MECHANISM

OF

THE HEAVENS

BY

Mary Fairfax Greig Somerville

1780-1872

Second Edition

Edited by Russell McNeil

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2001

Mary Fairfax Greig Somerville
1780-1872

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TO

HENRY, LORD BROUGHAM AND VAUX,

LORD HIGH CHANCELLOR OF GREAT BRITAIN,

———

This Work, undertaken at His Lordship's request, is inscribed as a testimony of the Author's esteem and regard.

Although it has unavoidably exceeded the limits of the Publications of the Society for the Diffusion of Useful Knowledge, for which it was originally intended, his Lordship still thinks it may tend to promote the views of the Society in its present form. To concur with that Society in the diffusion of useful knowledge, would be the highest ambition of the Author,

MARY SOMERVILLE.

Royal Hospital, Chelsea,
21st July, 1831.

To my three children
Liam, Bronwyn and Rose Siubhan

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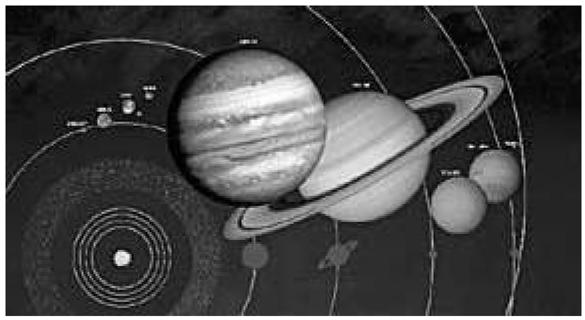
Solar System



The four planets closest to the Sun–Mercury, Venus, Earth, and Mars--are called the terrestrial planets because they have solid rocky surfaces. The four large planets beyond the orbit of Mars–Jupiter, Saturn, Uranus, and Neptune–are called gas giants. Tiny, distant, Pluto has a solid but icier surface than the terrestrial planets.

There are 67 natural satellites (also called moons) around the various planets in our solar system, ranging from bodies larger than our own Moon to small pieces of debris. Many of these were discovered by planetary spacecraft. Some of these have atmospheres (Saturn's Titan); some even have magnetic fields (Jupiter's Ganymede). Jupiter's moon Io is the most volcanically active body in the solar system. An ocean may lie beneath the frozen crust of Jupiter's moon Europa, while images of Jupiter's moon Ganymede show historical motion of icy crustal plates. Some planetary moons, such as Phoebe at Saturn may be asteroids that were captured by the planet's gravity.

From 1610 to 1977, Saturn was thought to be the only planet with rings. We now know that Jupiter, Uranus, and Neptune also have ring systems, although Saturn's is by far the largest. Particles in these ring systems range in size from dust to boulders to house sized, and may be rocky and/or icy.

Most of the planets also have magnetic fields which extend into space and form a "magnetosphere" around each planet. These magnetospheres rotate with the planet, sweeping charged particles with them. The Sun has a magnetic field, the heliosphere, which envelops our entire solar system. (Courtesy of NASA)

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The archivists, librarians and staff of the Bodleian Library, Oxford, were especially helpful and generous in answering many questions and providing free access to the Mary Somerville Collection and related documents. The more than 5,000 items in that Collection were sorted and catalogued by Elizabeth Chambers Patterson beginning in 1967. That archival work culminated in the publication in 1983 of her extraordinarily thorough *Mary Somerville and the Cultivation of Science 1815-1840*, an invaluable source for students of Mary Somerville.

The brief biographical summaries contained in this work are a synthesis of materials drawn from several sources. In addition to biographical materials published in the above-mentioned work, the Somerville Collection, and other sources listed in the *Basic Bibliography*, the writer is especially indebted to: the *MacTutor History of Mathematics Archive*, School of Mathematics and Statistics, University of St. Andrews, Scotland; and to *Encyclopædia Britannica*, and to *Britannica.com* and *Biography.com* for online materials.

Many of my colleagues at Malaspina University College have been very supportive throughout this project. I am especially grateful for the encouragement and interest provided by Dr. Deborah Hearn and Dr. William Weller in the Department of Physics, Engineering and Astronomy. I would also like to thank Mr. Ian Johnston (Department of English and Liberal Studies), Dr. Anne Leavitt (Liberal Studies and Philosophy), Dr. John Black (Liberal Studies and Philosophy), and Dr. Marni Stanley (English and Women's Studies).

Lastly, I would like to extend my deep gratitude to the members of the Malaspina University College Leave Committee for their approval and grant which enabled this project to go forward.

Russell McNeil, Ph.D.
September 1, 2001

FOREWORD TO THE SECOND EDITION

MARY Somerville (1780-1882)[1] wanted to produce a second edition (and indeed, a second volume) of this historically important work. She reveals in the first draft of her handwritten autobiography that it would be the *Mechanism of the Heavens* (1831), and nothing else for which future ages would remember her: "*All my other books will soon be forgotten, by this my name will be alone remembered…I heartily regret having written on popular science. The calculus was my strong point. I ought to have made a new edition of the "Mechanism of the Heavens"…*"[2] Somerville was in her 89th year when she penned these reflections. She understood that the *Mechanism of the Heavens*, written nearly four decades earlier, did more than introduce Laplace to the English speaking world. What was more important was the language Somerville chose to bring forth her rendition[3] (as Somerville always referred to her book) of the inspiration for Pierre Simon Laplace's "world formula" as expressed in his *Mécanique céleste*.[4] That language was the calculus in its highly evolved continental form, as developed initially by G. W. Leibniz[5] and brought to a high degree of perfection in its application to the problems of celestial mechanics by Euler,[6] Lacroix,[7] Lagrange,[8] Legendre,[9] Laplace,[10] and others. But the language of calculus did not flourish in the United Kingdom during the same period. As J. F. W. Herschel[11] remarks in his critique of Somerville's work in the *Quarterly Review*:[12] "*Whatever might be the causes* [of the decline of British science and mathematics] *however, it will hardly be denied by any one versed in this kind of reading, that the last twenty years of the eighteenth century were not more remarkable for the triumphs of both the pure and applied mathematics abroad, than for their decline, and, indeed, all but total extinction, at home*." In her autobiography Somerville identifies the reason for this decline as a "*reverence for Newton* [that] *had prevented scientific men from adopting the calculus which had enabled foreign mathematicians to carry astronomical and mechanical science to the highest perfection*."[13]

Somerville's work marked a significant turning point. As Herschel comments in his article in the *Reviews* section of this volume, a series of elementary texts designed to address this deficiency had been introduced to England during the first decades of the 19th century. And, as Somerville recalls in her autobiography, a letter she received from Professor Peacock on February 14, 1832 announced that, " '*Mr. Whewell and myself have already taken steps to introduce* [The *Mechanism of the Heavens*] *into the* [advanced mathematics] *Course of our studies at Cambridge, and I have little doubt that it will immediately become an essential work to those of our students who aspire to the highest places in our examinations.' Peacock,*[14] *Whewell*[15] *and Babbage*[16] *had only a few years earlier introduced the calculus as an essential branch of science at the University of Cambridge*."[17] Indeed, most of the 750 copies made for the first and only press run of the *Mechanism* were employed in the resuscitation of mathematics at the university that had taken the lead in reform and had the proudest mathematical tradition. The *Preliminary Dissertation* was printed separately both in England,[18] and as a pirate edition in the United States.[19] There are no records of the numbers of printed or sold copies of the independently produced *Preliminary Dissertation*.

While there was to be neither a second edition nor second volume of the *Mechanism of the Heavens* during her lifetime, Somerville did begin a second exercise in celestial mechanics shortly after finishing her first edition. As Herschel says in his review, topics not treated in depth in Somerville's work would be suited for a future project: "*The development of the theory of the tides, and the precession of the equinoxes, the attraction of spheroids and the figure of the earth, appear to be reserved for a second volume.*" Somerville indeed did leave an unpublished 408 page manuscript, *On the Figure of the Celestial Bodies,*[20] which may have been intended for that purpose. The idea for that manuscript had been suggested in an 1832 letter to Somerville[21] from the eminent French mathematician Siméon Poisson.[22]

Mary Somerville never regarded herself as an original thinker: "*I was conscious that I had made no discovery myself, that I had no originality. I have perseverance and intelligence but no genius, that spark from heaven is not granted to the sex, we are of the earth, earthy, whether higher powers may be allotted to us in another state of existence God knows, original genius in science at least is hopeless in this.*"[23] Ironically, it is in her popular writings–the works she "*regrets having written*"– that I find Somerville's most important historical contribution to astronomical science, and concrete evidence that belies her modest claim. In referring to the perturbations of the recently discovered Uranus, the outermost known planet when the *Mechanism of the Heavens* was published, Somerville makes this prediction based initially on an anomalous motion in the orbit of Uranus observed first by Alexis Bouvard (1767-1843) and noted in his tables published in 1821 (see note 11, *Bk. III, Chap. II*): "*Those of Uranus, however, are already defective, probably because the discovery of that planet in 1781 is too recent to admit of much precision in the determination of its motions, or that possibly it may be subject to disturbances from some unseen planet revolving about the sun beyond the present boundaries of our system. If, after a lapse of years, the tables formed from a combination of numerous observations should be still inadequate to represent the motions of Uranus, the discrepancies may reveal the existence, nay, even the mass and orbit, of a body placed for ever beyond the sphere of vision.*"[24] Four years after that 1842 prediction, astronomer John Adams[25] calculated the orbit of this unseen planet, Neptune. As Somerville's recalls in her autobiography, Adams acknowledged reading her prediction and it was this that led him to "*calculate the orbit of Neptune.*"[26] Somerville's confidence later extended to a second prediction. In subsequent editions of her *Connexion*[27] text she writes: "*The prediction may now be transferred from Uranus to Neptune, whose perturbations may reveal the existence of a planet still further removed, which may for ever remain beyond the reach of telescopic vision-yet its mass, the form and position of its orbit, and all the circumstances of its motion may become known, and the limits of the solar system may still be extended hundreds of millions of miles.*" The ninth planet, Pluto, remained undiscovered until 1930.[28]

After publication of the *Mechanism of the Heavens* Mary Somerville began to move in the highest scientific circles both in the United Kingdom and on the continent. Aside from the names mentioned above, a short list of distinguished contemporaries Somerville counted as peers, colleagues or acquaintances must also include:[29] Andre Ampère (1775-1836), Dominique Arago (1786-1853), Antoine Becquerel (1788-1878), Jean Biot (1774-1862), Sir David Brewster (1781-1868), Georges Cuvier (1769-1832), Charles Darwin (1809-1882), Michael Faraday (1791-1867), Joseph Gay-Lussac (1778-

1850), Sir William Hamilton (1805-1865), Joseph Henry (1797-1897), Caroline Herschel (1750-1848), Washington Irving (1783-1859), Lady Ada Byron Lovelace (1815-1852), Sir Charles Lyell (1797-1875), Harriet Martineau (1802-1876), James Clerk Maxwell (1831-1879), William Milne Edwards (c. 1776-1842), John Stuart Mill (1806-1873), Florence Nightingale (1820-1910), and Sir Charles Wheatstone (1802-1875).

How did a woman of modest means and with no formal training in mathematics achieve such recognition? The universities were closed to women–a brutal reality that Somerville always resented: "*From my earliest years my mind revolted against oppression and tyranny and resented the injustice of the world in denying those privileges of education which were denied to my sex which were so lavishly bestowed on men*."[30] For a time as a young lady Somerville pursued an interest in art under the direction of landscapist Alexander Nasmyth (1758-1840). A casual remark by Nasmyth set Somerville on the course of her life's work: "*…you should study Euclid's Elements of geometry, the foundations not only of perspective, but of astronomy and all mechanical science.*"[31] Somerville followed that advice and began to study on her own. While the pressures to conform to the social strictures of her day discouraged such interest–her father forbade her reading mathematics–Somerville persevered. After the death of her first husband in 1807, a chance meeting with Professor John Playfair (1748-1819),[32] a leading figure in Edinburgh mathematics, culminated in her introduction to, and a longstanding mentor relationship with, Edinburgh mathematician William Wallace.[33] Her exchanges with Wallace included studies of French mathematics and in particular Laplace's *Mécanique céleste*. It was during this period that Somerville, now in her late 20's, became part of the reform-minded Edinburgh intellectual scene[34] where she met some of the men associated with the liberal journal the *Edinburgh Review*. Somerville first encountered Henry Brougham[35] during this period. In 1827 Brougham approached her with a request to prepare an "account" of the *Mécanique céleste* for his newly established Society for the Diffusion of Useful Knowledge. The Society proposed to "*bring sound literature and self improvement within the reach of all by publishing cheap and worthy treatises*."[36] Although Somerville, now 47, had studied Laplace's work for 20 years, she accepted Brougham's request with reluctance. It took three years to complete her rendition. Unfortunately, the length of the final manuscript made it unsuitable for Brougham's popular series. After consultation with her longtime friend Sir John Herschel, she decided to publish the work independently.[37] The critical success of the first edition of *Mechanism of the Heavens*,[38] as documented in the *Reviews* section at the end of this volume, established Somerville's reputation as a brilliant scientific author. Her next book, *On the Connexion of the Physical Sciences*,[39] published in 1834, ran into ten editions, and sold over 15,000 copies. It was also translated into French, German and Italian, and a pirated copy was published in the United States.[40] Her other major work, *Physical Geography*,[41] first published in 1848, sold 16,000 copies in seven editions. Somerville began her last scientific work, *On Molecular and Microscopic Science*,[42] when she was 89, and completed the book shortly before her death at the age of 92.

This second edition of the *Mechanism of the Heavens* is designed to address not only its scarcity, but several deficiencies reflected in the first edition. More than 140 published errata were reported in the first edition. These are corrected in the second edition. In our review of the first edition at least twice as many unidentified printing errors were uncovered along with several page repeats, mislabeled chapters, and other errata. These have all been addressed and reflected in notes at the end of each chapter. But perhaps the most serious deficiency in the original work is one identified by J. Herschel in his critique at the end of this volume. Although lavish in his praise for Somerville's work, Herschel makes the following comment: "*…the most considerable fault we have to find with the work before us consists in an habitual laxity of language, evidently originating in so complete a familiarity with the quantities concerned, as to induce a disregard of the words by which they are designated, but which, to any one less intimately conversant with the actual analytical operations than its author, must have infallibly become a source of serious errors, and which at all events, renders it necessary for the reader to be constantly on his guard*."

This "*laxity of language*" criticism addresses a style reflected in the technical body of the work, but one not found in the *Preliminary Dissertation*. The *Dissertation* not only addresses a broader more general audience, it also reflects Somerville's lifelong curiosity and love of science and the "*mutual dependence and connection in many branches of science*."[43] Somerville carries this style and feeling for mutual dependence in her *Connexion of the Physical Sciences*. That work not only reflects its title in content, it *defines* the boundaries amongst the branches of the physical sciences (physical and descriptive astronomy, matter, sound, light, heat, and electricity and magnetism) at a time when such definitions were only beginning to emerge. The writing is clear, careful, and directed to the student of science.

James Clerk Maxwell,[44] the most influential scientist of the 19th century, cites Somerville's *Connexion* as one of those "*…suggestive books, which put into definite, intelligible and communicable form, the guiding ideas that are already working in the minds of men of science, so as to lead them to discoveries, but which they cannot yet shape into a definite statement*."[45] Over 100 pages of the *Connexion* covers material in celestial mechanics addressed in the *Mechanism* but in language more suited to the student. For that reason those topics in astronomy in her second book could serve, and do serve in this second edition, as introductory summaries for ideas and topics covered in the four books of the *Mechanism of the Heavens*.

Somerville says in her *Introduction* (p. 41), "*…the object of this work is rather to give the spirit of Laplace's method…*" I believe that the inclusion of Somerville's carefully crafted summaries, incorporated in this edition as forewords to each of her four books, not only conforms with Somerville's original objective, but also unifies the work stylistically, by carrying forward the enthusiasm embodied in the *Preliminary Dissertation* to the remainder of her work. The inclusion of this new material also addresses Herschel's concern about a "*laxity of language*." It should now be possible to capture "*the spirit of Laplace*" from Somerville's work by reading the *Preliminary Dissertation* together with the forewords to each of the four books, without recourse to the branches of higher mathematics.

The four books of the *Mechanism of the Heavens* address the topics of Dynamics, Universal Gravitation, Lunar Theory, and the Satellites. Except for the inclusion of the four forewords keyed to each of these books from materials drawn from the relevant sections of Somerville's *Connexion of the Physical Sciences* (10th edition, 1877), the addition of annotations (as notes placed at the end of each chapter so as not to disturb the integrity of the original work), short biographies of important figures referred to by Somerville in the text, the highlighting of articles and equation numbering, minor changes in the spacing of text and equations, spelling and punctuation, changes in pagination (which begins with the *Preliminary Dissertation* as part of the main text–the first edition uses roman numerals), and the correction of errata (as noted above), the structure of this second edition is identical to that of the first edition with respect to article and equation numbering, chapter and subsection headings, and the use of 116 figures (which have all been redrawn). Chapters II, III, and IV of Book IV were erroneously numbered VII, VIII and IX in the first edition. These have been renumbered to reflect the author's original intent. This volume also contains a Glossary of Symbols, a Basic Bibliography of key references, a Table of Contents, and a Name Index–none of which was incorporated in the first edition. The entries in the Subject Index (labeled "Index" in the first edition) are the same entries used by Somerville in the first edition, but refer to article numbers rather than page numbers. Finally, the name of the author, identified as "Mrs. Somerville" on the title page of the first edition, now reads "Mary Fairfax Greig Somerville."

Notes

[1] Somerville, Greig Fairfax Mary, (1780-1872), mathematician, born in Jedburgh and raised in Burntisland, Scotland. Mary was the daughter of Margaret Charters and vice-Admiral Sir William George Fairfax. Somerville married naval officer Samuel Greig in 1804. In Mary's words her first husband, "had a very low opinion of the capacity of my sex, and had neither knowledge of, nor interest in, science of any kind" (Martha Somerville, *Personal Recollections from Early Life to Old Age of Mary Somerville*, London, 1873). She married William Somerville in 1812 after the death of her first husband in 1807. William, an inspector of hospitals, was supportive of Mary's interest in science and played a leading role as her assistant. William and Mary lived in Edinburgh where she studied mathematics, botany, geology, French and Greek. Mary's circle of friends in Edinburgh included William Wallace (1768-1843), John Playfair (1748-1819), John Leslie (1766-1832), and Sir David Brewster (1781-1868). During this period Mary read Newton's *Principia* and Laplace's *Mécanique céleste*. After moving to London in 1816 Mary became acquainted with a range of leading figures in science including William Herschel (1738-1822), John Herschel (1792-1871), George Biddell Airy (1801-1892), George Peacock (1791-1858), and Charles Babbage (1791-1871). Through these acquaintances and in visits to Paris she met Jean-Baptiste Biot (1774-1862), Dominique Arago (1786-1853), Pierre-Simon Laplace (1749-1827), Siméon Poisson (1781-1840), Louis Poinsot (1777-1859) and Emile Mathieu (1835-1890). The many honours Somerville received included memberships in the Royal Astronomical Society, the Royal Irish Academy and the American and Italian Geographical Societies. She was also elected honorary Member of the Société de Physique et d'Histoire Naturelle de Genève. For her achievements she was awarded an annual pension of 200 pounds in 1834 (increased later to 300 pounds). In 1838 Mary and William moved to Italy, where she remained for the rest of her life. During her lifetime Mary wrote four significant scientific texts (*see notes 38-42 below*) and influenced many of the leading scientists of her day, including James Clerk Maxwell (1831-1879). In her writings Somerville predicted the existence of an unseen planet beyond the orbit of Uranus. John Adams (1819-1892) later calculated the exact position of the planet (Neptune) on the basis of Somerville's prediction (*See note 39, Bk. II, Foreword*). Somerville later predicted a ninth planet (Pluto), which remained undiscovered until 1930 (*see note 28 below*). Mary died in Naples in her ninety-second year on 29 November 1872. She is buried in the English Cemetery at Naples beneath a monument erected by her daughter Martha. Although informal consent from the Dean of Westminster Abbey was obtained for Mary's burial there, the formal request was denied by the then Astronomer Royal, who was not familiar with her works. Somerville Hall (now Somerville College) at Oxford University and the Mary Somerville scholarship in mathematics were established in 1879. (*Based on materials drawn from the School of Mathematics, University of St. Andrews, Scotland, and the references in notes 2, 29 and 34 below.*)

[2] Dep c.355, 22, MSAU-2: p.57, *Mary Somerville Autobiography (first draft)*, Mary Somerville Collection, Bodleian Library, Oxford University.
[3] A fully annotated five volume English *translation* of Laplace's work was undertaken between 1829-1839. (Bowditch, Nathaniel, (1773-1838), *Mécanique céleste. By the marquis de La Place ... Tr., with a commentary, by Nathaniel Bowditch*, Boston, Hillard, Gray, Little, and Wilkins, 1829-39.)
[4] Laplace, Pierre Simon, Marquis de, 1749-1827, mathematician and astronomer, born in Beaumont-en-Auge, France. Laplace was professor of mathematics at the Ecole Militaire, Paris. His five-volume *Mécanique céleste* (1799-1825) was considered the most important contribution to applied mathematics since Newton's *Principia*. In 1773 Laplace announced that the mean motions of the planetary motions were invariable in spite of perturbations. In 1786 he demonstrated the self-correcting nature of certain periodic planetary perturbations. In 1787 he removed what was the last theoretical threat to the stability of the earth-moon system by showing how the moon's acceleration depends upon eccentricity of the earth's orbit. The stability of the system impressed Laplace immensely and led to his famous and highly influential expression of a "world formula" stated in his *Essai philosophique sur les probabilités* (1814): "*A mind that in a given instance knew all the forces by which nature is animated and the position of all the bodies of which it is composed, if it were vast enough to include all these data within his analysis, could embrace in one single formula the movements of the largest bodies of the universe and the smallest atoms; nothing would be uncertain for him; the future and the past would be equally before his eyes.*" (Hayek, F.A. *The Counter Revolution of Science*, Liberty Fund, 2nd ed. p. 201, 1979.)
[5] Leibniz, Gottfried, Wilhelm, (1646-1716), philosopher and mathematician, born in Leipzig, Germany. Isaac Newton and Leibniz were involved in a bitter controversy over who first developed integral and differential calculus. Leibniz employed the now familiar notation used in calculus in a manuscript written in 1675. The first printed use of the "d" notation and the rules for differentiation appeared in the journal *Acta Eruditorum* in 1686. The first use in print of the $\int$ notation appeared in the same journal the following year. Newton's rival but equivalent method of "fluxions" was written much earlier, in 1671. However Newton's work did not appear in print until 1736. Leibniz is also considered the founder of dynamics, an approach in which kinetic energy is substituted for the conservation of movement or momentum. Leibniz also disputed Newton's idea of absolute space, advocating instead a complete relativism.
[6] Euler, Leonhard, 1707-1783, mathematician, born in Basel, Switzerland. Euler studied mathematics under Jean Bernoulli. Later he taught physics (1731) and mathematics (1733) at the St Petersburg Academy of Sciences. Euler published over 800 different books and papers on mathematics, physics and astronomy including his *Institutiones calculi differentialis* (1755) and *Institutiones calculi integralis* (1768-70). Euler made several important advances in integral calculus and in the theory of trigonometric and logarithmic functions. Euler also introduced much of the notation used in mathematics today, including the symbols Σ (sum), p, i for $\sqrt{-1}$ and e for the base of natural logarithms. Euler wrote works on the calculus of variations, the moon's motion and planetary orbits.
[7] See note 26, *Bk. I, Chap. II.*
[8] See note 16, *Preliminary Dissertation.*
[9] See note 60, *Bk. II. Chap. XIV.*
[10] See note 4.
[11] See note 64, *Preliminary Dissertation.*
[12] See the *Reviews* section at the end of this volume.
[13] Dep c.355, 22, MSAU-2: p. 57, *Mary Somerville Autobiography (first draft)*, Mary Somerville Collection, Bodleian Library, Oxford University.
[14] Peacock, George, (1791-1858), mathematician, born in Denton, England. In 1815, as an undergraduate at Cambridge, Peacock with John Herschel, and Charles Babbage established the Analytical Society with the goal of bringing advanced continental methods of analysis to Cambridge. The following year the Society produced a translation of a book on calculus by Lacroix. In 1817 Peacock became an examiner at Cambridge and Lowndean professor of astronomy and geometry (1836).
[15] Whewell, William, (1794-1866), scholar, born in Lancashire, England. Whewell held posts at Cambridge in mineralogy and moral theology. His works include his *History of the Inductive Sciences* (1837).
[16] Babbage, Charles, (1791-1871), mathematician, born in London, England. Babbage became Lucasian Professor of Mathematics at Cambridge in 1827, a post held originally by Newton and today (2000) by Stephen W. Hawking (1942-). Babbage is most remembered for his pioneering work on mechanical

computers. He constructed a "difference engine" in 1822, and in 1834 he completed the drawing for a more powerful "analytical engine," considered the prototype of the modern digital computer. The design included a capacity for memory storage and was intended to operate on modern programming principles by receiving instructions from punched cards. Although no operational version of this machine was ever constructed in his lifetime, the principles of its design were proven correct.

[17] Dep c.355, 22, MSAU-2: p. 165, *Mary Somerville Autobiography (first draft)*, Mary Somerville Collection, Bodleian Library, Oxford University.

[18] Somerville, Mary, *Preliminary Dissertation to 'The Mechanism of the Heavens'*, Clowes, London, 1831.

[19] Somerville, Mary, *A Preliminary Dissertation on the Mechanism of the Heavens*, Philadelphia, 1832.

[20] Dep b.207-8, *On the Figure of the Celestial Bodies*, Mary Somerville Collection, Bodleian Library, Oxford University.

[21] Dep c.355, 22, MSAU-2: p. 193, *Mary Somerville Autobiography (first draft)*, Mary Somerville Collection, Bodleian Library, Oxford University.

[22] See note 1, *Bk. I, Chap VI.*

[23] Dep c.355, 5, MSAU-3: p. 34, *Mary Somerville Autobiography (final draft)*, Mary Somerville Collection, Bodleian Library, Oxford University.

[24] See note 44, *Bk. I, Foreword.*

[25] See note 39, *Bk. II, Foreword.*

[26] Dep c.355, 22, MSAU-2: p. 222, *Mary Somerville Autobiography (first draft)*, Mary Somerville Collection, Bodleian Library, Oxford University.

[27] See note 39.

[28] As Somerville predicted, the planet Pluto was discovered, based on errors in the motions of Uranus and Neptune, in 1930 by Clyde W. Tombaugh (1906-) at Lowell Observatory in Arizona.

[29] Patterson, Elizabeth Chambers, *Mary Somerville and the Cultivation of Science*, International Archives of the History of Science, Martinus Nijhoff Pub., 1983. See also *Name Index* (p. 783) for short biographies.

[30] Dep c.355, 22, MSAU-2: p. 31, *Mary Somerville Autobiography (first draft)*, Mary Somerville Collection, Bodleian Library, Oxford University.

[31] *Op cit.*, p. 34.

[32] See note 17, *Preliminary Dissertation.*

[33] Wallace, William (1768-1843), mathematician, born in Dysart, Scotland. Wallace, like Somerville, was self taught. He was appointed professor of mathematics at Edinburgh University in 1819. He wrote two books including his *Geometrical Theorems and Analytical Formulae*. He also wrote articles on astronomy. Wallace and Somerville maintained a mathematical correspondence by mail.

[34] McKinley, Jane, *Mary Somerville 1780-1872*, Scotland Cultural Heritage, University of Edinburgh, 1987.

[35] Brougham, Henry Peter, Baron Brougham and Vaux, (1778-1868), jurist and politician, born in Edinburgh, Scotland. Brougham helped found the *Edinburgh Review.* As a peer he introduced several important reform measures. Brougham also established the *Society for the Diffusion of Useful Knowledge*.

[36] Patterson, Elizabeth Chambers, *Mary Somerville and the Cultivation of Science*, International Archives of the History of Science, Martinus Nijhoff Pub., p. 50, 1983.

[37] McKinley, Jane, *Mary Somerville 1780-1872*, Scotland Cultural Heritage, University of Edinburgh, p. 15, 1987.

[38] Somerville, Mrs., *Mechanism of the Heavens*, John Murray, London, 1831.

[39] Somerville, Mary, *On the Connexion of the Physical Sciences*, John Murray, London, 1834, 1835, 1836, 1837, 1840, 1842, 1846, 1849, 1858, 1977.

[40] Somerville, Mary, *On the Connexion of the Physical Sciences*, Philadelphia, 1834.

[41] Somerville, Mary, *Physical Geography*, John Murray, London, 1848, 1849, 1851, 1858, 1862, 1870, 1877.

[42] Somerville, Mary, *On Molecular and Microscopic Science*, John Murray, London, 1873, 1874.

[43] McKinley, Jane, *Mary Somerville 1780-1872*, Scotland Cultural Heritage, University of Edinburgh, p. 15, 1987.

[44] See note 5, *Bk. IV, Foreword.*

[45] Maxwell, James Clerk, 'Grove's Correlation of Physical Forces', *The Scientific Papers of J. Clerk Maxwell*, ii 401, Cambridge, 1890 .

