdot(D_o - D_i)}{D_i\cdot L\cdot\left[1 + \frac{(D_o - D_i)}{D_i\cdot L}\cdot x\right]^5}$$

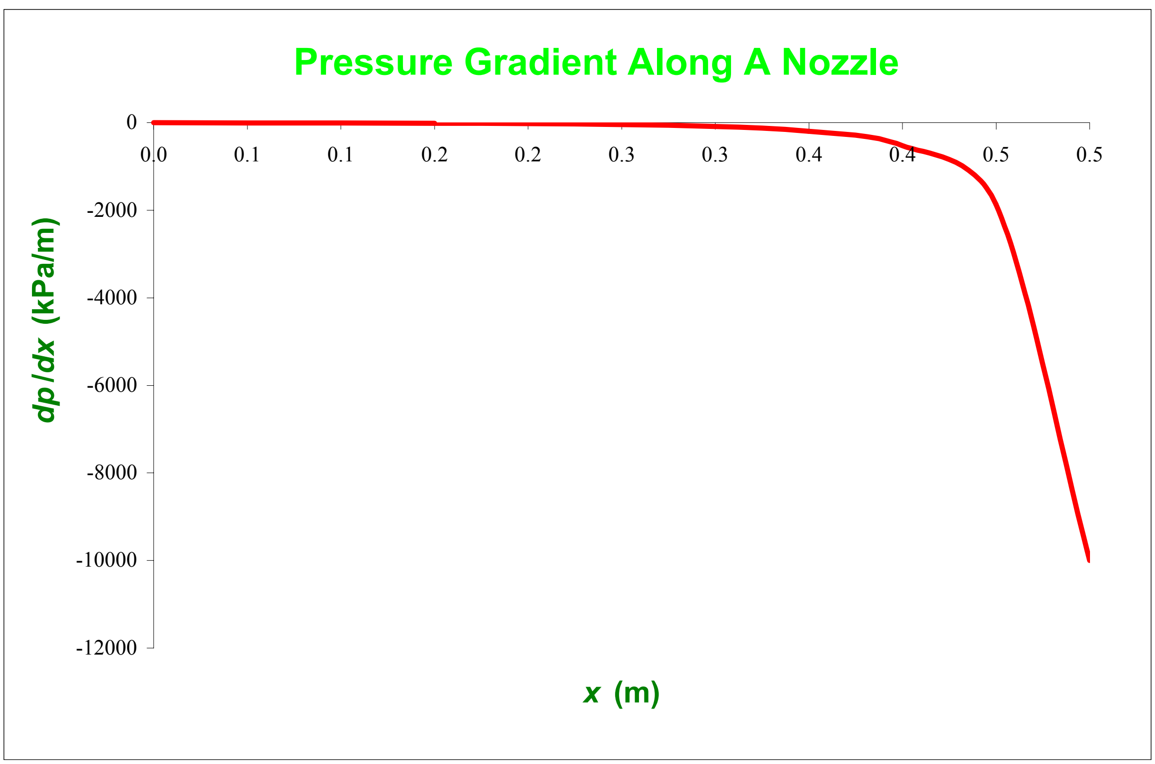
x **(m)**	***a*** **(m/s^2)**	***dp/dx*** **(kPa/m)**
0.00	3.20	-3.20
0.05	4.86	-4.86
0.10	7.65	-7.65
0.15	12.6	-12.6
0.20	22.0	-22.0
0.25	41.2	-41.2
0.30	84.2	-84.2
0.35	194	-194
0.40	529	-529
0.45	1859	-1859
0.50	10000	-10000

For the length L required for the pressure gradient to be less than 5 MPa/m (abs) use *Goal Seek*

L = 1.00 m

x **(m)**	***dp/dx*** **(kPa/m)**
1.00	-5000


Acceleration Through A Nozzle
12000
10000
8000
6000
4000
2000
0
0.0
0.1
0.1
0.2
0.2
0.3
0.3
0.4
0.4
0.5
0.5
a (m/s2)
x (m)

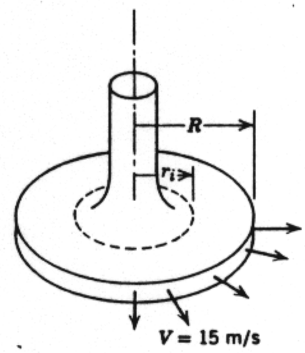

Pressure Gradient Along A Nozzle
0
-2000
-4000
-6000
-8000
-10000
-12000
0.0
0.1
0.1
0.2
0.2
0.3
0.3
0.4
0.4
0.5
0.5
dp/dx (kPa/m)
x (m)

Problem 6.17

Given: Steady, incompressible flow of air between parallel discs as shown

$$\vec{V} = V \frac{R}{r} \hat{e}_r \quad \text{for } r_i \le r \le R$$

where $V = 15$ m/s, $r_i = R/2$, $R = 75$ mm



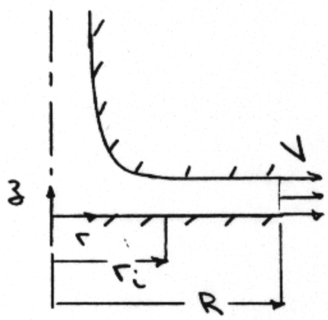
Find: magnitude and direction of the net pressure force that acts on the upper plate between r_i and R.

Solution:

Basic equations: $\rho \vec{g} - \nabla p = \rho \frac{D\vec{V}}{Dt}$ $\qquad \vec{F} = -\int p \, d\vec{A}$

Assumptions: (1) incompressible flow
(2) steady flow
(3) frictionless flow
(4) uniform flow at each section.

To determine the pressure distribution $p(r)$, apply Eulers equation in the r direction



$$-\frac{\partial p}{\partial r} + \rho g_r = \rho a_r = \rho V_r \frac{\partial V_r}{\partial r}$$

$$\frac{\partial p}{\partial r} = -\rho V_r \frac{\partial V_r}{\partial r} = -\rho V \frac{R}{r} \frac{\partial}{\partial r}\left(V \frac{R}{r}\right) = \rho V \frac{R}{r} \frac{VR}{r^2}$$

$$\frac{dp}{dr} = \rho V^2 \frac{R^2}{r^3}$$

$$dp = \rho V^2 \frac{R^2}{r^3} dr$$

Integrating we obtain

$$p - p_{atm} = \int_{p_{atm}}^{p} dp = \rho V^2 R^2 \int_R^r r^{-3} dr = \rho V^2 R^2 \left[-\frac{1}{2r^2}\right]_R^r = \frac{1}{2}\rho V^2 R^2 \left[\frac{1}{R^2} - \frac{1}{r^2}\right]$$

Then

$$F_z = \int (p - p_{atm}) dA = \int_{R/2}^{R} \frac{1}{2}\rho V^2 R^2 \left[\frac{1}{R^2} - \frac{1}{r^2}\right] 2\pi r \, dr = \rho V^2 R^2 \pi \left[\frac{r^2}{2R^2} - \ln r\right]_{R/2}^{R}$$

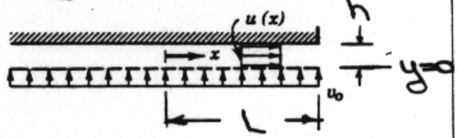
$$= \rho V^2 R^2 \pi \left[\frac{1}{2R^2}\left(R^2 - \frac{R^2}{4}\right) - \ln \frac{R}{R/2}\right] = \rho V^2 R^2 \pi \left[0.375 - \ln 2\right] = -0.318 \pi \rho V^2 R^2$$

$$= -0.318\pi \times 1.23 \frac{kg}{m^3} \times (15)^2 \frac{m^2}{s^2} \times (0.075)^2 m^2 \times \frac{N \cdot s^2}{kg \cdot m}$$

$F_z = -1.56$ N $\quad$ ($F_z < 0$, so force acts down) $\qquad F_z$

Problem 6.18

Given: Air flows into the narrow gap between closely spaced parallel plates through a porous surface as shown. The uniform velocity in the x direction is $u = v_0 x / h$. Assume the flow to incompressible with $\rho = 1.23\ kg/m^3$ and that friction is negligible.



$v_0 = 15 mm/s$, $L = 22 mm$, $h = 1.2 mm$

Find: (a) the pressure gradient at the point (L, h)
(b) an equation for the flow streamlines in the cavity

Solution:

Eulers equation, $\rho \vec{g} - \nabla p = \rho \frac{D\vec{V}}{Dt}$, can be used to determine the pressure gradient for incompressible frictionless flow.

We need first to determine the velocity field. With $u = v_0 x/h$, for 2-D, incompressible flow we can use the continuity equation to determine v.

Since $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$, then $\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -\frac{\partial}{\partial x}\left(\frac{v_0 x}{h}\right) = -\frac{v_0}{h}$

Then
$$v = \int \frac{\partial v}{\partial y}\, dy + f(x) = -\frac{v_0}{h} y + f(x)$$

But $v = v_0$ at $y = 0$ and hence $f(x) = v_0$ and $v = v_0\left(1 - \frac{y}{h}\right)$

Then
$$\nabla p = \rho \vec{g} - \rho \frac{D\vec{V}}{Dt} = \rho\left[\vec{g} - u\frac{\partial \vec{V}}{\partial x} - v\frac{\partial \vec{V}}{\partial y}\right] = \rho\left[-g\hat{j} - \frac{v_0 x}{h}\left(\frac{v_0}{h}\hat{i}\right) - v_0\left(1-\frac{y}{h}\right)\left(-\frac{v_0}{h}\hat{j}\right)\right]$$

$$\nabla p = \rho\left[-g\hat{j} - \frac{v_0^2 x}{h^2}\hat{i} - \frac{v_0^2}{h}\left(1 - \frac{y}{h}\right)\hat{j}\right]$$

At the point $(x, y) = (L, h)$
$$\nabla p = \rho\left[-\frac{v_0^2 L}{h^2}\hat{i} - g\hat{j}\right]$$

$$= 1.23 \frac{kg}{m^3}\left[-(15)^2 \frac{mm^2}{s^2} \times 0.022 m \times \frac{1}{(1.2)^2 mm^2}\hat{i} - 9.81 \frac{m}{s^2}\hat{j}\right] \cdot \frac{N \cdot s^2}{kg \cdot m}$$

$$\nabla p\Big]_{L,h} = -4.23\hat{i} - 12.1\hat{j}\ N/m^3$$ ← ∇p

(b) The slope of the streamlines is given by $\frac{dy}{dx} = \frac{v}{u}$

$\therefore \frac{dy}{dx} = \frac{v_0(1 - y/h)}{\frac{v_0 x}{h}}$ and separating variables, we can write

$$\frac{d\left(\frac{y}{h}\right)}{(1 - y/h)} = \frac{d\left(\frac{x}{h}\right)}{x/h}$$ Then integrating we obtain

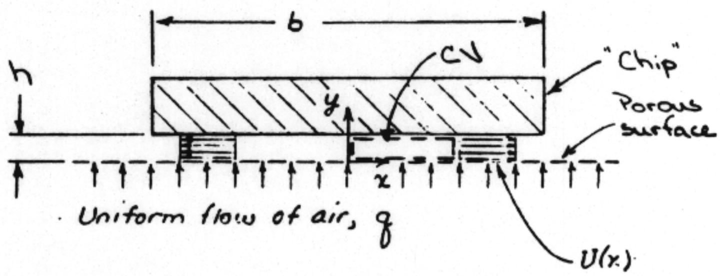
$$-\ln(1 - y/h) = \ln\frac{x}{h} - \ln c$$

or
$$\frac{x}{h}\left(1 - \frac{y}{h}\right) = \text{constant}$$ ←

Given: Rectangular "chip" floats on thin layer of air of thickness, $h = 0.5$ mm above a porous surface as shown. Chip width $b = 20$ mm; length L (perpendicular to diagram) $>> b$; no flow in z direction. Flow in x direction under chip may be assumed uniform; $p =$ constant; neglect frictional effects

Find: (a) Use a suitably chosen CV to show $U(x) = qx/h$ in the gap
(b) Find an expression for $\vec{a}_p$ in the gap
(c) Estimate the maximum value of $\vec{a}_p$
(d) Obtain an expression for $\partial p/\partial x$
(e) Sketch the pressure distribution under the chip
(f) Is the net pressure force on the chip directed up or down?
(g) Estimate the mass per unit length of the chip if $q = 0.06$ m³/sec/m²

Solution:



Assumptions:
(1) steady flow
(2) incompressible flow
(3) frictionless flow
(4) uniform flow at porous surface and in the gap at x.

(a) Apply continuity equation to CV, $\quad 0 = \cancel{\frac{\partial}{\partial t}\int \rho\, d\forall}^{\,0(1)} + \int_{CS} \rho \vec{V}\cdot d\vec{A}$

Then

$$0 = \{-|\rho q x L|\} + \{+|\rho U h L|\} \quad \text{or} \quad U = q\frac{x}{h} \longleftarrow U(x)$$

(b) Apply the substantial derivative definition

$$\vec{a}_p = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + \cancel{w}^{\,\text{given}}\frac{\partial \vec{V}}{\partial z} + \cancel{\frac{\partial \vec{V}}{\partial t}}^{\,0(1)}$$

Obtain v from differential continuity $\quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

$$\therefore \frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -\frac{q}{h} \quad \text{and} \quad v - v_0 = \int_0^y -\frac{q}{h}\,dy + f(x) = -\frac{q}{h}y + f(x)$$

$$\text{or} \quad v = q\left(1 - \frac{y}{h}\right) \qquad [f(x) = 0 \text{ since } v = v_0 = q = \text{const. along } y = 0]$$

$$a_{p_x} = u\frac{\partial u}{\partial x} + \cancel{v\frac{\partial u}{\partial y}} = q\frac{x}{h}\left(\frac{q}{h}\right) = \frac{q^2 x}{h^2}$$

$$a_{p_y} = \cancel{u\frac{\partial v}{\partial x}} + v\frac{\partial v}{\partial y} = q\left(1 - \frac{y}{h}\right)\left(-\frac{q}{h}\right) = \frac{q^2}{h}\left(\frac{y}{h} - 1\right)$$

$$\vec{a}_p = \frac{q^2 x}{h^2}\hat{i} + \frac{q^2}{h}\left(\frac{y}{h} - 1\right)\hat{j} = \frac{q^2}{h}\left[\frac{x}{h}\hat{i} + \left(\frac{y}{h} - 1\right)\hat{j}\right]$$

(c) The magnitude of $|\vec{a}_p| = \frac{q^2}{h}\left[\left(\frac{x}{h}\right)^2 + \left(\frac{y}{h} - 1\right)^2\right]^{1/2}$ is a maximum at $x = \frac{b}{2}$, $y = 0$

$$|\vec{a}_p|_{max} = \frac{q^2}{h}\left[\left(\frac{b}{2h}\right)^2 + 1\right]^{1/2} = 144 \text{ m/s}^2 \longleftarrow |\vec{a}_p|_{max}$$

(d) To obtain $\partial p/\partial x$ write the x component of the Euler equation

$$-\frac{\partial p}{\partial x} + \rho g_x = \rho a_{p_x} \quad \therefore \quad \frac{\partial p}{\partial x} = -\rho a_{p_x} = -\frac{\rho q_0^2 x}{h^2}$$ ← $\frac{\partial p}{\partial x}$

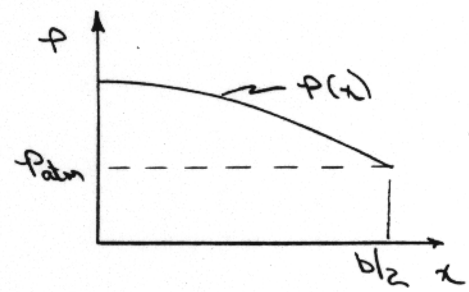
(e) To obtain an expression for the pressure distribution, $p(x)$ we need to separate variables and integrate noting that $p = p_{atm}$ at $x = b/2$. Thus.

$$p - p_{atm} = \int_{b/2}^{x} \frac{\partial p}{\partial x}\, dx = -\int_{b/2}^{x} \frac{\rho q_0^2}{h^2} x = -\frac{\rho q_0^2 x^2}{2 h^2}\Bigg]_{b/2}^{x}$$

$$p - p_{atm} = \frac{\rho q_0^2}{2h^2}\left[\left(\frac{b}{2}\right)^2 - x^2\right] = \frac{\rho q_0^2 b^2}{8 h^2}\left[1 - \left(\frac{2x}{b}\right)^2\right]$$

$$p = p_{atm} + \frac{\rho q_0^2 b^2}{8h^2}\left[1 - \left(\frac{2x}{b}\right)^2\right]$$ ← $p(x)$

(f) The net pressure force on the chip is up. Note that the pressure on the chip is greater than p_{atm} over the entire chip surface



(g) To estimate the mass per unit weight of the chip we must determine the net pressure force on the chip

$$F_{net} = \int_A (p - p_{atm})\, dA = 2\int_0^{b/2} \frac{\rho q_0^2 b^2}{8h^2}\left[1 - \left(\frac{2x}{b}\right)^2\right] L\, dx$$

$$= \frac{\rho q_0^2 b^2 L}{4h^2}\left[x - \frac{4}{3}\frac{x^3}{b^2}\right]_0^{b/2} = \frac{\rho q_0^2 b^2 L}{4h^2}\left[\frac{b}{2} - \frac{1}{3}\frac{b}{2}\right]$$

$$F_{net} = \frac{\rho q_0^2 b^3 L}{12 h^2}$$

The weight of the chip, $W = Mg$, must be balanced by the net pressure force. Hence

$$Mg = F_{net} = \frac{\rho q_0^2 b^3 L}{12 h^2}$$

$$\frac{M}{L} = \frac{\rho q_0^2 b^3}{12 h^2 g}$$

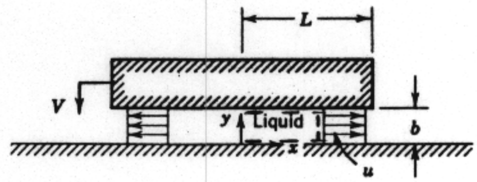
$$= \frac{1}{12} \times 1.23\,\frac{kg}{m^3} \times (0.06)^2 \frac{m^6}{s^2 \cdot m^4} \times (0.02)^3 m^3 \times \frac{1}{(0.0005)^2 m^2} \times \frac{s^2}{9.81 m}$$

$$\frac{M}{L} = 1.20 \times 10^{-3}\ kg/m$$ ← mass/length

Problem 6.20

Given: Upper plane surface moving downward at constant speed V causes incompressible liquid layer to be squeezed between surfaces as shown. Depth w in z direction and $w \gg L$.

Find: (a) Show that $u = Vx/b$ within the gap $(b = b_0 - Vt)$
(b) expression for a_x
(c) $\partial p/\partial x$
(d) $p(x)$
(e) net pressure force on upper surface



Solution:

Basic equations: $0 = \frac{\partial}{\partial t}\int_{cv} \rho\, d\forall + \int_{cs} \rho \vec{V}\cdot d\vec{A}$

$-\nabla p + \rho\vec{g} = \rho \frac{D\vec{V}}{Dt}$ $\qquad \vec{F} = -\int p\, d\vec{A}$

(a) For the deformable cv shown

$$0 = \frac{\partial}{\partial t}\int_0^y \rho w x\, dy + \rho u w y = \rho w x \frac{dy}{dt} + \rho u w y$$

But $dy/dt = -V$ and hence $u = \frac{Vx}{y}$

If $y = b_0$ at $t = 0$, then $y = b = b_0 - Vt$ at any time t

$$\therefore\ u = \frac{Vx}{b}$$ u(t)

(b) $a_x = \frac{Du}{Dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$ (the $v\,\partial u/\partial y$ and $w\,\partial u/\partial z$ terms struck out, each marked $\rightarrow 0\,(1)$)

Assumptions: (1) $u \neq u(y)$, $w = 0$

$$a_x = \frac{Vx}{b}\left(\frac{V}{b}\right) + \frac{\partial u}{\partial b}\frac{\partial b}{\partial t} = \frac{V^2 x}{b^2} + \left(-\frac{Vx}{b^2}\right)(-V) = \frac{2V^2 x}{b^2}$$ a_x

(c) From Euler's equation in the x direction with $g_x = 0$

$$\frac{\partial p}{\partial x} = -\rho a_x = -\frac{2\rho V^2 x}{b^2}$$ $\frac{\partial p}{\partial x}$

(d) $$p - p_{atm} = \int_L^x \frac{\partial p}{\partial x}\, dx = \int_L^x -\frac{2\rho V^2}{b^2} x\, dx = -\frac{\rho V^2 x^2}{b^2}\Big]_L^x = \frac{\rho V^2 L^2}{b^2}\left[1 - \left(\frac{x}{L}\right)^2\right]$$ $p(x)$

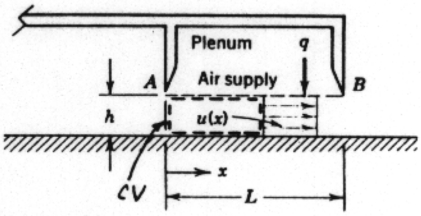
(e) $$F_y = \int_A (p - p_{atm})\, dA = 2\int_0^L \frac{\rho V^2 L^2}{b^2}\left[1 - \left(\frac{x}{L}\right)^2\right] w\, dx$$

$$= 2\int_0^1 \frac{\rho V^2 L^3}{b^2}\left[1 - \left(\frac{x}{L}\right)^2\right] w\, d\left(\frac{x}{L}\right) = \frac{2\rho V^2 L^3 w}{b^2}\left[\left(\frac{x}{L}\right) - \frac{1}{3}\left(\frac{x}{L}\right)^3\right]_0^1$$

$$F_y = \frac{4\rho V^2 L^3 w}{3b^2}$$ (upward, since $F_y > 0$) F_y

Problem 6.21

Given: Load pallet supported by air:



Flow is incompressible, uniform, and frictionless; $h \ll L$.

No flow across plane at $x = 0$.

Find: (a) Use a suitable CV to show $u(x) = qx/h$ in the gap.

(b) Calculate the acceleration of a fluid particle in the gap.

(c) Evaluate the pressure gradient, $\partial p/\partial x$.

(d) Sketch the pressure distribution; indicate pressure at $x = L$.

Solution: Choose a CV in the gap, from 0 to x, as shown.

Basic equations: $0 = \cancel{\frac{\partial}{\partial t}\int_{CV} \rho\, d\forall}^{=0(1)} + \int_{CS} \rho \vec{V} \cdot d\vec{A}$

$$a_x = u\frac{\partial u}{\partial x} + \cancel{v\frac{\partial u}{\partial y}}^{=0(3)} + \cancel{w\frac{\partial u}{\partial z}}^{=0(4)} + \cancel{\frac{\partial u}{\partial t}}^{=0(1)} \qquad -\frac{\partial p}{\partial x} + \cancel{\rho g_x}^{=0(5)} = \rho a_{p_x}$$

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Uniform flow at each section
(4) No variation with z
(5) Horizontal, so $g_x = 0$

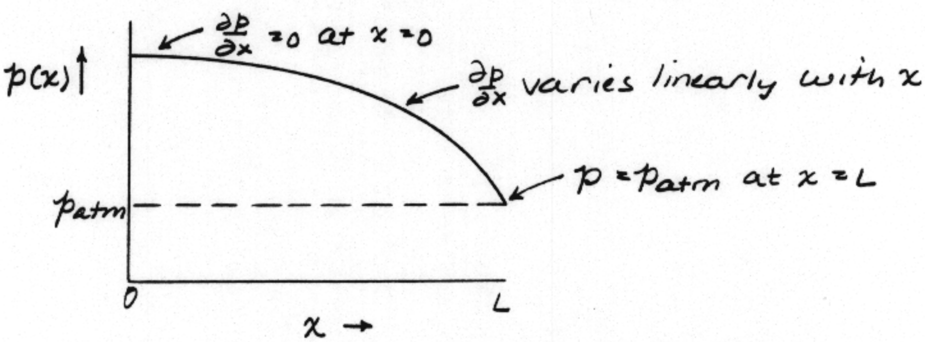
From continuity,

$$0 = \{-|\rho q w x|\} + \{+|\rho u(x) w h|\} \quad \text{so} \quad u(x) = q\frac{x}{h}$$ u(x)

The acceleration is $a_{p_x} = \left(q\frac{x}{h}\right)\left(q\frac{1}{h}\right) = \frac{q^2 x}{h^2}$ a_{p_x}

The pressure gradient is $\frac{\partial p}{\partial x} = -\rho a_{p_x} = -\frac{\rho q^2 x}{h^2}$ $\frac{\partial p}{\partial x}$

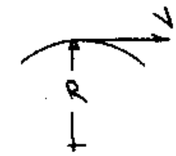
Sketching:



Sketch

Problem 6.22

Given: Air at 20 psia, 100°F flows around a smooth corner
Velocity = 150 ft/s
Radius of curvature of streamline is 3 in.



Find: (a) magnitude of centripetal acceleration in G's
(b) pressure gradient, $\frac{\partial p}{\partial r}$

Solution:
Basic equations: $\rho \vec{g} - \nabla p = \rho \frac{D\vec{V}}{Dt}$ ......(1)

$\frac{D\vec{V}}{Dt} = \vec{a}_p$ ....(2) $\qquad p = \rho R T$ .....(3)

Assumptions: (1) ρ = constant
(2) frictionless flow
(3) $\vec{g} = -g\hat{k}$

Writing the r component of equation (1)

$$\cancel{g_r}^{0} - \frac{1}{\rho}\frac{\partial p}{\partial r} = a_r = \cancel{\frac{\partial V_r}{\partial t}}^{0} + \cancel{V_r}^{0}\frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\cancel{\frac{\partial V_r}{\partial \theta}}^{0} + \cancel{V_z}^{0}\frac{\partial V_r}{\partial z} - \frac{V_\theta^2}{r}$$

$$a_r = -\frac{V_\theta^2}{r} \qquad \frac{a_r}{g} = -\frac{V_\theta^2}{rg} = -(150)^2\frac{ft^2}{s^2} \times \frac{1}{3\,in} \times \frac{12\,in}{ft} \times \frac{s^2}{32.2\,ft}$$

$$\frac{a_r}{g} = -2800 \text{ G's} \qquad \frac{a_r}{g}$$

Also

$$\frac{\partial p}{\partial r} = \rho\frac{V_\theta^2}{r}$$

where $\rho = \frac{p}{RT} = 20\frac{lbf}{in^2} \times \frac{lbm \cdot {}^\circ R}{53.3\,ft\cdot lbf} \times \frac{1}{560^\circ R} \times \frac{144\,in^2}{ft^2} \times \frac{slug}{32.2\,lbm}$

$$\rho = 0.003 \text{ slug}/ft^3$$

$$\frac{\partial p}{\partial r} = \rho\frac{V_\theta^2}{r} = 0.003\frac{slug}{ft^3} \times (150)^2\frac{ft^2}{s^2} \times \frac{1}{3\,in} \times \frac{12\,in}{ft} \times \frac{lbf\cdot s^2}{ft\cdot slug}$$

$$\frac{\partial p}{\partial r} = 270 \frac{lbf/ft^2}{ft} \qquad \frac{\partial p}{\partial r}$$

Given: The velocity field for steady, frictionless, incompressible flow (from right to left) over a stationary circular cylinder of radius, a, is given by

$$\vec{V} = U\left[\left(\frac{a}{r}\right)^2 - 1\right]\cos\theta\,\hat{e}_r + U\left[\left(\frac{a}{r}\right)^2 + 1\right]\sin\theta\,\hat{e}_\theta$$

Consider flow along the streamline forming the cylinder surface, i.e. $r = a$

Find: The pressure gradient along cylinder surface
Plot $V(r)$ along $\theta = \pi/2$ for $r > a$.

Solution:

Basic equation: $\rho\vec{g} - \nabla p = \rho \frac{D\vec{V}}{Dt}$

Assumptions: (1) neglect body force

Along the surface, $r = a$, $\vec{V} = 2U\sin\theta\,\hat{i}_\theta$.

Computing equations:

$$-\frac{1}{\rho}\frac{\partial p}{\partial r} = \frac{\partial V_r}{\partial t} + V_r\frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} + V_z\frac{\partial V_r}{\partial z} - \frac{V_\theta^2}{r}$$

$$-\frac{1}{\rho r}\frac{\partial p}{\partial \theta} = \frac{\partial V_\theta}{\partial t} + V_r\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + V_z\frac{\partial V_\theta}{\partial z} + \frac{V_r V_\theta}{r}$$

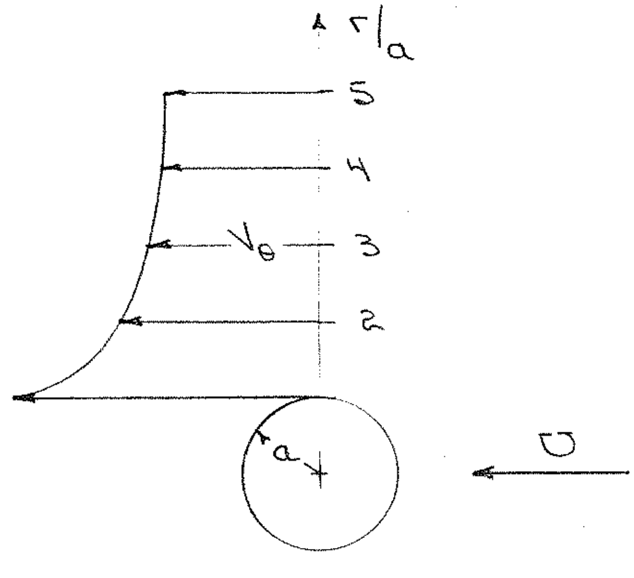
$$\frac{\partial p}{\partial r} = \rho\frac{V_\theta^2}{r} = \rho\frac{[2U\sin\theta]^2}{a} = \frac{4U^2\rho}{a}\sin^2\theta$$

$$\frac{1}{r}\frac{\partial p}{\partial \theta} = -\rho\frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} = -\rho\frac{[2U\sin\theta]}{a}(2U\cos\theta) = -\frac{4U^2\rho}{a}\sin\theta\cos\theta$$

$$\nabla p = \hat{i}_r\frac{\partial p}{\partial r} + \hat{i}_\theta\frac{1}{r}\frac{\partial p}{\partial \theta} = \frac{4\rho U^2}{a}\sin\theta\left(\hat{i}_r\sin\theta - \hat{i}_\theta\cos\theta\right) \quad \longleftarrow \nabla p$$

Along $\theta = \frac{\pi}{2}$, $\vec{V} = U\left[\left(\frac{a}{r}\right)^2 + 1\right]\hat{e}_\theta$

$\frac{r}{a}$	V_θ
1	$2U$
2	$1.25U$
3	$1.111U$
4	$1.063U$
5	$1.04U$

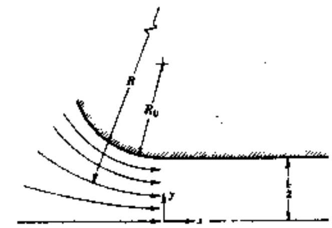


Problem 6.24

Given: Radius of curvature of streamlines at wind tunnel inlet is modeled as

$$R = \frac{L/2}{y} R_0$$

Speed along each streamline assumed constant at $V = 20$ m/s ; $L = 0.15$ m, $R = 0.6$ m.



Find: Δp between $y = 0$ and tunnel wall ($y = L/2$)

Solution:

Basic equation: $\frac{\partial p}{\partial n} = \rho \frac{V^2}{R}$

Assumptions: (1) steady flow (2) frictionless flow
(3) neglect body forces
(4) constant speed along each streamline

At the inlet section, $p = p(y)$

$$\therefore \quad \frac{dp}{dn} = -\frac{dp}{dy} = \rho \frac{V^2}{R} = \rho V^2 \frac{2y}{R_0 L}$$

$$\therefore \quad dp = -\frac{\rho V^2}{R_0 L} 2y \, dy$$

$$p_{L/2} - p_0 = \int_0^{L/2} dp = -\frac{2\rho V^2}{R_0 L} \int_0^{L/2} y \, dy = -\frac{2\rho V^2}{R_0 L} \frac{y^2}{2} \Big]_0^{L/2}$$

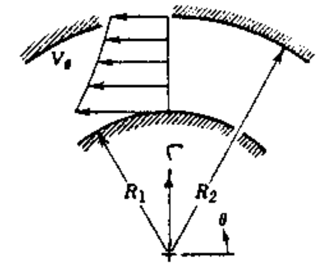
$$p_{L/2} - p_0 = -\frac{\rho V^2}{R_0 L} \frac{L^2}{4} = -\frac{\rho V^2 L}{4 R_0}$$

$$p_{L/2} - p_0 = -1.225 \frac{kg}{m^3} \times \left(20 \frac{m}{s}\right)^2 \times 0.15\,m \times \frac{1}{4} \times \frac{1}{0.6\,m} \times \frac{N \cdot s^2}{kg \cdot m}$$

$$p_{L/2} - p_0 = -30.6 \; N/m^2 \qquad \longleftarrow \; p_{L/2} - p_0$$

Problem 6.25

Given: Velocity variation at midsection of 180° bend is given by rV_θ = constant. Cross section of the bend is square.



Find: Derive an equation for the pressure difference, $p_2 - p_1$. (Express the answer in terms of $\dot{m}$, ρ, R_1, R_2, and the depth of the bend, h)

Solution:

Assumptions: (1) frictionless flow (Euler's equations apply)
(2) ρ = constant
(3) $V_\theta = V_\theta(r)$ only
(4) streamlines are circular in the bend.

Apply Eulers "n" equation, $\frac{1}{\rho}\frac{\partial p}{\partial n} = \frac{V^2}{R}$

Then we can write

$$\frac{dp}{dr} = \rho\frac{V^2}{R} = \rho\frac{V_\theta^2}{r} \quad \text{where } V_\theta = \frac{c}{r}$$

Separating variables, $dp = \rho\frac{V_\theta^2}{r}dr = \rho\frac{c^2}{r^3}dr$

$$p_2 - p_1 = \int_{p_1}^{p_2} dp = \rho c^2\int_{R_1}^{R_2}\frac{dr}{r^3} = \rho c^2\left(-\frac{1}{2}\right)\left[r^{-2}\right]_{R_1}^{R_2}$$

$$p_2 - p_1 = -\frac{1}{2}\rho c^2\left[\frac{1}{R_2^2} - \frac{1}{R_1^2}\right] = -\frac{1}{2}\rho c^2\left[\frac{R_1^2 - R_2^2}{R_1^2R_2^2}\right]$$

$$p_2 - p_1 = \frac{1}{2}\rho c^2\frac{(R_2^2 - R_1^2)}{R_1^2R_2^2}$$

The constant, c, can be written in terms of the mass flow rate, $\dot{m}$.

$$\dot{m} = \int\rho\vec{V}\cdot d\vec{A} = \int\rho V_\theta h\,dr = \rho h\int_{R_1}^{R_2}\frac{c}{r}dr = \rho hc\left[\ln r\right]_{R_1}^{R_2} = \rho hc\ln\frac{R_2}{R_1}$$

Solving for c, $c = \frac{\dot{m}}{\rho h\ln\frac{R_2}{R_1}}$

Substituting into the expression for $p_2 - p_1$,

$$p_2 - p_1 = \frac{1}{2}\rho\frac{\dot{m}^2}{\rho^2h^2}\frac{1}{\left(\ln\frac{R_2}{R_1}\right)^2}\frac{\left[R_2^2 - R_1^2\right]}{R_1^2R_2^2}$$

$$p_2 - p_1 = \frac{\dot{m}^2}{2\rho h^2}\frac{1}{\left(\ln\frac{R_2}{R_1}\right)^2}\frac{\left[R_2^2 - R_1^2\right]}{R_1^2R_2^2}$$

$p_2 - p_1$

Given: Velocity field $\vec{V} = (Ax + B)\hat{i} - Ay\hat{j}$ where $A = 1\ s^{-1}$, $B = 2\ m/s$ and coordinates are measured in meters.

Show: that streamlines are given by $(x + B/A)y = \text{constant}$.
Plot: streamlines through points $(x,y) = (1,1), (1,2), (2,2)$.

Find: (a) velocity vector & acceleration vector at (1,2); show these on streamline plot
(b) component of $\vec{a}_p$ along the streamline at (1,2); express as a vector.
(c) pressure gradient along streamline at (1,2) for air
(d) relative value of pressure at points (1,1) & (2,2)

Solution:

The slope of a streamline is $\left.\frac{dy}{dx}\right)_{s.l.} = \frac{v}{u} = \frac{-Ay}{Ax+B} = \frac{-y}{x+B/A}$

Then $\frac{dy}{y} + \frac{dx}{x+B/A} = 0$ and $\ln y + \ln(x + B/A) = \ln c$.

and $(x + B/A)y = \text{constant}$ ← Streamlines

For (1,1) $(x+2)y = 3$
(1,2) $(x+2)y = 6$
(2,2) $(x+2)y = 8$

} These streamlines are shown in the plot at the end of the problem solution

The particle acceleration $\vec{a}_p = \frac{D\vec{V}}{Dt} = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}$ (terms (1) and (2) crossed out)

Assumptions: (1) steady flow (given)
(2) 2-D (given) $\vec{V} \neq \vec{V}(z)$.

$\vec{a}_p = (Ax+B)\frac{\partial}{\partial x}\left[(Ax+B)\hat{i} - Ay\hat{j}\right] - Ay\frac{\partial}{\partial y}\left[(Ax+B)\hat{i} - Ay\hat{j}\right]$

$\vec{a}_p = (Ax+B)A\hat{i} - Ay(-A\hat{j}) = A(Ax+B)\hat{i} + A^2y\hat{j}$

At point (1,2).

$\vec{a}_p = \frac{1}{s}\left(\frac{1}{s}\times 1m + 2\frac{m}{s}\right)\hat{i} + \frac{1}{s^2}\times 2m\hat{j} = 3\hat{i} + 2\hat{j}\ m/s^2$ ← $\vec{a}_{(1,2)}$

$\vec{V} = \left(\frac{1}{s}\times 1m + 2\frac{m}{s}\right)\hat{i} - \frac{1}{s}\,2m\,\hat{j} = 3\hat{i} - 2\hat{j}\ m/s$ ← $\vec{V}_{(1,2)}$

$\vec{V}$ and $\vec{a}$ are shown on the streamline plot.

(b) The component of $\vec{a}_p$ along (tangent to) the streamline is given by $a_t = \vec{a}_p \cdot \hat{e}_t$ where $\hat{e}_t = \frac{\vec{V}}{|\vec{V}|}$

Thus $\hat{e}_t = \frac{3\hat{i} - 2\hat{j}}{\left[3^2 + (-2)^2\right]^{1/2}} = 0.832\hat{i} - 0.555\hat{j}$

and

$a_t = \vec{a}_p \cdot \hat{e}_t = (3\hat{i}+2\hat{j})\,m/s^2 \cdot (0.832\hat{i} - 0.555\hat{j}) = 1.39\ m/s^2$

$\vec{a}_t = 1.39\,\hat{e}_t = 1.16\hat{i} - 0.771\hat{j}\ m/s^2$ ← $\vec{a}_{t(1,2)}$

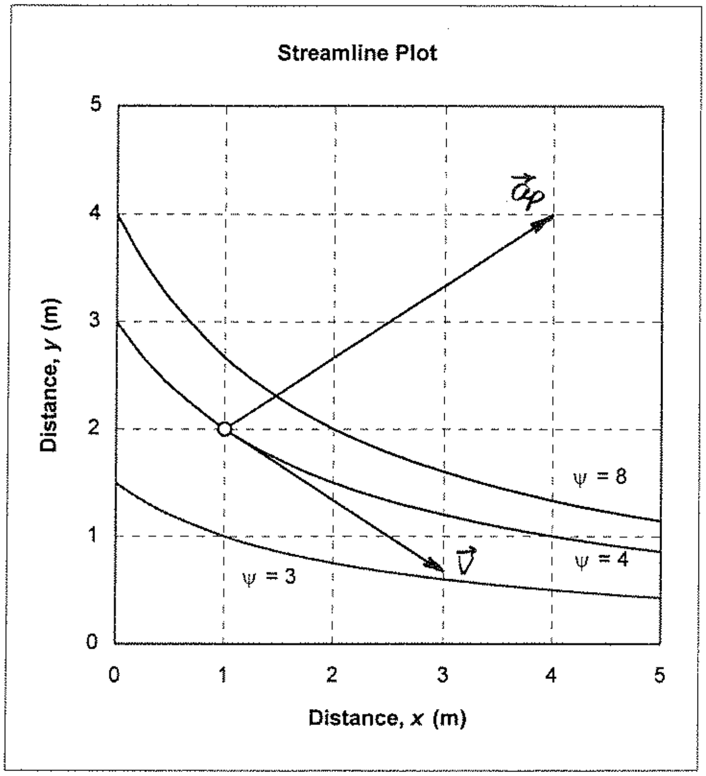
For frictionless flow, Euler's equation along a streamline (neglecting gravity, i.e. assuming flow in horizontal plane) is

$$\frac{\partial p}{\partial s} = -\rho V \frac{\partial V}{\partial s} = -\rho a_t = -1.23\,\frac{kg}{m^3} \times 1.39\,\frac{m}{s^2} \times \frac{N\cdot s^2}{kg\cdot m}$$

$\frac{\partial p}{\partial s} = -1.71\ N/m^2/m$ ← $\frac{\partial p}{\partial s}_{(1,2)}$

Looking at the streamline we would expect $p(2,2)$ to be less than $p(1,1)$ due to streamline curvature; Euler's equation normal to a streamline says

$\partial p/\partial n = \frac{\rho V^2}{R}$



Problem 6.27

Given: Velocity field $\vec{V} = Axy\hat{i} + By^2\hat{j}$; $A = 0.2\ m^{-1}\cdot s^{-1}$

B = constant

Find: (a) Value and units for B for incompressible flow.
(b) Acceleration of a fluid particle at point $(x, y) = (2, 1)$.
(c) Component of particle acceleration normal to velocity vector at this point.

Solution: Apply conservation of mass. For ρ = constant, $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$

But $w = 0$, so $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$. For this field, $\frac{\partial u}{\partial x} = Ay$ and $\frac{\partial v}{\partial y} = 2By$

Thus $Ay + 2By = 0$, or $B = -\frac{A}{2} = -0.1\ m^{-1}\cdot s^{-1}$ ← B

Acceleration of a particle is given by (since $\vec{V} = \vec{V}(x,y)$ only),

$$\vec{a}_p = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} = (Axy)Ay\hat{i} + (By^2)(Ax\hat{i} + 2By\hat{j})$$

$$\vec{a}_p = A^2xy^2\hat{i} - \frac{A^2}{2}xy^2\hat{i} + \frac{A^2}{2}y^3\hat{j} = \frac{A^2}{2}(xy^2\hat{i} + y^3\hat{j})$$

At point $(x,y) = (2,1)$ the acceleration is

$$\vec{a}_p = \frac{1}{2}\times(0.2)^2\frac{1}{m^2\cdot s^2}\left[2\,m\times(1)^2m^2\hat{i} + (1)^3m^3\hat{j}\right] = 0.04\hat{i} + 0.02\hat{j}\ m/s^2$$ ← $\vec{a}_p$ (1,2)

The velocity at point $(x,y) = (2,1)$ is

$$\vec{V} = \frac{0.2}{m\cdot s}\times 2\,m\times 1\,m\,\hat{i} - \frac{0.1}{m\cdot s}\times(1)^2m^2\hat{j} = 0.40\hat{i} - 0.10\hat{j}\ m/s$$

The unit vectors tangent and normal to the velocity vector are

$$\hat{e}_t = \frac{\vec{V}}{|\vec{V}|} = \frac{0.40\hat{i} - 0.10\hat{j}}{[(0.40)^2 + (0.10)^2]^{1/2}} = \frac{0.40\hat{i} - 0.10\hat{j}}{0.412} = 0.971\hat{i} - 0.243\hat{j}$$

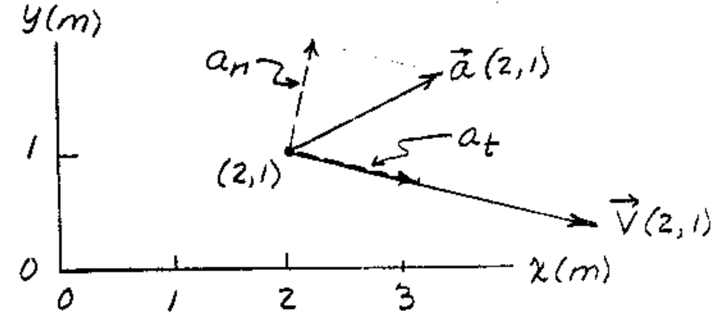
and

$$\hat{e}_n = 0.243\hat{i} + 0.971\hat{j}$$

Thus $a_n = \vec{a}\cdot\vec{e}_n = (0.04\hat{i} + 0.02\hat{j})\frac{m}{s^2}\cdot(0.243\hat{i} + 0.971\hat{j})$

$$a_n = [0.04(0.243) + 0.02(0.971)]\frac{m}{s^2} = 0.0291\ m/s^2$$ ← a_n

Plotting:



Given: The x component of velocity in a 2-D, incompressible flow field is
$u = Ax^2$ where $A = 1\ ft^{-1} \cdot s^{-1}$ and coordinates are in ft; $w = 0$ and $\partial/\partial z = 0$.

Find: (a) acceleration of fluid particle at $(x,y) = (1,2)$
(b) radius of curvature of streamline at $(1,2)$

Plot: streamline through $(1,2)$; show velocity and acceleration vectors on the plot.

Solution:

For 2-D incompressible flow $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$, so $\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}$

$$v = \int \frac{\partial v}{\partial y} dy + f(x) = \int -\frac{\partial u}{\partial x} dy + f(x) = -\int 2Ax\, dy + f(x) = -2Axy + f(x).$$

Choose the simplest solution, $f(x) = 0$, so $v = -2Axy$. Hence

$$\vec{V} = Ax^2\hat{i} - 2Axy\hat{j} = A\left[x^2\hat{i} - 2xy\hat{j}\right]$$

The acceleration of a fluid particle is

$$\vec{a}_p = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} = Ax^2\left[A(2x\hat{i} - 2y\hat{j})\right] - 2Axy\left[-2Ax\hat{j}\right]$$

$$\vec{a}_p = 2A^2x^3\hat{i} + 2A^2x^2y\hat{j} = 2A^2x^2\left[x\hat{i} + y\hat{j}\right]$$

At the point $(1,2)$

$$\vec{a}_p = 2 \times \frac{(1)^2}{ft^2 s^2} \times (1)^2 ft^2 \left[1\,ft\,\hat{i} + 2\,ft\,\hat{j}\right] = 2\hat{i} + 4\hat{j}\ ft/s^2 \longleftarrow \vec{a}_{(1,2)}$$

$$\vec{V} = \frac{1}{ft \cdot s}\left[(1)^2 ft^2\,\hat{i} - 2(1\,ft)(2\,ft)\hat{j}\right] = \hat{i} - 4\hat{j}\ ft/s$$

The unit vector tangent to the streamline is

$$\hat{e}_t = \frac{\vec{V}}{|\vec{V}|} = \frac{\hat{i} - 4\hat{j}}{\left[(1)^2 + (-4)^2\right]^{1/2}} = 0.243\hat{i} - 0.970\hat{j}$$

The unit vector normal to the streamline is

$$\hat{e}_n = \hat{e}_t \times \hat{k} = (0.243\hat{i} - 0.970\hat{j}) \times \hat{k} = -0.970\hat{i} - 0.243\hat{j}$$

The normal component of acceleration is

$$a_n = -\frac{V^2}{R} = \vec{a} \cdot \hat{e}_n = (2\hat{i} + 4\hat{j}) \cdot (-0.970\hat{i} - 0.243\hat{j})$$

$$-\frac{V^2}{R} = -2.91\ ft/s^2$$

$$R = \frac{V^2}{2.91} = \frac{17\ ft^2/s^2}{2.91\ ft/s^2} = 5.84\ ft \longleftarrow R$$

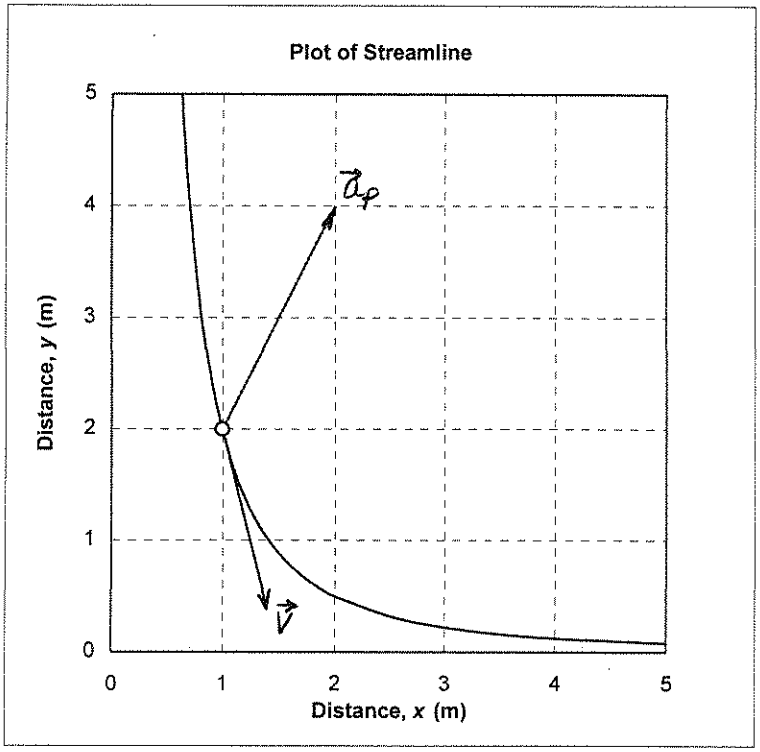
The slope of the streamline is given by

$$\left.\frac{dy}{dx}\right)_{s.l.} = \frac{v}{u} = \frac{-2Axy}{Ax^2} = \frac{-2y}{x}$$

Thus $\frac{dy}{y} + \frac{2\,dx}{x} = 0$ and $\ln y + \ln x^2 = \ln C$ or

$$x^2 y = C$$

The equation of the streamline through (1,2) is $x^2 y = 2$.



Given: Incompressible, 2-D flow with $u = Axy$, $v = 0$; $A = 2\ ft^{-1}s^{-1}$

Find: (a) Acceleration of particle at $(x, y) = (2, 1)$.
(b) Radius of curvature of streamline at that point.
(c) Plot streamline, show velocity vector and acceleration vector.

Solution: For two-d. incompressible flow, $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$, so

$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -Ay$; Integrating, $v = -\frac{1}{2}Ay^2$; $\vec{V} = Axy\hat{i} - \frac{1}{2}Ay^2\hat{j}$.

The acceleration is

$$a_{p_x} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = (Axy)(Ay) + (-\tfrac{1}{2}Ay^2)(Ax) = \tfrac{1}{2}A^2xy^2$$

$$a_{p_y} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = (Axy)(0) + (-\tfrac{1}{2}Ay^2)(-Ay) = \tfrac{1}{2}A^2y^3$$

$$\vec{a}_p = \tfrac{1}{2}A^2xy^2\hat{i} + \tfrac{1}{2}A^2y^3\hat{j}\text{; at } (2,1)\quad \vec{a}_p = 4\hat{i} + 2\hat{j}\ (ft/s^2)$$ $\vec{a}_p$

Note $a_n = \frac{V^2}{R}$, so $R = \frac{V^2}{a_n}$, where a_n is acceleration normal to $\vec{V}$

At $(2,1)$, $\vec{V} = 4\hat{i} - 1\hat{j}$ ft/s, so $V^2 = (4)^2 + (1)^2 = 17\ ft^2/s^2$

To find a_n, dot $\vec{a}_p$ with $\hat{e}_n$, the unit normal vector. To find $\hat{e}_n$, set

$$\hat{e}_n = -\frac{v}{V}\hat{i} + \frac{u}{V}\hat{j} = \frac{1}{\sqrt{17}}\hat{i} + \frac{4}{\sqrt{17}}\hat{j}$$

$$a_n = \hat{e}_n \cdot \hat{a}_p = \frac{4}{\sqrt{17}} + \frac{8}{\sqrt{17}} = \frac{12}{\sqrt{17}} = 2.91\ ft/s^2$$

Substituting

$$R = \frac{V^2}{a_n} = 17\frac{ft^2}{s^2} \times \frac{s^2}{2.91\ ft} = 5.84\ ft$$ R

The streamline is $\frac{dx}{u} = \frac{dy}{v} = \frac{dx}{Axy} = \frac{dy}{-\frac{1}{2}Ay^2}$ or $\frac{dx}{x} + 2\frac{dy}{y} = 0$

Integrating, $\ln x + 2\ln y = \ln C$ or $xy^2 = C$

For $(x,y) = (2,1)$, then $C = 2\ ft^3$.

The plot and streamlines are on the following page.

Components of Velocity and Acceleration:

Input Parameters:

A = 2 $ft^{-1}s^{-1}$

Calculated Values:

C = 2 ft^3

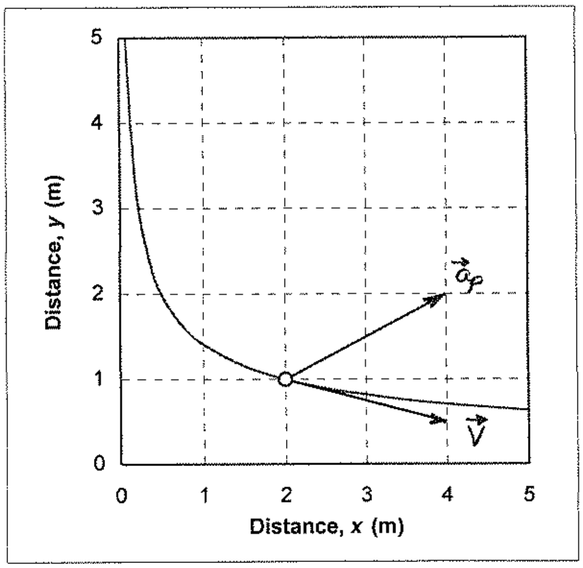
Coord. x	Coord. y	Velocity, V_x	Velocity, V_y	Velocity, V	Accel., a_x	Accel., a_y	Accel., a	Normal Accel., a_n
0.08	5.00							
0.2	3.16							
0.4	2.24							
0.5	2.00	2.00	-4.00	4.47	2.00	16.0	16.1	8.94
0.6	1.83							
0.8	1.58							
1.0	1.41	2.83	-2.00	3.46	2.83	5.66	6.32	6.25
1.5	1.15	3.46	-1.33	3.71	3.46	3.08	4.63	4.12
2.0	1.00	4.00	-1.00	4.12	4.00	2.00	4.47	2.91
2.5	0.89	4.47	-0.80	4.54	4.47	1.43	4.70	2.20
3.0	0.82	4.90	-0.67	4.94	4.90	1.09	5.02	1.74
3.5	0.76	5.29	-0.57	5.32	5.29	0.86	5.36	1.43
4.0	0.71	5.66	-0.50	5.68	5.66	0.71	5.70	1.20
4.5	0.67	6.00	-0.44	6.02	6.00	0.59	6.03	1.03
5.0	0.63	6.32	-0.40	6.34	6.32	0.51	6.34	0.90

Acceleration:

2	1
4	2

Velocity:

2	1
4	0.5



Given: The y component of velocity in a 2-D, incompressible flow field is
$v = -Axy$ where $A = 1\ m^{-1}s^{-1}$ and coordinates are in meters; $w = 0$ and $\partial/\partial z = 0$.

Find: (a) acceleration of fluid particle at $(x,y) = (1,2)$
(b) radius of curvature of streamline at $(1,2)$.

Plot: streamline through $(1,2)$; show velocity and acceleration vectors on the plot.

Solution:

For 2-D incompressible flow $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$, so $\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$

$$u = \int \frac{\partial u}{\partial x}dx + f(y) = \int -\frac{\partial v}{\partial y}dx + f(y) = -\int(-Ax)dx + f(y) = \frac{Ax^2}{2} + f(y)$$

Choose the simplest solution, $f(y) = 0$, so $u = \frac{Ax^2}{2}$. Hence

$$\vec{V} = \frac{Ax^2}{2}\hat{i} - Axy\hat{j} = A\left(\frac{x^2}{2}\hat{i} - xy\hat{j}\right)$$

The acceleration of a fluid particle is

$$\vec{a}_p = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} = \frac{Ax^2}{2}(Ax\hat{i} - Ay\hat{j}) - Axy(-Ax\hat{j})$$

$$\vec{a}_p = \frac{A^2x^3}{2}\hat{i} + \frac{A^2x^2y}{2}\hat{j} = \frac{A^2}{2}(x^3\hat{i} + x^2y\hat{j})$$

At the point $(1,2)$

$$\vec{a}_p = \frac{1}{2}\times(1)\frac{1}{m^2s^2}\left[(1)^3m^3\hat{i} + (1)^2(2)m^3\hat{j}\right] = 0.5\hat{i} + \hat{j}\ m/s^2 \quad \vec{a}(1,2)$$

$$\vec{V} = \frac{1}{m\cdot s}\left[\frac{1}{2}(1)^2m^2\hat{i} - (1)(2)m^2\hat{j}\right] = 0.5\hat{i} - 2\hat{j}\ m/s$$

The unit vector tangent to the streamline is

$$\hat{e}_t = \frac{\vec{V}}{|\vec{V}|} = \frac{0.5\hat{i} - 2\hat{j}}{[(0.5)^2 + (-2)^2]^{1/2}} = 0.243\hat{i} - 0.970\hat{j}$$

The unit vector normal to the streamline is

$$\vec{e}_n = \hat{e}_t \times \hat{k} = (0.243\hat{i} - 0.970\hat{j}) \times \hat{k} = -0.970\hat{i} - 0.243\hat{j}$$

The normal component of acceleration is

$$a_n = -\frac{V^2}{R} = \vec{a}\cdot\hat{e}_n = (0.5\hat{i} + \hat{j})\cdot(-0.970\hat{i} - 0.243\hat{j})$$

$$-\frac{V^2}{R} = -0.728\ m/s^2$$

$$R = \frac{V^2}{0.728} = \frac{4.25\ m^2/s^2}{0.728\ m/s^2} = 5.84\ m \quad R(1,2)$$

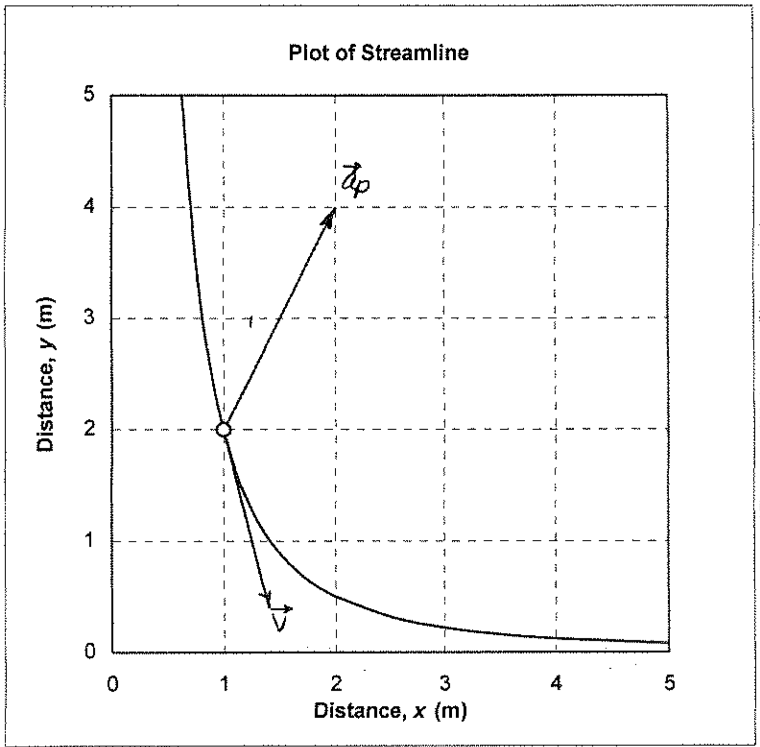
The slope of the streamlines is given by

$$\left.\frac{dy}{dx}\right)_{s\ell} = \frac{v}{u} = \frac{-Axy}{Ax^2/2} = -\frac{2y}{x}$$

Thus $\frac{dy}{y} + 2\frac{dx}{x} = 0$ and $\ln y + \ln x^2 = \ln c$ or

$$x^2 y = c$$

The equation of the streamline through (1,2) is $x^2 y = 2$



Problem 6.31

The x component of velocity in a two-dimensional incompressible flow field is given by $u = -\frac{\Lambda(x^2 - y^2)}{(x^2 + y^2)^2}$, where u is in m/s, the coordinates are measured in meters, and $\Lambda = 2\ m^3 \cdot s^{-1}$. Show that the simplest form of the y component of velocity is given by $v = -\frac{2\Lambda xy}{(x^2 + y^2)^2}$. There is no velocity component or variation in the z direction. Calculate the acceleration of fluid particles at points $(x, y) = (0, 1)$, $(0, 2)$ and $(0, 3)$. Estimate the radius of curvature of the streamlines passing through these points. What does the relation among the three points and their radii of curvature suggest to you about the flow field? Verify this by plotting these streamlines. [Hint: You will need to use an integrating factor.]

Given: *x* component of velocity field

Find: *y* component of velocity field; acceleration at several points; estimate radius of curvature; plot streamlines

Solution

The given data is $\Lambda = 2 \cdot \frac{m^3}{s}$ $\qquad u = -\frac{\Lambda \cdot \left(x^2 - y^2\right)}{\left(x^2 + y^2\right)^2}$

The governing equation (continuity) is $\frac{du}{dx} + \frac{dv}{dy} = 0$

Hence $$v = -\int \frac{du}{dx}\, dy = -\int \frac{2 \cdot \Lambda \cdot x \cdot \left(x^2 - 3 \cdot y^2\right)}{\left(x^2 + y^2\right)^3}\, dy$$

Integrating (using an integrating factor)

$$v = -\frac{2 \cdot \Lambda \cdot x \cdot y}{\left(x^2 + y^2\right)^2}$$

Alternatively, we could check that the given velocities *u* and *v* satisfy continuity

$$u = -\frac{\Lambda\cdot\left(x^2 - y^2\right)}{\left(x^2 + y^2\right)^2} \qquad \frac{du}{dx} = \frac{2\cdot\Lambda\cdot x\cdot\left(x^2 - 3\cdot y^2\right)}{\left(x^2 + y^2\right)^3}$$

$$v = -\frac{2\cdot\Lambda\cdot x\cdot y}{\left(x^2 + y^2\right)^2} \qquad \frac{dv}{dy} = -\frac{2\cdot\Lambda\cdot x\cdot\left(x^2 - 3\cdot y^2\right)}{\left(x^2 + y^2\right)^3}$$

so $$\frac{du}{dx} + \frac{dv}{dy} = 0$$

The governing equation for acceleration is

$$\vec{a}_p = \underbrace{\frac{D\vec{V}}{Dt}}_{\text{total acceleration of a particle}} = \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\text{convective acceleration}} + \underbrace{\frac{\partial \vec{V}}{\partial t}}_{\text{local acceleration}}$$

x - component $$a_x = u\cdot\frac{du}{dx} + v\cdot\frac{du}{dy}$$

$$a_x = \left[-\frac{\Lambda\cdot\left(x^2 - y^2\right)}{\left(x^2 + y^2\right)^2}\right]\cdot\left[\frac{2\cdot\Lambda\cdot x\cdot\left(x^2 - 3\cdot y^2\right)}{\left(x^2 + y^2\right)^3}\right] + \left[-\frac{2\cdot\Lambda\cdot x\cdot y}{\left(x^2 + y^2\right)^2}\right]\cdot\left[\frac{2\cdot\Lambda\cdot y\cdot\left(3\cdot x^2 - y^2\right)}{\left(x^2 + y^2\right)^3}\right]$$

$$a_x = -\frac{2\cdot\Lambda^2\cdot x}{\left(x^2+y^2\right)^3}$$

y - component $$a_y = u\cdot\frac{dv}{dx} + v\cdot\frac{dv}{dy}$$

$$a_y = \left[-\frac{\Lambda\cdot\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2}\right]\cdot\left[\frac{2\cdot\Lambda\cdot y\cdot\left(3\cdot x^2-y^2\right)}{\left(x^2+y^2\right)^3}\right] + \left[-\frac{2\cdot\Lambda\cdot x\cdot y}{\left(x^2+y^2\right)^2}\right]\cdot\left[\frac{2\cdot\Lambda\cdot y\cdot\left(3\cdot y^2-x^2\right)}{\left(x^2+y^2\right)^3}\right]$$

$$a_y = -\frac{2\cdot\Lambda^2\cdot y}{\left(x^2+y^2\right)^3}$$

Evaluating at point (0,1) $u = 2\cdot\frac{m}{s}$ $v = 0\cdot\frac{m}{s}$ $a_x = 0\cdot\frac{m}{s^2}$ $a_y = -8\cdot\frac{m}{s^2}$

Evaluating at point (0,2) $u = 0.5\cdot\frac{m}{s}$ $v = 0\cdot\frac{m}{s}$ $a_x = 0\cdot\frac{m}{s^2}$ $a_y = -0.25\cdot\frac{m}{s^2}$

Evaluating at point (0,3) $u = 0.222\cdot\frac{m}{s}$ $v = 0\cdot\frac{m}{s}$ $a_x = 0\cdot\frac{m}{s^2}$ $a_y = -0.0333\cdot\frac{m}{s^2}$

The instantaneous radius of curvature is obtained from $a_{radial} = -a_y = -\frac{u^2}{r}$ or $r = -\frac{u^2}{a_y}$

For the three points $y = 1\,m$ $\qquad r = \frac{\left(2\cdot\frac{m}{s}\right)^2}{8\cdot\frac{m}{s^2}}$ $\qquad r = 0.5\,m$

$y = 2\,m$ $\qquad r = \frac{\left(0.5\cdot\frac{m}{s}\right)^2}{0.25\cdot\frac{m}{s^2}}$ $\qquad r = 1\,m$

$y = 3\,m$ $\qquad r = \frac{\left(0.2222\cdot\frac{m}{s}\right)^2}{0.03333\cdot\frac{m}{s^2}}$ $\qquad r = 1.5\cdot m$

The radius of curvature in each case is 1/2 of the vertical distance from the origin. The streamli
form circles tangent to the *x* axis

The streamlines are given by $\frac{dy}{dx} = \frac{v}{u} = \frac{-\frac{2\cdot\Lambda\cdot x\cdot y}{\left(x^2+y^2\right)^2}}{-\frac{\Lambda\cdot\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2}} = \frac{2\cdot x\cdot y}{\left(x^2-y^2\right)}$

so $\qquad -2\cdot x\cdot y\cdot dx + \left(x^2 - y^2\right)\cdot dy = 0$

This is an inexact integral, so an integrating factor is needed

First we try $$R = \frac{1}{-2\cdot x\cdot y}\cdot\left[\frac{d}{dx}\left(x^2 - y^2\right) - \frac{d}{dy}(-2\cdot x\cdot y)\right] = -\frac{2}{y}$$

Then the integrating factor is $$F = e^{\int -\frac{2}{y}\,dy} = \frac{1}{y^2}$$

The equation becomes an exact integral $$-2\cdot\frac{x}{y}\cdot dx + \frac{\left(x^2 - y^2\right)}{y^2}\cdot dy = 0$$

So $$u = \int -2\cdot\frac{x}{y}\,dx = -\frac{x^2}{y} + f(y)$$ and $$u = \int \frac{\left(x^2 - y^2\right)}{y^2}\,dy = -\frac{x^2}{y} - y + g(x)$$

Comparing solutions $$\psi = \frac{x^2}{y} + y$$ or $$x^2 + y^2 = \psi\cdot y = \text{const}\cdot y$$

These form circles that are tangential to the *x* axis, as shown in the associated *Excel* workbook

Problem 6.31 (In Excel)

The x component of velocity in a two-dimensional incompressible flow field is given by $u = -\frac{\Lambda(x^2 - y^2)}{(x^2 + y^2)^2}$, where u is in m/s, the coordinates are measured in meters, and $\Lambda = 2\ \text{m}^3 \cdot \text{s}^{-1}$. Show that the simplest form of the y component of velocity is given by $v = -\frac{2\Lambda xy}{(x^2 + y^2)^2}$. There is no velocity component or variation in the z direction. Calculate the acceleration of fluid particles at points $(x, y) = (0, 1)$, $(0, 2)$ and $(0, 3)$. Estimate the radius of curvature of the streamlines passing through these points. What does the relation among the three points and their radii of curvature suggest to you about the flow field? Verify this by plotting these streamlines. [Hint: You will need to use an integrating factor.]

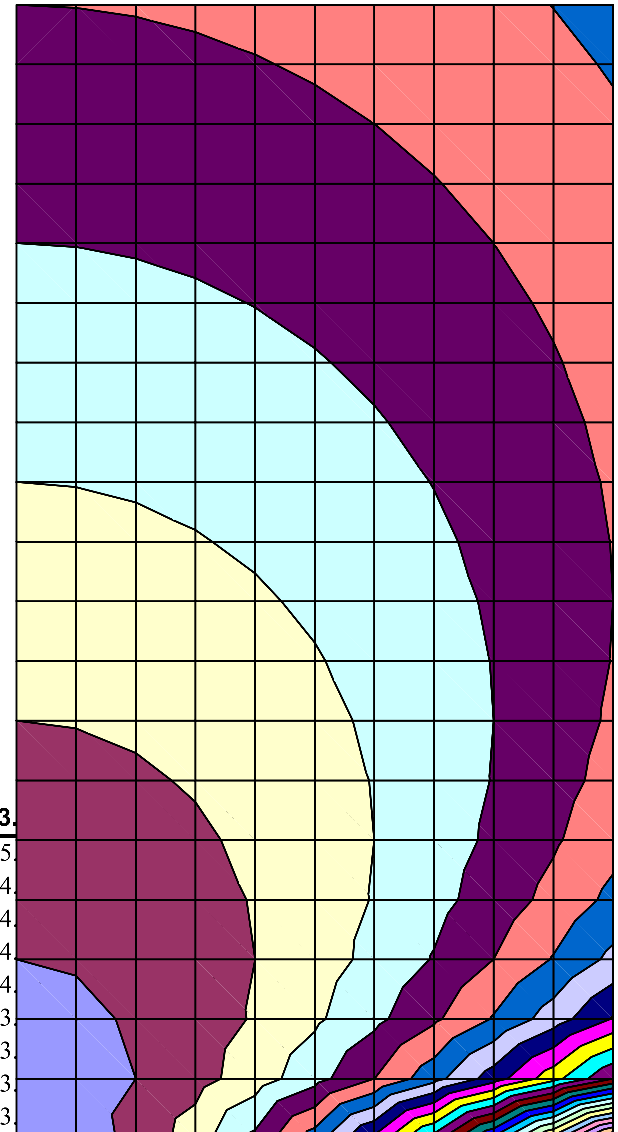
Given: x component of velocity

Find: Streamlines

The stream function is $\psi = \frac{x^2}{y} + y$

Solution

This function is computed and plotted below



y values

x values	0.10	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.25	2.50	2.75	3.00	3.25	3.						
2.50	62.6	25.3	13.0	9.08	7.25	6.25	5.67	5.32	5.13	5.03	5.00	5.02	5.08	5.17	5.						
2.25	50.7	20.5	10.6	7.50	6.06	5.30	4.88	4.64	4.53	4.50	4.53	4.59	4.69	4.81	4.						
2.00	40.1	16.3	8.50	6.08	5.00	4.45	4.17	4.04	4.00	4.03	4.10	4.20	4.33	4.48	4.						
1.75	30.7	12.5	6.63	4.83	4.06	3.70	3.54	3.50	3.53	3.61	3.73	3.86	4.02	4.19	4.						
1.50	22.6	9.25	5.00	3.75	3.25	3.05	3.00	3.04	3.13	3.25	3.40	3.57	3.75	3.94	4.						
1.25	15.7	6.50	3.63	2.83	2.56	2.50	2.54	2.64	2.78	2.94	3.13	3.32	3.52	3.73	3.						
1.00	10.1	4.25	2.50	2.08	2.00	2.05	2.17	2.32	2.50	2.69	2.90	3.11	3.33	3.56	3.						
0.75	5.73	2.50	1.63	1.50	1.56	1.70	1.88	2.07	2.28	2.50	2.73	2.95	3.19	3.42	3.						
0.50	2.60	1.25	1.00	1.08	1.25	1.45	1.67	1.89	2.13	2.36	2.60	2.84	3.08	3.33	3.						
0.25	0.73	0.50	0.63	0.83	1.06	1.30	1.54	1.79	2.03	2.28	2.53	2.77	3.02	3.27	3.52	3.77	4.02	4.26	4.51	4.76	5.01
0.00	0.10	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.25	2.50	2.75	3.00	3.25	3.50	3.75	4.00	4.25	4.50	4.75	5.00

Problem 6.32

Given: Velocity field $\vec{V} = Ax^2\hat{i} - Bxy\hat{j}$, where $A = 2\,m^{-1}s^{-1}$, $B = 4\,m^{-1}s^{-1}$ and coordinates are measured in meters

Show: that this is a possible incompressible flow

Find: (a) equation of streamline through point $(x,y) = (1,2)$
(b) expression for the acceleration of a fluid particle.
(c) radius of curvature of streamline at $(1,2)$

Solution:

For 2-D incompressible flow $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

For this flow $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 2Ax - Bx = 2(2)x - 4x = 0 \quad \therefore \rho = \text{const.}$

The slope of the streamline is given by

$$\left.\frac{dy}{dx}\right)_{s.l} = \frac{v}{u} = -\frac{Bxy}{Ax^2} = -\frac{By}{Ax} = -\frac{4y}{2x} = -\frac{2y}{x}$$

Thus $\frac{dy}{y} + 2\frac{dx}{x} = 0$ and $\ln y + \ln x^2 = \ln c$ or $x^2 y = c$

The streamline through point $(1,2)$ is $x^2 y = 2$ ——— streamline

The acceleration of a fluid particle is

$$\vec{a}_p = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} = Ax^2\left[2Ax\hat{i} - By\hat{j}\right] - Bxy\left[-Bx\hat{j}\right]$$

$$\vec{a}_p = 2A^2x^3\hat{i} + \left[B^2x^2y - ABx^2y\right]\hat{j} = 2A^2x^3\hat{i} + Bx^2y(B-A)\hat{j} \text{ ——— } \vec{a}_p$$

At the point $(1,2)$

$$\vec{a}_p = 2\times\frac{(2)^2}{m^2\cdot s^2}\times(1)^3m^3\hat{i} + \frac{4}{m\cdot s}\times(1)^2m^2\times 2m\left[4-2\right]\frac{1}{m\cdot s}\hat{j} = 8\hat{i} + 16\hat{j}\ m/s^2$$

$$\vec{V} = \frac{2}{m\cdot s}\times(1)m^2\hat{i} - \frac{4}{m\cdot s}\times(1m)\times(2m)\hat{j} = 2\hat{i} - 8\hat{j}\ m/s$$

The unit vector tangent to the streamline is

$$\hat{e}_t = \frac{\vec{V}}{|\vec{V}|} = \frac{2\hat{i} - 8\hat{j}}{[(2)^2 + (-8)^2]^{1/2}} = 0.243\hat{i} - 0.970\hat{j}$$

The unit vector normal to the streamline is

$$\hat{e}_n = \hat{e}_t \times \hat{k} = (0.243\hat{i} - 0.970\hat{j}) \times \hat{k} = -0.970\hat{i} - 0.243\hat{j}$$

The normal component of acceleration is

$$a_n = \vec{a}\cdot\hat{e}_n = (8\hat{i} + 16\hat{j})\cdot(-0.970\hat{i} - 0.243\hat{j}) = -11.6\ m/s^2$$

$$a_n = -\frac{V^2}{R} = -11.6 \quad \therefore R = \frac{V^2}{11.6} = \frac{68\ m^2/s^2}{11.6\ m/s^2}$$

$$R = 5.86\ m \text{ ——— } R(1,2)$$

Problem 6.33

Given: Flow of water with speed $V = 3$ m/s.

Find: Dynamic pressure, expressed in mm of mercury.

Solution:

Dynamic pressure is $p_d = \frac{1}{2}\rho V^2$

From hydrostatics, $p_d = \rho_{Hg} g \Delta h$

$$\therefore \Delta h = \frac{\rho V^2}{2\rho_{Hg} g} = \frac{V^2}{2\,SG_{Hg}\,g}$$

$$= \frac{1}{2} \times (3)^2 \frac{m^2}{s^2} \times \frac{1}{13.6} \times \frac{s^2}{9.81 m} \times 1000 \frac{mm}{m}$$

$$\Delta h = 33.7 \text{ mm Hg} \longleftarrow \Delta h$$

Problem 6.34

Given: Standard air

Find: Dynamic pressure that corresponds to $V = 100\ km/hr$

Solution: Dynamic pressure is $p_{dyn} = \frac{1}{2}\rho V^2$

For standard air, $\rho = 1.23\ kg/m^3$

Then $p_{dyn} = \frac{1}{2} \times 1.23 \frac{kg}{m^3} \times (100)^2 \frac{(km)^2}{(hr)^2} \times (1000)^2 \frac{m^2}{(km)^2} \times \frac{(hr)^2}{(3600)^2 s^2} \times \frac{N \cdot s^2}{kg \cdot m}$

$p_{dyn} = 475\ N/m^2$ ← p_{dyn}

This may be expressed conveniently as a water column height.

$p_{dyn} = \rho_{water}\, g\, h_{dyn}$

$h_{dyn} = \frac{p_{dyn}}{\rho_w g} = 475 \frac{N}{m^2} \times \frac{m^3}{999\ kg} \times \frac{s^2}{9.81\ m} \times \frac{kg \cdot m}{N \cdot s^2}$

$h_{dyn} = 0.0484\ m$ or $48.4\ mm$ ← h_{dyn}

Problem 6.35

You present your open hand out of the window of an automobile perpendicular to the airflow. Assuming for simplicity that the air pressure on the entire front surface is stagnation pressure (with respect to automobile coordinates), with atmospheric pressure on the rear surface, estimate the net force on your hand when driving at (a) 30 mph and (b) 60 mph. Do these results roughly correspond with your experience? Do the simplifications tend to make the calculated force an over- or underestimate?

Given: Velocity of automobile

Find: Estimates of aerodynamic force on hand

Solution

For air $\rho = 0.00238 \cdot \frac{\text{slug}}{\text{ft}^3}$

We need an estimate of the area of a typical hand. Personal inspection indicates that a good approximation is a square of sides 9 cm and 17 cm

$A = 9 \cdot \text{cm} \times 17 \cdot \text{cm}$ $A = 153\,\text{cm}^2$

The governing equation is the Bernoulli equation (in coordinates attached to the vehicle)

$$p_{atm} + \frac{1}{2} \cdot \rho \cdot V^2 = p_{stag}$$

where *V* is the free stream velocity

Hence, for p_{stag} on the front side of the hand, and p_{atm} on the rear, by assumption,

$$F = \left(p_{stag} - p_{atm}\right) \cdot A = \frac{1}{2} \cdot \rho \cdot V^2 \cdot A$$

(a) $V = 30 \cdot mph$

$$F = \frac{1}{2} \cdot \rho \cdot V^2 \cdot A = \frac{1}{2} \times 0.00238 \cdot \frac{slug}{ft^3} \times \left(30 \cdot mph \cdot \frac{22 \cdot \frac{ft}{s}}{15 \cdot mph} \right)^2 \times 153 \cdot cm^2 \times \left(\frac{\frac{1}{12} \cdot ft}{2.54 \cdot cm} \right)^2$$

$F = 0.379\,lbf$

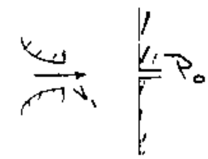
(a) $V = 60 \cdot mph$

$$F = \frac{1}{2} \cdot \rho \cdot V^2 \cdot A = \frac{1}{2} \times 0.00238 \cdot \frac{slug}{ft^3} \times \left(60 \cdot mph \cdot \frac{22 \cdot \frac{ft}{s}}{15 \cdot mph} \right)^2 \times 153 \cdot cm^2 \times \left(\frac{\frac{1}{12} \cdot ft}{2.54 \cdot cm} \right)^2$$

$F = 1.52\,lbf$

Problem 6.36

Given: Air discharging from a nozzle impinges on a wall as shown



$P_1 = 14.7$ psia $\quad T_1 = 40°F$

$P_o = 0.14$ in Hg gage

Find: the speed, V_1

Solution:

Basic equations: $\frac{P}{\rho} + \frac{V^2}{2} + gz = \text{constant}$ for flow

$\frac{dP}{dh} = \gamma$ for manometer reading P_o

Assumptions:
(1) steady flow
(2) incompressible flow
(3) frictionless flow
(4) flow along a streamline
(5) γ = constant for manometer
(6) air behaves as an ideal gas

From the Bernoulli equation

$$\frac{P_o}{\rho} = \frac{P_1}{\rho} + \frac{V_1^2}{2}$$

$$P_o - P_1 = \frac{1}{2}\rho V_1^2$$

For the manometer $\quad dP = \gamma\, dh$

$\therefore P_o - P_{atm} = \gamma \Delta h \quad$ where $\Delta h = 0.14$ in Hg

Since $P_1 = P_{atm}$

$$P_o - P_1 = \gamma \Delta h$$

$\therefore \gamma \Delta h = \frac{1}{2}\rho V_1^2 \quad$ and $\quad V_1 = \sqrt{\frac{2\gamma \Delta h}{\rho}}$

where

$$\rho = \frac{P}{RT} = 14.7\,\frac{lbf}{in^2} \times \frac{144\,in^2}{ft^2} \times \frac{lbm \cdot °R}{53.3\,ft \cdot lbf} \times \frac{1}{500\,°R} \times \frac{slug}{32.2\,lbm} = 0.00247\,\frac{slug}{ft^3}$$

$$V_1 = \sqrt{\frac{2\gamma \Delta h}{\rho}}$$

$$= \left[2 \times 13.6 \times 62.4\,\frac{lbf}{ft^3} \times 0.14\,in \times \frac{ft}{12\,in} \times \frac{ft^3}{0.00247\,slug} \times \frac{slug \cdot ft}{lbf \cdot s^2}\right]^{1/2}$$

$$V_1 = 89.5\ ft/s$$

Problem 6.37

Given: Pitot static probe is used to measure speed in standard air.

$V = 100$ m/s

Find: Manometer deflection in mm H_2O, corresponding to given conditions.

Solution:

Manometer reads $p_0 - p$ in mm of H_2O.

Basic equations: $\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$ for flow

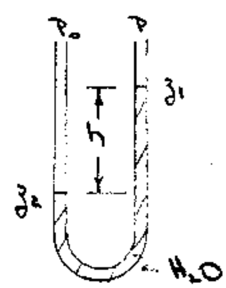
$\frac{dp}{dz} = -\rho g$ for manometer

Assumptions: (1) steady flow
(2) incompressible flow
(3) flow along a streamline
(4) frictionless deceleration to p_0
(5) ρ = constant for manometer

From the Bernoulli equation

$$\frac{p_0}{\rho} = \frac{p}{\rho} + \frac{V^2}{2}$$

$$p_0 - p = \rho \frac{V^2}{2}$$



For the manometer, $dp = -\rho g\, dz$

$$p_0 - p = \int_p^{p_0} dp = -\rho g (z_2 - z_1) = \rho g h$$

Then,

$$\rho_{H_2O}\, g h = \rho_{air} \frac{V^2}{2}$$

and

$$h = \frac{\rho_{air}}{\rho_{H_2O}} \frac{V^2}{2g} = \frac{1.23}{999} \times (100)^2 \frac{m^2}{s^2} \times \frac{1}{2} \times \frac{s^2}{9.81\,m} \times \frac{10^3\,mm}{m} = 62.8 \text{ mm}$$ ← h

Given: High-pressure hydraulic system subject to small leak.

Plot: Jet speed of a leak vs system pressure for system pressures up to 40 MPa gage; explain how a high-speed jet of hydraulic fluid can cause injury

Solution:

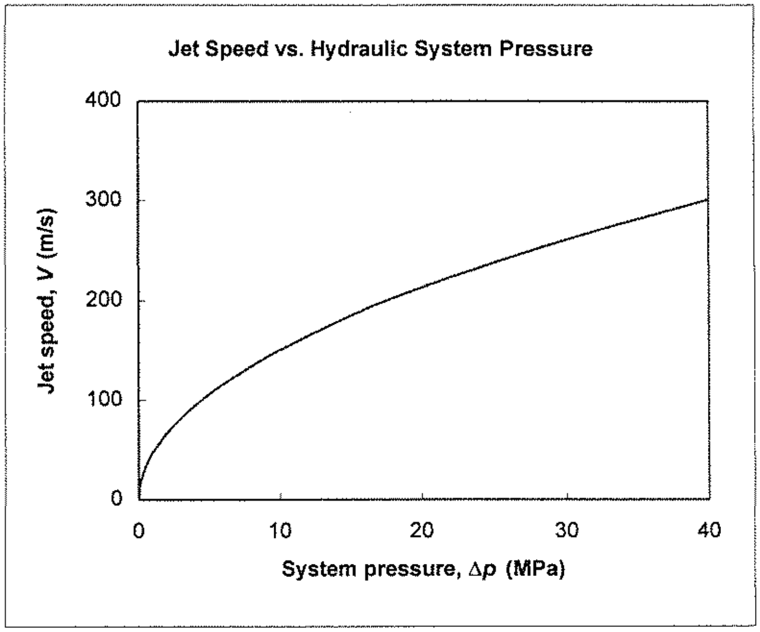
Basic equation: $\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$

Assumptions: (1) steady flow
(2) incompressible flow
(3) frictionless flow
(4) flow along a streamline.

The Bernoulli equation gives

$$V = \left[\frac{2(p_0 - p_{atm})}{\rho}\right]^{1/2}$$

From Table A.2 (Appendix A) for lubricating oil SG = 0.88



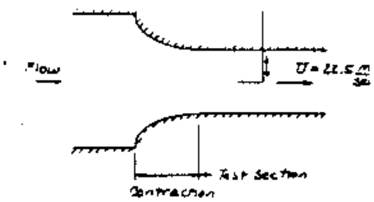
The high stagnation pressure ruptures the skin causing the jet to penetrate the tissue.

Problem 6.39

Given: Wind tunnel with inlet and test section as shown.

$\bar{U} = 22.5$ m/s, $p_{0s} = -6.0$ mm H_2O gage

$p_a = 99.1$ kPa (abs), $T_a = 23°C$



Find: (a) $p_{dynamic}$ on tunnel centerline

(b) p_{static} " " "

(c) compare p_{static} at tunnel wall with that measured at centerline

Solution:

(a) By definition $p_{dyn} = \frac{1}{2}\rho\bar{U}^2$

Assume: (1) air behaves as an ideal gas, and (2) incompressible flow

Then $\rho = \frac{p}{RT} = 99.1\times10^3\frac{N}{m^2}\times\frac{kg\cdot K}{287 N\cdot m}\times\frac{1}{(273+23)K} = 1.17\ kg/m^3$

and $p_{dyn} = \frac{1}{2}\rho\bar{U}^2 = \frac{1}{2}\times1.17\frac{kg}{m^3}\times(22.5)^2\frac{m^2}{s^2}\times\frac{N\cdot s^2}{kg\cdot m} = 296\ N/m^2$ ← p_{dyn}

(b) By definition $p_0 = p_s + p_{dyn}$

$\therefore p_s = p_0 - p_{dyn}$ where $p_0 = -6$ mm H_2O gage

then $p_0 - p_a = \rho g\Delta h = 999\frac{kg}{m^3}\times9.81\frac{m}{s^2}\times-6\times10^{-3}m\times\frac{N\cdot s^2}{kg\cdot m}$

$p_{0\,gage} = -58.8\ N/m^2$

$\therefore p_s = p_0 - p_{dyn} = -58.8 - 296 = -355\ N/m^2$ gage ← p_{static}

$\{$ or $p_s = -36.2$ mm H_2O (gage) $\}$

(c) Streamlines in the test section should be straight. Then in the test section the variation of static pressure is given by $\frac{\partial p}{\partial n} = 0$ and $p_{wall} = p_{centerline}$ ←

In the contraction section the streamlines are curved. The variation of static pressure normal to the streamlines is given by $\frac{\partial p}{\partial n} = \rho\frac{V^2}{R}$

and consequently the static pressure increases toward the centerline, i.e. $p_{wall} < p_{centerline}$

Problem 6.40

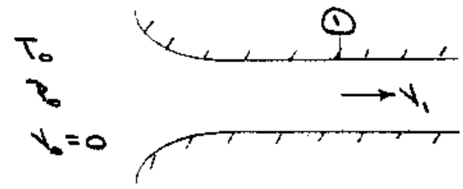
Given: Air flow in open circuit wind tunnel as shown.

$p_{atm} - p_1 = 45 \text{ mm } H_2O$

$T_0 = 25C$

$p_0 = p_{atm}$

Consider air to be incompressible.



Find: Air speed in tunnel at section ①

Solution:

Basic equations: $\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$

Assumptions: (1) steady flow
(2) incompressible flow
(3) frictionless flow
(4) flow along a streamline
(5) air behaves as an ideal gas
(6) stagnation pressure = p_{atm}

From the Bernoulli equation, $\frac{p_0}{\rho} = \frac{p_1}{\rho} + \frac{V_1^2}{2}$

$$p_0 - p_1 = p_{atm} - p_1 = \frac{1}{2}\rho V_1^2$$

$$V_1 = \left[\frac{2(p_{atm} - p_1)}{\rho}\right]^{1/2}$$

From the manometer reading, $p_{atm} - p_1 = \rho_{H_2O} g h$ then

$$V_1 = \left[\frac{2\rho_{H_2O} g h}{\rho}\right]^{1/2}$$

From the ideal gas equation of state

$$\rho = \frac{p}{RT} = 100\times10^3 \frac{N}{m^2} \times \frac{kg\cdot K}{287\, N\cdot m} \times \frac{1}{298\, K} = 1.17\, kg/m^3$$

$$V_1 = \left[2\frac{\rho_{H_2O}}{\rho} gh\right]^{1/2} = \left[2 \times \frac{999}{1.17} \times 9.81\frac{m}{s^2} \times 0.045\, m\right]^{1/2} = 27.5\, m/s$$

V_1

Problem 6.41

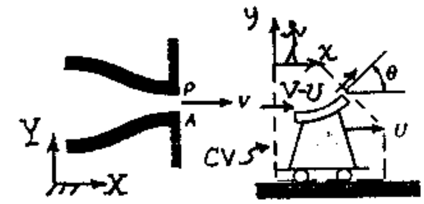
Given: Wheeled cart of Problem 4.106:

$V = 40$ m/s

$A = 25$ mm^2

Water, no friction on vane, $\theta = 120°$

Vane accelerates to the right



Find: At instant when $U = 15$ m/s,

(a) stagnation pressure leaving nozzle, relative to fixed observer.
(b) Stagnation pressure leaving nozzle, relative to observer on vane.
(c) Absolute velocity of jet leaving vane.
(d) Stagnation pressure of jet leaving vane, relative to fixed observer.
(e) How would viscous forces increase, decrease, or leave unchanged the stagnation pressure in (d). How can you justify this?

Solution: Stagnation pressure is $p_0 = p + \frac{1}{2}\rho V^2$ or $p_0 - p = \frac{1}{2}\rho V^2$

At jet, $p_{0_j} = \frac{1}{2}\rho V^2 = \frac{1}{2} \times 999 \frac{kg}{m^3} \times (40)^2 \frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m} = 799$ kPa (gage) $\qquad p_{0_j}$

At cart, $p_{o,rel} = \frac{1}{2}\rho (V-U)^2 = \frac{1}{2} \times 999 \frac{kg}{m^3} \times (40-15)^2 \frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m} = 312$ kPa (gage) $\qquad p_{o,rel}$

Leaving vane, $\vec{V}_{abs} = U\hat{i} + (V-U)(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$

$$\vec{V}_{abs} = [U + (V-U)\cos\theta]\hat{i} + (V-U)\sin\theta\,\hat{j}$$

$$= \left[15\,\frac{m}{s} + (40-15)\frac{m}{s} \times \left(-\frac{1}{2}\right)\right]\hat{i} + (40-15)\frac{m}{s} \times 0.866\,\hat{j}$$

$$\vec{V}_{abs} = 2.5\,\hat{i} + 21.7\,\hat{j} \text{ m/s} \qquad \vec{V}_{abs}$$

The magnitude $|\vec{V}_{abs}| = [(2.5)^2 + (21.7)^2]^{1/2}$ m/s $= 21.8$ m/s

Leaving vane, $p_0 = \frac{1}{2}\rho |\vec{V}_{abs}|^2$, relative to a fixed observer. Thus

$$p_{o,fixed} = \frac{1}{2} \times 999 \frac{kg}{m^3} \times (21.8)^2 \frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m} = 237 \text{ kPa (gage)} \qquad p_{o,fixed}$$

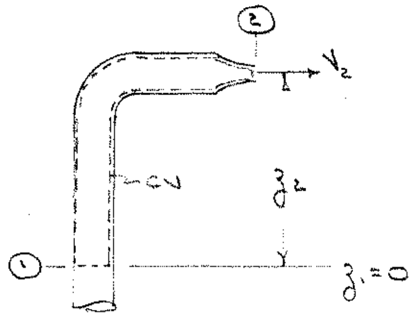
{The corresponding absolute pressures are 900, 413, and 338 kPa (abs).}

Discussion: Viscous forces would slow the jet speed relative to the vane. The jet would enter the vane with relative speed $(V - U)$; it would leave the vane with speed $\alpha\,(V - U)$, where $\alpha < 1$.

Friction would reduce both components of relative velocity leaving the vane. The absolute velocity of the jet leaving the vane, as seen by a fixed observer, would decrease. Thus the stagnation pressure of the flow leaving the vane, relative to a fixed observer, would decrease.

Problem 6.42

Given: Steady flow of water through elbow and nozzle as shown



$D_1 = 0.1 m$ $\quad D_2 = 0.05 m$

$p_2 = p_{atm}$ $\quad V_2 = 20 m/s$

$z_1 = 0$ $\quad z_2 = 4 m$

Find: Gage pressure, p_1; p_1 if device were inverted

Solution: Apply continuity to cv. shown to determine V_1; the Bernoulli equation is then applied along a streamline from ① to ② to determine p_1

Basic equations: $$0 = \frac{\partial}{\partial t}\int_{cv} \rho \, d\forall + \int_{cs} \rho \vec{V} \cdot d\vec{A} \quad (\text{first term} = 0(1))$$

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + g z_2 \quad (g z_1 = 0(6),\ p_2/\rho = 0(5))$$

Assumptions: (1) steady flow
(2) incompressible flow
(3) frictionless flow
(4) flow along a steamline
(5) p_2 gage $= 0$
(6) $z_1 = 0$

From the continuity equation, $0 = -|\rho V_1 A_1| + |\rho V_2 A_2|$

then, $$V_1^2 = \left(\frac{A_2}{A_1}\right)^2 V_2^2 = \left(\frac{D_2}{D_1}\right)^4 V_2^2$$

From the Bernoulli equation

$$p_1 = \rho\left[\frac{V_2^2 - V_1^2}{2} + g z_2\right] = \rho\left[\frac{V_2^2}{2}\left(1 - \frac{V_1^2}{V_2^2}\right) + g z_2\right] = \rho\left[\frac{V_2^2}{2}\left(1 - \left(\frac{D_2}{D_1}\right)^4\right) + g z_2\right]$$

$$p_1 = 999 \frac{kg}{m^3}\left[\frac{1}{2} \times (20)^2 \frac{m^2}{s^2} \times \left(1 - \left(\frac{1}{2}\right)^4\right) + 9.81 \frac{m}{s^2} \times 4 m\right] \times \frac{N \cdot s^2}{kg \cdot m}$$

$p_1 = 227 \ kN/m^2 = 227 \ kPa$ (gage) ⟵ p_1

If device is inverted, $z_2 = -4m$ with $z_1 = 0$

$$p_1 = \rho\left[\frac{V_2^2}{2}\left\{1 - \left(\frac{D_2}{D_1}\right)^4\right\} + g z_2\right]$$

$$= 999 \frac{kg}{m^3}\left[\frac{1}{2} \times (20)^2 \frac{m^2}{s^2}\left\{1 - \left(\frac{1}{2}\right)^4\right\}\right] + 9.81 \frac{m}{s^2} \times (-4 m)\Big] \frac{N \cdot s^2}{kg \cdot m}$$

$p_1 = 148 \ kN/m^2 = 148 \ kPa$ (gage) ⟵ p_1

Problem 6.43

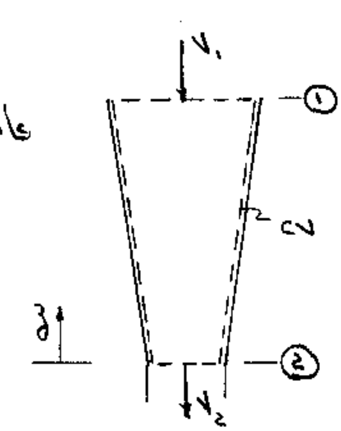
Given: Water flow in a circular duct

$D_1 = 0.3\,m \quad P_1 = 260\,kPa\,(gage) \quad \vec{V}_1 = -3\hat{k}\ m/s$

$z_1 = 10\,m$

$z_2 = 0 \quad D_2 = 0.15\,m$

Frictional effects may be neglected.



Find: Pressure, P_2

Solution: Apply continuity to CV shown to determine V_2; the Bernoulli equation is then applied along a streamline from ① to ② to determine P_2

Basic equations: $0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$ (the time-derivative term is crossed out, marked 0(1))

$$\frac{P_1}{\rho} + \frac{V_1^2}{2} + g z_1 = \frac{P_2}{\rho} + \frac{V_2^2}{2} + g z_2$$

Assumptions: (1) steady flow
(2) incompressible flow
(3) frictionless flow
(4) flow along a streamline
(5) uniform flow at sections ① and ②

From the continuity equation

$$0 = -|\rho V_1 A_1| + |\rho V_2 A_2|$$

Then,

$$V_2 = \frac{A_1}{A_2} V_1 = \left(\frac{D_1}{D_2}\right)^2 V_1 = \left(\frac{0.3}{0.15}\right)^2 \times 3\,\frac{m}{s} = 12\ m/s$$

From the Bernoulli equation,

$$P_2 = P_1 + \frac{\rho}{2}(V_1^2 - V_2^2) + \rho g (z_1 - z_2)$$

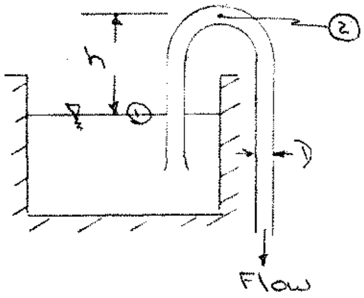
$$= 260\,\frac{kN}{m^2} + \frac{1}{2} \times 999\,\frac{kg}{m^3} \times \left[(3)^2 - (12)^2\right]\frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m} + 999\,\frac{kg}{m^3} \times 9.81\,\frac{m}{s^2} \times 10\,m \times \frac{N \cdot s^2}{kg \cdot m}$$

$P_2 = 291\ kN/m^2 = 291\ kPa\ (gage).$ P_2

Problem 6.44

Given: Water flow through siphon as shown

$Q = 0.02\ m^3/sec$, $T = 20°C$, $D = 50\ mm$



Find: Maximum allowable height, h, such that P_2 is above the vapor pressure of the water

Solution: Apply the Bernoulli equation along the streamline between locations ① and ② to determine h after employing the definition of volume flowrate to determine the flow speed in the tube

Basic equations: $Q = \int \vec{V} \cdot d\vec{A}$ (Q is volume flow rate)

$$\frac{P_1}{\rho} + \frac{V_1^2}{2}^{\approx 0(6)} + g z_1^{\,0(5)} = \frac{P_2}{\rho} + \frac{V_2^2}{2} + g z_2$$

Assumptions:
(1) steady flow
(2) incompressible flow
(3) frictionless flow
(4) flow along a streamline
(5) $z_1 = 0$
(6) $V_1 \approx 0$
(7) uniform flow in the tube

From the definition of Q and assumption 7, $Q = V_2 A_2$, and

$$V_2 = \frac{Q}{A_2} = \frac{4Q}{\pi D_2^2} = \frac{4}{\pi} \times 0.02\frac{m^3}{s} \times \frac{1}{(50\times10^{-3})^2\ m^2} = 10.2\ m/s$$

From the Bernoulli equation,

$$h = z_2 = \frac{1}{g}\left[\frac{P_1 - P_2}{\rho} - \frac{V_2^2}{2}\right]$$

For water at 20°C, $P_{vapor} = P_2 = 2.33\ kPa$. Then

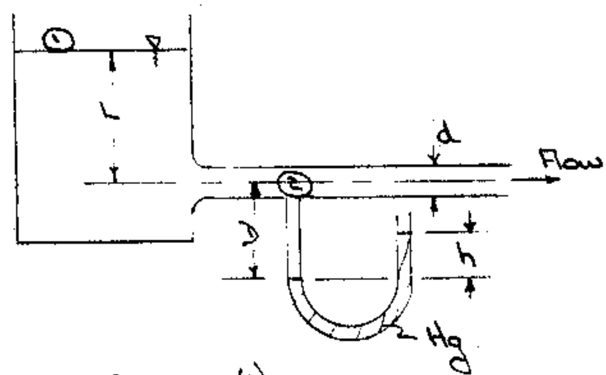
$$h = \frac{1}{g}\left[\frac{P_1 - P_2}{\rho} - \frac{V_2^2}{2}\right] = \frac{s^2}{9.81\,m}\left[(101-2.33)\times10^3\frac{N}{m^2} \times \frac{m^3}{999\,kg} \times \frac{kg\cdot m}{N\cdot s^2} - \frac{1}{2}(10.2)^2\frac{m^2}{s^2}\right]$$

$h = 4.78\ m$ ← h

Problem 6.45

Given: Water flow from a large tank as shown

$L = 12\ ft \quad D = 2\ ft \quad d = 2\ in$
$h = 6''$



Find: (a) Velocity in discharge pipe
(b) Rate of discharge

Solution:

Basic equations: $\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2$ (with $V_1^2/2 \approx 0$ (5), $gz_2 = 0$ (8))

$$Q = \int u\,dA$$

Assumptions:
(1) steady flow
(2) incompressible flow
(3) no friction
(4) flow along a streamline
(5) $V_1 \approx 0$, i.e. large tank
(6) $p_1 = p_{atm}$
(7) uniform flow at section ②
(8) $z_2 = 0$

From the Bernoulli equation, $V_2 = \left[2\frac{(p_1 - p_2)}{\rho} + 2gL\right]^{1/2} = \left[2\frac{(p_{atm} - p_2)}{\rho} + 2gL\right]^{1/2}$

From the conditions of the manometer,

$$p_{atm} + \gamma_{Hg} h - \gamma_{H_2O} D = p_2 \quad \text{and} \quad p_{atm} - p_2 = \gamma_{H_2O} D - \gamma_{Hg} h$$

Substituting into the expression for V_2,

$$V_2 = \left[\frac{2}{\rho}(\gamma_{H_2O} D - \gamma_{Hg} h) + 2gL\right]^{1/2} = \left[\frac{2}{\rho}\gamma_{H_2O}(D - SG_{Hg} h) + 2gL\right]^{1/2} = \left[2g(D - SG_{Hg} h + L)\right]^{1/2}$$

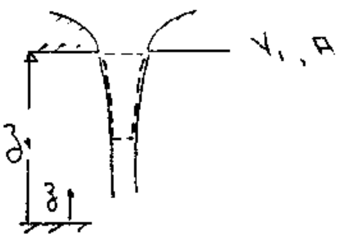
$$V_2 = \left[2 \times 32.2\,\frac{ft}{s^2} \times \left(2\,ft - 13.6 \times \tfrac{1}{2}\,ft + 12\,ft\right)\right]^{1/2} = 21.5\ ft/s$$ ← V_2

$$Q = \int u\,dA = V_2 A_2 \quad \text{(for uniform flow at ②)}$$

$$Q = V_2 \frac{\pi d^2}{4} = 21.5\,\frac{ft}{s} \times \frac{\pi}{4} \times \left(\frac{2}{12}\right)^2 ft^2 = 0.469\ ft^3/s$$ ← Q

Problem 6.46

Given: Liquid stream leaving a nozzle pointing downward as shown
Assume uniform flow
Neglect friction



Find: Variation in jet area for $z < z_0$

Solution:

Basic equations: $\frac{p_1}{\rho} + \frac{V_1^2}{2} + g z_1 = \frac{p}{\rho} + \frac{V^2}{2} + g z$

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$

Assumptions:
(1) steady flow
(2) incompressible flow
(3) frictionless flow
(4) flow along a streamline
(5) $p = p_1 = p_{atm}$
(6) uniform flow at a section

From the Bernoulli equation

$$V^2 = V_1^2 + 2g(z_1 - z)$$

From the continuity equation

$$0 = \int_{CS} \rho \vec{V} \cdot d\vec{A} = -\{|\rho V_1 A_1|\} + \{|\rho V A|\}$$

and

$$V_1 A_1 = V A \quad \text{or} \quad V = V_1 \frac{A_1}{A}$$

Thus

$$V_1^2 \left(\frac{A_1}{A}\right)^2 = V_1^2 + 2g(z_1 - z)$$

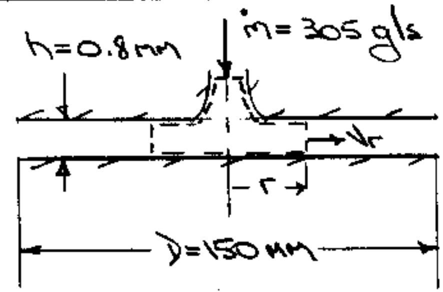
Solving for A,

$$A = A_1 \sqrt{\frac{1}{1 + \frac{2g(z_1 - z)}{V_1^2}}}$$

$A(z)$

{Note: jet area decreases as z decreases, owing to the higher velocity}

Problem 6.47



Given: Water flow between parallel disks discharging to atmosphere as shown.

Find: (a) theoretical static pressure between the disks at $r = 50\text{ mm}$.
(b) in actual laboratory situation, would the pressure be above or below the theoretical value?

Solution:

Basic equations:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2$$

Assumptions: (1) steady flow
(2) incompressible flow
(3) flow along a streamline
(4) neglect friction
(5) uniform flow at each section

Apply continuity to the CV shown

$$0 = \{-\dot{m}\} + \{\rho V_r 2\pi r h\} \quad \text{so} \quad V = \frac{\dot{m}}{2\pi \rho r h}$$

$$V_1 = V_{r=50\text{mm}} = \frac{1}{2\pi} \times 0.305\frac{\text{kg}}{\text{s}} \times \frac{\text{m}^3}{999\,\text{kg}} \times \frac{1}{0.050\,\text{m}} \times \frac{1}{8\times10^{-4}\,\text{m}} = 1.21\,\text{m/s}$$

$$V_2 = V_{r=R} = \frac{1}{2\pi} \times 0.305\frac{\text{kg}}{\text{s}} \times \frac{\text{m}^3}{999\,\text{kg}} \times \frac{1}{0.075\,\text{m}} \times \frac{1}{8\times10^{-4}\,\text{m}} = 0.810\,\text{m/s}$$

From the Bernoulli equation

$$p_1 - p_2 = p_{r=50\text{m}} - p_{atm} = \frac{1}{2}\rho V_2^2 - \frac{1}{2}\rho V_1^2 = \frac{\rho}{2}(V_2^2 - V_1^2)$$

$$p_{r=50\text{mm}} = \frac{1}{2} \times 999\frac{\text{kg}}{\text{m}^3}\left[(0.810)^2 - (1.21)^2\right]\frac{\text{m}^2}{\text{s}^2} \times \frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}}$$

$$p_{r=50\text{mm}} = -404\,\text{N/m}^2 \text{ (gage)} \qquad \longleftarrow p_{r=50\text{mm}}$$

Friction would cause a pressure drop in the flow direction. Since the discharge pressure is fixed at p_{atm}, the measured pressure would be greater than the theoretical value.

Problem 6.48

Given: Steady, frictionless, incompressible air flow over a wing as shown

$P_1 = 10$ psia

$T_1 = 40°F$

$V_1 = 200$ ft/s

$P_2 = -0.40$ psig

Find: V_2

Solution: Apply the Bernoulli equation along the streamline from the upstream conditions through point ②

Basic equations: $$\frac{P_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{P_2}{\rho} + \frac{V_2^2}{2} + gz_2$$

$$P = \rho RT$$

Assumptions:
(1) steady flow
(2) incompressible flow
(3) frictionless flow
(4) flow along a streamline
(5) ideal gas
(6) neglect Δz

Then from the Bernoulli equation

$$V_2^2 = V_1^2 + \frac{2}{\rho}(P_1 - P_2)$$

where $$\rho = \frac{P}{RT} = \frac{10\ \text{lbf}}{\text{in}^2} \times \frac{\text{lbm}\cdot°R}{53.3\ \text{ft}\cdot\text{lbf}} \times \frac{1}{500°R} \times \frac{\text{slug}}{32.2\ \text{lbm}} \times \frac{144\ \text{in}^2}{\text{ft}^2} = 1.68\times10^{-3}\ \frac{\text{slug}}{\text{ft}^3}$$

$$V_2^2 = (200)^2 \frac{\text{ft}^2}{\text{s}^2} + 2 \times \frac{\text{ft}^3}{1.68\times10^{-3}\ \text{slug}} \times 0.40 \frac{\text{lbf}}{\text{in}^2} \times \frac{144\ \text{in}^2}{\text{ft}^2} \times \frac{\text{slug}\cdot\text{ft}}{\text{lbf}\cdot\text{s}^2}$$

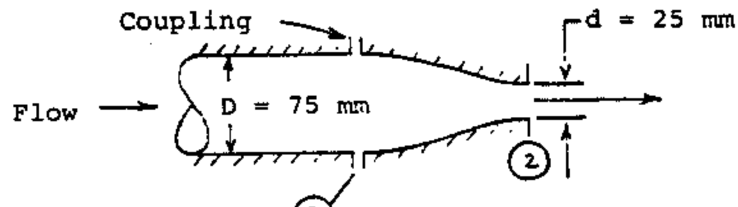
$$V_2^2 = 109{,}000\ \text{ft}^2/\text{s}^2$$

$$V_2 = 330\ \text{ft/s}$$ ← V_2

{ Note: this is about the upper limit on velocity for the assumption of incompressible flow to be valid }

Problem 6.49

Given: Fire hose nozzle shown.



$p_1 = 689$ kPa (gage)

Find: Maximum flow rate that could be delivered.

Solution: Apply Bernoulli equation. Assumptions needed are:

(1) Steady
(2) Incompressible
(3) Frictionless
(4) Flow along streamline
(5) Neglect Δz
(6) Uniform at ① and ②

Then $\frac{p_1}{\rho} + \frac{V_1^2}{2} + g\cancel{z_1} = \frac{p_2}{\rho} + \frac{V_2^2}{2} + g\cancel{z_2}$

But $V_1 A_1 = V_2 A_2$, so $V_1 = V_2 \frac{A_2}{A_1} = V_2\left(\frac{d}{D}\right)^2$; $V_1^2 = V_2^2\left(\frac{d}{D}\right)^4$

$$V_2 = \left[\frac{2(p_1 - p_2)}{\rho\left[1-(d/D)^4\right]}\right]^{1/2}$$

$$= \left[2 \times 689\times10^3 \frac{N}{m^2} \times \frac{m^3}{999\,kg} \times \frac{1}{1-(1/3)^4} \times \frac{kg \cdot m}{N \cdot s^2}\right]^{\frac{1}{2}}$$

$$V_2 = 37.4\ m/s$$

$$Q = V_2 A_2 = 37.4\frac{m}{s} \times \frac{\pi}{4}(0.025)^2 m^2 \times 3600\frac{s}{hr} = 66.1\ m^3/hr$$ Q

Problem 6.50

Given: Mercury barometer carried in car on windless day.

Outside: $T = 20°C$, $h_{bar} = 761$ mm Hg (corrected)

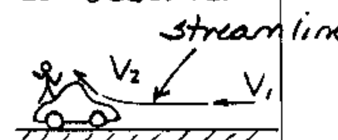
Inside: $V = 105$ km/hr, window open, $h_{bar} = 756$ mm Hg

Find: (a) Explain what is happening.
(b) Local speed of air flow past window, relative to car.

Solution: (a) Air speed relative to car is higher than in the freestream, thus lowering the pressure at window.

(b) Apply the Bernoulli equation in frame seen by an observer on the car:

Basic equation: $\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2$



Assumptions: (1) Steady flow (seen by observer on car)
(2) Incompressible flow
(3) Neglect friction
(4) Flow along a streamline
(5) Neglect Δz

Then $V_2^2 = \left[V_1^2 + 2\left(\frac{p_1 - p_2}{\rho}\right)\right]$ or $V_2 = \left[V_1^2 + \frac{2(p_1 - p_2)}{\rho}\right]^{1/2}$ (1)

From fluid statics

$p_1 - p_2 = \rho g(h_1 - h_2) = SG \rho_{H_2O}\, g\, \Delta h$

$= 13.6 \times 1000 \frac{kg}{m^3} \times 9.81 \frac{m}{s^2} \times 0.005\, m \times \frac{N \cdot s^2}{kg \cdot m}$

$p_1 - p_2 = 667\ N/m^2$

and from ideal gas

$\rho = \frac{p}{RT} = 13.6 \times 1000 \frac{kg}{m^3} \times 9.81 \frac{m}{s^2} \times 0.761 m \times \frac{kg \cdot K}{287\ N \cdot m} \times \frac{1}{(273 + 20) K} \times \frac{N \cdot s^2}{kg \cdot m}$

$\rho = 1.21\ kg/m^3$

Substituting into Eq. 1

$V_2 = \left[\left(105 \frac{km}{hr} \times 1000 \frac{m}{km} \times \frac{hr}{3600\, s}\right)^2 + 2 \times 667 \frac{N}{m^2} \times \frac{m^2}{1.21\ kg} \times \frac{kg \cdot m}{N \cdot s^2}\right]^{1/2}$

$V_2 = 44.2\ m/s$ (159 km/hr) relative to car V_2

Problem 6.51

Given: Indianapolis race car, $V_0 = 98.3$ m/s, on a straightaway.

Air inlet at location where $V = 25.5$ m/s along body surface.

Find: (a) Static pressure at inlet location.
(b) Express pressure rise as a fraction of the dynamic pressure.

Solution: Apply the Bernoulli equation, relative to the auto.

Basic equation: $\frac{p_\infty}{\rho} + \frac{V_\infty^2}{2} + gz_\infty = \frac{p}{\rho} + \frac{V^2}{2} + gz$

Assumptions: (1) Steady flow (as seen by observer on auto)
(2) Incompressible flow ($V_\infty < 100$ m/sec)
(3) No friction
(4) Flow along a streamline
(5) Neglect changes in z
(6) Standard air: $\rho = 1.23$ kg/m^3

Then

$$p - p_\infty = \frac{1}{2}\rho V_\infty^2 - \frac{1}{2}\rho V^2 = \frac{1}{2}\rho V_\infty^2\left[1-\left(\frac{V}{V_\infty}\right)^2\right] = q\left[1-\left(\frac{V}{V_\infty}\right)^2\right]$$

$$q = \frac{1}{2}\rho V_\infty^2 = \frac{1}{2} \times 1.23\,\frac{kg}{m^3} \times (98.3)^2\frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m} = 5.94 \text{ kPa}$$

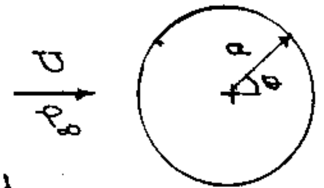
$$\frac{\Delta p}{q} = 1-\left(\frac{V}{V_\infty}\right)^2 = 1-\left(\frac{25.5}{98.3}\right)^2 = 0.933$$ ← $\Delta p/q$

and $\Delta p = 0.933\,q = 0.933 \times 5.94 \text{ kPa} = 5.54 \text{ kPa}$ ← $p - p_\infty$

Problem 6.52

Given: Steady, frictionless, incompressible flow over a stationary cylinder of radius, a.

$$\vec{V} = U\left[1-\left(\frac{a}{r}\right)^2\right]\cos\theta\,\hat{e}_r - U\left[1+\left(\frac{a}{r}\right)^2\right]\sin\theta\,\hat{e}_\theta$$



Find: (a) expression for pressure distribution along streamline forming cylinder, r = a.
(b) locations on cylinder where $p = p_\infty$.

Solution:

Basic equation: $\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$

Assumptions: (1) steady flow (given)
(2) incompressible flow (given)
(3) frictionless flow (given)
(4) flow along a streamline.

Along the cylinder surface $r = a$ and $\vec{V} = -2U\sin\theta\,\hat{e}_\theta$

Applying the Bernoulli equation along the streamline r=a,

$$\frac{p}{\rho} + \frac{V^2}{2} = \frac{p_\infty}{\rho} + \frac{U^2}{2}$$

$$p = p_\infty + \tfrac{1}{2}\rho(U^2 - V^2) = p_\infty + \tfrac{1}{2}\rho(U^2 - 4U^2\sin^2\theta)$$

$$p = p_\infty + \tfrac{1}{2}\rho U^2(1 - 4\sin^2\theta) \qquad p$$

For $p = p_\infty$, $1 - 4\sin^2\theta = 0$ and $\sin\theta = \pm 0.5$

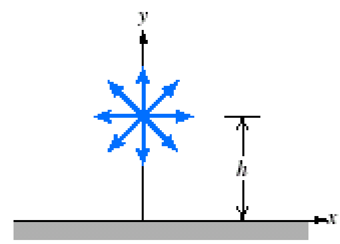
$$\therefore\ \theta = 30^\circ, 150^\circ, 210^\circ, 330^\circ \qquad \theta$$

Problem 6.53

The velocity field for a plane source at a distance *h* above an infinite wall aligned along the *x* axis was given in Problem 6.7. Using the data from that problem, plot the pressure distribution along the wall from *x* = - 10*h* to *x* = + 10*h* (assume the pressure at infinity is atmospheric). Find the net force on the wall if the pressure on the lower surface is atmospheric. Does the force tend to pull the wall towards the source, or push it away?

Given: Velocity field

Find: Pressure distribution along wall; plot distribution; net forc on wall



Solution

The given data is $q = 2\cdot\frac{\frac{m^3}{s}}{m}$ $\quad h = 1\cdot m \quad \rho = 1000\cdot\frac{kg}{m^3}$

$$u = \frac{q\cdot x}{2\cdot\pi\left[x^2 + (y-h)^2\right]} + \frac{q\cdot x}{2\cdot\pi\left[x^2 + (y+h)^2\right]}$$

$$v = \frac{q\cdot(y-h)}{2\cdot\pi\left[x^2 + (y-h)^2\right]} + \frac{q\cdot(y+h)}{2\cdot\pi\left[x^2 + (y+h)^2\right]}$$

The governing equation is the Bernoulli equation

$$\frac{p}{\rho} + \frac{1}{2}\cdot V^2 + g\cdot z = \text{const} \qquad \text{where} \qquad V = \sqrt{u^2 + v^2}$$

Apply this to point arbitrary point (*x*,0) on the wall and at infinity (neglecting gravity)

At $\quad |x| \to 0 \quad u \to 0 \quad v \to 0 \quad V \to 0$

At point (x,0) $\quad u = \dfrac{q \cdot x}{\pi \cdot (x^2 + h^2)} \quad v = 0 \quad V = \dfrac{q \cdot x}{\pi \cdot (x^2 + h^2)}$

Hence the Bernoulli equation becomes

$$\frac{p_{atm}}{\rho} = \frac{p}{\rho} + \frac{1}{2} \cdot \left[\frac{q \cdot x}{\pi \cdot (x^2 + h^2)} \right]^2$$

or (with pressure expressed as gage pressure)

$$p(x) = -\frac{\rho}{2} \cdot \left[\frac{q \cdot x}{\pi \cdot (x^2 + h^2)} \right]^2$$

(Alternatively, the pressure distribution could have been obtained from Problem 6.7, where

$$\frac{\partial}{\partial x} p = \frac{\rho \cdot q^2 \cdot x \cdot (x^2 - h^2)}{\pi^2 \cdot (x^2 + h^2)^3}$$

along the wall. Integration of this with respect to x leads to the same result for $p(x)$)

The plot of pressure is shown in the associated *Excel* workbook. From the plot it is clear that th wall experiences a negative gage pressure on the upper surface (and zero gage pressure on the lower), so the net force on the wall is upwards, towards the source

The force per width on the wall is given b: $F = \displaystyle\int_{-10 \cdot h}^{10 \cdot h} (p_{upper} - p_{lower})\, dx$

$$F = -\frac{\rho \cdot q^2}{2 \cdot \pi^2} \cdot \int_{-10 \cdot h}^{10 \cdot h} \frac{x^2}{(x^2 + h^2)^2}\, dx$$

The integral is $$\int \frac{x^2}{\left(x^2+h^2\right)^2}\,dx \rightarrow \frac{-1}{2}\cdot\frac{x}{\left(x^2+h^2\right)}+\frac{1}{2\cdot h}\cdot \operatorname{atan}\left(\frac{x}{h}\right)$$

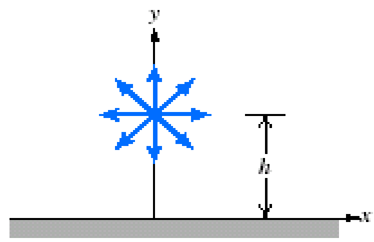
so $$F = -\frac{\rho\cdot q^2}{2\cdot\pi^2\cdot h}\cdot\left(-\frac{10}{101}+\operatorname{atan}(10)\right)$$

$$F = -\frac{1}{2\cdot\pi^2}\times 1000\cdot\frac{kg}{m^3}\times\left(2\cdot\frac{m^2}{s}\right)^2\times\frac{1}{1\cdot m}\times\left(-\frac{10}{101}+\operatorname{atan}(10)\right)\times\frac{N\cdot s^2}{kg\cdot m}$$

$$F = -278\,\frac{N}{m}$$

Problem 6.53 (In Excel)

The velocity field for a plane source at a distance h above an infinite wall aligned along the x axis was given in Problem 6.7. Using the data from that problem, plot the pressure distribution along the wall from $x = -10h$ to $x = +10h$ (assume the pressure at infinity is atmospheric). Find the net force on the wall if the pressure on the lower surface is atmospheric. Does the force tend to pull the wall towards the source, or push it away?



Given: Velocity field

Find: Pressure distribution along wall

Solution

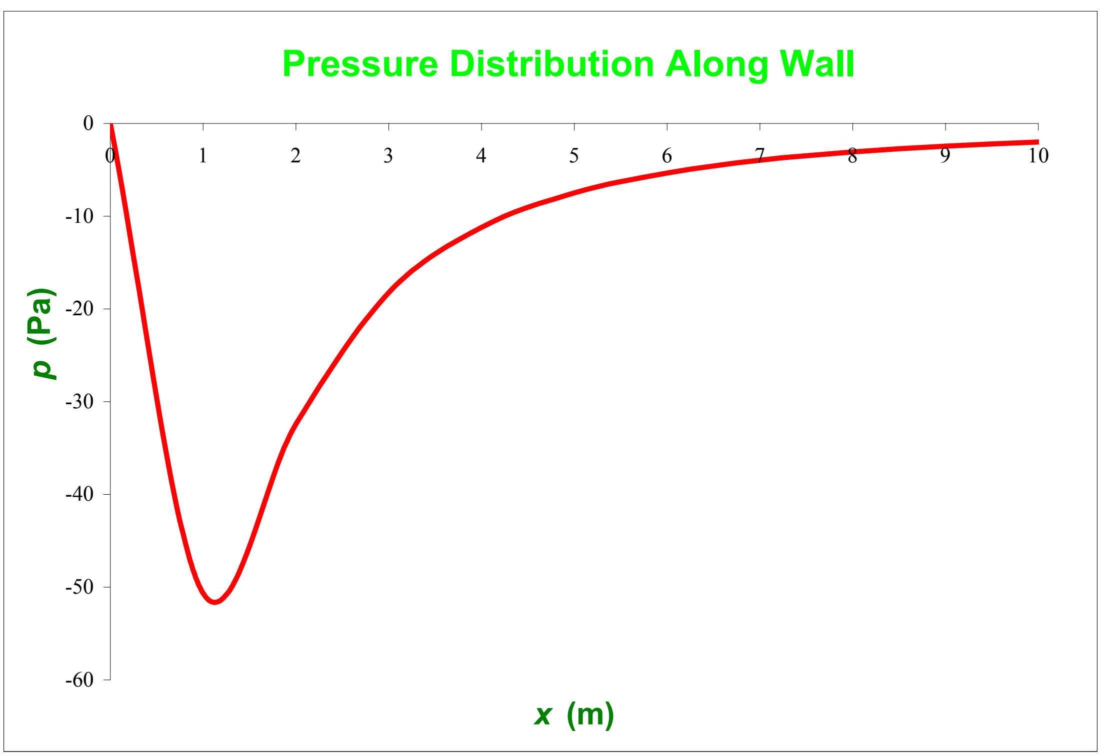
The given data is

$q =$ 2 $m^3/s/m$
$h =$ 1 m
$\rho =$ 1000 kg/m^3

The pressure distribution is

$$p(x) = -\frac{\rho}{2}\cdot\left[\frac{q\cdot x}{\pi\cdot\left(x^2+h^2\right)}\right]^2$$

x (m)	p (Pa)
0.0	0.00
1.0	-50.66
2.0	-32.42
3.0	-18.24
4.0	-11.22
5.0	-7.49
6.0	-5.33
7.0	-3.97
8.0	-3.07
9.0	-2.44
10.0	-1.99



Problem 6.54

The velocity field for a plane doublet is given in Table 6.1 (page S-27 on the CD). If $\Lambda = 3$ $m^3.s^{-1}$, the fluid density is $\rho = 1.5$ kg/m^3, and the pressure at infinity is 100 kPa, plot the pressure along the x axis from $x = -2.0$ m to -0.5 m and $x = 0.5$ m to 2.0 m.

Given: Velocity field for plane doublet

Find: Pressure distribution along x axis; plot distribution

Solution

The given data is $\Lambda = 3\cdot\frac{m^3}{s}$ $\qquad \rho = 1000\cdot\frac{kg}{m^3}$ $\qquad p_0 = 100\cdot kPa$

From Table 6.1 $\qquad V_r = -\frac{\Lambda}{r^2}\cdot\cos(\theta) \qquad V_\theta = -\frac{\Lambda}{r^2}\cdot\sin(\theta)$

where V_r and V_θ are the velocity components in cylindrical coordinates (r,θ). For points along t
x axis, $r = x$, $\theta = 0$, $V_r = u$ and $V_\theta = v = 0$

$$u = -\frac{\Lambda}{x^2} \qquad\qquad v = 0$$

The governing equation is the Bernoulli equation

$$\frac{p}{\rho} + \frac{1}{2}\cdot V^2 + g\cdot z = \text{const} \qquad \text{where} \qquad V = \sqrt{u^2 + v^2}$$

so (neglecting gravity) $\frac{p}{\rho} + \frac{1}{2}\cdot u^2 = \text{const}$

Apply this to point arbitrary point $(x,0)$ on the x axis and at infinity

At $|x| \to 0$ $u \to 0$ $p \to p_0$

At point (x,0) $u = -\frac{\Lambda}{x^2}$

Hence the Bernoulli equation becomes

$$\frac{p_0}{\rho} = \frac{p}{\rho} + \frac{\Lambda^2}{2 \cdot x^4}$$

or $$p(x) = p_0 - \frac{\rho \cdot \Lambda^2}{2 \cdot x^4}$$

The plot of pressure is shown in the associated *Excel* workbook

Problem 6.54 (In Excel)

The velocity field for a plane doublet is given in Table 6.1 (page S-27 on the CD). If $\Lambda = 3$ $m^3.s^{-1}$, the fluid density is $\rho = 1.5$ kg/m^3, and the pressure at infinity is 100 kPa, plot the pressure along the x axis from $x = -2.0$ m to -0.5 m and $x = 0.5$ m to 2.0 m.

Given: Velocity field

Find: Pressure distribution along x axis

Solution

The given data is

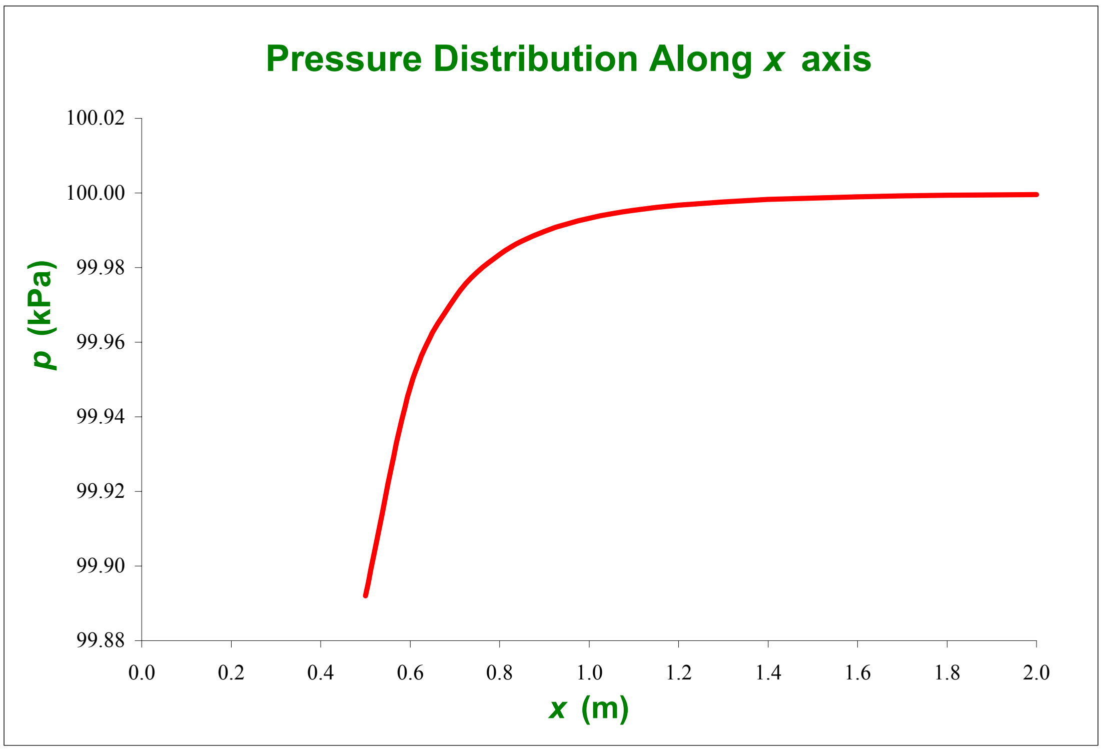
$\Lambda =$ 3 m^3/s

$\rho =$ 1.5 kg/m^3

$p_0 =$ 100 kPa

The pressure distribution is $$p(x) = p_0 - \frac{\rho \cdot \Lambda^2}{2 \cdot x^4}$$

x (m)	p (Pa)
0.5	99.89
0.6	99.95
0.7	99.97
0.8	99.98
0.9	99.99
1.0	99.99
1.1	100.00
1.2	100.00
1.3	100.00
1.4	100.00
1.5	100.00
1.6	100.00
1.7	100.00
1.8	100.00
1.9	100.00
2.0	100.00



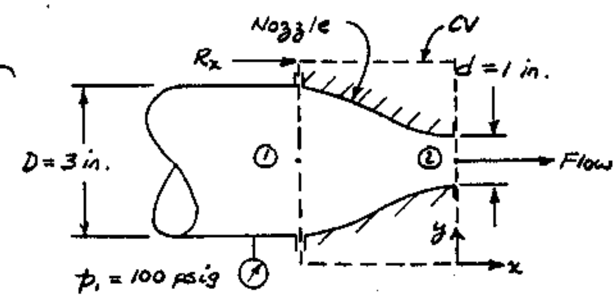
Problem 6.55

Given: A fire nozzle is attached to a hose of inside diameter, $D = 3$ in. The smoothly contoured nozzle is designed to operate at an inlet water pressure of $p_1 = 100$ psig. The outlet diameter is $d = 1$ in.

Find: (a) The design flow rate of the nozzle, in gpm.
(b) The force required to hold the nozzle in place.

Solution:

(a) To determine the design flow rate we apply the continuity equation and the Bernoulli equation.



Assume: (1) steady flow
(2) incompressible flow
(3) frictionless flow
(4) flow along a streamline
(5) neglect Δz
(6) uniform flow at each section

From the continuity equation $A_1V_1 = A_2V_2 \quad \therefore V_1 = V_2\dfrac{A_2}{A_1} = V_2\left(\dfrac{d}{D}\right)^2$

Bernoulli equation $\dfrac{p_1}{\rho} + \dfrac{V_1^2}{2} + \cancel{gy_1} = \cancel{\dfrac{p_2}{\rho}} + \dfrac{V_2^2}{2} + \cancel{gy_2}$

Then substituting for V_1 with $p_2 = p_{atm} = 0$ (gage).

$$\frac{p_{1g}}{\rho} + \frac{V_2^2}{2}\left(\frac{d}{D}\right)^4 = \frac{V_2^2}{2} \quad \text{and} \quad V_2 = \left\{\frac{2\,p_{1g}}{\rho\left[1-\left(\frac{d}{D}\right)^4\right]}\right\}^{1/2}$$

Substituting numerical values

$$V_2 = \left\{2 \times 100\,\frac{\text{lbf}}{\text{in}^2} \times \frac{\text{ft}^3}{1.94\,\text{slug}} \times \frac{1}{\left[1-\left(\frac{1}{3}\right)^4\right]} \times 144\,\frac{\text{in}^2}{\text{ft}^2} \times \frac{\text{slug}\cdot\text{ft}}{\text{lbf}\cdot\text{s}^2}\right\}^{1/2} = 123\ \text{ft/s}$$

and

$$Q = A_2V_2 = \frac{\pi d_2^2}{4}V_2 = \frac{\pi}{4}\left(\frac{1}{12}\right)^2\text{ft}^2 \times 123\,\frac{\text{ft}}{\text{s}} \times 7.48\,\frac{\text{gal}}{\text{ft}^3} \times 60\,\frac{\text{s}}{\text{min}} = 301\ \text{gpm} \qquad Q$$

(b) Apply the x component of the momentum equation to the CV shown

$$F_{S_x} + F_{B_x} = \cancel{\frac{\partial}{\partial t}\int_{CV} u\,\rho\,d\forall}^{\,0} + \int u\,\rho\vec{V}\cdot d\vec{A}$$

$$R_x + p_{1g}A_1 - \cancel{p_{2g}A_2}^{\,0} = u_1\{-|\rho_1V_1A_1|\} + u_2\{+|\rho V_2A_2|\}$$

$$u_1 = V_1 \qquad u_2 = V_2$$

$$R_x = -p_{1g}A_1 + \rho V_2A_2V_2 - \rho V_1A_1V_1 = -p_{1g}A_1 + \rho Q(V_2 - V_1)$$

$$R_x = -p_{1g}A_1 + \rho QV_2\left(1 - \frac{V_1}{V_2}\right) = -p_{1g}A_1 + \rho QV_2\left[1 - \left(\frac{d}{D}\right)^2\right]$$

$$\therefore R_x = -100\,\frac{\text{lbf}}{\text{in}^2} \times \frac{\pi}{4}(3)^2\,\text{in}^2 + 1.94\,\frac{\text{slug}}{\text{ft}^3} \times 301\,\frac{\text{gal}}{\text{min}} \times \frac{\text{ft}^3}{7.48\,\text{gal}} \times \frac{\text{min}}{60\,\text{s}} \times 123\,\frac{\text{ft}}{\text{s}}\left[1 - \left(\frac{1}{3}\right)^2\right] \times \frac{\text{lbf}\cdot\text{s}^2}{\text{ft}\cdot\text{slug}}$$

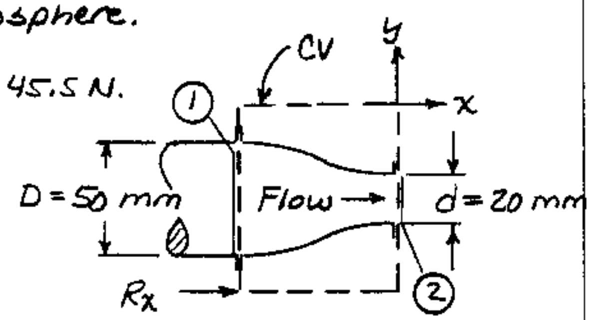
$$R_x = -707\ \text{lbf} + 142\ \text{lbf} = -565\ \text{lbf} \qquad R_x$$

The coupling is in tension.

Problem 6.56

Given: Nozzle coupled to straight pipe by flanges, bolts. Water flow discharges to atmosphere.

For steady, inviscid flow, $R_x = -45.5$ N.



Find: Volume flow rate.

Solution: Apply continuity, x momentum, and Bernoulli.

Basic equation: $0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$ (first term $= 0(1)$)

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + g z_2 \quad (p_2 = 0(7))$$

$$F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u \rho \, d\forall + \int_{CS} u \rho \vec{V} \cdot d\vec{A} \quad (F_{B_x} = 0(6),\ \text{unsteady term} = 0(1))$$

Assumptions: (1) Steady flow
(2) Uniform flow at each section
(3) Flow along a streamline
(4) Incompressible flow
(5) No friction
(6) Horizontal, $F_{B_x} = 0$, $z_1 = z_2$
(7) Use gage pressures

Then

$$0 = \{-V_1 A_1\} + \{+V_2 A_2\} \; ; \; V_2 = V_1 \frac{A_1}{A_2} = V_1 \left(\frac{D}{d}\right)^2 \; ; \; Q = V_1 A_1 = V_2 A_2$$

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} = \frac{V_2^2}{2} \; ; \; p_1 = \rho\left(\frac{V_2^2}{2} - \frac{V_1^2}{2}\right) = \rho \frac{V_1^2}{2}\left[\left(\frac{V_2}{V_1}\right)^2 - 1\right] = \rho \frac{V_1^2}{2}\left[\left(\frac{D}{d}\right)^4 - 1\right]$$

$$R_x + p_1 A_1 - p_2 A_2 = u_1 \{-|\rho V_1 A_1|\} + u_2 \{+|\rho V_2 A_2|\} = \rho V_1 A_1 (V_2 - V_1) \quad (p_2 = 0(7))$$

$$u_1 = V_1 \qquad u_2 = V_2$$

$$R_x + A_1 \rho \frac{V_1^2}{2}\left[\left(\frac{D}{d}\right)^4 - 1\right] = \rho V_1^2 A_1 \left(\frac{V_2}{V_1} - 1\right) = \rho V_1^2 A \left[\left(\frac{D}{d}\right)^2 - 1\right]$$

Thus

$$V_1^2 = \frac{-2R_x}{\rho A_1} \frac{1}{\left(\frac{D}{d}\right)^4 - 2\left(\frac{D}{d}\right)^2 + 1} \quad \text{so} \quad V_1 = \sqrt{\frac{-2R_x}{\rho A_1}} \frac{1}{\left(\frac{D}{d}\right)^2 - 1}$$

$$V_1 = \left[-2 \times -45.5\,\text{N} \times \frac{\text{m}^3}{999\,\text{kg}} \times \frac{4}{\pi (0.050)^2 \text{m}^2} \times \frac{\text{kg}\cdot\text{m}}{\text{N}\cdot\text{s}^2}\right]^{\frac{1}{2}} \frac{1}{\left(\frac{50}{20}\right)^2 - 1} = 1.30\ \text{m/s}$$

Finally,

$$Q = V_1 A_1 = 1.30\,\frac{\text{m}}{\text{s}} \times \frac{\pi}{4}(0.050)^2\,\text{m}^2 = 2.55 \times 10^{-3}\,\text{m}^3/\text{s}$$ Q

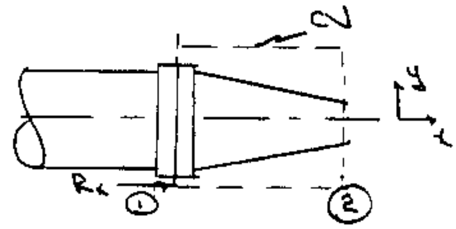
{Note: It is necessary to recognize that $R_x < 0$ for a nozzle, see Example Problem 4.7.}

Problem 6.57

Given: Water flows steadily through a pipe with diameter $D = 3.25$ in. and discharges through a nozzle ($d = 1.25$ in) to atmosphere. The flow rate is $Q = 24.5$ gal/min.

Find: (a) the minimum static pressure required in the pipe to produce this flowrate
(b) the horizontal force of the nozzle assembly on the pipe flange.

Solution:



Apply the Bernoulli equation along the central streamline between sections ① and ②

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + g\cancel{z_1} = \frac{p_2}{\rho} + \frac{V_2^2}{2} + g\cancel{z_2}$$

Assumptions: (1) steady flow (2) incompressible flow
(3) frictionless flow (4) flow along a streamline
(5) $\Delta z = 0$ (6) uniform flow at each section

Then $p_1 = p_2 + \frac{\rho}{2}(V_2^2 - V_1^2) = p_2 + \frac{\rho V_2^2}{2}\left[1 - \left(\frac{V_1}{V_2}\right)^2\right]$

$p_2 = p_{atm}$ and from continuity, $A_2V_2 = A_1V_1$

$$\therefore\ p_{1g} = \frac{\rho}{2}V_2^2\left[1 - \left(\frac{A_2}{A_1}\right)^2\right] = \frac{\rho V_2^2}{2}\left[1 - \left(\frac{D_2}{D_1}\right)^4\right]$$

$$V_2 = \frac{Q}{A} = \frac{4Q}{\pi d^2} = \frac{4}{\pi} \times 24.5\,\frac{\text{gal}}{\text{min}} \times \frac{\text{ft}^3}{7.48\,\text{gal}} \times \frac{\text{min}}{60\,\text{s}} \times \frac{1}{(1.25)^2\,\text{in}^2} \times \frac{144\,\text{in}^2}{\text{ft}^2}$$

$V_2 = 6.41$ ft/s and

$$p_{1g} = \frac{1}{2} \times 1.94\,\frac{\text{slug}}{\text{ft}^3} \times (6.41)^2\,\frac{\text{ft}^2}{\text{s}^2} \times \frac{\text{lbf}\cdot\text{s}^2}{\text{ft}\cdot\text{slug}}\left[1 - \left(\frac{1.25}{3.25}\right)^4\right] = 39.0\ \text{psf gage}$$ p_1

(b) Apply the x momentum equation to the CV

$$F_{Sx} + F_{Bx} = \cancel{\frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall}^{\,0} + \int_{CS} u\rho\vec{V}\cdot d\vec{A}$$

$$R_x + p_{1g}A_1 = u_1\{-\dot{m}\} + u_2\{\dot{m}\} = -V_1\dot{m} + V_2\dot{m}$$

$$R_x = -p_{1g}A_1 + \dot{m}(V_2 - V_1) = -p_{1g}A_1 + \rho Q V_2\left(1 - \frac{V_1}{V_2}\right)$$

$$= -39\,\frac{\text{lbf}}{\text{ft}^2} \times \frac{\pi}{4}\left(\frac{3.25}{12}\right)^2 \text{ft}^2 + 1.94\,\frac{\text{slug}}{\text{ft}^3} \times 24.5\,\frac{\text{gal}}{\text{min}} \times \frac{\text{ft}^3}{7.48\,\text{gal}} \times \frac{\text{min}}{60\,\text{s}} \times 6.41\,\frac{\text{ft}}{\text{s}}\left[1 - \left(\frac{1.25}{3.25}\right)^2\right]\frac{\text{lbf}\cdot\text{s}^2}{\text{ft}\cdot\text{slug}}$$

$R_x = -2.25 + 0.58 = -1.67$ lbf

Force of nozzle on flange $K_x = -R_x = 1.67$ lbf K_x

Problem 6.58

Given: Steady flow of water through elbow in horizontal plane.

Find: (a) Gage pressure at ①.
(b) x component of force exerted by elbow on supply pipe.

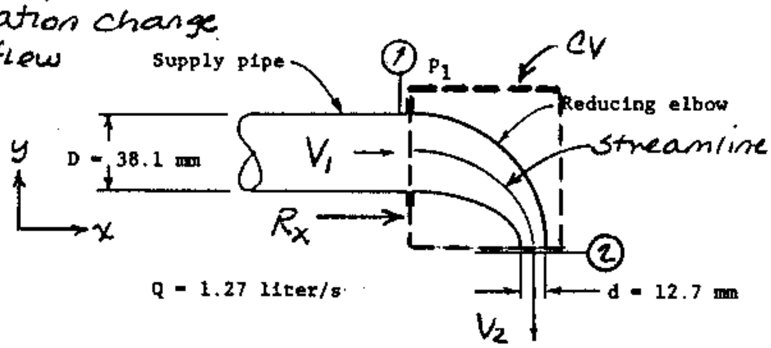
Solution: Apply Bernoulli and momentum equations using streamline and CV shown.

Basic equation: $\frac{p_1}{\rho} + \frac{V_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + g z_2$

$$F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u \rho \, d\forall + \int_{CS} u \rho \vec{V} \cdot d\vec{A}$$

($F_{B_x} = 0$ (6); $\frac{\partial}{\partial t} = 0$ (1))

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Neglect friction
(4) Flow along a streamline
(5) Neglect elevation change
(6) Horizontal flow
(7) $p_2 = p_{atm}$
(8) Uniform flow at each section



Then

$$p_{1_{gage}} = \frac{\rho}{2}\left(V_2^2 - V_1^2\right)$$

From continuity,

$$V_1 = \frac{Q}{A_1} = \frac{4Q}{\pi D_1^2}$$

$$V_1 = \frac{4}{\pi} \times 1.27 \frac{L}{s} \times \frac{1}{(0.0381)^2 \, m^2} \times \frac{m^3}{1000 \, L} = 1.11 \, m/s$$

and $V_1 A_1 = V_2 A_2$

$$V_2 = V_1 \frac{A_1}{A_2} = V_1 \left(\frac{D_1}{D_2}\right)^2 = 1.11 \frac{m}{s} \left(\frac{38.1}{12.7}\right)^2 = 9.99 \, m/s$$

Thus

$$p_{1g} = \frac{1}{2} \times 999 \frac{kg}{m^3} \left[(9.99)^2 - (1.11)^2\right] \frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m} = 49.2 \, kPa \, (gage)$$

p_{1g}

From momentum

$$R_x + p_{1g} A_1 = u_1\{-\dot{m}\} + u_2\{+\dot{m}\} = -\dot{m} V_1 = -\rho Q V_1$$

$$u_1 = V_1 \qquad u_2 = 0$$

$$R_x = -p_{1g} A_1 - \dot{m} V_1 = -49.2 \times 10^3 \frac{N}{m^2} \times \frac{\pi}{4} (0.0381)^2 m^2 - 999 \frac{kg}{m^3} \times 0.00127 \frac{m^3}{s} \times 1.11 \frac{m}{s} \times \frac{N \cdot s^2}{kg \cdot m}$$

$R_x = -57.5 \, N$ (force on CV)

The force on the supply pipe is

$K_x = -R_x = 57.5 \, N$ (on pipe to right)

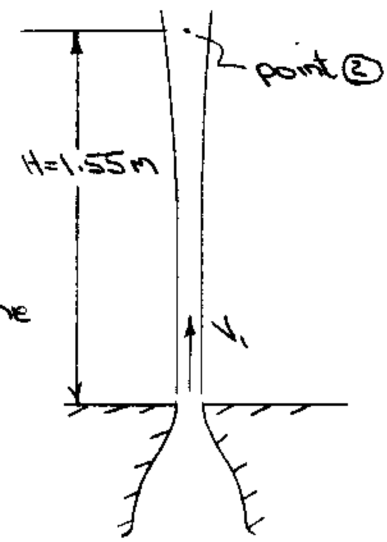
K_x

Problem 6.59

Given: A water jet is directed upward from a well-designed nozzle of area $A_1 = 600\ mm^2$; $V_1 = 6.3\ m/s$
The flow is steady and liquid stream does not break up. Point ② is $H = 1.55\ m$ above nozzle exit

Find: (a) V_2 (b) p_{o2}
(c) force on flat plate placed normal to the flow at ②
(d) Sketch pressure distribution on the plate

Solution: Apply Bernoulli and then y-momentum equation



Basic eq.: $\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2$

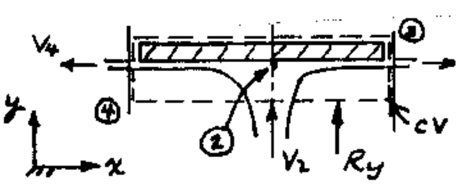
Assumptions: (1) steady flow
(2) incompressible flow
(3) frictionless flow
(4) flow along a streamline
(5) $p_1 = p_2 = p_{atm}$

Then
$$V_2 = \left[V_1^2 + 2g(z_1 - z_2)\right]^{1/2}$$
$$V_2 = \left[(6.3)^2 \frac{m^2}{s^2} + 2 \times 9.81 \frac{m}{s^2}(-1.55 m)\right]^{1/2}$$
$$V_2 = 3.05\ m/s \quad \leftarrow V_2$$

By definition, $p_{o2} = p_2 + \frac{1}{2}\rho V_2^2 = p_{atm} + \frac{1}{2}\rho V_2^2$, so
$$p_{o2\,gage} = \frac{1}{2} \times 999 \frac{kg}{m^3} \times (3.05)^2 \frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m} = 4.65\ kPa\,(g) \quad \leftarrow p_{o2}$$

Apply y-momentum equation to CV surrounding plate

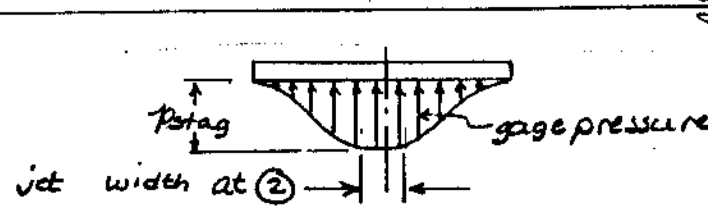
Basic eq.: $F_{Sy} + F_{By} = \frac{\partial}{\partial t}\int_{CV} v \rho\, dV + \int_{CS} v \rho \vec{V} \cdot d\vec{A}$ (with $F_{By} \approx 0$ (6), $\frac{\partial}{\partial t} = 0$ (1))



Assumptions: (6) neglect mass in CV
(7) V_2 enters CV uniformly
(8) $v_3 = v_4 = 0$

Then
$$R_y = v_2\{-\rho V_1 A_1\} + v_3\{\dot{m}_3\} + v_4\{\dot{m}_4\} = -\rho V_1 A_1 V_2 \quad \text{and}$$
(with $v_3 = 0$ (8), $v_4 = 0$ (8))
$$K_y = -R_y = \rho V_1 A_1 V_2 = 999 \frac{kg}{m^3} \times 6.3 \frac{m}{s} \times 600\ mm^2 \times 3.05 \frac{m}{s} \times \frac{m^2}{10^6\ mm^2} \times \frac{N \cdot s^2}{kg \cdot m}$$
$$K_y = 11.5\ N \text{ (force up)} \quad \leftarrow K_y$$

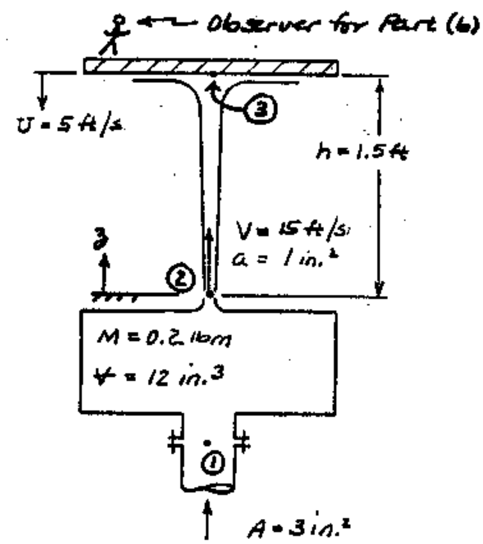
The pressure distribution on the plate is as shown.



Given: A flat object moves downward, at speed $U = 5$ ft/sec, into the water jet of the spray system shown. The spray system, of mass $M = 0.200$ lbm and internal volume $\forall = 12$ in³, operates under steady conditions

Find: (a) the minimum supply pressure required to produce the jet of the spray system.
(b) the maximum pressure exerted by the jet on the object when the object is at $z = 1.5$ ft.

Solution:



(a)
The minimum pressure occurs when friction is neglected, and so we apply the Bernoulli equation

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + \cancel{gz_1}^{(5)} = \frac{p_2}{\rho} + \frac{V_2^2}{2} + \cancel{gz_2}^{(5)}$$

Assume: (1) steady flow
(2) incompressible flow
(3) no friction
(4) flow along a streamline
(5) neglect $z_2 - z_1$
(6) $p_2 = p_{atm}$
(7) uniform flow at ① & ②

Then

$$p_{1g} = p_1 - p_{atm} = \frac{\rho}{2}(V_2^2 - V_1^2) = \frac{\rho V_2^2}{2}\left[1 - \left(\frac{V_1}{V_2}\right)^2\right]$$

From continuity, $A_1V_1 = A_2V_2$, and $\frac{V_1}{V_2} = \frac{A_2}{A_1} = \frac{a}{A}$. Then,

$$p_{1g} = \frac{\rho V_2^2}{2}\left[1 - \left(\frac{a}{A}\right)^2\right] = \frac{1}{2} \times 1.94\,\frac{\text{slug}}{\text{ft}^3} \times (15)^2\,\frac{\text{ft}^2}{\text{s}^2}\left[1 - \left(\frac{1}{3}\right)^2\right] \times \frac{\text{lbf}\cdot\text{s}^2}{\text{ft}\cdot\text{slug}} \times \frac{\text{ft}^2}{144\,\text{in}^2} = 1.35\text{ psig} \qquad p_{1g}$$

Frictional effects would cause this value to be higher.

