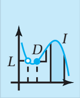
CHAPTER

1

Limits, Derivatives, Integrals, and Integrals

Automakers have recently begun to produce electric cars, which utilize electricity instead of gasoline to run their engines. Engineers are constantly looking for ways to design an electric car that can match the performance of a conventional gasoline-powered car. Engineers can predict a car's performance characteristics even before the first prototype is built. From information about the acceleration, they can calculate the car's velocity as a function of time. From the velocity, they can predict the distance it will travel while it is accelerating. Calculus provides the mathematical tools to analyze quantities that change at variable rates.



Mathematical Overview

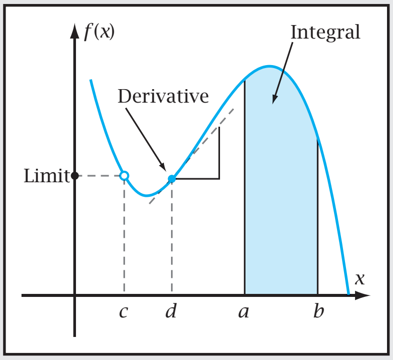
Calculus deals with calculating things that change at variable rates. The four concepts invented to do this are

- Limits
- Derivatives
- Integrals (one kind)
- Integrals (another kind)

In Chapter 1, you will study three of these concepts in four ways.

Graphically

The icon at the top of each even-numbered page of this chapter illustrates a limit, a derivative, and one type of integral.



Numerically

x	$x - d$	Slope
2.1	0.1	1.071666...
2.01	0.01	1.007466...
2.001	0.001	1.000749...
⋮	⋮	⋮

Algebraically

$$\text{Average rate of change} = \frac{f(x) - f(2)}{x - 2}$$

Verbally

I have learned that a definite integral is used to measure the product of x and f(x). For instance, velocity multiplied by time gives the distance traveled by an object. The definite integral is used to find this distance if the velocity varies.

1-1 The Concept of Instantaneous Rate

If you push open a door that has an automatic closer, it opens fast at first, slows down, stops, starts closing, then slams shut. As the door moves, the number of degrees, d, it is from its closed position depends on how many seconds it has been since you pushed it. Figure 1-1a shows such a door from above.

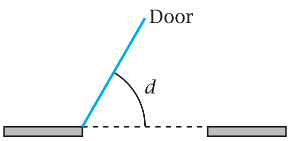


Figure 1-1a

The questions to be answered here are, "At any particular instant in time, is the door opening or closing?" and "How fast is it moving?" As you progress through this course, you will learn to write equations expressing the rate of change of one variable quantity in terms of another. For the time being, you will answer such questions graphically and numerically.

OBJECTIVE Given the equation for a function relating two variables, estimate the instantaneous rate of change of the dependent variable with respect to the independent variable at a given point.

Suppose that a door is pushed open at time $t = 0$ s and slams shut again at time $t = 7$ s. While the door is in motion, assume that the number of degrees, d, from its closed position is modeled by this equation.

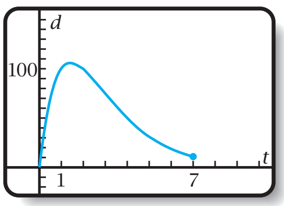


Figure 1-1b

$$d = 200t \cdot 2^{-t} \quad \text{for} \quad 0 \le t \le 7$$

How fast is the door moving at the instant when $t = 1$ s? Figure 1-1b shows this equation on a grapher (graphing calculator or computer). When t is 1, the graph is going *up* as t increases from left to right. So the angle is increasing and the door is opening. You can estimate the rate numerically by calculating values of d for values of t close to 1.

$$t = 1: \quad d = 200(1) \cdot 2^{-1} = 100^\circ$$
$$t = 1.1: \quad d = 200(1.1) \cdot 2^{-1.1} = 102.633629...^\circ$$

The door's angle increased by 2.633...° in 0.1 s, meaning that it moved at a rate of about (2.633...)/0.1, or 26.33... deg/s. However, this rate is an *average* rate, and the question was about an *instantaneous* rate. In an "instant" that is 0 s long, the door moves 0°. Thus, the rate would be 0/0, which is awkward because division by zero is undefined.

To get closer to the instantaneous rate at $t = 1$ s, find d at $t = 1.01$ s and at $t = 1.001$ s.

$$t = 1.01: \quad d = 200(1.01) \cdot 2^{-1.01} = 100.30234..., \text{ a change of } 0.30234...^\circ$$
$$t = 1.001: \quad d = 200(1.001) \cdot 2^{-1.001} = 100.03064..., \text{ a change of } 0.03064...^\circ$$

Here are the average rates for the time intervals 1 s to 1.01 s and 1 s to 1.001 s.

$$1 \text{ s to } 1.01 \text{ s:} \quad \text{average rate} = \frac{0.30234...}{0.01} = 30.234... \text{ deg/s}$$

$$1 \text{ s to } 1.001 \text{ s:} \quad \text{average rate} = \frac{0.03064...}{0.001} = 30.64... \text{ deg/s}$$

The important thing for you to notice is that as the time interval gets smaller and smaller, the number of degrees per second doesn't change much. Figure 1-1c shows why. As you zoom in on the point (1, 100), the graph appears to be straighter, so the change in d divided by the change in t becomes closer to the slope of a straight line.

If you list the average rates in a table, another interesting feature appears. The values stay the same for more and more decimal places.

t (s)	**Average Rate**
1 to 1.01	30.23420...
1 to 1.001	30.64000...
1 to 1.0001	30.68075...
1 to 1.00001	30.68482...
1 to 1.000001	30.68524...

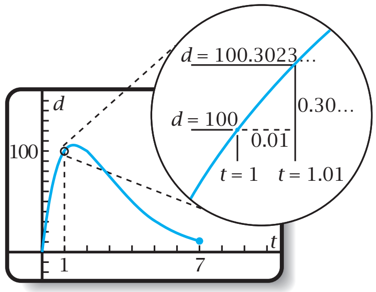


Figure 1-1c

There seems to be a *limiting* number that the values are approaching.

To estimate the instantaneous rate at $t = 3$ s, follow the same steps as for $t = 1$ s.

$$t = 3: \quad d = 200(3) \cdot 2^{-3} = 75^\circ$$
$$t = 3.1: \quad d = 200(3.1) \cdot 2^{-3.1} = 72.310056...^\circ$$
$$t = 3.01: \quad d = 200(3.01) \cdot 2^{-3.01} = 74.730210...^\circ$$
$$t = 3.001: \quad d = 200(3.001) \cdot 2^{-3.001} = 74.973014...^\circ$$

Here are the corresponding average rates.

$$\text{3 s to 3.1 s:} \quad \text{average rate} = \frac{72.310056... - 75}{3.1 - 3} = -26.899... \text{ deg/s}$$

$$\text{3 s to 3.01 s:} \quad \text{average rate} = \frac{74.730210... - 75}{3.01 - 3} = -26.978... \text{ deg/s}$$

$$\text{3 s to 3.001 s:} \quad \text{average rate} = \frac{74.973014... - 75}{3.001 - 3} = -26.985... \text{ deg/s}$$

Again, the rates seem to be approaching some limiting number, this time, around -27. So the instantaneous rate at $t = 3$ s should be somewhere close to -27 deg/s. The negative sign tells you that the number of degrees, d, is *decreasing* as time goes on. Thus, the door is closing when $t = 3$ s. It is opening when $t = 1$ because the rate of change is positive.

For the door example shown above, the angle is said to be a **function** of time. Time is the **independent variable** and angle is the **dependent variable**. These names make sense, because the number of degrees the door is open *depends* on the number of seconds since it was pushed. The instantaneous rate of change of the dependent variable is said to be the **limit** of the average rates as the time interval gets closer to zero. This limiting value is called the **derivative** of the dependent variable with respect to the independent variable.

Problem Set 1-1

1. *Pendulum Problem:* A pendulum hangs from the ceiling (Figure 1-1d). As the pendulum swings, its distance, d, in centimeters from one wall of the room depends on the number of seconds, t, since it was set in motion. Assume that the equation for d as a function of t is

$$d = 80 + 30\cos\frac{\pi}{3}t, \qquad t \geq 0$$

You want to find out how fast the pendulum is moving at a given instant, t, and whether it is approaching or going away from the wall.

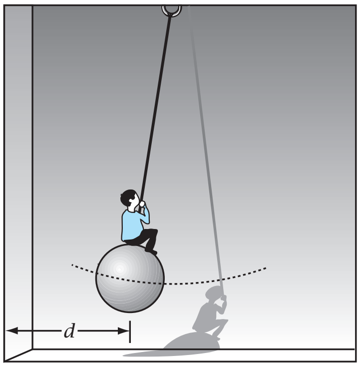


Figure 1-1d

a. Find d when $t = 5$. If you don't get 95 for the answer, make sure your calculator is in radian mode.

b. Estimate the instantaneous rate of change of d at $t = 5$ by finding the average rates from $t = 5$ to 5.1, $t = 5$ to 5.01, and $t = 5$ to 5.001.

c. Why can't the actual instantaneous rate of change of d with respect to t be calculated using the method in part b?

d. Estimate the instantaneous rate of change of d with respect to t at $t = 1.5$. At that time, is the pendulum approaching the wall or moving away from it? Explain.

e. How is the instantaneous rate of change related to the average rates? What name is given to the instantaneous rate?

f. What is the reason for the domain restriction $t \geq 0$? Can you think of any reason that there would be an *upper* bound to the domain?