GLOSSARY OF SYMBOLS & LIST OF IMAGES

(0.1), $\boxed{0.1}$	coefficients relative to action of m on m' (article 475)
A	apparent diameter
A_0, A_1, A_2	coefficients in expansion of R; lunar perturbation coefficients
$A_{/}^{(i,j)}$	coefficients in perturbation theory of Jupiter satellites i and j
A, B, C	moments of inertia
a	mean distance of planet m from the sun; half the greater axis of an ellipse
a_0, a_1, a_2	lunar perturbation coefficients (article 725)
$a_/, e_/, v_/$	$= -d\,\overline{a}, -d\,\overline{e}, -d\overline{v}$ (article 531)
$B_0, B_1, \ldots$	coefficients in expansion of R; lunar perturbation coefficients
B, B'	functions of K_i, L_i
b	distance
b_0, b_1, b_2	lunar perturbation coefficients (article 716)
$C, C', C_/$	arbitrary constant quantities
C_i	coefficients in the perturbations in radius vector; lunar perturbation coefficients (article 724)
c, c', c''	arbitrary constants
c	distance; the number 2.71828… whose hyperbolic logarithm is unity; quantity in the general form of $R = m'k\cos\{i'n't - int + c\}$
D	diameter; arbitrary constant quantity
D_i	coefficients in the perturbations in radius vector
D	perihelion distance of a comet
d	differential
E	mass of the earth
E_i	coefficients in the perturbations in radius vector
e	eccentricity (CS/CP in fig. 75)
F	resultant force
F'	resultant of forces F, F', F'', &c.
F_i	coefficients in the perturbations in longitude
f	function; centrifugal force
f, f', f''	distances; arbitrary constants
G	common centre of gravity of a planet and its satellites
G_i	coefficients in the perturbations in longitude
g	acceleration due to gravity
g, g_1, g_2	mean secular motions of the perihelia of m, m', m''; annual and sidereal motions of the apsides of the orbits of the four Jupiter satellites (article 831)

H	eccentricity of Jupiter's orbit (article 836)
H, H'	coefficients in theory of ethereal medium (article 790)
H_i	coefficients in the perturbations in longitude
h, h'	$e \sin v$, $e' \sin v'$
h, h_1, h_2, h_3	real eccentricities of the four Jupiter satellites (article 831)
I	inclination of the invariable plane (article 525)
i	integer; ratio of mean motion of planet to moon (article 780)
K, K'	coefficients in theory of ethereal medium (article 788)
K_i	coefficients in perturbations of radius vector
k	constant in Kepler's 3rd law $T^2 = k^2 a^3$; quantity in the general form of $R = m'k \cos\{i'n't - int + c\}$
L_i	coefficients in perturbations of radius vector
$L, L_/, -`L$	inclinations of Jupiter's equator and orbit on the fixed plane (article 869 & 872)
l, l'	$e \cos v$, $e' \cos v'$
l, l', l''	arbitrary constant quantities
l, l_1, l_2, l_3	inclinations of the orbits of the four Jupiter satellites (article 859)
$M_0, M_1, \ldots$	coefficients in expansion of R
m	mass
$m_/$	mass of any satellite (article 810) with co-ordinates x', y', z'
$N_0, N_1, \ldots$	coefficients in expansion of R
n	angular velocity $2p / T$ or $360^\circ / T$
nt	mean anomaly
o	origin of co-ordinate system
P	$P = R`A/D$; parallax; density of a shell of Jupiter's spheroid at a distance $\overline{R}$ from his centre; mass of a planet (article 780)
P, P'	functions of $Q_0, Q_1, \ldots$
p	(du/dx); pressure; quantity dependent on longitude of the nodes of the Jupiter satellites (article 861) in $\sin(v + pt + \Lambda)$
$`p$	quantity dependent on longitude of the nodes of the Jupiter satellites (article 872) in $-`L \sin(v + `pt + `\Lambda)$
$p, p_1, \ldots$	annual and sidereal motions of the nodes of the Jupiter satellites (article 868)
$Q_0, Q_1, \ldots$	coefficients in expansion of R
q	(du/dy)
r	(du/dz); radius vector
$r_/$	curtate distance (radius vector projected onto the plane of the ecliptic)
$d r$	periodic perturbation in radius vector of planet *m*
R	radius

$R\grave{}$	mean earth radius (article 743)
R	perturbation forces defined in article 347
R'	value of R when u, u', v, v', z and z' equal zero
$R_{/}$	surface resistance (re-action force); $R_{/} = \mathrm{d}R + d\,R + d\,'R$ (article 463); radius vector (article 780)
S, S'	values of $A_{0,}A_{1}$ when $s = +1/2$ and $s = -1/2$
S	mass of sun; heliocentric latitude (article 780)
S	sign of ordinary integrals
s	latitude of m in perturbed orbit above the fixed plane; $\tan f \sin(v - q)$; $q \sin v - p \cos v$; tangent of the moon's latitude (article 771)
$d\,s$	periodic perturbation in true latitude of planet *m*
T	period of a sidereal revolution of a planet *m*
t	time
U	longitude (article 780)
u	eccentric anomaly; vector from origin to particle m; angular velocity
V	linear velocity
v	true anomaly; true longitude of a planet *m*; velocity; angular velocity
$d\,v$	periodic perturbation in true longitude of planet *m*
$v_{/}$	true longitude projected onto the plane of the ecliptic
x_0, x_1, x_2	lunar perturbation coefficients (article 730)
$x,\ y,\ z$	spatial co-ordinates of particle *m*
x', y', z'	spatial co-ordinates of particle m' or $m_{/}$
$\overline{x},\ \overline{y},\ \overline{z}$	common centre of gravity of a system
$x_{/}, y_{/}, z_{/}$	co-ordinates of a planet in perihelio, designated $(x_{/}),(y_{/}),(z_{/})$ when $R = 0$
X, Y, Z	partial forces
a	a / a'; ratio of mean distance of planet from sun to mean distance of sun from earth (article 780)
b	$n' - nt + \in'$
$a\,, b\,, g$	angular measure
$\in$	longitude of the epoch; the mean place of a planet in its orbit at a given instant, assumed to be the origin of time; $\in$ mean longitude of planet *m*
$\in_{/}$	$\in$ referred to the plane of the ecliptic
g	equinoctial point; tangent of the inclination of the orbit of planet m' on the orbit of planet *m*; inclination of Jupiter's orbit on the fixed plane (article 863)
n	sine of the moon's declination
m	sum of the mass of the sun plus mass of a planet *S*+*m*
h	declination of a planet *m* relative to the sun's equator
r	density of planet *m*; ellipticity of the sun or earth
Ω	longitude of the ascending node of the invariable plane (see 525)
z	mean motion
$z\,, z_1, z_2$	mean longitudes of Jupiter's satellites (article 843)

y	the ratio of the centrifugal force to gravity at the solar equator; retrograde motion of the descending node of Jupiter's equator on the fixed plane
q	longitude of the ascending node; inclination of Jupiter's equator on the fixed plane (article 862)
q'	inclination of Jupiter's equator on his orbit (article 870)
f	inclination of orbit of planet *m* on the plane of the ecliptic; ratio of centrifugal force at the equator to gravity (article 771)
w	angular velocity; obliquity of the ecliptic (article 771)
v	longitude of the perihelion
x	longitude of the node estimated on the plane of the orbit; arbitrary quantity;
$x_i^{(j)}$	parameter in theory of Jupiter satellites $x_i^{(j)} = h_i / h_j$ (article 832)
l	function of $m, m', x, x', y, y', z, z'$, (see article 687); arbitrary quantity
$\Lambda, `\Lambda$	quantities dependent on longitude of the nodes of the Jupiter satellites (article 861 & 872) in $\sin(v + pt + \Lambda)$
Π	longitude of the line of intersection of the orbital planes of planets m and m'; longitude of the perihelion (article 836)
t	longitude of the ascending node of Jupiter's orbit (article 863)
$\Gamma, \Gamma_1, \ldots$	mean longitudes of the lower apsides of the orbits of the four Jupiter satellites at the epoch (article 833)

List of Images

PRELIMINARY DISSERTATION

IN order to convey some idea of the object of this work it may be useful to offer a few preliminary observations on the nature of the subject which it is intended to investigate, and of the means that have already been adopted with so much success to bring within the reach of our faculties, those truths which might seem to be placed so far beyond them.

All the knowledge we possess of external objects is founded upon experience, which furnishes a knowledge of facts, and the comparison of these facts establishes relations, from which, induction, the intuitive belief that like causes will produce like effects, leads us to general laws. Thus, experience teaches that bodies fall at the surface of the earth with an accelerated velocity, and proportional to their masses. Newton[1] proved, by comparison, that the force which occasions the fall of bodies at the earth's surface, is identical with that which retains the moon in her orbit; and induction led him to conclude that as the moon is kept in her orbit by the attraction of the earth, so the planets might be retained in their orbits by the attraction of the sun. By such steps he was led to the discovery of one of those powers with which the Creator has ordained that matter should reciprocally act upon matter.

Physical astronomy is the science which compares and identifies the laws of motion observed on earth with the motions that take place in the heavens, and which traces, by an uninterrupted chain of deduction from the great principle that governs the universe, the revolutions and rotations of the planets, and the oscillations of the fluids at their surfaces, and which estimates the changes the system has hitherto undergone or may hereafter experience, changes which require millions of years for their accomplishment.

The combined efforts of astronomers, from the earliest dawn of civilization, have been requisite to establish the mechanical theory of astronomy: the courses of the planets have been observed for ages with a degree of perseverance that is astonishing, if we consider the imperfection, and even the want of instruments. The real motions of the earth have been separated from the apparent motions of the planets; the laws of the planetary revolutions have been discovered; and the discovery of these laws has led to the knowledge of the gravitation of matter. On the other hand, descending from the principle of gravitation, every motion in the system of the world has been so completely explained, that no astronomical phenomenon can now be transmitted to posterity of which the laws have not been determined.

Science, regarded as the pursuit of truth, which can only be attained by patient and unprejudiced investigation, wherein nothing is too great to be attempted, nothing so minute as to be justly disregarded, must ever afford occupation of consummate interest and of elevated meditation. The contemplation of the worlds of creation elevates the mind to the admiration of whatever is great and noble, accomplishing the object of all study, which in the elegant language of Sir James Mackintosh[2] is to inspire the love of truth, of wisdom, of beauty, especially of goodness, the highest beauty, and of that supreme and eternal mind, which contains all truth and wisdom, all beauty and goodness. By the love or delightful contemplation and pursuit of these transcendent aims for their own sake only, the mind of man is raised from low and perishable objects, and prepared for those high destinies which are appointed for all those who are capable of them.

The heavens afford the most sublime subject of study which can be derived from science: the magnitude and splendour of the objects, the inconceivable rapidity with which they move, and the enormous distances between them, impress the mind with some notion of the energy that maintains them in their motions with a durability to which we can see no limits. Equally conspicuous is the goodness of the great First Cause in having endowed man with faculties by which he can not only appreciate the magnificence of his works, but trace, with precision, the operation of his laws, use the globe he inhabits as a base wherewith to measure the magnitude and distance of the sun and planets, and make the diameter of the earth's orbit the first step of a scale by which he may ascend to the starry firmament. Such pursuits, while they ennoble the mind, at the same time inculcate humility, by showing that there is a barrier, which no energy**,** mental or physical, can ever enable us to pass: that however profoundly we may penetrate the depths of space, there still remain innumerable systems, compared with which those which seem so mighty to us must dwindle into insignificance, or even become invisible; and that not only man, but the globe he inhabits, nay the whole system of which it forms so small a part, might be annihilated, and its extinction be unperceived in the immensity of creation.

A complete acquaintance with Physical Astronomy can only be attained by those who are well versed in the higher branches of mathematical and mechanical science: such alone can appreciate the extreme beauty of the results, and of the means by which these results are obtained. Nevertheless a sufficient skill in analysis to follow the general outline, to see the mutual dependence of the different parts of the system, and to comprehend by what means some of the most extraordinary conclusions have been arrived at, is within the reach of many who shrink from the task, appalled by difficulties, which perhaps are not more formidable than those incident to the study of the elements of every branch of knowledge, and possibly overrating them by not making a sufficient distinction between the degree of mathematical acquirement necessary for making discoveries, and that which is requisite for understanding what others have done. That the study of mathematics and their application to astronomy are full of interest will be allowed by all who have devoted their time and attention to these pursuits, and they only can estimate the delight of arriving at truth, whether it be in the discovery of a world, or of a new property of numbers.

It has been proved by Newton that a particle of matter placed without the surface of a hollow sphere is attracted by it as if its mass, or the whole matter it contains, were collected in its centre. The same is therefore true of a solid sphere which may be supposed to consist of an infinite number of concentric hollow spheres. This however is not the case with a spheroid, but the celestial bodies are so nearly spherical, and at such remote distances from each other, that they attract and are attracted as if each were a dense point situate in its centre of gravity, a circumstance which greatly facilitates the investigation of their motions.

The attraction of the earth on bodies at its surface in that latitude, the square of whose sine is 1/3, is the same as if it were a sphere; and experience shows that bodies there fall through 16.0697 feet in a second. The mean distance of the moon from the earth is about sixty times the mean radius of the earth. When the number 16.0697 is diminished in the ratio of 1 to 3,600, which is the square of the moon's distance from the earth, It is found to be exactly the space the moon would fall through in the first second of her descent to the earth, were she not prevented by her centrifugal force, arising from the velocity with which she moves in her orbit. So that the moon is retained in her orbit by a force having the same origin and regulated by the same law with that which causes a stone to fall at the earth's surface. The earth may therefore be regarded as the centre of a force which extends to the moon; but as experience shows that the action and reaction of matter are equal and contrary, the moon must attract the earth an equal and contrary force.

Newton proved that a body projected in space will move in a conic section, if it be attracted by a force directed towards a fixed point, and having an intensity inversely as the square of the distance; but that any deviation from that law will cause it to move in a curve of a different nature. Kepler[3] ascertained by direct observation that the planets describe ellipses round the sun, and later observations show that cornets also move in conic sections: it consequently follows that the sun attracts all the planets and comets inversely as the square of their distances from his centre; the sun therefore is the centre of a force extending indefinitely in space, and including all the bodies of the system in its action.

Kepler also deduced from observation, that the squares of the periodic times of the planets, or the times of their revolutions round the sun, are proportional to the cubes of their mean distances from his centre:[4] whence it follows, that the intensity of gravitation of all the bodies towards the sun is the same at equal distances; consequently gravitation is proportional to the masses, for if the planets and comets be supposed to be at equal distances from the sun and left to the effects of gravity, they would arrive at his surface at the same time. The satellites also gravitate to their primaries according to the same law that their primaries do to the sun. Hence, by the law of action and reaction, each body is itself the centre of an attractive force extending indefinitely in space, whence proceed all the mutual disturbances that render the celestial motions so complicated, and their investigation so difficult.

The gravitation of matter directed to a centre, and attracting directly as the mass, and inversely as the square of the distance, does not belong to it when taken in mass; particle acts on particle according to the same law when at sensible distances from each other. If the sun acted on the centre of the earth without attracting each of its particles, the tides would be very much greater than they now are, and in other respects they also would be very different. The gravitation of the earth to the sun results from the gravitation of all its particles, which in their turn attract the sun in the ratio of their respective masses. There is a reciprocal action likewise between the earth and every particle at its surface; were this not the case, and were any portion of the earth, however small, to attract another portion and not be itself attracted, the centre of gravity of the earth would be moved in space, which is impossible.

The form of the planets results from the reciprocal attraction of their component particles. A detached fluid mass, if at rest, would assume the form of a sphere, from the reciprocal attraction of its particles; but if the mass revolves about an axis it becomes flattened at the poles, and bulges at the equator, in consequence of the centrifugal force arising from the velocity of rotation. For, the centrifugal force diminishes the gravity of the particles at the equator, and equilibrium can only exist when these two forces are balanced by an increase of gravity; therefore, as the attractive force is the same on all particles at equal distances from the centre of a sphere, the equatorial particles would recede from the centre till their increase in number balanced the centrifugal force by their attraction, consequently the sphere would become an oblate spheroid; and a fluid partially or entirely covering a solid, as the ocean and atmosphere cover the earth, must assume that form in order to remain in equilibrio. The surface of the sea is therefore spheroidal, and the surface of the earth only deviates from that figure where it rises above or sinks below the level of the sea; but the deviation is so small that it is unimportant when compared with the magnitude of the earth. Such is the form of the earth and planets, but the compression and flattening at their poles is so small, that even Jupiter, whose rotation is the most rapid, differs but little from a sphere. Although the planets attract each other as if they were spheres on account of their immense distances, yet the satellites are near enough to be sensibly affected in their motions by the forms of their primaries. The moon for example is so near the earth, that the reciprocal attraction between each of her particles and each of the particles in the prominent mass at the terrestrial equator, occasions considerable disturbances in the motions of both bodies. For, the action of the moon on the matter at the earth's equator produces a nutation[5] in axis of rotation, and the reaction of that matter on the moon is the cause of a corresponding nutation in the lunar orbit.

If a sphere at rest in space receives an impulse passing through its centre of gravity, all its parts will move with an equal velocity in a straight line; but if the impulse does not pass through the centre of gravity, its particles having unequal velocities, will give it rotatory motion at the same time that it is translated in space. These motions are independent of one another, so that a contrary impulse passing through its centre of gravity will impede its progression, without interfering with its rotation. As the sun rotates about an axis it seems probable if an impulse in a contrary direction has not been given to his centre of gravity, that he moves in space accompanied by all those bodies which compose the solar system, a circumstance that would in no way interfere with their relative motions; for, in consequence of our experience that force is proportional to velocity, the reciprocal attractions of a system remain the same, whether its centre of gravity be at rest, or moving uniformly in space. It is computed that had the earth received its motion from a single impulse, such impulse must have passed through a point about twenty-five miles from its centre.

Since the motions of the rotation and translation of the planets are independent of each other, though probably communicated by the same impulse, they form separate subjects of investigation.

A planet moves in its elliptical orbit with a velocity varying every instant, in consequence of two forces, one tending to the centre of the sun, and the other in the direction of a tangent to its orbit, arising from the primitive impulse given at the time when it was launched into space: should the force in the tangent cease, the planet would fall to the sun by its gravity; were the sun not to attract it, the planet would fly off in the tangent. Thus, when a planet is in its aphelion[6] or at the point where the orbit is farthest from the sun, his action overcomes its velocity, and brings it towards him with such an accelerated motion, that it at last overcomes the sun's attraction, and shoots past him; then, gradually decreasing in velocity, it arrives at the aphelion where the sun's attraction again prevails. In this motion the radii vectores,[7] or imaginary lines joining the centers of the sun and planets, pass over equal areas in equal times.[8]

If the planets were attracted by the sun only, this would ever be their course; and because his action is proportional to his mass, which is immensely larger than that of all the planets put together, the elliptical is the nearest approximation to their true motions, which are extremely complicated, in consequence of their mutual attraction, so that they do not move in any known or symmetrical curve, but in paths now approaching to, and now receding from the elliptical form, and their radii vectores do not describe areas exactly proportional to the time. Thus the areas become a test of the existence of disturbing forces.

To determine the motion of each body when disturbed by all the rest is beyond the power of analysis; it is therefore necessary to estimate the disturbing action of one planet at a time, whence arises the celebrated problem of the three bodies, which originally was that of the moon, the earth, and the sun, namely,–the masses being given of three bodies projected from three given points, with velocities given both in quantity and direction; and supposing the bodies to gravitate to one another with forces that are directly as their masses, and inversely as the squares of the distances, to find the lines described by these bodies, and their position at any given instant.

By this problem the motions of translation of all the celestial bodies are determined. It is one of extreme difficulty, and would be of infinitely greater difficulty, if the disturbing action were not very small, when compared with the central force. As the disturbing influence of each body may be found separately, it is assumed that the action of the whole system in disturbing any one planet is equal to the sum of all the particular disturbances it experiences, on the general mechanical principle, that the sum of any number of small oscillations is nearly equal to their simultaneous and joint effect.

On account of the reciprocal action of matter, the stability of the system depends on the intensity of the primitive momentum of the planets, and the ratio of their masses to that of the sun: for the nature of the conic sections in which the celestial bodies move, depends on the velocity with which they were first propelled in space; had that velocity been such as to make the planets move in orbits of unstable equilibrium, their mutual attractions might have changed them into parabolas or even hyperbolas; so that the earth and planets might ages ago have been sweeping through the abyss of space: but as the orbits differ very little from circles, the momentum of the planets when projected, must have been exactly sufficient to ensure the permanency and stability of the system. Besides the mass of the sun is immensely greater than those of the planets; and as their inequalities bear the same ratio to their elliptical motions as their masses do to that of the sun, their mutual disturbances only increase or diminish the eccentricities of their orbits by very minute quantities; consequently the magnitude of the sun's mass is the principal cause of the stability of the system. There is not in the physical world a more splendid example of the adaptation of means to the accomplishment of the end, than is exhibited in the nice adjustment of these forces.

The orbits of the planets have a very small inclination to the plane of the ecliptic in which the earth moves; and on that account, astronomers refer their motions to it at a given epoch as a known and fixed position. The paths of the planets, when their mutual disturbances are omitted, are ellipses nearly approaching to circles, whose planes, slightly inclined to the ecliptic, cut it in straight lines passing through the centre of the sun; the points where the orbit intersects the plane of the ecliptic are its nodes.

The orbits of the recently discovered planets[9] deviate more from the ecliptic: that of Pallas[10] has an inclination of 35^{o} to it: on that account it will be more difficult to determine their motions. These little planets have no sensible effect in disturbing the rest, though their own motions are rendered very irregular by the proximity of Jupiter and Saturn.

The planets are subject to disturbances of two distinct kinds, both resulting from the constant operation of their reciprocal attraction, one kind depending upon their positions with regard to each other, begins from zero, increases to a maximum, decreases and becomes zero again, when the planets return to the same relative positions. In consequence of these, the troubled planet is sometimes drawn away from the sun, sometimes brought nearer to him; at one time it is drawn above the plane of its orbit, at another time below it, according to the position of the disturbing body. All such changes, being accomplished in short periods, some in a few months, others years, or in hundreds of years, are denominated Periodic Inequalities.

The inequalities of the other kind, though occasioned likewise by the disturbing energy of the planets, are entirely independent of their relative positions; they depend on the relative positions of the orbits alone, whose forms and places in space are altered by very minute quantities in immense periods of time, and are therefore called Secular Inequalities.

In consequence of disturbances of this kind, the apsides,[11] or extremities of the major axes of all the orbits, have a direct, but variable motion in space, excepting those of Venus, which are retrograde;[12] and the lines of the nodes move with a variable velocity in the contrary direction. The motions of both are extremely slow; it requires more than 109,770 years for the major axis of the earth's orbit to accomplish a sidereal[13] revolution, and 20,935 years to complete its tropical[14] motion. The major axis of Jupiter's orbit requires no less than 197,561 years to perform its revolution from the disturbing action of Saturn alone. The periods in which the nodes revolve are also very great. Beside these, the inclination and eccentricity of every orbit are in a state of perpetual, but slow change. At the present time, the inclinations of all the orbits are decreasing; but so slowly, that the inclination of Jupiter's orbit is only six minutes less now than it was in the age of Ptolemy.[15] The terrestrial eccentricity is decreasing at the rate of 3,914 miles in a century; and if it were to decrease equably, it would be 36,300 years before the earth's orbit became a circle. But in the midst of all these vicissitudes, the major axes and mean motions of the planets remain permanently independent of secular changes; they are so connected by Kepler's law of the squares of the periodic times being proportional to the cubes of the mean distances of the planets from the sun, that one cannot vary without affecting the other.

With the exception of these two elements, it appears, that all the bodies are in motion, and every orbit is in a state of perpetual change. Minute as these changes are, they might be supposed liable to accumulate in the course of ages sufficiently to derange the whole order of nature, to alter the relative positions of the planets, to put an end to the vicissitudes of the seasons, and to bring about collisions, which would involve our whole system, now so harmonious, in chaotic confusion. The consequences being so dreadful, it is natural to inquire, what proof exists that creation will be preserved from such a catastrophe? For nothing can be known from observation, since the existence of the human race has occupied but a point in duration, while these vicissitudes embrace myriads of ages. The proof is simple and convincing. All the variations of the solar system, as well secular as periodic, are expressed analytically by the sines and cosines of circular arcs, which increase with the time; and as a sine or cosine never can exceed the radius, but must oscillate between zero and unity, however much the time may increase, it follows, that when the variations have by slow changes accumulated in however long a time to a maximum, they decrease by the same slow degrees, till they arrive at their smallest value, and then begin a new course, thus forever oscillating about a mean value. This, however, would not be the case if the planets moved in a resisting medium, for then both the eccentricity and the major axes of the orbits would vary with the time, so that the stability of the system would be ultimately destroyed. But if the planets do move in an ethereal medium, it must be of extreme rarity, since resistance has hitherto been quite insensible.

Three circumstances have generally been supposed necessary to prove the stability of the system: the small eccentricities of the planetary orbits, their small inclinations, and the revolution of all the bodies, as well planets as satellites, in the same direction. These, however, are not necessary conditions: the periodicity of the terms in which the inequalities are expressed is sufficient to assure us, that though we do not know the extent of the limits, nor the period of that grand cycle which probably embraces millions of years, yet they never will exceed what is requisite for the stability and harmony of the whole, for the preservation of which every circumstance is so beautifully and wonderfully adapted.

The plane of the ecliptic itself, though assumed to be fixed at a given epoch for the convenience of astronomical computation, is subject to a minute secular variation of $52''.109$, occasioned by the reciprocal action of the planets; but as this is also periodical, the terrestrial equator, which is inclined to it at an angle of about $23^\circ\ 28'$, will never coincide with the plane of the ecliptic; so there never can be perpetual spring. The rotation of the earth is uniform; therefore day and night, summer and winter, will continue their vicissitudes while the system endures, or is untroubled by foreign causes.

Yonder starry sphere
Of planets, and of fix'd, in all her wheels
Resembles nearest, mazes intricate,
Eccentric, intervolved, yet regular
Then most, when most irregular they seem.

The stability of our system was established by Lagrange,[16] '*a discovery,*' says Professor Playfair,[17] '*that must render the name for ever memorable in science, and revered by those who delight in the contemplation of whatever is excellent and sublime. After Newton's discovery of the elliptical orbits of the planets, Lagrange's discovery of their periodical inequalities is without doubt the noblest truth in physical astronomy; and, in respect of the doctrine of final causes, it may be regarded as the greatest of all.*'

Notwithstanding the permanency of our system, the secular variations in the planetary orbits would have been extremely embarrassing to astronomers, when it became necessary to compare observations separated by long periods. This difficulty is obviated by Laplace[18], who has shown that whatever changes time may induce either in the orbits themselves, or in the plane of the ecliptic, there exists an invariable plane passing through the centre of gravity of the sun, about which the whole system oscillates within narrow limits, and which is determined by this property; that if every body in the system be projected on it, and if the mass of each be multiplied by the area described in a given time by its projection on this plane, the sum of all these products will be a maximum. This plane of greatest inertia, by no means peculiar to the solar system, but existing in every system of bodies submitted to their mutual attractions only, always remains parallel to itself, and maintains a fixed position, whence the oscillations of the system may be estimated through unlimited time. It is situate nearly half way between the orbits of Jupiter and Saturn, and is inclined to the ecliptic at an angle of about $1^\circ 35' 31''$.

All the periodic and secular inequalities deduced from the law of gravitation are so perfectly confirmed by observations, that analysis has become one of the most certain means of discovering the planetary irregularities, either when they are too small, or too long in their periods, to be detected by other methods. Jupiter and Saturn, however, exhibit inequalities which for a long time seemed discordant with that law. All observations, from those of the Chinese and Arabs down to the present day, prove that for ages the mean motions of Jupiter and Saturn have been affected by great inequalities of very long periods, forming what appeared an anomaly in the theory of the planets. It was long known by observation, that five times the mean motion of Saturn is nearly equal to twice that of Jupiter; a relation which the sagacity of Laplace perceived to be the cause of a periodic inequality in the mean motion of each of these planets, which completes its period in nearly 929 Julian years, the one being retarded, while the other is accelerated. These inequalities are strictly periodical, since they depend on the configuration of the two planets; and the theory is perfectly confirmed by observation, which shows that in the course of twenty centuries, Jupiter's mean motion has been accelerated by $3^\circ 23'$, and Saturn's retarded by $5^\circ 13'$.

It might be imagined that the reciprocal action of such planets as have satellites would be different from the influence of those that have none; but the distances of the satellites from their primaries are incomparably less than the distances of the planets from the sun, and from one another, so that the system of a planet and its satellites moves nearly as if all those bodies were united in their common centre of gravity; the action of the sun however disturbs in some degree the motion of the satellites about their primary.

The changes that take place in the planetary system are exhibited on a small scale by Jupiter and his satellites; and as the period requisite for the development of the inequalities of these little moons only extends to a few centuries, it may be regarded as an epitome of that grand cycle which will not be accomplished by the planets in myriads of centuries. The revolutions of the satellites about Jupiter are precisely similar to those of the planets about the sun; it is true they are disturbed by the sun, but his distance is so great, that their motions are nearly the same as if they were not under his influence. The satellites like the planets, were probably projected in elliptical orbits, but the consequence of Jupiter's spheroid is very great in consequence of his rapid rotation; and as the masses of the satellites are nearly 100,000 times less than that of Jupiter, the immense quantity of prominent matter at his equator must soon have given the circular form observed in the orbits of the first and second satellites, which its superior attraction will always maintain. The third and fourth satellites being further removed from its influence move in orbits with a very small eccentricity. The same cause occasions the orbits of the satellites to remain nearly in the plane of Jupiter's equator, on account of which they are always seen nearly in the same line; and the powerful action of that quantity of prominent matter is the reason why the motion of the nodes of these little bodies is so much more rapid than those of a planet. The nodes of the fourth satellite accomplish a revolution in 520 years, while those of Jupiter's orbit require no less than 50,673 years, a proof of the reciprocal attraction between each particle of Jupiter's equator and of the satellites. Although the two first satellites sensibly move in circles, they acquire a small ellipticity from the disturbances they experience.

The orbits of the satellites do not retain a permanent inclination, either to the plane of Jupiter's equator, or to that of his orbit, but to certain planes passing between the two, and through their intersection; these have a greater inclination to his equator the further the satellite is removed, a circumstance entirely owing to the influence of Jupiter's compression.