(b) The maximum pressure of the jet on the object is the stagnation pressure

$$p_0 = p + \frac{1}{2}\rho V_r^2$$

where V is the velocity of the impinging jet relative to the object

At $z = 1.5$ ft, the jet velocity, V_4, in the absence of the object can be calculated from $\frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2 = \frac{p_4}{\rho} + \frac{V_4^2}{2} + gz_4$

$$V_4 = \left[V_2^2 - 2g(z_4 - z_2)\right]^{1/2} = \left[(15)^2\,\frac{\text{ft}^2}{\text{s}^2} - 2 \times 32.2\,\frac{\text{ft}}{\text{s}^2}\,(1.5)\,\text{ft}\right]^{1/2} = 11.3\text{ ft/s}$$

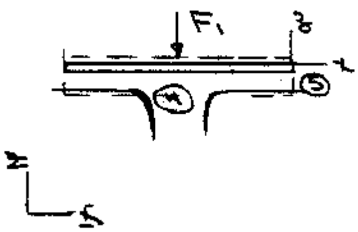
Then

$$V_{rel} = V_4 - (-U) = (11.3 + 5)\text{ ft/s} = 16.3\text{ ft/s}$$

and

$$p_0 - p_{atm} = p_{0g} = \frac{1}{2}\rho V_r^2 = \frac{1}{2} \times 1.94\,\frac{\text{slug}}{\text{ft}^3} \times (16.3)^2\,\frac{\text{ft}^2}{\text{s}^2} \times \frac{\text{lbf}\cdot\text{s}^2}{\text{ft}\cdot\text{slug}} \times \frac{\text{ft}^2}{144\,\text{in}^2} = 1.79\text{ psig} \qquad p_{0g}$$

(c) To determine the force of the water on the object we apply the z component of the momentum equation to the CV shown



$$F_{S_z} + \cancel{F_{B_z}} = \cancel{\frac{\partial}{\partial t}\int_{CV} w_{xyz}\,\rho\, d\forall} + \int_{CS} w_{xyz}\left(\rho \vec{V}_{xyz}\cdot d\vec{A}\right)$$

(the F_{B_z} term is struck through and marked ≈0(9); the $\frac{\partial}{\partial t}\int_{CV}$ term is struck through and marked ≈0(8))

Assumptions:
(8) neglect $\frac{\partial}{\partial t}\int_{CV}$
(9) neglect body forces
(10) uniform radial flow at ⑤
(11) uniform vertical flow at ④ with $z_4 = 1.5$ ft

Then $-F_1 = -w_{4xyz}\,|\rho V_{4xyz} A_4|$

where F_1 is applied force necessary to maintain motion of plate at constant speed U

$$V_{4xyz} = V_4 - (-U) = V_4 + U$$

$$w_{4xyz} = V_{4xyz} = V_4 + U$$

$$\therefore\ F_1 = \rho (V_4 + U)^2 A_4$$

From continuity $A_2 V_2 = A_4 V_4$

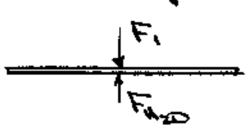
and $A_4 = \frac{V_2}{V_4} A_2 = \frac{15}{11.3} \times 1\,in^2 = 1.33\ in^2$

Then

$$F_1 = \rho (V_4 + U)^2 A_4 = 1.94\,\frac{slug}{ft^3}\,(11.3 + 5)^2\,\frac{ft^2}{s^2} \times 1.33\,in^2 \times \frac{ft^2}{144\,in^2} \times \frac{lbf \cdot s^2}{ft \cdot slug}$$

$F_1 = 4.76$ lbf (in the direction shown)

Since the plate is moving at constant speed, then



$$\Sigma \vec{F}_{plate} = m\vec{a} = 0$$ and

neglecting the weight of the plate then

$$F_{H_2O} = F_1 = 4.76\ lbf$$

$$\vec{F}_{H_2O} = 4.76\,\hat{k}\ lbf$$

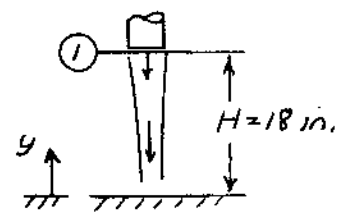
Problem 6.61

Given: Water flow from a kitchen faucet of 0.5 in. diameter at 2 gpm. Bottom of sink is 18 in. below faucet outlet.

Find: (a) If area of stream increases, decreases, or remains constant, and why.
(b) Expression for cross-section vs. y, measured above bottom.
(c) Force on plate held horizontal; variation with height, and why?

Solution: The water stream is accelerated by gravity. The area of the stream will decrease toward the sink bottom, because less area is needed to carry the same flow rate.

Apply Bernoulli to steady, incompressible, frictionless flow along a streamline:



Basic equation: $\frac{\cancel{p_1}}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{\cancel{p}}{\rho} + \frac{V^2}{2} + gz$

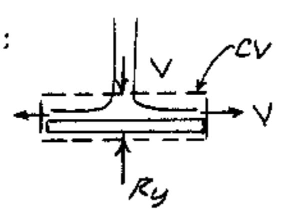
But $p_1 = p = p_{atm}$, so

$$\frac{V_1^2}{2} + gH = \frac{V^2}{2} + gy \; ; \; V = \left[V_1^2 + 2g(H-y)\right]^{\frac{1}{2}}$$

For uniform flow, continuity reduces to $V_1 A_1 = VA$

$$A = A_1 \frac{V_1}{V} = A_1 \frac{V_1}{\left[V_1^2 + 2g(H-y)\right]^{\frac{1}{2}}} = \frac{A_1}{\left[1 + \frac{2g}{V_1^2}(H-y)\right]^{\frac{1}{2}}}$$ A(y)

Predict force on plate from y component of momentum:



Basic equation: $F_{S_y} + F_{B_y} = \cancel{\frac{\partial}{\partial t}} \int_{CV} v \rho \, d\forall + \int_{CS} v \rho \vec{V} \cdot d\vec{A}$ (=0 (3))

Since uniform, $R_y - W = v\{-\rho V_1 A_1\} = -V\{-\rho Q\} = +V\rho Q$

$v = -V$

Thus $R_y = W + V\rho Q$ R_y

Since V increases as y decreases, R_y varies in the same manner.

Open-Ended Problem Statement: An old "parlor trick" uses an empty thread spool and a playing card. The playing card is placed against the bottom of the spool. Contrary to intuition, when one blows downward, through the central hole in the spool, the card is not blown away. Instead it is "sucked" up against the spool. Explain.

Discussion: The secret to this "parlor trick" lies in the velocity distribution, and hence the pressure distribution, that exists between the spool and the playing card.

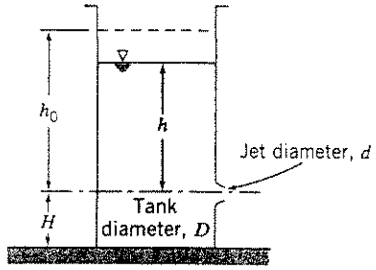
Neglect viscous effects for the purpose of initial discussion. Consider the space between the end of the spool and the playing card as a pair of parallel disks. Air from the hole in the spool enters the annular space surrounding the hole, then flows radially outward between the parallel disks. For a given flow rate of air the edge of the hole is the cross-section of minimum flow area and therefore the location of maximum air speed.

After entering the space between the parallel disks, air flows radially outward. The flow area becomes larger as the radius increases. Thus the air slows and its pressure increases. The largest flow area, slowest air speed, and highest pressure between the disks occur at the outer periphery of the spool where the air is discharged from an annular area.

The air leaving the annular space between the disk and card must be at atmospheric pressure. This is the location of the highest pressure in the space between the parallel disks. Therefore pressure at smaller radii between the disks must be lower, and hence the pressure between the disks is sub-atmospheric. Pressure above the card is less than atmospheric pressure; pressure beneath the card is atmospheric. Each portion of the card experiences a pressure difference acting upward. This causes a net pressure force to act upward on the whole card. The upward pressure force acting on the card tends to keep it from blowing off the spool when air is introduced through the central hole in the spool.

Viscous effects are present in the narrow space between the disk and card. However, they only reduce the pressure rise as the air flows outward, they do not dominate the flow behavior.

Given: Tank shown has well-rounded nozzle.
At time $t=0$, water level is h_0



Find: expression for h/h_0 as a function of time.

Plot: (a) h/h_0 vs t for $D/d = 10$, with h_0 as a parameter for $0.1 \le h/h_0 \le 1\,m$
(b) h/h_0 vs t for $h_0 = 1\,m$, with D/d as a parameter for $2 \le D/d \le 10$.

Solution:

Apply the Bernoulli equation along a streamline between the surface and the jet

Basic equation: $\frac{p_s}{\rho} + \frac{V_t^2}{2} + gz_s = \frac{p_j}{\rho} + \frac{V_j^2}{2} + gz_j$

Assumptions: (1) quasi-steady flow, ie neglect acceleration in tank.
(2) incompressible flow
(3) neglect frictional effects
(4) flow along a streamline
(5) $p_t = p_j = p_{atm}$.

From continuity, $V_t A_t = V_j A_j$ or $V_j = V_t \frac{A_t}{A_j} = V_t \left(\frac{D}{d}\right)^2$

Solving,

$$\frac{V_t^2}{2} - \frac{V_j^2}{2} = \frac{V_t^2}{2}\left[1 - \left(\frac{V_j}{V_t}\right)^2\right] = g(z_j - z_s) = g[H - (H+h)] = -gh$$

Then

$$V_t = \left[\frac{2gh}{(V_j/V_t)^2 - 1}\right]^{1/2} = \left[\frac{2gh}{(A_t/A_j)^2 - 1}\right]^{1/2} = \left[\frac{2gh}{(D/d)^4 - 1}\right]^{1/2} = -\frac{dh}{dt}$$

Separating variables,

$$\frac{dh}{h^{1/2}} = -\left[\frac{2g}{(D/d)^4 - 1}\right]^{1/2} dt$$

Integrating,

$$2h^{1/2} = -\left[\frac{2g}{(D/d)^4 - 1}\right]^{1/2} t + c$$

At $t=0$, $h = h_0$, so $c = 2h_0^{1/2}$ and

$$h = \left\{ h_0^{1/2} - \frac{1}{2}\left[\frac{2g}{(D/d)^4 - 1}\right]^{1/2} t \right\}^2$$

Nondimensionalize (divide by h_0) to obtain

$$\frac{h}{h_0} = \left\{1 - \sqrt{\frac{g}{2h_0\{(D/d)^4 - 1\}}}\, t\right\}^2 \quad \longleftarrow \quad h/h_0$$

Draining of a cylindrical liquid tank:

Plot of h/h_0 vs. t for $0.1 < h_0 < 1$ m

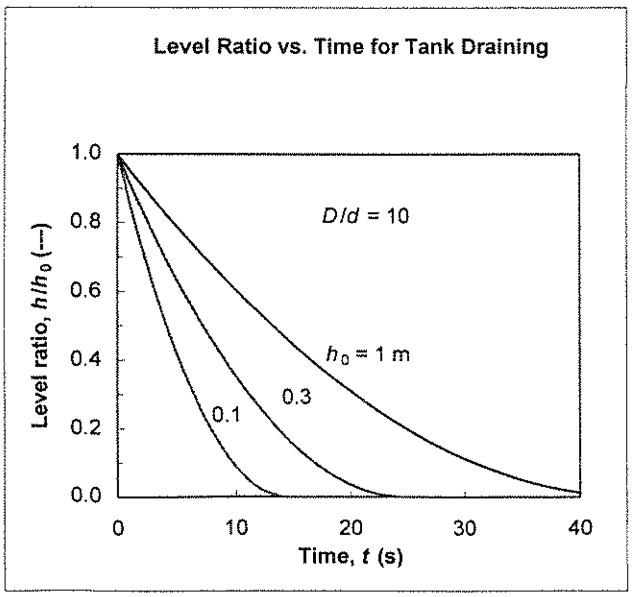
Input Data: D = 50 mm, d = 5 mm

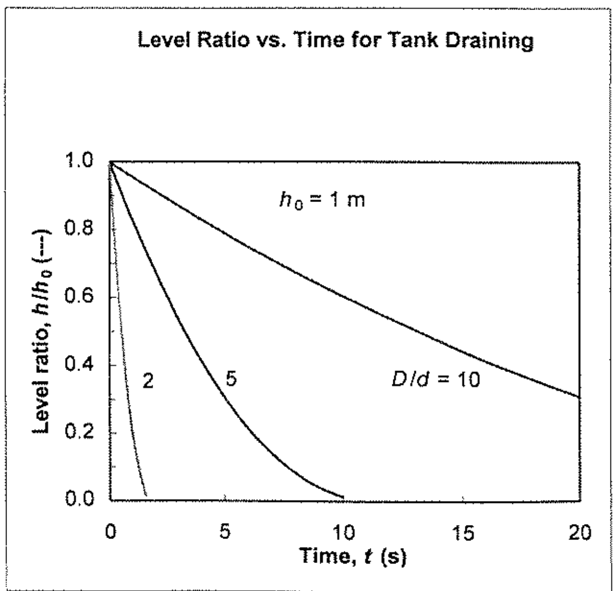
h_0 (m) =	0.1	0.3	1
Time, t (s)	**h/h_0 (---)**	**h/h_0 (---)**	**h/h_0 (---)**
0	1.00	1.00	1.00
2	0.739	0.845	0.913
4	0.518	0.703	0.831
6	0.336	0.574	0.752
8	0.193	0.458	0.677
10	0.090	0.355	0.606
12	0.025	0.265	0.539
14	0.000	0.188	0.476
16		0.125	0.417
18		0.074	0.362
20		0.037	0.310
22		0.012	0.263
24		0.001	0.219
26			0.180
28			0.144
30			0.113
32			0.085
34			0.061
36			0.041
38			0.025
40			0.013
45			0.000

Plot of h/h_0 vs. t for $10 < D/d < 2$

h_0 = 1 m

D/d (---) =	2	5	10
Time, t (s)	**h/h_0 (---)**	**h/h_0 (---)**	**h/h_0 (---)**
0	1.00	1.00	1.00
0.5	0.523	0.913	0.978
1	0.199	0.831	0.956
1.5	0.029	0.752	0.935
1.6	0.013	0.737	0.930
3		0.539	0.872
4		0.417	0.831
5		0.310	0.791
6		0.219	0.752
7		0.144	0.714
8		0.085	0.677
9		0.041	0.641
10		0.013	0.606
12			0.539
14			0.476
16			0.417
18			0.362
20			0.310

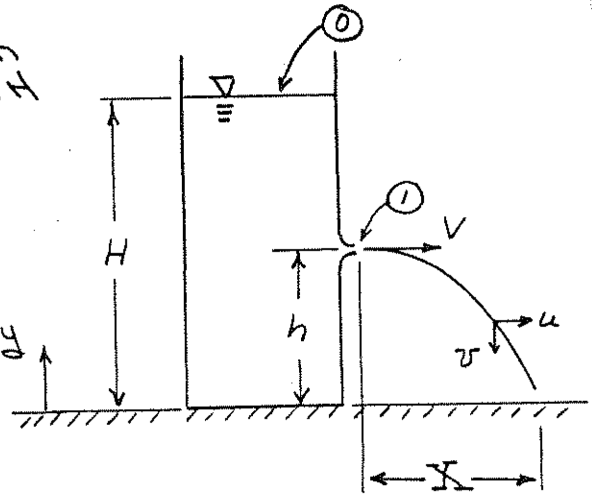




Given: Water level in tank shown is maintained at height H

Find: Elevation h to maximize range, X, of jet.

Plot: Jet speed, V, & distance, X as function of h, for $0<h<H$.



Solution:

Apply Bernoulli equation between tank surface and jet.

Basic equation: $\frac{p_0}{\rho} + \frac{V_0^2}{2} + gy_0 = \frac{p_1}{\rho} + \frac{V_1^2}{2} + gy_1$ (with p_0, $V_0^2 \approx 0$ and p_1 cancelled)

Assumptions: (1) steady flow (2) incompressible flow (3) flow along streamline (4) no friction

Then
$$gH = \frac{V^2}{2} + gh \quad \text{or} \quad V = \sqrt{2g(H-h)} \qquad (1)$$

Assume no air resistance in the stream. Then u = constant, and $X = ut = \sqrt{2g(H-h)}\,t$ — — — — — — (2)

The only force acting on the stream is gravity
$$\Sigma F_y = -mg = ma_y = m\frac{dv}{dt} \; ; \text{ thus } \frac{dv}{dt} = -g$$

Integrating we obtain $v = v_0 - gt$ (with $v_0 = 0$) and
$$y = y_0 + v_0 t - \tfrac{1}{2}gt^2 \quad (v_0 = 0)$$

Solving for t, $t = \left[\frac{2(y_0 - y)}{g}\right]^{1/2}$

The time of flight is then $t = \sqrt{\frac{2y_0}{g}} = \sqrt{\frac{2h}{g}}$

Substituting into Eq. 2
$$X = \sqrt{2g(H-h)}\sqrt{\frac{2h}{g}} = 2\sqrt{h(H-h)} \qquad (3)$$

X will be maximized when $h(H-h)$ is maximized, or when
$$\frac{d}{dh}\left[h(H-h)\right] = 0 = (H-h) + h(-1) = H - 2h \quad \text{or} \quad h = H/2 \quad \leftarrow h$$

The corresponding range is
$$X = 2\sqrt{\frac{H}{2}\cdot\frac{H}{2}} = H$$

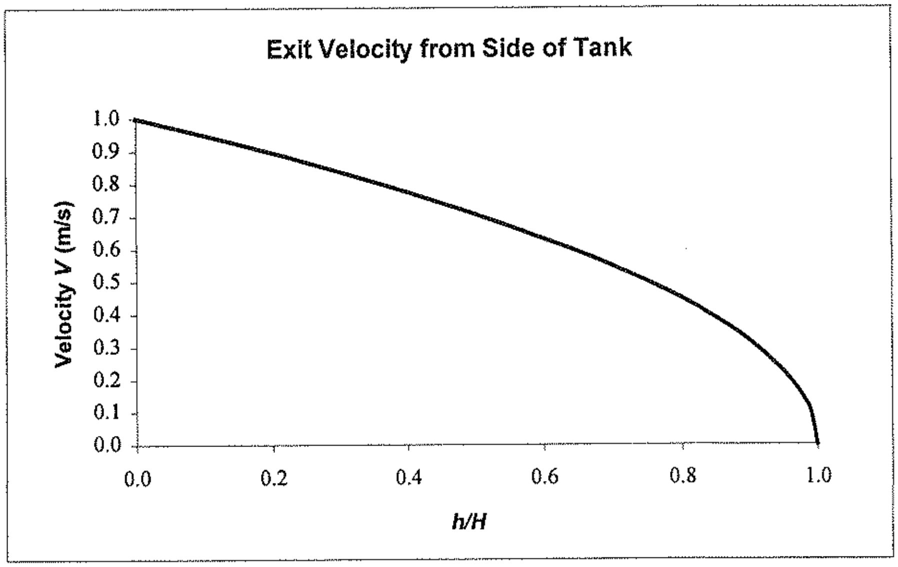
See the next page for plots

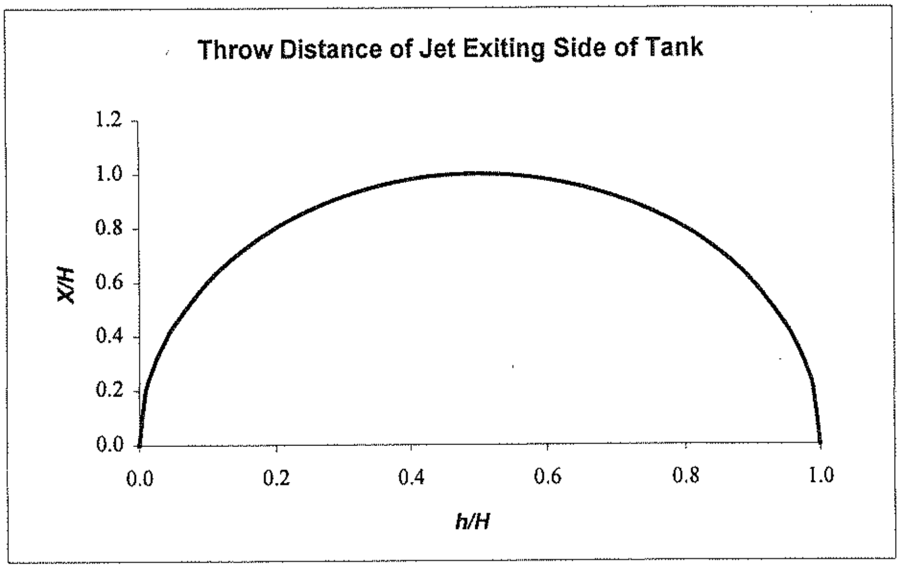
From Eq. 1, $\frac{V}{\sqrt{2gH}} = \sqrt{1 - \frac{h}{H}}$

From Eq. 2, $X/H = 2\sqrt{\frac{h}{H}\left(1 - \frac{h}{H}\right)}$

Exit velocity and throw distance from orifice in side of tank, versus height *h/H*

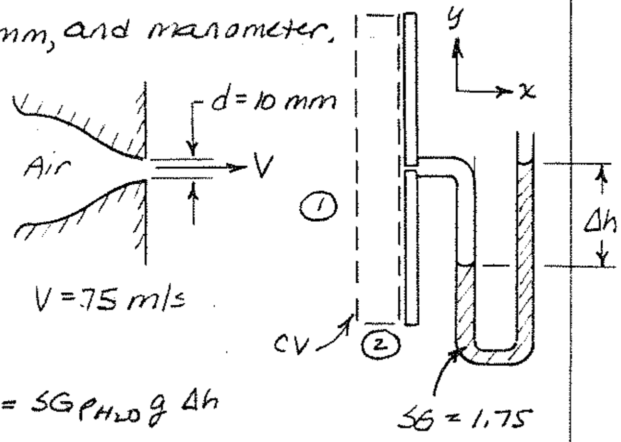
h/H	$V/(2gH)^{1/2}$	X/H
0.00	1.00	0.000
0.01	0.995	0.199
0.02	0.990	0.280
0.03	0.985	0.341
0.04	0.980	0.392
0.05	0.975	0.436
0.10	0.949	0.600
0.15	0.922	0.714
0.20	0.894	0.800
0.25	0.866	0.866
0.30	0.837	0.917
0.35	0.806	0.954
0.40	0.775	0.980
0.45	0.742	0.995
0.50	0.707	1.000
0.55	0.671	0.995
0.60	0.632	0.980
0.65	0.592	0.954
0.70	0.548	0.917
0.75	0.500	0.866
0.80	0.447	0.800
0.85	0.387	0.714
0.90	0.316	0.600
0.95	0.224	0.436
0.96	0.200	0.392
0.97	0.173	0.341
0.98	0.141	0.280
0.99	0.100	0.199
1.00	0.00	0.00





Given: Air jet, disk of diameter $D = 200$ mm, and manometer.

Find: (a) Δh
(b) Force on disk
(d) Sketch pressure distribution.
(c) Force on disk assuming $p_{disk} = p_o$



Solution: Apply Bernoulli, hydrostatic, and x component of momentum.

Basic equations: $\frac{p_o}{\rho} = \frac{p}{\rho} + \frac{V^2}{2}$ $\quad \Delta p = SG\,\rho_{H_2O}\, g\, \Delta h$

$$F_{S_x} + \cancel{F_{B_x}} = \cancel{\frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall} + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$$

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Flow along a streamline
(4) No friction
(5) Static liquid in manometer
(6) $F_{B_x} = 0$
(7) Uniform flow at each section

Then

$$\Delta p = p_o - p = \tfrac{1}{2}\rho V^2 = SG\,\rho_{H_2O}\, g\, \Delta h$$

$$\Delta h = \frac{\rho V^2}{2 SG \rho_{H_2O} g} = \frac{1}{2} \times 1.23\,\frac{kg}{m^3} \times (75)^2 \frac{m^2}{s^2} \times \frac{m^3}{(1.75)\,999\,kg} \times \frac{s^2}{9.81\,m} = 0.202\ m\ (202\ mm)$$ Δh

and

$$R_x = u_1\{-\rho V A\} + u_2\{\rho V A\} = -\rho V^2 A$$

$$u_1 = V \qquad u_2 = 0$$

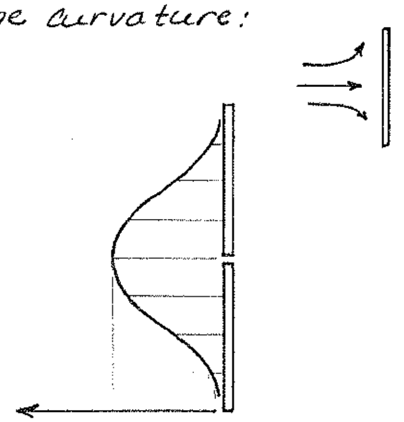
or

$$K_x = -R_x = \rho V^2 A = 1.23\,\frac{kg}{m^3} \times (75)^2 \frac{m^2}{s^2} \cdot \frac{\pi}{4}(0.010)^2 m^2 \times \frac{N\cdot s^2}{kg\cdot m} = 0.543\ N$$ K_x

The pressure distribution is caused by streamline curvature:

Pressure is p_o (gage) at center.

Pressure is zero (gage) at edges.



Assuming p_o acts on entire forward surface, then

$$K_x = (p_o - p_{atm})\, A_{disk} = \tfrac{1}{2}\rho V^2 \frac{\pi D^2_{disk}}{4}$$

$$K_x = \tfrac{1}{2} \times 1.23\,\frac{kg}{m^3} \times (75)^2 \frac{m^2}{s^2} \times \frac{\pi}{4}(0.20)^2 m^2 \times \frac{N\cdot s^2}{kg\cdot m}$$

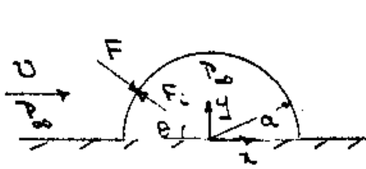
$K_x = 217$ N (a huge overestimate)

Problem 6.66

Given: Flow over a Quonset hut may be approximated by the velocity field

$\vec{V} = U\left[1-\left(\frac{a}{r}\right)^2\right]\cos\theta\,\hat{e}_r - U\left[1+\left(\frac{a}{r}\right)^2\right]\sin\theta\,\hat{e}_\theta$

with $0 \le \theta \le 2\pi$



The hut has a diameter, $D = 6m$, and a length, $L = 18m$

During a storm, $U = 100$ km/hr, $p_\infty = 720$ mm Hg, $T_\infty = 5C$

Find: The net force tending to lift the hut off its foundation.

Solution:

Basic equations: $\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{const}$ $\quad F = \int p\,dA$

Assumptions: (1) steady flow
(2) incompressible flow
(3) frictionless flow
(4) flow along a streamline

Along the top half of the cylinder, $r = a$ and $\vec{V} = -2U\sin\theta\,\hat{e}_\theta$, $0 \le \theta \le \pi$

Applying the Bernoulli equation along the streamline $(r = a)$

$$\frac{p}{\rho} + \frac{V^2}{2} = \frac{p_\infty}{\rho} + \frac{V_\infty^2}{2}$$

$$p - p_\infty = \frac{\rho}{2}(V_\infty^2 - V^2) = \frac{\rho}{2}(U^2 - 4U^2\sin^2\theta) = \rho\frac{U^2}{2}(1 - 4\sin^2\theta)$$

$$F_{R_y} = \int_A (p_\infty - p)\,dA\sin\theta = \int_0^\pi (p_\infty - p)\sin\theta\,La\,d\theta$$

$$= \int_0^\pi \rho\frac{U^2}{2}(4\sin^2\theta - 1)\sin\theta\,La\,d\theta = \rho\frac{U^2}{2}aL\left\{4\left[\frac{\cos^3\theta}{3} - \cos\theta\right]_0^\pi + \cos\theta\Big]_0^\pi\right\}$$

$$= \rho\frac{U^2}{2}aL\left\{4\left[\left(-\frac{1}{3}+1\right) - \left(\frac{1}{3}-1\right)\right] + (-1-1)\right\}$$

$$F_{R_y} = \rho\frac{U^2}{2}aL\left(\frac{10}{3}\right) = \frac{5}{3}\rho U^2 aL$$

From the ideal gas equation of state

$$\rho = \frac{p}{RT} = 720\,\text{mm Hg} \times \frac{\text{atm}}{760\,\text{mm}} \times 1.01\times10^5\frac{N}{m^2\cdot\text{atm}} \times \frac{kg\cdot K}{287\,N\cdot m} \times \frac{1}{278K} = 1.20\,kg/m^3$$

$$F_{R_y} = \frac{5}{3}\rho U^2 aL = \frac{5}{3}\times 1.20\frac{kg}{m^3}\times(10^5)^2\frac{m^2}{hr^2}\times\frac{hr^2}{(3600)^2 s^2}\times 3m \times 18m \times \frac{N\cdot s^2}{kg\cdot m}$$

$F_{R_y} = 83.3$ kN $\quad F_{R_y}$

Comment: The actual pressure distribution over the rear portion of the hut is not modelled well by ideal flow. The force calculated here is lower than the actual force.

Problem 6.67

Given: Inflatable "bubble" structure modelled as circular semi-cylinder

diameter, $D = 30\,m$

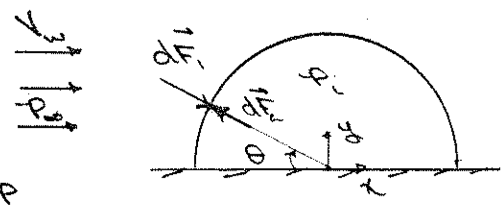
length, $L = 70\,m$

Pressure inside is $p_i = p_\infty + \Delta p$

where $\Delta p = \rho_{H_2O}\, g\, \Delta h$ and $\Delta h = 10\,mm$.

Pressure distribution over outer surface is given by

$$\frac{p - p_\infty}{\frac{1}{2}\rho V_w^2} = 1 - 4\sin^2\theta \qquad V_w = 60\,km/hr$$



Find: net vertical force exerted on the structure.

Solution:

The force due to pressure is $F = \int p\, dA$.

The vertical component of dF_1 is $dF_{1v} = -p\,dA\sin\theta = -p\,RL\,d\theta\sin\theta$

The vertical component of dF_2 is $dF_{2v} = p_i\,dA\sin\theta = p_i RL\,d\theta\sin\theta$

Then, neglecting end effects

$$dF_{v\,net} = (p_i - p)RL\sin\theta\, d\theta = (p_\infty + \Delta p - p)RL\sin\theta\, d\theta$$

$$F_v = \int dF_v = \int_0^\pi \left[\Delta p - (p - p_\infty)\right] RL\sin\theta\, d\theta$$

$$= \int_0^\pi \left[\Delta p - \tfrac{1}{2}\rho V_w^2 (1 - 4\sin^2\theta)\right] RL\sin\theta\, d\theta$$

$$= RL\left\{\Delta p\left[-\cos\theta\right]_0^\pi - \tfrac{1}{2}\rho V_w^2\left[-\cos\theta + 4\left(\cos\theta - \frac{\cos^3\theta}{3}\right)\right]_0^\pi\right\}$$

$$= RL\left\{2\Delta p - \tfrac{1}{2}\rho V_w^2\left[2 + 4\left(-2 + \tfrac{2}{3}\right)\right]\right\}$$

$$F_v = RL\left\{2\Delta p + \tfrac{5}{3}\rho V_w^2\right\} = RL\left\{2\rho_{H_2O}\, g\,\Delta h + \tfrac{5}{3}\rho V_w^2\right\}$$

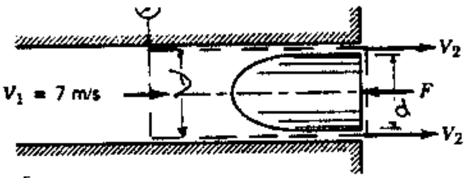
$$F_v = 15\,m \times 70\,m\left\{2 \times 999\,\frac{kg}{m^3} \times 9.81\,\frac{m}{s^2} \times 0.01\,m + \frac{5}{3} \times 1.23\,\frac{kg}{m^3} \times (60)^2\,\frac{km^2}{hr^2} \times 10^6\,\frac{m^2}{km^2} \times \frac{hr^2}{(3600\,s)^2}\right\} \times \frac{N\cdot s^2}{kg\cdot m}$$

$F_{v\,net} = 804\,kN$ ← $F_{v\,net}$

Problem 6.68

Given: Low speed water flow through a circular tube of diameter, $D = 50$ mm. Smoothly contoured plug of diameter, $d = 40$ mm, is held in the end of the tube where the water discharges to the atmosphere.
Frictional effects are to be neglected. Velocity profiles may be assumed uniform at each section.

Find: (a) pressure measured by the gage shown.
(b) force required to hold plug.



Solution:

Basic equation: $\frac{p_1}{\rho} + \frac{V_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + g z_2$ (with $p_2 = 0$ (gage); gz terms cancelled)

Assumptions: (1) steady flow
(2) incompressible flow
(3) no friction
(4) flow along a streamline
(5) $\Delta z = 0$

From the Bernoulli equation $p_1 = \frac{1}{2}\rho(V_2^2 - V_1^2)$

From continuity for uniform flow, $V_1 A_1 = V_2 A_2$

$$\therefore V_2 = V_1 \frac{A_1}{A_2} = V_1 \frac{D^2}{D^2 - d^2} = V_1 \frac{1}{1-(d/D)^2} = 7\frac{m}{s} \times \frac{1}{1-(0.8)^2} = 19.4 \text{ m/s}$$

and

$$p_1 = \frac{1}{2}\rho(V_2^2 - V_1^2) = \frac{1}{2} \times 999 \frac{kg}{m^3}\left[(19.4)^2 - (7)^2\right]\frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m} = 164 \text{ kPa (gage)}$$ p_1

To determine the force required to hold the plug, apply the x-component of the momentum equation to the CV shown.

$$F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u \rho \, d\forall + \int_{CS} u \rho \vec{V} \cdot d\vec{A}$$

($F_{B_x} = 0$; $\frac{\partial}{\partial t} = 0$ (1))

$$p_{1g} A_1 - F = u_1\{-\dot{m}\} + u_2\{\dot{m}\} = \dot{m}(u_2 - u_1) = \rho V_1 A_1 (V_2 - V_1)$$

$$F = p_{1g} A_1 - \rho V_1 A_1 (V_2 - V_1)$$

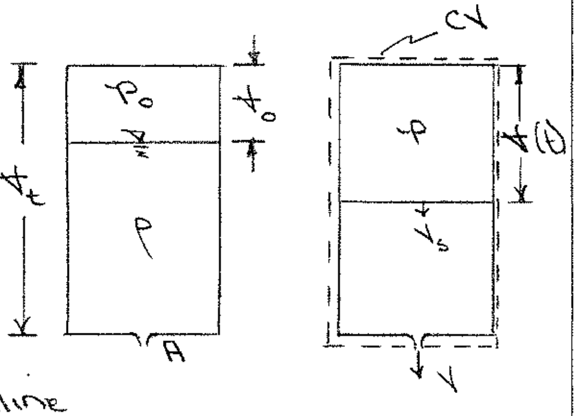
$$= 164 \times 10^3 \frac{N}{m^2} \times \frac{\pi}{4}(0.05)^2 m^2 - 999 \frac{kg}{m^3} \times 7\frac{m}{s} \times \frac{\pi}{4}(0.05)^2 m^2 (19.4 - 7)\frac{m}{s} \times \frac{N \cdot s^2}{kg \cdot m}$$

$F = 322$ N $- 170$ N $= 152$ N (in direction shown) F

Given: High-pressure air forces a stream of water from a tiny, rounded orifice, of area A, in a tank. The air expands slowly so the expansion may be considered isothermal.

Find: (a) algebraic expression for $\dot{m}$ leaving the tank
(b) " " " dM/dt in tank
(c) expression for $M_w(t)$
(d) plot $M_w(t)$ for $0 < t < 40$ min if $\mathcal{V}_o = 5\,m^3$, $\mathcal{V}_t = 10\,m^3$, $A = 25\,mm^2$, $p_o = 1$ MPa

Solution:



Basic equations: $\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{const}$

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\mathcal{V} + \int_{CS} \rho \vec{V}\cdot d\vec{A}$$

Assumptions: (1) quasi steady flow $V_s \ll V$
(2) frictionless
(3) incompressible
(4) flow along a streamline
(5) uniform flow at outlet
(6) neglect gravity
(7) $p \gg p_{atm}$ $\therefore$ $p_{abs} \approx p_{gage}$

Apply Bernoulli equation between liquid surface and orifice

$$V_j = \left[\frac{2(p - p_{atm})}{\rho}\right]^{1/2} \approx \sqrt{\frac{2p}{\rho}}$$

$$\dot{m} = \rho A V_j = \rho A\sqrt{\frac{2p}{\rho}} = \sqrt{2p\rho}\; A \longleftarrow \dot{m}$$

Rate of change of mass in tank is $\frac{dM}{dt} = \frac{\partial}{\partial t}\int \rho\, d\mathcal{V}$

$$\frac{dM}{dt} = \rho_w \frac{d\mathcal{V}_w}{dt} = -\rho_w \frac{d\mathcal{V}_{air}}{dt} \quad (\mathcal{V}_t = \mathcal{V}_{air} + \mathcal{V}_w) \longleftarrow \frac{dM}{dt}$$

For isothermal flow, $\frac{p}{\rho} = RT = \text{constant} = \frac{p_o}{\rho_o}$

where ρ is the air density and $\rho = M_{air}/\mathcal{V}_{air}$

thus

$$p\mathcal{V} = p_o\mathcal{V}_o \quad \text{or} \quad p = p_o\frac{\mathcal{V}_o}{\mathcal{V}}$$

From continuity

$$0 = \rho_w \frac{d\mathcal{V}_w}{dt} + \dot{m}$$

and

$$0 = -\rho_w \frac{d\mathcal{V}_{air}}{dt} + \sqrt{2p\rho_w}\; A$$

$$\frac{d\mathcal{V}}{dt} = \sqrt{\frac{2p}{\rho_w}} = \sqrt{\frac{2p_o\mathcal{V}_o}{\rho_w\mathcal{V}}}$$

Separating variables, $V^{1/2}\,dV = \sqrt{\frac{2p_0V_0}{\rho}}\,A\,dt$

Integrating $\frac{2}{3}V^{3/2}\Big]_{V_0}^{V} = \sqrt{\frac{2p_0V_0}{\rho_w}}\,At$

$$\frac{2}{3}\left(V^{3/2}-V_0^{3/2}\right) = \frac{2V_0^{3/2}}{3}\left[\left(\frac{V}{V_0}\right)^{3/2}-1\right] = \sqrt{\frac{2p_0V_0}{\rho_w}}\,At$$

Then

$$\left(\frac{V}{V_0}\right)^{3/2} = \left[1+\frac{3}{2V_0^{3/2}}\sqrt{\frac{2p_0V_0}{\rho_w}}\,At\right]$$

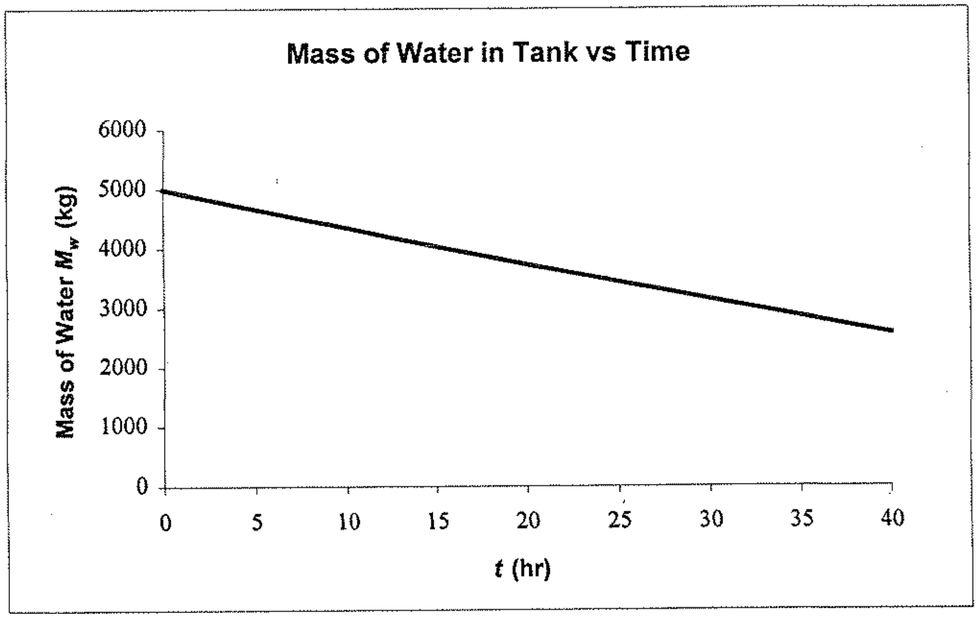
$$\frac{V}{V_0} = \left[1+1.5\sqrt{\frac{2p_0}{\rho_w}}\,\frac{At}{V_0}\right]^{2/3}$$

But $M_w = \rho_w(V_t - V) = \rho V_0\left\{\frac{V_t}{V_0}-\frac{V}{V_0}\right\}$

$$\therefore M_w = \rho_w V_0\left\{\frac{V_t}{V_0}-\left[1+1.5\sqrt{\frac{2p_0}{\rho_w}}\,\frac{At}{V_0}\right]^{2/3}\right\}$$

M_w

t (s)	M_w (kg)
0	4995
2	4862
4	4730
6	4600
8	4472
10	4345
12	4220
14	4096
16	3973
18	3851
20	3731
22	3612
24	3494
26	3377
28	3260
30	3145
32	3031
34	2918
36	2806
38	2695
40	2584



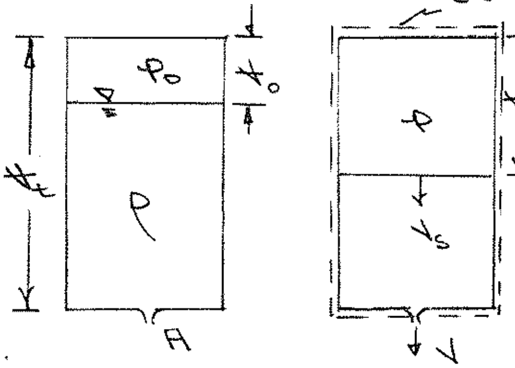
Given: High-pressure air forces a stream of water from a tiny rounded orifice, of area A, in a tank. The air expands rapidly so the expansion may be treated as adiabatic.

Find: (a) algebraic expression for $\dot{m}$ leaving the tank
(b) " " " dM/dt in the tank
(c) expression for $M_w(t)$; plot $M_w(t)$ for $0 \le t \le 40$ min if $\forall_0 = 5\,m^3$, $\forall_t = 10\,m^3$, $A = 25\,mm^2$, & $p_0 = 1$ MPa

Solution:

Basic equations: $\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{const}$

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$



Assumptions: (1) quasi steady flow $V_s \ll V$
(2) frictionless
(3) incompressible
(4) flow along a streamline
(5) uniform flow at outlet
(6) neglect gravity
(7) $p \Rightarrow p_{atm}$ $\therefore p_{abs} = p_{gage}$

Apply Bernoulli equation between liquid surface and orifice

$$V_j = \left[\frac{2(p - p_{atm})}{\rho}\right]^{1/2} = \sqrt{\frac{2p}{\rho}}$$

$$\dot{m} = \rho A V_j = \rho A \sqrt{\frac{2p}{\rho}} = \sqrt{2p\rho}\; A \qquad \dot{m}$$

Rate of change of mass in tank is $\frac{dM}{dt} = \frac{\partial}{\partial t}\int \rho\, d\forall$

$$\frac{dM}{dt} = \rho_w \frac{d\forall_w}{dt} = -\rho_w \frac{d\forall_{air}}{dt} \qquad (\forall_t = \forall_{air} + \forall_w) \qquad \frac{dM}{dt}$$

For adiabatic expansion of air $p/\rho^k = \text{constant}$

Since mass of air is constant, $p_0 \forall_0^k = p \forall^k$

From continuity, $-\rho_w \frac{d\forall_{air}}{dt} + \sqrt{2p\rho_w}\, A = 0$

$$\frac{d\forall_{air}}{dt} = \frac{A\sqrt{2}}{\sqrt{\rho_w}}\, p^{1/2} = A\sqrt{\frac{2}{\rho_w}}\left[\frac{p_0 \forall_0^k}{\forall^k}\right]^{1/2} = A\sqrt{\frac{2p_0\forall_0^k}{\rho_w}}\; \forall^{-k/2}$$

$$\forall^{k/2}\, d\forall = A\sqrt{\frac{2p_0\forall_0^k}{\rho_w}}\, dt = c\, dt \quad \text{where } c = A\sqrt{\frac{2p_0\forall_0^k}{\rho_w}}$$

Integrating

$$\frac{2}{(k+2)} \forall^{\frac{k}{2}+1}\Big]_{\forall_0}^{\forall} = ct$$

$$V^{\frac{k+2}{2}} - V_0^{\frac{k+2}{2}} = \frac{k+2}{2} c t$$

$$\left(\frac{V}{V_0}\right)^{\frac{k+2}{2}} = 1 + \frac{(k+2)}{2} c t \cdot \frac{1}{V_0^{\frac{k+2}{2}}}$$

$$= 1 + \frac{(k+2)}{2} A \left[\frac{2 P_0 V_0^{k}}{\rho_w}\right]^{1/2} \cdot \left[\frac{1}{V_0^{(k+2)}}\right]^{1/2} t$$

$$\left(\frac{V}{V_0}\right)^{\frac{k+2}{2}} = 1 + \frac{(k+2)}{2} A \left[\frac{2 P_0}{\rho_w V_0^2}\right]^{1/2} t = 1 + \frac{(k+2)}{2} \frac{A}{V_0} \left[\frac{2 P_0}{\rho_w}\right]^{1/2} t$$

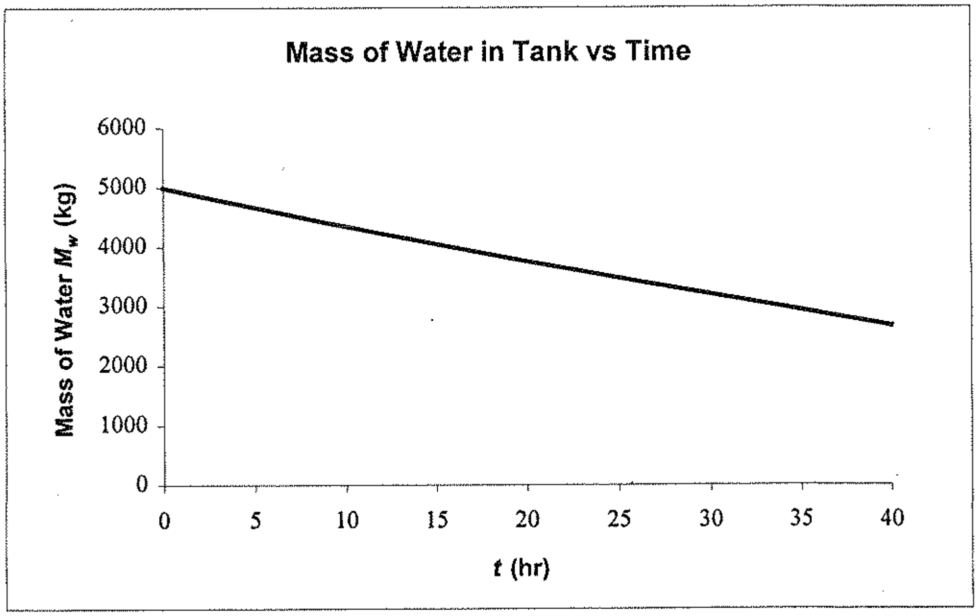
$$\frac{V}{V_0} = \left[1 + \frac{A}{V_0}\sqrt{\frac{2 P_0}{\rho_w}} \left(\frac{k+2}{2}\right) t\right]^{2/k+2}$$

$$M_w = \rho_w (V_t - V) = \rho V_0 \left\{\frac{V_t}{V_0} - \frac{V}{V_0}\right\}$$

$$M_w = \rho_w V_0 \left\{\frac{V_t}{V_0} - \left[1 + \frac{A}{V_0}\sqrt{\frac{2 P_0}{\rho_w}} \left(\frac{k+2}{2}\right) t\right]^{2/k+2}\right\}$$

$$M_w = \rho_w V_0 \left\{\frac{V_t}{V_0} - \left[1 + 1.70 \sqrt{\frac{2 P_0}{\rho_w}} \frac{A t}{V_0}\right]^{0.588}\right\} \quad M_w$$

t (s)	M_w (kg)
0	4995
2	4862
4	4732
6	4603
8	4477
10	4353
12	4231
14	4110
16	3991
18	3874
20	3759
22	3645
24	3532
26	3420
28	3310
30	3202
32	3094
34	2988
36	2882
38	2778
40	2675



Problem 6.71

Open-Ended Problem Statement: Describe the pressure distribution on the exterior of a multi-story building in a steady wind. Identify the locations of the maximum and minimum pressures on the outside of the building. Discuss the effect of these pressures on infiltration of outside air into the building.

Discussion: A multi-story building acts as a bluff-body obstruction in a thick atmospheric boundary layer. The boundary-layer velocity profile causes the air speed near the top of the building to be highest and that toward the ground to be lower.

Obstruction of air flow by the building causes regions of stagnation pressure on upwind surfaces. The stagnation pressure is highest where the air speed is highest. Therefore the maximum surface pressure occurs near the roof on the upwind side of the building. Minimum pressure on the upwind surface of the building occurs near the ground where the air speed is lowest.

The minimum pressure on the entire building will likely be in the low-speed, low-pressure wake region on the downwind side of the building.

Static pressure inside the building will tend to be an average of all the surface pressures that act on the outside of the building. It is never possible to seal all openings completely. Therefore air will tend to infiltrate into the building in regions where the outside surface pressure is above the interior pressure, and will tend to pass out of the building in regions where the outside surface pressure is below the interior pressure. Thus generally air will tend to move through the building from the upper floors toward the lower floors, and from the upwind side to the downwind side.

Open-Ended Problem Statement: Imagine a garden hose with a stream of water flowing out through a nozzle. Explain why the end of the hose may be unstable when held a half meter or so from the nozzle end.

Discussion: Water flowing out of the nozzle tends to exert a thrust force on the end of the hose. The thrust force is aligned with the flow from the nozzle and is directed toward the hose.

Any misalignment of the hose will lead to a tendency for the thrust force to bend the hose further. This will quickly become unstable, with the result that the free end of the hose will "flail" about, spraying water from the nozzle in all directions.

This instability phenomenon can be demonstrated easily in the backyard. However, it will tend to do least damage when the person demonstrating it is wearing a bathing suit!

Open-Ended Problem Statement: An aspirator provides suction by using a stream of water flowing through a venturi. Analyze the shape and dimensions of such a device. Comment on any limitations on its use.

Discussion: The basic shape of the aspirator channel should be a converging nozzle section to reduce pressure followed by a diverging diffuser section to promote pressure recovery. The basic shape is that of a venturi flow meter.

If the diffuser exhausts to atmosphere, the exit pressure will be atmospheric. The pressure rise in the diffuser will cause the pressure at the diffuser inlet (venturi throat) to be below atmospheric.

A small tube can be brought in from the side of the throat to aspirate another liquid or gas into the throat as a result of the reduced pressure there.

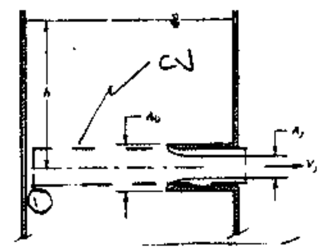
The following comments can be made about limitations on the aspirator:

1. It is desirable to minimize the area of the aspirator tube compared to the flow area of the venturi throat. This minimizes the disturbance of the main flow through the venturi and promotes the best possible pressure recovery in the diffuser.
2. It is desirable to avoid cavitation in the throat of the venturi. Cavitation alters the effective shape of the flow channel and destroys the pressure recovery in the diffuser. To avoid cavitation, the reduced pressure must always be above the vapor pressure of the driver liquid.
3. It is desirable to limit the flow rate of gas into the venturi throat. A large amount of gas can alter the flow pattern and adversely affect pressure recovery in the diffuser.

The best combination of specific dimensions could be determined experimentally by a systematic study of aspirator performance. A good starting point probably would be to use dimensions similar to those of a commercially available venturi flow meter.

Problem 6.74

Given: Reentrant orifice in the side of a large tank. Pressure along the tank walls is essentially hydrostatic.



Find: the contraction coefficient, $C_c = A_j / A_0$

Solution:

Apply the x-component of the momentum equation to the CV shown

$$F_{S_x} + \cancel{F_{B_x}} = \cancel{\frac{\partial}{\partial t}\int_{CV} u \rho \, d\forall}^{\,0(1)} + \int_{CS} u \rho \vec{V} \cdot d\vec{A}$$

Assumptions: (1) steady flow
(2) uniform flow at jet exit.
(3) hydrostatic pressure variation across CS ①, $V_1 \approx 0$
(4) x momentum flux across horizontal portion of CS is negligible.
(5) ρ = constant

then

$$\int_{A_0} p \, dA_1 = \dot{m} V_j = \rho V_j A_j V_j = \rho A_j V_j^2$$

$$\bar{p}_1 A_0 = \rho g h A_0 = \rho A_j V_j^2$$

$$\therefore \frac{A_0}{A_j} = \frac{V_j^2}{gh}$$

Apply the Bernoulli equation along the central streamline from ① to the jet exit.

$$\frac{p_1}{\rho} + \cancel{\frac{V_1^2}{2}}^{\,0(3)} + g z_1 = \cancel{\frac{p_j}{\rho}}^{\,0(gage)} + \frac{V_j^2}{2} + g z_j$$

Assumptions: (6) frictionless flow

$$p_1 = \rho g h = \rho \frac{V_j^2}{2}$$

$$\therefore \frac{V_j^2}{2} = gh$$

and

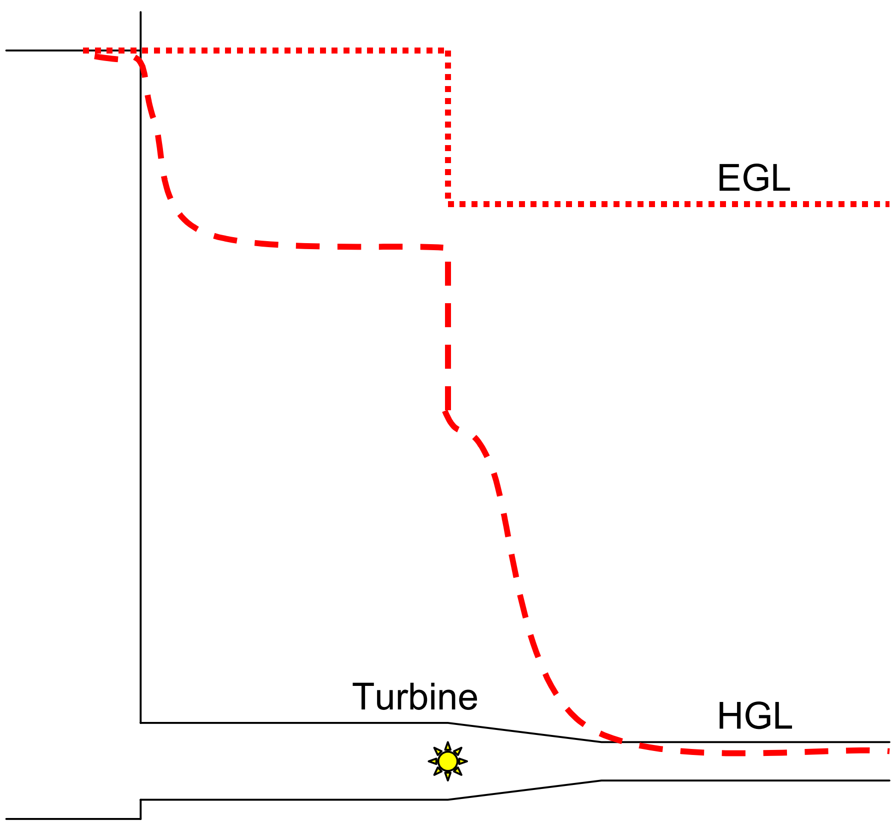
$$\frac{A_0}{A_j} = \frac{V_j^2}{gh} = 2$$

$$\therefore C_c = \frac{A_j}{A_0} = \frac{1}{2}$$ ← C_c

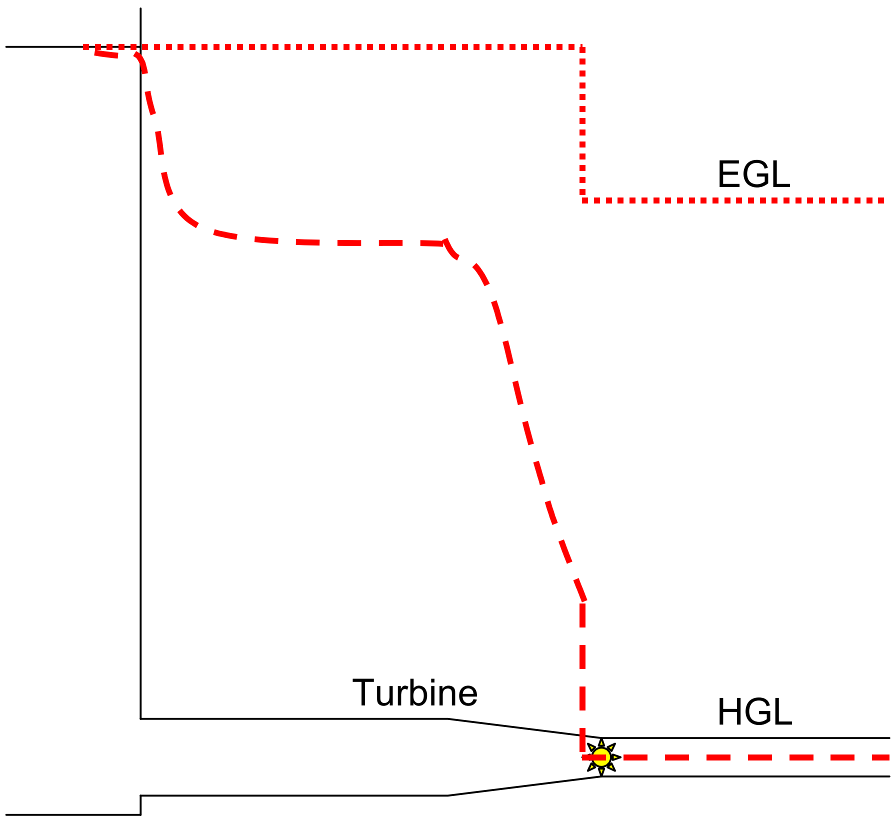
Problem 6.75

Carefully sketch the energy grade lines (EGL) and hydraulic grade lines (HGL) for the system shown in Fig. 6.6 if the pipe is horizontal (i.e., the outlet is at the base of the reservoir), and a water turbine (extracting energy) is located at (a) point ②, or (b) at point ③. In Chapter 8 we will investigate the effects of friction on internal flows. Can you anticipate and sketch the effect of friction on the EGL and HGL for cases (a) and (b)?

(a) Note that the effect of friction would be that the EGL would tend to drop: suddenly at the contraction, gradually in the large pipe, more steeply in the small pipe. The HGL would then "hang" below the HGL in a manner similar to that shown.



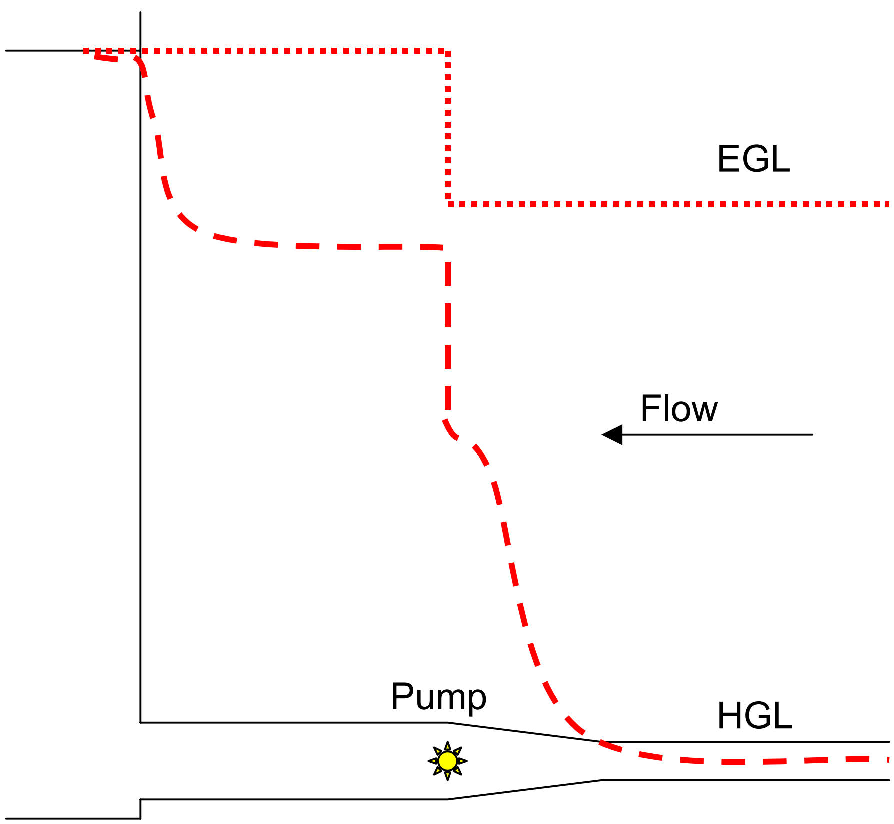
(b) Note that the effect of friction would be that the EGL would tend to drop: suddenly at the contraction, gradually in the large pipe, more steeply in the small pipe. The HGL would then "hang" below the HGL in a manner similar to that shown.



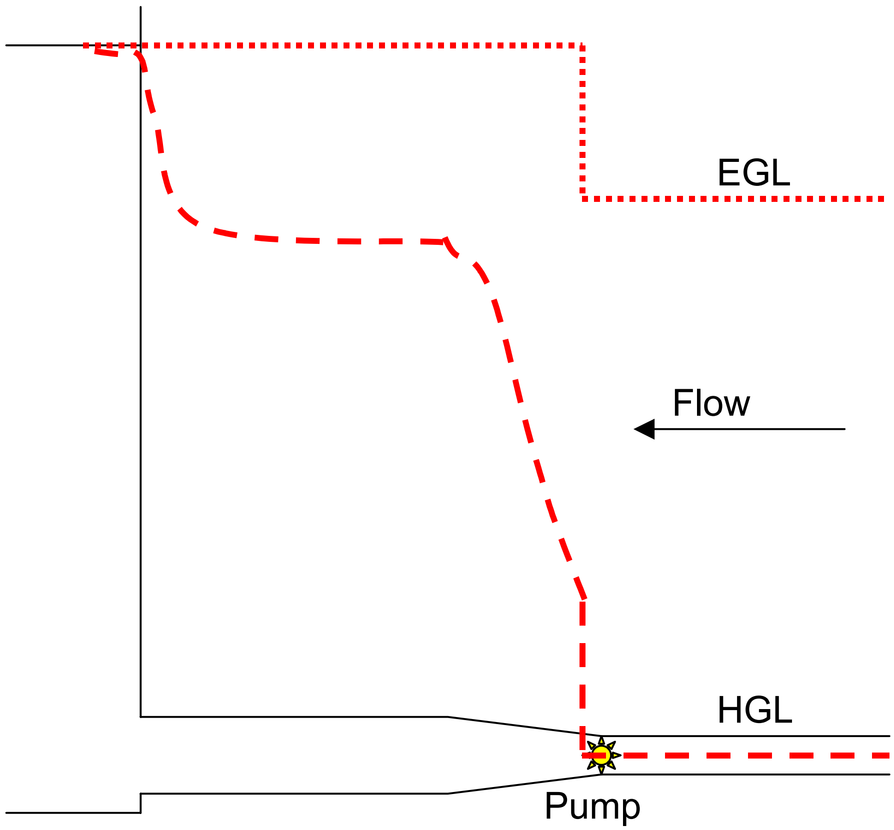
Problem 6.76

Carefully sketch the energy grade lines (EGL) and hydraulic grade lines (HGL) for the system shown in Fig. 6.6 if a pump (adding energy to the fluid) is located at (a) point ②, or (b) at point ③, such that flow is into the reservoir. In Chapter 8 we will investigate the effects of friction on internal flows. Can you anticipate and sketch the effect of friction on the EGL and HGL for cases (a) and (b)?

(a) Note that the effect of friction would be that the EGL would tend to drop from right to left: steeply in the small pipe, gradually in the large pipe, and suddenly at the expansion. The HGL would then "hang" below the HGL in a manner similar to that shown.

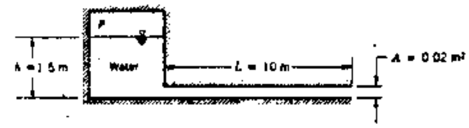


(b) Note that the effect of friction would be that the EGL would tend to drop from right to left: steeply in the small pipe, gradually in the large pipe, and suddenly at the expansion. The HGL would then "hang" below the HGL in a manner similar to that shown.



Problem *6.77

Given: Compressed air is used to accelerate water in tube. Velocity in tube is uniform at any section.
$V = 2\,m/s \qquad dV/dt = 2.5\,m/s^2$



Find: Pressure in tank for given conditions.

Solution:

Basic equation: $\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2 + \int_1^2 \frac{\partial V_s}{\partial t} ds$

Assumptions: (1) frictionless flow (2) incompressible flow
(3) flow along a streamline.
(4) $p_2 = p_{atm}$
(5) $V_1 \approx 0$

$p_{1g} = p_1 - p_{atm} = \rho\left[\frac{V_2^2}{2} - g(z_1 - z_2)\right] + \rho\int_1^2 \frac{\partial V_s}{\partial t} ds$

From continuity, for incompressible flow in a constant area tube, $V_2 = V_s = V$.

$\therefore p_{1g} = \rho\left[\frac{V_2^2}{2} - g(z_1 - z_2) + \left(\frac{dV}{dt}\right)\int_0^L ds\right]$

$= \rho\left[\frac{V_2^2}{2} - g(z_1 - z_2) + \left(\frac{dV}{dt}\right)L\right]$

$= 999\,\frac{kg}{m^3}\left[\frac{1}{2} \times \left(\frac{2\,m}{s}\right)^2 - 9.81\,\frac{m}{s^2} \times 1.5\,m + 2.5\,\frac{m}{s^2} \times 10\,m\right] \times \frac{N \cdot s^2}{kg \cdot m}$

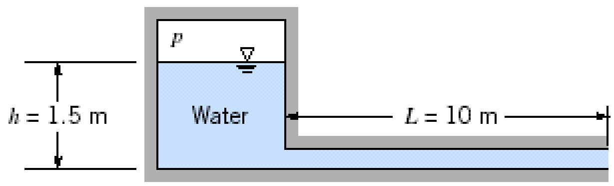
$p_{1gage} = 12.3\ kN/m^2$ ← p_{1g}

Problem *6.78

If the water in the pipe in Problem 6.77 is initially at rest and the air pressure is 20 kPa (gage), what will be the initial acceleration of the water in the pipe?

Given: Data on water pipe system

Find: Initial water acceleration



Solution

The given data is $h = 1.5 \cdot m$ $L = 10 \cdot m$ $p_{air} = 20 \cdot kPa$ $\rho = 999 \cdot \frac{kg}{m^3}$

The simplest approach is to apply Newton's 2nd law to the water in the pipe. The net horizontal force on the water in the pipe at the initial instant is $(p_L - p_L)A$ where p_L and p_R are the pressures at the left and right ends and A is the pipe cross section area (the water is initially at rest so there are no friction forces)

$$m \cdot a_x = \Sigma F_x \quad \text{or} \quad \rho \cdot A \cdot L \cdot a_x = (p_L - p_R) \cdot A$$

Also, for no initial motion $p_L = p_{air} + \rho \cdot g \cdot h$ $p_R = 0$ (gage pressures)

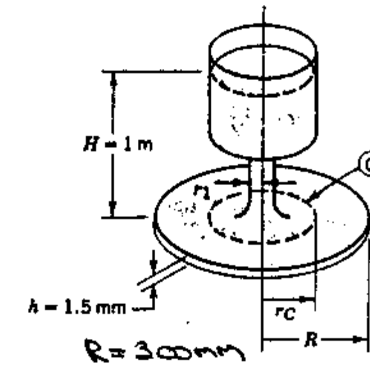
Hence

$$a_x = \frac{p_{air} + \rho \cdot g \cdot h}{\rho \cdot L} = \frac{p_{air}}{\rho \cdot L} + g \cdot \frac{h}{L} = 20 \cdot 10^3 \cdot \frac{N}{m^2} \times \frac{m^3}{999 \cdot kg} \times \frac{1}{10 \cdot m} \times \frac{kg \cdot m}{N \cdot s^2} + 9.81 \cdot \frac{m}{s^2} \times \frac{1.5}{10}$$

$$a_x = 3.47 \frac{m}{s^2}$$

Problem *6.79

Given: Flow between parallel disks shown is started from rest at $t=0$. The reservoir level is maintained constant; $r_1 = 50\text{mm}$.



Find: Rate of change of volume flow, dQ/dt, at $t=0$

Solution:

Apply the unsteady Bernoulli equation from the surface to the exit.

$$\frac{p_s}{\rho} + \frac{V_s^2}{2} + gz_s = \frac{p_e}{\rho} + \frac{V_e^2}{2} + gz_e + \int_1^2 \frac{\partial V_s}{\partial t}\,ds$$

$$gH = \frac{V_e^2}{2} + \int_1^2 \frac{\partial V_s}{\partial t}\,ds$$

Assumptions: (1) frictionless flow
(2) incompressible flow
(3) flow along a streamline

For uniform flow at any section between the plates, for $r \geq r_1$, the volume flow rate is given by

$$Q = \int \vec{V}\cdot d\vec{A} = V_r 2\pi r h \quad \text{and} \quad V_r = \frac{Q}{2\pi r h}$$

At the exit $V_e = Q/2\pi R h$

Assume that the rate of change of fluid velocity in the reservoir (out to $r = r_1$) is negligible. Then

$$\int_1^2 \frac{\partial V_s}{\partial t}\,ds = \frac{\partial}{\partial t}\int_1^2 V_r\,dr = \frac{\partial}{\partial t}\int_1^2 \frac{Q}{2\pi h}\frac{dr}{r} = \frac{\ln R/r_1}{2\pi h}\frac{dQ}{dt}$$

Then substituting into the unsteady Bernoulli equation, we obtain

$$gH = \frac{Q^2}{8\pi^2 R^2 h^2} + \frac{\ln R/r_1}{2\pi h}\frac{dQ}{dt}$$

At $t=0$, $Q=0$ and

$$\frac{dQ}{dt} = \frac{2\pi h g H}{\ln R/r_1}$$

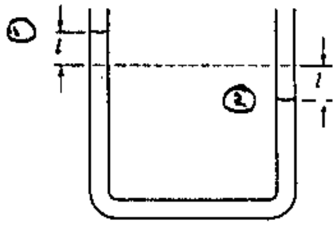
$$= 2\pi \times 0.0015\,\text{m} \times 9.81\,\frac{\text{m}}{\text{s}^2} \times 1\,\text{m} \times \frac{1}{\ln\frac{300}{50}}$$

$$\frac{dQ}{dt} = 0.0516\ \text{m}^3/\text{s}/\text{s}$$

$\left.\frac{dQ}{dt}\right)_{t=0}$

Problem *6.80

Given: U-tube manometer of constant area as shown
Manometer fluid is initially deflected and then released



Find: a differential equation for l as a function of time

Solution

Basic equation: $\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2 + \int_1^2 \frac{\partial V_s}{\partial t}\,ds$

Assumptions: (1) incompressible flow
(2) frictionless flow
(3) flow along a streamline

Since $P_1 = P_2 = P_{atm}$ and $V_1^2 = V_2^2$, then

$$g(z_1 - z_2) = \int_1^2 \frac{\partial V_s}{\partial t}\,ds$$

Let L = total length of column
l = deflection

Then $ds = dL$
$V_s = V = \frac{dl}{dt}$

$$\therefore\ 2gl = \int_1^2 \frac{\partial V}{\partial t}\,dL = \frac{\partial V}{\partial t}\int_1^2 dL = L\frac{\partial V}{\partial t}$$

Since $V = -\frac{dl}{dt}$

$$2gl = L\frac{\partial V}{\partial t} = -L\frac{d^2 l}{dt^2}$$

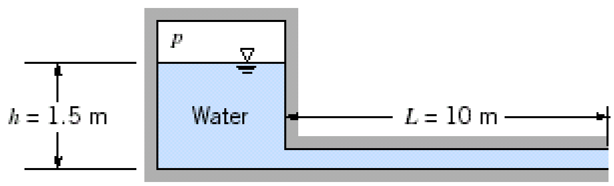
Finally $\frac{d^2 l}{dt^2} + \frac{2g}{L} l = 0$

Problem *6.81

If the water in the pipe of Problem 6.77 is initially at rest, and the air pressure is maintained at 10 kPa (gage), derive a differential equation for the velocity V in the pipe as a function of time, integrate, and plot V versus t for $t = 0$ to 5 s.

Given: Data on water pipe system

Find: Velocity in pipe; plot



Solution

The given data is $h = 1.5\cdot m$ $L = 10\cdot m$ $p_{air} = 10\cdot kPa$ $\rho = 999\cdot\frac{kg}{m^3}$

The governing equation for this flow is the unsteady Bernoulli equation

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2 + \int_1^2 \frac{\partial V}{\partial t} ds \qquad (6.21)$$

State 1 is the free surface; state 2 is the pipe exit. For state 1, $V_1 = 0$, $p_1 = p_{air}$ (gage), $z_1 = h$. For state 2, $V_2 = V$, $p_2 = 0$ (gage), $z_2 = 0$. For the integral, we assume V is negligible in the reservoir

Hence $$\frac{p_{air}}{\rho} + g\cdot h = \frac{V^2}{2} + \int_0^L \frac{\partial}{\partial t} V\, dx$$

At each instant V has the same value everywhere in the pipe, i.e., $V = V(t)$ only

Hence $$\frac{p_{air}}{\rho} + g\cdot h = \frac{V^2}{2} + L\cdot\frac{dV}{dt}$$

The differential equation for *V* is then

$$\frac{dV}{dt} + \frac{1}{2 \cdot L} \cdot V^2 - \frac{\left(\frac{p_{air}}{\rho} + g \cdot h\right)}{L} = 0$$

Separating variables

$$\frac{L \cdot dV}{\left(\frac{p_{air}}{\rho} + g \cdot h\right) - \frac{V^2}{2}} = dt$$

Integrating and applying the IC that V(0) = 0 yields, after some simplification

$$V(t) = \sqrt{2 \cdot \left(\frac{p_{air}}{\rho} + g \cdot h\right)} \cdot \tanh\left[\sqrt{\frac{\left(\frac{p_{air}}{\rho} + g \cdot h\right)}{2 \cdot L^2}} \cdot t\right]$$

This function is plotted in the associated Excel workbook. Note that as time increases V approa

$$V(t) = 7.03 \frac{m}{s}$$

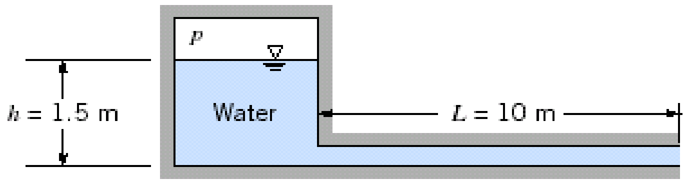
The flow approaches 95% of its steady state rate after about 5 s

Problem *6.81 (In Excel)

If the water in the pipe of Problem 6.77 is initially at rest, and the air pressure is maintained at 10 kPa (gage), derive a differential equation for the velocity V in the pipe as a function of time, integrate, and plot V versus t for t = 0 to 5 s.

Given: Data on water pipe system

Find: Plot velocity in pipe



Solution

The given data is

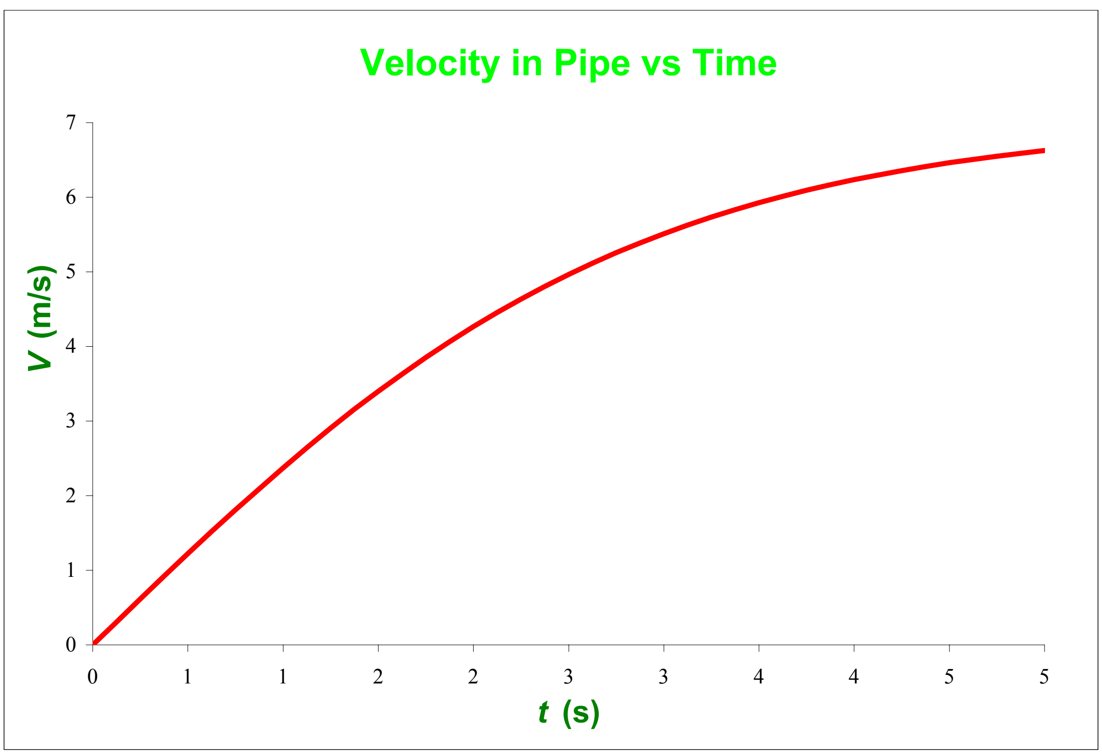
h =	1.5	m
L =	10	m
ρ =	999	kg/m^3
p_{air} =	10	kPa

The solution is

$$V(t) = \sqrt{2\cdot\left(\frac{p_{air}}{\rho} + g\cdot h\right)}\cdot\tanh\left[\sqrt{\frac{\left(\frac{p_{air}}{\rho} + g\cdot h\right)}{2\cdot L^2}}\cdot t\right]$$

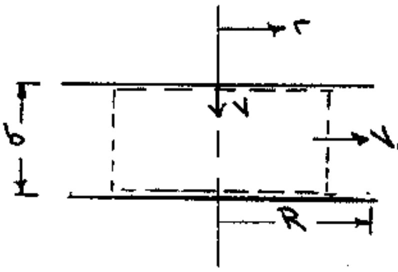
t (s)	V (m/s)
0.00	0.00
0.25	0.62
0.50	1.22
0.75	1.81
1.00	2.38
1.25	2.91
1.50	3.40
1.75	3.85
2.00	4.26
2.25	4.63
2.50	4.96
2.75	5.26
3.00	5.51
3.25	5.73
3.50	5.93
3.75	6.09
4.00	6.24
4.25	6.36
4.50	6.46
4.75	6.55
5.00	6.63

The flow approaches 95% of its steady state rate after about 5 s



Problem *6.82

Given: Two circular discs of radius, R, are separated by a distance, b.
Upper disc moves toward the lower one at speed, V.
Fluid between discs is incompressible and is squeezed out radially.
Assume frictionless flow and uniform radial flow and any radial section.
Pressure surrounding disc is at P_{atm}



Find: gage pressure at $r=0$

Solution:

Basic equation: $\frac{P_1}{\rho} + \frac{V_1^2}{2} + g z_1 = \frac{P_2}{\rho} + \frac{V_2^2}{2} + g z_2 + \int_1^2 \frac{\partial V_s}{\partial t}\, ds$

$$0 = \frac{\partial}{\partial t}\int_{cv} \rho\, d\forall + \int_{cs} \rho \vec{V}\cdot d\vec{A}$$

Assumptions: (1) incompressible flow
(2) frictionless flow
(3) flow along a streamline
(4) uniform radial flow at any r
(5) neglect elevation changes.

Then,

$$0 = \frac{\partial}{\partial t}\int_{cv} \rho\, d\forall + \int_{cs} \rho \vec{V}\cdot d\vec{A} = \frac{\partial}{\partial t}(\rho \pi r^2 b) + \rho V_r 2\pi r b$$

$$= \rho \pi r^2 \frac{\partial b}{\partial t} + \rho V_r 2\pi r b. \quad \text{But } \frac{\partial b}{\partial t} = -V$$

$$\therefore 0 = -\rho \pi r^2 V + \rho V_r 2\pi r b \qquad \text{and } V_r = V\frac{r}{2b}$$

Applying the Bernoulli equation between point ① $(r=r)$ and point ② $(r=R)$

$$P_1 - P_2 = \frac{\rho}{2}\left[V_2^2 - V_1^2\right] + \int_r^R \rho \frac{\partial V_r}{\partial t}\, dr \quad \text{Now, } \frac{\partial V_r}{\partial t} = \frac{\partial}{\partial t}\left(V\frac{r}{2b}\right) = \frac{rV}{2}\left(-\frac{1}{b^2}\frac{db}{dt}\right) = \frac{V^2 r}{2b^2}$$

$$= \frac{\rho}{2}\left[\left(\frac{VR}{2b}\right)^2 - \left(\frac{Vr}{2b}\right)^2\right] + \int_r^R \rho \frac{V^2 r}{2b^2}\, dr$$

$$= \frac{\rho V^2}{8b^2}\left[R^2 - r^2\right] + \left.\frac{\rho V^2}{4b^2} r^2\right]_r^R = \frac{\rho V^2}{8b^2}\left[R^2 - r^2\right] + \frac{\rho V^2}{4b^2}\left[R^2 - r^2\right]$$

$$P_1 - P_{atm} = \frac{3}{8}\frac{\rho V^2}{b^2}\left[R^2 - r^2\right] = \frac{3}{8}\frac{\rho V^2 R^2}{b^2}\left[1 - \left(\frac{r}{R}\right)^2\right]$$

When $r = 0$ $\quad P_1 = P_0$

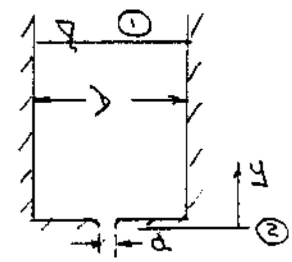
$$\therefore P_0 - P_{atm} = \frac{3}{8}\frac{\rho V^2 R^2}{b^2}$$

Given: A cylindrical tank of diameter, D = 50 mm, drains through an opening, d = 5 mm, in the body of the tank. If the flow is assumed to be quasi-steady, the speed of the liquid leaving the tank may be approximated by $V = \sqrt{2gy}$, where y is the height from the tank bottom to the free surface.

Find: Using the Bernoulli equation for unsteady flow along a streamline, evaluate the minimum diameter ratio, D/d, required to justify the assumption that flow from the tank is quasi-steady.

Solution:

For incompressible, frictionless flow along a streamline, the unsteady Bernoulli equation is



$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gy_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gy_2 + \int_1^2 \frac{\partial V_s}{\partial t}\,dy$$

$p_1 = p_2 = p_{atm}$, $y_2 = 0$

From continuity $V_1 A_1 = V_2 A_2 = V_j A_j$

$$\therefore \quad \frac{1}{2} V_j^2 \left(\frac{A_j}{A_1}\right)^2 + gy_1 = \frac{1}{2} V_j^2 + \int_1^2 \frac{\partial V_s}{\partial t}\,dy$$

or,

$$gy_1 = \frac{1}{2} V_j^2 \left[1 - \left(\frac{A_j}{A_1}\right)^2\right] + \int_1^2 \frac{\partial V_s}{\partial t}\,dy$$

If we assume quasi-steady flow, we say that

$\int_1^2 \frac{\partial V_s}{\partial t}\,dy$ is negligible and hence $\frac{2gy}{V_j^2\left[1 - AR^2\right]} = 1$ where $AR = \frac{A_j}{A_1}$

$$\text{Now,} \quad \int_1^2 \frac{\partial V_s}{\partial t} \approx y\frac{dV_s}{dt} = y\frac{dV_1}{dt} = y\frac{d}{dt}\left(V_j \frac{A_j}{A_1}\right) = y\frac{A_j}{A_1}\frac{dV_j}{dt}$$

Thus for the assumption to be reasonable we must have

$$\left| y \frac{A_j}{A_1}\frac{dV_j}{dt}\right| \ll gy \quad \text{or} \quad \left|\frac{A_j}{A_1}\frac{dV_j}{dt}\right| \ll g$$

Under the assumption of quasi-steady flow

$$V_j = \left[2gy \frac{1}{(1 - AR^2)}\right]^{1/2} \quad \text{where } AR = A_j/A_1$$

then,

$$\frac{dV_j}{dt} = \sqrt{\frac{2g}{(1 - AR^2)}}\,\frac{1}{2\sqrt{y}}\,\frac{dy}{dt} = \frac{dy}{dt}\sqrt{\frac{g}{2y(1 - AR^2)}}$$

Since

$$\frac{dy}{dt} = -V_1 = -V_j\frac{A_j}{A_1} \quad , \text{ then}$$

$$\frac{dV_j}{dt} = -V_j\frac{A_j}{A_1}\sqrt{\frac{g}{2y(1 - AR^2)}} = -\frac{A_j}{A_1}\sqrt{\frac{V_j^2(1 - AR^2)}{2gy}}\;\frac{g}{(1 - AR^2)}$$

and

$$\frac{dV_j}{dt} = -\frac{A_j}{A_1}\frac{g}{(1 - AR^2)}$$

For $\left| \frac{A_j}{A_1} \frac{dV_j}{dt} \right| \ll g$, then $\left(\frac{A_j}{A_1}\right)^2 \frac{1}{(1-AR^2)} \ll 1$

If we take $\left(\frac{A_j}{A_1}\right)^2 \frac{1}{(1-AR^2)} \approx 0.01$

then,
$$\left(\frac{A_j}{A_1}\right)^2 = 0.01(1-AR^2) = 0.01\left[1-\left(\frac{A_j}{A_1}\right)^2\right]$$

and
$$1.01\left(\frac{A_j}{A_1}\right)^2 = 0.01$$
$$\frac{A_j}{A_1} = 0.0995$$

or
$$\frac{D_j}{D_1} = \left(\frac{A_j}{A_1}\right)^{1/2} = 0.32 \longleftarrow$$

In problem 4.34, $D_j/D_1 = d/D = 0.1$ and hence the assumption of quasi-steady flow is valid.

Problem *6.84

Given: Two vortex flows with velocity fields

$$\vec{V}_1 = \omega r \hat{e}_\theta \qquad \vec{V}_2 = \frac{K}{2\pi r}\hat{e}_\theta$$

Determine: if the Bernoulli equation can be applied between different radii for each flow.

Solution: Since $V_r = 0$, the streamlines are concentric circles.

In order for it to be possible to apply the Bernoulli equation between different radii, it is necessary that the flow be irrotational.

Basic equation: $\vec{\omega} = \frac{1}{2}\nabla \times \vec{V}$

Flow (1)

$$\nabla \times \vec{V}_1 = \left(\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{k}\frac{\partial}{\partial z}\right) \times \omega r \hat{e}_\theta$$

$$= \hat{e}_r \times \hat{e}_\theta \frac{\partial}{\partial r}(\omega r) + \hat{e}_r \times \omega r \frac{\partial \hat{e}_\theta}{\partial r}^{\nearrow 0} + \hat{e}_\theta \times \hat{e}_\theta \frac{1}{r}\frac{\partial(\omega r)}{\partial \theta}^{\nearrow 0} + \hat{e}_\theta \times \frac{\omega r}{r}\frac{\partial \hat{e}_\theta}{\partial \theta}$$

$$= \hat{k}\omega + \hat{e}_\theta \times \omega(-\hat{e}_r)$$

$$\nabla \times \vec{V}_1 = 2\omega \hat{k}$$

$\therefore$ Flow (1) is rotational and Bernoulli equation cannot be applied between different radii.

Flow (2)

$$\nabla \times \vec{V}_2 = \left(\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{k}\frac{\partial}{\partial z}\right) \times \frac{K}{2\pi r}\hat{e}_\theta$$

$$= \hat{e}_r \times \hat{e}_\theta \frac{\partial}{\partial r}\left(\frac{K}{2\pi r}\right) + \hat{e}_r \times \left(\frac{K}{2\pi r}\right)\frac{\partial \hat{e}_\theta}{\partial r}^{\nearrow 0} + \hat{e}_\theta \times \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{K}{2\pi r}\right)^{\nearrow 0} + \hat{e}_\theta \times \frac{1}{r}\left(\frac{K}{2\pi r}\right)\frac{\partial \hat{e}_\theta}{\partial \theta}$$

$$= -\hat{k}\frac{K}{2\pi r^2} + \hat{e}_\theta \frac{K}{2\pi r^2} \times (-\hat{e}_r)$$

$$= -\hat{k}\frac{K}{2\pi r^2} + \hat{k}\frac{K}{2\pi r^2}$$

$$\nabla \times \vec{V}_2 = 0$$

Since the flow field is irrotational, Bernoulli equation can be applied between different radii if the flow is also incompressible and frictionless.

Problem *6.85

Given: Flow field represented by $\psi = Ax^2y$; $A = 2.5\ ft^{-1} \cdot s^{-1}$, $\rho = 2.45$ slug/ft³.

Find: (a) Is the flow irrotational?
(b) If possible, determine $p_1 - p_2$ if $(x_1, y_1) = (1,4)$ and $(x_2, y_2) = (2,1)$

Solution:

The velocity field is determined from the stream function

$$u = \frac{\partial \psi}{\partial y} = Ax^2, \quad v = -\frac{\partial \psi}{\partial x} = -2Axy \quad \therefore \vec{V} = Ax^2\hat{i} - 2Axy\hat{j}$$

Since $w = 0$ and $\frac{\partial}{\partial z} = 0$, then

$$\nabla \times \vec{V} = \hat{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \hat{k}(-2Ay) \neq 0$$

$\therefore$ flow is not irrotational

Note: For irrotational flow, $\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$

For $\psi = Ax^2y$, $\nabla^2 \psi = 2Ay \neq 0$ $\therefore$ flow is rotational

Since the flow is rotational, points ① and ② must be on the same streamline to apply the Bernoulli equation between the two points.

$$\psi_{x_1,y_1} = A(1)^2(4) = 4A, \quad \psi_{x_2 y_2} = A(2)^2(1) = 4A$$

Hence,

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2$$

Assume flow in horizontal plane, i.e. $z_1 = z_2$

$$\vec{V}_1 = Ax_1^2\hat{i} - 2Ax_1y_1\hat{j} = 2.5\frac{1}{m \cdot s}\left[1^2 m^2\hat{i} - 2 \times 1m \times 4m\hat{j}\right] = 2.5\hat{i} - 20\hat{j}\ \frac{m}{s}$$

$$\vec{V}_2 = Ax_2^2\hat{i} - 2Ax_2y_2\hat{j} = \frac{2.5}{m \cdot s}\left[(2)^2 m^2\hat{i} - 2 \times 2m \times 1m\hat{j}\right] = 10\hat{i} - 10\hat{j}\ \frac{m}{s}$$

Thus $V_1^2 = 406\ m^2/s^2$ $\quad V_2^2 = 200\ m^2/s^2$

and

$$p_1 - p_2 = \rho\left[\frac{V_2^2}{2} - \frac{V_1^2}{2}\right] = \frac{\rho}{2}(V_2^2 - V_1^2)$$

$$= \frac{1}{2} \times 2.45\ \frac{slug}{ft^3}(200 - 406)\frac{ft^2}{s^2} \times \frac{lbf \cdot s^2}{ft \cdot slug}$$

$$p_1 - p_2 = -252\ lbf/ft^2$$

Problem *6.86

Given: Two-dimensional flow represented by the velocity field $\vec{V} = (Ax - By)t\hat{i} - (Bx + Ay)t\hat{j}$, where $A = 1\,s^{-2}$, $B = 2\,s^{-2}$, t is in s, and coordinates are in meters.

Find: (a) Is this a possible incompressible flow?
(b) Is the flow steady or unsteady?
(c) Show that the flow is irrotational
(d) Derive an expression for the velocity potential

Solution: For incompressible flow, $\nabla \cdot \vec{V} = 0$

For given flow $\nabla \cdot \vec{V} = \frac{\partial}{\partial x}(Ax - By)t - \frac{\partial}{\partial y}(Bx + Ay)t = At - At = 0$

$\therefore$ velocity field represents a possible incompressible flow

The flow is unsteady since $\vec{V} = \vec{V}(x, y, t)$

The rotation is given by $\vec{\omega} = \frac{1}{2}\nabla \times \vec{V} = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k}$

$$\vec{\omega} = \frac{1}{2}\left[\frac{\partial}{\partial x} - (Bx + Ay)t - \frac{\partial}{\partial y}(Ax - By)t\right] = -Bt + Bt = 0$$

$\vec{\omega} = 0$, so flow is irrotational

From the definition of the velocity potential, $\vec{V} = -\nabla\phi$

$u = -\frac{\partial \phi}{\partial x}$ and $\phi = \int -u\,dx + f(y,t) = \int -(Ax - By)t\,dx + f(y,t)$

$$\phi = \left(-A\frac{x^2}{2} + Bxy\right)t + f(y,t)$$

$v = \frac{-\partial \phi}{\partial y}$ and $\phi = \int -v\,dy + g(x,t) = \int (Bx + Ay)t\,dy + g(x,t)$

$$\phi = \left(Bxy + A\frac{y^2}{2}\right)t + g(x,t)$$

Comparing the two expressions for ϕ we conclude

$$f(y,t) = \frac{A}{2}y^2 t \quad \text{and} \quad g(x,t) = -\frac{A}{2}x^2 t$$

Hence,

$$\phi = \left\{\frac{A}{2}(y^2 - x^2) + Bxy\right\}t$$ ϕ

Problem *6.87

The flow field for a plane source at a distance h above an infinite wall aligned along the x axis is given by

$$\vec{V} = \frac{q}{2\pi\left[x^2 + (y-h)^2\right]}\left[x\hat{i} + (y-h)\hat{j}\right] + \frac{q}{2\pi\left[x^2 + (y+h)^2\right]}\left[x\hat{i} + (y+h)\hat{j}\right]$$

where q is the strength of the source. The flow is irrotational and incompressible. Derive the stream function and velocity potential. By choosing suitable values for q and h, plot the streamlines and lines of constant velocity potential. (Hint: Use the *Excel* workbook of Example Problem 6.10.)

Find: Stream function and velocity potential; plot

Solution

The velocity field is $$u = \frac{q\cdot x}{2\cdot\pi\left[x^2 + (y-h)^2\right]} + \frac{q\cdot x}{2\cdot\pi\left[x^2 + (y+h)^2\right]}$$

$$v = \frac{q\cdot(y-h)}{2\cdot\pi\left[x^2 + (y-h)^2\right]} + \frac{q\cdot(y+h)}{2\cdot\pi\left[x^2 + (y+h)^2\right]}$$

The governing equations are

$$u = \frac{\partial}{\partial y}\psi \qquad v = -\frac{\partial}{\partial x}\psi$$

$$u = -\frac{\partial}{\partial x}\phi \qquad v = -\frac{\partial}{\partial y}\phi$$

Hence for the stream function

$$\psi = \int u(x,y)\,dy = \frac{q}{2\cdot\pi}\cdot\left(\operatorname{atan}\left(\frac{y-h}{x}\right) + \operatorname{atan}\left(\frac{y+h}{x}\right)\right) + f(x)$$

$$\psi = -\int v(x,y)\,dx = \frac{q}{2\cdot\pi}\cdot\left(\operatorname{atan}\left(\frac{y-h}{x}\right) + \operatorname{atan}\left(\frac{y+h}{x}\right)\right) + g(y)$$

The simplest expression for ψ is then

$$\psi(x,y) = \frac{q}{2\cdot\pi}\cdot\left(\operatorname{atan}\left(\frac{y-h}{x}\right) + \operatorname{atan}\left(\frac{y+h}{x}\right)\right)$$

For the stream function

$$\phi = -\int u(x,y)\,dx = -\frac{q}{4\cdot\pi}\cdot\ln\left[\left[x^2 + (y-h)^2\right]\cdot\left[x^2 + (y+h)^2\right]\right] + f(y)$$

$$\phi = -\int v(x,y)\,dy = -\frac{q}{4\cdot\pi}\cdot\ln\left[\left[x^2 + (y-h)^2\right]\cdot\left[x^2 + (y+h)^2\right]\right] + g(x)$$

The simplest expression for ϕ is then

$$\phi(x,y) = -\frac{q}{4\cdot\pi}\cdot\ln\left[\left[x^2 + (y-h)^2\right]\cdot\left[x^2 + (y+h)^2\right]\right]$$

Problem *6.87 (In Excel)

The flow field for a plane source at a distance h above an infinite wall aligned along the x axis is given by

$$\vec{V} = \frac{q}{2\pi\left[x^2 + (y-h)^2\right]}\left[x\hat{i} + (y-h)\hat{j}\right] + \frac{q}{2\pi\left[x^2 + (y+h)^2\right]}\left[x\hat{i} + (y+h)\hat{j}\right]$$

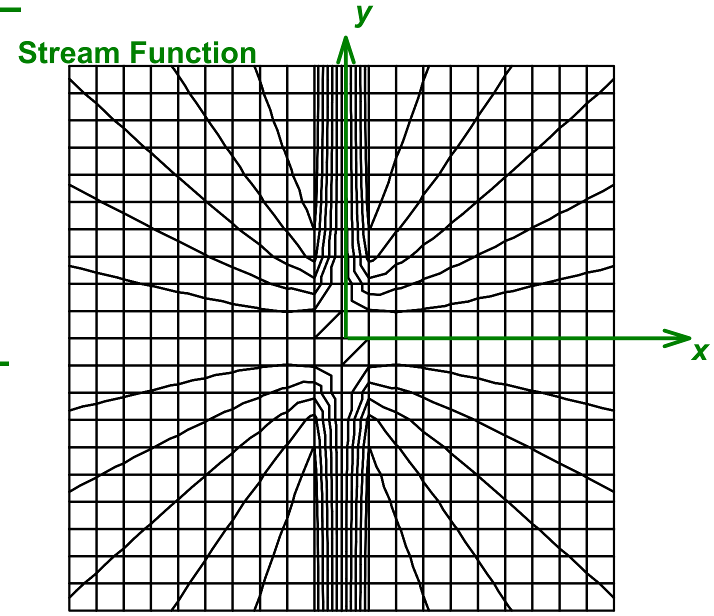
where q is the strength of the source. The flow is irrotational and incompressible. Derive the stream function and velocity potential. By choosing suitable values for q and h, plot the streamlines and lines of constant velocity potential. (Hint: Use the *Excel* workbook of Example Problem 6.10.)

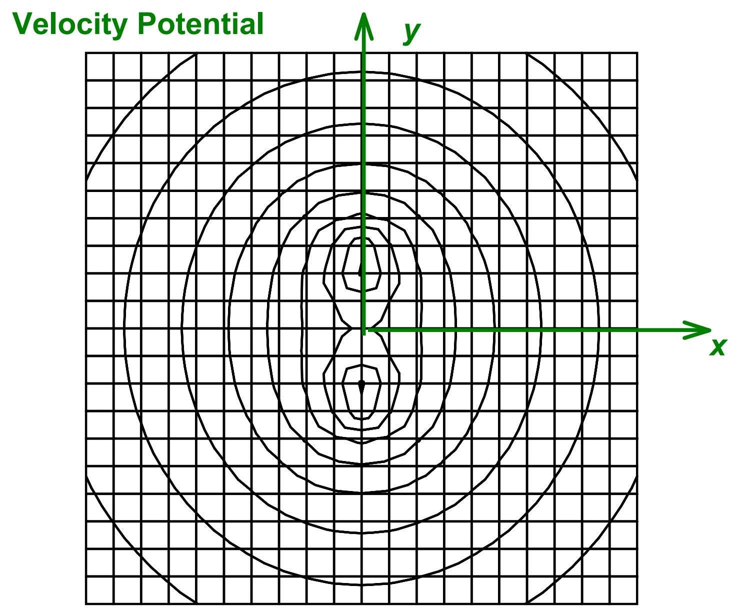
$$\psi(x,y) = \frac{q}{2\cdot\pi}\cdot\left(\operatorname{atan}\left(\frac{y-h}{x}\right) + \operatorname{atan}\left(\frac{y+h}{x}\right)\right)$$

$$\phi(x,y) = -\frac{q}{4\cdot\pi}\cdot\ln\left[\left[x^2 + (y-h)^2\right]\cdot\left[x^2 + (y+h)^2\right]\right]$$

#NAME?	**Stream Function**
#NAME?	**Velocity Potential**

Note that the plot is
from x = -5 to 5 and y = -5 to 5





Problem *6.88

Using Table 6.1, find the stream function and velocity potential for a plane source, of strength q, near a 90° corner. The source is equidistant h from each of the two infinite planes that make up the corner. Find the velocity distribution along one of the planes, assuming $p = p_0$ at infinity. By choosing suitable values for q and h, plot the streamlines and lines of constant velocity potential. (Hint: Use the *Excel* workbook of Example Problem 6.10.)

Given: Data from Table 6.1

Find: Stream function and velocity potential for a source in a corner; plot; velocity along one pla

Solution

From Table 6.1, for a source at the origin

$$\psi(r,\theta) = \frac{q}{2\cdot\pi}\cdot\theta \qquad \phi(r,\theta) = -\frac{q}{2\cdot\pi}\cdot\ln(r)$$

Expressed in Cartesian coordinates

$$\psi(x,y) = \frac{q}{2\cdot\pi}\cdot\text{atan}\left(\frac{y}{x}\right) \qquad \phi(x,y) = -\frac{q}{4\cdot\pi}\cdot\ln\left(x^2+y^2\right)$$

To build flow in a corner, we need image sources at three locations so that there is symmetry ab
both axes. We need sources at (h,h), (h,- h), (- h,h), and (- h,- h)

$$\psi(x,y) = \frac{q}{2\cdot\pi}\cdot\left(\text{atan}\left(\frac{y-h}{x-h}\right) + \text{atan}\left(\frac{y+h}{x-h}\right) + \text{atan}\left(\frac{y+h}{x+h}\right) + \text{atan}\left(\frac{y-h}{x+h}\right)\right)$$

$$\phi(x,y) = -\frac{q}{4\cdot\pi}\cdot\ln\left[\left[(x-h)^2+(y-h)^2\right]\cdot\left[(x-h)^2+(y+h)^2\right]\right]\cdots$$
$$+-\frac{q}{4\cdot\pi}\cdot\left[(x+h)^2+(y+h)^2\right]\cdot\left[(x+h)^2+(y-h)^2\right]$$

(Too long to fit on one line!)

By a similar reasoning the horizontal velocity is given by

$$u = \frac{q\cdot(x-h)}{2\cdot\pi\left[(x-h)^2+(y-h)^2\right]} + \frac{q\cdot(x-h)}{2\cdot\pi\left[(x-h)^2+(y+h)^2\right]}\cdots$$
$$+\frac{q\cdot(x+h)}{2\cdot\pi\left[(x+h)^2+(y+h)^2\right]} + \frac{q\cdot(x+h)}{2\cdot\pi\left[(x+h)^2+(y+h)^2\right]}$$

Along the horizontal wall ($y = 0$)

$$u = \frac{q\cdot(x-h)}{2\cdot\pi\left[(x-h)^2+h^2\right]} + \frac{q\cdot(x-h)}{2\cdot\pi\left[(x-h)^2+h^2\right]}\cdots$$
$$+\frac{q\cdot(x+h)}{2\cdot\pi\left[(x+h)^2+h^2\right]} + \frac{q\cdot(x+h)}{2\cdot\pi\left[(x+h)^2+h^2\right]}$$

or

$$u(x) = \frac{q}{\pi}\cdot\left[\frac{x-h}{(x-h)^2+h^2} + \frac{x+h}{(x+h)^2+h^2}\right]$$

Problem *6.88 (In Excel)

Using Table 6.1, find the stream function and velocity potential for a plane source, of strength q, near a 90° corner. The source is equidistant h from each of the two infinite planes that make up the corner. Find the velocity distribution along one of the planes, assuming $p = p_0$ at infinity. By choosing suitable values for q and h, plot the streamlines and lines of constant velocity potential. (Hint: Use the *Excel* workbook of Example Problem 6.10.)

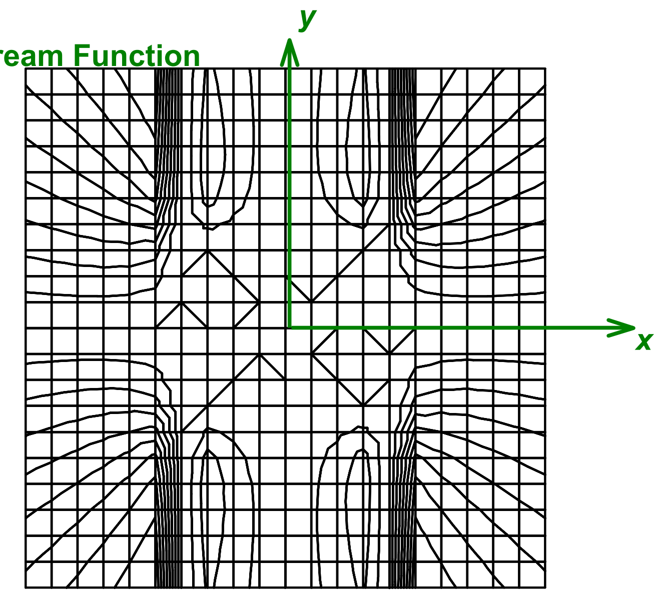
$$\psi(x,y) = \frac{q}{2\cdot\pi}\cdot\left(\operatorname{atan}\left(\frac{y-h}{x-h}\right) + \operatorname{atan}\left(\frac{y+h}{x-h}\right) + \operatorname{atan}\left(\frac{y+h}{x+h}\right) + \operatorname{atan}\left(\frac{y-h}{x+h}\right)\right)$$

$$\phi(x,y) = -\frac{q}{4\cdot\pi}\cdot\ln\left[\left[(x-h)^2 + (y-h)^2\right]\cdot\left[(x-h)^2 + (y+h)^2\right]\right] \ldots$$
$$+ -\frac{q}{4\cdot\pi}\cdot\left[(x+h)^2 + (y+h)^2\right]\cdot\left[(x+h)^2 + (y-h)^2\right]$$

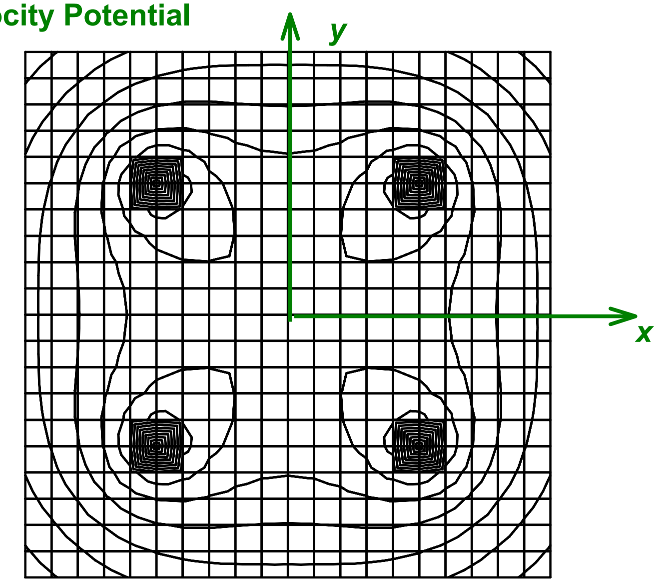
#NAME?	**Stream Function**
#NAME?	**Velocity Potential**

Note that the plot is from x = -5 to 5 and y = -5 to 5

Stream Function



Velocity Potential



Problem *6.89

Using Table 6.1, find the stream function and velocity potential for a plane vortex, of strength *K*, near a 90° corner. The vortex is equidistant *h* from each of the two infinite planes that make up the corner. Find the velocity distribution along one of the planes, assuming $p = p_0$ at infinity. By choosing suitable values for *K* and *h*, plot the streamlines and lines of constant velocity potential. (Hint: Use the *Excel* workbook of Example Problem 6.10.)

Given: Data from Table 6.1

Find: Stream function and velocity potential for a vortex in a corner; plot; velocity along one pla

Solution

From Table 6.1, for a vortex at the origin

$$\phi(r,\theta) = \frac{K}{2\cdot\pi}\cdot\theta \qquad \psi(r,\theta) = -\frac{K}{2\cdot\pi}\cdot\ln(r)$$

Expressed in Cartesian coordinates

$$\phi(x,y) = \frac{q}{2\cdot\pi}\cdot\operatorname{atan}\left(\frac{y}{x}\right) \qquad \psi(x,y) = -\frac{q}{4\cdot\pi}\cdot\ln\left(x^2 + y^2\right)$$

To build flow in a corner, we need image vortices at three locations so that there is symmetry ab both axes. We need vortices at (*h*,*h*), (*h*,- *h*), (- *h*,*h*), and (- *h*,- *h*). Note that some of them must have strengths of - *K*!

$$\phi(x,y) = \frac{K}{2\cdot\pi}\cdot\left(\operatorname{atan}\left(\frac{y-h}{x-h}\right) - \operatorname{atan}\left(\frac{y+h}{x-h}\right) + \operatorname{atan}\left(\frac{y+h}{x+h}\right) - \operatorname{atan}\left(\frac{y-h}{x+h}\right)\right)$$

$$\psi(x,y) = -\frac{K}{4\cdot\pi}\cdot\ln\left[\frac{(x-h)^2 + (y-h)^2}{(x-h)^2 + (y+h)^2}\cdot\frac{(x+h)^2 + (y+h)^2}{(x+h)^2 + (y-h)^2}\right]$$

By a similar reasoning the horizontal velocity is given by

$$u = -\frac{K\cdot(y-h)}{2\cdot\pi\left[(x-h)^2+(y-h)^2\right]} + \frac{K\cdot(y+h)}{2\cdot\pi\left[(x-h)^2+(y+h)^2\right]} \cdots$$
$$+ -\frac{K\cdot(y+h)}{2\cdot\pi\left[(x+h)^2+(y+h)^2\right]} + \frac{K\cdot(y-h)}{2\cdot\pi\left[(x+h)^2+(y-h)^2\right]}$$

Along the horizontal wall ($y = 0$)

$$u = \frac{K\cdot h}{2\cdot\pi\left[(x-h)^2+h^2\right]} + \frac{K\cdot h}{2\cdot\pi\left[(x-h)^2+h^2\right]} \cdots$$
$$+ -\frac{K\cdot h}{2\cdot\pi\left[(x+h)^2+h^2\right]} - \frac{K\cdot h}{2\cdot\pi\left[(x+h)^2+h^2\right]}$$

or

$$u(x) = \frac{K\cdot h}{\pi}\cdot\left[\frac{1}{(x-h)^2+h^2} - \frac{1}{(x+h)^2+h^2}\right]$$

Problem *6.89 (In Excel)

Using Table 6.1, find the stream function and velocity potential for a plane vortex, of strength K, near a 90° corner. The vortex is equidistant h from each of the two infinite planes that make up the corner. Find the velocity distribution along one of the planes, assuming $p = p_0$ at infinity. By choosing suitable values for K and h, plot the streamlines and lines of constant velocity potential. (Hint: Use the *Excel* workbook of Example Problem 6.10.)

$$\psi(x,y) = -\frac{K}{4\cdot\pi}\cdot\ln\left[\frac{(x-h)^2+(y-h)^2}{(x-h)^2+(y+h)^2}\cdot\frac{(x+h)^2+(y+h)^2}{(x+h)^2+(y-h)^2}\right]$$

$$\phi(x,y) = \frac{K}{2\cdot\pi}\cdot\left(\operatorname{atan}\left(\frac{y-h}{x-h}\right) - \operatorname{atan}\left(\frac{y+h}{x-h}\right) + \operatorname{atan}\left(\frac{y+h}{x+h}\right) - \operatorname{atan}\left(\frac{y-h}{x+h}\right)\right)$$

#NAME?

Stream Function

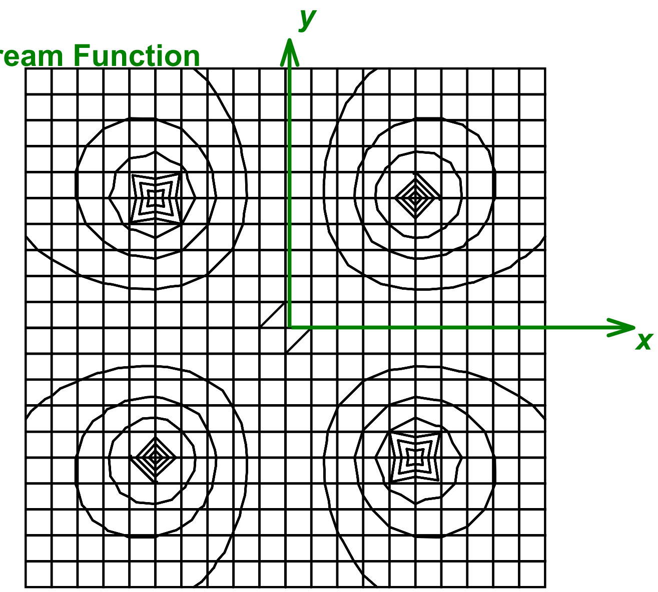
#NAME?

#NAME?

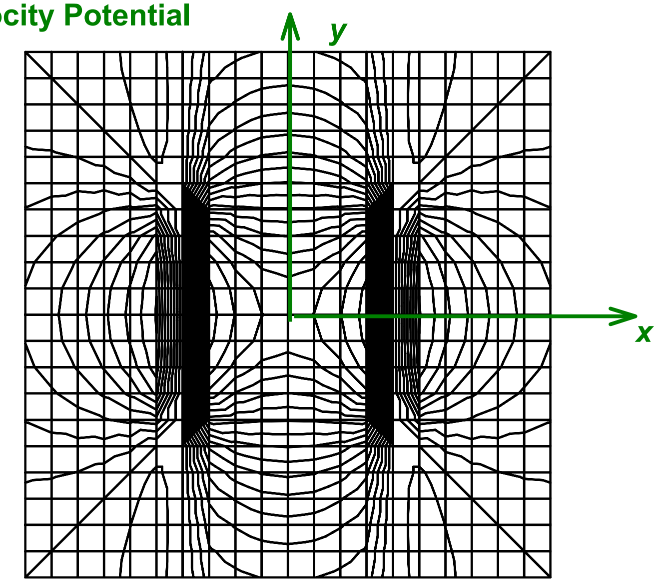
Velocity Potential

Note that the plot is
from x = -5 to 5 and y = -5 to 5

Stream Function



Velocity Potential



Problem *6.90

Given: Flow field represented by $\psi = Ax^2y - By^3$, where $A = 1\ m^{-1}\cdot s^{-1}$, $B = \frac{1}{3}\ m^{-1}\cdot s^{-1}$, and coordinates are in meters.

Find: an expression for the velocity potential, ϕ

Solution:

The velocity field is determined from the stream function

$$u = \partial\psi/\partial y = Ax^2 - 3By^2, \quad v = -\partial\psi/\partial x = -2Axy \quad \therefore \vec{V} = (Ax^2 - 3By^2)\hat{i} - 2Axy\hat{j}$$

The rotation is given by $\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$

$$\omega_z = \frac{1}{2}(-2Ay + 6By) = \frac{1}{2}\left(-2\times 1\, y + 6\times\frac{1}{3}y\right) = 0$$

Since $\omega_z = 0$, the flow is irrotational and $\vec{V} = -\nabla\phi$

Then

$$u = -\frac{\partial\phi}{\partial x} \text{ and } \phi = \int -u\,dx + f(y) = \int(-Ax^2 + 3By^2)\,dx + f(y)$$

$$\phi = -\frac{A}{3}x^3 + 3Bxy^2 + f(y)$$

$$v = -\frac{\partial\phi}{\partial y} \text{ and } \phi = \int -v\,dy + g(x) = \int 2Axy\,dy + g(x)$$

$$\phi = Axy^2 + g(x)$$

Comparing the two expressions for ϕ we

- note that $Axy^2 = 3Bxy^2$ $\quad (A=1, B=\frac{1}{3})$
- conclude that $g(x) = -\frac{A}{3}x^3$, $f(y) = 0$

Hence $\phi = Axy^2 - \frac{A}{3}x^3$ or $\phi = 3Bxy^2 - \frac{A}{3}x^3$ $\quad\phi$

Problem *6.91

Given: Flow field represented by $\psi = x^2 - y^2$

Find: (a) the velocity field
(b) show that the flow field is irrotational
(c) the potential function, ϕ.

Solution:

The velocity field is determined from the stream function.

$$u = \frac{\partial \psi}{\partial y} = -2y$$
$$v = -\frac{\partial \psi}{\partial x} = -2x$$
$$\therefore \vec{V} = -2y\hat{i} - 2x\hat{j} \qquad \vec{V}$$

If the flow is irrotational, then $\nabla \times \vec{V} = 0$

Since $w = 0$ and $\frac{\partial}{\partial z} = 0$,

$$\nabla \times \vec{V} = \hat{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = k\left[-2-(-2)\right] = 0 \qquad \text{Flow is irrotational}$$

From $\vec{V} = -\nabla\phi$

$$u = -\frac{\partial \phi}{\partial x} \quad \text{and} \quad \phi = \int -u\,dx + f(y) = 2xy + f(y)$$

$$v = -\frac{\partial \phi}{\partial y} \quad \text{and} \quad \phi = \int -v\,dy + g(x) = 2xy + g(x)$$

Comparing these expressions, we see that neither contains a function of x only or a function of y only.

Thus $f(y) = g(x) = 0$ and

$$\phi = 2xy \qquad \phi$$

Problem *6.92

Given: Flow field represented by the potential function,

$$\phi = x^2 - y^2$$

Find: (a) Verify that this is an incompressible flow.
(b) Corresponding stream function

Solution:

The velocity field is given by $\vec{V} = -\nabla\phi$

$$\vec{V} = -\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 - y^2) = -2x\hat{i} + 2y\hat{j}$$

If the flow is incompressible, then $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial}{\partial x}(-2x) + \frac{\partial}{\partial y}(2y) = -2 + 2 = 0$$

$\therefore$ Flow is incompressible

From the definition of ψ, $u = \frac{\partial \psi}{\partial y}$ and $v = -\frac{\partial \psi}{\partial x}$

Thus,

$$u = \frac{\partial \psi}{\partial y} = -2x \qquad \psi = \int -2x\, dy + f(x) = -2xy + f(x).$$

Then

$$v = 2y = -\frac{\partial \psi}{\partial x} = 2y + \frac{df}{dx}$$

and $\frac{df}{dx} = 0$ or $f = \text{constant}$

$$\therefore \psi = -2xy + c$$

Taking $c = 0$, then $\psi = -2xy$ $\quad\psi$

Problem *6.93

Given: Flow field represented by the potential function,

$$\phi = Ax^2 + Bxy - Ay^2$$

Find: (a) Verify that the flow is incompressible
(b) Determine the corresponding stream function, ψ

Solution:

The velocity field is given by $\vec{V} = -\nabla\phi$

$$\vec{V} = -\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(Ax^2 + Bxy - Ay^2) = -\hat{i}(2Ax + By) - \hat{j}(Bx - 2Ay)$$

If the flow is incompressible, then $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial}{\partial x}(-)(2Ax + By) + \frac{\partial}{\partial y}(-1)(Bx - 2Ay) = -2A + 2A = 0$$

$\therefore$ flow is incompressible

From the definition of ψ, $u = \frac{\partial \psi}{\partial y}$ and $v = -\frac{\partial \psi}{\partial x}$

Thus,

$$u = -2Ax - By = \frac{\partial \psi}{\partial y} \quad \text{and} \quad \psi = -\int (2Ax + By)\,dy + f(x)$$

$$\psi = -2Axy - B\frac{y^2}{2} + f(x)$$

Then,

$$v = -Bx + 2Ay = -\frac{\partial \psi}{\partial x} = 2Ay - \frac{df}{dx}$$

and $-\frac{df}{dx} = -Bx$ or $f = \frac{1}{2}Bx^2 + \text{constant}$

$$\therefore \psi = -2Axy - B\frac{y^2}{2} + B\frac{x^2}{2} + \text{constant}$$

Setting the constant equal to zero, we obtain

$$\psi = \frac{B}{2}(x^2 - y^2) - 2Axy$$

ψ

Problem *6.94

Given: Flow field represented by the velocity potential $\phi = Ax + Bx^2 - By^2$, where $A = 1\ m \cdot s^{-1}$, $B = 1\ s^{-1}$, and coordinates are measured in meters.

Find: (a) expression for the velocity field
(b) stream function
(c) pressure difference between points $(x_1, y_1) = (0,0)$ and $(x_2, y_2) = (1,2)$

Solution

The velocity field is determined from the velocity potential

$$u = -\partial\phi/\partial x = -A - 2Bx, \quad v = -\partial\phi/\partial y = 2By \qquad \vec{V} = -(A+2Bx)\hat{i} + 2By\hat{j}$$

From the definition of the stream function, $u = \frac{\partial\psi}{\partial y}$, $v = -\frac{\partial\psi}{\partial x}$

Then

$$\psi = \int u\,dy + f(x) = \int -(A+2Bx)\,dy + f(x)$$

$$\psi = -Ay - 2Bxy + f(x)$$

Also,

$$\psi = \int -v\,dx + g(y) = \int -2By\,dx + g(y)$$

$$\psi = -2Bxy + g(y)$$

Comparing the two expressions for ψ we conclude

$$f(x) = 0, \quad g(y) = -Ay$$

$$\therefore \psi = -(Ay + 2Bxy) \qquad \psi$$

Since $\nabla^2\phi = 2B - 2B = 0$, the flow is irrotational and the Bernoulli equation can be applied between any two points in the flow field

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2 \qquad \left\{ \text{Assume: } \rho = \text{constant},\ z_1 = z_2 \right\}$$

$$\vec{V}(0,0) = -A\hat{i} = -\hat{i}\ m/s \qquad V_{0,0} = 1\ m/s$$

$$\vec{V}(1,2) = -(A+2B)\hat{i} + 4B\hat{j} = -3\hat{i} + 4\hat{j}\ m/s \qquad \therefore V_{1,2} = 5\ m/s$$

$$\therefore p_1 - p_2 = \rho\left(\frac{V_2^2}{2} - \frac{V_1^2}{2}\right) = \frac{\rho}{2}(V_2^2 - V_1^2)$$

Assume fluid is water

$$p_1 - p_2 = \frac{1}{2} \times 999\ \frac{kg}{m^3}\ (25 - 1)\frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m} = 12\ kN/m^2$$

Given: Flow field represented by the velocity potential $\phi = Ay^3 - Bx^2y$, where $A = \frac{1}{3}\,m^{-1}\cdot s^{-1}$, $B = 1\,m^{-1}\cdot s^{-1}$, and the coordinates are measured in meters.

Find: (a) expression for the magnitude of the velocity vector
(b) the stream function.

Plot: streamlines and potential lines, and visually verify that they are orthogonal.

Solution:

The velocity field is determined from the velocity potential

$u = -\partial\phi/\partial x = 2Bxy = 2xy$

$v = -\partial\phi/\partial y = -3Ay^2 + Bx^2 = x^2 - y^2$

$\vec{V} = 2xy\hat{i} + (x^2 - y^2)\hat{j}$

$V = \left[u^2 + v^2\right]^{1/2} = \left[4x^2y^2 + (x^2-y^2)^2\right]^{1/2} = \left(4x^2y^2 + x^4 - 2x^2y^2 + y^4\right)^{1/2}$

$V = \left[x^4 + 2x^2y^2 + y^4\right]^{1/2} = \left[(x^2+y^2)^2\right]^{1/2} = x^2 + y^2 \longleftarrow V$

The stream function is defined such that $u = \frac{\partial\psi}{\partial y}$ and $v = -\frac{\partial\psi}{\partial x}$

Then, $\psi = \int u\,dy + f(x) = \int 2Bxy\,dy + f(x) = Bxy^2 + f(x)$ ---- (1)

Also, $\psi = \int -v\,dx + g(y) = \int (3Ay^2 - Bx^2)\,dx + g(y) = 3Axy^2 - \frac{B}{3}x^3 + g(y)$ ---(2)

Comparing the two expressions for ψ, we

- note that $Bxy^2 = 3Axy^2$ $(B=1, A=\frac{1}{3})$, and
- conclude that $f(x) = -\frac{B}{3}x^3$ and $g(y) = 0$

$\therefore\ \psi = Bxy^2 - \frac{B}{3}x^3$ or $\psi = 3Axy^2 - \frac{B}{3}x^3 \longleftarrow \psi$

with $A = \frac{1}{3}$, $B = 1$

$\psi = xy^2 - \frac{x^3}{3} = x\left(y^2 - \frac{x^2}{3}\right)$

For $\psi = 0$, $x = 0$ or $y = 0.577x$

For $\psi = -4$, $y^2 = \frac{x^2}{3} - \frac{4}{x}$

For $\psi = 4$, $y^2 = \frac{x^2}{3} + \frac{4}{x}$

See the next page for plots

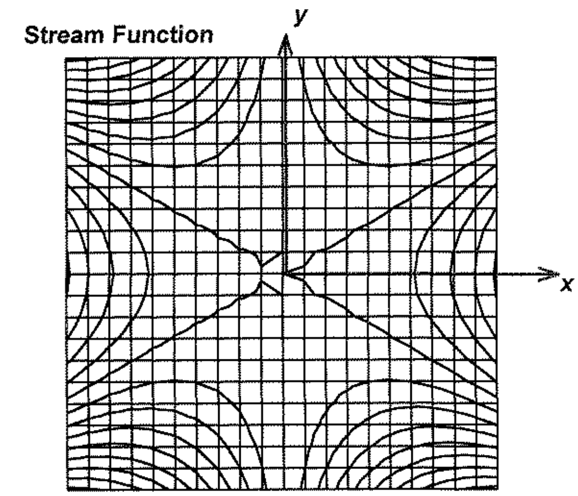
Problem*6.95 (Cont'd)

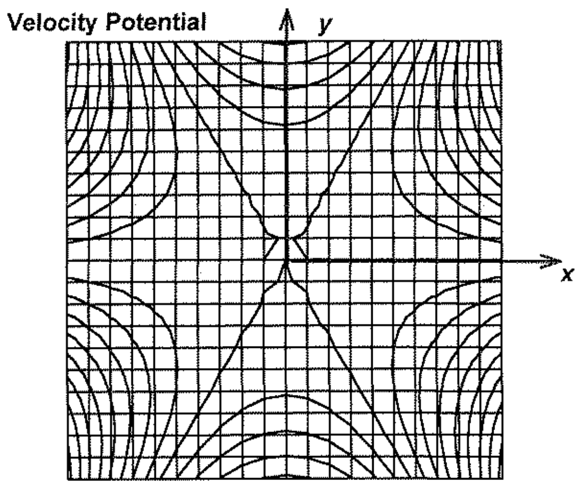
Using *Excel*, the stream function and velocity potential can be plotted.
The data below was obtained using the workbook for Example Problem 6.10.
Note the orthogonality of ψ and ϕ!

#NAME?	**Stream Function**

#NAME?	**Velocity Potential**

Note that the plot is
from $x = -5$ to 5 and $y = -5$ to 5





Problem* 6.96

Given: Incompressible flow field represented by $\psi = 3Ax^2y - Ay^3$ where $A = 1\ m^{-1}\cdot s^{-1}$

Show: that this flow field is irrotational

Find: the velocity potential ϕ

Plot: streamlines and potential lines, and visually verify that they are orthogonal

Solution:

For a 2-D incompressible, irrotational flow $\nabla^2\psi = 0$ (6.30)

For this flow field

$\nabla^2\psi = \frac{\partial^2}{\partial x^2}(3Ax^2y - Ay^3) + \frac{\partial^2}{\partial y^2}(3Ax^2y - Ay^3) = 6Ay - 6Ay = 0$ ← irrotational

The velocity field is given by $\vec{V} = u\hat{i} + v\hat{j}$

$u = \partial\psi/\partial y = 3Ax^2 - 3Ay^2 = 3A(x^2 - y^2)$

$v = -\partial\psi/\partial x = -6Axy$

$\vec{V} = 3A(x^2-y^2)\hat{i} - 6Axy\hat{j}$

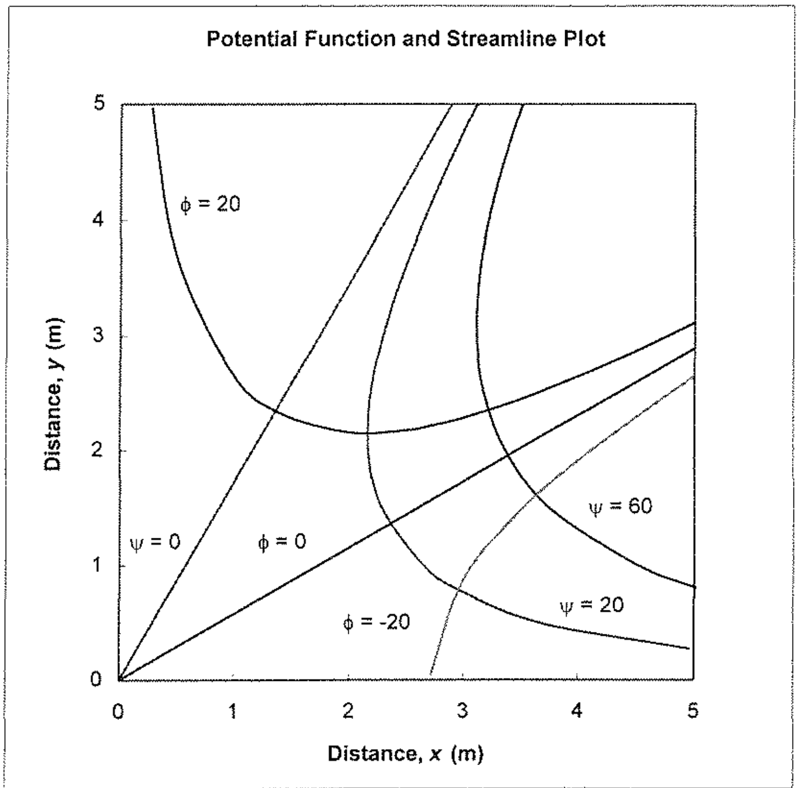
The velocity potential is defined such that $u = -\frac{\partial\phi}{\partial x}$, $v = -\frac{\partial\phi}{\partial y}$

Then, $\phi = -\int u\,dx + f(y) = -\int 3A(x^2-y^2)\,dx + f(y) = -Ax^3 + 3Axy^2 + f(y)$ ...(1)

Also, $\phi = -\int v\,dy + g(x) = \int 6Axy\,dy + g(x) = 3Axy^2 + g(x)$ ......(2)

Equating expressions for ϕ (Eqs 1 and 2) we see that

$g(x) = -Ax^3$ and $f(y) = 0$ $\therefore \phi = 3Axy^2 - Ax^3$ ← ϕ



Given: Two-dimensional, inviscid flow with velocity field $\vec{V} = (Ax+B)\hat{i} + (C-Ay)\hat{j}$, where $A = 3s^{-1}$, $B = 6 m/s$, $C = 4 m/s$ and the coordinates are measured in meters. The body force distribution is $\vec{B} = -g\hat{k}$; $\rho = 825 kg/m^3$.

Find: (a) if this is a possible incompressible flow
(b) stagnation point(s) of the flow field
(c) if the flow is irrotational
(d) the velocity potential (if one exists)
(e) pressure difference between origin and point $(x,y,z) = (2,2,2)$

Plot: a few streamlines in the upper half plane.

Solution:

For incompressible flow $\nabla\cdot\vec{V} = 0$. For this flow

$$\nabla\cdot\vec{V} = \frac{\partial}{\partial x}(Ax+B) + \frac{\partial}{\partial y}(C-Ay) = A - A = 0$$

∴ velocity field represents possible incompressible flow.

At the stagnation point $u = v = 0$. $(\vec{V} = 0)$

$$u = 0 = (Ax+B) \quad \therefore x = -B/A = \frac{-6 m.s^{-1}}{3s^{-1}} = -2m$$

$$v = 0 = (C-Ay) \quad \therefore y = C/A = \frac{4m.s^{-1}}{3s^{-1}} = 4/3\,m$$

Stagnation point is at $(x,y) = (-2, 4/3)m$.

The fluid rotation (for a 2-D flow) is given by $\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$

For this flow $\omega_z = \frac{1}{2}\left[\frac{\partial(C-Ay)}{\partial x} - \frac{\partial(Ax+B)}{\partial y}\right] = 0$

∴ flow is irrotational

Then, $\vec{V} = -\nabla\phi$ and $u = -\partial\phi/\partial x$ and $v = -\partial\phi/\partial y$.

and $\phi = \int -u\,dx + f(y) = -\int(Ax+B)dx + f(y) = -A\frac{x^2}{2} - Bx + f(y)$ ...(1)

Also $\phi = -\int v\,dy + g(x) = -\int(C-Ay)dy + g(x) = A\frac{y^2}{2} - Cy + g(x)$ ...(2)

Equating the two expressions for ϕ (Eq.s 1 and 2) we note that

$$g(x) = -\left(A\frac{x^2}{2} + Bx\right) \quad \text{and} \quad f(y) = A\frac{y^2}{2} - Cy$$

$$\therefore \phi = \frac{A}{2}(y^2 - x^2) - Bx - Cy \qquad \phi$$

Since the flow is irrotational we can apply the Bernoulli equation between any two points in the flow field.

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2$$

At point 1 $(0,0,0)$, $\vec{V} = B\hat{i} + C\hat{j} = 6\hat{i} + 4\hat{j}\ m/s$, $V_1^2 = 52\,m^2/s^2$

At point 2 (2,2,2) $\vec{V}_2 = [3\,s^{-1} \times 2\,m + 6\,m/s]\hat{i} + [4\,m/s - 3\,s^{-1} \times 2\,m]\hat{j}$

$\vec{V}_2 = 12\hat{i} - 2\hat{j}$ m/s , $V_2^2 = 148\ m^2/s^2$

$$p_1 - p_2 = \frac{\rho}{2}(V_2^2 - V_1^2) + \rho g(z_2 - z_1) = \rho\left[\frac{(V_2^2 - V_1^2)}{2} + g(z_2 - z_1)\right]$$

$$= 825\,\frac{kg}{m^3} \times \left[\frac{1}{2} \times (148 - 52)\,\frac{m^2}{s^2} + 9.81\,\frac{m}{s^2} \times (2\,m)\right] \times \frac{N \cdot s^2}{kg \cdot m}$$

$p_1 - p_2 = 55.8$ kPa ←——— Δp

The stream function is defined such that $u = \frac{\partial \psi}{\partial y}$, $v = -\frac{\partial \psi}{\partial x}$

Hence, $\psi = \int u\,dy + f(x) = \int (Ax + B)\,dy + f(x) = Axy + By + f(x)$ ----(1)

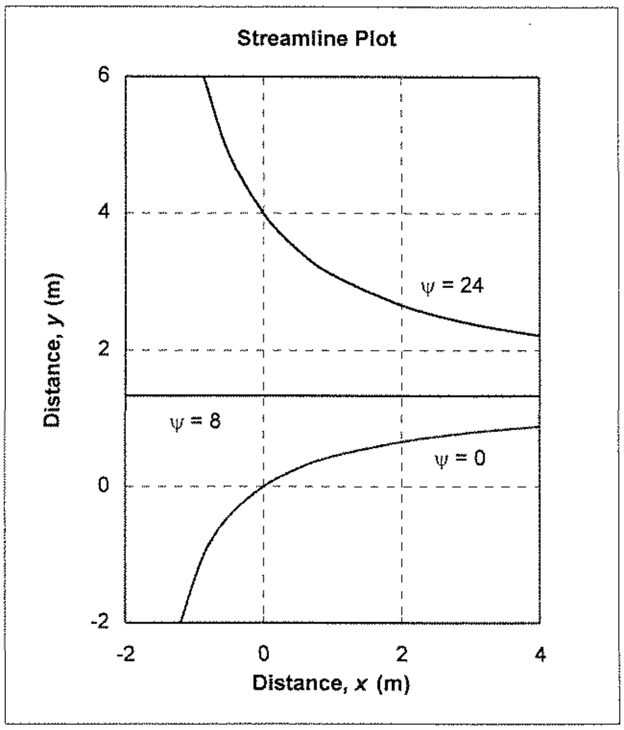
Also, $\psi = -\int v\,dx + g(y) = \int (-C + Ay)\,dx + g(y) = -Cx + Axy + g(y)$ ----(2)

Equating the two expressions for ψ (Eqs 1 and 2) we note that

$f(x) = -Cx$, $g(y) = By$ and $\therefore\ \psi = Axy + By - Cx$ ←— ψ

The stagnation streamline goes through the stagnation point $(-2, \frac{4}{3})$

$\psi_{stag} = 3\,s^{-1} \times (-2\,m) \times \frac{4}{3}\,m + 6\,m \cdot s^{-1} \times \frac{4}{3}\,m - 4\,m \cdot s^{-1} \times (-2\,m) = 8\ m^2/s$ ←— ψ_{stag}



Given: Irrotational flow represented by $\psi = Bxy$, where $B = 0.25\ s^{-1}$ and the coordinates are measured in meters.

Find: (a) the rate of flow between points $(x_1, y_1) = (2,2)$ and $(x_2, y_2) = (3,3)$
(b) the velocity potential for this flow

Plot: streamlines and potential lines, and visually verify that they are orthogonal.

Solution:

The volume flow rate (per unit depth) between points ① and ② is given by

$$Q_{12} = \psi_2 - \psi_1 = B[x_2 y_2 - x_1 y_1] = 0.25\ s^{-1}[3m \times 3m - 2m \times 2m]$$

$$Q_{12} = 1.25\ m^3/s/m \qquad Q_{12}$$

The velocity field is determined from the stream function

$$u = \partial\psi/\partial y = Bx \qquad v = -\partial\psi/\partial x = -By \qquad \therefore \vec{V} = Bx\hat{i} - By\hat{j}$$

For irrotational flow $\vec{V} = -\nabla\phi$ and $u = -\partial\phi/\partial x$, $v = -\partial\phi/\partial y$

and

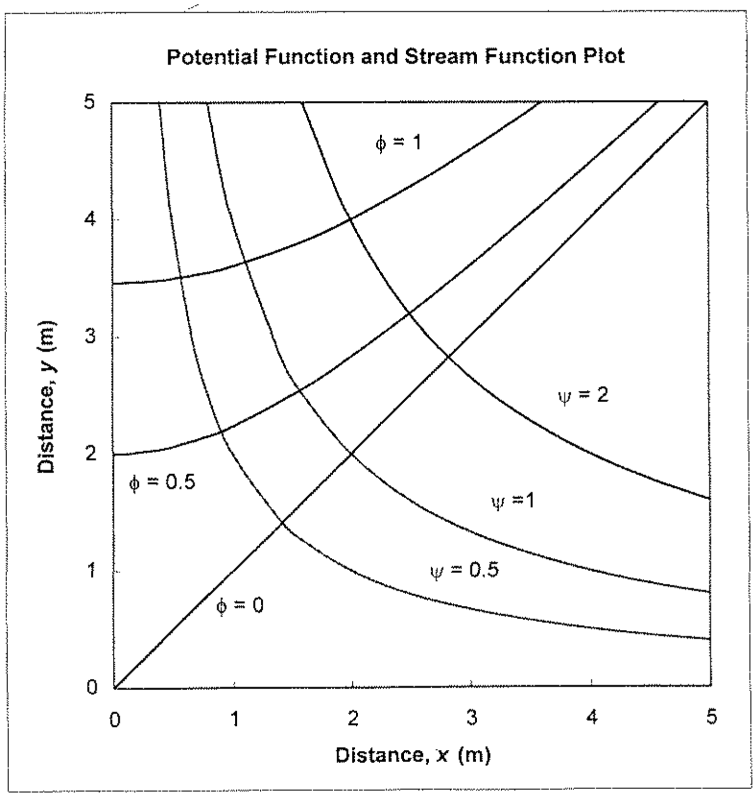
$$\phi = -\int u\,dx + f(y) = -\int Bx\,dx + f(y) = -\frac{B}{2}x^2 + f(y) \quad \text{....(1)}$$

Also

$$\phi = -\int v\,dy + g(x) = \int By\,dy + g(x) = \frac{B}{2}y^2 + g(x) \quad \text{.....(2)}$$

Equating expressions for ϕ (Eqs 1 and 2) we conclude that

$$f(y) = \frac{B}{2}y^2, \quad g(x) = -\frac{B}{2}x^2 \quad \text{and} \quad \phi = \frac{B}{2}(y^2 - x^2) \qquad \phi$$



Problem *6.99

Given: Flow past a circular cylinder of Example Problem 6.11.

Find: (a) Show that $V_r = 0$ along the lines $(r,\theta) = (r, \pm\pi/2)$
(b) Plot V_θ/U versus r for $r \geq a$ along line $(r, \pi/2)$.
(c) Find distance beyond which the influence of the cylinder on the velocity is less that 1% of U

Solution.

From Example Problem 6.11

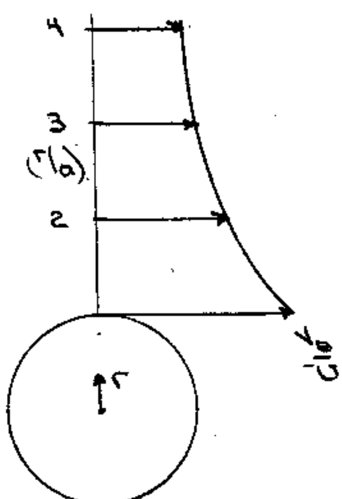
$$\vec{V} = \left(-\frac{\Lambda\cos\theta}{r^2} + U\cos\theta\right)\hat{i}_r + \left(-\frac{\Lambda\sin\theta}{r^2} - U\sin\theta\right)\hat{i}_\theta \quad \ldots\ldots (1)$$

then $V_r = \left(-\frac{\Lambda}{r^2} + U\right)\cos\theta$. For $\theta = \pm\frac{\pi}{2}$, $\cos\theta = 0$ and $V_r = 0$

$V_\theta = -\left(\frac{\Lambda}{r^2} + U\right)\sin\theta$, but $\frac{\Lambda}{U} = a^2$

$\therefore V_\theta = -\left(\frac{a^2}{r^2} + 1\right)U\sin\theta$ For $\theta = \pi/2$.

$$\frac{V_\theta}{U} = -\left(1 + \frac{a^2}{r^2}\right)$$



$$\vec{V} = U\cos\theta\left(1 - \frac{a^2}{r^2}\right)\hat{i} - U\sin\theta\left(1 + \frac{a^2}{r^2}\right)\hat{j}$$

For $\theta = \pi/2$

$\frac{V}{U} = 1 + \frac{a^2}{r^2}$ If $\frac{V}{U} = 1.01$ then $\frac{a^2}{r^2} = 0.01$ or $\frac{a}{r} = 0.1$

$\therefore \frac{V}{U} < 1\%$ for $r > 10a$

Problem *6.100

Given: Consider flow around a circular cylinder with freestream velocity from right to left and a counterclockwise free vortex.

Find: Show that the lift force on the cylinder (per unit width) can be expressed as $F_L = -\rho U\Gamma$, as illustrated in Example Problem 6.12.

Discussion: The only change in this flow from the flow of Example Problem 6.12 is that the directions of the freestream velocity and the vortex are changed. This changes the sign of the freestream velocity from U to $-U$ and the sign of the vortex strength from K to $-K$. Consequently the signs of both terms in the equation for lift are changed. Therefore the direction of the lift force remains unchanged.

The analysis of Example Problem 6.12 (see page 282) shows that only the term involving the vortex strength contributes to the lift force. Therefore the expression for lift obtained with the changed freestream velocity and vortex strength is identical to that derived in Example Problem 6.12. Thus the general solution of Example Problem 6.12 holds for any orientation of the freestream and vortex velocities. For the present case, $F_L = -\rho U\Gamma$, as shown for the general case in Example Problem 6.12.

Problem *6.101

Given: A tornado is modelled by the superposition of a sink (strength, $q = 2800\ m^2/sec$) and a free vortex (strength, $K = 5600\ m^2/sec$)

Find: (a) Expressions for ψ and ϕ
(b) Estimate the radius beyond which the flow may be treated as incompressible.
(c) Find the gage pressure at that radius.

Solution:

$$\psi = \psi_{si} + \psi_{vo} = -\frac{q}{2\pi}\theta - \frac{K}{2\pi}\ln r \qquad \psi$$

$$\phi = \phi_{si} + \phi_{vo} = \frac{q}{2\pi}\ln r - \frac{K}{2\pi}\theta \qquad \phi$$

$$\vec{V} = V_r\hat{i}_r + V_\theta\hat{i}_\theta \qquad V_{r\,si} = -\frac{q}{2\pi r},\ V_{r\,vo} = 0;\ V_{\theta\,si} = 0,\ V_{\theta\,vo} = \frac{K}{2\pi r}$$

$$\therefore \vec{V} = -\frac{q}{2\pi r}\hat{i}_r + \frac{K}{2\pi r}\hat{i}_\theta$$

Then

$$V = (V_r^2 + V_\theta^2)^{1/2} = \left[\left(-\frac{q}{2\pi}\right)^2 + \left(\frac{K}{2\pi}\right)^2\right]^{1/2}\frac{1}{r}$$

For incompressible flow $M \le 0.3$. For standard air this corresponds to $V < 102\ m/sec$

Then, for incompressible flow

$$V = 102\ m/sec < \left[q^2 + K^2\right]^{1/2}\frac{1}{2\pi r}$$

or

$$r > \left[q^2+K^2\right]^{1/2}\frac{1}{2\pi} \times \frac{s}{102\ m} = \left[(2800)^2 + (5600)^2\right]^{1/2}\frac{m^2}{s^2} \times \frac{1}{2\pi} \times \frac{s}{102\ m}$$

$$r > 9.77\ m \qquad r$$

To determine the gage pressure at this radius, apply the Bernoulli equation for irrotational flow

$$\frac{p_\infty}{\rho} + \frac{V_\infty^2}{2} + gz_\infty = \frac{p}{\rho} + \frac{V^2}{2} + gz \qquad \text{assume } \Delta z = 0$$

($V_\infty^2 \approx 0$)

Then

$$p_{gage} = p - p_\infty = -\frac{1}{2}\rho V^2 = -\frac{1}{2} \times 1.225\frac{kg}{m^3} \times (102)^2\frac{m^2}{s^2} \times \frac{N\,s^2}{kg\,m}$$

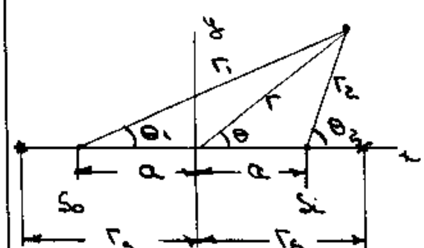
$$p_{gage} = -6.37\ kPa \quad \text{(for standard air)} \qquad p_{gage}$$

Problem *6.102

Given: Flow past a Rankine body is formed from the superposition of a uniform flow ($U = 20$ m/s) in the +x direction and a source and a sink of equal strengths ($q = 3\pi$ m²/s) located on the x axis at $x = -a$ and $x = a$, respectively.

Find: (a) expressions for ψ, ϕ, and $\vec{V}$
(b) the value of ψ = constant on the stagnation streamline.
(c) the stagnation points if $a = 0.3$ m.

Solution:



$$\psi = \psi_{so} + \psi_{si} + \psi_{uf} = \frac{q}{2\pi}\theta_1 - \frac{q}{2\pi}\theta_2 + Uy$$

$$\psi = \frac{q}{2\pi}(\theta_1 - \theta_2) + Ur\sin\theta \quad \longleftarrow \psi$$

$$\phi = \phi_{so} + \phi_{si} + \phi_{uf} = -\frac{q}{2\pi}\ln r_1 + \frac{q}{2\pi}\ln r_2 - Ux$$

$$\phi = \frac{q}{2\pi}\ln\frac{r_2}{r_1} - Ur\cos\theta \quad \longleftarrow \phi$$

$$u = u_{so} + u_{si} + u_{uf} = \frac{q}{2\pi r_1}\cos\theta_1 - \frac{q}{2\pi r_2}\cos\theta_2 + U$$

$$v = v_{so} + v_{si} + v_{uf} = \frac{q}{2\pi r_1}\sin\theta_1 - \frac{q}{2\pi r_2}\sin\theta_2$$

$$\vec{V} = u\hat{i} + v\hat{j} = \left\{\frac{q}{2\pi}\left(\frac{\cos\theta_1}{r_1} - \frac{\cos\theta_2}{r_2}\right) + U\right\}\hat{i} + \frac{q}{2\pi}\left(\frac{\sin\theta_1}{r_1} - \frac{\sin\theta_2}{r_2}\right)\hat{j} \quad \longleftarrow \vec{V}$$

At stagnation point $\vec{V} = 0$ $\quad y = 0 \quad \theta_1 = \theta_2 = 0$

$r_2 = r_s - a, \quad r_1 = r_s + a$

$$\therefore u = 0 = \frac{q}{2\pi}\left(\frac{1}{r_s + a} - \frac{1}{r_s - a}\right) + U = \frac{q}{2\pi}\left[\frac{(r_s - a) - (r_s + a)}{(r_s^2 - a^2)}\right] + U$$

$$0 = \frac{-qa}{\pi(r_s^2 - a^2)} + U \quad \text{or} \quad (r_s^2 - a^2) = \frac{qa}{\pi U}$$

$$r_s = \left(a^2 + \frac{qa}{\pi U}\right)^{1/2} = a\left(1 + \frac{q}{\pi U a}\right)^{1/2}$$

For $a = 0.3$ m

$$r = 0.3\,\text{m}\left[1 + \frac{3\pi}{\pi}\,\frac{\text{m}^2}{\text{s}} \cdot \frac{\text{s}}{20\,\text{m}} \cdot \frac{1}{0.3\,\text{m}}\right]^{1/2} = 0.367\,\text{m}$$

Stagnation points located at $\theta = 0, \pi \quad r = 0.367$ m $\longleftarrow$

Since $\psi = \frac{q}{2\pi}(\theta_1 - \theta_2) + Uy$ and $\theta_1 = \theta_2$ & $y = 0$ at stagnation

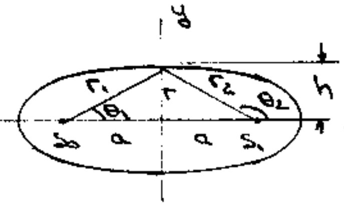
$$\psi_{stag} = 0 \quad \longleftarrow$$

Problem *6.103

Given: Flow past a Rankine body is formed from the superposition of a uniform flow ($U = 20$ m/s) in the $+x$ direction, and a source and a sink of equal strengths ($q = 3\pi$ m²/s) located on the x axis at $x = -a$ and $x = a$, respectively

Find: (a) the half width of the body
(b) $\vec{V}$ and p at the points $(0, \pm h)$

Solution:



$$\psi = \psi_{so} + \psi_{si} + \psi_{uf} = \frac{q}{2\pi}(\theta_1 - \theta_2) + Ur\sin\theta$$

At stagnation point $\theta_1 = \theta_2$ and $\theta = 0, \pi$

$\therefore \psi_{stag} = 0$ and equation of stag. streamline is

$$0 = \frac{q}{2\pi}(\theta_1 - \theta_2) + Ur\sin\theta$$

or $$r = \frac{q}{2\pi}\frac{(\theta_2 - \theta_1)}{U\sin\theta}$$

At half width, $\theta = \frac{\pi}{2}$, $\theta_2 = \pi - \theta_1$, and $r = h = \frac{q}{2\pi}\frac{[(\pi - \theta_1) - \theta_1]}{U}$

$\therefore hU = \frac{q}{2\pi}[\pi - 2\theta_1] = \frac{q}{2} - \frac{q\theta_1}{\pi}$ or $\theta_1 = \frac{\pi}{2} - \frac{Uh\pi}{q}$

Since $h = a\tan\theta_1$

$$\frac{h}{a} = \tan\left(\frac{\pi}{2} - \frac{Uh\pi}{q}\right) = \cot\left(\frac{Uh\pi}{q}\right)$$

Substituting values, $\frac{h}{0.3} = \cot\left(\frac{20}{3}h\right)$. Trial and error solution gives

$h = 0.1615$ m ← h

The velocity field is given by $\vec{V} = \hat{i}u + \hat{j}v$

$$\vec{V} = \left\{\frac{q}{2\pi}\left(\frac{\cos\theta_1}{r_1} - \frac{\cos\theta_2}{r_2}\right) + U\right\}\hat{i} + \frac{q}{2\pi}\left(\frac{\sin\theta_1}{r_1} - \frac{\sin\theta_2}{r_2}\right)\hat{j}$$

At $(0,h)$, $r_1 = r_2$, $\theta_2 = \pi - \theta_1$ $\therefore \sin\theta_2 = \sin\theta_1$, $\cos\theta_2 = -\cos\theta_1$

and $$\vec{V} = \left(\frac{q\cos\theta_1}{r_1} + U\right)\hat{i}$$

$\theta_1 = \tan^{-1}\frac{h}{a} = \tan^{-1}\frac{0.1615}{0.3} = 28.3°$ $r_1 = [a^2 + h^2]^{1/2} = [0.3^2 + 0.1615^2]^{1/2} = 0.341$ m

$$\vec{V} = \left(\frac{q\cos\theta_1}{r_1} + U\right)\hat{i} = \left(3\pi\,\frac{m^2}{sec} \times \frac{\cos 28.3°}{0.341\,m} + 20\,\frac{m}{s}\right)\hat{i} = 44.3\,\hat{i}\ \text{m/s}$$ ← V

To find the gage pressure apply the Bernoulli equation between the point at conditions at ∞

$$\frac{p_\infty}{\rho} + \frac{U^2}{2} = \frac{p}{\rho} + \frac{V^2}{2}$$

$$p_{gage} = p - p_\infty = \frac{1}{2}\rho(U^2 - V^2) = \frac{1}{2} \times 1.225\,\frac{kg}{m^3}\left[(20)^2 - (44.3)^2\right]\frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m}$$

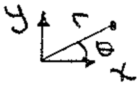
$p_{gag} = -957$ N/m² ← p_{gag}

Given: Flow field formed by superposition of a uniform flow in the +x direction ($U = 10$ m/s) and a counterclockwise vortex, with strength $K = 16\pi$ m²/s, located at the origin

Find: (a) ψ, ϕ, and $\vec{V}$ for the flow field
(b) stagnation point(s)

Plot: streamlines and lines of constant potential

Solution:



$$\psi = \psi_{uf} + \psi_v = Uy - \frac{K}{2\pi}\ln r = Ur\sin\theta - \frac{K}{2\pi}\ln r \quad \longleftarrow \quad \psi$$

$$\phi = \phi_{uf} + \phi_v = -Ux - \frac{K}{2\pi}\theta = -Ur\cos\theta - \frac{K}{2\pi}\theta \quad \longleftarrow \quad \phi$$

$$V_r = -\frac{\partial\phi}{\partial r} = U\cos\theta \quad , \quad V_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = -U\sin\theta + \frac{K}{2\pi r}$$

$$\vec{V} = U\cos\theta\,\hat{e}_r + \left(\frac{K}{2\pi r} - U\sin\theta\right)\hat{e}_\theta \quad \longleftarrow \quad \vec{V}$$

At stagnation point, $\vec{V} = 0$

$V_r = 0$ at $\theta = \pm\frac{\pi}{2}$; $V_\theta = 0$ on $r = \frac{K}{2\pi U\sin\theta}$

$\therefore \vec{V} = 0$ at $(r, \theta) = \frac{K}{2\pi U}, \pi/2$ $\longleftarrow$ Stagnation

See the next page for plots

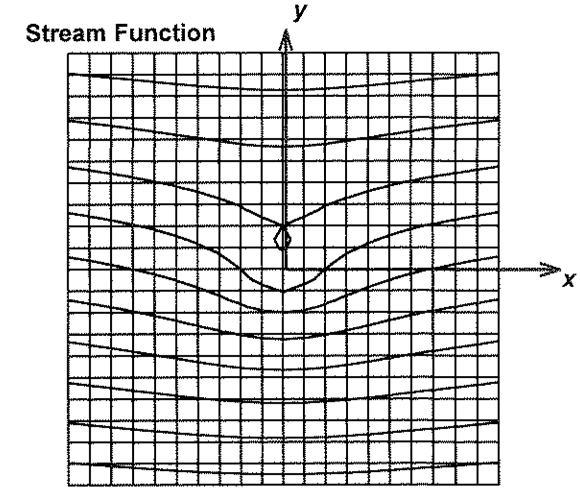
Problem*6.104 (Cont'd)

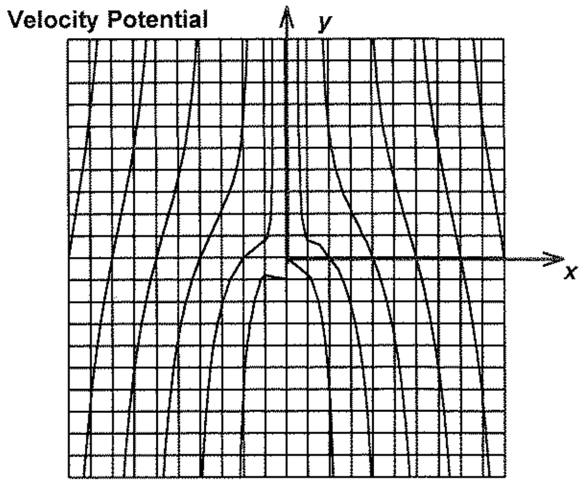
Using *Excel*, the stream function and velocity potential can be plotted.
The data below was obtained using the workbook for Example Problem 6.10.
Note the orthogonality of ψ and ϕ!