2. *Board Price Problem:* If you check the prices of various lengths of lumber, you will find that a board twice as long as another of the same type does not necessarily cost twice as much. Let x be the length, in feet, of a $2'' \times 6''$ board (Figure 1-1e) and let y be the price, in cents, that you pay for the board. Assume that y is given by

$$y = 0.2x^3 - 4.8x^2 + 80x$$

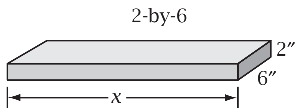


Figure 1-1e

a. Find the price of $2'' \times 6''$ boards that are 5 ft long, 10 ft long, and 20 ft long.

b. Find the average rate of change of the price in cents per foot for $x = 5$ to 5.1, $x = 5$ to 5.01, and $x = 5$ to 5.001.

c. The average number of cents per foot in part b is approaching an *integer* as the change in x gets smaller and smaller. What integer? What is the name given to this rate of change?

d. Estimate the instantaneous rate of change in price if x is 10 ft and if x is 20 ft. You should find that each of these rates is an integer.

e. One of the principles of marketing is that when you buy in larger quantities, you usually pay less per unit. Explain how the numbers in Problem 2 show that this principle does not apply to buying longer boards. Think of a reason why it does not apply.

1-2 Rate of Change by Equation, Graph, or Table

In Section 1-1, you explored functions for which an equation related two variable quantities. You found average rates of change of $f(x)$ over an interval of x-values, and used these to estimate the instantaneous rate of change at a particular value of x. The instantaneous rate is called the *derivative* of the function at that value of x. In this section you will estimate instantaneous rates for functions specified graphically or numerically, as well as algebraically (by equations).

OBJECTIVE Given a function $y = f(x)$ specified by a graph, a table of values, or an equation, describe whether the y-value is increasing or decreasing as x increases through a particular value, and estimate the instantaneous rate of change of y at that value of x.

Background: Function Terminology and Types of Functions

The price you pay for a certain type of board depends on how long it is. In mathematics the symbol $f(x)$ (pronounced "f of x" or "f at x") is often used for the dependent variable. The letter f is the name of the function, and the number in parentheses is either a value of the independent variable or the variable itself. If $f(x) = 3x + 7$, then $f(5)$ is $3(5) + 7$, or 22.

The equation $f(x) = 3x + 7$ is the **particular equation** for a linear function. The **general equation** for a linear function is written $y = mx + b$, or $f(x) = mx + b$, where m and b represent the constants. The following box shows the names of some types of functions and their general equations.

DEFINITIONS: Types of Functions

Linear: $f(x) = mx + b$; m and b stand for constants, $m \neq 0$

Quadratic: $f(x) = ax^2 + bx + c$; a, b, and c stand for constants, $a \neq 0$

Polynomial: $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots + a_nx^n$; $a_0, a_1, \ldots$ stand for constants, n is a positive integer, $a_n \neq 0$ (nth degree polynomial function)

Power: $f(x) = ax^n$; a and n stand for constants

Exponential: $f(x) = ab^x$; a and b stand for constants, $a \neq 0$, $b > 0$, $b \neq 1$

Rational Algebraic: $f(x) = $ (polynomial)/(polynomial)

Absolute value: $f(x)$ contains |(variable expression)|

Trigonometric or Circular: $f(x)$ contains $\cos x$, $\sin x$, $\tan x$, $\cot x$, $\sec x$, or $\csc x$

EXAMPLE 1 Figure 1-2a shows the graph of a function. At $x = a$, $x = b$, and $x = c$, state whether y is increasing, decreasing, or neither as x increases. Then state whether the rate of change is fast or slow.

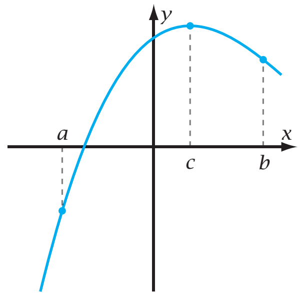


Figure 1-2a

Solution At $x = a$, y is increasing quickly as you go from left to right.

At $x = b$, y is decreasing slowly because y is dropping as x goes from left to right, but it's not dropping very quickly.

At $x = c$, y is neither increasing nor decreasing, as shown by the fact that the graph has leveled off at $x = c$. ◀

EXAMPLE 2 Figure 1-2b shows the graph of a function that could represent the height, $h(t)$, in feet, of a soccer ball above the ground as a function of the time, t, in seconds since it was kicked into the air.

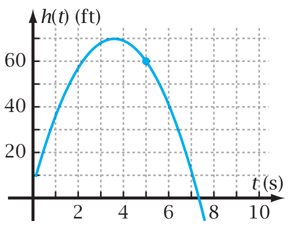


Figure 1-2b

a. Estimate the instantaneous rate of change of $h(t)$ at time $t = 5$.

b. Give the mathematical name of this instantaneous rate, and state why the rate is negative.

Solution

a. Draw a line tangent to the graph at $x = 5$ by laying a ruler against it, as shown in Figure 1-2c. You will be able to estimate the tangent line more accurately if you put the ruler on the *concave* side of the graph.

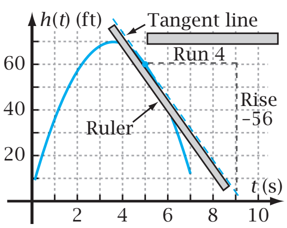


Figure 1-2c

The instantaneous rate is the slope of this tangent line. From the point where $t = 5$, run over a convenient distance in the t-direction, say 4 seconds. Then draw a vertical line to the tangent line. As shown in the figure, this rise is about 56 feet in the negative direction.

$$\text{Instantaneous rate} = \text{slope of tangent} \approx \frac{-56}{4} = -14 \text{ ft/s}$$

b. The mathematical name is *derivative.* The rate is negative because $h(t)$ is decreasing at $t = 5$. ◀

EXAMPLE 3 Figure 1-2d shows a graph of $P(x) = 40(0.6^x)$, the probability that it rains a number of inches, x, at a particular place during a particular thunderstorm.

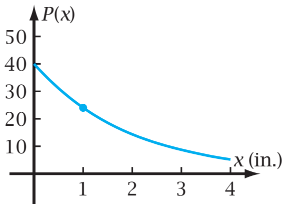


Figure 1-2d

a. The probability that it rains 1 inch is $P(1) = 24\%$. By how much, and in which direction, does the probability change from $x = 1$ to $x = 1.1$? What is the average rate of change from 1 inch to 1.1 inches? Make sure to include units in your answer. Why is the rate negative?

b. Write an equation for $r(x)$, the average rate of change of $P(x)$ from 1 to x. Make a table of values of $r(x)$ for each 0.01 unit of x from 0.97 to 1.03. Explain why $r(x)$ is undefined at $x = 1$.

c. The instantaneous rate at $x = 1$ is the limit that the average rate approaches as x approaches 1. Estimate the instantaneous rate using information from part b. Name the concept of calculus that is given to this instantaneous rate.

Solution a. To find the average rate, first you must find $P(1)$ and $P(1.1)$.

$$P(1) = 40(0.6^1) = 24$$

$$P(1.1) = 40(0.6^{1.1}) = 22.8048...$$

$$\text{Change} = 22.8048... - 24 = -1.1951...$$

Change is always final minus initial.

$$\text{Average rate} = \frac{-1.1951...}{0.1} = -11.1951\ \%/\text{in.}$$

The rate is negative because the probability decreases as the number of inches increases.

b. The average rate of change of $P(x)$ from 1 to x is equal to the change in $P(x)$ divided by the change in x.

$$r(x) = \frac{P(x) - 24}{x - 1} = \frac{40(0.6^x) - 24}{x - 1} \qquad \frac{\text{change in } P(x)}{\text{change in } x}$$

Store $P(x)$ as y_1 and $r(x)$ as y_2 in your grapher. Make a table of values of x and $r(x)$.

x	$r(x)$
0.97	−12.3542...
0.98	−12.3226...
0.99	−12.2911...
1.00	Error
1.01	−12.2285...
1.02	−12.1974...
1.03	−12.1663...

Note that $r(1)$ is undefined because you would be dividing by zero. When $x = 1$, $x - 1 = 0$.

c. Average $r(0.99)$ and $r(1.01)$, the values in the table closest to $x = 1$.

$$\text{Instantaneous rate} \approx \frac{1}{2}[r(0.99) + r(1.01)] = -12.2598...$$

The percentage is decreasing at about 12.26% per inch. (The percentage decreases because it is less likely to rain greater quantities.) The name is *derivative*. ◀

▶ EXAMPLE 4 A mass is bouncing up and down on a spring hanging from the ceiling (Figure 1-2e). Its distance, y, in feet, from the ceiling is measured by a calculator distance probe each 1/10 s, giving this table of values, in which t is time in seconds.

t (s)	y (ft)
0.2	3.99
0.3	5.84
0.4	7.37
0.5	8.00
0.6	7.48
0.7	6.01
0.8	4.16
0.9	2.63
1.0	2.00
1.1	2.52

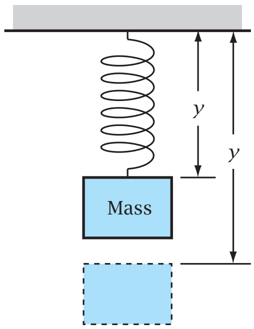


Figure 1-2e

a. How fast is y changing at each time?

i. $t = 0.3$

ii. $t = 0.6$

iii. $t = 1.0$

b. At time $t = 0.3$, is the mass going up or down? Justify your answer.