A singular law obtains among the mean motions and mean longitudes of the three first satellites. It appears from observation, that the mean motion of the first satellite, plus twice that of the third, is equal to three times that of the second, and that the mean longitude of the first satellite, minus three times that of the second, plus twice that of the third, is always equal to two right angles. It is proved by theory, that if these relations had only been approximate when the satellites were first launched into space, their mutual attractions would have established and maintained them. They extend to the synodic motions of the satellites, consequently they affect their eclipses, and have a very great influence on their whole theory. The satellites move so nearly in the plane of Jupiter's equator, which has a very small inclination to his orbit, that they are frequently eclipsed by the planet. The instant of the beginning or end of an eclipse of a satellite marks the same instant of absolute time to all the inhabitants of the earth; therefore the time of these eclipses observed by a traveler, when compared with the time of the eclipse computed for Greenwich or any other fixed meridian, gives the difference of the meridians in time, and consequently the longitude of the place of observation. It has required all the refinements of modern instruments to render the eclipses of these remote moons available to the mariner; now however, that system of bodies invisible to the naked eye, known to man by the aid of science alone, enables him to traverse the ocean, spreading the light of knowledge and the blessings of civilization over the most remote regions, and to return loaded with the productions of another hemisphere. Nor is this all: the eclipses of Jupiter's satellites have been the means of a discovery, which, though not so immediately applicable to the wants of man, unfolds a property of light, that medium, without whose cheering influence all the beauties of the creation would have been to us a blank. It is observed, that those eclipses of the first satellite which happen when Jupiter is near conjunction, are later by $16' 26''$ than those which take place when the planet is in opposition. But as Jupiter is nearer to us when in opposition by the whole breadth of the earth's orbit than when in conjunction, this circumstance was attributed to the time employed by the rays of light in crossing the earth's orbit, a distance of 192 millions of miles; whence it is estimated, that light travels at the rate of 192,000 miles in one second.[19] Such is its velocity, that the earth, moving at the rate of nineteen miles in a second, would take two months to pass through a distance which a ray of light would dart over in eight minutes. The subsequent discovery of the aberration of light confirmed this astonishing result.

Objects appear to be situate in the direction of the rays that proceed from them. Were light propagated instantaneously, every object, whether at rest or in motion, would appear in the direction of these rays; but as light takes some time to travel, when Jupiter is in conjunction, we see him by means of rays that left him $16' 26''$ before; but during that time we have changed our position, in consequence of the motion of the earth in its orbit; we therefore refer Jupiter to a place in which he is not. His true position is in the diagonal of the parallelogram, whose sides are in the ratio of the velocity of light to the velocity of the earth in its orbit, which is as 192,000 to 19. In consequence of aberration, none of the heavenly bodies are in the place in which they seem to be. In fact, if the earth were at rest, rays from a star would pass along the axis of a telescope directed to it; but if the earth were to begin to move in its orbit with its usual velocity, these rays would strike against the side of the tube; it would therefore be necessary to incline the telescope a little, in order to see the star. The angle contained between the axis of the telescope and a line drawn to the true place of the star, is its aberration, which varies in quantity and direction in different parts of the earth's orbit; but as it never exceeds twenty seconds, it is insensible in ordinary cases.

The velocity of light deduced from the observed aberration of the fixed stars, perfectly corresponds with that given by the eclipses of the first satellite. The same result obtained from sources so different, leaves not a doubt of its truth. Many such beautiful coincidences, derived from apparently the most unpromising and dissimilar circumstances, occur in physical astronomy, and prove dependencies which we might otherwise be unable to trace. The identity of the velocity of light at the distance of Jupiter and on the earth's surface shows that its velocity is uniform; and if light consists in the vibrations of an elastic fluid or ether filling space, which hypothesis accords best with observed phenomena, the uniformity of its velocity shows that the density of the fluid throughout the whole extent of the solar system, must be proportional to its elasticity.[20] Among the fortunate conjectures which have been confirmed by subsequent experience, that of Bacon[21] is not the least remarkable. '*It produces in me,*' says the restorer of true philosophy, '*a doubt, whether the face of the serene and starry heavens be seen at the instant it really exists, or not till some time later; and whether there be not, with respect to the heavenly bodies, a true time and an apparent time, no less than a true place and an apparent place, as astronomers say, on account of parallax. For it seems incredible that the species or rays of the celestial bodies can pass through the immense interval between them and us in an instant; or that they do not even require some considerable portion of time.*'

As great discoveries generally lead to a variety of conclusions, the aberration of light affords a direct proof of the motion of the earth in its orbit; and its rotation is proved by the theory of falling bodies, since the centrifugal force it induces retards the oscillations of the pendulum in going from the pole to the equator. Thus a high degree of scientific knowledge has been requisite to dispel the errors of the senses.

The little that is known of the theories of the satellites of Saturn and Uranus is in all respects similar to that of Jupiter. The great compression of Saturn occasions its satellites to move nearly in the plane of its equator. Of the situation of the equator of Uranus we know nothing, nor of its compression. The orbits of its satellites are nearly perpendicular to the plane of the ecliptic.

Our constant companion the moon next claims attention. Several circumstances concur to render her motions the most interesting, and at the same time the most difficult to investigate of all the bodies of our system. In the solar system planet troubles planet, but in the lunar theory the sun is the great disturbing cause; his vast distance being compensated by his enormous magnitude, so that the motions of the moon are more irregular than those of the planets; and on account of the great ellipticity of her orbit and the size of the sun, the approximations to her motions are tedious and difficult, beyond what those unaccustomed to such investigations could imagine. Neither the eccentricity of the lunar orbit, nor its inclination to the plane of the ecliptic, have experienced any changes from secular inequalities; but the mean motion, the nodes, and the perigee, are subject to very remarkable variations.

From an eclipse observed at Babylon by the Chaldeans, on the 19th of March, seven hundred and twenty-one years before the Christian era, the place of the moon is known from that of the sun at the instant of opposition; whence her mean longitude may be found; but the comparison of this mean longitude with another mean longitude, computed back for the instant of the eclipse from modern observations, shows that the moon performs her revolution round the earth more rapidly and in a shorter time now, than she did formerly; and that the acceleration in her mean motion has been increasing from age to age as the square of the time; all the ancient and intermediate eclipses confirm this result. As the mean motions of the planets have no secular inequalities, this seemed to be an unaccountable anomaly, and it was at one time attributed to the resistance of an ethereal medium pervading space; at another to the successive transmission of the gravitating force: but as Laplace proved that neither of these causes, even if they exist, have any influence on the motions of the lunar perigee or nodes, they could not affect the mean motion, a variation in the latter from such a cause being inseparably connected with variations in the two former of these elements. That great mathematician, however, in studying the theory of Jupiter's satellites, perceived that the secular variations in the elements of Jupiter's orbit, from the action of the planets, occasion corresponding changes in the motions of the satellites: this led him to suspect that the acceleration in the mean motion of the moon might be connected with the secular variation in the eccentricity of the terrestrial orbit; and analysis has proved that he assigned the true cause.

If the eccentricity of the earth's orbit were invariable, the moon would be exposed to a variable disturbance from the action of the sun, in consequence of the earth's annual revolution; but it would be periodic, since it would be the same as often as the sun, the earth, and the moon returned to the same relative positions: on account however of the slow and incessant diminution in the eccentricity of the terrestrial orbit, the revolution of our planet is performed at different distances from the sun every year. The position of the moon with regard to the sun, undergoes a corresponding change; so that the mean action of the sun on the moon varies from one century to another, and occasions the secular increase in the moon's velocity called the acceleration, a name which is very appropriate in the present age, and which will continue to be so for a vast number of ages to come; because, as long as the earth's eccentricity diminishes, the moon's mean motion will be accelerated; but when the eccentricity has passed its minimum and begins to increase, the mean motion will be retarded from age to age. At present the secular acceleration is about 10″, but its effect on the moon's place increases as the square of the time. It is remarkable that the action of the planets thus reflected by the sun to the moon, is much more sensible than their direct action, either on the earth or moon. The secular diminution in the eccentricity, which has not altered the equation of the centre of the sun by eight minutes since the earliest recorded eclipses, has produced a variation of $1^\circ\ 48'$ in the moon's longitude, and of $7^\circ\ 12'$ in her mean anomaly.

The action of the sun occasions a rapid but variable motion in the nodes and perigee of the lunar orbit; the former though they recede during the greater part of the moon's revolution, and advance during the smaller, perform their sidereal revolutions in $6793^{\text{days}}.4212$, and the latter, though its motion is sometimes retrograde and sometimes direct, in $3232^{\text{days}}.5807$, or a little more than nine years: but such is the difference between the disturbing energy of the sun and that of all the planets put together, that it requires no less than 109,770 years for the greater axis of the terrestrial orbit to do the same. It is evident that the same secular variation which changes the sun's distance from the earth, and occasions the acceleration in the moon's mean motion, must affect the motion of the nodes and perigee; and it consequently appears, from theory as well as observation, that both these elements are subject to a secular inequality, arising from the variation in the eccentricity of the earth's orbit, which connects them with the acceleration; so that both are retarded when the mean motion is anticipated. The secular variations in these three elements are in the ratio of the numbers 3, 0.735, and 1; whence the three motions of the moon, with regard to the sun, to her perigee, and to her nodes, are continually accelerated, and their secular equations are as the numbers 1, 4, and 0.265, or according to the most recent investigations as 1, 4, 6776 and 0.391. A comparison of ancient eclipses observed by the Arabs, Greeks, and Chaldeans, imperfect as they are, with modern observations, perfectly confirms these results of analysis.

Future ages will develop these great inequalities, which at some most distant period will amount to many circumferences. They are indeed periodic; but who shall tell their period? Millions of years must elapse before that great cycle is accomplished; but '*such changes, though rare in time, are frequent in eternity.*'

The moon is so near, that the excess of matter at the earth's equator occasions periodic variations in her longitude and latitude; and, as the cause must be proportional to the effect, a comparison of these inequalities, computed from theory, with the same given by observation, shows that the compression of the terrestrial spheroid, or the ratio of the difference between the polar and equatorial diameter to the diameter of the equator is 1/505.05. It is proved analytically, that if a fluid mass of homogeneous matter, whose particles attract each other inversely as the square of the distance, were to revolve about an axis, as the earth, it would assume the form of a spheroid, whose compression is 1/230. Whence it appears, that the earth is not homogeneous, but decreases in density from its centre to its circumference. Thus the moon's eclipses show the earth to be round, and her inequalities not only determine the form, but the internal structure of our planet; results of analysis which could not have been anticipated. Similar inequalities in Jupiter's satellites prove that his mass is not homogeneous, and that his compression is 1/13.8.

The motions of the moon have now become of more importance to the navigator and geographer than those of any other body, from the precision with which the longitude is determined by the occultations of stars and lunar distances. The lunar theory is brought to such perfection, that the times of these phenomena, observed under any meridian, when compared with that computed for Greenwich in the *Nautical Almanac,*[22] gives the longitude of the observer within a few miles. The accuracy of that work is obviously of extreme importance to a maritime nation; we have reason to hope that the new Ephemeris,[23] now in preparation, will be by far the most perfect work of the kind that ever has been published.

From the lunar theory, the mean distance of the sun from the earth, and thence the whole dimensions of the solar System are known; for the forces which retain the earth and moon in their orbits, are respectively proportional to the radii vectores of the earth and moon, each being divided by the square of its periodic time; and as the lunar theory gives the ratio of the forces, the ratio of the distance of the sun and moon from the earth is obtained: whence it appears that the sun's distance from the earth is nearly 396 times greater than that of the moon.

The method however of finding the absolute distances of the celestial bodies in miles, is in fact the same with that employed in measuring distances of terrestrial objects. From the extremities of a known base the angles which the visual rays from the object form with it, are measured; their sum subtracted from two right-angles gives the angle opposite the base; therefore by trigonometry, all the angles and sides of the triangle may be computed; consequently the distance of the object is found. The angle under which the base of the triangle is seen from the object, is the parallax[24] of that object; it evidently increases and decreases with the distance; therefore the base must be very great indeed, to be visible at all from the celestial bodies. But the globe itself whose dimensions are ascertained by actual admeasurement, furnishes a standard of measures, with which we compare the distances, masses, densities, and volumes of the sun and planets.

The courses of the great rivers, which are in general navigable to a considerable extent, prove that the curvature of the land differs but little from that of the ocean; and as the heights of the mountains and continents are, at any rate, quite inconsiderable when compared with the magnitude of the earth, its figure is understood to be determined by a surface at every point perpendicular to the direction of gravity, or of the plumbline, and is the same which the sea would have if it were continued all round the earth beneath the continents. Such is the figure that has been measured in the following manner:

A terrestrial meridian is a line passing through both poles, all the points of which have contemporaneously the same noon. Were the lengths and curvatures of different meridians known, the figure of the earth might be determined; but the length of one degree is sufficient to give the figure of the earth, if it be measured on different meridians, and in a variety of latitudes; for if the earth were a sphere, all degrees would be of the same length, but if not, the lengths of the degrees will be greatest where the curvature is least; a comparison of the length of the degrees in different parts of the earth's surface will therefore determine its size and form.

An arc of the meridian may be measured by observing the latitude of its extreme points, and then measuring the distance between them in feet or fathoms; the distance thus determined on the surface of the earth, divided by the degrees and parts of a degree contained in the difference of the latitudes, will give the exact length of one degree, the difference of the latitudes being the angle contained between the verticals at the extremities of the arc. This would be easily accomplished were the distance unobstructed, and on a level with the sea; but on account of the innumerable obstacles on the surface of the earth, it is necessary to connect the extreme points of the arc by a series of triangles, the sides and angles of which are either measured or computed, so that the length of the arc is ascertained with much laborious computation. In consequence of the inequalities of the surface, each triangle is in a different plane; they must therefore be reduced by computation to what they would have been, had they been measured on the surface of the sea; and as the earth is spherical, they require a correction to reduce them from plane to spherical triangles.

Arcs of the meridian have been measured in a variety of latitudes, both north and south, as well as arcs perpendicular to the meridian. From these measurements it appears that the length of the degrees increase from the equator to the poles, nearly as the square of the sine of the latitude; consequently, the convexity of the earth diminishes from the equator to the poles. Many discrepancies occur, but the figure that most nearly follows this law is an ellipsoid of revolution, whose equatorial radius is 3,962.6 miles, and the polar radius 3,949.7 miles; the difference, or 12.9

miles, divided by the equatorial radius, is 1/308.7, or 1/309 nearly; this fraction is called the compression of the earth, because, according as it is greater or less, the terrestrial ellipsoid is more or less flattened at the poles; it does not differ much from that given by the lunar inequalities. If we assume the earth to be a sphere, the length of a degree of the meridian is 69.044 British miles; therefore 360 degrees, or the whole circumference of the globe is 24,856 miles, and the diameter, which is something less than a third of the circumference, is 7,916 or 8,000 miles nearly. Eratosthenes,[25] who died 194 years before the Christian era, was the first to give an approximate value of the earth's circumference, by the mensuration of an arc between Alexandria and Syene.

But there is another method of finding the figure of the earth, totally independent of either of the preceding. If the earth were a homogeneous sphere without rotation, its attraction on bodies at its surface would be everywhere the same; if it be elliptical, the force of gravity theoretically ought to increase, from the equator to the pole, as the square of the sine of the latitude; but for a spheroid in rotation, by the laws of mechanics the centrifugal force varies as the square of the sine of the latitude from the equator where it is greatest, to the pole where it vanishes; and as it tends to make bodies fly off the surface, it diminishes the effects of gravity by a small quantity. Hence by gravitation, which is the difference of these two forces, the fall of bodies ought to be accelerated in going from the equator to the poles, proportionally to the square of the sine of the latitude; and the weight of the same body ought to increase in that ratio. This is directly proved by the oscillations of the pendulum; for if the fall of bodies be accelerated, the oscillations will be more rapid; and that they may always be performed in the same time, the length of the pendulum must be altered. Now, by numerous and very careful experiments, it is proved that a pendulum, which makes 86,400 oscillations in a mean day at the equator, will do the same at every point of the earth's surface, if its length be increased in going to the pole, as the square of the sine of the latitude. From the mean of these it appears that the compression of the terrestrial spheroid is about 1/342 which does not differ much from that given by the lunar inequalities, and from the arcs of the meridian. The near coincidence of these three values, deduced by methods so entirely independent of each other, shows that the mutual tendencies of the centres of the celestial bodies to one another, and the attraction of the earth for bodies at its surface, result from the reciprocal attraction of all their particles. Another proof may be added; the nutation of the earth's axis, and the precession of the equinoxes,[26] are occasioned by the action of the sun and moon on the protuberant matter at the earth's equator; and although these inequalities do not give the absolute value of the terrestrial compression, they show that the fraction expressing it is comprised between the limits 1/279 and 1/578.

It might be expected that the same compression should result from each, if the different methods of observation could be made without error. This, however, is not the case; for such discrepancies are found both in the degrees of the meridian and in the length of the pendulum, as show that the figure of the earth is very complicated; but they are so small when compared with the general results, that they may be disregarded. The compression deduced from the mean of the whole, appears to be about 1/320; that given by the lunar theory has the advantage of being independent of the irregularities at the earth's surface and of local attractions. The form and size of the earth being determined, it furnishes a standard of measure with which the dimensions of the solar system may be compared.

The parallax of a celestial body is the angle under which the radius of the earth would be seen if viewed from the centre of that body; it affords the means of ascertaining the distances of the sun, moon, and planets. Suppose that, when the moon is in the horizon at the instant of rising or setting, lines were drawn from her centre to the spectator and to the centre of the earth, these would form a right-angled triangle with the terrestrial radius, which is of a known length; and as the parallax or angle at the moon can be measured, all the angles and one side are given; whence the distance of the moon from the centre of the earth may be computed. The parallax of an object may he found, if two observers under the same meridian, but at a very great distance from one another, observe its zenith distances on the same day at the time of its passage over the meridian. By such contemporaneous observations at the Cape of Good Hope and at Berlin, the mean horizontal parallax of the moon was found to be $3{,}454''.2$; whence the mean distance of the moon is about sixty times the mean terrestrial radius, or 240,000 miles nearly. Since the parallax is equal to the radius of the earth divided by the distance of the moon; under the same parallel of latitude it varies with the distance of the moon from the earth, and proves the ellipticity of the lunar orbit; and when the moon is at her mean distance, it varies with the terrestrial radii, thus showing that the earth is not a sphere.

Although the method described is sufficiently accurate for finding the parallax of an object so near as the moon, it will not answer for the sun which is so remote, that the smallest error in observation would lead to a false result; but by the transits of Venus that difficulty is obviated. When that planet is in her nodes, or within $1^\circ.25$ of them, that is, in, or nearly in the plane of the ecliptic, she is occasionally seen to pass over the sun like a black spot. If we could imagine that the sun and Venus had no parallax, the line described by the planet on his disc, and the duration of the transit would be the same to all the inhabitants of the earth; but as the sun is not so remote but that the semidiameter of the earth has a sensible magnitude when viewed from his centre, the line described by the planet in its passage over his disc appears to be nearer to his centre or farther from it, according to the position of the observer; so that the duration of the transit varies with the different points of the earth's surface at which it is observed. This difference of time, being entirely the effect of parallax, furnishes the means of computing it from the known motions of the earth and Venus, by the same method as for the eclipses of the sun. In fact the ratio of the distances of Venus and the sun from the earth at the time of the transit, are known from the theory of their elliptical motion; consequently, the ratio of the parallaxes of these two bodies, being inversely as their distances, is given; and as the transit gives the difference of the parallaxes, that of the sun is obtained. In 1769, the parallax of the sun was determined by observations of a transit of Venus made at Wardhus in Lapland, and at Otaheite in the South Sea, the latter observation being the object of Cook's first voyage.[27] The transit lasted about six hours at Otaheite, and the difference in the duration at these two stations was eight minutes; whence the sun's parallax was found to be $8''.72$: but by other considerations it has subsequently been reduced to $8''.575$; from which the mean distance of the sun appears to be about 95,996,000 miles or ninety six millions of miles nearly. This is confirmed by an inequality in the motion of the moon, which depends on the parallax of the sun, and which when compared with observation gives $8''.6$ for the sun's parallax.

The parallax of Venus is determined by her transits, that of Mars by direct observation. The distances of these two planets from the earth are therefore known in terrestrial radii; consequently their mean distances from the sun may be computed; and as the ratios of the distances of the planets from the sun are known by Kepler's law, their absolute distances in miles are easily found.

Far as the earth seems to be from the sun, it is near to him when compared with Uranus; that planet is no less than 1,843 millions of miles from the luminary that warms and enlivens the world; to it, situate on the verge of the system, the sun must appear not much larger than Venus does to us. The earth cannot even be visible as a telescopic object to a body so remote; yet man, the inhabitant of the earth, soars beyond the vast dimensions of the system to which his planet belongs, and assumes the diameter of its orbit as the base of a triangle, whose apex extends to the stars.

Sublime as the idea is, this assumption proves ineffectual, for the apparent places of the fixed stars are not sensibly changed by the earth's annual revolution; and with the aid derived from the refinements of modern astronomy and the most perfect instruments, it is still a matter of doubt whether a sensible parallax has been detected, even in the nearest of these remote suns. If a fixed star had the parallax of one second, its distance from the sun would be 20,500,000 millions of miles.[28] At such a distance not only the terrestrial orbit shrinks to a point, but, where the whole solar system, when seen in the focus of the most powerful telescope, might be covered by the thickness of a spider's thread. Light, flying at the rate of 200,000 miles in a second, would take three years and seven days to travel over that space; one of the nearest stars may therefore have been kindled or extinguished more than three years before we could have been aware of so mighty an event. But this distance must be small when compared with that of the most remote of the bodies which are visible in the heavens. The fixed stars are undoubtedly luminous like the sun; it is therefore probable that they are not nearer to one another than the sun is to the nearest of them. In the milky way and the other starry nebulae, some of the stars that seem to us to be close to others, may be far behind them in the boundless depth of space; nay, may rationally be supposed to be situate many thousand times further off: light would therefore require thousands of years to come to the earth from those myriads of suns, of which our own is but *'the dim and remote companion.'*

The masses of such planets as have no satellites are known by comparing the inequalities they produce in the motions of the earth and of each other, determined theoretically, with the same inequalities given by observation, for the disturbing cause must necessarily be proportional to the effect it produces. But as the quantities of matter in any two primary planets are directly as the cubes of the mean distances at which their satellites revolve, and inversely as the squares of their periodic times, the mass of the sun and of any planets which have satellites, may be compared with the mass of the earth. In this manner it is computed that the mass of the sun is 354,936 times greater than that of the earth; whence the great perturbations of the moon and the rapid motion of the perigee and nodes of her orbit. Even Jupiter, the largest of the planets, is 1,070.5 times less than the sun. The mass of the moon is determined from four different sources, from her action on the terrestrial equator, which occasions the nutation in the axis of rotation; from her horizontal parallax, from an inequality she produces in the sun's longitude, and from her action on the tides. The three first quantities, computed from theory, and compared with their observed values, give her mass respectively equal to the 1/71, 1/74.2, and 1/69.2 part of that of the earth, which do not differ very much from each other; but, from her action in raising the tides, which furnishes the fourth method, her mass appears to be about the seventy-fifth part of that of the earth, a value that cannot differ much from the truth.

The apparent diameters of the sun, moon, and planets are determined by measurement; therefore their real diameters may be compared with that of the earth; for the real diameter of a planet is to the real diameter of the earth, or 8,000 miles, as the apparent diameter of the planet to the apparent diameter of the earth as seen from the planet, that is, to twice the parallax of the planet. The mean apparent diameter of the sun is $1920''$, and with the solar parallax $8''.65$ it will be found that the diameter of the sun is about 888,000 miles; therefore, if the centre of the sun were to coincide with the centre of the earth, his volume would not only include the orbit of the moon, but would extend. nearly as far again, for the moon's mean distance from the earth is about sixty times the earth's mean radius or 240,000 miles; so that twice the distance of the moon is

480,000 miles, which differs but little from the solar radius; his equatorial radius is probably not much less than the major axis of the lunar orbit.

The diameter of the moon is only 2,160 miles; and Jupiter's diameter of 88,000 miles is incomparably less than that of the sun. The diameter of Pallas does not much exceed 71 miles, so that an inhabitant of that planet, in one of our steam-carriages, might go round his world in five or six hours.

The oblate form of the celestial bodies indicates rotatory motion, and this has been confirmed, in most cases, by tracing spots on their surfaces, whence their poles and times of rotation have been determined. The rotation of Mercury is unknown, on account of his proximity to the sun; and that of the new planets has not yet been ascertained. The sun revolves in twenty-five days ten hours, about an axis that is directed towards a point half way between the pole star and Lyra, the plane of rotation being inclined a little more than 70° to that on which the earth revolves. From the rotation of the sun, there is every reason to believe that he has a progressive motion in space, although the direction to which he tends is as yet unknown: but in consequence of the reaction of the planets, he describes a small irregular orbit about the centre of inertia of the system, never deviating from his position by more than twice his own diameter, or about seven times the distance of the moon from the earth.

The sun and all his attendants rotate from west to east on axes that remain nearly parallel to themselves in every point of their orbit, and with angular velocities that are sensibly uniform. Although the uniformity in the direction of their rotation is a circumstance hitherto unaccounted for in the economy of Nature, yet from the design and adaptation of every other part to the perfection of the whole, a coincidence so remarkable cannot be accidental; and as the revolutions of the planets and satellites are also from west to east, it is evident that both must have arisen from the primitive causes which have determined the planetary motions.

The larger planets rotate in shorter periods than the smaller planets and the earth; their compression is consequently greater, and the action of the sun and of their satellites occasions a nutation in their axes, and a precession of their equinoxes, similar to that which obtains in the terrestrial spheroid from the attraction of the sun and moon on the prominent matter at the equator. In comparing the periods of the revolutions of Jupiter and Saturn with the times of their rotation, it appears that a year of Jupiter contains nearly ten thousand of his days, and that of Saturn about thirty thousand Saturnian days.

The appearance of Saturn is unparalleled in the system of the world; he is surrounded by a ring even brighter than himself, which always remains in the plane of his equator, and viewed with a very good telescope, it is found to consist of two concentric rings, divided by a dark band. By the laws of mechanics, it is impossible that this body can retain its position by the adhesion of its particles alone; it must necessarily revolve with a velocity that will generate a centrifugal force sufficient to balance the attraction of Saturn. Observation confirms the truth of these principles, showing that the rings rotate about the planet in 10.5 hours, which is considerably less than the time a satellite would take to revolve about Saturn at the same distance. Their plane is inclined to the ecliptic at an angle of 31°; and in consequence of this obliquity of position they always appear elliptical to us, but with an eccentricity so variable as even to be occasionally like a straight line drawn across the planet. At present the apparent axes of the rings are as 1,000 to 160; and on the 29th of September, 1832, the plane of the rings will pass through the centre of the earth when they will be visible only with superior instruments, and will appear like a fine line across the disc of Saturn. On the 1st of December in the same year, the plane of the rings will pass through the centre of the sun.

It is a singular result of the theory, that the rings could not maintain their stability of rotation if they were everywhere of uniform thickness; for the smallest disturbance would destroy the equilibrium, which would become more and more deranged, till at last they would be precipitated on the surface of the planet. The rings of Saturn must therefore be irregular solids of unequal breadth in the different parts of the circumference, so that their centres of gravity do not coincide with the centres of their figures.

Professor Struve[29] has also discovered that the centre of the ring is not concentric with the centre of Saturn; the interval between the outer edge of the globe of the planet and the outer edge of the ring on one side, is $11''.073$, and on the other side the interval is $11''.288$; consequently there is an eccentricity of the globe in the ring of $0''.215$.

If the rings obeyed different forces, they would not remain in the same plane, but the powerful attraction of Saturn always maintains them and his satellites in the plane of his equator. The rings, by their mutual action, and that of the sun and satellites, must oscillate about the centre of Saturn, and produce phenomena of light and shadow, whose periods extend to many years.

The periods of the rotation of the moon and the other satellites are equal to the times of their revolutions, consequently these bodies always turn the same face to their primaries; however, as the mean motion of the moon is subject to a secular inequality which will ultimately amount to many circumferences, if the rotation of the moon were perfectly uniform, and not affected by the same inequalities, it would cease exactly to counterbalance the motion of revolution; and the moon, in the course of ages, would successively and gradually discover every point of her surface to the earth. But theory proves that this never can happen; for the rotation of the moon, though it does not partake of the periodic inequalities of her revolution, is affected by the same secular variations, so that her motions of rotation and revolution round the earth will always balance each other, and remain equal. This circumstance arises from the form of the lunar spheroid, which has three principal axes of different lengths at right angles to each other. The moon is flattened at the poles from her centrifugal force, therefore her polar axis is least; the other two are in the plane of her equator, but that directed towards the earth is the greatest. The attraction of the earth, as if it had drawn out that part of the moon's equator, constantly brings the greatest axis, and consequently the same hemisphere towards us, which makes her rotation participate in the secular variations in her mean motion of revolution. Even if the angular velocities of rotation and revolution had not been nicely balanced in the beginning of the moon's motion, the attraction of the earth would have recalled the greatest axis to the direction of the line joining the centres of the earth and moon; so that it would vibrate on each side of that line in the same manner as a pendulum oscillates on each side of the vertical from the influence of gravitation.

No such libration[30] is perceptible; and as the smallest disturbance would make it evident, it is clear that if the moon has ever been touched by a comet, the mass of the latter must have been extremely small; for if it had been only the hundred-thousandth part of that of the earth, it would have rendered the libration sensible. A similar libration exists in the motions of Jupiter's satellites; but although the comet of 1767 and 1779 passed through the midst of them, their libration still remains insensible. It is true, the moon is liable to librations depending on the position of the spectator; at her rising, part of the western edge of her disc is visible, which is invisible at her setting, and the contrary takes place with regard to her eastern edge. There are also librations arising from the relative positions of the earth and moon in their respective orbits, but as they are only optical appearances, one hemisphere will be eternally concealed from the earth. For the same reason, the earth, which must be so splendid an object to one lunar hemisphere, will be for ever veiled from the other. On account of these circumstances, the remoter hemisphere of the moon has its day a fortnight long, and a night of the same duration not even enlightened by a moon, while the favoured side is illuminated by the reflection of the earth during its long night. A moon exhibiting a surface thirteen times larger than ours, with all the varieties of clouds, land, and water coming successively into view, would be a splendid object to a lunar traveler in a journey to his antipodes.

The great height of the lunar mountains probably has a considerable influence on the phenomena of her motion, the more so as her compression is small, and her mass considerable.