#NAME? **Stream Function**

#NAME? **Velocity Potential**

Note that the plot is
from x = -5 to 5 and y = -5 to 5



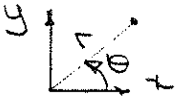


Given: Flow field obtained by superposing a uniform flow in the +x direction ($U = 25$ m/s) and a source (of strength q) at the origin. Stagnation point is at $x = -1.0$ m.

Find: (a) expressions for ψ, ϕ, $\vec{V}$
(b) source strength, q.

Plot: streamlines and potential lines.

Solution:



$\psi = \psi_{uf} + \psi_{so} = Uy + \frac{q}{2\pi}\theta = Ur\sin\theta + \frac{q}{2\pi}\theta$ ——— ψ

$\phi = \phi_{uf} + \phi_{so} = -Ux - \frac{q}{2\pi}\ln r = -Ur\cos\theta - \frac{q}{2\pi}\ln r$ ——— ϕ

$u = u_{uf} + u_{so}$; $u_{uf} = U$; $u_{so} = V_r\cos\theta = \frac{q}{2\pi r}\frac{x}{r}$ $\therefore u = U + \frac{q}{2\pi}\frac{x}{r^2}$

$v = v_{uf} + v_{so}$; $v_{uf} = 0$; $v_{so} = V_r\sin\theta = \frac{q}{2\pi r}\frac{y}{r}$ $\therefore v = \frac{q}{2\pi}\frac{y}{r^2}$

$\vec{V} = u\hat{i} + v\hat{j} = \left\{U + \frac{q}{2\pi}\frac{x}{(x^2+y^2)}\right\}\hat{i} + \frac{q}{2\pi}\frac{y}{(x^2+y^2)}\hat{j}$ ——— $\vec{V}$

At the stagnation point $\vec{V} = 0$ $x = -1.0$ m $y = 0$ ($v = 0$).

For $u = 0 = U + \frac{q}{2\pi}\frac{x}{(x^2+y^2)}$ $\therefore q = -2\pi U x_{stag}$

$q = -2\pi \times 25\frac{m}{s} \times (-1.0\,m) = 50\pi\ m^2/s$ ——— q

At the stagnation point, $\theta = \pi$ $\therefore \psi_{stag} = \frac{q}{2\pi}\theta = \frac{q}{2}$

The equation of the stagnation streamline is then

$q/2 = Ur\sin\theta + \frac{q}{2\pi}\theta$ and $r = \frac{q(\pi-\theta)}{2\pi U\sin\theta}$

At $\theta = \pi/2$, $r = \frac{q}{4U} = 50\pi\frac{m^2}{s} \times \frac{1}{4} \times \frac{s}{25m} = \pi/2$

Far downstream $\theta \to 0$ and the y coordinate of the body

$y = r\sin\theta = \frac{q(\pi-\theta)}{2\pi U}$ approaches $\frac{q}{2U} = \frac{50\pi}{2 \times 25} = \pm\pi$ m ——— $y_{\theta\to 0}$

See the next page for plots

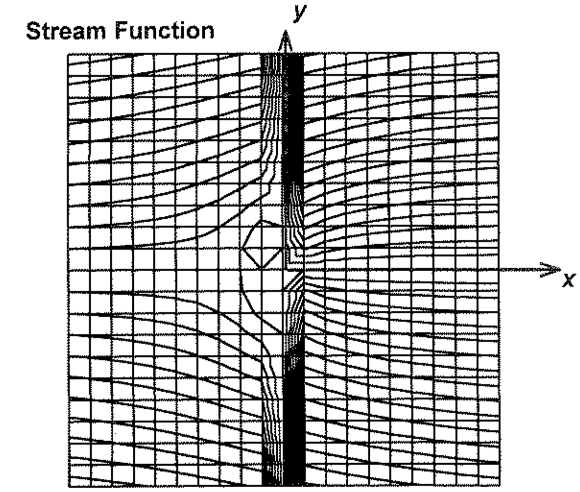
Problem*6.105 (Cont'd)

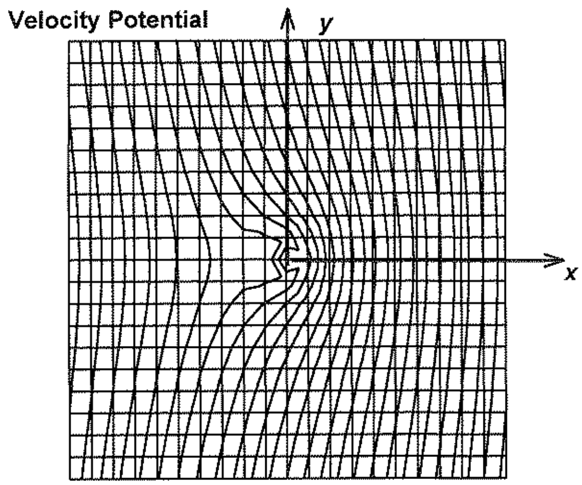
Using *Excel*, the stream function and velocity potential can be plotted.
The data below was obtained using the workbook for Example Problem 6.10.
Note the orthogonality of ψ and ϕ!

#NAME?	**Stream Function**

#NAME?	**Velocity Potential**

Note that the plot is
from x = -5 to 5 and y = -5 to 5





Given: Flow field obtained by combining a uniform flow in the +x direction ($U = 30\ m/s$) and a source (of strength $q = 150\ m^2/s$) located at the origin.

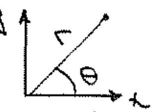
Plot: the ratio of the local velocity V to the free stream velocity U as a function of θ along the stagnation streamline.

Find: (a) points on the stagnation streamline where the velocity reaches its maximum value

(b) gage pressure at this location if $\rho = 1.2\ kg/m^3$

Solution:

Superposition of a uniform flow and source gives flow around a half body.



$$\psi = \psi_{u.f} + \psi_{so} = Uy + \frac{q}{2\pi}\theta = U r\sin\theta + \frac{q}{2\pi}\theta \quad \text{----(1)}$$

$$u = u_{u.f} + u_{so}\ ;\ u_{u.f} = U\ ;\ u_{so} = v_r\cos\theta = \frac{q}{2\pi r}\frac{x}{r} \qquad \therefore u = U + \frac{q}{2\pi}\frac{x}{r^2}$$

$$v = v_{u.f} + v_{so}\ ;\ v_{u.f} = 0\ ;\ v_{so} = v_r\sin\theta = \frac{q}{2\pi r}\frac{y}{r} \qquad \therefore v = \frac{q}{2\pi}\frac{y}{r^2}$$

$$\therefore \vec{V} = u\hat{i} + v\hat{j} = \left(U + \frac{q}{2\pi}\frac{x}{r^2}\right)\hat{i} + \frac{q}{2\pi}\frac{y}{r^2}\hat{j} \quad \text{--------(2)}$$

Then,

$$V^2 = u^2 + v^2 = \left(U + \frac{q}{2\pi r}\cos\theta\right)^2 + \left(\frac{q}{2\pi r}\sin\theta\right)^2$$

$$= U^2 + \left(\frac{q}{2\pi r}\right)^2\cos^2\theta + \frac{Uq}{\pi r}\cos\theta + \left(\frac{q}{2\pi r}\right)^2\sin^2\theta$$

$$V^2 = U^2 + \left(\frac{q}{2\pi r}\right)^2 + \frac{Uq}{\pi r}\cos\theta \quad \text{----------(3)}$$

To determine the equation of the stagnation streamline, we locate the stagnation point ($\vec{V} = 0$). From Eq. 2 $y = 0$ and

$$U + \frac{q}{2\pi}\frac{x}{r^2} = 0 = U + \frac{q}{2\pi}\frac{x}{(x^2+y^2)} = U + \frac{q}{2\pi x} \quad \text{and} \quad x_{stag} = -\frac{q}{2\pi U}$$

$$x_{stag} = -\frac{q}{2\pi U} = -\frac{1}{2\pi} \times 150\frac{m^2}{s} \times \frac{s}{30\ m} = -0.796\ m$$

At the stagnation point $y = 0$ and $\theta = \pi$. From Eq. 1 $\psi_{stag} = \frac{q}{2}$

the equation of the stagnation streamline is then:

$$\psi_{stag} = \frac{q}{2} = U r\sin\theta + \frac{q}{2\pi}\theta \quad . \text{ Solving for } r\text{, we obtain}$$

$$r = \frac{1}{U\sin\theta}\left(\frac{q}{2} - \frac{q}{2\pi}\theta\right) = \frac{q(\pi-\theta)}{2\pi U\sin\theta} \quad \text{----------(4)}$$

Substituting this value of r into the expression for V^2 [Eq. 3] we obtain

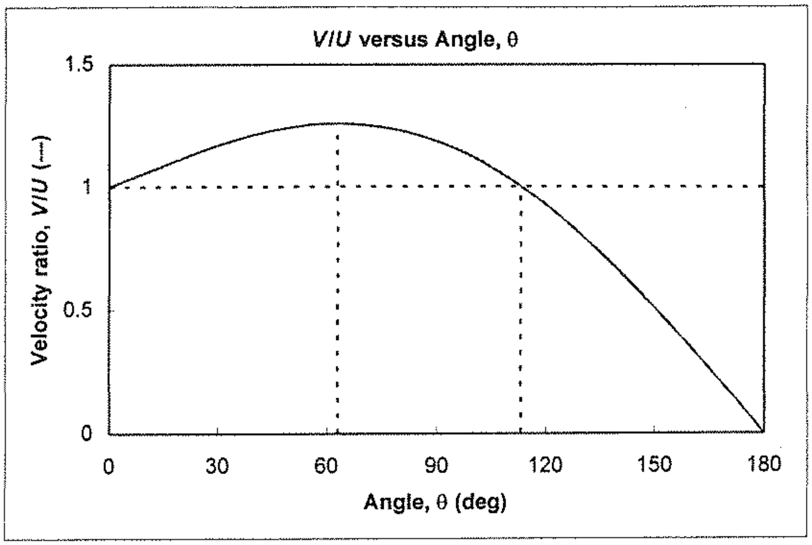
$$V^2 = U^2 + \left[\frac{q}{2\pi}\times\frac{2\pi U\sin\theta}{q(\pi-\theta)}\right]^2 + \frac{Uq\cos\theta}{\pi}\times\frac{2\pi U\sin\theta}{q(\pi-\theta)}$$

$$V^2 = U^2 + \frac{U^2\sin^2\theta}{(\pi-\theta)^2} + \frac{2U^2\sin\theta\cos\theta}{(\pi-\theta)} = U^2\left[1 + \frac{\sin^2\theta}{(\pi-\theta)^2} + \frac{2\sin\theta\cos\theta}{(\pi-\theta)}\right]$$

Along the stagnation streamline

$$\frac{V}{U} = \left[1 + \frac{\sin^2\theta}{(\pi-\theta)^2} + \frac{2\sin\theta\cos\theta}{(\pi-\theta)}\right]^{1/2} \quad \text{--------(5)}$$

V/U is plotted as a function of θ



From the plot we see that V/U is a maximum at $\theta = 63°$ (also at $\theta = 297°$ from symmetry with respect to the x axis.

At $\theta = 63°$, Eq 5 gives $V/U_{max} = 1.26$

, Eq. 4 gives $r = 150\frac{m^2}{s} \times \frac{(\pi - 0.35\pi)}{2\pi \sin 63°} \times \frac{s}{30\,m} = 1.82\,m$.

Thus $V = V_{max}$ at $r = 1.82\,m$ and $\theta = 63°, 297°$ ← $(r,\theta)_{V_{max}}$

To determine the gage pressure at this point, write the Bernoulli equation between a point upstream and the point of maximum velocity.

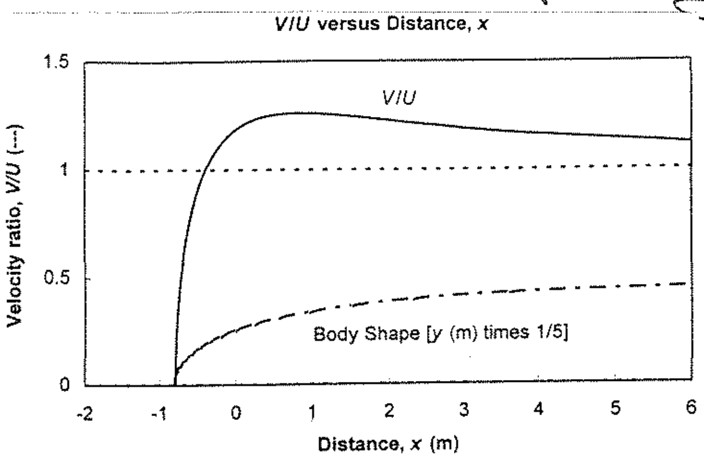
$$\frac{p_\infty}{\rho} + \frac{U^2}{2} = \frac{p}{\rho} + \frac{V_{max}^2}{2}$$

$$\therefore p - p_\infty = \frac{\rho}{2}\left[U^2 - V_{max}^2\right] = \frac{1}{2}\rho U^2\left[1 - \left(\frac{V_{max}}{U}\right)^2\right]$$

$$= \frac{1}{2} \times 1.2\frac{kg}{m^3} \times (30)^2 \frac{m^2}{s^2}\left[1 - (1.26)^2\right] \times \frac{N \cdot s^2}{kg \cdot m}$$

$p - p_\infty = 317\,N/m^2$ ← p_{gage}

Note: From the plot we see that $V/U = 1.0$, and hence $p = p_\infty$, at $\theta = 113°$. The corresponding r is 1.01 m.

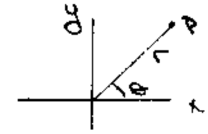


Problem *6.107

Given: Flow field formed by combining a uniform flow in the +x direction ($U = 50\,m/s$) and a sink (of strength, $q = 90\,m^2/s$) at the origin.

Find: the net force per unit depth needed to hold in place (in standard air) the surface shape formed by the stagnation streamline

Solution:



$\psi = \psi_{uf} + \psi_{si} = Uy - \frac{q}{2\pi}\theta = U r \sin\theta - \frac{q}{2\pi}\theta \quad ----(1)$

$u = u_{uf} + u_{si}\ ;\ u_{uf} = U,\ u_{si} = -V_r\cos\theta = -\frac{q}{2\pi r}\frac{x}{r} \qquad \therefore u = U - \frac{q}{2\pi}\frac{x}{r^2}$

$v = v_{uf} + v_{si}\ ;\ v_{uf} = 0,\ v_{si} = -V_r\sin\theta = -\frac{q}{2\pi r}\frac{y}{r^2} \qquad \therefore v = -\frac{q}{2\pi}\frac{y}{r^2}$

$\therefore \vec{V} = u\hat{i} + v\hat{j} = \left(U - \frac{q}{2\pi}\frac{x}{r^2}\right)\hat{i} - \frac{q}{2\pi}\frac{y}{r^2}\hat{j}$

At the stagnation point, $\vec{V} = 0$

$\therefore -\frac{q}{2\pi}\frac{y}{r^2} = 0$, i.e. $y = 0$. Also $U - \frac{q}{2\pi}\frac{x}{r^2} = 0 \quad \therefore U = \frac{q}{2\pi}\frac{x}{(x^2+y^2)}$

and $x_{stag} = \frac{q}{2\pi U} = 90\frac{m^2}{s} \times \frac{1}{2\pi} \times \frac{s}{50\,m} = 0.286\,m$

At stagnation point, $y = 0$ and $\theta = 0$. From eq (1), then $\psi_{stag} = 0$

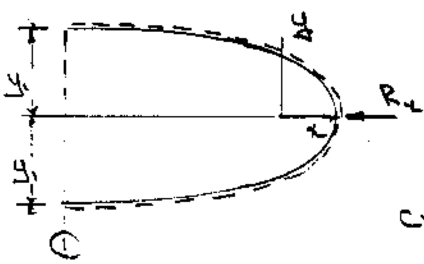
The equation of the stagnation streamline is then,

$\psi = 0 = U r\sin\theta - \frac{q}{2\pi}\theta \quad$ or $\quad r_{stag} = \frac{q\theta}{2\pi U\sin\theta}$

Since $y = r\sin\theta$, then along the stagnation streamline $y = \frac{q\theta}{2\pi U}$.

For upstream, $\theta \to \pi$ and $y = y_1 \to \frac{q}{2U}$.

The surface shape formed by the stagnation streamline is then as follows:



There is no flow across this streamline. The flow in through the left face must be equal to the flow (q) which leaves through the sink at the origin.

Applying the x momentum equation to the CV shown. R_x is force required to hold shape in place.

$$-R_x = \int u\,\rho\vec{V}\cdot d\vec{A} = -U\dot{m}_1 = -U\rho q b$$

$$\therefore \frac{R_x}{b} = \rho q U$$

For standard air $\rho = 1.225\,kg/m^3$ and

$$\frac{R_x}{b} = 1.225\frac{kg}{m^3} \times 90\frac{m^2}{s} \times 50\frac{m}{s} \times \frac{N\cdot s^2}{kg\cdot m} = 5.51\,kN/m$$

$\vec{R}_x/b = -5.51\,\hat{i}\ kN/m$ ← R_x/b

Problem 7.1:

Given: The propagation speed of small amplitude waves in a region of uniform depth is given by

$$c^2 = \left(\frac{\sigma}{\rho}\frac{2\pi}{\lambda} + \frac{g\lambda}{2\pi}\right)\tanh\frac{2\pi h}{\lambda}$$

where h is the depth of the undisturbed liquid
λ is the wavelength.

Find: Obtain the dimensionless groups that characterize the equation. (Use L as a characteristic length and V_0 as a characteristic velocity)

Solution:

$$c^2 = \left(\frac{\sigma}{\rho}\frac{2\pi}{\lambda} + \frac{g\lambda}{2\pi}\right)\tanh\frac{2\pi h}{\lambda}$$

To nondimensionalize the equation, all lengths are divided by L and all velocities are divided by V_0.

Denoting nondimensional quantities by an asterisk, then

$$\lambda^* = \frac{\lambda}{L} \qquad h^* = \frac{h}{L} \qquad c^* = \frac{c}{V_0}$$

then

$$c^{*2} V_0^2 = \left(\frac{\sigma}{\rho}\frac{2\pi}{L\lambda^*} + \frac{g\lambda^* L}{2\pi}\right)\tanh\frac{2\pi h^* L}{\lambda^* L}$$

$$c^{*2} = \left(\frac{\sigma}{\rho L V_0^2}\frac{2\pi}{\lambda^*} + \frac{gL}{V_0^2}\frac{\lambda^*}{2\pi}\right)\tanh\frac{2\pi h^*}{\lambda^*}$$

$\therefore$ Dimensionless groups are $\dfrac{\sigma}{\rho L V_0^2}$, $\dfrac{gL}{V_0^2}$ ←

Problem 7.2

Given: The slope of the free surface of a steady wave in one-dimensional flow in a shallow liquid layer is described by the equation

$$\frac{\partial h}{\partial x} = -\frac{u}{g}\frac{\partial u}{\partial x}$$

Find: Nondimensionalize the equation (using length scale, L, and velocity scale, V_0)
Obtain the dimensionless groups that characterize this flow.

Solution:

To nondimensionalize the equation, all lengths are divided by the reference length, L, and all vecolities are divided by the reference velocity, V_0.

Denoting the nondimensional quantities by an asterisk,

$$h^* = \frac{h}{L} \quad , \quad x^* = \frac{x}{L} \quad , \quad u^* = \frac{u}{V_0}$$

Substituting into the governing equation

$$\frac{\partial(h^* L)}{\partial(x^* L)} = -\frac{V_0 u^*}{g}\frac{\partial(V_0 u^*)}{\partial(L x^*)}$$

$$\frac{\partial h^*}{\partial x^*} = -\frac{V_0^2}{gL}\frac{\partial u^*}{\partial x^*}$$

The dimensionless group is $\frac{V_0^2}{gL}$. This is the square of the Froude number.

Problem 7.3

Given: One-dimensional, unsteady flow in a thin liquid layer is described by the equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -g\frac{\partial h}{\partial x}$$

Find: Nondimensionalize the equation (using length scale, L, and velocity scale, V_0)
Obtain the dimensionless groups that characterize this flow.

Solution:

To nondimensionalize the equation, all lengths are divided by the reference length, L, velocity is divided by the reference velocity, V_0, and time is divided by the ratio, L/V_0

Denoting the nondimensional quantities by an asterisk,

$$x^* = \frac{x}{L}, \quad h^* = \frac{h}{L}, \quad u^* = \frac{u}{V_0}, \quad t^* = \frac{t}{L/V_0}$$

Substituting into the governing equation

$$\frac{\partial(V_0 u^*)}{\partial(Lt^*/V_0)} + u^* V_0 \frac{\partial(V_0 u^*)}{\partial(x^* L)} = -g\frac{\partial(h^* L)}{\partial(x^* L)}$$

$$\frac{V_0^2}{L}\frac{\partial u^*}{\partial t^*} + \frac{V_0^2}{L}u^*\frac{\partial u^*}{\partial x^*} = -g\frac{\partial h^*}{\partial x^*}$$

Multiplying through by L/V_0^2,

$$\frac{\partial u^*}{\partial t^*} + u^*\frac{\partial u^*}{\partial x^*} = -\frac{gL}{V_0^2}\frac{\partial h^*}{\partial x^*}$$

The dimensionless group is $\frac{gL}{V_0^2}$. This is one over the square of the Froude number.

Problem 7.4

Given: For steady, incompressible, two-dimensional flow, the Prandtl boundary layer equations are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \text{-------- (1)}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \quad \text{...(2)}$$

Find: Nondimensionalize these equations (using L and V_0 as characteristic length and velocity, respectively) and identify the resulting similarity parameters.

Solution:

Denoting nondimensional quantities by an asterisk.

$$x^* = \frac{x}{L}, \quad y^* = \frac{y}{L} \quad u^* = \frac{u}{V_0} \quad v^* = \frac{v}{V_0}$$

Substituting into Eq. 1, we obtain

$$\frac{\partial(u^* V_0)}{\partial(x^* L)} + \frac{\partial(v^* V_0)}{\partial(y^* L)} = 0 = \frac{V_0}{L}\frac{\partial u^*}{\partial x^*} + \frac{V_0}{L}\frac{\partial v^*}{\partial y^*}$$

or

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0$$

Consider each term in Eq. 2.

$$u\frac{\partial u}{\partial x} = u^* V_0 \frac{\partial(u^* V_0)}{\partial(x^* L)} = \frac{V_0^2}{L} u^* \frac{\partial u^*}{\partial x^*}$$

$$v\frac{\partial u}{\partial y} = v^* V_0 \frac{\partial(u^* V_0)}{\partial(y^* L)} = \frac{V_0^2}{L} v^* \frac{\partial u^*}{\partial x^*}$$

Leave 3rd term as is for the moment

$$\nu\frac{\partial^2 u}{\partial y^2} = \nu\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \nu\frac{\partial}{\partial y}\left(\frac{\partial u^* V_0}{\partial y^* L}\right) = \nu\frac{V_0}{L}\frac{\partial}{\partial y}\left(\frac{\partial u^*}{\partial y^*}\right) = \nu\frac{V_0}{L}\frac{\partial}{\partial(y^* L)}\frac{\partial u^*}{\partial y^*} = \nu\frac{V_0}{L^2}\frac{\partial^2 u^*}{\partial y^{*2}}$$

Substituting into Eq. 2.

$$\frac{V_0^2}{L} u^* \frac{\partial u^*}{\partial x^*} + \frac{V_0^2}{L} v^* \frac{\partial u^*}{\partial y^*} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{V_0}{L^2}\frac{\partial^2 u^*}{\partial y^{*2}}$$

Multiplying through by $\frac{L}{V_0^2}$

$$u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = -\frac{L}{\rho V_0^2}\frac{\partial p}{\partial x} + \frac{\nu}{V_0 L}\frac{\partial^2 u^*}{\partial y^{*2}} = -\frac{L\,\partial\left(\frac{p}{\rho V_0^2}\right)}{\partial(x^* L)} + \frac{\nu}{V_0 L}\frac{\partial^2 u^*}{\partial y^{*2}}$$

Define the non-dimensional pressure $p^* = \frac{p}{\rho V_0^2}$, then

$$u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y} = -\frac{\partial p^*}{\partial x^*} + \frac{\nu}{V_0 L}\frac{\partial^2 u^*}{\partial y^{*2}}$$

The similarity parameter is $\frac{\nu}{V_0 L} = \frac{1}{Re}$

Problem 7.5

The equation describing motion of fluid in a pipe due to an applied pressure gradient, when the flow starts from rest, is

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right)$$

Use the average velocity $\bar{V}$, pressure drop Δp, pipe length L, and diameter D to nondimensionalize this equation. Obtain the dimensionless groups that characterize this flow.

Nondimensionalizing the velocity, pressure, spatial measures, and time:

$$u^* = \frac{u}{\overline{V}} \qquad p^* = \frac{p}{\Delta p} \qquad x^* = \frac{x}{L} \qquad r^* = \frac{r}{L} \qquad t^* = t\frac{\overline{V}}{L}$$

Hence

$$u = \overline{V}\,u^* \qquad p = \Delta p\; p^* \qquad x = L\,x^* \qquad r = D\,r^* \qquad t = \frac{L}{\overline{V}}t^*$$

Substituting into the governing equation

$$\frac{\partial u}{\partial t} = \overline{V}\frac{\overline{V}}{L}\frac{\partial u^*}{\partial t^*} = -\frac{1}{\rho}\Delta p\frac{1}{L}\frac{\partial p^*}{\partial x^*} + \nu\overline{V}\frac{1}{D^2}\left(\frac{\partial^2 u^*}{\partial r^{*2}} + \frac{1}{r^*}\frac{\partial u^*}{\partial r^*}\right)$$

The final dimensionless equation is

$$\frac{\partial u^*}{\partial t^*} = -\frac{\Delta p}{\rho\overline{V}^2}\frac{\partial p^*}{\partial x^*} + \left(\frac{\nu}{D\overline{V}}\right)\left(\frac{L}{D}\right)\left(\frac{\partial^2 u^*}{\partial r^{*2}} + \frac{1}{r^*}\frac{\partial u^*}{\partial r^*}\right)$$

The dimensionless groups are

$$\frac{\Delta p}{\rho\overline{V}^2} \qquad \frac{\nu}{D\overline{V}} \qquad \frac{L}{D}$$

Problem 7.6

In atmospheric studies the motion of the earth's atmosphere can sometimes be modeled with the equation

$$\frac{D\vec{V}}{Dt} + 2\vec{\Omega} \times \vec{V} = -\frac{1}{\rho}\nabla p$$

where $\vec{V}$ is the large-scale velocity of the atmosphere across the earth's surface, ∇p is the climatic pressure gradient, and $\vec{\Omega}$ is the earth's angular velocity. What is the meaning of the term $\vec{\Omega} \times \vec{V}$? Use the pressure difference, Δp, and typical length scale, L (which could, for example, be the magnitude of, and distance between, an atmospheric high and low, respectively), to nondimensionalize this equation. Obtain the dimensionless groups that characterize this flow.

Recall that the total acceleration is

$$\frac{D\vec{V}}{Dt} = \frac{\partial \vec{V}}{\partial t} + \vec{V} \cdot \nabla \vec{V}$$

Nondimensionalizing the velocity vector, pressure, angular velocity, spatial measure, and time, (using a typical velocity magnitude V and angular velocity magnitude Ω):

$$\vec{V}^* = \frac{\vec{V}}{V} \qquad p^* = \frac{p}{\Delta p} \qquad \vec{\Omega}^* = \frac{\vec{\Omega}}{\Omega} \qquad x^* = \frac{x}{L} \qquad t^* = t\frac{V}{L}$$

Hence

$$\vec{V} = V\,\vec{V}^* \qquad p = \Delta p\; p^* \qquad \vec{\Omega} = \Omega\vec{\Omega}^* \qquad x = L\,x^* \qquad t = \frac{L}{V}t^*$$

Substituting into the governing equation

$$V\frac{V}{L}\frac{\partial \vec{V}^*}{\partial t^*} + V\frac{V}{L}\vec{V}^* \cdot \nabla^* \vec{V}^* + 2\Omega V\,\vec{\Omega}^* \times \vec{V}^* = -\frac{1}{\rho}\frac{\Delta p}{L}\nabla p^*$$

The final dimensionless equation is

$$\frac{\partial \vec{V}^*}{\partial t^*} + \vec{V}^* \cdot \nabla^* \vec{V}^* + 2\left(\frac{\Omega L}{V}\right)\vec{\Omega}^* \times \vec{V} = -\frac{\Delta p}{\rho V^2}\nabla p^*$$

The dimensionless groups are

$$\frac{\Delta p}{\rho \overline{V}^2} \qquad \frac{\Omega L}{V}$$

The second term on the left of the governing equation is the Coriolis force due to a rotating coordinate system. This is a very significant term in atmospheric studies, leading to such phenomena as geostrophic flow.

Problem 7.7

Given: At low speeds, drag is independent of fluid density.

$$F = F(\mu, V, D)$$

Find: Appropriate dimensionless parameters.

Solution: Apply Buckingham Π procedure.

① $F \quad \mu \quad V \quad D$ $\qquad n = 4$ parameters

② Select primary dimensions M, L, t.

③

F	μ	V	D
$\frac{ML}{t^2}$	$\frac{M}{Lt}$	$\frac{L}{t}$	L

$r = 3$ primary dimensions

④ μ, V, D $\quad m = r = 3$ repeating parameters

⑤ Then $n - m = 1$ dimensionless group will result. Setting up a dimensional equation,

$$\Pi_1 = \mu^a V^b D^c F$$

$$= \left(\frac{M}{Lt}\right)^a \left(\frac{L}{t}\right)^b (L)^c \frac{ML}{t^2} = M^0 L^0 t^0$$

Summing exponents,

$$\begin{aligned} M&: \quad a + 1 = 0 \\ L&: \quad -a + b + c + 1 = 0 \\ t&: \quad -a - b - 2 = 0 \end{aligned} \quad \left| \quad \begin{aligned} a &= -1 \\ c &= -1 \\ b &= -1 \end{aligned} \right. \qquad \therefore \Pi_1 = \frac{F}{\mu V D}$$

Π_1

⑥ Check, using F, L, t primary dimensions.

$$\Pi_1 = F \frac{L^2}{Ft} \frac{t}{L} \frac{1}{L} = [1] \quad \checkmark$$

Since the procedure produces only one dimensionless group, it must be a constant. Thus

$$\Pi_1 = \frac{F}{\mu V D} \quad \text{or} \quad F \propto \mu V D$$

Problem 7.8

At relatively high speeds the drag on an object is independent of fluid viscosity. Thus the aerodynamic drag force, F, on an automobile, is a function only of speed, V, air density ρ, and vehicle size, characterized by its frontal area A. Use dimensional analysis to determine how the drag force F depends on the speed V.

Given: That drag depends on speed, air density and frontal area

Find: How drag force depend on speed

Apply the Buckingham Π procedure

① $F \quad V \quad \rho \quad A$ — $n = 4$ parameters

② Select primary dimensions M, L, t

③

F	V	ρ	A
$\dfrac{ML}{t^2}$	$\dfrac{L}{t}$	$\dfrac{M}{L^3}$	L^2

$r = 3$ primary dimensions

④ $V \quad \rho \quad A$ — $m = r = 3$ repeat parameters

⑤ Then $n - m = 1$ dimensionless groups will result. Setting up a dimensional equation,

$$\Pi_1 = V^a \rho^b A^c F$$

$$= \left(\frac{L}{t}\right)^a \left(\frac{M}{L^3}\right)^b \left(L^2\right)^c \frac{ML}{t^2} = M^0 L^0 t^0$$

Summing exponents,

$$\begin{array}{lc|l} M: & b+1=0 & b=-1 \\ L: & a-3b+2c+1=0 & c=-1 \\ t: & -a-2=0 & a=-2 \end{array}$$

Hence

$$\Pi_1 = \frac{F}{\rho V^2 A}$$

⑥ Check using *F*, *L*, *t* as primary dimensions

$$\Pi_1 = \frac{F}{\frac{Ft^2}{L^4}\frac{L^2}{t^2}L^2} = [1]$$

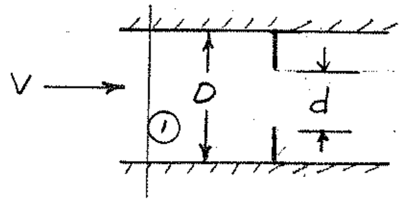
The relation between drag force *F* and speed *V* must then be

$$F \propto \rho V^2 A \propto V^2$$

The drag is proportional to the *square* of the speed.

Given: Flow through an orifice plate

$$\Delta p = p_1 - p_2 = f(\rho, \mu, V, D, d)$$



Find: Dimensionless parameters.

Solution: Choose ρ, V, and D as repeating variables.

① $\Delta p \quad \rho \quad \mu \quad V \quad D \quad d$ $\qquad n = 6$ parameters

② Select primary dimensions M, L, t

③ $\Delta p \quad \rho \quad \mu \quad V \quad D \quad d$

$\frac{M}{Lt^2} \quad \frac{M}{L^3} \quad \frac{M}{Lt} \quad \frac{L}{t} \quad L \quad L$ $\qquad r = 3$ primary dimensions

④ ρ, V, D $\qquad m = r = 3$ repeating parameters

⑤ Then $n - m = 3$ dimensionless groups will result. Setting up dimensional equations,

$$\Pi_1 = \rho^a V^b D^c \Delta p = \left(\frac{M}{L^3}\right)^a \left(\frac{L}{t}\right)^b L^c \left(\frac{M}{Lt^2}\right) = M^0 L^0 t^0$$

Summing exponents,

$M: a + 1 = 0 \qquad a = -1$
$L: -3a + b + c - 1 = 0$
$t: -b - 2 = 0 \qquad b = -2$

$c = 1 - b + 3a = 0$

$$\therefore \Pi_1 = \frac{\Delta p}{\rho V^2}$$

$$\Pi_2 = \rho^a V^b D^c \mu = \left(\frac{M}{L^3}\right)^a \left(\frac{L}{t}\right)^b L^c \left(\frac{M}{Lt}\right) = M^0 L^0 t^0$$

Summing exponents,

$M: a + 1 = 0 \qquad a = -1$
$L: -3a + b + c - 1 = 0$
$t: -b - 1 = 0 \qquad b = -1$

$c = 1 - b + 3a = -1$

$$\therefore \Pi_2 = \frac{\mu}{\rho V D}$$

$$\Pi_3 = \rho^a V^b D^c d = \left(\frac{M}{L^3}\right)^a \left(\frac{L}{t}\right)^b L^c L = M^0 L^0 t^0$$

$M: a + 0 = 0 \qquad a = 0$
$L: -3a + b + c + 1 = 0$
$t: -b + 0 = 0 \qquad b = 0$

$\Rightarrow c = -1 \; ; \; \Pi_3 = \frac{d}{D}$

Thus $\Pi_1 = f(\Pi_2, \Pi_3)$ or $\frac{\Delta p}{\rho V^2} = f\left(\frac{\mu}{\rho V D}, \frac{d}{D}\right)$ $\qquad \frac{\Delta p}{\rho V^2}$

⑥ Check, using F, L, t $\quad \Pi_1 = \frac{F}{L^2}\frac{L^4}{Ft^2}\frac{t^2}{L^2} = [1]$ ✓, $\Pi_2 = Re = [1]$ ✓, $\Pi_3 = \frac{L}{L} = [1]$ ✓

Problem 7.10

Given: The boundary layer thickness, δ, on a smooth flat plate in incompressible flow without pressure gradient is a function of U (free stream velocity), ρ, μ, and x (distance)

Find: suitable dimensionless parameters

Solution: Apply Buckingham π-theorem

① $\delta \quad U \quad \rho \quad \mu \quad x$ $\qquad n = 5$ parameters

② Select M, L, t as primary dimensions

③ $\delta \quad U \quad \rho \quad \mu \quad x$

$L \quad \frac{L}{t} \quad \frac{M}{L^3} \quad \frac{M}{Lt} \quad L$ $\qquad r = 3$ primary dimensions

④ ρ, U, x $\qquad m = r = 3$ repeating parameters

⑤ Then $n - m = 2$ dimensionless groups will result.

Setting up dimensional equations.

$$\pi_1 = \rho^a U^b x^c \delta$$

$$M^0 L^0 t^0 = \left(\frac{M}{L^3}\right)^a \left(\frac{L}{t}\right)^b L^c L$$

Equating exponents,

M: $0 = a$ $\quad \therefore a = 0$

L: $0 = -3a + b + c + 1$ $\quad c = -1$

t: $0 = -b$ $\quad \therefore b = 0$

$$\therefore \pi_1 = \frac{\delta}{x}$$

$$\pi_2 = \rho^a U^b x^c \mu$$

$$M^0 L^0 t^0 = \left(\frac{M}{L^3}\right)^a \left(\frac{L}{t}\right)^b L^c \frac{M}{Lt}$$

Equating exponents,

M: $0 = a + 1$ $\quad \therefore a = -1$

L: $0 = -3a + b + c - 1$ $\quad c = -1$

t: $0 = -b - 1$ $\quad \therefore b = -1$

$$\pi_2 = \frac{\mu}{\rho U x}$$

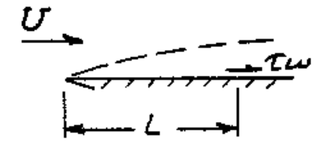
and $\frac{\delta}{x} = f\left(\frac{\rho U x}{\mu}\right)$ $\qquad \frac{\delta}{x}$

⑥ Check using F, L, t dimensions

$$\pi_1 = \frac{L}{L} = [1] \checkmark \qquad \pi_2 = \frac{Ft}{L^2} \cdot \frac{L^4}{Ft^2} \frac{t}{L} \frac{1}{L} = [1] \checkmark$$

Problem 7.11

Given: Wall shear stress, τ_w, in a boundary layer, depends on ρ, μ, L, and U.



Find: (a) Dimensionless groups.
(b) Express the functional relationship.

Solution: Step ① $\tau_w \quad \rho \quad \mu \quad L \quad U \qquad n=5$

Step ② Choose M, L, t. $\tau_w = \frac{F}{L^2} \times \frac{ML}{Ft^2} = \frac{M}{Lt^2}$

Step ③ $\frac{M}{Lt^2} \quad \frac{M}{L^3} \quad \frac{M}{Lt} \quad L \quad \frac{L}{t} \qquad r=3$

Step ④ Select ρ, L, U

Step ⑤ $\Pi_1 = \tau_w \rho^a L^b U^c = \frac{M}{Lt^2}\left(\frac{M}{L^3}\right)^a (L)^b \left(\frac{L}{t}\right)^c = M^0L^0t^0$

$M: 0 = 1 + a \qquad a = -1$
$L: 0 = -1 - 3a + b + c \qquad b = 3a - c + 1 = 0$
$t: 0 = -2 - c \qquad c = -2$

$\Pi_1 = \frac{\tau_w}{\rho U^2}$ — Π_1

$\Pi_2 = \mu \rho^a L^b U^c = \frac{M}{Lt}\left(\frac{M}{L^3}\right)^a (L)^b \left(\frac{L}{t}\right)^c = M^0L^0t^0$

$M: 0 = 1 + a \qquad a = -1$
$L: 0 = -1 - 3a + b + c \qquad b = 3a - c + 1 = -1$
$t: 0 = -1 - c \qquad c = -1$

$\Pi_2 = \frac{\mu}{\rho U L}$ — Π_2

Step ⑥: Check using F, L, t: $\rho = \frac{M}{L^3} \times \frac{Ft^2}{ML} = \frac{Ft^2}{L^4}$

$$\Pi_1 = \frac{\tau_w}{\rho U^2} = \frac{F}{L^2}\,\frac{L^4}{Ft^2}\,\frac{t^2}{L^2} = \frac{FL^4t^2}{FL^4t^2} = 1 \quad ✓✓$$

$$\Pi_2 = \frac{\mu}{\rho U L} = \frac{Ft}{L^2}\,\frac{L^4}{Ft^2}\,\frac{t}{L}\,\frac{1}{L} = \frac{FL^4t^2}{FL^4t^2} = 1 \quad ✓✓$$

The functional relationship is

$$\Pi_1 = f(\Pi_2)$$

f

Problem 7.12

Given: The mean velocity, $\bar{u}$, for turbulent pipe or boundary layer flow, may be correlated in terms of the wall shear stress, τ_w, distance from the wall, y, and fluid properties, ρ and μ.

Find: (a) dimensionless parameter containing $\bar{u}$ and one containing y that are suitable for organizing experimental data.
(b) show that the result may be written as

$$\frac{\bar{u}}{u_*} = f\left(\frac{y u_*}{\nu}\right) \quad \text{where } u_* = (\tau_w/\rho)^{1/2}$$

Solution: Apply the Buckingham π-Theorem

① $\bar{u}$ $\quad \tau_w$ $\quad y$ $\quad \rho$ $\quad \mu$ $\qquad n = 5$ parameters

② Select M, L, t as primary dimensions

③ $\frac{L}{t}$ $\quad \frac{M}{Lt^2}$ $\quad L$ $\quad \frac{M}{L^3}$ $\quad \frac{M}{Lt}$

④ τ_w, y, ρ $\qquad m = r = 3$ repeating parameters

⑤ Then $n - m = 2$ dimensionless groups will result

Setting up dimensional equations

$$\pi_1 = \tau_w^a\, y^b\, \rho^c\, \bar{u} \qquad\qquad \pi_2 = \tau_w^a\, y^b\, \rho^c\, \mu$$

$$M^0L^0t^0 = \left(\frac{M}{Lt^2}\right)^a L^b \left(\frac{M}{L^3}\right)^c \frac{L}{t} \qquad\qquad M^0L^0t^0 = \left(\frac{M}{Lt^2}\right)^a L^b \left(\frac{M}{L^3}\right)^c \frac{M}{Lt}$$

Summing exponents (for π_1):

M: $a + c = 0 \quad \therefore a = -c$

L: $-a + b - 3c + 1 = 0$

t: $-2a - 1 = 0 \quad \therefore a = -1/2$

$a = -1/2,\ c = 1/2,\ b = 0$

$$\pi_1 = \bar{u}\,\frac{\rho^{1/2}}{\tau_w^{1/2}} = \frac{\bar{u}}{\sqrt{\tau_w/\rho}}$$

Summing exponents (for π_2):

M: $a + c + 1 = 0 \quad \therefore c = -a - 1$

L: $-a + b - 3c - 1 = 0$

t: $-2a - 1 = 0 \quad \therefore a = -1/2$

$a = -1/2,\ c = -1/2,\ b = -1$

$$\pi_2 = \frac{\mu}{\tau_w^{1/2}\rho^{1/2}y} = \frac{\mu}{\rho y\sqrt{\frac{\tau_w}{\rho}}}$$

$$\pi_1 = f(\pi_2) \quad \text{or} \quad \frac{\bar{u}}{\sqrt{\tau_w/\rho}} = f\left(\frac{\mu}{\rho y\sqrt{\tau_w/\rho}}\right)$$

Since $\sqrt{\tau_w/\rho} = u_*$, then

$$\frac{\bar{u}}{u_*} = f\left(\frac{\mu}{\rho y u_*}\right) = f\left(\frac{\nu}{y u_*}\right) = g\left(\frac{y u_*}{\nu}\right) \longleftarrow \frac{\bar{u}}{u_*}$$

Problem 7.13

Given: Velocity, V, of a free surface gravity wave in deep water is a function of λ (wavelength) D, ρ, and g

Find: Dependence of V on other variables.

Solution: Apply Buckingham Π-theorem

① $V \quad \lambda \quad D \quad \rho \quad g$ $\qquad n = 5$ parameters

② Select M, L, t as primary dimensions

③ $V \quad \lambda \quad D \quad \rho \quad g$

$\frac{L}{t} \quad L \quad L \quad \frac{M}{L^3} \quad \frac{L}{t^2}$ $\qquad r = 3$ primary dimensions

④ ρ, D, g $\qquad m = r = 3$ repeating parameters

⑤ Then $n - m = 2$ dimensionless groups will result

Setting up dimensional equations

$\Pi_1 = \rho^a D^b g^c V$

$M^0 L^0 t^0 = \left(\frac{M}{L^3}\right)^a L^b \left(\frac{L}{t^2}\right)^c \frac{L}{t}$

Summing exponents,

M: $a = 0$

L: $-3a + b + c + 1 = 0$

t: $-2c - 1 = 0$

i.e. $a = 0$

$c = -\frac{1}{2}$

$b = 3a - c - 1 = -\frac{1}{2}$

$\therefore \Pi_1 = \frac{V}{\sqrt{gD}}$

$\Pi_2 = \rho^a D^b g^c \lambda$

$M^0 L^0 t^0 = \left(\frac{M}{L^3}\right)^a L^b \left(\frac{L}{t^2}\right)^c L$

Summing exponents,

M: $a = 0$

L: $-3a + b + c + 1 = 0$

t: $-2c = 0$

i.e. $a = 0$

$c = 0$

$b = 3a - c - 1 = -1$

$\therefore \Pi_2 = \frac{\lambda}{D}$

Thus $\frac{V}{\sqrt{gD}} = f\left(\frac{\lambda}{D}\right)$ or $V = \sqrt{gD}\, f\left(\frac{\lambda}{D}\right)$ $\qquad V$

⑥ Check using F, L, t

$\Pi_1 = \frac{L}{t} \cdot \frac{1}{\left(\frac{L}{t^2} L\right)^{1/2}} = [1]$ ✓ $\qquad \Pi_2 = \frac{L}{L} = [1]$

Problem 7.14

Given: Volume flow rate, Q, over a weir is a function of: upstream height, h, gravity, g, and channel width, b.

Find: Expression for Q (using dimensional analysis)

Solution: Apply Buckingham π - theorem

① List Q h g b $n = 4$ parameters

② Choose F, L, t as primary dimensions

③ Dimensions $\frac{L^3}{t}$ L $\frac{L}{t^2}$ L

④ Repeating variables g, h $m = r = 2$

⑤ Then $n - m = 2$ dimensionless groups will result

Setting up dimensional equations

$$\pi_1 = g^a h^b Q \qquad \pi_2 = g^a h^c b$$

$$L^0 t^0 = \left(\frac{L}{t^2}\right)^a L^b \left(\frac{L^3}{t}\right) \qquad L^0 t^0 = \left(\frac{L}{t^2}\right)^a L^c L$$

Equating exponents

L: $0 = a + b + 3$

t: $0 = -2a - 1$

$\therefore a = -\frac{1}{2}$

$b = -2\frac{1}{2}$

$$\therefore \pi_1 = \frac{Q}{g^{1/2} h^{2.5}}$$

$$\pi_1 = \frac{Q}{h^2\sqrt{gh}}$$

Equating exponents

L: $0 = a + c + 1$

t: $0 = -2a$

$\therefore a = 0$

$c = -1$

$$\pi_2 = \frac{b}{h}$$

(this is obvious by inspection)

Then

$$\frac{Q}{h^2\sqrt{gh}} = f\left(\frac{b}{h}\right)$$

$$Q = h^2\sqrt{gh}\; f\left(\frac{b}{h}\right) \longleftarrow Q$$

Problem 7.15

Given: Load-carrying capacity, W, (of a journal bearing) depends on: diameter, D; length, l; clearance, c; angular speed, ω; lubricant viscosity, μ

Find: Dimensionless parameters that characterize the problem.

Solution: Apply Buckingham π - theorem

① List W D l c ω μ n=6 parameters

② Choose F, L, t as primary dimensions

③ Dimensions F L L L $\frac{1}{t}$ $\frac{Ft}{L^2}$

④ Repeating variables D, ω, μ m=r=3

⑤ Then n-m = 3 dimensionless groups will result

By inspection, $\pi_1 = \frac{l}{D}$ $\pi_2 = \frac{c}{D}$

Set up dimensional equation to determine π_3

$$\pi_3 = D^a \omega^b \mu^e W$$

$$F^0 L^0 t^0 = L^a \left(\frac{1}{t}\right)^b \left(\frac{Ft}{L^2}\right)^e F$$

Equating exponents: F $0 = e+1$ $\therefore e = -1$

L $0 = a - 2e$ $\therefore a = -2$

t $0 = -b + e$ $\therefore b = -1$

and

$$\pi_3 = \frac{W}{D^2 \omega \mu}$$

⑥ Check using M, L, t dimensions

$$\pi_3 = \frac{ML}{t^2} \times \frac{1}{L^2} \times t \times \frac{Lt}{M} = [1] \checkmark$$

$$\therefore \frac{W}{D^2 \omega \mu} = f\left(\frac{l}{D}, \frac{c}{D}\right)$$

W

Problem 7.16

Given: Capillary waves form on a liquid free surface. The speed of the wave is a function of σ (surface tension), λ (the wave length) and ρ

Find: The wave speed as a function of the variables

Solution: Apply Buckingham π - theorem

① $V \quad \sigma \quad \lambda \quad \rho$ $\qquad n = 4$ parameters

② Select M, L, t as primary dimensions

③ $V \quad \sigma \quad \lambda \quad \rho$

$\frac{L}{t} \quad \frac{M}{t^2} \quad L \quad \frac{M}{L^3}$ $\qquad r = 3$ primary dimensions

④ σ, λ, ρ $\qquad m = r = 3$ repeating parameters

⑤ Then $n - m = 1$ dimensionless group will result

Setting up dimensional equation

$$\pi_1 = \sigma^a \lambda^b \rho^c V$$

$$M^0 L^0 t^0 = \left(\frac{M}{t^2}\right)^a L^b \left(\frac{M}{L^3}\right)^c \frac{L}{t}$$

Summing exponents

M: $a + c = 0$ $\qquad c = -a = \frac{1}{2}$

L: $b - 3c + 1 = 0$ $\qquad b = 3c - 1 = \frac{1}{2}$

t: $-2a - 1 = 0$ $\quad \therefore \quad a = -\frac{1}{2}$

$$\therefore \pi_1 = \left(\frac{\rho\lambda}{\sigma}\right)^{\frac{1}{2}} V = \text{constant} \qquad \therefore V \propto \sqrt{\frac{\sigma}{\rho\lambda}}$$

V

⑥ Check using F, L, t

$$\pi_1 = \left(\frac{Ft^2}{L^4} \cdot L \cdot \frac{L}{F}\right)^{1/2} \frac{L}{t} = [1] \checkmark$$

Problem 7.17 (In Excel)

The time, t, for oil to drain out of a viscosity calibration container depends on the fluid viscosity, μ, and density, ρ, the orifice diameter, d, and gravity, g. Use dimensional analysis to find the functional dependence of t on the other variables. Express t in the simplest possible form.

Given: That drain time depends on fluid viscosity and density, orifice diameter, and gravity

Find: Functional dependence of t on other variables

Solution

We will use the workbook of Example Problem 7.1, modified for the current problem

The number of parameters is:	$n = 5$
The number of primary dimensions is:	$r = 3$
The number of repeat parameters is:	$m = r = 3$
The number of Π groups is:	$n - m = 2$

Enter the dimensions (**M**, **L**, **t**) of
the repeating parameters, and of up to
four other parameters (for up to four Π groups).
The spreadsheet will compute the exponents a, b, and c for each.

REPEATING PARAMETERS: **Choose ρ, g, d**

	M	L	t
ρ	1	-3	
g		1	-2
d		1	

Π GROUPS:

	M	L	t
t	0	0	1

	M	L	t
μ	1	-1	-1

Π_1:

$a =$	**0**
$b =$	**0.5**
$c =$	**-0.5**

Π_2:

$a =$	**-1**
$b =$	**-0.5**
$c =$	**-1.5**

The following Π groups from Example Problem 7.1 are not used:

		M	L	t			M	L	t
		0	0	0			0	0	0
Π_3:	$a =$		**0**		Π_4:	$a =$		**0**	
	$b =$		**0**			$b =$		**0**	
	$c =$		**0**			$c =$		**0**	

Hence $\Pi_1 = t\sqrt{\frac{g}{d}}$ and $\Pi_2 = \frac{\mu}{\rho g^{\frac{1}{2}} d^{\frac{3}{2}}} \rightarrow \frac{\mu^2}{\rho^2 g d^3}$ with $\Pi_1 = f(\Pi_2)$

The final result is $t = \sqrt{\frac{d}{g}} f\left(\frac{\mu^2}{\rho^2 g d^3}\right)$

Problem 7.18

Given: Power per unit cross-sectional area, E, transmitted by a sound wave, depends on wave speed, V, amplitude, r, frequency, n, and medium density, ρ.

Find: General form of dependence of E on the other variables.

Solution: Step ① $E \quad V \quad r \quad n \quad \rho \qquad n = 5$

Step ② Choose M, L, t. $E = \frac{P}{L^2} = \frac{FL}{t} \times \frac{1}{L^2} = \frac{F}{Lt} \times \frac{ML}{Ft^2} = \frac{M}{t^3}$

Step ③ $\frac{M}{t^3} \quad \frac{L}{t} \quad L \quad \frac{1}{t} \quad \frac{M}{L^3} \qquad r = 3$

Step ④ Choose ρ, V, r

Step ⑤ $\Pi_1 = \rho^a V^b r^c E = \left(\frac{M}{L^3}\right)^a \left(\frac{L}{t}\right)^b (L)^c \frac{M}{t^3} = M^0 L^0 t^0$

$$\left.\begin{aligned} &M: a + 1 = 0 && a = -1 \\ &L: -3a + b + c = 0 && c = 3a - b = 3(-1) - (-3) = 0 \\ &t: -b - 3 = 0 && b = -3 \end{aligned}\right\} \Pi_1 = \frac{E}{\rho V^3}$$

Π_1

$\Pi_2 = \rho^a V^b r^c n = \left(\frac{M}{L^3}\right)^a \left(\frac{L}{t}\right)^b (L)^c \frac{1}{t} = M^0 L^0 t^0$

$$\left.\begin{aligned} &M: a + 0 = 0 && a = 0 \\ &L: -3a + b + c = 0 && c = 3a - b = 3(0) - (-1) = 1 \\ &t: -b - 1 = 0 && b = -1 \end{aligned}\right\} \Pi_2 = \frac{nr}{V}$$

Π_2

Step ⑥ Check using FLt: $\rho = \frac{M}{L^3} \times \frac{Ft^2}{ML} = \frac{Ft^2}{L^4}$

$$\Pi_1 = \frac{E}{\rho V^3} = \frac{FL}{tL^2} \frac{L^4}{Ft^2} \frac{t^3}{L^3} = \frac{FL^5 t^3}{FL^5 t^3} = 1 \quad ✓✓$$

$$\Pi_2 = \frac{nr}{V} = \frac{1}{t} L \times \frac{t}{L} = \frac{Lt}{Lt} = 1 \quad ✓✓$$

Problem 7.19

Given: Power, $\mathcal{P}$, required to drive a fan depends on ρ, Q, D and ω.

Find: Dependence of $\mathcal{P}$ on other parameters.

Solution: Apply Buckingham Π procedure.

① $\mathcal{P}$ ρ Q D ω $n=5$ parameters

② Choose primary dimensions M, L, t

③ $\mathcal{P}$ ρ Q D ω

$\frac{ML^2}{t^3}$ $\frac{M}{L^3}$ $\frac{L^3}{t}$ L $\frac{1}{t}$ $r=3$ primary dimensions

④ ρ, D, ω $m=r=3$ repeating parameters

⑤ Then $n-m=2$ dimensionless groups will result. Setting up dimensional equations,

$$\Pi_1 = \rho^a D^b \omega^c \mathcal{P} = \left(\frac{M}{L^3}\right)^a (L)^b \left(\frac{1}{t}\right)^c \left(\frac{ML^2}{t^3}\right) = M^0L^0t^0$$

Summing exponents,

$M: a+1=0$ $a=-1$
$L: -3a+b+2=0$ $b=-5$
$t: -c-3=0$ $c=-3$

$$\therefore \Pi_1 = \frac{\mathcal{P}}{\rho D^5 \omega^3}$$

$$\Pi_2 = \rho^d D^e \omega^f Q = \left(\frac{M}{L^3}\right)^d (L)^e \left(\frac{1}{t}\right)^f \left(\frac{L^3}{t}\right) = M^0L^0t^0$$

Summing exponents,

$M: d+0=0$ $d=0$
$L: -3d+e+3=0$ $e=-3$
$t: -f-1=0$ $f=-1$

$$\therefore \Pi_2 = \frac{Q}{D^3\omega}$$

⑥ Check using primary dimensions F, L, t

$$\Pi_1 = \frac{FL}{t}\frac{L^4}{Ft^2}\frac{1}{L^5}t^3 = [1] \checkmark \qquad \Pi_2 = \frac{L^3}{t}\frac{1}{L^3}t = [1] \checkmark$$

Thus $\Pi_1 = f(\Pi_2)$, or
$$\frac{\mathcal{P}}{\rho D^5\omega^3} = f\left(\frac{Q}{D^3\omega}\right)$$

$\mathcal{P}$

Problem 7.20

Given: Draining of a tank from initial level, h_0.

Time, τ, depends on tank diameter, D, orifice diameter, d, acceleration of gravity, g, density, ρ, and viscosity, μ.

Find: (a) Number of dimensionless parameters
(b) Number of repeating variables.
(c) Π-parameter containing viscosity.

Solution: Step ① $\tau \quad h_0 \quad D \quad d \quad g \quad \rho \quad \mu$

Step ② Choose MLt system $(n = 7)$

Step ③ $t \quad L \quad L \quad L \quad \frac{L}{t^2} \quad \frac{M}{L^3} \quad \frac{M}{Lt}$

$(r = 3)$

Then $n - r = 7 - 3 = 4$ parameters will result. Π's

Step ④ $r = 3$, so choose 3 variables: ρ, d, g

Step ⑤ $\Pi_1 = \rho^a d^b g^c \mu = \left(\frac{M}{L^3}\right)^a L^b \left(\frac{L}{t^2}\right)^c \frac{M}{Lt} = M^0 L^0 t^0$

$M: a + 1 = 0 \qquad a = -1$

$L: -3a + b + c - 1 = 0 \qquad b = 3a - c + 1 = 3(-1) - (-\tfrac{1}{2}) + 1$

$t: -2c - 1 = 0 \qquad c = -\tfrac{1}{2} \qquad b = -\tfrac{3}{2}$

$$\Pi_1 = \frac{\mu}{\rho\, d^{3/2} g^{1/2}}$$ Π_1

Step ⑥ Check, using FLt system.

$$\mu = \frac{Ft}{L^2}\,;\quad \rho = \frac{M}{L^3} \times \frac{Ft^2}{ML} = \frac{Ft^2}{L^4}$$

$$\Pi_1 = \frac{Ft}{L^2}\,\frac{L^4}{Ft^2}\,\frac{1}{L^{3/2}}\,\frac{t}{L^{1/2}} = \frac{FL^4t^2}{FL^4t^2} = 1 \quad ✓✓$$

Problem 7.21

Given: Water is drained from a tank of diameter D, through a smoothly rounded drain hole of diameter, d. The initial mass flow rate, $\dot{m}$, from the tank is written in functional form as

$$\dot{m} = \dot{m}(h_o, D, d, g, \rho, \mu)$$

where h_o is the initial water depth in the tank
g is the acceleration of gravity
ρ and μ are fluid properties.

Find: (a) the number of dimensionless groups required to correlate the data
(b) the number of repeating variables that must be selected to determine the dimensionless parameters.
(c) the π parameter that contains the fluid viscosity, μ.

Solution: Apply the Buckingham π-theorem

① List: $\dot{m}$ h_o D d g ρ μ — $n = 7$ parameters

② Select M, L, t as primary dimensions

③ Dimensions: $\frac{M}{t}$ L L L $\frac{L}{t^2}$ $\frac{M}{L^3}$ $\frac{M}{Lt}$ — $r = 3$ prim. dim

④ Choose repeating variables ρ, d, g — $m = 3$ repeating parameters ← m

$\therefore$ expect $n - m = 7 - 3 = 4$ dimensionless parameters ← n

⑤ $\pi_1 = \rho^a d^b g^c \mu$

$$M^0 L^0 t^0 = \left(\frac{M}{L^3}\right)^a L^b \left(\frac{L}{t^2}\right)^c \frac{M}{Lt}$$

t: $0 = -2c - 1$ $\therefore c = -\frac{1}{2}$
M: $0 = a + 1$ $\therefore a = -1$
L: $0 = -3a + b + c - 1$ $\therefore b = 3a - c + 1 = -\frac{3}{2}$

$$\pi_1 = \frac{\mu}{\rho d^{3/2} g^{1/2}}$$ ← π

⑥ Check $\pi_1 = \frac{Ft}{L^2} \times \frac{L^4}{Ft^2} \times \frac{1}{L^{3/2}} \times \frac{t}{L^{1/2}} = [1]$ ✓

Problem 7.22

Given: Continuous belt moving vertically through a viscous liquid bath

The volume rate of liquid loss, Q, is a function of μ, ρ, g, h (thickness of liquid layer), and V

Find: form of dependence of Q on other variables.

Solution: Apply Buckingham π-theorem.

① $Q \quad \mu \quad \rho \quad g \quad h \quad V$ $\qquad n = 6$ parameters

② Select M, L, t as primary dimensions

③ $Q: \frac{L^3}{t} \quad \mu: \frac{M}{Lt} \quad \rho: \frac{M}{L^3} \quad g: \frac{L}{t^2} \quad h: L \quad V: \frac{L}{t}$ $\qquad r = 3$ primary dimensions

④ ρ, V, h $\qquad m = r = 3$ repeating parameters

⑤ Then $n - m = 3$ dimensionless groups will result.

Setting up dimensional equations.

$\pi_1 = \rho^a V^b h^c Q$

$M^0L^0t^0 = \left(\frac{M}{L^3}\right)^a \left(\frac{L}{t}\right)^b L^c \frac{L^3}{t}$

Equating exponents,

M: $0 = a$

L: $0 = -3a + b + c + 3$

t: $0 = -b - 1$

i.e. $a = 0$, $b = -1$, $c = -2$

$\therefore \pi_1 = \frac{Q}{Vh^2}$

$\pi_2 = \rho^a V^b h^c \mu$

$M^0L^0t^0 = \left(\frac{M}{L^3}\right)^a \left(\frac{L}{t}\right)^b L^c \frac{M}{Lt}$

Equating exponents,

M: $0 = a + 1$

L: $0 = -3a + b + c - 1$

t: $0 = -b - 1$

i.e. $a = -1$, $b = -1$, $c = -1$

$\therefore \pi_2 = \frac{\mu}{\rho V h}$

$\pi_3 = \rho^a V^b h^c g$

$M^0L^0t^0 = \left(\frac{M}{L^3}\right)^a \left(\frac{L}{t}\right)^b L^c \frac{L}{t^2}$

Equating exponents,

M: $0 = a$

L: $0 = -3a + b + c + 1$

t: $0 = -b - 2$

i.e. $a = 0$, $b = -2$, $c = 1$

$\therefore \pi_3 = \frac{gh}{V^2}$

Then
$$\frac{Q}{Vh^2} = f\left(\frac{\rho V h}{\mu}, \frac{V^2}{gh}\right)$$ Q

⑥ Check using F, L, t dimensions

$\pi_1 = \frac{L^3}{t} \cdot \frac{t}{L} \cdot \frac{1}{L^2} = [1]$ ✓ $\qquad \pi_2 = \frac{Ft}{L^2} \cdot \frac{L^4}{Ft^2} \frac{t}{L} \frac{1}{L} = [1]$ ✓ $\qquad \pi_3 = \frac{L}{t^2} \cdot L \cdot \frac{t^2}{L^2} = [1]$ ✓

Problem 7.23

Given: Diameter, d, of liquid droplets formed in fuel injection process is a function of ρ, μ, σ (surface tension), V, D.

Find: (a) number of dimensionless ratios required to characterize the process
(b) the dimensionless ratios.

Solution: Apply Buckingham π-theorem

① $d \quad \rho \quad \mu \quad \sigma \quad V \quad D \qquad n = 6$ parameters

② Select M, L, t as primary dimensions

③ $d \quad \rho \quad \mu \quad \sigma \quad V \quad D$

$L \quad \frac{M}{L^3} \quad \frac{M}{Lt} \quad \frac{M}{t^2} \quad \frac{L}{t} \quad L \qquad r = 3$ primary dimensions

④ $\rho, D, V \qquad m = r = 3$ repeating parameters

⑤ Then $n - m = 3$ dimensionless groups will result.

Setting up dimensional equations

$\Pi_1 = \rho^a D^b V^c d$	$\Pi_2 = \rho^a D^b V^c \mu$	$\Pi_3 = \rho^a D^b V^c \sigma$
$M^0L^0t^0 = \left(\frac{M}{L^3}\right)^a L^b \left(\frac{L}{t}\right)^c L$	$M^0L^0t^0 = \left(\frac{M}{L^3}\right)^a L^b \left(\frac{L}{t}\right)^c \frac{M}{Lt}$	$M^0L^0t^0 = \left(\frac{M}{L^3}\right)^a L^b \left(\frac{L}{t}\right)^c \frac{M}{t^2}$
Summing exponents,	Summing exponents	Summing exponents
$M: \; a = 0$	$M: \; a + 1 = 0$	$M: \; a + 1 = 0$
$L: \; -3a + b + c + 1 = 0$	$L: \; -3a + b + c - 1 = 0$	$L: \; -3a + b + c = 0$
$t: \; -c = 0$	$t: \; -c - 1 = 0$	$t: \; -c - 2 = 0$
i.e. $a = 0$, $c = 0$, $b = -1$	i.e. $a = -1$, $c = -1$, $b = 3a - c + 1 = -1$	i.e. $a = -1$, $c = -2$, $b = 3a - c = -1$
$\therefore \Pi_1 = \frac{d}{D}$	$\therefore \Pi_2 = \frac{\mu}{\rho V D}$	$\therefore \Pi_3 = \frac{\sigma}{\rho D V^2}$

⑥ Check using F, L, t dimensions

$\Pi_1 = \frac{L}{L} = [1]$ ✓ $\qquad \Pi_2 = \frac{Ft}{L^2} \cdot \frac{L^4}{Ft^2} \cdot \frac{t}{L} \cdot \frac{1}{L} = [1]$ ✓ $\qquad \Pi_3 = \frac{F}{L} \cdot \frac{L^4}{Ft^2} \cdot \frac{1}{L} \cdot \frac{t^2}{L^2} = [1]$ ✓

Problem 7.24 (In Excel)

The diameter, d, of the dots made by an ink jet printer depends on the ink viscosity μ, density ρ, and surface tension, σ, the nozzle diameter, D, the distance, L, of the nozzle from the paper surface, and the ink jet velocity V. Use dimensional analysis to find the Π parameters that characterize the ink jet's behavior.

Given: That dot size depends on ink viscosity, density, and surface tension, and geometry

Find: Π groups

Solution

We will use the workbook of Example Problem 7.1, modified for the current problem

The number of parameters is:	$n = 7$
The number of primary dimensions is:	$r = 3$
The number of repeat parameters is:	$m = r = 3$
The number of Π groups is:	$n - m = 4$

Enter the dimensions (**M**, **L**, **t**) of
the repeating parameters, and of up to
four other parameters (for up to four Π groups).
The spreadsheet will compute the exponents a, b, and c for each.

REPEATING PARAMETERS: Choose ρ, V, D

	M	L	t
ρ	1	-3	
V		1	-1
D		1	

Π GROUPS:

	M	L	t
d	0	1	0

Π_1: $a =$ **0**, $b =$ **0**, $c =$ **-1**

	M	L	t
μ	1	-1	-1

Π_2: $a =$ **-1**, $b =$ **-1**, $c =$ **-1**

	M	L	t
σ	1	0	-2

Π_3:

$a =$ **-1**
$b =$ **-2**
$c =$ **-1**

	M	L	t
L	0	1	0

Π_4:

$a =$ **0**
$b =$ **0**
$c =$ **-1**

Hence $\Pi_1 = \dfrac{d}{D}$ $\quad \Pi_2 = \dfrac{\mu}{\rho V D} \to \dfrac{\rho V D}{\mu}$ $\quad \Pi_3 = \dfrac{\sigma}{\rho V^2 D}$ $\quad \Pi_4 = \dfrac{L}{D}$

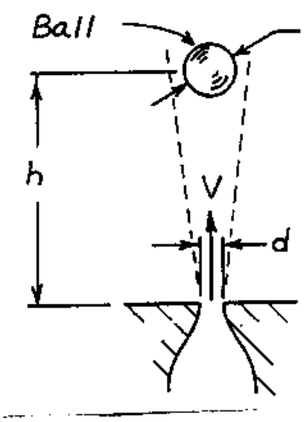
Note that groups Π_1 and Π_4 can be obtained by inspection

Problem 7.25

Given: Ball in jet

$$h = h(d, D, \rho, V, \mu, W)$$

Find: Pi parameters



Solution: Apply Buckingham procedure

① $h \quad d \quad D \quad \rho \quad V \quad \mu \quad W \qquad n = 7$

② M, L, t

③ $L \quad L \quad L \quad \frac{M}{L^3} \quad \frac{L}{t} \quad \frac{M}{Lt} \quad \frac{ML}{t^2} \qquad m = 3 \qquad n - m = 7 - 3 = 4$ parameters

④ Choose ρ, V, d as repeating parameters.

⑤ $\rho^a V^b d^c W = \left(\frac{M}{L^3}\right)^a \left(\frac{L}{t}\right)^b (L)^c \frac{ML}{t^2} = M^0 L^0 t^0$

$M: a + 1 = 0 \qquad a = -1$

$L: -3a + b + c + 1 = 0 \qquad c = -2$

$t: -b - 2 = 0 \qquad b = -2$

$$\Pi_1 = \frac{W}{\rho V^2 d^2}$$

Π_1

⑥ Check: $F \times \frac{L^4}{Ft^2} \frac{t^2}{L^2} \times \frac{1}{L^2} = 1$ ✓✓

$$\Pi_2 = \rho^a V^b d^c \mu = \frac{\mu}{\rho V d}$$

$$\Pi_3 = \rho^a V^b d^c h = \frac{h}{d}$$

$$\Pi_4 = \rho^a V^b d^c D = \frac{D}{d}$$

Problem 7.26 (In Excel)

The diameter, d, of bubbles produced by a bubble-making toy depends on the soapy water viscosity μ, density ρ, and surface tension, σ, the ring diameter, D, and the pressure differential, Δp, generating the bubbles. Use dimensional analysis to find the Π parameters that characterize this phenomenon.

Given: Bubble size depends on viscosity, density, surface tension, geometry and pressure

Find: Π groups

Solution

We will use the workbook of Example Problem 7.1, modified for the current problem

The number of parameters is:	$n = 6$
The number of primary dimensions is:	$r = 3$
The number of repeat parameters is:	$m = r = 3$
The number of Π groups is:	$n - m = 3$

Enter the dimensions (**M**, **L**, **t**) of
the repeating parameters, and of up to
four other parameters (for up to four Π groups).
The spreadsheet will compute the exponents a, b, and c for each.

REPEATING PARAMETERS: Choose ρ, Δp, D

	M	**L**	**t**
ρ	1	-3	
Δp	1	-1	-2
D		1	

Π GROUPS:

	M	**L**	**t**
d	0	1	0

Π_1: $a =$ **0**, $b =$ **0**, $c =$ **-1**

	M	**L**	**t**
μ	1	-1	-1

Π_2: $a =$ **-0.5**, $b =$ **-0.5**, $c =$ **-1**

	M	L	t		M	L	t
σ	1	0	-2		0	0	0

Π_3: $a =$ **0**, $b =$ **-1**, $c =$ **-1**

Π_4: $a =$ **0**, $b =$ **0**, $c =$ **0**

Hence $\Pi_1 = \dfrac{d}{D}$ $\quad \Pi_2 = \dfrac{\mu}{\rho^{\frac{1}{2}} \Delta p^{\frac{1}{2}} D} \rightarrow \dfrac{\mu^2}{\rho \Delta p D^2}$ $\quad \Pi_3 = \dfrac{\sigma}{D \Delta p}$

Note that the Π_1 group can be obtained by inspection

Problem 7.27 (In Excel)

The terminal speed V of shipping boxes sliding down an incline on a layer of air (injected through numerous pinholes in the incline surface) depends on the box mass, m, and base area, A, gravity, g, the incline angle, θ, the air viscosity, μ, and the air layer thickness, δ. Use dimensional analysis to find the Π parameters that characterize this phenomenon.

Given: Speed depends on mass, area, gravity, slope, and air viscosity and thickness

Find: Π groups

Solution

We will use the workbook of Example Problem 7.1, modified for the current problem

The number of parameters is:	$n = 7$
The number of primary dimensions is:	$r = 3$
The number of repeat parameters is:	$m = r = 3$
The number of Π groups is:	$n - m = 4$

Enter the dimensions (**M**, **L**, **t**) of
the repeating parameters, and of up to
four other parameters (for up to four Π groups).
The spreadsheet will compute the exponents a , b , and c for each.

REPEATING PARAMETERS: Choose g , δ, m

	M	**L**	**t**
g		1	-2
δ		1	
m	1		

Π GROUPS:

	M	**L**	**t**
V	0	1	-1

Π_1: a = **-0.5**, b = **-0.5**, c = **0**

	M	**L**	**t**
μ	1	-1	-1

Π_2: a = **-0.5**, b = **1.5**, c = **-1**

	M	L	t
θ	0	0	0

Π_3: $a = 0$, $b = 0$, $c = 0$

	M	L	t
A	0	2	0

Π_4: $a = 0$, $b = -2$, $c = 0$

Hence $\Pi_1 = \dfrac{V}{g^{\frac{1}{2}}\delta^{\frac{1}{2}}} \rightarrow \dfrac{V^2}{g\delta}$ $\quad \Pi_2 = \dfrac{\mu\delta^{\frac{3}{2}}}{g^{\frac{1}{2}}m} \rightarrow \dfrac{\mu^2\delta^3}{m^2 g}$ $\quad \Pi_3 = \theta$ $\quad \Pi_4 = \dfrac{A}{\delta^2}$

Note that the Π_1 , Π_3 and Π_4 groups can be obtained by inspection

Problem 7.28 (In Excel)

The time, t, for a flywheel, with moment of inertia I, to reach angular velocity ω, from rest, depends on the applied torque, T, and the following flywheel bearing properties: the oil viscosity μ, gap δ, diameter D, and length L. Use dimensional analysis to find the Π parameters that characterize this phenomenon.

Given: Time to speed up depends on inertia, speed, torque, oil viscosity and geometry

Find: Π groups

Solution

We will use the workbook of Example Problem 7.1, modified for the current problem

The number of parameters is:	$n = 8$
The number of primary dimensions is:	$r = 3$
The number of repeat parameters is:	$m = r = 3$
The number of Π groups is:	$n - m = 5$

Enter the dimensions (**M**, **L**, **t**) of
the repeating parameters, and of up to
four other parameters (for up to four Π groups).
The spreadsheet will compute the exponents a, b, and c for each.

REPEATING PARAMETERS: Choose ω, D, T

	M	**L**	**t**
ω			-1
D		1	
T	1	2	-2

Π GROUPS:

Two Π groups can be obtained by inspection: δ/D and L/D. The others are obtained below

	M	**L**	**t**
t	0	0	1

Π_1: $a =$ **1**, $b =$ **0**, $c =$ **0**

	M	**L**	**t**
μ	1	-1	-1

Π_2: $a =$ **1**, $b =$ **3**, $c =$ **-1**

	M	L	t		M	L	t
I	1	2	0		0	0	0

Π_3: $a =$ **2**, $b =$ **0**, $c =$ **-1**

Π_4: $a =$ **0**, $b =$ **0**, $c =$ **0**

Hence the Π groups are

$$t\omega \qquad \frac{\delta}{D} \qquad \frac{L}{D} \qquad \frac{\mu\omega D^3}{T} \qquad \frac{I\omega^2}{T}$$

Note that the Π_1 group can also be easily obtained by inspection

Problem 7.29

Given: Pressurized tank drained through a smooth nozzle, area A.

$$\dot{m} = \dot{m}(\Delta p, h, \rho, A, g)$$

Find: (a) Number of independent dimensionless parameters.
(b) Obtain the parameters.
(c) State the functional relationship for $\dot{m}$.

Solution: Apply the Buckingham Π-theorem.

① $\dot{m}$ Δp h ρ A g $n=6$ parameters

② Select M, L, t as primary dimensions

③ $\frac{M}{t}$ $\frac{M}{Lt^2}$ L $\frac{M}{L^3}$ L^2 $\frac{L}{t^2}$ $r=3$ primary dimensions

$m=r=3$

④ Choose ρ, A, g as repeating parameters.

⑤ Then $n-m = 6-3 = 3$ dimensionless parameters result. $n-m$

Set up dimensional equations:

$\Pi_1 = \rho^a A^b g^c \dot{m}$	$\Pi_2 = \rho^a A^b g^c \Delta p$	$\Pi_3 = \rho^a A^b g^c h$
$M^0L^0t^0 = \left(\frac{M}{L^3}\right)^a (L^2)^b \left(\frac{L}{t^2}\right)^c \frac{M}{t}$	$M^0L^0t^0 = \left(\frac{M}{L^3}\right)^a (L^2)^b \left(\frac{L}{t^2}\right)^c \frac{M}{Lt^2}$	$M^0L^0t^0 = \left(\frac{M}{L^3}\right)^a (L^2)^b \left(\frac{L}{t^2}\right)^c L$
Equating exponents:	Equating exponents:	Equating exponents:
$M: a+1=0 \quad a=-1$	$M: a+1=0 \quad a=-1$	$M: a=0 \quad a=0$
$L: -3a+2b+c=0$	$L: -3a+2b+c-1=0$	$L: -3a+2b+c+1=0$
$t: -2c-1=0 \quad c=-\frac{1}{2}$	$t: -2c-2=0 \quad c=-1$	$t: -2c+0=0 \quad c=0$
$\therefore b = \frac{1}{2}(3a-c) = -\frac{5}{4}$	$\therefore b = \frac{1}{2}(1+3a-c) = -\frac{1}{2}$	$\therefore b = \frac{1}{2}(-1+3a-c) = -\frac{1}{2}$
$\Pi_1 = \frac{\dot{m}}{\rho A^{5/4} g^{1/2}}$	$\Pi_2 = \frac{\Delta p}{\rho A^{1/2} g}$	$\Pi_3 = \frac{h}{A^{1/2}}$

Π's

⑥ Check using FLt dimensions: $\dot{m} = \frac{M}{t}\frac{Ft^2}{ML} = \frac{Ft}{L}$; $\rho = \frac{M}{L^3}\frac{Ft^2}{ML} = \frac{Ft^2}{L^4}$

$\Pi_1 = \frac{Ft}{L}\frac{L^4}{Ft^2}\frac{1}{L^{5/2}}\frac{t}{L^{1/2}} = [1]$ ✓✓ | $\Pi_2 = \frac{F}{L^2}\frac{L^4}{Ft^2}\frac{1}{L}\frac{t^2}{L} = [1]$ ✓✓ | $\Pi_3 = \frac{L}{L} = [1]$ ✓✓

Thus

$$\Pi_1 = f(\Pi_2, \Pi_3) \qquad \frac{\dot{m}}{\rho A^{5/4} g^{1/2}} = f\left(\frac{\Delta p}{\rho A^{1/2} g}, \frac{h}{A^{1/2}}\right)$$

or

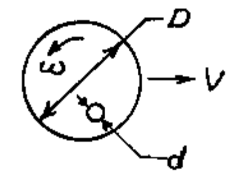
$$\dot{m} = \rho A^{5/4} g^{1/2} f\left(\frac{\Delta p}{\rho A^{1/2} g}, \frac{h}{A^{1/2}}\right)$$

$\dot{m}$

Problem 7.30

Given: Aerodynamic torque on spinning ball,

$$T = f(V, \rho, \mu, D, \omega, d)$$



Find: Dimensionless parameters

Solution: Apply Buckingham procedure.

① List: $T \quad V \quad \rho \quad \mu \quad D \quad \omega \quad d \qquad n = 7$

② Choose M, L, t

③ $\frac{ML^2}{t^2} \quad \frac{L}{t} \quad \frac{M}{L^3} \quad \frac{M}{Lt} \quad L \quad \frac{1}{t} \quad L \qquad m = 3$

④ Choose ρ, V, D $\qquad n - m = 4$ parameters

⑤ $\Pi_1 = \rho^a V^b D^c T = \left(\frac{M}{L^3}\right)^a \left(\frac{L}{t}\right)^b (L)^c \frac{ML^2}{t^2} = M^0 L^0 t^0$

M: $a + 1 = 0 \qquad a = -1$

L: $-3a + b + c + 2 = 0 \qquad c = -3 \qquad \Pi_1 = \frac{T}{\rho V^2 D^3}$

t: $-b - 2 = 0 \qquad b = -2$

Π_1

⑥ Check: $\Pi_1 = FL \times \frac{L^4}{Ft^2} \times \frac{t^2}{L^2} \times \frac{1}{L^3} = 1$ ✓✓

$$\Pi_2 = \frac{\mu}{\rho V D}$$

$$\Pi_3 = \frac{\omega D}{V}$$

$$\Pi_4 = \frac{d}{D}$$

$$\Pi_1 = f(\Pi_2, \Pi_3, \Pi_4)$$

$$\frac{T}{\rho V^2 D^3} = f\left(\frac{\mu}{\rho V D}, \frac{\omega D}{V}, \frac{d}{D}\right)$$

T

Problem 7.31

Given: Power loss, $\mathcal{P}$, depends on: length, ℓ, diameter, D; clearance, c; angular speed, ω; viscosity, μ; mean pressure, p.

Find: (a) Dimensionless parameters that characterize the problem
(b) Functional form of dependence of $\mathcal{P}$ on these parameters

Solution: Apply Buckingham π - theorem

① $\mathcal{P}$ ℓ D c ω μ p $n = 7$ parameters

② Select F, L, t as primary dimensions

③

$\mathcal{P}$	ℓ	D	c	ω	μ	p
$\frac{FL}{t}$	L	L	L	$\frac{1}{t}$	$\frac{Ft}{L^2}$	$\frac{F}{L^2}$

④ D, ω, p $m = r = 3$ repeating parameters

⑤ Then $n - m = 4$ dimensionless groups will result.

Setting up dimensional equations

$\pi_1 = D^a \omega^b p^e \mathcal{P}$	$\pi_2 = D^a \omega^b p^e \ell$	$\pi_3 = D^a \omega^b p^e c$	$\pi_4 = D^a \omega^b p^e \mu$
$F^0L^0t^0 = L^a\left(\frac{1}{t}\right)^b\left(\frac{F}{L^2}\right)^e\frac{FL}{t}$	$F^0L^0t^0 = L^a\left(\frac{1}{t}\right)^b\left(\frac{F}{L^2}\right)^e L$	$F^0L^0t^0 = L^a\left(\frac{1}{t}\right)^b\left(\frac{F}{L^2}\right)^e L$	$F^0L^0t^0 = L^a\left(\frac{1}{t}\right)^b\left(\frac{F}{L^2}\right)^e \frac{Ft}{L^2}$
Equating exponents,	Equating exponents,	Equating exponents,	Equating exponents,
$F: \; 0 = e + 1$	$F: \; 0 = e$	$F: \; 0 = e$	$F: \; 0 = e + 1$
$L: \; 0 = a - 2e + 1$	$L: \; 0 = a - 2e + 1$	$L: \; 0 = a - 2e + 1$	$L: \; 0 = a - 2e - 2$
$t: \; 0 = -b - 1$	$t: \; 0 = -b$	$t: \; 0 = -b$	$t: \; 0 = -b + 1$
$\therefore \; e = -1$	$\therefore \; e = 0$	$\therefore \; e = 0$	$\therefore \; e = -1$
$a = -3$	$a = -1$	$a = -1$	$a = 0$
$b = -1$	$b = 0$	$b = 0$	$b = 1$
$\pi_1 = \frac{\mathcal{P}}{p\omega D^3}$	$\pi_2 = \frac{\ell}{D}$	$\pi_3 = \frac{c}{D}$	$\pi_4 = \frac{\mu\omega}{p}$

Then, $$\frac{\mathcal{P}}{p\omega D^3} = f\left(\frac{\mu\omega}{p}, \frac{c}{D}, \frac{\ell}{D}\right)$$ $\mathcal{P}$

⑥ Check using M, L, t dimensions

$$\pi_1 = \frac{ML^2}{t^3} \times \frac{Lt^2}{M} \times t \times \frac{1}{L^3} = [1] \checkmark$$

$$\pi_2 = \frac{L}{L} = [1] \checkmark \qquad \pi_3 = \frac{L}{L} = [1] \checkmark$$

$$\pi_4 = \frac{M}{Lt} \times \frac{1}{t} \times \frac{Lt^2}{M} = [1] \checkmark$$

Problem 7.32

Given: Thrust, F_t, of a marine propeller is thought to depend on: ρ (water density), D (diameter), V (speed of advance), g (acceleration of gravity), ω (angular speed of propeller), p (pressure in the liquid), and μ (liquid viscosity)

Find: Dimensionless parameters that characterize propeller performance.

Solution: Apply Buckingham π - theorem

① List: F_t ρ D V g ω p μ $(n=8)$

② Choose M, L, t as primary dimensions

③ Dimensions: $\frac{ML}{t^2}$ $\frac{M}{L^3}$ L $\frac{L}{t}$ $\frac{L}{t^2}$ $\frac{1}{t}$ $\frac{M}{Lt^2}$ $\frac{M}{Lt}$

④ Repeating variables ρ, V, D $m = r = 3$

⑤ Then $n - m = 5$ dimensionless groups will result

Setting up dimensional equations

$$\pi_1 = \rho^a V^b D^c F_t$$

$$M^0L^0t^0 = \left(\frac{M}{L^3}\right)^a \left(\frac{L}{t}\right)^b L^c \frac{ML}{t^2} \quad \begin{cases} M: 0 = a+1 \\ t: 0 = -b-2 \\ L: 0 = -3a+b+c+1 \end{cases} \quad \begin{matrix} a=-1 \\ b=-2 \\ c=-2 \end{matrix} \quad \therefore \pi_1 = \frac{F_t}{\rho V^2 D^2}$$

$$\pi_2 = \rho^a V^b D^c g$$

$$M^0L^0t^0 = \left(\frac{M}{L^3}\right)^a \left(\frac{L}{t}\right)^b L^c \left(\frac{L}{t^2}\right) \quad \begin{cases} M: 0 = a \\ t: 0 = -b-2 \\ L: 0 = -3a+b+c+1 \end{cases} \quad \begin{matrix} a=0 \\ b=-2 \\ c=1 \end{matrix} \quad \therefore \pi_2 = \frac{gD}{V^2}$$

$$\pi_3 = \rho^a V^b D^c \omega$$

$$M^0L^0t^0 = \left(\frac{M}{L^3}\right)^a \left(\frac{L}{t}\right)^b L^c \left(\frac{1}{t}\right) \quad \begin{cases} M: 0 = a \\ t: 0 = -b-1 \\ L: 0 = -3a+b+c \end{cases} \quad \begin{matrix} a=0 \\ b=-1 \\ c=1 \end{matrix} \quad \therefore \pi_3 = \frac{\omega D}{V}$$

$$\pi_4 = \rho^a V^b D^c p$$

$$M^0L^0t^0 = \left(\frac{M}{L^3}\right)^a \left(\frac{L}{t}\right)^b L^c \frac{M}{Lt^2} \quad \begin{cases} M: 0 = a+1 \\ t: 0 = -b-2 \\ L: 0 = -3a+b+c-1 \end{cases} \quad \begin{matrix} a=-1 \\ b=-2 \\ c=0 \end{matrix} \quad \therefore \pi_4 = \frac{p}{\rho V^2}$$

$$\pi_5 = \rho^a V^b D^c \mu$$

$$M^0L^0t^0 = \left(\frac{M}{L^3}\right)^a \left(\frac{L}{t}\right)^b L^c \frac{M}{Lt} \quad \begin{cases} M: 0 = a+1 \\ t: 0 = -b-1 \\ L: 0 = -3a+b+c-1 \end{cases} \quad \begin{matrix} a=-1 \\ b=-1 \\ c=-1 \end{matrix} \quad \therefore \pi_5 = \frac{\mu}{\rho V D}$$

Dimensionless parameters are $\frac{F_t}{\rho V^2 D^2}$, $\frac{gD}{V^2}$, $\frac{\omega D}{V}$, $\frac{p}{\rho V^2}$, $\frac{\mu}{\rho V D}$

⑥ Check using F, L, t

$\pi_1 = F \times \frac{L^4}{Ft^2} \times \frac{t^2}{L^2} \times \frac{1}{L^2} = [1]$ ✓ $\quad \pi_2 = \frac{L}{t^2} \times L \times \frac{t^2}{L^2} = [1]$ ✓

$\pi_3 = \frac{1}{t} \times L \times \frac{t}{L} = [1]$, $\pi_4 = \frac{F}{L^2} \times \frac{L^4}{Ft^2} \times \frac{t^2}{L^2} = [1]$

$\pi_5 = \frac{1}{Re} = [1]$ ✓

Problem 7.33

Given: Power, $\mathcal{P}$, required to drive a propeller is a function of V, D, ω (angular velocity), μ, ρ, and c (speed of sound)

Find: (a) number of dimensionless groups required to characterize situation
(b) the dimensionless groups

Solution: Apply Buckingham Π-theorem

① $\mathcal{P} \quad V \quad D \quad \omega \quad \mu \quad \rho \quad c$ $\qquad n = 7$ parameters

② Select M, L, t as primary dimensions

③ $\mathcal{P} \quad V \quad D \quad \omega \quad \mu \quad \rho \quad c$

$\frac{ML^2}{t^3} \quad \frac{L}{t} \quad L \quad \frac{1}{t} \quad \frac{M}{Lt} \quad \frac{M}{L^3} \quad \frac{L}{t}$ $\qquad r = 3$ primary dimensions

④ V, D, ρ $\qquad m = r = 3$ repeating parameters

⑤ Then $n - m = 4$ dimensionless groups will result.

Setting up dimensional equations:

$\Pi_1 = V^a D^b \rho^c \mathcal{P}$

$M^0L^0t^0 = \left(\frac{L}{t}\right)^a L^b \left(\frac{M}{L^3}\right)^c \frac{ML^2}{t^3}$

Summing exponents,

M: $c + 1 = 0 \quad \therefore c = -1$

L: $a + b - 3c + 2 = 0$

t: $-a - 3 = 0 \quad \therefore a = -3$

$b = 3c - 2 - a = -2$

$\therefore \Pi_1 = \frac{\mathcal{P}}{\rho D^2 V^3}$

$\Pi_2 = V^a D^b \rho^c \omega$

$M^0L^0t^0 = \left(\frac{L}{t}\right)^a L^b \left(\frac{M}{L^3}\right)^c \frac{1}{t}$

Summing exponents,

M: $c = 0$

L: $a + b - 3c = 0$

t: $-a - 1 = 0 \quad \therefore a = -1$

$b = 3c - a = 1$

$\therefore \Pi_2 = \frac{\omega D}{V}$

$\Pi_3 = V^a D^b \rho^c \mu$

$M^0L^0t^0 = \left(\frac{L}{t}\right)^a L^b \left(\frac{M}{L^3}\right)^c \frac{M}{Lt}$

Summing exponents,

M: $c + 1 = 0 \quad \therefore c = -1$

L: $a + b - 3c - 1 = 0$

t: $-a - 1 = 0 \quad \therefore a = -1$

$b = 3c + 1 - a = -1$

$\therefore \Pi_3 = \frac{\mu}{\rho D V}$

$\Pi_4 = V^a D^b \rho^c c$

$M^0L^0t^0 = \left(\frac{L}{t}\right)^a L^b \left(\frac{M}{L^3}\right)^c \frac{L}{t}$

Summing exponents,

M: $c = 0$

L: $a + b - 3c + 1 = 0$

t: $-a - 1 = 0 \quad \therefore a = -1$

$b = 3c - a - 1 = 0$

$\therefore \Pi_4 = \frac{c}{V}$

Dimensionless groups are: $\frac{\mathcal{P}}{\rho D^2 V^3}, \frac{\omega D}{V}, \frac{\mu}{\rho D V}, \frac{c}{V}$

⑥ Check using F, L, t

$\Pi_1 = \frac{FL}{t} \cdot \frac{L^4}{Ft^2} \frac{1}{L^2} \frac{t^3}{L^3} = [1]$ ✓

$\Pi_2 = \frac{1}{t} L \frac{t}{L} = [1]$ ✓

$\Pi_3 = \frac{1}{Re} = [1]$ ✓

$\Pi_4 = \frac{L}{t} \cdot \frac{t}{L} = [1]$ ✓

Problem 7.34

Given: Fan-assisted convection oven; $\dot{Q}$ = heat transfer rate (energy/time).

$$\dot{Q} = f(c_p, \theta, L, \rho, \mu, V)$$

Find: (a) Number of basic dimensions included in these variables.
(b) Number of Π-parameters.
(c) Obtain the parameters.

Solution: Apply the Buckingham Π-theorem.

① $\dot{Q}$ c_p Θ L ρ μ V $n = 7$ parameters

② Select F, L, t, T (temperature) as primary dimensions.

③ $\frac{FL}{t}$ $\frac{L^2}{t^2T}$ T L $\frac{Ft^2}{L^4}$ $\frac{Ft}{L^2}$ $\frac{L}{t}$ $r = 4$ primary dimensions

④ Choose ρ, V, L, Θ as repeating parameters. $m = r = 4$

⑤ Then $n - m = 7 - 4 = 3$ dimensionless parameters result. $n - m$

Set up dimensional equations:

$\Pi_1 = \rho^a V^b L^c \theta^d \dot{Q}$

$F^0L^0t^0T^0 = \left(\frac{Ft^2}{L^4}\right)^a \left(\frac{L}{t}\right)^b (L)^c (T)^d \frac{FL}{t}$

Equating exponents:

F: $a + 1 = 0$ $a = -1$
L: $-4a + b + c + 1 = 0$
t: $2a - b - 1 = 0$ $b = -3$
T: $d = 0$

$\therefore c = -1 + 4a - b = -2$

$$\Pi_1 = \frac{\dot{Q}}{\rho V^3 L^2}$$

$\Pi_2 = \rho^a V^b L^c \theta^d c_p$

$F^0L^0t^0T^0 = \left(\frac{Ft^2}{L^4}\right)^a \left(\frac{L}{t}\right)^b (L)^c (T)^d \frac{L^2}{t^2T}$

Equating exponents:

F: $a = 0$ $a = 0$
L: $-4a + b + c + 2 = 0$
t: $2a - b - 2 = 0$ $b = -2$
T: $d - 1 = 0$ $d = 1$

$\therefore c = -2 + 4a - b = 0$

$$\Pi_2 = \frac{c_p \theta}{V^2}$$

$\Pi_3 = \rho^a V^b L^c \theta^d \mu$

$F^0L^0t^0T^0 = \left(\frac{Ft^2}{L^4}\right)^a \left(\frac{L}{t}\right)^b (L)^c (T)^d \frac{Ft}{L^2}$

Equating exponents:

F: $a + 1 = 0$ $a = -1$
L: $-4a + b + c - 2 = 0$
t: $2a - b + 1 = 0$ $b = -1$
T: $d = 0$ $d = 0$

$\therefore c = 2 + 4a - b = -1$

$$\Pi_3 = \frac{\mu}{\rho V L}$$

⑥ Check, using MLtT dimensions: $\dot{Q} = ML^2/t^3$; $\mu = M/Lt$

$\Pi_1 = \frac{ML^2}{t^3} \frac{L^3}{M} \frac{t^3}{L^3} \frac{1}{L^2} = [1]$ ✓✓

$\Pi_2 = \frac{L^2}{t^2T} T \frac{t^2}{L^2} = [1]$ ✓✓

$\Pi_3 = \frac{M}{Lt} \frac{L^3}{M} \frac{t}{L} \frac{1}{L} = [1]$ ✓✓

Thus

$$\Pi_1 = f(\Pi_2, \Pi_3) \qquad \frac{\dot{Q}}{\rho V^3 L^2} = f\left(\frac{c_p \Theta}{V^2}, \frac{\mu}{\rho V L}\right)$$

or

$$\dot{Q} = \rho V^3 L^2 f\left(\frac{c_p \Theta}{V^2}, \frac{\mu}{\rho V L}\right)$$

$\dot{Q}$

Problem 7.35

The rate dT/dt at which the temperature T at the center of a rice kernel falls during a food technology process is critical—too high a value leads to cracking of the kernel, and too low a value makes the process slow and costly. The rate depends on the rice specific heat, c, thermal conductivity, k, and size, L, as well as the cooling air specific heat, c_p, density, ρ, viscosity, μ, and speed, V. How many basic dimensions are included in these variables? Determine the Π parameters for this problem.

Given: That the cooling rate depends on rice properties and air properties

Find: The Π groups

Apply the Buckingham Π procedure

① $dT/dt \quad c \quad k \quad L \quad c_p \quad \rho \quad \mu \quad V$ $\qquad n = 8$ parameters

② Select primary dimensions M, L, t and T (temperature)

③

$$\begin{array}{cccccccc} dT/dt & c & k & L & c_p & \rho & \mu & V \\ \frac{T}{t} & \frac{L^2}{t^2 T} & \frac{ML}{t^2 T} & L & \frac{L^2}{t^2 T} & \frac{M}{L^3} & \frac{M}{Lt} & \frac{L}{t} \end{array}$$

$r = 4$ primary dimensions

④ $V \quad \rho \quad L \quad c_p$ $\qquad m = r = 4$ repeat parameters

⑤ Then $n - m = 4$ dimensionless groups will result.

By inspection, one Π group is c/c_p

Setting up a dimensional equation,

$$\Pi_1 = V^a \rho^b L^c c_p^d \frac{dT}{dt}$$

$$= \left(\frac{L}{t}\right)^a \left(\frac{M}{L^3}\right)^b (L)^c \left(\frac{L^2}{t^2 T}\right)^d \frac{T}{t} = T^0 M^0 L^0 t^0$$

Summing exponents,

$$\begin{array}{lc|c} T: & -d+1=0 & d=1 \\ M: & b=0 & b=0 \\ L: & a-3b+c+2d=0 & a+c=-2 \rightarrow c=1 \\ t: & -a-2d-1=0 & a=-3 \end{array}$$

Hence

$$\Pi_1 = \frac{dT}{dt}\frac{Lc_p}{V^3}$$

By a similar process, find

$$\Pi_2 = \frac{k}{\rho L^2 c_p}$$

and

$$\Pi_3 = \frac{\mu}{\rho L V}$$

Hence

$$\frac{dT}{dt}\frac{Lc_p}{V^3} = f\left(\frac{c}{c_p}, \frac{k}{\rho L^2 c_p}, \frac{\mu}{\rho L V}\right)$$

Problem 7.36

Given: Water hammer caused by sudden closure of valve in pipeline.

$$p_{max} = f(\rho, V_0, E_v)$$

Find: (a) How many dimensionless groups needed to characterize?
(b) Functional relationship in terms of Π groups.

Solution: Step ①: List p_{max} ρ V_0 E_v $n = 4$

Step ②: Choose M, L, t

Step ③: $\frac{M}{Lt^2}$ $\frac{M}{L^3}$ $\frac{L}{t}$ $\frac{M}{Lt^2}$

Check dimensional matrix:

	p_{max}	ρ	V_0	E_v
M	1	1	0	1
L	−1	−3	1	−1
t	−2	0	−1	−2

For this matrix, $r = 2$

Step ④: Choose ρ, V_0

Step ⑤: $\Pi_1 = \rho^a V_0^b p_{max} = \left(\frac{M}{L^3}\right)^a \left(\frac{L}{t}\right)^b \frac{M}{Lt^2} = M^0 L^0 t^0$

M: $a + 1 = 0$ $a = -1$
L: $-3a + b - 1 = 0$
t: $-b - 2 = 0$ $b = -2$

✓ $\Pi_1 = \frac{p_{max}}{\rho V_0^2}$ Π_1

$\Pi_2 = \rho^a V_0^b E_v = \left(\frac{M}{L^3}\right)^a \left(\frac{L}{t}\right)^b \frac{M}{Lt^2} = M^0 L^0 t^0$

By inspection $\Pi_2 = \frac{E_v}{\rho V_0^2}$ Π_2

Step ⑥: Check using FLt: $\rho = \frac{M}{L^3} \times \frac{Ft^2}{ML} = \frac{Ft^2}{L^4}$

$\Pi_1 = \frac{F}{L^2} \frac{L^4}{Ft^2} \frac{t^2}{L^2} = \frac{FL^4t^2}{FL^4t^2} = 1$ ✓✓

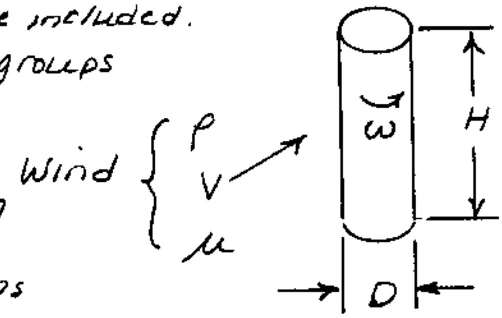
The functional relationship is $\Pi_1 = f(\Pi_2)$. Thus

$$\frac{p_{max}}{\rho V_0^2} = f\left(\frac{E_v}{\rho V_0^2}\right)$$ f

Problem 7.37

Given: Vessel to be powered by rotating cylinder. Model to be tested to estimate power needed to rotate cylinder.

Find: (a) Parameters that should be included.
(b) Important dimensionless groups



Solution: $\mathcal{P} = f(\rho, \omega, D, \mu, H, V)$

① $\rho \quad \omega \quad D \quad \mu \quad H \quad V \quad \mathcal{P} \qquad n = 7$

② Choose M, L, t as primary dimensions

③ $\frac{M}{L^3} \quad \frac{1}{t} \quad L \quad \frac{M}{Lt} \quad L \quad \frac{L}{t} \quad \frac{ML^2}{t^3} \qquad r = 3$ primary dimensions

④ $\rho, \omega, D \quad m = 3 \qquad m = r = 3$ repeating parameters

⑤ Then expect $n - m = 4$ dimensionless groups

$\Pi_1 = \rho^a \omega^b D^c \mathcal{P} = \left(\frac{M}{L^3}\right)^a \left(\frac{1}{t}\right)^b (L)^c \frac{ML^2}{t^3}$

M: $a + 1 = 0$ $\qquad a = -1$
L: $-3a + c + 2 = 0$ $\qquad c = -5$
t: $-b - 3 = 0$ $\qquad b = -3$

$$\Pi_1 = \frac{\mathcal{P}}{\rho \omega^3 D^5} \qquad \Pi_1$$

$\Pi_2 = \rho^a \omega^b D^c V = \left(\frac{M}{L^3}\right)^a \left(\frac{1}{t}\right)^b (L)^c \frac{L}{t}$

M: $a + 0 = 0$ $\qquad a = 0$
L: $-3a + c + 1 = 0$ $\qquad c = -1$
t: $-b - 1 = 0$ $\qquad b = -1$

$$\Pi_2 = \frac{V}{\omega D} \qquad \Pi_2$$

$\Pi_3 = \rho^a \omega^b D^c H$

By inspection $\Pi_3 = \frac{H}{D}$ $\qquad \Pi_3$

$\Pi_4 = \rho^a \omega^b D^c \mu = \left(\frac{M}{L^3}\right)^a \left(\frac{1}{t}\right)^b (L)^c \frac{M}{Lt}$

M: $a + 1 = 0$ $\qquad a = -1$
L: $-3a + c - 1 = 0$ $\qquad c = -2$
t: $-b - 1 = 0$ $\qquad b = -1$

$$\Pi_4 = \frac{\mu}{\rho \omega D^2}$$

Thus

$$\Pi_1 = f(\Pi_2, \Pi_3, \Pi_4) \quad \text{or} \quad \frac{\mathcal{P}}{\rho \omega^3 D^5} = f\left(\frac{V}{\omega D}, \frac{H}{D}, \frac{\mu}{\rho \omega D^2}\right) \qquad \frac{\mathcal{P}}{\rho \omega^3 D^5}$$

⑥ Check, using F, L, t

$\Pi_1 = \frac{FL}{t} \frac{L^4}{Ft^2} \frac{t^3}{1} \frac{1}{L^5} = [1]$ ✓ $\qquad \Pi_2 = \frac{L}{t} \frac{t}{1} \frac{1}{L} = [1]$ ✓

$\Pi_3 = \frac{L}{L} = [1]$ ✓ $\qquad \Pi_4 = \frac{Ft}{L^2} \frac{L^4}{Ft^2} \frac{t}{1} \frac{1}{L^2} = [1]$ ✓

Problem 7.38

Given: Airship to operate at 20 m/sec in standard air.
Model built to 1/20 scale tested at same air temperature.
Model is tested at 75 m/sec

Find: (a) Criterion for dynamic similarity.
(b) Wind tunnel pressure.
(c) Prototype drag if drag force on model is 250 N.

Solution

Dimensional analysis predicts $\frac{F}{\rho V^2 L^2} = f\left(\frac{\rho V L}{\mu}\right)$

Consequently for similarity, $\left(\frac{\rho V L}{\mu}\right)_m = \left(\frac{\rho V L}{\mu}\right)_p$

Since L is fixed, and $\mu_p = \mu_m$ (because T is the same)

$$\rho_m = \rho_p \frac{V_p}{V_m} \frac{L_p}{L_m} \frac{\mu_m}{\mu_p} = \rho_p \frac{20}{75}(20)(1) = 5.33\,\rho_p$$

From ideal gas law, $P = \rho R T$

$$\therefore \frac{P_m}{P_p} = \frac{\rho_m}{\rho_p} = 5.33 \quad \text{and} \quad P_m = 5.33\,P_p = 5.33 \times 101\ \text{kPa} = 5.39 \times 10^5\ \text{Pa}$$ P_m

From the force ratios,

$$F_p = F_m \frac{\rho_p}{\rho_m} \frac{V_p^2}{V_m^2} \frac{L_p^2}{L_m^2} = F_m \frac{1}{5.33}\left(\frac{20}{75}\right)^2 (20)^2 = 5.34\,F_m$$

Thus

$$F_p = 5.34\,F_m = 5.34 \times 250\ \text{N} = 1.34\ \text{kN}$$ F_p

Problem 7.39

Given: Desire to match Reynolds number in two flows: one of air and one of water, using the same size model.

Find: Which flow must have the higher speed, and by how much.

Solution: Set $Re_w = \frac{\rho_w V_w L_w}{\mu_w} = Re_a = \frac{\rho_a V_a L_a}{\mu_a}$

Since $L_w = L_a$, then $\frac{V_a}{V_w} = \frac{\rho_w}{\rho_a}\frac{\mu_a}{\mu_w} = \frac{\nu_a}{\nu_w}$

From Tables A.8 and A.10, at 20°C, $\nu_w = 1.00 \times 10^{-6}\ m^2/s$ and $\nu_a = 1.51 \times 10^{-5}\ m^2/s$.

Thus $\frac{V_a}{V_w} = 1.51 \times 10^{-5} \frac{m^2}{s} \times \frac{s}{1.00 \times 10^{-6}\ m^2} = 15.1$

Therefore V_a must be larger than V_w. ← V_a

In fact, to match Re,

$V_a = 15.1\ V_w$ ← V_a

Problem 7.40

The designers of a large tethered pollution-sampling balloon wish to know what the drag will be on the balloon for the maximum anticipated wind speed of 5 m/s (the air is assumed to be at 20°C). A $\frac{1}{20}$-scale model is built for testing in water at 20°C. What water speed is required to model the prototype? At this speed the model drag is measured to be 2 kN. What will be the corresponding drag on the prototype?

Given: Model scale for on balloon

Find: Required water model water speed; drag on protype based on model drag

Solution

From Appendix A (inc. Fig. A.2) $\rho_{air} = 1.24 \cdot \frac{kg}{m^3}$ $\mu_{air} = 1.8 \times 10^{-5} \cdot \frac{N \cdot s}{m^2}$

$\rho_w = 999 \cdot \frac{kg}{m^3}$ $\mu_w = 10^{-3} \cdot \frac{N \cdot s}{m^2}$

The given data is $V_{air} = 5 \cdot \frac{m}{s}$ $L_{ratio} = 20$ $F_w = 2 \cdot kN$

For dynamic similarity we assume $\frac{\rho_w \cdot V_w \cdot L_w}{\mu_w} = \frac{\rho_{air} \cdot V_{air} \cdot L_{air}}{\mu_{air}}$

Then

$$V_w = V_{air} \cdot \frac{\mu_w}{\mu_{air}} \cdot \frac{\rho_{air}}{\rho_w} \cdot \frac{L_{air}}{L_w} = V_{air} \cdot \frac{\mu_w}{\mu_{air}} \cdot \frac{\rho_{air}}{\rho_w} \cdot L_{ratio} = 5 \cdot \frac{m}{s} \times \left(\frac{10^{-3}}{1.8 \times 10^{-5}} \right) \times \left(\frac{1.24}{999} \right) \times 20$$

$$V_w = 6.9 \frac{m}{s}$$

For the same Reynolds numbers, the drag coefficients will be the same so

$$\frac{F_{air}}{\frac{1}{2}\cdot\rho_{air}\cdot A_{air}\cdot V_{air}^2} = \frac{F_w}{\frac{1}{2}\cdot\rho_w\cdot A_w\cdot V_w^2}$$

where $$\frac{A_{air}}{A_w} = \left(\frac{L_{air}}{L_w}\right)^2 = L_{ratio}^{\ 2}$$

Hence the prototype drag is

$$F_{air} = F_w\cdot\frac{\rho_{air}}{\rho_w}\cdot L_{ratio}^{\ 2}\cdot\left(\frac{V_{air}}{V_w}\right)^2 = 2000\cdot N \times \left(\frac{1.24}{999}\right) \times 20^2 \times \left(\frac{5}{6.9}\right)^2$$

$$F_{air} = 522\,N$$

Problem 7.41

Given: Measurements of drag force are made on a model car in a towing tank filled with freshwater; $L_m/L_p = 1/5$. The dimensionless force ratio becomes constant at model test speeds above $V_m = 4$ m/s. At this speed the drag force on the model is $F_{Dm} = 182$ N

Find: (a) State conditions required to assure dynamic similarity between model and prototype
(b) Determine required speed ratio V_m/V_p to assure dynamically similar conditions
(c) Calculate expected prototype drag when operating in air at speed, $V_p = 90$ km/hr

Solution:

(a) The flows must be geometrically and kinematically similar, and have equal Reynolds numbers to be dynamically similar.

- geometric similarity requires true model in all respects
- kinematic similarity requires same flow pattern, i.e. no free-surface effects or cavitation.
- the problem may be stated as $F_D = f(\rho, V, L, \mu)$. Dimensional analysis gives

$$\frac{F_D}{\rho V^2 L^2} = f\left(\frac{\mu}{\rho V L}\right) = g(Re).$$

(b) Matching Reynolds numbers between model & prototype flows gives

$$\frac{V_m L_m}{\nu_m} = \frac{V_p L_p}{\nu_p} \qquad \text{Assume } T = 20°C$$

$$\frac{V_m}{V_p} = \frac{\nu_m}{\nu_p} \times \frac{L_p}{L_m} = 1\times 10^{-6}\,\frac{m^2}{s} \times \frac{1}{1.51\times 10^{-5}}\,\frac{s}{m^2} \times 5 = 0.331$$

$\frac{V_m}{V_p}$

(c) For dynamically similar conditions, $\left.\frac{F_D}{\rho V^2 L^2}\right)_m = \left.\frac{F_D}{\rho V^2 L^2}\right)_p$

$$\therefore\ F_{D_p} = F_{D_m}\,\frac{\rho_p}{\rho_m} \times \left(\frac{V_p}{V_m}\right)^2 \times \left(\frac{L_p}{L_m}\right)^2$$

$$= 182\,N \times \frac{1.20}{999} \times \left(90\,\frac{km}{hr} \times 1000\,\frac{m}{km} \times \frac{hr}{3600\,s} \times \frac{s}{4\,m}\right)^2 (5)^2$$

$$F_{D_p} = 214\ N$$

F_{D_p}

Problem 7.42

Given: Prototype torpedo, $D = 533$ mm, $l = 6.7$ m operates in water at a speed of 28 m/s. Model (1/5 scale) is to be tested in a wind tunnel. Maximum wind tunnel speed is 110 m/sec; $T = 20°C$; pressure is variable.
At dynamically similar test conditions, $F_{D\,model} = 618$ N

Find: (a) required wind tunnel pressure for dynamically similar test
(b) expected drag force on prototype

Solution:

Assume $F_D = F_D(V, D, \rho, \mu)$. From the Buckingham π-theorem, for $n = 5$, with $m = r = 3$, we would expect two dimensionless groups.

$$\frac{F}{\rho V^2 D^2} = f\left(\frac{\rho V D}{\mu}\right)$$

To attain dynamically similar model test, $\left.\frac{\rho V D}{\mu}\right)_m = \left.\frac{\rho V D}{\mu}\right)_p$

$$\therefore \quad \rho_m = \rho_p \frac{V_p}{V_m} \frac{D_p}{D_m} \frac{\mu_m}{\mu_p}$$

For air at 20°C $\mu_m = 1.81 \times 10^{-5}$ N·s/m²
water at 20°C $\mu_p = 1 \times 10^{-3}$ N·s/m²

$$\rho_m = 998 \frac{kg}{m^3} \times \frac{28}{110} \times 5 \times \frac{1.81 \times 10^{-5}}{1 \times 10^{-3}} = 23.0 \ kg/m^3$$

From the ideal gas equation of state,

$$p = \rho_m R T_m = 23.0 \frac{kg}{m^3} \times 287 \frac{N \cdot m}{kg \cdot K} \times 293 K = 1.93 \text{ MPa (abs)}$$ ← p

For dynamically similar flows,

$$\left.\frac{F_D}{\rho V^2 D^2}\right)_m = \left.\frac{F_D}{\rho V^2 D^2}\right)_p$$

$$\therefore \quad F_{D_p} = F_{D_m} \frac{\rho_p}{\rho_m} \left(\frac{V_p}{V_m}\right)^2 \left(\frac{D_p}{D_m}\right)^2$$

$$= 618 \text{ N} \times \frac{998}{23.0} \left(\frac{28}{110}\right)^2 (5)^2$$

$$F_{D_p} = 43.4 \text{ kN}$$ ← F_{D_p}

Given: Drag force, F, of an airfoil at zero angle of attack is a function of ρ, μ, V, and L.

Model test conditions:

$\frac{L_m}{L_p} = \frac{1}{10}$ $Re_m = 5.5 \times 10^6$ based on chord length

$T = 15^\circ C$, $P = 10$ atmospheres

Prototype data: chord length, $L = 2\,m$

$T = 15^\circ C$ $P = 101$ kPa

Find: (a) velocity, V_m, of model test
(b) corresponding prototype velocity.

Solution

Dimensional analysis predicts $\frac{F}{\rho V^2 L^2} = f\left(\frac{\rho V L}{\mu}\right)$

$Re_m = \left(\frac{\rho V L}{\mu}\right)_m$ and hence $V_m = \frac{Re_m \mu_m}{\rho_m L_m}$

To determine ρ_m assume air behaves as an ideal gas.

$\rho_m = \frac{P_m}{R T_m} = 10 \times 101 \times 10^3 \frac{N}{m^2} \times \frac{kg \cdot K}{287\, N \cdot m} \times \frac{1}{288\, K} = 12.2\ kg/m^3$

From Table A.10, Appendix A, $\mu_m = 1.79 \times 10^{-5}\ N \cdot s/m^2$

$V_m = \frac{Re_m \mu_m}{\rho_m L_m} = 5.5 \times 10^6 \times 1.79 \times 10^{-5} \frac{N \cdot sec}{m^2} \times \frac{m^3}{12.2\, kg} \times \frac{1}{0.2\, m} \times \frac{kg \cdot m}{N \cdot sec^2}$

$V_m = 40.3\ m/s$ ——— V_m

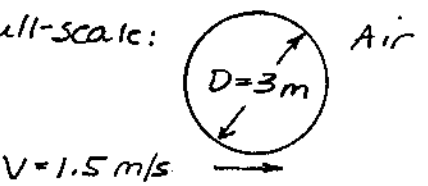
For dynamic similarity $\left(\frac{\rho V L}{\mu}\right)_m = \left(\frac{\rho V L}{\mu}\right)_p$

$V_p = V_m \frac{\mu_p}{\mu_m} \frac{\rho_m}{\rho_p} \frac{L_m}{L_p} = V_m \frac{\mu_p}{\mu_m} \frac{P_m}{P_p} \frac{T_p}{T_m} \frac{L_m}{L_p}$

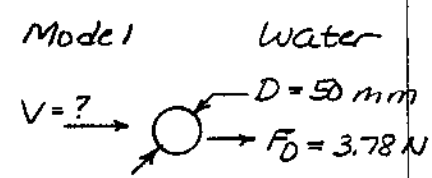
$V_p = 40.3 \frac{m}{s} \times (1) \times (10) \times (1) \times \left(\frac{1}{10}\right) = 40.3\ m/s$ ——— V_p

Problem 7.44

Given: Model test of weather balloon. Full-scale:



$$F_D = f(\rho, V, D, \mu, c)$$



Find: (a) Model test speed.
(b) Drag force on full-scale balloon.

Solution: Apply Buckingham procedure to obtain

$$\frac{F_D}{\rho V^2 D^2} = f\left(\frac{\mu}{\rho V D}, \frac{V}{c}\right) = f(Re, M)$$

For similarity $Re_p = Re_m$ and $M_p = M_m$. (Mach number criterion satisfied automatically because $M \approx 0$.) Assume $T = 20°C$.

$$Re_p = \frac{V_p D_p}{\nu_p} = Re_m = \frac{V_m D_m}{\nu_m}$$

{ Water (Table A.8)
{ Air (Table A.10)

$$V_m = V_p \frac{\nu_m}{\nu_p} \frac{D_p}{D_m} = 1.5 \frac{m}{s} \times 1 \times 10^{-6} \frac{m^2}{s} \times \frac{s}{1.51 \times 10^{-5} m^2} \times \frac{3\ m}{0.05\ m}$$

$$V_m = 5.96\ m/s$$ ← V_m

Then $\left(\frac{F_D}{\rho V^2 D^2}\right)_m = \left(\frac{F_D}{\rho V^2 D^2}\right)_p$

$$F_{D_p} = F_{D_m} \frac{\rho_p}{\rho_m} \frac{V_p^2}{V_m^2} \frac{D_p^2}{D_m^2}$$

$$= 3.78\ N \times 1.23 \frac{kg}{m^3} \times \frac{m^3}{999\ kg} \times \left(\frac{1.5\ m/s}{5.96\ m/s}\right)^2 \times \left(\frac{3.0\ m}{0.05\ m}\right)^2$$

$$F_{D_p} = 1.06\ N$$ ← F_{D_p}

Problem 7.45

Given: Airplane wing with chord length, $l = 5$ ft and span, $s = 30$ ft, is designed to move through standard air at speed, $V = 230$ ft/s. A model (1/10 scale) is to be tested in a water tunnel.

Find: (a) speed necessary in water tunnel to achieve dynamic similarity.
(b) ratio of forces measured in the model flow to those on the prototype airfoil.

Solution:

For an airfoil at a given angle of attack, we would expect the forces (e.g. drag) to be depended on l, s, V, ρ, and μ.

$F = F(l, s, V, \rho, \mu)$ From the Buckingham π theorem, with $n=6$, and $m = r = 3$, we would expect three dimensionless parameters.

$$\frac{F}{\rho V^2 l s} = f\left(\frac{\rho V l}{\mu}, \frac{l}{s}\right)$$

Thus for dynamically similar flows over geometrically similar airfoils (at the same angle of attack), then

$$\left.\frac{\rho V l}{\mu}\right)_m = \left.\frac{\rho V l}{\mu}\right)_p \qquad \text{(Assume } T = 59°\text{ F)}.$$

$$V_m = V_p \frac{\rho_p}{\rho_m} \frac{l_p}{l_m} \frac{\mu_m}{\mu_p} = 230 \frac{\text{ft}}{\text{sec}} \times \frac{0.00238}{1.94} \times 10 \times \frac{2.37 \times 10^{-5}}{3.74 \times 10^{-7}}$$

$$V_m = 179 \text{ ft/s}$$

For dynamically similar flows,

$$\left.\frac{F}{\rho V^2 l s}\right)_m = \left.\frac{F}{\rho V^2 l s}\right)_p$$

$$\therefore \frac{F_m}{F_p} = \frac{\rho_m}{\rho_p}\left(\frac{V_m}{V_p}\right)^2 \frac{l_m s_m}{l_p s_p} = \frac{1.94}{0.00238} \times \left(\frac{179}{230}\right)^2 \frac{1}{10} \times \frac{1}{10} = 4.94$$

This speed is high. The tunnel would have to be pressurized to minimize the chances of cavitation

Problem 7.46

Given: Fluid dynamic characteristics of a golf ball are to be tested using a model in a wind tunnel:
dependent variables: F_D, F_L
independent variables should include ω + d (dimple depth)
Golf pro can hit prototype ($D = 1.68$ in) at $V = 240$ ft/s and $\omega = 9000$ rpm. Prototype is to be modeled in wind tunnel with $V = 80$ ft/s.

Find: (a) suitable dimensionless parameters
(b) required diameter of model
(c) required rotational speed of model

Solution: Assume the functional dependence to be given by

$$F_D = F_D(D, V, \omega, d, \rho, \mu) \quad \text{and} \quad F_L = F_L(D, V, \omega, d, \rho, \mu)$$

From the Buckingham π-theorem, for $n = 7$ and $m = r = 3$, we would expect four dimensionless groups

$$\frac{F_D}{\rho V^2 D^2} = f\left(\frac{\rho V D}{\mu}, \frac{\omega D}{V}, \frac{d}{D}\right) \quad \text{and} \quad \frac{F_L}{\rho V^2 D^2} = g\left(\frac{\rho V D}{\mu}, \frac{\omega D}{V}, \frac{d}{D}\right)$$ π's

To determine the required diameter of the model,

$$\left.\frac{\rho V D}{\mu}\right)_m = \left.\frac{\rho V D}{\mu}\right)_p \qquad \therefore D_m = \frac{\rho_p}{\rho_m}\frac{V_p}{V_m}\frac{\mu_m}{\mu_p}D_p = 1 \times \frac{240}{80} \times 1 \times D_p$$

$$D_m = 3\,D_p = 3 \times 1.68 \text{ in} = 5.04 \text{ in.}$$ D_m

To determine the required rotational speed of the model,

$$\left.\frac{\omega D}{V}\right)_m = \left.\frac{\omega D}{V}\right)_p \qquad \therefore \omega_m = \omega_p \frac{D_p}{D_m}\frac{V_m}{V_p} = \omega_p \frac{1}{3} \times \frac{80}{240} = \frac{1}{9}\omega_p$$

$$\omega_m = \frac{1}{9}\omega_p = \frac{1}{9} \times 9000 \text{ rpm} = 1000 \text{ rpm}$$ ω_m

Problem 7.47

Given: Flight characteristics of a Frisbee are to be determined via a model test:
dependent parameters: F_D, F_L
independent parameters should include ω + h (roughness height).
Test is to be performed (using air) on a model (1/4 scale), which is to be geometrically, kinematically, and dynamically similar to the prototype. For prototype, $V_p = 20$ ft/s, $\omega_p = 100$ rpm

Find: (a) suitable dimensionless parameters.
(b) values of V_m and ω_m.

Solution: Assume the functional dependence is given by

$$F_D = F_D(D, V, \omega, h, \rho, \mu) \quad \text{and} \quad F_L = F_L(D, V, \omega, h, \rho, \mu)$$

From the Buckingham π-theorem, for n=7 and m=r=3, we would expect four dimensionless groups

$$\frac{F_D}{\rho V^2 D^2} = f\left(\frac{\rho V D}{\mu}, \frac{\omega D}{V}, \frac{h}{D}\right) \quad \text{and} \quad \frac{F_L}{\rho V^2 D^2} = g\left(\frac{\rho V D}{\mu}, \frac{\omega D}{V}, \frac{h}{D}\right)$$

To determine the required air speed, V_m,

$$\left.\frac{\rho V D}{\mu}\right)_m = \left.\frac{\rho V D}{\mu}\right)_p \qquad \therefore V_m = V_p \frac{\rho_p}{\rho_m} \frac{D_p}{D_m} \frac{\mu_m}{\mu_p} = V_p (1) \times 4 \times 1 = 4 V_p$$

$$V_m = 4 \times 20 \frac{ft}{s} = 80 \text{ ft/s}$$

To determine the required rotational speed, ω_m,

$$\left.\frac{\omega D}{V}\right)_m = \left.\frac{\omega D}{V}\right)_p \qquad \therefore \omega_m = \omega_p \frac{D_p}{D_m} \frac{V_m}{V_p} = \omega_p \times 4 \times 4 = 16\,\omega_p$$

$$\omega_m = 16 \times 100 \text{ rpm} = 1600 \text{ rpm}$$

Problem 7.48

Given: Model of hydrofoil boat (1:20 scale) is to be tested in water at 130°F. Prototype operates at speed of 60 knots in water at 45°F.

To model cavitation correctly, cavitation number must be duplicated.

Find: ambient pressure at which model test must be run.

Solution:

To duplicate the Froude number between model and prototype requires

$$\frac{V_m}{\sqrt{gL_m}} = \frac{V_p}{\sqrt{gL_p}} \qquad \text{or} \qquad \frac{V_m}{V_p} = \left(\frac{L_m}{L_p}\right)^{1/2} = \frac{1}{\sqrt{20}}$$

$$\text{and } V_m = \frac{1}{\sqrt{20}} V_p = \frac{1}{\sqrt{20}} 60 \text{ knot} = 13.4 \text{ knot}$$

For $Ca_m = Ca_p$, then

$$\left.\frac{p - p_v}{\frac{1}{2}\rho V^2}\right)_m = \left.\frac{p - p_v}{\frac{1}{2}\rho V^2}\right)_p$$

or

$$p_m = p_{v_m} + (p - p_v)_p \frac{V_m^2}{V_p^2} \qquad (\text{assuming } \rho_m \approx \rho_p)$$

and

$$p_m = p_{v_m} + (p - p_v)_p \cdot \frac{1}{20}$$

From the Table A.7, at $T = 130°F$ $\quad p_{v_m} = 2.23 \text{ psia}$

$T = 45 F \quad p_{v_p} = 0.15 \text{ psia}$

$$\therefore p_m = 2.23 \text{ psia} + (14.7 - 0.15) \text{ psia} \cdot \frac{1}{20}$$

$$p_m = 2.96 \text{ psia}$$

p_m

Problem 7.49

Given: SAE 10W oil at 80F flows in a horizontal pipe of diameter, D = 1 in. at an average speed $\bar{V} = 3$ ft/sec. The pressure drop, ΔP, is 65.3 psig over a length of 500 ft.
Water at 60°F flows through the same pipe under dynamically similar conditions.

Find: (a) the average speed of the water.
(b) the corresponding pressure drop.

Solution:

From Example Problem 7.2, we learn that pressure drop data for flow in a pipe are correlated by the functional relationship

$$\frac{\Delta P}{\rho \bar{V}^2} = f\left(\frac{\mu}{\rho \bar{V} D}, \frac{l}{D}, \frac{e}{D}\right)$$

For water flow and oil flow in the same pipe to be dynamically similar requires that

$$\left.\frac{\mu}{\rho \bar{V} D}\right)_{H_2O} = \left.\frac{\mu}{\rho \bar{V} D}\right)_{oil}$$

or $\bar{V}_{H_2O} = \left(\frac{\mu}{\rho}\right)_{H_2O} \left(\frac{\rho}{\mu}\right)_{oil} \bar{V}_{oil} = \frac{\nu_{H_2O}}{\nu_{oil}} \times \bar{V}_{oil}$

From Fig A 3 $\quad \nu_{oil}$ at 80°F (26.7°C) = 7×10^{-5} m²/s = 7.53×10^{-4} ft²/s

From Table A.7 ν_{H_2O} at 60°F = 1.21×10^{-5} ft²/s

$$\therefore \bar{V}_{H_2O} = 1.21 \times 10^{-5}\ \text{ft}^2/\text{s} \times \frac{1}{7.53 \times 10^{-4}\ \text{ft}^2/\text{s}} \times 3\ \frac{\text{ft}}{\text{sec}} = 0.0482\ \text{ft/sec} \quad \longleftarrow \bar{V}_{H_2O}$$

Then

$$\left.\frac{\Delta P}{\rho \bar{V}^2}\right)_{oil} = \left(\frac{\Delta P}{\rho \bar{V}^2}\right)_{H_2O}$$

$$\Delta P_{H_2O} = \frac{\rho_{H_2O}}{\rho_{oil}} \times \frac{\bar{V}^2_{H_2O}}{\bar{V}^2_{oil}} \times \Delta P_{oil}$$

From Table A.2 Appendix A, S.G lubricating oil = 0.88

$$\therefore \Delta P_{H_2O} = \frac{1}{0.88} \times \frac{(0.0482)^2}{3^2} \times 65.3\ \text{psig} = 0.019\ \text{psig} \quad \longleftarrow \Delta P_{H_2O}$$

Problem 7.50

Given: $\frac{1}{8}$-scale model of tractor-trailer rig tested in pressurized wind tunnel.

$$W = 0.305\ m \qquad V = 75.0\ m/s$$
$$H = 0.476\ m \qquad F_D = 128\ N$$
$$L = 2.48\ m \qquad \rho = 3.23\ kg/m^3$$

Find: (a) Aerodynamic drag coefficient of model.
(b) Compare Reynolds number for model with prototype at $V = 55$ mph.
(c) Aerodynamic drag on prototype at $V = 55$ mph, with headwind, $V_w = 10$ mph.

Solution: Defining equations: $F_D = C_D A \frac{1}{2}\rho V^2$; $Re = \frac{\rho V L}{\mu}$

Then $C_{D_m} = \dfrac{F_{D_m}}{\frac{1}{2}\rho_m V_m^2 A_m}$

Assume $A_m = W_m H_m = 0.305\ m \times 0.476\ m = 0.145\ m^2$

$$C_{D_m} = 2 \times 128\ N \times \frac{m^3}{3.23\ kg} \times \frac{s^2}{(75)^2 m^2} \times \frac{1}{0.145\ m^2} \times \frac{kg \cdot m}{N \cdot s^2} = 0.0972$$ C_{D_m}

$$\frac{Re_m}{Re_p} = \frac{\rho_m V_m L_m}{\mu_m} \times \frac{\mu_p}{\rho_p V_p L_p} = \frac{\rho_m}{\rho_p}\frac{V_m}{V_p}\frac{L_m}{L_p} \quad (\text{assume air: } \mu_m = \mu_p)$$

For the prototype, $V_p = 55 \frac{mi}{hr} \times 5280 \frac{ft}{mi} \times \frac{hr}{3600\ s} \times 0.305 \frac{m}{ft} = 24.6\ m/s$

$$\frac{Re_m}{Re_p} = \left(\frac{3.23}{1.23}\right)\left(\frac{75.0}{24.6}\right)\left(\frac{1}{8}\right) = 1.00 \qquad \therefore Re_m = Re_p$$ Re

Since $Re_m = Re_p$, then $C_{D_p} = C_{D_m}$, assuming geometric and kinematic similarity, so

$$F_{D_p} = C_{D_p} A_p \frac{1}{2}\rho_p (V_p + V_w)^2$$

With $V_w = 10$ mph, $V_p + V_w = \frac{65}{55} \times 24.6\ m/s = 29.1\ m/s$

Thus

$$F_{D_p} = 0.0972 \times (8)^2 0.145\ m^2 \times \frac{1}{2} \times 1.23 \frac{kg}{m^3} (29.1)^2 \frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m}$$

$$F_{D_p} = 470\ N$$ F_{D_p}

Problem 7.51

Given: The frequency, f, of vortex shedding from the rear of a bluff cylinder is a function of ρ, V, d, μ

Two cylinders in standard air, $\frac{d_1}{d_2} = 2$

Find: (a) functional relationship for f, using dimensional analysis
(b) V_1/V_2 for dynamic similarity
(c) f_1/f_2

Solution: Apply Buckingham π theorem.

① $f \quad \rho \quad V \quad d \quad \mu$ $\qquad n = 5$ parameters

② Select M, L, t as primary dimensions

③ $f \quad \rho \quad V \quad d \quad \mu$

$\frac{1}{t} \quad \frac{M}{L^3} \quad \frac{L}{t} \quad L \quad \frac{M}{Lt}$ $\qquad r = 3$ primary dimensions

④ ρ, V, d $\qquad m = r = 3$ repeating parameters

⑤ Then $n - m = 2$ dimensionless groups will result

Setting up dimensional equations

$\pi_1 = \rho^a V^b d^c f$	$\pi_2 = \rho^a V^b d^c \mu$
$M^0L^0t^0 = \left(\frac{M}{L^3}\right)^a \left(\frac{L}{t}\right)^b L^c \frac{1}{t}$	$M^0L^0t^0 = \left(\frac{M}{L^3}\right)^a \left(\frac{L}{t}\right)^b L^c \frac{M}{Lt}$
Equating exponents,	Equating exponents,
$M: \quad 0 = a$	$M: \quad 0 = a+1 \quad \therefore a = -1$
$L: \quad 0 = -3a+b+c \quad c = 1$	$L: \quad 0 = -3a+b+c-1 \quad c = -1$
$t: \quad 0 = -b-1 \quad \therefore b = -1$	$t: \quad 0 = -b-1 \quad \therefore b = -1$
$\therefore \pi_1 = \frac{fd}{V}$	$\therefore \pi_2 = \frac{\mu}{\rho V d}$

⑥ Check using F, L, t dimensions

$\pi_1 = \frac{1}{t} \cdot L \cdot \frac{t}{L} = [1]$ ✓ $\qquad \pi_2 = \frac{Ft}{L^2} \cdot \frac{L^4}{Ft^2} \cdot \frac{t}{L} \cdot \frac{1}{L} = [1]$ ✓

$$\therefore \quad \frac{fd}{V} = g\left(\frac{\rho V d}{\mu}\right) \qquad \longleftarrow \frac{fd}{V}$$

To achieve dynamic similarity between geometrically similar flows, we must duplicate all but one of the dimensionless groups

$$\left.\frac{\rho V d}{\mu}\right)_1 = \left.\frac{\rho V d}{\mu}\right)_2 \Rightarrow \frac{V_1}{V_2} = \frac{\rho_2}{\rho_1} \frac{\mu_1}{\mu_2} \frac{d_2}{d_1} = 1 \times 1 \times \frac{1}{2} = \frac{1}{2} \qquad \longleftarrow V_1/V_2$$

If $\left.\frac{\rho V d}{\mu}\right)_1 = \left.\frac{\rho V d}{\mu}\right)_2$, then $\left.\frac{fd}{V}\right)_1 = \left.\frac{fd}{V}\right)_2$

$$\text{and} \quad \frac{f_1}{f_2} = \frac{V_1}{V_2} \frac{d_2}{d_1} = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} \qquad \longleftarrow f_1/f_2$$

Problem 7.52

The aerodynamic behavior of a flying insect is to be investigated in a wind tunnel using a ten-times scale model. If the insect flaps its wings 50 times a second when flying at 1.25 m/s, determine the wind tunnel air speed and wing oscillation frequency required for dynamic similarity. Do you expect that this would be a successful or practical model for generating an easily measurable wing lift? If not, can you suggest a different fluid (e.g., water, or air at a different pressure and/or temperature) that would produce a better modeling?

Given: 10-times scale model of flying insect

Find: Required model speed and oscillation frequency

Solution

From Appendix A (inc. Fig. A.3) $\rho_{air} = 1.24 \cdot \frac{kg}{m^3}$ $\nu_{air} = 1.5 \times 10^{-5} \cdot \frac{m^2}{s}$

The given data is $\omega_{insect} = 50\,Hz$ $V_{insect} = 1.25 \cdot \frac{m}{s}$ $L_{ratio} = \frac{1}{10}$

For dynamic similarity the following dimensionless groups must be the same in the insect and m

$$\frac{V_{insect} \cdot L_{insect}}{\nu_{air}} = \frac{V_m \cdot L_m}{\nu_{air}} \qquad \frac{\omega_{insect} \cdot L_{insect}}{V_{insect}} = \frac{\omega_m \cdot L_m}{V_m}$$

Hence

$$V_m = V_{insect} \cdot \frac{L_{insect}}{L_m} = V_{insect} \cdot L_{ratio} = 1.25 \cdot \frac{m}{s} \times \frac{1}{10} \qquad V_m = 0.125 \frac{m}{s}$$

Also $\omega_m = \omega_{insect} \cdot \frac{V_m}{V_{insect}} \cdot \frac{L_{insect}}{L_m} = \omega_{insect} \cdot \frac{V_m}{V_{insect}} \cdot L_{ratio} = 50 \cdot Hz \times \frac{0.125}{1.25} \times \frac{1}{10}$

$$\omega_m = 0.5 \cdot Hz$$

It is unlikely measurable wing lift can be measured at such a low wing frequency (unless the measured lift was averaged, using an integrator circuit). Maybe try hot air for the model

For hot air try $\nu_{hot} = 2 \times 10^{-5} \cdot \frac{m^2}{s}$ instead of $\nu_{air} = 1.5 \times 10^{-5} \cdot \frac{m^2}{s}$

Hence $\frac{V_{insect} \cdot L_{insect}}{\nu_{air}} = \frac{V_m \cdot L_m}{\nu_{hot}}$

$$V_m = V_{insect} \cdot \frac{L_{insect}}{L_m} \cdot \frac{\nu_{hot}}{\nu_{air}} = 1.25 \cdot \frac{m}{s} \times \frac{1}{10} \times \frac{2}{1.5} \qquad V_m = 0.167 \frac{m}{s}$$

Also $\omega_m = \omega_{insect} \cdot \frac{V_m}{V_{insect}} \cdot \frac{L_{insect}}{L_m} = \omega_{insect} \cdot \frac{V_m}{V_{insect}} \cdot L_{ratio} = 50 \cdot Hz \times \frac{0.167}{1.25} \times \frac{1}{10}$

$$\omega_m = 0.67 \cdot Hz$$

Hot air does not improve things much

Finally, try modeling in water $\nu_w = 9 \times 10^{-7} \cdot \frac{m^2}{s}$

Hence $$\frac{V_{insect} \cdot L_{insect}}{\nu_{air}} = \frac{V_m \cdot L_m}{\nu_w}$$

$$V_m = V_{insect} \cdot \frac{L_{insect}}{L_m} \cdot \frac{\nu_w}{\nu_{air}} = 1.25 \cdot \frac{m}{s} \times \frac{1}{10} \times \frac{9 \times 10^{-7}}{1.5 \times 10^{-5}}$$

$$V_m = 0.0075 \frac{m}{s}$$

Also $$\omega_m = \omega_{insect} \cdot \frac{V_m}{V_{insect}} \cdot \frac{L_{insect}}{L_m} = \omega_{insect} \cdot \frac{V_m}{V_{insect}} \cdot L_{ratio} = 50 \cdot Hz \times \frac{0.0075}{1.25} \times \frac{1}{10}$$

$$\omega_m = 0.03 \cdot Hz$$

This is even worse! It seems the best bet is hot (very hot) air for the wind tunnel.

Problem 7.53

Given: Model test of tractor-trailer rig in standard air.

$F_D = f(A, V, \rho, \mu)$; scale is 1:4 ; $A_m = 0.625\ m^2$

At $V_m = 89.6$ m/s, $F_D = 2.46$ kN

Find: (a) Dimensionless parameters.
(b) Conditions for dynamic similarity.
(c) Drag force on prototype at $V_p = 22.4$ m/s (no wind).
(d) Power to overcome aero drag.

Solution: ① F_D A V ρ μ | ② $M\,L\,t$

③ $\frac{ML}{t^2}$ L^2 $\frac{L}{t}$ $\frac{M}{L^3}$ $\frac{M}{Lt}$ | ④ $\rho V A$

⑤ $\Pi_1 = \rho^a V^b A^c F_D = M^0 L^0 t^0$ $\qquad$ $\Pi_2 = \rho^a V^b A^c \mu$

M: $a + 1 = 0$	$a = -1$
L: $-3a + b + 2c + 1 = 0$	$c = -1$
t: $-b - 2 = 0$	$b = -2$

M: $a + 1 = 0$	$a = -1$
L: $-3a + b + 2c - 1 = 0$	$c = -\frac{1}{2}$
t: $-b - 1 = 0$	$b = -1$

$$\Pi_1 = \frac{F_D}{\rho V^2 A} \qquad \Pi_2 = \frac{\mu}{\rho V A^{1/2}}$$

Π_1, Π_2

⑥ $\Pi_1 = F \times \frac{L^4}{Ft^2} \times \frac{t^2}{L^2} \times \frac{1}{L^2} = 1$ ✓✓ $\qquad$ $\Pi_2 = \frac{Ft}{L^2} \times \frac{L^4}{Ft^2} \times \frac{t}{L} \times \frac{1}{L} = 1$ ✓✓

For dynamic similarity, must have geometric and kinematic similarity and $Re_m = Re_p$. Then $\left.\frac{F_D}{\rho V^2 A}\right)_m = \left.\frac{F_D}{\rho V^2 A}\right)_p$

For the prototype,

$$F_{D_p} = F_{D_m} \frac{\rho_p}{\rho_m} \left(\frac{V_p}{V_m}\right)^2 \frac{A_p}{A_m} = F_{D_m} \left(\frac{1.23}{1.23}\right)\left(\frac{22.4}{89.6}\right)^2 (4)^2 = F_{D_m} = 2.46\ kN$$

F_{D_p}

The power requirement is

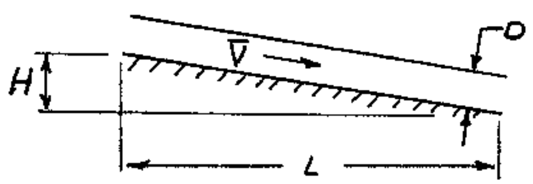
$$P = F_{D_p} V_p = 2.46\ kN \times 22.4\ \frac{m}{s} \times \frac{W \cdot s}{N \cdot m} = 55.1\ kW\ (73.9\ hp)$$

P

Problem 7.54

Given: Model glacier using glycerine. Assume ice is Newtonian and 10^6 x as viscous.

$D = 15\ m$, $H = 1.5\ m$, $L = 1850\ m$ } model



ρ, g, μ

In lab test, model instructor reappears in $\tau = 9.6\ hr$.

Find: (a) Develop suitable dimensionless parameters.
(b) Estimate time when instructor will reappear.

Solution: ① $\bar{V}$ ρ g μ D H L $n = 7$

② MLt $\frac{L}{t}$ $\frac{M}{L^3}$ $\frac{L}{t^2}$ $\frac{M}{Lt}$ L L L $m = r = 3$

④ Choose ρ, g, D as repeating variables: $n - m = 7 - 3 = 4$ parameters

⑤ $\Pi_1 = \rho^a g^b D^c \bar{V} = M^0 L^0 t^0$

M: $a + 0 = 0$ | $a = 0$
L: $-3a + b + c + 1 = 0$ | $c = -b - 1 = -\frac{1}{2}$
t: $-2b - 1 = 0$ | $b = -\frac{1}{2}$

$\Pi_1 = \frac{\bar{V}}{\sqrt{gD}}$ (Froude no.)

$\Pi_2 = \rho^a g^b D^c \mu = M^0 L^0 t^0$

M: $a + 1 = 0$ | $a = -1$
L: $-3a + b + c - 1 = 0$ | $c = 3a - b + 1 = -\frac{3}{2}$
t: $-2b - 1 = 0$ | $b = -\frac{1}{2}$

$\Pi_2 = \frac{\mu}{\rho g^{1/2} D^{3/2}} \sim \frac{\mu}{\rho \sqrt{gD}\, D}$ (Reynolds no)

$\Pi_3 = \frac{H}{D}$, $\Pi_4 = \frac{L}{D}$ (by inspection) ← geometric similarity

⑥ Check: obvious from forms above. $\Pi_1 = f(\Pi_2, \Pi_3, \Pi_4)$

For dynamic similarity, $\Pi_{2m} = \Pi_{2p} = \frac{\mu_m}{\rho_m g_m^{1/2} D_m^{3/2}} = \frac{\mu_p}{\rho_p g_p^{1/2} D_p^{3/2}}$, so

$$\frac{D_m}{D_p} = \left(\frac{\mu_m}{\mu_p}\frac{\rho_p}{\rho_m}\right)^{2/3} = \left(\frac{\mu_m}{\mu_p}\frac{SG_p}{SG_m}\right)^{2/3} = \left(\frac{1}{10^6} \times \frac{0.92}{1.26}\right)^{2/3} = 8.11 \times 10^{-5}$$

{ $SG_{ice} = 0.92$ (Table A.1) ; $SG_{glycerin} = 1.26$ (A.2) }

So $\frac{L_m}{L_p} = 8.11 \times 10^{-5}$; $L_m = 8.11 \times 10^{-5} L_p = 8.11 \times 10^{-5} \times 1850\ m = 0.150\ m$

From Π_1, $\frac{\bar{V}_m}{\bar{V}_p} = \sqrt{\frac{D_m}{D_p}} = 9.00 \times 10^{-3}$

The time to reappear is $\tau = L/\bar{V}$, so $\tau_p = L_p/\bar{V}_p$, $\tau_m = L_m/\bar{V}_m$

$$\frac{\tau_p}{\tau_m} = \frac{L_p}{L_m}\frac{\bar{V}_m}{\bar{V}_p} = \frac{D_p}{D_m}\sqrt{\frac{D_m}{D_p}} = \sqrt{\frac{D_p}{D_m}} = \frac{1}{9.00 \times 10^{-3}} = 111$$

Thus $\tau_p = 111\, \tau_m = 111 \times 9.6\ hr = 1070\ hr$ (~45 days) ← τ_p

{ The instructor will reappear before the semester ends! }

Problem 7.55

Given: Submarine model (1:30 scale) to be tested in fresh water under two conditions:

(1) on the surface at 20 kt (prototype)

(2) far below the surface at 0.5 kt (prototype)

Find: (a) Speed for model test on surface

(b) Speed for model test submerged

(c) Ratio of full-scale to model drag force.

Solution: On the surface, match the Froude number, $Fr = \frac{V}{\sqrt{gL}}$

Thus $Fr_m = \frac{V_m}{\sqrt{gL_m}} = Fr_p = \frac{V_p}{\sqrt{gL_p}}$ or $V_m = V_p\sqrt{\frac{L_m}{L_p}}$

For 1:30 scale,

$$V_m = 20\,kt\sqrt{\frac{1}{30}} = 3.65\,kt$$

$$V_m = 3.65\,\frac{nm}{hr} \times 1852\,\frac{m}{nm} \times \frac{hr}{3600\,s} = 1.88\,m/s$$ V_m

Submerged, match the Reynolds number, $Re = \frac{\rho VL}{\mu} = \frac{VL}{\nu}$

Thus $Re_m = \frac{V_m L_m}{\nu_m} = Re_p = \frac{V_p L_p}{\nu_p}$ or $V_m = V_p \frac{L_p}{L_m}\frac{\nu_m}{\nu_p}$

From Table A.2, for seawater, $SG = 1.025$ and $\mu = 1.08 \times 10^{-3}\,N \cdot s/m^2$ at 20°C, Thus

$$\nu_p = \frac{\mu_p}{\rho_p} = \frac{\mu_p}{SG\,\rho_{H_2O}} = 1.08 \times 10^{-3}\,\frac{N \cdot s}{m^2} \times \frac{m^3}{(1.025)\,1000\,kg} \times \frac{kg \cdot m}{N \cdot s^2} = 1.05 \times 10^{-6}\,\frac{m^2}{s}$$

From Table A.8, fresh water at 20°C has $\nu = 1.00 \times 10^{-6}\,m^2/s$.

For 1:30 scale

$$V_m = 0.5\,kt \times \frac{30}{1} \times 1.00 \times 10^{-6}\,\frac{m^2}{s} \times \frac{s}{1.05 \times 10^{-6}\,m^2} = 14.3\,kt$$

$$V_m = 14.3\,\frac{nm}{hr} \times 1852\,\frac{m}{nm} \times \frac{hr}{3600\,s} = 7.36\,m/s$$ V_m

Under dynamically similar conditions the drag coefficients, $C_D = \frac{F_D}{\rho V^2 A}$, will be identical. Thus

$$\frac{F_p}{\rho_p V_p^2 L_p^2} = \frac{F_m}{\rho_m V_m^2 L_m^2} \quad \text{or} \quad F_p = F_m \frac{\rho_p}{\rho_m}\frac{V_p^2}{V_m^2}\frac{L_p^2}{L_m^2}$$

$$F_p = F_m\,\frac{1.025}{0.999} \times \left(\frac{0.5}{14.3}\right)^2\left(\frac{30}{1}\right)^2 = 1.13 \text{ (submerged)},\ 2.77 \times 10^4 \text{ (surface)}$$ F_p/F_m

Problem 7.56

Given: Automobile (prototype) to travel at 100 km/hr through standard air. Model, $L_m/L_p = \frac{1}{5}$, to be tested in water.
The lowest pressure coefficient is $C_p = -1.4$ at the location of minimum static pressure on the surface.
Onset of cavitation occurs at $Ca = 0.5$

Find: (a) factors necessary to ensure kinematic similarity in tests.
(b) water speed to be used.
(c) corresponding ratio of drag forces
(d) minimum tunnel pressure to avoid cavitation.