Solution a. If data are given in numerical form, you cannot get better estimates of the rate by taking values of t closer and closer to 0.3. However, you can get a better estimate by using the closest t-values on both sides of the given value. A time-efficient way to do the computations is shown in the following table. If you like, do the computations mentally and write only the final answer.

t	y	Difference	Rate	Average Rate
0.2	3.99	1.85	1.85/0.1 = 18.5	
0.3	5.84	1.53	1.53/0.1 = 15.3	16.9
0.4	7.37			
0.5	8.00	−0.52	−0.52/0.1 = −5.2	
0.6	7.48	−1.47	−1.47/0.1 = −14.7	−9.95
0.7	6.01			
0.8	4.16			
0.9	2.63	−0.63	−0.63/0.1 = −6.3	
1.0	2.00	0.52	0.52/0.1 = 5.2	−0.55
1.1	2.52			

All you need to write on your paper are the results, as shown here.

i. $t = 0.3$: increasing at about 16.9 ft/s

ii. $t = 0.6$: decreasing at about 9.95 ft/s

iii. $t = 1.0$: decreasing at about 0.55 ft/s

Write real-world answers with units.

b. At $t = 0.3$, the rate is about 16.9 ft/s, a *positive* number. This fact implies that y is *increasing*. As y increases, the mass goes *downward*. ◀

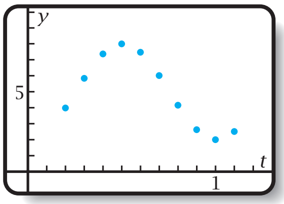


Figure 1-2f

Note that although a graph is not asked for in Example 4, plotting the data either on graph paper or by scatter plot on the grapher will help you understand what is happening. Figure 1-2f shows such a scatter plot.

The technique in Example 4 for estimating instantaneous rates by going forward and backward from the given value of x can also be applied to functions specified by an equation. The result is usually more accurate than the rate estimated by only going forward as you did in the last section.

As you learned in Section 1-1, the instantaneous rate of change of $f(x)$ at $x = c$ is the limit of the average rate of change over the interval from c to x as x approaches c. The value of the instantaneous rate is called the **derivative** of $f(x)$ with respect to x at $x = c$. The meaning of the word *derivative* is shown here. You will learn the precise definition when it is time to calculate derivatives exactly.

Meaning of Derivative

The derivative of function $f(x)$ at $x = c$ is the *instantaneous rate of change* of $f(x)$ with respect to x at $x = c$. It is found

- numerically, by taking the *limit* of the average rate over the interval from c to x as x approaches c
- graphically, by finding the slope of the line tangent to the graph at $x = c$

Note that "with respect to x" implies that you are finding how fast y changes *as x changes*.

Preview: Definition of Limit

In Section 1-1, you saw that the average rate of change of the y-value of a function got closer and closer to some fixed number as the change in the x-value got closer and closer to zero. That fixed number is called the **limit** of the average rate as the change in x approaches zero. The following is a verbal definition of limit. The full meaning will become clearer to you as the course progresses.

Verbal Definition of Limit

L is the limit of $f(x)$ as x approaches c
if and only if
L is the *one* number you can keep $f(x)$ arbitrarily close to
just by keeping x close enough to c, but not equal to c.

Problem Set 1-2

Quick Review

From now on, there will be ten short problems at the beginning of most problem sets. Some of the problems will help you review skills from previous sections or chapters. Others will test your general knowledge. Speed is the key here, not detailed work. You should be able to do all ten problems in less than five minutes.

Q1. Name the type of function: $f(x) = x^3$.

Q2. Find $f(2)$ for the function in Problem Q1.

Q3. Name the type of function: $g(x) = 3^x$.

Q4. Find $g(2)$ for the function in Problem Q3.

Q5. Sketch the graph: $h(x) = x^2$.

Q6. Find $h(5)$ for the function in Problem Q5.

Q7. Write the general equation for a quadratic function.

Q8. Write the particular equation for the function in Figure 1-2g.

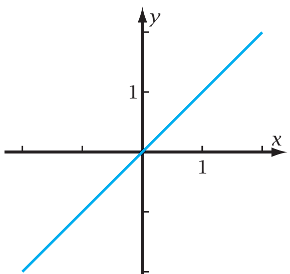


Figure 1-2g

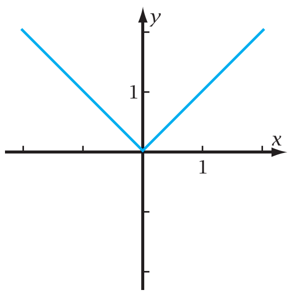


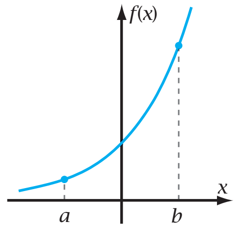
Figure 1-2h

Q9. Write the particular equation for the function in Figure 1-2h.

Q10. What name is given to the instantaneous rate of change of a function?

Problems 1–10 show graphs of functions with values of x marked a, b, and so on. At each marked value, state whether the function is increasing, decreasing, or neither as x increases from left to right, and also whether the rate of increase or decrease is fast or slow.

1.



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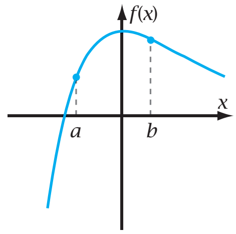

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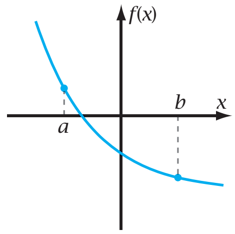

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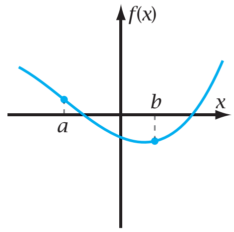
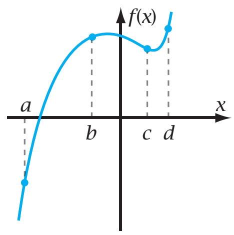


5.



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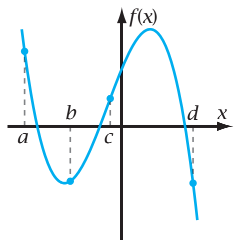

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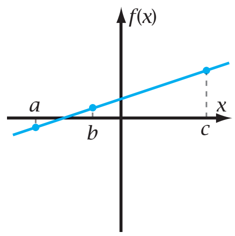

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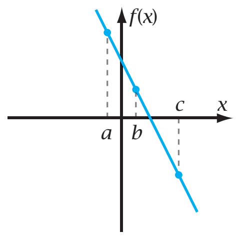

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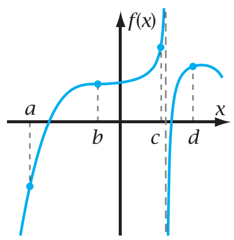

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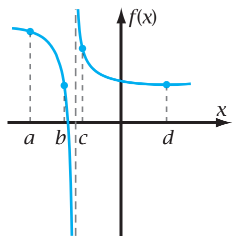


11. *Boiling Water Problem:* Figure 1-2i shows the temperature, $T(x)$, in degrees Celsius, of a kettle of water at time x, in seconds, since the burner was turned on.

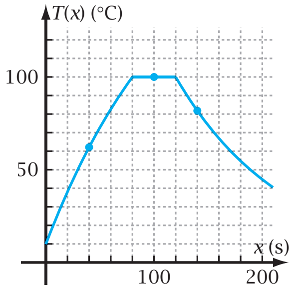


Figure 1-2i

a. On a copy of the figure, draw tangent lines at the points where $x = 40$, 100, and 140. Use the tangent lines to estimate the instantaneous rate of change of temperature at these times.

b. What do you suppose is happening to the water for $0 < x < 80$? For $80 < x < 120$? For $x > 120$?

12. *Roller Coaster Velocity Problem:* Figure 1-2j shows the velocity, $v(x)$, in ft/s, of a roller coaster car at time x, in seconds, after it starts down the first hill.

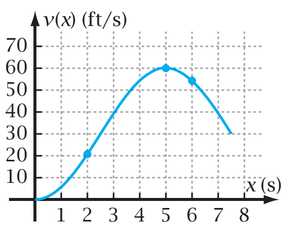


Figure 1-2j

a. On a copy of the figure, draw tangent lines at the points where $x = 2$, 5, and 6. Use the tangent lines to estimate the instantaneous rate of change of velocity at these times.

b. The instantaneous rates in part a are derivatives of $v(x)$ with respect to x. What units must you include in your answers? What physical quantity is this?

13. *Rock in the Air Problem:* A small rock is tied to an inflated balloon, then the rock and balloon are thrown into the air. While the rock and balloon are moving, the height of the rock is given by

$$h(x) = -x^2 + 8x + 2$$

where $h(x)$ is in feet above the ground at time x, in seconds, after the rock was thrown.

a. Plot the graph of function h. Sketch the result. Based on the graph, is $h(x)$ increasing or decreasing at $x = 3$? At $x = 7$?

b. How high is the rock at $x = 3$? At $x = 3.1$? What is the average rate of change of its height from 3 to 3.1 seconds?

c. Find the average rate of change from 3 to 3.01 seconds, and from 3 to 3.001 seconds. Based on the answers, what limit does the average rate seem to be approaching as the time interval gets shorter and shorter?

d. The limit of the average rates in part c is called the *instantaneous* rate at $x = 3$. It is also called the *derivative* of $h(x)$ at $x = 3$. Estimate the derivative of $h(x)$ at $x = 7$. Make sure to include units in your answer. Why is the derivative *negative* at $x = 7$?