In the curve passing through the poles, and that diameter of the moon which always points to the earth, nature has furnished it permanent meridian, to which the different spots on her surface have been referred, and their positions determined with as much accuracy as those of many of the most remarkable places on the surface of our globe.

The rotation of the earth which determines the length of the day may be regarded as one of the most important elements in the system of the world. It serves as a measure of time, and forms the standard of comparison for the revolutions of the celestial bodies, which by their proportional increase or decrease would soon disclose any changes it might sustain. Theory and observation concur in proving, that among the innumerable vicissitudes that prevail throughout creation, the period of the earth's diurnal rotation is immutable. A fluid, as Mr. Babbage[31] observes, in falling from a higher to a lower level, carries with it the velocity due to its revolution with the earth at a greater distance from its centre. It will therefore accelerate, although to an almost infinitesimal extent, the earth's daily rotation. The sum of all these increments of velocity, arising from the descent of all the rivers on the earth's surface, would in time become perceptible, did not nature, by the process of evaporation, raise the waters back to their sources; and thus again by removing matter to a greater distance from the centre, destroy the velocity generated by its previous approach; so that the descent of the rivers does not affect the earth's rotation. Enormous masses projected by volcanoes from the equator to the poles, and the contrary, would indeed affect it, but there is no evidence of such convulsions. The disturbing action of the moon and planets, which has so powerful an effect on the revolution of the earth, in no way influences its rotation: the constant friction of the trade winds on the mountains and continents between the tropics does not impede its velocity, which theory even proves to be the same, as if the sea together with the earth formed one solid mass. But although these circumstances be inefficient, a variation in the mean temperature would certainly occasion a corresponding change in the velocity of rotation: for in the science of dynamics, it is a principle in a system of bodies or of particles revolving about a fixed centre, that the momentum, or sum of the products of the mass of each into its angular velocity and distance from the centre is a constant quantity, if the system be not deranged by an external cause. Now since the number of particles in the system is the same whatever its temperature may be, when their distances from the centre are diminished, their angular velocity must be increased in order that the preceding quantity may still remain constant. It follows then, that as the primitive momentum of rotation with which the earth was projected into space must necessarily remain the same, the smallest decrease in heat, by contracting the terrestrial spheroid, would accelerate its rotation, and consequently diminish the length of the day. Notwithstanding the constant accession of heat from the sun's rays, geologists have been induced to believe from the nature of fossil remains, that the mean temperature of the globe is decreasing.

The high temperature of mines, hot springs, and above all, the internal fires that have produced, and do still occasion such devastation on our planet, indicate an augmentation of heat towards its centre; the increase of density in the strata corresponding to the depth and the form of the spheroid, being what theory assigns to a fluid mass in rotation, concur to induce the idea that the temperature of the earth was originally so high as to reduce all the substances of which it is composed to a state of fusion, and that in the course of ages it has cooled down to its present state; that it is still becoming colder, and that it will continue to do so, till the whole mass arrives at the temperature of the medium in which it is placed, or rather at a state of equilibrium between this temperature, the cooling power of its own radiation, and the beating effect of the sun's rays. But even if this cause be sufficient to produce the observed effects, it must be extremely slow in its operation; for in consequence of the rotation of the earth being a measure of the periods of the celestial motions, it has been proved, that if the length of the day had decreased by the three hundredth part of a second since the observations of Hipparchus[32] two thousand years ago, it would have diminished the secular equation of the moon by $4''.4$. It is therefore beyond a doubt, that the mean temperature of the earth cannot have sensibly varied during that time; if then the appearances exhibited by the strata are really owing to a decrease of internal temperature, it either shows the immense periods requisite to produce geological changes to which two thousand years are as nothing, or that the mean temperature of the earth had arrived at a state of equilibrium before these observations. However strong the indications of the primitive fluidity of the earth, as there is no direct proof, it can only be regarded as a very probable hypothesis; but one of the most profound philosophers and elegant writers of modern times has found, in the secular variation of the eccentricity of the terrestrial orbit, an evident cause of decreasing temperature. That accomplished author, in pointing out the mutual dependencies of phenomena, says: *'It is evident that the mean temperature of the whole surface of the globe, in so far as it is maintained by the action of the sun at a higher degree than it would have were the sun extinguished, must depend on the mean quantity of the sun's rays which it receives, or, which comes to the same thing, on the total quantity received in a given invariable time: and the length of the year being unchangeable in all the fluctuations of the planetary system, it follows, that the total amount of solar radiation will determine*, *caeteris paribus,*[33] *the general climate of the earth. Now it is not difficult to show, that this amount is inversely proportional to the minor axis of the ellipse described by the earth about the sun, regarded as slowly variable; and that, therefore, the major axis remaining, as we know it to be, constant, and the orbit being actually in a state of approach to a circle, and consequently the minor axis being on the increase, the mean annual amount of solar radiation received by the whole earth must be actually on the decrease. We have, therefore, an evident real cause to account for the phenomenon.*' The limits of the variation in the eccentricity of the earth's orbit are unknown; but if its ellipticity has ever been as great as that of the orbit of Mercury or Pallas, the mean temperature of the earth must have been sensibly higher than it is at present; whether it was great enough to render our northern climates fit for the production of tropical plants, and for the residence of the elephant, and the other inhabitants of the torrid zone, it is impossible to say.

The relative quantity of heat received by the earth at different moments during a single revolution, varies with the position of the perigee of its orbit, which accomplishes a tropical revolution in 20,935 years. In the year 1250 of our era, and 29,653 years before it, the perigee coincided with the summer solstice; at both these periods the earth was nearer the sun during the summer, and farther from him in the winter than in any other position of the apsides: the extremes of temperature must therefore have been greater than at present; but as the terrestrial orbit was probably more elliptical at the distant epoch, the heat of the summers most have been very great, though possibly compensated by the rigour of the winters; at all events, none of these changes affect the length of the day.

It appears from the marine shells found on the tops of the highest mountains, and in almost every part of the globe, that immense continents have been elevated above the ocean, which must have engulfed others. Such a catastrophe would be occasioned by a variation in the position of the axis of rotation on the surface of the earth; for the seas tending to the new equator would leave some portions of the globe, and overwhelm others.

But theory proves that neither nutation, precession, nor any of the disturbing forces that affect the system, have the smallest influence on the axis of rotation, which maintains a permanent position on the surface, if the earth be not disturbed in its rotation by some foreign cause, as the collision of a comet which may have happened in the immensity of time. Then indeed, the equilibrium could only have been restored by the rushing of the seas to the new equator, which they would continue to do, till the surface was every where perpendicular to the direction of gravity. But it is probable that such an accumulation of the waters would not be sufficient to restore equilibrium if the derangement had been great; for the mean density of the sea is only about a fifth part of the mean density of the earth, and the mean depth even of the Pacific ocean is not more than four miles, whereas the equatorial radius of the earth exceeds the polar radius by twenty-five or thirty miles; consequently the influence of the sea on the direction of gravity is very small; and as it appears that a great change in the position of the axes is incompatible with the law of equilibrium, the geological phenomena must be ascribed to an internal cause. Thus amidst the mighty revolutions which have swept innumerable races of organized beings from the earth, which have elevated plains, and buried mountains in the ocean, the rotation of the earth, and the position of the axis on its surface, have undergone but slight variations.

It is beyond a doubt that the strata increase in density from the surface of the earth to its centre, which is even proved by the lunar inequalities; and it is manifest from the mensuration of arcs of the meridian and the lengths of the seconds pendulum that the strata are elliptical and concentric. This certainly would have happened if the earth had originally been fluid, for the denser parts must have subsided towards the centre, as it approached a state of equilibrium; but the enormous pressure of the superincumbent mass is a sufficient cause for these phenomena. Professor Leslie[34] observes, that air compressed into the fiftieth part of its volume has its elasticity fifty times augmented; if it continue to contract at that rate, it would, from its own incumbent weight, acquire the density of water at the depth of thirty-four miles. But water itself would have its density doubled at the depth of ninety-three miles, and would even attain the density of quicksilver at a depth of 362 miles. In descending therefore towards the centre through 4,000 miles, the condensation of ordinary materials would surpass the utmost powers of conception. But a density so extreme is not borne out by astronomical observation. It might seem therefore to follow, that our planet must have a widely cavernous structure, and that we tread on a crust or shell, whose thickness bears a very small proportion to the diameter of its sphere. Possibly too this great condensation at the central regions may be counterbalanced by the increased elasticity due to a very elevated temperature. Dr. Young[35] says that steel would be compressed into one-fourth, and stone into one-eighth of its bulk at the earth's centre. However we are yet ignorant of the laws of compression of solid bodies beyond a certain limit; but, from the experiments of Mr. Perkins,[36] they appear to be capable of a greater degree of compression than has generally been imagined.

It appears then, that the axis of rotation is invariable on the surface of the earth, and observation shows, that were it not for the action of the sun and moon on the matter at the equator, it would remain parallel to itself in every point of its orbit.

The attraction of an exterior body not only draws a spheroid towards it; but, as the force varies inversely as the square of the distance, it gives it a motion about its centre of gravity, unless when the attracting body is situated in the prolongation of one of the axes of the spheroid.

The plane of the equator is inclined to the plane of the ecliptic at an angle of about $23^\circ\ 28'$, and the inclination of the lunar orbit on the same is nearly 5°; consequently, from the oblate figure of the earth, the sun and moon acting obliquely and unequally on the different parts of the terrestrial spheroid, urge the plane of the equator from its direction, and force it to move from east to west, so that the equinoctial points have a slow retrograde motion on the plane of the ecliptic of about $50''.512$ annually. The direct tendency of this action would be to make the planes of the equator and ecliptic coincide; but in consequence of the rotation of the earth, the inclination of the two planes remains constant, as a top in spinning preserves the same inclination to the plane of the horizon. Were the earth spherical this effect would not be produced, and the equinoxes would always correspond to the same points of the ecliptic, at least as far as this kind of action is concerned. But another and totally different cause operates on this motion, which has already been mentioned. The action of the planets on one another and on the sun, occasions a very slow variation in the position of the plane of the ecliptic, which affects its inclination on the plane of the equator, and gives the equinoctial points a slow but direct motion on the ecliptic of $0''.312$ annually, which is entirely independent of the figure of the earth, and would be the same if it were a sphere. Thus the sun and moon, by moving the plane of the equator, cause the equinoctial points to retrograde on the ecliptic; and the planets, by moving the plane of the ecliptic, give them a direct motion, but much less than the former; consequently the difference of the two is the mean precession, which is proved, both by theory and observation, to be about $50''.1$ annually. As the longitudes of all the fixed stars are increased by this quantity, the effects of precession are soon detected; it was accordingly discovered by Hipparchus, in the year 128 before Christ, from a comparison of his own observations with those of Timocharis,[37] 155 years before. In the time of Hipparchus the entrance of the sun into the constellation Aries was the beginning of spring, but since then the equinoctial points have receded 30°; so that the constellations called the signs of the zodiac are now at a considerable distance from those divisions of the ecliptic which bear their names. Moving at the rate of $50''.1$ annually, the equinoctial points will accomplish a revolution in 25,868 years; but as the precession varies in different centuries, the extent of this period will be slightly modified. Since the motion of the sun is direct, and that of the equinoctial points retrograde, he takes a shorter time to return to the equator than to arrive at the same stars; so that the tropical year of 365.242264 days must be increased by the time be takes to move through an arc of $50''.1$, in order to have the length of the sidereal year. By simple proportion it is the 0.014119th part of a day, so that the sidereal year is 365.256383.

The mean annual precession is subject to a secular variation; for although the change in the plane of the ecliptic which is the orbit of the sun, be independent of the form of the earth, yet by bringing the sun, moon and earth into different relative positions from age to age, it alters the direct action of the two first on the prominent matter at the equator; on this account the motion of the equinox is greater by $0''.455$ now than it was in the time of Hipparchus; consequently the actual length of the tropical year is about $4''.154$ shorter than it was at that time. The utmost change that it can experience from this cause amounts to $43''$.

Such is the secular motion of the equinoxes, but it is sometimes increased and sometimes diminished by periodic variations, whose periods depend on the relative positions of the sun and moon with regard to the earth, and occasioned by the direct action of these bodies on the equator.

Dr. Bradley[38] discovered that by this action the moon causes the pole of the equator to describe a small ellipse in the heavens, the diameters of which are $16''$ and $20''$. The period of this inequality is nineteen years, the time employed by the nodes of the lunar orbit to accomplish a revolution. The sun causes a small variation in the description of this ellipse; it runs through its period in half a year. This nutation in the earth's axis affects both the precession and obliquity with small periodic variations; but in consequence of the secular variation in the position of the terrestrial orbit, which is chiefly owing to the disturbing energy of Jupiter on the earth, the obliquity of the ecliptic is annually diminished by $0''.52109$. With regard to the fixed stars, this variation in the course of ages may amount to ten or eleven degrees; but the obliquity of the ecliptic to the equator can never vary more than two or three degrees, since the equator will follow in some measure the motion of the ecliptic.

It is evident that the places of all the celestial bodies are affected by precession and nutation, and therefore all observations of them must be corrected for these inequalities.

The densities of bodies are proportional to their masses divided by their volumes; hence if the sun and planets be assumed to be spheres, their volumes will be as the cubes of their diameters. Now the apparent diameters of the sun and earth at their mean distance, are $1922''$ and $17''.08$, and the mass of the earth is the 1/354936th part of that of the sun taken as the unit; it follows therefore, that the earth is nearly four times as dense as the sun; but the sun is so large that his attractive force would cause bodies to fall through about 450 feet in a second; consequently if he were even habitable by human beings, they would be unable to move, since their weight would be thirty times as great as it is here. A moderate sized man would weigh about two tons at the surface of the sun. On the contrary, at the surface of the four new planets we should be so light, that it would be impossible to stand from the excess of our muscular force, for a man would only weigh a few pounds. All the planets and satellites appear to be of less density than the earth. The motions of Jupiter's satellites show that his density increases towards his centre; and the singular irregularities in the form of Saturn, and the great compression of Mars, prove the internal structure of these two planets to be very far from uniform.

Astronomy has been of immediate and essential use in affording invariable standards for measuring duration, distance, magnitude, and velocity. The sidereal day, measured by the time elapsed between two consecutive transits of any star at the same meridian, and the sidereal year, are immutable units with which to compare all great periods of time; the oscillations of the isochronous pendulum measure its smaller portions. By these invariable standards alone we can judge of the slow changes that other elements of the system may have undergone in the lapse of ages.

The returns of the sun to the same meridian, and to the same equinox or solstice, have been universally adopted as the measure of our civil days and years. The solar or astronomical day is the time that elapses between two consecutive noons or midnights; it is consequently longer than the sidereal day, on account of the proper motion of the sun during a revolution of the celestial sphere; but as the sun moves with greater rapidity at the winter than at the summer solstice, the astronomical day is more nearly equal to the sidereal day in summer than in winter. The obliquity of the ecliptic also affects its duration, for in the equinoxes the arc of the equator is less than the corresponding arc of the ecliptic, and in the solstices it is greater. The astronomical day is therefore diminished in the first case, and increased in the second. If the sun moved uniformly in the equator at the rate of $59' 8''.3$ every day, the solar days would be all equal; the time therefore, which is reckoned by the arrival of an imaginary sun at the meridian, or of one which is supposed to move in the equator, is denominated mean solar time, such as is given by clocks and watches in common life: when it is reckoned by the arrival of the real sun at the meridian, it is apparent time, such as is given by dials. The difference between the time shown by a clock. and a dial is the equation of time given in the *Nautical Almanac*, and sometime amounts to as much as sixteen minutes. The apparent and mean time coincide four times in the year.

Astronomers begin the day at noon, but in common reckoning the day begins at midnight. In England it is divided into twenty-four hours, which are counted by twelve and twelve; but in France, astronomers adopting decimal division, divide the day into ten hours, the hour into one hundred minutes, and the minute into a hundred seconds, because of the facility in computation, and in conformity with their system of weights and measures. This subdivision is not used in common life, nor has it been adopted in any other country, though their scientific writers still employ that division of time. The mean length of the day, though accurately determined, is not sufficient for the purposes either of astronomy or civil life. The length of the year is pointed out by nature as a measure of long periods; but the incommensurability that exists between the lengths of the day, and the revolutions of the sun, renders it difficult to adjust the estimation of both in whole numbers. If the revolution of the sun were accomplished in 365 days, all the years would be of precisely the same number of days, and would begin and end with the sun at the same point of the ecliptic; but as the sun's revolution includes the fraction of a day, a civil year and a revolution of the sun have not the same duration. Since the fraction is nearly the fourth of a day, four years are nearly equal to four revolutions of the sun, so that the addition of a supernumerary day every fourth year nearly compensates the difference; but in process of time further correction will be necessary, because the fraction is less than the fourth of a day. The period of seven days, by far the most permanent division of time, and the most ancient monument of astronomical knowledge, was used by the Brahmins in India[39] with the same denominations employed by us, and was alike found in the Calendars of the Jews, Egyptians, Arabs, and Assyrians; it has survived the fall of empires, and has existed among all successive generations, a proof of their common origin.

The new moon immediately following the winter solstice in the 707th year of Rome was made the 1st of January of the first year of Caesar; the 25th of December in his 45th year, is considered as the date of Christ's nativity; and Caesar's 46th year is assumed to be the first of our era. The preceding year is called the first year before Christ by chronologists, but by astronomers it is called the year 0. The astronomical year begins on the 31st of December at noon; and the date of an observation expresses the days and hours which actually elapsed since that time.

Some remarkable astronomical eras are determined by the position of the major axis of the solar ellipse. Moving at the rate of 61″.906 annually, it accomplishes a tropical revolution in 20,935 years. It coincided with the line of the equinoxes 4000 or 4089 years before the Christian era, much about the time chronologists assign for the creation of man.[40] In 6485 the Major axis will again coincide with the line of the equinoxes, but then the solar perigee will coincide with the equinox of spring; whereas at the creation of man it coincided with the autumnal equinox. In the year 1250 the major axis was perpendicular to the line of the equinoxes, and then the solar perigee coincided with the solstice of winter, and the apogee with the solstice of summer. On that account Laplace proposed the year 1250 as a universal epoch, and that the vernal equinox of that year should be the first day of the first year.

The variations in the positions of the solar ellipse occasion corresponding changes in the length of the seasons. In its present position spring is shorter than summer, and autumn longer than winter; and while the solar perigee continues as it now is, between the solstice of winter and the equinox of spring, the period including spring and summer will be longer than that including autumn and winter: in this century the difference is about seven days. These intervals will be equal towards the year 6485, when the perigee comes to the equinox of spring. Were the earth's orbit circular, the seasons would be equal; their differences arise from the eccentricity of the earth's orbit, small as it is; but the changes are so gradual as to be imperceptible in the short space of human life.

No circumstance in the whole science of astronomy excites a deeper interest than its application to chronology. '*Whole nations*,' says Laplace, '*have been swept from the earth, with their language, arts and sciences, leaving but confused masses of ruin to mark the place where mighty cities stood; their history, with the exception of it few doubtful traditions, has perished; but the perfection of their astronomical observations marks their high antiquity, fixes the periods of their existence, and proves that even at that early period they must have made considerable progress in science.*'

The ancient state of the heavens may now be computed with great accuracy; and by comparing the results of computation with ancient observations, the exact period at which they were made may be verified if true, or if false, their error may be detected. If the date be accurate, and the observation good, it will verify the accuracy of modern tables, and show to how many centuries they may be extended, without the fear of error. A few examples will show the importance of this subject.

At the solstices the sun is at his greatest distance from the equator, consequently his declination at these times is equal to the obliquity of the ecliptic, which in former times was determined from the meridian length of the shadow of the style of a dial on the day of the solstice. The lengths of the meridian shadow at the summer and winter solstice are recorded to have been observed at the city of Layang, in China, 1100 years before the Christian era. From these, the distances of the sun from the zenith of the city of Layang are known. Half the sum of these zenith distances determines the latitude, and half their difference gives the obliquity of the ecliptic at the period of the observation; and as the law of the variation in the obliquity is known, both the time and place of the observations have been verified by computation from modern tables. Thus the Chinese had made some advances in the science of astronomy at that early period; the whole chronology of the Chinese is founded on the observations of eclipses, which prove the existence of that empire for more than 4700 years. The epoch of the lunar tables of the Indians, supposed by Bailly[41] to be 3000 before the Christian era, was proved by Laplace from the acceleration of the moon, not to be more ancient than the time of Ptolemy. The great inequality of Jupiter and Saturn whose cycle embraces 929 years, is peculiarly fitted for marking the civilization of a people. The Indians had determined the mean motions of these two planets in that part of their periods when the apparent mean motion of Saturn was at the slowest, and that of Jupiter the most rapid. The periods in which that happened were 3102 years before the Christian era, and the year 1491 after it.

The returns of comets to their perihelia may possibly mark the present state of astronomy to future ages.

The places of the fixed stars are affected by the precession of the equinoxes; and as the law of that variation is known, their positions at any time may be computed. Now Eudoxus,[42] a contemporary of Plato, mentions a star situate in the pole of the equator, and from computation it appears that *c* Draconis was not very far from that place about 3000 years ago; but as Eudoxus lived only about 2150 years ago, be must have described an anterior state of the heavens, supposed to be the same that was determined by Chiron,[43] about the time of the siege of Troy. Every circumstance concurs in showing that astronomy was cultivated in the highest ages of antiquity.

A knowledge of astronomy leads to the interpretation of hieroglyphical characters, since astronomical signs are often found on the ancient Egyptian monuments, which were probably employed by the priests to record dates. On the ceiling of the portico of a temple among the ruins of Tentyris,[44] there is a long row of figures of men and animals, following each other in the same direction; among these are the twelve signs of the zodiac, placed according to the motion of the sun: it is probable that the first figure in the procession represents the beginning of the year. Now the first is the Lion as if coming out of the temple; and as it is well known that the agricultural year of the Egyptians commenced at the solstice of summer, the epoch of the inundations of the Nile, if the preceding hypothesis be true, the solstice at the time the temple was built must have happened in the constellation of the lion; but as the solstice now happens $21^{o}.6$ north of the constellation of the Twins, it is easy to compute that the zodiac of Tentyris must have been made 4000 years ago.

The author had occasion to witness an instance of this most interesting application of astronomy, in ascertaining the date of a papyrus sent from Egypt by Mr. Salt,[45] in the hieroglyphical researches of the late Dr. Thomas Young,[46] whose profound and varied acquirements do honour not only to his country, but to the age in which he lived. The manuscript was found in a mummy case; it proved to be a horoscope of the age of Ptolemy, and its antiquity was determined from the configuration of the heavens at the time of its construction.

The form of the earth furnishes a standard of weights and measures for the ordinary purposes of life, as well as for the determination of the masses and distances of the heavenly bodies. The length of the pendulum vibrating seconds in the latitude of London forms the standard of the British measure of extension. Its length oscillating in vacuo at the temperature of 62° of Fahrenheit, and reduced to the level of the sea, was determined by Captain Kater,[47] in parts of the imperial standard yard, to be 39.1387 inches. The weight of a cubic inch of water at the temperature of 62° Fahrenheit, barometer 30, was also determined in parts of the imperial troy pound, whence a standard both of weight and capacity is deduced. The French have adopted the metre for their unit of linear measure, which is the ten millionth part of that quadrant of the meridian passing through Formentera[48] and Greenwich, the middle of which is nearly in the forty-fifth degree of latitude. Should the national standards of the two countries be lost in the vicissitudes of human affairs, both may be recovered, since they are derived from natural standards presumed to be invariable. The length of the pendulum would be found again with more facility than the metre; but as no measure is mathematically exact, an error in the original standard may at length become sensible in measuring a great extent, whereas the error that must necessarily arise in measuring the quadrant of the meridian is rendered totally insensible by subdivision in taking its ten millionth part. The French have adopted the decimal division not only in time, but in their degrees, weights, and measures, which affords very great facility in computation. It has not been adopted by any other people; though nothing is more desirable than that all nations should concur in using the same division and standards, not only on account of the convenience, but as affording a more definite idea of quantity. It is singular that the decimal division of the day, of degrees, weights and measures, was employed in China 4000 years ago; and that at the time Ibn Junis[49] made his observations at Cairo, about the year 1000, the Arabians were in the habit of employing the vibrations of the pendulum in their astronomical observations.

One of the most immediate and striking effects of a gravitating force external to the earth is the alternate rise and fall of the surface of the sea twice in the course of a lunar day, or $24^{h}\ 50^{m}\ 48^{s}$, of mean solar time. As it depends on the action of the sun and moon, it is classed among astronomical problems, of which it is by far the most difficult and the least satisfactory. The form of the surface of the ocean in equilibrio, when revolving with the earth round its axis, is an ellipsoid flattened at the poles; but the action of the sun and moon, especially of the moon, disturbs the equilibrium of the ocean.

If the moon attracted the centre of gravity of the earth and all its particles with equal and parallel forces, the whole system of the earth and the waters that cover it, would yield to these forces with a common motion, and the equilibrium of the seas would remain undisturbed. The difference of the forces, and the inequality of their directions, alone trouble the equilibrium.

It is proved by daily experience, as well as by strict mechanical reasoning, that if a number of waves or oscillations be excited in a fluid by different forces, each pursues its course, and has its effect independently of the rest. Now in the tides there are three distinct kinds of oscillations, depending on different causes, producing their effects independently of each other, which may therefore be estimated separately.

The oscillations of the first kind which are very small, are independent of the rotation of the earth; and as they depend on the motion of the disturbing body in its orbit, they are of long periods. The second kind of oscillations depends on the rotation of the earth, therefore their period is nearly a day: and the oscillations of the third kind depend on an angle equal to twice the angular rotation of the earth; and consequently happen twice in twenty-four hours. The first afford no particular interest, and are extremely small; but the difference of two consecutive tides depends on the second. At the time of the solstices, this difference which, according to Newton's theory, ought to be very great, is hardly sensible on our shores. Laplace has shown that this discrepancy arises from the depth of the sea, and that if the depth were uniform, there would be no difference in the consecutive tides, were it not for local circumstances: it follows therefore, that as this difference is extremely small, the sea, considered in a large extent, must be nearly of uniform depth, that is to say, there is a certain mean depth from which the deviation is not great. The mean depth of the Pacific ocean is supposed to be about four miles, that of the Atlantic only three. From the formulae which determine the difference of the consecutive tides it is also proved that the precession of the equinoxes, and the nutation in the earth's axis, are the same as if the sea formed one solid mass with the earth.

The third kind of oscillations are the semidiurnal tides, so remarkable on our coasts; they are occasioned by the combined action of the sun and moon, but as the effect of each is independent of the other, they may be considered separately.

The particles of water under the moon are more attracted than the centre of gravity of the earth, in the inverse ratio of the square of the distances; hence they have a tendency to leave the earth, but are retained by their gravitation, which this tendency diminishes. On the contrary, the moon attracts the centre of the earth more powerfully than she attracts the particles of water in the hemisphere opposite to her; so that the earth has a tendency to leave the waters but is retained by gravitation, which this tendency again diminishes. Thus the waters immediately under the moon are drawn from the earth at the same time that the earth is drawn from those which are diametrically opposite to her; in both instances producing an elevation of the ocean above the surface of equilibrium of nearly the same height; for the diminution of the gravitation of the particles in each position is almost the same, on account of the distance of the moon being great in comparison of the radius of the earth. Were the earth entirely covered by the sea, the water thus attracted by the moon would assume the form of an oblong spheroid, whose greater axis would point towards the moon, since the columns of water under the moon and in the direction diametrically opposite to her are rendered lighter, in consequence of the diminution of their gravitation; and in order to preserve the equilibrium, the axes 90^{o} distant would be shortened. The elevation, on account of the smaller space to which it is confined, is twice as great as the depression, because the contents of the spheroid always remain the same. The effects of the sun's attraction are in all respects similar to those of the moon's, though greatly less in degree, on account of his distance; he therefore only modifies the form of this spheroid it little. If the waters were capable of instantly assuming the form of equilibrium, that is, the form of the spheroid, its summit would always point to the moon, notwithstanding the earth's rotation; but on account of their resistance, the rapid motion produced in them by rotation prevents them from assuming at every instant the form which the equilibrium of the forces acting on them requires. Hence, on account of the inertia of the waters, if the tides be considered relatively to the whole earth and open sea, there is a meridian about 30° eastward of the moon, where it is always high water both in the hemisphere where the moon is, and in that which is opposite. On the west side of this circle the tide is flowing, on the east it is ebbing, and on the meridian at 90° distant, it is everywhere low water. It is evident that these tides must happen twice in a day, since in that time the rotation of the earth brings the same point twice tinder the meridian of the moon, once under the superior and once under the inferior meridian.

In the semidiurnal tides there are two phenomena particularly to be distinguished, one that happens twice in a month, and the other twice in a year.

The first phenomenon is, that the tides are much increased in the syzigies,[50] or at the time of new and full moon. In both cases the sun and moon are in the same meridian, for when the moon is new they are in conjunction, and when she is full they are in opposition. In each of these positions their action is combined to produce the highest or spring tides under that meridian, and the lowest in those points that are 90° distant. It is observed that the higher the sea rises in the full tide, the lower it is in the ebb. The neap tides take place when the moon is in quadrature, they neither rise so high nor sink so low it's the spring tides. The spring tides are much increased when the moon is in perigee. It is evident that the spring tides must happen twice a month, since in that time the moon is once new and once full.

The second phenomenon in the tides is the augmentation which occurs at the time of the equinoxes when the sun's declination is zero, which happens twice every year. The greatest tides take place when a new or full moon happens near the equinoxes while the moon is in perigee. The inclination of the moon's orbit on the ecliptic is $5^\circ 9'$; hence in the equinoxes the action of the moon would be increased if her node were to coincide with her perigee. The equinoctial gales often raise these tides to a great height. Beside these remarkable variations, there are others arising from the declination of the moon, which has a great influence on the ebb and flow of the waters.

Both the height and time of high water are thus perpetually changing; therefore, in solving the problem, it is required to determine the heights to which they rise, the times at which they happen, and the daily variations.