Solution:

To assure kinematic similarity:
(1) model and protype must be geometrically similar.
(2) model must be submerged in flow to avoid surface effects.
(3) cavitation effects must be absent in model test

To determine model test speed, note that flows will be dynamically similar if

$$Re_m = Re_p \quad , \text{ i.e } \left.\frac{\rho D V}{\mu}\right)_m = \left.\frac{\rho D V}{\mu}\right)_p \quad \text{or} \quad \left.\frac{DV}{\nu}\right)_m = \left.\frac{DV}{\nu}\right)_p$$

Hence, $V_m = V_p \frac{\nu_m}{\nu_p}\frac{D_p}{D_m}$. For standard air, $\nu_p = 1.46\times10^{-5}\ m^2/s$.
Assume water at 20°C. Table A.8, Appendix A gives $\nu_m = 1.0\times10^{-6}\ m^2/s$

$$V_m = 100\frac{km}{hr}\times\frac{hr}{3600s}\times\frac{1.0\times10^{-6}}{1.46\times10^{-5}}\times 5\times10^{3}\frac{m}{km} = 9.51\ m/s \qquad V_m$$

Then $\left.\frac{F_D}{\rho V^2 D^2}\right)_m = \left.\frac{F_D}{\rho V^2 D^2}\right)_p$ and $\frac{F_{Dp}}{F_{Dm}} = \frac{(\rho V^2 D^2)_p}{(\rho V^2 D^2)_m}$

$$\frac{F_{Dp}}{F_{Dm}} = \frac{1.23}{999}\times\left(\frac{27.8}{9.51}\right)^2\times\left(\frac{5}{1}\right)^2 = 0.262 \qquad F_{Dp}/F_{Dm}$$

For $Ca = 0.5$, then $\frac{p - p_v}{\frac{1}{2}\rho V^2} = 0.5$ and local pressure $p = p_v + \frac{1}{4}\rho V^2$

For water at 20°C, $p_v = 2.34$ kPa and

$$p_{min} = 2.34\ kPa + \frac{1}{4}\times 999\frac{kg}{m^3}\times(9.51)^2\frac{m^2}{s^2}\times\frac{N\cdot s^2}{kg\cdot m} = 24.9\ kPa$$

For $C_{p_{min}} = -1.4 = \frac{p_{min} - p_\infty}{\frac{1}{2}\rho V^2}$, then

$$p_\infty = p_{min} + 0.7\rho V^2$$

$$p_\infty = 24.9\ kPa + 0.7\times 999\frac{kg}{m^3}\times(9.51)^2\frac{m^2}{s^2} = 88.1\ kPa \qquad p_\infty$$

Given: The drag force on a circular cylinder immersed in a water flow can be expressed as

$$F_D = f(D, l, V, \rho, \mu)$$

The static pressure distribution on a circular cylinder can be expressed in terms of the dimensionless pressure coefficient

$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho V^2}$$

At the location of minimum static pressure on the cylinder surface, $C_p = -2.4$. The onset of cavitation occurs at $Ca = 0.5$

Find: (a) expression for dimensionless drag force
(b) an estimate of maximum speed V at which cylinder could be towed in water (at p_{atm}) without causing cavitation

Solution:

$F_D = f(D, l, V, \rho, \mu)$. From the Buckingham π-theorem, for $n=6$, with $m = r = 3$, we would expect three dimensionless groups.

$$\frac{F}{\rho V^2 D^2} = f_1\left(\frac{l}{D}, \frac{\rho V D}{\mu}\right)$$

$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho V^2} \qquad Ca = \frac{p - p_v}{\frac{1}{2}\rho V^2}$$

For $C_{p_{min}} = -2.4$, $p_{min} - p_\infty = \frac{1}{2}\rho V_{max}^2 (C_{p_{min}})$ $\therefore p_{min} = p_\infty + \frac{1}{2}\rho V_{max}^2 C_{p_{min}}$

For $Ca = \frac{1}{2}$, $p_{min} - p_v = \frac{1}{2}\rho V_{max}^2 Ca$ $\therefore p_{min} = p_v + \frac{1}{2}\rho V_{max}^2 Ca$

Equation expressions for p_{min},

$$p_\infty + \frac{1}{2}\rho V_{max}^2 C_{p_{min}} = p_v + \frac{1}{2}\rho V_{max}^2 Ca$$

$$\frac{1}{2}\rho V_{max}^2 \left[Ca - C_{p_{min}}\right] = p_\infty - p_v$$

$$V_{max} = \left\{\frac{2(p_\infty - p_v)}{\rho\left[Ca - C_{p_{min}}\right]}\right\}^{1/2}$$

For water at 68°F (Table A.7), $p_v = 0.339$ psia

$$\therefore V_{max} = \left\{2 \times (14.7 - 0.339)\frac{lbf}{in^2} \times \frac{ft^3}{1.94\,slug} \times \frac{1}{[0.5 - (-2.4)]} \times 144\frac{in^2}{ft^2} \times \frac{slug \cdot ft}{lbf \cdot s^2}\right\}^{1/2}$$

$V_{max} = 27.1$ ft/s (8.26 m/s) ⟵

Problem 7.58

Given: A model ($\frac{1}{10}$ scale) of a tractor-trailer rig is tested in a wind tunnel; $A_m = 1.08\ ft^2$. For $V_m = 250\ ft/s$, $F_{D_m} = 76.3\ lbf$

Find: (a) drag coefficient for the model.
(b) F_{D_p} at $V_p = 55\ mi/hr$ if $C_{D_p} = C_{D_m}$
(c) V_m if $V_p = 55\ mi/hr$
(d) Is answer to part (c) reasonable.

Solution:

$C_D = \frac{F_D}{\frac{1}{2}\rho V^2 A}$ For the model assuming air at STP,

$$C_D = \frac{F_D}{\frac{1}{2}\rho V^2 A} = \frac{76.3\ lbf}{0.5} \times \frac{ft^3}{0.002377\ slug} \times \frac{s^2}{(250)^2 ft^2} \times \frac{1}{1.08 ft^2} \times \frac{slug \cdot ft}{lbf \cdot s^2}$$

$C_{D_m} = 0.951$ ← C_{D_m}

$F_{D_p} = \frac{1}{2}\rho V_p^2 A_p C_{D_p}$ $C_{D_p} = C_{D_m} = 0.951$ $A_p = \left(\frac{L_p}{L_m}\right)^2 A_m = 100 A_m$

$$F_{D_p} = \frac{1}{2} \times 0.002377 \frac{slug}{ft^3} \times \left(55 \frac{mi}{hr} \times 5280 \frac{ft}{mi} \times \frac{hr}{3600 s}\right)^2 \times 100 \times 1.08 ft^2 \times 0.951 \times \frac{lbf \cdot s^2}{ft \cdot slug}$$

$F_{D_p} = 794\ lbf$ ← F_{D_p}

For dynamic similarity between model and prototype

$$\left.\frac{\rho V L}{\mu}\right)_m = \left.\frac{\rho V L}{\mu}\right)_p \quad \text{or} \quad V_m = V_p \frac{\rho_p}{\rho_m} \frac{L_p}{L_m} \frac{\mu_m}{\mu_p} = V_p \times 1 \times 10 \times 1$$

$V_m = 10 V_p = 550\ mi/hr$ ← V_m

$$V_m = 550 \frac{mi}{hr} \times 5280 \frac{ft}{mi} \times \frac{hr}{3600 s} = 807\ ft/s$$

For air at standard conditions, the speed of sound, $c = \sqrt{kRT}$

$$c = \left(1.4 \times 53.3 \frac{ft \cdot lbf}{lbm \cdot R} \times 519 R \times \frac{32.2\ lbm}{slug} \times \frac{slug \cdot ft}{lbf \cdot s^2}\right)^{1/2} = 1117\ ft/s$$

$$M = \frac{V}{c} = \frac{807}{1117} = 0.72$$

At this value of M, compressibility would be important in the model test. Thus, the speed is not practical. ←

Problem 7.59

Given: Recommended procedures for wind tunnel tests of trucks & buses suggest:

$A_{model} / A_{test\ section} < 0.05$

$h_{model} / h_{test\ section} < 0.30$ (h = height)

w_{model} at max yaw (20°) $/ w_{test\ sect} < 0.30$ (w = projected width)

$V_{max} < 300$ ft/s

Wind tunnel test section is $h = 1.5$ ft, $w = 2$ ft.

Prototype has: $h = 13.5$ ft, $w = 8$ ft, Length = 65 ft.

Find: (a) scale ratio of largest model that meets the recommended criteria.
(b) Use results of Ex. Prob 7.5 to assess whether an adequate value of Re can be achieved in the test facility.

Solution:

Let s = scale ratio. Then $h_m = s h_p$, $w_m = s w_p$, $l_m = s l_p$

(1) height criteria

$$h_m = 0.30\, h_{test\ section} = 0.3(1.5\text{ ft}) = 0.45\text{ ft}$$

$$s = \frac{h_m}{h_p} = \frac{0.45\text{ ft}}{13.5\text{ ft}} = 0.0333 \qquad \left\{ \frac{1}{s} = 30 \right\}$$

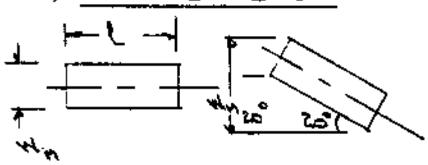
(2) frontal area criteria

$$A_{model} = 0.05\, A_{test\ sect} = 0.05 \times 1.5\text{ ft} \times 2\text{ ft} = 0.15\text{ ft}^2$$

$$A_{model} = s^2 A_p = s^2 [13.5\text{ ft} \times 8\text{ ft}] = s^2 (108)\text{ ft}^2 = 0.15$$

$$\therefore s = \left(\frac{0.15}{108}\right)^{1/2} = 0.0373 \qquad \left\{ \frac{1}{s} = 26.8 \right\}$$

(3) width criteria



$$w_{m_{20^\circ}} = l_m \sin 20^\circ + w_m \cos 20^\circ$$

$$= s(l_p \sin 20^\circ + w_p \cos 20^\circ)$$

$$w_{m_{20^\circ}} = s[65 \sin 20^\circ + 8 \cos 20^\circ]\text{ ft} = 29.7\, s\text{ ft}$$

From constraint, $w_{m_{20^\circ}} = 0.30\, w_{test\ sect} = 0.30(2\text{ ft}) = 0.6\text{ ft}$

$\therefore 0.6\text{ ft} = 29.7 s$ ft and $s = 0.0202$ $\left\{ \frac{1}{s} = 49.5 \right\}$

The width criteria is the most stringent $\therefore s = \frac{1}{50}$

Model $= \frac{1}{50}$ Prototype.

From Ex. Prob 7.5, C_D = const for $Re > 4 \times 10^5$

with $Re = \dfrac{\rho V w}{\mu} = \dfrac{V w}{\nu}$ Standard air $\nu = 1.57 \times 10^{-4}$ ft²/s

For current model test, $Re = 300\frac{\text{ft}}{\text{sec}} \times \left(\frac{1}{50} \times 8\text{ ft}\right) \times \frac{\text{s}}{1.57 \times 10^{-4}\text{ ft}^2} = 3.06 \times 10^5$

$\therefore$ Adequate Re cannot be achieved

Problem 7.60

Given: Circular container partially filled with water is rotated about its axis at constant angular velocity, ω.

The velocity V_θ is a function of: location, r, time from start, τ, angular velocity, ω, density, ρ, and viscosity, μ.

Water is replaced with honey and cylinder is rotated at the same value of ω.

Find: (a) dimensionless parameters that characterize the problem.
(b) Determine whether honey will attain steady state motion as quickly as water.
(c) Explain why Re would not be an important parameter in scaling the steady state motion of the liquid.

Solution:

$$V_\theta = V_\theta(\omega, r, \tau, \rho, \mu)$$

From the Buckingham π-theorem, for $n=6$ and $m=r=3$, we would expect three dimensionless groups.

$$\frac{V_\theta}{\omega r} = f\left(\frac{\mu}{\rho\omega r^2}, \omega\tau\right)$$ $\frac{V_\theta}{\omega r}$

From the above results $\pi_2 = \frac{\mu}{\rho\omega r^2}$ contains the fluid properties ρ, μ.

$\pi_3 = \omega\tau$ contains the time τ

$$\pi_2\pi_3 = \frac{\mu}{\rho\omega r^2}\omega\tau = \frac{\mu\tau}{\rho r^2} = \frac{\nu\tau}{r^2} \qquad \text{where } \nu = \mu/\rho$$

For steady flow at the same radius

$$\left.\frac{\nu\tau}{r^2}\right)_{Honey} = \left.\frac{\nu\tau}{r^2}\right)_{water}$$

$$\therefore \tau_H = \frac{\nu_{water}}{\nu_{Honey}}\tau_{water}$$

Since $\nu_{Honey} > \nu_{water}$ ($\mu_{Honey} > \mu_{water}$ and $\rho_H \approx \rho_w$)

$$\tau_H < \tau_{water}$$ τ_H

At steady state conditions, we have solid body rotation there are no viscous forces. Hence Re is not important.

Given: Power, P, to drive a fan depends on ρ, Q, D, and ω.

Condition	D(mm)	Q (m³/s)	ω(rpm)
1	200	0.4	2400
2	400	?	1850

Find: Volume flow rate at condition 2, for dynamic similarity.

Solution: step ① $P \quad \rho \quad Q \quad D \quad \omega$

step ② MLt ③: $\frac{ML^2}{t^3} \quad \frac{M}{L^3} \quad \frac{L^3}{t} \quad L \quad \frac{1}{t}$ ④ ρ, ω, D

⑤ $\Pi_1 = \rho^a \omega^b D^c P = M^0 L^0 t^0$

$M: a + 1 = 0 \quad | \quad a = -1$

$L: -3a + c + 2 = 0 \quad | \quad c = 3a - 2 = -5$

$t: -b - 3 = 0 \quad | \quad b = -3$

$$\Pi_1 = \frac{P}{\rho \omega^3 D^5}$$

$\Pi_2 = \rho^a \omega^b D^c Q = M^0 L^0 t^0$

$M: a + 0 = 0 \quad | \quad a = 0$

$L: -3a + c + 3 = 0 \quad | \quad c = -3$

$t: -b - 1 = 0 \quad | \quad b = -1$

$$\Pi_2 = \frac{Q}{\omega D^3}$$

⑥ $\Pi_1 = \frac{FL}{t} \times \frac{L^4}{Ft^2} \times t^3 \times \frac{1}{L^5} = \frac{FL^5t^3}{FL^5t^3} = 1$ ✓✓ $\quad \Pi_2 = \frac{L^3}{t} \times t \times \frac{1}{L^3} = \frac{L^3 t}{L^3 t} = 1$ ✓✓

Thus $\Pi_1 = f(\Pi_2)$ or $\frac{P}{\rho \omega^3 D^5} = f\left(\frac{Q}{\omega D^3}\right)$

For dynamic similarity, need geometric and kinematic similarity and

$$\frac{Q_1}{\omega_1 D_1^3} = \frac{Q_2}{\omega_2 D_2^3}$$

Thus

$$Q_2 = Q_1 \frac{\omega_2}{\omega_1}\left(\frac{D_2}{D_1}\right)^3 = 0.4\ m^3/s\ \frac{1850\ rpm}{2400\ rpm}\left(\frac{200\ mm}{400\ mm}\right)^3 = 2.47\ m^3/s$$

Q_2

Problem 7.62

Over a certain range of air speeds, V, the lift, F_L, produced by a model of a complete aircraft in a wind tunnel depends on the air speed, air density, ρ, and a characteristic length (the wing base chord length, c = 150 mm). The following experimental data is obtained for air at standard atmospheric conditions:

V (m/s)	10	15	20	25	30	35	40	45	50
F_L (N)	2.2	4.8	8.7	13.3	19.6	26.5	34.5	43.8	54

Plot the lift versus speed curve. By using *Excel* to perform a trendline analysis on this curve, generate and plot data for the lift produced by the prototype, which has a wing base chord length of 5 m, over a speed range of 75 m/s to 250 m/s.

Given: Data on model of aircraft

Find: Plot of lift vs speed of model; also of prototype

Solution

For high Reynolds number, the drag coefficient of model and prototype agree

$$C_D = \frac{F_p}{\frac{1}{2}\cdot\rho\cdot A_p\cdot V_p^{\,2}} = \frac{F_m}{\frac{1}{2}\cdot\rho\cdot A_m\cdot V_m^{\,2}}$$

The problem we have is that we do not know the area that can be used for the entire model or prototype (we only know their chords).

We have $F_p = \frac{1}{2}\cdot\rho\cdot A_p\cdot C_D\cdot V_p^{\,2}$ and $F_m = \frac{1}{2}\cdot\rho\cdot A_m\cdot C_D\cdot V_m^{\,2}$

or $F_p = k_p\cdot V_p^{\,2}$ and $F_m = k_m\cdot V_m^{\,2}$

where $k_p = \frac{1}{2}\cdot\rho\cdot A_p\cdot C_D$ and $k_m = \frac{1}{2}\cdot\rho\cdot A_m\cdot C_D$

Note that the area ratio A_p/A_m is given by $(L_p/L_m)^2$ where L_p and L_m are length scales, e.g., chord lengths. Hence

$$k_p = \frac{A_p}{A_m} \cdot k_m = \left(\frac{L_p}{L_m}\right)^2 \cdot k_m = \left(\frac{5}{0.15}\right)^2 \cdot k_m = 1110 \cdot k_m$$

We can use *Excel*'s *Trendline* analysis to fit the data of the model to find k_m, and then find k_p from the above equation to use in plotting the prototype lift vs velocity curve. This is done in the corresponding *Excel* workbook

An alternative and equivalent approach would be to find the area-drag coefficient $A_m C_D$ for the model and use this to find the area-drag coefficient $A_p C_D$ for the prototype.

Problem 7.62 (In Excel)

Over a certain range of air speeds, V, the lift, F_L, produced by a model of a complete aircraft in a wind tunnel depends on the air speed, air density, ρ, and a characteristic length (the wing base chord length, c = 150 mm). The following experimental data is obtained for air at standard atmospheric conditions:

V (m/s)	10	15	20	25	30	35	40	45	50
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Plot the lift versus speed curve. By using *Excel* to perform a trendline analysis on this curve, generate and plot data for the lift produced by the prototype, which has a wing base chord length of 5 m, over a speed range of 75 m/s to 250 m/s.

Given: Data on model of aircraft

Find: Plot of lift vs speed of model; also of prototype

Solution

V_m (m/s)	10	15	20	25	30	35	40	45	50
F_m (N)	2.2	4.8	8.7	13.3	19.6	26.5	34.5	43.8	54.0

This data can be fit to

$$F_m = \frac{1}{2}\cdot\rho\cdot A_m\cdot C_D\cdot V_m^2 \qquad \text{or} \qquad F_m = k_m\cdot V_m^2$$

From the trendline, we see that

$k_m = 0.0219$ N/(m/s)^2

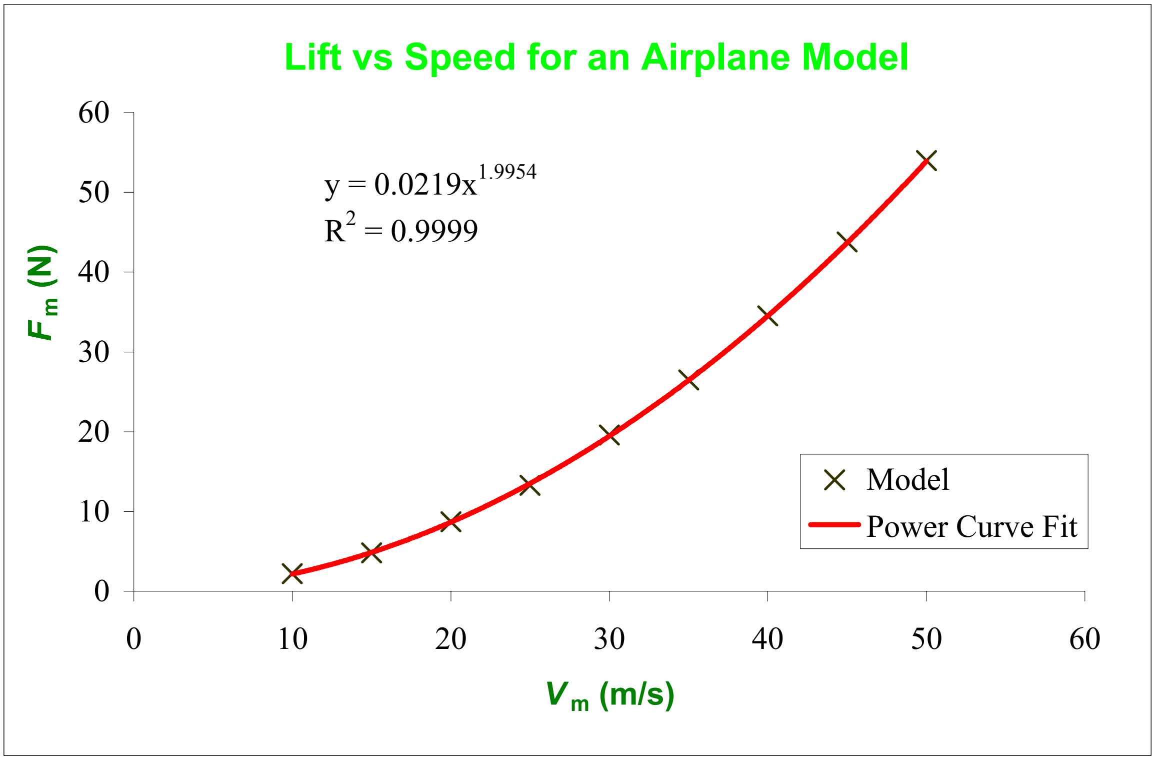
(And note that the power is 1.9954 or 2.00 to three signifcant figures, confirming the relation is quadratic)

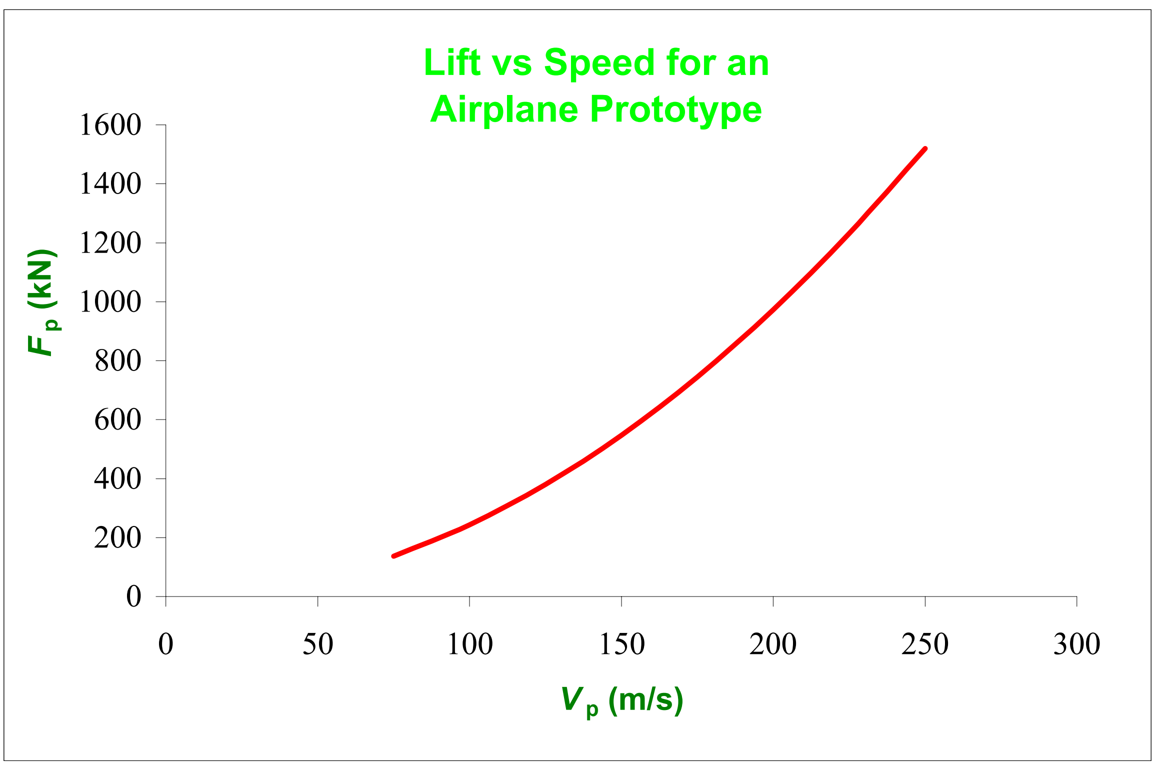
Also, $k_p = 1110\, k_m$

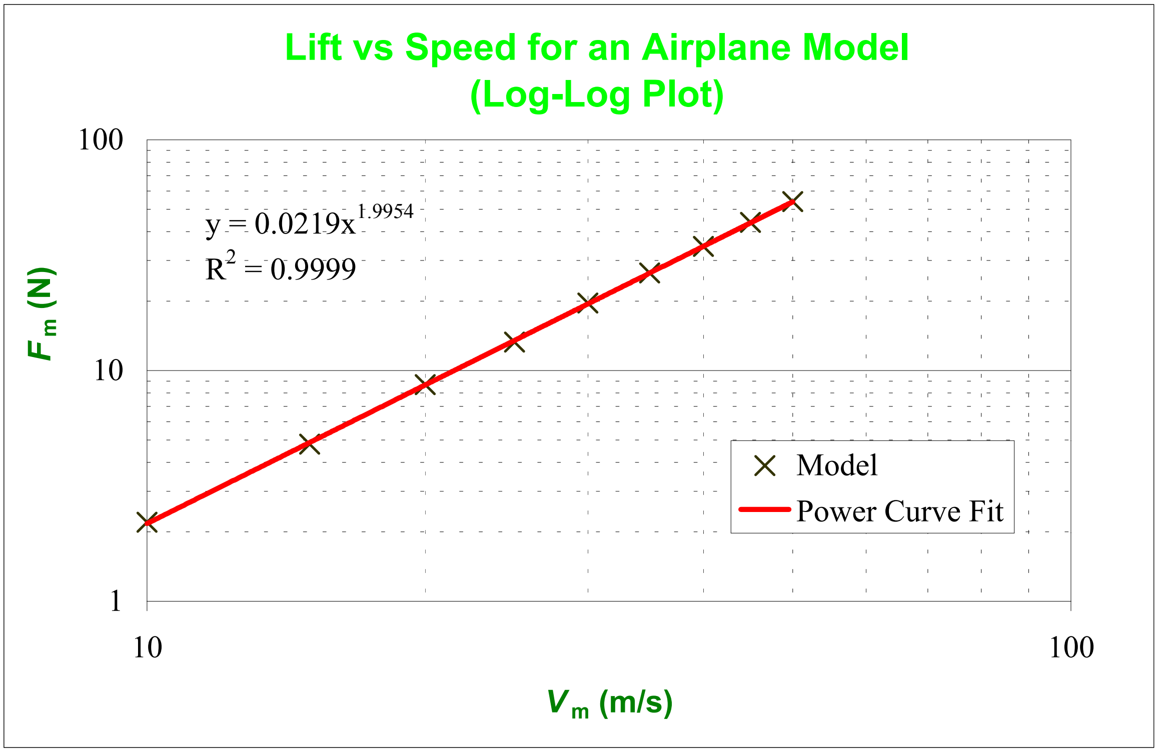
Hence,

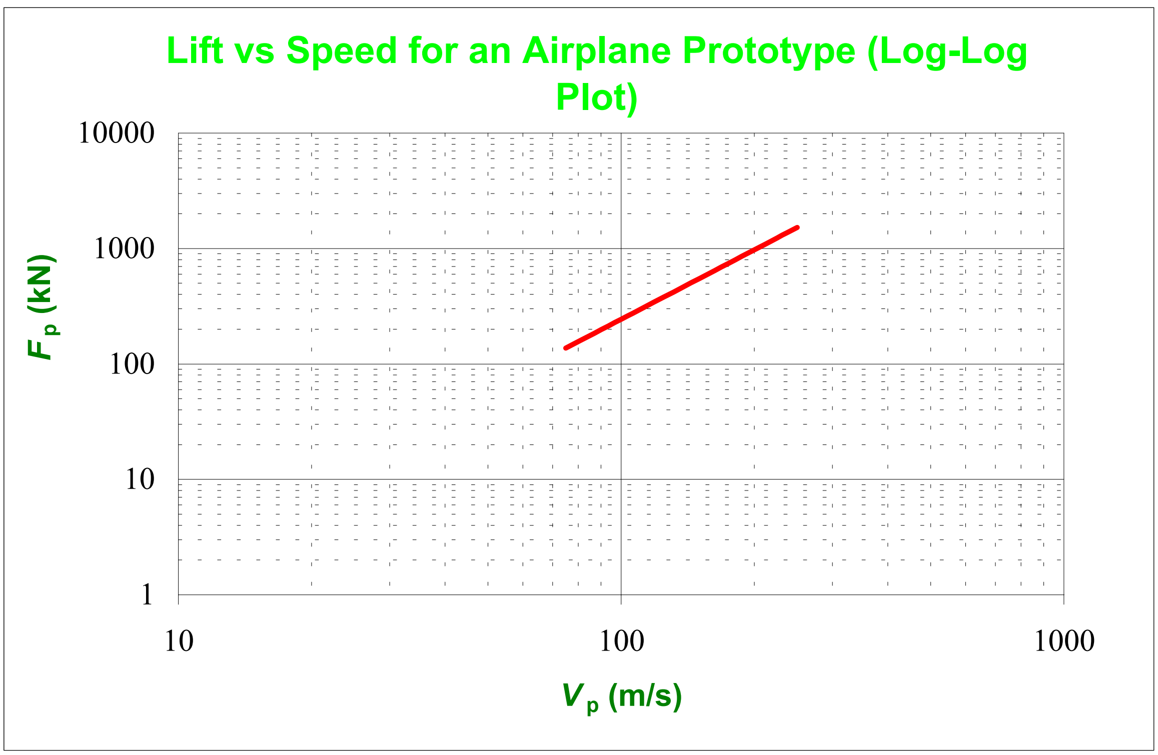
$k_p =$ 24.3 N/(m/s)^2 $\qquad F_p = k_p V_m^{\ 2}$

V_p (m/s)	75	100	125	150	175	200	225	250
F_p (kN) (Trendline)	137	243	380	547	744	972	1231	1519






Lift vs Speed for an Airplane Model (Log-Log Plot)
y = 0.0219x^1.9954
R^2 = 0.9999
Model
Power Curve Fit
Fm (N)
Vm (m/s)
100
10
1
10
100


Lift vs Speed for an Airplane Prototype (Log-Log Plot)
Fp (kN)
Vp (m/s)
10000
1000
100
10
1
10
100
1000

Problem 7.63

Given: Information relating to geometrically similar model test of centrifugal pump:

	Variable	Prototype	Model
Pressure Rise	Δp		29.3 kPa
Volume Flow Rate	Q	1.25 m³/min	
Density	ρ	800 kg/m³	999 kg/m³
Angular Speed	ω	183 rad/s	367 rad/s
Diameter	D	150 mm	50 mm

Find: Missing values for dynamically similar conditions.

Solution: Apply Buckingham Π-theorem. Assume $\Delta p = f(Q, \rho, \omega, D)$

① $\Delta p \quad Q \quad \rho \quad \omega \quad D \qquad n = 5$ parameters

② Choose M, L, t as fundamental dimensions.

③ $\frac{M}{Lt^2} \quad \frac{L^3}{t} \quad \frac{M}{L^3} \quad \frac{L}{t} \quad L \qquad r = 3$ primary dimensions

④ Let ρ, ω, and D be repeating variables. $\quad m = r = 3$

⑤ Then $n - m = 5 - 3 = 2$ dimensionless parameters result. ⑥ Check:

$$\Pi_1 = \rho^a \omega^b D^c \Delta p = \left(\frac{M}{L^3}\right)^a \left(\frac{1}{t}\right)^b (L)^c \frac{M}{Lt^2} = M^0 L^0 t^0$$

$$\left.\begin{aligned} M&: a+1=0 & a&=-1 \\ L&: -3a+c-1=0 & c&=-2 \\ t&: -b-2=0 & b&=-2 \end{aligned}\right\} \Pi_1 = \frac{\Delta p}{\rho \omega^2 D^2}$$

$$\Pi_1 = \frac{F}{L^2} \frac{L^4}{Ft^2} \frac{t^2}{1} \frac{1}{L^2} = [1] \checkmark\checkmark$$

$$\Pi_2 = \rho^a \omega^b D^c Q = \left(\frac{M}{L^3}\right)^a \left(\frac{1}{t}\right)^b (L)^c \frac{L^3}{t} = M^0 L^0 t^0$$

$$\left.\begin{aligned} M&: a=0 & a&=0 \\ L&: -3a+c+3=0 & c&=-3 \\ t&: -b-1=0 & b&=-1 \end{aligned}\right\} \Pi_2 = \frac{Q}{\omega D^3}$$

$$\Pi_2 = \frac{L^3}{t} \frac{t}{1} \frac{1}{L^3} = [1] \checkmark\checkmark$$

Thus $\Pi_1 = f(\Pi_2)$ for this situation. Flows are geometrically similar. Assume kinematic similarity. Then for dynamic similarity, if $\Pi_{2m} = \Pi_{2p}$ then $\Pi_{1m} = \Pi_{1p}$.

$$\Pi_{2m} = \frac{Q_m}{\omega_m D_m^3} = \Pi_{2p} = \frac{Q_p}{\omega_p D_p^3}; \quad Q_m = Q_p \left(\frac{\omega_m}{\omega_p}\right)\left(\frac{D_m}{D_p}\right)^3 = Q_p \left(\frac{367}{183}\right)\left(\frac{50}{150}\right)^3 = 0.0743\, Q_p$$

$$Q_m = 0.0743 \times 1.25 \frac{m^3}{min} = 0.0928 \, m^3/min$$ Q_m

$$\Pi_{1m} = \frac{\Delta p_m}{\rho_m \omega_m^2 D_m^2} = \Pi_{1p} = \frac{\Delta p_p}{\rho_p \omega_p^2 D_p^2}; \quad \Delta p_p = \Delta p_m \frac{\rho_p}{\rho_m} \left(\frac{\omega_p}{\omega_m}\right)^2 \left(\frac{D_p}{D_m}\right)^2$$

$$\Delta p_p = \Delta p_m \left(\frac{800}{999}\right)\left(\frac{183}{367}\right)^2 \left(\frac{150}{50}\right)^2 = 1.79 \times 29.3 \text{ kPa} = 52.5 \text{ kPa}$$ Δp_p

{This result neglects any effect of viscosity.}

Problem 7.64 (In Excel)

A centrifugal water pump running at speed ω = 750 rpm has the following data for flow rate Q and pressure head Δp:

Q (m^3/hr)	0	100	150	200	250	300	325	350
Δp (kPa)	361	349	328	293	230	145	114	59

The pressure head Δp is a function of flow rate, Q, and speed, ω, and also impeller diameter, D, and water density, ρ. Plot the pressure head versus flow rate curve. Find the two Π parameters for this problem, and from the above data plot one against the other. By using *Excel* to perform a trendline analysis on this latter curve, generate and plot data for pressure head versus flow rate for impeller speeds of 500 rpm and 1000 rpm.

Given: Data on centrifugal water pump

Find: Π groups; plot pressure head vs flow rate for range of speeds

Solution

We will use the workbook of Example Problem 7.1, modified for the current problem

The number of parameters is:	***n*** **= 5**
The number of primary dimensions is:	***r*** **= 3**
The number of repeat parameters is:	***m*** **=** ***r*** **= 3**
The number of Π groups is:	***n*** **-** ***m*** **= 2**

Enter the dimensions (**M**, **L**, **t**) of
the repeating parameters, and of up to
four other parameters (for up to four Π groups).
The spreadsheet will compute the exponents *a* , *b* , and *c* for each.

REPEATING PARAMETERS: Choose ρ, *g* , *d*

	M	L	t
ρ	1	-3	
ω			-1
D		1	

Π GROUPS:

	M	L	t
Δp	1	-1	-2

Π_1: *a* = **-1**, *b* = **-2**, *c* = **-2**

	M	L	t
Q	0	3	-1

Π_2: *a* = **0**, *b* = **-1**, *c* = **-3**

The following Π groups from Example Problem 7.1 are not used:

		M	L	t			M	L	t
		0	0	0			0	0	0
Π_3:	a =	**0**			Π_4:	a =	**0**		
	b =	**0**				b =	**0**		
	c =	**0**				c =	**0**		

Hence $\Pi_1 = \dfrac{\Delta p}{\rho \omega^2 D^2}$ and $\Pi_2 = \dfrac{Q}{\omega D^3}$ with $\Pi_1 = f(\Pi_2)$.

Based on the plotted data, it looks like the relation between Π_1 and Π_2 may be parabolic

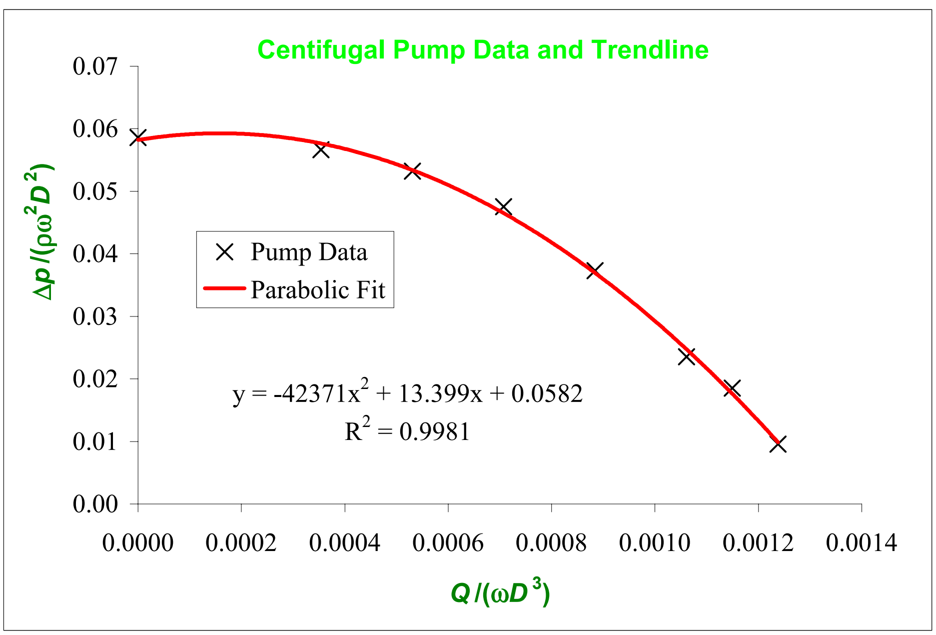
Hence $$\frac{\Delta p}{\rho \omega^2 D^2} = a + b\left(\frac{Q}{\omega D^3}\right) + c\left(\frac{Q}{\omega D^3}\right)^2$$

The data is

Q **(m³/hr)**	0	100	150	200	250	300	325	350
Δ*p* **(kPa)**	361	349	328	293	230	145	114	59

ρ = 999 kg/m³
ω = 750 rpm
D = 1 m (*D* is not given; use *D* = 1 m as a scale)

$Q/(\omega D^3)$	0.00000	0.000354	0.000531	0.000707	0.000884	0.00106	0.00115	0.00124
$\Delta p/(\rho\omega^2 D^2)$	0.0586	0.0566	0.0532	0.0475	0.0373	0.0235	0.0185	0.00957



From the *Trendline* analysis

a = 0.0582
b = 13.4
c = -42371

and $$\Delta p = \rho \omega^2 D^2 \left[a + b\left(\frac{Q}{\omega D^3}\right) + c\left(\frac{Q}{\omega D^3}\right)^2 \right]$$

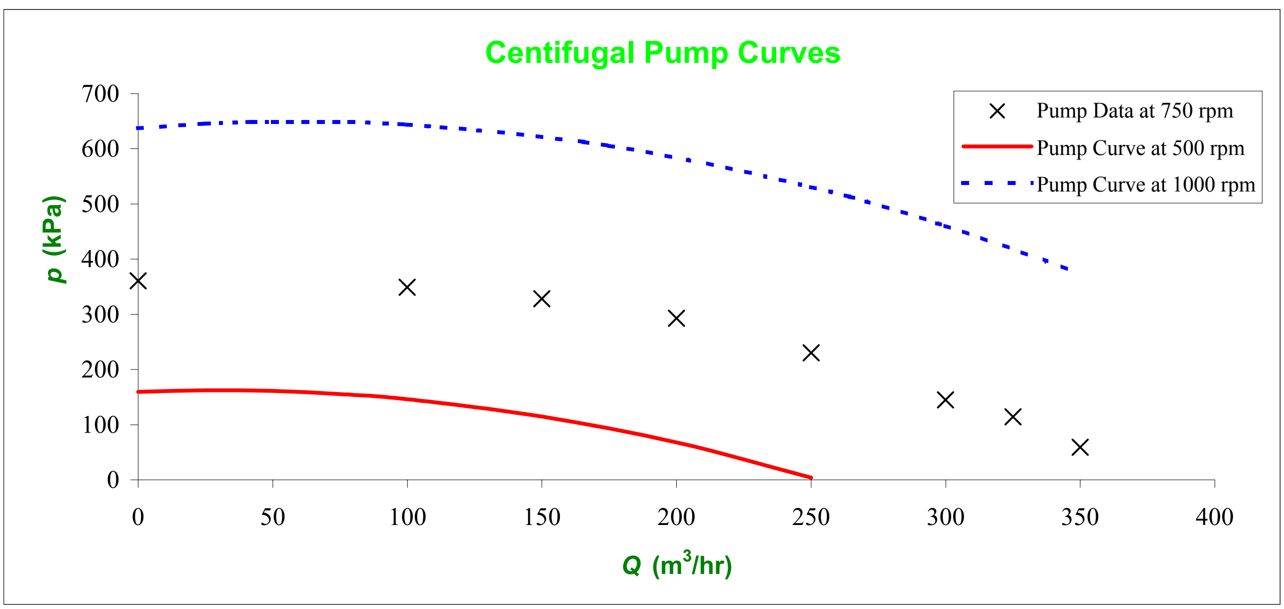
Finally, data at 500 and 1000 rpm can be calculated and plotted

ω = 500 rpm

Q (m^3/hr)	0	25	50	75	100	150	200	250
Δp (kPa)	159	162	161	156	146	115	68	4

ω = 1000 rpm

Q (m^3/hr)	0	25	50	100	175	250	300	350
Δp (kPa)	638	645	649	644	606	531	460	374



Problem 7.65

Given: Axial-flow pump:

$Q = 25\ \text{ft}^3/\text{s}$ (water) $h = 150\ \text{ft}\cdot\text{lbf/slug}$

$D = 1\ \text{ft}$ $\omega = 500\ \text{rpm}$

Model:

$\mathcal{P} = 3\ \text{hp}$, $\omega = 1000\ \text{rpm}$

Find: For similar performance between prototype and model, calculate the head, volume flow rate, and the diameter of the model

Solution:

$$\frac{h}{\omega^2 D^2} = f_1\left(\frac{Q}{\omega D^3}, \frac{\rho \omega D^2}{\mu}\right) \quad \text{and} \quad \frac{\mathcal{P}}{\rho \omega^3 D^5} = f_2\left(\frac{Q}{\omega D^3}, \frac{\rho \omega D^2}{\mu}\right)$$

Neglecting viscous effects,

$$\text{if} \quad \left(\frac{Q}{\omega D^3}\right)_m = \left(\frac{Q}{\omega D^3}\right)_p \quad \text{then} \quad \left(\frac{h}{\omega^2 D^2}\right)_m = \left(\frac{h}{\omega^2 D^2}\right)_p$$

$$\text{and} \quad \left(\frac{\mathcal{P}}{\rho \omega^3 D^5}\right)_m = \left(\frac{\mathcal{P}}{\rho \omega^3 D^5}\right)_p$$

or

$$\text{if} \quad \frac{Q_m}{Q_p} = \frac{\omega_m}{\omega_p}\frac{D_m^3}{D_p^3} = \frac{1000}{500}\left(\frac{D_m}{D_p}\right)^3 = 2\left(\frac{D_m}{D_p}\right)^3 \quad \text{-----(1)}$$

$$\text{then} \quad \frac{h_m}{h_p} = \frac{\omega_m^2}{\omega_p^2}\frac{D_m^2}{D_p^2} = \frac{(1000)^2}{(500)^2}\left(\frac{D_m}{D_p}\right)^2 = 4\left(\frac{D_m}{D_p}\right)^2 \quad \text{----(2)}$$

$$\text{and} \quad \frac{\mathcal{P}_m}{\mathcal{P}_p} = \frac{\rho_m}{\rho_p}\frac{\omega_m^3}{\omega_p^3}\frac{D_m^5}{D_p^5} = \left(\frac{1000}{500}\right)^3\left(\frac{D_m}{D_p}\right)^5 = 8\left(\frac{D_m}{D_p}\right)^5 \quad \text{----(3)}$$

We can determine $\mathcal{P}_p$ from the energy equation applied to the prototype. From footnote, page 316

$$\mathcal{P}_p = \rho Q h = 1.94\ \frac{\text{slug}}{\text{ft}^3} \times 25\ \frac{\text{ft}^3}{\text{s}} \times 150\ \frac{\text{ft}\cdot\text{lbf}}{\text{slug}} \times \frac{\text{s}\cdot\text{hp}}{550\ \text{ft}\cdot\text{lbf}}$$

$$\mathcal{P}_p = 13.2\ \text{hp}$$

From Eq. 3

$$\frac{\mathcal{P}_m}{\mathcal{P}_p} = 8\left(\frac{D_m}{D_p}\right)^5 \quad \therefore \frac{D_m}{D_p} = \left[\frac{1}{8}\frac{\mathcal{P}_m}{\mathcal{P}_p}\right]^{1/5} = \left[\frac{1}{8}\times\frac{3}{13.2}\right]^{1/5} = 0.491$$

$$\therefore D_m = 0.491\, D_p = 0.491\ \text{ft} \quad \longleftarrow D_m$$

From Eq. 1

$$Q_m = 2\left(\frac{D_m}{D_p}\right)^3 Q_p = 2(0.491)^3 \times 25\ \frac{\text{ft}^3}{\text{s}} = 5.92\ \text{ft}^3/\text{s} \quad \longleftarrow Q_m$$

From Eq. 2

$$h_m = 4\left(\frac{D_m}{D_p}\right)^2 h_p = 4(0.491)^2 \times 150\ \frac{\text{ft}\cdot\text{lbf}}{\text{slug}} = 145\ \frac{\text{ft}\cdot\text{lbf}}{\text{slug}} \quad \longleftarrow h_m$$

Problem 7.66

Given: For a marine propeller (from prob 7.22) the thrust force, F_t, is

$$F_t = F_t(\rho, D, V, g, \omega, p, \mu)$$

Neglecting viscous effects, and pressure, then

$$F_t = F_t(\rho, D, V, g, \omega)$$

Assume that Torque, T, and power, P, depend on same parameters

$$T = T(\rho, D, V, g, \omega)$$

$$P = P(\rho, D, V, g, \omega)$$

Find: Derive scaling "laws" for propellers that relate F_t, T, and P to other variables.

Solution: Apply Buckingham π - theorem

① $\rho \quad D \quad V \quad g \quad \omega \quad F_t \quad T \quad P$

② Choose F, L, t as primary dimensions

③ $\frac{Ft^2}{L^4} \quad L \quad \frac{L}{t} \quad \frac{L}{t^2} \quad \frac{1}{t} \quad F \quad FL \quad \frac{FL}{t}$

④ Repeating variables ρ, ω, D

⑤ Then $n - m = 5$ dimensionless groups (2 independent, 3 dependent)

Setting up dimensional equations

$$\pi_1 = \rho^a \omega^b D^c V \qquad F^0L^0t^0 = \left(\frac{Ft^2}{L^4}\right)^a \left(\frac{1}{t}\right)^b L^c \frac{L}{t} \qquad \begin{Bmatrix} F: & 0 = a \\ t: & 0 = 2a - b - 1 \\ L: & 0 = -4a + c + 1 \end{Bmatrix} \quad \begin{matrix} a = 0 \\ b = -1 \\ c = -1 \end{matrix} \qquad \therefore \pi_1 = \frac{V}{\omega D}$$

$$\pi_2 = \rho^a \omega^b D^c g \qquad F^0L^0t^0 = \left(\frac{Ft^2}{L^4}\right)^a \left(\frac{1}{t}\right)^b L^c \frac{L}{t^2} \qquad \begin{Bmatrix} F: & 0 = a \\ t: & 0 = 2a - b - 2 \\ L: & 0 = -4a + c + 1 \end{Bmatrix} \quad \begin{matrix} a = 0 \\ b = -2 \\ c = -1 \end{matrix} \qquad \therefore \pi_2 = \frac{g}{\omega^2 D}$$

$$\pi_3 = \rho^a \omega^b D^c F_t \qquad F^0L^0t^0 = \left(\frac{Ft^2}{L^4}\right)^a \left(\frac{1}{t}\right)^b L^c F \qquad \begin{Bmatrix} F: & 0 = a + 1 \\ t: & 0 = 2a - b \\ L: & 0 = -4a + c \end{Bmatrix} \quad \begin{matrix} a = -1 \\ b = -2 \\ c = -4 \end{matrix} \qquad \therefore \pi_3 = \frac{F_t}{\rho \omega^2 D^4}$$

$$\pi_4 = \rho^a \omega^b D^c T \qquad F^0L^0t^0 = \left(\frac{Ft^2}{L^4}\right)^a \left(\frac{1}{t}\right)^b L^c FL \qquad \begin{Bmatrix} F: & 0 = a + 1 \\ t: & 0 = 2a - b \\ L: & 0 = -4a + c + 1 \end{Bmatrix} \quad \begin{matrix} a = -1 \\ b = -2 \\ c = -5 \end{matrix} \qquad \therefore \pi_4 = \frac{T}{\rho \omega^2 D^5}$$

$$\pi_5 = \rho^a \omega^b D^c P \qquad F^0L^0t^0 = \left(\frac{Ft^2}{L^4}\right)^a \left(\frac{1}{t}\right)^b L^c \frac{FL}{t} \qquad \begin{Bmatrix} F: & 0 = a + 1 \\ t: & 0 = 2a - b - 1 \\ L: & 0 = -4a + c + 1 \end{Bmatrix} \quad \begin{matrix} a = -1 \\ b = -3 \\ c = -5 \end{matrix} \qquad \therefore \pi_5 = \frac{P}{\rho \omega^3 D^5}$$

Then scaling "laws" are

$$\frac{F_t}{\rho \omega^2 D^4} = f_1\left(\frac{V}{\omega D}, \frac{g}{\omega^2 D}\right)$$

$$\frac{T}{\rho \omega^2 D^5} = f_2\left(\frac{V}{\omega D}, \frac{g}{\omega^2 D}\right)$$

$$\frac{P}{\rho \omega^3 D^5} = f_3\left(\frac{V}{\omega D}, \frac{g}{\omega^2 D}\right)$$

Given: Thrust and torque of propeller depend on D, ω, V, μ, ρ

Model:		Prototype:	
$D =$	600 mm	$D =$	6 m
$\omega =$	2000 rpm	$\omega =$	?
$V =$	45 m/s	$V =$	120 m/s
$F_t =$	110 N (thrust)	$F_t =$	?
$T =$	10 N·m	$T =$	?

Find: (a) ω, (b) F_t and (c) T for prototype, neglecting effects of viscosity, under dynamically similar conditions.

<u>Solution</u>: There are two problems here. (1) Determine $F_t = f_1(D, \omega, V, \mu, \rho)$ and (2) $T = f_2(D, \omega, V, \mu, \rho)$. Since μ is to be ignored, do not select it as a repeating parameter. Instead, select D, ω, ρ as repeating variables.

(1) $F_t = f_1(D, \omega, V, \mu)$

① $F_t \quad D \quad \omega \quad V \quad \mu \quad \rho \qquad n = 6$ parameters

② Select F, L, t as primary dimensions.

③
$$\begin{array}{cccccc} F_t & D & \omega & V & \mu & \rho \\ F & L & \frac{1}{t} & \frac{L}{t} & \frac{Ft}{L^2} & \frac{Ft^2}{L^4} \end{array} \qquad r = 3 \text{ primary dimensions}$$

④ Choose $D, \omega, V \qquad m = r = 3$ repeating parameters

⑤ Then $n - m = 3$ dimensionless groups will result. Setting up dimensional equations,

$$\Pi_1 = D^a \omega^b \rho^c F_t = (L)^a \left(\frac{1}{t}\right)^b \left(\frac{Ft^2}{L^4}\right)^c F = F^0 L^0 t^0$$

$F: \; c + 1 = 0 \qquad c = -1$
$L: \; a - 4c = 0 \qquad a = -4$
$t: \; -b + 2c = 0 \qquad b = -2$

$$\Pi_1 = \frac{F_t}{\rho \omega^2 D^4}$$

$$\Pi_2 = D^a \omega^b \rho^c V = (L)^a \left(\frac{1}{t}\right)^b \left(\frac{Ft^2}{L^4}\right)^c \frac{L}{t} = F^0 L^0 t^0$$

$F: \; c = 0 \qquad c = 0$
$L: \; a - 4c + 1 = 0 \qquad a = -1$
$t: \; -b + c - 1 = 0 \qquad b = -1$

$$\Pi_2 = \frac{V}{\omega D}$$

$$\Pi_3 = D^a \omega^b \rho^c \mu = (L)^a \left(\frac{1}{t}\right)^b \left(\frac{Ft^2}{L^4}\right)^c \frac{Ft}{L^2} = F^0 L^0 t^0$$

$F: \; c + 1 = 0 \qquad c = -1$
$L: \; a - 4c - 2 = 0 \qquad a = 4c + 2 = -2$
$t: \; -b + 2c + 1 = 0 \qquad b = 2c + 1 = -1$

$$\Pi_3 = \frac{\mu}{\rho \omega D^2}$$

Then $\Pi_1 = f_1(\Pi_2, \Pi_3)$ or $\dfrac{F_t}{\rho \omega^2 D^4} = f_1\left(\dfrac{V}{\omega D}, \dfrac{\mu}{\rho \omega D^2}\right)$

If viscous effects are neglected, then $\Pi_1 = g_1(\Pi_2)$ or $\frac{F_t}{\rho \omega^2 D^4} = g_1\left(\frac{V}{\omega D}\right)$

For dynamic similarity, $\Pi_2)_{model} = \Pi_2)_{prototype}$, or

$$\frac{V_m}{\omega_m D_m} = \frac{V_p}{\omega_p D_p}$$

Thus $\omega_p = \omega_m \frac{V_p}{V_m}\frac{D_m}{D_p} = (2000\ rpm)\left(\frac{120}{45}\right)\left(\frac{1}{10}\right) = 533\ rpm$ ω_p

When $\Pi_2)_{model} = \Pi_2)_{prototype}$, then neglecting μ, $\Pi_1)_{model} = \Pi_1)_{prototype}$, or

$$\frac{F_{tm}}{\rho_m \omega_m^2 D_m^4} = \frac{F_{tp}}{\rho_p \omega_p^2 D_p^4}$$; assume $\rho_m = \rho_p$

Then $F_{tp} = F_{tm}\left(\frac{\omega_p}{\omega_m}\right)^2\left(\frac{D_p}{D_m}\right)^4 = 110\ N \cdot \left(\frac{533}{2000}\right)^2 (10)^4 = 78.1\ kN$ F_{tp}

(2) The analysis of Π_2 and Π_3 for the second problem is identical to that for problem (1). Combining T with D, ω and ρ gives

$$\Pi_4 = D^a \omega^b \rho^c T = (L)^a\left(\frac{1}{t}\right)^b\left(\frac{Ft^2}{L^4}\right)^c (FL) = M^0 L^0 t^0$$

$F: c + 1 = 0 \qquad c = -1$

$L: a - 4c + 1 = 0 \qquad a = 4c - 1 = -5$

$t: -b + 2c = 0 \qquad b = 2c = -2$

$$\Pi_4 = \frac{T}{\rho \omega^2 D^5}$$

Thus $\Pi_4 = f_2(\Pi_2, \Pi_3)$ or neglecting μ, $\Pi_4 = g_2(\Pi_2)$. For dynamic similarity, $\Pi_4)_{model} = \Pi_4)_{prototype}$, or

$$\frac{T_m}{\rho_m \omega_m^2 D_m^5} = \frac{T_p}{\rho_p \omega_p^2 D_p^5}$$; assume $\rho_m = \rho_p$

Then $T_p = T_m\left(\frac{\omega_p}{\omega_m}\right)^2\left(\frac{D_p}{D_m}\right)^5 = 10\ N\cdot m \cdot \left(\frac{533}{2000}\right)^2 (10)^5 = 71\ kN\cdot m$ T_p

⑥ Check, using M, L, t:

$$\Pi_1 = \frac{ML}{t^2}\frac{L^3}{M}\frac{t^2}{1}\frac{1}{L^4} = [1] \checkmark$$

$$\Pi_2 = \frac{L}{t}\frac{t}{1}\frac{1}{L} = [1] \checkmark$$

$$\Pi_3 = \frac{M}{Lt}\frac{L^3}{M}\frac{t}{1}\frac{1}{L^2} = [1] \checkmark$$

$$\Pi_4 = \frac{ML^2}{t^2}\frac{L^3}{M}\frac{t^2}{1}\frac{1}{L^5} = [1] \checkmark$$

Problem 7.68

Given: The kinetic energy ratio is a figure of merit defined as the ratio of kinetic energy flux in a wind tunnel test section to the drive power.

Find: an estimate of the kinetic energy ratio for the 40×80 wind tunnel at NASA-Ames.

Solution:

From text (p.319) for NASA-Ames tunnel:

$$A = 40\,ft \times 80\,ft = 3200\,ft^2 \quad , \quad \mathcal{P} = 125{,}000\,hp$$

$$V_{max} = 300\,\frac{kmi}{hr} \times 6080\,\frac{ft}{k\text{-}mi} \times \frac{hr}{3600\,s} = 507\ ft/s$$

$$K.E\ ratio = \frac{K.E\ flux}{Power\ in} = \frac{\dot{m}\frac{V^2}{2}}{\mathcal{P}} = \frac{\rho V A V^2}{2\mathcal{P}} = \frac{\rho V^3 A}{2\mathcal{P}}$$

Assuming standard air,

$$K.E.\ ratio = \frac{1}{2} \times 0.00238\,\frac{slug}{ft^3} \times (507)^3\,\frac{ft^3}{s^3} \times 3200\,ft^2 \times \frac{1}{125{,}000\,hp} \times \frac{hp \cdot s}{550\,ft \cdot lbf} \times \frac{lbf \cdot s^2}{slug \cdot ft}$$

$$K.E\ ratio = 7.22$$

Problem 7.69

Given: Wind tunnel test of 1:16 model bus in standard air.

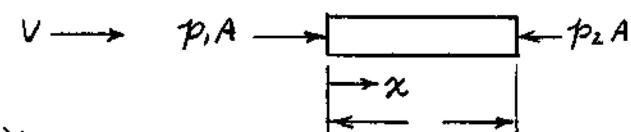
$W = 152$ mm $V = 26.5$ m/s Pressure gradient:
$H = 200$ mm $F_D = 6.09$ N $\frac{dp}{dx} = -11.8$ N/m²/m
$L = 762$ mm (measured)

Find: (a) Estimate the horizontal buoyancy correction.
(b) Calculate the corrected model drag coefficient.
(c) Evaluate the drag force on the prototype at 100 km/hr on a calm day.

Solution: Apply definitions

Computing equations: $C_D \equiv \frac{F_D}{\frac{1}{2}\rho V^2 A}$ Assume $A = WH$

The buoyancy force will be



$$F_B = p_1 A - p_2 A = (p_1 - p_2)A$$

But $p_2 = p_1 + \frac{\partial p}{\partial x}\Delta x + \cdots \approx p_1 + \frac{\partial p}{\partial x} L$

Therefore $p_1 - p_2 = -\frac{\partial p}{\partial x} L$, and $F_B \approx -\frac{\partial p}{\partial x} LA = -\frac{\partial p}{\partial x} LWH$

$$F_B \approx -(-11.8)\frac{N}{m^3} \times 0.762\ m \times 0.152\ m \times 0.200\ m = 0.273\ N \text{ (to right)}$$ F_B

The corrected drag force is

$$F_{Dc} = F_{Dm} - F_B = (6.09 - 0.273)\ N = 5.82\ N$$

The corrected model drag coefficient is

$$C_{Dm} = \frac{F_{Dc}}{\frac{1}{2}\rho V^2 A} = 2 \times 5.82\ N \times \frac{m^3}{1.23\ kg} \times \frac{s^2}{(26.5)^2 m^2} \times \frac{1}{(0.200)(0.152) m^2} \times \frac{kg \cdot m}{N \cdot s^2} = 0.443$$ C_D

Assume the test was conducted at high enough Reynolds number so $C_{Dp} = C_{Dm}$. Then

$$F_{Dp} = C_{Dp} A_p \frac{1}{2}\rho V_p^2$$

$$= \frac{1}{2} \times 0.443 \times 0.200(16)\ m \times 0.152(16)\ m \times 1.23 \frac{kg}{m^3} \times \left[100 \frac{km}{hr} \times 1000 \frac{m}{km} \times \frac{hr}{3600\ s}\right]^2 \frac{N \cdot s^2}{kg \cdot m}$$

$$F_{Dp} = 1.64\ kN \text{ (prototype at 100 km/hr)}$$ F_{Dp}

{Rolling resistance must be included to obtain the total tractive effort needed to propel the full-scale vehicle.}

Problem 7.70

Given: A 1:16 scale model of a 20 m long truck is tested in a wind tunnel at speed $V_m = 80$ m/s. The axial pressure gradient at this speed is $dh/dx = -1.2$ mm H_2O/m. The frontal area of the prototype is $A_p = 10\ m^2$. $C_D = 0.85$

Find: (a) Estimate the horizontal buoyancy correction
(b) Express the correction as a fraction of the measured C_D.

Solution:

The horizontal buoyancy force, F_B, is the difference in the pressure force between the front and back of the model due to the pressure gradient in the tunnel

$$F_B = (p_f - p_b)A = \rho_w g \frac{dh}{dx} L_m A_m \qquad (\Delta p = \rho_w g\, \Delta h)$$

$$L_m = \frac{L_p}{16} \qquad A_m = \frac{A_p}{(16)^2}$$

$$\therefore F_B = 999\ \frac{kg}{m^3} \times 9.81\ \frac{m}{s^2} \times (-1.2)\times 10^{-3}\ \frac{m}{m} \times \frac{20\ m}{16} \times \frac{10\ m^2}{(16)^2} \times \frac{N \cdot s^2}{kg \cdot m}$$

$$F_B = -0.574\ N$$ F_B

The horizontal buoyancy correction should be added to the measured drag force on the model

The measured drag force on the model is given by

$$F_{D_m} = \frac{1}{2}\rho V^2 A_m C_D = \frac{1}{2}\rho V^2 \frac{A_p}{(16)^2} C_D$$

Assume air at standard conditions, $\rho = 1.23\ kg/m^3$

$$F_{D_m} = \frac{1}{2} \times 1.23\ \frac{kg}{m^3} \times (80)^2\ \frac{m^2}{s^2} \times \frac{10\ m^2}{(16)^2} \times 0.85 \times \frac{N \cdot s^2}{kg \cdot m}$$

$$F_{D_m} = 131\ N$$

$$\frac{F_B}{F_{D_m}} = \frac{-0.574}{131} = -4.38 \times 10^{-3} = -0.44\%$$ F_B/F_{D_m}

Problem 7.71

Open-Ended Problem Statement: During a recent stay at a motel, a hanging lamp was observed to oscillate in the air stream from the air conditioning unit. Explain why this might occur.

Discussion: Minor fluctuations occur in the speed and direction of the air blowing from the air conditioning unit. These tend to move the hanging lamp from the vertical, steady-state position.

If the fluctuations in air flow speed and direction are large enough, they can cause significant random motions of the hanging lamp.

If the fluctuations in air flow speed and direction contain a periodic frequency content that is close to the natural frequency of the lamp's motion, they can excite the resonant frequency, leading to quite large oscillations in the lamp motion. These periodic motions may occur in combination with the smaller, random motions.

Problem 7.72

Open-Ended Problem Statement: Frequently one observes a flag on a pole "flapping" in the wind. Explain why this occurs. What dimensionless parameters might characterize the phenomenon? Why?

Discussion: The natural wind contains significant fluctuations in air speed and direction. These fluctuations tend to disturb the flag from an initially plane position.

When the flag is bent or curved from the plane position, the flow nearby must follow its contour. Flow over a convex curved surface tends to be faster, and have lower pressure, than flow over a concave curved surface. The resulting pressure forces tend to exaggerate the curvature of the flag. The result is a seemingly random, "flapping" motion of the flag.

The rope or chain used to raise the flag may also flap in the wind. It is much more likely to exhibit a periodic motion than the flag itself. The rope is quite close to the flag pole, where it is influenced by any vortices shed from the pole. If the Reynolds number is such that periodic vortices are shed from the pole, they will tend to make the rope move with the same frequency. This accounts for the periodic thump of a rope or clank of a chain against the pole.

The vortex shedding phenomenon is characterized by the Strouhal number, $St = fD/V_\infty$, where f is the vortex shedding frequency, D the pole diameter, and V_∞ the wind speed. The Strouhal number is constant at approximately 0.2 over a broad range of Reynolds number.

Problem 7.73

Open-Ended Problem Statement: Explore the variation in wave propagation speed given by the equation of Problem 7.61 for a free-surface flow of water. Find the operating depth to minimize the speed of capillary waves (waves with small wavelength, also called *ripples*). First assume wavelength is much smaller than water depth. Then explore the effect of depth. What depth do you recommend for a water table used to visualize compressible-flow wave phenomena? What is the effect of reducing surface tension by adding a surfactant?

Discussion: The equation given in Problem 7.61 contains three terms. The first term contains surface tension and gives a speed inversely proportional to wavelength. This term will be important when small wavelengths are considered.

The second term contains gravity and gives a speed proportional to wavelength. This term will be important when long wavelengths are considered.

The argument of the hyperbolic tangent is proportional to water depth and inversely proportional to wavelength. For small wavelengths, this term should approach unity since the hyperbolic tangent of a large number approaches one.

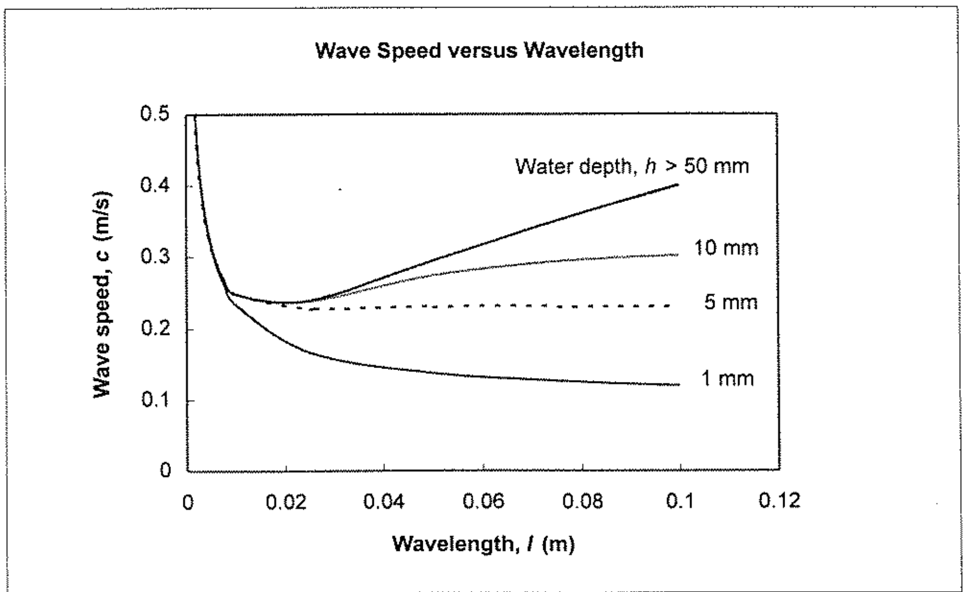
See the spreadsheet for numerical values and a plot.

Input Parameters:

g =	9.81	m/s^2	Acceleration of gravity
h =	0.01	m	Liquid depth (for hyperbolic tangent calculation)
ρ =	999	kg/m^3	Liquid density
σ =	0.0728	N/m	Surface tension

Calculated Values:

Wavelength, λ (m)	**tanh (---)** (h = 10 mm)	h (m) = 0.001	0.005	0.01	0.05	0.1	0.5
		Wave Speed, c (m/s)					
0.00185	1.00	0.500	0.500	0.500	0.500	0.500	0.500
0.003	1.00	0.396	0.397	0.397	0.397	0.397	0.397
0.005	1.00	0.313	0.315	0.315	0.315	0.315	0.315
0.0075	1.00	0.263	0.270	0.270	0.270	0.270	0.270
0.01	1.00	0.233	0.248	0.248	0.248	0.248	0.248
0.025	0.987	0.167	0.227	0.238	0.239	0.239	0.239
0.05	0.850	0.138	0.229	0.275	0.295	0.295	0.295
0.075	0.685	0.126	0.229	0.294	0.351	0.351	0.351
0.1	0.557	0.120	0.228	0.303	0.400	0.401	0.401
0.2	0.304	0.110	0.226	0.312	0.537	0.560	0.561
0.5	0.125	0.104	0.223	0.314	0.660	0.815	0.884
0.75	0.0836	0.102	0.223	0.314	0.681	0.896	1.08
1	0.0627	0.101	0.222	0.314	0.690	0.933	1.25
2	0.0314	0.100	0.222	0.314	0.698	0.975	1.69
5	0.0126	0.100	0.222	0.313	0.700	0.988	2.09
7.5	0.00838	0.0994	0.222	0.313	0.700	0.989	2.15
10	0.00628	0.0993	0.222	0.313	0.700	0.990	2.18
Froude Speed, $(gh)^{1/2}$ (m/s)		0.0990	0.221	0.313	0.700	0.990	2.21



Problem 8.1

Given: Incompressible flow in a circular channel.
$Re = 1800$ in a section where the channel diameter is $D = 10\text{ mm}$.

Find: (i) general expression for Re in terms of
(a) volume flow rate, Q, and channel diameter, D
(b) mass flow rate, $\dot{m}$, and channel diameter, D.
(ii) Re for same flow rate and $D = 6\text{ mm}$.

Solution:

Assume steady, incompressible flow

Definitions: $Re = \frac{\rho D \bar{V}}{\mu}$, $Q = A\bar{V}$, $\dot{m} = \rho A \bar{V}$ and $A = \frac{\pi D^2}{4}$

Then,

$$Re = \frac{\rho D \bar{V}}{\mu} = \frac{\rho D}{\mu}\frac{Q}{A} = \frac{\rho D}{\mu}\frac{Q4}{\pi D^2} = \frac{4Q}{\pi D}\frac{\rho}{\mu} = \frac{4Q}{\pi D \nu}$$ ← Re

Also

$$Re = \frac{\rho D \bar{V}}{\mu} = \frac{D}{\mu}\frac{\rho \bar{V} A}{A} = \frac{D}{\mu}\frac{\dot{m}4}{\pi D^2} = \frac{4\dot{m}}{\pi D \mu}$$ ← Re

From Eq (i) a

$$Q = \frac{\pi D \nu Re}{4}$$

Then for same flow rate in sections with different channel diameter,

$$D_1 Re_1 = D_2 Re_2$$

$$Re_2 = \frac{D_1}{D_2} Re_1 = \frac{10\text{ mm}}{6\text{ mm}} \times 1800 = 3000$$ ← Re_2

Problem 8.2

Standard air enters a 0.25 m diameter duct. Find the volume flow rate at which the flow becomes turbulent. At this flow rate, estimate the entrance length required to establish fully developed flow.

Given: Data on air flow in duct

Find: Volume flow rate for turbulence; entrance length

Solution

The given data is $D = 0.25 \cdot m$

From Fig. A.3 $\nu = 1.46 \cdot 10^{-5} \cdot \frac{m^2}{s}$

The governing equations are

$$Re = \frac{V \cdot D}{\nu} \qquad Re_{crit} = 2300 \qquad Q = \frac{\pi}{4} \cdot D^2 \cdot V$$

$$L_{laminar} = 0.06 \cdot Re_{crit} \cdot D \qquad \text{or, for turbulent,} \qquad L_{turb} = 25 \cdot D - 40 \cdot D$$

Hence $Re_{crit} = \frac{\frac{Q}{\frac{\pi}{4} \cdot D^2} \cdot D}{\nu}$ or $Q = \frac{Re_{crit} \cdot \pi \cdot \nu \cdot D}{4}$ $Q = 0.396 \frac{m^3}{min}$

$L_{laminar} = 0.06 \cdot Re_{crit} \cdot D$ $\quad$ $L_{laminar} = 34.5\,m$

or, for turbulent, $\quad L_{min} = 25 \cdot D$ $\quad$ $L_{min} = 6.25\,m$

$L_{max} = 40 \cdot D$ $\quad$ $L_{max} = 10\,m$

Problem 8.3

For flow in circular tubes, transition to turbulence usually occurs around $Re \approx 2300$. Investigate the circumstances under which the flows of (a) standard air and (b) water at 15°C become turbulent. On log-log graphs, plot: the average velocity, the volume flow rate, and the mass flow rate, at which turbulence first occurs, as functions of tube diameter.

Given: That transition to turbulence occurs at about $Re = 2300$

Find: Plots of average velocity and volume and mass flow rates for turbulence for air and water

Solution

From Tables A.8 and A.10 $\qquad \rho_{air} = 1.23 \cdot \frac{kg}{m^3} \qquad \nu_{air} = 1.45 \times 10^{-5} \cdot \frac{m^2}{s}$

$$\rho_w = 999 \cdot \frac{kg}{m^3} \qquad \nu_w = 1.14 \times 10^{-6} \cdot \frac{m^2}{s}$$

The governing equations are $\qquad Re = \frac{V \cdot D}{\nu} \qquad Re_{crit} = 2300$

For the average velocity $\qquad V = \frac{Re_{crit} \cdot \nu}{D}$

Hence for air $\qquad V_{air} = \frac{2300 \times 1.45 \times 10^{-5} \cdot \frac{m^2}{s}}{D} \qquad V_{air} = \frac{0.0334 \cdot \frac{m^2}{s}}{D}$

For water $V_w = \dfrac{2300 \times 1.14 \times 10^{-6} \cdot \dfrac{m^2}{s}}{D}$ $V_w = \dfrac{0.00262 \cdot \dfrac{m^2}{s}}{D}$

For the volume flow rates

$$Q = A \cdot V = \frac{\pi}{4} \cdot D^2 \cdot V = \frac{\pi}{4} \cdot D^2 \cdot \frac{Re_{crit} \cdot \nu}{D} = \frac{\pi \cdot Re_{crit} \cdot \nu}{4} \cdot D$$

Hence for air $Q_{air} = \dfrac{\pi}{4} \times 2300 \times 1.45 \cdot 10^{-5} \cdot \dfrac{m^2}{s} \cdot D$ $Q_{air} = 0.0262 \cdot \dfrac{m^2}{s} \times D$

For water $Q_w = \dfrac{\pi}{4} \times 2300 \times 1.14 \cdot 10^{-6} \cdot \dfrac{m^2}{s} \cdot D$ $Q_w = 0.00206 \cdot \dfrac{m^2}{s} \times D$

Finally, the mass flow rates are obtained from volume flow rates

$m_{air} = \rho_{air} \cdot Q_{air}$ $m_{air} = 0.0322 \cdot \dfrac{kg}{m \cdot s} \times D$

$m_w = \rho_w \cdot Q_w$ $m_w = 2.06 \cdot \dfrac{kg}{m \cdot s} \times D$

These results are plotted in the associated *Excel* workbook

Problem 8.3 (In Excel)

For flow in circular tubes, transition to turbulence usually occurs around $Re \approx 2300$. Investigate the circumstances under which the flows of (a) standard air and (b) water at 15°C become turbulent. On log-log graphs, plot: the average velocity, the volume flow rate, and the mass flow rate, at which turbulence first occurs, as functions of tube diameter.

Given: That transition to turbulence occurs at about $Re = 2300$

Find: Plots of average velocity and volume and mass flow rates for turbulence for air and water

Solution

The relations needed are

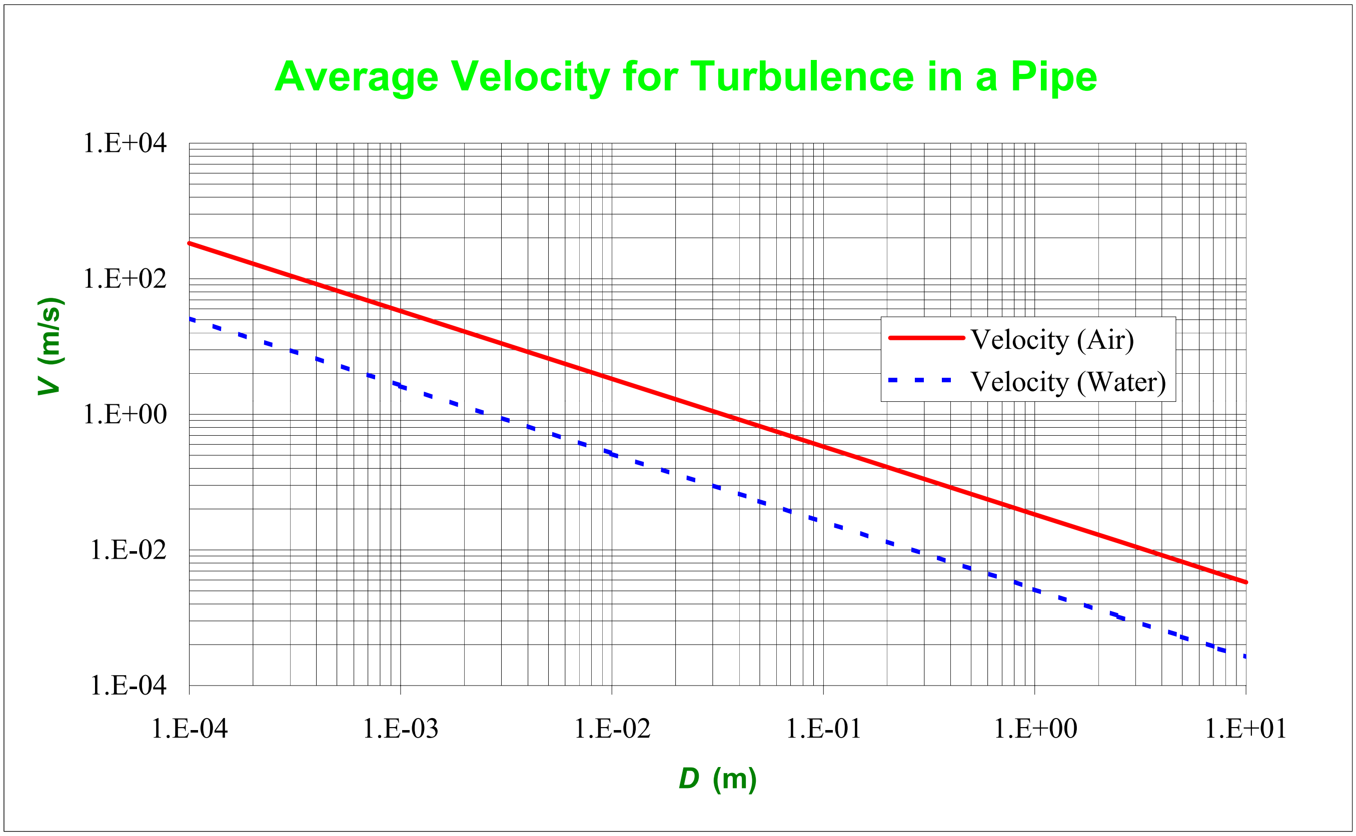
$$Re_{crit} = 2300 \qquad V = \frac{Re_{crit} \cdot \nu}{D} \qquad Q = \frac{\pi \cdot Re_{crit} \cdot \nu}{4} \cdot D \qquad m_{rate} = \rho \cdot Q$$

From Tables A.8 and A.10 the data required is

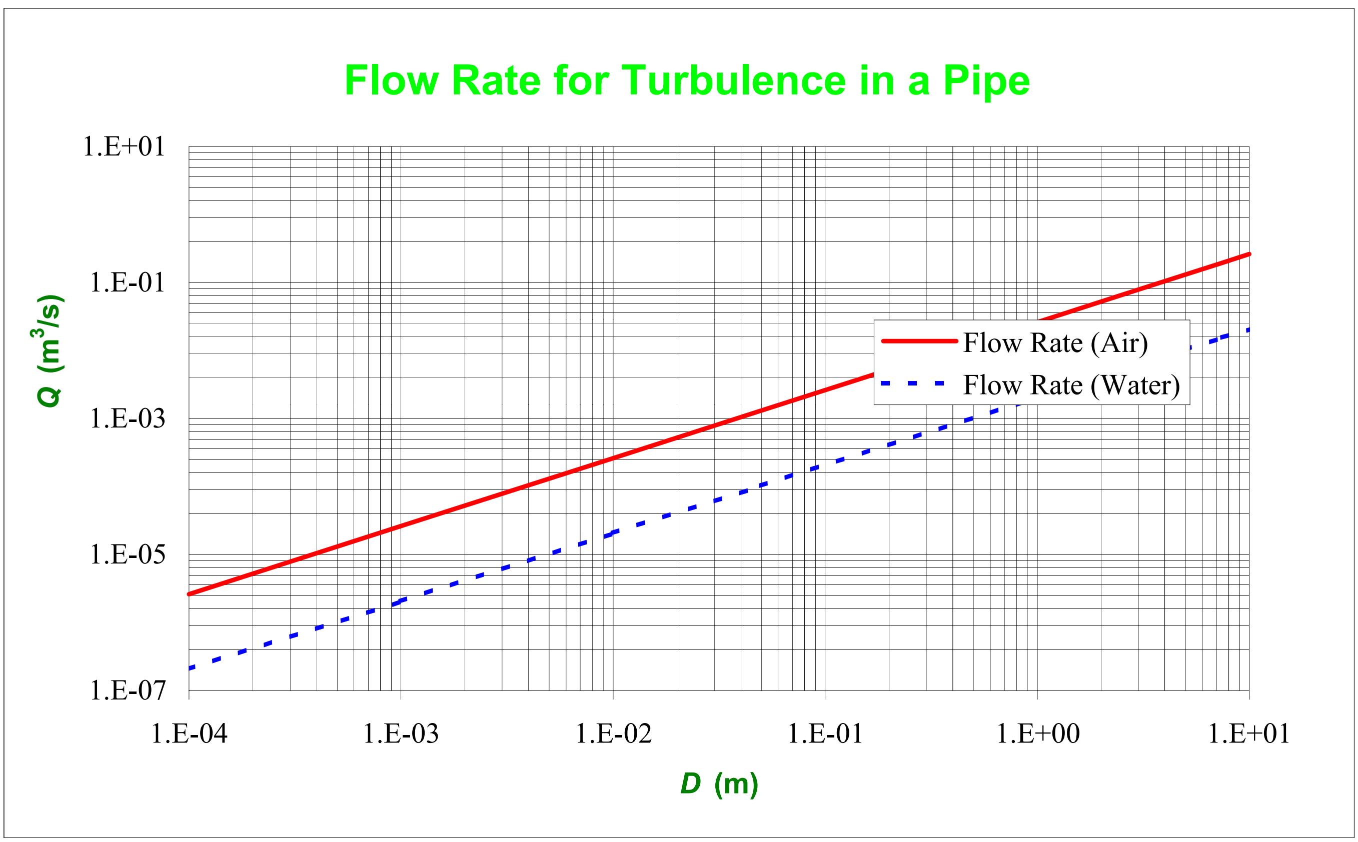
$\rho_{air} =$ 1.23 kg/m^3 $\qquad \nu_{air} =$ 1.45E-05 m^2/s

$\rho_w =$ 999 kg/m^3 $\qquad \nu_w =$ 1.14E-06 m^2/s

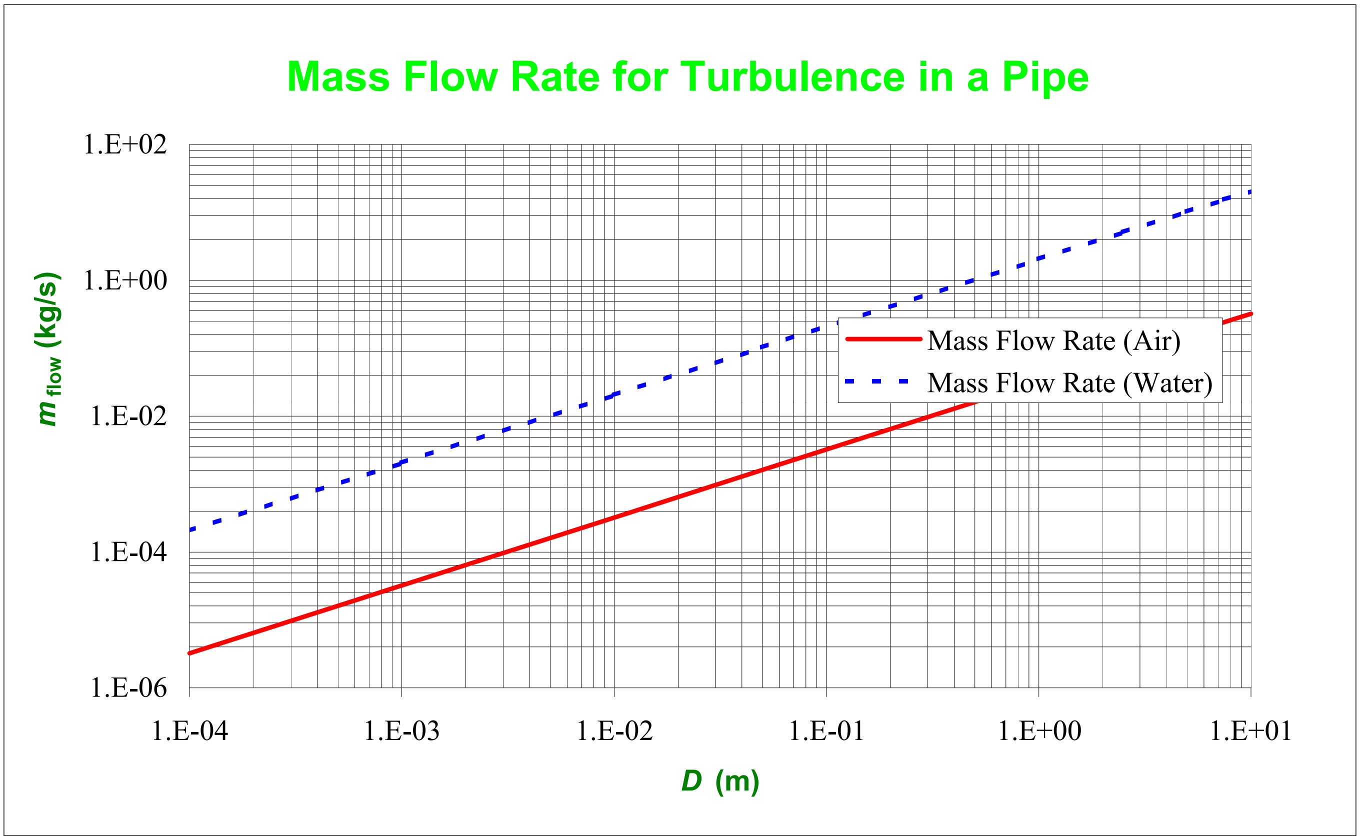
D **(m)**	0.0001	0.001	0.01	0.05	1.0	2.5	5.0	7.5	10.0
V_{air} **(m/s)**	333.500	33.350	3.335	0.667	3.34E-02	1.33E-02	6.67E-03	4.45E-03	3.34E-03
V_w **(m/s)**	26.2	2.62	0.262	5.24E-02	2.62E-03	1.05E-03	5.24E-04	3.50E-04	2.62E-04
Q_{air} **(m^3/s)**	2.62E-06	2.62E-05	2.62E-04	1.31E-03	2.62E-02	6.55E-02	1.31E-01	1.96E-01	2.62E-01
Q_w **(m^3/s)**	2.06E-07	2.06E-06	2.06E-05	1.03E-04	2.06E-03	5.15E-03	1.03E-02	1.54E-02	2.06E-02
m_{air} **(kg/s)**	3.22E-06	3.22E-05	3.22E-04	1.61E-03	3.22E-02	8.05E-02	1.61E-01	2.42E-01	3.22E-01
m_w **(kg/s)**	2.06E-04	2.06E-03	2.06E-02	1.03E-01	2.06E+00	5.14E+00	1.03E+01	1.54E+01	2.06E+01



Average Velocity for Turbulence in a Pipe
1.E+04
1.E+02
1.E+00
1.E-02
1.E-04
V (m/s)
1.E-04
1.E-03
1.E-02
1.E-01
1.E+00
1.E+01
D (m)
Velocity (Air)
Velocity (Water)



Flow Rate for Turbulence in a Pipe
1.E+01
1.E-01
1.E-03
1.E-05
1.E-07
Q (m3/s)
1.E-04
1.E-03
1.E-02
1.E-01
1.E+00
1.E+01
D (m)
Flow Rate (Air)
Flow Rate (Water)

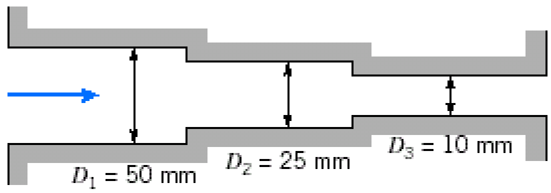


Mass Flow Rate for Turbulence in a Pipe
1.E+02
1.E+00
1.E-02
1.E-04
1.E-06
m flow (kg/s)
1.E-04
1.E-03
1.E-02
1.E-01
1.E+00
1.E+01
D (m)
Mass Flow Rate (Air)
Mass Flow Rate (Water)

Problem 8.4

Standard air flows in a pipe system in which the area is decreased in two stages from 50 mm, to 25 mm, to 10 mm. Each section is 1 m long. As the flow rate is increased, which section will become turbulent first? Determine the flow rates at which one, two, then all three sections first become turbulent. At each of these flow rates, determine which sections, if any, attain fully developed flow.

Given: Pipe geometry



Find: Flow rates for turbuence to start; which sections have fully developed flow

Solution

From Table A.10 $\nu = 1.45 \times 10^{-5} \cdot \frac{m^2}{s}$

The given data is $L = 1 \cdot m$ $\quad D_1 = 50 \cdot mm$ $\quad D_2 = 25 \cdot mm$ $\quad D_3 = 10 \cdot mm$

The critical Reynolds number is $Re_{crit} = 2300$

Writing the Reynolds number as a function of flow rate

$$Re = \frac{V \cdot D}{\nu} = \frac{Q}{\frac{\pi}{4} \cdot \pi \cdot D^2} \cdot \frac{D}{\nu} \qquad \text{or} \qquad Q = \frac{Re \cdot \pi \cdot \nu \cdot D}{4}$$

Then the flow rates for turbulence to begin in each section of pipe are

$$Q_1 = \frac{Re_{crit} \cdot \pi \cdot \nu \cdot D_1}{4} \qquad Q_1 = 0.0786 \frac{m^3}{min}$$

$$Q_2 = \frac{Re_{crit} \cdot \pi \cdot \nu \cdot D_2}{4} \qquad Q_2 = 0.0393 \frac{m^3}{min}$$

$$Q_3 = \frac{Re_{crit} \cdot \pi \cdot \nu \cdot D_3}{4} \qquad Q_3 = 0.0157 \frac{m^3}{min}$$

Hence, smallest pipe becomes turbulent first, then second, then the largest.

For the smallest pipe transitioning to turbulence (*Q_3*)

For pipe 3 $$Re_3 = \frac{4 \cdot Q_3}{\pi \cdot \nu \cdot D_3} \qquad Re_3 = 2300$$

$$L_{laminar} = 0.06 \cdot Re_3 \cdot D_3 \qquad L_{laminar} = 1.38\,m$$

If the flow is still laminar — Not fully developed flow

or, for turbulent, $$L_{min} = 25 \cdot D_3 \qquad L_{min} = 0.25\,m$$

$$L_{max} = 40 \cdot D_3 \qquad L_{max} = 0.4\,m$$

Fully developed flow

For pipes 1 and 2 $$L_{laminar} = 0.06 \cdot \left(\frac{4 \cdot Q_3}{\pi \cdot \nu \cdot D_1}\right) \cdot D_1 \qquad L_{laminar} = 1.38\,m$$

$$L_{laminar} = 0.06 \cdot \left(\frac{4 \cdot Q_3}{\pi \cdot \nu \cdot D_2}\right) \cdot D_2 \qquad L_{laminar} = 1.38\,m$$

Pipes 1 and 2 are laminar, not fully developed.

For the middle pipe transitioning to turbulence (Q_2)

For pipe 2 $$Re_2 = \frac{4 \cdot Q_2}{\pi \cdot \nu \cdot D_2}$$ $$Re_2 = 2300$$

$$L_{laminar} = 0.06 \cdot Re_2 \cdot D_2$$ $$L_{laminar} = 3.45\,m$$

If the flow is still laminar — Not fully developed flow

or, for turbulent, $$L_{min} = 25 \cdot D_2$$ $$L_{min} = 0.625\,m$$

$$L_{max} = 40 \cdot D_2$$ $$L_{max} = 1\,m$$

Fully developed flow

For pipes 1 and 3 $$L_1 = 0.06 \cdot \left(\frac{4 \cdot Q_2}{\pi \cdot \nu \cdot D_1}\right) \cdot D_1$$ $$L_1 = 3.45\,m$$

$$L_{3min} = 25 \cdot D_3$$ $$L_{3min} = 0.25\,m$$

$$L_{3max} = 40 \cdot D_3$$ $$L_{3max} = 0.4\,m$$

Pipe 1 (Laminar) is not fully developed; pipe 3 (turbulent) is fully developed

For the large pipe transitioning to turbulence (Q_1)

For pipe 1

$$Re_1 = \frac{4 \cdot Q_1}{\pi \cdot \nu \cdot D_1}$$ $$Re_1 = 2300$$

$$L_{laminar} = 0.06 \cdot Re_1 \cdot D_1$$ $$L_{laminar} = 6.9\,m$$

If the flow is still laminar — Not fully developed flow

or, for turbulent,

$$L_{min} = 25 \cdot D_1$$ $$L_{min} = 1.25\,m$$

$$L_{max} = 40 \cdot D_1$$ $$L_{max} = 2\,m$$

Not fully developed flow

For pipes 2 and 3

$$L_{2min} = 25 \cdot D_2$$ $$L_{2min} = 0.625\,m$$

$$L_{2max} = 40 \cdot D_2$$ $$L_{2max} = 1\,m$$

$$L_{3min} = 25 \cdot D_3$$ $$L_{3min} = 0.25\,m$$

$$L_{3max} = 40 \cdot D_3$$ $$L_{3max} = 0.4\,m$$

Pipes 2 and 3 (turbulent) are fully developed

Problem 8.5

Given: Laminar flow in the entrance section of a pipe shown schematically in Fig. 8.1.

Find: Sketch centerline velocity, static pressure, and wall shear stress as functions of distance along the pipe. Explain significant features of the plots, comparing them with fully developed flow. Can the Bernoulli equation be applied anywhere in the flow field? If so, where? Explain briefly.

Discussion: The centerline velocity, static pressure, and wall shear stress variations are sketched on the next page. Each variation sketch is aligned vertically with the corresponding sections of the developing pipe flow in Fig. 8.1.

Boundary layers grow on the tube wall, reducing the velocity near the wall. The velocity reduction becomes more pronounced farther downstream. Consequently the centerline velocity must increase in the streamwise direction to carry the same mass flow rate across each section of the tube. (When laminar flow becomes fully developed, the centerline velocity becomes twice the average velocity at any cross-section.)

Frictional effects are concentrated within the boundary layers. The boundary layers do not join at the tube centerline for some distance along the tube. Therefore in the center region outside the boundary layers flow may still be considered to behave as though it were inviscid.

Flow outside the boundary layers is steady, frictionless, incompressible, and along a streamline. These are the restrictions required to apply the Bernoulli equation. Therefore the Bernoulli equation may be applied as a reasonable model for the actual flow outside the boundary layers. The Bernoulli equation predicts that pressure decreases as flow speed increases.

After the boundary layers merge at the centerline of the channel the entire flow is affected by friction. Therefore it is no longer possible to apply the Bernoulli equation.

When flow becomes fully developed the rate of change of pressure with distance becomes constant. In the entrance region the pressure falls more rapidly; the increased pressure gradient is caused by increased shear stress at the wall (larger than for fully developed flow) and by the developing velocity profile, which causes momentum flux to increase.

In fully developed flow the pressure curve becomes linear; the pressure drops the same amount for each length along the tube. The pressure distribution curve at the end of the entrance length becomes asymptotic to the linear variation for fully developed flow.

The wall shear stress initially is large, because the boundary layers are thin. The shear stress decreases as the boundary layers become thicker. At the end of the entrance length the shear stress asymptotically approaches the constant value for fully developed flow.

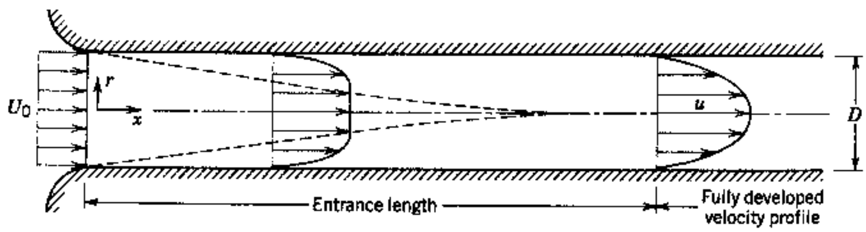
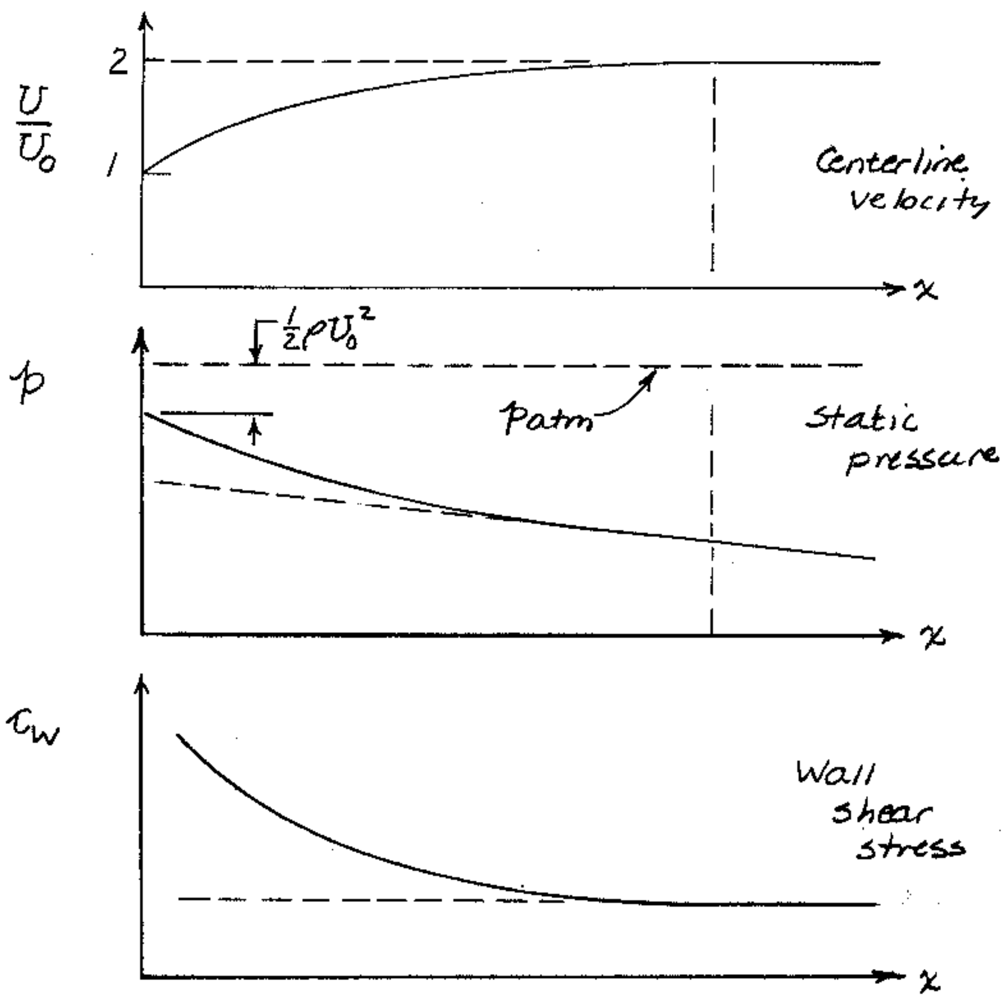


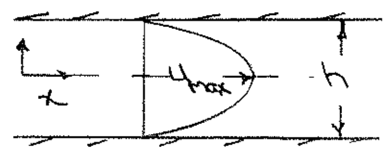
Fig. 8.1 Flow in the entrance region of a pipe.



Given: Velocity profile for flow between stationary parallel plates.

$$u = a(h^2/4 - y^2)$$

where a = constant



Find: Ratio $\bar{V}/u_{max}$

Solution: First find u_{max}, by setting $\frac{du}{dy} = 0$

$$\frac{du}{dy} = -2ay \quad ; \quad \frac{du}{dy} = 0 \text{ at } y = 0$$

$$u_{max} = u(0) = a\frac{h^2}{4}$$

From the definition of $\bar{V}$,

$$\bar{V} = \frac{Q}{A} = \frac{1}{A}\int u\,dA = \frac{1}{h}\int_{-h/2}^{h/2} u\,dy$$

$$= \frac{1}{h}\int_{-h/2}^{h/2} a\left(\frac{h^2}{4} - y^2\right) = \frac{a}{h}\left[\frac{h^2 y}{4} - \frac{y^3}{3}\right]_{-h/2}^{h/2}$$

$$\bar{V} = \frac{a}{h}\left[\left(\frac{h^3}{8} - \frac{h^3}{24}\right) - \left(-\frac{h^3}{8} + \frac{h^3}{24}\right)\right] = \frac{a}{h}\left[\frac{h^3}{4} - \frac{h^3}{12}\right]$$

$$\bar{V} = \frac{1}{6}ah^2$$

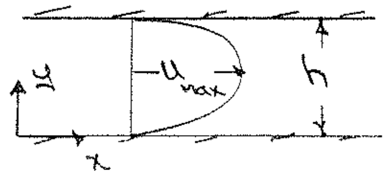
and

$$\frac{\bar{V}}{u_{max}} = \frac{ah^2}{6}\,\frac{4}{ah^2} = \frac{2}{3}$$

$\frac{\bar{V}}{u_{max}}$

Given: Incompressible flow between parallel plates with

$$u = U_{max}(Ay^2 + By + C)$$



Find: (a) constants A, B, C using appropriate boundary conditions
(b) Q per unit depth b.
(c) $\bar{V}/U_{max}$

Solution:

(a) Available boundary conditions: (1) $y=0,\ u=0$
(2) $y=h,\ u=0$
(3) $y=h/2,\ u=U_{max}$

From B.C (1) $u(0) = 0 = U_{max} C \quad \therefore \quad C = 0$ ← C

From B.C (2) $u(h) = 0 = U_{max}(Ah^2 + Bh)$ ---(i)

From B.C (3) $u(h/2) = U_{max} = U_{max}\left(A\frac{h^2}{4} + B\frac{h}{2}\right)$ ---(ii)

From Eq (i), $B = -Ah$. Substituting into Eq (ii) gives

$$\cancel{U_{max}} = \cancel{U_{max}}\left(A\frac{h^2}{4} - A\frac{h^2}{2}\right) \quad \therefore \quad A = -\frac{4}{h^2}$$ ← A

and $B = -Ah = \frac{4}{h}$ ← B

Then

$$u = U_{max}(Ay^2 + By + C) = U_{max}\left(-4\frac{y^2}{h^2} + 4\frac{y}{h}\right) = 4U_{max}\left[\frac{y}{h} - \left(\frac{y}{h}\right)^2\right]$$

(b) $$Q = \int_0^h u b\,dy = \int_0^h 4U_{max}\left[\frac{y}{h} - \frac{y^2}{h^2}\right] b\,dy = 4U_{max} b\left[\frac{y^2}{2h} - \frac{y^3}{3h^2}\right]_0^h$$

$$Q = 4bU_{max}\left[\frac{h}{2} - \frac{h}{3}\right] = \frac{2}{3}U_{max} bh$$

$Q/b = \frac{2}{3}U_{max} h$ ← Q/b

(c) Since $Q = \bar{V}A = \bar{V}bh$

$$\frac{Q}{b} = \bar{V}h = \frac{2}{3}U_{max} h$$

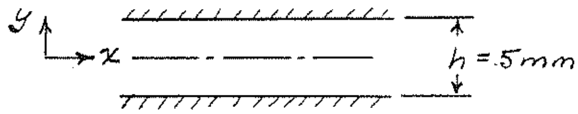
and

$$\frac{\bar{V}}{U_{max}} = \frac{2}{3}$$ ← $\frac{\bar{V}}{U_{max}}$

Problem 8.8

Given: Laminar, fully developed flow between parallel plates

$$\mu = 0.5\ \frac{N \cdot s}{m^2} \quad ; \quad \frac{\partial p}{\partial x} = -1000\ \frac{N}{m^3}$$



Find: (a) Shear stress on upper plate.
(b) Volume flow rate per unit width.

width = b

Solution: From Eq. 8.7 with $a = h$,

$$u = -\frac{h^2}{8\mu}\frac{\partial p}{\partial x}\left[1 - \left(\frac{2y}{h}\right)^2\right]$$

Then

$$\tau_{yx} = \mu \frac{du}{dy} = -\frac{h^2}{8}\frac{\partial p}{\partial x}\left(-\frac{8y}{h^2}\right) = y\frac{\partial p}{\partial x}$$

At upper surface, $y = h/2$, and

$$\tau_{yx} = \frac{0.005\ m}{2} \times -1000\ \frac{N}{m^3} = -2.5\ N/m^2$$

The upper plate is a negative y surface. Thus since $\tau_{yx} < 0$, stress acts to <u>right</u>, in $+x$ direction. ⟵ τ_{yx}

The volume flow rate is

$$Q = \int_A u\,dA = \int_{-h/2}^{h/2} u\,b\,dy = 2\int_0^{h/2} u\,b\,dy = 2\left(\frac{h}{2}\right)b\int_0^1 u\,d\left(\frac{2y}{h}\right)$$

or

$$\frac{Q}{b} = h\int_0^1 u\,d\eta \quad \text{where } \eta = \frac{2y}{h} \text{ and } u = -\frac{h^2}{8\mu}\frac{\partial p}{\partial x}(1-\eta^2)$$

Thus

$$\frac{Q}{b} = h\int_0^1 -\frac{h^2}{8\mu}\frac{\partial p}{\partial x}(1-\eta^2)\,d\eta = -\frac{h^3}{8\mu}\frac{\partial p}{\partial x}\left(\eta - \frac{1}{3}\eta^3\right)\Big|_0^1 = -\frac{h^3}{12\mu}\frac{\partial p}{\partial x}$$

$$\frac{Q}{b} = -\frac{1}{12} \times (0.005)^3 m^3 \times \frac{m^2}{0.5\ N \cdot s} \times -1000\ \frac{N}{m^3} = 20.8 \times 10^{-6}\ m^2/s$$

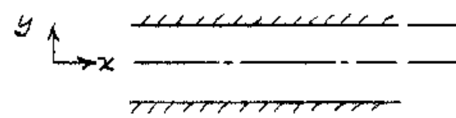
⟵ Q/b

Note $u > 0$, so flow is from left to right.

Problem 8.9

Given: Fully developed laminar flow between parallel plates.

$$\mu = 0.01\ \frac{\text{lbf}\cdot\text{s}}{\text{ft}^2}\ ; \quad \frac{\partial p}{\partial x} = -8\ \frac{\text{lbf}}{\text{ft}^3}$$



Find: (a) Shear stress on upper plate.
(b) Volume flow rate per unit width.

<u>Solution</u>: From Eq. 8.7 with $a = 2h$, $\quad u = -\frac{h^2}{2\mu}\frac{\partial p}{\partial x}\left[1 - \left(\frac{y}{h}\right)^2\right]$

Then
$$\tau_{yx} = \mu\frac{du}{dy} = -\frac{h^2}{2}\frac{\partial p}{\partial x}\left(-\frac{2y}{h^2}\right) = y\frac{\partial p}{\partial x}$$

At upper surface, $y = h$, and

$$\tau_{yx} = 0.06\ \text{in.} \times -8\ \frac{\text{lbf}}{\text{ft}^3} \times \frac{\text{ft}}{12\ \text{in.}} = -0.0400\ \text{lbf}/\text{ft}^2$$

The upper plate is a negative y surface. Thus since $\tau_{yx} < 0$, stress acts to <u>right</u>, in $+x$ direction. τ_{yx}

The volume flow rate is

$$Q = \int_A u\,dA = \int_{-h}^{h} u\,b\,dy = 2\int_0^h u\,b\,dy = 2hb\int_0^1 u\,d(y/h)$$

or

$$\frac{Q}{b} = 2h\int_0^1 u\,d\eta \quad \text{where } \eta = y/h \text{ and } u = -\frac{h^2}{2\mu}\frac{\partial p}{\partial x}(1-\eta^2)$$

Thus
$$\frac{Q}{b} = 2h\int_0^1 -\frac{h^2}{2\mu}\frac{\partial p}{\partial x}(1-\eta^2)\,d\eta = -\frac{h^3}{\mu}\frac{\partial p}{\partial x}\left(\eta - \frac{1}{3}\eta^3\right)\Big|_0^1 = -\frac{2h^3}{3\mu}\frac{\partial p}{\partial x}$$

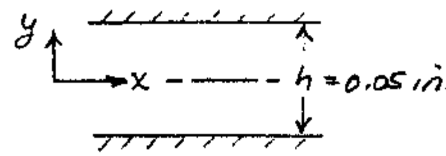
$$\frac{Q}{b} = -\frac{2}{3}\times\left(\frac{0.06}{12}\right)^3 \text{ft}^3 \times \frac{\text{ft}^2}{0.01\ \text{lbf}\cdot\text{s}} \times -8\ \frac{\text{lbf}}{\text{ft}^3} = 6.67\times10^{-5}\ \text{ft}^2/\text{s}$$ Q/b

Note $u > 0$, so flow is from left to right.

Problem 8.10

Given: Fully developed laminar flow between parallel plates.

$$\mu = 2.40 \times 10^{-5}\ \frac{lbf \cdot s}{ft^2}\ ; \quad \frac{\partial p}{\partial x} = -4\ \frac{lbf}{ft^3}$$



Find: (a) Derive and plot equation for shear stress versus y.

(b) Maximum shear stress.

Solution: From Eq. 8.7, with a = h, $u = -\frac{h^2}{8\mu}\frac{\partial p}{\partial x}\left[1 - \left(\frac{2y}{h}\right)^2\right]$

By symmetry, the origin for y must be located at the channel centerline. Apply Newton's law of viscosity.

$$\tau_{yx} = \mu \frac{du}{dy}$$

Assumption: Newtonian fluid

Then

$$\tau_{yx} = \mu \frac{d}{dy}\left\{-\frac{h^2}{8\mu}\frac{\partial p}{\partial x}\left[1 - \left(\frac{2y}{h}\right)^2\right]\right\} = y\frac{\partial p}{\partial x}$$

For $u > 0$, $\partial p / \partial x < 0$. Thus $\tau_{yx} < 0$ for $y > 0$ and $\tau_{yx} > 0$ for $y < 0$.

On the upper plate (a minus y surface), $\tau_{yx} < 0$, so shear stress acts to the right.

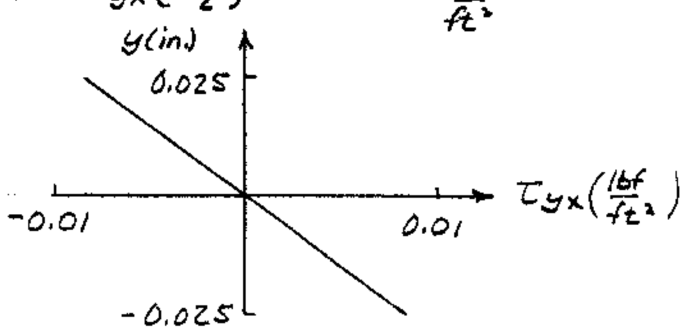
On the lower plate (a plus y surface), $\tau_{yx} > 0$, so shear stress acts to the right.

The maximum stress occurs when $y = \pm h/2$. Thus

$$\tau_{max} = \tau_{yx}\left(\frac{h}{2}\right) = \frac{h}{2}\frac{\partial p}{\partial x} = \frac{1}{2} \times 0.05\ in. \times \frac{ft}{12\ in.} \times \left(-4.0\ \frac{lbf}{ft^3}\right) = -0.00835\ \frac{lbf}{ft^2}$$

or $\quad \tau_{max} = \tau_{yx}\left(-\frac{h}{2}\right) = 0.00835\ \frac{lbf}{ft^2}$

Plot:

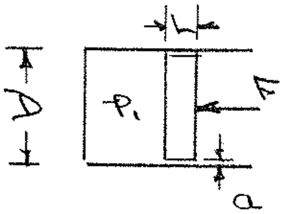


Problem 8.11

Given: Oil is confined in a cylinder of diameter $D = 100$ mm, by a piston with radial clearance $a = 0.025$ mm, and length $L = 50$ mm. A steady force, $F = 20$ kN, is applied to the piston. The oil has properties of SAE 30 oil at 50°C.

Find: Leakage rate of oil past the piston

Solution:



Model the flow as steady, fully developed laminar flow between stationary parallel plates, i.e., neglect motion of the piston.

Then the leakage flow rate can be evaluated from Eq. 8.6c (of the text).

$$\frac{Q}{l} = \frac{a^3 \Delta p}{12 \mu L} \qquad \text{where } l = \pi D$$

From Fig. A.2 at $T = 50°C$, $\mu = 5.9 \times 10^{-2}$ N·s/m²

$\Delta p = p_1 - p_{atm}$ and $\Delta p = \frac{F}{A} = \frac{4F}{\pi D^2} = \frac{4}{\pi} \times 20\,\text{kN} \times \frac{1}{(0.1)^2\,\text{m}^2} = 2.55\,\text{MPa}$

Then

$$Q = \frac{\pi D a^3 \Delta p}{12 \mu L} = \frac{\pi}{12} \times 0.1\,\text{m} \times (2.5 \times 10^{-5}\,\text{m})^3 \times 2.55 \times 10^6 \frac{\text{N}}{\text{m}^2} \times \frac{1}{5.9 \times 10^{-2}} \frac{\text{m}^2}{\text{N s}} \times \frac{1}{0.05\,\text{m}}$$

$$Q = 3.54 \times 10^{-7}\,\text{m}^3/\text{s} = 3.54 \times 10^{-4}\,\text{L/s} \qquad Q$$

Check $Re = \frac{\rho a \bar{V}}{\mu} = \frac{a \bar{V}}{\nu}$ $\qquad \nu = 6 \times 10^{-5}$ m²/s (Fig. A.3)

$$\bar{V} = \frac{Q}{A} = \frac{Q}{a l} = \frac{Q}{a \pi D} = \frac{1}{\pi} \times 3.54 \times 10^{-7} \frac{\text{m}^3}{\text{s}} \times \frac{1}{2.5 \times 10^{-5}\,\text{m}} \times \frac{1}{0.1\,\text{m}} = 0.045\,\text{m/s}$$

$$Re = \frac{a \bar{V}}{\nu} = 2.5 \times 10^{-5}\,\text{m} \times 0.045 \frac{\text{m}}{\text{s}} \times \frac{1}{6 \times 10^{-5}} \frac{\text{s}}{\text{m}^2} = 0.0188$$

and flow is definitely laminar

Piston moving down at speed v displaces liquid at rate Q, where

$$Q = \frac{\pi D^2}{4} v$$

Then

$$v = \frac{4Q}{\pi D^2} = \frac{4}{\pi} \times 3.54 \times 10^{-7} \frac{\text{m}^3}{\text{s}} \times \frac{1}{(0.1\,\text{m})^2} = 4.51 \times 10^{-5}\,\text{m/s}$$

Since $\frac{v}{\bar{V}} = \frac{4.51 \times 10^{-5}\,\text{m/s}}{0.045\,\text{m/s}} = 10^{-3}$, motion of piston can be neglected.

Problem 8.12

Given: Hydraulic jack supports supports a load of 9000 kg

piston diameter $D = 100$ mm

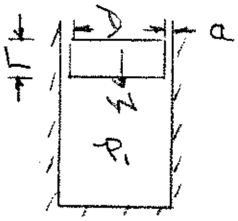
radial clearance $a = 0.05$ mm

piston length $L = 120$ mm

Fluid has viscosity of SAE 30 oil at 30°C

Find: Leakage rate of fluid past the piston

Solution:



Model the flow as steady, fully developed laminar flow between stationary parallel plates, i.e., neglect motion of the piston.

Then, the leakage flow rate can be evaluated from Eq. 8.6c (in the text)

$$\frac{Q}{l} = \frac{a^3 \Delta p}{12 \mu L} \quad \text{where } l = \pi D$$

From Fig. A.2 at $T = 30°C$, $\mu = 3.0 \times 10^{-1}$ N.s/m²

$\Delta p = p_1 - p_{atm}$ and $p_1 = \frac{W}{A} = \frac{mg}{A} = \frac{4mg}{\pi D^2}$

$$p_1 = \frac{4}{\pi} \times 9000\,kg \times 9.81\,\frac{m}{s^2} \times \frac{1}{(0.1\,m)^2} \times \frac{N \cdot s^2}{kg \cdot m} = 11.2\,MPa$$

$$Q = \frac{\pi D a^3 \Delta p}{12 \mu L} = \frac{\pi}{12} \times (0.1\,m) \times (5 \times 10^{-5}\,m)^3 \times 11.2 \times 10^6\,\frac{N}{m^2} \times \frac{1}{0.3}\,\frac{m^2}{N.s} \times \frac{1}{0.12\,m}$$

$$Q = 1.01 \times 10^{-6}\,m^3/s = 1.01 \times 10^{-3}\,L/s \quad \text{——— } Q$$

Check $Re = \frac{\rho a \bar{V}}{\mu} = \frac{a \bar{V}}{\nu}$ where $\nu = 2.8 \times 10^{-4}$ m²/s (Fig. A.3)

$$\bar{V} = \frac{Q}{A} = \frac{Q}{a l} = \frac{Q}{a \pi D} = \frac{1}{\pi} \times 1.01 \times 10^{-6}\,\frac{m^3}{s} \times \frac{1}{5 \times 10^{-5}\,m} \times \frac{1}{0.1\,m} = 0.0643\,m/s$$

$$Re = \frac{a \bar{V}}{\nu} = 5 \times 10^{-5}\,m \times 0.0643\,\frac{m}{s} \times \frac{1}{2.8 \times 10^{-4}}\,\frac{s}{m^2} = 0.011$$

∴ flow is definitely laminar

Piston moving down at speed v displaces liquid at rate Q where $Q = \frac{\pi D^2}{4} v$

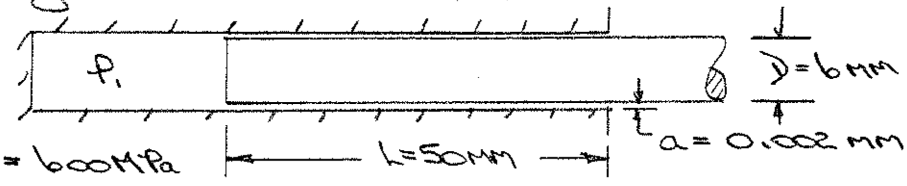
Then

$$v = \frac{4Q}{\pi D^2} = \frac{4}{\pi} \times 1.01 \times 10^{-6}\,\frac{m^3}{s} \times \frac{1}{(0.1\,m)^2} = 1.29 \times 10^{-4}\,m/s$$

Since $\frac{v}{\bar{V}} = \frac{1.29 \times 10^{-4}\,m/s}{0.0643\,m/s} = 2.0 \times 10^{-3}$, motion of piston can be neglected.

Problem 8.13

Given: Piston-cylinder device with SAE 10W oil at 35°



Find: Leakage flow rate

Solution: Computing equation: $\frac{Q}{l} = \frac{a^3 \Delta p}{12 \mu L}$ (8.6c)

Assumptions: (1) Laminar flow
(2) Fully developed flow ($L \gg a$)

For SAE 10W oil at 35°C, $\mu = 3.8 \times 10^{-2}$ N·s/m² (Fig. A.2)

For this configuration, $l = \pi D$, since $a \ll D$. Then

$$Q = \frac{a^3 \Delta p \, l}{12 \mu L} = \frac{\pi a^3 \Delta p D}{12 \mu L}$$

$$Q = \frac{\pi}{12} \times (2 \times 10^{-6}\,m)^3 \times 6 \times 10^8 \frac{N}{m^2} \times 0.006\,m \times \frac{1}{3.8 \times 10^{-2}} \frac{m^2}{N \cdot s} \times \frac{1}{0.05\,m}$$

$$Q = 3.97 \times 10^{-9}\,m^3/s = 3.97 \times 10^{-6}\,L/s \quad Q$$

Check Re to assure laminar flow

$$\bar{V} = \frac{Q}{A} = \frac{Q}{\pi D a} = \frac{1}{\pi} \times 3.97 \times 10^{-9} \frac{m^3}{s} \times \frac{1}{0.006\,m} \times \frac{1}{2 \times 10^{-6}\,m} = 0.105\,m/s$$

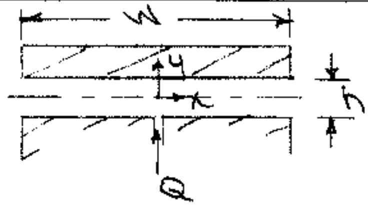
SG = 0.88 (Table A.2); $\rho = SG \rho_{H_2O}$

$$Re = \frac{\rho \bar{V} a}{\mu} = \frac{SG \rho_{H_2O} \bar{V} a}{\mu}$$

$$= 0.88 \times 999 \frac{kg}{m^3} \times 0.105 \frac{m}{s} \times 2 \times 10^{-6}\,m \times \frac{1}{3.8 \times 10^{-2}} \frac{m^2}{N \cdot s}$$

Re = 0.005 ≪ 2300 so flow is definitely laminar!

Problem 8.14

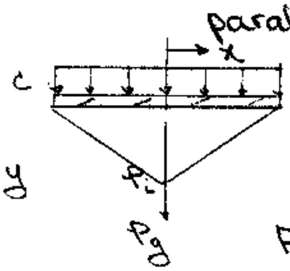


Given: Hydrostatic bearing is to support a load of $c = 3600$ lbf per ft. of length perpendicular to the diagram. Bearing is supplied with SAE 30 oil at 100°F and 100 psig.

Find: (a) Required width of the bearing pad
(b) Resulting pressure gradient
(c) Gap height, h, for $\frac{Q}{\ell} = 6.0 \times 10^{-4}$ ft³/min/ft

Solution:

Assume steady, fully developed, laminar flow between infinite, parallel plates.



Then the pressure over the bearing is linear, varying from p_i at $x=0$ to 0 gage at $x = \frac{W}{2}$

Let b = length perpendicular to diagram.
From the freebody diagram of the pad, $\Sigma F_y = 0$

$$\therefore \; cb = 2\int p_g \, dA = 2\int_0^{W/2} p_g \, b\, dx = 2\int_0^{W/2} p_i \left(1 - \frac{x}{W/2}\right) b\, dx$$

$$c = 2\int_0^{W/2} p_i\left(1 - \frac{2x}{W}\right)dx = 2\left[p_i\left(x - \frac{x^2}{W}\right)\right]_0^{W/2} = 2p_i\left(\frac{W}{2} - \frac{W}{4}\right) = p_i \frac{W}{2}$$

$$\therefore \; W = \frac{2c}{p_i} = 2 \times \frac{3600\,\text{lb}}{\text{ft}} \times \frac{\text{in}^2}{100\,\text{lb}} \times \frac{\text{ft}^2}{144\,\text{in}^2} = 0.50\,\text{ft} \qquad W$$

The corresponding pressure gradient, $\frac{dp}{dx}$, is given by

$$\frac{dp}{dx} = -\frac{\Delta p}{\Delta x} = -\frac{2\Delta p}{W} = -2 \times \left(100\,\frac{\text{lb}}{\text{in}^2}\right) \times \frac{1}{0.50\,\text{ft}} = -400\,\text{psi/ft} \qquad \frac{dp}{dx}$$

The flow rate is given by Eq 8.6b $\quad \frac{Q}{\ell} = -\frac{1}{12\mu}\left(\frac{\partial p}{\partial x}\right)h^3$

Then
$$h = \left[-\frac{12\mu (Q/\ell)}{\partial p/\partial x}\right]^{1/3} \qquad \text{From Fig. A.2, } \mu = 2.30 \times 10^{-3}\,\frac{\text{lbf}\cdot\text{s}}{\text{ft}^2}$$

$$h = \left[12 \times 2.30 \times 10^{-3}\,\frac{\text{lbf}\cdot\text{s}}{\text{ft}^2} \times 6.0 \times 10^{-4}\,\frac{\text{ft}^2}{\text{min}} \times \frac{\text{in}^2\,\text{ft}}{-400\,\text{lbf}} \times \frac{\text{min}}{60\,\text{s}} \times \frac{12\,\text{in}}{\text{ft}}\right]^{1/3} = 2.02 \times 10^{-3}\,\text{in} \qquad h$$

Check Re

$$Re = \frac{\rho \bar{V} D}{\mu} = \frac{\bar{V}}{\nu}\frac{Q}{A} = \frac{h}{\nu}\frac{Q}{bh} = \frac{1}{\nu}\left(\frac{Q}{\ell}\right)$$

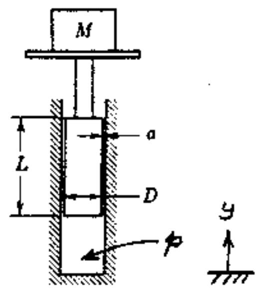
From Fig. A.3 $\quad \nu = 1.29 \times 10^{-3}$ ft²/s

$$\therefore \; Re = \frac{1}{\nu}\left(\frac{Q}{\ell}\right) = \frac{1}{1.29 \times 10^{-3}}\,\frac{\text{s}}{\text{ft}^2} \times 6.0 \times 10^{-4}\,\frac{\text{ft}^2}{\text{min}} \times \frac{\text{min}}{60\,\text{s}} = 7.75 \times 10^{-3}$$

$\therefore$ Flow is definitely laminar.

Problem 8.15

Given: Piston-cylinder device, as shown.



$D = 6\ mm \quad L = 25\ mm$

Liquid is SAE 30 oil at 20°C.

Find: (a) M to develop $p = 1.5$ MPa (gage)
(b) Leakage flow rate in terms of a
(c) Maximum a to provide < 1 mm/min movement.

Solution: The mass may be found from a force balance on the piston.

$$\Sigma F_y = \frac{\pi D^2}{4}(p - p_{atm}) - Mg = 0 \quad \text{so} \quad M = \frac{\pi D^2}{4g} p_{gage}$$

$$M = \frac{\pi}{4} \times (0.006)^2 m^2 \times 1.5 \times 10^6 \frac{N}{m^2} \times \frac{s^2}{9.81\ m} \times \frac{kg \cdot m}{N \cdot s^2} = 4.32\ kg$$ M

The leakage flow rate may be evaluated for flow between flat plates. From Eq. 8.6c, neglecting motion of the piston,

$$\frac{Q}{l} = \frac{a^3 \Delta p}{12 \mu L} \quad \text{or, since } l = \pi D, \quad Q = \frac{\pi}{12} \frac{a^3 \Delta p D}{\mu L} \sim a^3$$ Q

The piston, moving downward at speed, V, displaces liquid at rate

$$Q = \frac{\pi D^2}{4} V = \frac{\pi}{4} (0.006)^2 m^2 \times 0.001 \frac{m}{min} \times \frac{min}{60\ s} = 4.71 \times 10^{-10}\ m^3/s$$

Then, with $\mu = 0.42\ N \cdot sec/m^2$ (at 20°C, Fig. A.2),

$$a = \left[\frac{12 \mu Q L}{\pi D \Delta p}\right]^{1/3} = \left[\frac{12}{\pi} \times 0.42 \frac{N \cdot s}{m^2} \times 4.71 \times 10^{-10} \frac{m^3}{s} \times 0.025\ m \times \frac{1}{0.006\ m} \times \frac{m^2}{1.5 \times 10^6\ N}\right]^{1/3}$$

$$a = 1.28 \times 10^{-5}\ m \quad (12.8\ \mu m)$$ a

Check assumptions: $\bar{V} = \frac{Q}{A} = \frac{Q}{\pi D a} = \frac{1}{\pi} \times 4.71 \times 10^{-10} \frac{m^3}{s} \times \frac{1}{0.006\ m} \times \frac{1}{1.28 \times 10^{-5}\ m} = 1.95 \frac{mm}{s}$

Thus $\frac{V}{\bar{V}} = 1 \frac{mm}{min} \times \frac{sec}{1.95\ mm} \times \frac{min}{60\ s} = 0.00855 < 0.01$

Therefore piston motion is negligible.

Also $Re = \frac{\bar{V} a}{\nu}$; $\nu = \frac{\mu}{\rho} = \frac{\mu}{SG\, \rho_{H_2O}}$. From Table A.2 (Appendix A), SG = 0.92

$$\nu = 0.42 \frac{N \cdot s}{m^2} \times \frac{m^3}{(0.92) 1000\ kg} \times \frac{kg \cdot m}{N \cdot s^2} = 4.57 \times 10^{-4}\ m^2/s$$

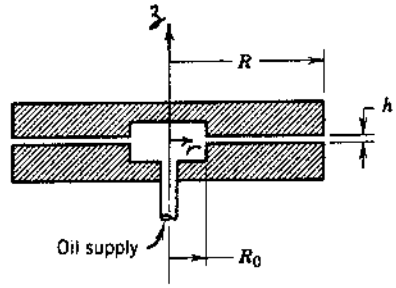
$$Re = 1.95 \times 10^{-3} \frac{m}{s} \times 1.28 \times 10^{-5} m \times \frac{s}{4.57 \times 10^{-4}\ m^2} = 5.46 \times 10^{-5} \lll 1$$

Therefore flow is surely laminar!

Problem 8.16

Given: Viscous flow in narrow gap between parallel disks, as shown.

Flow rate is Q, accelerations are small.
Velocity profile same as fully developed.



Find: (a) Expression for $\bar{V}(r)$, (b) dp/dr in gap
(c) Expression for $p(r)$.
(d) Show net force to hold upper plate is

$$F = \frac{3\mu Q R^2}{h^3}\left[1-\left(\frac{R_0}{R}\right)^2\right]$$

Solution: From the definition of mean velocity, $Q = \bar{V}\,2\pi r h$ so $\bar{V} = \dfrac{Q}{2\pi r h}$ ← $\bar{V}(r)$

The pressure change with radius can be evaluated by analogy to Eq. 8.6b

$$\frac{Q}{l} = -\frac{1}{12\mu}\left(\frac{\partial p}{\partial x}\right)h^3 \quad \text{with } l = 2\pi r \quad \text{so } \frac{Q}{2\pi r} = -\frac{1}{12\mu}\left(\frac{\partial p}{\partial r}\right)h^3$$

Thus

$$\frac{dp}{dr} = -\frac{6\mu Q}{\pi h^3 r}$$ ← $\frac{dp}{dr}$

Integrating to find $p(r)$,

$$\int_p^{p_{atm}} dp = p_{atm} - p = \int_r^R -\frac{6\mu Q}{\pi h^3 r}\,dr = -\frac{6\mu Q}{\pi h^3}\ln r\Big]_r^R = \frac{6\mu Q}{\pi h^3}\ln(r/R)$$

Thus $p(r) = p_{atm} - \dfrac{6\mu Q}{\pi h^3}\ln(r/R)$ $\quad (R_0 < r < R)$; $p = p_0 \quad r < R_0$ ← $p(r)$

The force on the upper plate is $dF_z = (p(r) - p_{atm})\,2\pi r\,dr$

Integrating and using gage pressures (note $p_{0g} = -\dfrac{6\mu Q}{\pi h^3}\ln\left(\dfrac{R_0}{R}\right)$)

$$F_z = p_0\pi R_0^2 + \int_{R_0}^R p(r)\,2\pi r\,dr = p_0\pi R_0^2 + 2\pi R^2\int_{R_0/R}^1 p(r)\left(\frac{r}{R}\right)d\left(\frac{r}{R}\right)$$

$$= p_0\pi R_0^2 + 2\pi R^2\int_{R_0/R}^1 -\frac{6\mu Q}{\pi h^3}\ln\left(\frac{r}{R}\right)\left(\frac{r}{R}\right)d\left(\frac{r}{R}\right) = p_0\pi R_0^2 - \frac{12\mu Q R^2}{h^3}\left(\frac{r}{R}\right)^2\left[\frac{1}{2}\ln\left(\frac{r}{R}\right) - \frac{1}{4}\right]_{R_0/R}^1$$

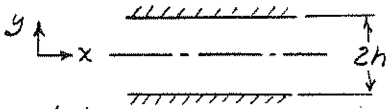
$$= p_0\pi R_0^2 - \frac{12\mu Q R^2}{h^3}\left\{(1)\left[\frac{1}{2}(0) - \frac{1}{4}\right] - \left(\frac{R_0}{R}\right)^2\left[\frac{1}{2}\ln\left(\frac{R_0}{R}\right) - \frac{1}{4}\right]\right\}$$

$$= -\frac{6\mu Q R^2}{h^3}\left(\frac{R_0}{R}\right)^2\ln\left(\frac{R_0}{R}\right) - \frac{6\mu Q R^2}{h^3}\left[-\frac{1}{2} - \left(\frac{R_0}{R}\right)^2\ln\left(\frac{R_0}{R}\right) + \frac{1}{2}\left(\frac{R_0}{R}\right)^2\right]$$

$$F_z = \frac{3\mu Q R^2}{h^3}\left[1 - \left(\frac{R_0}{R}\right)^2\right]$$ ← F_z

Given: Power-law model for non-Newtonian liquid, $\tau_{yx} = k\left(\frac{du}{dy}\right)^n$

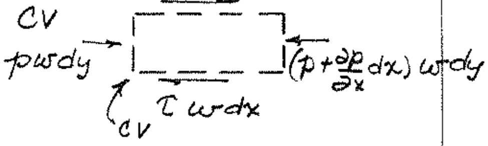
Find: Show $u = \left(\frac{h}{k}\frac{\Delta p}{L}\right)^{1/n} \frac{nh}{n+1}\left[1-\left(\frac{y}{h}\right)^{\frac{n+1}{n}}\right]$



for fully developed laminar flow between plates.

Plot: Profiles u/U vs. y/h for $n = 0.7, 1.0$, and 1.3 ($U = u_{max}$).

Solution: Apply momentum equation to differential CV



Basic equation:

$$F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$$

($F_{B_x} = 0(1)$, $\frac{\partial}{\partial t} = 0(2)$, $\int_{CS} = 0(3)$)

Assumptions: (1) Horizontal flow
(2) Steady flow
(3) Fully developed flow

Then

$$p\,w\,dy + \left(\tau + \frac{\partial \tau}{\partial y}dy\right)w\,dx - \left(p + \frac{\partial p}{\partial x}dx\right)w\,dy - \tau w\,dx = 0 \quad \text{or} \quad \frac{\partial \tau}{\partial y} = \frac{\partial p}{\partial x}$$

Since $\tau = \tau(y)$ and $p = p(x)$, then $\frac{d\tau}{dy} = \frac{\partial p}{\partial x} = \text{constant}$ and $\tau = y\frac{\partial p}{\partial x}$ or

$$\tau_{yx} = k\left(\frac{du}{dy}\right)^n = y\frac{\partial p}{\partial x} = -y\frac{\Delta p}{L}$$

Thus

$$\frac{du}{dy} = -\left(\frac{1}{k}\frac{\Delta p}{L}\right)^{1/n} y^{1/n}$$

Integrating

$$u = -\left(\frac{1}{k}\frac{\Delta p}{L}\right)^{1/n}\frac{1}{1/n+1}\, y^{1/n+1} + c = -\left(\frac{1}{k}\frac{\Delta p}{L}\right)^{\frac{1}{n}}\frac{n}{n+1}\, y^{\frac{n+1}{n}} + c$$

But $u = 0$ at $y = h$, so

$$c = \left(\frac{1}{k}\frac{\Delta p}{L}\right)^{\frac{1}{n}}\frac{n}{n+1}\, h^{\frac{n+1}{n}}$$

and

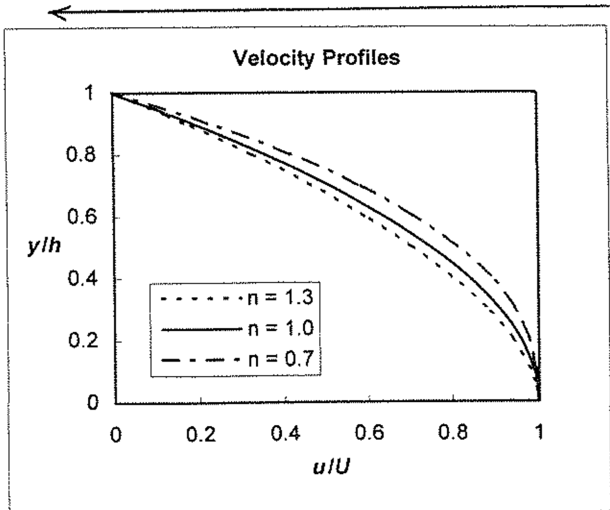
$$u = \left(\frac{1}{k}\frac{\Delta p}{L}\right)^{\frac{1}{n}}\frac{n}{n+1}\, h^{\frac{n+1}{n}}\left[1-\left(\frac{y}{h}\right)^{\frac{n+1}{n}}\right]$$

or

$$u = \left(\frac{h}{k}\frac{\Delta p}{L}\right)^{\frac{1}{n}}\frac{nh}{n+1}\left[1-\left(\frac{y}{h}\right)^{\frac{n+1}{n}}\right]$$

u

	n = 0.7	n = 1.0	n = 1.3
y/h	u/U	u/U	u/U
0	1	1	1
0.03	1.000	0.999	0.998
0.06	0.999	0.996	0.993
0.1	0.996	0.990	0.983
0.2	0.980	0.960	0.942
0.3	0.946	0.910	0.881
0.4	0.892	0.840	0.802
0.5	0.814	0.750	0.707
0.6	0.711	0.640	0.595
0.7	0.580	0.510	0.468
0.8	0.418	0.360	0.326
0.9	0.226	0.190	0.170
1	0	0	0



Problem 8.18

Using the profile of Problem 8.17, show that the flow rate for fully developed laminar flow of a power-law fluid between stationary parallel plates may be written as

$$Q = \left(\frac{h}{k}\frac{\Delta p}{L}\right)^{\frac{1}{n}} \frac{2nwh^2}{2n+1}$$

Here w is the plate width. In such an experimental setup the following data on applied pressure difference Δp and flow rate Q were obtained:

Δp (kPa)	10	20	30	40	50	60	70	80	90	100
Q (L/min)	0.451	0.759	1.01	1.15	1.41	1.57	1.66	1.85	2.05	2.25

Determine if the fluid is pseudoplastic or dilatant, and obtain an experimental value for n.

Given: Laminar velocity profile of power-law fluid flow between parallel plates

Find: Expression for flow rate; from data determine the type of fluid

Solution

The velocity profile is $$u = \left(\frac{h}{k}\cdot\frac{\Delta p}{L}\right)^{\frac{1}{n}} \cdot \frac{n\cdot h}{n+1}\cdot\left[1 - \left(\frac{y}{h}\right)^{\frac{n+1}{n}}\right]$$

The flow rate is then $$Q = w\cdot\int_{-h}^{h} u\,dy$$ or, because the flow is symmetric

$$Q = 2\cdot w\cdot\int_{0}^{h} u\,dy$$

The integral is computed as

$$\int 1-\left(\frac{y}{h}\right)^{\frac{n+1}{n}} dy = y\cdot\left[1-\frac{n}{2\cdot n+1}\cdot\left(\frac{y}{h}\right)^{\frac{2\cdot n+1}{n}}\right]$$

Using this with the limits

$$Q = 2\cdot w\cdot\left(\frac{h}{k}\cdot\frac{\Delta p}{L}\right)^{\frac{1}{n}}\cdot\frac{n\cdot h}{n+1}\cdot h\cdot\left[1-\frac{n}{2\cdot n+1}\cdot(1)^{\frac{2\cdot n+1}{n}}\right]$$

$$Q = \left(\frac{h}{k}\cdot\frac{\Delta p}{L}\right)^{\frac{1}{n}}\cdot\frac{2\cdot n\cdot w\cdot h^{2}}{2\cdot n+1}$$

Problem 8.18 (In Excel)

Using the profile of Problem 8.17, show that the flow rate for fully developed laminar flow of a power-law fluid between stationary parallel plates may be written as

$$Q = \left(\frac{h}{k}\frac{\Delta p}{L}\right)^{\frac{1}{n}} \frac{2nwh^2}{2n+1}$$

Here w is the plate width. In such an experimental setup the following data on applied pressure difference Δp and flow rate Q were obtained:

Δp (kPa)	10	20	30	40	50	60	70	80	90	100
Q (L/min)	0.451	0.759	1.01	1.15	1.41	1.57	1.66	1.85	2.05	2.25

Determine if the fluid is pseudoplastic or dilatant, and obtain an experimental value for n.

Given: Laminar velocity profile of power-law fluid flow between parallel plates

Find: Expression for flow rate; from data determine the type of fluid

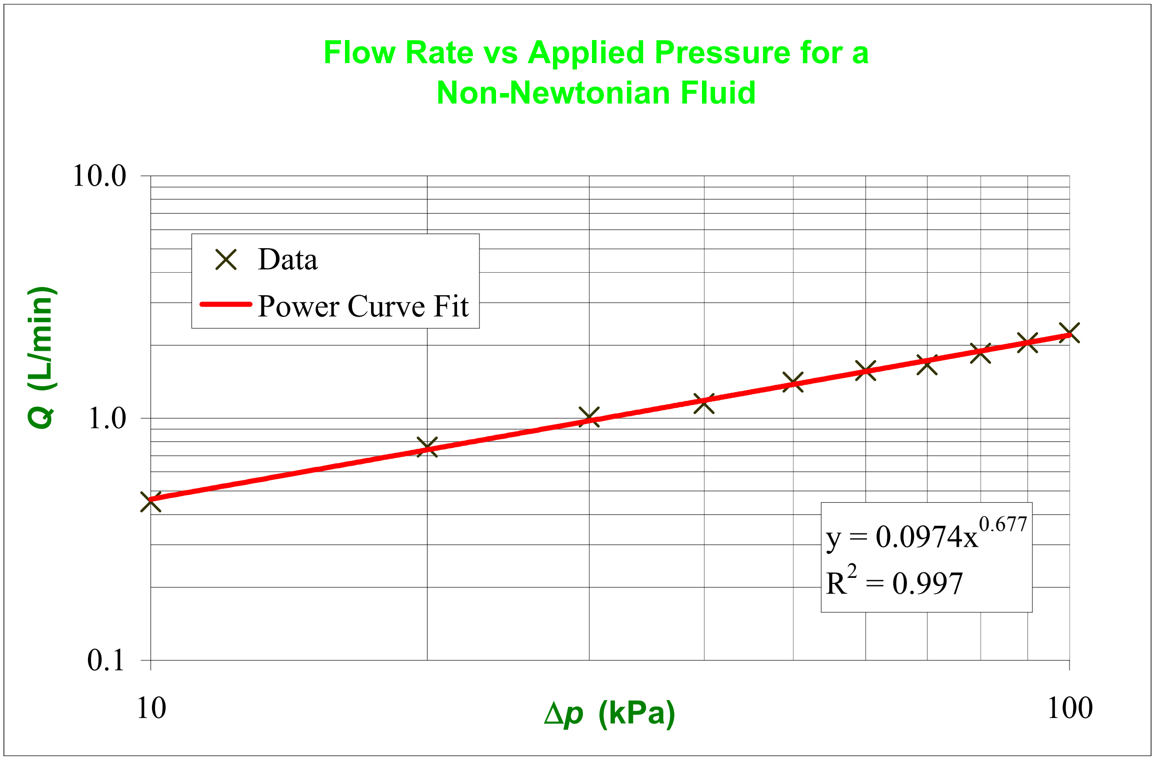
Solution

The data is

Δp (kPa)	10	20	30	40	50	60	70	80	90	100
Q (L/min)	0.451	0.759	1.01	1.15	1.41	1.57	1.66	1.85	2.05	2.25

This must be fitted to $Q = \left(\frac{h}{k}\cdot\frac{\Delta p}{L}\right)^{\frac{1}{n}} \cdot \frac{2\cdot n\cdot w\cdot h^2}{2\cdot n+1}$ or $Q = k\cdot\Delta p^{\frac{1}{n}}$

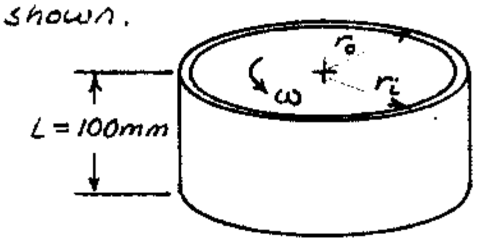
We can fit a power curve to the data



Hence $1/n = 0.677$ $n = 1.48$

Problem 8.19

Given: Sealed journal bearing rotating as shown.



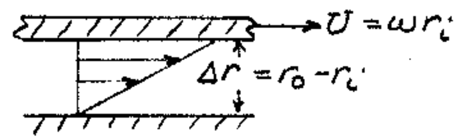
$r_o = 26 mm$, $r_i = 25 mm$

Gap contains oil in laminar motion with linear velocity profile.

$\omega = 2800 rpm$ and Torque, $T = 0.2 N\cdot m$

Find: (a) Viscosity of oil
(b) Will torque increase or decrease with time? Why?

Solution: "Unfold" bearing since gap is small, and consider as flow between parallel plates. Apply Newton's law of viscosity.



Basic equation: $\tau_{yx} = \mu \frac{du}{dy}$

Assumption: Linear velocity profile

Then $\tau_{yx} = \mu \frac{U}{\Delta r} = \frac{\mu \omega r_i}{\Delta r}$

and

$$T = r_i \left(2\pi r_i L \tau_{yx}\right) = 2\pi r_i^2 L \tau_{yx} = \frac{2\pi \mu \omega r_i^3 L}{\Delta r}$$

Solving, $\mu = \frac{\Delta r\, T}{2\pi \omega r_i^3 L}$

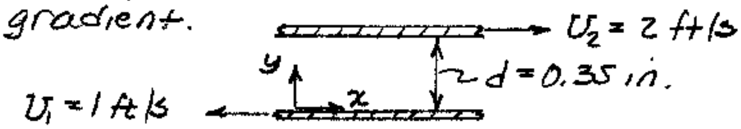
$$\mu = \frac{1}{2\pi} \times 0.001 m \times 0.2 N\cdot m \times \frac{min}{2800 rev} \times \frac{1}{(0.025)^3 m^3} \times \frac{1}{0.1 m} \times \frac{rev}{2\pi rad} \times 60 \frac{s}{min}$$

$\mu = 0.0695 N\cdot s / m^2$ ← μ

Bearing is sealed, so oil temperature will increase as energy is dissipated by friction. For liquids, μ decreases as T increases. Thus torque will decrease, since it is proportional to μ.

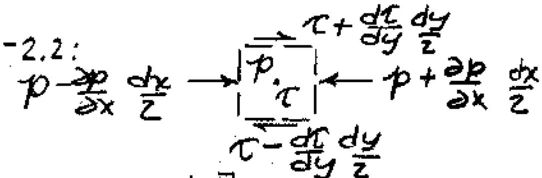
Problem 8.20

Given: Fully developed laminar flow between parallel plates with no pressure gradient.



Find: (a) Expression for velocity profile in gap.
(b) Volume flow rate per unit depth passing cross-section.

Solution: Use analysis of Section 8-2.2:



Sum forces in x direction:

$$\left[\tau + \frac{d\tau}{dy}\frac{dy}{2} - \left(\tau - \frac{d\tau}{dy}\frac{dy}{2}\right)\right] b\,dx + \left[p - \frac{\partial p}{\partial x}\frac{dx}{2} - \left(p + \frac{\partial p}{\partial x}\frac{dx}{2}\right)\right] b\,dy = 0$$

Simplifying $\quad \frac{d\tau}{dy} = \frac{dp}{dx} = 0 \quad$ so $\quad \mu \frac{d^2u}{dy^2} = 0$

Integrating twice $\quad u = C_1 y + C_2$

Boundary conditions: $\quad y = 0,\ u = -U_1\ ;\ C_2 = -U_1$

$\quad y = d,\ u = U_2\ ;\ U_2 = C_1 d - U_1$, so $C_1 = \frac{U_1 + U_2}{d}$

Thus $\quad u = (U_1 + U_2)\frac{y}{d} - U_1$

$\quad u(\text{m/sec}) = 3\frac{y}{d} - 1$ ← Profile $u(y)$

The volume flow rate is

$$Q = \int_A u\,dA = \int_0^d u\,b\,dy = \int_0^d \left[(U_1 + U_2)\frac{y}{d} - U_1\right] b\,dy = b\left[(U_1 + U_2)\frac{y^2}{2d} - U_1 y\right]_0^d$$

$$Q = b\left[(U_1 + U_2)\frac{d}{2} - U_1 d\right] = b\,(U_2 - U_1)\frac{d}{2} = bd\left(\frac{U_2 - U_1}{2}\right)$$

So

$$\frac{Q}{b} = \frac{1}{2} \times 0.35\ \text{in.} \times (2-1)\,\frac{\text{ft}}{\text{s}} \times \frac{\text{ft}}{12\ \text{in.}} = 0.0146\ \text{ft}^3/\text{s}\ /\ \text{ft}$$ ← $\frac{Q}{b}$

Problem 8.21

Two immiscible fluids are contained between infinite parallel plates. The plates are separated by distance $2h$, and the two fluid layers are of equal thickness h; the dynamic viscosity of the upper fluid is three times that of the lower fluid. If the lower plate is stationary and the upper plate moves at constant speed $U = 5$ m/s, what is the velocity at the interface? Assume laminar flows, and that the pressure gradient in the direction of flow is zero.

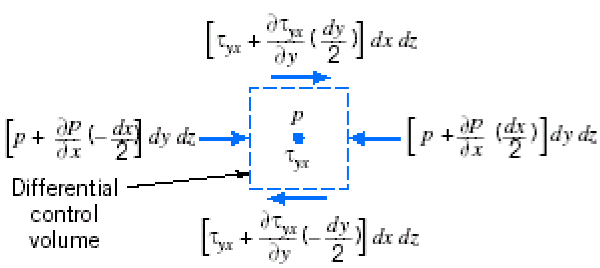
Given: Properties of two fluids flowing between parallel plates; upper plate has velocity of 5 m/s

Find: Velocity at the interface

Solution

Given data $\quad U = 5\cdot\frac{m}{s} \qquad \mu_2 = 3\cdot\mu_1 \qquad$ (Lower fluid is fluid 1; upper is fluid 2)

Following the analysis of Section 8-2, analyse the forces on a differential CV of either fluid



The net force is zero for steady flow, so

$$\left[\tau + \frac{d\tau}{dy}\cdot\frac{dy}{2} - \left(\tau - \frac{d\tau}{dy}\cdot\frac{dy}{2}\right)\right]\cdot dx\cdot dz + \left[p - \frac{dp}{dx}\cdot\frac{dx}{2} - \left(p + \frac{dp}{dx}\cdot\frac{dx}{2}\right)\right]\cdot dy\cdot dz = 0$$

Simplifying

$$\frac{d\tau}{dy} = \frac{dp}{dx} = 0 \qquad \text{so for each fluid} \qquad \mu \cdot \frac{d^2}{dy^2} u = 0$$

Applying this to fluid 1 (lower fluid) and fluid 2 (upper fluid), integrating twice yields

$$u_1 = c_1 \cdot y + c_2 \qquad\qquad u_2 = c_3 \cdot y + c_4$$

We need four BCs. Three are obvious

$$y = 0 \qquad u_1 = 0 \qquad (1)$$

$$y = h \qquad u_1 = u_2 \qquad (2)$$

$$y = 2 \cdot h \qquad u_2 = U \qquad (3)$$

The fourth BC comes from the fact that the stress at the interface generated by each fluid is the s

$$y = h \qquad \mu_1 \cdot \frac{du_1}{dy} = \mu_2 \cdot \frac{du_2}{dy} \qquad (4)$$

Using these four BCs

$$0 = c_2$$

$$c_1 \cdot h + c_2 = c_3 \cdot h + c_4$$

$$U = c_3 \cdot 2 \cdot h + c_4$$

$$\mu_1 \cdot c_1 = \mu_2 \cdot c_3$$

Hence

$$c_2 = 0$$

Eliminating c_4 from the second and third equations

$$c_1 \cdot h - U = -c_3 \cdot h$$

and

$$\mu_1 \cdot c_1 = \mu_2 \cdot c_3$$

Hence

$$c_1 \cdot h - U = -c_3 \cdot h = -\frac{\mu_1}{\mu_2} \cdot h \cdot c_1$$

$$c_1 = \frac{U}{h \cdot \left(1 + \frac{\mu_1}{\mu_2}\right)}$$

Hence for fluid 1 (we do not need to complete the analysis for fluid 2)

$$u_1 = \frac{U}{h \cdot \left(1 + \frac{\mu_1}{\mu_2}\right)} \cdot y$$

Evaluating this at $y = h$, where $u_1 = u_{\text{interface}}$

$$u_{\text{interface}} = \frac{5 \cdot \frac{m}{s}}{\left(1 + \frac{1}{3}\right)}$$

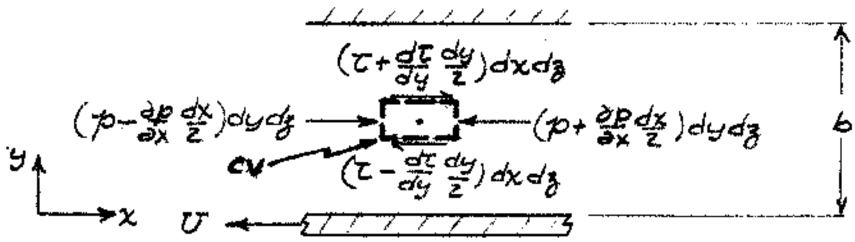
$$u_{\text{interface}} = 3.75 \frac{m}{s}$$

Problem 8.22

Given: Water at 60°C flows between large flat plates.

$U = 0.3\, m/s$

$b = 3\, mm$



Find: Pressure gradient required for zero net flow at a section.

Solution: Apply momentum equation using CV and coordinates shown.