14. *Fox Population Problem:* The population of foxes in a particular region varies periodically due to fluctuating food supplies. Assume that the number of foxes, $f(t)$, is given by

$$f(t) = 300 + 200 \sin t$$

where t is time in years after a certain date.

a. Store the equation for $f(t)$ as y_1 in your grapher, and plot the graph using a window with [0, 10] for t. Sketch the graph. On the sketch, show a point where $f(t)$ is increasing, a point where it is decreasing, and a point where it is not changing much.

b. The change in $f(t)$ from 1 year to t is $(f(t) - f(1))$. So for the time interval $[1, t]$, $f(t)$ changes at the average rate $r(t)$ given by

$$r(t) = \frac{f(t) - f(1)}{t - 1}$$

Enter $r(t)$ as y_2 in your grapher. Then make a table of values of $r(t)$ for each 0.01 year from 0.97 through 1.03.

c. The instantaneous rate of change of $f(t)$ at $t = 1$ is the limit $f(t)$ approaches as t

approaches 1. Explain why your grapher gives an error message if you try to calculate $r(1)$. Find an estimate for the instantaneous rate by taking values of t closer and closer to 1. What special name is given to this instantaneous rate?

d. At approximately what instantaneous rate is the fox population changing at $t = 4$? Explain why the answer is negative.

15. *Bacteria Culture Problem:* Bacteria in a laboratory culture are multiplying in such a way that the surface area of the culture, $a(t)$, in mm^2, is given by

$$a(t) = 200(1.2^t)$$

where t is the number of hours since the culture was started.

a. Find the average rate of increase of bacteria from $t = 2$ to $t = 2.1$.

b. Write an equation for $r(t)$, the average rate of change of $a(t)$, from 2 hours to t. Plot the graph of r using a friendly window that includes $t = 2$ as a grid point. What do you notice when you trace the graph of r to $t = 2$?

c. The instantaneous rate of change (the derivative) of $a(t)$ at $t = 2$ is $52.508608...\text{ mm}^2/\text{h}$. How close to this value is $r(2.01)$? How close must t be kept to 2 on the positive side so that the average rate is within 0.01 unit of this derivative?

16. *Sphere Volume Problem:* Recall from geometry that the volume of a sphere is

$$V(x) = \frac{4}{3}\pi x^3$$

where $V(x)$ is volume in cubic centimeters and x is the radius in centimeters.

a. Find $V(6)$. Write the answer as a multiple of π.

b. Find the average rate of change of $V(x)$ from $x = 6$ to $x = 6.1$. Find the average rate from $x = 5.9$ to $x = 6$. Use the answers to find an estimate of the instantaneous rate at $x = 6$.

c. Write an equation for $r(x)$, the average rate of change of $V(x)$ from 6 to x. Plot the graph of r using a friendly window that has $x = 6$ as a grid point. What do you notice when you trace the graph to $x = 6$?

d. The derivative of $V(x)$ at $x = 6$ equals $4\pi 6^2$, the surface area of a sphere of radius 6 cm. How close is $r(6.1)$ to this derivative? How close to 6 on the positive side must the radius be kept for $r(x)$ to be within 0.01 unit of this derivative?

17. *Rolling Tire Problem:* A pebble is stuck in the tread of a car tire (Figure 1-2k). As the wheel turns, the distance, y, in inches, between the pebble and the road at various times, t, in seconds, is given by the table below.

t (s)	**y (in.)**
1.2	0.63
1.3	0.54
1.4	0.45
1.5	0.34
1.6	0.22
1.7	0.00
1.8	0.22
1.9	0.34
2.0	0.45

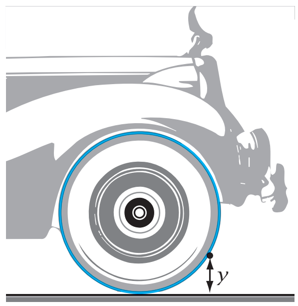


Figure 1-2k

a. About how fast is y changing at each time?

i. $t = 1.4$

ii. $t = 1.7$

iii. $t = 1.9$

b. At what time does the stone strike the pavement? Justify your answer.

18. *Flat Tire Problem:* A tire is punctured by a nail. As the air leaks out, the distance, y, in inches, between the rim and the pavement (Figure 1-2l) depends on the time, t, in minutes, since the tire was punctured. Values of t and y are given in the table below.

t (min)	**y (in.)**
0	6.00
2	4.88
4	4.42
6	4.06
8	3.76
10	3.50
12	3.26
14	3.04
16	2.84

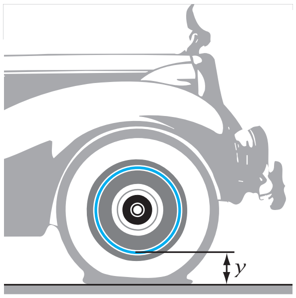


Figure 1-2l

a. About how fast is y changing at each time?

i. $t = 2$ ii. $t = 8$ iii. $t = 14$

b. How do you interpret the *sign* of the rate at which y is changing?

For Problems 19–28,

a. Give the type of function (linear, quadratic, and so on).

b. State whether $f(x)$ is increasing or decreasing at $x = c$, and how you know this.

19. $f(x) = x^2 + 5x + 6, c = 3$

20. $f(x) = -x^2 + 8x + 5, c = 1$

21. $f(x) = 3^x, c = 2$

22. $f(x) = 2^x, c = -3$

23. $f(x) = \dfrac{1}{x - 5}, c = 4$

24. $f(x) = -\dfrac{1}{x}, c = -2$

25. $f(x) = -3x + 7, c = 5$

26. $f(x) = 0.2x - 5, c = 8$

27. $f(x) = \sin x, c = 2$ (Radian mode!)

28. $f(x) = \cos x, c = 1$ (Radian mode!)

29. *Derivative Meaning Problem:* What is the physical meaning of the derivative of a function? How can you estimate the derivative graphically? Numerically? How does the numerical computation of a derivative illustrate the meaning of limit?

30. *Limit Meaning Problem:* From memory, write the verbal meaning of limit. Compare it with the statement in the text. If you did not state all parts correctly, try writing it again until you get it completely correct. How do the results of Problems 13 and 14 of this problem set illustrate the meaning of limit?

1-3 One Type of Integral of a Function

The title of this chapter is *Limits, Derivatives, Integrals, and Integrals*. In Section 1-2, you estimated the derivative of a function, which is the instantaneous rate of change of y with respect to x. In this section you will learn about one type of integral, the definite integral.

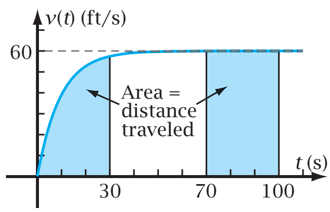


Figure 1-3a

Suppose you start driving your car. The velocity increases for a while, then levels off. Figure 1-3a shows the velocity, $v(t)$, increasing from zero, then approaching and leveling off at 60 ft/s.

In the 30 seconds between time $t = 70$ and $t = 100$, the velocity is a constant 60 ft/s. Because distance = rate × time, the distance you go in this time interval is

$$60 \text{ ft/s} \times 30 \text{ s} = 1800 \text{ ft}$$

Geometrically, 1800 is the area of the rectangle shown in Figure 1-3a. The width is 30 and the length is 60. Between 0 s and 30 s, where the velocity is changing, the area of the region under the graph also equals the distance traveled. Because the length varies, you cannot find the area simply by multiplying two numbers.

The process of evaluating a product in which one factor varies is called finding a **definite integral**. You can evaluate definite integrals by finding the corresponding area. In this section you will find the approximate area by counting squares on graph paper (by "brute force"!). Later, you will apply the concept of limit to calculate definite integrals *exactly*.

OBJECTIVE Given the equation or the graph for a function, estimate on a graph the definite integral of the function between $x = a$ and $x = b$ by counting squares.

If you are given only the equation, you can plot it with your grapher's grid-on feature, estimating the number of squares in this way. However, it is more accurate to use a plot on graph paper to count squares. You can get plotting data by using your grapher's TRACE or TABLE feature.

EXAMPLE 1 Estimate the definite integral of the exponential function $f(x) = 8(0.7)^x$ from $x = 1$ to $x = 7$.

Solution You can get reasonable accuracy by plotting $f(x)$ at each integer value of x (Figure 1-3b).

x	$f(x)$
0	$f(0) = 8$
1	$f(1) = 5.6$
2	$f(2) \approx 3.9$
3	$f(3) \approx 2.7$
4	$f(4) \approx 1.9$
5	$f(5) \approx 1.3$
6	$f(6) \approx 0.9$
7	$f(7) \approx 0.7$

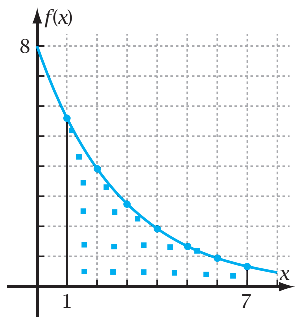


Figure 1-3b

The integral equals the area under the graph from $x = 1$ to $x = 7$. "Under" the graph means "between the graph and the x-axis." To find the area, first count the whole squares. Put a dot in each square as you count it to keep track, then estimate the area of each partial square to the nearest 0.1 unit. For instance, less than half a square is 0.1, 0.2, 0.3, or 0.4. You be the judge. You should get about 13.9 square units for the area, so the definite integral is approximately 13.9. Answers anywhere from 13.5 to 14.3 are reasonable. ◀

If the graph is already given, you need only count the squares. Be sure you know how much area each square represents! Example 2 shows you how to do this.