The periodic motions of the waters of the ocean on the hypothesis of an ellipsoid of revolution entirely covered by the sea, are very far from according with observation; this arises from the very great irregularities in the surface of the earth, which is but partially covered by the sea, the variety in the depths of the ocean, the manner in which it is spread out on the earth, the position and inclination of the shores, the currents, the resistance the waters meet with, all of them causes which it is impossible to estimate, but which modify the oscillations of the great mass of the ocean. However, amidst all these irregularities, the ebb and flow of the sea maintain a ratio to the forces producing them sufficient to indicate their nature, and to verify the law of the attraction of the sun and moon on the sea. Laplace observes, that the investigation of such relations between cause and effect is no less useful in natural philosophy than the direct solution of problems, either to prove the existence of the causes, or trace the laws of their effects. Like the theory of probabilities, it is a happy supplement to the ignorance and weakness of the human mind. Thus the problem of the tides does not admit of a general solution; it is certainly necessary to analyze the general phenomena which ought to result from the attraction of the sun and moon, but these must be corrected in each particular case by those local observations which are modified by the extent and depth of the sea, and the peculiar circumstances of the port.

Since the disturbing action of the sun and moon can only become sensible in a very great extent of water, it is evident that the Pacific ocean is one of the principal sources of our tides; but in consequence of the rotation of the earth, and the inertia of the ocean, high water does not happen till some time after the moon's southing. The tide raised in that world of waters is transmitted to the Atlantic, and from that sea it moves in a northerly direction along the coasts of Africa and Europe, arriving later and later at each place. This great wave however is modified by the tide raised in the Atlantic, which sometimes combines with that from the Pacific in raising the sea, and sometimes is in opposition to it, so that the tides only rise in proportion to their difference. This great combined wave, reflected by the shores of the Atlantic, extending nearly from pole to pole, still coming northward, pours through the Irish and British channels into the North sea, so that the tides in our ports are modified by those of another hemisphere. Thus the theory of the tides in each port, both as to their height and the times at which they take place, is really a matter of experiment, and can only be perfectly determined by the mean of a very great number of observations including several revolutions of the moon's nodes.

The height to which the tides rise is much greater in narrow channels than in the open sea, on account of the obstructions they meet with. In high latitudes where the ocean is less directly under the influence of the luminaries, the rise and fall of the sea is inconsiderable, so that, in all probability, there is no tide at the poles, or only a small annual and monthly one. The ebb and flow of the sea are perceptible in rivers to a very great distance from their estuaries. In the straits of Pauxis, in the river of the Amazons, more than five hundred miles from the sea, the tides are evident. It requires so many days for the tide to ascend this mighty stream, that the returning tides meet a succession of those which are coming up; so that every possible variety occurs in some part or other of its shores, both as to magnitude and time. It requires a very wide expanse of water to accumulate the impulse of the sun and moon, so as to render their influence sensible; on that account the tides in the Mediterranean and Black Sea are scarcely perceptible.

These perpetual commotions in the waters of the ocean are occasioned by forces that bear a very small proportion to terrestrial gravitation: the sun's action in raising the ocean is only the 1/38,448,000 of gravitation at the earth's surface, and the action of the moon is little more than twice as much, these forces being in the ratio of 1 to 2.35333. From this ratio the mass of the moon is found to be only 1/75th part of that of the earth. The initial state of the ocean has no influence on the tides; for whatever its primitive conditions may have been, they must soon have vanished by the friction and mobility of the fluid. One of the most remarkable circumstances in the theory of the tides is the assurance that in consequence of the density of the sea being only one-fifth of the mean density of the earth, the stability of the equilibrium of the ocean never can be subverted by any physical cause whatever. A general inundation arising from the mere instability of the ocean is therefore impossible.

The atmosphere when in equilibrio is an ellipsoid flattened at the poles from its rotation with the earth: in that state its strata are of uniform density at equal heights above the level of the sea, and it is sensibly of finite extent, whether it consists of particles infinitely divisible or not. On the latter hypothesis it must really be finite; and even if the particles of matter be infinitely divisible, it is known by experience to be of extreme tenuity at very small heights. The barometer rises in proportion to the superincumbent pressure. Now at the temperature of melting ice, the density of mercury is to that of air as 10,320 to 1; and as the mean height of the barometer is 29.528 inches, the height of the atmosphere by simple proportion is 30,407 feet, at the mean temperature of 62°, or 34,153 feet, which is extremely small, when compared with the radius of the earth. The action of the sun and moon disturbs the equilibrium of the atmosphere, producing oscillations similar to those in the ocean, which occasion periodic variations in the heights of the barometer. These, however, are so extremely small, that their existence in latitudes so far removed from the equator is doubtful; a series of observations within the tropics can alone decide this delicate point. Laplace seems to think that the flux and reflux distinguishable at Paris may be occasioned by the rise and fall of the ocean, which forms a variable base to so great a portion of the atmosphere.

The attraction of the sun and moon has no sensible effect on the trade winds; the beat of the sun occasions these aerial currents by rarefying the air at the equator, which causes the cooler and more dense part of the atmosphere to rush along the surface of the earth to the equator, while that which is heated is carried along the higher strata to the poles, forming two currents in the direction of the meridian. But the rotatory velocity of the air corresponding to its geographical situation decreases towards the poles; in approaching the equator it must therefore revolve more slowly than the corresponding parts of the earth, and the bodies on the surface of the earth must strike against it with the excess of their velocity, and by its reaction they will meet with a resistance contrary to their motion of rotation; so that the wind will appear, to a person supposing himself to be at rest, to blow in a contrary direction to the earth's rotation, or from east to west, which is the direction of the trade winds. The atmosphere scatters the sun's rays, and gives all the beautiful tints and cheerfulness of day. It transmits the blue light in greatest abundance; the higher we ascend, the sky assumes a deeper hue, but in the expanse of space the sun and stars must appear like brilliant specks in profound blackness.

The sun and most of the planets appear to be surrounded with atmospheres of considerable density. The attraction of the earth has probably deprived the moon of hers, for the refraction of the air at the surface of the earth is at least a thousand times as great as at the moon. The lunar atmosphere, therefore, must be of a greater degree of rarity than can be produced by our best air-pumps; consequently no terrestrial animal could exist in it. Many philosophers of the highest authority concur in the belief that light consists in the undulations of a highly elastic ethereal medium pervading space, which, communicated to the optic nerves, produce the phenomena of vision. The experiments of our illustrious countryman, Dr. Thomas Young, and those of the celebrated Fresnel,[51] show that this theory accords better with all the observed phenomena than that of the emission of particles from the luminous body. As sound is propagated by the undulations of the air, its theory is in a great many respects similar to that of light. The grave or low tones are produced by very slow vibrations, which increase in frequency progressively as the note becomes more acute. When the vibrations of a musical chord, for example, are less than sixteen in a second, it will not communicate a continued sound to the ear; the vibrations or pulses increase in number with the acuteness of the note, till at last all sense of pitch is lost. The whole extent of human hearing, from the lowest notes of the organ to the highest known cry of insects, as of the cricket, includes about nine octaves.

The undulations of light are much more rapid than those of sound, but they are analogous in this respect, that as the frequency of the pulsations in sound increases from the low tones to the higher, so those of light augment in frequency, from the red rays of the solar spectrum to the extreme violet. By the experiments of Sir William Herschel,[52] it appears that the heat communicated by the spectrum increases from the violet to the red rays; but that the maximum of the hot invisible rays is beyond the extreme red. Heat in all probability consists, like light and sound, in the undulations of an elastic medium. All the principal phenomena of heat may actually be illustrated by a comparison with those of sound. The excitation of heat and sound are not only similar, but often identical, as in friction and percussion; they are both communicated by contact and by radiation; and Dr. Young observes, that the effect of radiant beat in raising the temperature of a body upon which it falls, resembles the sympathetic agitation of a string, when the sound of another string, which is in unison with it, is transmitted to it through the air. Light, heat, sound, and the waves of fluids are all subject to the same laws of reflection, and, indeed, their undulating theories are perfectly similar. If, therefore, we may judge from analogy, the undulations of the heat producing rays must be less frequent than those of the extreme red of the solar spectrum; but if the analogy were perfect, the interference of two hot rays ought to produce cold, since darkness results from the interference of two undulations of light, silence ensues from the interference of two undulations of sound; and still water, or no tide, is the consequence of the interference of two tides.

The propagation of sound requires a much denser medium than that of either light or heat; its intensity diminishes as the rarity of the air increases; so that, at a very small height above the surface of the earth, the noise of the tempest ceases, and the thunder is heard no more in those boundless regions where the heavenly bodies accomplish their periods in eternal and sublime silence.

What the body of the sun may be, it is impossible to conjecture; but he seems to be surrounded by an ocean of flame, through which his dark nucleus appears like black spots, often of enormous size. The solar rays, which probably arise from the chemical processes that continually take place at his surface, are transmitted through space in all directions; but, notwithstanding the sun's magnitude, and the inconceivable heat that must exist where such combustion is going on, as the intensity both of his light and heat diminishes with the square of the distance, his kindly influence can hardly be felt at the boundaries of our system. Much depends on the manner in which the rays fall, as we readily perceive from the different climates on our globe. In winter the earth is nearer the sun by 1/30th than in summer, but the rays strike the northern hemisphere more obliquely in winter than in the other half of the year. In Uranus the sun must be seen like a small but brilliant star, not above the hundred and fiftieth part so bright as he appears to us; that is however 2,000 times brighter than our moon to us, so that he really is a sun to Uranus, and probably imparts some degree of warmth. But if we consider that water would not remain fluid in any part of Mars, even at his equator, and that in the temperate zones of the same planet even alcohol and quicksilver would freeze, we may form some idea of the cold that must reign in Uranus, unless indeed the ether has a temperature. The climate of Venus more nearly resembles that of the earth, though, excepting perhaps at her poles, much too hot for animal and vegetable life as they exist here; but in Mercury the mean heat, arising only from the intensity of the sun's rays, must be above that of boiling quicksilver, and water would boil even at his poles. Thus the planets, though kindred with the earth in motion and structure, are totally unfit for the habitation of such a being as man.

The direct light of the sun has been estimated to be equal to that of 5,563 wax candles of a moderate size, supposed to be placed at the distance of one foot from the object: that of the moon is probably only equal to the light of one candle at the distance of twelve feet; consequently the light of the sun is more than three hundred thousand times greater than that of the moon; for which reason the light of the moon imparts no heat, even when brought to a focus by a mirror.

In adverting to the peculiarities in the form and nature of the earth and planets, it is impossible to pass in silence the magnetism of the earth, the director of the mariner's compass, and his guide through the ocean. This property probably arises from metallic iron in the interior of the earth, or from the circulation of currents of electricity round it: its influence extends over every part of its surface, but its accumulation and deficiency determine the two poles of this great magnet, which are by no means the same as the poles of the earth's rotation. In consequence of their attraction and repulsion, a needle freely suspended, whether it be magnetic or not, only remains in equilibrio when in the magnetic meridian, that is, in the plane which passes through the north and south magnetic poles. There are places where the magnetic meridian coincides with the terrestrial meridian; in these a magnetic needle freely suspended, points to the true north, but if it be carried successively to different places on the earth's surface, its direction will deviate sometimes to the east and sometimes to the west of north. Lines drawn on the globe through all the places where the needle points due north and south, are called lines of no variation, and are extremely complicated. The direction of the needle is not even constant in the same place, but changes in a few years, according to a law not yet determined. In 1657, the line of no variation passed through London. In the year 1819, Captain Parry,[53] in his voyage to discover the north-west passage round America, sailed directly over the magnetic pole; and in 1824, Captain Lyon, when on an expedition for the same purpose, found that the variation of the compass was $37^\circ\ 30'$ west, and that the magnetic pole was then situate in $63^\circ\ 26' 51''$ north latitude, and in $80^\circ\ 51' 25''$ west longitude. It appears however from later researches that the law of terrestrial magnetism is of considerable complication, and the existence of more than one magnetic pole in either hemisphere has been rendered highly probable. The needle is also subject to diurnal variations; in our latitudes it moves slowly westward from about three in the morning till two, and returns to its former position in the evening.

A needle suspended so as only to be moveable in the vertical plane, dips or becomes more and more inclined to the horizon the nearer it is brought to the magnetic pole. Captain Lyon found that the dip in the latitude and longitude mentioned was $86^\circ\ 32'$. What properties the planets may have in this respect, it is impossible to know, but it is probable that the moon has become highly magnetic, in consequence of her proximity to the earth, and because her greatest diameter always points towards it.

The passage of comets has never sensibly disturbed the stability of the solar system; their nucleus is rare, and their transit so rapid, that the time has not been long enough to admit of a sufficient accumulation of impetus to produce a perceptible effect. The comet of 1770 passed within 80,000 miles of the earth without even affecting our tides, and swept through the midst of Jupiter's satellites without deranging the motions of those little moons. Had the mass of that comet been equal to the mass of the earth, its disturbing action would have shortened the year by the ninth of a day; but, as Delambre's[54] computations from the Greenwich observations of the sun, show that the length of the year has not been sensibly affected by the approach of the comet, Laplace proved that its mass could not be so much as the 5,000th part of that of the earth. The paths of comets have every possible inclination to the plane of the ecliptic, and unlike the planets, their motion is frequently retrograde. Comets are only visible when near their perihelia. Then their velocity is such that its square is twice as great as that of a body moving in a circle at the same distance; they consequently remain a very short time within the planetary orbits; and as all the conic sections of the same focal distance sensibly coincide through a small arc on each side of the extremity of their axis, it is difficult to ascertain in which of these curves the comets move, from observations made, as they necessarily must be, at their perihelia: but probably they all move in extremely eccentric ellipses, although, in most cases, the parabolic curve coincides most nearly with their observed motions. Even if the orbit be determined with all the accuracy that the case admits of, it may be difficult, or even impossible, to recognize a comet on its return, because its orbit would be very much changed if it passed near any of the large planets of this or of any other system, in consequence of their disturbing energy, which would be very great on bodies of so rare a nature. Halley[55] and Clairaut[56] predicted that, in consequence of the attraction of Jupiter and Saturn, the return of the comet of 1759[57] would be retarded 618 days, which was verified by the event as nearly as could be expected.

The nebulous appearance of comets is perhaps occasioned by the vapours which the solar heat raises at their surfaces in their passage at the perihelia, and which are again condensed as they recede from the sun. The comet of 1680[58] when in its perihelion was only at the distance of one-sixth of the sun's diameter, or about 148,000 miles from its surface; it consequently would be exposed to a heat 27,500 times greater than that received by the earth. As the sun's heat is supposed to be in proportion to the intensity of his light, it is probable that a degree of heat so very intense would be sufficient to convert into vapour every terrestrial substance with which we are acquainted.

In those positions of comets where only half of their enlightened hemisphere ought to be seen, they exhibit no phases even when viewed with high magnifying powers. Some slight indications however were once observed by Hevelius[59] and Lahire[60] in 1682; and in 1811 Sir William Herschel discovered a small luminous point, which he concluded to be the disc of the comet. In general their masses are so minute, that they have no sensible diameters, the nucleus being principally formed of denser strata of the nebulous matter, but so rare that stars have been seen through them. The transit of a comet over the sun's disc would afford the best information on this point. It was computed that such an event was to take place in the year 1827; unfortunately the sun was hid by clouds in this country, but it was observed at Viviers and at Marseilles at the time the comet must have been on it, but no spot was seen. The tails are often of very great length, and are generally situate in the planes of their orbits; they follow them in their descent towards the sun, but precede them in their return, with a small degree of curvature; but their extent and form must vary in appearance, according to the position of their orbits with regard to the ecliptic. The tail of the comet of 1680[61] appeared, at Paris, to extend over sixty-two degrees. The matter of which the tail is composed must be extremely buoyant to precede a body moving with such velocity; indeed the rapidity of its ascent cannot be accounted for. The nebulous part of comets diminishes every time they return to their perihelia; after frequent returns they ought to lose it altogether, and present the appearance of a fixed nucleus; this ought to happen sooner in comets of short periods. Laplace supposes that the comet of 1682 must be approaching rapidly to that state. Should the substances be altogether or even to a great degree evaporated, the comet will disappear for ever. Possibly comets may have vanished from our view sooner than they otherwise would have done from this cause. Of about six hundred comets that have been seen at different times, three are now perfectly ascertained to form part of our system; that is to say, they return to the sun at intervals of 76, 6.33, and 3.25 years nearly.

A hundred and forty comets have appeared within the earth's orbit during the last century that have not again been seen; if a thousand years be allowed as the average period of each, it may be computed by the theory of probabilities, that the whole number that range within the earth's orbit must be 1,400; but Uranus being twenty times more distant, there may be no less than

11,200,000 comets that come within the known extent of our system. In such a multitude of wandering bodies it is just possible that one of them may come in collision with the earth; but even if it should, the mischief would be local, and the equilibrium soon restored. It is however more probable that the earth would only be deflected a little from its course by the near approach of the comet, without being touched. Great as the number of comets appears to be, it is absolutely nothing when compared to the number of the fixed stars. About two thousand only are visible to the naked eye, but when we view the heavens with a telescope, their number seems to be limited only by the imperfection of the instrument. In one quarter of an hour Sir William Herschel estimated that 116,000 stars passed through the field of his telescope, which subtended an angle of 15′. This however was stated as a specimen of extraordinary crowding; but at an average the whole expanse of the heavens must exhibit about a hundred millions of fixed stars that come within the reach of telescopic vision.

Many of the stars have a very small progressive motion, especially *m* Cassiopeia and 61 Cygni, both small stars; and, as the sun is decidedly, a star, it is an additional reason for supposing the solar system to be in motion. The distance of the fixed stars is too great to admit of their exhibiting a sensible disc; but in all probability they are spherical, and must certainly be so, if gravitation pervades all space. With a fine telescope they appear like a point of light; their twinkling arises from sudden changes in the refractive power of the air, which would not be sensible if they had discs like the planets. Thus we can learn nothing of the relative distances of the stars from us and from one another, by their apparent diameters; but their annual parallax being insensible, shows that we must be one hundred millions of millions of miles from the nearest; many of them however must be vastly more remote, for of two stars that appear close together, one may be, far beyond the other in the depth of space. The light of Sirius, according to the observations of Mr. Herschel, is 324 times greater than that of a star of the sixth magnitude; if we suppose the two to be really of the same size, their distances from us must be in the ratio of 57.3 to 1, because light diminishes as the square of the distance of the luminous body increases.

Of the absolute magnitude of the stars, nothing is known, only that many of them must be much larger than the sun, from the quantity of light emitted by them. Dr. Wollaston[62] determined the approximate ratio that the light of a wax candle bears to that of the sun, moon, and stars, by comparing their respective images reflected from small glass globes filled with mercury, whence a comparison was established between the quantities of light emitted by the celestial bodies themselves. By this method he found that the light of the sun is about twenty millions of millions of times greater than that of Sirius, the brightest, and supposed to be the nearest of the fixed stars. If Sirius had a parallax of half a second, its distance from the earth would be 525,481 times the distance of the sun from the earth; and therefore Sirius, placed where the sun is, would appear to us to be 3.7 times as large as the sun, and would give 13.8 times more light; but many of the fixed stars must be immensely greater than Sirius. Sometimes stars have all at once appeared, shone with a brilliant light, and then vanished. In 1572 a star was discovered in Cassiopeia, which rapidly increased in brightness till it even surpassed that of Jupiter;[63] it then gradually diminished in splendour, and after exhibiting all the variety of tints that indicates the changes of combustion, vanished sixteen months after its discovery, without altering its position. It is impossible to imagine any thing more tremendous than a conflagration that could be visible at such a distance. Some stars are periodic, possibly from the intervention of opaque bodies revolving about them, or from extensive spots on their surfaces. Many thousands of stars that seem to be only brilliant points, when carefully examined are found to be in reality systems of two or more suns revolving about a common centre. These double and multiple stars are extremely remote, requiring the most powerful telescopes to show them separately.

The first catalogue of double stars in which their places and relative positions are determined, was accomplished by the talents and industry of Sir William Herschel, to whom astronomy is indebted for so many brilliant discoveries, and with whom originated the idea of their combination in binary and multiple systems, an idea which his own observations had completely established, but which has since received additional confirmation from those of his son[64] and Sir James South,[65] the former of whom, as well as Professor Struve of Dorpat,[66] have added many thousands to their numbers. The motions of revolution round a common centre of many have been clearly established, and their periods determined with considerable accuracy. Some have already since their first discovery accomplished nearly a whole revolution, and one, if the latest observations can be depended on, is actually considerably advanced in its second period. These interesting systems thus present a species of sidereal chronometer, by which the chronology of the heavens will be marked out to future ages by epochs of their own, liable to no fluctuations from planetary disturbances such as obtain in our system.

Possibly among the multitudes of small stars, whether double or insulated, some may be found near enough to exhibit distinct parallactic motions, or perhaps something, approaching to planetary motion, which may prove that solar attraction is not confined to our system, or may lead to the discovery of the proper motion of the sun. The double stars are of various hues, but most frequently exhibit the contrasted colours. The large star is generally yellow, orange, or red; and the small star blue, purple, or green. Sometimes a white star is combined with a blue or purple, and more rarely a red and white are united. In many cases, these appearances are due to the influences of contrast on our judgment of colours. For example, in observing a double star where the large one is of a full ruby red, or almost blood colour, and the small one a fine green, the latter lost its colour when the former was hid by the cross wires of the telescope. But there are a vast number of instances where the colours are too strongly marked to be merely imaginary. Mr. Herschel observes in one of his papers in the *Philosophical Transactions*, as a very remarkable fact, that although red single stars are common enough, no example of an insulated blue, green, or purple one has as yet been produced.

In some parts of the heavens, the stars are so near together as to form clusters, which to the unassisted eye appear like thin white clouds: such is the milky way, which has its brightness from the diffused light of myriads of stars. Many of these clouds, however, are never resolved into separate stars, even by the highest magnifying powers. This nebulous matter exists in vast abundance in space. No fewer than 2,500 nebulae were observed by Sir William Herschel, whose places have been computed from his observations, reduced to a common epoch, and arranged into a catalogue in order of right ascension by his sister Miss Caroline Herschel,[67] a lady so justly celebrated for astronomical knowledge and discovery. The nature and use of this matter scattered over the heavens in such a variety of forms is involved in the greatest obscurity. That it is a self-luminous, phosphorescent material substance, in a highly dilated or gaseous state, but gradually subsiding by the mutual gravitation of its particles into stars and sidereal systems, is the hypothesis which seems to be most generally received; but the only way that any real knowledge on this mysterious subject can be obtained, is by the determination of the form, place, and present state of each individual nebula, and a comparison of these with future observations will show generations to come the changes that may now be going on in these rudiments of future systems. With this view, Mr. Herschel is now engaged in the difficult and laborious investigation, which is understood to be nearly approaching its completion, and the results of which we may therefore hope ere long to see made public. The most conspicuous of these appearances are found in Orion, and in the girdle of Andromeda. It is probable that light must be millions of years travelling to the earth from some of the nebulae.

So numerous are the objects which meet our view in the heavens, that we cannot imagine a part of space where some light would not strike the eye: but as the fixed stars would not be visible at such distances, if they did not shine by their own light, it is reasonable to infer that they are suns; and if so, they are in all probability attended by systems of opaque bodies, revolving about them as the planets do about ours. But although there be no proof that planets not seen by us revolve about these remote suns, certain it is, that there are many invisible bodies wandering in space, which, occasionally coming within the sphere of the earth's attraction are ignited by the velocity with which they pass through the atmosphere, and are precipitated with great violence on the earth. The obliquity of the descent of meteorites, the peculiar matter of which they are composed, and the explosion with which their fall is invariably accompanied, show that they are foreign to our planet. Luminous spots altogether independent of the phases have occasionally appeared on the dark part of the moon, which have been ascribed to the light arising from the eruption of volcanoes; whence it has been supposed that meteorites have been projected from the moon by the impetus of volcanic eruption; it has even been computed, that if a stone were projected from the moon in a vertical line, and with an initial velocity of 10,992 feet in a second, which is more than four times the velocity of a ball when first discharged from a cannon, instead of falling back to the moon by the attraction of gravity, it would come within the sphere of the earth's attraction, and revolve about it like a satellite. These bodies, impelled either by the direction of the primitive impulse, or by the disturbing action of the sun, might ultimately penetrate the earth's atmosphere, and arrive at its surface. But from whatever source meteoric stones may come, it seems highly probable, that they have a common origin, from the uniformity, we may almost say identity, of their chemical composition.

The known quantity of matter bears a very small proportion to the immensity of space. Large as the bodies are, the distances that separate them are immeasurably greater; but as design is manifest in every part of creation, it is probable that if the various systems in the universe had been nearer to one another, their mutual disturbances would have been inconsistent with the harmony and stability of the whole. It is clear that space is not pervaded by atmospheric air, since its resistance would long ere this have destroyed the velocity of the planets; neither can we affirm it to be void, when it is traversed in all directions by light, heat, gravitation, and possibly by influences of which we can form no idea; but whether it be replete with an ethereal medium, time alone will show.

Though totally ignorant of the laws which obtain in the more distant regions of creation, we are assured, that one alone regulates the motions of our own system; and as general laws form the ultimate object of philosophical research, we cannot conclude these remarks without considering the nature of that extraordinary power, whose effects we have been endeavouring to trace through some of their mazes. It was at one time imagined, that the acceleration in the moon's mean motion was occasioned by the successive transmission of the gravitating force; but it has been proved, that, in order to produce this effect, its velocity must be about fifty millions of times greater than that of light, which flies at the rate of 200,000 miles in a second: its action even at the distance of the sun may therefore be regarded as instantaneous; yet so remote are the nearest of the fixed stars, that it may be doubted whether the sun has any sensible influence on them.

The analytical expression for the gravitating force is a straight line; the curves in which the celestial bodies move by the force of gravitation are only lines of the second order; the attraction of spheroids according to any other law would be much more complicated; and as it is easy to prove that matter might have been moved according to an infinite variety of laws, it may be concluded, that gravitation must have been selected by Divine wisdom out of an infinity of other laws, as being the most simple, and that which gives the greatest stability to the celestial motions.

It is a singular result of the simplicity of the laws of nature, which admit only of the observation and comparison of ratios, that the gravitation and theory of the motions of the celestial bodies are independent of their absolute magnitudes and distances; consequently if all the bodies in the solar system, their mutual distances, and their velocities, were to diminish proportionally, they would describe curves in all respects similar to those in which they now move; and the system might be successively reduced to the smallest sensible dimensions, and still exhibit the same appearances. Experience shows that a very different law of attraction prevails when the particles of matter are placed within inappreciable distances from each other, as in chemical and capillary attractions, and the attraction or cohesion; whether it be a modification of gravity, or that some new and unknown power comes into action, does not appear; but as a change in the law of the force takes place at one end of the scale, it is possible that gravitation may not remain the same at the immense distance of the fixed stars. Perhaps the day may come when even gravitation, no longer regarded as an ultimate principle, may be resolved into a yet more general cause, embracing every law that regulates the material world.

The action of the gravitating force is not impeded by the intervention even of the densest substances. If the attraction of the sun for the centre of the earth, and for the hemisphere diametrically opposite to him, was diminished by a difficulty in penetrating the interposed matter, the tides would be more obviously affected. Its attraction is the same also, whatever the substances of the celestial bodies may be, for if the action of the sun on the earth differed by a millionth part from his action on the moon, the difference would occasion a variation in the sun’s parallax mounting to several seconds, which is proved to be impossible by the agreement of theory with observation. Thus all matter is pervious to gravitation, and is equally attracted by it.

As far as human knowledge goes, the intensity of gravitation has never varied within the limits of the solar system; nor does even analogy lead us to expect that it should; on the contrary, there is every reason to be assured, that the great laws of the universe are immutable like their Author. Not only the sun and planets, but the minutest particles in all the varieties of their attractions and repulsions, nay even the imponderable matter of the electric, galvanic, and magnetic fluids are obedient to permanent laws, though we may not be able in every case to resolve their phenomena into general principles. Nor can we suppose the structure of the globe alone to be exempt from the universal fiat, though ages may pass before the changes it has undergone, or that are now in progress, can be referred to existing causes with the same certainty with which the motions of the planets and all their secular variations are referable to the law of gravitation. The traces of extreme antiquity perpetually occurring to the geologist, give that information as to the origin of things which we in vain look for in the other parts of the universe. They date the beginning of time; since there is every reason to believe, that the formation of the earth was contemporaneous with that of the rest of the planets, but they show that creation is the work of Him with whom ‘*a thousand years are as one day, and one day as a thousand years.*’

Notes

[1] Newton, Sir Isaac, 1642-1727, physicist and mathematician. Born in Woolsthorpe, England. Newton's major work, the *Principia* was published in 1687. The *Principia* demonstrated that celestial bodies follow the laws of dynamics and formulated the law of universal gravitation. Newton also independently developed the methods of integral and

differential calculus which he called his "method of Fluxions." The Fluxions method was documented in his *Methodis Serierum et Fluxionum* written in 1671, but unpublished until 1736. This delay led to a bitter rivalry between Newton and Leibniz who independently developed both the foundations of and notation of calculus in use today.

2 Mackintosh, Sir James, 1765-1832, physician, lawyer, and philosopher. Born near Inverness, Scotland.

3 Kepler, Johannes, 1571-1630, astronomer, born in Weil-der-Stadt, Germany. Kepler worked with Tycho Brahe, showing that planetary motions were far simpler than had been imagined. He announced his first and second laws of planetary motion in *Astronomia nova* (1609). Kepler's third law was formulated in *Harmonice mundi* (1619). Kepler also made several important contributions to mathematics including a method for finding the volume of a solid of revolution (1616) which contributed to the later development of calculus.

4 *…proportional to the cubes of their mean distance from his centre.* Kepler's 3rd law.

5 *nutation.* An oscillatory movement of the axis of a rotating body (as the earth). *Merriam-Webster's Dictionary.*

6 *aphelion.* The point that is farthest from the sun in the orbit of a planet or comet.

7 *vectores.* Archaic spelling for "vectors."

8 *equal areas in equal times.* Kepler's 2nd law.