Basic equations:

$$F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\rho\, dV + \int_{CS} u\rho \vec{V}\cdot d\vec{A}, \quad \tau = \tau_{yx} = \mu \frac{du}{dy}$$

($F_{B_x} = 0(1)$, $\frac{\partial}{\partial t} = 0(2)$, $\int_{CS} = 0(3)$)

Assumptions: (1) $F_{B_x} = 0$
(2) Steady flow
(3) Fully-developed flow
(4) Newtonian fluid

Then $F_{S_x} = 0$. Substituting the force terms (see page 315 for details) gives

$$\frac{\partial p}{\partial x} = \frac{d\tau_{yx}}{dy} = \frac{d}{dy}\left(\mu\frac{du}{dy}\right) = \mu\frac{d^2u}{dy^2} \quad \text{or} \quad \frac{d^2u}{dy^2} = \frac{1}{\mu}\frac{\partial p}{\partial x}$$

Integrating twice,

$$u = \frac{1}{2\mu}\frac{\partial p}{\partial x}y^2 + c_1 y + c_2$$

To evaluate the constants c_1 and c_2, we must use the boundary conditions. At $y = 0$, $u = -U$, so $c_2 = -U$. At $y = b$, $u = 0$, so

$$0 = \frac{1}{2\mu}\frac{\partial p}{\partial x}b^2 + c_1 b - U \quad \text{or} \quad c_1 = \frac{U}{b} - \frac{1}{2\mu}\frac{\partial p}{\partial x}b$$

Thus

$$u = \frac{1}{2\mu}\frac{\partial p}{\partial x}(y^2 - by) + U\left(\frac{y}{b} - 1\right)$$

To find the flowrate, we integrate

$$\frac{Q}{w} = \int_0^b u\,dy = \int_0^b\left[\frac{1}{2\mu}\frac{\partial p}{\partial x}(y^2 - by) + U\left(\frac{y}{b} - 1\right)\right]dy = -\frac{1}{12\mu}\frac{\partial p}{\partial x}b^3 - \frac{Ub}{2}$$

For $Q = 0$, with $\mu = 4.63\times10^{-4}\ \frac{N\cdot s}{m^2}$ from Table A.8,

$$\frac{\partial p}{\partial x} = -\frac{6U\mu}{b^2} = -6 \times 0.3\frac{m}{s} \times 4.63\times10^{-4}\frac{N\cdot s}{m^2} \times \frac{1}{(0.003)^2 m^2} = -92.6\ N/m^2\cdot m$$ $\frac{\partial p}{\partial x}$

Thus pressure must decrease in x direction for zero net flowrate.

Problem 8.23

Two immiscible fluids are contained between infinite parallel plates. The plates are separated by distance $2h$, and the two fluid layers are of equal thickness $h = 2.5$ mm. The dynamic viscosity of the upper fluid is twice that of the lower fluid, which is $\mu_{lower} = 0.5\ N \cdot s/m^2$. If the plates are stationary and the applied pressure gradient is $-1000\ N/m^2/m$, find the velocity at the interface. What is the maximum velocity of the flow? Plot the velocity distribution.

Given: Properties of two fluids flowing between parallel plates; applied pressure gradient

Find: Velocity at the interface; maximum velocity; plot velocity distribution

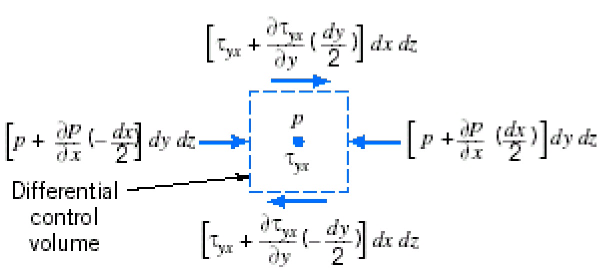
Solution

Given data $\quad k = \frac{dp}{dx} = -1000 \cdot \frac{Pa}{m} \qquad h = 2.5 \cdot mm$

$$\mu_1 = 0.5 \cdot \frac{N \cdot s}{m^2} \qquad \mu_2 = 2 \cdot \mu_1 \qquad \mu_2 = 1 \frac{N \cdot s}{m^2}$$

(Lower fluid is fluid 1; upper is fluid 2)

Following the analysis of Section 8-2, analyse the forces on a differential CV of either fluid



The net force is zero for steady flow, so

$$\left[\tau + \frac{d\tau}{dy}\cdot\frac{dy}{2} - \left(\tau - \frac{d\tau}{dy}\cdot\frac{dy}{2}\right)\right]\cdot dx\cdot dz + \left[p - \frac{dp}{dx}\cdot\frac{dx}{2} - \left(p + \frac{dp}{dx}\cdot\frac{dx}{2}\right)\right]\cdot dy\cdot dz = 0$$

Simplifying

$$\frac{d\tau}{dy} = \frac{dp}{dx} = k \qquad \text{so for each fluid} \qquad \mu\cdot\frac{d^2}{dy^2}u = k$$

Applying this to fluid 1 (lower fluid) and fluid 2 (upper fluid), integrating twice yields

$$u_1 = \frac{k}{2\cdot\mu_1}\cdot y^2 + c_1\cdot y + c_2 \qquad\qquad u_2 = \frac{k}{2\cdot\mu_2}\cdot y^2 + c_3\cdot y + c_4$$

For convenience the origin of coordinates is placed at the centerline

We need four BCs. Three are obvious $y = -h \qquad u_1 = 0 \qquad (1)$

$y = 0 \qquad u_1 = u_2 \qquad (2)$

$y = h \qquad u_2 = 0 \qquad (3)$

The fourth BC comes from the fact that the stress at the interface generated by each fluid is the s

$$y = 0 \qquad \mu_1\cdot\frac{du_1}{dy} = \mu_2\cdot\frac{du_2}{dy} \qquad (4)$$

Using these four BCs $\quad 0 = \frac{k}{2\cdot\mu_1}\cdot h^2 - c_1\cdot h + c_2$

$$c_2 = c_4$$

$$0 = \frac{k}{2\cdot\mu_2}\cdot h^2 + c_3\cdot h + c_4$$

$$\mu_1\cdot c_1 = \mu_2\cdot c_3$$

Hence, after some algebra

$$c_1 = \frac{k\cdot h}{2\cdot\mu_1}\cdot\frac{(\mu_2 - \mu_1)}{(\mu_2 + \mu_1)} \qquad c_2 = c_4 = -\frac{k\cdot h^2}{\mu_2 + \mu_1} \qquad c_3 = \frac{k\cdot h}{2\cdot\mu_2}\cdot\frac{(\mu_2 - \mu_1)}{(\mu_2 + \mu_1)}$$

The velocity distributions are then

$$u_1 = \frac{k}{2\cdot\mu_1}\cdot\left[y^2 + y\cdot h\cdot\frac{(\mu_2 - \mu_1)}{(\mu_2 + \mu_1)}\right] - \frac{k\cdot h^2}{\mu_2 + \mu_1}$$

$$u_2 = \frac{k}{2\cdot\mu_2}\cdot\left[y^2 + y\cdot h\cdot\frac{(\mu_2 - \mu_1)}{(\mu_2 + \mu_1)}\right] - \frac{k\cdot h^2}{\mu_2 + \mu_1}$$

Evaluating either velocity at $y = 0$, gives the velocity at the interface

$$u_{interface} = -\frac{k\cdot h^2}{\mu_2 + \mu_1} \qquad u_{interface} = 4.17 \times 10^{-3}\,\frac{m}{s}$$

The plots of these velocity distributions are shown in the associated *Excel* workbook, as is the determination of the maximum velocity.

From *Excel* $\qquad u_{max} = 4.34 \times 10^{-3}\cdot\frac{m}{s}$

Problem 8.23 (In Excel)

Two immiscible fluids are contained between infinite parallel plates. The plates are separated by distance $2h$, and the two fluid layers are of equal thickness $h = 2.5$ mm. The dynamic viscosity of the upper fluid is twice that of the lower fluid, which is $\mu_{lower} = 0.5\ \text{N} \cdot \text{s/m}^2$. If the plates are stationary and the applied pressure gradient is $-1000\ \text{N/m}^2/\text{m}$, find the velocity at the interface. What is the maximum velocity of the flow? Plot the velocity distribution.

Given: Properties of two fluids flowing between parallel plates; applied pressure gradient

Find: Velocity at the interface; maximum velocity; plot velocity distribution

Solution

The data is

$k =$	-1000	Pa/m
$h =$	2.5	mm
$\mu_1 =$	0.5	N.s/m^2
$\mu_2 =$	1.0	N.s/m^2

The velocity distribution is

$$u_1 = \frac{k}{2 \cdot \mu_1} \cdot \left[y^2 + y \cdot h \cdot \frac{(\mu_2 - \mu_1)}{(\mu_2 + \mu_1)} \right] - \frac{k \cdot h^2}{\mu_2 + \mu_1}$$

$$u_2 = \frac{k}{2 \cdot \mu_2} \cdot \left[y^2 + y \cdot h \cdot \frac{(\mu_2 - \mu_1)}{(\mu_2 + \mu_1)} \right] - \frac{k \cdot h^2}{\mu_2 + \mu_1}$$

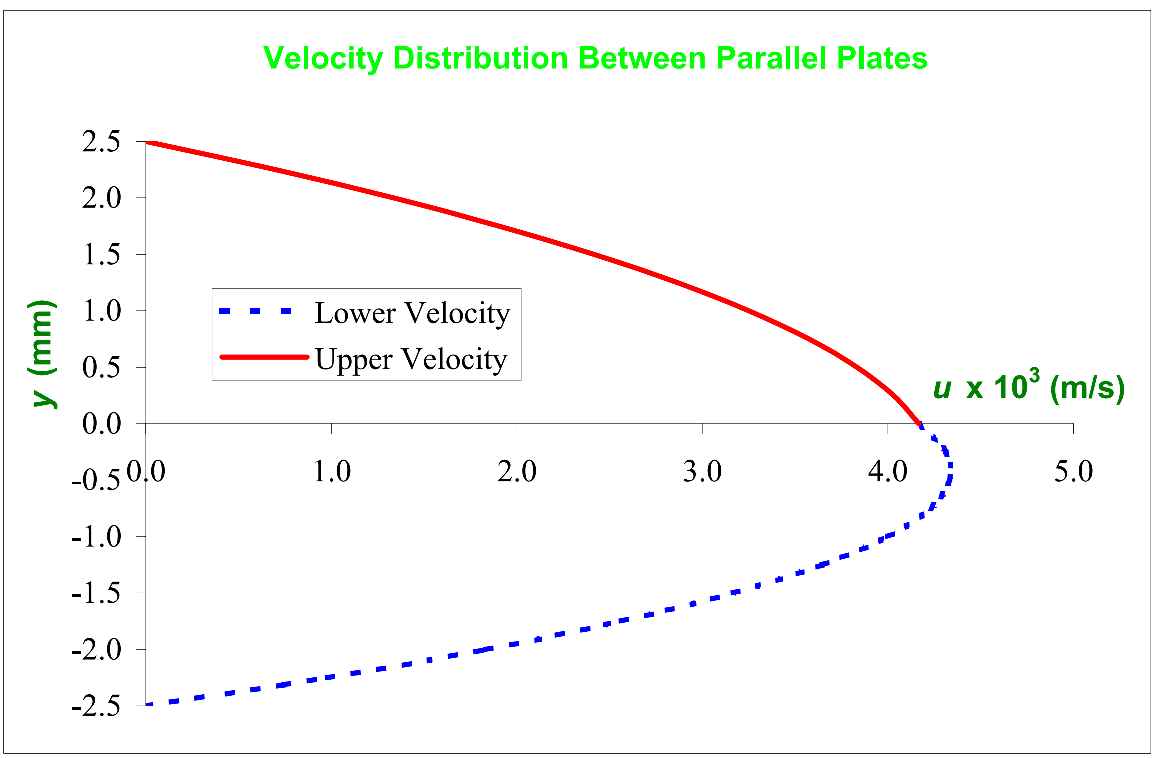
y (mm)	$u_1 \times 10^3$ (m/s)	$u_2 \times 10^3$ (m/s)
-2.50	0.000	NA
-2.25	0.979	NA
-2.00	1.83	NA
-1.75	2.56	NA
-1.50	3.17	NA
-1.25	3.65	NA
-1.00	4.00	NA
-0.75	4.23	NA
-0.50	4.33	NA
-0.25	4.31	NA
0.00	4.17	4.17
0.25	NA	4.03
0.50	NA	3.83
0.75	NA	3.57
1.00	NA	3.25
1.25	NA	2.86
1.50	NA	2.42
1.75	NA	1.91
2.00	NA	1.33
2.25	NA	0.698
2.50	NA	0.000

The lower fluid has the highest velocity

We can use *Solver* to find the maximum

(Or we could differentiate to find the maximum)

y (mm)	$u_{max} \times 10^3$ (m/s)
-0.417	4.34



Problem 8.24

The dimensionless velocity profile for fully developed laminar flow between infinite parallel plates with the upper plate moving at constant speed U is shown in Fig. 8.5. Find the pressure gradient $\partial p/\partial x$ at which (a) the upper plate and (b) the lower plate experience zero shear stress, in terms of U, a, and μ. Plot the dimensionless velocity profiles for these cases.

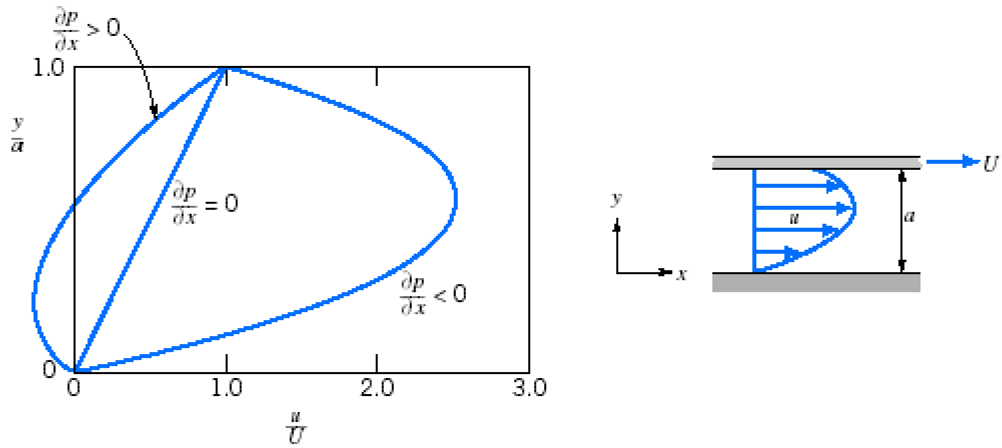


Fig. 8.5 Dimensionless velocity profile for fully developed laminar flow between infinite parallel plates: upper plate moving with constant speed, U.

Given: Velocity profile between parallel plates

Find: Pressure gradients for zero stress at upper/lower plates; plot

Solution

From Eq. 8.8, the velocity distribution i $u = \frac{U \cdot y}{a} + \frac{a^2}{2 \cdot \mu} \cdot \left(\frac{\partial}{\partial x} p\right) \cdot \left[\left(\frac{y}{a}\right)^2 - \frac{y}{a}\right]$

The shear stress is $\tau_{yx} = \mu \cdot \frac{du}{dy} = \mu \cdot \frac{U}{a} + \frac{a^2}{2} \cdot \left(\frac{\partial}{\partial x} p\right) \cdot \left(2 \cdot \frac{y}{a^2} - \frac{1}{a}\right)$

(a) For $\tau_{yx} = 0$ at $y = a$ $\quad 0 = \mu \cdot \frac{U}{a} + \frac{a}{2} \cdot \frac{\partial}{\partial x} p \quad \frac{\partial}{\partial x} p = -\frac{2 \cdot U \cdot \mu}{a^2}$

The velocity distribution is then $u = \frac{U \cdot y}{a} - \frac{a^2}{2 \cdot \mu} \cdot \frac{2 \cdot U \cdot \mu}{a^2} \cdot \left[\left(\frac{y}{a} \right)^2 - \frac{y}{a} \right]$

$$\frac{u}{U} = 2 \cdot \frac{y}{a} - \left(\frac{y}{a} \right)^2$$

(b) For $\tau_{yx} = 0$ at $y = 0$ $\quad 0 = \mu \cdot \frac{U}{a} - \frac{a}{2} \cdot \frac{\partial}{\partial x} p \qquad \frac{\partial}{\partial x} p = \frac{2 \cdot U \cdot \mu}{a^2}$

The velocity distribution is then $u = \frac{U \cdot y}{a} + \frac{a^2}{2 \cdot \mu} \cdot \frac{2 \cdot U \cdot \mu}{a^2} \cdot \left[\left(\frac{y}{a} \right)^2 - \frac{y}{a} \right]$

$$\frac{u}{U} = \left(\frac{y}{a} \right)^2$$

The velocity distributions are plotted in the associated *Excel* workbook

Problem 8.24 (In Excel)

The dimensionless velocity profile for fully developed laminar flow between infinite parallel plates with the upper plate moving at constant speed U is shown in Fig. 8.5. Find the pressure gradient $\partial p/\partial x$ at which (a) the upper plate and (b) the lower plate experience zero shear stress, in terms of U, a, and μ. Plot the dimensionless velocity profiles for these cases.

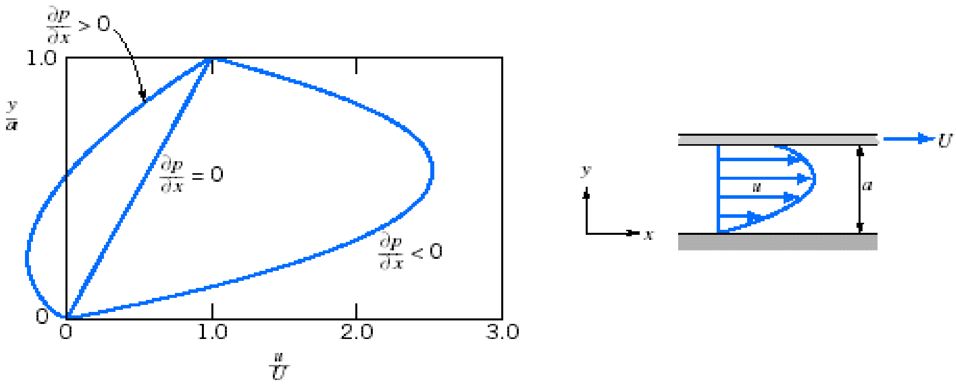


Fig. 8.5 Dimensionless velocity profile for fully developed laminar flow between infinite parallel plates: upper plate moving with constant speed, U.

Given: Velocity profile between parallel plates

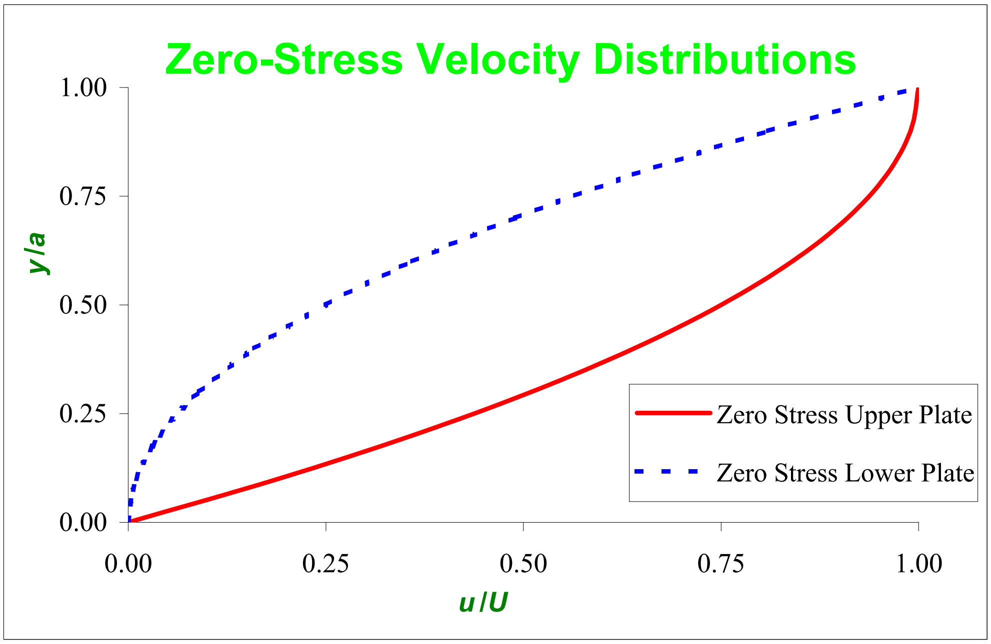
Find: Pressure gradients for zero stress at upper/lower plates; plot

Solution

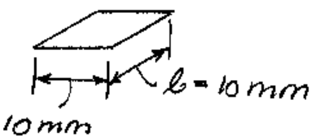
(a) For zero shear stress at upper plate $\frac{u}{U} = 2\cdot\frac{y}{a} - \left(\frac{y}{a}\right)^2$

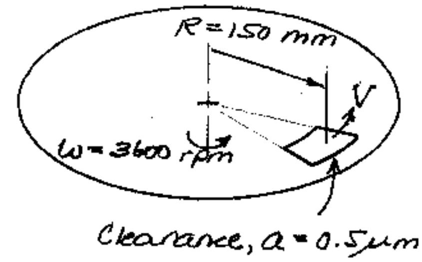
(b) For zero shear stress at lower plate $\frac{u}{U} = \left(\frac{y}{a}\right)^2$

y/a	(a) u/U	(b) u/U
0.0	0.000	0.000
0.1	0.190	0.010
0.2	0.360	0.040
0.3	0.510	0.090
0.4	0.640	0.160
0.5	0.750	0.250
0.6	0.840	0.360
0.7	0.910	0.490
0.8	0.960	0.640
0.9	0.990	0.810
1.0	1.00	1.000



Given: Record-write head for computer disk-storage system.





Find: (a) Reynolds number in gap
(b) Viscous shear stress
(c) Power to overcome viscous shear.

Solution: $V = R\omega = 0.15\,m \times 3600\,\frac{rev}{min} \times 2\pi\,\frac{rad}{rev} \times \frac{min}{60\,sec} = 56.5\,m/s$

$Re = \frac{\rho V a}{\mu} = \frac{Va}{\nu} = 56.5\,\frac{m}{s} \times 0.5\times10^{-6}\,m \times \frac{s}{1.46\times10^{-5}\,m^2} = 1.94$ ← Re

(Table A.10 at $T = 15°C$)

$\tau = \mu\frac{du}{dy} = \mu\frac{V}{a}$ for small gap

Assuming standard conditions, $\mu = 1.79\times10^{-5}\,kg/m\cdot s$

$\tau = 1.79\times10^{-5}\,\frac{kg}{m\cdot s} \times 56.5\,\frac{m}{s} \times \frac{1}{0.5\times10^{-6}\,m} = 2.02\,kN/m^2$ ← τ

The force is $F = \tau A = \tau w l$, and the torque is $T = FR = \tau w l R$.

The power dissipation rate is

$P = T\omega = \tau l w R \omega$

$= 2.02\times10^3\,\frac{N}{m^2} \times 0.01\,m \times 0.01\,m \times 0.150\,m \times 3600\,\frac{rev}{min} \times 2\pi\,\frac{rad}{rev} \times \frac{min}{60\,s} \times \frac{W\cdot s}{N\cdot m}$

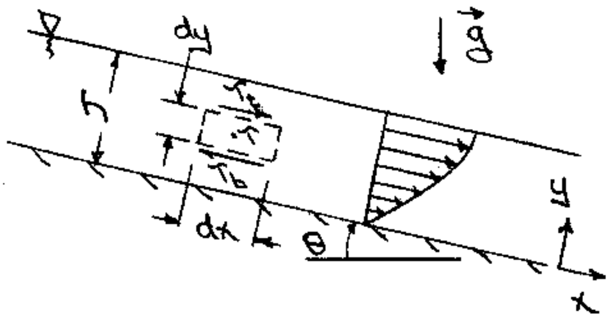
$P = 11.4\,W$ ← P

Problem 8.26

Given: Fully developed, laminar flow of an incompressible liquid down an inclined surface. The thickness, h, of the liquid layer is constant.

Find: (a) the velocity profile by use of a suitably chosen differential control volume. (b) volume flow rate, Q/w

Solution: Flow is fully developed, so $u = u(y)$ and $\tau = \tau(y)$.



Expand τ in a Taylor series about the center of the differential CV

$$\tau_t = \tau + \frac{d\tau}{dy}\frac{dy}{2}$$

$$\tau_b = \tau + \frac{d\tau}{dy}\left(-\frac{dy}{2}\right)$$

The boundary conditions on the velocity profile are:

@ $y = 0$, $u = 0$ (no slip).

@ $y = h$, $\frac{du}{dy} = 0$ (no shear stress).

Apply the x component of the momentum equation to the differential CV shown

$$F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u \rho d\forall + \int_{CS} u \rho \vec{V} \cdot d\vec{A}$$

(terms marked ≈ 0 by (1) and (2))

Assumptions: (1) steady flow
(2) fully developed flow, so u and τ are functions of y only
(3) no variation of pressure in the x direction

Then

$$F_{S_x} + F_{B_x} = 0 = \left(\tau + \frac{d\tau}{dy}\frac{dy}{2}\right)dx\,dz - \left(\tau - \frac{d\tau}{dy}\frac{dy}{2}\right)dx\,dz + \rho g \sin\theta\, dx\,dy\,dz$$

or

$$\frac{d\tau}{dy} = -\rho g \sin\theta$$

Integrating, $\tau = -\rho g \sin\theta\, y + c_1$

But $\tau = 0$ @ $y = h$, $\therefore$ $c_1 = \rho g \sin\theta\, h$, and

$$\frac{du}{dy} = \frac{\rho g \sin\theta}{\mu}(h - y)$$

Integrating again,

$$u = \frac{\rho g \sin\theta}{\mu}\left(hy - \frac{y^2}{2}\right) + c_2$$

At $y = 0$, $u = 0$, so $c_2 = 0$ and hence

$$u = \frac{\rho g \sin\theta}{\mu}\left(hy - \frac{y^2}{2}\right) \longleftarrow u$$

$$Q/w = \int_0^h u\,dy = \frac{\rho g \sin\theta}{\mu}\int_0^h \left(hy - \frac{y^2}{2}\right)dy = \frac{\rho g \sin\theta}{\mu}\left[\frac{hy^2}{2} - \frac{y^3}{6}\right]_0^h$$

$$Q/w = \rho g \sin\theta\, h^3 / 3\mu \longleftarrow Q/w$$

Given: Steady, incompressible, fully developed laminar flow down an incline (of angle θ).
Velocity profile (Example Problem 5.9) is

$$u = \frac{\rho g \sin\theta}{\mu}\left(hy - y^2/2\right)$$

Find: Kinematic viscosity ν of liquid for h = 0.8 mm, θ = 30° and $u_{max} = 15.7$ mm/s

Plot: the velocity profile

Solution:

$$u = \frac{\rho g \sin\theta}{\mu}\left(hy - y^2/2\right) = \frac{g\sin\theta}{\nu}\left(hy - y^2/2\right)$$

$u = u_{max}$ at $y = h$

$$\therefore\ u_{max} = \frac{g\sin\theta}{\nu}\left(h^2 - h^2/2\right) = \frac{g\sin\theta h^2}{2\nu}$$

and

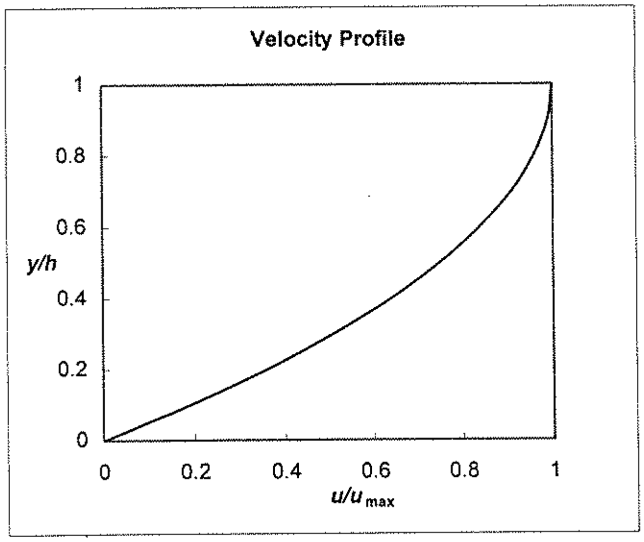
$$\nu = \frac{g\sin\theta h^2}{2u_{max}} = \frac{\sin 30^\circ}{2} \times 9.81\frac{m}{s^2} \times (0.8\times10^{-3}m)^2 \times \frac{s}{15.7\times10^{-3}m}$$

$$\nu = 1.00\times10^{-4}\ m^2/s \qquad \nu$$

Plot

$$\frac{u}{u_{max}} = \frac{g\sin\theta}{\nu}\left(hy - y^2/2\right) \times \frac{2\nu}{g\sin\theta h^2} = 2\frac{y}{h} - \left(\frac{y}{h}\right)^2$$

u/u_{max}	y/h
0	0
0.0396	0.02
0.098	0.05
0.190	0.1
0.360	0.2
0.510	0.3
0.640	0.4
0.750	0.5
0.840	0.6
0.910	0.7
0.960	0.8
0.990	0.9
1.00	1.0



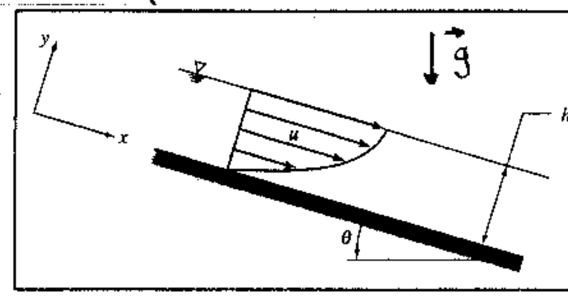
Problem 8.28

Given: Velocity distribution for flow of a thin viscous film down a plane surface, inclined at angle θ is given (from Example 5.9) as

$$u = \frac{\rho g \sin\theta}{\mu}\left(hy - y^2/2\right)$$

$h = 5.63\ \text{mm}$, SG liquid $= 1.26$, $\mu = 1.40\ \text{N}\cdot\text{s/m}^2$

Find: (a) expression for shear stress distribution in film
(b) maximum shear stress in film; indicate direction
(c) volume flow rate (mm^3/s) per mm of width.
(d) film Re based on $\bar{V}$



Solution:

(a) $\tau_{yx} = \mu \dfrac{du}{dy} = \rho g \sin\theta\,(h - y)$

(b) τ_{yx} is a maximum at $y = 0$

$\tau_{yx_{max}} = \rho g \sin\theta\, h = SG\,\rho_{H_2O}\, g \sin\theta\, h$

$\tau_{yx_{max}} = 1.26\times10^{3}\dfrac{\text{kg}}{\text{m}^3}\times 9.81\dfrac{\text{m}}{\text{s}^2}\times \sin 30^\circ \times 5.63\times10^{-3}\,\text{m} = 34.8\ \text{N/m}^2$ ____ $\tau_{yx_{max}}$

- stress on wall (+y surface) is in + x direction
- " " film (−y surface) in in − x direction.

(c) $Q = \int \vec{u}\cdot d\vec{A} = \int_0^h \dfrac{\rho g \sin\theta}{\mu}\left(hy - y^2/2\right) w\,dy$

$= \dfrac{\rho g \sin\theta\, w}{\mu}\left[\dfrac{hy^2}{2} - \dfrac{y^3}{6}\right]_0^h = \dfrac{\rho g \sin\theta\, w\, h^3}{3\mu}$

$\dfrac{Q}{w} = \dfrac{1}{3}\times 1.26\times10^{3}\dfrac{\text{kg}}{\text{m}^3}\times 9.81\dfrac{\text{m}}{\text{s}^2}\times \sin 30^\circ \times (5.63\,\text{mm})^3 \times \dfrac{\text{m}^2}{1.40\,\text{N}\cdot\text{s}}\times\dfrac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}}\times\dfrac{\text{m}}{10^3\,\text{mm}}$

$\dfrac{Q}{w} = 263\ \text{mm}^3/\text{s}/\text{mm}$ ____ Q/w

(d) $\bar{V} = \dfrac{Q}{A} = \dfrac{Q}{wh} = 263\dfrac{\text{mm}^2}{\text{s}}\times\dfrac{1}{5.63\,\text{mm}} = 46.7\ \text{mm/s}$

$Re = \dfrac{\rho \bar{V} h}{\mu} = 1.26\times10^{3}\dfrac{\text{kg}}{\text{m}^3}\times 46.7\times10^{-3}\dfrac{\text{m}}{\text{s}}\times 5.63\times10^{-3}\,\text{m}\times\dfrac{\text{m}^2}{1.40\,\text{N}\cdot\text{s}}\times\dfrac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}}$

$Re = 0.236$ ____ Re

Problem 8.29

Two immiscible fluids of equal density are flowing down a surface inclined at a 30° angle. The two fluid layers are of equal thickness $h = 2.5$ mm; the kinematic viscosity of the upper fluid is twice that of the lower fluid, which is $\nu_{lower} = 2 \times 10^{-4}$ m²/s. Find the velocity at the interface and the velocity at the free surface. Plot the velocity distribution.

Given: Data on flow of liquids down an incline

Find: Velocity at interface; velocity at free surface; plot

Solution

Given data $\quad h = 2.5 \cdot mm \quad \theta = 30 \cdot deg \quad \nu_1 = 2 \times 10^{-4} \cdot \frac{m^2}{s} \quad \nu_2 = 2 \cdot \nu_1$

(The lower fluid is designated fluid 1, the upper fluid 2)

From Example Problem 5.9 (or Exanple Problem 8.3 with *g* replaced with *g*sinθ),
a free body analysis leads to (for either fluid)

$$\frac{d^2}{dy^2}u = -\frac{\rho \cdot g \cdot \sin(\theta)}{\mu}$$

Applying this to fluid 1 (lower fluid) and fluid 2 (upper fluid), integrating twice yields

$$u_1 = -\frac{\rho \cdot g \cdot \sin(\theta)}{2 \cdot \mu_1} \cdot y^2 + c_1 \cdot y + c_2 \qquad u_2 = -\frac{\rho \cdot g \cdot \sin(\theta)}{2 \cdot \mu_2} \cdot y^2 + c_3 \cdot y + c_4$$

We need four BCs. Two are obvious $\quad y = 0 \quad u_1 = 0 \quad (1)$

$$y = h \quad u_1 = u_2 \quad (2)$$

The third BC comes from the fact that there is no shear stress at the free surface

$$y = 2 \cdot h \qquad \mu_2 \cdot \frac{du_2}{dy} = 0 \qquad (3)$$

The fourth BC comes from the fact that the stress at the interface generated by each fluid is the s

$$y = h \qquad \mu_1 \cdot \frac{du_1}{dy} = \mu_2 \cdot \frac{du_2}{dy} \qquad (4)$$

Using these four BCs $\quad c_2 = 0$

$$-\frac{\rho \cdot g \cdot \sin(\theta)}{2 \cdot \mu_1} \cdot h^2 + c_1 \cdot h + c_2 = -\frac{\rho \cdot g \cdot \sin(\theta)}{2 \cdot \mu_2} \cdot h^2 + c_3 \cdot h + c_4$$

$$-\rho \cdot g \cdot \sin(\theta) \cdot 2 \cdot h + \mu_2 \cdot c_3 = 0$$

$$-\rho \cdot g \cdot \sin(\theta) \cdot h + \mu_1 \cdot c_1 = -\rho \cdot g \cdot \sin(\theta) \cdot h + \mu_2 \cdot c_3$$

Hence, after some algebra

$$c_1 = \frac{2 \cdot \rho \cdot g \cdot \sin(\theta) \cdot h}{\mu_1} \qquad c_2 = 0$$

$$c_3 = \frac{2 \cdot \rho \cdot g \cdot \sin(\theta) \cdot h}{\mu_2} \qquad c_4 = 3 \cdot \rho \cdot g \cdot \sin(\theta) \cdot h^2 \cdot \frac{(\mu_2 - \mu_1)}{2 \cdot \mu_1 \cdot \mu_2}$$

The velocity distributions are then

$$u_1 = \frac{\rho \cdot g \cdot \sin(\theta)}{2 \cdot \mu_1} \cdot \left(4 \cdot y \cdot h - y^2\right)$$

$$u_2 = \frac{\rho \cdot g \cdot \sin(\theta)}{2 \cdot \mu_2} \cdot \left[3 \cdot h^2 \cdot \frac{(\mu_2 - \mu_1)}{\mu_1} + 4 \cdot y \cdot h - y^2\right]$$

Rewriting in terms of ν_1 and ν_2 (ρ is constant and equal for both fluids)

$$u_1 = \frac{g \cdot \sin(\theta)}{2 \cdot \nu_1} \cdot \left(4 \cdot y \cdot h - y^2\right)$$

$$u_2 = \frac{g \cdot \sin(\theta)}{2 \cdot \nu_2} \cdot \left[3 \cdot h^2 \cdot \frac{(\nu_2 - \nu_1)}{\nu_1} + 4 \cdot y \cdot h - y^2\right]$$

(Note that these result in the same expression if $\nu_1 = \nu_2$, i.e., if we have one fluid)

Evaluating either velocity at $y = h$, gives the velocity at the interface

$$u_{interface} = \frac{3 \cdot g \cdot h^2 \cdot \sin(\theta)}{2 \cdot \nu_1} \qquad u_{interface} = 0.23 \frac{m}{s}$$

Evaluating u_2 at $y = 2h$ gives the velocity at the free surface

$$u_{freesurface} = g \cdot h^2 \cdot \sin(\theta) \cdot \frac{(3 \cdot \nu_2 + \nu_1)}{2 \cdot \nu_1 \cdot \nu_2} \qquad u_{freesurface} = 0.268 \frac{m}{s}$$

The velocity distributions are plotted in the associated *Excel* workbook

Problem 8.29 (In Excel)

Two immiscible fluids of equal density are flowing down a surface inclined at a 30° angle. The two fluid layers are of equal thickness $h = 2.5$ mm; the kinematic viscosity of the upper fluid is twice that of the lower fluid, which is $\nu_{lower} = 2 \times 10^{-4}$ m²/s. Find the velocity at the interface and the velocity at the free surface. Plot the velocity distribution.

Given: Data on flow of liquids down an incline

Find: Velocity at interface; velocity at free surface; plot

Solution

$h =$ 2.5 mm

$\theta =$ 30 deg

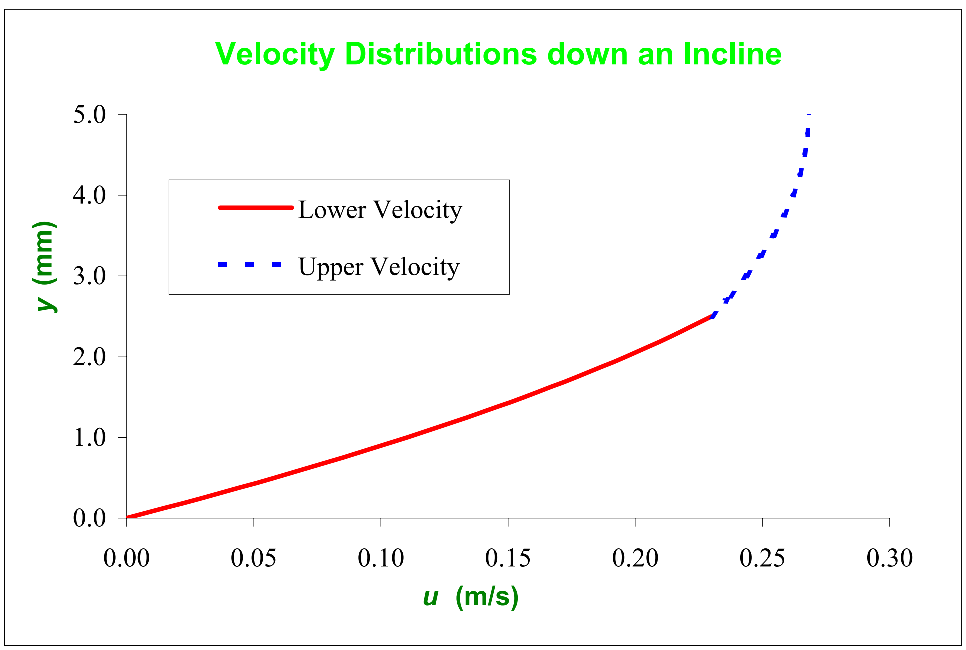
$\nu_1 =$ 2.00E-04 m^2/s

$\nu_2 =$ 4.00E-04 m^2/s

$$u_1 = \frac{g \cdot \sin(\theta)}{2 \cdot \nu_1} \cdot \left(4 \cdot y \cdot h - y^2\right)$$

$$u_2 = \frac{g \cdot \sin(\theta)}{2 \cdot \nu_2} \cdot \left[3 \cdot h^2 \cdot \frac{(\nu_2 - \nu_1)}{\nu_1} + 4 \cdot y \cdot h - y^2\right]$$

y (mm)	*u*₁ (m/s)	*u*₂ (m/s)
0.000	0.000	
0.250	0.0299	
0.500	0.0582	
0.750	0.0851	
1.000	0.110	
1.250	0.134	
1.500	0.156	
1.750	0.177	
2.000	0.196	
2.250	0.214	
2.500	0.230	0.230
2.750		0.237
3.000		0.244
3.250		0.249
3.500		0.254
3.750		0.259
4.000		0.262
4.250		0.265
4.500		0.267
4.750		0.268
5.000		0.268



Problem 8.30

Given: Fully developed flow between parallel plates with the upper plate moving (Fig. 8.5). $U = 2$ m/s; $a = 2.5$ mm.

Find: (a) Q/l for $\partial p/\partial x = 0$
(b) τ_{yx} at $y = 0$ for $\partial p/\partial x = 0$
(c) Plot τ_{yx} vs. y for $\partial p/\partial x = 0$
(d) Will Q increase or decrease if $\partial p/\partial x > 0$?
(e) $\partial p/\partial x$ for $\tau_{yx} = 0$ at $y = 0.25a$, if fluid is air
(f) Plot τ_{yx} vs. y for this case.

Solution: The velocity profile is given by Eq. 8.8: $u = \frac{Uy}{a} + \frac{a^2}{2\mu}\left(\frac{\partial p}{\partial x}\right)\left[\left(\frac{y}{a}\right)^2 - \left(\frac{y}{a}\right)\right]$

(a) For $\frac{\partial p}{\partial x} = 0$, $u = \frac{Uy}{a}$; $\frac{Q}{l} = \int_0^a u\,dy = \int_0^a \frac{Uy}{a}\,dy = \frac{U}{a}\frac{y^2}{2}\Big]_0^a = \frac{Ua}{2}$

$$\frac{Q}{l} = \frac{1}{2} \times 2\,\frac{m}{s} \times 0.0025\,m = 0.00250\ m^3/s/m$$ Q/l

(b) $\tau_{yx} = \mu\frac{du}{dy}$; for $\frac{\partial p}{\partial x} = 0$, $\tau = \frac{\mu U}{a}$. For air at STP, $\mu = 1.79\times10^{-5}\ N\cdot s/m^2$.

$$\tau_{yx} = 1.79\times10^{-5}\,\frac{N\cdot s}{m^2} \times 2\,\frac{m}{s} \times \frac{1}{0.0025\,m} = 0.0143\ N/m^2$$ τ_{yx}

(c) Shear stress is constant for $\frac{\partial p}{\partial x} = 0$; see plot below.

(d) Q will <u>decrease</u> if $\frac{\partial p}{\partial x} > 0$; Q will <u>increase</u> if $\frac{\partial p}{\partial x} < 0$. Q

The shear stress is given by Eq. 8.9a: $\tau_{yx} = \frac{\mu U}{a} + a\left(\frac{\partial p}{\partial x}\right)\left[\frac{y}{a} - \frac{1}{2}\right]$

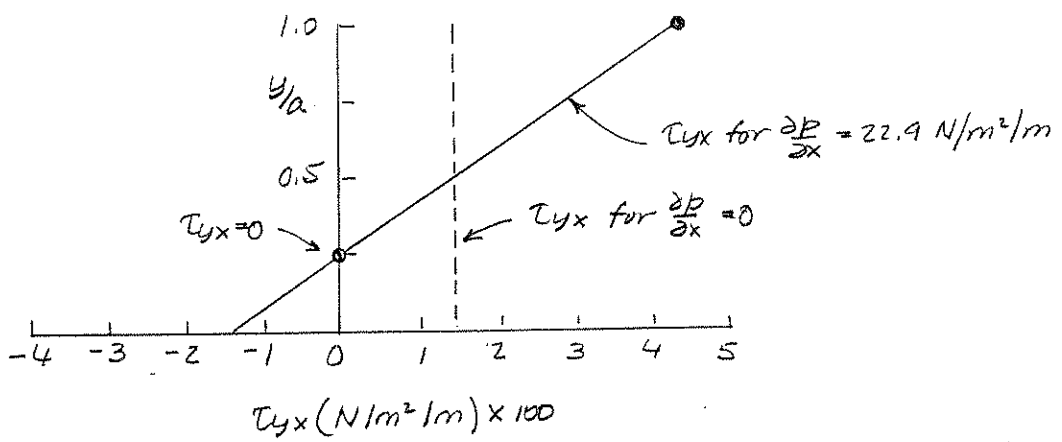
(e) For $\tau = 0$ at $y = 0.25a$, $0 = \mu\frac{U}{a} + a\left(\frac{\partial p}{\partial x}\right)\left[\frac{1}{4} - \frac{1}{2}\right]$ or $\frac{\partial p}{\partial x} = \frac{4\mu U}{a^2}$

$$\frac{\partial p}{\partial x} = 4 \times 1.79\times10^{-5}\,\frac{N\cdot s}{m^2} \times 2\,\frac{m}{s} \times \frac{1}{(0.0025)^2\,m^2} = 22.9\ N/m^2/m$$ $\frac{\partial p}{\partial x}$

(f) To plot, calculate τ_{yx} at $y = a$:

$$\tau_{yx} = \frac{\mu U}{a} + a\left(\frac{\partial p}{\partial x}\right)\left[1 - \frac{1}{2}\right] = \frac{\mu U}{a} + \frac{a}{2}\left(\frac{\partial p}{\partial x}\right) = 0.0143\,\frac{N}{m^2} + \frac{1}{2}\times 0.0025\,m \times 22.9\,\frac{N}{m^3} = 0.0429\,\frac{N}{m^2}$$

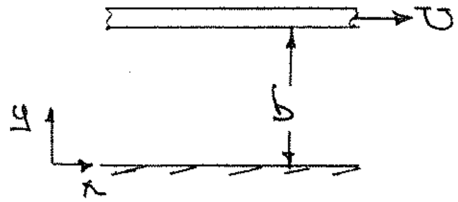
Plotting:



Given: Water at 60°F flows between parallel plates as shown.

$U = 1\ ft/s \quad b = 0.01\ ft$

$\partial p/\partial x = -1.20\ lbf/ft^2/ft$



Find: (a) location of point of maximum velocity
(b) value of u_{max}
(c) volume of flow passing a section in 10 s.

Plot: the velocity and shear stress distributions.

Solution:

Computing equation: $u = \frac{Uy}{b} + \frac{b^2}{2\mu}\left(\frac{\partial p}{\partial x}\right)\left[\left(\frac{y}{b}\right)^2 - \frac{y}{b}\right]$ (8.8)

To locate u_{max}, set $du/dy = 0$

$$\frac{du}{dy} = 0 = \frac{U}{b} + \frac{b^2}{2\mu}\left(\frac{\partial p}{\partial x}\right)\left[\frac{2y}{b^2} - \frac{1}{b}\right] = \frac{U}{b} + \frac{1}{2\mu}\frac{\partial p}{\partial x}(2y - b)$$

when $y = \frac{b}{2} - \frac{\mu U}{b(\partial p/\partial x)}$ From Table A.7, $\mu = 2.34 \times 10^{-5}\ lbf \cdot s/ft^2$

$$y = \frac{0.01\ ft}{2} - 2.34 \times 10^{-5}\ \frac{lbf \cdot s}{ft^2} \times 1\frac{ft}{s} \times \frac{1}{0.01\ ft} \times \frac{1}{(-1.20)}\frac{ft^3}{lbf} =$$

$u = u_{max}$ at $y = 0.00695\ ft$ ← y for u_{max}

So

$$u_{max} = \frac{Uy}{b} + \frac{b^2}{2\mu}\left(\frac{\partial p}{\partial x}\right)\left[\left(\frac{y}{b}\right)^2 - \frac{y}{b}\right] \quad \text{where } y = 0.695 \times 10^{-2}\ ft$$

$$= 1\frac{ft}{s} \times \frac{0.695}{1.0} + \frac{(0.01\ ft)^2}{2} \times \frac{ft^2}{2.34 \times 10^{-5}\ lbf \cdot s} \times \left(-1.20\frac{lbf}{ft^3}\right)\left[(0.695)^2 - (0.695)\right]$$

$u_{max} = 1.24\ ft/s$ ← u_{max}

To find the volume of flow, evaluate $\forall = Q\Delta t$

$$Q = \int_A u\,dA = \int_0^b u w\,dy = w\int_0^b \left[\frac{Uy}{b} + \frac{b^2}{2\mu}\left(\frac{\partial p}{\partial x}\right)\left\{\left(\frac{y}{b}\right)^2 - \frac{y}{b}\right\}\right] dy$$

$$\frac{Q}{w} = \left[U\frac{y^2}{2b} + \frac{b^2}{2\mu}\left(\frac{\partial p}{\partial x}\right)\left\{\frac{y^3}{3b^2} - \frac{y^2}{2b}\right\}\right]_0^b = \frac{Ub}{2} + \frac{b^2}{2\mu}\left(\frac{\partial p}{\partial x}\right)\left\{\frac{b}{3} - \frac{b}{2}\right\}$$

$$\frac{Q}{w} = \frac{Ub}{2} - \frac{b^3}{12\mu}\left(\frac{\partial p}{\partial x}\right) = \frac{1}{2} \times 1\frac{ft}{s} \times 0.01\ ft - \frac{(0.01\ ft)^3}{12} \times \frac{ft^2}{2.34 \times 10^{-5}\ lbf \cdot s}\left(-1.20\frac{lbf}{ft^3}\right)$$

$Q/w = 9.27 \times 10^{-3}\ ft^2/s$

$\forall/w = \frac{Q}{w}\Delta t = 9.27 \times 10^{-3}\ \frac{ft^2}{s} \times 10\ s = 0.0927\ ft^3/ft$ ← $\forall/w$

The velocity distribution is given by

$$\frac{u}{U} = \frac{y}{b} + \frac{b^2}{2\mu}\left(\frac{\partial p}{\partial x}\right)\left[\left(\frac{y}{b}\right)^2 - \frac{y}{b}\right]$$

$$\frac{u}{U} = \frac{y}{b} - 2.56\left[\left(\frac{y}{b}\right)^2 - \frac{y}{b}\right] \leftarrow$$

The shear stress is $\tau_{yx} = \mu \frac{du}{dy} = \frac{\mu}{b}\frac{du}{d(y/b)}$

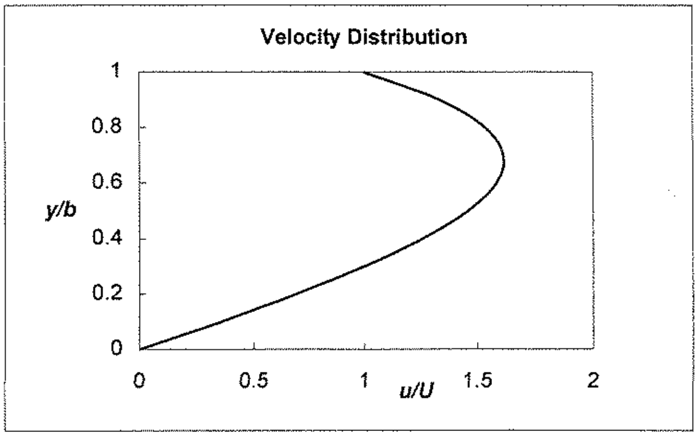
$$\tau_{yx} = \frac{\mu U}{b} + \frac{b}{2}\left(\frac{\partial p}{\partial x}\right)\left[2\left(\frac{y}{b}\right) - 1\right]$$

$$= 2.34\times10^{-5}\,\frac{\text{lbf}\cdot\text{s}}{\text{ft}^2}\times\frac{1\,\text{ft}}{\text{s}}\times\frac{1}{0.01\,\text{ft}} + \frac{0.01\,\text{ft}}{2}\left(-1.20\,\frac{\text{lbf}}{\text{ft}^3}\right)\left[2\left(\frac{y}{b}\right)-1\right]$$

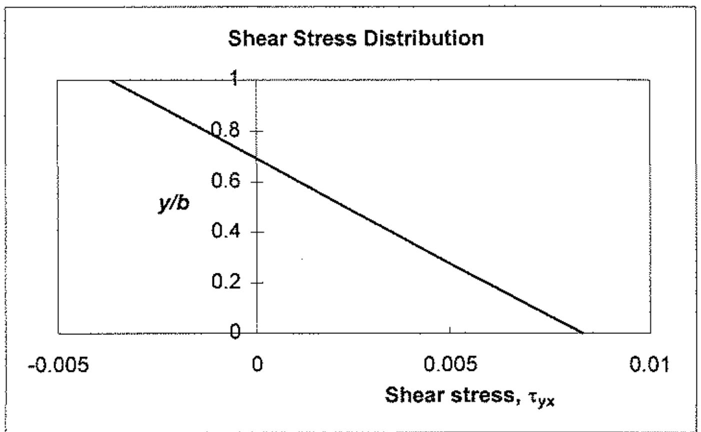
$$\tau_{yx} = 2.34\times10^{-3}\,\frac{\text{lbf}}{\text{ft}^2} - 6.00\times10^{-3}\,\frac{\text{lbf}}{\text{ft}^2}\left[2\left(\frac{y}{b}\right)-1\right] \leftarrow$$

The velocity and shear stress distributions are plotted below.

u/U	*y/b*
0	0
0.353	0.1
0.692	0.2
0.999	0.3
1.26	0.4
1.46	0.5
1.58	0.6
1.61	0.7
1.54	0.8
1.34	0.9
1.00	1



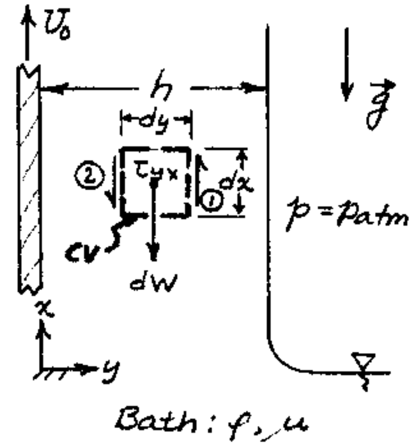
τ_{yx}	*y/b*
0.00834	0
0.00714	0.1
0.00594	0.2
0.00474	0.3
0.00354	0.4
0.00234	0.5
0.00114	0.6
-0.00006	0.7
-0.00126	0.8
-0.00246	0.9
-0.00366	1.0



Problem 8.32

Given: Belt moving steadily through bath as shown.

Assume zero shear at film/air surface, and no pressure forces.



Find: (a) Boundary conditions for velocity at $y=0$, $y=h$.
(b) Velocity profile.

Solution: Choose CV $dx\,dy\,dz$ as shown.
Apply x component of momentum equation.

Basic equations:

$$F_{S_x} + F_{B_x} = \overset{=0(2)}{\frac{\partial}{\partial t}\int_{CV} u\rho\, dV} + \overset{=0(3)}{\int_{CS} u\rho\vec{V}\cdot d\vec{A}} \quad ; \quad \tau_{yx} = \mu\frac{du}{dy} = \tau$$

Assumptions: (1) F_{S_x} due to shear forces only
(2) Steady flow
(3) Fully-developed flow

Then

$$F_{S_x} + F_{B_x} = F_{①} - F_{②} + F_{B_x} = \left(\tau + \frac{d\tau}{dy}\frac{dy}{2}\right)dx\,dz - \left(\tau - \frac{d\tau}{dy}\frac{dy}{2}\right)dx\,dz - \rho g\, dx\,dy\,dz = 0$$

or $\frac{d\tau}{dy} = \rho g$. Integrating

$$\tau = \rho g y + c_1 = \mu\frac{du}{dy} \quad \text{or} \quad \frac{du}{dy} = \frac{\rho g y}{\mu} + \frac{c_1}{\mu}.$$ Integrating again,

$$u = \frac{\rho g y^2}{2\mu} + \frac{c_1}{\mu}y + c_2$$

To evaluate the constants c_1 and c_2, apply the boundary conditions:

At $y=0$, $u = U_0$, so $c_2 = U_0$

At $y=h$, $\tau = 0$, so $\frac{du}{dy} = 0$, and $c_1 = -\rho g h$ ← BC

Substituting,

$$u = \frac{\rho g y^2}{2\mu} - \frac{\rho g h y}{\mu} + U_0 = \frac{\rho g}{\mu}\left(\frac{y^2}{2} - hy\right) + U_0$$ ← u

{Note that at $y=h$,

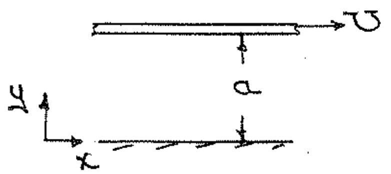
$$u = \frac{\rho g}{\mu}\left(-\frac{h^2}{2}\right) + U_0 \neq 0$$

Thus the solution is determined only when U_0 and h are known.}

Given: Velocity profile for fully developed laminar flow of air between parallel plates

$$u = \frac{Uy}{a} + \frac{a^2}{2\mu}\left(\frac{\partial p}{\partial x}\right)\left[\left(\frac{y}{a}\right)^2 - \left(\frac{y}{a}\right)\right]$$

$U = 2\ m/s \qquad a = 2.5\ mm$



Find: (a) pressure gradient for which net flow is zero; plot expected $u(y)$ and $\tau_{yx}(y)$

(b) expected $u(y)$ and $\tau_{yx}(y)$ for case where $u = 2U$ at $y/a = 0.5$

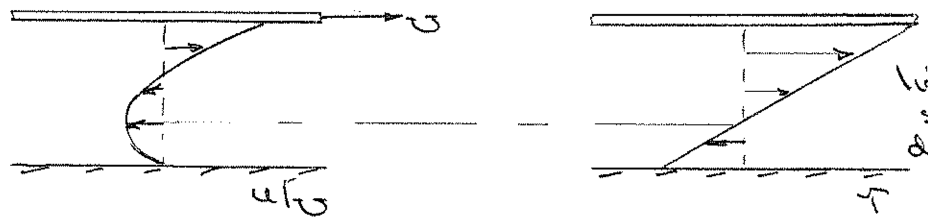
Solution:

Computing equations:
$$Q/l = \frac{Ua}{2} - \frac{a^3}{12\mu}\left(\frac{\partial p}{\partial x}\right) \qquad (8.9b)$$
$$\tau_{yx} = \mu\frac{U}{a} + a\left(\frac{\partial p}{\partial x}\right)\left[\frac{y}{a} - \frac{1}{2}\right] \qquad (8.9a)$$

For $Q = 0$, from Eq. 8.9b (assuming $T = 15°C$)

$$\frac{\partial p}{\partial x} = \frac{6\mu U}{a^2} = 6 \times 1.79\times10^{-5}\ \frac{N\cdot s}{m^2} \times 2\ \frac{m}{s} \times \frac{1}{(2.5\times10^{-3}\ m)^2} = 34.4\ N/m^2/m \qquad \frac{\partial p}{\partial x}$$

For this adverse pressure gradient



(b) For $u = 2U$ at $y/a = 0.5$

$$2U = 0.5U + \frac{a^2}{2\mu}\left(\frac{\partial p}{\partial x}\right)\left[\frac{1}{4} - \frac{1}{2}\right] \quad \text{and} \quad \frac{3}{2}U = -\frac{a^2}{8\mu}\left(\frac{\partial p}{\partial x}\right)$$

$$\frac{\partial p}{\partial x} = -\frac{12U\mu}{a^2} = -12 \times 2\frac{m}{s} \times 1.79\times10^{-5}\ \frac{N\cdot s}{m^2} \times \frac{1}{(2.5\times10^{-3}\ m)^2} = -68.7\ N/m^2/m$$

$$\tau = \mu\frac{U}{a} + a\left(\frac{\partial p}{\partial x}\right)\left[\frac{y}{a} - \frac{1}{2}\right] \qquad \{\text{shear stress is linear}\}$$

$y = 0$: $$\tau = \mu\frac{U}{a} - \frac{a}{2}\left(\frac{\partial p}{\partial x}\right) = 1.79\times10^{-5}\ \frac{N\cdot s}{m^2} \times \frac{2m}{s} \times \frac{1}{2.5\times10^{-3}\ m} - \frac{2.5\times10^{-3}\ m}{2}\left(-68.7\ \frac{N}{m^3}\right) = 0.10\ \frac{N}{m^2}$$

$y = a$: $$\tau = \mu\frac{U}{a} + \frac{a}{2}\left(\frac{\partial p}{\partial x}\right) = -0.0716\ N/m^2$$

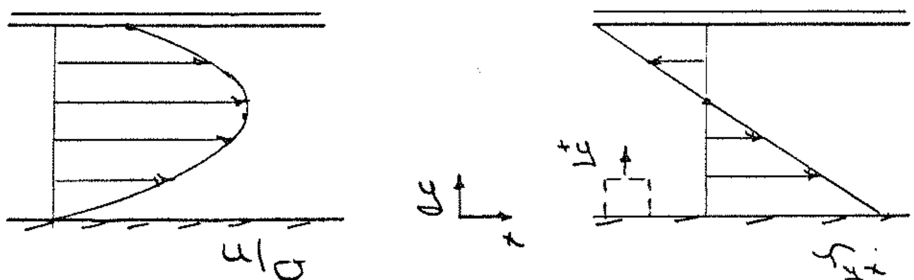
Note that the point of zero shear stress is not at $y/a = 0.5$ and hence $y/a = 0.5$ is not the location of maximum velocity. Maximum velocity occurs at $y/a > 0.5$.

To find the location of zero shear set $\tau_{yx} = 0$, then

$$0 = \frac{\mu U}{a} + a\left(\frac{\partial p}{\partial x}\right)\left(\frac{y}{a} - \frac{1}{2}\right) \quad \text{and} \quad \frac{y}{a} = \frac{1}{2} - \frac{\mu U}{a^2\left(\frac{\partial p}{\partial x}\right)}.$$

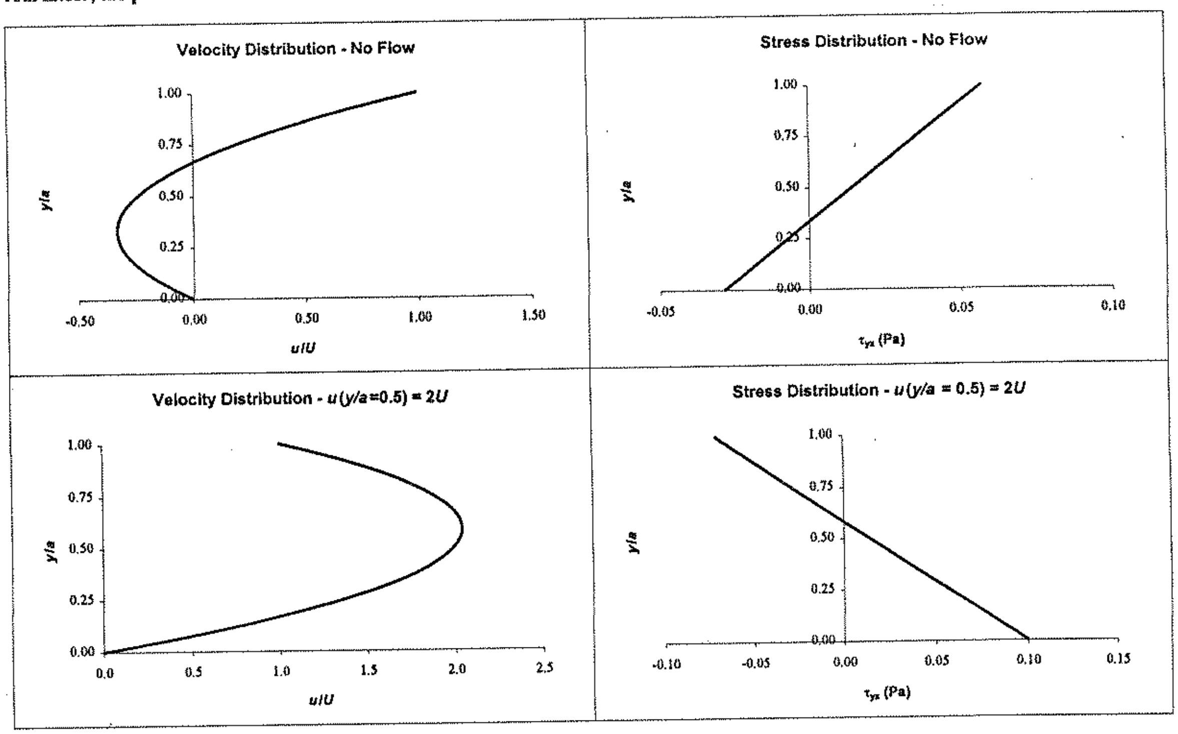
$$\frac{y}{a} = 0.5 - 1.79 \times 10^{-5}\,\frac{N\cdot s}{m^2} \times 2\,\frac{m}{s} \times \frac{1}{(2.5\times10^{-3}\,m)^2} \times \frac{1}{(-68.7)}\,\frac{m^3}{N} = 0.583$$

For this case ($u = 2U$ at $y/a = 0.5$) the velocity and shear stress distributions would be as follows.



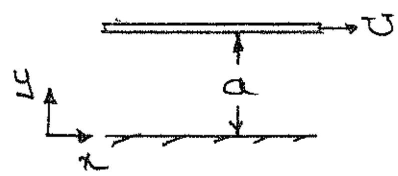
The shear stress is positive ($du/dy > 0$) below $y/a = 0.583$; positive stress acts in positive x direction on a positive y surface.
The shear stress is negative ($du/dy < 0$) above $y/a = 0.583$; negative stress acts in the negative x direction on a positive y surface.

From *Excel*, the plots are



Problem 8.34

Given: Velocity profile for fully developed laminar flow of water between parallel plates



$$u = \frac{Uy}{a} + \frac{a^2}{2\mu}\left(\frac{\partial p}{\partial x}\right)\left[\left(\frac{y}{a}\right)^2 - \frac{y}{a}\right]$$

$U = 2\ m/s \qquad a = 2.5\ mm$

Find: (a) Volume flow rate for zero pressure gradient.
(b) shear stress on lower plate; sketch $\tau(y)$
(c) effect of mild adverse pressure gradient on Q
(d) pressure gradient for zero shear at $y/a = 0.25$; sketch $\tau(y)$.

Solution:

Computing equations: $\tau_{yx} = \mu\frac{U}{a} + a\left(\frac{\partial p}{\partial x}\right)\left[\frac{y}{a} - \frac{1}{2}\right]$ (8.9a)

$Q/l = \frac{Ua}{2} - \frac{1}{12\mu}\left(\frac{\partial p}{\partial x}\right)a^3$ (8.9b)

For $\partial p/\partial x = 0$, $Q/l = \frac{Ua}{2} = \frac{1}{2} \times 2\frac{m}{s} \times 2.5\times10^{-3}\ m = 2.50\times10^{-3}\ m^3/s/m$ — Q

The shear stress is $\tau_{yx} = \mu\frac{U}{a}$ {At 15°C, $\mu = 1.14\times10^{-3}\ N\cdot s/m^2$ (Table A.8)}

$\tau_{yx} = 1.14\times10^{-3}\ \frac{N\cdot s}{m^2} \times 2\frac{m}{s} \times \frac{1}{2.5\times10^{-3}\ m} = 0.912\ N/m^2$ — τ_{yx}

The shear stress is constant across the channel (curve 1 below)

For $\partial p/\partial x > 0$, Eq. 8.9b indicates that Q will decrease.

For $\tau = 0$ at $y/a = 0.25$

$\tau_{yx} = 0 = \mu\frac{U}{a} + a\left(\frac{\partial p}{\partial x}\right)\left[\frac{1}{4} - \frac{1}{2}\right] = \mu\frac{U}{a} - \frac{a}{4}\left(\frac{\partial p}{\partial x}\right)$

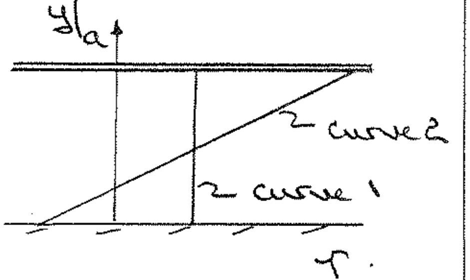
$\frac{\partial p}{\partial x} = \frac{4\mu U}{a^2} = 4 \times 1.14\times10^{-3}\ \frac{N\cdot s}{m^2} \times 2\frac{m}{s} \times \frac{1}{(2.5\times10^{-3}\ m)^2} = 1.46\ kN/m^2/m$ — $\frac{\partial p}{\partial x}$

For this pressure gradient

$\tau_{yx} = 1.14\times10^{-3}\ \frac{N\cdot s}{m^2} \times 2\frac{m}{s} \times \frac{1}{2.5\times10^{-3}\ m} + 2.5\times10^{-3}\ m \times 1.46\times10^{3}\ \frac{N}{m^3}\left[\frac{y}{a} - 0.5\right]$

$\tau_{yx} = 0.912\ N/m^2 + 3.65\ \frac{N}{m^2}\left(\frac{y}{a} - 0.5\right)$

$\left.\tau_{yx}\right)_{y=0} = -0.913\ N/m^2$
$\left.\tau_{yx}\right)_{y=a} = 2.74\ N/m^2$
} curve 2

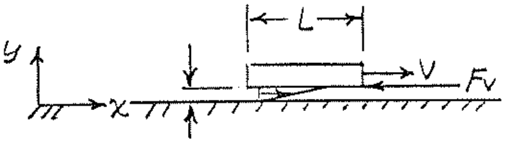


Problem 8.35

Given: Microchip supported on air film, on a horizontal surface. Chips are $L = 11.7$ mm long, $w = 9.35$ mm wide, and have mass $m = 0.325$ g. The air film is $h = 0.125$ mm thick. The initial speed of the chips is $V_0 = 1.75$ mm/s; they slow from viscous shear.

Find: (a) Differential equation for chip motion during deceleration.
(b) Time required for chip to lose 5 percent of V_0.
(c) Plot of chip speed vs. time, with labels and comments.

Solution: Apply Newton's law of viscosity



Basic equations: $\tau_{yx} = \mu \dfrac{du}{dy}$

$$F_V = \tau A \qquad \sum F = m a_x$$

Assume: (1) Newtonian fluid
(2) Linear velocity profile in narrow gap
(3) Air at STP

Then

$$\tau_{yx} = \mu \frac{du}{dy} = \mu \frac{V}{h}; \quad F_V = \tau A = \mu \frac{V}{h} w L = \frac{\mu V w L}{h}$$

The free-body diagram for the chip is

$$\sum F_x = -F_V = -\frac{\mu V w L}{h} = m \frac{dV}{dt} \; ; \quad \frac{dV}{V} = -\frac{\mu w L}{m h} dt$$ D.E.

Integrating, $\displaystyle\int_{V_0}^{V} \frac{dV}{V} = \ln \frac{V}{V_0} = -\frac{\mu w L}{m h} t$

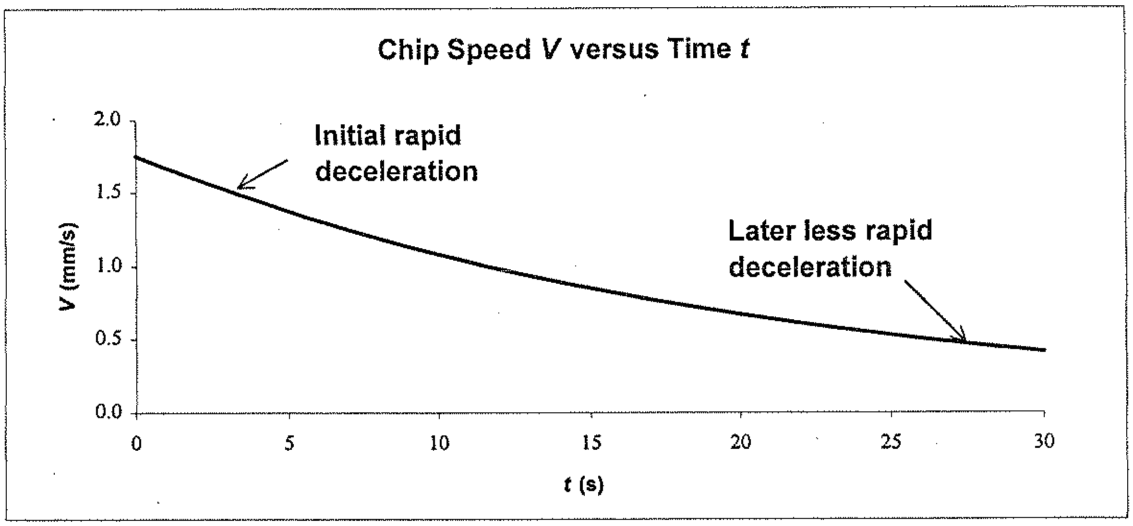
Thus

$$t = -\frac{m h}{\mu w L} \ln \frac{V}{V_0}$$

$$t = -0.325\,\text{g} \times 0.125\,\text{mm} \times \frac{\text{m} \cdot \text{s}}{1.79 \times 10^{-5}\,\text{kg}} \times \frac{1}{9.35\,\text{mm}} \times \frac{1}{11.7\,\text{mm}} \times \ln 0.95 \times \frac{\text{kg}}{1000\,\text{g}} \times \frac{1000\,\text{mm}}{\text{m}}$$

$t = 1.06$ s   t

From *Excel*, the plot of speed vs. time is:



Problem 8.36

Given: Free-surface waves begin to form on a laminar liquid film flowing down an inclined surface whenever the Reynolds number, based on mass flow per unit width of film, is larger than about 33.

Find: Estimate of the maximum thickness of a laminar film of water that remains free from waves while flowing down a vertical surface.

Solution: The mass flow rate is $\dot{m} = \rho \bar{V} A = \rho \bar{V} w \delta$, so $\dot{m}/w = \rho \bar{V} \delta$.

Thus

$$Re = \frac{\rho \bar{V} \delta}{\mu} = \frac{\bar{V}\delta}{\nu} = 33 \text{ (maximum)}$$

From Example Problem 8.3 (pp. 343-345),

$$\bar{V} = \frac{\rho g \delta^2}{3\mu}$$

Thus

$$\frac{\rho \bar{V} \delta}{\mu} = \frac{\rho^2 g \delta^3}{3\mu^2} = 33$$

Solving for δ,

$$\delta = \left[\frac{99\mu^2}{\rho^2 g}\right]^{1/3}$$

At $T = 20°C$, $\mu = 1.00 \times 10^{-3}\ kg/m \cdot s$ and $\rho = 998\ kg/m^3$ (Table A.8). Substituting,

$$\delta = \left[99 \times (1.00 \times 10^{-3})^2 \frac{kg^2}{m^2 \cdot s^2} \times \frac{m^6}{(998)^2\ kg^2} \times \frac{s^2}{9.81\ m}\right]^{1/3}$$

$$\delta = 2.16 \times 10^{-4}\ m \quad \text{or} \quad 0.216\ mm$$

δ_{max}

Open-Ended Problem Statement: Hold a flat sheet of paper 50 to 75 mm above a smooth desktop. Propel the sheet smoothly parallel to the desk surface as you release it. Comment on the motion you observe. Explain the fluid dynamic phenomena involved in the motion.

Discussion: After some practice, one can release the sheet so that it continues to move parallel to the desktop for a considerable distance before finally slowing and stopping. The slowing of the paper sheet is so gradual that the motion appears to be almost frictionless.

The thin layer of air trapped under the paper sheet acts to "lubricate" the motion as the sheet moves parallel to the tabletop. Kinetic sliding friction between the sheet and the desktop is prevented by the fluid layer. Instead the motion is resisted by the much smaller viscous shear stress caused by the motion of the sheet (see Section 8-2.2). Thus the sheet appears to move across the desktop almost without friction.

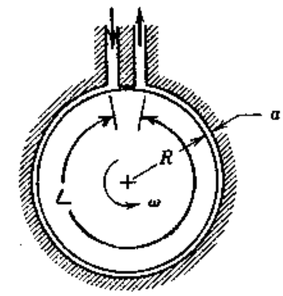
The same phenomena are involved in hydrodynamic lubrication. Detailed analysis of lubrication is beyond the scope of this text, but contact between two solid surfaces can be prevented, even with large normal loads, by properly shaping the clearance space between the two surfaces. To analyze the phenomenon, the Navier-Stokes equations for incompressible flow (Equations 5.27) are simplified further to a "thin layer" form. These equations are used to predict the load carrying capacity of a lubricated bearing.

The NCFMF video *Low-Reynolds-Number Flows* shows further examples of flows in which viscous effects are dominant.

Problem 8.38

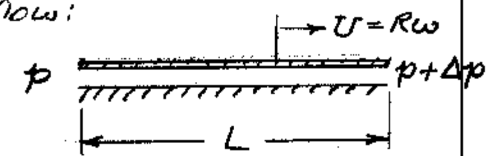
Given: Viscous-shear pump, as shown.

b = width normal to diagram; $a \ll R$



Find: Performance characteristics
(a) Pressure differential
(b) Input power
(c) Efficiency
as functions of volume flow rate.

Solution: Since $a \ll R$, unwrap to form flow between parallel plates. Apply Eqs. 8.9 to fully developed flow:



Volume flow rate is $\frac{Q}{b} = \frac{Ua}{2} - \frac{1}{12\mu}\left(\frac{\partial p}{\partial x}\right)a^3$

Substituting $U = R\omega$ and $\frac{\partial p}{\partial x} = \frac{\Delta p}{L}$, then

$$\Delta p = \frac{12\mu L}{a^3}\left(\frac{\omega R a}{2} - \frac{Q}{b}\right) = \frac{6\mu L R\omega}{a^2}\left(1 - \frac{2Q}{abR\omega}\right)$$ Δp

Torque is $T = \tau R(bL) = RLb\tau$. Power is $P = T\omega$. From Eq. 8.9a, at $y = a$,

$$P = RLb\omega\left[\frac{\mu R\omega}{a} + \frac{\Delta p}{L}\frac{a}{2}\right] = RLb\omega\left[\frac{\mu R\omega}{a} + \frac{6\mu L R\omega}{a^2}\left(1 - \frac{2Q}{abR\omega}\right)\frac{a}{2L}\right]$$

$$P = RLb\omega\left[\frac{\mu R\omega}{a}\left(4 - \frac{6Q}{abR\omega}\right)\right] = \frac{\mu Lb(R\omega)^2}{a}\left(4 - \frac{6Q}{abR\omega}\right)$$ P

Output power is $Q\Delta p$, so efficiency is

$$\eta = \frac{Q\Delta p}{P} = \frac{6\mu Q L R\omega}{a^2}\left(1 - \frac{2Q}{abR\omega}\right)\frac{a}{\mu Lb(R\omega)^2}\frac{1}{\left(4 - \frac{6Q}{abR\omega}\right)}$$

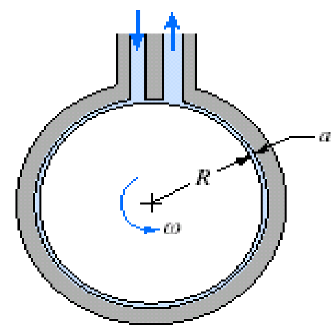
$$\eta = \frac{6Q}{abR\omega}\frac{\left(1 - \frac{2Q}{abR\omega}\right)}{\left(4 - \frac{6Q}{abR\omega}\right)}$$ η

Problem 8.39 (In Excel)

The efficiency of the viscous-shear pump of Fig. P8.39 is given by

$$\eta = 6q\frac{(1-2q)}{(4-6q)}$$

where $q = Q/abR\omega$ is a dimensionless flow rate (Q is the flow rate at pressure differential Δp, and b is the depth normal to the diagram). Plot the efficiency versus dimensionless flow rate, and find the flow rate for maximum efficiency. Explain why the efficiency peaks, and why it is zero at certain values of q.



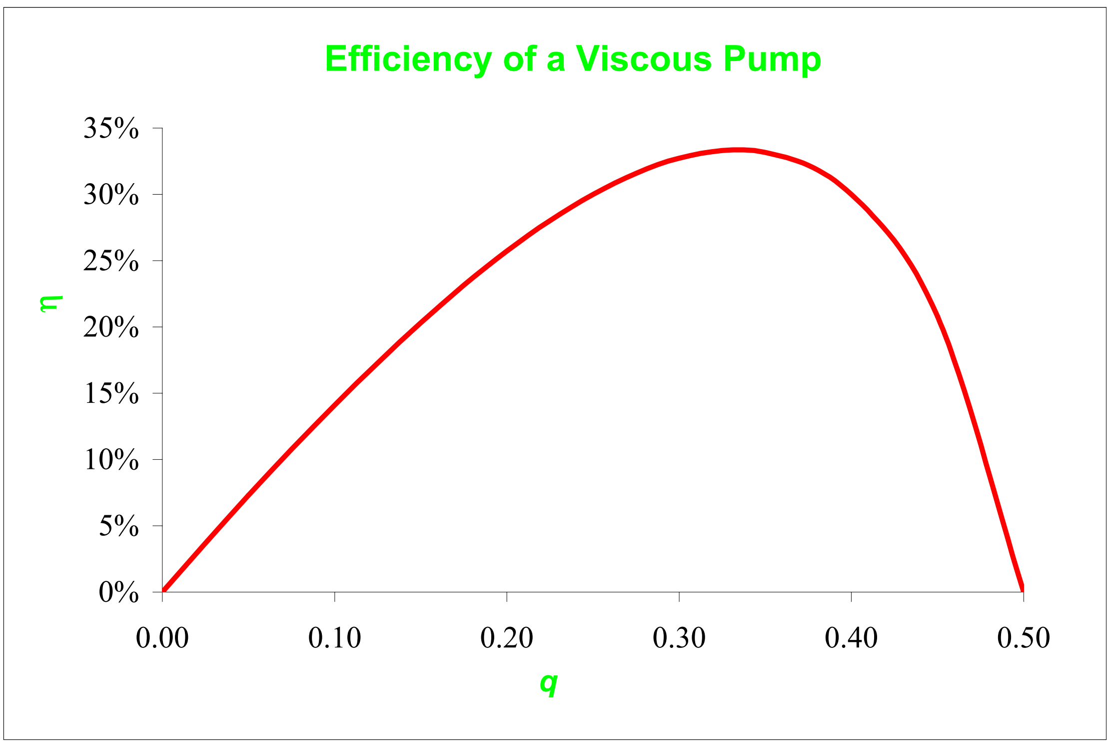
P8.38, P8.39

Given: Expression for efficiency

Find: Plot; find flow rate for maximum efficiency; explain curve

Solution

q	η
0.00	0.0%
0.05	7.30%
0.10	14.1%
0.15	20.3%
0.20	25.7%
0.25	30.0%
0.30	32.7%
0.35	33.2%
0.40	30.0%
0.45	20.8%
0.50	0.0%



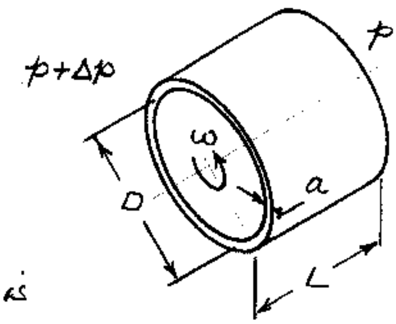
For the maximum efficiency point we can use *Solver* (or alternatively differentiate)

q	η
0.333	33.3%

The efficiency is zero at zero flow rate because there is no output at all
The efficiency is zero at maximum flow rate $\Delta p = 0$ so there is no output
The efficiency must therefore peak somewhere between these extremes

Problem 8.40

Given: Annular gap seal as shown.



Power required to pump oil, P_p.

Power to overcome viscous dissipation, P_V.

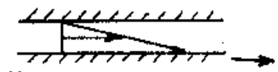
Find: (a) Expressions for P_p, P_V

(b) Show total power minimized when a is chosen so that $P_V = 3P_p$.

Solution: Apply Eqs. 8.6 and 8.9 for flow between parallel plates.

Assumptions: (1) $a \ll D$, so unfold to flat plates

(2) No pressure gradient circumferentially



The viscous power is the product of viscous torque times ω:

$$P_V = T\omega = \tau(2\pi RL)R\omega = \mu\frac{V}{a}\left(2\pi\frac{D}{2}L\right)\frac{D}{2}\omega = \mu\frac{\omega D}{2a}\pi DL\frac{D}{2}\omega = \frac{\pi\mu\omega^2D^3L}{4a}$$ P_V

The pump power is the product of flow rate times pressure drop.

$$P_p = Q\Delta p$$

From Eq. 8.6c, $Q = \dfrac{la^3\Delta p}{12\mu L} = \dfrac{\pi Da^3\Delta p}{12\mu L}$, so $P_p = \dfrac{\pi Da^3\Delta p^2}{12\mu L}$ P_p

The total power required is $P_T = P_V + P_p = \dfrac{\pi\mu\omega^2D^3L}{4a} + \dfrac{\pi Da^3\Delta p^2}{12\mu L}$

It may be minimized by setting $\dfrac{dP_T}{da} = 0$. Thus

$$\frac{dP_T}{da} = -\frac{\pi\mu\omega^2D^3L}{4a^2} + \frac{\pi Da^2\Delta p^2}{4\mu L} = 0 \qquad (1)$$

This can be written

$$\frac{dP_T}{da} = -\frac{1}{a}P_V + \frac{3}{a}P_p = 0$$

which is satisfied when $3P_p - P_V = 0$ or $P_V = 3P_p$ Optimum

Equation 1 also can be solved for a at optimum conditions:

$$a^4 = \frac{\mu^2\omega^2D^2L^2}{\Delta p^2} \quad \text{or} \quad a^2 = \frac{\mu\omega DL}{\Delta p} \quad \text{or} \quad \frac{a}{D} = \sqrt{\frac{\mu\omega L}{D\Delta p}} \text{ (optimum)}$$

Problem 8.41

A journal bearing consists of a shaft of diameter $D = 50$ mm and length $L = 1$ m (moment of inertia $I = 0.055 \text{ kg} \cdot \text{m}^2$) installed symmetrically in a stationary housing such that the annular gap is $\delta = 1$ mm. The fluid in the gap has viscosity $\mu = 0.1 \text{ N} \cdot \text{s/m}^2$. If the shaft is given an initial angular velocity of $\omega = 60$ rpm, determine the time for the shaft to slow to 10 rpm.

Given: Data on a journal bearing

Find: Time for the bearing to slow to 10 rpm

Solution

The given data is $D = 50 \cdot \text{mm}$ $\quad L = 1 \cdot \text{m}$ $\quad I = 0.055 \cdot \text{kg} \cdot \text{m}^2$ $\quad \delta = 1 \cdot \text{mm}$

$\mu = 0.1 \cdot \dfrac{\text{N} \cdot \text{s}}{\text{m}^2}$ $\quad \omega_i = 60 \cdot \text{rpm}$ $\quad \omega_f = 10 \cdot \text{rpm}$

The equation of motion for the slowing bearing is

$$I \cdot \alpha = \text{Torque} = -\tau \cdot A \cdot \frac{D}{2}$$

where α is the angular acceleration and τ is the viscous stress, and $A = \pi \cdot D \cdot L$ is the surface area of the bearing

As in Example Problem 8.2 the stress is given by $\tau = \mu \cdot \dfrac{U}{\delta} = \dfrac{\mu \cdot D \cdot \omega}{2 \cdot \delta}$

where *U* and ω are the instantaneous linear and angular velocities.

Hence
$$I \cdot \alpha = I \cdot \frac{d\omega}{dt} = -\frac{\mu \cdot D \cdot \omega}{2 \cdot \delta} \cdot \pi \cdot D \cdot L \cdot \frac{D}{2} = -\frac{\mu \cdot \pi \cdot D^3 \cdot L}{4 \cdot \delta} \cdot \omega$$

Separating variables

$$\frac{d\omega}{\omega} = -\frac{\mu \cdot \pi \cdot D^3 \cdot L}{4 \cdot \delta \cdot I} \cdot dt$$

Integrating and using IC $\omega = \omega_0$

$$\omega(t) = \omega_i \cdot e^{-\frac{\mu \cdot \pi \cdot D^3 \cdot L}{4 \cdot \delta \cdot I} \cdot t}$$

The time to slow down to $\omega_f = 10$ rpm is obtained from solving

$$\omega_f = \omega_i \cdot e^{-\frac{\mu \cdot \pi \cdot D^3 \cdot L}{4 \cdot \delta \cdot I} \cdot t}$$

so

$$t = -\frac{4 \cdot \delta \cdot I}{\mu \cdot \pi \cdot D^3 \cdot L} \cdot \ln\left(\frac{\omega_f}{\omega_i}\right)$$

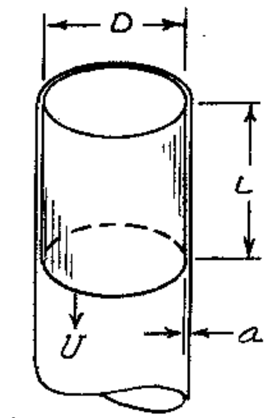
$t = 10\,s$

Problem 8.42

Given: "Viscous timer," consisting of a cylindrical mass inside a circular tube filled with viscous liquid, creating a narrow annular gap.

Find: (a) The flow field created when the mass falls under gravity.
(b) Whether this would make a satisfactory timer, and if so, for what range of time intervals.
(c) Effect of temperature change on measured time interval.

Solution: Apply conservation of mass to a CV enclosing the cylinder and the moving mass:

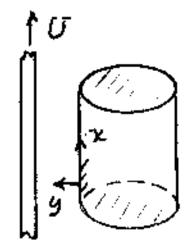


Then: $Q = U\frac{\pi D^2}{4} = \bar{V}\,\pi D a = \bar{V} l a$ (1)

Assume: (1) Gap is narrow, $a \ll D$
(2) Unroll gap so flat, $l = \pi D$
(3) Steady flow
(4) Fully developed laminar flow

Under these assumptions, the flow field in the gap is that for flow between parallel plates with one plate moving. ← Flow Field

Place coordinates on the moving mass:

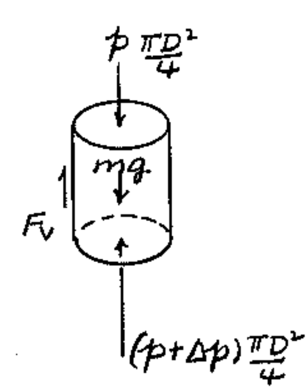


Then the volume flow rate (Eq. 8.9b) is

$$\frac{Q}{l} = \frac{Q}{\pi D} = \frac{Ua}{2} - \frac{1}{12\mu}\left(\frac{\partial p}{\partial x}\right)a^3$$

But $\frac{\partial p}{\partial x} = -\frac{\Delta p_v}{L}$, where Δp_v is the pressure drop driving viscous flow, so

$$\frac{Q}{l} = \frac{Ua}{2} - \frac{1}{12\mu}\left(-\frac{\Delta p_v}{L}\right)a^3 = \frac{Ua}{2} + \frac{\Delta p_v a^3}{12\mu L} \quad (2)$$



The pressure change across the moving mass is

$$\Delta p = \rho_l g L + \Delta p_v \quad (3)$$

Summing forces on the moving mass gives

$$\Sigma F_x = \Delta p\frac{\pi D^2}{4} - mg + F_v = m\frac{dU}{dt} \overset{=0(3)}{}$$

But $mg = \rho_m\frac{\pi D^2}{4}L$ and $F_v = \tau_s \pi D L$

From Eq. 8.9a, $\tau_s = \mu\frac{U}{a} - \frac{a}{2}\left(\frac{\partial p}{\partial x}\right) = \mu\frac{U}{a} + \frac{a}{2}\frac{\Delta p_v}{L}$

Substituting, $\Delta p\frac{\pi D^2}{4} - \rho_m\frac{\pi D^2}{4}Lg + \left[\mu\frac{U}{a} + \frac{a}{2}\frac{\Delta p_v}{L}\right]\pi D L = 0$

or $$\Delta p = \rho_m g L - \left[\mu\frac{U}{a} + \frac{a}{2}\frac{\Delta p_v}{L}\right]\frac{4L}{D} \quad (4)$$

Combining Eqs. 1 and 2 gives $\frac{UD}{4} = \frac{Ua}{2} + \frac{\Delta p_v a^3}{12\mu L}$

Thus

$$\Delta p_v = \frac{12\mu L}{a^3}\left[\frac{UD}{4} - \frac{Ua}{2}\right] = \frac{3\mu U L D}{a^3} \qquad (5)$$

(annotation: $a \ll D$, so the term $\frac{Ua}{2}$ is struck out)

Δp_v

Combining Eqs. 3 and 4 gives $\Delta p = \rho_\ell g L + \Delta p_v = \rho_m g L - \left[\mu\frac{U}{a} + \frac{a}{2}\frac{\Delta p_v}{L}\right]\frac{4L}{D}$

Using Eq. 5,

$$\rho_\ell g L + \frac{3\mu ULD}{a^3} = \rho_m g L - \mu\frac{U}{a}\frac{4L}{D} - \frac{a}{2}\,\frac{3\mu ULD}{La^3}\,\frac{4L}{D}$$

Simplifying and re-arranging,

$$(\rho_m - \rho_\ell) g L = \frac{3\mu ULD}{a^3} + \frac{4\mu UL}{aD} + \frac{6\mu UL}{a^2} \approx \frac{3\mu ULD}{a^3}$$

Finally, using $\rho = SG\,\rho_{H_2O}$,

$$U = \frac{(SG_m - SG_\ell)\,\rho_{H_2O}\, g\, a^3}{3\mu D}$$

U

The time interval for the mass to move distance H is

$$\Delta t = \frac{H}{U} = \frac{3\mu D}{(SG_m - SG_\ell)\,\rho_{H_2O}\, g\, a^3} \qquad (6)$$

Δt

Equation 6 shows that the time interval for the mass to fall any distance H is proportional to liquid viscosity μ and inversely proportional to gap width a cubed. A temperature change would affect the diameter of the measuring tube and the diameter of the falling mass. A temperature change also would affect the viscosity of the liquid in the tube.

Speed of the falling mass is proportional to the cube of gap width. If the coefficient of thermal expansion of the falling mass were greater than that of the glass measuring tube (which seems likely), then the width of the annular gap would decrease with increasing temperature. This would tend to slow the falling mass. The total amount of thermal expansion would depend on the diameter of the mass and tube. The effect on gap width would be greater, the larger the tube diameter compared to the initial gap width.

It might be possible to "tailor" the thermal expansion coefficient of the cylinder, by using a suitable material, to closely match that of the falling mass. Then there would be no differential thermal expansion between the mass and tube, and changes in temperature would not affect the gap width.

Speed of the falling mass is inversely proportional to liquid viscosity. Liquid viscosity decreases sharply as temperature increases (the viscosity of SAE 30 oil drops more than 10 percent as its temperature increases from 20°C to 25°C, see Fig. A.2). This would tend to increase the speed of the falling mass.

The entire device could be maintained at constant temperature.

Open-Ended Design Problem: Automotive design is tending toward all-wheel drive to improve vehicle performance and safety when traction is poor. An all-wheel drive vehicle must have an interaxle differential to allow operation on dry roads. Numerous vehicles are being built using multiplate viscous drives for interaxle differentials. Perform the analysis and design needed to define the torque transmitted by the differential for a given speed difference, in terms of the design parameters. Identify suitable dimensions for a viscous differential to transmit a torque of 150 N · m at a speed loss of 125 rpm, using lubricant with the properties of SAE 30 oil. Discuss how to find the minimum material cost for the differential, if the plate cost per square meter is constant.

Solution: From Problem 2.45, $dT = r\,dF = r\tau\,dA$

But $\tau = \mu \frac{du}{dy} = \mu \frac{u}{h} = \mu \frac{r\Delta\omega}{h}$; $dA = 2\pi r\,dr$

Thus $dT = r\mu \frac{r\Delta\omega}{h} 2\pi r\,dr = \frac{2\pi\mu\Delta\omega}{h} r^3 dr$; $T = \frac{\pi\mu\Delta\omega}{2h}\left[R_o^4 - R_i^4\right]$

or $T = \frac{\pi\mu\Delta\omega}{2h} R^4(1-\alpha^4)$ where $\alpha = R_i/R$ — T

This value is per gap. Each rotor has 2 gaps to a housing. For n gaps

$$T_n = \frac{n\pi\mu\Delta\omega}{2h} R^4(1-\alpha^4) \qquad (1)$$ — T_n

From Eq. 1, assuming $\mu = 0.18$ kg/m·s (Fig. A.2) and $\alpha = \frac{1}{2}$, so $1-\alpha^4 = 1 - \frac{1}{16} \approx 1$, then

$$\frac{nR^4}{h} = \frac{2T_n}{\pi\mu\Delta\omega} = \frac{2}{\pi} \times 150\ \text{N}\cdot\text{m} \times \frac{\text{m}^2}{0.18\ \text{N}\cdot\text{s}} \times \frac{\text{min}}{125\ \text{rev}} \times \frac{\text{rev}}{2\pi\ \text{rad}} \times \frac{60\ \text{s}}{\text{min}} = 40.5\ \text{m}^3 = C$$

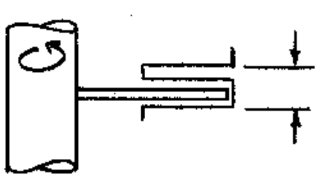
or

$$R^4 = C\frac{h}{n}$$

For $n = 100$ and $h = 0.2$ mm, $R^4 = 40.5\ \text{m}^3 \times 0.0002\ \text{m} \times \frac{1}{100} = 8.11 \times 10^{-5}\ \text{m}^4$

$$R = \left[8.11\times10^{-5}\right]^{1/4}\text{m} = 0.0949\ \text{m} \quad (\text{or } D \approx 190\ \text{mm})$$ — R

The stack length might be



≈ 2.5 mm for $n = 2$, or 125 mm for $n = 100$

Problem 8.44

Given: Fully developed laminar flow in a pipe, with

$$u = -\frac{R^2}{4\mu}\frac{\partial p}{\partial x}\left[1-\left(\frac{r}{R}\right)^2\right]$$

Find: Radius from pipe axis at which u equals the average velocity, $\bar{V}$.

Solution: First determine $\bar{V}$.

$$\bar{V} = \frac{Q}{A} = \frac{1}{\pi R^2}\int_A u\,dA = \frac{1}{\pi R^2}\int_0^R \left\{-\frac{R^2}{4\mu}\frac{\partial p}{\partial x}\left[1-\left(\frac{r}{R}\right)^2\right]\right\} 2\pi r\,dr$$

$$= -\frac{R^2}{2\mu}\frac{\partial p}{\partial x}\int_0^1 \left[1-\left(\frac{r}{R}\right)^2\right]\left(\frac{r}{R}\right) d\left(\frac{r}{R}\right) = -\frac{R^2}{2\mu}\frac{\partial p}{\partial x}\left[\frac{1}{2}\left(\frac{r}{R}\right)^2 - \frac{1}{4}\left(\frac{r}{R}\right)^4\right]_0^1$$

$$\bar{V} = -\frac{R^2}{8\mu}\frac{\partial p}{\partial x}$$

Then $u = \bar{V}$ when

$$u = -\frac{R^2}{4\mu}\frac{\partial p}{\partial x}\left[1-\left(\frac{r}{R}\right)^2\right] = \bar{V} = -\frac{R^2}{8\mu}\frac{\partial p}{\partial x}$$

or

$$1-\left(\frac{r}{R}\right)^2 = \frac{1}{2}$$

or

$$\left(\frac{r}{R}\right)^2 = \frac{1}{2}$$

$$r = \frac{R}{\sqrt{2}} = 0.707\,R \longleftarrow r$$

Problem 8.45

Given: Water and SAE 10W oil flowing at 40°C through a 6 mm tube.

Find, for each fluid:

(a) The maximum flowrate for laminar flow.
(b) The corresponding pressure gradient.

Solution: Laminar flow is expected for $Re \le 2300$. Expressing this in terms of flowrate,

$$Re = \frac{\rho \bar{V} D}{\mu} = \frac{\bar{V} D}{\nu} = \frac{Q D}{A \nu} = \frac{4}{\pi D^2}\frac{QD}{\nu} = \frac{4Q}{\pi \nu D} \quad \text{or} \quad Q = \frac{\pi \nu D Re}{4}$$

Thus

$$Q_{max} = \frac{\pi \nu D Re_{max}}{4} = \frac{\pi}{4} \times 2300 \times 0.006\,m \times \nu \frac{m^2}{s} = 10.8\,\nu \left(\frac{m^3}{s}\right)$$

Also, $Q = -\frac{\pi R^4}{8\mu}\frac{\partial p}{\partial x}$ for laminar flow, according to Eq. 8.13b. Then

$$\frac{\partial p}{\partial x} = -\frac{8 \mu Q}{\pi R^4} = -\frac{128 \mu Q}{\pi D^4}$$

so

$$\frac{\partial p}{\partial x} = -\frac{128}{\pi} \times \mu \frac{N \cdot s}{m^2} \times Q \frac{m^3}{s} \times \frac{1}{(0.006)^4 m^4} = -3.14 \times 10^{10} \mu Q \left(\frac{N}{m^3}\right)$$

Using data from Appendix A, at 40°C,

Fluid	$\nu\left(\frac{m^2}{s}\right)$	$Q\left(\frac{m^3}{s}\right)$	$\mu\left(\frac{N \cdot s}{m^2}\right)$	$\mu Q (N \cdot m)$	$\frac{\partial p}{\partial x}\left(\frac{N}{m^3}\right)$
Water	6.57×10^{-7}	7.10×10^{-6}	6.51×10^{-4}	4.62×10^{-9}	-145
SAE 10W oil	3.8×10^{-5}	4.10×10^{-4}	3.4×10^{-2}	1.39×10^{-5}	-4.36×10^{5}

$\frac{\partial p}{\partial x}$

Q_{ma}

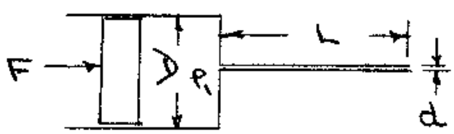
{Note $Q \sim \nu = \frac{\mu}{\rho}$ and $\frac{\partial p}{\partial x} \sim \mu Q \sim \frac{\mu^2}{\rho}$.}

Problem 8.46

Given: Hypodermic needle of diameter, d = 0.100 mm and length L = 25.0 mm is attached to a syringe of diameter, D = 10.0 mm. The syringe is filled with saline solution of viscosity, $\mu = 5\mu_{H_2O}$. The maximum force on the plunger is F = 45.0 N.

Find: the maximum flow rate at which saline can be delivered.

Solution:



Model the flow as steady, fully developed laminar flow in a circular tube.

Assume: (1) discharge is to p_{atm}
(2) fluid is at T = 20C.

Then, the volume flow rate Q can be evaluated from Eq. 8.13c

$$Q = \frac{\pi \Delta p D^4}{128 \mu L}$$

where $\Delta p = p_1 - p_{atm}$ and $\Delta p = \frac{F}{A} = \frac{4F}{\pi D^2} = \frac{4}{\pi} \times 45\,N \times \frac{1}{(0.01\,m)^2} = 573\,kPa$

$\mu = 5\mu_{H_2O}$ and from Table A.8, $\mu_{H_2O} = 1 \times 10^{-3}\,kg/m\cdot s$

Then $Q = \frac{\pi \Delta p D^4}{128 \mu L} = \frac{\pi}{128} \times 573 \times 10^3 \frac{N}{m^2} \times (10^{-4}\,m)^4 \times \frac{1}{5\times10^{-3}} \frac{m\cdot s}{kg} \times \frac{1}{0.025\,m} \times \frac{kg\cdot m}{N\cdot s^2}$

$Q = 11.3\ mm^3/s$ ← Q

Check Re

$$Re = \frac{\rho D \bar{V}}{\mu} = \frac{\rho D}{\mu}\frac{Q}{A} = \frac{\rho D}{\mu}\frac{4Q}{\pi D^2} = \frac{4\rho Q}{\pi \mu D}$$

Assume $\rho_{saline} = \rho_{H_2O}$, then

$Re = \frac{4}{\pi}\frac{\rho Q}{\mu D} = \frac{4}{\pi} \times 999 \frac{kg}{m^3} \times 9.27 \times 10^{-9} \frac{m^3}{s} \times \frac{1}{5\times10^{-3}} \frac{m\cdot s}{kg} \times \frac{1}{10^{-4}\,m}$

$Re = 28.8$ (flow is definitely laminar)

Problem 8.47

Given: Viscosity of water is to be determined by measuring pressure drop and flow rate through tygon tubing of length, $L = 50$ ft and diameter, $D = 0.125 \pm 0.010$ in

Find: (a) Maximum volume flow rate at which flow would be laminar
(b) Pressure drop corresponding to this Q
(c) Estimate of experimental uncertainty in "measured" viscosity
(d) Way in which set up might be improved.

Solution:

Assume: steady, fully developed laminar flow in the tube

Flow is expected to remain laminar up to $Re = 2300$.

$$Re = \frac{\rho D \bar{V}}{\mu} = \frac{D}{\nu}\frac{Q}{A} = \frac{D}{\nu}\frac{Q\,4}{\pi D^2} = \frac{4Q}{\pi D \nu}$$

To determine Q, we need to know ν. Assume $T = 70°F$. Then

$\nu = 1.05 \times 10^{-5}$ ft²/s (Table A.7)

$\mu = 2.04 \times 10^{-5}$ lbf·s/ft² (Table A.7)

$$\therefore Q_{max} = \frac{\pi D \nu Re}{4} = \frac{\pi}{4} \times 0.125\,\text{in} \times 1.08 \times 10^{-5}\,\frac{\text{ft}^2}{\text{sec}} \times 2300 \times \frac{\text{ft}}{12\,\text{in}} = 2.03 \times 10^{-4}\,\frac{\text{ft}^3}{\text{sec}}$$ ← Q_{max}

The corresponding pressure drop, $\Delta p = p_1 - p_2$, can be determined from Eq. 8.13c

$$Q = \frac{\pi}{128}\frac{\Delta p D^4}{\mu L} \quad \text{.....(8.13c)}$$

$$\therefore \Delta p = \frac{128 \mu L Q}{\pi D^4} = \frac{128}{\pi} \times 2.04 \times 10^{-5}\,\frac{\text{lbf}\cdot\text{s}}{\text{ft}^2} \times 50\,\text{ft} \times 2.03 \times 10^{-4}\,\frac{\text{ft}^3}{\text{s}} \times \frac{1}{(0.125\,\text{in})^4}\left(\frac{12\,\text{in}}{\text{ft}}\right)^4$$

$\Delta p = 716$ lbf/ft² $= 4.97$ lbf/in² ← Δp

Equation 8.13c can be used to determine μ from measurements of Δp and Q. Thus

$$\mu = \frac{\pi}{128}\frac{\Delta p D^4}{L Q} \quad \text{or} \quad \mu = \mu(\Delta p, D, L, Q)$$

From uncertainty analysis

$$u_\mu = \pm\left[\left(\frac{\Delta p}{\mu}\frac{\partial \mu}{\partial \Delta p}u_{\Delta p}\right)^2 + \left(\frac{D}{\mu}\frac{\partial \mu}{\partial D}u_D\right)^2 + \left(\frac{L}{\mu}\frac{\partial \mu}{\partial L}u_L\right)^2 + \left(\frac{Q}{\mu}\frac{\partial \mu}{\partial Q}u_Q\right)^2\right]^{1/2}$$

Evaluating, $\frac{\Delta p}{\mu}\frac{\partial \mu}{\partial \Delta p} = \frac{\Delta p}{\mu}\frac{\pi}{128}\frac{D^4}{LQ} = 1$, $\frac{D}{\mu}\frac{\partial \mu}{\partial D} = \frac{D}{\mu}\,4D^3\,\frac{\pi}{128}\frac{\Delta p}{LQ} = 4$

$\frac{L}{\mu}\frac{\partial \mu}{\partial L} = \frac{L}{\mu}(-1)\frac{\pi}{128}\frac{\Delta p D^4}{Q L^2} = -1$, $\frac{Q}{\mu}\frac{\partial \mu}{\partial Q} = \frac{Q}{\mu}(-1)\frac{\pi}{128}\frac{\Delta p D^4}{L Q^2} = -1$

Thus $u_\mu = \left[(u_{\Delta p})^2 + (4u_D)^2 + (-u_L)^2 + (-u_Q)^2\right]^{1/2}$

Since $u_D = \frac{\delta D}{D} = \pm\frac{0.01}{0.125} = \pm 8\%$, $u_\mu \approx 4u_D = 32\%$ ← u_μ

The set up could be improved by reducing u_D. Use somewhat larger diameter tube and/or more uniform diameter tube.

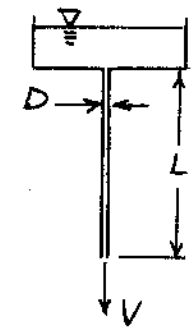
Problem 8.48

Given: Commercial viscometer, $D = 0.31$ mm and $L = 73$ mm.

Size chosen so $\Delta t \approx 200$ s, and $u_{\Delta V}$ is negligible.

Find: Estimate the maximum uncertainty in D to allow measurement of μ within ± 1 percent.

Solution: Computing equation: $Q = \dfrac{\pi \Delta p D^4}{128 \mu L}$ (8.13c)



Assumptions: (1) Laminar flow
(2) Fully developed flow

Solving for viscosity, $\mu = \dfrac{\pi \Delta p D^4}{128 Q L}$

But $\Delta p \sim \rho g L$ and $Q \sim \Delta V/\Delta t$. Thus

$$\mu \sim \rho g L \frac{\Delta t}{\Delta V} \frac{D^4}{L} = \frac{\rho g \Delta t D^4}{\Delta V}$$

From Appendix E,

$$u_\mu = \pm\left[(u_{\Delta t})^2 + (4u_D)^2 + (u_{\Delta V})^2\right]^{1/2}$$

Neglecting $u_{\Delta t}$ and $u_{\Delta V}$ compared to u_D,

$$u_\mu \simeq \pm\left[(4u_D)^2\right]^{1/2} = \pm 4 u_D = \pm 4 \frac{\delta D}{D}$$

Thus

$$\delta D = \pm \frac{u_\mu}{4} D = \pm \frac{0.01}{4} \times 0.31 \text{ mm} \times \frac{\text{m}}{1000 \text{ mm}}$$

$$\delta D = \pm 0.775 \, \mu\text{m}$$ δD

{Such a small tolerance would be impossible to hold in any manufacturing operation. Therefore capillary viscometers are calibrated using a liquid of known viscosity in the range of interest.}

Problem 8.49

In engineering science there are often analogies to be made between disparate phenomena. For example, the applied pressure difference Δp and corresponding volume flow rate Q in a tube can be compared to the applied DC voltage V across and current I through an electrical resistor, respectively. By analogy, find a formula for the "resistance" of laminar flow of fluid of viscosity μ in a tube length of L and diameter D, corresponding to electrical resistance R. For a tube 100 mm long with inside diameter 0.3 mm, find the maximum flow rate and pressure difference for which this analogy will hold for (a) kerosine and (b) castor oil (both at 40°C). When the flow exceeds this maximum, why does the analogy fail?

Given: Data on a tube

Find: "Resistance" of tube; maximum flow rate and pressure difference for which electrical anal holds for (a) kerosine and (b) castor oil

Solution

The given data is $L = 100 \cdot mm$ $\quad D = 0.3 \cdot mm$

From Fig. A.2 and Table A.2

Kerosene: $\mu = 1.1 \times 10^{-3} \cdot \frac{N \cdot s}{m^2}$ $\quad \rho = 0.82 \times 990 \cdot \frac{kg}{m^3} = 812 \cdot \frac{kg}{m^3}$

Castor oil: $\mu = 0.25 \cdot \frac{N \cdot s}{m^2}$ $\quad \rho = 2.11 \times 990 \cdot \frac{kg}{m^3} = 2090 \cdot \frac{kg}{m^3}$

For an electrical resistor $V = R \cdot I$ $\quad (1)$

The governing equation for the flow rate for laminar flow in a tube is Eq. 8.13c

$$Q = \frac{\pi \cdot \Delta p \cdot D^4}{128 \cdot \mu \cdot L}$$

or

$$\Delta p = \frac{128 \cdot \mu \cdot L}{\pi \cdot D^4} \cdot Q \qquad (2)$$

By analogy, current *I* is represented by flow rate *Q*, and voltage *V* by pressure drop Δ*p*. Comparing Eqs. (1) and (2), the "resistance" of the tube is

$$R = \frac{128 \cdot \mu \cdot L}{\pi \cdot D^4}$$

The "resistance" of a tube is directly proportional to fluid viscosity and pipe length, and strongly dependent on the inverse of diameter

The analogy is only valid for $Re < 2300$ or $\frac{\rho \cdot V \cdot D}{\mu} < 2300$

Writing this constraint in terms of flow rate

$$\frac{\rho \cdot \frac{Q}{\frac{\pi}{4} \cdot D^2} \cdot D}{\mu} < 2300 \quad \text{or} \quad Q_{max} = \frac{2300 \cdot \mu \cdot \pi \cdot D}{4 \cdot \rho}$$

The corresponding maximum pressure gradient is then obtained from Eq. (2)

$$\Delta p_{max} = \frac{128 \cdot \mu \cdot L}{\pi \cdot D^4} \cdot Q_{max} = \frac{32 \cdot 2300 \cdot \mu^2 \cdot L}{\rho \cdot D^3}$$

(a) For kerosine $Q_{max} = 7.34 \times 10^{-7} \frac{m^3}{s}$ $\Delta p_{max} = 406\,kPa$

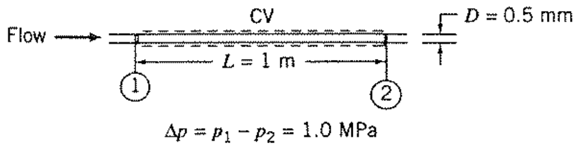
(b) For castor oil $Q_{max} = 6.49 \times 10^{-5} \frac{m^3}{s}$ $\Delta p_{max} = 8156\,MPa$

The analogy fails when Re > 2300 because the flow becomes turbulent, and "resistance" to flow then no longer linear with flow rate

Problem 8.50

Given: Capillary viscometer of Example Problem 8.4. $Q = 880\ mm^3/s$.

Least counts are: ± 0.01 MPa for Δp, ± 0.01 mm for D, $\pm 5\ mm^3/s$ for Q, and ± 1.00 mm for L. SG of test liquid is 0.82.



Find: (a) Effect of tube diameter on experimental uncertainty.
(b) If proper choice of diameter can minimize uncertainty.

Solution: Viscosity is given by $\mu = \dfrac{\pi \Delta p D^4}{128 L Q}$, so the uncertainty is

$$u_\mu = \pm\left[u_{\Delta p}^2 + (4u_D)^2 + u_L^2 + u_Q^2\right]^{\frac{1}{2}} \qquad (1)$$

$$= \pm\left[\left(\pm\frac{0.01}{1.00}\right)^2 + \left(\pm 4\,\frac{0.01}{0.50}\right)^2 + \left(\pm\frac{0.001}{1.00}\right)^2 + \left(\pm\frac{5}{880}\right)^2\right]^{\frac{1}{2}}$$

$$= \pm\left[(\pm 0.01)^2 + (\pm 0.08)^2 + (\pm 0.001)^2 + (\pm 0.00568)^2\right]^{\frac{1}{2}}$$

$4u_D$ is the largest influence on u_μ — u_D

$u_{\Delta p}$ is the second largest influence on u_μ

$$u_\mu = \pm[0.00653]^{\frac{1}{2}} = \pm 0.0808 \text{ or } \pm 8.08 \text{ percent}$$ — u_μ

To reduce u_D, increase diameter. Then reduce Q to maintain Re = constant

$Re = \dfrac{\rho \bar{V} D}{\mu}$, so $\bar{V}D$ = constant. But $\bar{V} = \dfrac{Q}{A} = \dfrac{4Q}{\pi D^2}$, so $\bar{V}D \sim \dfrac{Q}{D}$ = constant

But Δp will be affected. $\Delta p = \dfrac{128 \mu L Q}{\pi D^4}$, so $\Delta p \sim \dfrac{Q}{D^4} \sim \dfrac{1}{D^3}$ (since $\dfrac{Q}{D}$ = const.)

Representative values may be computed as follows:

D	Q	Δp	$u_{\Delta p}$	$4u_D$	u_L	u_Q	u_μ
(mm)	(mm³/s)	(MPa)	(---)	(---)	(---)	(---)	(---)
0.50	880	1.00	0.0100	0.0800	0.001	0.00568	0.0808
0.55	968	0.751	0.0133	0.0727	0.001	0.00517	0.0741
0.60	1056	0.579	0.0173	0.0667	0.001	0.00473	0.0690
0.65	1144	0.455	0.0220	0.0615	0.001	0.00437	0.0655
0.70	1232	0.364	0.0274	0.0571	0.001	0.00406	0.0635
0.75	1320	0.296	0.0338	0.0533	0.001	0.00379	0.0632
0.80	1408	0.244	0.0410	0.0500	0.001	0.00355	0.0647
0.85	1496	0.204	0.0491	0.0471	0.001	0.00334	0.0681
0.90	1584	0.171	0.0583	0.0444	0.001	0.00316	0.0734
0.95	1672	0.146	0.0686	0.0421	0.001	0.00299	0.0805
1.00	1760	0.125	0.0800	0.0400	0.001	0.00284	0.0895

u_μ (min.)

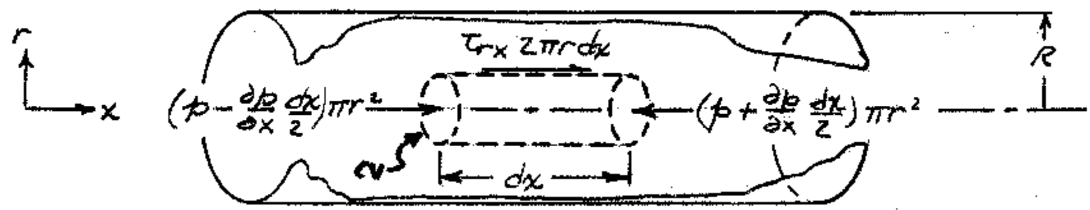
The uncertainty in D drops quickly as D increases. Although $u_{\Delta p}$ increases, there is a diameter that minimizes u_μ.

The optimum diameter is $D \approx 0.75$ mm. — $D_{opt.}$

(Note that the entrance length would increase, since $L_e/D = 0.06\,Re$.)

Problem 8.51

Given: Fully-developed laminar flow in a circular pipe, with cylindrical control volume as shown.



Find: (a) Forces acting on CV.
(b) Expression for velocity distribution.

Solution: The forces on a CV of radius r are shown above.

Apply the x component of momentum to CV shown.

Basic equations:

$$F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho \vec{V}\cdot d\vec{A}, \quad \tau_{rx} = \mu \frac{du}{dr}$$

(F_{Bx} = 0(1), $\frac{\partial}{\partial t}$ term = 0(2), CS term = 0(3))

Assumptions: (1) $F_{Bx} = 0$
(2) Steady flow
(3) Fully-developed flow

Then

$$F_{Sx} = (p - \frac{\partial p}{\partial x}\frac{dx}{2})\pi r^2 + \tau_{rx}\, 2\pi r dx - (p + \frac{\partial p}{\partial x}\frac{dx}{2})\pi r^2 = 0$$

Cancelling and combining terms,

$$-r\frac{\partial p}{\partial x} + 2\tau_{rx} = 0 \quad \text{or} \quad \tau_{rx} = \mu\frac{du}{dr} = \frac{r}{2}\frac{\partial p}{\partial x}$$

Thus $\frac{du}{dr} = \frac{r}{2\mu}\frac{\partial p}{\partial x}$

and

$$u = \frac{r^2}{4\mu}\frac{\partial p}{\partial x} + c_1$$

To evaluate c_1, apply the boundary condition $u = 0$ at $r = R$. Thus

$$c_1 = -\frac{R^2}{4\mu}\frac{\partial p}{\partial x}$$

and

$$u = \frac{1}{4\mu}\frac{\partial p}{\partial x}(r^2 - R^2) = -\frac{R^2}{4\mu}\frac{\partial p}{\partial x}\left[1 - \left(\frac{r}{R}\right)^2\right]$$

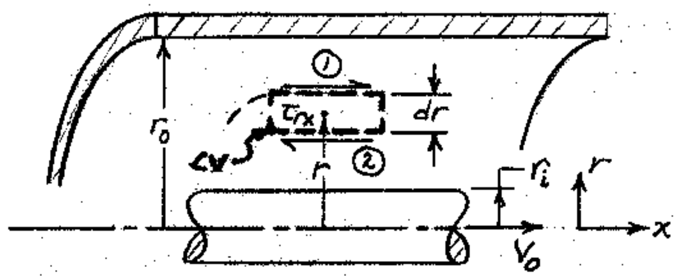
u

which is identical to Eq. 8.12.

Problem 8.52

Given: Fully-developed laminar flow in an annulus as shown. The inner section is stationary; the outer moves at V_0.

Assume $\frac{\partial p}{\partial x} = 0$.



Find: (a) $\tau(r)$ in terms of c_1.
(b) $V(r)$ in terms of c_1, c_2.
(c) Evaluate c_1, c_2.

Solution: Apply x component of momentum equation, using annular CV shown.

Basic Equations: $F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho\vec{V}\cdot d\vec{A}$; $\tau_{rx} = \mu \frac{du}{dr} = \tau$

($F_{Bx} = 0(1)$, $\frac{\partial}{\partial t} = 0(2)$, $\int_{CS} = 0(3)$)

Assumptions: (1) $F_{Bx} = 0$
(2) Steady flow
(3) Fully-developed flow

Then

$$F_{Sx} = F_{①} - F_{②} = \left(\tau + \frac{d\tau}{dr}\frac{dr}{2}\right)2\pi\left(r + \frac{dr}{2}\right)dx - \left(\tau - \frac{d\tau}{dr}\frac{dr}{2}\right)2\pi\left(r - \frac{dr}{2}\right)dx = 0$$

Neglecting products of differentials, this reduces to

$$\tau + r\frac{d\tau}{dr} = 0 \qquad \text{or} \qquad \frac{d}{dr}(r\tau) = 0$$

Thus $r\tau = c_1$ or $\tau = \frac{c_1}{r}$ ← $\tau(r)$

But $\tau = \mu \frac{du}{dr}$, so $\frac{du}{dr} = \frac{c_1}{\mu r}$

and $u = \frac{c_1}{\mu}\ln r + c_2$ ← $u(r)$

To evaluate constants c_1 and c_2, use boundary conditions.

At $r = r_i$, $u = V_0$, so $V_0 = \frac{c_1}{\mu}\ln r_i + c_2$

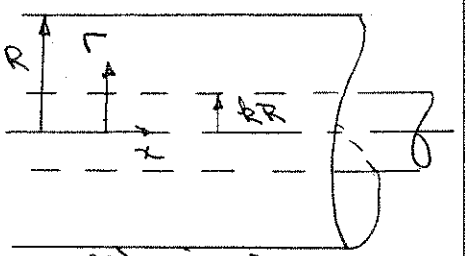
At $r = r_0$, $u = 0$, so $0 = \frac{c_1}{\mu}\ln r_0 + c_2$ and $c_2 = -\frac{c_1}{\mu}\ln r_0$

Thus, subtracting, $V_0 = \frac{c_1}{\mu}\ln\left(\frac{r_i}{r_0}\right)$ or $c_1 = \frac{\mu V_0}{\ln(r_i/r_0)}$ so $c_2 = -\frac{V_0 \ln r_0}{\ln(r_i/r_0)}$

Finally

$$u = \frac{V_0}{\ln(r_i/r_0)}(\ln r - \ln r_i) = V_0 \frac{\ln(r/r_0)}{\ln(r_i/r_0)}$$ ← $u(r)$

Given: Fully developed laminar flow with pressure gradient, $\partial p/\partial x$, in the annulus shown



(a) Show that the velocity profile is given by $u = -\frac{R^2}{4\mu}\left(\frac{\partial p}{\partial x}\right)\left[1-\left(\frac{r}{R}\right)^2 + \frac{(1-k^2)}{\ln(1/k)}\ln\frac{r}{R}\right]$

(b) Obtain an expression for the location ($\alpha = r/R$) of maximum u as a function of k.

(c) Plot α vs k.

(d) Compare limiting case, $k \to 0$, with flow in circular pipe.

Solution: We may use the results of the differential control volume analysis of Section 8-3 to write

$$u = \frac{r^2}{4\mu}\frac{\partial p}{\partial x} + \frac{c_1}{\mu}\ln r + c_2 \quad \text{----(1)}$$

The boundary conditions are $u=0$ at $r=R$
$u=0$ at $r=kR$

Substituting the boundary conditions

$$0 = \frac{R^2}{4\mu}\frac{\partial p}{\partial x} + \frac{c_1}{\mu}\ln R + c_2 \quad \text{----(2)}$$

$$0 = \frac{k^2R^2}{4\mu}\frac{\partial p}{\partial x} + \frac{c_1}{\mu}\ln kR + c_2 \quad \text{----(3)}$$

Subtracting, $0 = \frac{R^2}{4\mu}\frac{\partial p}{\partial x}(1-k^2) + \frac{c_1}{\mu}(\ln R - \ln kR)$

$$\therefore \quad c_1 = -\frac{R^2}{4}\frac{\partial p}{\partial x}\frac{(1-k^2)}{\ln(1/k)}$$

From Eq. 2

$$c_2 = -\frac{R^2}{4\mu}\frac{\partial p}{\partial x} + \frac{R^2}{4\mu}\frac{\partial p}{\partial x}\frac{(1-k^2)}{\ln(1/k)}\ln R$$

Substituting for c_1 and c_2 into Eq. 1 gives

$$u = \frac{r^2}{4\mu}\frac{\partial p}{\partial x} - \frac{R^2}{4\mu}\frac{\partial p}{\partial x}\frac{(1-k^2)}{\ln(1/k)}\ln r - \frac{R^2}{4\mu}\frac{\partial p}{\partial x} + \frac{R^2}{4\mu}\frac{\partial p}{\partial x}\frac{(1-k^2)}{\ln(1/k)}\ln R$$

$$u = \frac{1}{4\mu}\frac{\partial p}{\partial x}\left[r^2 - R^2 - \frac{R^2(1-k^2)}{\ln(1/k)}(\ln r - \ln R)\right]$$

$$u = -\frac{R^2}{4\mu}\frac{\partial p}{\partial x}\left[1-\left(\frac{r}{R}\right)^2 + \frac{(1-k^2)}{\ln(1/k)}\ln\frac{r}{R}\right] \longleftarrow u$$

To locate max u, set $\tau_{rx} = \mu\frac{du}{dr} = 0$

$$\tau_{rx} = \mu\frac{du}{dr} = -\frac{R^2}{4}\frac{\partial p}{\partial x}\left[-\frac{2r}{R^2} + \frac{(1-k^2)}{\ln(1/k)}\frac{1}{r}\right]$$

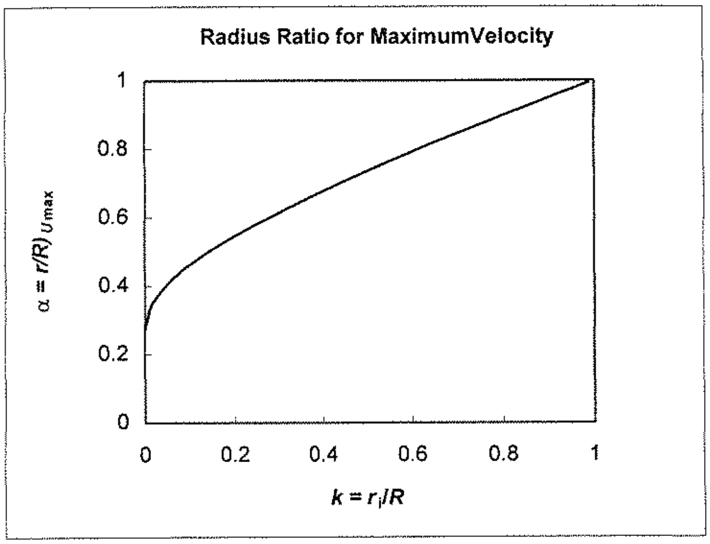
Set $\tau_{rx} = 0$ at $r = \alpha R$. Then

$$0 = \frac{-2\alpha R}{R^2} + \frac{(1-k^2)}{\ln(1/k)} \frac{1}{\alpha R}$$

and

$$\alpha = \left[\frac{1}{2} \frac{(1-k^2)}{\ln(1/k)} \right]^{1/2} \qquad \alpha = \alpha(k)$$

$k = r_i/R$	$\alpha = r/R)_{u\,max}$
0.001	0.269
0.01	0.329
0.02	0.357
0.05	0.408
0.08	0.444
0.1	0.464
0.2	0.546
0.4	0.677
0.6	0.791
0.8	0.898
0.95	0.975
0.99	0.995

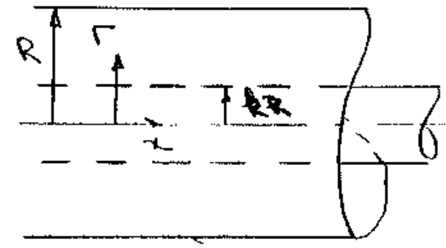


For $k \to 0$, $\alpha \to 0$ and $u = -\frac{R^2}{4\mu}\left(\frac{\partial p}{\partial x}\right)\left[1 - \left(\frac{r}{R}\right)^2\right]$

This agrees with the results for flow in a circular pipe

As $k \to 1.0$, $\alpha \to 1.0$ and the flow behaves like flow between stationary infinite parallel plates.

Given: Fully developed laminar flow in the annulus shown, with pressure gradient $\partial p/\partial x$. The velocity profile is given by



$$u = -\frac{R^2}{4\mu}\frac{\partial p}{\partial x}\left[1-\left(\frac{r}{R}\right)^2 + \frac{(1-k^2)}{\ln(1/k)}\ln\frac{r}{R}\right]$$

(a) Show that the volume flow rate is given by

$$Q = -\frac{\pi R^4}{8\mu}\frac{\partial p}{\partial x}\left[(1-k^4) - \frac{(1-k^2)^2}{\ln(1/k)}\right]$$

(b) Obtain an expression for the average velocity

(c) Compare limiting case, $k \to 0$, with flow in a circular pipe.

<u>Solution</u>: The volume flow rate is given by

$$Q = \int u\,dA = \int_{kR}^{R} u\,2\pi r\,dr = 2\pi\int_{kR}^{R} u\,r\,dr$$

$$= 2\pi\left(-\frac{R^2}{4\mu}\frac{\partial p}{\partial x}\right)\int_{kR}^{R}\left[r - \frac{r^3}{R^2} + \frac{(1-k^2)}{\ln(1/k)}\,r\ln\frac{r}{R}\right]dr$$

$$= -\frac{\pi R^4}{2\mu}\frac{\partial p}{\partial x}\int_{k}^{1}\left[\frac{r}{R} - \left(\frac{r}{R}\right)^3 + \frac{(1-k^2)}{\ln(1/k)}\frac{r}{R}\ln\frac{r}{R}\right]d\left(\frac{r}{R}\right)$$

$$= -\frac{\pi R^4}{2\mu}\frac{\partial p}{\partial x}\left[\frac{1}{2}\left(\frac{r}{R}\right)^2 - \frac{1}{4}\left(\frac{r}{R}\right)^4 + \frac{(1-k^2)}{\ln(1/k)}\left\{\left(\frac{r}{R}\right)^2\left[\frac{1}{2}\ln\left(\frac{r}{R}\right) - \frac{1}{4}\right]\right\}\right]_k^1$$

$$= -\frac{\pi R^4}{2\mu}\frac{\partial p}{\partial x}\left[\frac{1}{2} - \frac{k^2}{2} - \frac{1}{4} + \frac{k^4}{4} + \frac{(1-k^2)}{\ln(1/k)}\left\{-\frac{1}{4} - k^2\left[\frac{1}{2}\ln k - \frac{1}{4}\right]\right\}\right]$$

$$= -\frac{\pi R^4}{2\mu}\frac{\partial p}{\partial x}\left[\frac{1}{4} - \frac{k^2}{2} + \frac{k^4}{4} + \frac{(1-k^2)}{\ln(1/k)}\left\{-\frac{1}{4} + \frac{k^2}{4} - k^2\frac{1}{2}\ln k\right\}\right]$$

$$= -\frac{\pi R^4}{2\mu}\frac{\partial p}{\partial x}\left[\frac{1-2k^2+k^4}{4} + \frac{(1-k^2)}{\ln(1/k)}\frac{(k^2-1)}{4} - \frac{(1-k^2)}{\ln(1/k)}k^2\frac{1}{2}\ln k\right]$$

$$= -\frac{\pi R^4}{2\mu}\frac{\partial p}{\partial x}\left[\frac{1-2k^2+k^4}{4} - \frac{(1-k^2)^2}{4\ln(1/k)} + \frac{k^2-k^4}{2}\right]$$

$$= -\frac{\pi R^4}{2\mu}\frac{\partial p}{\partial x}\left[\frac{1-2k^2+k^4+2k^2-2k^4}{4} - \frac{(1-k^2)^2}{4\ln(1/k)}\right]$$

$$Q = -\frac{\pi R^4}{8\mu}\frac{\partial p}{\partial x}\left[(1-k^4) - \frac{(1-k^2)^2}{\ln(1/k)}\right] \longleftarrow Q$$

The average velocity, $\bar{V} = \frac{Q}{A}$

The area is given by

$$A = \int dA = \int 2\pi r\,dr = 2\pi R^2 \int_k^1 \frac{r}{R}\, d\left(\frac{r}{R}\right)$$

$$A = 2\pi R^2 \left[\frac{1}{2}\left(\frac{r}{R}\right)^2 \right]_k^1 = 2\pi R^2 \times \frac{1}{2}(1-k^2) = \pi R^2(1-k^2)$$

Thus

$$\bar{V} = \frac{Q}{A} = -\frac{\pi R^4}{8\mu}\frac{\partial p}{\partial x} \times \frac{1}{\pi R^2}\left[\frac{(1-k^4)}{(1-k^2)} - \frac{(1-k^2)}{\ln(1/k)} \right]$$

$$\bar{V} = -\frac{R^2}{8\mu}\frac{\partial p}{\partial x}\left[\frac{(1-k^4)}{(1-k^2)} - \frac{(1-k^2)}{\ln(1/k)} \right] \longleftarrow \bar{V}$$

For $k \to 0$

$$Q = -\frac{\pi R^4}{8\mu}\frac{\partial p}{\partial x} \quad \text{and} \quad \bar{V} = -\frac{R^2}{8\mu}\frac{\partial p}{\partial x}$$

These agree with the results for flow in a circular pipe.

Problem 8.55

In a food industry plant two immiscible fluids are pumped through a tube such that fluid 1 (μ_1 = 1 N · s/m^2) forms an inner core and fluid 2 (μ_2 = 1.5 N · s/m^2) forms an outer annulus. The tube has D = 5 mm diameter and length L = 10 m. Derive and plot the velocity distribution if the applied pressure difference, Δp, is 10 kPa.

Given: Data on tube, applied pressure, and on two fluids in annular flow

Find: Velocity distribution; plot

Solution

Given data $\quad D = 5\cdot mm \quad L = 10\cdot m \quad \Delta p = -10\cdot kPa$

$$\mu_1 = 1\cdot\frac{N\cdot s}{m^2} \qquad \mu_2 = 1.5\cdot\frac{N\cdot s}{m^2}$$

From Section 8-3 for flow in a pipe, Eq. 8.11 can be applied to either fluid

$$u = \frac{r^2}{4\cdot\mu}\cdot\left(\frac{\partial}{\partial x}p\right) + \frac{c_1}{\mu}\cdot\ln(r) + c_2$$

Applying this to fluid 1 (inner fluid) and fluid 2 (outer fluid)

$$u_1 = \frac{r^2}{4\cdot\mu_1}\cdot\frac{\Delta p}{L} + \frac{c_1}{\mu_1}\cdot\ln(r) + c_2 \qquad u_2 = \frac{r^2}{4\cdot\mu_2}\cdot\frac{\Delta p}{L} + \frac{c_3}{\mu_2}\cdot\ln(r) + c_4$$

We need four BCs. Two are obvious $\quad r = \frac{D}{2} \qquad u_2 = 0 \qquad (1)$

$$r = \frac{D}{4} \qquad u_1 = u_2 \qquad (2)$$

The third BC comes from the fact that the axis is a line of symmetry

$$r = 0 \qquad \frac{du_1}{dr} = 0 \qquad (3)$$

The fourth BC comes from the fact that the stress at the interface generated by each fluid is the s

$$r = \frac{D}{4} \qquad \mu_1 \cdot \frac{du_1}{dr} = \mu_2 \cdot \frac{du_2}{dr} \qquad (4)$$

Using these four BCs $\quad \frac{\left(\frac{D}{2}\right)^2}{4 \cdot \mu_2} \cdot \frac{\Delta p}{L} + \frac{c_3}{\mu_2} \cdot \ln\left(\frac{D}{2}\right) + c_4 = 0$

$$\frac{\left(\frac{D}{4}\right)^2}{4 \cdot \mu_1} \cdot \frac{\Delta p}{L} + \frac{c_1}{\mu_1} \cdot \ln\left(\frac{D}{4}\right) + c_2 = \frac{\left(\frac{D}{4}\right)^2}{4 \cdot \mu_2} \cdot \frac{\Delta p}{L} + \frac{c_3}{\mu_2} \cdot \ln\left(\frac{D}{4}\right) + c_4$$

$$\lim_{r \to 0} \frac{c_1}{\mu_1 \cdot r} = 0$$

$$\frac{D}{8} \cdot \frac{\Delta p}{L} + \frac{4 \cdot c_1}{D} = \frac{D}{8} \cdot \frac{\Delta p}{L} + \frac{4 \cdot c_3}{D}$$

Hence, after some algebra

$$c_1 = 0 \qquad \text{(To avoid singularity)} \qquad c_2 = -\frac{D^2 \cdot \Delta p}{64 \cdot L} \frac{\left(\mu_2 + 3 \cdot \mu_1\right)}{\mu_1 \cdot \mu_2}$$

$$c_3 = 0 \qquad c_4 = -\frac{D^2 \cdot \Delta p}{16 \cdot L \cdot \mu_2}$$

The velocity distributions are then

$$u_1 = \frac{\Delta p}{4 \cdot \mu_1 \cdot L} \cdot \left[r^2 - \left(\frac{D}{2}\right)^2 \cdot \frac{\left(\mu_2 + 3 \cdot \mu_1\right)}{4 \cdot \mu_2} \right]$$

$$u_2 = \frac{\Delta p}{4 \cdot \mu_2 \cdot L} \cdot \left[r^2 - \left(\frac{D}{2}\right)^2 \right]$$

(Note that these result in the same expression if $\mu_1 = \mu_2$, i.e., if we have one fluid)

Evaluating either velocity at $r = D/4$ gives the velocity at the interface

$$u_{interface} = -\frac{3 \cdot D^2 \cdot \Delta p}{64 \cdot \mu_2 \cdot L} \qquad u_{interface} = 7.81 \times 10^{-4} \frac{m}{s}$$

Evaluating u_1 at $r = 0$ gives the maximum velocity

$$u_{max} = -\frac{D^2 \cdot \Delta p \cdot \left(\mu_2 + 3 \cdot \mu_1\right)}{64 \cdot \mu_1 \cdot \mu_2 \cdot L} \qquad u_{max} = 1.17 \times 10^{-3} \frac{m}{s}$$

The velocity distributions are plotted in the associated *Excel* workbook

Problem 8.55 (In Excel)

In a food industry plant two immiscible fluids are pumped through a tube such that fluid 1 ($\mu_1 = 1\ \text{N}\cdot\text{s/m}^2$) forms an inner core and fluid 2 ($\mu_2 = 1.5\ \text{N}\cdot\text{s/m}^2$) forms an outer annulus. The tube has $D = 5$ mm diameter and length $L = 10$ m. Derive and plot the velocity distribution if the applied pressure difference, Δp, is 10 kPa.

Given: Data on tube, applied pressure, and on two fluids in annular flow

Find: Velocity distribution; plot

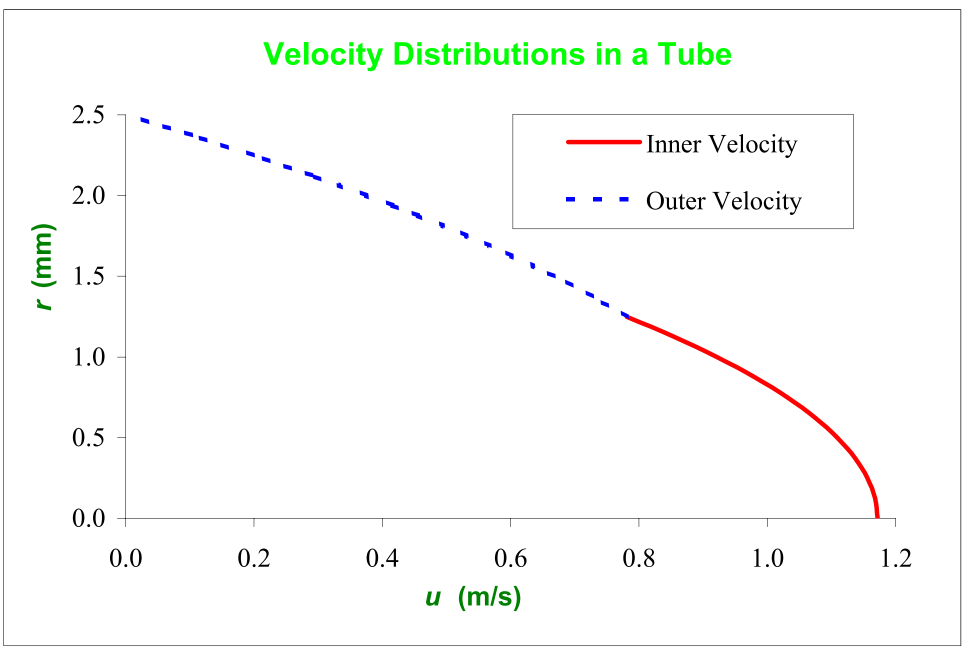
Solution

L =	10	m
D =	5	mm
μ_1 =	1	N.s/m^2
μ_2 =	1.5	N.s/m^2
Δp =	-10	kPa

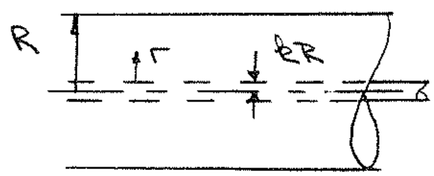
$$u_1 = \frac{\Delta p}{4\cdot\mu_1\cdot L}\cdot\left[r^2 - \left(\frac{D}{2}\right)^2\cdot\frac{(\mu_2 + 3\cdot\mu_1)}{4\cdot\mu_2}\right]$$

$$u_2 = \frac{\Delta p}{4\cdot\mu_2\cdot L}\cdot\left[r^2 - \left(\frac{D}{2}\right)^2\right]$$

r (mm)	u_1 (m/s)	u_2 (m/s)
0.00	1.172	
0.13	1.168	
0.25	1.156	
0.38	1.137	
0.50	1.109	
0.63	1.074	
0.75	1.031	
0.88	0.980	
1.00	0.922	
1.13	0.855	
1.25	0.781	0.781
1.38		0.727
1.50		0.667
1.63		0.602
1.75		0.531
1.88		0.456
2.00		0.375
2.13		0.289
2.25		0.198
2.38		0.102
2.50		0.000



Given: Fully developed laminar flow in a circular pipe is converted to flow in an annulus by insertion of a thin wire along the centerline



(a) Use results of Problem 8.51 to obtain an expression for the percent change in pressure drop as a function of radius ratio k.

(b) Plot percent change in Δp vs k for $0.001 \le k \le 0.10$

Solution: The results of problem 8.48 give

$$Q = -\frac{\pi R^4}{8\mu}\frac{\partial p}{\partial x}\left[(1-k^4) - \frac{(1-k^2)^2}{\ln(1/k)}\right]$$

Thus

$$\frac{\Delta p}{L} = -\frac{\partial p}{\partial x} = \frac{8\mu Q}{\pi R^4} \times \frac{1}{\left[(1-k^4) - \frac{(1-k^2)^2}{\ln(1/k)}\right]}$$

For $k=0$, $\frac{\Delta p}{L} = \frac{8\mu Q}{\pi R^4}$

$$\text{Percent change} = \frac{\Delta p/L - \Delta p/L)_{k=0}}{\Delta p/L)_{k=0}} = \frac{1}{\left[(1-k^4) - \frac{(1-k^2)^2}{\ln(1/k)}\right]} - 1$$

$$\%\ \text{change} = \frac{1 - \left[(1-k^4) - \frac{(1-k^2)}{\ln(1/k)}\right]}{\left[(1-k^4) - \frac{(1-k^2)}{\ln(1/k)}\right]}$$

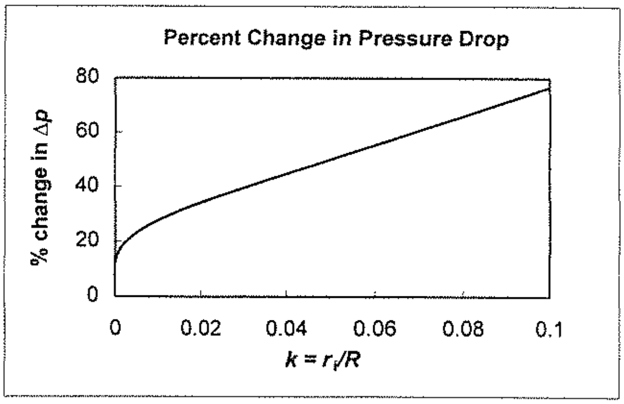
For small k,

$$\%\ \text{change} = \frac{1-\left[1 - \frac{1}{\ln(1/k)}\right]}{\left[1 - \frac{1}{\ln(1/k)}\right]} = \frac{1-\left[1+\frac{1}{\ln k}\right]}{\left[1+\frac{1}{\ln k}\right]} = \frac{-\frac{1}{\ln k}}{\left[1+\frac{1}{\ln k}\right]}$$

$$\%\ \text{change} = -\frac{1}{\ln k\left(1+\frac{1}{\ln k}\right)} \times 100$$

% change

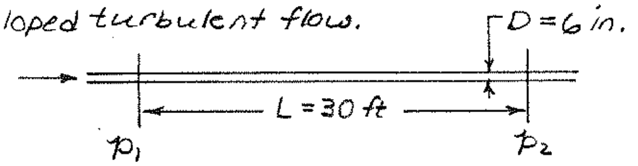
$k = r_i/R$	% change in Δp
0.0001	12.2
0.0002	13.3
0.0005	15.1
0.001	16.9
0.002	19.2
0.005	23.3
0.01	27.7
0.02	34.3
0.05	50.1
0.1	76.8



The plot shows that even the smallest of wires causes a significant increase in pressure drop for a given flow rate.

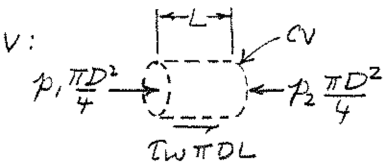
Given: Horizontal pipe with fully developed turbulent flow.

$\Delta p = p_1 - p_2 = 5\ \text{psi}$



Find: Wall shear stress, τ_w.

Solution: Apply momentum equation to cylindrical CV:



Basic equation:

$$F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\rho\, dV + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$$

($F_{B_x} = 0(1)$, $\frac{\partial}{\partial t} = 0(2)$, $\int_{CS} = 0(3)$; axes x, r)

Assumptions: (1) Horizontal pipe
(2) Steady flow
(3) Fully developed flow

Then

$$F_{S_x} = p_1 \frac{\pi D^2}{4} + \tau_w \pi D L - p_2 \frac{\pi D^2}{4} = 0 \quad \text{or} \quad \tau_w = \frac{p_2 - p_1}{4}\frac{D}{L} = -\frac{\Delta p D}{4L}$$

or

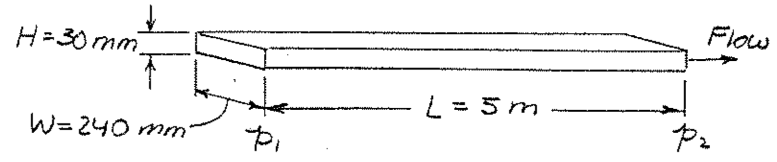
$$\tau_w = -\frac{1}{4} \times 5\ \frac{\text{lbf}}{\text{in.}^2} \times 6\ \text{in.} \times \frac{1}{30\ \text{ft}} \times 12\ \frac{\text{in.}}{\text{ft}} = -3.0\ \text{lbf/ft}^2$$ τ_w

Since $\tau_w < 0$, it acts to left on fluid, to right on pipe wall.

Problem 8.58

Given: Horizontal rectangular channel with fully developed flow of water.

$\Delta p = p_1 - p_2 = 3.0$ kPa



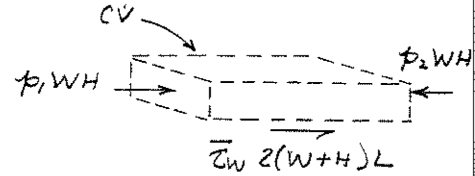
Find: Average wall shear stress, $\bar{\tau}_w$.

<u>Solution</u>: Apply momentum equation to CV inside duct surface $\mapsto x$

Basic equation:

$$F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$$

(with $F_{B_x} = 0(1)$, $\frac{\partial}{\partial t} = 0(2)$, $\int_{CS} = 0(3)$)



Assumptions: (1) Horizontal channel
(2) Steady flow
(3) Fully developed flow

Then

$$F_{S_x} = p_1 WH + \bar{\tau}_w 2(W+H)L - p_2 WH = 0 \quad \text{or} \quad \bar{\tau}_w = (p_2 - p_1)\frac{WH}{2(W+H)L} = -\Delta p \frac{H}{2(1+\frac{H}{W})L}$$

or

$$\bar{\tau}_w = -\frac{3.0\ \text{kPa}}{2} \times 0.03\ \text{m} \times \frac{1}{(1+30/240)} \times \frac{1}{5\ \text{m}} = -8.0\ \text{Pa} \quad (-8.0\ \text{N/m}^2)$$

$\bar{\tau}_w$

Since $\tau_w < 0$, it acts to <u>left</u> on fluid, to <u>right</u> on channel wall.

Problem 8.59

Kerosine is pumped through a smooth tube with inside diameter $D = 30$ mm at close to the critical Reynolds number. The flow is unstable and fluctuates between laminar and turbulent states, causing the pressure gradient to intermittently change from approximately -4.5 kPa/m to -11 kPa/m. Which pressure gradient corresponds to laminar, and which to turbulent, flow? For each flow, compute the shear stress at the tube wall, and sketch the shear stress distributions.

Given: Data on pressure drops in flow in a tube

Find: Which pressure drop is laminar flow, which turbulent

Solution

Given data $\qquad \frac{\partial}{\partial x}p_1 = -4.5\cdot\frac{kPa}{m} \qquad \frac{\partial}{\partial x}p_2 = -11\cdot\frac{kPa}{m} \qquad D = 30\cdot mm$

From Section 8-4, a force balance on a section of fluid leads to

$$\tau_w = -\frac{R}{2}\cdot\frac{\partial}{\partial x}p = -\frac{D}{4}\cdot\frac{\partial}{\partial x}p$$

Hence for the two cases

$$\tau_{w1} = -\frac{D}{4}\cdot\frac{\partial}{\partial x}p_1 \qquad \tau_{w1} = 33.8\,Pa$$

$$\tau_{w2} = -\frac{D}{4}\cdot\frac{\partial}{\partial x}p_2 \qquad \tau_{w2} = 82.5\,Pa$$

Because both flows are at the same nominal flow rate, the higher pressure drop must correspond to the turbulent flow, because, as indicated in Section 8-4, turbulent flows experience additional stresses. Also indicated in Section 8-4 is that for both flows the shear stress varies from zero at the centerline to the maximums computed above at the walls.

Problem 8.60

Given: Liquid with viscosity and density of water in laminar flow in a smooth capillary tube. $D = 0.25$ mm, $L = 50$ mm.

Find: (a) Maximum volume flow rate.
(b) Pressure drop to produce this flow rate.
(c) Corresponding wall shear stress.

Solution: Flow will be laminar for $Re < 2300$.

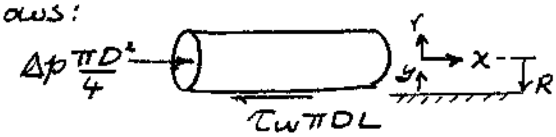
$$Re = \frac{\rho \bar{V} D}{\mu} = \frac{\bar{V} D}{\nu} = \frac{Q}{A}\frac{D}{\nu} = \frac{4Q}{\pi D^2}\frac{D}{\nu} = \frac{4Q}{\pi \nu D} < 2300$$

Thus (at $T = 20°C$)

$$Q < \frac{2300\pi \nu D}{4} = \frac{2300\pi}{4} \times 1.0\times10^{-6}\frac{m^2}{s} \times 0.00025\,m = 4.52\times10^{-7}\,m^3/s$$ Q

(This flow rate corresponds to 27.1 mL/min.)