EXAMPLE 2 Figure 1-3c shows the graph of the velocity function $v(t) = -100t^2 + 90t + 14$, where t is in seconds and $v(t)$ is in feet per second. Estimate the definite integral of $v(t)$ with respect to t from $t = 0.1$ to $t = 1$.

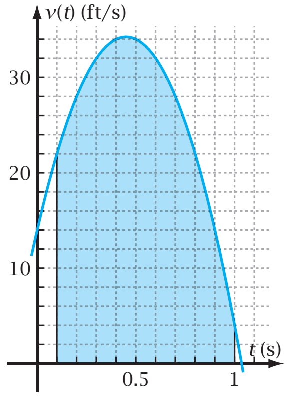


Figure 1-3c

Solution Notice that each space in the t-direction is 0.1 s and each space in the direction of $v(t)$ is 2 ft/s. Thus, each square represents (0.1)(2), or 0.2 ft. You should count about 119.2 squares for the area. So, the definite integral will be about

$$(119.2)(0.2) \approx 23.8 \text{ ft}$$

The following box gives the meaning of definite integral. The precise definition is given in Chapter 5, where you will learn an algebraic technique for calculating *exact* values of definite integrals.

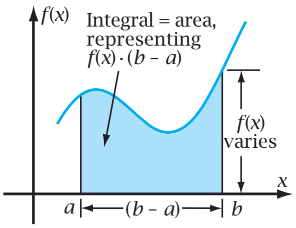


Figure 1-3d

Meaning of Definite Integral

The definite integral of the function f from $x = a$ to $x = b$ gives a way to find the product of $(b - a)$ and $f(x)$, even if $f(x)$ is not a constant. See Figure 1-3d.

Problem Set 1-3

Quick Review

5 min

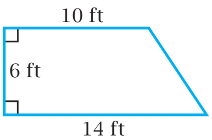


Figure 1-3e

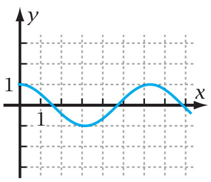


Figure 1-3f

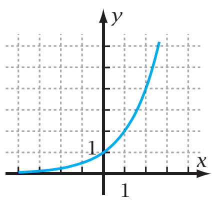


Figure 1-3g

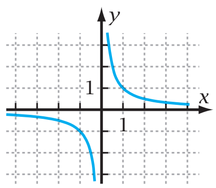


Figure 1-3h

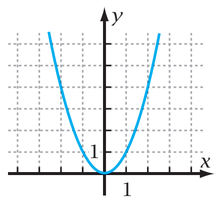


Figure 1-3i

Q1. Find the area of the trapezoid in Figure 1-3e.

Q2. Write the particular equation for the function graphed in Figure 1-3f.

Q3. Write the particular equation for the function graphed in Figure 1-3g.

Q4. Write the particular equation for the function graphed in Figure 1-3h.

Q5. Write the particular equation for the function graphed in Figure 1-3i.

Q6. Find $f(5)$ if $f(x) = x - 1$.

Q7. Sketch the graph of a linear function with positive y-intercept and negative slope.

Q8. Sketch the graph of a quadratic function opening downward.

Q9. Sketch the graph of a decreasing exponential function.

Q10. At what value(s) of x is $f(x) = (x - 4)/(x - 3)$ undefined?

For Problems 1–4, estimate the definite integral by counting squares on a graph.

1. $f(x) = -0.1x^2 + 7$
   a. $x = 0$ to $x = 5$
   b. $x = -1$ to $x = 6$
2. $f(x) = -0.2x^2 + 8$
   a. $x = 0$ to $x = 3$
   b. $x = -2$ to $x = 5$
3. $h(x) = \sin x$
   a. $x = 0$ to $x = \pi$
   b. $x = 0$ to $x = \pi/2$
4. $g(x) = 2^x + 5$
   a. $x = 1$ to $x = 2$
   b. $x = -1$ to $x = 1$
5. In Figure 1-3j, a car is slowing down from velocity $v = 60$ ft/s. Estimate the distance it travels from time $t = 5$ s to $t = 25$ s by finding the definite integral.

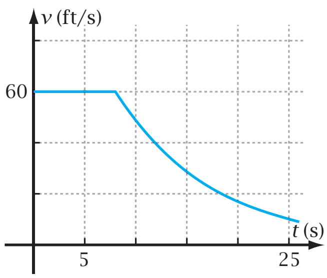


Figure 1-3j

6. In Figure 1-3k, a car slowly speeds up from $v = 55$ mi/h during a long trip. Estimate the distance it travels from time $t = 0$ h to $t = 4$ h by finding the definite integral.

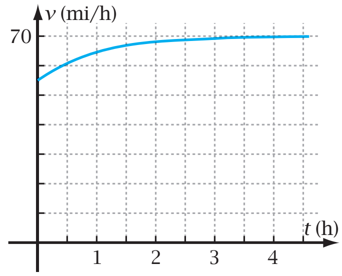


Figure 1-3k

For Problems 7 and 8, estimate the derivative of the function at the given value of x.

7. $f(x) = \tan x$, $x = 1$
8. $h(x) = -7x + 100$, $x = 5$
9. *Electric Car Problem:* You have been hired by an automobile manufacturer to analyze the predicted motion of a new electric car they are building. When accelerated hard from a standing start, the velocity of the car, $v(t)$, in ft/s, is expected to vary exponentially with time, t, in seconds, according to the equation

   $v(t) = 50(1 - 0.9^t)$

   a. Plot the graph of function v in the domain [0, 10]. What is the corresponding range of the function?
   b. Approximately how many seconds will it take the car to reach a velocity of 30 ft/s?
   c. Approximately how far will the car have traveled when it reaches 30 ft/s? Which of the four concepts of calculus is used to find this distance?
   d. At approximately what rate is the velocity changing when $t = 5$? Which of the four concepts of calculus is used to find this rate? What is the physical meaning of the rate of change of velocity?

10. *Slide Problem:* Phoebe sits atop the swimming pool slide (Figure 1-3l). At time $t = 0$ s she pushes off. Calvin finds that her velocity, $v(t)$, in ft/s, is given by

$$v(t) = 10 \sin 0.3t$$

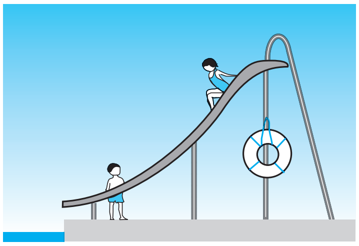


Figure 1-3l

Phoebe splashes into the water at time $t = 4$ s.

a. Plot the graph of function v. Use radian mode.

b. How fast was Phoebe going when she hit the water? What, then, are the domain and range of the velocity function?

c. Find, approximately, the definite integral of the velocity function from $t = 0$ to $t = 4$. What are the units of the integral? What real-world quantity does this integral give you?

d. What, approximately, was the derivative of the velocity function when $t = 3$? What are the units of the derivative? What is the physical meaning of the derivative in this case?

11. *Negative Velocity Problem:* Velocity differs from speed in that it can be *negative*. If the velocity of a moving object is negative, then its distance from its starting point is *decreasing* as time increases. The graph in Figure 1-3m shows $v(t)$, in cm/s, as a function of t, in seconds, after its motion started. How far is the object from its starting point when $t = 9$?

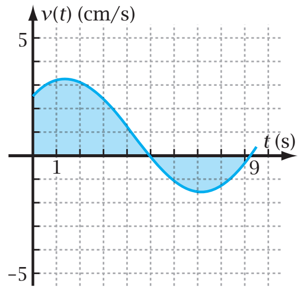


Figure 1-3m

12. Write the meaning of derivative.

13. Write the meaning of definite integral.

14. Write the verbal definition of limit.

1-4 Definite Integrals by Trapezoids, from Equations and Data

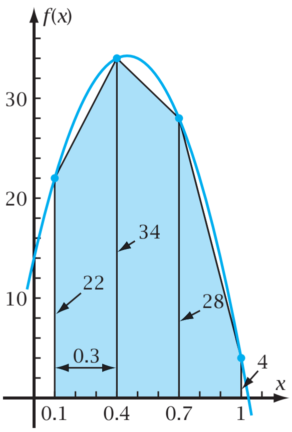


Figure 1-4a

In Section 1-3, you learned that the definite integral of a function is the product of x- and y-values, where the y-values may be different for various values of x. Because the integral is represented by the area of a region under the graph, you were able to estimate it by counting squares. In this section you will learn a more efficient way of estimating definite integrals.

Figure 1-4a shows the graph of

$$f(x) = -100x^2 + 90x + 14$$

which is the function in Example 2 of Section 1-3 using $f(x)$ instead of $v(t)$. Instead of counting squares to find the area of the region under the graph, the region is divided into vertical strips. Line segments connect the points where the strip boundaries meet the graph. The result is a set of trapezoids whose areas add up to a number approximately equal to the area of the region.

Recall from geometry that the area of a trapezoid is the altitude times the average of the lengths of the parallel sides. Figure 1-4a shows that for three trapezoids, the parallel sides are $f(0.1) = 22$, $f(0.4) = 34$, $f(0.7) = 28$, and $f(1) = 4$. The "altitude" of each trapezoid is the change in x, which equals 0.3 in this case. Using T_3 to represent the sum of the areas of the three trapezoids,

$$T_3 = \tfrac{1}{2}(22 + 34)(0.3) + \tfrac{1}{2}(34 + 28)(0.3) + \tfrac{1}{2}(28 + 4)(0.3) = 22.5$$

which is approximately equal to the definite integral. The answer is slightly smaller than the 23.8 found by counting squares in Example 2 of Section 1-3 because the trapezoids are inscribed, leaving out small parts of the region. You can make the approximation more accurate by using more trapezoids. You can also do the procedure numerically instead of graphically.