9 *Pallas.* The "recently discovered planets" are the first four asteroids (Ceres, Pallas, Juno and Vesta) discovered in the first decade of the 19th century. Ceres was discovered by the Italian monk Giuseppe Piazzi (1746-1826) at the Palermo Observatory on January 1, 1801. Pallas was discovered by Olbers in 1802, Juno by Harding in 1804, and Vesta by Olbers in 1807. The four asteroids are all over 200 km in diameter with Ceres the largest (1,003 km). Several hundred thousand asteroids are now known and 26 of those larger than 200 km in diameter.

10 See previous note.

11 *apsides.* The point in an astronomical orbit at which the distance of the body from the center of attraction is either greatest or least. *Merriam-Webster's Collegiate Dictionary.*

12 *retrograde.* In astronomy, moving in a direction opposite to that of the movement of the earth around the sun.

13 *Sidereal revolution (or year)*. In astronomy, the period in which the fixed stars apparently complete a revolution and come to the same point in the heavens. *Webster's 1828 Dictionary.*

14 *tropical year.* The space or period of time in which the sun moves through the twelve signs of the ecliptic, or whole circle, and returns to the same point. This is the solar year, and the year, in the strict and proper sense of the word. It is called also the tropical year. This period comprehends what are called the twelve calendar months, or 365 days, 5 hours, and 49 minutes, within a small fraction. But in popular usage, the year consists of 365 days, and every fourth year of 366; a day being added to February, on account of the 5 hours and 49 minutes. *Webster's 1828 Dictionary.*

15 Ptolemy, fl.127-145, Greek astronomer and geographer. Ptolemy's major work, the *Almagest* formulated the principles of the geocentric system of the world. Ptolemy also compiled works in geography and music.

16 Lagrange, Joseph Louis, 1736-1813 an Italian-French mathematician and astronomer, born in Turin, Italy. His major work was the *Mécanique analytique* (1788). Lagrange taught at the École Polytechnique, and was instrumental in the reform of the metric system (1795). Lagrange made important contributions to number theory, the calculus of variations, the principle of least action, kinetic energy, and on the stability of the solar system.

17 Playfair, John, 1748-1819, mathematician and geologist, born in Benvie, Dundee, Scotland, UK. He taught at Edinburgh (1785), where he held a joint appointment in mathematics and natural philosophy. He wrote on geometry.

18 See note 4, *Foreword to the Second Edition.*

19 *speed of light.* The modern value is nearly 186,000 miles/s or 300,000 km/s.

20 Book III, Chapter VI considers the properties of an ethereal medium.

21 Bacon, Francis, 1561-1626, philosopher and scientist, born in London, England. He studied at Cambridge and Gray's Inn (1576), and was called to the bar in 1582. His major works *are The Advancement of Learning* (1605) and *Novum Organum* (1620). Bacon is considered by many as the founder of the modern scientific method.

22 The Royal Greenwich Observatory (RGO) was founded in 1675 by decree of Charles II for the sole purpose of improving the level of astronomical knowledge required to support navigation at sea. Almost a century was to elapse between the founding of the RGO and the first edition of *The Nautical Almanac* (1767) under the fifth Astronomer Royal, Nevil Maskelyne. This almanac contained tabulations of the distances of the Moon's centre from the Sun and from the bright stars for every three hours, so that the navigator could determine Greenwich time and hence his longitude from observations of such lunar distances. *HM Nautical Almanac Office, Rutherford Appleton Laboratory.*

23 *Ephemeris.* Tables giving the projected positions of celestial bodies for every day of a certain period.

24 *parallax.* The apparent change in the position of an observed object when seen from two different points.

25 Eratosthenes, c. 194 BC, Greek astronomer, born in Cyrene. He was chief librarian at Alexandria, and is remembered for the first scientific calculation of the Earth's circumference.

[26] *precession of the equinoxes.* The earlier occurrence of the equinoxes in each sidereal year because of a slow variation in the rotation of the earth, caused by the gravitational pull of the sun and moon. *The Wordsmyth Educational Dictionary-Thesaurus.*
[27] Cook, James, 1728-1779, navigator, born in Marton, England. In an expedition to Tahiti for the Royal Society he observed a transit of Venus across the Sun (1768-71). In his second voyage he sailed round Antarctica (1772-75). His third voyage (1776-79) attempted to find a Northwest passage from the Pacific.
[28] At 4.2 light years, the nearest star *Proxima Centuri* is approximately 10 times farther than this distance.
[29] Struve, Friedrich Georg Wilhelm, 1793-1864, astronomer, born in Altona, Germany. He taught at Dorpat (1813), and directed the observatory there. In 1838 he made one of the first measurements of the parallax of the star Vega. Friedrich Bessel was the first to measure stellar parallax on star 61 Cygni (see also note 37, *bk. II, Chap. XIV*). Struve is rememb ered most for his pioneering studies of binary stars. In 1837 he published a catalogue of over 3000 binary systems in his *Stellarum Duplicium Mensurae Micrometricae*.
[30] *libration.* An oscillation in the apparent aspect of a secondary body (as a planet or a satellite) as seen from the primary object around which it revolves. *Merriam-Webster's Collegiate Dictionary.*
[31] See note 14, *Foreword to the Second Edition.*.
[32] Hipparchus, c. 150 BC, astronomer, born in Nicaea, Rhodes. He discovered the precession of the equinoxes, the eccentricity of the Sun's path, and the length of the solar year. He also drew up a catalogue of 1080 stars.
[33] *caeteris paribus.* New Latin, "other things being equal." *Merriam-Webster's Collegiate Dictionary.*
[34] Leslie, Sir John, 1766-1832, physicist, born in Largo, Scotland, UK. He taught mathematics and natural philosophy at Edinburgh. His inventions included a differential thermometer, a hygrometer, and a photometer.
[35] Young, Thomas, 1773-1829, physicist, physician, and Egyptologist (he helped decipher the Rosetta stone), born in Milverton, Somerset, England. He was professor of natural philosophy to the Royal Institution (1801). Young's research on interference established the wave theory of light. In the first draft of her autobiography Mary Somerville expresses great sympathy for Young who she says was forced to suffer the scorn of British science [which supported Newton's corpuscular theory of light] in reacting to Young's original results on the diffraction of light: "Young had to endure this for 14 years until Fresnel [and Arago] in France verified the results (Dep c.355, 22, MSAU-2: p. 118-120, *Mary Somerville Autobiography (first draft)*, Mary Somerville Collection, Bodleian Library, Oxford University)." Young was also the first to use the word energy in its modern sense. Young's modulus, a constant in the expression for elasticity, is named after him.
[36] Perkins, Jacob, 1766-1849, inventor; born in Newburyport, USA. An apprentice goldsmith. He developed steel plates as a replacement for copper in the bank note engraving process. Perkins moved to England (1818) where he produced the first penny postage stamps (1840).
[37] Timocharis, Aristarchus and Aristyllus were three astronomers who all worked at Alexandria. Timocharis observed around 290 BC while Aristyllus observed a generation later around 260 BC. We know that Aristarchus measured the ratio of the distances to the moon and to the sun and, although his methods could never yield accurate results, they did show that the sun was much further from the earth than was the moon. There is some evidence that the observations of Timocharis and Aristyllus were made to determine the constants in the heliocentric theory of Aristarchus and that Timocharis certainly began his observations some time before Aristarchus proposed his heliocentric universe. *The MacTutor History of Mathematics archive.*
[38] Bradley, James, 1693-1762, astronomer, born in Sherborne, England. Bradley taught astronomy at Oxford from 1721. He later succeeded Edmond Halley as professor of astronomy at Greenwich. His work on the aberration of light (1729) provided the first direct evidence of the Copernican hypothesis.
[39] *Brahmins.* A Hindu of the highest caste traditionally assigned to the priesthood. *Merriam-Webster's Collegiate Dictionary.*
[40] According to traditional literal interpretations of biblical chronologies.
[41] Bailly, Jean Sylvain, 1736-1793, astronomer, born in Paris, France. Bailly studied Halley's comet and the satellites of Jupiter. His major work, the *Histoire de l'astronomie* was published between 1775 and 1787.
[42] Eudoxus of Cnidus, c. 353 BC, Greek astronomer and mathematician, born in Cnidus, Asia Minor. Eudoxus introduced a system of 27 nested spheres to account for planetary motion. He also established geometric principles that helped in laying the foundations for Euclid.
[43] Chiron, a wise centaur. In Greek mythology Chiron received the legendary Achilles as a disciple.
[44] Known today as Dendera, Egypt. Site of the ancient temple of Aset and Hathor.
[45] Salt, Henry, 1780-1828, Essay on Dr. Young's and M. Champollion's phonetic system of hieroglyphics; with some additional discoveries, by which it may be applied to decipher the names of the ancient kings of Egypt and Ethiopia. London, Longman, Hurst, Rees, Orme, Brown, and Green, 1825.

[46] See note 35.
[47] Kater, Captain Henry, 1777-1835, *Mechanics*, London : Printed for Longman, Rees, Orme, Brown, and Green, and John Taylor, 1830.
[48] Formentera is the smallest inhabited island in the archipelago of the Balears (Balearic Islands) and also the closest to the meridian. It is situated at 38l N and 1l E.
[49] Ibn Junis, c. 1009, Muslim astronomer, author of the "Hakimite Tables."
[50] *sysigies (or sysygies).* in astronomy, the alignment, either in conjunction or opposition, of three celestial bodies within the same gravitational system, esp. the sun, moon, and earth. *The Wordsmyth Educational Dictionary-Thesaurus.*
[51] Fresnel, Augustin Jean, 1788-1827, physicist, born in Broglie, France. Fresnel's investigations on interference were instrumental in establishing the wave theory of light demonstrated earlier by Thomas Young in England.
[52] Herschel, Sir William (Frederick), 1738-1822, astronomer, born in Hanover, Germany. Herschel moved to England in 1757 and discovered Uranus, the first planet to be discovered since antiquity, in 1781. His fame spread quickly and was appointed private astronomer to King George III. He continued his research, assisted by his sister Caroline and his son John. In 1785 Herschel developed a cosmogony, a theory about the origin of the universe based on his hypothesis that nebulae were really large clusters of stars.
[53] Parry, Sir William Edward, 1790-1855, navigator, born in Bath, England. Parry commanded five Arctic expeditions (1818-27). Mary Somerville presented Parry with 'a large quantity of marmalade' for his 1824 Arctic expedition. Parry later named an island ($74^\circ\ 44'$N $96^\circ\ 10'$W) in the Canadian Arctic after her.
[54] Delambre, Jean Baptiste Joseph, 1749-1822, *Astronomie théorique et pratique*, Paris, Ve. Courcier, 1814.
[55] Halley, Edmund, 1656-1742, astronomer and mathematician, born in London, England. Halley correctly predicted the return in 1758 of the comet now named after him (see note 57). He encouraged and financed Isaac Newton's *Principia* (1687).
[56] Clairaut, Alexis Claude, 1713-1765, *Theorie de la figure de la terre, tirée des principes de l'hydrostratique*, Paris, 1743.
[57] In 1705 Edmund Halley predicted, using Newton's newly formulated laws of motion, that the comet seen in 1531, 1607, and 1682 would return in 1758 (after his death in 1742).
[58] Isaac Newton in the *Principia* (1687), applied his theory of gravitation to show that the 1680 comet moved in an elliptical very nearly parabolic orbit.
[59] Hevelius, Johannes, 1611-1687, astronomer, born in Gdansk, Poland. Hevelius catalogued over 1,500 stars in *Prodromus Astronomiae* (1690), and is credited with the discovery of four comets. Many features of the Moon were named by him in his lunar atlas, *Selenographia* (1647).
[60] Lahire, [Philippe] de, 1640-1718, *Memoires de mathematique et de physique, contenant un traité des epicycloïde, & de leurs usages dans les mechanique*, Paris, De l'Imprimerie royale, 1694.
[61] See note 58.
[62] Wollaston, William Hyde, 1766-1828, chemist, born in East Dereham, England. After an eleven year practice as a physician, Wollaston devoted his time to studies in chemistry, optics, and physiology. He discovered palladium and rhodium, and is credited with several important inventions.
[63] Known also as Cassiopeiae 1572. The Danish astronomer Tycho Brahe's precise measurements of parallax demonstrated that this Nova was not a nearby phenomenon. The Nova remained visible to the naked eye until 1574.
[64] Herschel, Sir John Frederick William, 1792-1871, a contemporary and close astronomer friend of Mary Somerville. Herschel was born in Slough, England. He was the son of Sir William Herschel. John Herschel was instrumental in the introduction of Leibniz notation into English mathematics. He was also a strong advocate of the wave nature of light, an active observational astronomer, and a pioneer in celestial photography. In 1849 he wrote a book for educated lay persons *Outlines of Astronomy*. Herschel also spent four years observing stars in the southern hemisphere where he recorded the locations of nearly 69,000 stars and other important objects.
[65] South, Sir James, Practical observations on the Nautical Almanac and Astronomical Ephemeris, London : William Sams, 1822.
[66] see note 29.
[67] Herschel, Caroline Lucretia, 1750-1848, astronomer, born in Hanover, Germany. She was the sister and assistant of Sir William Herschel. She is credited with the discovery of eight comets. She published a catalogue of 2500 nebulae and star clusters in 1822.

INTRODUCTION

PHYSICAL ASTRONOMY

THE infinite varieties of motion in the heavens, and on the earth, obey a few laws, so universal in their application, that they regulate the curve traced by an atom which seems to be the sport of the winds, with as much certainty as the orbits of the planets. These laws, on which the order of nature depends, remained unknown till the sixteenth century, when Galileo,[1] by investigating the circumstances of falling bodies, laid the foundation of the science of mechanics, which Newton,[2] by the discovery of gravitation, afterwards extended from the earth to the farthest limits of our system.

This original property of matter, by means of which we ascertain the past and anticipate the future, is the link which connects our planet with remote worlds, and enables us to determine distances, and estimate magnitudes, that might seem to be placed beyond the reach of human faculties. To discern and deduce from ordinary and apparently trivial occurrences the universal laws of nature, as Galileo and Newton have done, is a mark of the highest intellectual power.

Simple as the law of gravitation is, its application to the motions of the bodies of the solar system is a problem of great difficulty, but so important and interesting, that the solution of it has engaged the attention and exercised the talents of the most distinguished mathematicians; among whom Laplace[3] holds a distinguished place by the brilliancy of his discoveries, as well as from having been the first to trace the influence of this property of matter from the elliptical motions of the planets, to its most remote effects on their mutual perturbations. Such was the object contemplated by him in his splendid work on the Mechanism of the Heavens;[4] a work which may be considered as a great problem of dynamics, wherein it is required to deduce all the phenomena of the solar system from the abstract laws of motion, and to confirm the truth of those laws, by comparing theory with observation.

Tables of the motions of the planets, by which their places may be determined at any instant for thousands of years, are computed from the analytical formulae of Laplace. In a research so profound and complicated, the most abstruse analysis is required, the higher branches of mathematical science are employed from the first, and approximations are made to the most intricate series. Easier methods, and more convergent series, may probably be discovered in process of time, which will supersede those now in use; but the work of Laplace, regarded as embodying the results of not only his own researches, but those of so many of his illustrious predecessors and contemporaries, must ever remain, as he himself expressed it to the writer of these pages, a monument to the genius of the age in which it appeared.

Although physical astronomy is now the most perfect of sciences, a wide range is still left for the industry of future astronomers. The whole system of comets is a subject involved in mystery; they obey, indeed, the general law of gravitation, but many generations must be swept from the earth before their paths can be traced through the regions of space, or the periods of their return can be determined. A new and extensive field of investigation has lately been opened in the discovery of thousands of double stars, or, to speak more strictly, of systems of double stars, since many of them revolve round centres in various and long periods. Who can venture to predict when their theories shall be known, or what laws may be revealed by the knowledge of their motions?–but, perhaps, *Veniet tempus, in quo ista quae nunc latent, in lucem dies extrahat et longioris aevi diligentia: ad inquisitionem tantorum aetas una non sufficit. Veniet tempus, quo posteri nostri tam aperta nos nescisse mirentur.*[5]

It must, however, be acknowledged that many circumstances seem to be placed beyond our reach. The planets are so remote, that observation discloses but little of their structure; and although their similarity to the earth, in the appearance of their surfaces, and in their annual and diurnal revolutions producing the vicissitudes of seasons, and of day and night, may lead us to fancy that they are peopled with inhabitants like ourselves; yet, were it even permitted to form an analogy from the single instance of the earth, the only one known to us, certain it is that the physical nature of the inhabitants of the planets, if such there be, must differ essentially from ours, to enable them to endure every gradation of temperature, from the intensity of heat in Mercury, to the extreme cold that probably reigns in Uranus. Of the use of Comets in the economy of nature it is impossible to form an idea; still less of the Nebulae, or cloudy appearances that are scattered through the immensity of space; but instead of being surprised that much is unknown, we have reason to be astonished that the successful daring of man has developed so much.

In the following pages it is not intended to limit the account of the *Mécanique Céleste* to a detail of results, but rather to endeavour to explain the methods by which these results are deduced from one general equation of the motion of matter. To accomplish this, without having recourse to the higher branches of mathematics, is impossible; many subjects, indeed, admit of geometrical demonstration; but as the object of this work is rather to give the spirit of Laplace's method than to pursue a regular system of demonstration, it would be a deviation from the unity of his plan to adopt it in the present case.

Diagrams are not employed in Laplace's works, being unnecessary to those versed in analysis; some, however, will be occasionally introduced for the convenience of the reader.

Notes

[1] Galilei, Galileo, 1564-1642, Galilei was born in Pisa in 1564, the son of Vincenzo Galilei, well known for his studies of music, and Giulia Ammannati. He studied at Pisa, where he later held the chair in mathematics from 1589 - 1592. He was then appointed to the chair of mathematics at the University of Padua, where he remained until 1610. During these years he carried out studies and experiments in mechanics, and also built a thermoscope. He devised and constructed a geometrical and military compass, and wrote a handbook which describes how to use this instrument. In 1594 he obtained the patent for a machine to raise water levels. He invented the microscope, and built a telescope with which he made celestial observations, the most spectacular of which was his discovery of the satellites of Jupiter. In 1610 he was nominated the foremost Mathematician of the University of Pisa and given the title of mathematician to the Grand Duke of Tuscany. He studied Saturn and observed the phases of Venus. In 1611 he went to Rome. He became a member of the Accademia dei Lincei and observed the sunspots. In 1612 he began to encounter serious

opposition to his theory of the motion of the earth that he taught after Copernicus. In 1614, Father Tommaso Caccini denounced the opinions of Galileo on the motion of the Earth from the pulpit of Santa Maria Novella, judging them to be erroneous. Galileo therefore went to Rome, where he defended himself against charges that had been made against him but, in 1616, he was admonished by Cardinal Bellarmino and told that he could not defend Copernican astronomy because it went against the doctrine of the Church. In 1622 he wrote the *Saggiatore* (*The Assayer*) which was approved and published in 1623. In 1630 he returned to Rome to obtain the right to publish his *Dialogue on the two chief world systems* which was eventually published in Florence in 1632. In October of 1632 he was summoned by the Holy Office to Rome. The tribunal passed a sentence condemning him and compelled Galileo to solemnly abjure his theory. He was sent to exile in Siena and finally, in December of 1633, he was allowed to retire to his villa in Arcetri, the Gioiello. His health condition was steadily declining, - by 1638 he was completely blind, and also by now bereft of the support of his daughter, Sister Maria Celeste, who died in 1634. Galileo died in Arcetri on 8 January 1642. I*nstitute and Museum of the History of Science of Florence, Italy.*

[2] See note 1, *Preliminary Dissertation.*

[3] See note 18, *Preliminary Dissertation.*

[4] Laplace, Pierre Simon, marquis de, *Traité de mécanique céleste*, Paris, Chez J.B.M. Duprat, an VII [1798]-1823 [i.e. 1825].

[5] *The time will come in which the diligence of a longer age draws into the daylight those things which are now concealed. For the enquiry into so many things one lifetime is not sufficient. There will come a time in which our ancestors are astonished that we did not know such obvious things.* Translated by Ian Johnston, Malaspina University-College.

Pluto

This is the clearest view yet of the distant planet Pluto and its moon, Charon, as revealed by NASA's Hubble Space Telescope (HST). The image was taken by the European Space Agency's Faint Object Camera on February 21, 1994 when the planet was 2.6 billion miles (4.4 billion kilometers) from Earth; or nearly 30 times the separation between Earth and the Sun. (Courtesy of NASA)

BOOK I - DYNAMICS

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FOREWORD[1]

The Figure of the Earth

THE theoretical investigation of the figure of the earth and planets is so complicated, that neither the geometry of Newton[2] nor the refined analysis of Laplace[3] has attained more than an approximation. The solution of that difficult problem was greatly advanced by the late Mr. Ivory.[4] The investigation has been conducted by successive steps, beginning with a simple case, and then proceeding to the more difficult. But, in all, the forces which occasion the revolutions of the earth and planets are omitted, because, by acting equally on all the particles, they do not disturb their mutual relations. A fluid mass of uniform density, whose particles mutually gravitate to one another, will assume the form of a sphere when at rest. But, if the sphere begins to revolve, every particle will describe a circle having its centre in the axis of revolution. The planes of all these circles will be parallel to one another and perpendicular to the axis, and the particles will have a tendency to fly from that axis in consequence of the centrifugal force arising from the velocity of rotation. The force of gravity is everywhere perpendicular to the surface, and tends to the interior of the fluid mass; whereas the centrifugal force, acts perpendicularly to the axis of rotation, and is directed to the exterior. And, as its intensity diminishes with the distance from the axis of rotation, it decreases from the equator to the poles, where it ceases. Now it is clear that these two forces are in direct opposition to each other in the equator alone, and that gravity is there diminished by the whole effect of the centrifugal force, whereas, in every other part of the fluid, the centrifugal force is resolved into two parts, one of which, being perpendicular to the surface, diminishes the force of gravity; but the other, being at a tangent to the surface, urges the particles toward the equator, where they accumulate till their numbers compensate the diminution of gravity, which makes the mass bulge at the equator, and become flattened at the poles. It appears, then, that the influence of the centrifugal force is most powerful at the equator, not only because it is actually greater there than elsewhere, but because its whole effect is employed in diminishing gravity, whereas, in every other point of the fluid mass, it is only a part that is so employed. For both these reasons, it gradually decreases toward the poles, where it ceases. On the contrary, gravity is least at the equator, because the particles are farther from the centre of the mass, and increases toward the poles, where it is greatest. It is evident, therefore, that, as the centrifugal force is much less than gravity–gravitation, which is the difference between the two, is least at the equator, and continually increases towards the poles, where it is a maximum. On these principles Sir Isaac Newton proved that a homogeneous fluid

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[1] The material in this and the subsequent forewords to Books II, III and IV is drawn from the 10th and last edition of Mary Somerville's *On the Connexion of the Physical Sciences*, (corrected and revised by Arabella B. Buckley), p. 4-106, London : John Murray, 1877.

[2] See note 1, *Preliminary Dissertation.*

[3] See note 18, *Preliminary Dissertation.*

[4] Ivory, James, Sir, (1765-1842), a mathematician who, like Somerville, was an exponent of the French analysis. Ivory did extensive work on the figure of the earth. He also wrote a critical commentary on Laplace's *Mécanique céleste* that Laplace praised. Ivory's mathematical research focussed on the gravitational attraction of ellipsoids, cometary orbits, and atmospheric refraction.

mass in rotation assumes the form of an ellipsoid of revolution. whose compression is $\frac{1}{230}$. Such, however, cannot be the form of the earth, because the strata increase in density towards the centre. The lunar inequalities also prove the earth to be so constructed: It was requisite, therefore, to consider the fluid mass to be of variable density, in which case the compression or flattening would be less than in the case of the homogeneous fluid. Moreover the compression is still less when the mass is considered to be, as it probably is, a solid nucleus decreasing regularly in density from the centre to the surface, and partially covered by the ocean, because the solid parts, by their cohesion, nearly destroy that part of the centrifugal force which gives the particles a tendency to accumulate at the equator, though not altogether; otherwise the sea, by the superior mobility of its particles, would flow towards the equator, and leave the poles dry. Besides, it is well known that the continents at the equator are more elevated than they are in higher latitudes. It is also necessary for the equilibrium of the ocean that its density should be less than the mean density of the earth, otherwise the continents would be perpetually liable to inundations from storms and other causes. Taking all these elements into consideration, it appears from theory, that a horizontal line passing round the earth through both poles must be nearly an ellipse, having its major axis in the plane of the equator, and its minor axis coincident with the axis of the earth's rotation, and Clairaut (see note 14, *Bk. III, Chap II*) and others have determined by mathematical analysis that the equatorial diameter of the spheroid exceeds the polar by $\frac{1}{1152}^{th}$ of its length, agreeing completely with the fraction deduced from the inequalities of the motion of the moon, and also with the results of actual measurement. It is easy to show in a spheroid whose strata are elliptical, that the increase in the length of the radii, the decrease of gravitation, and the increase in the length of the arcs of the meridian, corresponding to angles of one degree, from the poles to the equator, are all proportional to the square of the cosine of the latitude. These quantities are so connected with the ellipticity of the spheroid, that the total increase in the length of the radii is equal to the compression or flattening, and the total diminution in the length of the arcs is equal to the compression, multiplied by three, times the length of an arc of one degree at the equator. Hence, by measuring the meridian curvature of the earth, the compression, and consequently its figure, become known. This, indeed, is assuming the earth to be an ellipsoid of revolution; but the actual measurement of the globe, will show how far it corresponds with that solid in figure and constitution.

The courses of the great rivers, which are in general navigable to a considerable extent, prove that the curvature of the land differs but little from that of the ocean; and, as the heights of the mountains and continents are inconsiderable when compared with the magnitude of the earth, its figure is understood to be determined by a surface at every point perpendicular to the direction of gravitation, or of the plumb-line, and is the same which the sea would have if it were continued all round the earth beneath the continents. Such is the figure that has been measured in the following manner:–

A terrestrial meridian is a line passing through both poles, all the points of which have their noon contemporaneously. Were the lengths and curvatures of different meridians known, the figure of the earth might be determined. But the length of one degree is sufficient to give the figure of the earth, if it be measured on different meridians, and in a variety of latitudes. For, if the earth were a sphere, all degrees would be of the same length; but, if not, the lengths of the degrees would be greater, exactly in proportion as the curvature is less. It may appear at first sight a paradox to assert that the lengths of the radii of a spheroidal body increase from the poles to the equator and at the same time to state that the length of the degrees marked on the surface increases from the equator to the poles. This apparent contradiction has proved a stumbling-block to many students, but the solution lies in the fact that in the first statement the measurement is supposed to be made from the centre of the spheroid, whereas that point is no longer the centre of equilibrium for all parts of the surface, as it would be in a true sphere. Consequently a plumb-line perpendicular to the surface of the globe will not point to the earth's centre at all, but to a different centre for every point on a meridian of the spheroid, and an angle of one degree of latitude at the poles having a much longer radius, will also have a longer arc than an angle of one degree at the equator. A comparison of the length of a degree in different parts of the earth's surface will therefore determine its size and form.

An arc of the meridian may be measured by determining the latitude of its extreme points by astronomical observations, and then measuring the distance between them in feet or fathoms. The distance thus determined on the surface of the earth, divided by the degrees and parts of a degree contained in the difference of the latitudes, will give the exact length of one degree, the difference of the latitudes being the angle contained between the verticals at the extremities of the arc. This would be easily accomplished were the distance unobstructed and on a level with the sea. But, on account of the innumerable obstacles on the surface of the earth, it is necessary to connect the extreme points of the arc by a series of triangles the sides and angles of which are either measured or computed, so that the length of the arc is ascertained with much laborious calculation. In consequence of the irregularities of the surface each triangle is in a different plane. They must therefore be reduced by computation to what they would have been had they been measured on the surface of the sea. And, as the earth may in this case be esteemed spherical, they require a correction to reduce them to spherical triangles. The officers who conducted the trigonometrical survey in measuring 500 feet of a base in Ireland twice over, found that the difference in the two measurements did not amount to the 800th part of an inch; and in the General Survey of Great Britain, five bases were measured from 5 to 7 miles long and some of them 400 miles apart, yet, when connected by series of triangles, the measured and computed lengths did not differ by more than 3 inches, a degree of accuracy which shows with what care these operations are conducted.

Arcs of the meridian have been measured in a variety of latitudes in both hemispheres, as well as arcs perpendicular to the meridian. From these measurements it appears that the length of the degrees increases from the equator to the poles, nearly in proportion to the square of the sine of the latitude. Consequently the convexity of the earth diminishes from the equator to the poles.

Were the earth a homogeneous ellipsoid of revolution, the meridians would be ellipses whose lesser axes would coincide with the axis of rotation, and all the degrees measured between the pole and the equator would give the same compression when combined two and two. That, however, is far from being the case. Scarcely any of the measurements give exactly the same results, chiefly on account of local attractions, which cause the plumb-line to deviate from the vertical. The vicinity of mountains produces that effect. One of the most remarkable anomalies of this kind has been observed in certain localities of northern Italy, where the action of some dense subterraneous matter causes the plumb-line to deviate seven or eight times more than it did from the attraction of Chimborazo,[5] in the observations of Bouguer (see note 51, *Bk. II, Chap. XIV*), while measuring a degree of the meridian at the equator. In consequence of this local attraction, the degrees of the meridian in that part of Italy seem to increase towards the equator

[5] Chimborazo, Ecuador Location: 1.46S 78.82W. Chimborazo is located in the Inter-Andean Graben, a north-northeast trend structural depression that separates the Western and Eastern Cordillera of the Andes in Ecuador.

through a small space, instead of decreasing, as if the earth was drawn out at the poles, instead of being flattened.