A force balance on a fluid element shows:



$$\sum F_x = \Delta p \frac{\pi D^2}{4} - \tau_w \pi D L = 0$$

or

$$\Delta p = \tau_w \frac{4L}{D}$$

For laminar pipe flow, $u = u_{max}\left[1 - \left(\frac{r}{R}\right)^2\right]$, from Eq. 8.14. Thus

$$\tau_w = \mu \frac{\partial u}{\partial y}\Big)_{y=0} = -\mu \frac{\partial u}{\partial r}\Big)_{r=R} = -\mu u_{max}\left(-\frac{2r}{R^2}\right)_{r=R} = \frac{2\mu u_{max}}{R}$$

But $u_{max} = 2\bar{V}$, so $\tau_w = \frac{2\mu 2\bar{V}}{D/2} = \frac{8\mu\bar{V}}{D} = 8\rho\frac{\nu\bar{V}}{D}$

Also

$$\bar{V} = \frac{Q}{A} = \frac{4Q}{\pi D^2} = \frac{4}{\pi} \times 4.52\times10^{-7}\frac{m^3}{s} \times \frac{1}{(0.00025)^2 m^2} = 9.21\,m/s$$

Thus

$$\tau_w = 8 \times 999\frac{kg}{m^3} \times 1.0\times10^{-6}\frac{m^2}{s} \times 9.21\frac{m}{s} \times \frac{1}{0.00025\,m} \times \frac{N\cdot s^2}{kg\cdot m} = 294\,N/m^2\ (294\,Pa)$$ τ_w

and

$$\Delta p = 4 \times 0.05\,m \times \frac{1}{0.00025\,m} \times 294\frac{N}{m^2} = 235\,kPa$$ Δp

Problem 8.61 (In Excel)

Laufer [5] measured the following data for mean velocity in fully developed turbulent pipe flow at $Re_U = 50{,}000$:

$\bar{u}/U$	0.996	0.981	0.963	0.937	0.907	0.866	0.831
y/r	0.898	0.794	0.691	0.588	0.486	0.383	0.280
$\bar{u}/U$	0.792	0.742	0.700	0.650	0.619	0.551	
y/R	0.216	0.154	0.093	0.062	0.041	0.024	

In addition, Laufer measured the following data for mean velocity in fully developed turbulent pipe flow at $Re_U = 500{,}000$:

$\bar{u}/U$	0.997	0.988	0.975	0.959	0.934	0.908
y/R	0.898	0.794	0.691	0.588	0.486	0.383
$\bar{u}/U$	0.874	0.847	0.818	0.771	0.736	0.690
y/R	0.280	0.216	0.154	0.093	0.062	0.037

Using *Excel*'s trendline analysis, fit each set of data to the "power-law" profile for turbulent flow, Eq. 8.22, and obtain a value of n for each set. Do the data tend to confirm the validity of Eq. 8.22? Plot the data and their corresponding trendlines on the same graph.

Given: Data on mean velocity in fully developed turbulent flow

Find: Trendlines for each set; values of n for each set; plot

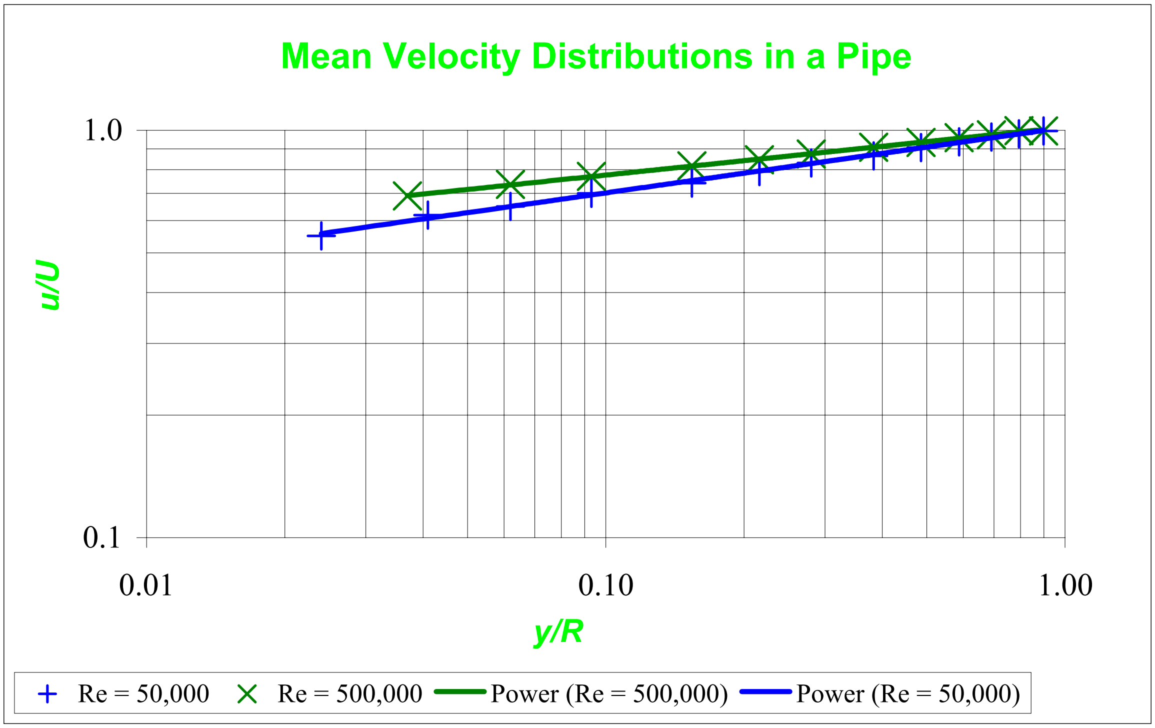
Solution

y/R	*u/U*
0.898	0.996
0.794	0.981
0.691	0.963
0.588	0.937
0.486	0.907
0.383	0.866
0.280	0.831
0.216	0.792
0.154	0.742
0.093	0.700
0.062	0.650
0.041	0.619
0.024	0.551

y/R	*u/U*
0.898	0.997
0.794	0.998
0.691	0.975
0.588	0.959
0.486	0.934
0.383	0.908
0.280	0.874
0.216	0.847
0.154	0.818
0.093	0.771
0.062	0.736
0.037	0.690

Equation 8.22 is

$$\frac{\bar{u}}{U} = \left(\frac{y}{R}\right)^{1/n} = \left(1 - \frac{r}{R}\right)^{1/n}$$



Applying the *Trendline* analysis to each set of data:

At Re = 50,000

$u/U = 1.017(y/R)^{0.161}$
with $R^2 = 0.998$ (high confidence)

Hence $1/n = 0.161$
$n = 6.21$

At Re = 500,000

$u/U = 1.017(y/R)^{0.117}$
with $R^2 = 0.999$ (high confidence)

Hence $1/n = 0.117$
$n = 8.55$

Both sets of data tend to confirm the validity of Eq. 8.22

Problem 8.62

Given: Velocity profiles for pipe flow

$\frac{u}{U} = \left(1 - \frac{r}{R}\right)^{\frac{1}{n}}$ (turbulent); $\frac{u}{U} = 1 - \left(\frac{r}{R}\right)^2$ (laminar)

Find: (a) value of r/R at which $u = \bar{V}$ for each profile.

Plot: r/R vs n for $6 \le n \le 10$

Solution:

Definition: $\bar{V} = \frac{Q}{A} = \frac{1}{A}\int u\,dA$

For laminar flow, $\bar{V} = \frac{1}{\pi R^2}\int_0^R U\left[1 - \left(\frac{r}{R}\right)^2\right]2\pi r\,dr = 2U\int_0^1\left[1-\left(\frac{r}{R}\right)^2\right]\frac{r}{R}\,d\left(\frac{r}{R}\right)$

$$\bar{V} = 2U\left[\frac{1}{2}\left(\frac{r}{R}\right)^2 - \frac{1}{4}\left(\frac{r}{R}\right)^4\right]_0^1 = \frac{U}{2}$$

Thus $u = \bar{V}$ when $1 - \left(\frac{r}{R}\right)^2 = \frac{\bar{V}}{U} = \frac{1}{2}$ or $\frac{r}{R} = 0.707$ ← laminar

For turbulent flow, $\bar{V} = \frac{1}{\pi R^2}\int_0^R U\left(1 - \frac{r}{R}\right)^{\frac{1}{n}} 2\pi r\,dr$

$$\bar{V} = 2U\int_0^1 \left(1 - \frac{r}{R}\right)^{\frac{1}{n}} \frac{r}{R}\,d\left(\frac{r}{R}\right)$$

To integrate let $m = 1 - \frac{r}{R}$. Then $\frac{r}{R} = 1 - m$, $d\left(\frac{r}{R}\right) = -dm$

and
$$\bar{V} = 2U\int_1^0 m^{\frac{1}{n}}(1-m)(-dm) = 2U\int_0^1 \left(m^{\frac{1}{n}} - m^{1+\frac{1}{n}}\right)dm$$

$$= 2U\left[\frac{n}{(n+1)}m^{\frac{1}{n}+1} - \frac{n}{2n+1}m^{2+\frac{1}{n}}\right]_0^1 = 2U\left[\frac{n}{n+1} - \frac{n}{2n+1}\right]$$

$$\bar{V} = 2U\left[\frac{n(2n+1) - n(n+1)}{(n+1)(2n+1)}\right] = U\frac{2n^2}{(n+1)(2n+1)} \quad \text{.........(8.24)}$$

For $n = 7$, $\bar{V} = U\frac{2(7)^2}{8 \times 15} = 0.817U$

Thus $u = \bar{V}$ when $\left(1 - \frac{r}{R}\right)^{1/7} = 0.817$ or $\frac{r}{R} = 1 - (0.817)^7 = 0.758$ ← turb

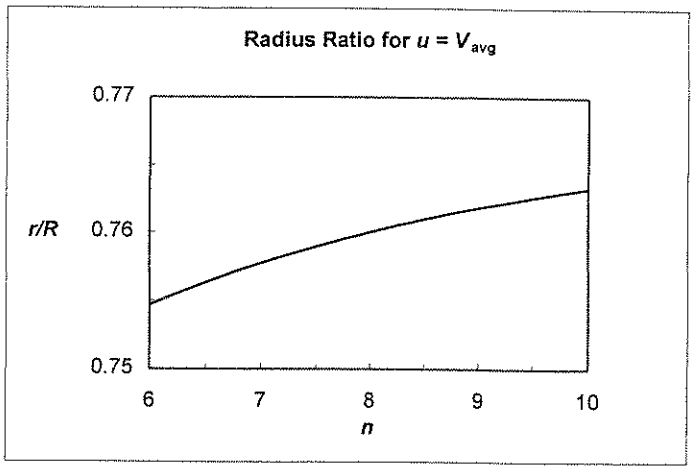
From Eq 8.24, $u = \bar{V}$ when

$$\left(1 - \frac{r}{R}\right)^{\frac{1}{n}} = \frac{2n^2}{(n+1)(2n+1)}$$

or

$$\frac{r}{R} = 1 - \left[\frac{2n^2}{(n+1)(2n+1)}\right]^n$$

r/R is plotted vs n.



Problem 8.63

Given: Power-law exponent n as a function of Re_U and ratio $\bar{V}/U$ as a function of n.

$$n = -1.7 + 1.8 \log Re_U \qquad (8.23)$$

$$\bar{V}/U = \frac{2n^2}{(n+1)(2n+1)} \qquad (8.24)$$

Plot: $\bar{V}/U$ vs $Re_{\bar{V}}$

Solution:

Prepare a Table of values

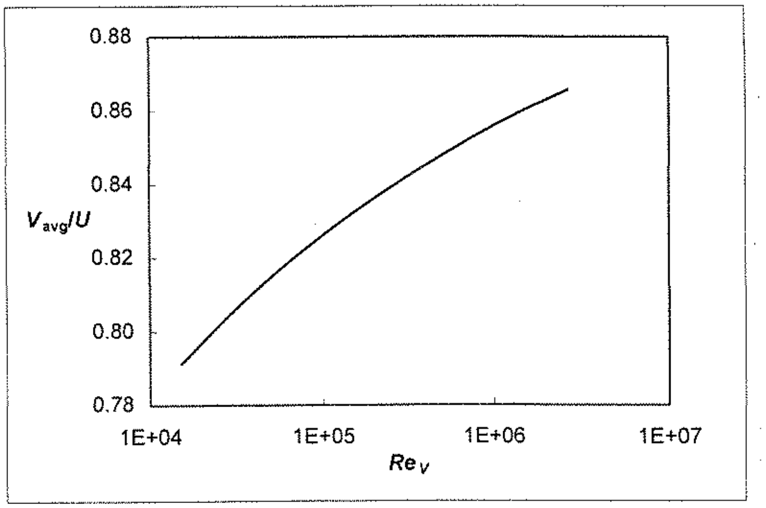
Re_U

n from Eq 8.23

$\bar{V}/U$ from Eq 8.24

$$Re_{\bar{V}} = \frac{\bar{V}}{U} \times Re_U$$

Re_U	n	Re_V	V_{avg}/U
1.90E+04	6.00	1.50E+04	0.791
3.60E+04	6.50	2.90E+04	0.805
6.85E+04	7.00	5.59E+04	0.817
1.29E+05	7.50	1.07E+05	0.827
2.45E+05	8.00	2.05E+05	0.837
4.65E+05	8.50	3.93E+05	0.845
8.80E+05	9.00	7.50E+05	0.853
1.67E+06	9.50	1.44E+06	0.860
3.16E+06	10.0	2.74E+06	0.866



Given: Velocity profiles for pipe flow:

$\frac{u}{U} = 1 - \left(\frac{r}{R}\right)^2$ (laminar) ; $\frac{u}{U} = \left(1 - \frac{r}{R}\right)^{\frac{1}{n}}$ (turbulent)

Momentum coefficient, β, where $\beta \dot{m} \bar{V} = \int_A u \rho u dA$

Find: (a) β for laminar profile.

(b) β for turbulent profile with $n=7$

Plot: β vs n for turbulent profile over range $6 \le n \le 10$, and compare with laminar profile

Solution:

$$\beta = \frac{1}{\dot{m}\bar{V}} \int_A u \rho u dA = \frac{1}{\rho \bar{V} \pi R^2 \bar{V}} \int_0^R u \rho u 2\pi r dr$$

Noting that $\frac{u}{U} = f(r/R)$

$$\beta = \frac{1}{\rho \pi R^2} \left[\frac{U}{\bar{V}}\right]^2 \int_0^R \left(\frac{u}{U}\right)^2 2\rho\pi r dr = 2\left[\frac{U}{\bar{V}}\right]^2 \int_0^1 \left(\frac{u}{U}\right)^2 \left(\frac{r}{R}\right) d\left(\frac{r}{R}\right) \quad \text{--- } \beta$$

For laminar flow, $\frac{u}{U} = 1 - \left(\frac{r}{R}\right)^2$, so $\left(\frac{u}{U}\right)^2 = 1 - 2\left(\frac{r}{R}\right)^2 + \left(\frac{r}{R}\right)^4$, and

$$\beta = 2\left[\frac{U}{\bar{V}}\right]^2 \int_0^1 \left[\left(\frac{r}{R}\right) - 2\left(\frac{r}{R}\right)^3 + \left(\frac{r}{R}\right)^5\right] d\left(\frac{r}{R}\right) = 2\left[\frac{U}{\bar{V}}\right]^2 \left[\frac{1}{2} - \frac{1}{2} + \frac{1}{6}\right]$$

$\beta = \frac{1}{3}\left[\frac{U}{\bar{V}}\right]^2$. For this case $U = 2\bar{V}$ so

$$\beta = \frac{1}{3}[2]^2 = \frac{4}{3} \quad \longleftarrow \quad \beta_{laminar}$$

For turbulent flow, $\frac{u}{U} = \left(1 - \frac{r}{R}\right)^{\frac{1}{n}}$, so $\left(\frac{u}{U}\right)^2 = \left(1 - \frac{r}{R}\right)^{\frac{2}{n}}$, and

$$\beta = 2\left[\frac{U}{\bar{V}}\right]^2 \int_0^1 \left(1 - \frac{r}{R}\right)^{\frac{2}{n}} \left(\frac{r}{R}\right) d\left(\frac{r}{R}\right)$$

To integrate, let $m = 1 - \frac{r}{R}$. Then $\frac{r}{R} = 1 - m$, $d\left(\frac{r}{R}\right) = -dm$, so

$$\beta = 2\left[\frac{U}{\bar{V}}\right]^2 \int_1^0 m^{\frac{2}{n}} (1-m)(-dm) = 2\left[\frac{U}{\bar{V}}\right]^2 \int_0^1 \left(m^{\frac{2}{n}} - m^{1+\frac{2}{n}}\right) dm$$

$$\beta = 2\left[\frac{U}{\bar{V}}\right]^2 \left[\frac{n}{(n+2)} m^{\frac{2}{n}+1} - \frac{n}{(2n+2)} m^{2+\frac{2}{n}}\right]_0^1 = 2\left[\frac{U}{\bar{V}}\right]^2 \left[\frac{n}{(n+2)} - \frac{n}{(2n+2)}\right]$$

$$\beta = 2\left[\frac{U}{\bar{V}}\right]^2 \left[\frac{(2n+2)n - (n+2)n}{(n+2)(2n+2)}\right] = 2\left[\frac{U}{\bar{V}}\right]^2 \left[\frac{n^2}{(n+2)(2n+2)}\right] \quad \text{--- (1)}$$

From Eq. 8.24, $\frac{\bar{V}}{U} = \frac{2n^2}{(n+1)(2n+1)}$

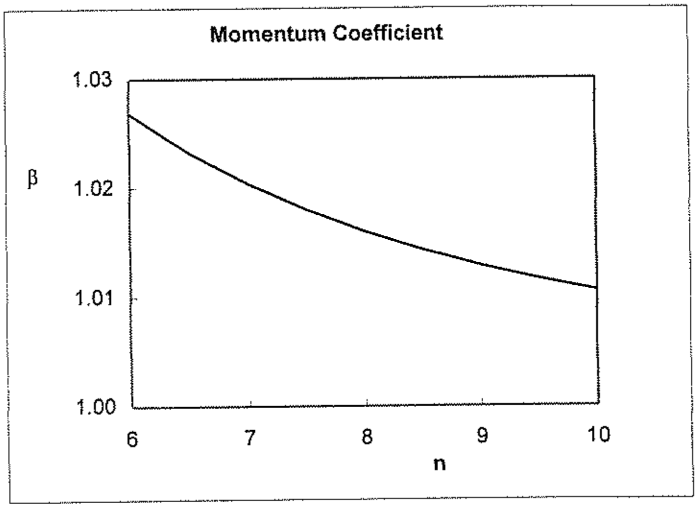
For $n = 7$, $\frac{\bar{V}}{U} = 0.817$, so

$$\beta = \left[\frac{1}{0.817}\right]^2 \frac{2(7)^2}{(9)(16)} = 1.02 \quad \longleftarrow \quad \beta_{turb}$$

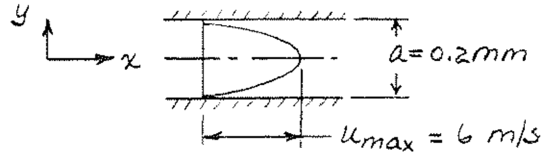
To plot β vs n

- $\frac{\bar{V}}{U} = \frac{2n^2}{(n+1)(2n+1)}$ .... (8.24)
- $\beta = \left[\frac{U}{\bar{V}}\right]^2 \frac{n^2}{(n+2)(n+1)}$
- $\beta = \frac{(n+1)(2n+1)^2}{4n^2(n+2)}$

n	β
6.0	1.027
6.5	1.023
7.0	1.020
7.5	1.018
8.0	1.016
8.5	1.014
9.0	1.013
9.5	1.012
10.0	1.011



Given: Fully developed, laminar flow of water between parallel plates.



$w = 30$ mm

Find: Kinetic energy coefficient, α

Solution: Apply definition of kinetic energy coefficient,

$$\alpha = \frac{\int_A \rho V^3 dA}{\dot{m}\bar{V}^2} \qquad \dot{m} = \rho \bar{V} A \qquad (8.26b)$$

From the analysis of Section 8-2, for flow between parallel plates,

$$u = u_{max}\left[1-\left(\frac{y}{a/2}\right)^2\right] = \frac{3}{2}\bar{V}\left[1-\left(\frac{y}{a/2}\right)^2\right] \quad \text{since } u_{max} = \frac{3}{2}\bar{V} \qquad (8.6c)$$

Substituting into Eq. 8.26b,

$$\alpha = \frac{\int_A \rho V^3 dA}{\dot{m}\bar{V}^2} = \frac{\int_A \rho u^3 dA}{\rho \bar{V} A \bar{V}^2} = \frac{1}{A}\int_A \left(\frac{u}{\bar{V}}\right)^3 dA = \frac{1}{wa}\int_{-a/2}^{a/2}\left(\frac{u}{\bar{V}}\right)^3 w\,dy = \frac{2}{a}\int_0^{a/2}\left(\frac{u}{\bar{V}}\right)^3 dy$$

Then

$$\alpha = \frac{2}{a}\frac{a}{2}\int_0^1\left(\frac{u}{u_{max}}\right)^3\left(\frac{u_{max}}{\bar{V}}\right)^3 d\left(\frac{y}{a/2}\right) = \left(\frac{3}{2}\right)^3\int_0^1 (1-\eta^2)^3 d\eta \quad \text{where } \eta = \frac{y}{a/2}.$$

Evaluating,

$$(1-\eta^2)^3 = 1 - 3\eta^2 + 3\eta^4 - \eta^6$$

The integral is

$$\int_0^1 (1-\eta^2)^3 d\eta = \left[\eta - \frac{3}{3}\eta^3 + \frac{3}{5}\eta^5 - \frac{1}{7}\eta^7\right]_0^1 = \frac{3}{5} - \frac{1}{7} = \frac{21-5}{35} = \frac{16}{35}$$

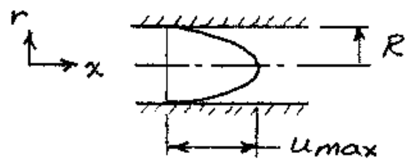
Substituting,

$$\alpha = \left(\frac{3}{2}\right)^3\int_0^1 (1-\eta^2)^3 d\eta = \frac{27}{8}\frac{16}{35} = \frac{54}{35} = 1.54$$

α

Problem 8.66

Given: Fully developed laminar flow in a circular tube.



Find: Kinetic energy coefficient, α.

Solution: Apply definition of kinetic energy coefficient,

$$\alpha = \frac{\int_A \rho V^3 dA}{\dot{m} \bar{V}^2}, \quad \dot{m} = \rho \bar{V} A \qquad (8.26b)$$

From the analysis of Section 8-3, for flow in a circular tube,

$$u = u_{max}\left[1-\left(\frac{r}{R}\right)^2\right] = 2\bar{V}\left[1-\left(\frac{r}{R}\right)^2\right] \quad \text{since} \quad u_{max} = 2\bar{V}$$

Substituting into Eq. 8.25b,

$$\alpha = \frac{\int_A \rho V^3 dA}{\dot{m}\bar{V}^2} = \frac{\int_A \rho u^3 dA}{\rho \bar{V} A \bar{V}^2} = \frac{1}{A}\int_A \left(\frac{u}{\bar{V}}\right)^3 dA = \frac{1}{\pi R^2}\int_0^R \left(\frac{u}{\bar{V}}\right)^3 2\pi r\,dr = 2\int_0^1 \left(\frac{u}{\bar{V}}\right)^3 \left(\frac{r}{R}\right) d\left(\frac{r}{R}\right)$$

Then

$$\alpha = 2\int_0^1 \left(\frac{u}{u_{max}}\right)^3 \left(\frac{u_{max}}{\bar{V}}\right)^3 \left(\frac{r}{R}\right) d\left(\frac{r}{R}\right) = 2(2)^3 \int_0^1 (1-\eta^2)^3 \eta\, d\eta \quad \text{where } \eta = \frac{r}{R}$$

Evaluating,

$$(1-\eta^2)^3 \eta = \eta - 3\eta^3 + 3\eta^5 - \eta^7$$

The integral is

$$\int_0^1 (1-\eta^2)^3 \eta\, d\eta = \left[\frac{\eta^2}{2} - \frac{3}{4}\eta^4 + \frac{3}{6}\eta^6 - \frac{1}{8}\eta^8\right]_0^1 = \frac{1}{2} - \frac{3}{4} + \frac{1}{2} - \frac{1}{8} = \frac{1}{8}$$

Substituting,

$$\alpha = 16\int_0^1 (1-\eta^2)^3 \eta\, d\eta = 16 \times \frac{1}{8} = 2$$

α

Problem 8.67

Show that the kinetic energy coefficient, α, for the "power law" turbulent velocity profile of Eq. 8.22 is given by Eq. 8.27. Plot α as a function of $Re_{\bar{V}}$, for $Re_{\bar{V}} = 1 \times 10^4$ to 1×10^7. When analyzing pipe flow problems it is common practice to assume $\alpha \approx 1$. Plot the error associated with this assumption as a function of $Re_{\bar{V}}$, for $Re_{\bar{V}} = 1 \times 10^4$ to 1×10^7.

Given: Definition of kinetic energy correction coefficient α

Find: α for the power-law velocity profile; plot

Solution

Equation 8.26b is

$$\alpha = \frac{\int \rho \cdot V^3\, dA}{m_{rate} \cdot V_{av}^2}$$

where V is the velocity, m_{rate} is the mass flow rate and V_{av} is the average velocity

For the power-law profile (Eq. 8.22)

$$V = U \cdot \left(1 - \frac{r}{R}\right)^{\frac{1}{n}}$$

For the mass flow rate $\quad m_{rate} = \rho \cdot \pi \cdot R^2 \cdot V_{av}$

Hence the denominator of Eq. 8.26b is

$$m_{rate} \cdot V_{av}^2 = \rho \cdot \pi \cdot R^2 \cdot V_{av}^3$$

We next must evaluate the numerator of Eq. 8.26b

$$\int \rho \cdot V^3\, dA = \int \rho \cdot 2 \cdot \pi \cdot r \cdot U^3 \cdot \left(1 - \frac{r}{R}\right)^{\frac{3}{n}} dr$$

$$\int_0^R \rho \cdot 2 \cdot \pi \cdot r \cdot U^3 \cdot \left(1 - \frac{r}{R}\right)^{\frac{3}{n}} dr = \frac{2 \cdot \pi \cdot \rho \cdot R^2 \cdot n^2 \cdot U^3}{(3 + n) \cdot (3 + 2 \cdot n)}$$

To integrate substitute $\quad m = 1 - \frac{r}{R} \qquad dm = -\frac{dr}{R}$

Then $\quad r = R \cdot (1 - m) \qquad dr = -R \cdot dm$

$$\int_0^R \rho \cdot 2 \cdot \pi \cdot r \cdot U^3 \cdot \left(1 - \frac{r}{R}\right)^{\frac{3}{n}} dr = -\int_1^0 \rho \cdot 2 \cdot \pi \cdot R \cdot (1 - m) \cdot m^{\frac{3}{n}} \cdot R\, dm$$

Hence $\quad \int \rho \cdot V^3\, dA = \int_0^1 \rho \cdot 2 \cdot \pi \cdot R \cdot \left(m^{\frac{3}{n}} - m^{\frac{3}{n}+1}\right) \cdot R\, dm$

$$\int \rho \cdot V^3\, dA = \frac{2 \cdot R^2 \cdot n^2 \cdot \rho \cdot \pi \cdot U^3}{(3 + n) \cdot (3 + 2 \cdot n)}$$

Putting all these results togethe $\alpha = \dfrac{\int \rho \cdot V^3\, dA}{m_{rate} \cdot V_{av}^{\ 2}} = \dfrac{\dfrac{2 \cdot R^2 \cdot n^2 \cdot \rho \cdot \pi \cdot U^3}{(3+n) \cdot (3+2 \cdot n)}}{\rho \cdot \pi \cdot R^2 \cdot V_{av}^{\ 3}}$

$$\alpha = \left(\frac{U}{V_{av}}\right)^3 \cdot \frac{2 \cdot n^2}{(3 + n) \cdot (3 + 2 \cdot n)}$$

To plot α versus Re_{Vav} we use the following parametric relations

$$n = -1.7 + 1.8 \cdot \log\left(Re_u\right) \qquad \text{(Eq. 8.23)}$$

$$\frac{V_{av}}{U} = \frac{2 \cdot n^2}{(n + 1) \cdot (2 \cdot n + 1)} \qquad \text{(Eq. 8.24)}$$

$$Re_{Vav} = \frac{V_{av}}{U} \cdot Re_U$$

$$\alpha = \left(\frac{U}{V_{av}}\right)^3 \cdot \frac{2 \cdot n^2}{(3 + n) \cdot (3 + 2 \cdot n)} \qquad \text{(Eq. 8.27)}$$

A value of Re_U leads to a value for n; this leads to a value for V_{av}/U;
these lead to a value for Re_{Vav} and α

The plots of α, and the error in assuming $\alpha = 1$, versus Re_{Vav} are shown in the associated *Excel* workbook

Problem 8.67 (In Excel)

Show that the kinetic energy coefficient, α, for the "power law" turbulent velocity profile of Eq. 8.22 is given by Eq. 8.27. Plot α as a function of $Re_{\bar{V}}$, for $Re_{\bar{V}} = 1 \times 10^4$ to 1×10^7. When analyzing pipe flow problems it is common practice to assume $\alpha \approx 1$. Plot the error associated with this assumption as a function of $Re_{\bar{V}}$, for $Re_{\bar{V}} = 1 \times 10^4$ to 1×10^7.

Given: Definition of kinetic energy correction coefficient α

Find: α for the power-law velocity profile; plot

Solution

$$n = -1.7 + 1.8 \cdot \log(Re_u) \qquad \text{(Eq. 8.23)}$$

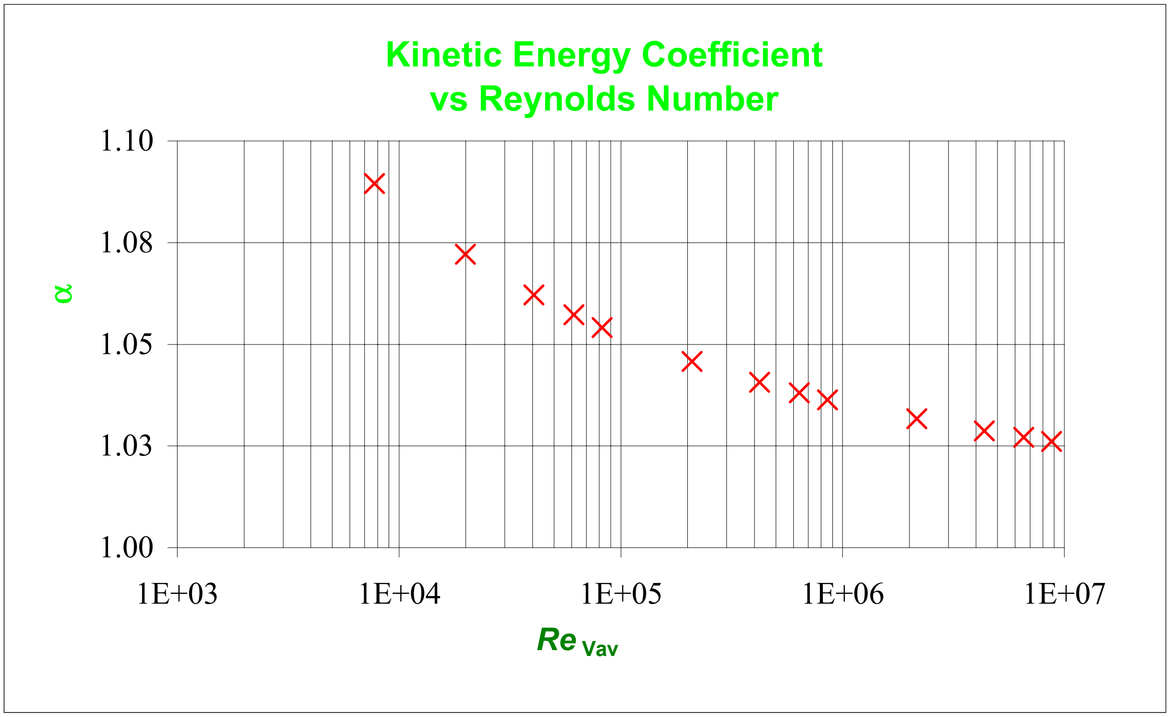
$$\frac{V_{av}}{U} = \frac{2 \cdot n^2}{(n+1) \cdot (2 \cdot n + 1)} \qquad \text{(Eq. 8.24)}$$

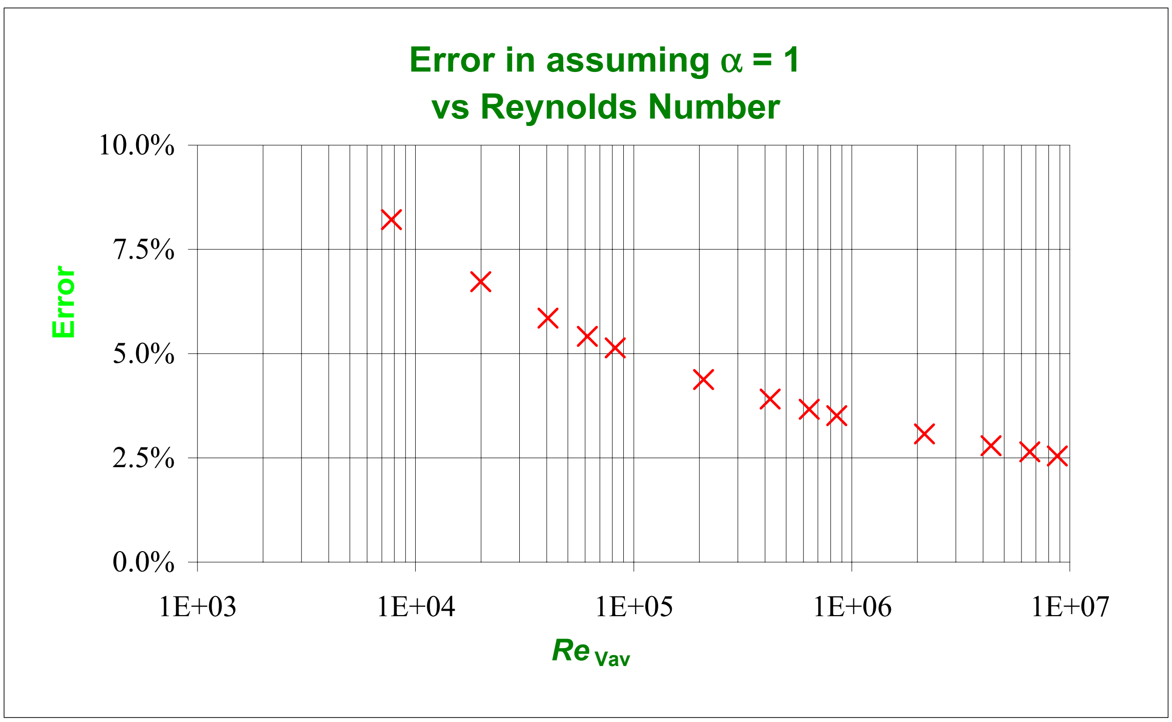
$$Re_{Vav} = \frac{V_{av}}{U} \cdot Re_U$$

$$\alpha = \left(\frac{U}{V_{av}}\right)^3 \cdot \frac{2 \cdot n^2}{(3+n) \cdot (3 + 2 \cdot n)} \qquad \text{(Eq. 8.27)}$$

A value of Re_U leads to a value for n;
this leads to a value for V_{av}/U;
these lead to a value for Re_{Vav} and α

Re_U	n	V_{av}/U	Re_{Vav}	α	α Error
1.00E+04	5.50	0.776	7.76E+03	1.09	8.2%
2.50E+04	6.22	0.797	1.99E+04	1.07	6.7%
5.00E+04	6.76	0.811	4.06E+04	1.06	5.9%
7.50E+04	7.08	0.818	6.14E+04	1.06	5.4%
1.00E+05	7.30	0.823	8.23E+04	1.05	5.1%
2.50E+05	8.02	0.837	2.09E+05	1.05	4.4%
5.00E+05	8.56	0.846	4.23E+05	1.04	3.9%
7.50E+05	8.88	0.851	6.38E+05	1.04	3.7%
1.00E+06	9.10	0.854	8.54E+05	1.04	3.5%
2.50E+06	9.82	0.864	2.16E+06	1.03	3.1%
5.00E+06	10.4	0.870	4.35E+06	1.03	2.8%
7.50E+06	10.7	0.873	6.55E+06	1.03	2.6%
1.00E+07	10.9	0.876	8.76E+06	1.03	2.5%


Kinetic Energy Coefficient
vs Reynolds Number
1.10
1.08
1.05
1.03
1.00
α
1E+03
1E+04
1E+05
1E+06
1E+07
Re Vav


Error in assuming α = 1
vs Reynolds Number
10.0%
7.5%
5.0%
2.5%
0.0%
Error
1E+03
1E+04
1E+05
1E+06
1E+07
Re Vav

Problem 8.68

Water flows in a horizontal constant-area pipe; the pipe diameter is 50 mm and the average flow speed is 1.5 m/s. At the pipe inlet the gage pressure is 588 kPa, and the outlet is at atmospheric pressure. Determine the head loss in the pipe. If the pipe is now aligned so that the outlet is 25 m above the inlet, what will the inlet pressure need to be to maintain the same flow rate? If the pipe is now aligned so that the outlet is 25 m below the inlet, what will the inlet pressure need to be to maintain the same flow rate? Finally, how much lower than the inlet must the outlet be so that the same flow rate is maintained if both ends of the pipe are at atmospheric pressure (i.e., gravity feed)?

Given: Data on flow in a pipe

Find: Head loss for horizontal pipe; inlet pressure for different alignments; slope for gravity feed

Solution

Given or available data $D = 50 \cdot mm$ $\rho = 999 \cdot \frac{kg}{m^3}$

The governing equation between inlet (1) and exit (2) is

$$\left(\frac{p_1}{\rho} + \alpha_1 \cdot \frac{V_1^2}{2} + g \cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \cdot \frac{V_2^2}{2} + g \cdot z_2\right) = h_{lT} \qquad (8.29)$$

Horizontal pipe data $p_1 = 588 \cdot kPa$ $p_2 = 0 \cdot kPa$ (Gage pressures)

$z_1 = z_2$ $V_1 = V_2$

Equation 8.29 becomes $h_{lT} = \frac{p_1 - p_2}{\rho}$ $h_{lT} = 589 \frac{J}{kg}$

For an inclined pipe with the same flow rate, the head loss will be the same as above; in additior we have the following new data

$z_1 = 0\cdot m$ $z_2 = 25\cdot m$

Equation 8.29 becomes $p_1 = p_2 + \rho\cdot g\cdot(z_2 - z_1) + \rho\cdot h_{lT}$ $p_1 = 833\,kPa$

For an declined pipe with the same flow rate, the head loss will be the same as above; in addition we have the following new data

$z_1 = 0\cdot m$ $z_2 = -25\cdot m$

Equation 8.29 becomes $p_1 = p_2 + \rho\cdot g\cdot(z_2 - z_1) + \rho\cdot h_{lT}$ $p_1 = 343\,kPa$

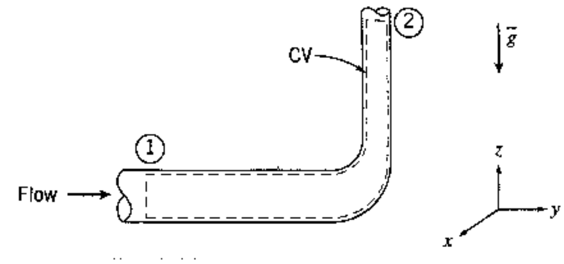
For a gravity feed with the same flow rate, the head loss will be the same as above; in addition we have the following new data

$p_1 = 0\cdot kPa$ (Gage)

Equation 8.29 becomes $z_2 = z_1 - \frac{h_{lT}}{g}$ $z_2 = -60\,m$

Problem 8.69

Given: Flow in configuration shown:



Section	p (psig)	$\bar{V}$ (ft/s)	z (ft)
1	10.2	5.5	7.5
2	6.5	11.2	10.5

Find: (a) Head loss (in ft)
(b) Head loss in energy per unit mass.

Solution: Apply the energy equation for pipe flow, Eq. 8.30:

Computing Equation: $$\left(\frac{p_1}{\rho g} + \alpha_1 \frac{\bar{V}_1^2}{2g} + z_1\right) - \left(\frac{p_2}{\rho g} + \alpha_2 \frac{\bar{V}_2^2}{2g} + z_2\right) = \frac{h_{lT}}{g} = H_{lT}$$

Assumptions: (1) Steady flow
(2) Incompressible flow (water)
(3) α_1 and α_2 approximately 1

Then

$$H_{lT} = \frac{p_1 - p_2}{\rho g} + \frac{\bar{V}_1^2 - \bar{V}_2^2}{2g} + z_1 - z_2$$

$$= (10.2 - 6.5)\frac{\text{lbf}}{\text{in.}^2} \times \frac{\text{ft}^3}{1.94\,\text{slug}} \times \frac{\text{s}^2}{32.2\,\text{ft}} \times 144\frac{\text{in.}^2}{\text{ft}^2} \times \frac{\text{slug}\cdot\text{ft}}{\text{lbf}\cdot\text{s}^2}$$

$$+ \frac{(5.5)^2 - (11.2)^2}{2}\frac{\text{ft}^2}{\text{s}^2} \times \frac{\text{s}^2}{32.2\,\text{ft}} + (7.5 - 10.5)\,\text{ft}$$

$$H_{lT} = 4.05\ \text{ft}$$ H_{lT}

and

$$h_{lT} = g\,H_{lT} = 32.2\frac{\text{ft}}{\text{s}^2} \times 4.05\,\text{ft} = 130\ \text{ft}^2/\text{s}^2$$

or

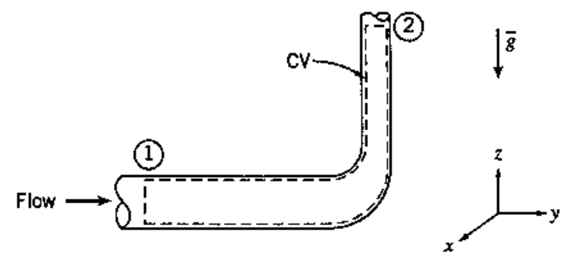
$$h_{lT} = 130\frac{\text{ft}^2}{\text{s}^2} \times \frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}} = 130\ \frac{\text{lbf}\cdot\text{ft}}{\text{slug}}$$ h_{lT}

Problem 8.70

Given: Flow through 90° reducing elbow.

$H_{\ell_T} = 1.7$ ft, $p_1 - p_2 = 3.7$ psi

$\bar{V}_2 = 1.75\,\bar{V}_1$, $z_2 - z_1 = 5.5$ ft.



Find: inlet velocity, $\bar{V}_1$

Solution:

Computing equation: $\left(\frac{p_1}{\rho g} + \alpha_1 \frac{\bar{V}_1^2}{2g} + z_1\right) - \left(\frac{p_2}{\rho g} + \alpha_2 \frac{\bar{V}_2^2}{2g} + z_2\right) = H_{\ell_T}$ (8.30)

Assumption: (1) $\alpha_1 = \alpha_2 = 1.0$, (2) fluid is water, $\rho = 1.94$ slug/ft³

$$\bar{V}_2^2 - \bar{V}_1^2 = \frac{2(p_1 - p_2)}{\rho} + 2g(z_1 - z_2) - gH_{\ell_T}$$

$$(1.75\bar{V}_1)^2 - \bar{V}_1^2 = 2 \times 3.7\frac{\text{lbf}}{\text{in}^2} \times 144\frac{\text{in}^2}{\text{ft}^2} \times \frac{\text{ft}^3}{1.94\text{ slug}} \times \frac{\text{slug}\cdot\text{ft}}{\text{lbf}\cdot\text{s}^2} + 2 \times 32.2\frac{\text{ft}}{\text{s}^2} \times (-5.5\text{ ft})$$

$$- 32.2\frac{\text{ft}}{\text{s}^2} \times 1.7\text{ ft}$$

$$2.063\,\bar{V}_1^2 = 140\ \text{ft}^2/\text{s}^2$$

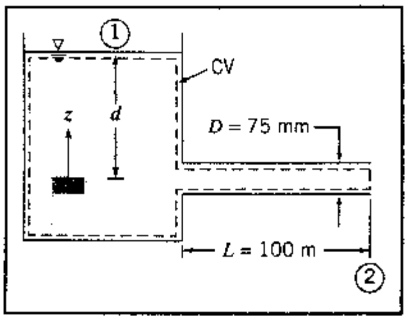
$$\bar{V}_1 = 8.25\ \text{ft/s}$$

$\bar{V}_1$

Problem 8.71

Given: Water flow from a reservoir through system shown

When $Q = 0.0067\ m^3/s$, $H_{l_T} = 2.85\ m$



Find: reservoir depth, d, to maintain this flow rate.

Solution:

Computing equation: $\left(\frac{p_1}{\rho g} + \alpha_1 \frac{\bar{V}_1^2}{2g} + z_1\right) - \left(\frac{p_2}{\rho g} + \alpha_2 \frac{\bar{V}_2^2}{2g} + z_2\right) = H_{l_T}$ (8.30)

Assumptions: (1) steady, incompressible flow

(2) $\bar{V}_1 = 0$, $\alpha_2 \approx 1.0$

(3) $p_1 = p_2 = p_{atm}$

Then,

$$z_1 - z_2 = d = H_{l_T} + \frac{\bar{V}_2^2}{2g}$$

$$\bar{V}_2 = \frac{Q}{A_2} = \frac{4Q}{\pi D^2} = \frac{4}{\pi} \times 0.0067\ \frac{m^3}{s} \times \frac{1}{(0.075\ m)^2} = 1.52\ m/s$$

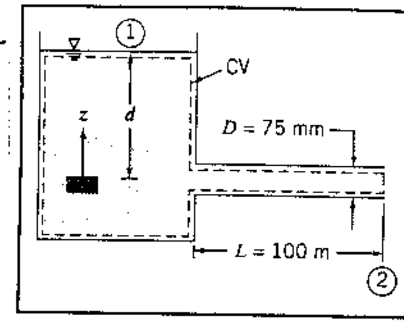
and

$$d = 2.85\ m + \frac{(1.52)^2}{2}\ \frac{m^2}{s^2} \times \frac{s^2}{9.81\ m} = 2.97\ m \qquad d$$

Problem 8.72

Given: Water flow from a reservoir through system shown.

When $d = 3.60\ m$, $H_{l_T} = 1.75\ m$



Find: Volume flow rate, Q

Solution:

Computing equation: $$\left(\frac{p_1}{\rho g} + \alpha_1 \frac{\bar{V}_1^2}{2g} + z_1\right) - \left(\frac{p_2}{\rho g} + \alpha_2 \frac{\bar{V}_2^2}{2g} + z_2\right) = H_{l_T} \qquad (8.30)$$

(with $\frac{p_1}{\rho g}$, $\alpha_1 \frac{\bar{V}_1^2}{2g}$ and $\frac{p_2}{\rho g}$ cancelled; $\bar{V}_1^2 \approx 0$ (2))

Assumptions: (1) steady, incompressible flow
(2) $\bar{V}_1 \approx 0$, $\alpha_2 = 1.0$
(3) $p_1 = p_2 = p_{atm}$

Then,

$$\bar{V}_2^2 = 2g\left[(z_1 - z_2) - H_{l_T}\right]$$

$$\bar{V}_2 = 2 \times 9.81 \frac{m}{s^2}\left[3.60 - 1.75\right] m$$

$$\bar{V}_2 = 6.03\ m/s$$

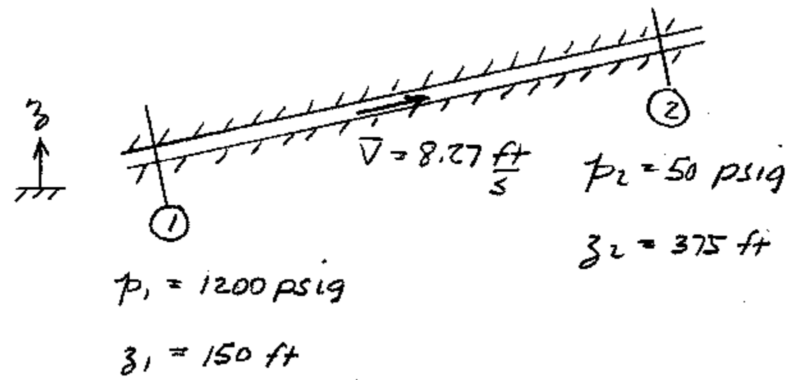
$$Q = A_2 \bar{V}_2 = \frac{\pi D^2}{4} \bar{V} = \frac{\pi}{4} \times (0.075\,m)^2 \times 6.03 \frac{m}{s} = 2.66 \times 10^{-2}\ m^3/s \quad Q$$

Problem 8.73

Given: section of Alaskan pipeline with conditions shown.

Find: Head loss.

Solution:



Computing equation:

$$h_{lT} = \left(\frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2} + g z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \frac{\bar{V}_2^2}{2} + g z_2\right)$$

(the kinetic energy terms are struck out, marked (1))

Assumptions: (1) Incompressible flow, so $\bar{V}_1 = \bar{V}_2$ (2) Fully developed so $\alpha_1 = \alpha_2$

(3) $SG = 0.9$ (Table A.2)

Then $h_{lT} = \frac{p_1 - p_2}{SG\,\rho_{H_2O}} + g(z_1 - z_2)$

$$h_{lT} = (1200 - 50)\ \frac{lbf}{in.^2} \times 144\ \frac{in.^2}{ft^2} \times \frac{ft^3}{(0.9)\ 1.94\ slug} \times \frac{slug \cdot ft}{lbf \cdot s^2} + 32.2\ \frac{ft}{s^2}\ (150 - 375)\ ft$$

$h_{lT} = 8.76 \times 10^4\ ft^2/s^2$ ← h_{lT}

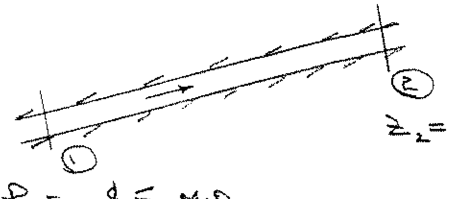
Also

$H_{lT} = \frac{h_{lT}}{g} = 8.76 \times 10^4\ \frac{ft^2}{s^2} \times \frac{s^2}{32.2\ ft} = 2{,}720\ ft$ ← H_{lT}

Given: Section of Alaskan pipeline with conditions shown

$h_{l_{1\to 2}} = 6.9$ kJ/kg

Find: outlet pressure, p_2



$z_2 = 115$ m

$p_1 = 8.5$ MPa

$z_1 = 45$ m

Solution:

Computing equation: $\left(\frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2} + g z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \frac{\bar{V}_2^2}{2} + g z_2\right) = h_{l_T}$ (8.29)

Assumptions: (1) incompressible flow, so $\bar{V}_1 = \bar{V}_2$
(2) fully developed so $\alpha_1 = \alpha_2$
(3) SG crude oil = 0.90 (Table A.2)

Then

$$p_2 = p_1 + \rho g (z_1 - z_2) - \rho h_{l_T}$$

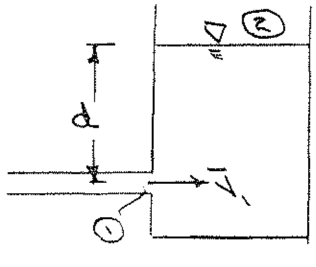
$$= 8.5 \times 10^6 \, \text{N/m}^2 + 0.9 \times 999 \, \frac{\text{kg}}{\text{m}^3} \times 9.81 \, \frac{\text{m}}{\text{s}^2} \times (-70 \, \text{m}) \times \frac{\text{N} \cdot \text{s}^2}{\text{kg} \cdot \text{m}}$$

$$- 0.9 \times 999 \, \frac{\text{kg}}{\text{m}^3} \times 6.9 \times 10^3 \, \frac{\text{N} \cdot \text{m}}{\text{kg}}$$

$p_2 = 1.68$ MPa ← p_2

Given: Water flows from a horizontal tube into a very large tank as shown.

$d = 2.5\,m$, $h_e = 2\,J/kg$



Find: Average flow speed in tube

Solution:

Apply definition of head loss, Eq 8.29,

$$\left(\frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2} + g z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \frac{\bar{V}_2^2}{2} + g z_2\right) = h_{l_T}$$

At free surface, $V_2 = 0$, $p_2 = p_{atm}$

At tube discharge $p_1 = \rho g d$, $z_1 = 0$. Assume $\alpha_1 \approx 1$

Then

$$gd + \frac{\bar{V}_1^2}{2} - gd = h_{l_T}$$

$$\bar{V}_1^2 = 2 h_{l_T} = 2 \times 2 \frac{N \cdot m}{kg} \times \frac{kg \cdot m}{N \cdot s^2} = 4\ m^2/s^2$$

$$\bar{V}_1 = 2\ m/s$$

$\bar{V}_1$

Problem 8.76

Given: Water flow at $Q = 3$ gpm through a horizontal 5/8 in. diameter garden hose. Pressure drop in $L = 50$ ft is 12.3 psi.

Find: Head loss

Solution: Computing equation is

$$h_{l_T} = \left(\frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2} + g z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \frac{\bar{V}_2^2}{2} + g z_2\right)$$

(terms in $\bar{V}^2$ cancelled by (1), terms in gz cancelled by (3))

Assumptions: (1) Incompressible flow, so $\bar{V}_1 = \bar{V}_2$

(2) Fully developed so $\alpha_1 = \alpha_2$

(3) Horizontal, so $z_1 = z_2$

Then $h_{l_T} = \frac{p_1 - p_2}{\rho} = 12.3 \frac{lbf}{in.^2} \times \frac{ft^3}{1.94\ slug} \times 144 \frac{in.^2}{ft^2} \times \frac{slug \cdot ft}{lbf \cdot s^2}$

$h_{l_T} = 913\ ft^2/s^2$ ← h_{l_T}

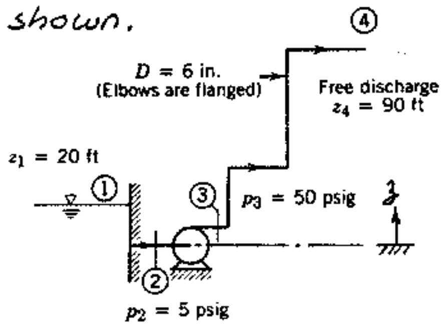
Also

$H_{l_T} = \frac{h_{l_T}}{g} = 913 \frac{ft^2}{s^2} \times \frac{s^2}{32.2\ ft} = 28.4\ ft$ ← H_{l_T}

Problem 8.77

Given: Water pumped through flow system shown.

$Q = 2\ ft^3/s$



Find: (a) Head supplied by pump.
(b) Head loss between pump outlet and free discharge.

Solution: Apply energy equation to CV around pump for steady flow:

Computing equation:

$$\dot{W}_{in} = \dot{m}\left[\left(\frac{p_3}{\rho} + \alpha_3\frac{\bar{V}_3^2}{2} + gz_3\right) - \left(\frac{p_2}{\rho} + \alpha_2\frac{\bar{V}_2^2}{2} + gz_2\right)\right]$$

Assumptions: (1) Incompressible flow
(2) $\alpha_2 \bar{V}_2^2 = \alpha_3 \bar{V}_3^2 = \alpha_4 \bar{V}_4^2$
(3) $z_2 = z_3$

Head is energy per unit mass (or per unit weight). On a unit mass basis,

$$\Delta h_{pump} = \frac{\dot{W}_{in}}{\dot{m}} = \frac{1}{\rho}(p_3 - p_2) = (50-5)\frac{lbf}{in.^2} \times \frac{ft^3}{1.94\ slug} \times 144\frac{in.^2}{ft^2} = 3{,}340\ ft\cdot lbf/slug$$

Δh_{pump}

Apply energy equation for steady, incompressible pipe flow between ③, ④:

Computing equation:

$$\left(\frac{p_3}{\rho} + \alpha_3\frac{\bar{V}_3^2}{2} + gz_3\right) - \left(\frac{p_4}{\rho} + \alpha_4\frac{\bar{V}_4^2}{2} + gz_4\right) = h_{lT} \quad (8.29)$$

Assumptions: (4) $p_4 = p_{atm}$
(5) $\alpha_3 \bar{V}_3^2 = \alpha_4 \bar{V}_4^2$

Then

$$h_{lT} = \frac{p_3}{\rho} - gz_4 = 50\frac{lbf}{in.^2} \times \frac{ft^3}{1.94\ slug} \times 144\frac{in.^2}{ft^2} - 32.2\frac{ft}{s^2} \times 90\ ft \times \frac{lbf \cdot s^2}{slug \cdot ft}$$

$$h_{lT} = 813\ ft \cdot lbf/slug$$

h_{lT}

On a per unit weight basis,

$$\Delta H = \frac{\dot{W}_{in}}{\dot{m}g} = \frac{p_3 - p_2}{\rho g} = (50-5)\frac{lbf}{in.^2} \times \frac{ft^3}{62.4\ lbf} \times 144\frac{in.^2}{ft^2} = 104\ ft$$

and

$$H_{lT} = \frac{h_{lT}}{g} = 813\frac{ft \cdot lbf}{slug} \times \frac{s^2}{32.2\ ft} \times \frac{slug \cdot ft}{lbf \cdot s^2} = 25.2\ ft$$

Problem 8.78

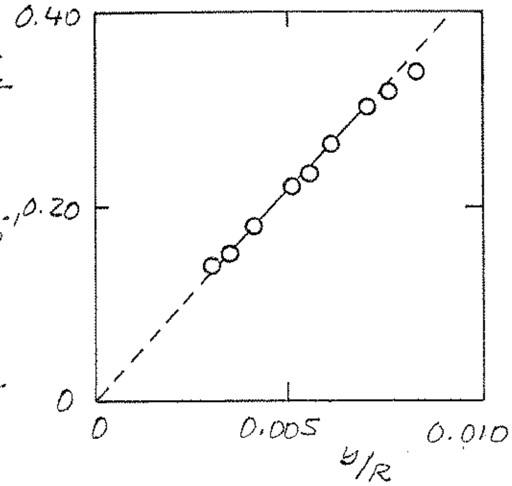
Given: Data measured in fully developed turbulent pipe flow at $Re_U = 50{,}000$ in air:

$\frac{\bar{u}}{U}$	0.343	0.318	0.300	0.264	0.228	0.221	0.179	0.152	0.140
$\frac{y}{R}$	0.0082	0.0075	0.0071	0.0061	0.0055	0.0051	0.0041	0.0034	0.0030

$U = 9.8$ ft/s and $R = 4.86$ in.

Find: (a) Evaluate best-fit value of $d\bar{u}/dy$ from plot.
(b) $\tau_w = \mu\, d\bar{u}/dy$
(c) τ_w calculated from friction factor.

Solution: "Best-fit" slope is {from analysis in Excel file}



$$\frac{d(\bar{u}/U)}{d(y/R)} \approx \frac{\Delta(\bar{u}/U)}{\Delta(y/R)} = 39.8$$

$$\frac{d\bar{u}}{dy} = \frac{U\, d(\bar{u}/U)}{R\, d(y/R)} = 39.8 \times 9.8\frac{ft}{s} \times \frac{1}{4.86\,in.} \times \frac{12\,in.}{ft} = 963\ s^{-1}$$ $\frac{d\bar{u}}{dy}$

For standard air, $\mu = 3.72 \times 10^{-7}$ lbf·s / ft², so

$$\tau_w = \mu\frac{d\bar{u}}{dy} = 3.72\times 10^{-7}\frac{lbf\cdot s}{ft^2} \times \frac{963}{s} = 3.58\times 10^{-4}\ lbf/ft^2$$ τ_w

Friction factor is $f = f(Re, e/D)$. For $Re_U = 50{,}000$, $n = 6.8$ from Eq. 8.23. Then from Eq. 8.24,

$$\frac{\bar{V}}{U} = \frac{2n^2}{(n+1)(2n+1)} = 0.812 \text{ and } Re_{\bar{V}} = 0.812\, Re_U = 0.812 \times 50{,}000 = 40{,}600$$

Assuming smooth pipe, $f = 0.0219$ from Eq. 8.37

Balancing forces on a fluid element: $(p+\Delta p)\frac{\pi D^2}{4} \rightarrow$ [ ] $\leftarrow p\frac{\pi D^2}{4}$, $\tau_w \pi D L$

Then $(p+\Delta p)\frac{\pi D^2}{4} - \tau_w \pi D L - p\frac{\pi D^2}{4} = 0$

$$\tau_w = \frac{R}{2}\frac{\Delta p}{L} = \frac{D}{4L} f \frac{L}{D} \rho \frac{\bar{V}^2}{2} = \frac{f}{8}\rho\bar{V}^2\ ;\ \bar{V} = 0.812U = 0.812 \times 9.8\frac{ft}{sec} = 7.96\ ft/sec$$

Substituting,

$$\tau_w = \frac{0.0219}{8} \times 0.00238\frac{slug}{ft^3} \times (7.96)^2\frac{ft^2}{s^2} \times \frac{lbf\cdot s^2}{slug\cdot ft} = 4.13\times 10^{-4}\ lbf/ft^2$$ τ_w

The result calculated from the friction factor is 15% higher than that evaluated graphically!

Problem 8.78 (In Excel)

Laufer [5] measured the following data for mean velocity near the wall in fully developed turbulent pipe flow at $Re_U = 50{,}000$ ($U = 9.8$ ft/s and $R = 4.86$ in.) in air:

$\frac{\bar{u}}{U}$	0.343	0.318	0.300	0.264	0.228	0.221	0.179	0.152	0.140
$\frac{y}{R}$	0.0082	0.0075	0.0071	0.0061	0.0055	0.0051	0.0041	0.0034	0.0030

Plot the data and obtain the best-fit slope, $d\bar{u}/dy$. Use this to estimate the wall shear stress from $\tau_w = \mu\, d\bar{u}/dy$. Compare this value to that obtained using the friction factor f computed using (a) the Colebrook formula (Eq. 8.37), and (b) the Blasius correlation (Eq. 8.38).

Given: Data on mean velocity in fully developed turbulent flow

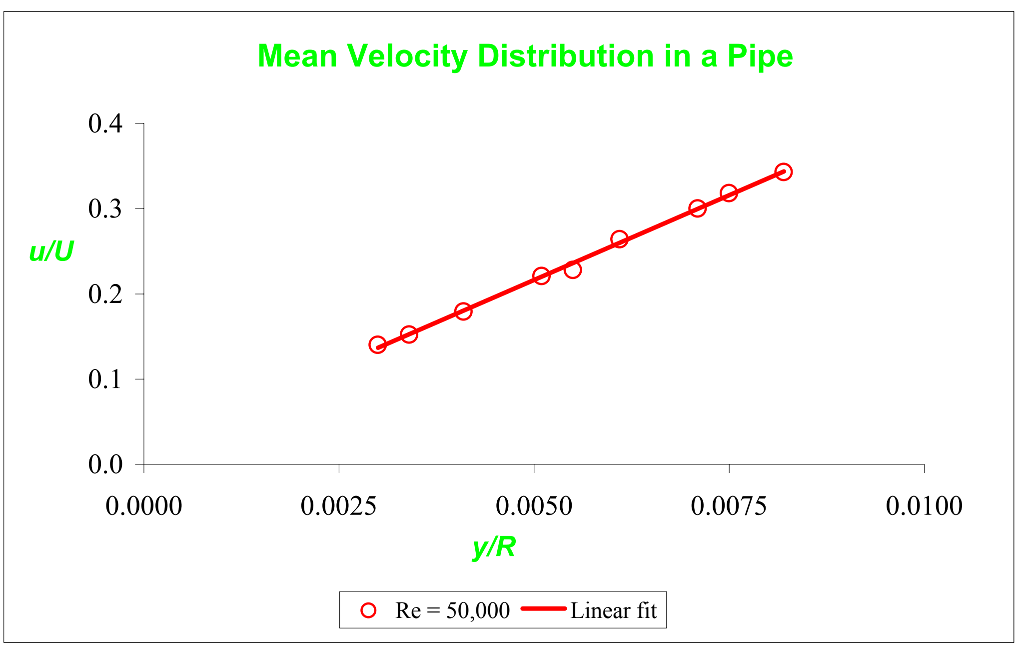
Find: Best fit value of du/dy from plot

Solution

y/R	*u/U*
0.0082	0.343
0.0075	0.318
0.0071	0.300
0.0061	0.264
0.0055	0.228
0.0051	0.221
0.0041	0.179
0.0034	0.152
0.0030	0.140

Using *Excel*'s built-in *Slope* function:

$d(u/U)/d(y/R) =$ **39.8**



Problem 8.79

Given: Small-diameter (i.d. = 0.5 mm) capillary tube made from drawn aluminum is used in place of an expansion valve in a home refrigerator

Find: corresponding relative roughness; with regard to fluid flow, can tube be considered "smooth"?

Solution:

For drawn tubing, from Table 8.1, $e = 0.0015$ mm

Then with $D = 0.5$ mm, $\frac{e}{D} = \frac{0.0015}{0.5} = 0.003$

Looking at the Moody diagram (Fig. 8.13), it is clear that this tube cannot be considered smooth for turbulent flow through the tube.

For laminar flow (Re < 2300) the relative roughness has no effect on the flow.

Problem 8.80

A smooth, 75 mm diameter pipe carries water (65°C) horizontally. When the mass flow rate is 0.075 kg/s, the pressure drop is measured to be 7.5 Pa per 100 m of pipe. Based on these measurements, what is the friction factor? What is the Reynolds number? Does this Reynolds number generally indicate laminar or turbulent flow? Is the flow actually laminar or turbulent?

Given: Data on flow in a pipe

Find: Friction factor; Reynolds number; if flow is laminar or turbulent

Solution

Given data $D = 75 \cdot mm$ $\frac{\Delta p}{L} = 0.075 \cdot \frac{Pa}{m}$ $m_{rate} = 0.075 \cdot \frac{kg}{s}$

From Appendix A $\rho = 999 \cdot \frac{kg}{m^3}$ $\mu = 4 \cdot 10^{-4} \cdot \frac{N \cdot s}{m^2}$

The governing equations between inlet (1) and exit (2) are

$$\left(\frac{p_1}{\rho} + \alpha_1 \cdot \frac{V_1^2}{2} + g \cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \cdot \frac{V_2^2}{2} + g \cdot z_2\right) = h_l \quad (8.29)$$

$$h_l = f \cdot \frac{L}{D} \cdot \frac{V^2}{2} \quad (8.34)$$

For a constant area pipe $V_1 = V_2 = V$

Hence Eqs. 8.29 and 8.34 become

$$f = \frac{2 \cdot D}{L \cdot V^2} \cdot \frac{(p_1 - p_2)}{\rho} = \frac{2 \cdot D}{\rho \cdot V^2} \cdot \frac{\Delta p}{L}$$

For the velocity $V = \dfrac{m_{rate}}{\rho \cdot \frac{\pi}{4} \cdot D^2}$ $\qquad V = 0.017 \frac{m}{s}$

Hence $f = \dfrac{2 \cdot D}{\rho \cdot V^2} \cdot \dfrac{\Delta p}{L}$ $\qquad f = 0.039$

The Reynolds number is $Re = \dfrac{\rho \cdot V \cdot D}{\mu}$ $\qquad Re = 3183$

This Reynolds number indicates the flow is Turbulent

(From Eq. 8.37, at this Reynolds number the friction factor for a smooth pipe is *f* = 0.043; the friction factor computed above thus indicates that, within experimental error, the flow correspon to trubulent flow in a smooth pipe)

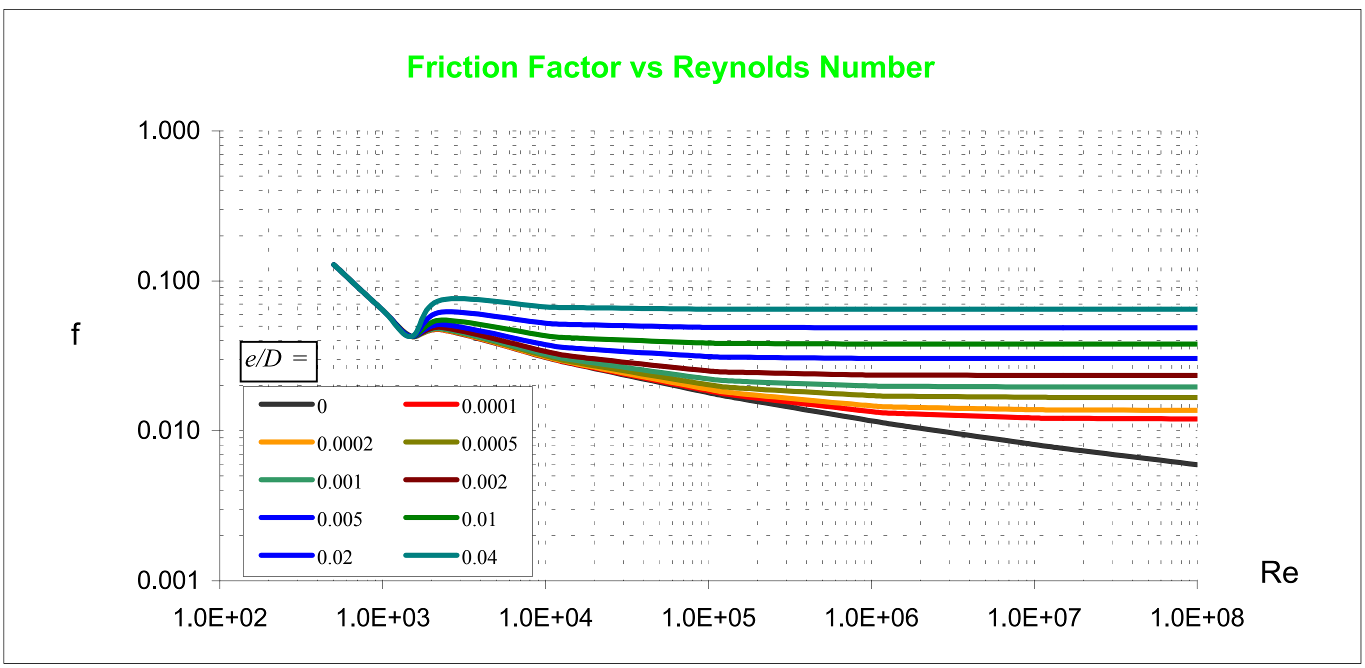
Problem 8.81 (In Excel)

Using Eqs. 8.36 and 8.37, generate the Moody chart of Fig. 8.12.

Solution

Using the add-in function *Friction factor* from the CD

e/D =	0	0.0001	0.0002	0.0005	0.001	0.002	0.005	0.01	0.02	0.04
Re	*f*									
500	0.1280	0.1280	0.1280	0.1280	0.1280	0.1280	0.1280	0.1280	0.1280	0.1280
1.00E+03	0.0640	0.0640	0.0640	0.0640	0.0640	0.0640	0.0640	0.0640	0.0640	0.0640
1.50E+03	0.0427	0.0427	0.0427	0.0427	0.0427	0.0427	0.0427	0.0427	0.0427	0.0427
2.30E+03	0.0473	0.0474	0.0474	0.0477	0.0481	0.0489	0.0512	0.0549	0.0619	0.0747
1.00E+04	0.0309	0.0310	0.0312	0.0316	0.0324	0.0338	0.0376	0.0431	0.0523	0.0672
1.50E+04	0.0278	0.0280	0.0282	0.0287	0.0296	0.0313	0.0356	0.0415	0.0511	0.0664
1.00E+05	0.0180	0.0185	0.0190	0.0203	0.0222	0.0251	0.0313	0.0385	0.0490	0.0649
1.50E+05	0.0166	0.0172	0.0178	0.0194	0.0214	0.0246	0.0310	0.0383	0.0489	0.0648
1.00E+06	0.0116	0.0134	0.0147	0.0172	0.0199	0.0236	0.0305	0.0380	0.0487	0.0647
1.50E+06	0.0109	0.0130	0.0144	0.0170	0.0198	0.0235	0.0304	0.0379	0.0487	0.0647
1.00E+07	0.0081	0.0122	0.0138	0.0168	0.0197	0.0234	0.0304	0.0379	0.0486	0.0647
1.50E+07	0.0076	0.0121	0.0138	0.0167	0.0197	0.0234	0.0304	0.0379	0.0486	0.0647
1.00E+08	0.0059	0.0120	0.0137	0.0167	0.0196	0.0234	0.0304	0.0379	0.0486	0.0647



Problem 8.82 (In Excel)

The turbulent region of the Moody chart of Fig, 8.12 is generated from the empirical correlation given by Eq. 8.37. As noted in Section 8-7, an initial guess for f_0, given by

$$f_0 = 0.25\left[\log\left(\frac{e/D}{3.7} + \frac{5.74}{Re^{0.9}}\right)\right]^{-2}$$

produces results accurate to 1 percent with a single iteration [10]. Investigate the validity of this claim by plotting the error of this approach as a function of Re, with e/D as a parameter. Plot curves over a range of $Re = 10^4$ to 10^8, for $e/D = 0$, 0.0001, 0.001, 0.01, and 0.05.

Solution

Using the above formula for f_0, and Eq. 8.37 for f_1

e/D =	0	0.0001	0.0002	0.0005	0.001	0.002	0.005	0.01	0.02	0.05

Re	f_1									
1.00E+04	0.0309	0.0310	0.0312	0.0316	0.0323	0.0337	0.0376	0.0431	0.0522	0.0738
2.50E+04	0.0245	0.0248	0.0250	0.0257	0.0268	0.0288	0.0337	0.0402	0.0501	0.0725
5.00E+04	0.0209	0.0213	0.0216	0.0226	0.0240	0.0265	0.0322	0.0391	0.0494	0.0720
7.50E+04	0.0191	0.0196	0.0200	0.0212	0.0228	0.0256	0.0316	0.0387	0.0492	0.0719
1.00E+05	0.0180	0.0185	0.0190	0.0203	0.0222	0.0251	0.0313	0.0385	0.0490	0.0718
2.50E+05	0.0150	0.0159	0.0166	0.0185	0.0208	0.0241	0.0308	0.0381	0.0488	0.0716
5.00E+05	0.0132	0.0144	0.0154	0.0177	0.0202	0.0238	0.0306	0.0380	0.0487	0.0716
7.50E+05	0.0122	0.0138	0.0149	0.0174	0.0200	0.0237	0.0305	0.0380	0.0487	0.0716
1.00E+06	0.0116	0.0134	0.0147	0.0172	0.0199	0.0236	0.0305	0.0380	0.0487	0.0716
5.00E+06	0.0090	0.0123	0.0139	0.0168	0.0197	0.0235	0.0304	0.0379	0.0486	0.0716
1.00E+07	0.0081	0.0122	0.0138	0.0168	0.0197	0.0234	0.0304	0.0379	0.0486	0.0716
5.00E+07	0.0065	0.0120	0.0138	0.0167	0.0196	0.0234	0.0304	0.0379	0.0486	0.0716
1.00E+08	0.0059	0.0120	0.0137	0.0167	0.0196	0.0234	0.0304	0.0379	0.0486	0.0716

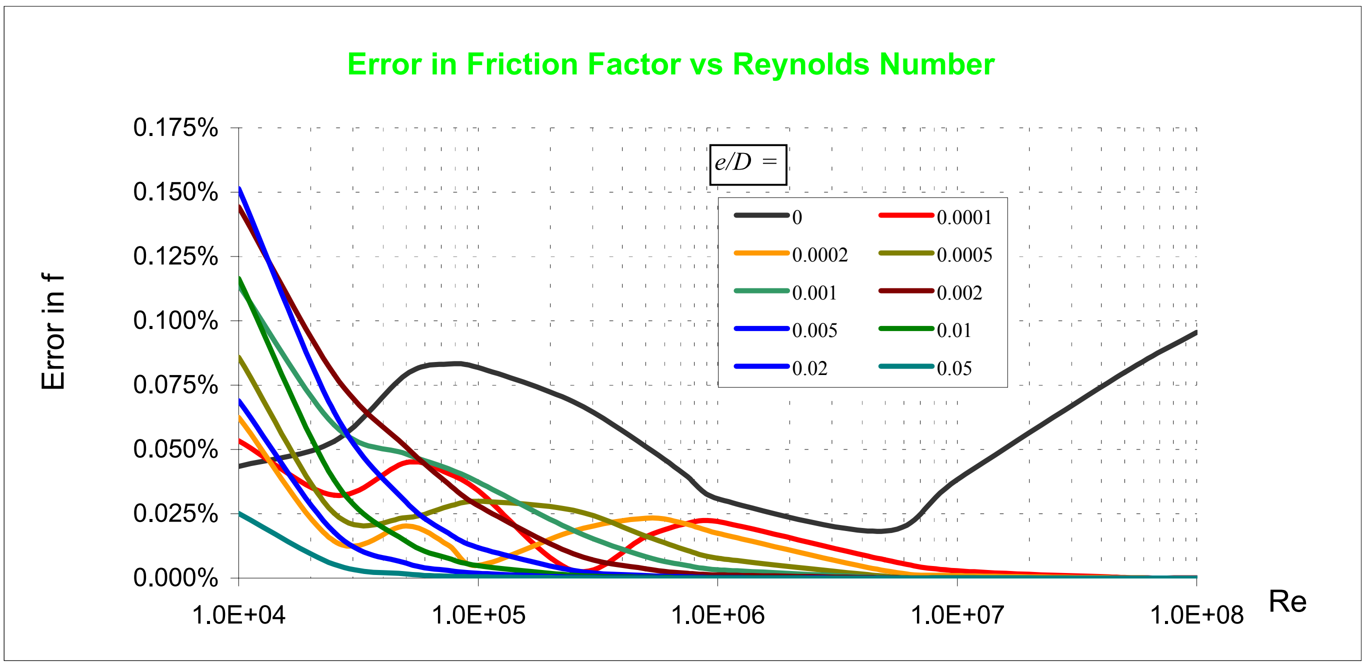
Using the add-in function *Friction factor* from the CD

e/D =	0	0.0001	0.0002	0.0005	0.001	0.002	0.005	0.01	0.02	0.05

Re	f									
1.00E+04	0.0309	0.0310	0.0312	0.0316	0.0324	0.0338	0.0376	0.0431	0.0523	0.0738
2.50E+04	0.0245	0.0248	0.0250	0.0257	0.0268	0.0288	0.0337	0.0402	0.0502	0.0725
5.00E+04	0.0209	0.0212	0.0216	0.0226	0.0240	0.0265	0.0322	0.0391	0.0494	0.0720
7.50E+04	0.0191	0.0196	0.0200	0.0212	0.0228	0.0256	0.0316	0.0387	0.0492	0.0719
1.00E+05	0.0180	0.0185	0.0190	0.0203	0.0222	0.0251	0.0313	0.0385	0.0490	0.0718
2.50E+05	0.0150	0.0158	0.0166	0.0185	0.0208	0.0241	0.0308	0.0381	0.0488	0.0716
5.00E+05	0.0132	0.0144	0.0154	0.0177	0.0202	0.0238	0.0306	0.0380	0.0487	0.0716
7.50E+05	0.0122	0.0138	0.0150	0.0174	0.0200	0.0237	0.0305	0.0380	0.0487	0.0716
1.00E+06	0.0116	0.0134	0.0147	0.0172	0.0199	0.0236	0.0305	0.0380	0.0487	0.0716
5.00E+06	0.0090	0.0123	0.0139	0.0168	0.0197	0.0235	0.0304	0.0379	0.0486	0.0716
1.00E+07	0.0081	0.0122	0.0138	0.0168	0.0197	0.0234	0.0304	0.0379	0.0486	0.0716
5.00E+07	0.0065	0.0120	0.0138	0.0167	0.0196	0.0234	0.0304	0.0379	0.0486	0.0716
1.00E+08	0.0059	0.0120	0.0137	0.0167	0.0196	0.0234	0.0304	0.0379	0.0486	0.0716

The error can now be computed

e/D =	0	0.0001	0.0002	0.0005	0.001	0.002	0.005	0.01	0.02	0.05
Re	**Error (%)**									
1.00E+04	0.0434%	0.0533%	0.0624%	0.0858%	0.1138%	0.1443%	0.1513%	0.1164%	0.0689%	0.0251%
2.50E+04	0.0531%	0.0322%	0.0144%	0.0252%	0.0596%	0.0793%	0.0646%	0.0382%	0.0179%	0.0054%
5.00E+04	0.0791%	0.0449%	0.0202%	0.0235%	0.0482%	0.0510%	0.0296%	0.0143%	0.0059%	0.0016%
7.50E+04	0.0833%	0.0407%	0.0129%	0.0278%	0.0426%	0.0367%	0.0175%	0.0077%	0.0030%	0.0008%
1.00E+05	0.0818%	0.0339%	0.0050%	0.0298%	0.0374%	0.0281%	0.0118%	0.0049%	0.0019%	0.0005%
2.50E+05	0.0685%	0.0029%	0.0183%	0.0264%	0.0186%	0.0095%	0.0029%	0.0011%	0.0004%	0.0001%
5.00E+05	0.0511%	0.0160%	0.0232%	0.0163%	0.0084%	0.0036%	0.0010%	0.0003%	0.0001%	0.0000%
7.50E+05	0.0394%	0.0213%	0.0209%	0.0107%	0.0049%	0.0019%	0.0005%	0.0002%	0.0001%	0.0000%
1.00E+06	0.0308%	0.0220%	0.0175%	0.0077%	0.0032%	0.0012%	0.0003%	0.0001%	0.0000%	0.0000%
5.00E+06	0.0183%	0.0071%	0.0029%	0.0008%	0.0003%	0.0001%	0.0000%	0.0000%	0.0000%	0.0000%
1.00E+07	0.0383%	0.0029%	0.0010%	0.0002%	0.0001%	0.0000%	0.0000%	0.0000%	0.0000%	0.0000%
5.00E+07	0.0799%	0.0002%	0.0001%	0.0000%	0.0000%	0.0000%	0.0000%	0.0000%	0.0000%	0.0000%
1.00E+08	0.0956%	0.0001%	0.0000%	0.0000%	0.0000%	0.0000%	0.0000%	0.0000%	0.0000%	0.0000%



Problem 8.83

Given: Moody diagram gives Darcy friction factor, f.

Fanning friction factor is $f_F \equiv \dfrac{\tau_w}{\frac{1}{2}\rho \bar{V}^2}$

Find: Relate Darcy and Fanning friction factors for fully developed pipe flow. Show $f = 4f_F$.

Solution: Consider cylindrical CV containing fluid in pipe; apply force balance, definition of f.

Basic equations: $\Sigma F_x = 0$

$$\Delta p = f \frac{L}{D} \rho \frac{\bar{V}^2}{2}$$

From the force balance,

$$(p+\Delta p)\frac{\pi D^2}{4} - \tau_w \pi D L - p\frac{\pi D^2}{4} = 0 \quad \text{or} \quad \tau_w = \frac{D}{4}\frac{\Delta p}{L}$$

Substituting,

$$\tau_w = \frac{D}{4L} f \frac{L}{D} \rho \frac{\bar{V}^2}{2} = f \rho \frac{\bar{V}^2}{8}$$

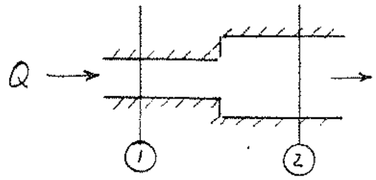
But

$$f_F \equiv \frac{\tau_w}{\frac{1}{2}\rho\bar{V}^2} = \frac{f\rho\bar{V}^2}{8}\frac{2}{\rho\bar{V}^2} = \frac{f}{4}$$

f_F

Given: Water flow through sudden enlargement from 25 mm to 50 mm diameter. $Q = 1.25$ liters per minute.

Find: Pressure rise across enlargement. Comparison with value for frictionless flow.



$D_1 = 25\,mm \quad D_2 = 50\,mm$

Solution: Apply energy equation for pipe flow.

Computing equation: $\frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \alpha_2 \frac{\bar{V}_2^2}{2} + g z_2 + h_{l_T}$

$h_{l_T} = K \frac{\bar{V}_1^2}{2}$

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Uniform flow at each section: $\alpha_1 = \alpha_2 = 1$
(4) Horizontal section

Then

$$p_2 - p_1 = \frac{\rho}{2}\left(\bar{V}_1^2 - \bar{V}_2^2\right) - \rho h_{l_{T_{12}}}$$

From continuity, $\bar{V}_1 A_1 = \bar{V}_2 A_2$, so $\bar{V}_2 = \bar{V}_1 \frac{A_1}{A_2} = \bar{V}_1\left(\frac{D_1}{D_2}\right)^2$; $\bar{V}_2^2 = \bar{V}_1^2\left(\frac{D_1}{D_2}\right)^4$

From Fig. 8.14, at $AR = \left(\frac{D_1}{D_2}\right)^2 = \frac{1}{4}$, $K = 0.56$.

$$\bar{V}_1 = \frac{Q}{A_1} = \frac{4Q}{\pi D_1^2} = \frac{4}{\pi} \times 1.25 \frac{L}{s} \times 10^{-3} \frac{m^3}{L} \times \frac{1}{(25 \times 10^{-3})^2 m^2} = 2.55\ m/s$$

Substituting,

$$p_2 - p_1 = \frac{\rho \bar{V}_1^2}{2}\left[1 - \left(\frac{D_1}{D_2}\right)^4\right] - K\frac{\rho \bar{V}_1^2}{2} = \frac{1}{2}\rho \bar{V}_1^2\left[1 - \left(\frac{D_1}{D_2}\right)^4 - K\right]$$

$$= \frac{1}{2} \times 999 \frac{kg}{m^3} \times (2.55)^2 \frac{m^2}{s^2}\left[1 - \left(\frac{1}{2}\right)^4 - 0.56\right] \frac{N \cdot s^2}{kg \cdot m}$$

$p_2 - p_1 = 1.22\ kPa$ ← Δp_a

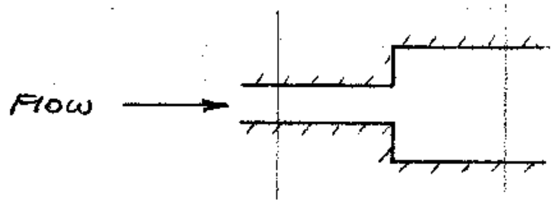
For frictionless flow, $K = 0$, and

$p_2 - p_1 = \frac{1}{2}\rho \bar{V}_1^2\left[1 - \left(\frac{D_1}{D_2}\right)^4\right] = 3.04\ kPa$ ← Δp_f

Thus $\frac{\Delta p_{actual}}{\Delta p_{frictionless}} = \frac{1.22}{3.04} = 0.403$ or 40.3% ← Ratio

Problem 8.85

Given: Air flow at standard conditions through a sudden expansion in a circular duct, as shown.



Find: (a) Average velocity of air at inlet
(b) Volume flow rate.

Solution: Apply the energy and continuity equations for steady, incompressible flow that is uniform at each section.

Basic equations: $$\frac{p_1}{\rho} + \frac{\bar{V}_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \frac{\bar{V}_2^2}{2} + g z_2 + h_{\ell T}$$ (1) (1)

$$h_{\ell T} = h_\ell + K\frac{\bar{V}_1^2}{2} \quad ; \quad \bar{V}_1 A_1 = \bar{V}_2 A_2$$ (h_ℓ = 0 (2))

Assumptions: (1) $z_1 = z_2$
(2) $h_\ell = 0$ between sections ① and ②.

Then
$$\frac{p_1}{\rho} + \frac{\bar{V}_1^2}{2} = \frac{p_2}{\rho} + \frac{\bar{V}_2^2}{2} + K\frac{\bar{V}_1^2}{2}$$

From continuity, $\bar{V}_2 = \bar{V}_1 \frac{A_1}{A_2} = \bar{V}_1 AR$, so
$$\frac{p_1}{\rho} + \frac{\bar{V}_1^2}{2} = \frac{p_2}{\rho} + \frac{\bar{V}_1^2}{2}AR^2 + K\frac{\bar{V}_1^2}{2}$$

or $$\frac{\bar{V}_1^2}{2}(1 - AR^2 - K) = \frac{p_2 - p_1}{\rho}$$ so that $$\bar{V}_1 = \sqrt{\frac{2(p_2 - p_1)}{\rho(1 - AR^2 - K)}}$$

Now $AR = \left(\frac{D_1}{D_2}\right)^2 = 0.11$, so from Fig. 8.15, $K \simeq 0.80$. Also

$$p_2 - p_1 = \gamma_{H_2O}\Delta h = 62.4\,\frac{lbf}{ft^3} \times 0.25\,in. \times \frac{ft}{12\,in.} = 1.3\,\frac{lbf}{ft^2}$$

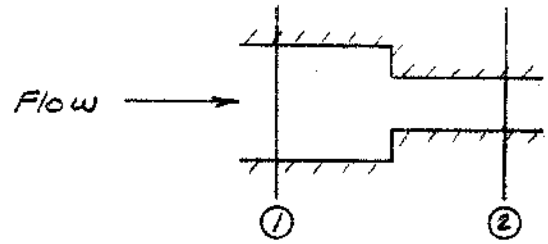
Thus
$$\bar{V}_1 = \left[2 \times 1.3\,\frac{lbf}{ft^2} \times \frac{ft^3}{0.00238\,slug} \times \frac{1}{(1-(0.11)^2-0.8)} \times \frac{slug \cdot ft}{lbf \cdot s^2}\right]^{\frac{1}{2}} = 76.2\ ft/s$$ $\bar{V}_1$

and

$$Q = \bar{V}_1 A_1 = 76.2\,\frac{ft}{s} \times \frac{\pi}{4}(0.25)^2\,\frac{ft^2}{s^2} \times 60\,\frac{s}{min} = 224\ ft^3/min$$ Q

Problem 8.86

Given: Water flow through a circular tube, as shown.



$p_1 - p_2 = 3.4$ kPa

$D_1 = 50$ mm $D_2 = 25$ mm

Find: Flow rate, Q.

Solution: Apply the energy and continuity equations for steady, incompressible flow that is uniform at each section.

Basic equations:
$$\frac{p_1}{\rho} + \frac{\bar{V}_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{\bar{V}_2^2}{2} + gz_2 + h_{\ell T}$$

$$h_{\ell T} = h_\ell + K\frac{\bar{V}_2^2}{2} \; ; \quad \bar{V}_1 A_1 = \bar{V}_2 A_2$$

Assumptions: (1) $z_1 = z_2$
(2) $h_\ell = 0$ between sections ① and ②

From Fig. 8.17, at $AR = \frac{A_2}{A_1} = \left(\frac{D_2}{D_1}\right)^2 = \frac{1}{4}$, $K_c = 0.4$. Then

$$\frac{p_1}{\rho} + \frac{\bar{V}_1^2}{2} = \frac{p_2}{\rho} + \frac{\bar{V}_2^2}{2} + K_c\frac{\bar{V}_2^2}{2} \quad \text{and} \quad \frac{\bar{V}_1^2}{2} = \left(\frac{A_2}{A_1}\right)^2 \frac{\bar{V}_2^2}{2} = \frac{1}{AR^2}\frac{\bar{V}_2^2}{2}$$

and
$$\frac{p_1}{\rho} + \frac{1}{AR^2}\frac{\bar{V}_2^2}{2} = \frac{p_2}{\rho} + \frac{\bar{V}_2^2}{2} + K_c\frac{\bar{V}_2^2}{2}$$

or
$$\frac{p_1 - p_2}{\rho} = \frac{\bar{V}_2^2}{2}\left[1 + K_c - \frac{1}{AR^2}\right] = \frac{\bar{V}_2^2}{2}\left[1 + 0.4 - 0.06\right] = \frac{1.34\bar{V}_2^2}{2}$$

Solving,
$$\bar{V}_2 = \sqrt{\frac{2(p_1 - p_2)}{1.34\rho}} = \left[\frac{2}{1.34} \times 3.4 \times 10^3 \frac{N}{m^2} \times \frac{m^3}{999\, kg} \times \frac{kg \cdot m}{N \cdot s^2}\right]^{\frac{1}{2}}$$

$$\bar{V}_2 = 2.25 \text{ m/s}$$

Then
$$Q = \bar{V}_2 A_2 = \frac{\pi \bar{V}_2 D_2^2}{4} = \frac{\pi}{4} \times 2.25\frac{m}{s} \times (0.025)^2 m^2 = 1.10 \times 10^{-3} \text{ m}^3/\text{s}$$

Q

Problem 8.87

In an undergraduate laboratory you have been assigned the task of developing a crude flow meter for measuring the flow in a 400 mm diameter water pipe system. You are to install a 200 mm diameter section of pipe, and a water manometer to measure the pressure drop at the sudden contraction. Derive an expression for the theoretical calibration constant k in $Q = k\sqrt{\Delta h}$, where Q is the volume flow rate in L/min, and Δh is the manometer deflection in mm. Plot the theoretical calibration curve for a flow rate range of 0 to 200 L/min. Would you expect this to be a practical device for measuring flow rate?

Given: Data on a pipe sudden contraction

Find: Theoretical calibration constant; plot

Solution

Given data $\quad D_1 = 400 \cdot mm \qquad D_2 = 200 \cdot mm$

The governing equations between inlet (1) and exit (2) are

$$\left(\frac{p_1}{\rho} + \alpha_1 \cdot \frac{V_1^2}{2} + g \cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \cdot \frac{V_2^2}{2} + g \cdot z_2\right) = h_l \qquad (8.29)$$

where

$$h_l = K \cdot \frac{V_2^2}{2} \qquad (8.40a)$$

Hence the pressure drop is (assuming $\alpha = 1$)

$$\Delta p = p_1 - p_2 = \rho \cdot \left(\frac{V_2^2}{2} - \frac{V_1^2}{2} + K \cdot \frac{V_2^2}{2}\right)$$

For the sudden contraction $$V_1 \cdot \frac{\pi}{4} \cdot D_1^{\ 2} = V_2 \cdot \frac{\pi}{4} \cdot D_2^{\ 2} = Q$$

or $$V_2 = V_1 \cdot \left(\frac{D_1}{D_2}\right)^2$$

so $$\Delta p = \frac{\rho \cdot V_1^{\ 2}}{2} \cdot \left[\left(\frac{D_1}{D_2}\right)^4 (1 + K) - 1\right]$$

For the pressure drop we can use the manometer equation

$$\Delta p = \rho \cdot g \cdot \Delta h$$

Hence $$\rho \cdot g \cdot \Delta h = \frac{\rho \cdot V_1^{\ 2}}{2} \cdot \left[\left(\frac{D_1}{D_2}\right)^4 (1 + K) - 1\right]$$

In terms of flow rate *Q* $$\rho \cdot g \cdot \Delta h = \frac{\rho}{2} \cdot \frac{Q^2}{\left(\frac{\pi}{4} \cdot D_1^{\ 2}\right)^2} \cdot \left[\left(\frac{D_1}{D_2}\right)^4 (1 + K) - 1\right]$$

or $$g \cdot \Delta h = \frac{8 \cdot Q^2}{\pi^2 \cdot D_1^{\ 4}} \cdot \left[\left(\frac{D_1}{D_2}\right)^4 (1 + K) - 1\right]$$

Hence for flow rate *Q* we find $Q = k \cdot \sqrt{\Delta h}$

where $$k = \sqrt{\frac{g \cdot \pi^2 \cdot D_1^{\,4}}{8 \cdot \left[\left(\frac{D_1}{D_2}\right)^4 (1 + K) - 1\right]}}$$

For K, we need the aspect ratio AR

$$AR = \left(\frac{D_2}{D_1}\right)^2 \qquad AR = 0.25$$

From Fig. 8.14 $K = 0.4$

Using this in the expression for k, with the other given values

$$k = \sqrt{\frac{g \cdot \pi^2 \cdot D_1^{\,4}}{8 \cdot \left[\left(\frac{D_1}{D_2}\right)^4 (1 + K) - 1\right]}} = 0.12 \cdot \frac{m^{\frac{5}{2}}}{s}$$

For Δh in mm and Q in L/min $k = 228 \frac{\frac{L}{min}}{mm^{\frac{1}{2}}}$

The plot of theoretical Q versus flow rate Δh is shown in the associated *Excel* workbook

Problem 8.87 (In Excel)

In an undergraduate laboratory you have been assigned the task of developing a crude flow meter for measuring the flow in a 400 mm diameter water pipe system. You are to install a 200 mm diameter section of pipe, and a water manometer to measure the pressure drop at the sudden contraction. Derive an expression for the theoretical calibration constant k in $Q = k\sqrt{\Delta h}$, where Q is the volume flow rate in L/min, and Δh is the manometer deflection in mm. Plot the theoretical calibration curve for a flow rate range of 0 to 200 L/min. Would you expect this to be a practical device for measuring flow rate?

Given: Data on a pipe sudden contraction

Find: Theoretical calibration constant; plot

Solution

$D_1 =$ 400 mm

$D_1 =$ 200 mm

$K =$ 0.4

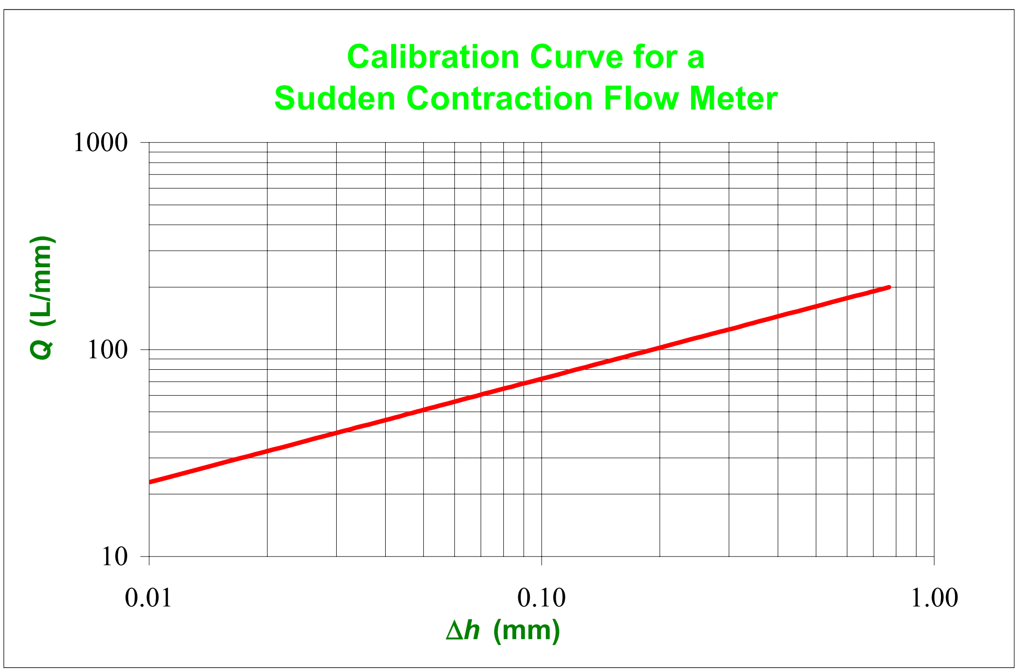
$k =$ 228 L/min/mm$^{1/2}$

$$Q = k\cdot\sqrt{\Delta h}$$

$$k = \sqrt{\frac{g\cdot\pi^2\cdot D_1^{\,4}}{8\cdot\left[\left(\frac{D_1}{D_2}\right)^4(1+K)-1\right]}}$$

The values for Δh are quite low; this would not be a good meter - it is not sensitive enough.
In addition, it is non-linear.

Δh (mm)	Q (L/min)
0.010	23
0.020	32
0.030	40
0.040	46
0.050	51
0.075	63
0.100	72
0.150	88
0.200	102
0.250	114
0.300	125
0.400	144
0.500	161
0.600	177
0.700	191
0.767	200



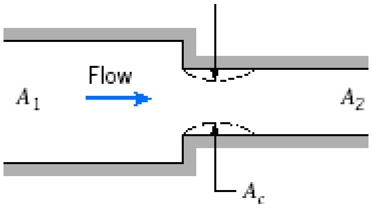
Problem 8.88

Flow through a sudden contraction is shown. The minimum flow area at the vena contracta is given in terms of the area ratio by the contraction coefficient [30],

$$C_c = \frac{A_c}{A_2} = 0.62 + 0.38\left(\frac{A_2}{A_1}\right)^3$$

The loss in a sudden contraction is mostly a result of the vena contracta: The fluid accelerates into the contraction, there is flow separation (as shown by the dashed lines), and the vena contracta acts as a miniature sudden expansion with significant secondary flow losses. Use these assumptions to obtain and plot estimates of the minor loss coefficient for a sudden contraction, and compare with the data presented in Fig. 8.14.

Given: Contraction coefficient for sudden contraction



Find: Expression for minor head loss; compare with Fig. 8.14; plot

Solution

We analyse the loss at the "sudden expansion" at the vena contracta

The governing CV equations (mass, momentum, and energy) are

$$\frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A} = 0 \qquad (4.12)$$

$$F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u \, \rho \, d\forall + \int_{CS} u \, \rho \vec{V} \cdot d\vec{A} \qquad (4.18a)$$

$$\dot{Q} - \dot{W}_s - \dot{W}_{\text{shear}} - \dot{W}_{\text{other}} = \frac{\partial}{\partial t}\int_{CV} e \, \rho \, d\forall + \int_{CS}\left(u + pv + \frac{V^2}{2} + gz\right)\rho \vec{V} \cdot d\vec{A} \qquad (4.56)$$

Assume
1. Steady flow
2. Incompressible flow
3. Uniform flow at each section
4. Horizontal: no body force

5. No shaft work
6. Neglect viscous friction
7. Neglect gravity

The mass equation becomes $$V_c \cdot A_c = V_2 \cdot A_2 \qquad (1)$$

The momentum equation becomes $$p_c \cdot A_2 - p_2 \cdot A_2 = V_c \cdot \left(-\rho \cdot V_c \cdot A_c\right) + V_2 \cdot \left(\rho \cdot V_2 \cdot A_2\right)$$

or (using Eq. 1) $$p_c - p_2 = \rho \cdot V_c \cdot \frac{A_c}{A_2} \cdot \left(V_2 - V_c\right) \qquad (2)$$

The energy equation becomes $$Q_{rate} = \left(u_c + \frac{p_c}{\rho} + V_c^{\,2}\right) \cdot \left(-\rho \cdot V_c \cdot A_c\right) \dots + \left(u_2 + \frac{p_2}{\rho} + V_2^{\,2}\right) \cdot \left(\rho \cdot V_2 \cdot A_2\right)$$

or (using Eq. 1) $$h_{lm} = u_2 - u_c - \frac{Q_{rate}}{m_{rate}} = \frac{V_c^{\,2} - V_2^{\,2}}{2} \dots + \frac{p_c - p_2}{\rho} \qquad (3)$$

Combining Eqs. 2 and 3 $$h_{lm} = \frac{V_c^{\,2} - V_2^{\,2}}{2} + V_c \cdot \frac{A_c}{A_2} \cdot \left(V_2 - V_c\right)$$

$$h_{lm} = \frac{V_c^{\,2}}{2} \cdot \left[1 - \left(\frac{V_2}{V_c}\right)^2\right] + V_c^{\,2} \cdot \frac{A_c}{A_2} \cdot \left[\left(\frac{V_2}{V_c}\right) - 1\right]$$

From Eq. 1

$$C_c = \frac{A_c}{A_2} = \frac{V_2}{V_c}$$

Hence

$$h_{lm} = \frac{V_c^{\,2}}{2}\cdot\left(1 - C_c^{\,2}\right) + V_c^{\,2}\cdot C_c\cdot\left(C_c - 1\right)$$

$$h_{lm} = \frac{V_c^{\,2}}{2}\cdot\left(1 - C_c^{\,2} + 2\cdot C_c^{\,2} - 2\cdot C_c\right)$$

$$h_{lm} = \frac{V_c^{\,2}}{2}\cdot\left(1 - C_c\right)^2 \qquad (4)$$

But we have

$$h_{lm} = K\cdot\frac{V_2^{\,2}}{2} = K\cdot\frac{V_c^{\,2}}{2}\cdot\left(\frac{V_2}{V_c}\right)^2 = K\cdot\frac{V_c^{\,2}}{2}\cdot C_c^{\,2} \qquad (5)$$

Hence, comparing Eqs. 4 and 5

$$K = \frac{\left(1 - C_c\right)^2}{C_c^{\,2}}$$

So, finally

$$K = \left(\frac{1}{C_c} - 1\right)^2$$

where

$$C_c = 0.62 + 0.38\cdot\left(\frac{A_2}{A_1}\right)^3$$

This result, and the curve of Fig. 8.14, are shown in the associated *Excel* workbook.
The agreement is reasonable

Problem 8.88 (In Excel)

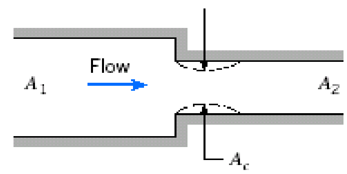
Flow through a sudden contraction is shown. The minimum flow area at the vena contracta is given in terms of the area ratio by the contraction coefficient [30],

$$C_c = \frac{A_c}{A_2} = 0.62 + 0.38\left(\frac{A_2}{A_1}\right)^3$$

The loss in a sudden contraction is mostly a result of the vena contracta: The fluid accelerates into the contraction, there is flow separation (as shown by the dashed lines), and the vena contracta acts as a miniature sudden expansion with significant secondary flow losses. Use these assumptions to obtain and plot estimates of the minor loss coefficient for a sudden contraction, and compare with the data presented in Fig. 8.14.

Given: Sudden contraction

Find: Expression for minor head loss; compare with Fig. 8.14; plot



Solution

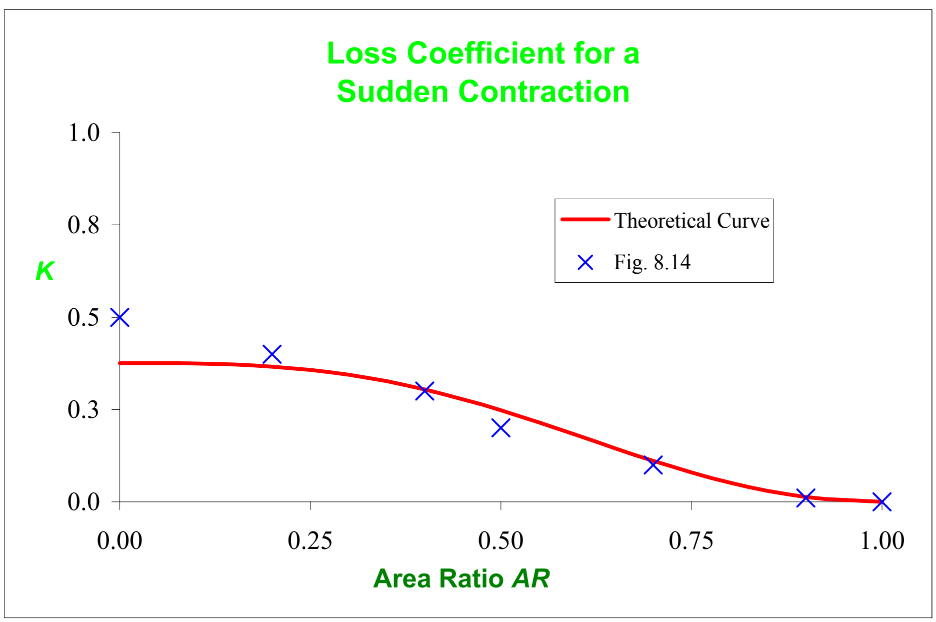
The CV analysis leads to

$$K = \left(\frac{1}{C_c} - 1\right)^2$$

$$C_c = 0.62 + 0.38 \cdot \left(\frac{A_2}{A_1}\right)^3$$

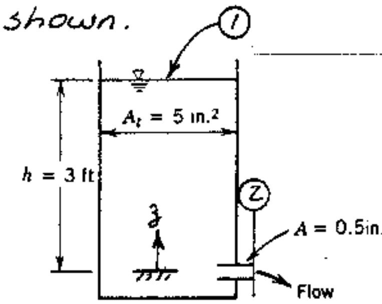
A_2/A_1	K_{CV}	$K_{Fig.\ 8.14}$
0.0	0.376	0.50
0.1	0.374	
0.2	0.366	0.40
0.3	0.344	
0.4	0.305	0.30
0.5	0.248	0.20
0.6	0.180	
0.7	0.111	0.10
0.8	0.052	
0.9	0.013	0.01
1.0	0.000	0.00

(Data from Fig. 8.14 is "eyeballed")
Agreement is reasonable



Given: Quasi-steady flow of water from tank shown.

$A_0 = 0.5 \text{ in.}^2 = A_2$



Find: (a) Flow rate at instant shown.

(b) How could system be improved?

<u>Solution</u>: Apply the energy equation for steady, incompressible pipe flow.

Computing equation:

$$\left(\frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2} + g z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \frac{\bar{V}_2^2}{2} + g z_2\right) + h_{l_T}$$

$$h_{l_T} = h_l + h_{l_m} = f \frac{L}{D} \frac{\bar{V}^2}{2} + K \frac{\bar{V}^2}{2}$$

Assumptions: (1) $p_1 = p_2 = p_{atm}$

(2) Neglect friction in short tube; $L \approx 0$

(3) Reentrant entrance, $K = 0.78$ (Table 8.2)

(4) Uniform flow at each section, so $\alpha = 1$

Then

$$\frac{\bar{V}_1^2}{2} + g z_1 = \frac{\bar{V}_2^2}{2} + K \frac{\bar{V}_2^2}{2} = (1+K)\frac{\bar{V}_2^2}{2}$$

But from continuity, $\bar{V}_1 A_1 = \bar{V}_2 A_2$, so $\bar{V}_1^2 = \bar{V}_2^2 \left(\frac{A_2}{A_1}\right)^2$ and

$$\frac{\bar{V}_2^2}{2}\left[1 + K - \left(\frac{A_2}{A_1}\right)^2\right] = g z_1 \quad \text{or} \quad \bar{V}_2 = \sqrt{\frac{2 g z_1}{1 + K - (A_2/A_1)^2}}$$

Thus

$$\bar{V}_2 = \left[2 \times 32.2 \frac{\text{ft}}{\text{s}^2} \times 3 \text{ ft} \times \frac{1}{1 + 0.78 - (0.1)^2}\right]^{1/2} = 10.4 \text{ ft/s}$$

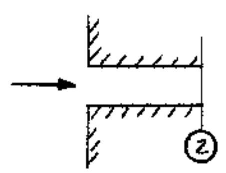
and

$$Q = \bar{V}_2 A_2 = 10.4 \frac{\text{ft}}{\text{s}} \times 0.5 \text{ in.}^2 \times \frac{\text{ft}^2}{144 \text{ in.}^2} = 0.0361 \text{ ft}^3/\text{s} \quad (16.2 \text{ gpm})$$ Q

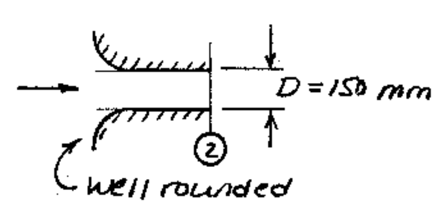
The flow system could be improved by (1) rounding the entrance and (2) adding a diffuser.

Given: Air flow from a clean room through a duct of 150 mm diameter.

Original:



Modified:



$h_1 - h_2 = 2.5 \text{ mm } H_2O$

Friction losses negligible, compared to inlet and exit losses.

Find: Increase in volume flow rate for modified duct.

Solution: Apply the energy equation for steady, incompressible pipe flow.

Computing equations:

$$\frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \alpha_2 \frac{\bar{V}_2^2}{2} + g z_2 + h_{l_T} \quad (\bar{V}_1 \approx 0 \; (1), \; z_1 = z_2 \; (2))$$

$$h_{l_T} = h_l + h_{l,m} \; (h_l \approx 0 \; (4)); \quad h_{l_m} = K_{ent} \frac{\bar{V}_2^2}{2}; \quad \Delta p = \rho_{H_2O}\, g\, \Delta h$$

Assumptions: (1) $\bar{V}_1 \approx 0$ (2) Neglect elevation changes (3) Uniform flow at exit (4) Neglect frictional losses

Then

$$\frac{\Delta p}{\rho} = \frac{p_1 - p_2}{\rho} = \frac{\bar{V}_2^2}{2} + K_{ent}\frac{\bar{V}_2^2}{2} = \frac{\bar{V}_2^2}{2}(1 + K_{ent}) = \frac{\rho_{H_2O}\, g\, \Delta h}{\rho}$$

or

$$\bar{V}_2 = \sqrt{\frac{2(p_1 - p_2)}{\rho(1 + K_{ent})}} = \sqrt{\frac{2\rho_{H_2O}\, g\, \Delta h}{\rho(1 + K_{ent})}}$$

From Table 8.2, $K_{ent} = 0.5$ for square-edged, $K_{ent} = 0.04$ for rounded entrance.

$$\bar{V}_2 = \sqrt{\frac{2}{1.50} \times 999 \frac{kg}{m^3} \times 9.81 \frac{m}{s^2} \times 0.0025 \, m \times \frac{m^3}{1.23 \, kg}} = 5.15 \text{ m/s}$$

$$\bar{V}_2 (\text{modified}) = \sqrt{\frac{2}{1.04} \times 999 \frac{kg}{m^3} \times 9.81 \frac{m}{s^2} \times 0.0025 \, m \times \frac{m^3}{1.23 \, kg}} = 6.19 \text{ m/s}$$

Since $Q = \bar{V} A$, then

$$\Delta Q = (\bar{V}_{2m} - \bar{V}_2) A = (6.19 - 5.15) \frac{m}{s} \times \frac{\pi}{4}(0.150)^2 \, m^2 = 0.0184 \text{ m}^3/\text{s}$$

ΔQ

The percentage improvement is

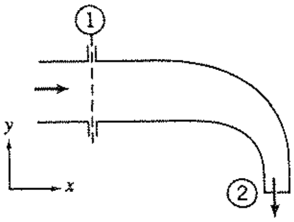
$$\% = \frac{\Delta Q}{Q} \times 100 = \frac{\bar{V}_{2m} - \bar{V}_2}{\bar{V}_2} \times 100 = \frac{6.19 - 5.15}{5.15} \times 100 = 20.2 \text{ percent}$$

Given: Consider again flow through the elbow analyzed in Example Problem 4.6

$p_1 = 221\ \text{kPa} \qquad A_1 = 0.01\ \text{m}^2$

$V_2 = 16\ \text{m/s} \qquad A_2 = 0.0025\ \text{m}^2$

$p_2 = p_{atm}$



Find: Minor head loss coefficient for the elbow

Solution: Apply the energy equation for steady, incompressible pipe flow.

Computing equation:

$$\left(\frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2} + g z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \frac{\bar{V}_2^2}{2} + g z_2\right) = h_{l_T} = h_l + h_{l_m}$$

($p_2/\rho = 0$ (4); gz_1, gz_2 crossed out; $h_l = 0$)

Assumptions: (1) $\alpha_1 = \alpha_2 = 1$

(2) neglect Δz

(3) uniform, incompressible flow so $\bar{V}_1 A_1 = \bar{V}_2 A_2$

(4) use gage pressures

From continuity $\bar{V}_1 = \bar{V}_2 \frac{A_2}{A_1} = 16\ \frac{\text{m}}{\text{s}} \times \frac{0.0025\ \text{m}^2}{0.01\ \text{m}^2} = 4\ \text{m/s}$

Then

$$h_{l_m} = \frac{p_{1_g}}{\rho} + \frac{\bar{V}_1^2}{2} - \frac{\bar{V}_2^2}{2} = (221-101)10^3\ \frac{\text{N}}{\text{m}^2} \times \frac{\text{m}^3}{999\ \text{kg}} \times \frac{\text{kg}\cdot\text{m}}{\text{N}\cdot\text{s}^2} + \frac{1}{2}\left[(4)^2 - (16)^2\right]\frac{\text{m}^2}{\text{s}^2}$$

$$h_{l_m} = 0.120\ \text{m}^2/\text{s}^2$$

But $h_{l_m} = K \frac{\bar{V}_2^2}{2}$; $K = \frac{2 h_{l_m}}{\bar{V}_2^2} = 2 \times 0.120\ \frac{\text{m}^2}{\text{s}^2} \times \frac{\text{s}^2}{(16)^2\ \text{m}^2} = 9.38 \times 10^{-4}$ **K**

Problem 8.92

A conical diffuser is used to expand a pipe flow from a diameter of 100 mm to a diameter of 150 mm. Find the minimum length of the diffuser if we want a loss coefficient (a) $K_{diffuser} \leq 0.2$, (b) $K_{diffuser} \leq 0.35$.

Given: Data on inlet and exit diameters of diffuser

Find: Minimum lengths to satisfy requirements

Solution

Given data $D_1 = 100 \cdot mm$ $D_2 = 150 \cdot mm$

The governing equations for the diffuser are

$$h_{lm} = K \cdot \frac{V_1^2}{2} = \left(C_{pi} - C_p\right) \cdot \frac{V_1^2}{2} \quad (8.44)$$

and

$$C_{pi} = 1 - \frac{1}{AR^2} \quad (8.42)$$

Combining these we obtain an expression for the loss coefficient *K*

$$K = 1 - \frac{1}{AR^2} - C_p \quad (1)$$

The area ratio *AR* is $AR = \left(\frac{D_2}{D_1}\right)^2$ $AR = 2.25$

The pressure recovery coefficient C_p is obtained from Eq. 1 above once we select *K*; then, with C_p and *AR* specified, the minimum value of N/R_1 (where *N* is the length and R_1 is the inlet radius) can be read from Fig. 8.15

(a) $K = 0.2$ $C_p = 1 - \frac{1}{AR^2} - K$ $C_p = 0.602$

From Fig. 8.15 $\frac{N}{R_1} = 5.5$ $R_1 = \frac{D_1}{2}$ $R_1 = 50\,mm$

$N = 5.5 \cdot R_1$ $N = 275\,mm$

(b) $K = 0.35$ $C_p = 1 - \frac{1}{AR^2} - K$ $C_p = 0.452$

From Fig. 8.15 $\frac{N}{R_1} = 3$

$N = 3 \cdot R_1$ $N = 150\,mm$

Problem 8.93

A conical diffuser of length 150 mm is used to expand a pipe flow from a diameter of 75 mm to a diameter of 100 mm. For a water flow rate of 0.1 m^3/s, estimate the static pressure rise. What is the approximate value of the loss coefficient?

Given: Data on geometry of conical diffuser; flow rate

Find: Static pressure rise; loss coefficient

Solution

Given data $D_1 = 75\cdot mm$ $\quad D_2 = 100\cdot mm$ $\quad N = 150\cdot mm$ $\quad$ (N = length)

$\rho = 999\cdot\frac{kg}{m^3}$ $\quad Q = 0.1\cdot\frac{m^3}{s}$

The governing equations for the diffuser are

$$C_p = \frac{p_2 - p_1}{\frac{1}{2}\cdot\rho\cdot V_1^{\,2}} \qquad (8.41)$$

$$h_{lm} = K\cdot\frac{V_1^{\,2}}{2} = \left(C_{pi} - C_p\right)\cdot\frac{V_1^{\,2}}{2} \qquad (8.44)$$

and

$$C_{pi} = 1 - \frac{1}{AR^2} \qquad (8.42)$$

From Eq. 8.41 $\Delta p = p_2 - p_1 = \frac{1}{2} \cdot \rho \cdot V_1^{\,2} \cdot C_p$ (1)

Combining Eqs. 8.44 and 8.42 we obtain an expression for the loss coefficient *K*

$$K = 1 - \frac{1}{AR^2} - C_p \qquad (2)$$

The pressure recovery coefficient C_p for use in Eqs. 1 and 2 above is obtained from Fig. 8.15 once compute *AR* and the dimensionless length N/R_1 (where R_1 is the inlet radius)

The aspect ratio *AR* is $AR = \left(\frac{D_2}{D_1}\right)^2$ $AR = 1.78$

$$R_1 = \frac{D_1}{2} \qquad R_1 = 37.5\,\mathrm{mm}$$

Hence $\frac{N}{R_1} = 4$

From Fig. 8.15, with *AR* = 1.78 and the dimensionless length $N/R_1 = 4$, we find

$$C_p = 0.5$$

To complete the calculations we need V_1

$$V_1 = \frac{Q}{\frac{\pi}{4} \cdot D_1^{\,2}} \qquad V_1 = 22.6\,\frac{m}{s}$$

We can now compute the pressure rise and loss coefficient from Eqs. 1 and 2

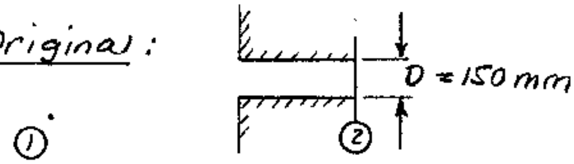
$$\Delta p = \frac{1}{2}\cdot\rho\cdot V_1^{\,2}\cdot C_p \qquad \Delta p = 128\,\text{kPa}$$

$$K = 1 - \frac{1}{AR^2} - C_p \qquad K = 0.184$$

Problem 8.94

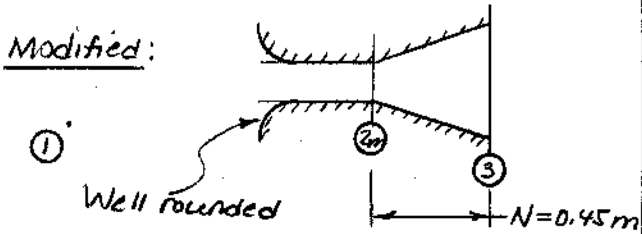
Given: Air flow from a clean room through a duct of 150 mm diameter.

Original:



$h_1 - h_2 = 2.5\ mm\ H_2O$

Modified:



$h_1 - h_3 = 2.5\ mm\ H_2O$

Neglect friction losses compared to "minor" losses.

Find: (a) Area ratio and angle for optimum conical diffuser.
(b) Flow rate for modified system.

Solution: Apply the energy equation for steady, incompressible pipe flow.

Computing equations:

$$\frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \alpha_2 \frac{\bar{V}_2^2}{2} + g z_2 + h_{\ell T} \quad \text{(or to section 3)}$$

($\bar{V}_1^2/2 \approx 0$ (1); $g z_1$, $g z_2$ cancelled)

$$h_{\ell T} = h_\ell + h_{\ell m}; \quad h_{\ell m} = K_{ent} + h_{\ell\,diffuser}; \quad \Delta p = \rho_{H_2O}\, g\, \Delta h$$

($h_\ell \approx 0$ (4))

$$\text{From Eq. 8.42,} \quad h_{\ell\,diffuser} = \frac{\bar{V}_2^2}{2}\left[1 - \frac{1}{AR^2} - C_p\right]$$

Assumptions: (1) $\bar{V}_1 \approx 0$
(2) Neglect Δz
(3) Uniform flow at each section
(4) Neglect frictional losses

For the original system, $\frac{p_1 - p_2}{\rho} = \frac{\bar{V}_2^2}{2} + K_{ent}\frac{\bar{V}_2^2}{2} = 1.5\frac{\bar{V}_2^2}{2} = \frac{\rho_{H_2O}\, g\, \Delta h}{\rho}$ $(K_{ent} = 0.5)$

Thus

$$\bar{V}_2 = \sqrt{\frac{2}{1.5}\frac{\rho_{H_2O}\, g\, \Delta h}{\rho}} = \sqrt{\frac{2}{1.5} \times 999\frac{kg}{m^3} \times 9.81\frac{m}{s^2} \times 0.0025\,m \times \frac{m^3}{1.23\,kg}} = 5.15\ m/s$$

For the modified system, $\frac{p_1 - p_3}{\rho} = \frac{\bar{V}_3^2}{2} + K_{ent}\frac{\bar{V}_2^2}{2} + \frac{\bar{V}_2^2}{2}\left[1 - \frac{1}{AR^2} - C_p\right] = \frac{\bar{V}_2^2}{2}\left[1 + K_{ent} - C_p\right]$

since $\bar{V}_3^2 = \bar{V}_2^2 \frac{1}{AR^2}$. Thus the best diffuser has the highest C_p.

From Fig. 8.16, $C_p = f(N/R_1, AR)$. $N/R_1 = 2N/D_1 = 2 \times \frac{0.45\,m}{0.15\,m} = 6$. From the figure, the best diffuser is

$$C_p \approx 0.62 \text{ at } AR \approx 2.7 \text{ and } 2\phi \approx 12 \text{ deg}$$

AR, ...

For the modified system,

$$\bar{V}_2 = \sqrt{\frac{2}{1 + K_{ent} - C_p}\frac{\rho_{H_2O}\, g\, \Delta h}{\rho}} = \sqrt{\frac{2}{1 + 0.04 - 0.62} \times 999\frac{kg}{m^3} \times 9.81\frac{m}{s^2} \times 0.0025\,m \times \frac{m^3}{1.23\,kg}} = 9.74\ m/s$$

and

$$Q = \bar{V}_2 A_2 = 9.74\frac{m}{s} \times \frac{\pi}{4}(0.15)^2 m^2 = 0.172\ m^3/s$$

Q

{The improvement is $\frac{Q_m - Q}{Q} \times 100 = \frac{\bar{V}_m - \bar{V}}{\bar{V}} \times 100 = \frac{9.74 - 5.15}{5.15} \times 100 = 89.1$ percent more }

Problem 8.95

By applying the basic equations to a control volume starting at the expansion and ending downstream, analyze flow through a sudden expansion (assume the inlet pressure p_1 acts on the area A_2 at the expansion). Develop an expression for and plot the minor head loss across the expansion as a function of area ratio, and compare with the data of Fig. 8.14.

Given: Sudden expansion

Find: Expression for minor head loss; compare with Fig. 8.14; plot

Solution

The governing CV equations (mass, momentum, and energy) are

$$\frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A} = 0 \qquad (4.12)$$

$$F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u \, \rho \, d\forall + \int_{CS} u \, \rho \vec{V} \cdot d\vec{A} \qquad (4.18a)$$

$$\dot{Q} - \dot{W}_s - \dot{W}_{shear} - \dot{W}_{other} = \frac{\partial}{\partial t}\int_{CV} e \, \rho \, d\forall + \int_{CS}\left(u + pv + \frac{V^2}{2} + gz\right)\rho \vec{V} \cdot d\vec{A} \qquad (4.56)$$

Assume
1. Steady flow
2. Incompressible flow
3. Uniform flow at each section
4. Horizontal: no body force
5. No shaft work
6. Neglect viscous friction
7. Neglect gravity

The mass equation becomes $\quad V_1 \cdot A_1 = V_2 \cdot A_2 \quad (1)$

The momentum equation becomes $p_1 \cdot A_2 - p_2 \cdot A_2 = V_1 \cdot \left(-\rho \cdot V_1 \cdot A_1\right) + V_2 \cdot \left(\rho \cdot V_2 \cdot A_2\right)$

or (using Eq. 1) $$p_1 - p_2 = \rho \cdot V_1 \cdot \frac{A_1}{A_2} \cdot \left(V_2 - V_1\right) \qquad (2)$$

The energy equation becomes $$Q_{rate} = \left(u_1 + \frac{p_1}{\rho} + V_1^2\right) \cdot \left(-\rho \cdot V_1 \cdot A_1\right) \ldots + \left(u_2 + \frac{p_2}{\rho} + V_2^2\right) \cdot \left(\rho \cdot V_2 \cdot A_2\right)$$

or (using Eq. 1) $$h_{lm} = u_2 - u_1 - \frac{Q_{rate}}{m_{rate}} = \frac{V_1^2 - V_2^2}{2} \ldots + \frac{p_1 - p_2}{\rho} \qquad (3)$$

Combining Eqs. 2 and 3 $$h_{lm} = \frac{V_1^2 - V_2^2}{2} + V_1 \cdot \frac{A_1}{A_2} \cdot \left(V_2 - V_1\right)$$

$$h_{lm} = \frac{V_1^2}{2} \cdot \left[1 - \left(\frac{V_2}{V_1}\right)^2\right] + V_1^2 \cdot \frac{A_1}{A_2} \cdot \left[\left(\frac{V_2}{V_1}\right) - 1\right]$$

From Eq. 1 $$AR = \frac{A_1}{A_2} = \frac{V_2}{V_1}$$

Hence $$h_{lm} = \frac{V_1^2}{2} \cdot \left(1 - AR^2\right) + V_1^2 \cdot AR \cdot (AR - 1)$$

$$h_{lm} = \frac{V_1^{\,2}}{2}\cdot\left(1 - AR^2 + 2\cdot AR^2 - 2\cdot AR\right)$$

$$h_{lm} = K\cdot\frac{V_1^{\,2}}{2} = (1 - AR)^2\cdot\frac{V_1^{\,2}}{2}$$

Finally $$K = (1 - AR)^2$$

This result, and the curve of Fig. 8.14, are shown in the associated *Excel* workbook. The agreement is excellent

Problem 8.95 (In Excel)

By applying the basic equations to a control volume starting at the expansion and ending downstream, analyze flow through a sudden expansion (assume the inlet pressure p_1 acts on the area A_2 at the expansion). Develop an expression for and plot the minor head loss across the expansion as a function of area ratio, and compare with the data of Fig. 8.14.

Given: Sudden expansion

Find: Expression for minor head loss; compare with Fig. 8.14; plot

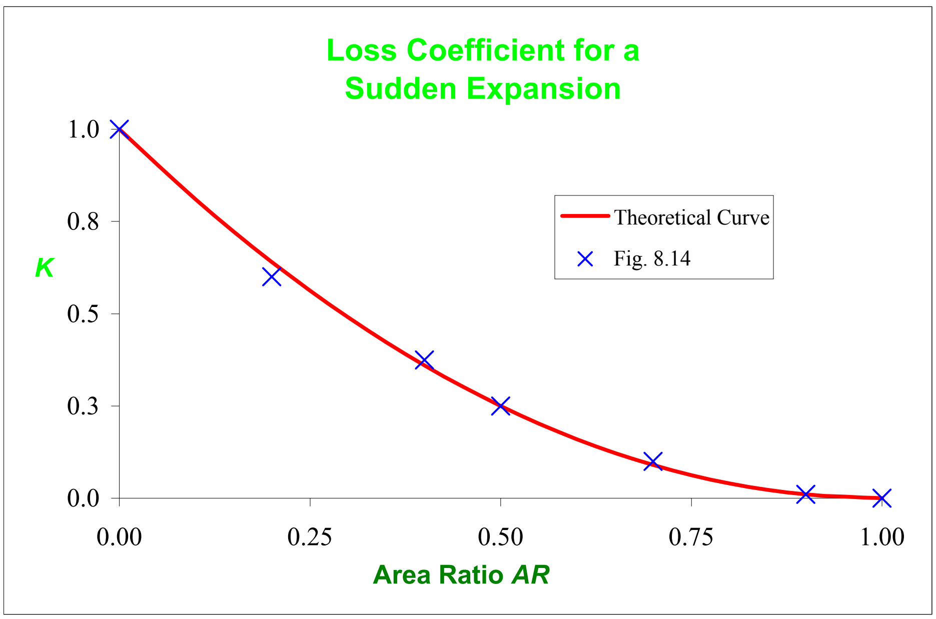
Solution

From the CV analysis

$K = (1 - AR)^2$

AR	K_{CV}	$K_{Fig.\ 8.14}$
0.0	1.00	1.00
0.1	0.81	
0.2	0.64	0.60
0.3	0.49	
0.4	0.36	0.38
0.5	0.25	0.25
0.6	0.16	
0.7	0.09	0.10
0.8	0.04	
0.9	0.01	0.01
1.0	0.00	0.00

(Data from Fig. 8.14 is "eyeballed")
Agreement is excellent



Problem 8.96

Analyze flow through a sudden expansion to obtain an expression for the upstream average velocity $\bar{V}_1$ in terms of the pressure change $\Delta p = p_2 - p_1$, area ratio AR, fluid density ρ, and loss coefficient K. If the flow were frictionless, would the flow rate indicated by a measured pressure change be higher or lower than a real flow, and why? Conversely, if the flow were frictionless, would a given flow generate a larger or smaller pressure change, and why?

Given: Sudden expansion

Find: Expression for upstream average velocity

Solution

The governing equation is

$$\left(\frac{p_1}{\rho} + \alpha_1 \cdot \frac{V_1^{\ 2}}{2} + g \cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \cdot \frac{V_2^{\ 2}}{2} + g \cdot z_2\right) = h_{lT} \qquad (8.29)$$

$$h_{lT} = h_l + K \cdot \frac{V^2}{2}$$

Assume

1. Steady flow
2. Incompressible flow
3. $h_l = 0$
4. $\alpha_2 = \alpha_2 = 1$
5. Neglect gravity

The mass equation is $$V_1 \cdot A_1 = V_2 \cdot A_2$$

so $$V_2 = V_1 \cdot \frac{A_1}{A_2}$$

$$V_2 = AR \cdot V_1 \qquad (1)$$

Equation 8.29 becomes

$$\frac{p_1}{\rho} + \frac{V_1^{\ 2}}{2} = \frac{p_1}{\rho} + \frac{V_1^{\ 2}}{2} + K \cdot \frac{V_1^{\ 2}}{2}$$

or (using Eq. 1)

$$\frac{\Delta p}{\rho} = \frac{p_2 - p_1}{\rho} = \frac{V_1^{\ 2}}{2} \cdot \left(1 - AR^2 - K\right)$$

Solving for V_1

$$V_1 = \sqrt{\frac{2 \cdot \Delta p}{\rho \cdot \left(1 - AR^2 - K\right)}}$$

If the flow were frictionless, $K = 0$, so

$$V_{inviscid} = \sqrt{\frac{2 \cdot \Delta p}{\rho \cdot \left(1 - AR^2\right)}} < V_1$$

Hence. the flow rate indicated by a given Δp would be lower

If the flow were frictionless, $K = 0$, so

$$\Delta p_{invscid} = \frac{V_1^{\ 2}}{2} \cdot \left(1 - AR^2\right)$$

compared to

$$\Delta p = \frac{V_1^{\ 2}}{2} \cdot \left(1 - AR^2 - K\right)$$

Hence. a given flow rate would generate a larger Δp for inviscid flow

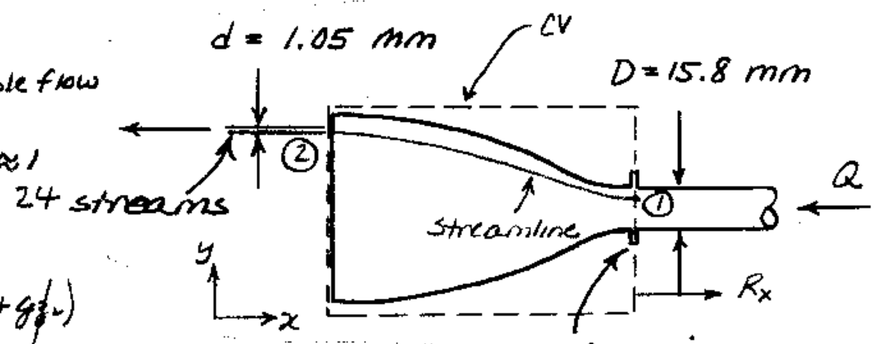
Problem 8.97

Given: Water at 45°C enters a shower head through a circular tube with 15.8 mm inside diameter. The water leaves in 24 streams, each of 1.05 mm diameter. The volume flow rate is 5.67 L/min.

Find:
(a) Estimate of the minimum water pressure needed at the inlet to the shower head.
(b) Force needed to hold the shower head onto the end of the circular tube, indicating clearly whether this is a compression or a tension force.

Solution: Apply the energy equation for steady, incompressible pipe flow, and the x component of momentum, using the CV shown.

Assume: (1) Steady flow
(2) Incompressible flow
(3) Neglect changes in z
(4) Uniform flow: $\alpha_1 = \alpha_2 \approx 1$
(5) Use gage pressures



Then

$$\frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2} + g z_1 - \left(\frac{p_2}{\rho} + \alpha_2 \frac{\bar{V}_2^2}{2} + g z_2 \right) = h_{l_T} = h_l + h_{l_m}$$

($p_2 = 0$ (gage), $h_l = 0$)

$$A_2 = 24 \frac{\pi D_2^2}{4} = 2.08 \times 10^{-5}\ m^2 \qquad A_1 = \frac{\pi D_1^2}{4} = 1.96 \times 10^{-4}\ m^2$$

$$\bar{V}_1 = \frac{Q}{A_1} = 5.67 \frac{L}{min} \times \frac{1}{1.96 \times 10^{-4}\ m^2} \times \frac{m^3}{1000\ L} \times \frac{min}{60\ s} = 0.487\ m/s$$

$$\bar{V}_2 = \bar{V}_1 \frac{A_1}{A_2} = 0.487 \frac{m}{s} \times \frac{1.96 \times 10^{-4}\ m^2}{2.08 \times 10^{-5}\ m^2} = 4.59\ m/s$$

Use $K = 0.5$, for a square-edged orifice, $\rho = 990\ kg/m^3$ (Table A.8). Then

$$p_1 = \frac{\rho}{2}\left(\bar{V}_2^2 + K \bar{V}_2^2 - \bar{V}_1^2 \right) = \frac{\rho}{2}\left[(1+K)\bar{V}_2^2 - \bar{V}_1^2 \right]$$

$$p_1 = \frac{1}{2} \times 990 \frac{kg}{m^3} \left[(1+0.5)(4.59)^2 - (0.487)^2 \right] \frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m} = 15.5\ kPa\ (gage)$$ p_1

Use momentum to find force:

Basic equation: $$F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u \rho\, d\forall + \int_{CS} u \rho \vec{V} \cdot d\vec{A}$$

($F_{B_x} = 0$ (6), $\frac{\partial}{\partial t} = 0$ (1))

Assume: (6) $F_{Bx} = 0$

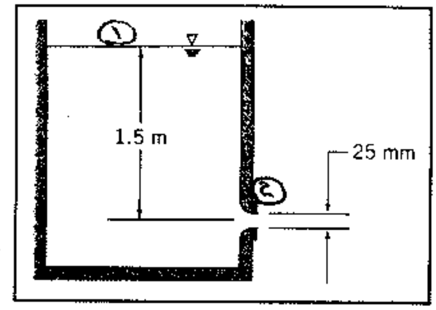
Then $R_x - p_{1g} A_1 = u_1\{-\rho Q\} + u_2\{+\rho Q\} = -V_1\{-\rho Q\} + (-V_2)\{+\rho Q\} = \rho Q (V_1 - V_2)$ (Step ①)

Step ②: $u_1 = -V_1 \qquad u_2 = -V_2$

$$R_x = p_{1g} A_1 + \rho Q (V_1 - V_2) = 15.5 \times 10^3 \frac{N}{m^2} \times 1.96 \times 10^{-4} m^2 + 990 \frac{kg}{m^3} \times 5.67 \frac{L}{min} \times (0.487 - 4.59) \frac{m}{s} \times \frac{m^3}{1000\ L} \times \frac{min}{60\ s}$$

$R_x = 2.65\ N$ (in direction shown, i.e., tension) R_x

Given: Water discharges to atmosphere from a large reservoir through a moderately round horizontal nozzle as shown.
A short section of 50 mm pipe is attached to the nozzle to form a sudden expansion.



Find: (a) the change in flow rate when the short section is added
(b) magnitude of the minimum pressure

Question: (a) If the flow were frictionless (with the sudden expansion in place) would the minimum pressure be higher, lower, or the same as in (b) above?
(b) Would the flow rate be higher, lower, or the same?

Solution:

Basic equations: $\left(\frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2} + g z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \frac{\bar{V}_2^2}{2} + g z_2\right) = h_{l_T}$ (8.29)

$$h_{l_T} = \cancel{h_l}^{(2)} + K \frac{\bar{V}^2}{2}$$

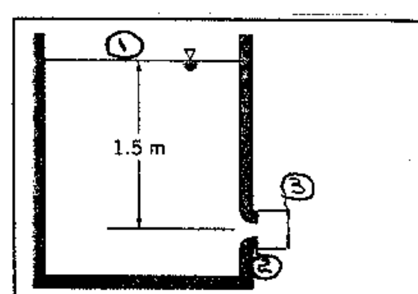
Assumptions: (1) steady, incompressible flow
(2) $h_l = 0$, $K_{nozzle} = 0.28$ (Table 8.2)
(3) $\bar{V}_1 = 0$, $\alpha_2 = 1.0$
(4) $p_1 = p_2 = p_{atm}$

Applying Eq. 8.29 between ① and ② gives

$$g(z_1 - z_2) = K_{nozzle} \frac{\bar{V}_2^2}{2} + \frac{\bar{V}_2^2}{2} = \frac{\bar{V}_2^2}{2}(K_{nozzle} + 1), \text{ and}$$

$$\bar{V}_2 = \left[\frac{2g(z_1 - z_2)}{K_{noz} + 1}\right]^{1/2} = \left[\frac{2}{(0.28+1)} \times 9.81 \frac{m}{s} \times 1.5 m\right]^{1/2} = 4.8 \ m/s$$

Add the short section of pipe as shown.



$$A_3/A_2 = (D_3/D_2)^2 = (2)^2 = 4$$

From Fig. 8.15 with $A_2/A_3 = 0.25$, $K_e = 0.6$

Applying Eq. 8.29 between ① and ③ with $p_1 = p_3 = p_{atm}$ and $\alpha_3 = 1.0$ gives

$$g(z_1 - z_3) = K_{nozzle} \frac{\bar{V}_2^2}{2} + K_e \frac{\bar{V}_2^2}{2} + \frac{\bar{V}_3^2}{2}$$

From continuity $A_2 \bar{V}_2 = A_3 \bar{V}_3$

and

$$g(z_1 - z_3) = \frac{\bar{V}_2^2}{2}\left[K_{noz} + K_e + AR^2\right] \text{ where } AR = 0.25$$

Then

$$\bar{V}_2 = \left[\frac{2g(z_1 - z_2)}{(K_{noz} + K_e + AR^2)}\right]^{1/2} \quad \text{------------} \quad (1)$$

Problem 8.99

Given: Steady flow of water from a large tank through a length of smooth plastic tubing, with $D = 3.18$ mm and $L = 15.3$ m.

Find: (a) Maximum volume flow rate for laminar flow.
(b) Estimate maximum water level in tank for laminar flow ($\alpha = 2$ and $K_{ent} = 1.4$)

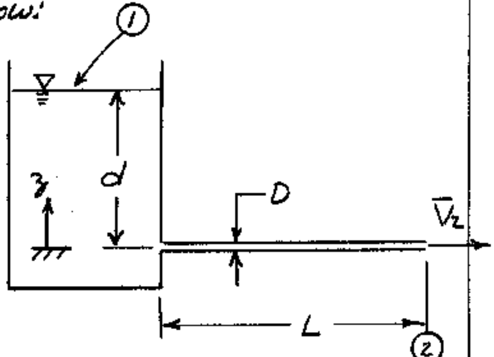
Solution: Assume water at 20°C. From Table A.8, $\rho = 998\ kg/m^3$, $\nu = 1.00\times10^{-6}\ m^2/s$.

$$Re = \frac{\rho \bar{V} D}{\mu} = \frac{\bar{V} D}{\nu} \le 2300\ ;\ \bar{V}_{max} = \frac{2300\nu}{D} = 2300 \times 1.00\times10^{-6}\frac{m^2}{s} \times \frac{1}{0.00318\ m} = 0.723\ m/s$$ $\bar{V}$

$$Q = \bar{V}A\ ;\ A = \frac{\pi D^2}{4} = \frac{\pi}{4}(0.00318)^2 m^2 = 7.94\times10^{-6}\ m^2$$

$$Q = 0.723\frac{m}{s}\times 7.94\times10^{-6}\ m^2 = 5.74\times10^{-6}\frac{m^3}{s}\times 10^3\frac{L}{m^3}\times 60\frac{s}{min} = 0.345\ L/min$$ Q

Apply energy equation for steady, ρ = constant pipe flow:



Computing Equation:
$$\left(\frac{p_1}{\rho} + \alpha_1\frac{\bar{V}_1^2}{2} + gz_1\right) - \left(\frac{p_2}{\rho} + \alpha_2\frac{\bar{V}_2^2}{2} + gz_2\right) = h_{l_T}$$
(with $\bar{V}_1 \approx 0$, $z_2 = 0$)

$$h_{l_T} = h_{l_m} + h_l$$

Assumptions: (1) $p_1 = p_2 = p_{atm}$
(2) $\bar{V}_1 \approx 0$
(3) $K_{ent} = 1.4$ (given)

Then $gd = \alpha_2 \frac{\bar{V}_2^2}{2} + K_{ent}\frac{\bar{V}_2^2}{2} + f\frac{L}{D}\frac{\bar{V}_2^2}{2}$ or $d = \frac{\bar{V}_2^2}{2g}\left(\alpha_2 + K_{ent} + f\frac{L}{D}\right)$

For laminar flow, $f = \frac{64}{Re} = \frac{64}{2300} = 0.0278$. Substituting

$$d = \frac{1}{2}\times(0.723)^2\frac{m^2}{s^2}\times\frac{s^2}{9.81\ m}\left(2.0 + 1.4 + 0.0278\ \frac{15.3\ m}{0.00318\ m}\right)$$

$$d = 3.65\ m$$ d

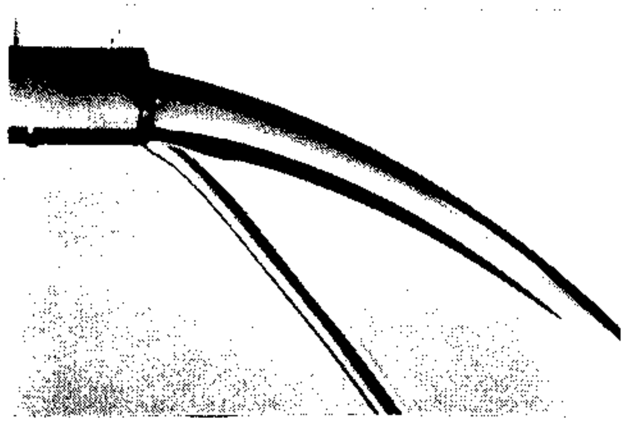
Problem 8.100

Open-Ended Problem Statement: You are asked to compare the behavior of fully developed laminar flow and fully developed turbulent flow in a horizontal pipe under different conditions. For the same flow rate, which will have the larger centerline velocity? Why? If the pipe discharges to atmosphere what would you expect the trajectory of the discharge stream to look like (for the same flow rate)? Sketch your expectations for each case. For the same flow rate, which flow would give the larger wall shear stress? Why? Sketch the shear stress distribution τ/τ_w as a function of radius for each flow. For the same Reynolds number, which flow would have the larger pressure drop per unit length? Why? For a given imposed pressure differential, which flow would have the larger flow rate? Why?

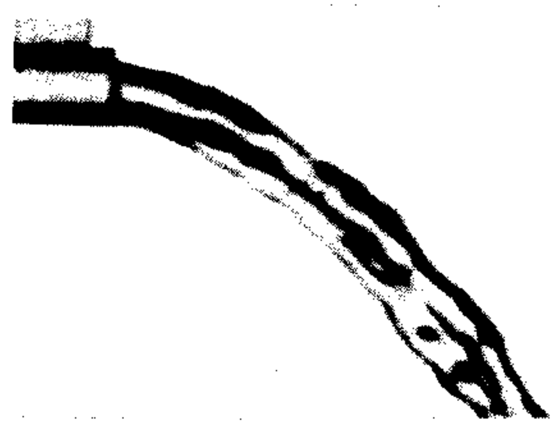
Discussion: In the following fully developed laminar flow and fully developed turbulent flow in a pipe are compared:

(a) For the same flow rate, laminar flow has the higher maximum velocity, because the turbulent velocity profile is more blunt.

(b) The trajectory of the discharge stream spreads out for laminar flow because of the large variation in velocity across the pipe exit. For turbulent flow the exit profile is more nearly uniform (except for the region adjacent to the wall) and hence the trajectory is more uniform. Since centerline velocity is larger for laminar flow, liquid travels the greatest horizontal distance. Trajectories for the two flow cases are shown below:



(i) Laminar flow



(ii) Turbulent flow

(c) For the same flow rate (same mean velocity), turbulent flow has larger wall shear stress because of the larger velocity gradient at the pipe wall. For fully developed flow the pressure force driving the flow is balanced by the shear force at the wall.

(d) Shear stress varies linearly with radius for both flow cases, from its maximum value at the wall to zero at the pipe centerline.

(e) For the same Reynolds number, turbulent flow has a larger pressure drop per unit length because the friction factor is larger.

(f) For a given pressure drop (per unit length), laminar flow has the larger flow rate (larger mean velocity), because it has the smaller friction factor.

The two flow cases are compared in the NCFMF video *Turbulence*, in which R. W. Stewart uses a clever experimental setup to contrast the two flow regimes at constant volume flow rate by varying the liquid viscosity. The trajectories of the liquid streams leaving the end of the pipe are particularly well shown.

Problem 8.101 (In Excel)

Estimate the minimum level in the water tank of Problem 8.99 such that the flow will be turbulent.

Given: Data on water flow from a tank/tubing system

Find: Minimum tank level for turbulent flow

Solution

Governing equations:

$$Re = \frac{\rho \cdot V \cdot D}{\mu}$$

$$\left(\frac{p_1}{\rho} + \alpha_1 \cdot \frac{{V_1}^2}{2} + g \cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \cdot \frac{{V_2}^2}{2} + g \cdot z_2\right) = h_{lT} = \sum_{major} h_l + \sum_{minor} h_{lm} \qquad (8.29)$$

$$h_l = f \cdot \frac{L}{D} \cdot \frac{V^2}{2} \qquad (8.34)$$

$$h_{lm} = K \cdot \frac{V^2}{2} \qquad (8.40a)$$

$$h_{lm} = f \cdot \frac{L_e}{D} \cdot \frac{V^2}{2} \qquad (8.40b)$$

$$f = \frac{64}{Re} \qquad (8.36) \quad \text{(Laminar)}$$

$$\frac{1}{f^{0.5}} = -2.0 \cdot \log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re \cdot f^{0.5}}\right) \qquad (8.37) \quad \text{(Turbulent)}$$

The energy equation (Eq. 8.29) becomes

$$g \cdot d - \alpha \cdot \frac{V^2}{2} = f \cdot \frac{L}{D} \cdot \frac{V^2}{2} + K \cdot \frac{V^2}{2}$$

This can be solved expicitly for height *d*, or solved using *Solver*

Given data:

L = 15.3 m

D = 3.18 mm

K_{ent} = 1.4

α = 2

Tabulated or graphical data:

ν = 1.00E-06 m^2/s

ρ = 998 kg/m^3

(Appendix A)

Computed results:

Re = 2300 (Transition *Re*)

V = 0.723 m/s

α = 1 (Turbulent)

f = 0.0473 (Turbulent)

d = 6.13 m (Vary *d* to minimize error in energy equation)

Energy equation:
(Using *Solver*)

Left (m^2/s)	Right (m^2/s)	Error
59.9	59.9	0.00%

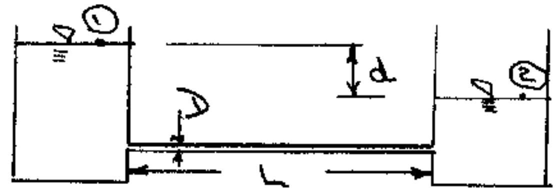
Note that we used α = 1 (turbulent); using α = 2 (laminar) gives *d* = 6.16 m

Problem 8.102

Given: System for measuring pressure drop for water flow in smooth tube as shown

$D = 15.9\,mm \quad L = 3.56\,m$

square-edged entrance to pipe



Find: (a) Volume flow rate needed for turbulent flow in pipe
(b) reservoir height differential needed for turbulent pipe flow

Solution:

Flow will be turbulent for $Re_D > 2300$

$$Re_D = \frac{\rho \bar{V} D}{\mu} = \frac{\bar{V} D}{\nu} = \frac{Q}{A}\frac{D}{\nu} = \frac{Q D}{\nu}\frac{4}{\pi D^2} = \frac{4Q}{\pi \nu D} \quad \text{so} \quad Q = \frac{\pi \nu D\, Re}{4}$$

Assume $T = 20°C$, $\nu = 1.00 \times 10^{-6}\,m^2/s$ (Table A.8)

For $Re = 2300$,

$$Q = \frac{\pi}{4} \times 1.0 \times 10^{-6}\,\frac{m^2}{s} \times 15.9 \times 10^{-3}\,m \times 2300 = 2.87 \times 10^{-5}\,m^3/s \qquad Q$$

Basic equations:

$$\left(\frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2} + g z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \frac{\bar{V}_2^2}{2} + g z_2\right) = h_{\ell_T} \qquad (8.29)$$

$$h_{\ell_T} = h_\ell + h_{\ell_m} \qquad h_\ell = f \frac{L}{D}\frac{\bar{V}^2}{2}, \quad h_{\ell_m} = K \frac{\bar{V}^2}{2}$$

Assumptions: (1) $p_1 = p_2 = p_{atm}$ (2) $\bar{V}_1 = \bar{V}_2 = 0$
(3) $K_{ent} = 0.5$ (Table 8.2), $K_{exit} = 1.0$

Then,

$$z_1 - z_2 = \frac{\bar{V}^2}{2g}\left[f \frac{L}{D} + K_{ent} + K_{exit}\right] \qquad (1)$$

$$\bar{V} = \frac{Q}{A} = \frac{4Q}{\pi D^2} = \frac{4}{\pi} \times 2.87 \times 10^{-5}\,\frac{m^3}{s} \times \frac{1}{(15.9 \times 10^{-3}\,m)^2} = 0.145\,m/s$$

For turbulent flow in a smooth pipe at $Re = 2300$,

$f = 0.05$ (Fig 8.13)

From Eq. 1

$$d = z_1 - z_2 = \frac{(0.145)^2}{2}\frac{m^2}{s^2} \times \frac{s^2}{9.81\,m}\left[0.05 \times \frac{3.56 \times 10^3}{15.9} + 0.5 + 1.0\right]$$

$d = 0.0136\,m$ or $13.6\,mm$ $\qquad d$

Problem 8.103 (In Excel)

Plot the required reservoir depth of water to create flow in a smooth tube of diameter 10 mm and length 100 m, for a flow rate range of 1 L/s through 10 L/s.