OBJECTIVE Estimate the value of a definite integral by dividing the region under the graph into trapezoids and summing the areas.

To accomplish the objective in a time-efficient way, observe that each y-value in the sum appears *twice,* except for the first and the last values. Factoring out 0.3 leads to

$$T_3 = 0.3\left[\tfrac{1}{2}(22) + 34 + 28 + \tfrac{1}{2}(4)\right] = 22.5$$

Inside the brackets is the sum of the y-values at the boundaries of the vertical strips, using half of the first value and half of the last value. (There is one more boundary than there are strips.) The answer is multiplied by the width of each strip.

EXAMPLE 1 Use trapezoids to estimate the definite integral of $f(x) = -100x^2 + 90x + 14$ from $x = 0.1$ to $x = 1$. Use 9 increments (that is, 9 strips, with 9 trapezoids).

Solution From $x = 0.1$ to $x = 1$, there is 0.9 x-unit. So the width of each strip is $0.9/9 = 0.1$. An efficient way to compute this is to make a list of the y-values in your grapher, taking half of the first value and half of the last value. Then sum the list.

$L_1 = x$	$L_2 = f(x)$	
0.1	11	Half of $f(0.1)$
0.2	28	
0.3	32	
0.4	34	
0.5	34	
0.6	32	
0.7	28	
0.8	22	
0.9	14	
1.0	2	Half of $f(1)$
	237	

$$\text{Integral} \approx T_9 = 0.1(237) = 23.7$$

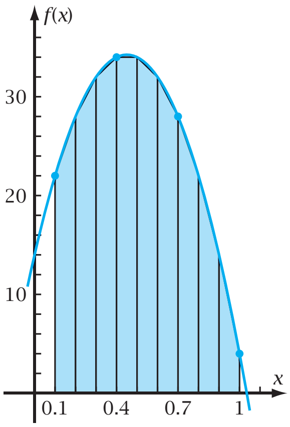


Figure 1-4b

Note that using more increments gives an answer closer to 23.8 than counting squares, which you used in Example 2 of Section 1-3. This is to be expected. As Figure 1-4b shows, using more trapezoids reduces the area of the region left out by the trapezoids.

You can generalize the preceeding example to any number of increments, n. The result is called the *trapezoidal rule*.

PROPERTY: The Trapezoidal Rule

The definite integral of $f(x)$ from $x = a$ to $x = b$ is approximately equal to

$$T_n = \Delta x \left(\tfrac{1}{2} f(a) + f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_{n-1}) + \tfrac{1}{2} f(b) \right)$$

where n is the number of increments (trapezoids), $\Delta x = (b - a)/n$ is the width of each increment, and the values of $x_1, x_2, x_3, \ldots$ are spaced Δx units apart.

Verbally: "Add the values of $f(x)$, taking half of the first value and half of the last value, then multiply by the width of each increment."

The exact value of the integral is the *limit* of the areas of the trapezoids as the number of increments becomes very large.

$$T_9 = 23.7$$
$$T_{20} = 23.819625$$
$$T_{100} = 23.8487850\ldots$$
$$T_{1000} = 23.84998785\ldots$$

The answers appear to be approaching 23.85 as n becomes very large. The exact value of the integral is the *limit* of the sum of the areas of the trapezoids as the number of trapezoids becomes infinitely large. In Chapter 5, you will learn how to calculate this limit algebraically.

PROPERTY: Exact Value of a Definite Integral

The exact value of a definite integral equals the limit of the trapezoidal rule sum T_n as n approaches infinity, provided the limit exists. The exact value can be estimated numerically by taking trapezoidal sums with more and more increments and seeing whether the sums approach a particular number.

The trapezoidal rule is advantageous if you must find the definite integral of a function specified by a table, rather than by equation. Example 2 shows you how to do this.

EXAMPLE 2 On a ship at sea, it is easier to measure how fast you are going than it is to measure how far you have gone. Suppose you are the navigator aboard a supertanker. The velocity of the ship is measured every 15 min and recorded in the table. Estimate the distance the ship has gone between 7:30 p.m. and 9:15 p.m.

Time	mi/h	Time	mi/h
7:30	28	8:30	7
7:45	25	8:45	10
8:00	20	9:00	21
8:15	22	9:15	26

Solution Figure 1-4c shows the given points. Because no information is known for times between the given ones, the simplest thing to assume is that the graph is a sequence of line segments. Because distance equals (miles/hour)(hours), the answer will equal a definite integral. You can find the integral from the area of the shaded region in Figure 1-4c, using the trapezoidal rule.

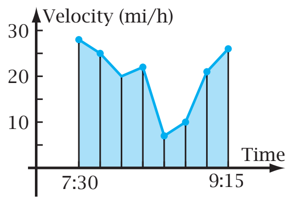


Figure 1-4c

$$T_7 = 0.25\left[\tfrac{1}{2}(28) + 25 + 20 + 22 + 7 + 10 + 21 + \tfrac{1}{2}(26)\right] = 33$$

Why 0.25? Why 7 increments?

$\therefore$ the distance is about 33 mi.

Problem Set 1-4

Quick Review

Q1. The value of y changes by 3 units when x changes by 0.1 unit. About how fast is y changing?

Q2. The value of y changes by -5 units when x changes by 0.01 unit. Approximately what does the derivative equal?

Q3. Sketch the graph of the absolute value function, $y = |x|$.

Q4. Find $f(3)$ if $f(x) = x^2$.

Q5. What is 50 divided by 1/2?

Q6. Evaluate: $\sin(\pi/2)$

Q7. How many days are there in a leap year?

Q8. The instantaneous rate of change of a function y is called the —?— of function y.

Q9. The product of x and y for a function is called the —?— of the function.

Q10. At what value(s) of x is $f(x) = (x - 4)/(x - 3)$ equal to zero?

1. *Spaceship Problem:* A spaceship is launched from Cape Canaveral. As the last stage of the rocket motor fires, the velocity is given by

   $$v(t) = 1600 \times 1.1^t$$

   where $v(t)$ is in feet per second and t is the number of seconds since the last stage started.

   a. Plot the graph of $v(t)$ versus t, from $t = 0$ to $t = 30$. Sketch trapezoids with parallel sides at 5-s intervals, extending from the t-axis to the graph.

b. Find, approximately, the definite integral of $v(t)$ with respect to t from $t = 0$ to $t = 30$ by summing the areas of the trapezoids. Will the sum overestimate the integral or underestimate it? How can you tell?

c. Based on the units of the definite integral, explain why it represents the distance the spaceship traveled in the 30-s interval.

d. To go into orbit around Earth, the spaceship must be traveling at least 27,000 ft/s. Will it be traveling this fast when $t = 30$? How can you tell?

2. *Walking Problem:* Pace Walker enters an AIDS walkathon. She starts off at 4 mi/h, speeds up as she warms up, then slows down as she gets tired. She estimates that her speed is given by

$$v(t) = 4 + \sin 1.4t$$

where t is the number of hours since she started and $v(t)$ is in miles per hour.

a. Pace walks for 3 h. Plot the graph of $v(t)$ as a function of t for these three hours. Sketch the result on your paper. (Be sure your calculator is in radian mode!)

b. Explain why a definite integral is used to find the distance Pace has gone in 3 h.

c. Estimate the integral in part b, using six trapezoids. Show these trapezoids on your graph. About how far did Pace walk in the 3 h?

d. How fast was Pace walking at the end of the 3 h? When did her maximum speed occur? What was her maximum speed?

3. *Aircraft Carrier Landing Problem:* Assume that as a plane comes in for a landing on an aircraft carrier, its velocity, in ft/s, takes on the values shown in the table. Find, approximately, how far the plane travels as it comes to a stop.

t (s)	**y (ft/s)**
0.0	300
0.6	230
1.2	150
1.8	90
2.4	40
3.0	0

In 1993, Kara Hultgreen became one of the first female pilots authorized to fly navy planes in combat.

4. *Water over the Dam Problem:* The amount of water that has flowed over the spillway on a dam can be estimated from the flow rate and the length of time the water has been flowing. Suppose that the flow rate has been recorded every 3 h for a 24-h period, as shown in the table. Estimate the number of cubic feet of water that has flowed over the dam in this period.

Time	**ft^3/h**	**Time**	**ft^3/h**
12:00 a.m.	5,000	12:00 p.m.	11,000
3:00 a.m.	8,000	3:00 p.m.	7,000
6:00 a.m.	12,000	6:00 p.m.	4,000
9:00 a.m.	13,000	9:00 p.m.	6,000
		12:00 a.m.	9,000

5. *Program for Trapezoidal Rule Problem:* Download or write a program for your grapher to evaluate definite integrals using the

trapezoidal rule. Store the equation as y_1. For the input, use a and b, the initial and final values of x, and n, the number of increments. The output should be the value of n that you used, and the approximate value of the integral. Test your program by using it to find T_3 from Example 1. Then find T_{20} and see if you get 23.819625.

6. *Program for Trapezoidal Rule from Data Problem:* Download or modify the program from Problem 5 to evaluate an integral, approximately, for a function specified by a table of data. Store the data points in L_1 on your grapher. For the input, use n, the number of increments, and Δx, the spacing between x-values. It is not necessary to input the actual x-values. Be aware that there will be $n + 1$ data points for n increments. Test your program by finding T_7 for the function in Example 2.