Many other discrepancies occur, but from the mean of the five principal measurements of arcs in Peru, India, France, England, and Lapland, Mr. Ivory deduced that the figure which most nearly follows this law is an ellipsoid of revolution whose equatorial radius would be 3902.824 miles, and the polar radius, 3949.585 miles. The difference, or 13.239 miles, divided by the equatorial radius, would be $\frac{1}{299}$ nearly. This fraction is called the compression of the earth, and does not differ much from that given by the lunar inequalities.[6] Since the preceding quantities were determined, arcs of the meridian have been measured in various parts of the globe, of which the most extensive are the Russian arc of $25^\circ 20'$ between the Glacial Sea and the Danube, conducted under the superintendence of M. Struve,[7] and the Indian arc extended to $21^\circ\ 21'$ by Colonel Everest.[8] All these measurements executed in various parts of the world were compared by Captain A. R. Clarke in an elaborate memoir to the Astronomical Society in 1860, in which he arrived at the following result: 'The earth is not exactly an ellipsoid of revolution. The equator itself is slightly elliptic, the longer and shorter diameters being respectively 41,852,864 and 41,843,096 feet. The ellipticity of the equatorial circumference is therefore $\frac{1}{4283}$ and the excess of its longer over its shorter diameter about two miles. The vertices of the longer diameter are situated in longitudes $14^\circ\ 23'$ and $194^\circ\ 23'$ E of Greenwich, and of its shorter in $104^\circ\ 23'$ and $284^\circ\ 23'$ E. The polar axis of the earth is 41,707,796 feet in length, and consequently the most elliptic meridian (that of long. $14^\circ\ 23'$ and $194^\circ\ 23'$) has for its ellipticity $\frac{1}{287.5}$ and the least–(that of long. $104^\circ\ 23'$ and $284^\circ\ 23'$) an ellipticity of $\frac{1}{308.3}$.' It appears, therefore, that our globe is not only flattened at the poles, but that the protuberance at the equator is slightly compressed in one direction, so that a line drawn through the centre of the earth from Loango[9] on the West Coast of Africa to the Centre of the Polynesian islands would be two miles longer than a similar line drawn from Sumatra to Equador on the West Coast of South America.

Eratosthenes,[10] who died 194 years before the Christian era, was the first to give an approximate value of the earth's circumference, by the measurement of an arc between Alexandria and Syrene.

There is another method of finding the figure of the earth, totally different from the preceding, solely depending upon the increase of gravitation from the equator to the poles. The force of gravitation at any place is measured by the descent of a heavy body during the first second of its fall. And the intensity of the centrifugal force is measured by the deflection of any point from the tangent in a second. For, since the centrifugal force balances the attraction of the earth, it is an exact measure of the gravitating force. Were the attraction to cease, a body on the surface of the earth would fly off in the tangent by the centrifugal force, instead of bending round in the circle of rotation. Therefore, the deflection of the circle from the tangent in a second measures the intensity of the earth's attraction, and is equal to the versed sine of the arc

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[6] See Book III.
[7] See note 29, *Preliminary Dissertation*.
[8] Everest, George, Sir, (1790-1866), military engineer, born in Gwernvale, Wales. He worked on the trigonometrical survey of India (1818-43). Mt Everest was renamed in his honour.
[9] Loango, Kingdom of, also called Brama Kingdom, former African state in the basin of the Kouilou and Niari rivers (now largely in southwestern Congo). *Encyclopaedia Britannica*
[10] See note 29, *Preliminary Dissertation*.

described during that time, a quantity easily determined from the known velocity of the earth's rotation. Whence it has been found that at the equator the centrifugal force is equal to the 289th part of gravity. Now it is proved by analysis that, whatever the constitution of the earth and planets may be, if the intensity of gravitation at the equator be taken equal to unity, the sum of the compression of the ellipsoid, and the whole increase of gravitation from the equator to the pole, is equal to five halves of the ratio of the centrifugal force to gravitation at the equator. This quantity with regard to the earth is $\frac{5}{2}$ of $\frac{1}{289}$ or $\frac{1}{115.2}$. Consequently the compression of the earth is equal to $\frac{1}{115.2}$ diminished by the whole increase of gravitation. So that its form will be known, if the whole increase of gravitation from the equator to the pole can be determined by experiment. This has been accomplished by a method founded upon the following considerations:–If the earth were a homogeneous sphere without rotation, its attraction on bodies at its surface would be everywhere the same. If it be elliptical and of variable density the force of gravity, theoretically ought to increase from the equator to the pole, as unity plus a constant quantity multiplied into the square of the sine of the latitude. But for a spheroid in rotation the centrifugal force varies, by the laws of mechanics, as the square of the sine of the latitude, from the equator, where it is greatest, to the pole, where it vanishes. And, as it tends to make bodies fly off the surface, it diminishes the force of gravity by a small quantity. Hence, by gravitation, which is the difference of these two forces, the fall of bodies ought to be accelerated from the equator to the poles proportionally to the square of the sine of the latitude; and the weight of the same body ought to increase in that ratio. This is directly proved by the oscillations of the pendulum, which, in fact, is a falling body; for, if the fall of bodies be accelerated, the oscillations will be more rapid: in order, therefore, that they may always be performed in the same time, the length of the pendulum must be altered. By numerous and careful experiments it is proved that a pendulum, which oscillates 86,400 times in a mean day at the equator, will do the same at every point of the earth's surface, if its length be increased progressively to the pole, as the square of the sine of the latitude.

From the mean of these it appears that the whole decrease of gravitation from the poles to the equator is 0.0051, which, subtracted from $\frac{1}{115.2}$, shows that the compression of the terrestrial spheroid is about $\frac{1}{285.26}$. This value was deduced by the late Mr. Baily,[11] President of the Astronomical Society, who devoted much attention to this subject; at the same time, it may be observed that no two sets of pendulum experiments give the same result, probably from local attractions. The compression obtained by this method does not differ much from that given by the lunar inequalities nor from the arcs in the direction of the meridian, and those perpendicular to it. The near coincidence of these three values, deduced by methods so entirely independent of each other, shows that the mutual tendencies of the centres of the celestial bodies to one another, and the attraction of the earth for bodies at its surface, result from the reciprocal attraction of all their particles. Another proof may be added. The nutation[12] of the earth's axis, and the precession of the equinoxes,[13] are occasioned by the action of the sun and moon on the protuberant matter at the earth's equator. And, although these inequalities do not give the absolute value of the

[11] Baily, Francis, (1774-1844), astronomer, born in Newbury, England. he is known for a phenomenon known as Baily's beads detected during an eclipse of the Sun in 1836. He also calculated the mean density and elliptical shape of the earth by repeating Henry Cavendish's experiments.
[12] see note 5, *Preliminary Dissertation.*
[13] see note 26, *Preliminary Dissertation.*

terrestrial compression, they show that the fraction expressing it is comprised between the limits $\frac{1}{279}$ and $\frac{1}{573}$.

It might be expected that the same compression should result from each, if the different methods of observation could be made without error. This, however, is not the case; for after allowance has been made for every cause of error, such discrepancies are found, both in the degrees of the meridian and in the length of the pendulum, as show that the figure of the earth is very complicated. But they are so small, when compared with the general results, that may be disregarded. The compression deduced from the mean of the whole appears not to differ much from $\frac{1}{300}$; that given by the lunar theory has the advantage of being independent of the irregularities of the earth's surface and of local attractions. The irregularity with which the observed variation in the length of the pendulum follows the law of the square of the sine of the latitude proves the strata to be elliptical, and symmetrically disposed round the centre of gravity of the earth, which affords a strong presumption in favour of its original fluidity. It is remarkable how little influence the sea has on the variation of the lengths of the arcs of the meridian, or on gravitation; neither does it much affect the lunar inequalities, from its density being only about a fifth of the mean density of the earth. For, if the earth were to become fluid after being stripped of the ocean, it would assume the form of an ellipsoid of revolution whose compression is $\frac{1}{304.8}$, which differs very little from that determined by observation, and proves, not only that the density of the ocean is inconsiderable, but that its mean depth is very small. There are, it is true, profound cavities in the bottom of the sea, for recent soundings by the 'Challenger' and other vessels show that the North Pacific is very deep, sometimes exceeding four or five miles over a large portion of its area. The Atlantic Ocean is also more than four miles deep in places, and there is a great central trough extending from near the coast of New York to near the Cape of Good Hope. But the central portion of the principal basin of the North Atlantic is occupied by a plateau, the greater part of which is less than two miles in depth. The South Pacific also, although not thoroughly explored, seems to be decidedly shallower than the North Pacific. On the whole, Dr. Carpenter estimates that the mean depth of the ocean generally may be taken at about two miles.[14] This depth is so insignificant when compared with the size of the earth that immense tracts of land might rise above or sink below the ocean level, as appears really to have been the case, without any great change in the form of the terrestrial spheroid. The variation in the length of the pendulum was first remarked by Richter in 1672, while observing transits of the fixed stars across the meridian at Cayenne,[15] about five degrees north of the equator. He found that his clock lost at the rate of $2^m\ 28^s$ daily, which induced him to determine the length of a pendulum beating seconds in that latitude; and, repeating the experiments on his return to Europe, be found the seconds' pendulum at Paris to be more than the twelfth of an inch longer than that at Cayenne. The form and size of the earth being determined, a standard of measure is furnished with which the dimensions of the solar system may be compared.

The Rotation of the Earth

The rotation of the earth which determines the length of the day, may he regarded as one of the most important elements in the system of the world. It serves as a measure of time, and

[14] *Encyclopaedia Britannica*, 9th edit., article 'Atlantic.' (Somerville's footnote.)
[15] Capital of French Guiana, on Cayenne Island, at the mouth of the Cayenne River.

forms the standard of comparison for the revolutions of the celestial bodies, which, by their proportional increase or decrease, would soon disclose any changes it might sustain. Theory and observation concur in proving that, among the innumerable vicissitudes which prevail throughout creation, the period of the earth's diurnal rotation has remained practically unchanged. The water of rivers, falling from a higher to a lower level, carries with it the velocity due to its revolution with the earth at a greater distance from the centre; it will therefore accelerate, although to an almost infinitesimal extent, the earth's daily rotation. The sum of all these increments of velocity, arising from the descent of all the rivers on the earth's surface, would in time become perceptible, did not nature, by the process of evaporation, raise the waters back to their sources, and thus, by again removing matter to a greater distance from the centre, destroy the velocity generated by its previous approach; so that the descent of rivers does not affect the earth's rotation. The disturbing action of the moon and planets, which has so powerful an effect on the revolution of the earth, has no effect upon its rotation beyond that produced by the friction of the tides, and this is so exceedingly slight as to require thousands of millions of years to produce effects of any magnitude.[16] The constant friction of the trade-winds on the mountains and continents between the tropics does not impede its velocity, which theory even proves to be the same as if the sea, together with the earth, formed one solid mass. But, although these circumstances be insufficient, a variation in the mean temperature would certainly occasion a corresponding change in the velocity of rotation. In the science of dynamics it is a principle in a system of bodies or of particles revolving about a fixed centre, that the momentum or sum of the products of the mass of each into its angular velocity and distance from the centre is a constant quantity, if the system be not deranged by a foreign cause. Now, since the number of particles in the system is the same whatever its temperature may be, when their distances from the centre are diminished, their angular velocity must be increased, in order that the preceding quantity may still remain constant. It follows, then, that, as the primitive momentum of rotation with which the earth was projected into space must necessarily remain the same, the smallest decrease in heat, by contracting the terrestrial spheroid, would accelerate its rotation, and consequently diminish the length of the day [and reciprocally increase the length of the day if the mean temperature were to increase–*ed. note*]. Notwithstanding the constant accession of heat from the sun's rays, geologists have been induced to believe, from the formation of mountain chains and the contorted strata occurring in them, that the mean temperature of the globe is decreasing.

The high temperature of mines, hot springs, and above all the internal fires which have produced, and do still occasion, such devastation on our planet, indicate a gradual augmentation of heat below the surface. The increase of density corresponding to the depth and the form of the spheroid, being what theory assigns to a fluid mass in rotation, concurs to induce the idea that the temperature of the earth was originally so high as to reduce all the substances of which it is composed to a state of fusion or of vapour, and that in the course of ages it has cooled down to its present state; that it is still becoming colder; and that it will continue to do so till the whole mass arrives at the temperature of the medium in which it is placed or rather at a state of equilibrium between this temperature, the cooling power of its own radiation, and the heating effect of the sun's rays.

Previous to the formation of ice at the poles, the ancient lands of northern latitudes might, no doubt, have been capable of producing those tropical plants preserved in the coal-measures, if indeed such plants could flourish without the intense light of a tropical sun. But, even if the decreasing temperature of the earth be sufficient to produce the observed effects, it must be

[16] Stone, *Astronomical Monthly Notices*, vol. xxvii. p. 197. (Somerville's note.)

extremely slow in its operation; for, in consequence of the rotation of the earth being a measure of the periods of the celestial motions, it has been proved that, if the length of the day had decreased by the three-thousandth part of a second since the observations of Hipparchus[17] two thousand years ago, it would have diminished the secular equation of the moon[18] by $44''.4$. It is, therefore, beyond a doubt that the mean temperature of the earth cannot have sensibly varied during that time. If, then, the appearances exhibited by the strata are really owing to a decrease in internal temperature, it either shows the immense periods requisite to produce geological changes, to which two thousand years are as nothing, or that the mean temperature of the earth had arrived at a state of equilibrium before these observations of Hipparchus.

Another cause of decrease of temperature which has been suggested is that of the secular variation of the eccentricity of the earth's orbit,[19] for as Sir John Herschel pointed out in 1835, the total quantity of heat received by the earth from the sun is inversely proportional to the minor axis; or in other words as the minor axis grows longer and the earth's orbit approaches more nearly to a circle we receive less heat from the sun. The utmost difference of heat, however, arising from this cause can never exceed the ratio of 1003 to 1000, and is therefore of very little importance.[20]

Of the decrease in temperature of the northern hemisphere there is abundant evidence in the fossil plants discovered in very high latitudes, which could only have existed in a tropical climate, and which must have near the spot where they are found, from the delicacy of their structure and the perfect state of their preservation. This change of temperature, has again been ascribed to an excess in the duration of spring and summer in the northern hemisphere, in consequence of the eccentricity of the solar ellipse. The length of the season varies with the position of the perihelion of the earth's orbit[21] for two reasons. On account of the eccentricity, small as it is, any line passing through the centre of the sun divides the terrestrial ellipse into two unequal parts, and by the laws of elliptical motion the earth moves through these two portions with unequal velocities.[22] The perihelion always lies in the smaller portion, and there the earth's motion is the most rapid. In the present position of the perihelion, spring and summer north of the equator exceed by about eight days the duration of the same seasons south of it. And 10,500 years ago the southern hemisphere enjoyed the advantage we now possess from the secular variation of the perihelion, Yet Sir John Herschel has shown that by this alternation neither hemisphere receives any excess of light or heat above the other; for, although the earth is nearer to the sun while moving through that part of its orbit in which the perihelion lies than in the other part, and consequently receives a greater quantity of light and heat, yet as it moves faster it is exposed to the heat for a shorter time. In the other part of the orbit, on the contrary, the earth being farther from the sun, receives fewer of his rays; but because its motion is slower, it is exposed to them for a longer time; and, as in both cases the quantity of heat and the angular velocity vary exactly in the same proportion, a perfect compensation takes place in the quantity of heat received from the sun.

Although, however, the mean temperature of the earth as a whole must have remained for ages a constant quantity, yet there is a way in which this unequal division of the heat during

[17] See note 32, *Preliminary Dissertation.*
[18] See *Book III, Chapter I, Article 720.*
[19] See *Book II, Chapter XIV, Article 646.*
[20] Sir J. Herschel, *Trans. of Geol. Society*, 2nd series, vol. iii. (Somerville's note.)
[21] See *Book II, Chapter II, Article 316.*
[22] See *Book II, Chapter II, fig, 63.*

different parts of the year may affect climate. Sir John Herschel pointed out in 1858, that the climates of the southern hemisphere are more extreme than those of the northern on account of the long cold winter endured by the south pole when the earth is in aphelion,[23] the south pole being then turned away from the sun, and the short fierce summer when the earth is in perihelion[24] and the south pole is turned towards the sun. These extremes he showed would become still more marked when the eccentricity of the earth's orbit was greater, as it has been in past ages. Still it seemed that in the course of each year these extremes must compensate each other; but in 1864, Mr. Croll[25] pointed out that such need not necessarily be the case, because the south pole during its long winter would have become covered with immense thicknesses of snow and ice which must be melted before the hot summer sun could warm the earth and raise the general temperature.

So long as water and aqueous vapour remain in their liquid and gaseous state, no difference of temperature in different parts of the earth during successive seasons can produce any cumulative effect, because this very difference causes a circulation throughout the globe which continually tends to bring about equilibrium. But when vapour is changed into snow, and water into ice, a totally different state of things is brought about. Circulation is comparatively stopped, especially if, as Mr. Croll contends, the heat of the succeeding summer is greatly neutralised by the dense fogs which must arise from the condensation of the vapour in contact with the ice covering. Thus the snow and ice are only partially converted into water and vapour, and when the long and cold winter returns an additional accumulation occurs, and thus cold is stored up in each succeeding winter while there is no corresponding storing up of heat.

In this way, when the eccentricity of the earth's orbit was at its maximum, and the extremes of climate greatly increased, that hemisphere which had its winter in aphelion might go on adding to its cap of snow and ice each successive year, till effects might be produced such as would account for the Glacial Period, of which we find so many traces in Europe and America. We have already seen that the conditions would be reversed for each hemisphere every 10,500 years, so that in this way glacial phenomena might be brought about in each hemisphere in succession, while the other hemisphere was enjoying a warm or temperate climate.

But there is another powerful cause which must probably combine with these astronomical changes in order to produce the required effects. Sir Charles Lyell, in his *Principles of Geology* refers the increased cold of the Glacial Period in great part to changes in the position of land and sea, such as we know to have taken place since the earliest geological periods. The loftiest mountains would be represented by a grain of sand on a globe six feet in diameter, and the depth of the ocean by a scratch on its surface. Consequently the gradual elevation of a continent or chain of mountains above the surface of the ocean, or their depression below it, is no very great event compared with the magnitude of the earth and the energy of its subterranean fires, if we take into account the immense periods of time during which these changes must have been in progress, as shown by the successive and various races of extinct beings entombed in the earth's crust. '*Continents, therefore, although permanent for whole geological epochs, shift their positions entirely in the course of ages, and it is not too much to say that every spot which is now dry land has been sea at some former period, and every part of the space now covered by the deepest ocean has been land.*'[26] Now such changes in the disposition of land upon the globe

[23] Point A, *Book II, Chapter II, fig, 63*.
[24] Point P, *Book II, Chapter II, fig, 63.*
[25] *Philosophical Magazine*, August, 1864. (Somerville's note.)
[26] *Principles of Geology*, 12th edition, pp. 258, 260. (Somerville's note.)

must affect climate, for variations of temperature are always more intense in the interior of continents than in islands or sea-coasts. An increase of land within the tropics would therefore augment the general heat, and an increase in the temperate and frigid zones would render the cold more severe.

There is at the present time an abnormal quantity of land in polar and circumpolar regions, the proportion of land to sea being as 1 to $2\frac{1}{2}$, whereas in the tropical regions it is only as 1 to 4. Sir C. Lyell,[27] therefore, considered that this might partly account for the amount of ice and snow now accumulated at both poles, and that the milder climates indicated by the fossil remains of some geological formations might be due to a more equable disposition of land and sea; while periods of intense cold, such as the Glacial Period, would be brought about by a still greater preponderance of land towards the poles than now exists.

In the present state of our knowledge it is very difficult to decide how far these different causes of change of climate have aided or counteracted each other, but it is certain that all of them must have had some influence. One more disturbing cause still remains to be mentioned, namely, the variation in the obliquity of the ecliptic,[28] or in the angle of the earth's axis of rotation to the plane of its orbit, which causes the poles to be turned more or less directly towards the sun. At the present time this angle amounts to $23^\circ\ 28'$, but it is decreasing at the rate of $48''$ per century, and will continue to do so for a long time yet to come; after that it will again increase, and will thus continue to oscillate as much as $1^\circ\ 21'$ on one side or the other of a mean position. This range of obliquity is so small that it makes very little difference in the presentation of the polar regions to the sun, and it is therefore of very little importance to climate, but Sir J. Herschel was of the opinion that in the course of millions of years, the deviation might become as great as 3° or 4° on each side of the mean, and if this be so it would be of great assistance to the geologist in helping to account for the tropical Miocene plants found in the polar regions, which must have required not only warmth but light, such as they cannot receive in the present position of our axis of rotation.[29]

It is evident from the marine shells found on the tops of the highest mountains and in almost every part of the globe, that immense continents have been elevated above the ocean, while others have sunk below it. If it were possible for the axis of rotation to alter with reference to the surface of the earth, the seas tending to a new equator would leave some portions of the globe and overwhelm others. Now it is found by the laws of mechanics that in every body, be its form or density what it may, there are at least three axes at right angles to each other, round any one of which, if the solid begins to rotate, it will continue to revolve for ever, provided it be not disturbed by a foreign cause, but that the rotation about any other axis will only be for an instant, and consequently the poles or extremities of the instantaneous axis of rotation would perpetually change their position on the surface of the body. In an ellipsoid of revolution the polar diameter and every diameter in the plane of the equator are the only permanent axes of rotation. Hence, if the ellipsoid were to begin to revolve about any diameter between the pole and the equator, the motion would be so unstable that the axis of rotation and the position of the poles would change every instant. Therefore, as the earth does not differ much from this figure, if it did not turn

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[27] Lyell, Charles, Sir, (1797-1875), Geologist, born in Kinnordy, Scotland. His famous *Principles of Geology* (1830-3) argued that geological change was brought about by factors that were still at play (uniformitarianism). Lyell communicated frequently with Somerville, Charles Darwin (who praised Lyell lavishly), and Sir John Herschel. Lyell was buried in Westminster Abbey.

[28] Angle P*np*, fig. 81, *Book II, Chapter V, Article 410*.

[29] Lyell, *Principles of Geology*, 12th edition, p. 293. (Somerville's note.)

round one of its principal axes, the position of the poles would change daily; the equator, which is 90° distant, would undergo corresponding variations; and the geographical latitudes of all places, being estimated from the equator, assumed to be fixed, would be perpetually changing. A displacement in the position of the poles of only two hundred miles would be sufficient to produce these effects, and would immediately be detected. But, as the latitudes are found to be invariable, it may be concluded that the terrestrial spheroid must have revolved about the same axis for ages.[30] The earth and planets differ so little from ellipsoids of revolution, that in all probability any libration from one axis to another, produced by the primitive impulse which put them in motion, must have ceased soon after their creation from the friction of the fluids at their surface.

Theory also proves that neither nutation, precession, nor any of the disturbing forces that affect the system, have the smallest influence on the axis of rotation, which maintains a permanent position, if the earth be not disturbed in its rotation by a foreign cause, as the collision of a comet, which might have happened in the immensity of time. But, had that been the case, its effects would still have been perceptible in the variations of the geographical latitudes. If we suppose that such an event had taken place, and that the disturbance had been very great, equilibrium could then only have been restored with regard to a new axis of rotation by the rushing of the seas to the new equator, which they must have continued to do till the surface was everywhere perpendicular to the direction of gravity. But it is probable that such an accumulation of the waters would not be sufficient to restore equilibrium if the derangement had been great, for the mean density of the sea is only about a fifth part of the mean density of the earth, and the mean depth of the Pacific Ocean is supposed not to be more than four or five miles, whereas the equatorial diameter of the earth exceeds the polar diameter by about $26\frac{1}{2}$ miles. Consequently the influence of the sea on the direction of gravity is very small. And, as it thus appears that a great change in the position of the axis is incompatible with the law of equilibrium, the geological phenomena in question must be ascribed to an internal cause. Indeed it is now demonstrated that the strata containing marine diluvia which are in lofty situations must have been formed at the bottom of the ocean, and afterwards upheaved by the action of subterraneous fires. Besides, it is clear, from the mensuration of the arcs of the meridian and the length of the seconds' pendulum, as well as from the lunar theory, that the internal strata and also the external outline of the globe are elliptical, their centres being coincident and their axes identical with that of the surface–a state of things which is incompatible with a subsequent accommodation of the surface to a new and different state of rotation from that which determined the original distribution of the component matter. Thus, amidst the mighty revolutions which have swept innumerable races of organized beings from the earth, which have elevated plains and buried mountains in the ocean, the rotation of the earth and the position of the axes on its surface can have undergone but slight variations.

The question of the nature of the interior of our planet has long been a matter of dispute, both among geologists and astronomers; and it is probable that we shall never arrive at a very definite answer. There can, however, be no doubt that the phenomena of precession and nutation, as they now occur, are almost exactly such as ought to take place if the earth were a continuous solid. Mr. Hopkins, therefore, came to the conclusion that the crust of the earth must be at least

[30] See *Book I, Chapter V, Article 216.*

from 800 to 1,000 miles in thickness; and Sir William Thomson[31] believes that 2,000 or 2,500 miles would not be an over-estimate of the depth of solid matter absolutely necessary to meet the observed facts. But Mr. Hopkins also suggested that if our globe, in the beginning, was condensed from a nebulous vapour into an incandescent fluid, solidification would begin at the centre and advance towards the surface; and that when the remaining liquid matter became of no great thickness, the surface also would begin to solidify by radiation into space: after which time the further solidification would proceed simultaneously from the outside inwards, and from the inside outwards. The result of this would be that in time the whole would be solidified, or that some portions or pockets of liquefied matter, as Mr. Scrope[32] calls them, only would remain here and there, wherever more intense heat was generated.

The rigidity of the planet as a whole is proved, according to Sir W. Thomson, by the tides; for if the solid parts of the earth had so little rigidity as to yield like a fluid, there would be no tides at all; and this would be equally the case if a thin solid crust rested on a fluid beneath. Therefore he comes to the conclusion that the earth's upper crust is probably nearly as rigid as glass, and that *the earth as a whole must be far more rigid than glass*, and probably even more rigid than steel.[33] This conclusion is, however, still questioned by some physicists. The lunar inequalities of themselves show that the strata of the earth increase in density from the surface to the centre, and the enormous pressure of the superincumbent mass is a sufficient cause for this phenomenon. Professor Leslie[34] calculated that air compressed into the fiftieth part of its volume has its elasticity fifty times augmented. If it continues to contract at that rate, it would, from its own incumbent weight, acquire the density of water at the depth of thirty-four miles. But water itself would have its density doubled at the depth of ninety-three miles, and would even attain the density of quicksilver at a depth of 262 miles. Descending therefore towards the centre through nearly 4,000 miles, the condensation of ordinary substances would surpass the utmost powers of conception. Dr. Young[35] computed that steel would be compressed into one-fourth and stone into one-eighth of its bulk at the earth's centre. We are yet ignorant of the laws of compression of solid bodies beyond a certain limit. But while astronomical considerations contradict the assumption of a fluid nucleus covered only by a thin shell, the opposite conclusion of a continuous solid nucleus of such extreme density as would result from the above experiments is rendered impossible by the fact that the mean density of the earth is only about six times that of water, or a little more than twice that of many rocks near the surface. On the whole, therefore, it would seem that our planet must have a cavernous structure of great rigidity, the extreme density of the central parts being to a considerable extent neutralized by expansion due to great heat.

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[31] Kelvin (of Largs), William Thomson, Baron (1824-1907), Scottish mathematician and physicist, born in Belfast, N. Ireland. Known for his work in thermodynamics, energy conservation and hydrodynamics. The unit of absolute temperature is named after him. Thomson laid foundations for modern physics in his development of the basis of electromagnetism. He was an early advocate of the idea that all forces in nature would ultimately be related in one unified theory. Thomson later became James Clerk Maxwell's mentor. His many other interests included questions on the shape of the earth.

[32] Scrope, George Julius Poulett, (1797-1776), born London England, geologist whose volcanic theories helped undermine the Neptunist theory: that the world's oldest rocks were sedimentary in origin. His treatise *Considerations on Volcanoes* was published in 1825.

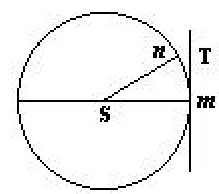
[33] Sir W. Thomson, *Phil. Trans.*, vol. cliii. 1863. (Somerville's note.)

[34] Leslie, John, Sir, (1766-1832), mathematician and physicist, born in Largo, Scotland. Leslie was Chair of mathematics at Edinburgh (1805) and later Physics (1819). Known primarily for his studies on heat, Leslie also invented a differential thermometer, a hygrometer, and a photometer. Leslie published 10 books including *The Philosophy of Arithmetic*.

[35] See note 35, *Preliminary Dissertation.*

The Sun, Moon, Planets and Satellites

The masses of such planets as have no satellites are known by comparing the inequalities they produce in the motions of the earth and of each other, determined theoretically, with the same inequalities given by observation; for the disturbing cause must necessarily be proportional to the effect it produces. The masses of the satellites themselves may also be compared with that of the sun by their perturbations. Thus, it is found, from the comparison of a vast number of observations with Laplace's theory of Jupiter's satellites, that the mass of the sun is more than 60,000,000 times greater than the least of these moons. But, as the quantities of matter in any two primary planets are directly as the cubes of the mean distances at which their satellites revolve, and inversely as the squares of their periodic times,[36] the mass of the sun and of any planets which have satellites may be compared with the mass of the earth.[37] In this manner it is computed that the mass of the sun is 315,000 times that of the earth; whence the great perturbations of the moon, and the rapid motion of the perigee and nodes of her orbit. Even Jupiter, the largest of the planets, has been found by Professor Airy[38] to be 1046.77 times less than the sun; and, indeed, the mass of the whole Jovial system is not more than the 1054.4th part of that of the sun. So that the mass of the satellites bears a very small proportion to that of their primary. The mass



[36] *Inversely &c.* The quantities of matter in any two primary planets are greater in proportion as the cubes of the numbers representing the mean distances of their satellites are greater, and also in proportion as the squares of their periodic times are less. (Somerville's note.)