Given: Data on tube geometry

Find: Plot of reservoir depth as a function of flow rate

Solution

Governing equations:

$$Re = \frac{\rho \cdot V \cdot D}{\mu}$$

$$\left(\frac{p_1}{\rho} + \alpha_1 \cdot \frac{V_1^2}{2} + g \cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \cdot \frac{V_2^2}{2} + g \cdot z_2\right) = h_{lT} = \sum_{major} h_l + \sum_{minor} h_{lm} \qquad (8.29)$$

$$h_l = f \cdot \frac{L}{D} \cdot \frac{V^2}{2} \qquad (8.34)$$

$$h_{lm} = K \cdot \frac{V^2}{2} \qquad (8.40a)$$

$$f = \frac{64}{Re} \qquad (8.36) \quad \text{(Laminar)}$$

$$\frac{1}{f^{0.5}} = -2.0 \cdot \log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re \cdot f^{0.5}}\right) \qquad (8.37) \quad \text{(Turbulent)}$$

The energy equation (Eq. 8.29) becomes

$$g \cdot d - \alpha \cdot \frac{V^2}{2} = f \cdot \frac{L}{D} \cdot \frac{V^2}{2} + K \cdot \frac{V^2}{2}$$

This can be solved explicitly for reservoir height *d*, or solved using (*Solver*)

$$d = \frac{V^2}{2 \cdot g} \cdot \left(\alpha + f \cdot \frac{L}{D} + K\right)$$

Given data:

L = 100 m

D = 10 mm

α = 1 (All flows turbulent)

Tabulated or graphical data:

μ = 1.01E-03 $N.s/m^2$

ρ = 998 kg/m^3

(Table A.8)

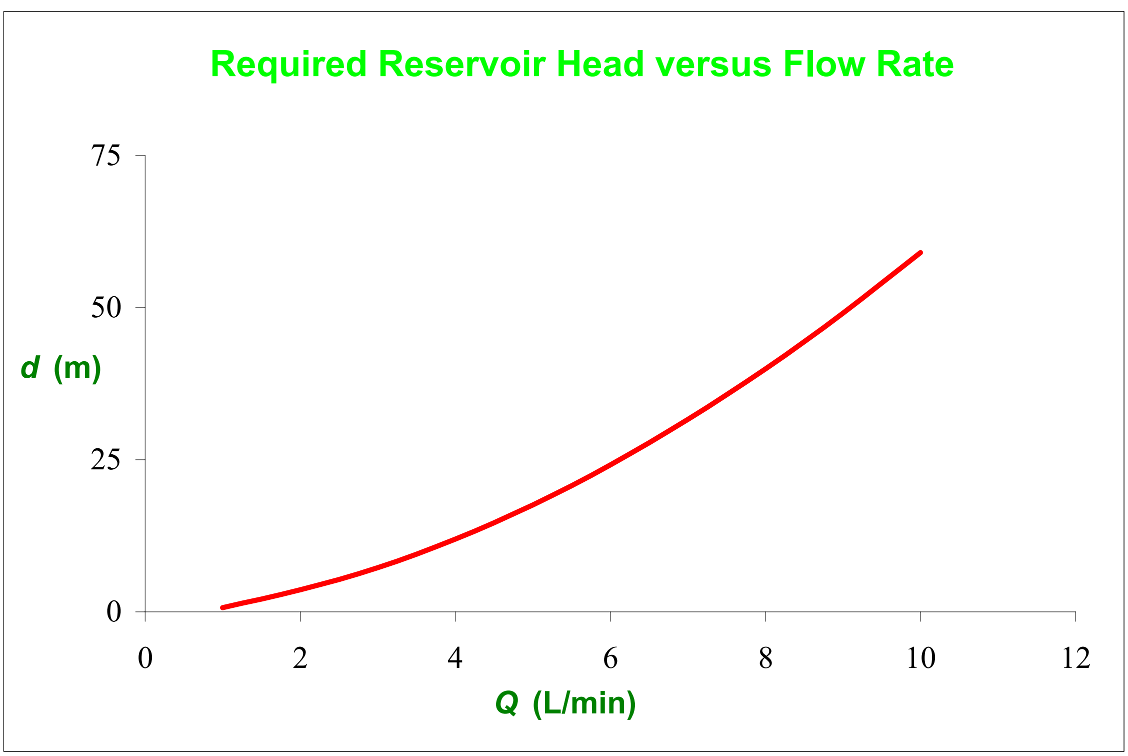
K_{ent} = 0.5 (Square-edged)

(Table 8.2)

Computed results:

Q (L/min)	V (m/s)	Re	f	d (m)
1	0.2	2.1E+03	0.0305	0.704
2	0.4	4.2E+03	0.0394	3.63
3	0.6	6.3E+03	0.0350	7.27
4	0.8	8.4E+03	0.0324	11.9
5	1.1	1.0E+04	0.0305	17.6
6	1.3	1.3E+04	0.0291	24.2
7	1.5	1.5E+04	0.0280	31.6
8	1.7	1.7E+04	0.0270	39.9
9	1.9	1.9E+04	0.0263	49.1
10	2.1	2.1E+04	0.0256	59.1

The flow rates given (L/s) are unrealistic!
More likely is L/min. Results would otherwise be multiplied by 3600!



Problem 8.104

As discussed in Problem 8.49, the applied pressure difference, Δp, and corresponding volume flow rate, Q, for laminar flow in a tube can be compared to the applied DC voltage V across, and current I through, an electrical resistor, respectively. Investigate whether or not this analogy is valid for turbulent flow by plotting the "resistance" $\Delta p/Q$ as a function of Q for turbulent flow of kerosine (at 40°C) in a tube 100 mm long with inside diameter 0.3 mm.

Given: Data on a tube

Find: "Resistance" of tube for flow of kerosine; plot

Solution

The given data is $L = 100 \cdot mm$ $\qquad D = 0.3 \cdot mm$

From Fig. A.2 and Table A.2

Kerosene: $\mu = 1.1 \times 10^{-3} \cdot \frac{N \cdot s}{m^2}$ $\qquad \rho = 0.82 \times 990 \cdot \frac{kg}{m^3} = 812 \cdot \frac{kg}{m^3}$

For an electrical resistor $\quad V = R \cdot I \qquad (1)$

The governing equations for turbulent flow are

$$\left(\frac{p_1}{\rho} + \alpha_1 \cdot \frac{V_1^2}{2} + g \cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \cdot \frac{V_2^2}{2} + g \cdot z_2\right) = h_l \qquad (8.29)$$

$$h_l = f \cdot \frac{L}{D} \cdot \frac{V^2}{2} \qquad (8.34)$$

$$\frac{1}{f^{0.5}} = -2.0\cdot\log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re\cdot f^{0.5}}\right) \qquad (8.37)$$

Simplifying Eqs. 8.29 and 8.34 for a horizontal, constant-area pipe

$$\frac{p_1 - p_2}{\rho} = f\cdot\frac{L}{D}\cdot\frac{V^2}{2} = f\cdot\frac{L}{D}\cdot\frac{\left(\frac{Q}{\frac{\pi}{4}\cdot D^2}\right)^2}{2}$$

or $$\Delta p = \frac{8\cdot\rho\cdot f\cdot L}{\pi^2\cdot D^5}\cdot Q^2 \qquad (2)$$

By analogy, current *I* is represented by flow rate *Q*, and voltage *V* by pressure drop Δ*p*. Comparing Eqs. (1) and (2), the "resistance" of the tube is

$$R = \frac{\Delta p}{Q} = \frac{8\cdot\rho\cdot f\cdot L\cdot Q}{\pi^2\cdot D^5}$$

The "resistance" of a tube is not constant, but is proportional to the "current" *Q*! Actually, the dependence is not quite linear, because *f* decreases slightly (and nonlinearly) with *Q*. The analo fails!

The analogy is hence invalid for $Re > 2300$ or $\frac{\rho\cdot V\cdot D}{\mu} > 2300$

Writing this constraint in terms of flow rate

$$\frac{\rho \cdot \frac{Q}{\frac{\pi}{4} \cdot D^2} \cdot D}{\mu} > 2300 \quad \text{or} \quad Q > \frac{2300 \cdot \mu \cdot \pi \cdot D}{4 \cdot \rho}$$

Flow rate above which analogy fails $\qquad Q = 7.34 \times 10^{-7} \frac{m^3}{s}$

The plot of "resistance" versus flow rate is shown in the associated *Excel* workbook

Problem 8.104 (In Excel)

As discussed in Problem 8.49, the applied pressure difference, Δp, and corresponding volume flow rate, Q, for laminar flow in a tube can be compared to the applied DC voltage V across, and current I through, an electrical resistor, respectively. Investigate whether or not this analogy is valid for turbulent flow by plotting the "resistance" $\Delta p/Q$ as a function of Q for turbulent flow of kerosine (at 40°C) in a tube 100 mm long with inside diameter 0.3 mm.

Given: Data on a tube

Find: "Resistance" of tube for flow of kerosine; plot

Solution

By analogy, current I is represented by flow rate Q, and voltage V by pressure drop Δp.
The "resistance" of the tube is

$$R = \frac{\Delta p}{Q} = \frac{8\cdot\rho\cdot f\cdot L\cdot Q}{\pi^2\cdot D^5}$$

The "resistance" of a tube is not constant, but is proportional to the "current" Q! Actually, the dependence is not quite linear, because f decreases slightly (and nonlinearly) with Q. The analogy fails!

Given data:

L = 100 mm
D = 0.3 mm

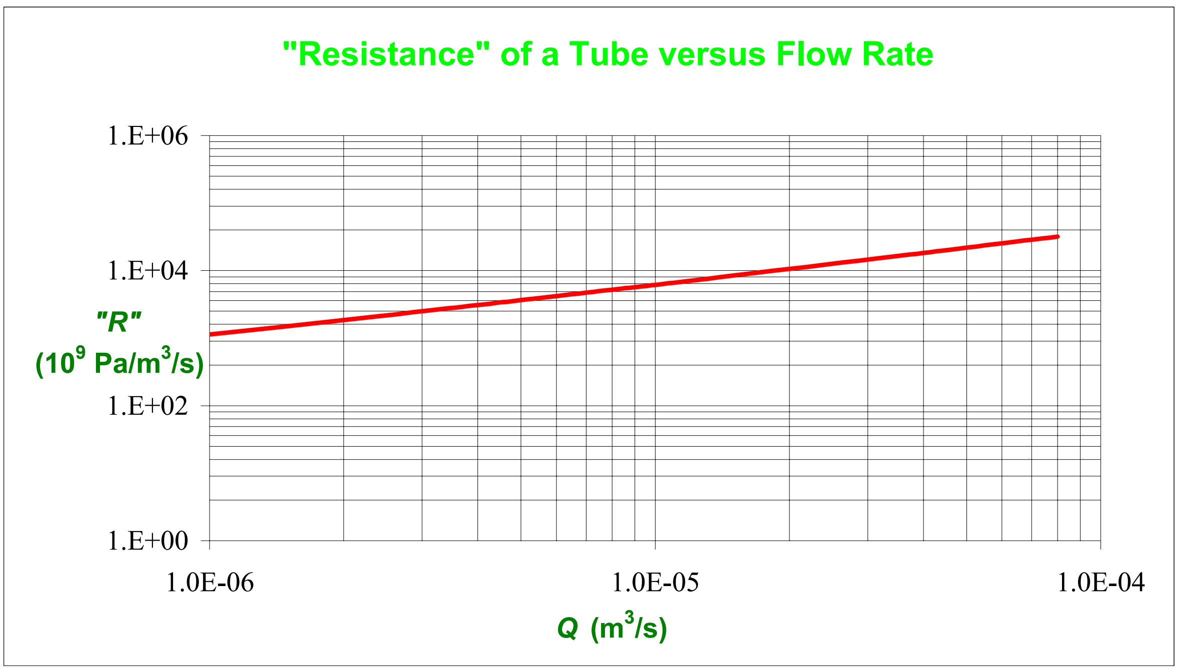
Tabulated or graphical data:

μ = 1.01E-03 $N.s/m^2$
SG_{ker} = 0.82
ρ_w = 990 kg/m^3
ρ = 812 kg/m^3
(Appendix A)

Computed results:

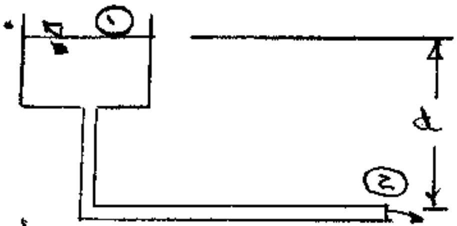
Q (m^3/s)	V (m/s)	Re	f	"R" (10^9 $Pa/m^3/s$)
1.0E-06	14.1	3.4E+03	0.0419	1133
2.0E-06	28.3	6.8E+03	0.0343	1855
4.0E-06	56.6	1.4E+04	0.0285	3085
6.0E-06	84.9	2.0E+04	0.0257	4182
8.0E-06	113.2	2.7E+04	0.0240	5202
1.0E-05	141.5	3.4E+04	0.0228	6171
2.0E-05	282.9	6.8E+04	0.0195	10568
4.0E-05	565.9	1.4E+05	0.0169	18279
6.0E-05	848.8	2.0E+05	0.0156	25292
8.0E-05	1131.8	2.7E+05	0.0147	31900

The "resistance" is not constant; the analogy is invalid for turbulent flow



Problem 8.105

Given: System for measuring pressure drop for water flow in smooth pipe supplies water from an overhead constant-head tank.
System includes:
- square-edged entrance
- two 45° standard elbows
- two 90° standard elbows
- fully open gate valve
- pipe length $L = 9.8$ m, diameter $D = 15.9$ mm



Find: elevation of water surface in supply tank above pipe discharge needed to achieve $Re_D = 10^5$.

Solution:

$Re = \frac{\rho D \bar{V}}{\mu} = \frac{D\bar{V}}{\nu}$ Assume $T = 20°C$, $\nu = 1.00 \times 10^{-6}\ m^2/s$ (Table A.8).

For $Re = 10^5$, $\bar{V} = \frac{Re\,\nu}{D} = 10^5 \times 1.0 \times 10^{-6} \frac{m^2}{s} \times \frac{1}{15.9 \times 10^{-3}\ m} = 6.29\ m/s$

Basic equations: $\left(\frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2} + gz_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \frac{\bar{V}_2^2}{2} + gz_2\right) = h_{l_T}$ (8.29)

$h_{l_T} = h_l + h_{l_m}$, $h_l = f \frac{L}{D}\frac{\bar{V}^2}{2}$, $h_{l_m} = f \frac{\bar{V}^2}{2} \sum \frac{L_e}{D} + K_{ent}\frac{\bar{V}^2}{2}$

Assumptions: (1) $p_1 = p_2 = p_{atm}$ (2) $\bar{V}_1 = 0$ (3) $\alpha_2 = 1.0$

Then,

$$g(z_1 - z_2) = \frac{\bar{V}_2^2}{2} + f\frac{L}{D}\frac{\bar{V}^2}{2} + f\frac{\bar{V}^2}{2}\sum\left(\frac{L_e}{D}\right) + K_{ent}\frac{\bar{V}^2}{2}$$

$$d = (z_1 - z_2) = \frac{\bar{V}_2^2}{2g}\left[1 + f\frac{L}{D} + 2f\left(\frac{L_e}{D}\right)_{45°el} + 2f\left(\frac{L_e}{D}\right)_{90°el} + f\left(\frac{L_e}{D}\right)_{gv} + K_{ent}\right]$$

From Table 8.2 $K_{ent} = 0.5$

From Table 8.4 $(L_e/D)_{45°el} = 16$, $(L_e/D)_{90°el} = 30$, $(L_e/D)_{gv} = 8$

For $Re = 10^5$ in smooth pipe, $f = 0.018$ (Fig. 8.13)

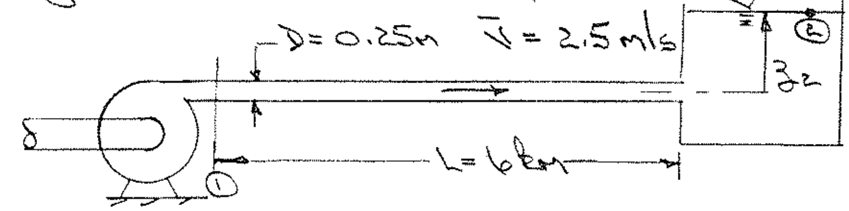
Thus

$$d = \frac{1}{2} \times (6.29)^2 \frac{m^2}{s^2} \times \frac{s^2}{9.81\ m}\left[1 + 0.018 \times \frac{9.8 \times 10^3}{15.9} + 2(0.018)16 + 2(0.018)30 + 0.018(8) + 0.5\right]$$

$d = 29.0$ m ← d

This value of d indicates that it will not be possible to obtain a value of $Re = 10^5$ in the flow system. The maximum value of Re will be considerably less than 10^5.

Given: Water flow from a pump to an open reservoir through commercial steel pipe as shown.



$z_2 = 10\,m$

Find: the pressure at the pump discharge

Solution:

Apply the energy equation for steady, incompressible flow that is uniform at each section

Basic equations: $\left(\frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2} + g z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \frac{\bar{V}_2^2}{2} + g z_2\right) = h_{l_T}$ (8.29)

$h_{l_T} = h_l + h_{l_m}$, $h_l = f \frac{L}{D} \frac{\bar{V}^2}{2}$, $h_{l_m} = K_{exit} \frac{\bar{V}^2}{2}$

Assumptions: (1) $z_1 = 0$ (2) $p_2 = p_{atm} = 0$ gage

(3) $\bar{V}_2 = 0$, $\alpha_1 = 1.0$

(4) $T = 20°C$, $\nu = 1.00 \times 10^{-6}\ m^2/s$ (Table A.8)

$Re = \frac{\rho D \bar{V}}{\mu} = \frac{D\bar{V}}{\nu} = 0.25\,m \times 2.5\,\frac{m}{s} \times \frac{1}{1.00 \times 10^{-6}}\,\frac{s}{m^2} = 6.25 \times 10^5$

For commercial steel pipe, $e = 0.046\ mm$

$\therefore e/D = \frac{0.046}{250} = 0.000184$

From Eq. 8.37, $f = 0.015$. Also $K_{exit} = 1.0$

Then $p_1 = \rho\left[g z_2 - \frac{\bar{V}_1^2}{2} + f \frac{L}{D} \frac{\bar{V}^2}{2} + K_{exit} \frac{\bar{V}^2}{2}\right]$

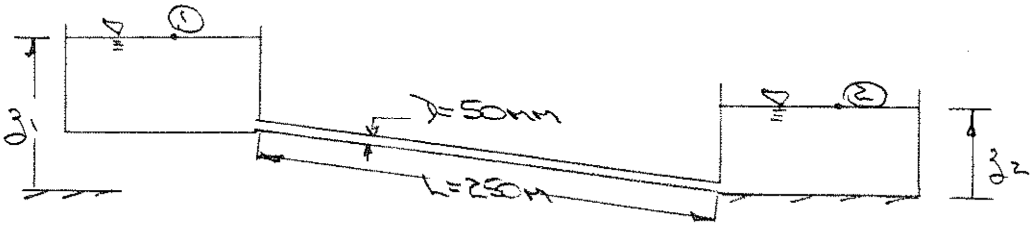
$p_1 = \rho\left[g z_2 + f \frac{L}{D} \frac{\bar{V}^2}{2}\right]$

$p_1 = 998\,\frac{kg}{m^3}\left[9.81\,\frac{m}{s^2} \times 10\,m + 0.015 \times \frac{6 \times 10^3}{0.25} \times \frac{1}{2} \times \frac{(2.5)^2\,m^2}{s^2}\right] \frac{N \cdot s^2}{kg \cdot m}$

$p_1 = 1.22\ MPa$ (gage) ← p_1

Problem 8.107

Given: Water flow by gravity between two reservoirs through straight galvanized iron pipe. Required flow rate is Q



Plot: required elevation difference Δz vs Q for $0 \le Q \le 0.01\ m^3/s$
Estimate: fraction of Δz due to minor losses

Plot: (a) Δz and (b) minor loss/total loss versus Q

Solution:

Apply the energy equation for steady incompressible flow between sections ① and ②.

Basic equations: $\left(\frac{p_1}{\rho} + \alpha_1\frac{\bar{V}_1^2}{2} + g z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2\frac{\bar{V}_2^2}{2} + g z_2\right) = h_{l_T}$ (8.29)

$h_{l_T} = h_l + h_{l_m}$; $h_l = f\frac{L}{D}\frac{\bar{V}^2}{2}$; $h_{l_m} = K_{ent}\frac{\bar{V}^2}{2} + K_{exit}\frac{\bar{V}^2}{2}$

Assumptions: (1) $p_1 = p_2 = p_{atm}$ (given)
(2) $\bar{V}_1 = \bar{V}_2 \approx 0$
(3) square edged entrance

For square edged entrance (Table 8.2) $K_{ent} = 0.5$; also $K_{exit} = 1.0$
For water at $20^\circ C$, $\nu = 1.00 \times 10^{-6}\ m^2/s$ (Table A.8)

$$Re = \frac{\rho \bar{V} D}{\mu} = \frac{\bar{V} D}{\nu} = \frac{Q}{A}\frac{D}{\nu} = \frac{QD}{\nu}\cdot\frac{4}{\pi D^2} = \frac{4Q}{\pi \nu D} \quad (1)$$

To plot Δz vs Q

$$\bar{V} = \frac{4Q}{\pi D^2} = 509\ Q\ (m^3/s)$$

$$\Delta z = \frac{\bar{V}^2}{2g}\left[K_{ent} + K_{exit} + f\frac{L}{D}\right] = \frac{\bar{V}^2}{2g}\left[1.5 + 5000 f\right]$$

where $f = f(Re,\ e/D = 0.003)$

$$\frac{h_{l_m}}{h_{l_T}} = \frac{K_{ent} + K_{exit}}{K_{ent} + K_{exit} + f\frac{L}{D}} = \frac{1.5}{1.5 + 5000f}$$

The ratio h_{l_m}/h_{l_T} increases with increasing Re because f decreases with increasing Re.

Problem 8.107 (In Excel)

Water is to flow by gravity from one reservoir to a lower one through a straight, inclined galvanized iron pipe. The pipe diameter is 50 mm, and the total length is 250 m. Each reservoir is open to the atmosphere. Plot the required elevation difference Δz as a function of flow rate Q, for Q ranging from 0 to 0.01 m^3/s. Estimate the fraction of Δz due to minor losses.

Given: Data on reservoir/pipe system

Find: Plot elevation as a function of flow rate; fraction due to minor losses

Solution

L = 250 m

D = 50 mm

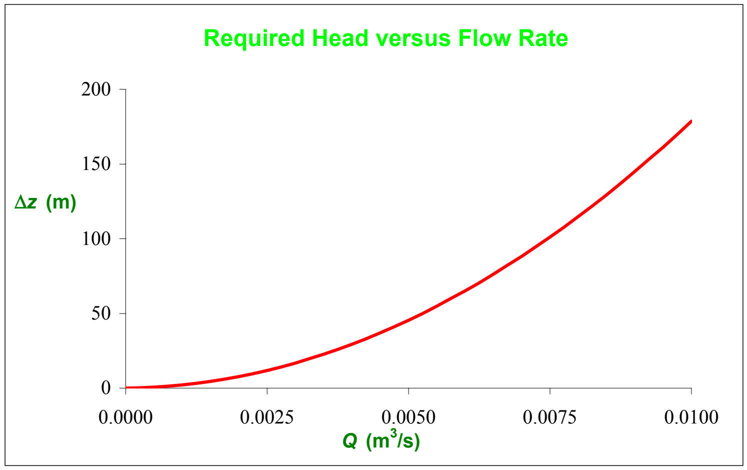
e/D = 0.003

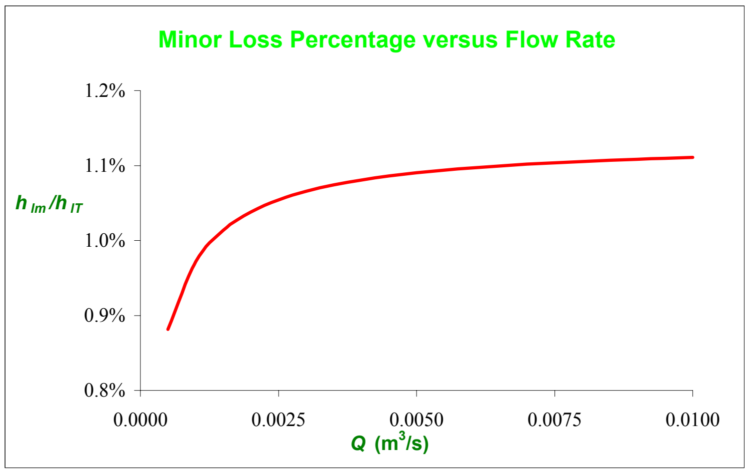
K_{ent} = 0.5

K_{exit} = 1.0

ν = 1.01E-06 m^2/s

Q (m^3/s)	V (m/s)	Re	f	Δz (m)	h_{lm}/h_{lT}
0.0000	0.000	0.00E+00		0.000	
0.0005	0.255	1.26E+04	0.0337	0.562	0.882%
0.0010	0.509	2.52E+04	0.0306	2.04	0.972%
0.0015	0.764	3.78E+04	0.0293	4.40	1.01%
0.0020	1.02	5.04E+04	0.0286	7.64	1.04%
0.0025	1.27	6.30E+04	0.0282	11.8	1.05%
0.0030	1.53	7.56E+04	0.0279	16.7	1.07%
0.0035	1.78	8.82E+04	0.0276	22.6	1.07%
0.0040	2.04	1.01E+05	0.0275	29.4	1.08%
0.0045	2.29	1.13E+05	0.0273	37.0	1.09%
0.0050	2.55	1.26E+05	0.0272	45.5	1.09%
0.0055	2.80	1.39E+05	0.0271	54.8	1.09%
0.0060	3.06	1.51E+05	0.0270	65.1	1.10%
0.0065	3.31	1.64E+05	0.0270	76.2	1.10%
0.0070	3.57	1.76E+05	0.0269	88.2	1.10%
0.0075	3.82	1.89E+05	0.0269	101	1.10%
0.0080	4.07	2.02E+05	0.0268	115	1.11%
0.0085	4.33	2.14E+05	0.0268	129	1.11%
0.0090	4.58	2.27E+05	0.0268	145	1.11%
0.0095	4.84	2.40E+05	0.0267	161	1.11%
0.0100	5.09	2.52E+05	0.0267	179	1.11%





Problem 8.108

Given: Air at a flowrate of 35 m^3/min at standard conditions in a smooth duct 0.3 m square.

Find: Pressure drop in mm H_2O per 30 m of horizontal duct.

Solution: Apply the energy equation for steady, incompressible flow that is uniform at each section. Use hydraulic diameter.

Basic equation: $$\frac{p_1}{\rho} + \frac{\bar{V}_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \frac{\bar{V}_2^2}{2} + g z_2 + f\frac{L}{D_h}\frac{\bar{V}^2}{2} + h_{\ell m}\,; \quad D_h = \frac{4A}{P_w}$$

Assumptions: (1) $\bar{V}_1 = \bar{V}_2$
(2) Horizontal
(3) $h_{\ell m} = 0$

Then
$$\Delta p = p_1 - p_2 = f\frac{L}{D_h}\rho\frac{\bar{V}^2}{2}$$

From continuity, $$\bar{V} = \frac{Q}{A} = 35\frac{m^3}{min} \times \frac{1}{(0.3)^2 m^2} \times \frac{min}{60\, sec} = 6.48\ m/s$$

$$D_h = \frac{4A}{P_w} = 4 \times (0.3)^2 m^2 \times \frac{1}{4(0.3)\,m} = 0.3\ m\,; \quad \nu = 1.45 \times 10^{-5}\ m^2/s \text{ (Table A.10)}$$

$$Re = \frac{\bar{V} D_h}{\nu} = 6.48\frac{m}{s} \times 0.3\,m \times \frac{s}{1.46 \times 10^{-5} m^2} = 1.33 \times 10^5$$

$f = 0.017$ (Fig. 8.13)

Then $$\Delta p = \frac{0.017}{2} \times \frac{30\,m}{0.3\,m} \times 1.23\frac{kg}{m^3} \times (6.48)^2\frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m} = 43.9\ N/m^2$$ Δp

For a manometer, $\Delta p = \rho_{H_2O}\, g\, \Delta h$

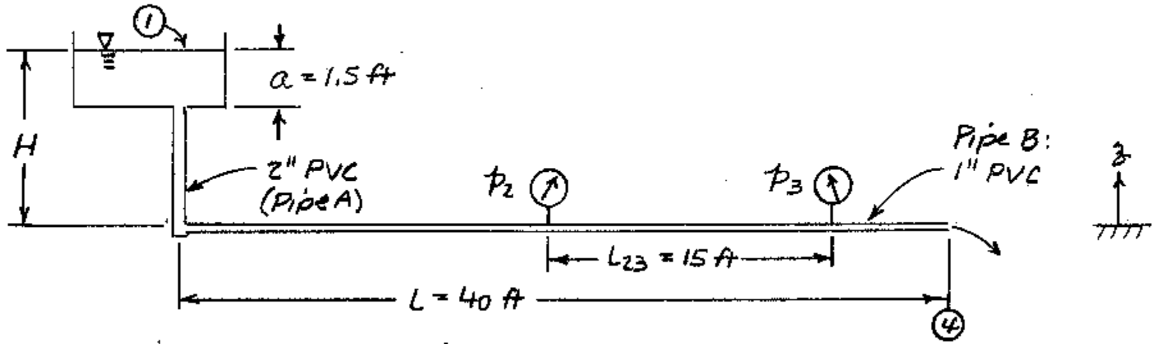
$$\Delta h = \frac{\Delta p}{\rho_{H_2O}\, g} = 43.9\frac{N}{m^2} \times \frac{m^3}{999\ kg} \times \frac{s^2}{9.81\,m} \times \frac{kg \cdot m}{N \cdot s^2} = 0.00448\ m$$

Thus
$\Delta h = 4.48$ mm H_2O (per 30 m of duct) Δh

(This is Δp expressed in mm of water.)

Problem 8.109

Given: Pipe friction experiment, using water, to reach $Re = 100{,}000$.



Find: (a) Required average speed in 1 in. pipe.
(b) Feasibility of using a constant-head tank.
(c) Pressure difference between taps 2 and 3.

Solution: Assume water at 68 F. From Table A.7, $\rho = 1.94\ \text{slug/ft}^3$, $\nu = 1.08\times10^{-5}\ \text{ft}^2/\text{s}$.

$$Re = \frac{\bar{V}D}{\nu} = 100{,}000;\ \bar{V} = \frac{100{,}000\,\nu}{D_B} = 10^5 \times 1.08\times10^{-5}\frac{\text{ft}^2}{\text{s}}\times\frac{1}{1\ \text{in.}}\times 12\frac{\text{in.}}{\text{ft}} = 13.0\ \text{ft/s}$$ $\bar{V}$

Apply energy equation for steady, incompressible pipe flow:

Computing equation:
$$\left(\frac{p_1}{\rho} + \alpha_1\frac{\bar{V}_1^2}{2} + gz_1\right) - \left(\frac{p_4}{\rho} + \alpha_4\frac{\bar{V}_4^2}{2} + gz_4\right) = h_{l_T} = h_l + h_{lm}$$
(with $\bar{V}_1^2/2 \approx 0$, $z_4 = 0$)

Assumptions: (1) $p_1 = p_4 = p_{atm}$
(2) $\bar{V}_1 \approx 0$
(3) $\alpha_4 \approx 1$
(4) Neglect minor losses
(5) Smooth pipes

$$\bar{V}_A = \bar{V}_B\left(\frac{D_B}{D_A}\right)^2 = \frac{\bar{V}_B}{4} = 3.25\ \text{ft/s}$$

$$Re_A = \frac{\bar{V}_A D_A}{\nu} = 50{,}000;\ f_A = 0.021;\ f_B = 0.018$$

{From Moody chart (Fig. 8.13)}

Then
$$gH - \alpha_4\frac{\bar{V}_4^2}{2} = f_A\frac{L_A}{D_A}\frac{\bar{V}_A^2}{2} + f_B\frac{L_B}{D_B}\frac{\bar{V}_B^2}{2}$$

or, since $L_A = H - A$,

$$H = \alpha_4\frac{\bar{V}_B^2}{2g} + f_A\frac{L_A}{D_A}\frac{\bar{V}_B^2}{2g}\left(\frac{D_A}{D_B}\right)^4 + f_B\frac{L_B}{D_B}\frac{\bar{V}_B^2}{2g} = \left[\alpha_4 + \underbrace{f_A\frac{H-a}{D_A}\left(\frac{D_A}{D_B}\right)^4}_{\text{neglect}} + f_B\frac{L}{D_B}\right]\frac{\bar{V}_B^2}{2g}$$

$$H \approx \left[1.0 + 0.018\,\frac{40\ \text{ft}}{(1/12)\ \text{ft}}\right]\frac{1}{2}\times(13.0)^2\frac{\text{ft}^2}{\text{s}^2}\times\frac{\text{s}^2}{32.2\ \text{ft}} = 25.3\ \text{ft}$$ H

Most ceilings are 12 feet high or less. A constant-head tank is impractical.

Between ② and ③
$$\frac{p_2 - p_3}{\rho} = f_B\frac{L_{23}}{D_B}\frac{\bar{V}_B^2}{2} \quad \text{or} \quad \Delta p = f\frac{L}{D}\rho\frac{\bar{V}^2}{2}$$

$$\Delta p = 0.018\times\frac{15\ \text{ft}}{(1/12)\ \text{ft}}\times\frac{1}{2}\times 1.94\frac{\text{slug}}{\text{ft}^3}\times(13.0)^2\frac{\text{ft}^2}{\text{s}^2}\times\frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}}\times\frac{\text{ft}^2}{144\ \text{in.}^2} = 3.69\ \text{psi}$$ Δp

Since $\Delta p = \rho g\,\Delta h$, then

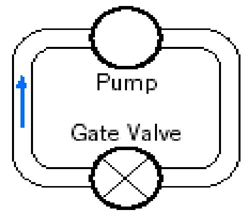
$$\Delta h = \frac{\Delta p}{\rho g} = 3.69\frac{\text{lbf}}{\text{in.}^2}\times 144\frac{\text{in.}^2}{\text{ft}^2}\times\frac{\text{ft}^3}{62.4\ \text{lbf}} = 8.51\ \text{ft}\ \ (102\ \text{in.})\ \text{of water}$$ Δh

Problem 8.110 (In Excel)

A system for testing variable-output pumps consists of the pump, four standard elbows, and an open gate valve forming a closed circuit as shown. The circuit is to absorb the energy added by the pump. The tubing is 75 mm diameter cast iron, and the total length of the circuit is 20 m. Plot the pressure difference required from the pump for water flow rates Q ranging from 0.01 m^3/s to 0.06 m^3/s.

Given: Data on circuit

Find: Plot pressure difference for a range of flow rates



Solution

Governing equations:

$$Re = \frac{\rho \cdot V \cdot D}{\mu}$$

$$\left(\frac{p_1}{\rho} + \alpha_1 \cdot \frac{V_1^{\,2}}{2} + g \cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \cdot \frac{V_2^{\,2}}{2} + g \cdot z_2\right) = h_{lT} = \sum_{major} h_l + \sum_{minor} h_{lm} \qquad (8.29)$$

$$h_l = f \cdot \frac{L}{D} \cdot \frac{V^2}{2} \qquad (8.34)$$

$$h_{lm} = f \cdot \frac{L_e}{D} \cdot \frac{V^2}{2} \qquad (8.40b)$$

$$f = \frac{64}{Re} \qquad (8.36) \quad \text{(Laminar)}$$

$$\frac{1}{f^{0.5}} = -2.0 \cdot \log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re \cdot f^{0.5}}\right) \qquad (8.37) \quad \text{(Turbulent)}$$

The energy equation (Eq. 8.29) becomes for the circuit (1 = pump outlet, 2 = pump inlet)

$$\frac{p_1 - p_2}{\rho} = f \cdot \frac{L}{D} \cdot \frac{V^2}{2} + 4 \cdot f \cdot L_{elbow} \cdot \frac{V^2}{2} + f \cdot L_{valve} \cdot \frac{V^2}{2}$$

or

$$\Delta p = \rho \cdot f \cdot \frac{V^2}{2} \cdot \left(\frac{L}{D} + 4 \cdot \frac{L_{elbow}}{D} + \frac{L_{valve}}{D}\right)$$

Given data:

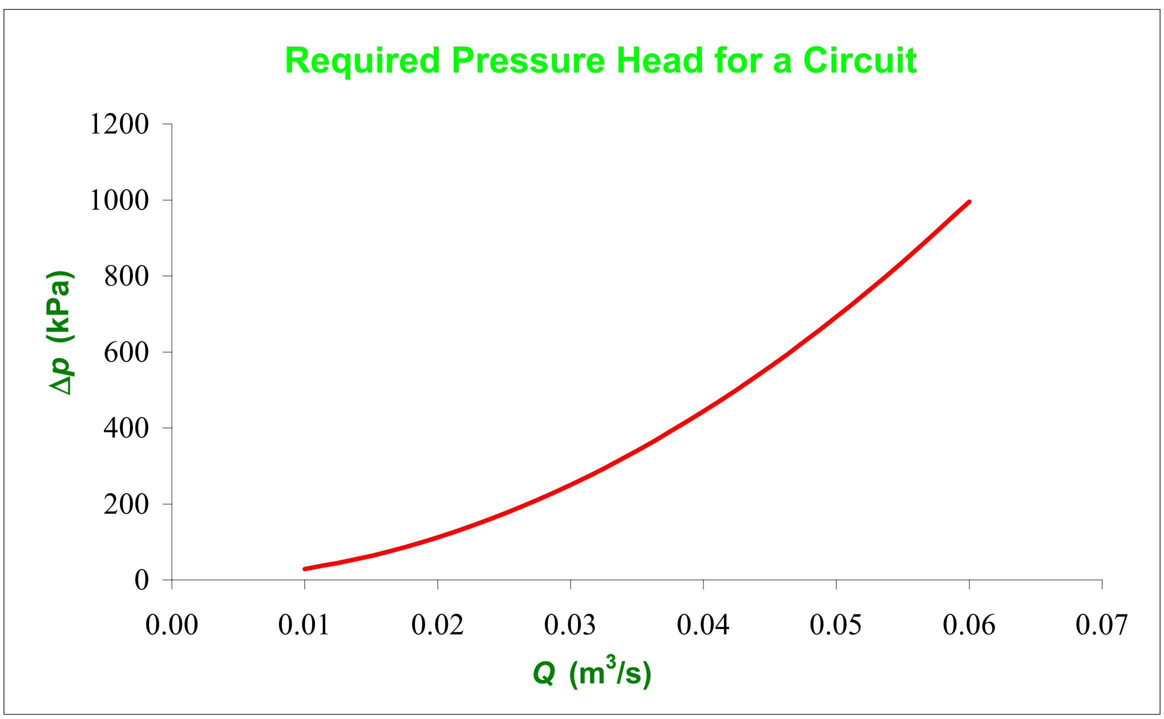
$L =$ 20 m
$D =$ 75 mm

Tabulated or graphical data:

$e =$ 0.26 mm
(Table 8.1)
$\mu =$ 1.00E-03 N.s/m^2
$\rho =$ 999 kg/m^3
(Appendix A)
Gate valve $L_e/D =$ 8
Elbow $L_e/D =$ 30
(Table 8.4)

Computed results:

Q (m^3/s)	V (m/s)	Re	f	Δp (kPa)
0.010	2.26	1.70E+05	0.0280	28.3
0.015	3.40	2.54E+05	0.0277	63.1
0.020	4.53	3.39E+05	0.0276	112
0.025	5.66	4.24E+05	0.0276	174
0.030	6.79	5.09E+05	0.0275	250
0.035	7.92	5.94E+05	0.0275	340
0.040	9.05	6.78E+05	0.0274	444
0.045	10.2	7.63E+05	0.0274	561
0.050	11.3	8.48E+05	0.0274	692
0.055	12.4	9.33E+05	0.0274	837
0.060	13.6	1.02E+06	0.0274	996



Problem 8.111

Given: Flow of standard air at 35 m^3/min, in smooth ducts of area, $A = 0.1\ m^2$.

Find: Compare pressure drop per unit length of a round duct with that for rectangular ducts of aspect ratio 1, 2 and 3.

Solution: Apply the energy equation for steady, incompressible flow that is uniform at each section. Use hydraulic diameter.

Basic equation: $\frac{p_1}{\rho} + \frac{\bar{V}_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \frac{\bar{V}_2^2}{2} + g z_2 + f\frac{L}{D_h}\frac{\bar{V}^2}{2} + h_{\ell m}$; $D_h = \frac{4A}{P_w}$

(terms marked: (1) on $\bar{V}_1^2/2$, $\bar{V}_2^2/2$; (2) on gz_1, gz_2; $=0(3)$ on $h_{\ell m}$)

Assumptions: (1) $\bar{V}_1 = \bar{V}_2$
(2) $z_1 = z_2$
(3) $h_{\ell m} = 0$

Then

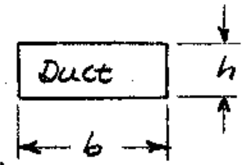
$$\Delta p = p_1 - p_2 = f\frac{L}{D_h}\frac{\rho \bar{V}^2}{2} \quad \text{or} \quad \frac{\Delta p}{L} = \frac{f}{D_h}\frac{\rho \bar{V}^2}{2}$$

But $\bar{V} = \frac{Q}{A} = 35\frac{m^3}{min} \times \frac{1}{0.1\ m^2} \times \frac{min}{60\ sec} = 5.83\ m/s$; $\nu = 1.46 \times 10^{-5}\ m^2/s$ (Table A.10)

$$Re = \frac{\bar{V} D_h}{\nu} = 5.83\frac{m}{s} \times D_h(m) \times \frac{s}{1.46 \times 10^{-5}\ m^2} = 3.99 \times 10^5 D_h(m) ; \frac{\rho \bar{V}^2}{2} = 20.9\ N/m^2$$

For a round duct, $D_h = D = \left(\frac{4A}{\pi}\right)^{\frac{1}{2}} = \left[\frac{4}{\pi} \times 0.1\ m^2\right]^{\frac{1}{2}} = 0.357\ m$

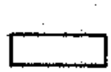
For a rectangular duct, $D_h = \frac{4A}{P_w} = \frac{4bh}{2(b+h)} = \frac{2h\,ar}{1+ar}$ where $ar = \frac{b}{h}$.



But $h = \frac{b}{ar}$, so $h^2 = \frac{bh}{ar} = \frac{A}{ar}$, or $h = \sqrt{\frac{A}{ar}}$ and $D_h = \frac{2ar^{1/2}}{1+ar} A^{1/2}$

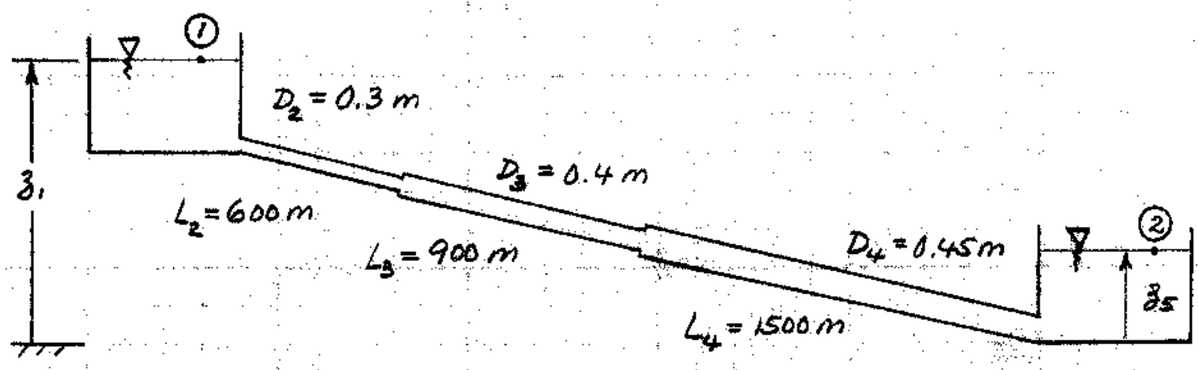
For smooth ducts, use Fig. 8.13 (or Blasius correlation, $f = \frac{0.316}{Re^{1/4}}$) to find f.

Tabulate results:

Duct Section	D_h (m)	Sketch	Re (−)	f (−)	$\Delta p/L$ (N/m^3)	Percent Increase
Round	0.357		1.43×10^5	0.0162	0.948	—
Square (ar=1)	0.316		1.26×10^5	0.0167	1.11	14.6
ar = 2	0.298		1.19×10^5	0.0170	1.19	20.3
ar = 3	0.274		1.09×10^5	0.0173	1.32	28.2

{Note that f varies only about 7 percent. The large change in $\Delta p/L$ is due primarily to the factor $\frac{f}{D_h}$.}

Given: Reservoirs connected by three clean, cast iron pipes in series. The flow is water at 0.11 m³/s and 15°C.



Find: Elevation difference, $z_1 - z_5$

Solution: Apply the energy equation for steady, incompressible flow that is uniform at each section.

Basic equations: $\frac{p_1}{\rho} + \frac{\bar{V}_1^2}{2} + gz_1 = \frac{p_5}{\rho} + \frac{\bar{V}_5^2}{2} + gz_5 + h_{\ell T}$ (terms cancelled by assumptions (1), (2))

$$h_{\ell T} = \Sigma f \frac{L}{D}\frac{\bar{V}^2}{2} + h_{\ell m}\ ;\ h_{\ell m} = K_{ent}\frac{\bar{V}_2^2}{2} + \Sigma h_{\ell exp} + K_{exit}\frac{\bar{V}_4^2}{2}$$ ($\Sigma h_{\ell exp} \approx 0$ (3))

Assumptions: (1) $p_1 = p_5 = p_{atm}$
(2) $\bar{V}_1 = \bar{V}_5 \simeq 0$
(3) Neglect $h_{\ell exp}$ at pipe joints (note all minor losses are probably small due to long lengths of straight pipe sections, but we will check)

For non-smooth pipe, $f = f(Re, e/D)$. $\mu = 1.1 \times 10^{-3}\ N\cdot s/m^2$ from Table A.8.

Section ②: $e/D_2 = 0.26\ mm/300\ mm = 0.00087$ (for cast iron, $e = 0.26$ mm, Table 8.1)

$$\bar{V}_2 = \frac{Q}{A_2} = 0.11\frac{m^3}{s} \times \frac{4}{\pi}\frac{1}{(0.3)^2 m^2} = 1.56\ m/s$$

$$Re_2 = \frac{\rho \bar{V}_2 D_2}{\mu} = 999\frac{kg}{m^3} \times 1.56\frac{m}{s} \times 0.3\,m \times \frac{m^2}{1.14\times10^{-3} N\cdot s} \times \frac{N\cdot s^2}{kg\cdot m} = 4.10\times10^5$$

From Fig 8.13, $f_2 = 0.020$

Section ③: $e/D_3 = 0.00065$

$$\bar{V}_3 = \frac{Q}{A_3} = 0.11\frac{m^3}{s} \times \frac{4}{\pi}\frac{1}{(0.4)^2 m^2} = 0.875\ m/s$$

$$Re_3 = \frac{\rho \bar{V}_3 D_3}{\mu} = 999\frac{kg}{m^3} \times 0.875\frac{m}{s} \times 0.4\,m \times \frac{m^2}{1.14\times10^{-3} N\cdot s} \times \frac{N\cdot s^2}{kg\cdot m} = 3.07\times10^5$$

From Fig. 8.13, $f_3 = 0.019$

Section ④: $e/D_4 = 0.00058$

$$\overline{V}_4 = \frac{Q}{A_4} = 0.11\,\frac{m^3}{s} \times \frac{4}{\pi}\,\frac{1}{(0.45)^2 m^2} = 0.692\ m/s$$

$$Re_4 = \frac{\rho \overline{V}_4 D_4}{\mu} = 999\,\frac{kg}{m^3} \times 0.692\,\frac{m}{s} \times 0.45\,m \times \frac{m^2}{1.14\times 10^{-3}\,N\cdot s} \times \frac{N\cdot s^2}{kg\cdot m} = 2.73\times 10^5$$

From Fig. 8.13, $f_4 = 0.0185$

Then
$$\sum f\frac{L}{D}\frac{\overline{V}^2}{2} = 0.020 \times \frac{600\,m}{0.3\,m} \times \frac{1}{2}\,(1.56)^2\frac{m^2}{s^2} + 0.019 \times \frac{900\,m}{0.4\,m} \times \frac{1}{2}\,(0.875)^2\frac{m^2}{s^2}$$

$$+\, 0.0185 \times \frac{1500\,m}{0.45\,m} \times \frac{1}{2}\,(0.692)^2\frac{m^2}{s^2} = 79.8\ m^2/s^2$$

The minor loss coefficients are $K_{ent} = 0.5$ (Table 8.2) and $K_{exit} = 1.0$. Thus,

$$h_{lm} = K_{ent}\frac{\overline{V}_2^2}{2} + K_{exit}\frac{\overline{V}_4^2}{2}$$

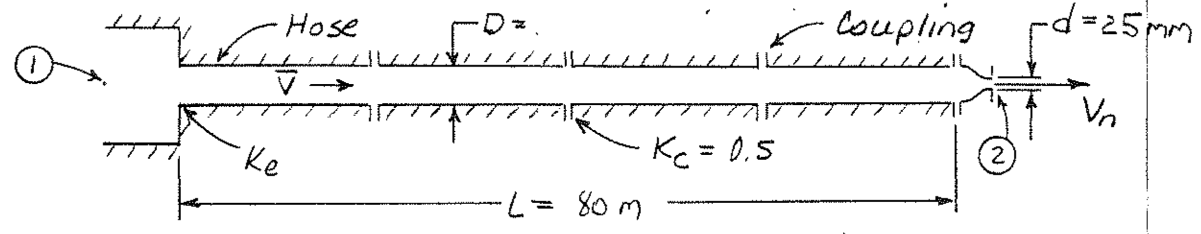
$$h_{lm} = 0.5 \times \frac{1}{2} \times (1.56)^2\frac{m^2}{s^2} + 1.0 \times \frac{1}{2} \times (0.692)^2\frac{m^2}{s^2} = 0.848\ m^2/s^2$$

Therefore minor losses are roughly 1 percent of the frictional losses, so they may be neglected. Thus from the energy equation

$$z_1 - z_5 = \sum f\frac{L}{D}\frac{\overline{V}^2}{2g} = 79.8\,\frac{m^2}{s^2} \times \frac{s^2}{9.81\,m} = 8.13\ m$$

$z_1 - z_5$

Given: Water at $Q = 20$ L/s in hose and nozzle assembly.



$e/D = 0.004$

Find: supply pressure required.

Solution: Apply the energy equation for steady, incompressible pipe flow.

Computing equation: $$\frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \alpha_2 \frac{\bar{V}_2^2}{2} + gz_2 + h_{l_T}$$

(with $\bar{V}_1 \approx 0$ (1), z_1, z_2 (2), $p_2 = 0$ (4) crossed out)

Assumptions: (1) $\bar{V}_1 \approx 0$, (2) $\Delta z = 0$, (3) $\alpha_2 \simeq 1.0$, (4) $p_2 = p_{atm}$, so $p_{gage} = 0$

(5) $h_{l_T} = h_l + h_{l_m} = f\frac{L}{D}\frac{\bar{V}^2}{2} + (K_e + 4K_c)\frac{\bar{V}^2}{2} + K_n\frac{\bar{V}_n^2}{2}$

From continuity, $\bar{V}_n = (\frac{D}{d})^2 \bar{V}$, so $\bar{V}_n^2 = (\frac{D}{d})^4 \bar{V}^2$. Thus

$$p_1 = \frac{\rho \bar{V}^2}{2}\left[f\frac{L}{D} + 4K_c + K_e + (\frac{D}{d})^4(1+K_n)\right]$$

To find f, we need Re. From continuity

$$\bar{V} = \frac{Q}{A} = \frac{4Q}{\pi D^2} = \frac{4}{\pi} \times 20\,\frac{L}{s} \times \frac{1}{(0.075\,m)^2} \times \frac{10^{-3}\,m^3}{L} = 4.53\ m/s$$

Assuming $T = 15°C$, $\nu = 1.0 \times 10^{-6}\ m^2/s$ (Table A.8), so

$$Re = \frac{\bar{V}D}{\nu} = 4.53\frac{m}{s} \times 0.075\,m \times \frac{s}{1.0\times10^{-6}\,m^2} = 3.39\times10^5$$

From Equation 8.37, with $e/D = 0.004$, $f = 0.0287$. Substituting,

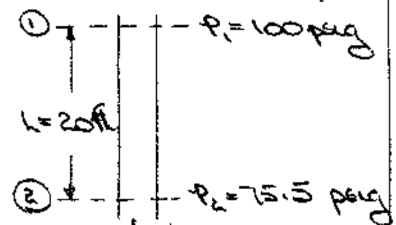
$$p_1 = \frac{1}{2} \times 999\frac{kg}{m^3} \times (4.53)^2\frac{m^2}{s^2}\left[0.0287 \times \frac{80\,m}{0.075\,m} + 4(0.5) + 0.5 + (\frac{3}{1})^4(1.02)\right]\frac{N\cdot s^2}{kg\cdot m}$$

$p_1 = 593$ kPa

p_1

Problem 8.114

Given: Water flow, $Q = 0.11\ \text{ft}^3/\text{s}$, through a corroded section of galvanized 1.0 in i.d. pipe with pressure readings as shown



Find: (a) estimate of relative roughness in the pipe section
(b) percent savings in pumping power if e/D value were that for clean pipe.

Solution: Apply the energy equation for steady, incompressible pipe flow

Computing equation:

$$\left(\frac{p_1}{\rho} + \alpha_1\frac{\bar{V}_1^2}{2} + gz_1\right) - \left(\frac{p_2}{\rho} + \alpha_2\frac{\bar{V}_2^2}{2} + gz_2\right) = h_{l_T} \quad \text{---(1)}$$

$$h_{l_T} = h_l + h_{l_m} = f\frac{L}{D}\frac{\bar{V}^2}{2} + K\frac{\bar{V}^2}{2} \;(\approx 0\ (4)) \quad \text{---(2)}$$

Assumptions:
(1) $\bar{V}_1 = \bar{V}_2$ from continuity
(2) $\alpha_1 = \alpha_2$
(3) $z_1 - z_2 = 20$ ft
(4) no minor losses

Since $f = f(e/D, Re)$, solve for f from eqs (1) & (2), calculate Re, and then determine e/D from Fig. 8.13

From eq's (1) & (2)

$$\frac{p_1 - p_2}{\rho} + g(z_1 - z_2) = f\frac{L}{D}\frac{\bar{V}^2}{2} \qquad \therefore f = \frac{D2}{L\bar{V}^2}\left[\frac{p_1 - p_2}{\rho} + g(z_1 - z_2)\right]$$

$$\bar{V} = \frac{Q}{A} = \frac{Q4}{\pi D^2} = \frac{4}{\pi} \times 0.11\ \frac{\text{ft}^3}{\text{s}} \times \left(\frac{12}{1\ \text{ft}}\right)^2 = 20.2\ \text{ft/s}. \quad \text{Then,}$$

$$f = 2 \times \frac{1\ \text{ft}}{12} \times \frac{1}{20\ \text{ft}} \times \left(\frac{\text{s}}{20.2\ \text{ft}}\right)^2 \left[(100 - 75.5)\frac{\text{lbf}}{\text{in}^2} \times \frac{144\ \text{in}^2}{\text{ft}^2} \times \frac{\text{ft}^3}{1.94\ \text{slug}} \times \frac{\text{slug}\cdot\text{ft}}{\text{lbf}\cdot\text{s}^2} + 32.2\ \frac{\text{ft}}{\text{s}^2} \times 20\ \text{ft}\right] = 0.050$$

Assume $T = 70°$ F, then $\nu = 1.05 \times 10^{-5}\ \text{ft}^2/\text{s}$ (Table A.7)

$$Re = \frac{\rho D\bar{V}}{\mu} = \frac{D\bar{V}}{\nu} = \frac{1}{12}\text{ft} \times 20.2\ \frac{\text{ft}}{\text{s}} \times 1.05 \times 10^{-5}\ \frac{\text{s}}{\text{ft}^2} = 1.60 \times 10^5$$

For $f = 0.050$ and $Re = 1.60 \times 10^5$, from Fig. 8.13, $\frac{e}{D} \approx 0.021$ ← $\frac{e}{D}$

For a 1.0 in. diameter clean galvanized iron pipe, $\frac{e}{D} = 0.006$ (Table 8.1)

Then, from Fig. 8.13 $f = 0.0325$ and for the clean pipe

$$h_{l\,clean} = f\frac{L}{D}\frac{\bar{V}^2}{2} = 0.0325 \times \frac{20\ \text{ft}}{2 \times 1\ \text{in}} \times \frac{12\ \text{in}}{\text{ft}} \times (20.2)^2\frac{\text{ft}^2}{\text{s}^2} = 1590\ \frac{\text{ft}^2}{\text{s}^2}$$

$$(p_1 - p_2)_{clean} = \rho\left[h_{l\,clean} + g(z_2 - z_1)\right] = 1.94\ \frac{\text{slug}}{\text{ft}^3}\left[1590\ \frac{\text{ft}^2}{\text{s}^2} + 32.2\ \frac{\text{ft}}{\text{s}^2}(-20\ \text{ft})\right]\frac{\text{lbf}\cdot\text{s}^2}{\text{ft}\cdot\text{slug}}\frac{\text{ft}^2}{144\ \text{in}^2}$$

$$\Delta p_{clean} = 12.7\ \text{lbf/in}^2$$

$$\%\ \text{Saving in pump power} = \frac{\Delta p_{dirty} - \Delta p_{clean}}{\Delta p_{dirty}} = \frac{24.5 - 12.7}{24.5} = 48.2\% \quad \text{← Power Saving}$$

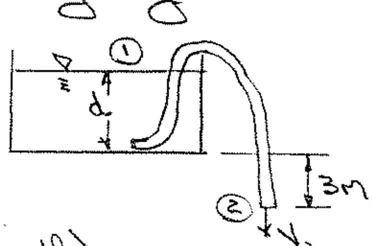
Problem 8.115

Given: Small swimming pool is drained using a garden hose.

Hose: $D = 20\,mm$, $L = 30\,m$

$e = 0.2\,mm$

$\bar{V}_2 = 1.2\,m/s$



Find: Water depth at instant shown.
If the flow were inviscid (at this depth) what would be the velocity

Solution:

Apply the energy equation for steady incompressible flow between sections ① and ②

Basic equations: $\left(\frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2} + g z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \frac{\bar{V}_2^2}{2} + g z_2\right) = h_{l_T}$ (8.29)

$h_{l_T} = h_l + h_{l_m}$; $h_l = f \frac{L}{D} \frac{\bar{V}^2}{2}$; $h_{l_m} = K_{ent} \frac{\bar{V}^2}{2}$

Assumptions: (1) $p_1 = p_2 = p_{atm}$

(2) $\bar{V}_1 = 0$, $\alpha_2 = 1.0$

(3) square edged entrance

Then

$$z_1 - z_2 = d + 3m = f \frac{L}{D} \frac{\bar{V}^2}{2g} + K_{ent} \frac{\bar{V}^2}{2g} + \frac{\bar{V}_2^2}{2g} = \frac{\bar{V}^2}{2g}\left[f \frac{L}{D} + K_{ent} + 1\right]$$

$$\therefore d = \frac{\bar{V}^2}{2g}\left[f \frac{L}{D} + K_{ent} + 1\right] - 3m \qquad (1)$$

For square edged entrance (Table 8.2) $K_{ent} = 0.5$

$Re = \frac{D\bar{V}}{\nu} = 0.020\,m \times 1.2\,\frac{m}{s} \times \frac{s}{1.00 \times 10^{-6}\,m^2} = 2.4 \times 10^4$ {assume $T = 20°C$, Table A.8}

$e/D = 0.2/20 = 0.01$. From Fig. 8.13, $f = 0.04$

Then from Eq. 1

$$d = \frac{(1.2)^2}{2} \frac{m^2}{s^2} \times \frac{s^2}{9.81\,m}\left[0.04 \times \frac{30}{0.02} + 0.5 + 1\right] - 3m = 1.51\,m \qquad d$$

For frictionless flow, $h_{l_T} = f \frac{L}{D} \frac{\bar{V}^2}{2} + K_{ent} \frac{\bar{V}^2}{2} = 0$ and Eq. 1 gives $d = \frac{\bar{V}^2}{2g} - 3m$

and $\bar{V} = \left[2g(d + 3m)\right]^{1/2} = \left[2 \times 9.81\,\frac{m}{s^2}(1.51 + 3)\,m\right]^{1/2}$

$\bar{V} = 9.41\,m/s$ $\qquad \bar{V}_{inviscid}$

Problem 8.116 (In Excel)

Flow in a tube may alternate between laminar and turbulent states for Reynolds numbers in the transition zone. Design a bench-top experiment consisting of a constant-head cylindrical transparent plastic tank with depth graduations, and a length of plastic tubing (assumed smooth) attached at the base of the tank through which the water flows to a measuring container. Select tank and tubing dimensions so that the system is compact, but will operate in the transition zone range. Design the experiment so that you can easily increase the tank head from a low range (laminar flow) through transition to turbulent flow, and vice versa. (Write instructions for students on recognizing when the flow is laminar or turbulent.) Generate plots (on the same graph) of tank depth against Reynolds number, assuming laminar or turbulent flow.

Solution

Governing equations:

$$Re = \frac{\rho \cdot V \cdot D}{\mu}$$

$$\left(\frac{p_1}{\rho} + \alpha_1 \cdot \frac{V_1^{\,2}}{2} + g \cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \cdot \frac{V_2^{\,2}}{2} + g \cdot z_2\right) = h_{lT} = \sum_{major} h_l + \sum_{minor} h_{lm} \qquad (8.29)$$

$$h_l = f \cdot \frac{L}{D} \cdot \frac{V^2}{2} \qquad (8.34)$$

$$h_{lm} = K \cdot \frac{V^2}{2} \qquad (8.40a)$$

$$f = \frac{64}{Re} \qquad (8.36) \quad \text{(Laminar)}$$

$$\frac{1}{f^{0.5}} = -2.0 \cdot \log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re \cdot f^{0.5}}\right) \qquad (8.37) \quad \text{(Turbulent)}$$

The energy equation (Eq. 8.29) becomes

$$g \cdot H - \alpha \cdot \frac{V^2}{2} = f \cdot \frac{L}{D} \cdot \frac{V^2}{2} + K \cdot \frac{V^2}{2}$$

This can be solved explicity for reservoir height *H*

$$H = \frac{V^2}{2 \cdot g} \cdot \left(\alpha + f \cdot \frac{L}{D} + K\right)$$

Choose data:

L =	1.0	m
D =	3.0	mm
e =	0.0	mm
α =	2	(Laminar)
=	1	(Turbulent)

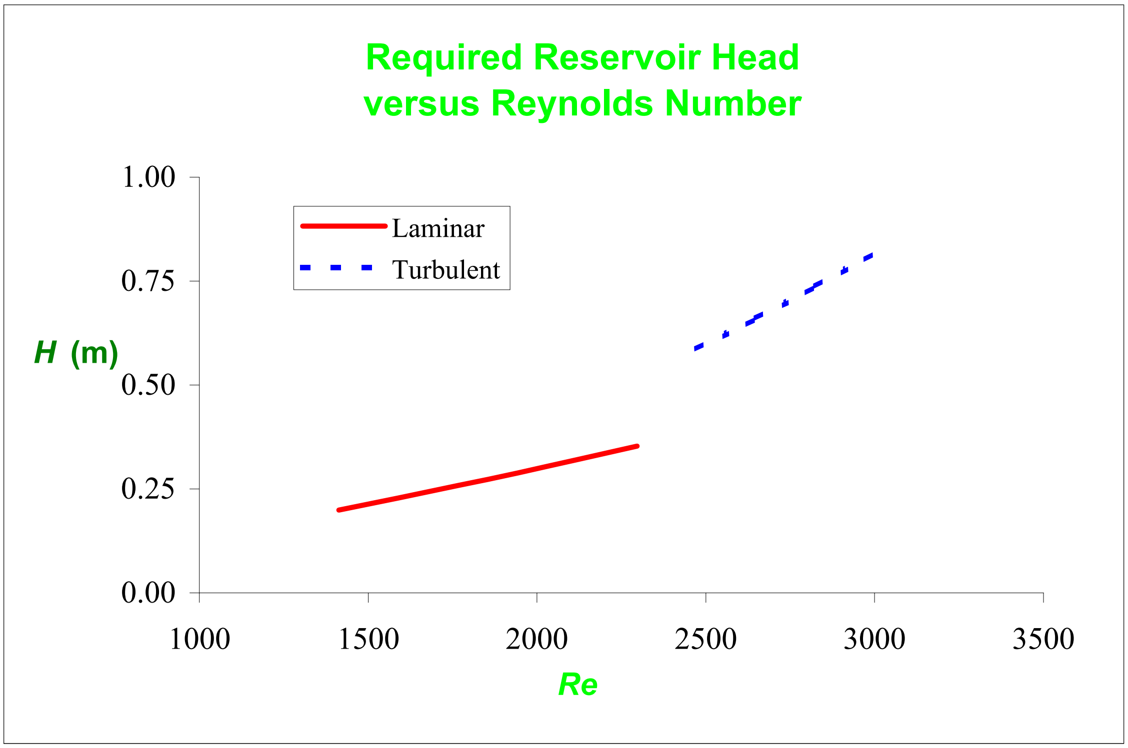
Tabulated or graphical data:

μ =	1.00E-03	$N.s/m^2$
ρ =	999	kg/m^3
	(Appendix A)	
K_{ent} =	0.5	(Square-edged)
	(Table 8.2)	

Computed results:

Q **(L/min)**	***V*** **(m/s)**	***Re***	**Regime**	***f***	***H*** **(m)**
0.200	0.472	1413	Laminar	0.0453	0.199
0.225	0.531	1590	Laminar	0.0403	0.228
0.250	0.589	1767	Laminar	0.0362	0.258
0.275	0.648	1943	Laminar	0.0329	0.289
0.300	0.707	2120	Laminar	0.0302	0.320
0.325	0.766	2297	Laminar	0.0279	0.353
0.350	0.825	2473	Turbulent	0.0462	0.587
0.375	0.884	2650	Turbulent	0.0452	0.660
0.400	0.943	2827	Turbulent	0.0443	0.738
0.425	1.002	3003	Turbulent	0.0435	0.819
0.450	1.061	3180	Turbulent	0.0428	0.904

The flow rates are realistic, and could easily be measured using a tank/timer system
The head required is also realistic for a small-scale laboratory experiment
Around *Re* = 2300 the flow may oscillate between laminar and turbulent:
Once turbulence is triggered (when *H* > 0.353 m), the resistance to flow increases requiring *H* >0.587 m to maintain; hence the flow reverts to laminar, only to trip over again to turbulent! This behavior will be visible: the exit flow will switch back and forth between smooth (laminar) and chaotic (turbulent)



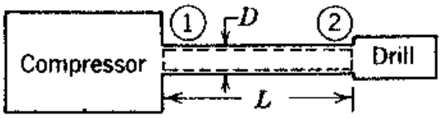
Problem 8.117

Given: Air flow through a line, of length L and diameter $D = 40\,mm$

$p_1 = 670\ kPa(g)$ $p_2 = 650\ kPa(g)$

$T_1 = 40°C$ $\dot{m} = 0.25\ kg/s$

$\rho \approx$ constant



Find: Allowable length of hose

Solution:

Computing equation: $\left(\frac{p_1}{\rho} + \alpha_1\frac{\bar{V}_1^2}{2} + gz_1\right) - \left(\frac{p_2}{\rho} + \alpha_2\frac{\bar{V}_2^2}{2} + gz_2\right) = h_{l_T} = h_l + h_{l_m}$ (8.29)

where $h_l = f\frac{L}{D}\frac{\bar{V}^2}{2}$ $\quad h_{l_m} = K\frac{\bar{V}^2}{2}$

For $\rho = c$, then $\bar{V}_1 = \bar{V}_2$, since $A_1 = A_2$. Since p_1 and p_2 are given, neglect minor losses. Assume $\alpha_1 = \alpha_2$ and neglect elevation changes. Then Eq. 8.29 can be written as

$$\frac{p_1 - p_2}{\rho} = f\frac{L}{D}\frac{\bar{V}^2}{2} \qquad \text{or} \qquad L = \frac{(p_1 - p_2)}{\rho}\frac{2D}{f\bar{V}^2}$$

The density is

$$\rho = \rho_1 = \frac{p_1}{RT_1} = 7.91\times10^5\frac{N}{m^2}\times\frac{kg\cdot K}{287\,N\cdot m}\times\frac{1}{313\,K} = 8.81\ kg/m^3$$

From continuity

$$\bar{V} = \frac{\dot{m}}{\rho A} = \frac{4\dot{m}}{\pi\rho D^2} = \frac{4}{\pi}\times 0.25\frac{kg}{sec}\times\frac{m^3}{8.81\,kg}\times\frac{1}{(0.04)^2 m^2} = 22.6\ m/sec$$

For air at 40°C, $\mu = 1.91\times10^{-5}\ kg/m\cdot s$ (Table A.10), so

$$Re = \frac{\rho\bar{V}D}{\mu} = 8.81\frac{kg}{m^3}\times 22.6\frac{m}{sec}\times 0.04\,m\times\frac{m\cdot sec}{1.91\times10^{-5}\,kg} = 4.17\times10^5$$

Assume smooth pipe; then from Fig. 8.13, $f = 0.0134$

Substituting gives

$$L = \frac{(p_1 - p_2)}{\rho}\frac{2D}{f\bar{V}^2}$$

$$= 20\times10^3\frac{N}{m^2}\times 2\times 0.04\,m\times\frac{m^3}{8.81\,kg}\times\frac{1}{0.0134}\times\frac{sec^2}{(22.6)^2 m^2}\times\frac{kg\cdot m}{N\cdot sec^2}$$

$L = 26.5\ m$ ← L

Problem 8.118

Given: Gasoline flow in a horizontal pipeline at 15°C. The distance and pressure drop between pumping stations are 13 km and 1.4 MPa, respectively. The pipe is 0.6 m in diameter. Its roughness corresponds to galvanized iron.

Find: Volume flow rate.

Solution: Apply the energy equation for steady, incompressible flow that is uniform at each section.

Basic equation: $$\frac{p_1}{\rho} + \alpha_1\frac{\bar{V}_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \alpha_2\frac{\bar{V}_2^2}{2} + gz_2 + h_{l_T}\,; \quad h_{l_T} = f\frac{L}{D}\frac{\bar{V}^2}{2} + h_{l_m}$$

(marked: $\alpha_1 = \alpha_2$, $\bar{V}_1 = \bar{V}_2$ (1); $z_1 = z_2$ (2); $h_{l_m} = 0$ (1))

Assumptions: (1) Constant area pipe, so $\bar{V}_1 = \bar{V}_2$, $h_{l_m} = 0$
(2) Level, so $z_1 = z_2$

Thus
$$\frac{p_1 - p_2}{\rho} = f\frac{L}{D}\frac{\bar{V}^2}{2} \quad \text{or} \quad \bar{V} = \left[\frac{2D(p_1 - p_2)}{\rho f L}\right]^{\frac{1}{2}}$$

But $f = f(Re, e/D)$, and the Reynolds number is not known. Therefore iteration is required. Choose f in the fully-rough zone. From Table 8.1, $e = 0.15$ mm; $e/D = 0.00025$. Then from Fig. 8.13, $f \simeq 0.014$. {From Eq. 8.37, using Excel's solver, $f = 0.014$} Then,

$$\bar{V} = \left[2 \times 0.6\,\text{m} \times 1.4\times10^{6}\,\frac{\text{N}}{\text{m}^2} \times \frac{\text{m}^3}{(0.72)1000\,\text{kg}} \times \frac{1}{0.014} \times \frac{1}{13\times10^{3}\,\text{m}} \times \frac{\text{kg}\cdot\text{m}}{\text{N}\cdot\text{s}^2}\right]^{\frac{1}{2}}$$

{SG = 0.72, Table A.2}

$\bar{V} = 3.58$ m/s

Now compute Re and check on guess for f. Choose $\mu \simeq 5\times10^{-4}\,\text{N}\cdot\text{s}/\text{m}^2$ (Fig. A.2).*

$$Re = \frac{\rho\bar{V}D}{\mu} = (0.72)1000\,\frac{\text{kg}}{\text{m}^3} \times 3.58\,\frac{\text{m}}{\text{s}} \times 0.6\,\text{m} \times \frac{\text{m}^2}{5\times10^{-4}\,\text{N}\cdot\text{s}} \times \frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}} = 3.09\times10^{6}$$

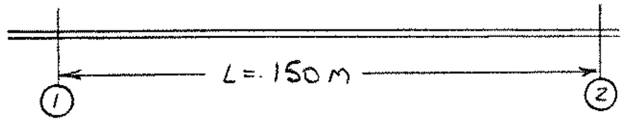
Checking on Fig. 8.13, flow is essentially in the fully-rough zone, and initial guess for f was okay. Thus

$$Q = \bar{V}A = 3.58\,\frac{\text{m}}{\text{s}} \times \frac{\pi}{4}(0.6)^2\,\text{m}^2 = 1.01\ \text{m}^3/\text{s}$$ Q

* Note gasoline is between heptane and octane.

Given: Steady flow of water in 5 in. diameter, horizontal, cast-iron pipe.

$D = 125\,mm$



$\Delta p = p_1 - p_2 = 150\,kPa$

Find: Volume flow rate.

Solution: Apply the energy equation for steady, incompressible pipe flow.

Computing equation: $$\left(\frac{p_1}{\rho} + \alpha_1\frac{\bar{V}_1^2}{2} + gz_1\right) - \left(\frac{p_2}{\rho} + \alpha_2\frac{\bar{V}_2^2}{2} + gz_2\right) + h_{l_T}$$

$$h_{l_T} = h_l + h_{lm} = f\frac{L}{D}\frac{\bar{V}^2}{2} + K\frac{\bar{V}^2}{2}$$

Assumptions: (1) Fully developed flow: $\alpha_1\bar{V}_1^2 = \alpha_2\bar{V}_2^2$
(2) Horizontal: $z_1 = z_2$
(3) Constant area, so $K = 0$

Then
$$\frac{\Delta p}{\rho} = h_{l_T} = f\frac{L}{D}\frac{\bar{V}^2}{2} \quad \text{so} \quad \bar{V} = \sqrt{\frac{2\Delta p D}{\rho f L}}$$

Since flow rate (hence Re and f) are unknown, must iterate. Guess a trial value of f in the fully rough zone. From Table 8.1, e = 0.26 mm

Then $e/D = \frac{0.26}{125} = 0.0021$. Then from Eq. 8.37*, $f = 0.0237$ for $Re \geq 6\times10^5$

$$\bar{V} = \left[2\times150\times10^3\,\frac{N}{m^2}\times0.125\,m\times\frac{m^3}{999\,kg}\times\frac{1}{0.0237}\times\frac{1}{150\,m}\times\frac{kg\cdot m}{N\cdot s^2}\right]^{1/2} = 3.25\,m/s$$

and, checking Re, with $\nu = 1.14\times10^{-6}\,m^2/s$ at $T = 15°C$ (Table A.8),

$$Re = \frac{\bar{V}D}{\nu} = 3.25\frac{m}{s}\times0.125\,m\times\frac{s}{1.14\times10^{-6}\,m^2} = 3.56\times10^5$$

The friction factor at this Re is still $f = 0.0242$ (2% error), so convergence is ok.

$$Q = \bar{V}A = 3.25\,\frac{m}{s}\times\frac{\pi}{4}\times(0.125\,m)^2 = 0.0399\,m^3/s$$ Q

Using $f = .0242$, $\bar{V} = 3.22\,m/s$ and $Q = 0.0395\,m^3/s$

* Value of $f = 0.237$ obtained using Excel's Solver (or Goal Seek)

Problem 8.120

Given: Steady flow of water through a cast iron pipe of diameter $D = 125\,mm$. The pressure drop over a length of pipe, $L = 150\,m$ is $p_1 - p_2 = 150\,kPa$. Section 2 is located 15 m above section 1.

Find: the volume flow rate, Q.

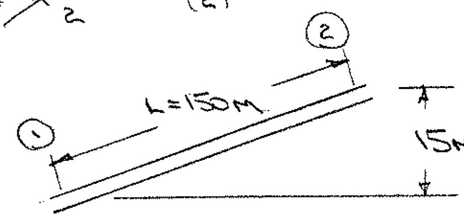
<u>Solution:</u> Apply the energy equation for steady, incompressible pipe flow

Computing equation:

$$\left(\frac{p_1}{\rho} + \alpha_1\frac{\bar{V}_1^2}{2} + gz_1\right) - \left(\frac{p_2}{\rho} + \alpha_2\frac{\bar{V}_2^2}{2} + gz_2\right) = h_{l_T} \quad (1)$$

$$h_{l_T} = h_l + h_{l_m} = f\frac{L}{D}\frac{\bar{V}^2}{2} + K\frac{\bar{V}^2}{2} \rightarrow 0\,(4) \quad (2)$$

Assumptions:
(1) $\bar{V}_1 = \bar{V}_2$ from continuity
(2) $\alpha_1 = \alpha_2$
(3) $z_2 - z_1 = 15\,m$
(4) neglect minor losses



For cast iron pipe with $D = 125\,mm$ $\frac{e}{D} = 0.0021$ ($e = 0.26\,mm$, Table 8.1)

Since $f = f(Re)$ and $\bar{V}$ is unknown, iteration will be required

From Eqs (1) and (2)

$$\left(\frac{p_1}{\rho} + gz_1\right) - \left(\frac{p_2}{\rho} + gz_2\right) = f\frac{L}{D}\frac{\bar{V}^2}{2}$$

Then

$$f\bar{V}^2 = \frac{2D}{L}\left[\frac{(p_1 - p_2)}{\rho} + g(z_1 - z_2)\right]$$

$$f\bar{V}^2 = 2 \times \frac{0.125\,m}{150\,m}\left[150 \times 10^3\,\frac{N}{m^2} \times \frac{m^3}{999\,kg} \times \frac{kg \cdot m}{N \cdot s^2}\right] + 9.81\,\frac{m}{s^2} \times (-15\,m)$$

$$f\bar{V}^2 = 0.005\,m^2/s^2$$

Assume flow in fully rough region, $f = 0.0237$, then $\bar{V} = 0.46\,m/s$

Check Re. Assume $T = 15°C$, $\nu = 1.14 \times 10^{-6}\,m^2/s$ (Table A.8)

Then $Re = \frac{D\bar{V}}{\nu} = 0.125\,m \times 0.46\,\frac{m}{s} \times \frac{1}{1.14 \times 10^{-6}}\,\frac{s}{m^2} = 50{,}400$

From Eq. 8.37 with $Re = 50{,}400$, $e/D = 0.0021$, then using Excel's solver (or Goal Seek)

$$f = 0.0267 \text{ and } \bar{V} = 0.433\,m/s$$

With this value of $\bar{V}$, $Re = 47{,}500$, $f = 0.0268$, $\bar{V} = 0.432\,m/s$

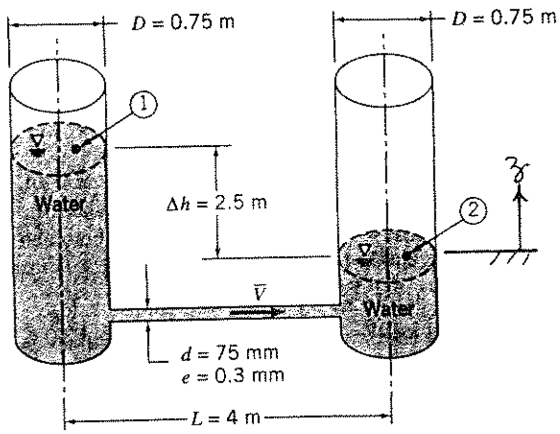
Then

$$Q = A\bar{V} = \frac{\pi D^2}{4}\bar{V} = \frac{\pi}{4}(0.125\,m)^2 \times 0.432\,\frac{m}{s} = 0.0053\,m^3/s \qquad Q$$

Problem 8.121

Given: Two open standpipes shown. Water flows by gravity.

Find: Estimate of rate of change of water level in left standpipe.



Solution: Apply the energy equation for quasi-steady, incompressible pipe flow.

Computing equation:

$$\left(\frac{p_1}{\rho} + \alpha_1 \frac{V_1^2}{2} + g z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \frac{V_2^2}{2} + g z_2\right) = h_{\ell T}; \quad z_2 = 0; \quad h_{\ell T} = h_\ell + h_{\ell m} = \left[f\frac{(L-D)}{d} + K_{ent} + K_{exit}\right]\frac{\bar{V}^2}{2}$$

From continuity, $A_1 V_1 = A_p \bar{V}$

Assumptions: (1) Neglect unsteady effects
(2) Incompressible flow
(3) $p_1 = p_2 = p_{atm}$
(4) $\bar{V}_1 = \bar{V}_2$ since diameters are equal

Then $g\,\Delta h = h_{\ell T} = \left[f\frac{(L-D)}{d} + K_{ent} + K_{exit}\right]\frac{\bar{V}^2}{2}$

Flow rate (hence speed) is unknown, so assume flow is in fully rough zone.

$\frac{e}{D} = \frac{0.3}{75} = 0.004$, so $f \approx 0.0285$ from Eq. 8.37 (using Excel's Solver or Goal Seek)

From Table 8.2, $K_{ent} = 0.5$; from Fig. 8.15, $K_{exit} = 1$. Then

$$\bar{V} = \left[\frac{2g\,\Delta h}{f\frac{(L-D)}{d} + K_{ent} + K_{exit}}\right]^{\frac{1}{2}} = \left[\frac{2 \times 9.81\,\frac{m}{s^2} \times 2.5\,m}{0.028\,\left(\frac{4-0.75}{0.075}\right) + 0.5 + 1.0}\right]^{\frac{1}{2}} = 4.23\ m/s$$

Check Re and f. For water at 20°C, $\nu = 1.00 \times 10^{-6}\ m^2/s$ (Table A.8)

$$Re = \frac{\bar{V}d}{\nu} = 4.23\,\frac{m}{s} \times 0.075\,m \times \frac{s}{1.00 \times 10^{-6}\,m^2} = 3.18 \times 10^5$$

From Equation 8.37, $f \approx 0.0288$, so this is satisfactory agreement. (~1%)

$$V_1 = \frac{A_p}{A_1} V_p = \left(\frac{d}{D}\right)^2 V_p = \left(\frac{0.075}{0.75}\right)^2 \times 4.23\,\frac{m}{s} = 0.0423\ m/s \text{ (down)}$$

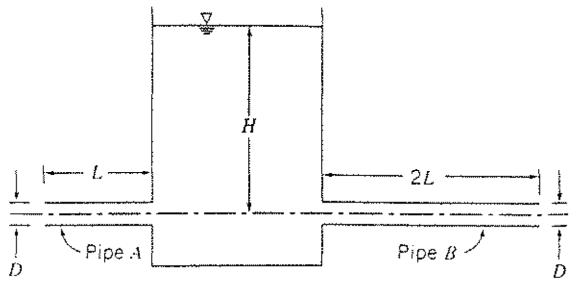
The water level in the left tank <u>falls</u> at about 42.3 mm/s

Problem 8.122

Given: Two galvanized iron pipes connected to large water reservoir as shown.

Determine: (a) which pipe will pass the larger flow rate (without calculations); justify

(b) the larger flow rate if $H = 10\,m$, $D = 50\,mm$, $L = 50\,m$



Solution:

Flow through each pipe is governed by the energy equation for steady incompressible flow

Basic equations: $\left(\frac{p_1}{\rho} + \alpha_1\frac{\bar{V}_1^2}{2} + gz_1\right) - \left(\frac{p_2}{\rho} + \alpha_2\frac{\bar{V}_2^2}{2} + gz_2\right) = h_{l_T}$ (8.29)

$$h_{l_T} = h_l + h_{l_m} = f\frac{L}{D}\frac{\bar{V}^2}{2} + K_{ent}\frac{\bar{V}^2}{2}$$

Assumptions: (1) $p_1 = p_2 = p_3 = p_{atm}$

(2) $\bar{V}_1 = 0$, $\alpha_2 = \alpha_3 = 1.0$

Then

$$g(z_1 - z_2) = h_{l_T} + \frac{\bar{V}_2^2}{2} = \frac{\bar{V}_2^2}{2}\left[f\frac{L}{D} + K_{ent} + 1\right] \quad \text{(pipe A)}$$

$$g(z_1 - z_3) = h_{l_T} + \frac{\bar{V}_3^2}{2} = \frac{\bar{V}_3^2}{2}\left[f\frac{2L}{D} + K_{ent} + 1\right] \quad \text{(pipe B)}$$

Since $z_1 - z_2 = z_1 - z_3$, then $\bar{V}_2 > \bar{V}_3$ and $Q_A > Q_B$ ←

Through pipe A $\quad gH = \frac{\bar{V}_2^2}{2}\left[f\frac{L}{D} + K_{ent} + 1\right]$

From Table 8.1 $e = 0.15\,m$ $\therefore e/D = 0.15/50 = 0.003$

Assume water at 20°C, $\nu = 1.00 \times 10^{-6}\,m^2/s$ (Table A.8)

Choose friction factor $f = 0.0263^*$ (in fully rough region)

Then

$$\bar{V}_2 = \left\{\frac{2gH}{\left[f\frac{L}{D} + K_{ent} + 1\right]}\right\}^{1/2} = \left\{2 \times 9.81\frac{m}{s^2} \times 10\,m \times \frac{1}{\left[0.0263 \times \frac{50}{0.05} + 0.5 + 1.0\right]}\right\}$$

$$\bar{V}_2 = 2.66\,m/s$$

Check $Re = \frac{D\bar{V}}{\nu} = 0.05\,m \times 2.66\frac{m}{s} \times \frac{1}{1.00\times10^{-6}}\frac{s}{m^2} = 1.33 \times 10^5$

At this Re, $f = 0.0272^*$ and $\bar{V}_2 = 2.62\,m/s$

$$Q = A\bar{V} = \frac{\pi D^2}{4}\bar{V} = \frac{\pi}{4}\times(0.05\,m)^2 \times 2.62\frac{m}{s} = 5.14 \times 10^{-3}\,m^3/s \leftarrow Q_A$$

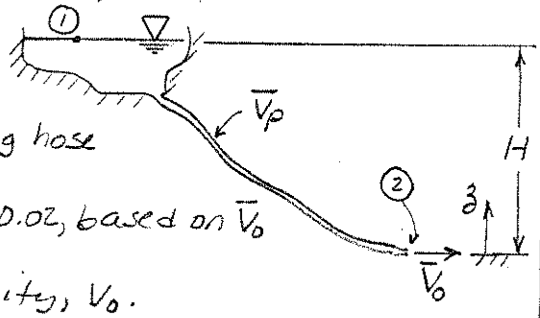
* Value obtained from Eq. 8.37, using Excel Solver (or Goal Seek)

Given: Site for hydraulic mining, $H = 300\ m$, $L = 900\ m$.

Hose with $D = 75\ mm$, $e/D = 0.01$.

Couplings, $\frac{Le}{D} = 20$, every 10 m along hose

Nozzle diameter, $d = 25\ mm$; $K = 0.02$, based on $\bar{V}_0$



Find: (a) Estimate maximum outlet velocity, V_0.
(b) Determine maximum force of jet on rock face.

<u>Solution</u>: Apply the energy equation for steady, incompressible pipe flow.

Computing equation: $\left(\frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2} + g z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \frac{\bar{V}_2^2}{2} + g z_2\right) = h_{\ell T}$ $\quad$ ($p_1 = 0(1)$, $\bar{V}_1 = 0(2)$, $p_2 = 0(3)$)

Assume: (1) $p_1 = 0$; (2) $\bar{V}_1 = 0$; (3) $p_2 = 0$; (4) $\alpha_2 = 1$; (5) $z_2 = 0$; (6) Fully-rough zone

Then $gH = h_{\ell T} + \frac{\bar{V}_2^2}{2} = f \frac{L}{D}\frac{\bar{V}_p^2}{2} + f \times 90 \frac{Le}{D}\frac{\bar{V}_p^2}{2} + K\frac{\bar{V}_0^2}{2} + \frac{\bar{V}_0^2}{2}$

From continuity $\bar{V}_p A_p = \bar{V}_0 A_0$; $\bar{V}_2 = \bar{V}_0 \frac{A_0}{A_2}$; $\bar{V}_2^2 = \bar{V}_0^2 \left(\frac{A_0}{A_2}\right)^2 = \bar{V}_0^2 \left(\frac{d}{D}\right)^4$

Substituting, $gH = \left[f\left(\frac{L}{D} + 90\frac{Le}{D}\right)\left(\frac{d}{D}\right)^4 + 1 + K\right]\frac{\bar{V}_0^2}{2}$

$$\bar{V}_0 = \left[\frac{2gH}{f\left(\frac{L}{D} + 90\frac{Le}{D}\right)\left(\frac{d}{D}\right)^4 + 1 + K}\right]^{1/2}$$

; in fully-rough zone ($\frac{e}{D} = 0.01$), $f = 0.038^*$ (Eq. 8.37)

$$\bar{V}_0 = \left[2 \times 9.81\frac{m}{s^2} \times 300\ m \times \frac{1}{0.038\left(\frac{900\ m}{0.075\ m} + 90(20)\right)\left(\frac{0.025}{0.075}\right)^4 + 1 + 0.02}\right]^{1/2} = 28.0\ m/s\ (\text{est.})$$

Check for fully-rough flow zone:

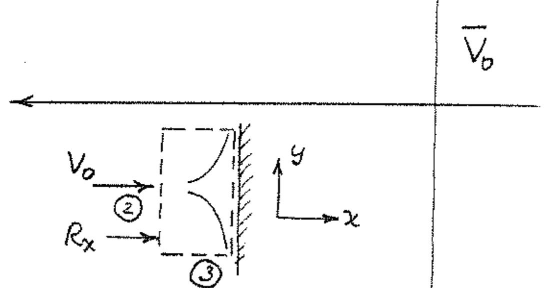
$Re = \frac{\bar{V}_p D}{\nu}$; $\bar{V}_p = \bar{V}_0\left(\frac{d}{D}\right)^4 = 28.0\frac{m}{s}\left(\frac{1}{3}\right)^4 = 0.346\ m/s$ $\quad$ {Assume $T = 20°C$}

$Re = 0.346\frac{m}{sec} \times 0.075\ m \times \frac{1}{1 \times 10^{-6}\ m^2} = 2.60 \times 10^4$; at $\frac{e}{D} = 0.01$, $f = 0.040$ (Eq. 8.37)

The new estimate is

$$\bar{V}_0 = \sqrt{\frac{0.038}{0.040}}\ \bar{V}_0\ (\text{est}) = \sqrt{\frac{0.038}{0.040}}\ 28.0\frac{m}{s} = 27.3\ m/s$$

Apply momentum to find force: CV is shown.



$$F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho\vec{V}\cdot d\vec{A}$$

Assumptions: (1) No pressure forces
(2) $F_{B_x} = 0$
(3) Steady flow

(4) Uniform flow at each cross-section

Then

$$R_x = u_2\{-\rho \bar{V}_0 A_0\} + u_3\{+\rho \bar{V}_0 A_0\}$$

$$u_2 = \bar{V}_0 \qquad u_3 = 0$$

$$R_x = -\rho \bar{V}_0^2 A_0$$

The force on the rock face is

$$K_x = -R_x = \rho \bar{V}_0^2 A_0$$

$$= 999 \frac{kg}{m^3} \times (27.3)^2 \frac{m^2}{s^2} \times \frac{\pi}{4}(0.025)^2 m^2 \times \frac{N \cdot s^2}{kg \cdot m}$$

$K_x = 365$ N (to right) K_x

* Values of f obtained from Eq. 8.37 using Excel's Solver (or Goal Seek)

Problem 8.124 (In Excel)

Investigate the effect of tube length on flow rate by computing the flow generated by a pressure difference $\Delta p = 100$ kPa applied to a length L of smooth tubing, of diameter $D = 25$ mm. Plot the flow rate against tube length for flow ranging from low speed laminar to fully turbulent.

Solution

Governing equations:

$$Re = \frac{\rho \cdot V \cdot D}{\mu}$$

$$\left(\frac{p_1}{\rho} + \alpha_1 \cdot \frac{V_1^2}{2} + g \cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \cdot \frac{V_2^2}{2} + g \cdot z_2\right) = h_l \qquad (8.29)$$

$$h_l = f \cdot \frac{L}{D} \cdot \frac{V^2}{2} \qquad (8.34)$$

$$f = \frac{64}{Re} \qquad (8.36) \quad \text{(Laminar)}$$

$$\frac{1}{f^{0.5}} = -2.0 \cdot \log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re \cdot f^{0.5}}\right) \qquad (8.37) \quad \text{(Turbulent)}$$

The energy equation (Eq. 8.29) becomes for flow in a tube

$$p_1 - p_2 = \Delta p = \rho \cdot f \cdot \frac{L}{D} \cdot \frac{V^2}{2}$$

This cannot be solved explicitly for velocity V, (and hence flow rate Q) because f depends on V; solution for a given L requires iteration (or use of *Solver*)

Fluid is not specified: use water

Given data:

Δp =	100	m
D =	25	mm

Tabulated or graphical data:

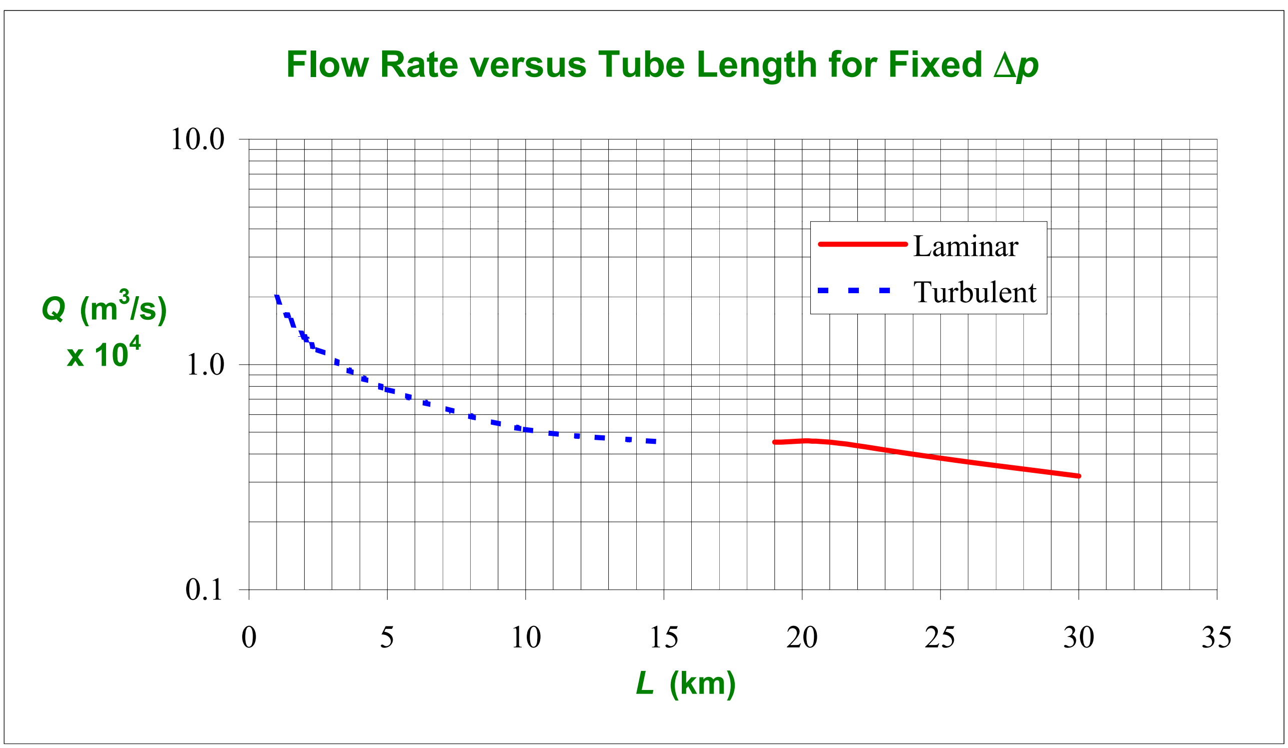
μ =	1.00E-03	N.s/m^2
ρ =	999	kg/m^3

(Water - Appendix A)

Computed results:

L (km)	V (m/s)	Q (m^3/s) x 10^4	Re	Regime	f	Δp (kPa)	Error
1.0	0.40	1.98	10063	Turbulent	0.0308	100	0.0%
1.5	0.319	1.56	7962	Turbulent	0.0328	100	0.0%
2.0	0.270	1.32	6739	Turbulent	0.0344	100	0.0%
2.5	0.237	1.16	5919	Turbulent	0.0356	100	0.0%
5.0	0.158	0.776	3948	Turbulent	0.0401	100	0.0%
10	0.105	0.516	2623	Turbulent	0.0454	100	0.0%
15	0.092	0.452	2300	Turbulent	0.0473	120	20.2%
19	0.092	0.452	2300	Laminar	0.0278	90	10.4%
21	0.092	0.452	2300	Laminar	0.0278	99	1.0%
25	0.078	0.383	1951	Laminar	0.0328	100	0.0%
30	0.065	0.320	1626	Laminar	0.0394	100	0.0%

The "critical" length of tube is between 15 and 20 km.
For this range, the fluid is making a transition between laminar
and turbulent flow, and is quite unstable. In this range the flow oscillates
between laminar and turbulent; no consistent solution is found
(i.e., an Re corresponding to turbulent flow needs an f assuming laminar to produce
the Δp required, and vice versa!)
More realistic numbers (e.g., tube length) are obtained for a fluid such as SAE 10W oil
(The graph will remain the same except for scale)



Problem 8.125 (In Excel)

Investigate the effect of tube roughness on flow rate by computing the flow generated by a pressure difference $\Delta p = 100$ kPa applied to a length $L = 100$ m of tubing, with diameter $D = 25$ mm. Plot the flow rate against tube relative roughness e/D for e/D ranging from 0 to 0.05 (this could be replicated experimentally by progressively roughening the tube surface). Is it possible that this tubing could be roughened so much that the flow could be slowed to a laminar flow rate?

Solution

Governing equations:

$$Re = \frac{\rho \cdot V \cdot D}{\mu}$$

$$\left(\frac{p_1}{\rho} + \alpha_1 \cdot \frac{V_1^2}{2} + g \cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \cdot \frac{V_2^2}{2} + g \cdot z_2\right) = h_l \qquad (8.29)$$

$$h_l = f \cdot \frac{L}{D} \cdot \frac{V^2}{2} \qquad (8.34)$$

$$f = \frac{64}{Re} \qquad (8.36) \quad \text{(Laminar)}$$

$$\frac{1}{f^{0.5}} = -2.0 \cdot \log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re \cdot f^{0.5}}\right) \qquad (8.37) \quad \text{(Turbulent)}$$

The energy equation (Eq. 8.29) becomes for flow in a tube

$$p_1 - p_2 = \Delta p = \rho \cdot f \cdot \frac{L}{D} \cdot \frac{V^2}{2}$$

This cannot be solved explicitly for velocity V, (and hence flow rate Q) because f depends on V; solution for a given relative roughness e/D requires iteration (or use of *Solver*)

Fluid is not specified: use water

Given data:

Δp =	100	kPa
D =	25	mm
L =	100	m

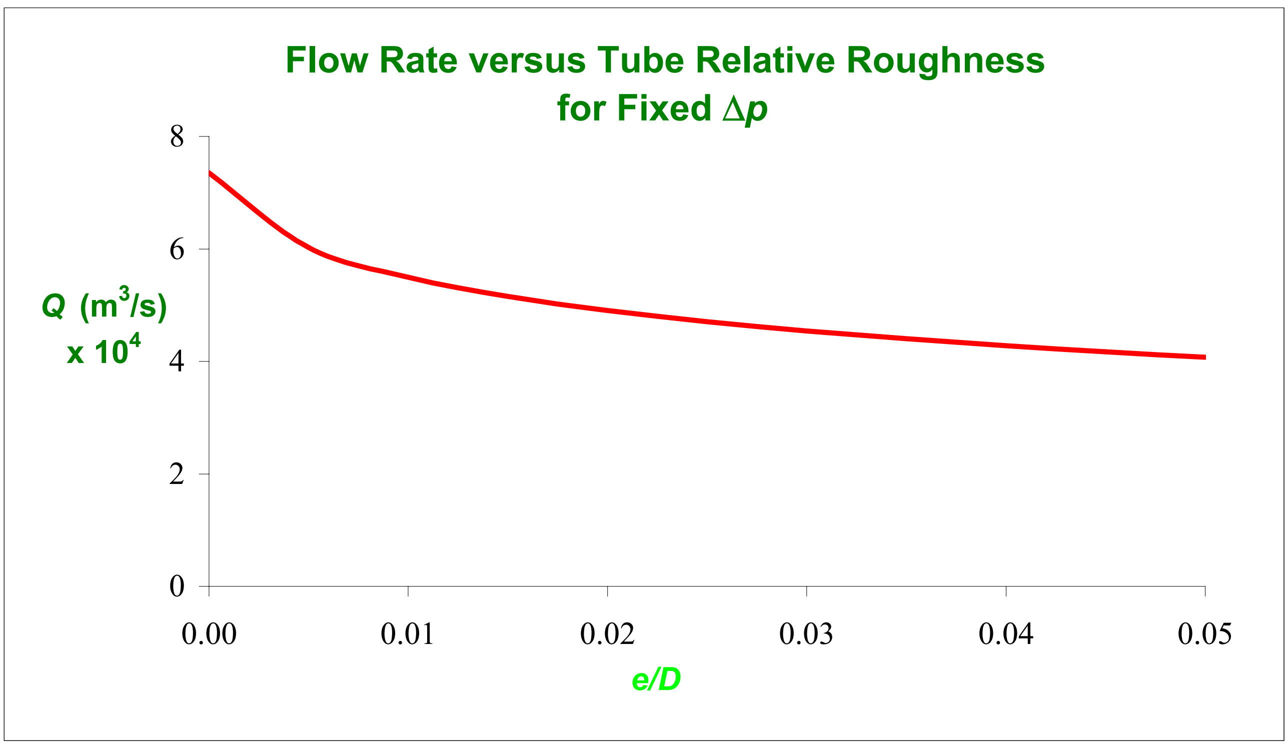
Tabulated or graphical data:

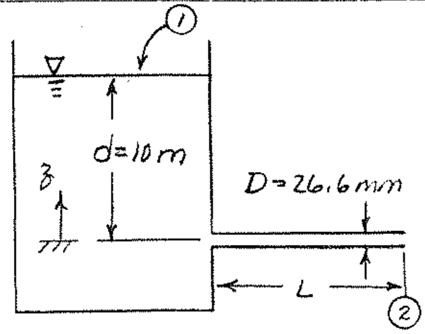
μ =	1.00E-03	N.s/m^2
ρ =	999	kg/m^3
	(Water - Appendix A)	

Computed results:

e/D	*V* (m/s)	*Q* (m^3/s) x 10^4	*Re*	Regime	*f*	Δp (kPa)	Error
0.000	1.50	7.35	37408	Turbulent	0.0223	100	0.0%
0.005	1.23	6.03	30670	Turbulent	0.0332	100	0.0%
0.010	1.12	5.49	27953	Turbulent	0.0400	100	0.0%
0.015	1.05	5.15	26221	Turbulent	0.0454	100	0.0%
0.020	0.999	4.90	24947	Turbulent	0.0502	100	0.0%
0.025	0.959	4.71	23939	Turbulent	0.0545	100	0.0%
0.030	0.925	4.54	23105	Turbulent	0.0585	100	0.0%
0.035	0.897	4.40	22396	Turbulent	0.0623	100	0.0%
0.040	0.872	4.28	21774	Turbulent	0.0659	100	0.0%
0.045	0.850	4.17	21224	Turbulent	0.0693	100	0.0%
0.050	0.830	4.07	20730	Turbulent	0.0727	100	0.0%

It is not possible to roughen the tube sufficiently to slow the flow down to a laminar flow for this Δp. Even a relative roughness of 0.5 (a physical impossibility!) would not work.





Given: Flow configuration of Example Problem 8.5, but with a well rounded inlet, $d = 10\,m$, and $D = 26.6\,mm$.

Find: (a) Volume flow rate if $L = 170\,m$.
(b) Length for laminar flow.
(c) What happens to flow rate when flow changes from laminar to turbulent.
(d) Length to reduce flow rate to 1 gal/hr.

Solution: Apply the energy equation for steady, incompressible pipe flow.

Computing equation:
$$\left(\frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2} + g z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \frac{\bar{V}_2^2}{2} + g z_2\right) = h_{l_T} = \left(f\frac{L}{D} + K_{ent}\right)\frac{\bar{V}^2}{2}$$
($\bar{V}_1^2 \approx 0$ (2); $z_2 = 0$)

Assumptions: (1) $p_1 = p_2 = p_{atm}$
(2) Reservoir large, so $\bar{V}_1 \approx 0$
(3) $\alpha_2 \approx 1$

$K_{ent} \approx 0.04$ (Table 8.2)
$\nu = 1\times10^{-6}\,\frac{m^2}{s}$ ($T = 20°C$, Table A.8)

Then
$$g d = \left(f\frac{L}{D} + K_{ent} + 1\right)\frac{\bar{V}^2}{2} \quad \text{or} \quad \bar{V} = \left[\frac{2gd}{f\frac{L}{D} + K_{ent} + 1}\right]^{\frac{1}{2}}$$

For smooth pipe, $f = f(Re)$, but $\bar{V}$ is not known. Guess $Re = 10^5$, so $f = 0.018^*$ (Eq 8.37).

$$\bar{V} \approx \left[\frac{2\times9.81\,\frac{m}{s^2}\times10\,m}{0.018\,\frac{170}{0.0266} + 0.04 + 1}\right]^{\frac{1}{2}} = 1.30\,m/s\,; \quad Re = 1.30\,\frac{m}{s}\times0.0266\,m\times\frac{s}{1\times10^{-6}\,m^2} = 3.46\times10^4$$

Try again with $f = 0.0227^*$

$$\bar{V} \approx \left[\frac{196\,m^2/s^2}{0.0227\,\frac{170}{0.0266} + 1.04}\right]^{\frac{1}{2}} = 1.16\,m/s\,; \quad Re = 3.08\times10^4\,;\; f = 0.0233 \checkmark\checkmark \text{ (okay) } (3\%\text{ error})$$

Thus $Q = \bar{V}A = 1.16\,\frac{m}{s}\times\frac{\pi}{4}(0.0266)^2\,m^2\times1000\,\frac{L}{m^3}\times60\,\frac{s}{min} = 38.6\,L/min$ $(L = 170\,m)$ — Q

For laminar flow, $f = 64/Re = 64/2300 = 0.0278$; $Re = \frac{\bar{V}D}{\nu}$; $\bar{V} = \frac{\nu Re}{D} = \frac{(1\times10^{-6})(2300)}{0.0266} = 0.0865\,m/s$.

From above, $f\frac{L}{D} = \frac{2gd}{\bar{V}^2} - K_{ent} - 1 = 196\,\frac{m^2}{s^2}\times\frac{s^2}{(0.0865)^2\,m^2} - 1.04 = 2.62\times10^4$

$L = 2.62\times10^4\,\frac{D}{f} = 2.62\times10^4\times0.0266\,m\times\frac{1}{0.0278} = 2.51\times10^4\,m = 25.1\,km$ — L

Flow rate would decrease at transition because friction factor increases. — Q

At $Q = 1\,gal/hr$, $\bar{V} = \frac{Q}{A} = \frac{4Q}{\pi D^2} = \frac{4}{\pi}\times1\,\frac{gal}{hr}\times\frac{1}{(0.0266)^2\,m^2}\times3.78\,\frac{L}{gal}\times\frac{m^3}{1000\,L}\times\frac{hr}{3600\,s} = 1.89\times10^{-3}\,m/s$

$Re = \frac{\bar{V}D}{\nu} = \frac{(1.89\times10^{-3})(0.0266)}{1\times10^{-6}} = 50.3\,;\; f = \frac{64}{Re} = \frac{64}{50.3} = 1.27$

$f\frac{L}{D} = \frac{2gd}{\bar{V}^2} - K_{ent} - 1 = 196\,\frac{m^2}{s^2}\times\frac{s^2}{(1.89\times10^{-3})^2\,m^2} - 1.04 = 5.49\times10^7$

$L = 5.49\times10^7\,\frac{D}{f} = 5.49\times10^7\times0.0266\,m\times\frac{1}{1.27} = 1.15\times10^6\,m$ or $1{,}150\,km$ — L

* Values of f obtained from Eq. 8.37 using Excel's Solver (or Goal Seek)

Problem 8.127 (In Excel)

Water for a fire protection system is supplied from a water tower through a 150 mm cast-iron pipe. A pressure gage at a fire hydrant indicates 600 kPa when no water is flowing. The total pipe length between the elevated tank and the hydrant is 200 m. Determine the height of the water tower above the hydrant. Calculate the maximum volume flow rate that can be achieved when the system is flushed by opening the hydrant wide (assume minor losses are 10 percent of major losses at this condition). When a fire hose is attached to the hydrant, the volume flow rate is 0.75 m^3/min. Determine the reading of the pressure gage at this flow condition.

Given: Some data on water tower system

Find: Water tower height; maximum flow rate; hydrant pressure at 0.75 m^3/min

Solution

Governing equations:

$$Re = \frac{\rho \cdot V \cdot D}{\mu}$$

$$\left(\frac{p_1}{\rho} + \alpha_1 \cdot \frac{V_1^2}{2} + g \cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \cdot \frac{V_2^2}{2} + g \cdot z_2\right) = h_{lT} \qquad (8.29)$$

$$h_l = f \cdot \frac{L}{D} \cdot \frac{V^2}{2} \qquad (8.34)$$

$$h_{lm} = 0.1 \times h_l$$

$$f = \frac{64}{Re} \qquad (8.36)\ \text{(Laminar)}$$

$$\frac{1}{f^{0.5}} = -2.0 \cdot \log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re \cdot f^{0.5}}\right) \qquad (8.37)\ \text{(Turbulent)}$$

For no flow the energy equation (Eq. 8.29) applied between the water tower free surface (state 1; height *H*) and the pressure gage is

$$g \cdot H = \frac{p_2}{\rho} \quad \text{or} \quad H = \frac{p_2}{\rho \cdot g} \qquad (1)$$

The energy equation (Eq. 8.29) becomes, for maximum flow (and $\alpha = 1$)

$$g \cdot H - \frac{V^2}{2} = h_{lT} = (1 + 0.1) \cdot h_l$$

$$g \cdot H = \frac{V^2}{2} \cdot \left(1 + 1.1 \cdot f \cdot \frac{L}{D}\right) \qquad (2)$$

This can be solved for *V* (and hence *Q*) by iterating, or by using *Solver*

The energy equation (Eq. 8.29) becomes, for maximum flow (and $\alpha = 1$)

$$g \cdot H - \frac{V^2}{2} = h_{lT} = (1 + 0.1) \cdot h_l$$

$$g \cdot H = \frac{V^2}{2} \cdot \left(1 + 1.1 \cdot f \cdot \frac{L}{D}\right) \qquad (2)$$

This can be solved for *V* (and hence *Q*) by iterating, or by using *Solver*

The energy equation (Eq. 8.29) becomes, for restricted flow

$$g \cdot H - \frac{p_2}{\rho} + \frac{V^2}{2} = h_{lT} = (1 + 0.1) \cdot h_l$$

$$p_2 = \rho \cdot g \cdot H - \rho \cdot \frac{V^2}{2} \cdot \left(1 + 1.1 \cdot \rho \cdot f \cdot \frac{L}{D}\right) \qquad (3)$$

Given data:

p_2 = 600 kPa (Closed)

D = 150 mm

L = 200 m

Q = 0.75 m^3/min (Open)

Tabulated or graphical data:

e = 0.26 mm (Table 8.1)

μ = 1.00E-03 $N.s/m^2$

ρ = 999 kg/m^3 (Water - Appendix A)

Computed results:

Closed:

H = 61.2 m (Eq. 1)

Fully open:

V = 5.91 m/s

Re = 8.85E+05

f = 0.0228

Eq. 2, solved by varying V using *Solver*:

Left (m^2/s)	Right (m^2/s)	Error
601	601	0%

Q = 0.104 m^3/s

Partially open:

Q = 0.75 m^3/min

V = 0.71 m/s

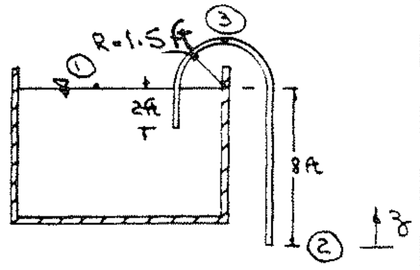
Re = 1.06E+05

f = 0.0243

p_2 = 591 kPa (Eq. 3)

Problem 8.128

Given: Siphon shown is fabricated from 2 in. i.d. drawn aluminum. The liquid is water at 60°F.



Find: Compute the volume flow rate. Estimate p_{min} inside the tube

Solution: Apply the energy equation for steady, incompressible flow that is uniform at each section

Basic equations: $\frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \alpha_2 \frac{\bar{V}_2^2}{2} + g z_2 + h_{l_T}$ (terms cancelled by (1), (2), with $z_2 = 0$)

$$h_{l_T} = f \frac{L}{D}\frac{\bar{V}^2}{2} + h_{l_m} \; ; \quad h_{l_m} = K_{ent}\frac{\bar{V}^2}{2} + f\left(\frac{L_e}{D}\right)_{bend}\frac{\bar{V}^2}{2}$$

Assumptions: (1) $p_1 = p_2 = p_{atm}$

(2) $V_1 \approx 0$

(3) uniform flow at ②, $\alpha_2 = 1.0$

(4) reentrant entrace

Then $g z_1 = \frac{\bar{V}_2^2}{2} + f\frac{L}{D}\frac{\bar{V}^2}{2} + K_{ent}\frac{\bar{V}^2}{2} + f\left(\frac{L_e}{D}\right)_{bend}\frac{\bar{V}^2}{2}$

$$2 g z_1 = \bar{V}_2^2\left[1 + K_{ent} + f\left\{\left(\frac{L_e}{D}\right)_{bend} + \frac{L}{D}\right\}\right]$$

An iterative solution for $\bar{V}$ is required

From Table 8.2 for reentrant entrace $K_{ent} = 0.78$

For bend $R/d = 1.5\text{ ft}/0.167\text{ ft} = 9$

- from Fig. 8.16, $L_e/D = 28$ for 90° bend
  - as first approximation assume $L_e/D = 56$ for 180° bend

For straight pipe $L = 10$ ft, $L/D = 60$

Then

$$2 \times 32.2\frac{\text{ft}}{\text{s}^2} \times 8\text{ ft} = \bar{V}_2^2\left[1 + 0.78 + f\{56 + 60\}\right] = \bar{V}_2^2\left[1.78 + 116 f\right]$$

For 2" drawn aluminum tubing, $e = 5\times10^{-6}$ ft (Table 8.1), $e/D = 0.00003$

Assume $Re = 5\times10^5$, then $f = 0.0138^*$, and $\bar{V}_2 = 12.3$ ft/s

Then $Re = \frac{D\bar{V}}{\nu} = \frac{\text{ft}}{6} \times 12.3\frac{\text{ft}}{\text{s}} \times \frac{\text{s}}{1.2\times10^{-5}\text{ ft}^2} = 1.71\times10^5$

With $Re = 1.71\times10^5$, then $f = 0.016^*$, and $\bar{V}_2 = 11.9$ ft/s

Then $Re = \frac{D\bar{V}}{\nu} = \frac{\text{ft}}{6}\times 11.9\frac{\text{ft}}{\text{s}} \times \frac{\text{s}}{1.2\times10^{-5}\text{ ft}} = 1.65\times10^5 \Rightarrow f = 0.016^*$ ✓

$$\therefore Q = A\bar{V} = \frac{\pi D^2}{4}\bar{V} = \frac{\pi}{4}\left(\frac{1}{6}\right)^2\text{ft}^2 \times 11.9\frac{\text{ft}}{\text{s}} = 0.260\ \text{ft}^3/\text{s}$$ ← Q

The minimum pressure occurs at point ③ in the pipe; $V_3 = V_2$

$$\frac{p_1}{\rho} + \alpha\frac{\bar{V}_1^2}{2} + g z_1 = \frac{p_3}{\rho} + \alpha_3\frac{\bar{V}_3^2}{2} + g z_3 + K_{ent}\frac{\bar{V}_3^2}{2} + f\left(\frac{L_e}{D}\right)_{bend}\frac{\bar{V}_3^2}{2} + f\frac{L}{D}\frac{\bar{V}_3^2}{2}$$

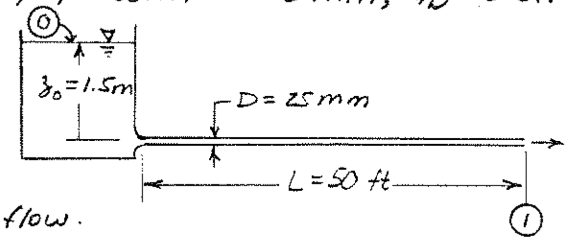
$$p_3 = \rho\left[g(z_1 - z_3) - \frac{\bar{V}_3^2}{2}\left(1 + K_{ent} + f\left\{\left(\frac{L_e}{D}\right)_{bend} + \left(\frac{L_e}{D}\right)_{pipe}\right\}\right)\right]$$

$$= 1.94\frac{\text{slug}}{\text{ft}^3}\left[32.2\frac{\text{ft}}{\text{s}^2}\times(-1.5\text{ ft}) - \frac{(11.9)^2}{2}\frac{\text{ft}^2}{\text{s}^2}\left(1 + 0.78 + 0.016\{28 + 12\}\right)\right]$$

$p_3 = -2.96$ psig ← p_{min}

* Values of f obtained from Eq. 8.37 using Excel's Solver (or Goal Seek)

Given: Roman water supply system from Example Problem 8.10, but with 50 foot length of straight pipe with D = 25 mm, e/D = 0.01.



Find: (a) Flow rate delivered.
(b) Effect of adding a diffuser.

Solution: Apply the energy equation for steady, incompressible pipe flow.

Computing equation:

$$\frac{p_0}{\rho} + \alpha_0\frac{\bar{V}_0^2}{2} + gz_0 = \frac{p_1}{\rho} + \alpha_1\frac{\bar{V}_1^2}{2} + gz_1 + h_{l_T}\ ;\quad h_{l_T} = \left(f\frac{L}{D} + K_{ent}\right)\frac{\bar{V}_1^2}{2}$$

Assumptions: (1) $p_0 = p_1 = p_{atm}$ (2) $\bar{V}_1 \approx 0$ (3) $\alpha_1 \approx 1$ (4) $K_{ent} = 0.04$

Then

$$gz_0 = \frac{\bar{V}_1^2}{2} + \left(f\frac{L}{D} + K_{ent}\right)\frac{\bar{V}_1^2}{2} \quad \text{or} \quad \bar{V}_1 = \sqrt{\frac{2gz_0}{1 + f\frac{L}{D} + K}}$$

For $e/D = 0.01$, $f = 0.038$ from Eq. 8.37*, so

$$\bar{V}_1 = \left[2 \times 9.81\,\frac{m}{s^2} \times 1.5\,m \times \frac{1}{1 + 0.038 \times 50\,ft \times \frac{1}{25\,mm} \times 304.8\,\frac{mm}{ft} + 0.04}\right]^{1/2} = 1.10\ m/s$$

Checking, assuming T = 20°C,

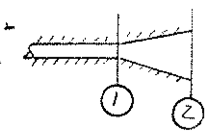
$$Re = \frac{\bar{V}D}{\nu} = 1.10\,\frac{m}{sec} \times 0.025\,m \times \frac{sec}{1.0 \times 10^{-6}\,m^2} = 2.75 \times 10^4\ ;\ \text{from Eq. 8.37*},\ f \approx 0.040,\ \text{so}$$

$$\bar{V}_1 = \left[2 \times 9.81\,\frac{m}{s^2} \times 1.5\,m \times \frac{1}{1 + 0.040 \times 50\,ft \times \frac{1}{25\,mm} \times 304.8\,\frac{mm}{ft} + 0.04}\right]^{1/2} = 1.08\ m/s$$

$$Q = \bar{V}_1 A = 1.08\,\frac{m}{s} \times \frac{\pi}{4}(0.025)^2\,m^2 = 5.30 \times 10^{-4}\ m^3/s \quad \text{(no diffuser)}$$ Q

The diffuser would increase head loss by $K_{diffuser} = 0.3$ (see Example 8.10), but would reduce $\bar{V}_2$ to $\frac{1}{2}\bar{V}_1$. The energy equation would be

$$gz_0 = \frac{\bar{V}_2^2}{2} + \left(f\frac{L}{D} + K_{ent} + K_{diff}\right)\frac{\bar{V}_1^2}{2} = \left(\frac{1}{4} + f\frac{L}{D} + K_{ent} + K_d\right)\frac{\bar{V}_1^2}{2}$$



Thus

$$\bar{V}_1 = \sqrt{\frac{2gz_0}{0.25 + f\frac{L}{D} + K_{ent} + K_{diff}}}$$

$$\bar{V}_1 = \left[2 \times 9.81\,\frac{m}{s^2} \times 1.5\,m \times \frac{1}{0.25 + 0.040 \times 50\,ft \times \frac{1}{25\,mm} \times 304.8\,\frac{mm}{ft} + 0.04 + 0.3}\right]^{1/2} = 1.09\ m/s$$

and

$$Q = \bar{V}_1 A = 1.09\,\frac{m}{s} \times \frac{\pi}{4}(0.025)^2\,m^2 = 5.35 \times 10^{-4}\ m^3/s \quad \text{(with diffuser)}$$ Q

{The diffuser increases flow rate only slightly (~1 percent), because loss is dominated by fL/D.}

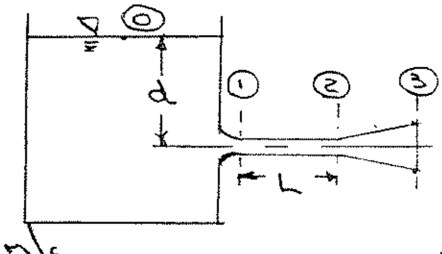
* Values of f obtained using Excel's Solver (or Goal Seek)

Given: Pipe of length L inserted between the nozzle (attached to the water main) and diffuser of Example Problem 8.10.

$D_1 = 25\text{ mm}$, $K_{ent} = 0.04$, $d = 1.5\text{ m}$

Diffuser: $N/R_1 = 3.0$, $AR = 2.0$

$K_{diff} = 0.3$



Flow with nozzle alone:

$Q_i = 2.61 \times 10^{-3}\ \text{m}^3/\text{s}$, $\bar{V}_1 = 5.32\ \text{m/s}$

Flow with nozzle and diffuser ($L=0$) $Q_d = 3.47 \times 10^{-3}\ \text{m}^3/\text{s}$

Find: Length (L) of pipe with $e/D = 0.01$ required to give flow rate Q_i with diffuser in place; compare with commissioner's requirement of $L = 50$ ft (15.2 m)

Plot: Q/Q_i vs L/D

Solution:

Apply the energy equation for steady, incompressible flow between the water surface and the diffuser discharge.

Basic equations: $\left(\frac{p_0}{\rho} + \alpha_0 \frac{\bar{V}_0^2}{2} + g z_0\right) - \left(\frac{p_3}{\rho} + \alpha_3 \frac{\bar{V}_3^2}{2} + g z_3\right) = h_{l_T}$ (8.29)

$h_{l_T} = h_l + h_{l_m}$; $h_l = f \frac{L}{D} \frac{\bar{V}_2^2}{2}$; $h_{l_m} = (K_{ent} + K_{diff}) \frac{\bar{V}_2^2}{2}$

Assumptions: (1) $p_0 = p_3 = p_{atm}$

(2) $\bar{V}_0 = 0$, $\alpha_3 = 1.0$

(3) water @ 20°C, $\nu = 1.00 \times 10^{-6}\ \text{m}^2/\text{s}$

Then,

$$g(z_0 - z_3) = gd = f \frac{L}{D} \frac{\bar{V}_2^2}{2} + (K_{ent} + K_{diff}) \frac{\bar{V}_2^2}{2} + \frac{\bar{V}_3^2}{2}$$

From continuity $A_2 \bar{V}_2 = A_3 \bar{V}_3$ $\therefore V_3 = \frac{\bar{V}_2}{AR}$

$$gd = f \frac{L}{D} \frac{\bar{V}_2^2}{2} + \left(K_{ent} + K_{diff} + \frac{1}{AR^2}\right) \frac{\bar{V}_2^2}{2} = f \frac{L}{D} \frac{\bar{V}_2^2}{2} + 0.59 \frac{\bar{V}_2^2}{2} \quad (1)$$

and

$$L = \frac{D}{f}\left[\frac{2gd}{\bar{V}_2^2} - 0.590\right] \quad (2)$$

For $\bar{V}_1 = 5.32\ \text{m/s}$, $Re = \frac{D\bar{V}}{\nu} = 0.025\text{ m} \times 5.32 \frac{\text{m}}{\text{s}} \times \frac{\text{s}}{1.00 \times 10^{-6}\ \text{m}^2} = 1.33 \times 10^5$

with $e/D = 0.01$, $f = 0.038$ (Fig. 8.13) and

$$L = \frac{0.025\text{ m}}{0.038}\left[2 \times 9.81 \frac{\text{m}}{\text{s}^2} \times 1.5\text{ m} \times \frac{\text{s}^2}{(5.32)^2\ \text{m}^2} - 0.590\right] = 0.296\text{ m} \qquad L$$

(0.971 ft, $L/D = 11.8$)

This is significantly less than the 50 ft required by the water commissioner. He was extremely conservative.

Note that $Q/Q_i = \bar{V}/\bar{V}_i$ where $V_i = 5.32\ \text{m/s}$

Increasing L reduces $\bar{V}$

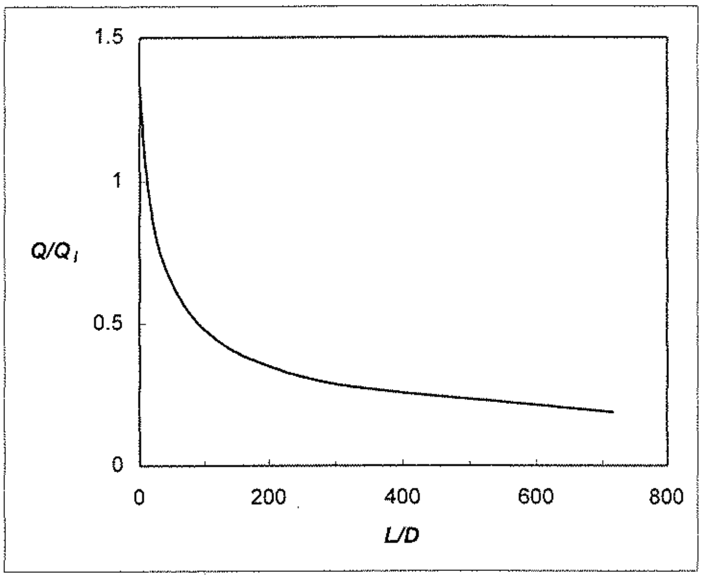
With $L = 0$ $\quad Q/Q_i = 1.33$

$L = 0.296$ m $\quad (L/D = 11.8) \quad Q/Q_i = 1.00$

As L is increased $\bar{V}_2$ (and hence Re) will decrease; the friction factor will increase slightly from 0.038.

The plot of Q/Q_i ($\bar{V}/\bar{V}_i$) is best done by assuming values of $\bar{V}$, and solving Eq. 2 for L.

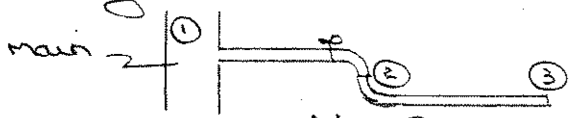
V_{avg}	Re	f	L/D	V/V$_i$
(m/s)	(---)	(---)	(---)	(---)
7.06	1.77E+05	0.0382	0.0	1.33
5.32	1.33E+05	0.0384	11.7	1.00
5.0	1.25E+05	0.0384	15.3	0.940
4.5	1.13E+05	0.0384	22.5	0.846
4.0	1.00E+05	0.0385	32.4	0.752
3.5	8.75E+04	0.0386	47.0	0.658
3.0	7.50E+04	0.0387	69.3	0.564
2.5	6.25E+04	0.0388	106	0.470
2.0	5.00E+04	0.0391	173	0.376
1.5	3.75E+04	0.0394	317	0.282
1.0	2.50E+04	0.0402	718	0.188



Given: Water flow from spigot (at 60°F) through an old hose with $D = 0.75$ in and $e = 0.022$ in. Pressure at main remains constant at 50 psig; pressure at spigot varies with flow rate.
One 50 ft. length of hose delivers 15 gpm

Find: (a) pressure at spigot (psig) for this case.
(b) delivery with two 50-ft lengths of hose connected.

Solution:



Apply the energy equation for steady, incompressible flow between the spigot ② and the hose discharge ③

Basic equations: $\left(\frac{p_2}{\rho} + \alpha_2 \frac{\bar{V}_2^2}{2} + g z_2\right) - \left(\frac{p_3}{\rho} + \alpha_3 \frac{\bar{V}_3^2}{2} + g z_3\right) = h_{l_T}$ (8.29)

$h_{l_T} = h_l + h_{l_m}$, $h_l = f \frac{L}{D} \frac{\bar{V}^2}{2}$

Assumptions: (1) $p_3 = p_{atm}$
(2) $\bar{V}_2 = \bar{V}_3$, $\alpha_2 = \alpha_3 = 1.0$
(3) $z_2 = z_3$
(4) Turbulent flow so $\Delta p_{1\to 2} \propto Q^2$

Then $p_2 = \rho f \frac{L}{D} \frac{\bar{V}^2}{2}$ — — — — — — — — — — — — (1)

$\bar{V} = \frac{Q}{A} = \frac{4Q}{\pi D^2} = \frac{4}{\pi} \times 15 \frac{gal}{min} \times \frac{min}{60 s} \times \frac{ft^3}{7.48 gal} \times \left(\frac{12}{0.75}\right)^2 \frac{1}{ft^2} = 10.9 \text{ ft/s}$

$Re = \frac{D\bar{V}}{\nu} = \frac{0.75}{12} ft \times 10.9 \frac{ft}{s} \times \frac{s}{1.21 \times 10^{-5} ft^2} = 5.63 \times 10^4$ {ν from Table A.7}

$e/D = 0.022 / 0.75 = 0.0293$

From Eq. 8.37 $f = 0.056$*. From Eq. 1,

$p_2 = 1.94 \frac{slug}{ft^3} \times 0.056 \times \frac{50 ft}{0.75 in} \times \frac{12 in}{ft} \times \frac{1}{2} \times (10.9)^2 \frac{ft^2}{s^2} \times \frac{lbf \cdot s^2}{slug \cdot ft} \times \frac{ft^2}{144 in^2}$

$p_2 = 35.9$ psi gage ← p_2

The pressure drop from the main ① to the spigot ② is proportional to the square of the flow rate. Obtain the loss coefficient using the energy equation between ① and ②.

$\left(\frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2} + g z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \frac{\bar{V}_2^2}{2} + g z_2\right) = K \frac{\bar{V}_2^2}{2}$

Assumptions: (4) $\bar{V}_1 = 0$ (5) $z_1 = z_2$

$p_1 - p_2 = \rho\left[K \frac{\bar{V}_2^2}{2} + \frac{\bar{V}_2^2}{2}\right] = \rho \frac{\bar{V}_2^2}{2}[K + 1]$

$K = \frac{\Delta p}{\frac{1}{2}\rho \bar{V}^2} - 1 = 2(50 - 35.9) 144 \frac{lbf}{ft^2} \times \frac{ft^3}{1.94 slug} \times \frac{s^2}{(10.9)^2 ft^2} \times \frac{slug \cdot ft}{lbf \cdot s^2} - 1$

$K = 16.6$

* Value of f obtained using Excel's Solver (or Goal Seek).

To find the delivery with two hoses, again apply the energy equation from the main ① to the end ④ of the second hose

$$\left(\frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2} + g z_1\right) - \left(\frac{p_4}{\rho} + \alpha_4 \frac{\bar{V}_4^2}{2} + g z_4\right) = f \frac{L_{24}}{D} \frac{\bar{V}_4^2}{2} + K \frac{\bar{V}_4^2}{2}$$

$p_4 = p_{atm}$, $z_1 = z_4$, $\bar{V}_1 = 0$, $\alpha_4 = 1$.

$$\frac{p_1 - p_{atm}}{\rho} = \frac{p_{1g}}{\rho} = \frac{\bar{V}_4^2}{2}\left(f \frac{L_{24}}{D} + K + 1\right)$$ and

$$\bar{V}_4 = \left[\frac{2 p_{1g}}{\rho\left(2f\frac{L}{D} + K + 1\right)}\right]^{1/2}$$

Delivery will be reduced somewhat with two lengths of hose, but f will not change much. Assume $f \approx 0.056$ and check

$$\bar{V}_4 = \left[2 \times 50 \frac{lbf}{in^2} \times 144 \frac{in^2}{ft^2} \times \frac{ft^3}{1.94\, slug} \times \frac{1}{\left(0.056 \times \frac{100\, ft}{0.75\, in} \times 12 \frac{in}{ft} + 16.6 + 1\right)} \times \frac{slug \cdot ft}{lbf \cdot s^2}\right]^{1/2}$$

$\bar{V}_4 = 8.32\ ft/s.$

Checking,

$$Re = \frac{D\bar{V}}{\nu} = \frac{0.75}{12} ft \times 8.32 \frac{ft}{s} \times \frac{1}{1.21 \times 10^{-5}} \frac{s}{ft^2} = 4.30 \times 10^4, \text{ so } f \approx 0.56$$

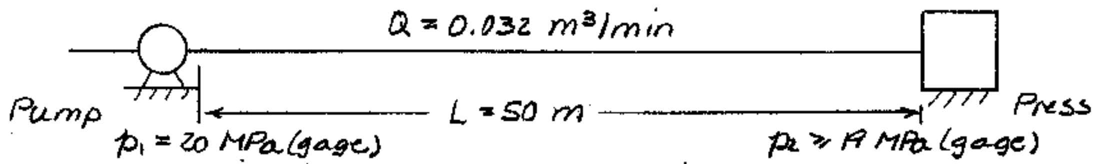
Thus with two hoses,

$$Q = \bar{V} A = 8.32 \frac{ft}{s} \times \frac{\pi}{4} \times \left(\frac{0.75}{12}\right)^2 ft^2 \times 7.48 \frac{gal}{ft^3} \times 60 \frac{s}{min} = 11.5\ gpm$$ ← Q

{Similar calculations could be performed using any desired number of hose lengths.}

Problem 8.132

Given: Hydraulic press powered by remote high-pressure pump.



Find: Minimum diameter drawn steel tubing for SAE 10W oil at 40°C.

Solution: Apply the energy equation for steady, incompressible pipe flow.

Computing equation: $\frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \alpha_2 \frac{\bar{V}_2^2}{2} + g z_2 + h_{l_T}$; $h_{l_T} = \left[f\left(\frac{L}{D} + \frac{L_e}{D}\right) + K \right] \frac{\bar{V}^2}{2}$ ($\frac{L_e}{D} \approx 0$ (3), $K \approx 0$ (3))

Assumptions: (1) Fully developed flow, $\alpha_1 \bar{V}_1^2 = \alpha_2 \bar{V}_2^2$, (2) $z_1 = z_2$, (3) No minor losses.

Then $\Delta p = f \frac{L}{D} \rho \frac{\bar{V}^2}{2}$

D is not known, so we cannot compute $\bar{V}$ and Re to find f. Q is small, so try laminar flow. For fully developed laminar flow, from Eq. 8.13c,

$$\Delta p = \frac{128 \mu Q L}{\pi D^4} \quad \text{so} \quad D = \left[\frac{128 \mu Q L}{\pi \Delta p} \right]^{\frac{1}{4}}$$

For SAE 10W oil at 40°C, $\mu = 3.3 \times 10^{-2}$ N·sec/m² (Fig. A.2)

$$D = \left[128 \times 3.3 \times 10^{-2} \frac{\text{N} \cdot \text{s}}{\text{m}^2} \times 0.032 \frac{\text{m}^3}{\text{min}} \times 50\ \text{m} \times \frac{\text{m}^2}{\pi (1 \times 10^6)\ \text{N}} \times \frac{\text{min}}{60\ \text{s}} \right]^{\frac{1}{4}} = 0.0138\ \text{m}$$

Check Re to assure flow is laminar:

$$\bar{V} = \frac{Q}{A} = \frac{4Q}{\pi D^2} = \frac{4}{\pi} \times 0.032 \frac{\text{m}^3}{\text{min}} \times \frac{1}{(0.0138)^2\ \text{m}^2} \times \frac{\text{min}}{60\ \text{s}} = 3.57\ \text{m/s}$$

$$Re = \frac{\bar{V} D}{\nu} = \frac{\rho \bar{V} D}{\mu}$$

For SAE 10 W oil, SG = 0.92 (Table A.2), so

$$Re = (0.92)\, 1000 \frac{\text{kg}}{\text{m}^3} \times 3.57 \frac{\text{m}}{\text{s}} \times 0.0138\ \text{m} \times \frac{\text{m}^2}{3.3 \times 10^{-2}\ \text{N} \cdot \text{s}} \times \frac{\text{N} \cdot \text{s}^2}{\text{kg} \cdot \text{m}} = 1370$$

Therefore flow is laminar since Re < 2300.

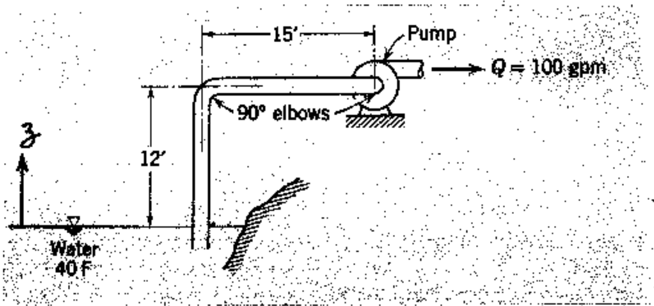
The minimum allowable tubing diameter is D = 13.8 mm. D

The next largest standard size should be chosen.

Problem 8.133

Given: Pump drawing water from reservoir as shown. For satisfactory operation, the suction head (p_2/γ) must not be less than −20 feet of water.

Section ① is at reservoir surface.



Section ② is at pump inlet.

Find: Smallest standard commercial steel pipe that will give the required performance.

Solution: Apply the energy equation for steady, incompressible flow that is uniform at each section.

Basic equation: $\frac{p_1}{\rho} + \frac{\bar{V}_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \frac{\bar{V}_2^2}{2} + g z_2 + f\frac{L}{D}\frac{\bar{V}^2}{2} + h_{lm}$ (terms 1, 2, 3 ≈ 0 by assumptions (1), (2), (3))

Assumptions: (1) $p_1 = 0$ psig
(2) $\bar{V}_1 \approx 0$
(3) $z_1 = 0$
(4) $h_{lm} = \left(K_{ent} + 2f\frac{L_{elbow}}{D}\right)\frac{\bar{V}^2}{2}$, $K_{ent} = 0.78$ for reentrant configuration (Table 8.2), $L_{elbow}/D \approx 12$ (Fig. 8.17)

Then

$$h_2 = \frac{p_2}{\gamma} = -z_2 - \left(1 + f\frac{L}{D} + K_{ent} + 2f\frac{L_e}{D}\right)\frac{\bar{V}_2^2}{2g} = -z_2 - \left[1.78 + f\left(\frac{L}{D}+24\right)\right]\frac{\bar{V}_2^2}{2g}$$

Since D is unknown, iteration is required. Set up calculating equations:

$$\bar{V} = \frac{Q}{A} = \frac{4Q}{\pi D^2} = \frac{4}{\pi}\times\frac{1}{D^2\,in.^2}\times\frac{144\,in.^2}{ft^2}\times 100\frac{gal}{min}\times\frac{ft^3}{7.48\,gal}\times\frac{min}{60\,s} = \frac{40.9}{D^2}\ ft/s$$

$$Re = \frac{\bar{V}D}{\nu} = \frac{4Q}{\pi\nu D} = \frac{4}{\pi}\times 100\frac{gal}{min}\times\frac{min}{60\,sec}\times\frac{ft^3}{7.48\,gal}\times\frac{s}{1.66\times10^{-5}\,ft^2}\times\frac{1}{D\,in.}\times\frac{12\,in.}{ft} = \frac{208{,}000}{D}$$

$e = 0.00015$ ft (Table 8.1), f from Fig. 8.13. $L = 27$ ft. D(in.) from Table 8.5.

D (nominal)	D (in.)	$\bar{V}$ (ft/s)	Re	e/D	f	L/D	h_2 (ft)
3	3.068	4.34	67,700	0.0006	0.0226	106	−13.4
2½	2.469	6.71	84,100	0.0007	0.0220	131	−15.6
2	2.067	9.57	100,000	0.0009	0.0215	157	−20.1

Recognizing that pipe friction calculations are only good to ± 10 percent, recommend

$D = 2\frac{1}{2}$ in. (nominal) pipe ← Dmin

Given: Flow of standard air at 80 m^3 /min through a smooth duct of aspect ratio 2.

Find: Minimum size duct for a head loss of 30 mm of water per 30 m of length.

Solution: Apply the energy equation for steady, incompressible flow that is uniform at each section. Use hydraulic diameter.

Basic equation: $$\frac{p_1}{\rho} + \frac{\bar{V}_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \frac{\bar{V}_2^2}{2} + g z_2 + f\frac{L}{D_h}\frac{\bar{V}^2}{2} + h_{\ell m} \; ; \; D_h = \frac{4A}{P_u}$$

(terms cancelled: $\bar{V}_1^2/2$ and $\bar{V}_2^2/2$ by (1), gz_1 and gz_2 by (2), $h_{\ell m} = 0$ (3))

Assumptions: (1) $\bar{V}_1 = \bar{V}_2$
(2) $z_1 = z_2$
(3) $h_{\ell m} = 0$

Then
$$\Delta p = p_1 - p_2 = f\frac{L}{D_h}\frac{\rho\bar{V}^2}{2} = \frac{f}{2}\frac{L\rho}{D_h}\left(\frac{Q}{A}\right)^2 = \frac{fLQ^2\rho}{2D_hA^2}$$

For a rectangular duct, $A = bh = h^2\left(\frac{b}{h}\right) = h^2 ar$, and

$$D_h = \frac{4bh}{2(b+h)} = \frac{2h^2ar}{h(1+ar)} = \frac{2har}{1+ar}$$

Substituting,

$$\Delta p = \frac{fLQ^2\rho}{2}\frac{1+ar}{2har}\frac{1}{h^4ar^2} = \frac{f\rho LQ^2}{4}\frac{1+ar}{ar^3}\frac{1}{h^5}$$

or
$$h = \left[\frac{f\rho LQ^2}{4\Delta p}\frac{1+ar}{ar^3}\right]^{\frac{1}{5}} \; ; \; \Delta p = \rho_{H_2O} g \Delta h = 999\frac{kg}{m^3} \times 9.81\frac{m}{s^2} \times 0.03\,m \times \frac{N \cdot s^2}{kg \cdot m}$$

Thus
$$\Delta p = 294\ N/m^2$$

$$h = (f)^{1/5}\left[\frac{1}{4} \times 1.23\frac{kg}{m^3} \times 30\,m \times (80)^2\frac{m^6}{min^2} \times \frac{m^2}{294\,N} \times \frac{1+2}{(2)^3} \times \frac{min^2}{3600\,s^2} \times \frac{N \cdot s^2}{kg \cdot m}\right]^{1/5}$$

$$h = (f)^{1/5}\,0.461\,m$$

Guess $f = 0.01$, then $h = 0.184$ m. $\bar{V} = \frac{Q}{A} = \frac{Q}{h^2ar} = 19.7$ m/s, and $D_h = \frac{2har}{1+ar} = \frac{4}{3}h = 0.245$ m and $Re = 3.33 \times 10^5$. For a smooth duct at this Reynolds number, $f = 0.013$ (Eq. 8.37 using Excel's Solver {or Goal Seek})

With $f = 0.013$, then $h = 0.193$ m. $\bar{V} = \frac{Q}{h^2ar} = 17.9$ m/s, and $D_h = \frac{4}{3}h = 0.257$ m and $Re = 3.17 \times 10^5$. This value of Re gives $f = 0.0133$.

With $f = 0.0133$, then $h = 0.194$ m, and $b = ar\,h = (2)\,0.194\,m = 0.388\,m$ ← b, h

Check: $\bar{V} = \frac{Q}{h^2ar} = 17.7$ m/s

$$\Delta h = \frac{\Delta p}{\rho_{H_2O}g} = f\frac{L}{D_h}\frac{\rho_a}{\rho_{H_2O}}\frac{\bar{V}^2}{2g} = 0.0303\text{ m or } 30.3\text{ mm} \checkmark$$

Problem 8.135

Given: New industrial plant requires water supply of 5.7 m³/min. The gage pressure at the main, 50 m from the plant, is 800 kPa. The supply line will have 4 elbows in a total length of 65 m. Pressure in the plant must be at least 500 kPa (gage).

Find: Minimum line size of galvanized iron to install.

Solution: Apply the energy equation for steady, incompressible flow that is uniform at each section ($\alpha \approx 1$).

Basic equation: $$\frac{p_1}{\rho} + \frac{\bar{V}_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \frac{\bar{V}_2^2}{2} + g z_2 + f \frac{L}{D}\frac{\bar{V}^2}{2} + h_{lm}$$
(velocity terms cancel by (2); elevation terms cancel by (3))

Assumptions: (1) $p_1 - p_2 \le 300\ \text{kPa} = \Delta p$

(2) Fully developed flow in constant-area pipe, $\bar{V}_1 = \bar{V}_2 = \bar{V}$

(3) $z_1 = z_2$

(4) $h_{lm} = 4\left(\frac{L_e}{D}\right)_{elbow} \frac{\bar{V}^2}{2} = 120\frac{\bar{V}^2}{2}$ ($\frac{L_e}{D} = 30$, from Table 8.5)

Then
$$\frac{\Delta p}{\rho} = f\left(\frac{L}{D} + 120\right)\frac{\bar{V}^2}{2} \quad \text{or} \quad \Delta p = \rho f\left(\frac{L}{D} + 120\right)\frac{\bar{V}^2}{2}$$

Since D is unknown, iteration is required. The calculating equations are:
$$\bar{V} = \frac{Q}{A} = \frac{4Q}{\pi D^2} = \frac{4}{\pi} \times 5.7\frac{m^3}{min} \times \frac{1}{D^2 m^2} \times \frac{min}{60\ s} = \frac{0.121}{D^2}\ (m/s)$$

$$Re = \frac{\bar{V}D}{\nu} = \frac{4Q}{\pi \nu D} = \frac{4}{\pi} \times 5.7\frac{m^3}{min} \times \frac{s}{1.14\times10^{-6} m^2} \times \frac{1}{D\,m} \times \frac{min}{60\ s} = \frac{1.06\times10^5}{D} \quad (T = 15°C)$$

$e = 0.15$ mm (Table 8.1), f from Eq. 8.37*, $L = 65$ m. D from Table 8.5.

D (nom.)	D (m)	$\bar{V}$ (m/s)	Re (−)	e/D (−)	f (−)	L/D (−)	Δp (kPa)
3	0.0779	19.9	1.36×10^6	0.0019	0.024	834	4530
5	0.128	7.39	8.29×10^5	0.0012	0.021	508	360
6	0.154	5.10	6.89×10^5	0.001	0.020	422	141

Pipe friction calculations are accurate only within about ±10 percent. Line resistance (and consequently Δp) will increase with age.

Recommend installation of 6 in. (nominal) line. D

* Values of f obtained using Excel's Solver (or Goal Seek)

Problem 8.136 (In Excel)

Investigate the effect of tube diameter on flow rate by computing the flow generated by a pressure difference, $\Delta p = 100$ kPa, applied to a length $L = 100$ m of smooth tubing. Plot the flow rate against tube diameter for a range that includes laminar and turbulent flow.

Given: Pressure drop per unit length

Find: Plot flow rate versus diameter

Solution

Governing equations:

$$Re = \frac{\rho \cdot V \cdot D}{\mu}$$

$$\left(\frac{p_1}{\rho} + \alpha_1 \cdot \frac{V_1^2}{2} + g \cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \cdot \frac{V_2^2}{2} + g \cdot z_2\right) = h_l \qquad (8.29)$$

$$h_l = f \cdot \frac{L}{D} \cdot \frac{V^2}{2} \qquad (8.34)$$

$$f = \frac{64}{Re} \qquad (8.36) \quad \text{(Laminar)}$$

$$\frac{1}{f^{0.5}} = -2.0 \cdot \log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re \cdot f^{0.5}}\right) \qquad (8.37) \quad \text{(Turbulent)}$$

The energy equation (Eq. 8.29) becomes for flow in a tube

$$p_1 - p_2 = \Delta p = \rho \cdot f \cdot \frac{L}{D} \cdot \frac{V^2}{2}$$

This cannot be solved explicitly for velocity *V* (and hence flow rate *Q*), because *f* depends on *V*; solution for a given diameter *D* requires iteration (or use of *Solver*)

Fluid is not specified: use water (basic trends in plot apply to any fluid)

Given data:

Δp = 100 kPa

L = 100 m

Tabulated or graphical data:

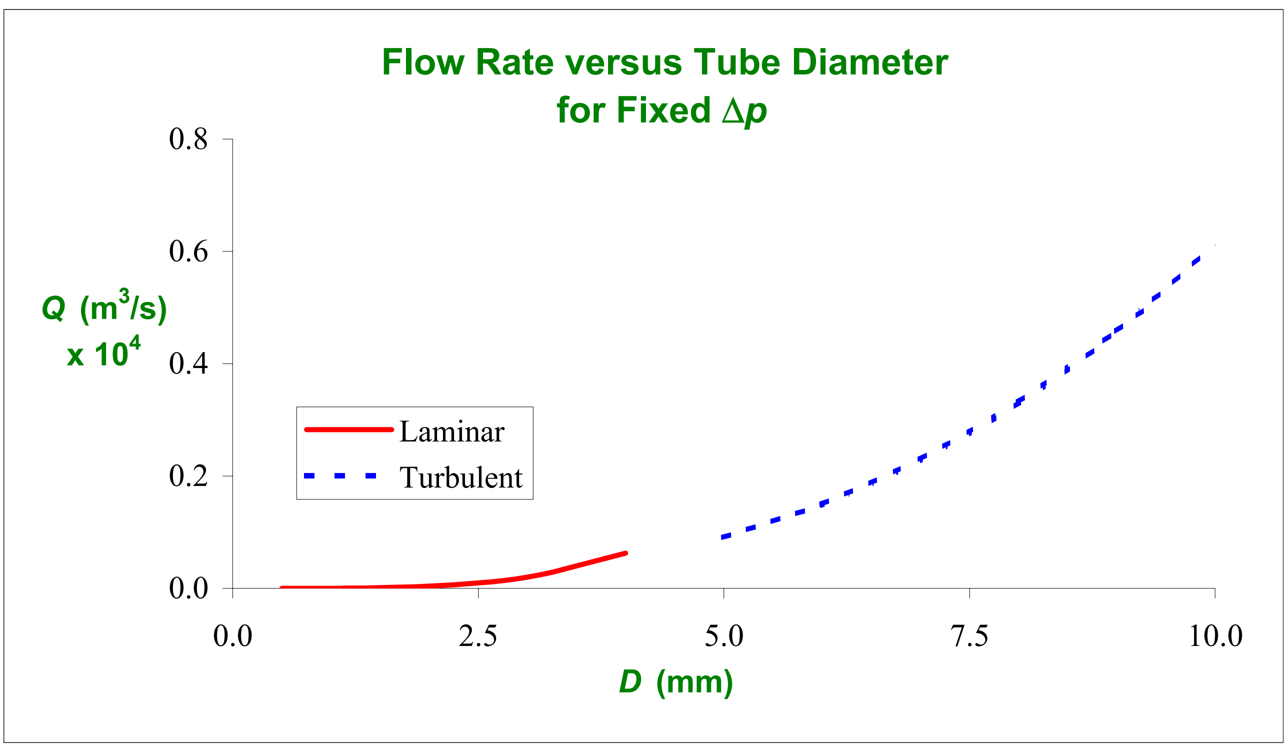
μ = 1.00E-03 $N.s/m^2$

ρ = 999 kg/m^3

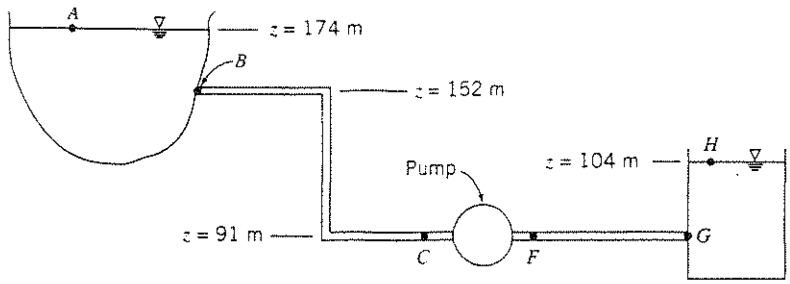
(Water - Appendix A)

Computed results:

D **(mm)**	***V*** **(m/s)**	***Q*** **(m^3/s) x 10^4**	***Re***	**Regime**	***f***	**Δp (kPa)**	**Error**
0.5	0.00781	0.0000153	4	Laminar	16.4	100	0.0%
1.0	0.0312	0.000245	31	Laminar	2.05	100	0.0%
2.0	0.125	0.00393	250	Laminar	0.256	100	0.0%
3.0	0.281	0.0199	843	Laminar	0.0759	100	0.0%
4.0	0.500	0.0628	1998	Laminar	0.0320	100	0.0%
5.0	0.460	0.0904	2300	Turbulent	0.0473	100	0.2%
6.0	0.530	0.150	3177	Turbulent	0.0428	100	0.0%
7.0	0.596	0.229	4169	Turbulent	0.0394	100	0.0%
8.0	0.659	0.331	5270	Turbulent	0.0368	100	0.0%
9.0	0.720	0.458	6474	Turbulent	0.0348	100	0.0%
10.0	0.778	0.611	7776	Turbulent	0.0330	100	0.0%



Given: Portion of water supply system designed to provide $Q = 1310$ L/s at $T = 20°C$.



System B→C
- square edged entrance
- 3 gate valves
- 4 45° elbows
- 2 90° elbows
- 760 m pipe
- $p_C = 197$ kPa gage

System F→G
- 760 m pipe
- 2 gate valves
- 4 90° elbows.

All pipe is cast iron, $D = 508$ mm

Find: (a) average velocity in pipe line
(b) gage pressure p_F
(c) shear stress on pipe centerline at C
(d) power input to pump if efficiency $\eta = 80\%$.
(e) wall shear stress at G.