7. For the definite integral of $f(x) = -0.1x^2 + 7$ from $x = 1$ to $x = 4$,
   a. Sketch the region corresponding to the integral.
   b. Approximate the integral by finding T_{10}, T_{20}, and T_{50} using the trapezoidal rule. Do these values overestimate the integral or underestimate it? How do you know?
   c. The exact value of the integral is 18.9. How close do T_{10}, T_{20}, and T_{50} come to this value? How many increments, n, do you need to use until T_n is first within 0.01 unit of the limit? Give evidence to suggest that T_n is within 0.01 unit of 18.9 for all values of n greater than this.

8. For the definite integral of $g(x) = 2^x$ from $x = 1$ to $x = 3$,
   a. Sketch the region corresponding to the integral.
   b. Approximate the integral by finding T_{10}, T_{20}, and T_{50} using the trapezoidal rule. Do these values overestimate the integral or underestimate it? How do you know?
   c. The exact value of the integral is 8.65617024... . How close do T_{10}, T_{20}, and T_{50} come to this value? How many increments, n, do you need to use until T_n is first within 0.01 unit of the limit? Give evidence to suggest that T_n is within 0.01 unit of the exact value for all values of n greater than this.

9. *Elliptical Table Problem:* Figure 1-4d shows the top of a coffee table in the shape of an ellipse. The ellipse has the equation

$$\left(\frac{x}{110}\right)^2 + \left(\frac{y}{40}\right)^2 = 1$$

where x and y are in centimeters. Use the trapezoidal rule to estimate the area of the table. Will this estimate be too high or too low? Explain. What is the *exact* area of the ellipse?

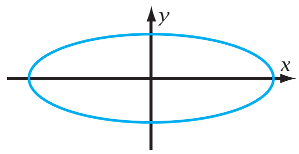


Figure 1-4d

10. *Football Problem:* The table shows the cross-sectional area, A, of a football at various distances, d, from one end. The distances are in inches and the areas are in square inches. Use the trapezoidal rule to find, approximately, the integral of area with respect to distance. What are the units of this integral? What, then, do you suppose the integral represents?

d (in.)	A (in.^{2})	d (in.)	A (in.^{2})
0	0.0	7	29.7
1	2.1	8	23.8
2	7.9	9	15.9
3	15.9	10	7.9
4	23.8	11	2.1
5	29.7	12	0.0
6	31.8		

11. *Exact Integral Conjecture Problem 1:* Now that you have a program to calculate definite integrals approximately, you can see what happens to the value of the integral as you use narrower trapezoids. Estimate the definite integral of $f(x) = x^2$ from $x = 1$ to $x = 4$, using 10, 100, and 1000 trapezoids. What number do the values seem to be approaching as the number of trapezoids gets larger and larger? Make a conjecture about the *exact* value of the definite integral as the width of each trapezoid

approaches zero. This number is the —?— of the areas of the trapezoids as their widths approach zero. What word goes in the blank?

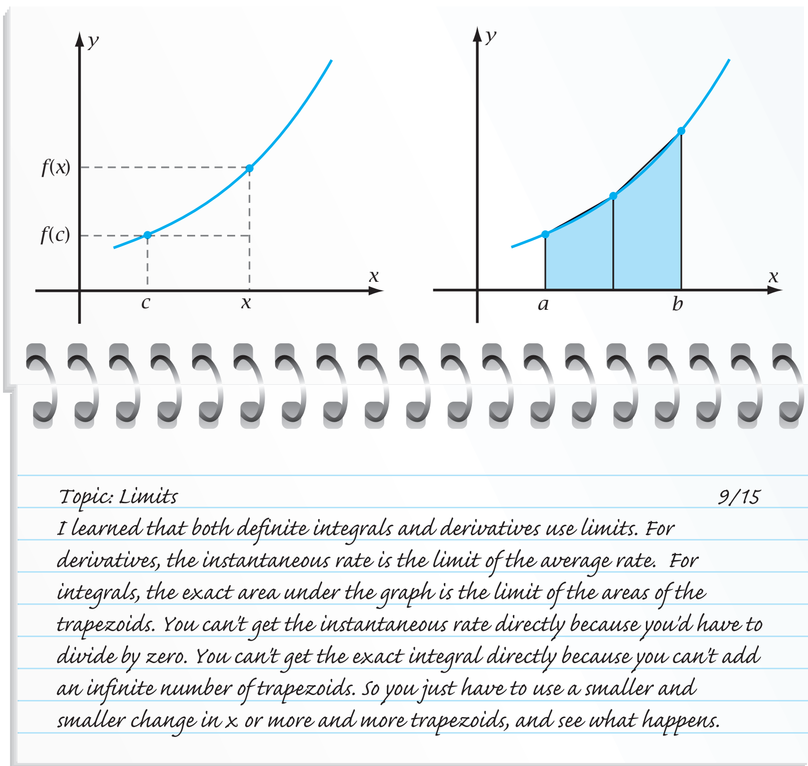
12. *Exact Integral Conjecture Problem 2:* The *exact* definite integral of $g(x) = x^3$ from $x = 1$ to $x = 5$ is an *integer*. Make a conjecture about what this integer is equal to. Justify your answer.

13. *Trapezoidal Rule Error Problem:* How can you tell whether the trapezoidal rule overestimates or underestimates an integral? Draw a sketch to justify your answer.

1-5 Calculus Journal

You have been learning calculus by reading, listening, and discussing, and also by working problems. An important ability to develop for any subject you study is the ability to *write* about it. To gain practice in this technique, you will be asked to record what you've been learning in a **journal**. (*Journal* comes from the same source as the French word *jour*, meaning "day." *Journey* comes from the same source and means "a day's travel.")

OBJECTIVE Start writing a journal in which you can record things you've learned about calculus and questions you still have about certain concepts.



Use a bound notebook or spiral-bound notebook with large index cards for pages. You can write narrative and equations on the lined side of the card and draw graphs on the facing blank side. A typical entry might look like the index card on the previous page.

Your journal should *not* be a simple transcription of your class notes. Nor should you take notes directly in it. Save it for concise summaries of things you have learned, conjectures that have been made, and topics about which you are not yet certain.

Problem Set 1-5

1. Start a journal in which you will record your understandings about calculus. The first entry should include such things as
   - The four concepts of calculus
   - The distinctions among derivative, definite integral, and limit
   - The fact that you still don't know what the other kind of integral is
   - The techniques you know for calculating derivatives, definite integrals, and limits
   - Any questions that still aren't clear in your mind

1-6 Chapter Review and Test

In this chapter you have had a brief introduction to the major concepts of calculus.

Limits

Derivatives

Definite integrals

Another type of integrals

The derivative of a function is its instantaneous rate of change. A definite integral of a function involves a product of the dependent and independent variables, such as (rate)(time). A limit is a number that y can be kept close to, just by keeping x suitably restricted. The other type of integral is called an *indefinite integral,* also known as an *antiderivative*. You will see why two different concepts use the word "integral" when you learn the fundamental theorem of calculus in Chapter 5.

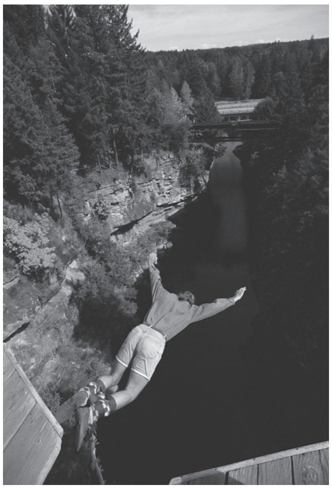
You have learned how to calculate approximate values of derivatives by dividing small changes in y by the corresponding change in x. Definite integrals can be found using areas under graphs and can thus be estimated by counting squares. Limits of functions can be calculated by finding the y-value of a removable discontinuity in the graph. Along the way you have refreshed your memory about the shapes of certain graphs.

The Review Problems are numbered according to the sections of this chapter. Answers are provided at the back of the book. The Concept Problems allow you to apply your knowledge to new situations. Answers are not provided, and in later chapters you may be required to do research to find answers to open-ended problems. The Chapter Test resembles a typical classroom test your instructor might give you. It has a calculator part and a no-calculator part, and the answers are not provided.

Review Problems

R1. *Bungee Problem:* Lee Per attaches himself to a strong bungee cord and jumps off a bridge. At time $t = 3$ s, the cord first becomes taut. From that time on, Lee's distance, d, in feet, from the river below the bridge is given by the equation

$$d = 90 - 80 \sin[1.2(t-3)]$$



a. How far is Lee from the water when $t = 4$?

b. Find the average rate of change of d with respect to t for the interval $t = 3.9$ to $t = 4$, and for the interval $t = 4$ to $t = 4.1$. Approximately what is the instantaneous rate of change at $t = 4$? Is Lee going up or going down at time $t = 4$? Explain.

c. Estimate the instantaneous rate of change of d with respect to t when $t = 5$.

d. Is Lee going up or down when $t = 5$? How fast is he going?

e. Which concept of calculus is the instantaneous rate of change?

R2. a. What is the physical meaning of the derivative of a function? What is the graphical meaning?

b. For the function in Figure 1-6a, explain how $f(x)$ is changing (increasing or decreasing, quickly or slowly) when x equals -4, 1, 3, and 5.