Solution:

Since $Q = A\bar{V}$, $\bar{V} = \frac{Q}{A} = \frac{4Q}{\pi D^2} = \frac{4}{\pi} \times 1310 \frac{L}{s} \times \frac{1}{(0.508 m)^2} \times 10^{-3} \frac{m^3}{L} = 6.46$ m/s $\quad \bar{V}$

To determine the pressure at point F, apply the energy equation for steady, incompressible flow between F and G.

Basic equation: $\left(\frac{p_F}{\rho} + \alpha \frac{\bar{V}_F^2}{2} + g z_F\right) - \left(\frac{p_H}{\rho} + \alpha \frac{\bar{V}_H^2}{2} + g z_H\right) = h_{l_T} \quad (8.29)$

$h_{l_T} = h_l + h_{l_m}$, $h_l = f \frac{L}{D} \frac{\bar{V}^2}{2}$, $h_{l_m} = \frac{\bar{V}^2}{2} \sum f \frac{L_e}{D} + \frac{\bar{V}^2}{2} K_{exit}$

Assume: (1) $\bar{V}_H = 0$ (large storage tank) (2) $p_H = p_{atm}$
(3) $\alpha_F \approx 1.0$

Then $\frac{p_F}{\rho} = h_{l_T} + g(z_H - z_F) - \frac{\bar{V}_F^2}{2}$

$$\frac{p_F}{\rho} = h_{l_{F-G}} + 2h_{l_{m_{gv}}} + 4h_{l_{m_{90°el}}} + h_{l_{m_{exit}}} + g(z_H - z_F) - \frac{\bar{V}_F^2}{2}$$

$$\frac{p_F}{\rho} = f\frac{L}{D}\frac{\bar{V}^2}{2} + 2f\left(\frac{L_e}{D}\right)_{gv}\frac{\bar{V}^2}{2} + 4f\left(\frac{L_e}{D}\right)_{90°el}\frac{\bar{V}^2}{2} + K_{exit}\frac{\bar{V}^2}{2} + g(z_H - z_F) - \frac{\bar{V}_F^2}{2} \quad ----(1)$$

From Table 8.4 $\left(\frac{L_e}{D}\right)_{gv} = 8$, $\left(\frac{L_e}{D}\right)_{90°el} = 30$; also $K_{exit} = 1$

$Re = \frac{D\bar{V}}{\nu} = 0.508 m \times 6.46 \frac{m}{s} \times \frac{s}{1.00 \times 10^{-6} m^2} = 3.28 \times 10^6$ (ν from Table A.8)

From Table 8.1, $e = 0.26$ mm $\therefore e/D = 0.00051$

From Eq. 8.37, $f = 0.017$ (using Excel's Solver [or Goal Seek])

From Eq (1)

$$\frac{p_F}{\rho} = f\frac{\bar{V}^2}{2}\left[\frac{L}{D} + 2\left(\frac{L_e}{D}\right)_{gv} + 4\left(\frac{L_e}{D}\right)_{90°el}\right] + g(z_H - z_F)$$

$$\frac{p_F}{\rho} = f\frac{\bar{V}^2}{2}\left[\frac{760}{0.508} + 2(8) + 4(30)\right] + g(z_H - z_F) = f\frac{\bar{V}^2}{2}(1630) + g(z_H - z_F)$$

$$p_F = \rho\left[1630\, f\frac{\bar{V}^2}{2} + g(z_H - z_F)\right]$$

$$= 999\frac{kg}{m^3}\left[\frac{1630}{2}\times 0.017\times(6.46)^2\frac{m^2}{s^2} + 9.81\frac{m}{s^2}(104-91)m\right]\times\frac{N\cdot s^2}{kg\cdot m}$$

$p_F = 705$ kPa (gage) ← p_F

For fully developed flow in a pipe $\tau = \frac{r}{2}\frac{\partial p}{\partial x}$ (8.15)

At the pipe centerline, $\tau = 0$ ← $\tau_{\mathbb{C}}$

To determine the power input to the fluid apply the energy equation across the pump. Assuming 100% efficiency

$$\frac{\dot{W}_{pump}}{\dot{m}} = \left(\frac{p}{\rho} + \frac{\bar{V}^2}{2} + gz\right)_{discharge} - \left(\frac{p}{\rho} + \frac{\bar{V}^2}{2} + gz\right)_{suction} \quad (8.47)$$

$$\dot{W}_{pump} = \left(\frac{p_F}{\rho} - \frac{p_C}{\rho}\right)\rho A\bar{V} = (p_F - p_C)Q$$

$$\dot{W}_{pump} = (705-197)\times 10^3\frac{N}{m^2}\times 1310\frac{L}{s}\times 10^{-3}\frac{m^3}{L} = 6.65\times 10^5\ \frac{N\cdot m}{s}$$

The actual pump input, $\dot{W}_{pump})_{act} = \dot{W}_{pump})_{ideal}/\eta$

$\dot{W}_{pump})_{actual} = 8.32\times 10^5$ N·m/s $= 832$ kW ← $\dot{W}_{actual}$

From Eq. 8.15 $\tau_w = \frac{R}{2}\frac{\partial p}{\partial x}$

Along the pipe from F to G $\frac{\Delta p}{\rho} = f\frac{L}{D}\frac{\bar{V}^2}{2}$

$$\therefore \frac{\partial p}{\partial x} = \frac{\Delta p}{L} = \rho\frac{f}{D}\frac{\bar{V}^2}{2} = 999\frac{kg}{m^3}\times\frac{0.017}{0.508\,m}\times\frac{1}{2}\times(6.46)^2\frac{m^2}{s^2}\times\frac{N\cdot s^2}{kg\cdot m}$$

$$\frac{\partial p}{\partial x} = 698\ N/m^2/m$$

$$\therefore \tau_w = \frac{R}{2}\frac{\partial p}{\partial x} = \frac{0.254\,m}{2}\times 698\frac{N}{m^3} = 88.6\ N/m^2$$ ← τ_w

Given: An air-pipe friction experiment utilizes smooth brass tube, $D = 63.5$ mm, $L = 1.52$ m.
At one flow condition $\Delta p = 12.3$ mm Meriam red oil, $U_{\not{c}} = 23.1$ m/s.

Find: (a) $Re_{\bar{V}}$
(b) friction factor f; compare with value for Fig. 8.13.

Solution:

Apply the energy equation for steady, incompressible flow along the pipe

Basic equation: $$\left(\frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2} + g z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \frac{\bar{V}_2^2}{2} + g z_2\right) = h_{l_T} \qquad (8.29)$$

$$h_l = f \frac{L}{D} \frac{\bar{V}_1^2}{2}$$

Computing equation: $$\frac{\bar{V}}{U} = \frac{2n^2}{(n+1)(2n+1)} \qquad (8.24)$$

Assumptions: (1) power law profile, $n = 7$
(2) $\alpha_1 = \alpha_2$, $z_1 = z_2$
(3) air at $T = 15^\circ C$ $\nu = 1.46 \times 10^{-5}$ m²/s (Table A.10).

From Eq. 8.24 with $n = 7$

$$\frac{\bar{V}}{U} = \frac{2(7)^2}{(8)(15)} = 0.817$$

$$Re_{\bar{V}} = \frac{D\bar{V}}{\nu} = 0.0635\,\text{m} \times 0.817 \times 23.1\,\frac{\text{m}}{\text{s}} \times \frac{\text{s}}{1.45 \times 10^{-5}\,\text{m}^2} = 8.26 \times 10^4 \quad \longleftarrow Re_{\bar{V}}$$

From Eq. 8.29 $$\Delta p / \rho = f \frac{L}{D} \frac{\bar{V}^2}{2}$$

and

$$f = \frac{\Delta p}{\rho \frac{L}{D} \frac{\bar{V}^2}{2}} = \frac{\rho_{oil}\, g\, \Delta h}{\rho_{air} \frac{L}{D} \frac{\bar{V}^2}{2}} = \frac{2 \rho_{H_2O}\, SG_{oil}\, g\, \Delta h}{\rho_{air} \frac{L}{D} \bar{V}^2} \qquad \begin{Bmatrix} SG = 0.827 \\ \text{Table A.1} \end{Bmatrix}$$

$$f = 2 \times \frac{10^3}{1.23} \times 0.827 \times 9.81\,\frac{\text{m}}{\text{s}^2} \times 0.0123\,\text{m} \times \frac{0.0635\,\text{m}}{1.52\,\text{m}} \times \frac{1}{(0.817 \times 23.1)^2}\,\frac{\text{s}^2}{\text{m}^2}$$

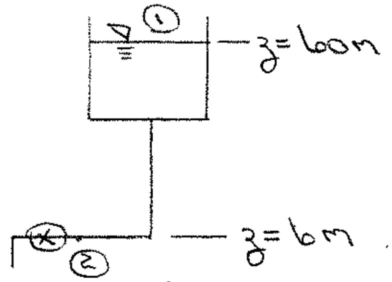
$$f = 0.0190 \quad \longleftarrow f$$

From Eq. 8.37 at $Re = 8.26 \times 10^4$ for smooth tube, $f = 0.0187$

The value of f is obtained using Excel's Solver (or Goal Seek)

Given: Oil flowing from a large tank on a hill to a tanker at the wharf. In stopping the flow, valve on wharf at such a rate that $p_2 = 1$ MPa is maintained in the line immediately upstream of the valve. Assume:

Length of line from tank to valve	3 km
Inside diameter of line	200 mm
Elevation of oil surface in tank	60 m
Elevation of valve on wharf	6 m
Instantaneous flow rate	2.5 m³/min
Head loss in line (exclusive of valve being closed) at this rate of flow	23 m of oil
Specific gravity of oil	0.88



Find: the initial instantaneous rate of change of volume flow rate.

Solution: For unsteady flow with friction, we modify the unsteady Bernoulli equation (Eq. 6.21) to include a head loss term.

Computing equation: $\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2 + \int_1^2 \frac{\partial V_s}{\partial t} ds + h_\ell$

Assume: (1) $V_1 \approx 0$ (3) ρ = constant
(2) $p_1 = p_{atm}$

Then
$$\int_1^2 \frac{\partial V_s}{\partial t} ds = \frac{p_1 - p_2}{\rho} + g(z_1 - z_2) - h_\ell - \frac{V_2^2}{2}$$

If we neglect velocity in the tank except for small region near the inlet to the pipe, then

$\int_1^2 \frac{\partial V_s}{\partial t} ds \approx \int_0^L \frac{\partial V_s}{\partial t} ds$. Since $V_s = V_2$ everywhere, then

$$\int_0^L \frac{\partial V_s}{\partial t} ds = L \frac{dV_2}{dt} \quad \text{and}$$

$$\frac{dV_2}{dt} = \frac{1}{L}\left[\frac{p_1 - p_2}{\rho} + g(z_1 - z_2) - h_\ell - \frac{V_2^2}{2}\right], \quad V_2 = \frac{Q}{A} = \frac{4Q}{\pi D^2}$$

Note $h_\ell = h_\ell(V)$ and hence this result can only be used to obtain the initial instantaneous rate of change of flow velocity

$$\therefore \left.\frac{dV_2}{dt}\right)_{initial} = \frac{1}{3\times10^3\,\text{m}}\left[10^6 \frac{\text{N}}{\text{m}^2} \times \frac{\text{m}^3}{999\,\text{kg}} \times \frac{1}{0.88}\,\frac{\text{kg}\cdot\text{m}}{\text{N}\cdot\text{s}^2} + 9.81\frac{\text{m}}{\text{s}^2}\times 54\,\text{m} - 23\,\text{m}\times 9.81\frac{\text{m}}{\text{s}^2} - \frac{1}{2}\left\{\frac{4}{\pi}\times 2.5\frac{\text{m}^3}{\text{min}}\times\frac{1}{(0.2\,\text{m})^2}\times\frac{1\,\text{min}}{60\,\text{s}}\right\}^2\right]$$

$$\left.\frac{dV_2}{dt}\right)_{initial} = -0.278\ \text{m/s/s}$$

The instantaneous rate of change of volume flow rate is

$$dQ/dt = \frac{d}{dt}(AV) = A\frac{dV}{dt} = \frac{\pi D^2}{4}\frac{dV}{dt}$$

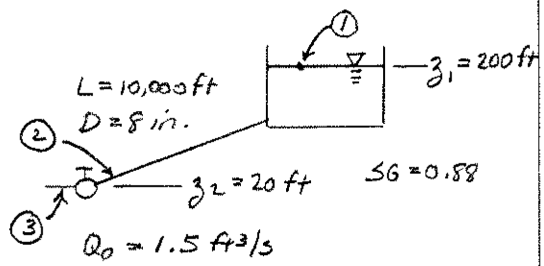
$$dQ/dt = \frac{\pi}{4}(0.2\,\text{m})^2 \times \left(-0.278\,\frac{\text{m/s}}{\text{s}}\right) \times 60\frac{\text{s}}{\text{min}} = -0.524\ \text{m}^3\text{/s/min} \leftarrow dQ/dt$$

Problem *8.140

Given: Problem 8.139 describes a situation in which flow in a long pipeline from a hilltop tank is slowed gradually to avoid a large pressure rise.

Find: Expansion of this analysis to predict and plot the closing schedule (valve loss coefficient versus time) needed to maintain the maximum pressure at the valve at or below a given value throughout the process of stopping the flow from the tank.

Solution: Apply the unsteady Bernoulli equation with a head loss term added.



$Q_0 = 1.5\ ft^3/s$

Line loss = 75 ft of oil

Computing equation:

$$\frac{p_1}{\rho}^{\approx 0} + \frac{V_1^2}{2}^{\approx 0(1)} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2 + \int_1^2 \frac{\partial V}{\partial t}\, ds + h_{lT}$$

Assume: (1) $V_1 \approx 0$ (2) $p_1 = p_{atm}$ (3) ρ = constant

At the initial condition, $V = \frac{Q}{A} = \frac{4Q}{\pi D^2} = \frac{4}{\pi} \times \left(\frac{12}{8}\right)^2 \frac{1}{ft^2} \times 1.5 \frac{ft^3}{s} = 4.30\ ft/s$

$$H_{lT} = 75\ ft = \frac{h_{lT}}{g} = f \frac{L}{D} \frac{V^2}{2g}\ ;\quad f\frac{L}{D} = H_{lT} \frac{2g}{V^2} = 2 \times 75\ ft \times 32.2 \frac{ft}{s^2} \times \frac{s^2}{(4.30)^2\ ft^2} = 261$$

Neglecting velocity in tank, $\int_1^2 \frac{\partial V}{\partial t} ds \approx \int_0^L \frac{\partial V}{\partial t} ds = \frac{dV}{dt} L$

Thus
$$\frac{dV}{dt} = \frac{1}{L}\left[-\frac{p_2}{\rho} + g(z_1 - z_2) - f\frac{L}{D}\frac{V^2}{2} - \frac{V^2}{2}\right]$$

Substituting values,

$$\frac{dV}{dt} = \frac{1}{10{,}000\ ft}\left[-150 \frac{lbf}{in.^2} \times \frac{ft^3}{(0.88)\,1.94\ slug} \times 144 \frac{in.^2}{ft^2} \times \frac{slug \cdot ft}{lbf \cdot s^2} + 32.2 \frac{ft}{s^2}(200-20)\ ft - (261+1)\frac{V^2}{2}\right]$$

$$\frac{dV}{dt} = -0.686 \frac{ft}{s^2} - 0.0131\, V^2 = -(a^2 + b^2 V^2)\ ; \quad a = \sqrt{0.686} = 0.828\ ; \ V \text{ in ft/s}$$
$$b = \sqrt{0.0131} = 0.114$$

Separating variables and integrating

$$\int_{V_0}^{V} \frac{dV}{a^2 + b^2 t} = \frac{1}{ab} \tan^{-1} \frac{bV}{a}\Big]_{V_0}^{V} = \frac{1}{ab}\left[\tan^{-1}\frac{bV}{a} - \tan^{-1}\frac{bV_0}{a}\right] = -\int_0^t dt = -t$$

Thus

$$\tan^{-1}\frac{bV}{a} = -abt + \tan^{-1}\frac{bV_0}{a} \quad \text{or} \quad V = \frac{a}{b}\tan\left[\tan^{-1}\frac{bV_0}{a} - abt\right]$$ $V(t)$

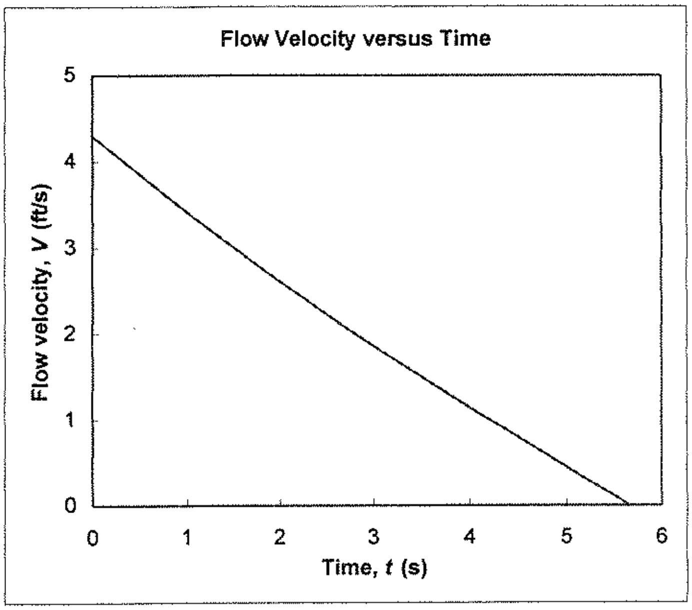
The pressure must drop across the valve:

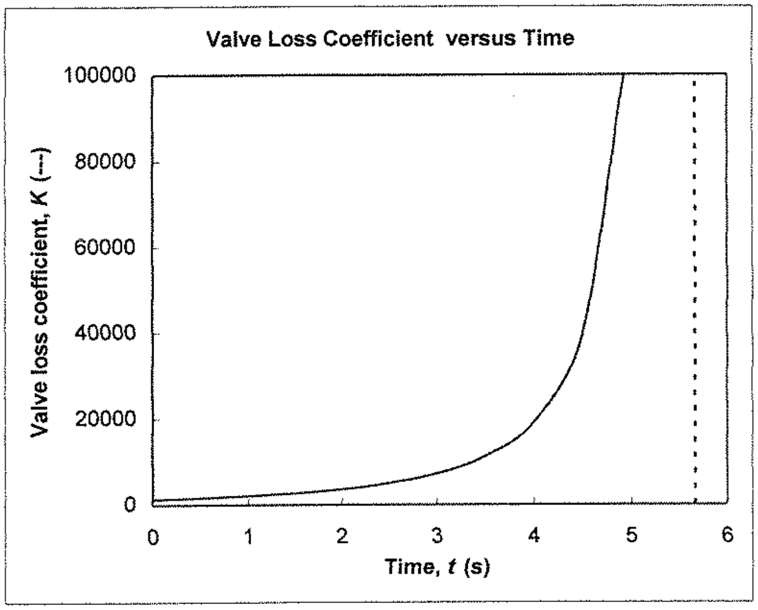
$$\frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2 - \left(\frac{p_3}{\rho} + \frac{V_3^2}{2} + gz_3\right) = h_{lT} = K_V \frac{V^2}{2} \quad \text{or} \quad K_V = \frac{2(p_2 - p_3)}{\rho V^2} \approx \frac{2p_2}{\rho V^2}$$

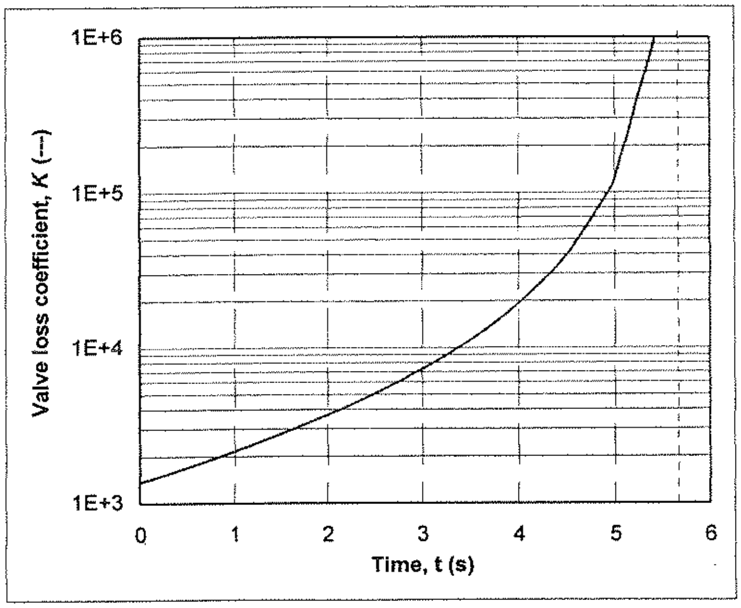
At $t=0$,
$$K_V = 2 \times 150 \frac{lbf}{in.^2} \times \frac{ft^3}{(0.88)\,1.94\ slug} \times \frac{s^2}{(4.3)^2\ ft^2} \times 144 \frac{in.^2}{ft^2} \times \frac{slug \cdot ft}{lbf \cdot s^2} = 1{,}370 \quad (t=0)$$ $K_V(0)$

Calculations and plots are shown on the spreadsheet, next page.

	a =	0.828	(---)
	b =	0.114	(---)
	V_0 =	4.3	(ft/s)
t (s)	V (ft/s)	K (---)	
0.0	4.300	1.37E+3	
0.5	3.849	1.71E+3	
1.0	3.421	2.16E+3	
1.5	3.011	2.79E+3	
2.0	2.616	3.70E+3	
2.5	2.235	5.07E+3	
3.0	1.865	7.28E+3	
3.5	1.504	1.12E+4	
4.0	1.150	1.92E+4	
4.5	0.801	3.95E+4	
4.93	0.503	1.00E+5	
5.0	0.455	1.22E+5	
5.431	0.159	1.00E+6	
5.5	0.112	2.03E+6	
5.663	0.000	2.92E+13	







Problem 8.141

Given: The pressure rise, Δp, across a water pump is 9.5 psi when the volume flow rate is $Q = 300$ gpm. The pump efficiency is $\eta = 0.80$

Find: the power input to the pump.

Solution:

Apply the first law of thermodynamics across the pump.

Basic equation: $\dot{W}_{pump} = \dot{m}\left[\left(\frac{p}{\rho} + \frac{\bar{V}^2}{2} + gz\right)_{discharge} - \left(\frac{p}{\rho} + \frac{\bar{V}^2}{2} + gz\right)_{suction}\right]$ (8.47)

where $\dot{W}_{pump}$ is energy added to fluid by the pump

Assume: (1) ρ = constant (2) $z_1 = z_2$
(3) uniform properties at inlet & outlet
(4) $A_1 = A_2$, $\therefore \bar{V}_2 = \bar{V}_1$

Then

$$\dot{W}_{pump} = \dot{m}\,\frac{\Delta p}{\rho} = \rho A V \frac{\Delta p}{\rho} = Q\,\Delta p$$

$$\dot{W}_{pump} = 300\,\frac{gal}{min} \times \frac{ft^3}{7.48\,gal} \times \frac{min}{60\,s} \times 9.5\,\frac{lbf}{in^2} \times \frac{144\,in^2}{ft^2} \times \frac{hp \cdot s}{550\,ft \cdot lbf}$$

$$\dot{W}_{pump} = 1.66\ hp$$

The pump efficiency, η is defined as

$$\eta = \frac{\dot{W}_{pump}}{\dot{W}_{in}} \quad \therefore \dot{W}_{in} = \frac{\dot{W}_{pump}}{\eta} = 2.08\ hp$$ ← $\dot{W}_{input}$

Problem 8.142

Given: Pump moves $\dot{m} = 10$ kg/s through a piping system

$p_{discharge} = 300$ kPa, $p_{suction} = -20$ kPa

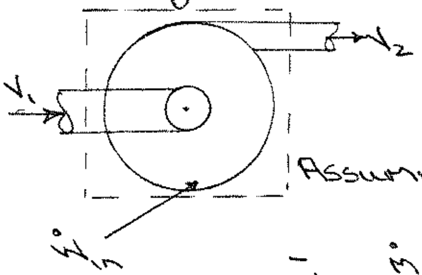
$D_{suction} = 75$ mm, $D_{discharge} = 50$ mm $\eta_{pump} = 0.70$

Find: Power required to drive the pump.

Solution:

Apply the first law of thermodynamics across the pump.

Basic equation: $$\frac{\dot{W}_{pump}}{\dot{m}} = \left(\frac{p}{\rho} + \frac{\bar{V}^2}{2} + gz\right)_{discharge} - \left(\frac{p}{\rho} + \frac{\bar{V}^2}{2} + gz\right)_{suction} \quad (8.45)$$



$$\eta_{pump} = \frac{\dot{W}_{pump}}{\dot{W}_{in}}$$

Assume: (1) ρ = constant (2) $z_1 = z_2$
(3) uniform properties at inlet & outlet.

$$\bar{V} = \frac{\dot{m}}{\rho A} = \frac{4\dot{m}}{\rho \pi D^2}$$

$$\bar{V}_1 = \frac{4}{\pi} \times 10\frac{kg}{s} \times 999\frac{m^3}{kg} \times \frac{1}{(0.075m)^2} = 2.27 \text{ m/s}$$

$$\bar{V}_2 = \frac{A_1 \bar{V}_1}{A_2} = \left(\frac{D_1}{D_2}\right)^2 \bar{V}_1 = \left(\frac{3}{2}\right)^2 \times 2.27 \text{ m/s} = 5.10 \text{ m/s}$$

From Eq. 8.45

$$\dot{W}_{pump} = \dot{m}\left[\frac{p_2 - p_1}{\rho} + \frac{\bar{V}_2^2 - \bar{V}_1^2}{2}\right]$$

$$\dot{W}_{pump} = 10\frac{kg}{s}\left[320 \times 10^3 \frac{N}{m^2} \times 999\frac{m^3}{kg} \times \frac{kg \cdot m}{N \cdot s^2} + \frac{(5.10)^2 - (2.27)^2}{2}\frac{m^2}{s^2}\right]\frac{N \cdot s^2}{kg \cdot m}$$

$$\dot{W}_{pump} = 3{,}310 \text{ N}\cdot\text{m/s} = 3.31 \text{ kW}$$

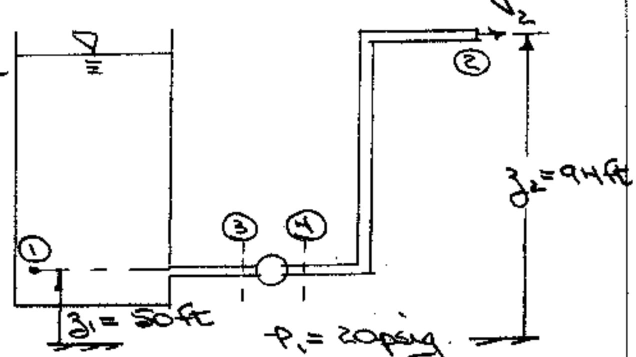
$$\dot{W}_{in} = \frac{\dot{W}_{pump}}{\eta} = \frac{3.31 \text{ kW}}{0.7} = 4.72 \text{ kW} \quad \dot{W}_{in}$$

Problem 8.143

Given: Pump in piping system shown moves $Q = 0.439\ ft^3/s$ of water.

System includes:

- $L = 290\ ft$ galvanized pipe, $D = 2.5\ in.$ (nominal)
- 2 gate valves (open)
- 1 angle valve (open)
- 7 standard 90° elbows
- 1 square edge entrance
- 1 free discharge



Find: pressure rise, $p_4 - p_3$, across pump.

Solution:

Computing equation:
$$\left(\frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2} + g z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \frac{\bar{V}_2^2}{2} + g z_2\right) + \Delta h_{pump} = h_{l_T} \qquad (8.48)$$

(with $\bar{V}_1^2/2 \to 0(1)$ and $p_2/\rho \to 0(2)$ crossed out)

$$h_{l_T} = h_l + h_{l_m}, \quad h_l = f\frac{L}{D}\frac{\bar{V}^2}{2}, \quad h_{l_m} = \frac{\bar{V}^2}{2}\left\{\Sigma f\left(\frac{L_e}{D}\right) + \Sigma K\right\}$$

Assumptions: (1) $\bar{V}_1 = 0$ (2) $p_2 = p_{atm}$ (3) $\alpha_2 = 1.0$ (4) $T = 60°F$

Then,
$$\Delta h_{pump} = h_{l_T} + g(z_2 - z_1) + \frac{\bar{V}_2^2}{2} - \frac{p_1}{\rho} \quad \text{------------} (1)$$

$$h_{l_T} = \frac{\bar{V}_2^2}{2}\left[f\frac{L}{D} + 2f\left(\frac{L_e}{D}\right)_{gv} + f\left(\frac{L_e}{D}\right)_{av} + 7f\left(\frac{L_e}{D}\right)_{90°el} + K_{ent}\right] \quad \text{------} (2)$$

From Table 8.4 $L_e/D)_{gv} = 8$, $L_e/D)_{av} = 150$, $L_e/D)_{90°el} = 30$

From Table 8.2 $K_e = 0.5$. From Table 8.5 $D = 2.47\ in$

From Table 8.1 $e = 0.0005\ ft \quad \therefore e/D = 0.0005 \times \frac{12}{2.47} = 0.0024$

$$\bar{V} = \frac{Q}{A} = \frac{4Q}{\pi D^2} = \frac{4}{\pi} \times 0.439\frac{ft^3}{s} \times \left(\frac{12}{2.47\ ft}\right)^2 = 13.2\ ft/s$$

$$Re = \frac{\bar{V}D}{\nu} = \frac{2.47\ ft}{12} \times 13.2\frac{ft}{s} \times \frac{s}{1.21 \times 10^{-5}\ ft^2} = 2.25 \times 10^5 \quad \{\nu \text{ from Table A.7}\}$$

From Fig. 8.13, $f = 0.025$.

From Eq. 2

$$h_{l_T} = \frac{1}{2} \times (13.2)^2 \frac{ft^2}{s^2}\left[0.025 \times \frac{290 \times 12}{2.47} + 2(0.025)(8) + (0.025)(150) + 7(0.025)(30) + 0.5\right]$$

$h_{l_T} = 3930\ ft^2/s^2$. Then from Eq. 1

$$\Delta h_{pump} = 3930\frac{ft^2}{s^2} + 32.2\frac{ft}{s^2}(44\ ft) + \frac{1}{2}(13.2)^2\frac{ft^2}{s^2} - 20\frac{lbf}{in^2} \times \frac{ft^3}{1.94\ slug} \times \frac{144\ in^2}{ft^2} \times \frac{slug \cdot ft}{lbf \cdot s^2}$$

$\Delta h_{pump} = 3950\ ft^2/s^2$

Apply the energy equation across the pump

$$\Delta h_{pump} = \left(\frac{p}{\rho} + \frac{\bar{V}^2}{2} + gz\right)_{discharge} - \left(\frac{p}{\rho} + \frac{\bar{V}^2}{2} + gz\right)_{suction} \qquad (8.47)$$

$$\Delta p = \rho\,\Delta h_{pump} = 1.94\frac{slug}{ft^3} \times 3950\frac{ft^2}{s^2} \times \frac{lbf \cdot s^2}{ft \cdot slug} \times \frac{ft^2}{144\ in^2} = 53.2\ lbf/in^2 \quad (p_4 - p_3)$$

Problem 8.144

Given: Water supply system requires Q = 100 gpm pumped to reservoir at elevation of 340 m. Water pressure at pump inlet (street level) is 400 kPa (gage). Piping is to be commercial steel; $\bar{V}_{max} = 3.5$ m/s

Find: (a) Minimum pipe diameter
(b) pressure rise across the pump
(c) minimum power needed to drive the pump.

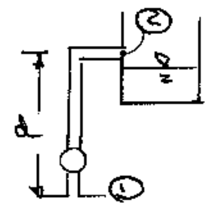
Solution:

$$\bar{V} = \frac{Q}{A} = \frac{4Q}{\pi D^2} \quad \therefore D = \left[\frac{4Q}{\pi \bar{V}}\right]^{1/2} = \left[\frac{4}{\pi} \times 100\,\frac{gal}{min} \times \frac{ft^3}{7.48\,gal} \times \frac{min}{60\,s} \times \left(\frac{0.305\,m}{ft}\right)^3 \times \frac{s}{3.5\,m}\right]^{1/2}$$

$$D = 0.048\,m = 48\,mm \quad \longleftarrow D$$

Computing equations:
$$\left(\frac{p_1}{\rho} + \alpha_1\frac{\bar{V}_1^2}{2} + gz_1\right) - \left(\frac{p_2}{\rho} + \alpha_2\frac{\bar{V}_2^2}{2} + gz_2\right) + \Delta h_{pump} = h_{l_T} \quad (8.48)$$
(with p_2 = 0 (2), and the $\alpha\bar{V}^2/2$ terms struck out (1))

$$h_{l_T} = h_l + h_{lm}\ (=0,\ (3)), \quad h_l = f\frac{L}{D}\frac{\bar{V}^2}{2}$$



Assume: (1) $\alpha_1 = \alpha_2$ (2) $p_2 = p_{atm}$
(3) minor losses are negligible.
(4) water at 20°C ($\nu = 1\times10^{-6}$ m²/s, Table A.8).

Then
$$\Delta h_{pump} = h_l - \frac{p_1}{\rho} + g(z_2 - z_1) = f\frac{L}{D}\frac{\bar{V}^2}{2} - \frac{p_1}{\rho} + gd \quad \ldots\ldots\ (1)$$

From Table 8.1, e = 0.046 mm $\therefore e/D = 0.046/48 = 0.00096$

$$Re = \frac{D\bar{V}}{\nu} = 0.048\,m \times 3.5\,\frac{m}{s} \times \frac{1}{1\times10^{-6}}\frac{s}{m^2} = 1.68\times10^5$$

From Fig. 8.13, f = 0.021. Then from Eq. 1

$$\Delta h_{pump} = 0.021 \times \frac{340}{0.048} \times \frac{1}{2}(3.5)^2\frac{m^2}{s^2} - 400\times10^3\,\frac{N}{m^2} \times \frac{m^3}{998\,kg} \times \frac{kg\cdot m}{N\cdot s^2} + 9.81\,\frac{m}{s^2} \times 340\,m$$

$$\Delta h_{pump} = 3{,}850\ m^2/s^2$$ (This is head added to fluid).

$$\Delta h_{pump} = \frac{\dot{W}_{pump}}{\dot{m}} = \left(\frac{p}{\rho} + \frac{\bar{V}^2}{2} + gz\right)_{discharge} - \left(\frac{p}{\rho} + \frac{\bar{V}^2}{2} + gz\right)_{suction} \quad (8.47)$$

Assume: (5) $\bar{V}_{disch} = \bar{V}_{suct}$; $z_{discharge} \approx z_{suction}$

$$\Delta p = \rho\,\Delta h = 998\,\frac{kg}{m^3} \times 3850\,\frac{m^2}{s^2} \times \frac{N\cdot s^2}{kg\cdot m} = 3840\ kPa \quad \Delta p$$

Also from Eq. 8.47

$$\dot{W}_{pump} = \dot{m}\,\Delta h_{pump} = \rho Q\,\Delta h_{pump}$$

$$\dot{W}_{pump} = 998\,\frac{kg}{m^3} \times 100\,\frac{gal}{min} \times \frac{ft^3}{7.48\,gal}\left(\frac{0.305\,m}{ft}\right)^3 \times \frac{min}{60\,s} \times 3850\,\frac{m^2}{s^2} \times \frac{N\cdot s^2}{kg\cdot m}$$

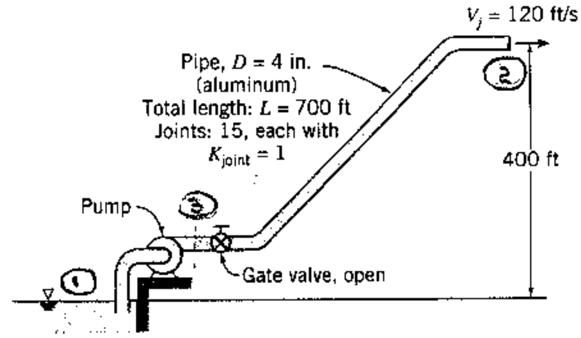
$$\dot{W}_{pump} = 24.3\ kW\ (32.6\ hp) \quad \longleftarrow \dot{W}_{pump}$$

Problem 8.145

Given: Cooling water supply system

$Q = 600$ gpm

$\eta_{pump} = 0.70$



Find: (a) minimum pressure needed at pump outlet

(b) power requirement

Solution:

Computing equations: $\left(\frac{p_1}{\rho} + \alpha_1\frac{\bar{V}_1^2}{2} + gz_1\right) - \left(\frac{p_2}{\rho} + \alpha_2\frac{\bar{V}_2^2}{2} + gz_2\right) + \Delta h_{pump} = h_{l_T}$ (8.48)

$h_{l_T} = h_l + h_{l_m}$, $h_l = f\frac{L}{D}\frac{\bar{V}^2}{2}$, $h_{l_m} = \frac{\bar{V}^2}{2}\left(\Sigma K + \Sigma f\left(\frac{L_e}{D}\right)\right)$

Assumptions: (1) $\bar{V}_1 = 0$ (2) $\alpha_2 = \alpha_3 = 1$ (3) $p_1 = p_2 = p_{atm}$

Then

$$\Delta h_{pump} = gz_2 + \frac{\bar{V}_2^2}{2} + f\frac{L}{D}\frac{\bar{V}^2}{2} + \frac{\bar{V}^2}{2}\left[K_{ent} + f\left(\frac{L_e}{D}\right)_{g.v.} + 2f\left(\frac{L_e}{D}\right)_{45^\circ el} + 15K_j\right] \quad (1)$$

$$\bar{V} = \frac{Q}{A} = \frac{4Q}{\pi D^2} = \frac{4}{\pi} \times \frac{600\,gal}{min} \times \frac{ft^3}{7.48\,gal} \times \frac{min}{60\,sec} \times \frac{1}{(0.333\,ft)^2} = 15.3\ ft/s$$

$$Re = \frac{D\bar{V}}{\nu} = \frac{1}{3}ft \times 15.3\frac{ft}{s} \times \frac{s}{1.24\times10^{-5}\,ft^2} = 4.11\times10^5 \quad \{\nu \text{ at } T = 60^\circ F, \text{ Table A.7}\}$$

Table 8.1, $e = 5\times10^{-6}$ ft (drawn tubing) $\therefore e/D = 1.5\times10^{-5}$

From Fig. 8.13, $f = 0.0135$

From Table 8.1, $K_{ent} = 0.78$

From Table 8.4, $L_e/D)_{g.v.} = 8$, $L_e/D)_{90^\circ el} = 30$, $L_e/D)_{45^\circ el} = 16$

Then from Eq. (1)

$$\Delta h_{pump} = 32.2\frac{ft}{s^2}\times 400\,ft + \frac{1}{2}(120)^2\frac{ft^2}{s^2} + 0.0135\times\frac{700}{0.333}\,\frac{1}{2}\times(15.3)^2\frac{ft^2}{s^2}$$
$$+ \frac{1}{2}(15.3)^2\frac{ft^2}{s^2}\left[0.78 + 0.0135(30) + 2\times0.0135(16) + 15(1)\right]$$

$\Delta h_{pump} = 2.53\times10^4\ ft^2/s^2$

The theoretical power input to the pump is given by $\dot{W}_{pump} = \dot{m}\,\Delta h_{pump}$

From the definition of efficiency, $\eta = \dot{W}_{theor}/\dot{W}_{act}$, then

$$\dot{W}_{act} = \frac{\dot{m}\,\Delta h_{pump}}{\eta} = \frac{\rho Q\,\Delta h_{pump}}{\eta}$$

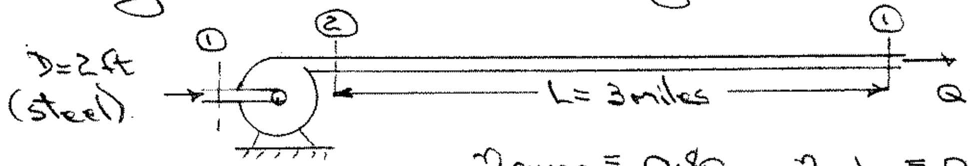
$$\dot{W}_{act} = \frac{1.94}{0.7}\frac{slug}{ft^3} \times \frac{600\,gal}{min} \times \frac{ft^3}{7.48\,gal} \times \frac{min}{60\,s} \times 2.53\times10^4\frac{ft^2}{s^2} \times \frac{lbf\cdot s^2}{ft\cdot slug} \times \frac{hp\cdot s}{550\,ft\cdot lbf} = 170\ hp \quad \dot{W}_{in}$$

The discharge pressure from the pump is obtained by applying Eq. 8.48 between sections ① and ③ neglecting losses in the inlet section, elevation change, and kinetic energy at ③

$$p_3 - p_A = \rho\,\Delta h_{pump} = 1.94\frac{slug}{ft^3} \times 2.53\times10^4\frac{ft^2}{s^2} \times \frac{lbf\cdot s^2}{ft\cdot s} \times \frac{ft^2}{144\,in^2} = 341\ psi \quad p_3$$

Problem 8.146

Given: Chilled-water pipe system for campus air conditioning makes a loop of length $L = 3$ miles.



$\eta_{pump} = 0.80$, $\eta_{motor} = 0.90$

$C = \$0.12 / (kw \cdot hr)$

Find: (a) the pressure drop, $p_2 - p_1$
(b) rate of energy addition to the water
(c) daily cost of electrical energy for pumping

Solution:

Apply energy equation for steady, incompressible pipe flow from pump discharge around loop to pump inlet

Computing equations: $$\left(\frac{p_2}{\rho} + \alpha_2 \frac{\bar{V}_2^2}{2} + gz_2\right) - \left(\frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2} + gz_1\right) = h_{l_T} \quad (8.29)$$

$$h_{l_T} = h_l + \cancel{h_{l_m}}^{(3)} \qquad h_l = f \frac{L}{D} \frac{\bar{V}^2}{2}$$

Assumptions: (1) $\alpha_1 = \alpha_2$, (2) $z_1 = z_2$ (3) neglect minor losses

Then
$$p_2 - p_1 = f \frac{L}{D} \rho \frac{\bar{V}^2}{2} \quad , \quad \bar{V} = \frac{Q}{A} = 11{,}200 \frac{gal}{min} \times \frac{4}{\pi (2)^2 ft^2} \times \frac{ft^3}{7.48\,gal} \times \frac{min}{60\,s} = 7.94\ ft/s$$

Assume $T = 50°F$, so $\nu = 1.40 \times 10^{-5}\ ft^2/s$.

$$Re = \frac{D\bar{V}}{\nu} = 2\,ft \times 7.94 \frac{ft}{s} \times \frac{s}{1.40 \times 10^{-5}\,ft^2} = 1.13 \times 10^6$$

From Table 8.1, $e = 0.00015\,ft$; $\therefore e/D = 0.000075$. Then, from Fig. 8.13, $f = 0.013$, and

$$\Delta p = p_2 - p_1 = 0.013 \times \frac{3\,mi}{2\,ft} \times \frac{5280\,ft}{mi} \times 1.94 \frac{slug}{ft^3} \times \frac{1}{2} \times (7.94)^2 \frac{ft^2}{s^2} \times \frac{lbf \cdot s^2}{ft \cdot slug} \times \frac{ft^2}{144\,in^2}$$

$\Delta p = 43.7$ psi. ← Δp

To determine the energy per unit mass applied by the pump

$$\frac{\dot{W}_{pump}}{\dot{m}} = \left(\frac{p}{\rho} + \cancel{\frac{\bar{V}^2}{2}} + \cancel{gz}\right)_{discharge} - \left(\frac{p}{\rho} + \cancel{\frac{\bar{V}^2}{2}} + \cancel{gz}\right)_{suction} \quad (8.45)$$

$$\dot{W}_{pump} = \dot{m} \frac{\Delta p}{\rho} = Q \Delta p$$

$$\dot{W}_{pump} = 11{,}200 \frac{gal}{min} \times \frac{ft^3}{7.48\,gal} \times \frac{min}{60\,s} \times \frac{43.7\,lbf}{in^2} \times \frac{144\,in^2}{ft^2} \times \frac{hp \cdot s}{550\,ft \cdot lbf} = 286\ hp \leftarrow \dot{W}_{pump}$$

The actual energy required to run the pump is

$$P = \frac{\dot{W}_{pump}}{\eta_{pump}\,\eta_{motor}} = 286\,hp \times \frac{1}{0.80} \times \frac{1}{0.90} = 397\ hp.$$

The daily cost is

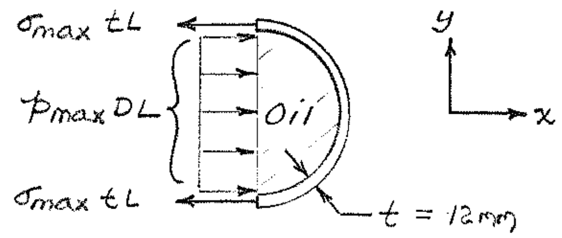
$$C = \frac{\$0.12}{kw \cdot hr} \times 397\,hp \times 0.746 \frac{kw}{hp} \times \frac{24\,hr}{day} = \$853$$

Given: Heavy crude oil (SG = 0.925) pumped through a level pipeline at a rate of 400,000 barrels per day (1 bbl = 42 gal). Pipe is 600 mm in diameter with 12 mm wall thickness. Maximum allowable stress in pipe wall is 275 MPa. Minimum pressure in oil is 500 kPa ($\nu = 1.0\times10^{-4}$ m²/s) Pipeline is steel.

Find: (a) Maximum allowable spacing between pumping stations.
(b) Power added to oil at each pumping station.

Solution: First find the maximum pressure allowable in pipe. Consider a free body diagram of a segment of length, L:

Basic equation: $\sum F_x = 0$



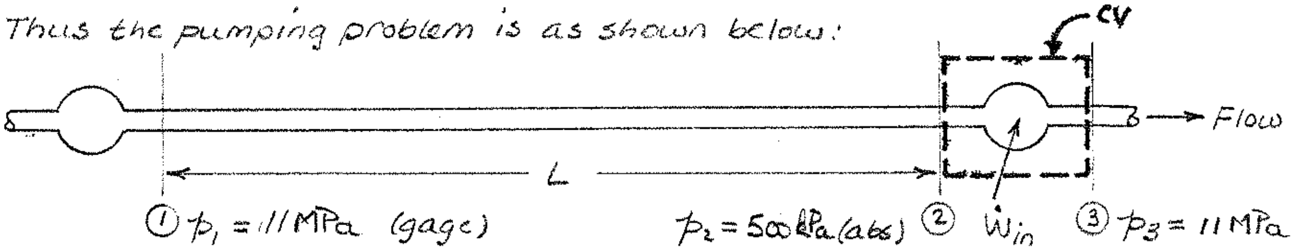
Assumption: Neglect hydrostatic pressure variation, and atmospheric pressure

Then

$$\sum F_x = p_{max} DL - 2\sigma_{max} tL = 0$$

$$p_{max} = 2\sigma_{max}\frac{t}{D} = 2\times 275\text{ MPa}\times\frac{12\text{mm}}{600\text{mm}} = 11\text{ MPa (gage)}$$

Thus the pumping problem is as shown below:



To find L, apply the energy equation for steady, incompressible flow that is uniform at each section.

Basic equation: $$\frac{p_1}{\rho} + \frac{\bar{V}_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{\bar{V}_2^2}{2} + gz_2 + h_{l_T}\ ;\ h_{l_T} = f\frac{L}{D}\frac{\bar{V}^2}{2} + h_{l_m}$$

Assumptions: (1) $\bar{V}_1 = \bar{V}_2$
(2) $z_1 = z_2$ (level)
(3) $h_{l_m} = 0$, since straight, constant area pipe

Then

$$f\frac{L}{D}\frac{\bar{V}^2}{2} = \frac{p_1 - p_2}{\rho} \quad \text{or} \quad L = \frac{D}{f}\left(\frac{p_1-p_2}{\rho}\right)\frac{2}{\bar{V}^2} \qquad (1)$$

$$\bar{V} = \frac{Q}{A} = 4\times10^5\frac{\text{bbl}}{\text{day}}\times\frac{\text{day}}{24\text{hr}}\times\frac{\text{hr}}{3600\text{s}}\times 42\frac{\text{gal}}{\text{bbl}}\times 4\frac{\text{qt}}{\text{gal}}\times 9.46\times10^{-4}\frac{\text{m}^3}{\text{qt}}\times\frac{4}{\pi}\frac{1}{(0.6\text{m})^2} = 2.60\text{ m/s}$$

$f = f(Re, e/D)$. From Table 8.1, $e = 0.046$ mm, so $e/D = 7.7\times10^{-5}$. Reynolds number is

$$Re = \frac{\rho\bar{V}D}{\mu} = \frac{\bar{V}D}{\nu} = 2.6\frac{\text{m}}{\text{s}}\times 0.6\text{m}\times\frac{\text{s}}{1.0\times10^{-4}\text{ m}^2} = 1.56\times10^4$$

From Eq 8.37, $f = 0.0277$ (Using Excel's Solver or Goal Seek)

Thus, substituting into Eq. 1

$$L = \frac{0.6\,m}{0.0277}\left[11\times10^6\,\frac{N}{m^2} - (500-101)\times10^3\,\frac{N}{m^2}\right]\times\frac{m^3}{(0.925)\,999\,kg}\times 2\times\frac{s^2}{(2.16)^2\,m^2}\times\frac{kg\cdot m}{N\cdot s^2}$$

$$L = 72.8\ km$$

L

To find pump power delivered to the oil, apply the energy equation to the CV shown, between sections ② and ③

$$\left(\frac{p}{\rho}+\alpha\frac{\bar{V}^2}{2}+gz\right)_{discharge} - \left(\frac{p}{\rho}+\alpha\frac{\bar{V}^2}{2}+gz\right)_{suction} = \frac{\dot{W}_{pump}}{\dot{m}} = \Delta h_{pump} \qquad (8.45)$$

Since $\bar{V}$ = constant and elevation change is small, this reduces to

$$\Delta h_{pump} = \frac{p_3 - p_2}{\rho}$$

$$= [11\times10^6 - (500-101)\times10^3]\,\frac{N}{m^2}\times\frac{m^3}{(0.925)\,999\,kg}\times\frac{kg\cdot m}{N\cdot s^2}$$

$$\Delta h_{pump} = 1.15\times10^4\ m^2/s^2$$

The mass flow rate is

$$\dot{m} = \rho Q = (0.925)\,999\,\frac{kg}{m^3}\times 400{,}000\,\frac{bbl}{day}\times 42\,\frac{gal}{bbl}\times 9.46\times10^{-4}\,\frac{m^3}{qt}\times 4\,\frac{qt}{gal}\times\frac{day}{24\,hr}\times\frac{hr}{3600\,s}$$

$$\dot{m} = 680\ kg/s$$

The power added to the oil is

$$\dot{W}_{pump} = \dot{m}\,\Delta h_{pump}$$

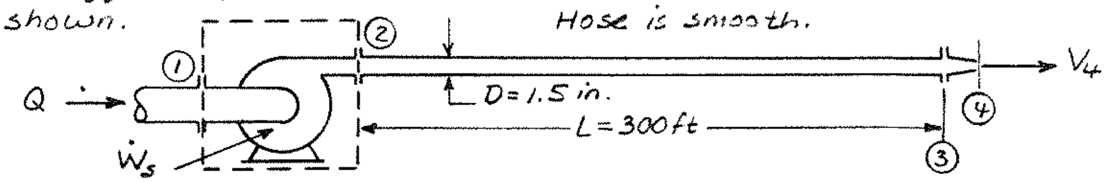
$$= 680\,\frac{kg}{s}\times 1.15\times10^4\,\frac{m^2}{s^2}\times\frac{N\cdot s^2}{kg\cdot m}$$

$$\dot{W}_{pump} = 7{,}730\ kW$$

$\dot{W}_{pump}$

Note pump efficiency does not affect the power that must be added to the oil.

Given: Fire nozzle supplied by booster pump at design conditions as shown. Hose is smooth.



$p_1 = 50$ psig $\qquad \dfrac{\Delta p}{L} = \dfrac{33\text{ psig}}{100\text{ ft}} \qquad p_3 = 100$ psig

Find: (a) Design flow rate.
(b) Exit velocity, assuming no losses in nozzle.
(c) Power to pump if $\eta = 0.70$.

Solution: Apply the energy equation to fully developed flow in the hose, the Bernoulli equation through the nozzle and the control volume form to the pump.

Basic equations:

$$\frac{p_2}{\rho} + \alpha_2 \frac{\bar{V}_2^2}{2} + g z_2 = \frac{p_3}{\rho} + \alpha_3 \frac{\bar{V}_3^2}{2} + g z_3 + h_{l_{T23}} \quad (3),(6)$$

$$\frac{p_3}{\rho} + \frac{V_3^2}{2} + g z_3 = \frac{p_4}{\rho} + \frac{V_4^2}{2} + g z_4 \quad (6)$$

(Eq. 8.45) $$\left(\frac{p}{\rho} + \alpha \frac{\bar{V}^2}{2} + g z\right)_{discharge} - \left(\frac{p}{\rho} + \alpha \frac{\bar{V}^2}{2} + g z\right)_{suction} = \frac{\dot{W}_{pump}}{\dot{m}} = \Delta h_{pump}$$

Assume water is at $T = 60°F$. Then $\nu = 1.21 \times 10^{-5}\ ft^2/s$ (Table A.7)

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Fully developed flow from ② to ③
(4) Flow along a streamline in nozzle
(5) No friction in nozzle
(6) Neglect elevation changes
(7) Uniform flow at each section across CV
(8) Neglect kinetic energy change across CV

For the hose,

$$\frac{\Delta p}{\rho} = \frac{p_2 - p_3}{\rho} = f \frac{L}{D} \frac{\bar{V}^2}{2} \quad \text{or} \quad \bar{V} = \left[\frac{2 \Delta p D}{\rho f L}\right]^{\frac{1}{2}} \quad ; \ \Delta p = 99 \text{ psi}$$

Since Re is not known, f is not known. Guess $f \simeq 0.015$. Then

$$\bar{V} = \left[2 \times 99 \frac{lbf}{in.^2} \times 1.5\,in. \times \frac{ft^3}{1.94\,slug} \times \frac{1}{0.015} \times \frac{1}{300\,ft} \times 12 \frac{in.}{ft} \times \frac{slug \cdot ft}{lbf \cdot s^2}\right]^{\frac{1}{2}} = 20.2\ ft/s$$

$$Re = \frac{\rho \bar{V} D}{\mu} = \frac{\bar{V} D}{\nu} = 20.2 \frac{ft}{sec} \times \frac{1.5}{12} ft \times \frac{s}{1.21 \times 10^{-5}\,ft^2} = 2.09 \times 10^5$$

Checking Eq. 8.37 gives $f = 0.016$ (using Excel's Solver or Goal Seek). Then

$$\bar{V} = \sqrt{\frac{0.015}{0.016}}\ 20.2\ ft/s = 19.6\ ft/s, \quad \text{and} \quad Re = \frac{19.6}{20.2} \times 2.09 \times 10^5 = 2.03 \times 10^5$$

This is satisfactory convergence.

$$Q = \bar{V}A = 19.6\,\frac{ft}{sec} \times \frac{\pi}{4}\left(\frac{1.5}{12}\right)^2 ft^2 \times 7.48\,\frac{gal}{ft^3} \times 60\,\frac{sec}{min} = 108\ gpm$$ Q

For the nozzle,

$$\frac{V_4^2}{2} = \frac{p_3 - p_4}{\rho} + \frac{V_3^2}{2} \quad \text{or} \quad V_4 = \sqrt{\frac{2(p_3 - p_4)}{\rho} + V_3^2}$$

Thus

$$V_4 = \left[2 \times 100\,\frac{lbf}{in.^2} \times \frac{ft^3}{1.94\,slug} \times 144\,\frac{in.^2}{ft^2} \times \frac{slug \cdot ft}{lbf \cdot sec^2} + (19.6)^2\,\frac{ft^2}{sec^2}\right]^{\frac{1}{2}} = 124\ ft/sec$$ V_4

Applying Eq. 8.45 across the pump,

$$\Delta h_{pump} = \frac{p_2 - p_1}{\rho}$$

(This is the head added to the fluid.)

$p_1 = 50$ psig

$p_2 = p_3 + 3 \times 33$ psi

$p_2 = 100 + 3(33) = 199$ psig

The theoretical power input to the pump is $\dot{W}_{pump} = \dot{m}\,\Delta h_{pump}$

The actual power input to the pump is $\mathcal{P}_{act} = \dot{W}_{pump}/\eta = \dot{m}\,\frac{\Delta h_{pump}}{\eta}$

Thus $\mathcal{P}_{act} = \frac{\dot{m}}{\eta}\frac{(p_2 - p_1)}{\rho} = \frac{Q(p_2 - p_1)}{\eta}$

$$= 108\,\frac{gal}{min} \times (199 - 50)\,\frac{lbf}{in.^2} \times \frac{1}{0.7} \times \frac{ft^3}{7.48\,gal} \times 144\,\frac{in.^2}{ft^2} \times \frac{min}{60\,s} \times \frac{hp \cdot s}{550\,ft \cdot lbf}$$

$\mathcal{P}_{act} = 13.4$ hp $\mathcal{P}_{act}$

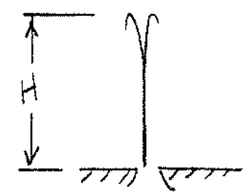
Problem 8.149

Given: Fountain on Purdue's Engineering Mall has

$Q = 550$ gpm and $H = 10$ m (32.8 ft)

Find: Estimate of annual cost to operate the fountain.

<u>Solution</u>: Model fountain as a vertical jet (this will give maximum cost).



Computing equations:

$$C(\$/yr) = C_e\left(\frac{\$}{kW\cdot hr}\right) \times P_{motor}(kW) \times N(hr/yr)$$

Assume $C_e = \$0.12/kW\cdot hr$

$$P_{motor} = \frac{P_{hydraulic}}{\eta_{pump}\eta_{motor}} \; ; \; \eta_{motor} = 0.9, \; \eta_{pump} = 0.8$$

$$P_{hydraulic} = Q\Delta p$$

$$N = 365\,\frac{days}{yr} \times 24\,\frac{hr}{day} = 8{,}760\ hr/yr$$

The minimum required Δp is $\rho g H$, so

$$\Delta p = 1.94\,\frac{slug}{ft^3} \times 32.2\,\frac{ft}{s^2} \times 32.8\,ft \times \frac{lbf\cdot s^2}{slug\cdot ft} = 2.05\times10^3\ lbf/ft^2$$

Combining,

$$C = \frac{\$0.12}{kW\cdot hr} \times \frac{1}{0.8(0.9)} \times 550\,\frac{gal}{min} \times 2.05\times10^3\,\frac{lbf}{ft^2} \times 8{,}760\,\frac{hr}{yr} \times \frac{ft^3}{7.48\,gal} \times \frac{hp\cdot min}{33{,}000\,ft\cdot lbf} \times 0.746\,\frac{kW}{hp}$$

$$C = \$4980/yr$$

The fountain does not operate year-round. It might be more fair to say $C \approx \$13$ per day of operation.

Problem 8.150

Given: Petroleum products transported long distances by pipeline, e.g., the Alaskan pipeline (see Example Problem 8.6).

Find: (a) Estimate of energy needed to pump typical petroleum product, expressed as a fraction of throughput energy carried by pipeline.

(b) Statement and critical assessment of assumptions.

Solution: From Example Problem 8.6, for the Alaskan pipeline, $Q = 1.6 \times 10^6$ bpd.

Thus $Q = 1.6\times10^6 \frac{bbl}{day} \times 42 \frac{gal}{bbl} \times \frac{ft^3}{7.48\, gal} \times \frac{day}{24\, hr} \times \frac{hr}{3600\, s} = 104\, ft^3/s$

and

$$\dot{m} = \rho Q = SG\,\rho_{H_2O} Q = 0.93 \times 1.94 \frac{slug}{ft^3} \times 104 \frac{ft^3}{s} = 188\ slug/s$$

The energy content of a typical petroleum product is about 18,000 Btu/lbm, so the throughput energy is

$$\dot{E} = e\dot{m} = 18{,}000 \frac{Btu}{lbm} \times 188 \frac{slug}{s} \times 32.2 \frac{lbm}{slug} = 1.09 \times 10^8\ Btu/s$$ $\dot{E}$

From Example Problem 8.6, each pumping station requires 36,800 hp, and they are located $L = 120$ mi apart.

The entire pipeline is about 750 mi long. Thus there must be $N = 750/120$ or about $N = 7$ pumping stations. Thus the total energy required to pump must be

$$\mathcal{P} = N\dot{W} = 7\ stations \times 36{,}800 \frac{hp}{station} = 258{,}000\ hp$$ $\mathcal{P}$

Expressed as a fraction of throughput energy

$$\frac{\mathcal{P}}{\dot{E}} = 258{,}000\ hp \times \frac{s}{1.09\times10^8\ Btu} \times 2545 \frac{Btu}{hp \cdot hr} \times \frac{hr}{3600\ s} = 1.67\times10^{-3} \text{ or } 0.00167$$

Thus about 0.167% of energy is used for transporting petroleum.

The assumptions outlined above appear reasonable. The computed result is probably accurate within ±10%.

A more universal metric would be energy per unit mass and distance, e.g., energy per ton-mile of transport.

$$\frac{E}{M \cdot L} = \frac{E/t}{M/t \cdot L} = \frac{\mathcal{P}}{\dot{m} L} = 36{,}800\ hp \times \frac{s}{188\ slug} \times \frac{1}{120\ mi} \times 2545 \frac{Btu}{hp \cdot hr} \times \frac{slug}{32.2\ lbm} \times 2000 \frac{lbm}{ton} \times \frac{hr}{3600\ s}$$

Thus

$$e = \frac{\mathcal{P}}{\dot{m} L} = 71.6\ Btu/ton \cdot mi$$ e

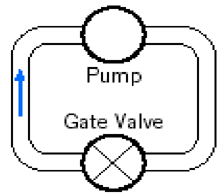
This specific metric allows direct comparison with other modes of transport.

Problem 8.151 (In Excel)

The pump testing system of Problem 8.110 is run with a pump that generates a pressure difference given by $\Delta p = 750 - 15 \times 10^4 Q^2$ where Δp is in kPa, and the generated flow rate is Q m³/s. Find the water flow rate, pressure difference, and power supplied to the pump if it is 70 percent efficient.

Given: Data on circuit and pump

Find: Flow rate, pressure difference, and power supplied



Solution

Governing equations:

$$Re = \frac{\rho \cdot V \cdot D}{\mu}$$

$$\left(\frac{p_1}{\rho} + \alpha_1 \cdot \frac{V_1^2}{2} + g \cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \cdot \frac{V_2^2}{2} + g \cdot z_2\right) = h_{lT} = \sum_{major} h_l + \sum_{minor} h_{lm} \qquad (8.29)$$

$$h_l = f \cdot \frac{L}{D} \cdot \frac{V^2}{2} \qquad (8.34)$$

$$h_{lm} = f \cdot \frac{L_e}{D} \cdot \frac{V^2}{2} \qquad (8.40b)$$

$$f = \frac{64}{Re} \qquad (8.36) \quad \text{(Laminar)}$$

$$\frac{1}{f^{0.5}} = -2.0 \cdot \log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re \cdot f^{0.5}}\right) \qquad (8.37) \quad \text{(Turbulent)}$$

The energy equation (Eq. 8.29) becomes for the circuit (1 = pump outlet, 2 = pump inlet)

$$\frac{p_1 - p_2}{\rho} = f \cdot \frac{L}{D} \cdot \frac{V^2}{2} + 4 \cdot f \cdot L_{elbow} \cdot \frac{V^2}{2} + f \cdot L_{valve} \cdot \frac{V^2}{2}$$

or

$$\Delta p = \rho \cdot f \cdot \frac{V^2}{2} \cdot \left(\frac{L}{D} + 4 \cdot \frac{L_{elbow}}{D} + \frac{L_{valve}}{D}\right) \qquad (1)$$

This must be matched to the pump characteristic equation; at steady state, the pressure generated by the pump just equals that lost to friction in the circuit

$$\Delta p = 750 - 15 \times 10^4 \cdot Q^2 \qquad (2)$$

Finally, the power supplied to the pump, efficiency η, is

$$Power = \frac{Q \cdot \Delta p}{\eta} \qquad (3)$$

Given data:

L = 20 m

D = 75 mm

η_{pump} = 70%

Tabulated or graphical data:

e = 0.26 mm

(Table 8.1)

μ = 1.00E-03 $N.s/m^2$

ρ = 999 kg/m^3

(Appendix A)

Gate valve L_e/D = 8

Elbow L_e/D = 30

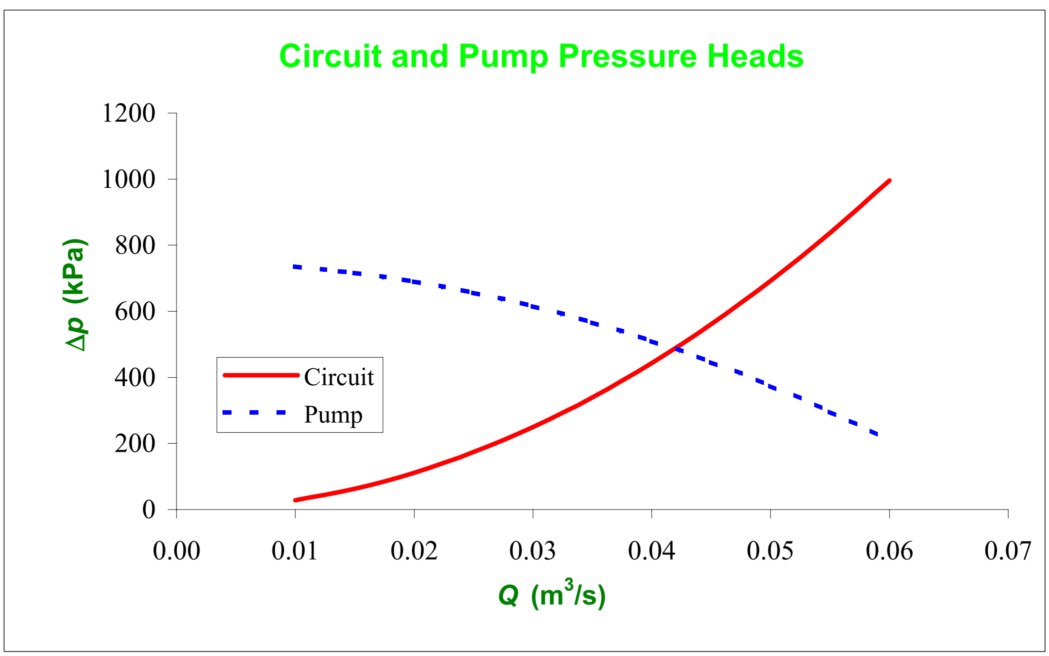
(Table 8.4)

Computed results:

Q (m^3/s)	V (m/s)	*Re*	*f*	Δp (kPa) (Eq 1)	Δp (kPa) (Eq 2)	
0.010	2.26	1.70E+05	0.0280	28.3	735	
0.015	3.40	2.54E+05	0.0277	63.1	716	
0.020	4.53	3.39E+05	0.0276	112	690	
0.025	5.66	4.24E+05	0.0276	174	656	
0.030	6.79	5.09E+05	0.0275	250	615	
0.035	7.92	5.94E+05	0.0275	340	566	
0.040	9.05	6.78E+05	0.0274	444	510	
0.045	10.2	7.63E+05	0.0274	561	446	
0.050	11.3	8.48E+05	0.0274	692	375	
0.055	12.4	9.33E+05	0.0274	837	296	
0.060	13.6	1.02E+06	0.0274	996	210	
						Error
0.0419	9.48	7.11E+05	0.0274	487	487	0

Using *Solver*!

Power = 29.1 kW (Eq. 3)



Problem 8.152 (In Excel)

A water pump can generate a pressure difference Δp (kPa) given by $\Delta p = 1000 - 800Q^2$, where the flow rate is Q m³/s. It supplies a pipe of diameter 500 mm, roughness 10 mm, and length 750 m. Find the flow rate, pressure difference, and the power supplied to the pump if it is 70 percent efficient. If the pipe were replaced with one of roughness 5 mm, how much would the flow increase, and what would the required power be?

Given: Data on pipe and pump

Find: Flow rate, pressure difference, and power supplied; repeat for smoother pipe

Solution

Governing equations:

$$Re = \frac{\rho \cdot V \cdot D}{\mu}$$

$$\left(\frac{p_1}{\rho} + \alpha_1 \cdot \frac{V_1^2}{2} + g \cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \cdot \frac{V_2^2}{2} + g \cdot z_2\right) = h_{lT} - \Delta h_{pump} \qquad (8.49)$$

$$h_{lT} = f \cdot \frac{L}{D} \cdot \frac{V^2}{2} \qquad (8.34)$$

$$f = \frac{64}{Re} \qquad (8.36) \quad \text{(Laminar)}$$

$$\frac{1}{f^{0.5}} = -2.0 \cdot \log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re \cdot f^{0.5}}\right) \qquad (8.37) \quad \text{(Turbulent)}$$

The energy equation (Eq. 8.49) becomes for the system (1 = pipe inlet, 2 = pipe outlet)

$$\Delta h_{pump} = f \cdot \frac{L}{D} \cdot \frac{V^2}{2}$$

or

$$\Delta p_{pump} = \rho \cdot f \cdot \frac{L}{D} \cdot \frac{V^2}{2} \qquad (1)$$

This must be matched to the pump characteristic equation; at steady state, the pressure generated by the pump just equals that lost to friction in the circuit

$$\Delta p_{pump} = 1000 - 800 \cdot Q^2 \qquad (2)$$

Finally, the power supplied to the pump, efficiency η, is

$$Power = \frac{Q \cdot \Delta p}{\eta} \qquad (3)$$

Tabulated or graphical data:

$\mu =$ 1.00E-03 $N.s/m^2$

$\rho =$ 999 kg/m^3

(Appendix A)

Given data:

$L =$ 750 m

$D =$ 500 mm

$\eta_{pump} =$ 70%

Computed results: $e =$ 10 mm

Q **(m^3/s)**	***V*** **(m/s)**	***Re***	***f***	**Δ*p* (kPa) (Eq 1)**	**Δ*p* (kPa) (Eq 2)**	**Error**	
0.1	0.509	2.54E+05	0.0488	9.48	992		
0.2	1.02	5.09E+05	0.0487	37.9	968		
0.3	1.53	7.63E+05	0.0487	85.2	928		
0.4	2.04	1.02E+06	0.0487	151	872		
0.5	2.55	1.27E+06	0.0487	236	800		
0.6	3.06	1.53E+06	0.0487	340	712		
0.7	3.57	1.78E+06	0.0487	463	608		
0.8	4.07	2.04E+06	0.0487	605	488		
0.9	4.58	2.29E+06	0.0487	766	352		
1.0	5.09	2.54E+06	0.0487	946	200		
1.1	5.60	2.80E+06	0.0487	1144	32.0		
0.757	3.9	1.93E+06	0.0487	542	542	0	Using *Solver* !

Power = 586 kW (Eq. 3)

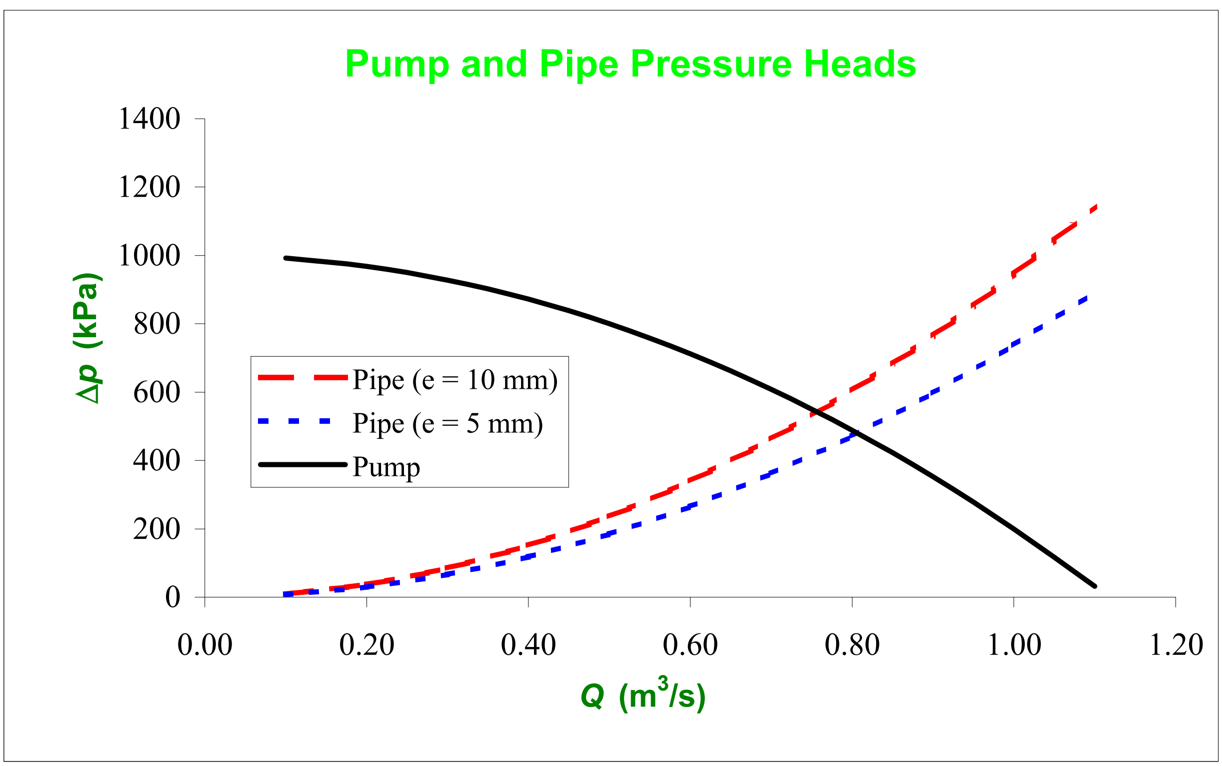
Repeating, with smoother pipe

Computed results: *e* = 5 mm

Q (m^3/s)	V (m/s)	*Re*	*f*	Δp (kPa) (Eq 1)	Δp (kPa) (Eq 2)	
0.1	0.509	2.54E+05	0.0381	7.41	992	
0.2	1.02	5.09E+05	0.0380	29.6	968	
0.3	1.53	7.63E+05	0.0380	66.4	928	
0.4	2.04	1.02E+06	0.0380	118	872	
0.5	2.55	1.27E+06	0.0380	184	800	
0.6	3.06	1.53E+06	0.0379	265	712	
0.7	3.57	1.78E+06	0.0379	361	608	
0.8	4.07	2.04E+06	0.0379	472	488	
0.9	4.58	2.29E+06	0.0379	597	352	
1.0	5.09	2.54E+06	0.0379	737	200	
1.1	5.60	2.80E+06	0.0379	892	32.0	
						Error
0.807	4.1	2.05E+06	0.0379	480	480	0

Using *Solver* !

Power = 553 kW (Eq. 3)



Given: Fan with outlet dimensions of 8×16 in. Head vs Capacity curve is approximately

$$H\,(\text{in. } H_2O) = 30 - 10^{-7}\left[Q\,(\text{ft}^3/\text{min})\right]^2$$

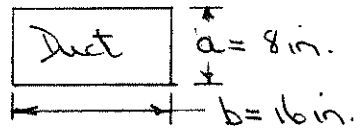
Find: Air flow rate delivered into a 200 ft. length of straight 8×16 in. duct.

Solution:

Basic equation:
$$\left(\frac{p_1}{\rho g} + \alpha_1\frac{\bar{V}_1^2}{2g} + z_1\right) - \left(\frac{p_2}{\rho g} + \alpha_2\frac{\bar{V}_2^2}{2g} + z_2\right) = H_{l_T} \qquad (8.30)$$
(terms $\alpha\frac{\bar{V}^2}{2g}$ crossed out by (1), z terms crossed out by (2))

$$H_{l_T} = f\frac{L}{D_h}\frac{\bar{V}^2}{2g} + h_{l_m} \quad ; \quad D_h = \frac{4A}{P_w}$$

Assumptions: (1) $\bar{V}_1 = \bar{V}_2$, $\alpha_1 = \alpha_2 \simeq 1$
(2) $z_1 = z_2$
(3) $h_{l_m} = 0$



$$A = ab = \frac{8}{12}\text{ft} \times \frac{16}{12}\text{ft} = 0.889\ \text{ft}^2$$

$$D_h = \frac{4A}{P_w} = \frac{4A}{2(a+b)} = \frac{2\times 0.889\ \text{ft}^2}{(2/3 + 4/3)\ \text{ft}} = 0.889\ \text{ft}$$

From Eq. 8.30 $\quad \Delta p = f\frac{L}{D}\rho_{air}\frac{\bar{V}^2}{2} = f\frac{L}{D}\frac{\rho}{2}\frac{Q^2}{A^2} = \gamma_{H_2O} H_{duct}$

where H_{duct} is the pressure drop in head of water

$$H_{duct} = \frac{fL\rho_{air}Q^2}{2\gamma_{H_2O}D_h A^2} = \frac{f}{2}\times\frac{200\ \text{ft}}{0.889\ \text{ft}}\times 0.00238\frac{\text{slug}}{\text{ft}^3}\times\frac{\text{ft}^3}{62.4\ \text{lbf}}\times\frac{1}{(0.889)^2\ \text{ft}^4}\times\left[\frac{Q\ \text{ft}^3}{\text{min}}\times\frac{\text{min}}{60\ \text{s}}\right]^2\times\frac{12\ \text{in}}{\text{ft}}$$

$$H_{duct} = 1.81\times 10^{-5}\, f Q^2 \quad (\text{where } H \text{ is in in. } H_2O) \quad \ldots\ldots\ldots (1)$$

For a smooth duct, $f = f(Re)$

$Re = \frac{\bar{V}D_h}{\nu} = \frac{D_h Q}{\nu A}$. For $T = 68°F$, from Table A.9, $\nu = 1.62\times 10^{-4}\ \text{ft}^2/\text{s}$

$$Re = \frac{0.889\ \text{ft}}{0.889\ \text{ft}^2}\times\frac{1}{1.62\times 10^{-4}}\frac{\text{s}}{\text{ft}^2}\times Q\frac{\text{ft}^3}{\text{min}}\times\frac{\text{min}}{60\ \text{s}} = 103\ Q$$

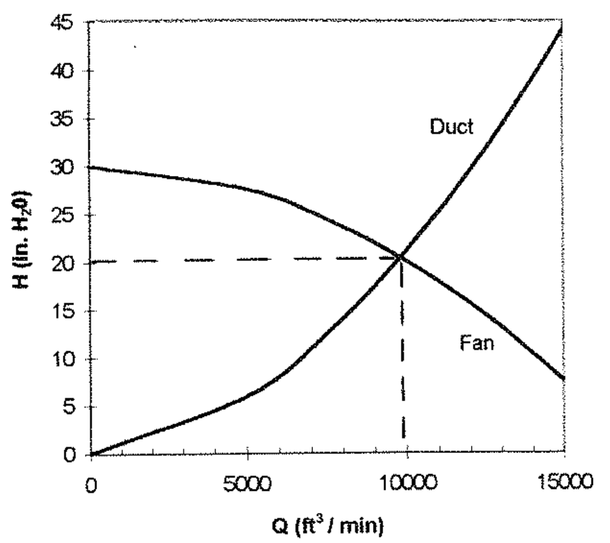
To determine the air flow rate delivered we need to determine the operating point of the fan.

The operating point is at the intersection of the
- fan head capacity curve, and the
- system curve (the head loss in the duct)

This is shown on the plot below.

Note that the friction factor f is determined from the Colebrook equation (8.37a) using Eq. 8.37b for the initial estimate of f.

Q	Re	H (fan)	f_0	f	H (duct)
(ft^3/min)	(-)	(in.H_2O)	(-)	(-)	(in.H_2O)
0		30			0
5,000	5.15E+05	27.5	0.0130	0.0131	5.9
7,500	7.73E+05	24.4	0.0121	0.0122	12.4
10,000	1.03E+06	20.0	0.0115	0.0116	21.0
12,500	1.29E+06	14.4	0.0111	0.0112	31.6
15,000	1.55E+06	7.5	0.0108	0.0108	44.1



Operating point
$Q \approx 10{,}000\ ft^3/min$
$H = 20\ in.\ H_2O$

Problem *8.154 (In Excel)

A cast-iron pipe system consists of a 50 m section of water pipe, after which the flow branches into two 50 m sections, which then meet in a final 50 m section. Minor losses may be neglected. All sections are 45 mm diameter, except one of the two branches, which is 25 mm diameter. If the applied pressure across the system is 300 kPa, find the overall flow rate and the flow rates in each of the two branches.

Given: Data on pipe system and applied pressure

Find: Flow rates in each branch

Solution

Governing equations:

$$\left(\frac{p_1}{\rho} + \alpha_1 \cdot \frac{V_1^{\,2}}{2} + g \cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \cdot \frac{V_2^{\,2}}{2} + g \cdot z_2\right) = h_l \qquad (8.29)$$

$$h_{lT} = f \cdot \frac{L}{D} \cdot \frac{V^2}{2} \qquad (8.34)$$

$$f = \frac{64}{Re} \qquad \text{(Laminar)} \qquad (8.36)$$

$$\frac{1}{f^{0.5}} = -2.0 \cdot \log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re \cdot f^{0.5}}\right) \qquad \text{(Turbulent)} \qquad (8.37)$$

The energy equation (Eq. 8.29) can be simplified to

$$\Delta p = \rho \cdot f \cdot \frac{L}{D} \cdot \frac{V^2}{2}$$

This can be written for each pipe section

In addition we have the following contraints

$$Q_A = Q_D \qquad (1)$$

$$Q_A = Q_B + Q_B \qquad (2)$$

$$\Delta p = \Delta p_A + \Delta p_B + \Delta p_D \qquad (3)$$

$$\Delta p_B = \Delta p_C \qquad (4)$$

We have 4 unknown flow rates (or, equivalently, velocities) and four equations

The workbook for Example Problem 8.11 is modified for use in this problem

Pipe Data:

Pipe	L (m)	D (mm)	e (mm)
A	50	45	0.26
B	50	45	0.26
C	50	25	0.26
D	50	45	0.26

Fluid Properties:

$\rho =$ 999 kg/m^3

$\mu =$ 0.001 $N.s/m^2$

Available Head:

$\Delta p =$ 300 kPa

Flows:

Q_A (m^3/s)	Q_B (m^3/s)	Q_C (m^3/s)	Q_D (m^3/s)
0.00396	0.00328	0.000681	0.00396

V_A (m/s)	V_B (m/s)	V_C (m/s)	V_D (m/s)
2.49	2.06	1.39	2.49

Re_A	Re_B	Re_C	Re_D
1.12E+05	9.26E+04	3.46E+04	1.12E+05

f_A	f_B	f_C	f_D
0.0325	0.0327	0.0400	0.0325

Heads:

Δp_A (kPa)	Δp_B (kPa)	Δp_C (kPa)	Δp_D (kPa)
112	77	77	112

Constraints:

(1) $Q_A = Q_D$
0.03%

(2) $Q_A = Q_B + Q_C$
0.01%

(3) $\Delta p = \Delta p_A + \Delta p_B + \Delta p_D$
0.01%

(4) $\Delta p_B = \Delta p_C$
0.01%

Error: **0.06%**

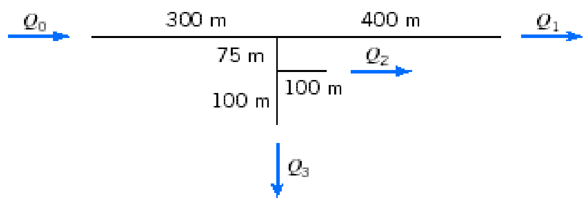
Vary Q_A, Q_B, Q_C, and Q_D using *Solver* to minimize total error

Problem *8.155 (In Excel)

The water pipe system shown is constructed from 75 mm galvanized iron pipe. Minor losses may be neglected. The inlet is at 250 kPa (gage), and all exits are at atmospheric pressure. Find the flow rates Q_0, Q_1, Q_2, and Q_3. If the flow in the 400 m branch is closed off ($Q_1 = 0$), find the increase in flows Q_2, and Q_3.

Given: Data on pipe system and applied pressure

Find: Flow rates in each branch



Solution

Governing equations:

$$\left(\frac{p_1}{\rho} + \alpha_1 \cdot \frac{V_1^2}{2} + g \cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \cdot \frac{V_2^2}{2} + g \cdot z_2\right) = h_l \qquad (8.29)$$

$$h_{lT} = f \cdot \frac{L}{D} \cdot \frac{V^2}{2} \qquad (8.34)$$

$$f = \frac{64}{Re} \qquad \text{(Laminar)} \qquad (8.36)$$

$$\frac{1}{f^{0.5}} = -2.0 \cdot \log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re \cdot f^{0.5}}\right) \qquad \text{(Turbulent)} \qquad (8.37)$$

The energy equation (Eq. 8.29) can be simplified to

$$\Delta p = \rho \cdot f \cdot \frac{L}{D} \cdot \frac{V^2}{2}$$

This can be written for each pipe section

In addition we have the following contraints

$$Q_0 = Q_1 + Q_4 \qquad (1)$$

$$Q_4 = Q_2 + Q_3 \qquad (2)$$

$$\Delta p = \Delta p_0 + \Delta p_1 \qquad (3)$$

$$\Delta p = \Delta p_0 + \Delta p_4 + \Delta p_2 \qquad (4)$$

$$\Delta p_2 = \Delta p_3 \qquad (5)$$

(Pipe 4 is the 75 m unlabelled section)

We have 5 unknown flow rates (or, equivalently, velocities) and five equations

The workbook for Example Problem 8.11 is modified for use in this problem

Pipe Data:

Pipe	L (m)	D (mm)	e (mm)
0	300	75	0.15
1	400	75	0.15
2	100	75	0.15
3	100	75	0.15
4	75	75	0.15

Fluid Properties:

$\rho =$ 999 kg/m^3

$\mu =$ 0.001 $N.s/m^2$

Available Head:

$\Delta p =$ 250 kPa

Flows:

Q_0 (m^3/s)	Q_1 (m^3/s)	Q_2 (m^3/s)	Q_3 (m^3/s)	Q_4 (m^3/s)
0.00928	0.00306	0.00311	0.00311	0.00623

V_0 (m/s)	V_1 (m/s)	V_2 (m/s)	V_3 (m/s)	V_4 (m/s)
2.10	0.692	0.705	0.705	1.41

Re_0	Re_1	Re_2	Re_3	Re_4
1.57E+05	5.18E+04	5.28E+04	5.28E+04	1.06E+05

f_0	f_1	f_2	f_3	f_4
0.0245	0.0264	0.0264	0.0264	0.0250

Heads:

Δp_0 (kPa)	Δp_1 (kPa)	Δp_2 (kPa)	Δp_3 (kPa)	Δp_4 (kPa)
216.4	33.7	8.7	8.7	24.8

Constraints:

(1) $Q_0 = Q_1 + Q_4$
0.00%

(2) $Q_4 = Q_2 + Q_3$
0.01%

(3) $\Delta p = \Delta p_0 + \Delta p_1$
0.03%

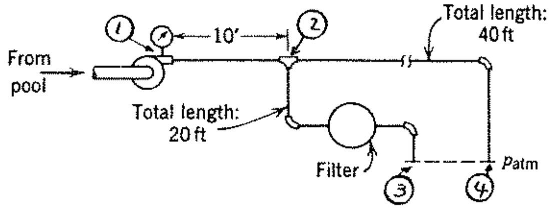
(4) $\Delta p = \Delta p_0 + \Delta p_4 + \Delta p_2$
0.01%

(5) $\Delta p_2 = \Delta p_3$
0.00%

Error: **0.05%** Vary Q_0, Q_1, Q_2, Q_3 and Q_4 using *Solver* to minimize total error

Problem *8.156

Given: Partial-flow filtration system:



Pipes are 3/4 in. nominal PVC (smooth plastic) with $D = 0.824$ in.

Pump delivers 30 gpm at 75°F.

Filter pressure drop is $\Delta p(\text{psi}) = 0.6\,[Q(\text{gpm})]^2$.

Find: (a) Pressure at pump outlet.
(b) Flow rate through each branch of system.

Solution: Apply the energy equation for steady, incompressible pipe flow.

Computing equation: $\frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \alpha_2 \frac{\bar{V}_2^2}{2} + gz_2 + h_{\ell T}$; $h_{\ell T} = \left[f\left(\frac{L}{D} + \frac{L_e}{D}\right) + K\right]\frac{\bar{V}^2}{2}$

Assumptions: (1) $\alpha_1 \bar{V}_1^2 = \alpha_2 \bar{V}_2^2$; (2) $z_1 = z_2$; (3) $h_{\ell m} = 0$ for $1 \to 2$; (4) Ignore "tee" at ②

The flow rate is $Q_{12} = 30$ gpm ($0.0668\ \text{ft}^3/\text{sec}$), so $\bar{V} = \frac{Q}{A} = 18.0$ ft/sec. Then

$$Re = \frac{\bar{V}D}{\nu} = 18.0\,\frac{\text{ft}}{\text{sec}} \times \left(\frac{0.824}{12}\right)\text{ft} \times \frac{\text{sec}}{1.0 \times 10^{-5}\,\text{ft}^2} = 1.24 \times 10^5, \text{ so } f = 0.017$$

$$\Delta p_{12} = f\frac{L}{D}\rho\frac{\bar{V}^2}{2} = 0.017 \times \frac{10\ \text{ft}}{0.824\ \text{in.}} \times \frac{1}{2} \times 1.94\,\frac{\text{slug}}{\text{ft}^3} \times (18.0)^2\,\frac{\text{ft}^2}{\text{sec}^2} \times \frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}} \times \frac{\text{ft}}{12\ \text{in.}} = 5.40\ \text{psi}$$

Branch flow rates are unknown, but flow split must produce the same drop in each branch. Solve by iteration to obtain

$Q_{23} = 5.2$ gpm; $\bar{V}_{23} = 3.12$ ft/s; $Re = 2.15 \times 10^4$, and $f = 0.025$*

$$\Delta p_{23} = f\left(\frac{L}{D} + 2\frac{L_e}{D}\right)\rho\frac{\bar{V}^2}{2} + 0.6Q^2$$

$$\Delta p_{23} = 0.025\left[\frac{240}{0.824} + 2(30)\right]\frac{1}{2} \times 1.94\,\frac{\text{slug}}{\text{ft}^3} \times (3.12)^2\,\frac{\text{ft}^2}{\text{s}^2} \times \frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}} \times \frac{\text{ft}^2}{144\ \text{in.}^2} + 0.6(5.2)^2\,\frac{\text{lbf}}{\text{in.}^2} = 16.8\ \text{psi}$$

$Q_{24} = 24.8$ gpm; $\bar{V}_{24} = 14.9$ ft/s; $Re = 1.03 \times 10^5$, and $f = 0.018$

$$\Delta p_{24} = f\left(\frac{L}{D} + \frac{L_e}{D}\right)\rho\frac{\bar{V}^2}{2} = 0.018\left(\frac{480}{0.824} + 30\right)\frac{1}{2} \times 1.94\,\frac{\text{slug}}{\text{ft}^3} \times (14.9)^2\,\frac{\text{ft}^2}{\text{s}^2} \times \frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}} \times \frac{\text{ft}^2}{144\ \text{in.}^2} = 16.5\ \text{psi}$$

The pump outlet pressure is

$$\Delta p_{pump} = \Delta p_{12} + \Delta p_{23} = (5.4 + 16.8)\ \text{psi} = 22.2\ \text{psi}$$ Δp

The branch flow rates are

$Q_{23} \approx 5.2$ gpm Q_{23}

$Q_{24} \approx 24.8$ gpm Q_{24}

* Value of f obtained from Eq 8.37 using Excel's solver (or Goal Seek)

Open-Ended Problem Statement: Why does the shower temperature change when a toilet is flushed? Sketch pressure curves for the hot and cold water supply systems to explain what happens.

Discussion: Assume the pressure in the water main servicing the dwelling remains constant. The hot and cold water flow rates reaching the shower are controlled by valve(s) in the shower. Assuming a water heater temperature of 140°F, a cold water temperature of 60°F, and a shower water temperature of 100°F, the hot and cold flow rates must be equal. The two water streams mix before reaching the shower head, then spray out into the shower itself at 100°F.

Supply curves and system curves for the hot and cold water streams are shown below. Diagram *a* is the cold water system and diagram *b* is the hot water system. The numerical values are representative of an actual system.

In general the supply curves for the hot and cold streams are not the same. The difference is caused by the two systems having different pipe lengths and different fittings.

Each stream operates at the flow rate where the curves intersect. An equal flow split is accomplished by adjusting the shower valves to vary their resistances.

Flushing the toilet temporarily increases the flow rate of cold water to the bathroom. This reduces the cold water supply pressure reaching the shower. The system curves do not change because the valve settings stay the same. Therefore the flow rate of cold water must decrease to again match the supply and system curves (diagram *c*).

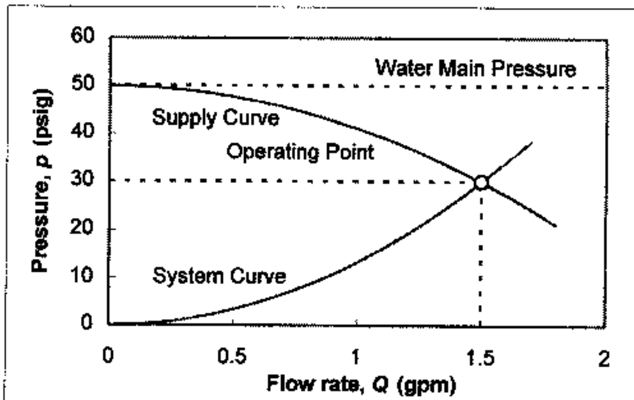
When the flow rate of cold water decreases the shower temperature increases, as experience testifies!

Problem 8.157 (cont'd)

2/2

(*a*) Cold water system:

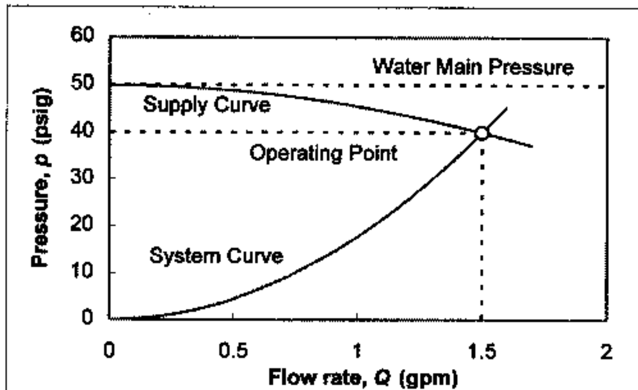
Q (gpm)	System Curve p (psig)	Supply Curve p (psig)
0	0.00	50.0
0.2	0.53	49.6
0.4	2.13	48.6
0.6	4.80	46.8
0.8	8.53	44.3
1.0	13.3	41.1
1.2	19.2	37.2
1.4	26.1	32.6
1.6	34.1	27.2
1.7	38.5	24.3
1.8		21.2



(*a*) Cold water system curves

(*b*) Hot water system:

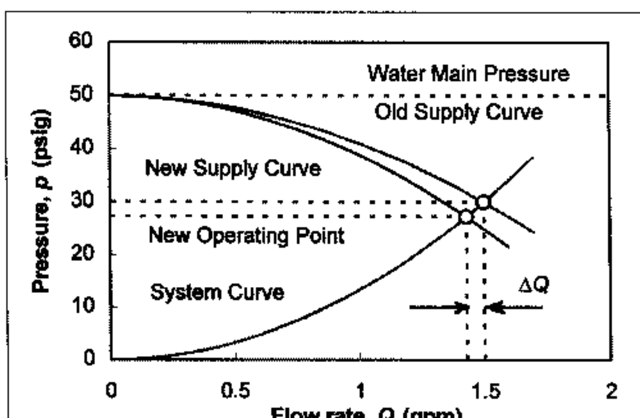
Q (gpm)	System Curve p (psig)	Supply Curve p (psig)
0	0.00	50.0
0.2	0.71	49.8
0.4	2.84	49.3
0.6	6.40	48.4
0.8	11.38	47.2
1.0	17.78	45.6
1.2	25.60	43.6
1.4	34.84	41.3
1.6	45.51	38.6
1.7		37.2
1.8		



(*b*) Hot water system curves

(*c*) Cold water system: toilet flush

Q (gpm)	System Curve p (psig)	Old Supply Curve p (psig)	New Supply Curve p (psig)
0	0.00	50.0	50.0
0.2	0.53	49.6	49.6
0.4	2.13	48.6	48.2
0.6	4.80	46.8	46.0
0.8	8.53	44.3	42.9
1.0	13.3	41.1	38.9
1.2	19.2	37.2	34.0
1.4	26.1	32.6	28.2
1.430	27.27	31.8	27.28
1.6	34.1	27.2	21.6
1.7	38.5	24.3	
1.8			



(*c*) Cold water system curves: toilet flush

Problem 8.158

Given: Water flow at 300 gpm (150°F) through a 3 in. diameter orifice installed in a 6 in. i.d. pipe.

Find: Pressure drop across corner taps.

Solution: Apply analysis of Section 8-10; data from Fig. 8.21 apply.

Computing equation:

$$\dot{m}_{actual} = K A_t \sqrt{2\rho(p_1 - p_2)} \qquad (8.51)$$

Flow coefficient is $K = K\left(Re_{D_1}, \frac{D_t}{D_1}\right)$. At 150°F, $\nu = 4.69 \times 10^{-6}\ ft^2/s$ (Table A.7). Thus,

$$\bar{V}_1 = \frac{Q}{A} = 300\,\frac{gal}{min} \times \frac{4}{\pi(0.5)^2 ft^2} \times \frac{ft^3}{7.48\,gal} \times \frac{min}{60\,s} = 3.40\ ft/s$$

$$Re_{D_1} = \frac{\bar{V}D_1}{\nu} = 3.40\,\frac{ft}{s} \times 0.5\,ft \times \frac{s}{4.69 \times 10^{-6}\,ft^2} = 3.62 \times 10^5$$

$$\beta = \frac{D_t}{D_1} = \frac{3\ in.}{6\ in.} = 0.5$$

From Fig. 8.21, $K = 0.624$. Then, from Eq. 8.51,

$$\Delta p = \left(\frac{\dot{m}}{KA_t}\right)^2 \frac{1}{2\rho} = \left(\frac{\rho Q}{KA_t}\right)^2 \frac{1}{2\rho} = \frac{\rho}{2}\left(\frac{Q}{KA_t}\right)^2$$

$$= \frac{1}{2} \times 1.94\,\frac{slug}{ft^3}\left[300\,\frac{gal}{min} \times \frac{1}{0.624} \times \frac{4}{\pi(0.25)^2 ft^2} \times \frac{ft^3}{7.48\,gal} \times \frac{min}{60\,s}\right]^2 \frac{lbf \cdot s^2}{slug \cdot ft}$$

$$\Delta p = 462\ lbf/ft^2 \quad (3.21\ psi)$$

Δp

Problem 8.159

Given: Square-edged orifice, $d_t = 100$ mm, used to meter air flow in a 150 mm i.d. line. The pressure upstream of the orifice is $p_1 = 600$ kPa. The pressure drop across the orifice is $\Delta p = 750$ mm H_2O. The air temperature is 25°C

Find: the volume flow rate of air in the line.

<u>Solution:</u> Apply analysis of section 8-10; data from Fig. 8.23 apply

Computing equation: $\dot{m}_{actual} = KA_t\sqrt{2\rho(p_1 - p_2)}$ (8.56)

Since $\dot{m} = \rho Q$, then

$$Q = \frac{\dot{m}}{\rho} = KA_t\sqrt{\frac{2(p_1-p_2)}{\rho}}$$

$$p_1 - p_2 = 750 \text{ mm } H_2O = \rho g \Delta h_{H_2O} = 999\frac{kg}{m^3} \times 9.81\frac{m}{s^2} \times 0.75\,m \times \frac{N\cdot s^2}{kg\cdot m} = 7.35\,kPa$$

For this small Δp, the assumption of incompressible flow is certainly valid

$$\rho = \frac{p_1}{RT} = 701\times10^3\frac{N}{m^2} \times \frac{kg\cdot K}{287\,J} \times \frac{1}{298\,K} \times \frac{J}{N\cdot m} = 8.20\,kg/m^3$$

The flow coefficient $K = K\left(Re_{D_1}, \frac{D_t}{D_1}\right)$

Assume $Re > 2\times10^5$. For $\beta = \frac{D_t}{D_1} = \frac{2}{3}$, from Fig. 8.20, $K = 0.675$

$$Q = KA_t\sqrt{\frac{2(p_1-p_2)}{\rho}} = 0.675\,\frac{\pi}{4}(0.1\,m)^2\left[2\times 7350\frac{N}{m^2} \times \frac{m^3}{8.20\,kg} \times \frac{kg\cdot m}{N\cdot s^2}\right]^{1/2}$$

$Q = 0.224\ m^3/s$ ← Q

Check Re. At $T = 25°C$ $\mu = 1.84\times10^{-5}\ N\cdot s/m^2$ (Table A.10)

$$Re = \frac{\rho D V}{\mu} = \frac{\rho D Q}{\mu A} = \frac{\rho D Q\,4}{\mu \pi D^2} = \frac{4\rho Q}{\pi \mu D}$$

$$Re = \frac{4}{\pi} \times 8.2\frac{kg}{m^3} \times 0.224\frac{m^3}{s} \times \frac{1}{1.84\times10^{-5}}\frac{m^2}{N\cdot s} \times \frac{1}{0.15\,m} \times \frac{N\cdot s^2}{kg\cdot m}$$

$Re = 8.47\times10^5$ ✓ assumption is valid

Problem 8.160 (In Excel)

A smooth 200 m pipe, 100 mm diameter connects two reservoirs (the entrance and exit of the pipe are sharp-edged). At the midpoint of the pipe is an orifice plate with diameter 40 mm. If the water levels in the reservoirs differ by 30 m, estimate the pressure differential indicated by the orifice plate and the flow rate.

Given: Data on pipe-reservoir system and orifice plate

Find: Pressure differential at orifice plate; flow rate

Solution

Governing equations:

$$\left(\frac{p_1}{\rho} + \alpha_1\cdot\frac{V_1^2}{2} + g\cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2\cdot\frac{V_2^2}{2} + g\cdot z_2\right) = h_{lT} = h_l + \Sigma h_{lm} \qquad (8.29)$$

$$h_l = f\cdot\frac{L}{D}\cdot\frac{V^2}{2} \qquad (8.34)$$

There are three minor losses: at the entrance; at the orifice plate; at the exit. For each

$$h_{lm} = K\cdot\frac{V^2}{2}$$

$$f = \frac{64}{Re} \qquad \text{(Laminar)} \qquad (8.36)$$

$$\frac{1}{f^{0.5}} = -2.0\cdot\log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re\cdot f^{0.5}}\right) \qquad \text{(Turbulent)} \qquad (8.37)$$

The energy equation (Eq. 8.29) becomes ($\alpha = 1$)

$$g\cdot\Delta H = \frac{V^2}{2}\cdot\left(f\cdot\frac{L}{D} + K_{ent} + K_{orifice} + K_{exit}\right) \qquad (1)$$

(ΔH is the difference in reservoir heights)

This cannot be solved for V (and hence Q) because f depends on V; we can solve by manually iterating, or by using *Solver*

The tricky part to this problem is that the orifice loss coefficient $K_{orifice}$ is given in Fig. 8.23 as a percentage of pressure differential Δp across the orifice, which is unknown until V is known!

The mass flow rate is given by

$$m_{rate} = K\cdot A_t\cdot\sqrt{2\cdot\rho\cdot\Delta p} \qquad (2)$$

where K is the orifice flow coefficient, A_t is the orifice area, and Δp is the pressure drop across the orifice

where K is the orifice flow coefficient, A_t is the orifice area,
and Δp is the pressure drop across the orifice

Equations 1 and 2 form a set for solving for TWO unknowns: the pressure drop Δp across the orifice (leading to a value for $K_{orifice}$) and the velocity V. The easiest way to do this is by using *Solver*

Given data:

ΔH =	30	m
L =	200	m
D =	100	mm
D_t =	40	mm
β =	0.40	

Tabulated or graphical data:

K_{ent} =	0.50	(Fig. 8.14)
K_{exit} =	1.00	(Fig. 8.14)
Loss at orifice =	80%	(Fig. 8.23)
μ =	0.001	$N.s/m^2$
ρ =	999	kg/m^3

(Water - Appendix A)

Computed results:

Orifice loss coefficient:

K = 0.61
(Fig. 8.20
Assuming high *Re*)

Flow system:

V =	2.25	m/s
Q =	0.0176	m^3/s
Re =	2.24E+05	
f =	0.0153	

Orifice pressure drop

Δp = 265 kPa

Eq. 1, solved by varying V AND Δp, using *Solver* :

Left (m^2/s)	Right (m^2/s)	Error
294	293	0.5%

Eq. 2 and $m_{rate} = \rho Q$ compared, varying V AND Δp

	(From Q)	(From Eq. 2)	Error
m_{rate} (kg/s) =	17.6	17.6	0.0%

Total Error	0.5%

Procedure using *Solver* :
a) Guess at V and Δp
b) Compute error in Eq. 1
c) Compute error in mass flow rate
d) Minimize total error
e) Minimize total error by varying V and Δp

Problem 8.161

Given: Venturi meter with 75 mm throat, installed in 150 mm diameter line carrying water at 25°C. Pressure drop between upstream and throat taps is 300 mm Hg.

Find: Flow rate of water.

Solution: Apply analysis of Section 8-10.3.

Computing equation:

$$\dot{m}_{actual} = \frac{C A_t}{\sqrt{1-\beta^4}} \sqrt{2\rho(p_1 - p_2)} \qquad (8.52)$$

For $Re_{D_1} > 2 \times 10^5$, $0.980 < C < 0.995$. Assume $C = 0.99$, then check Re.

$$\beta = \frac{D_t}{D_1} = \frac{75\ mm}{150\ mm} = 0.5$$

$$\Delta p = p_1 - p_2 = \rho_{Hg}\, g\, \Delta h = SG\, \rho_{H_2O}\, g\, \Delta h$$

$\dot{m} = \rho Q$, so

$$Q = \frac{C A_t}{\sqrt{1-\beta^4}} \sqrt{2\, SG\, g\, \Delta h}$$

$$Q = \frac{0.99}{\sqrt{1-(0.5)^4}} \frac{\pi}{4}(0.075)^2 m^2 \sqrt{2 \times 13.6 \times 9.81 \frac{m}{s^2} \times 0.3\, m} = 0.0404\ m^3/s$$

Thus

$$\bar{V} = \frac{Q}{A_1} = 0.0404 \frac{m^3}{s} \times \frac{4}{\pi (0.15)^2 m^2} = 2.29\ m/s$$

$$Re_{D_1} = \frac{\bar{V} D_1}{\nu} = 2.29 \frac{m}{s} \times 0.15\, m \times \frac{s}{8.93 \times 10^{-7} m^2} = 3.85 \times 10^5 \quad (\nu \text{ from Table A.8})$$

Thus $Re_{D_1} > 2 \times 10^5$. The volume flow rate of water is

$Q = 0.0404\ m^3/s$ Q

Problem 8.162

Given: Flow of gasoline through a venturi meter.

$SG = 0.73$, $D_1 = 2.0$ in., $D_t = 1.0$ in., $\Delta h = 380$ mm Hg

Find: Volume flow rate of gasoline.

Solution: Apply the analysis of Section 8-10.3.

Computing equations:

$$\dot{m}_{actual} = \frac{C A_t}{\sqrt{1-\beta^4}} \sqrt{2\rho (p_1 - p_2)} \qquad (8.52)$$

$$C = 0.99 \text{ for } Re_{D_1} > 2 \times 10^5$$

For the manometer, $\Delta p = \rho_{Hg} g \Delta h = SG_{Hg} \rho_{H_2O} g \Delta h$

Then

$$Q = \frac{\dot{m}}{\rho} = \frac{C A_t}{\sqrt{1-\beta^4}} \sqrt{\frac{2\Delta p}{\rho}} = \frac{C A_t}{\sqrt{1-\beta^4}} \sqrt{\frac{2 SG_{Hg} \rho_{H_2O} g \Delta h}{SG_{gas} \rho_{H_2O}}} = \frac{C A_t}{\sqrt{1-\beta^4}} \sqrt{\frac{2 SG_{Hg} g \Delta h}{SG_{gas}}}$$

$$Q = \frac{0.99}{\sqrt{1-(0.5)^4}} \frac{\pi (0.0254)^2 m^2}{4} \sqrt{2 \times \frac{13.6}{0.73} \times 9.81 \frac{m}{s^2} \times 0.38\, m} = 0.00611\ m^3/s$$

Now check Reynolds number:

$$\bar{V}_1 = \frac{Q}{A_1} = 0.00611 \frac{m^3}{s} \times \frac{4}{\pi (0.0508)^2 m^2} = 3.01\ m/s$$

Assume viscosity midway between octane and heptane at 20°C. From Fig. A.1,

$\mu \approx 5.0 \times 10^{-4}\ N \cdot s / m^2$

$$Re_{D_1} = \frac{\rho \bar{V}_1 D_1}{\mu} = (0.73)\, 1000 \frac{kg}{m^3} \times 3.01 \frac{m}{s} \times 0.0508\, m \times \frac{m^2}{5.0 \times 10^{-4} N \cdot s} \times \frac{N \cdot s^2}{kg \cdot m} = 2.23 \times 10^5$$

Thus assumption that $C = 0.99$ is okay

$Q = 0.00611\ m^3/s$ ← Q

Problem 8.163

Given: Flow of water through venturi meter.

$D_1 = 2$ in. $\quad D_t = 1$ in. $\quad \Delta p = 20$ psi

Find: Volume flow rate of water.

Solution: Apply the analysis of Section 8-10.3.

Computing equations:

$$\dot{m}_{actual} = \frac{C A_t}{\sqrt{1-\beta^4}} \sqrt{2\rho(p_1 - p_2)} \qquad (8.52)$$

$$C = 0.99 \text{ for } Re_{D_1} > 2 \times 10^5$$

Then

$$Q = \frac{\dot{m}}{\rho} = \frac{C A_t}{\sqrt{1-\beta^4}} \sqrt{\frac{2 \Delta p}{\rho}}$$

$$Q = \frac{0.99}{\sqrt{1-(0.5)^4}} \frac{\pi}{4}\left(\frac{1}{12}\right)^2 ft^2 \sqrt{2 \times 20 \frac{lbf}{in.^2} \times \frac{ft^3}{1.94\ slug} \times 144 \frac{in.^2}{ft^2} \times \frac{slug \cdot ft}{lbf \cdot s^2}} \times 7.48 \frac{gal}{ft^3} \times 60 \frac{s}{min}$$

$Q = 136$ gal/min $\quad$ Q

The Reynolds number (with ν from Table A.7) is

$$Re_{D_1} = \frac{\bar{V}_1 D_1}{\nu} = \frac{Q}{A} \frac{D_1}{\nu} = \frac{4Q}{\pi D_1^2} \frac{D_1}{\nu} = \frac{4Q}{\pi \nu D_1}$$

$$= \frac{4}{\pi} \times 136 \frac{gal}{min} \times \frac{s}{1.08 \times 10^{-5}\ ft^2} \times \frac{1}{(2/12)\ ft} \times \frac{ft^3}{7.48\ gal} \times \frac{min}{60\ s}$$

$$Re_{D_1} = 2.14 \times 10^5$$

Therefore $C = 0.99$ may be used.

Problem 8.164

Given: Test of 1.6 ℓ internal combustion engine at 6000 rpm.
Meter air with flow nozzle, $\Delta h \le 0.25$ m. Manometer reads to ± 0.5 mm of water.

Find: (a) Flow nozzle diameter required.
(b) Minimum rate of air flow that can be measured ±2 percent.

Solution: Apply computing equation for flow nozzle.

Computing equation: $\dot{m} = K A_t \sqrt{2\rho(p_1 - p_2)}$ (8.54)

Assumptions: (1) $K = 0.97$ (Section 8-10.2b.)
(2) $\beta = 0$ (nozzle inlet is from atmosphere)
(3) Four-stroke cycle engine with 100 percent volumetric efficiency ($\forall$/rev = displacement/2)
(4) Standard air

Then

$$\dot{m} = \rho Q = 1.23 \frac{kg}{m^3} \times \frac{1.6 \ell}{2 rev} \times 6000 \frac{rev}{min} \times \frac{m^3}{1000 \ell} \times \frac{min}{60 s} = 0.0984 \; kg/s$$

Solving for A_t,

$$A_t = \frac{\dot{m}}{K\sqrt{2\rho \Delta p}} = \frac{\dot{m}}{K\sqrt{2\rho \rho_{H_2O} g \Delta h}}$$

$$A_t = 0.0984 \frac{kg}{s} \times \frac{1}{0.97}\left[\frac{1}{2} \times \frac{m^3}{1.23\,kg} \times \frac{m^3}{999\,kg} \times \frac{s^2}{9.81\,m} \times \frac{1}{0.25\,m}\right]^{\frac{1}{2}} = 1.31 \times 10^{-3} m^2$$

$$A_t = \frac{\pi D_t^2}{4} \; ; \; D_t = \sqrt{\frac{4A_t}{\pi}} = 40.8 \; mm$$ D_t

The allowable error is ±2 percent, or ±0.02. As discussed in Appendix E, the square-root relationship halves the experimental uncertainty. Thus

$$e = \pm 0.02 \text{ when } e_{\Delta h} = \pm 0.04 \; ; \; \Delta h_{min} = \frac{\pm 0.5\,mm}{\pm 0.04} = 12.5 \; mm$$

$$\dot{m}_{min} \simeq \dot{m}\sqrt{\frac{\Delta h_{min}}{\Delta h}} = 0.0984 \frac{kg}{s}\sqrt{\frac{12.5\,mm}{250\,mm}} = 0.0220 \; kg/s$$ $\dot{m}_{min}$

The air flow rate could be measured with ±2 percent accuracy down to about

$$\omega = 6000 \; rpm \; \frac{0.0220}{0.0984} = 1340 \; rpm$$

with this setup.

Problem 8.165

Given: Venturi meter with 75mm diameter throat installed in a 150mm diameter line. Upstream air pressure is 400 kPa and the temperature is 20°C.

Find: (a) Maximum mass flow rate for incompressible assumption.
(b) Corresponding pressure drop on mercury manometer.

Solution: Use analysis of Section 8-10.3.

Computing equation: $\dot{m}_{actual} = \frac{CA_t}{\sqrt{1-\beta^4}}\sqrt{2\rho(p_1-p_2)}$ (8.52)

Assumptions: (1) Neglect change in density
(2) Ideal gas

Then

$$\rho = \frac{p}{RT} = 4.0\times10^5\frac{N}{m^2}\times\frac{kg\cdot K}{287\,N\cdot m}\times\frac{1}{293\,K} = 4.76\ kg/m^3$$

For incompressible flow, $V \lesssim 100$ m/s at the throat. Thus

$$\dot{m} = \rho V_2 A_2 = 4.76\frac{kg}{m^3}\times 100\frac{m}{s}\times\frac{\pi}{4}(0.075)^2 m^2$$

$$\dot{m} = 2.10\ kg/s \quad \text{(maximum mass flow rate)}$$

$\dot{m}$

The pressure drop may be calculated by solving Eq. 8.52:

$$\Delta p = p_1 - p_2 = \rho_{Hg}\, g\, \Delta h = \frac{1}{2\rho}\left(\frac{\dot{m}}{CA_t}\right)^2(1-\beta^4)$$

Thus

$$\Delta h = \frac{\Delta p}{\rho_{Hg}\, g} = \frac{1}{2\rho\rho_{Hg}\, g}\left(\frac{\dot{m}}{CA_t}\right)^2(1-\beta^4)$$

For $Re_{D_1} \geq 2\times10^5$, $C = 0.99$ may be used. Substituting,

$$\Delta h = \frac{1}{2}\times\frac{m^3}{4.76\,kg}\times\frac{m^3}{(13.6)1000\,kg}\times\frac{s^2}{9.81\,m}\left[2.10\frac{kg}{s}\times\frac{1}{0.99}\times\frac{4}{\pi(0.075)^2m^2}\right]^2\left[1-(0.5)^4\right]$$

$$\Delta h = 0.170\ m \quad (170\ mm\ Hg)$$

Δh

Checking the Reynolds number ($\mu = 1.81\times10^{-5}\,N\cdot s/m^2$, Table A.10)

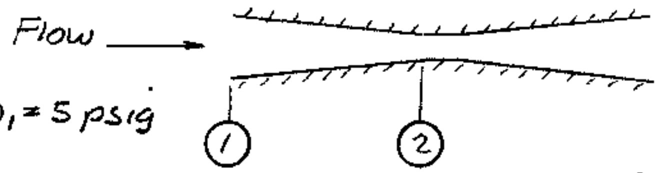
$$Re_{D_1} = \frac{\bar{V}D_1}{\nu} = \frac{\rho\bar{V}D_1}{\mu} = \rho\bar{V}\frac{\pi D_1^2}{4}\frac{4}{\pi\mu D_1} = \frac{4\dot{m}}{\pi\mu D_1}$$

$$Re_{D_1} = \frac{4}{\pi}\times 2.10\frac{kg}{s}\times\frac{m^2}{1.81\times10^{-5}\,N\cdot s}\times\frac{1}{0.15\,m}\times\frac{N\cdot s^2}{kg\cdot m} = 9.85\times10^5 > 2\times10^5$$

Therefore use of $C = 0.99$ is appropriate.

Problem 8.166

Given: Water at 70°F flows through a Venturi.



$p_1 = 5$ psig

$A_1 = 0.10\ ft^2$ $A_2 = 0.025\ ft^2$

Find: Estimate the maximum flow rate with no cavitation. (Express answer in cfs.)

Solution: Apply flowmeter equation.

Computing equation: $\dot{m} = \frac{C A_t}{\sqrt{1-\beta^4}}\sqrt{2\rho(p_1 - p_2)}$; $\beta^2 = A_t/A_1$

Assume $C = 0.99$ for $Re_{D_1} \geq 2\times10^5$.

Cavitation occurs when $p_2 \leq p_v$. From Steam table, $p_v = 0.363$ psia at 70 F. Thus

$$p_1 - p_2 = (14.7 + 5.0) - 0.363 = 19.3 \text{ psi}$$

and

$$\dot{m} = 0.99 \times 0.025\ ft^2 \times \frac{1}{\sqrt{1-(0.025/0.1)^2}}\left[2\times 1.94\frac{slug}{ft^3}\times 19.3\frac{lbf}{in.^2}\times 144\frac{in.^2}{ft^2}\times\frac{slug\cdot ft}{lbf\cdot s^2}\right]^{1/2}$$

$$\dot{m} = 2.65\ slug/s$$

But $\dot{m} = \rho \bar{V} A = \rho Q$, so

$$Q = \frac{\dot{m}}{\rho} = 2.65\frac{slug}{sec}\times\frac{ft^3}{1.94\ slug} = 1.37\ ft^3/s$$ ← Q

{Note $Q = 1.37\frac{ft^3}{s}\times 7.48\frac{gal}{ft^3}\times 60\frac{s}{min} = 613$ gpm.}

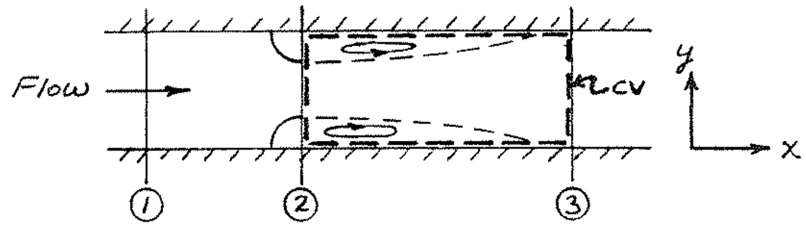
At 70 F, $\nu = 1.05\times10^{-5}\ ft^2/s$ (Table A.7). $Re_{D_1} = \frac{\bar{V}_1 D_1}{\nu}$ $A_1 = \frac{\pi D_1^2}{4}$; so

$D_1 = \sqrt{\frac{4A_1}{\pi}} = \sqrt{\frac{4}{\pi}\times 0.1\ ft^2} = 0.357$ ft (4.28 in.) ; $\bar{V}_1 = \frac{Q}{A_1} = 1.37\frac{ft^3}{s}\times\frac{1}{0.1\ ft^2} = 13.7$ ft/s

Then

$Re_{D_1} = 13.7\frac{ft}{s}\times 0.357\ ft\times\frac{s}{1.05\times10^{-5}\ ft^2} = 4.66\times10^5$, so $C = 0.99$ is okay. ✓✓

Given: Flow nozzle installation in pipe as shown.



Find: Head loss between sections ① and ③, expressed in coefficient form, $C_\ell = \frac{p_1 - p_3}{p_1 - p_2}$. Show $C_\ell = \frac{1 - A_2/A_1}{1 + A_2/A_1}$

Plot: C_ℓ vs. D_2/D_1

Solution: Apply the Bernoulli, continuity, momentum and energy equations, using the CV shown.

Basic equations:
$$\frac{p_1}{\rho} + \frac{\bar{V}_1^2}{2} + gz_1^{\,(4)} = \frac{p_2}{\rho} + \frac{\bar{V}_2^2}{2} + gz_2^{\,(4)}$$

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A} \quad (=0(1))$$

$$F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u \rho \, d\forall + \int_{CS} u \rho \vec{V} \cdot d\vec{A} \quad (F_{B_x}=0(5),\ \tfrac{\partial}{\partial t}=0(1))$$

$$\dot{Q} + \dot{W}_s = \frac{\partial}{\partial t}\int_{CV} e \rho \, d\forall + \int_{CS} \left(u + \frac{\bar{V}^2}{2} + gz + \frac{p}{\rho}\right) \rho \vec{V} \cdot d\vec{A} \quad (\dot{W}_s=0(6),\ \tfrac{\partial}{\partial t}=0(1),\ gz\ (4))$$

Assumptions:
(1) Steady flow
(2) Incompressible flow
(3) No friction between ① and ②
(4) Neglect elevation terms
(5) $F_{B_x} = 0$
(6) $\dot{W}_s = 0$
(7) Uniform flow at each section

From continuity,
$$Q = \bar{V}_1 A_1 = \bar{V}_2 A_2 = \bar{V}_3 A_3$$

Apply Bernoulli along a streamline from ① to ②, noting $A_1 = A_3$,
$$\frac{p_1 - p_2}{\rho} = \frac{\bar{V}_2^2 - \bar{V}_1^2}{2} = \frac{\bar{V}_2^2}{2}\left[1 - \left(\frac{A_2}{A_1}\right)^2\right] = \frac{\bar{V}_2^2}{2}\left[1 - \left(\frac{A_2}{A_3}\right)^2\right]$$

From momentum, and using continuity,
$$F_{S_x} = p_2 A_1 - p_3 A_3 = \bar{V}_2\{-|\rho \bar{V}_2 A_2|\} + \bar{V}_3\{+|\rho \bar{V}_3 A_3|\} = (\bar{V}_3 - \bar{V}_2)\rho \bar{V}_3 A_3$$

or
$$\frac{p_3 - p_2}{\rho} = \bar{V}_3(\bar{V}_2 - \bar{V}_3) = \bar{V}_2 \frac{A_2}{A_3}\left[\bar{V}_2 - \bar{V}_2 \frac{A_2}{A_3}\right] = \bar{V}_2^2 \frac{A_2}{A_3}\left(1 - \frac{A_2}{A_3}\right)$$

From energy,
$$\dot{Q} = \left(u_2 + \frac{\bar{V}_2^2}{2} + \frac{p_2}{\rho}\right)\{-|\rho \bar{V}_2 A_2|\} + \left(u_3 + \frac{\bar{V}_3^2}{2} + \frac{p_3}{\rho}\right)\{|\rho \bar{V}_3 A_3|\}$$

or $h_{l_{23}} = u_3 - u_2 - \frac{\dot{Q}}{\dot{m}} = \frac{\bar{V}_2^2 - \bar{V}_3^2}{2} - \frac{p_3 - p_2}{\rho} = \frac{\bar{V}_2^2}{2}\left[1 - \left(\frac{A_2}{A_3}\right)^2\right] - \frac{p_3 - p_2}{\rho}$

But $h_{l_{12}} \simeq 0$ by assumption (3), so $h_{l_{13}} \simeq h_{l_{23}}$ and using momentum

$$h_{l_{13}} \simeq \frac{\bar{V}_2^2}{2}\left[1 - \left(\frac{A_2}{A_3}\right)^2\right] - \bar{V}_2^2 \frac{A_2}{A_3}\left(1 - \frac{A_2}{A_3}\right)$$

After a little algebra, this may be written

$$h_{l_{13}} \simeq \frac{\bar{V}_2^2}{2}\left(1 - \frac{A_2}{A_3}\right)^2$$

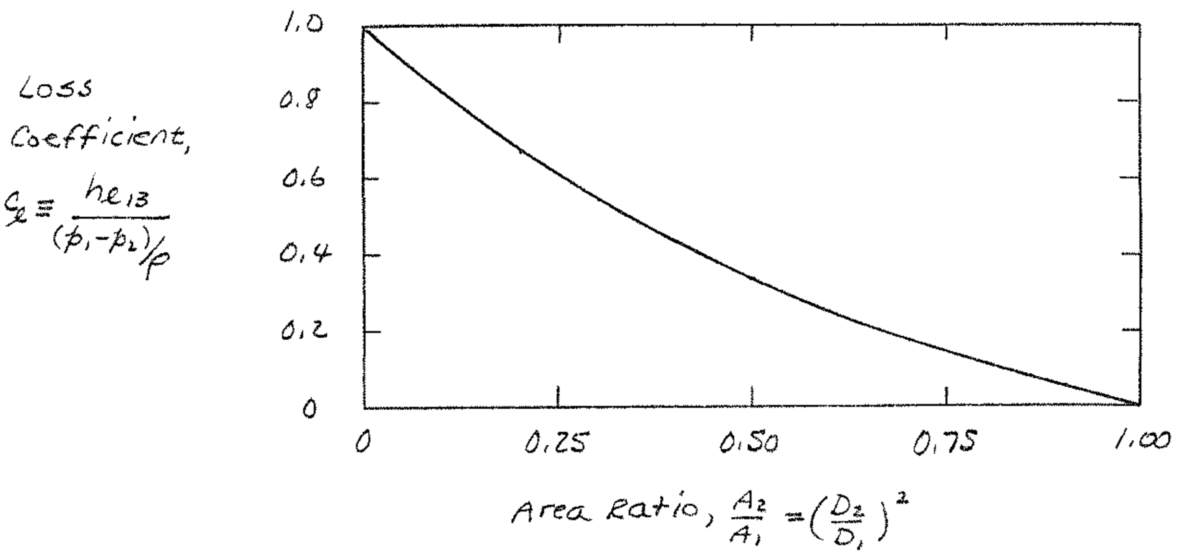
Dividing by $(p_1 - p_2)/\rho$, a loss coefficient is derived as

$$C_l = \frac{h_{l_{13}}}{(p_1 - p_2)/\rho} = \frac{\frac{\bar{V}_2^2}{2}\left(1 - \frac{A_2}{A_3}\right)^2}{\frac{\bar{V}_2^2}{2}\left[1 - \left(\frac{A_2}{A_3}\right)^2\right]} = \frac{\left(1 - A_2/A_3\right)^2}{\left[1 - \left(A_2/A_3\right)^2\right]}$$

But $1 - \left(\frac{A_2}{A_3}\right)^2 = \left(1 + \frac{A_2}{A_3}\right)\left(1 - \frac{A_2}{A_3}\right)$, so

$$C_l = \frac{h_{l_{13}}}{(p_1 - p_2)/\rho} = \frac{1 - A_2/A_3}{1 + A_2/A_3}$$ C_l

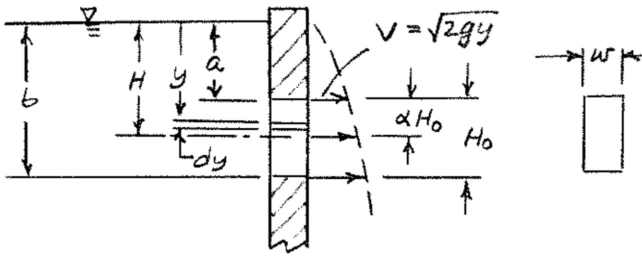
Plotting:



Problem 8.168

Open-Ended Problem Statement: In some western states, water for mining and irrigation was sold by the "miner's inch," the rate at which water flows through an opening in a vertical plank of 1 in.^{2} area, up to 4 in. tall, under a head of 6 to 9 in. Develop an equation to predict the flow rate through such an orifice. Specify clearly the aspect ratio of the opening, thickness of the plank, and datum level for measurement of head (top, bottom, or middle of the opening). Show that the unit of measure varies from 38.4 (in Colorado) to 50 (in Arizona, Idaho, Nevada, and Utah) miner's inches equal to 1 ft^3/s.

Analysis: The geometry of the opening in a vertical plank is shown. The analysis includes the effect on flow speed of the variation in water depth vertically across the opening.



Aspect ratio, $ar \equiv \frac{H_o}{w}$; area, $A = wH_o = 1\ \text{in.}^2$

$\alpha = 0, \frac{1}{2}, 1$

$$Q_{geom} = \int V\,dA = \int_a^b \sqrt{2gy}\, w\,dy = w\sqrt{2g}\left[\frac{2}{3}y^{3/2}\right]_a^b = \frac{2}{3} w a \sqrt{2ga}\left[\left(\frac{b}{a}\right)^{3/2} - 1\right]$$

For $ar = 1$, $\alpha = 0$, $a = H = 9$ in, $b = 10.0$ in., $w = 1.0$ in.

$$Q_{geom} = \frac{2}{3} \times 1.0\ \text{in.} \times 9\ \text{in.}\left[2 \times 32.2\ \frac{\text{ft}}{\text{s}^2} \times 9\ \text{in.} \times \frac{\text{ft}}{12\ \text{in.}}\right]^{\frac{1}{2}}\left[\left(\frac{10}{9}\right)^{3/2} - 1\right]\frac{\text{ft}^2}{144\ \text{in.}^2} = 0.0496\ \frac{\text{ft}^3}{\text{s}}$$

$Q_{actual} = 0.6\, Q_{geom} = 0.0297\ \text{ft}^3/\text{s}$; thus $1/0.0297 = 33.6\ \text{MI} = 1\ \text{cfs}$ MI

Numerical results are presented in the spread sheet on the next page.

Discussion: All results assume a *vena contracta* in the liquid jet leaving the opening, reducing the effective flow area to 60 percent of the geometric area of the opening.

The calculated unit of measure varies from 31.3 to 52.4 miner's inch per cubic foot of water flow per second. This range encompasses the 38.4 and 50 values given in the problem statement.

Trends may be summarized as follows. The largest flow rate occurs when datum H is measured to the top of the opening in the vertical plank. This gives the deepest submergence and thus the highest flow speeds through the opening.

When $ar = 1$, the opening is square; when $ar = 16$, the opening is 4 inches tall and 1/4 inch wide. Increasing ar from 1 to 16 increases the flow rate through the opening when H is measured to the top of the opening, because it increases the submergence of the lower portion of the opening, thus increasing the flow speeds. When H is measured to the center of the opening ar has almost no effect on flow rate. When H is measured to the bottom of the opening, increasing ar reduces the flow rate. For this case, the depth of the opening decreases as ar becomes larger.

Plank thickness does not affect calculated flow rates since a *vena contracta* is assumed. In this flow model, water separates from the interior edges of the opening in the vertical plank. Only if the plank were several inches thick might the stream reattach and affect the flow rate.

The actual relationship between Q_{flow} and Q_{geom} might be a weak function of aspect ratio. The flow separates from all four edges of the opening in the vertical plank. At large ar, contraction on the narrow ends of the stream has a relatively small effect on flow area. As ar approaches 1 the effect becomes more pronounced, but would need to be measured experimentally. Assuming a constant 60 percent area fraction certainly gives reasonable trends.

Computation of "Miner's Inch" in Engineering Units:

a =	depth to top of opening	(in.)
ar =	aspect ratio of opening	(---)
A =	area of opening	1 in.^{2}
b =	depth to bottom of opening	(in.)
H =	nominal head	(in.)
H_0 =	height of opening	(in.)
MI =	"miner's inch"	(mixed)
Q =	volume flow rate	(ft^3/s)
w =	width of opening	(in.)

Assume $Q_{flow} = 0.6 \times Q_{geometric}$ to account for contraction of the stream leaving the opening.

(a) Measure *H* to top of opening:

H	ar	H_0	a	b	w	Q_{geom}	Q_{flow}	MI/cfs
9	1	1.00	9.00	10.0	1.00	0.0496	0.0297	33.6
9	2	1.41	9.00	10.4	0.707	0.0501	0.0301	33.3
9	4	2.00	9.00	11.0	0.500	0.0509	0.0305	32.8
9	8	2.83	9.00	11.8	0.354	0.0519	0.0311	32.1
9	16	4.00	9.00	13.0	0.250	0.0533	0.0320	31.3
6	1	1.00	6.00	7.00	1.00	0.0410	0.0246	40.6
6	2	1.41	6.00	7.41	0.707	0.0416	0.0250	40.0
6	4	2.00	6.00	8.00	0.500	0.0425	0.0255	39.2
6	8	2.83	6.00	8.83	0.354	0.0437	0.0262	38.1
6	16	4.00	6.00	10.0	0.250	0.0454	0.0272	36.7

(b) Measure *H* to middle of opening:

H	ar	H_0	a	b	w	Q_{geom}	Q_{flow}	MI/cfs
9	1	1.00	8.50	9.50	1.00	0.0483	0.0290	34.5
9	2	1.41	8.29	9.71	0.707	0.0483	0.0290	34.5
9	4	2.00	8.00	10.0	0.500	0.0482	0.0289	34.6
9	8	2.83	7.59	10.4	0.354	0.0482	0.0289	34.6
9	16	4.00	7.00	11.0	0.250	0.0482	0.0289	34.6
6	1	1.00	5.50	6.50	1.00	0.0394	0.0236	42.3
6	2	1.41	5.29	6.71	0.707	0.0394	0.0236	42.3
6	4	2.00	5.00	7.00	0.500	0.0394	0.0236	42.3
6	8	2.83	4.59	7.41	0.354	0.0393	0.0236	42.4
6	16	4.00	4.00	8.00	0.250	0.0392	0.0235	42.5

(c) Measure *H* to bottom of opening:

H	ar	H_0	a	b	w	Q_{geom}	Q_{flow}	MI/cfs
9	1	1.00	8.00	9.00	1.00	0.0469	0.0281	35.5
9	2	1.41	7.59	9.00	0.707	0.0463	0.0278	36.0
9	4	2.00	7.00	9.00	0.500	0.0455	0.0273	36.7
9	8	2.83	6.17	9.00	0.354	0.0442	0.0265	37.7
9	16	4.00	5.00	9.00	0.250	0.0424	0.0254	39.3
6	1	1.00	5.00	6.00	1.00	0.0377	0.0226	44.2
6	2	1.41	4.59	6.00	0.707	0.0370	0.0222	45.1
6	4	2.00	4.00	6.00	0.500	0.0359	0.0215	46.4
6	8	2.83	3.17	6.00	0.354	0.0343	0.0206	48.6
6	16	4.00	2.00	6.00	0.250	0.0318	0.0191	52.4

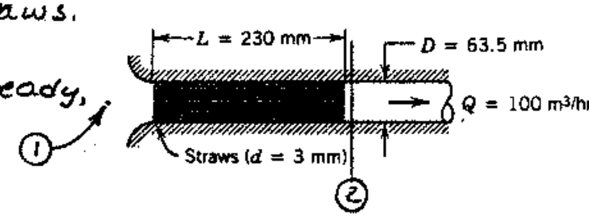
Problem 8.169

Given: Pipe-flow experiment with flow straightener made from straws.

Find: (a) Reynolds number for flow in each straw. $K_{ent} = 1.4$

(b) Friction factor for flow in each straw. $\alpha = 2.0$

(c) Gage pressure at exit from straws.



Solution: Apply energy equation for steady, incompressible pipe flow.

Computing equation:

$$\frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \alpha_2 \frac{\bar{V}_2^2}{2} + g z_2 + h_{lT} \qquad (\bar{V}_1 \approx 0 \; (1))$$

$$h_{lT} = h_l + h_{lm} = f \frac{L}{D}\frac{\bar{V}^2}{2} + K_{ent}\frac{\bar{V}^2}{2} = \left(f\frac{L}{D} + K_{ent}\right)\frac{\bar{V}^2}{2}$$

Assumptions: (1) Flow from atmosphere: $p_1 = p_{atm}$, $\bar{V}_1 \approx 0$

(2) Horizontal

(3) Neglect thickness of straws

Then

$$\bar{V}_2 = \frac{Q}{A} = 100\,\frac{m^3}{hr} \times \frac{4}{\pi (0.0635)^2 m^2} \times \frac{hr}{3600\,s} = 8.77\ m/s$$

$$Re_d = \frac{\bar{V}_2 d}{\nu} = 8.77\,\frac{m}{sec} \times 0.003\,m \times \frac{sec}{1.46 \times 10^{-5}\,m^2} = 1800$$ Re_d

For laminar flow,

$$f = \frac{64}{Re} = \frac{64}{1800} = 0.0356$$ f

The gage pressure at ② is

$$p_{2g} = -\frac{\rho \bar{V}_2^2}{2}\left(\alpha_2 + K_{ent} + f\frac{L}{D}\right)$$

$$= -\frac{1}{2} \times 1.23\,\frac{kg}{m^3} \times (8.77)^2 \frac{m^2}{s^2}\left(2.0 + 1.4 + 0.0356 \times \frac{230\,mm}{3\,mm}\right)\frac{N \cdot s^2}{kg \cdot m}$$

$$p_{2g} = -290\ N/m^2\ (gage)$$ p_{2g}

This pressure drop is equivalent to

$$\Delta h = \frac{\Delta p}{\rho_{H_2O}\, g} = 290\,\frac{N}{m^2} \times \frac{m^3}{999\,kg} \times \frac{s^2}{9.81\,m} \times \frac{kg \cdot m}{N \cdot s^2} = 29.6\ mm\ H_2O$$

Comments: (1) This pressure drop is large enough to measure readily. The straws could be used as a flowmeter.

(2) Straws would eliminate any swirl from the flow.

Problem 8.170

Given: Volume flow rate in a circular duct is to be measured using a "Pitot traverse," by measuring the velocity in each of several area segments across the duct, then summing.

Find: Comment on the way the traverse should be set up. Quantify and plot the expected error in measurement of flow rate as a function of the number of radial locations used in the traverse.

Solution: First divide the duct cross section into segments of equal area. Then measure velocity at the mean area of each segment.

Assume flow is turbulent, and that the velocity profile is well represented by the 1/7-power profile. From Eq. 8.24 the ratio of average flow velocity to centerline velocity is 0.817.

Distinguish two cases, depending on whether velocity is measured at the centerline.

Case 1: Measure velocity at the duct centerline, plus at (k - 1) other locations.

For $k = 1$, the sole measurement is at the duct centerline. This measures the centerline velocity U, which is 1/0.817 = 1.22 times the average flow velocity $\bar{\ }$. Thus the volume flow rate estimated by this 1-point measurement is 22 percent larger than the true value.

For $k = 2$, the duct is divided into two segments of equal area. The centerline velocity is measured and assigned the half of the duct area surrounding the centerline. The second measurement point is located at the midpoint of the remaining half of the duct area. Thus this point is located at the radius that encloses 3/4 of the duct area, or $r_2/R = (3/4)^{1/2} = 0.866$, as shown on the attached spreadsheet. The velocity ratio at this point is $\bar{u}/U = 0.92$. Averaging the segmental flow rates gives (1.22 + 0.92)/2 = 1.07. Thus the volume flow rate estimated by this 2-point measurement is 7 percent high.

For $k = 3$, the duct is divided into three portions of equal area. The centerline velocity is measured and assigned the one-third of the duct area surrounding the centerline. The second measurement point is located at the midpoint of the second one-third of the duct area. This point is located at the radius that encloses half the duct area, or at $r_2/R = (1/2)^{1/2} = 0.707$. The third measurement point is located at the midpoint of the third one-third of the duct area. This point is located at the radius enclosing 5/6 of the duct area, or at $r_3/R = (5/6)^{1/2} = 0.913$.

Results of calculations for $k = 4$ and 5 are also given on the spreadsheet.

Case 2: Measure velocity at k locations, not including the centerline.

For $k = 1$, the radius is chosen at half the duct area. Thus $r_1/R = (1/2)^{1/2} = 0.707$, $\bar{u}/U = 0.839$, and $\bar{u}/\bar{\ } = 1.03$, or about 3 percent too high, as shown on the spreadsheet.

For $k = 2$, the duct is divided into two equal areas. The first measurement is made at the midpoint of the inner area, where the radius includes one fourth of the total area. The second is made at the midpoint of the outer area, where the radius includes three fourths of the total duct area. The results are shown; the flow rate estimate is high by about 1.4 percent.

For $k = 3$, the duct is divided into three equal areas. The first measurement is made at the midpoint of the inner 1/3 of the duct area, where the radius includes 1/6 of the total area. The second is made at the midpoint of the second 1/3 of the duct area, where the radius includes 1/2 of the total duct area. The third is made at the midpoint of the third 1/3 of the duct area, where the radius includes 5/6 of the total duct area. The results are shown; the flow rate estimate is high by about 0.9 percent.

Results of calculations for $k = 4$ and 5 also are given on the spreadsheet.

Remarkably, Case 2 gives less than 2 percent error for any number of locations.

V_{bar}/U = 0.817 n = 7 k = Number of measurement points

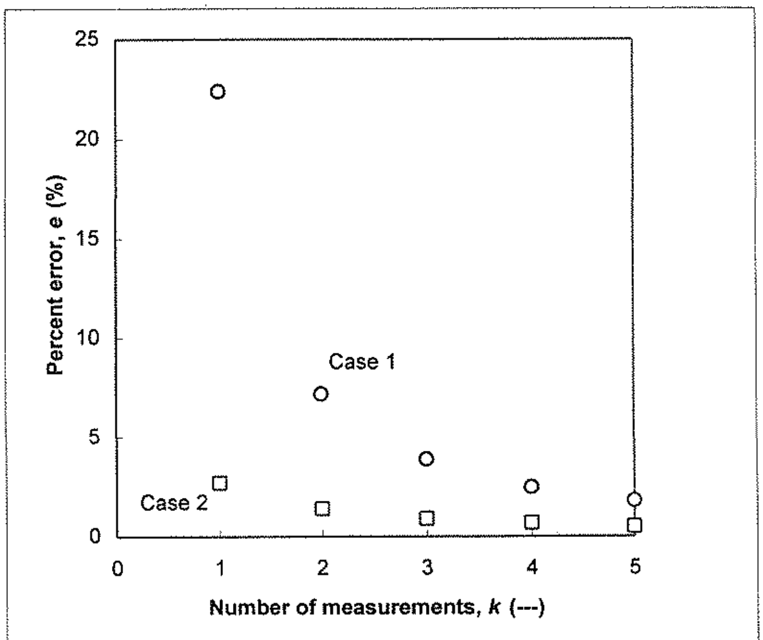
Case 1: Measure at centerline plus at (k - 1) other locations

k	i	r_i/R	u/U	u/V_{bar}	(%) Error
1	1	0.000	1.000	1.22	22.4
2	1	0.000	1.000	1.22	
	2	0.866	0.750	0.92	
				1.07	7.2
3	1	0.000	1.000	1.22	
	2	0.707	0.839	1.03	
	3	0.913	0.706	0.864	
				1.04	3.9
4	1	0.000	1.000	1.22	
	2	0.612	0.873	1.07	
	3	0.791	0.800	0.98	
	4	0.935	0.676	0.828	
				1.03	2.5
5	1	0.000	1.000	1.22	
	2	0.548	0.893	1.09	
	3	0.707	0.839	1.03	
	4	0.837	0.772	0.945	
	5	0.949	0.654	0.801	
				1.02	1.8

Case 2: Measure at k locations not including the centerline

k	i	r_i/R	u/U	u/V_{bar}	(%) Error
1	1	0.707	0.839	1.03	2.7
2	1	0.500	0.906	1.11	
	2	0.866	0.750	0.92	
				1.01	1.4
3	1	0.408	0.928	1.14	
	2	0.707	0.839	1.03	
	3	0.913	0.706	0.86	
				1.01	0.9
4	1	0.354	0.940	1.15	
	2	0.612	0.873	1.07	
	3	0.791	0.800	0.98	
	4	0.935	0.676	0.83	
				1.01	0.7
5	1	0.316	0.947	1.16	
	2	0.548	0.893	1.09	
	3	0.707	0.839	1.03	
	4	0.837	0.772	0.95	
	5	0.949	0.654	0.80	
				1.01	0.5

k	Case 1 e (%)	Case 2 e (%)
1	22.4	2.7
2	7.2	1.4
3	3.9	0.9
4	2.5	0.7
5	1.8	0.5



Problem 8.171

Open-Ended Problem Statement: The chilled-water pipeline system that provides air conditioning for the Purdue University campus is described in Problem 8.140. The pipe diameter is selected to minimize total cost (capital cost plus operating cost). Annualized costs are compared, since capital cost occurs once and operating cost continues for the life of the system. The optimum diameter depends on both cost factors and operating conditions; the analysis must be repeated when these variables change. Perform a pipeline optimization analysis. Solve Problem 8.140 arranging your calculations to study the effect of pipe diameter on annual pumping cost. (Assume friction factor remains constant.) Obtain an expression for total annual cost per unit delivery (e.g., dollars per cubic meter), assuming construction cost varies as the square of pipe diameter. Obtain an analytic relation for the pipe diameter that yields minimum total cost per unit delivery. Assume the present chilled-water pipeline was optimized for a 20-year life with 5 percent annual interest. Repeat the optimization for a design to operate at 30 percent larger flow rate. Plot the annual cost for electrical energy for pumping and the capital cost, using the flow conditions of Problem 8.140, with pipe diameter varied from 300 to 900 mm. Show how the diameter may be chosen to minimize total cost. How sensitive are the results to interest rate?

(From Problem 8.140: The pipe makes a loop 3 miles in length. The pipe diameter is 2 ft and the material is steel. The maximum design volume flow rate is 11,200 gpm. The circulating pump is driven by an electric motor. The efficiencies of pump and motor are $\eta_p = 0.80$ and $\eta_m = 0.90$, respectively. Electricity cost is \$0.067/(kW·hr).)

Analysis: From Problem 8.140, the electrical energy for pumping costs \$174,000 per year for 11,200 gallons per minute circulation. The present line, with D = 24 in., is optimized for this flow rate. $\dot{W} = Q\Delta p$, so $\dot{W}/Q = \Delta p$.

The optimum pipe diameter minimizes total annualized cost, for construction and operation of the pipeline, $C_t = C_c + C_p$. Construction cost C_c is a one-time cost. Annualized pumping cost C_p is computed by summing the present worth of each annual pumping cost over the lifetime of the pipeline. For 20 years at 5 percent per year, spwf = 13.1 (see spreadsheet). Costs may be expressed in terms of diameter as

$$C_t = C_c + C_p = K_c D^2 + \frac{K_p}{D^5} \qquad (1)$$

For the optimum diameter, $dC_t/dD = 2K_c D - 5K_p D^{-6} = 0$, so

$$K_c = \frac{5K_p}{2D^7} = \frac{5C_p}{2D^2} = \frac{5}{2} \times (13.1)\$174{,}000 \times \frac{1}{(24)^2\,\text{in.}^2} = \$9890/\text{in.}^2$$

K_c

From Eq. 1,

$$K_p = C_p D^5 = (13.1)\$174{,}000 \times (24)^5\,\text{in.}^5 = 1.81 \times 10^{13}\,\$\cdot\text{in.}^5$$

K_p

Calculations with these values are shown on the spreadsheet.

To optimize at a new, larger flow rate, note $C_p \sim \Delta p \sim f\frac{L}{D}\frac{\rho\bar{V}^2}{2} = f\frac{L}{D}\frac{\rho}{2}\left(\frac{Q}{A}\right)^2 \sim f\frac{Q^2}{D^5}$

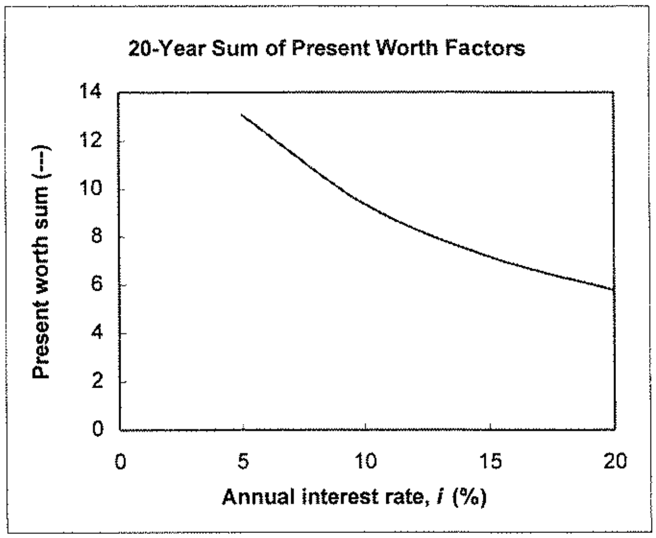
Thus

$$K_p(\text{new}) = K_p(\text{old})\left(\frac{Q_{new}}{Q_{old}}\right)^2 = (1.3)^2 K_p(\text{old}) = 3.06 \times 10^{13}\,\$\cdot\text{in.}^5$$ The new optimum is at

$D \approx 25.9$ in., as shown on the second plot.

Results are not too sensitive to interest rate; only K_p varies. $D_{opt} \to 25$ in. for $i = 15\%$.

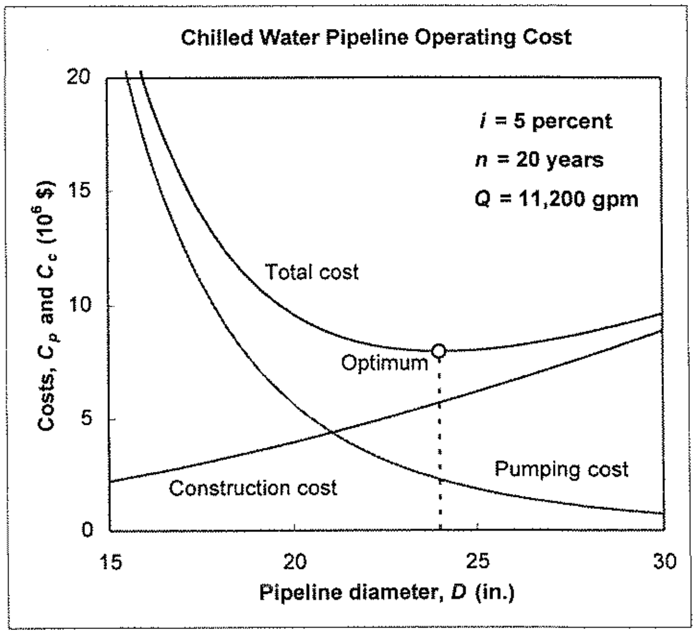
i =	5	10	15	20
	Annual interest rate (%)			
Year	*pwf*	*pwf*	*pwf*	*pwf*
1	1.00	1.00	1.00	1.00
2	0.952	0.909	0.870	0.833
3	0.907	0.826	0.756	0.694
4	0.864	0.751	0.658	0.579
5	0.823	0.683	0.572	0.482
6	0.784	0.621	0.497	0.402
7	0.746	0.564	0.432	0.335
8	0.711	0.513	0.376	0.279
9	0.677	0.467	0.327	0.233
10	0.645	0.424	0.284	0.194
11	0.614	0.386	0.247	0.162
12	0.585	0.350	0.215	0.135
13	0.557	0.319	0.187	0.112
14	0.530	0.290	0.163	0.0935
15	0.505	0.263	0.141	0.0779
16	0.481	0.239	0.123	0.0649
17	0.458	0.218	0.107	0.0541
18	0.436	0.198	0.0929	0.0451
19	0.416	0.180	0.0808	0.0376
20	0.396	0.164	0.0703	0.0313
Sum:	**13.1**	**9.4**	**7.2**	**5.8**



K_c = 9,890 \$/in.^{2} Cost of construction per diameter squared

K_p = 1.81E+13 \$*in.^{5} Present worth 20-yr cost of pumping 11,200 gpm

Pipe Diameter, D (in.)	Cost of Pumping, C_p (10^6 \$)	Cost to Construct, C_c (10^6 \$)	Total Cost, C_t (10^6 \$)
15	23.9	2.23	26.1
16	17.3	2.53	19.8
17	12.8	2.86	15.6
18	9.59	3.20	12.8
19	7.32	3.57	10.9
20	5.67	3.96	9.62
21	4.44	4.36	8.80
22	3.52	4.79	8.30
23	2.82	5.23	8.05
24	2.28	5.70	7.97
25	1.86	6.18	8.04
26	1.53	6.69	8.21
27	1.26	7.21	8.47
28	1.05	7.75	8.81
29	0.884	8.32	9.20
30	0.746	8.90	9.65



K_c = 9,890 $/in.^{2} Cost of construction per diameter squared

K_p = 3.06E+13 $*in.^{5} Present worth 20-yr cost of pumping 14,600 gpm

Pipe Diameter, D (in.)	Cost of Pumping, C_p (10^6 $)	Cost to Construct, C_c (10^6 $)	Total Cost, C_t (10^6 $)
15	40.3	2.23	42.6
16	29.2	2.53	31.8
17	21.6	2.86	24.4
18	16.2	3.20	19.4
19	12.4	3.57	15.9
20	9.57	3.96	13.53
21	7.50	4.36	11.86
22	5.95	4.79	10.73
23	4.76	5.23	9.99
24	3.85	5.70	9.54
25	3.14	6.18	9.32
26	2.58	6.69	9.26
27	2.14	7.21	9.35
28	1.78	7.75	9.53
29	1.49	8.32	9.81
30	1.26	8.90	10.2