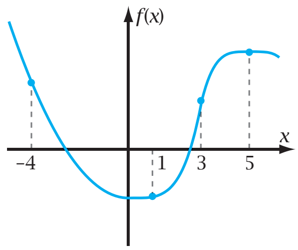


Figure 1-6a

c. If $f(x) = 5^x$, find the average rate of change of $f(x)$ from $x = 2$ to $x = 2.1$, from $x = 2$ to $x = 2.01$, and from $x = 2$ to $x = 2.001$. How close are these average rates to the instantaneous rate, 40.235947...? Do the average rates seem to be approaching this instantaneous rate as the second value of x approaches 2? Which concept of calculus is the instantaneous rate? Which concept of calculus is used to find the instantaneous rate?

d. Mary Thon runs 200 m in 26 s! Her distance, d, in meters from the start at various times t, in seconds, is given in the table. Estimate her instantaneous velocity in m/s when $t = 2$, $t = 18$, and $t = 24$. For which time intervals did her velocity stay relatively constant? Why is the velocity at $t = 24$

reasonable in relation to the velocities at other times?

t (s)	d (m)	t (s)	d (m)
0	0	14	89
2	7	16	103
4	13	18	119
6	33	20	138
8	47	22	154
10	61	24	176
12	75	26	200

R3. Izzy Sinkin winds up his toy boat and lets it run on the pond. Its velocity is given by

$$v(t) = (2t)(0.8^t)$$

as shown in Figure 1-6b. Find, approximately, the distance the boat travels between $t = 2$ and $t = 10$. Which concept of calculus is used to find this distance?

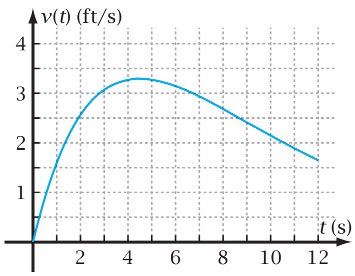


Figure 1-6b

R4. The graph in Figure 1-6c shows

$$f(x) = -0.5x^2 + 1.8x + 4$$

a. Plot the graph of f. Sketch your results. Does your graph agree with Figure 1-6c?

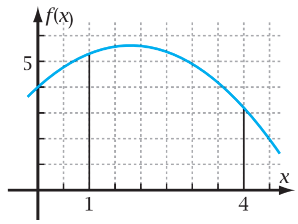


Figure 1-6c

b. Estimate the definite integral of $f(x)$ with respect to x from $x = 1$ to $x = 4$ by counting squares.

c. Estimate the integral in part b by drawing trapezoids each 0.5 unit of x and summing their areas. Does the trapezoidal sum overestimate the integral or underestimate it? How can you tell?

d. Use your trapezoidal rule program to estimate the integral using 50 increments and 100 increments. How close do T_{50} and T_{100} come to 15, the exact value of the integral? Do the trapezoidal sums seem to be getting closer to 15 as the number of increments increases? Which concept of calculus is used to determine this?

R5. In Section 1-5, you started a calculus journal. In what ways do you think keeping this journal will help you? How can you use the completed journal at the end of the course? What is your responsibility throughout the coming year to ensure that keeping the journal will be a worthwhile project?

Concept Problems

C1. *Exact Value of a Derivative Problem:* You have been calculating approximate values of derivatives by finding the change in y for a given change in x, then dividing. In this problem you will apply the concept of limit to the concept of derivative to find the *exact* value of a derivative. Let $y = f(x) = x^2 - 7x + 11$.

a. Find $f(3)$.

b. Suppose that x is slightly different from 3. Find an expression in terms of x for the amount by which y changes, $f(x) - f(3)$.

c. Divide the answer to part b by $x - 3$ to get an expression for the approximate rate of change of y. Simplify.

d. Find the limit of the fraction in part c as x approaches 3. The answer is the *exact* rate of change at $x = 3$.

C2. *Tangent to a Graph Problem:* If you worked Problem C1 correctly, you found that the instantaneous rate of change of $f(x)$ at $x = 3$ is exactly -1 y-unit per x-unit. Plot the graph of function f. On the same screen, plot a line through the point $(3, f(3))$ with slope -1. What do you notice about the line and the curve as you zoom in on the point $(3, f(3))$?

C3. *Formal Definition of Limit Problem:* In Chapter 2, you will learn that the formal definition of limit is

$L = \lim_{x \to c} f(x)$ if and only if

for any positive number epsilon, no matter how small

there is a positive number delta such that

if x is within delta units of c, but not equal to c,

then $f(x)$ is within epsilon units of L.

Notes: "$\lim_{x \to c} f(x)$" is read "the limit of $f(x)$ as x approaches c." Epsilon is the Greek lowercase letter ϵ. Delta is the Greek lowercase letter δ.

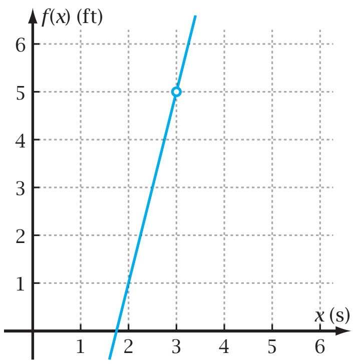


Figure 1-6d

Figure 1-6d shows the graph of the average velocity in ft/s for a moving object from 3 s to x s given by the function

$$f(x) = \frac{4x^2 - 19x + 21}{x - 3}$$

From the graph you can see that 5 is the limit of $f(x)$ as x approaches 3 (the instantaneous velocity at $x = 3$), but that $r(3)$ is undefined because of division by zero.

a. Show that the $(x - 3)$ in the denominator can be canceled out by first factoring the numerator, and that 5 is the value of the simplified expression when $x = 3$.

b. If $\epsilon = 0.8$ unit, on a copy of Figure 1-6d show the range of permissible values of $f(x)$ and the corresponding interval of x-values that will keep $f(x)$ within 0.8 unit of 5.

c. Calculate the value of δ to the right of 3 in part b by substituting $3 + \delta$ for x and 5.8 for $f(x)$, then solving for δ. Show that you get the same value of δ to the left of 3 by substituting $3 - \delta$ for x and 4.2 for $f(x)$.

d. Suppose you must keep $f(x)$ within ϵ units of 5, but you haven't been told the value of ϵ. Substitute $3 + \delta$ for x and $5 + \epsilon$ for $f(x)$. Solve for δ in terms of ϵ. Is it true that there is a positive value of δ for each positive value of ϵ, no matter how small, as required by the definition of limit?

e. In this problem, what are the values of L and c in the definition of limit? What is the reason for the clause ". . . but not equal to c" in the definition?

Chapter Test

PART 1: No calculators allowed (T1–T8)

T1. Write the four concepts of calculus.

T2. Write the verbal definition of limit.

T3. Write the physical meaning of derivative.

T4. Sketch the graph of a function that is increasing quickly at (2, 3) and decreasing slowly at (5, 6).

T5. Figure 1-6e shows the graph of the velocity, $v(t)$, in feet per second of a roller coaster as a function of time, t, in seconds since it started. Which concept of calculus is used to find the distance the roller coaster travels from $t = 0$ to $t = 35$? Estimate this distance graphically by counting squares.

T6. On a copy of Figure 1-6e, sketch trapezoids that you would use to find T_7, the distance in Problem T5 using the trapezoidal rule with 7 increments. Estimate T_7 by trapezoidal rule. Will T_7 overestimate or underestimate the actual distance? How do you know?

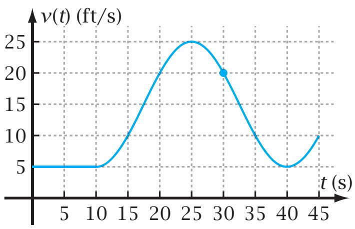


Figure 1-6e

T7. In Problem T5, which concept of calculus is used to find the rate of change of velocity at the instant when $t = 30$? Estimate this rate graphically. Give the units of this rate of change, and the physical name for this quantity.

T8. At what time is the roller coaster in Problem T5 first at the bottom of a hill? How do you explain the fact that the graph is horizontal between $t = 0$ and $t = 10$?

PART 2: Graphing calculators allowed (T9–T18)

On the no-calculator part of this test, you estimated graphically the distance a roller coaster traveled between 0 s and 35 s. The equation is

$$y = 5, \quad \text{if } 0 \le x \le 10$$
$$y = 15 + 10\cos\frac{\pi}{15}(x - 25), \quad \text{if } 10 \le x \le 35$$

where x is in seconds and y is in ft/s. Use this information for Problems T9–T15.

T9. How far did the roller coaster go from $x = 0$ to $x = 10$?

T10. Use your trapezoidal rule program (radian mode) to estimate the integral of y with respect to x from $x = 10$ to $x = 35$ by finding the trapezoidal sums T_5, T_{50}, and T_{100}.

T11. The exact value of the integral in Problem T10 is 416.349667..., the *limit* of T_n as n approaches infinity. Give numerical evidence that T_n is getting closer to this limit as n increases.

T12. Find (without rounding) the average rate of change of y with respect to x from

$x = 30$ to $x = 31$

$x = 30$ to $x = 30.1$

$x = 30$ to $x = 30.01$

T13. Explain why the average rates in Problem T12 are *negative*.

T14. The instantaneous rate of change of y at 30 s is $-1.81379936...$, the *limit* of the average rates from 30 to x as x approaches 30. Find the difference between the average rate and this limit for the three values in Problem T12. How does the result confirm that the average rate is approaching the instantaneous rate as x approaches 30?

T15. About how close would you have to keep x to 30 (on the positive side) so that the average rates are within 0.01 unit of the limit given in Problem T14?

T16. Name the concept of calculus that means instantaneous rate of change.

T17. You can estimate derivatives numerically from tables of data. Estimate $f'(4)$ (read "f-prime of four"), the derivative of $f(x)$ at $x = 4$.

x	$f(x)$
3.4	24
3.7	29
4.0	31
4.3	35
4.7	42

T18. What did you learn from this test that you did not know before?