[37] As hardly anything appears more impossible than that man should have been able to weigh the sun as it were in scales and the earth in a balance, the method of doing so may have some interest. The attraction of the sun is to the attraction of the earth as the quantity of matter in the sun to the quantity of matter in the earth; and, as the force of this reciprocal attraction is measured by its effects, the space the earth would fall through in a second by the sun's attraction is to the space which the sun would fall through by the earth's attraction as the mass of the sun to the mass of the earth. Hence, as many times as the fall of the earth to the sun in a second exceeds the fall of the sun to the earth in the same time, so many times does the mass of the sun exceed the mass of the earth. Thus the weight of the sun will be known if the length of these two spaces can be found in miles or parts of a mile. Nothing can be easier. A heavy body falls through 16.0697 feet in a second at the surface of the earth by the earth's attraction; and, as the force of gravity is inversely as the square of the distance, it is clear that 16.0697 feet are to the space a body would fall through at the distance of the sun by the earth's attraction, as the square of the distance of the sun from the earth to the square of the distance of the centre of the earth from its surface; that is, as the square of 91,600,000 miles to the square of 4000 miles. And thus, by a simple question in proportion the space which the sun would fall through in a second by the attraction of the earth may be found in parts of a mile. The space the earth would fall through in a second, by the attraction of the sun, must now be found in miles also. Suppose *mn*, in the figure above, to be the arc which the earth describes round the sun in S, by the joint action of the sun and the velocity of the planet in a second of time. By the planet's velocity alone the earth would move from *m* to T in a second, and by the sun's attraction alone it would fall through T*n* in the same time. Hence, the length of T*n*, in miles, is the space the earth would fall through in a second by the sun's attraction. Now, as the earth's orbit is very nearly a circle, if 360 degrees be divided by the number of seconds in a sidereal year of 366.25 days, it will give *mn*, the arc which the earth moves through in a second, and then the tables will give the length of the line ST in numbers corresponding to that angle; but, as the radius S*n* is assumed to be unity in the tables, if 1 be subtracted from the number representing ST, the length of T*n* will be obtained; and when multiplied by 91,600,000, to reduce it to miles, the space which the earth falls through, will be obtained in miles. By this simple process it is found that, if the sun were placed in one scale of a balance, it would require 315,000 earths to form a counterpoise. (Somerville's note.)

[38] Airy, George Biddell, Sir, (1801-1892), astronomer, born in Alnwick, England. Airy was professor of mathematics at Cambridge. He calculated the mass of the earth from gravity measurements.

of the moon is determined from several sources–from her action on the terrestrial equator, which occasions the nutation in the axis of rotation; from her horizontal parallax; from an inequality she produces in the sun's longitude; and from her action on the tides. The three first quantities, computed from theory and compared with their observed values, give her mass respectively equal to the $\frac{1}{71}$, $\frac{1}{74.2}$, and $\frac{1}{69.2}$ part of that of the earth, which do not differ much from each other. Dr. Brinkley[39] has found it to be $\frac{1}{80}$ from the constant of lunar nutation: but, from the moon's action in raising the tides, her mass appears to be about the $\frac{1}{75}$ part of that of the earth–a value that cannot differ much from the truth.

The apparent diameters of the sun, moon, and planets are determined by measurement; therefore their real diameters may be compared with that of the earth; for the real diameter of the planet is to the real diameter of the earth, or 7899 miles, as the apparent diameter of the planet to the apparent diameter of the earth as seen from the planet, that is, to twice the parallax of the planet. According to Bessel,[40] the mean apparent diameter of the sun is $1923''.64$, and with the solar parallax $8''.9$, it will be found that the diameter of the sun is about 852,900 miles. Therefore, if the centre of the sun were to coincide with the centre of the earth, his volume would not only include the orbit of the moon, but would extend nearly as far again; for the moon's mean distance from the earth is about sixty times the earth's equatorial radius, or 238,793 miles: so that twice the distance of the moon is 477,586 miles, which does not greatly exceed the solar radius. The diameter of the moon is only 2164 miles; and Jupiter's diameter of 84,800 miles is very much less than that of the sun; the diameter of Ceres, the largest of the minor planets, is only 196 miles, and that of Pallas does not much exceed 171 miles, so that an inhabitant of that planet, in one of our steam carriages, might go round these worlds in a few hours, and the whole of the 158 telescopic planets[41] are so small, that their united mass is almost inappreciable in affecting the movements of the heavenly bodies.

The densities of bodies are proportional to their masses, divided by their volumes. Hence, if the sun and planets be assumed to be spheres, their volumes will be as the cubes of their diameters. Now, the apparent diameters of the sun and earth, at their mean distance, are $1923''.6$ and $17''.1552$, and the mass of the earth is the 315,000th part of that of the sun taken as the unit. It follows, therefore, that the earth is four times as dense as the sun. But the sun is so large that his attractive force is 27.9 times as great as that of the earth. Consequently, if he were habitable by human beings, they would be unable to move, since their weight would be nearly twenty-eight times as great as it is here. A man of moderate size would weigh about two tons at the surface of the sun; whereas at the surface of some of the new planets he would be so light that it would be impossible to stand steady, since he would only weigh a few pounds. The mean density of the earth has been determined by the following method. Since a comparison of the action of two planets upon a third gives the ratio of the masses of these two planets, it is clear that, if we can compare the effect of the whole earth with the effect of any part of it, a comparison may be instituted between the mass of the whole earth and the mass of that part of it. Now a leaden ball was weighed against the earth by comparing the effects of each upon a pendulum; the nearness of the smaller mass making it produce a sensible effect as compared with that of the larger: for by the laws of attraction the whole earth must be considered as collected in its centre. By this

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39 Brinkley, John, *Elements of Astronomy*, Dublin, 1813.
40 See note 37, *Bk. II, Chap. XIV.*
41 See note 9, *Preliminary Dissertation.*

method it has been found that the mean density of the earth is 5.660 times greater than that of water at the temperature of 62° of the Fahrenheit's thermometer. The late Mr. Baily,[42] whose accuracy as an experimental philosopher is acknowledged, was unremittingly occupied nearly four years in accomplishing this very important object. In order to ascertain the mean density of the earth still more perfectly, Sir G. B. Airy made a series of experiments to compare the simultaneous oscillations of two pendulums, one at the bottom of the Harton coal-pit, 1260 feet deep, in Northumberland, and the other on the surface of the earth immediately above it. The oscillations of the pendulums were compared with an astronomical clock at each station, and the time was instantaneously transmitted from one to the other by a telegraphic wire. The oscillations were observed for more than 100 hours continuously, when it was found that the lower pendulum made $2\frac{1}{2}$ oscillations more in 24 hours than the upper one. The experiment was repeated for the same length of time with the same result; but on this occasion the upper pendulum was taken to the bottom of the mine and the lower brought to the surface. From the difference between the oscillations at the two stations it appears that gravitation at the bottom of the mine exceeds that at the surface by the $\frac{1}{19190}$ part, and that the mean density of the earth is 6.565, which is greater than that obtained by Mr. Baily by 0.89, but there are many reasons why this result cannot be so exact as could be wished. While employed on the trigonometrical survey of Scotland, Colonel James determined the mean density of the earth to be 5.316, from a deviation of the plumb-line amounting to $2''$, caused by the attraction of Arthur's Seat[43] and the heights east of Edinburgh: it agrees more nearly with the density found by Mr. Baily than with that deduced from Sir G. B. Airy's experiments, and upon the whole 5.6 seems to be the nearest estimate we can assume for the earth's density. All the planets and satellites, except perhaps Mercury, appear to be of less density than the earth.

Before entering on the theory of rotation, it may not be foreign to the subject to give some idea of the methods of computing the places of the planets, and of forming astronomical tables. Astronomy is now divided into the three distinct departments of theory, observation, and computation. Since the problem of the three bodies can only be solved by approximation, the analytical astronomer determines the position of a planet in space by a series of corrections. Its place in its circular orbit is first found, then the addition or subtraction of the equation of the centre (see art. 382, *Bk. II, Chap. IV*) to or from its mean place gives its position in the ellipse. This again is corrected by the application of the principal periodic inequalities. But, as these are determined for some particular position of the three bodies, they require to be corrected to suit other relative positions. This process is continued till the corrections become less than the errors of observation, when it is obviously unnecessary to carry the approximation further. The true latitude and distance of the planet from the sun are obtained by methods similar to those employed for the longitude.

As the earth revolves equably about its axis in 24 hours, at the rate of 15° in an hour, time becomes a measure of angular motion, and the principal element in astronomy, where the object is to determine the exact state of the heavens and the successive changes it undergoes in all ages, past, present, and to come. Now, the longitude, latitude, and distance of a planet from the sun are given in terms of the time, by general analytical formulae. These formulae will consequently give the exact place of the body in the heavens, for any time assumed at pleasure, provided they can be reduced to numbers. But before the calculator begins his task the observer must furnish

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[42] See note 11, *Bk. I, Foreword.*

[43] A well known ridge near Edinburgh.

the necessary data, which are, obviously, the forms of the orbits, and their positions with regard to the plane of the ecliptic. It is therefore necessary to determine by observation, for each planet, the length of the major axis of its orbit, the eccentricity, the inclination of the orbit to the plane of the ecliptic, the longitudes of its perihelion and ascending node at a given time, the periodic time of the planet, and its longitude at any instant arbitrarily assumed, as an origin from whence all its subsequent and antecedent longitudes are estimated. Each of these quantities is determined from that position of the planet on which it has most influence. For example, the sum of the greatest and least distances of the planet from the sun is equal to the major axis of the orbit, and their difference is equal to twice the eccentricity. The longitude of the planet, when at its least distance from the sun, is the same with the longitude of the perihelion: the greatest latitude of the planet is equal to the inclination of the orbit: the longitude of the planet, when in the plane of the ecliptic in passing towards the north, is the longitude of the ascending node, and the periodic time is the interval between two consecutive passages of the planet through the same node, a small correction being made for the precession of the node during the revolution of the planet. Notwithstanding the excellence of instruments and the accuracy of modern observers, unavoidable errors of observation can only be compensated by finding the value of each element from the mean of a thousand, or even many thousands of observations. For as it is probable that the errors are not all in one direction, but that some are in excess and others in defect, they will compensate each other when combined.

However, the values of the elements determined separately can only be regarded as approximate, because they are so connected that the estimation of any one independently will induce errors in the others. The eccentricity depends upon the longitude of the perihelion, the mean motion depends upon the major axis, the longitude of the node upon the inclination of the orbit, and *vice versâ*. Consequently, the place of a planet computed with the approximate data will differ from its observed place. Then the difficulty is to ascertain what elements are most in fault, since the difference in question is the error of all; that is obviated by finding the errors of some thousands of observations, and combining them, so as to correct the elements simultaneously, and to make the sum of the squares of the errors a minimum with regard to each element. The method of accomplishing this depends upon the Theory of Probabilities; a subject fertile in most important results in the various departments of science and of civil life, and quite indispensable in the determination of astronomical data. A series of observations continued for some years will give approximate values of the secular and periodic inequalities, which must be corrected from time to time, till theory and observation agree. And these again will give values of the masses of the bodies forming the solar system, which are important data in computing their motions. The periodic inequalities derived from a great number of observations are employed for the determination of the values of the masses till such time as the secular inequalities shall be perfectly known, which will then give them with all the necessary precision. When all these quantities are determined in numbers, the longitude, latitude, and distance of the planet from the sun are computed for stated intervals, and formed into tables, arranged according to the time estimated from a given epoch, so that the place of the body may be determined from them by inspection alone, at any instant for perhaps a thousand years before and after that epoch. By this tedious process, tables have been computed for all the great planets, and several of the small, besides the moon and the satellites of Jupiter. In the present state of astronomy the masses and elements of the orbits are pretty well known, so that the tables only require to be corrected from time to time as observations become more accurate. Those containing the motions of Jupiter, Saturn, and Uranus have already been three times constructed within the last fifty years, and the tables of Jupiter and Saturn agree almost perfectly with modern observation. The following prediction will be found in the sixth edition of *Connexion of the Physical Sciences*,[44] published in the year 1842: *'Those of Uranus, however, are already defective, probably because the discovery of that planet in 1781 is too recent to admit of much precision in the determination of its motions, or that possibly it may be subject to disturbances from some unseen planet revolving about the sun beyond the present boundaries of our system. If, after a lapse of years, the tables formed from a combination of numerous observations should be still inadequate to represent the motions of Uranus, the discrepancies may reveal the existence, nay, even the mass and orbit, of a body placed for ever beyond the sphere of vision.'*

That prediction has been fulfilled since the seventh edition of that book was published.[45] Not only the existence of Neptune, revolving at the distance of two thousand seven hundred millions of miles from the sun, has been discovered from his disturbing action on Uranus, but his mass, the form and position of his orbit in space, and his periodic time had been determined before the planet had been seen, and the planet itself was discovered in the very point of the heavens which had been assigned to it. It had been noticed for years that the perturbation of Uranus had increased in an unaccountable manner.[46] After the disturbing action of all the known planets had been determined, it was found that, between the years 1833 and 1837, the observed and computed distance of Uranus from the sun differed by 24,000 miles, which is about the mean distance of the moon from the earth, while, in 1841, the error in the geocentric longitude of the planet amounted to $96''$. These discrepancies were therefore attributed to the attraction of some unseen and unknown planet, consequently they gave rise to a case altogether unprecedented in the history of astronomy. Heretofore it was required to determine the disturbing action of one known planet upon another. Whereas the inverse problem had now to be solved, in which it was required to find the place of an unknown body in the heavens, at a given time, together with its mass, and the form and position of its orbit, from the disturbance it produced on the motions of another. The difficulty was extreme, because all the elements of the orbit of Uranus were erroneous from the action of Neptune, and those of Neptune's orbit were unknown. In this dilemma it was necessary to form some hypothesis with regard to the unknown planet; it was therefore assumed, according to Bode's empirical law on the mean distances of the planets, that it was revolving at twice the distance of Uranus from the sun. In fact, the periodic time of Uranus is about 84 years, and, as the discrepancies in his motions increased slowly and regularly it was evident that it would require a planet with a much longer periodic time to produce them–moreover, it was clear that the new planet must be exterior to Uranus, otherwise it would have disturbed the motions of Saturn.

Another circumstance tended to lessen the difficulty; the latitude of Uranus was not much affected, therefore it was concluded that the inclination of the orbit of the unknown body must be very small, and, as that of the orbit of Uranus is only $46'28''.4$, both planets were assumed to be moving in the plane of the ecliptic, and thus the elements of the orbit of the unknown planet were reduced from six to four. Having thus assumed that the unknown body was revolving in a circle in the plane of the ecliptic, the analytical expression of its action on the motion of Uranus, when

[44] Somerville, Mary, *On the Connexion of the Physical Sciences*, 6th edition, London : John Murray, 1842

[45] Neptune was discovered in the year 1846. (Somerville note.)

[46] The true longitude of Uranus was in advance of the tables previous to 1795, and continued to advance till 1822, after which it diminished rapidly till 1830-1, when the observed and calculated longitudes agreed, but then the planet fell behind the calculated place so rapidly that it was clear the tables could no longer represent its motion. (Somerville's note.)

in numerous points of its orbit, was compared with the observed longitude of Uranus, through a regular series of years, by means of which the faulty elements of the orbit of Uranus were eliminated, or got rid of, and there only remained a relation between the mass of the new planet and three of the elements of its orbit; and it then was necessary to assume such a value for two of them as would suit the rest. That was accomplished so dexterously, that the perturbations of Uranus were perfectly conformable to the motions of Neptune, moving in the orbit thus found, and the place of the new planet exactly agreed with observation. Subsequently its orbit and motions have been determined more accurately.

The honour of this admirable effort of genius is shared by Mr. Adams and M. Leverrier,[47] who, independently of each other, arrived at these wonderful results. Mr. Adams had determined the mass and apparent diameter of Neptune, with all the circumstances of its motion, eight months before M. Leverrier had terminated his results, and had also pointed out the exact spot where the planet would be found; but the English observers neglected to look for it till M. Leverrier made known his researches, and communicated its position to Dr. Galle,[48] at Berlin, who found it the very first night he looked for it, and then it was evident that it would have been seen in the place Mr. Adams had assigned to it eight months before had it been looked for. So closely did the results of these two great mathematicians agree.

Neptune has a diameter of 37,314 miles,[49] consequently he is nearly 200 times larger than the earth, and may be seen with a telescope of moderate power. His motion is retrograde at present, and six times slower than that of the earth. At so great a distance from the sun it can only have the $\frac{1}{1300}$ part of the light and heat the earth receives; but having a satellite,[50] the deficiency of light may in some measure be supplied.

The prediction may now be transferred from Uranus to Neptune, whose perturbations may reveal the existence of a planet still further removed, which may for ever remain beyond the reach of telescopic vision—yet its mass, the form and position of its orbit, and all the circumstances of its motion may become known, and the limits of the solar system may still be extended hundreds of millions of miles.[51]

The mean distance of Neptune from the sun has subsequently proved to be only 2746 millions of miles, and the period of his revolution 164 years, so that Baron Bode's law, of the interval between the orbits of any two planets being twice as great as the inferior interval and half of the superior, fails in the case of Neptune, though it was useful on the first approximation to his motions; and since Bode's time it has led to the discovery of 150 telescopic planets revolving between the orbits of Mars and Jupiter.

The tables of Mars, Venus, and even those of the sun, have been greatly improved, and, together with those of Jupiter, Saturn, and Uranus, form the basis of a grand work by M. Leverrier, which is only now (1876) approaching completion, after having occupied that eminent

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[47] See note 28, *Bk. II, Foreword.*

[48] Galle, Johann Gottfried, (1812-1910), astronomer who observed the planet Neptune at the Berlin Observatory on September 23, 1846. Galle looked for and located the planet a few days after a request from the French astronomer Leverrier who had independently calculated the location of the planet eight months after J. C. Adams had made a similar request in September 1845 to the Cambridge Observatory. (see also *Introduction to the Second Edition*; notes 28 & 39, *Bk. II, Foreword*; and note 38, *Bk. II, Chap. XIV*.)

[49] The modern value for Neptune's equatorial diameter is 30,777 miles (49,528 km). The polar diameter is 30,200 (48,600 km).

[50] The largest of Neptune's eight known satellites, *Triton*, was discovered by William Lassell (1799-1880), a few weeks after the discovery of Neptune in 1846.

[51] See note 23, *Foreword to the Second Edition.*

astronomer during twenty years. We are chiefly indebted to the German astronomers for tables of the four older telescopic planets, Vesta, Juno, Ceres, and Pallas, the others have only been discovered since the year 1845.

The determination of the path of a planet when disturbed by all the others, a problem which has employed the talents of the greatest astronomers, from Newton to the present day is only completely accomplished with regard to the older planets, which revolve in nearly circular orbits, but little inclined to the plane of the ecliptic. When the eccentricity and inclination of the orbits are great, analysis becomes almost impossible, because the series expressing the co-ordinates of the bodies become extremely complicated, and do not converge when applied to comets and the telescopic planets. This difficulty has, at last, been overcome by mathematicians, and many of the secular variations and mutual relations of the orbits of the asteroids or minor planets have been worked out. The problem is, however, one of extreme difficulty, and it must be long before they can all be included together with the larger planets in one comprehensive theory of planetary motion.

BOOK I

CHAPTER I

DEFINITIONS, AXIOMS, &c.

1. THE activity of matter seems to be a law of the universe, as we know of no particle that is at rest. Were a body absolutely at rest, we could not prove it to be so, because there are no fixed points to which it could be referred; consequently, if only one particle of matter were in existence, it would be impossible to ascertain whether it were at rest or in motion. Thus, being totally ignorant of absolute motion, relative motion alone forms the subject of investigation: a body is, therefore, said to be in motion, when it changes its position with regard to other bodies which are assumed to be at rest.

2. The cause of motion is unknown, force being only a name given to a certain set of phenomena preceding the motion of a body, known by the experience of its effects alone. Even after experience, we cannot prove that the same consequents will invariably follow certain antecedents; we only believe that they will, and experience tends to confirm this belief.

3. No idea of force can be formed independent of matter; all the forces of which we have any experience are exerted by matter; as gravity, muscular force, electricity, chemical attractions and repulsions, &c. &c., in all which cases, one portion of matter acts upon another.

4. When bodies in a state of motion or rest are not acted upon by matter under any of these circumstances, we know by experience that they will remain in that state: hence a body will continue to move uniformly in the direction of the force which caused its motion, unless in some of the cases enumerated, in which we have ascertained by experience that a change of motion will take place, then a force is said to act.

5. Force is proportional to the differential of the velocity, divided by the differential of the time, or analytically $F = \frac{dv}{dt}$, which is all we know about it.

6. The direction of a force is the straight line in which it causes a body to move. This is known by experience only.

7. In dynamics, force is proportional to the indefinitely small space caused to be moved over in a given indefinitely small time.

8. Velocity is the space moved over in a given time, how small soever the parts may be into which the interval is divided.

9. The velocity of a body moving uniformly, is the straight line or space over which it moves in a given interval of time; hence if the velocity v be the space moved over in one second or unit of time, vt is the space moved over in t seconds or units of time; or representing the space by s, $s = vt$.

10. Thus it is proved that the space described with a uniform motion is proportional to the product of the time and the velocity.

11. Conversely, v, the space moved over in one second of time, is equal to s, the space moved over in t seconds of time, multiplied by $\frac{1''}{t}$, or $v = s\left(\frac{1''}{t}\right) = \frac{s}{t}$.

12. Hence the velocity varies directly as the space, and inversely as the time; and because $t = \frac{s}{v}$,

13. The time varies directly as the space, and inversely as the velocity.

14. Forces are proportional to the velocities they generate in equal times.

The intensity of forces can only be known by comparing their effects under precisely similar circumstances. Thus two forces are equal, which in a given time will generate equal velocities in bodies of the same magnitude; and one force is said to be double of another which, in a given time, will generate double the velocity in one body that it will do in another body of the same magnitude.

15. The intensity of a force may therefore be expressed by the ratios of numbers, or both its intensity and direction by the ratios of lines, since the direction of a force is the straight line in which it causes the body to move.

16. In general, a line expressing the intensity of a force is taken in the direction of the force, beginning from the point of application.

17. Since motion is the change of rectilinear distance between two points, it appears that force, velocity, and motion are expressed by the ratios of spaces; we are acquainted with the ratios of quantities only.

Uniform Motion

18. A body is said to move uniformly, when, in equal successive intervals of time, how short soever, it moves over equal intervals of space.

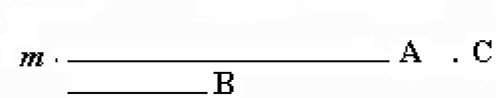
19. Hence in uniform motion the space is proportional to the time.

20. The only uniform motion that comes under our observation is the rotation of the earth upon its axis; all other motions in nature are accelerated or retarded. The rotation of the earth forms the only standard of time to which all recurring periods are referred. To be certain of the uniformity of its rotation is, therefore, of the greatest importance. The descent of materials from a higher to a lower level at its surface, or a change of internal temperature, would alter the length of the radius, and consequently the time of rotation: such causes of disturbance do take place; but it will be shown that their effects are so minute as to be insensible, and that the earth's rotation has suffered no sensible change from the earliest times recorded.

21. The equality of successive intervals of time may he measured by the recurrence of an event under circumstances as precisely similar as possible: for example, from the oscillations of a pendulum. When dissimilarity of circumstances takes place, we rectify our conclusions respecting the presumed equality of the intervals, by introducing an equation, which is a quantity to be added or taken away, in order to obtain the equality.

Composition and Resolution of Forces

fig. 1.



22. Let *m* be a particle of matter which is free to move in every direction; if two forces, represented both in intensity and direction by the lines *m*A and *m*B, be applied to it, and urge it towards C, the particle will move by the combined action of these two forces, and it will require a force equal to their sum, applied in a contrary direction, to keep it at rest. It is then said to be in a state of equilibrium.

fig. 2.

23. If the forces *m*A, *m*B, be applied to a particle *m* in contrary directions, and if *m*B be greater than *m*A, the particle *m* will be put in motion by the difference of these forces, and a force equal to their difference acting in a contrary direction will be required to keep the particle at rest.

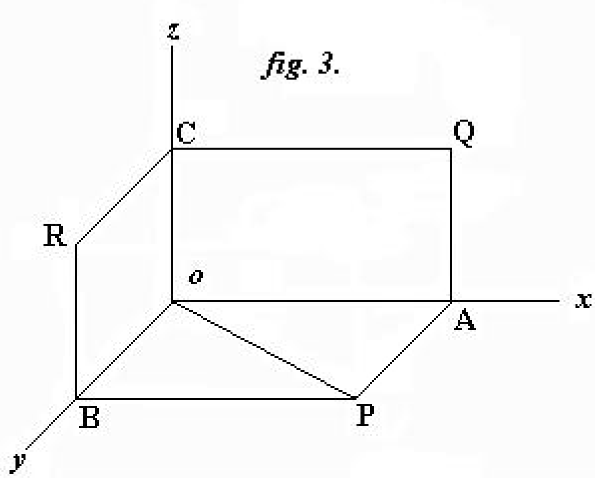


fig. 3.

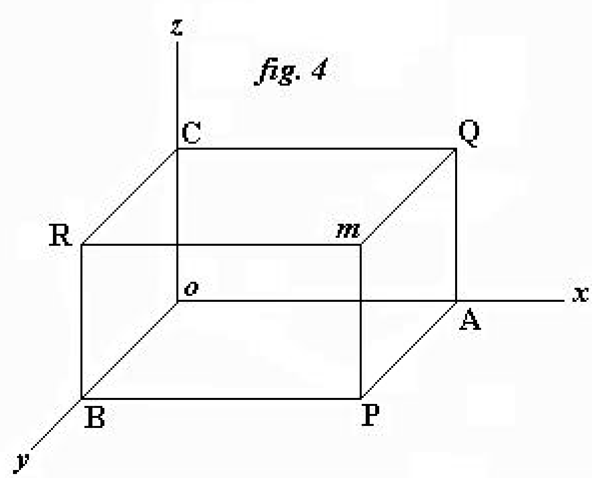
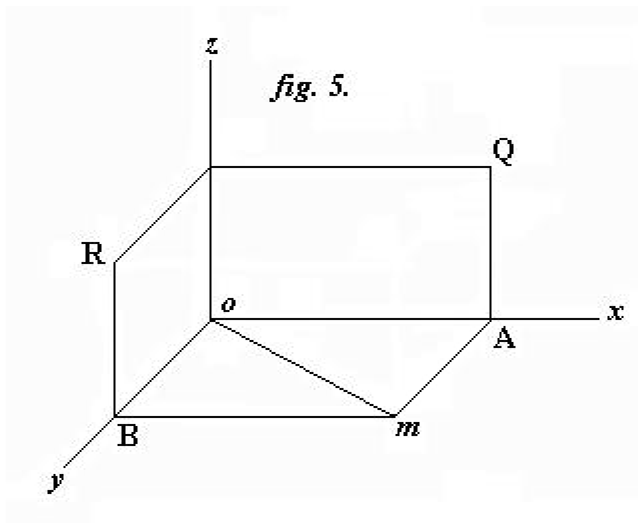


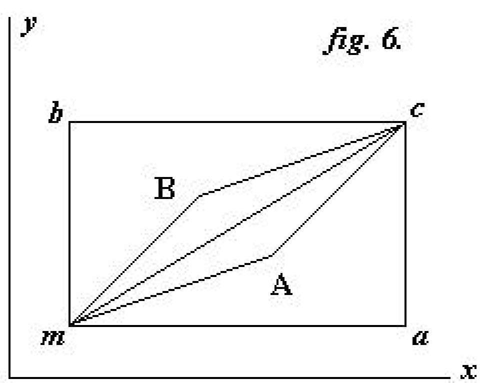
fig. 4

24. When the forces mA, mB are equal, and in contrary directions, the particle will remain at rest.

25. It is usual to determine the position of points, lines, surfaces, and the motions of bodies in space, by means of three plane surfaces, *o*P, *o*Q, *o*R, fig. 3, intersecting at given angles. The intersecting or coordinate planes are generally assumed to be perpendicular to each other, so that *xoy, xoz, yoz,* are right angles. The position of *ox*, *oy*, *oz*, the axes of the co-ordinates, and their origin *o*, are arbitrary; that is, they may be placed where we please, and are therefore always assumed to be known. Hence the position of a point *m* in space is determined, if its distance from each co-ordinate plane be given; for by taking *o*A, *o*B, *o*C, fig. 4, respectively equal to the given distances, and drawing three planes through A, B, and C, parallel to the co-ordinate planes, they will intersect in *m*.

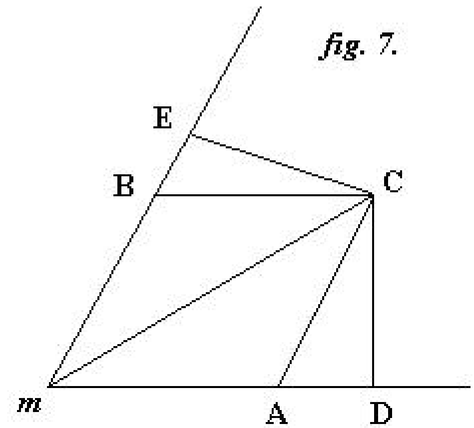


26. If a force applied to a particle of matter at *m*, (fig. 5,) make it approach to the plane *o*Q uniformly by the space *m*A, in a given time *t*; and if another force applied to *m* cause it to approach the plane *o*R uniformly by the space *m*B, in the same time *t*, the particle will move in the diagonal *mo*, by the simultaneous action of these two forces. For, since the forces are proportional to the spaces, if *a* be the space described in one second, *at* will be the space described in *t* seconds; hence if *at* be equal to the space *m*A, and *bt* equal to the space *m*B, we have $t=\frac{m\text{A}}{a}=\frac{m\text{B}}{b}$; whence $m\text{A}=\left(\frac{a}{b}\right)m\text{B}$ which is the equation to a straight line *mo*, passing through *o*, the origin of the co-ordinates. If the co-ordinates be rectangular, $\frac{a}{b}$ is the tangent of the angle *mo*A, for *m*B = *o*A, and *o*A*m* is a right angle; hence[1] $o\text{A}:\text{A}m::1:\tan \text{A}om$; whence $m\text{A}=o\text{A}\times\tan\text{A}om=m\text{B}.\tan\text{A}om$. As this relation is the same for every point of the straight line *mo*, it is called its equation. Now since forces are proportional to the velocities they generate in equal times, *m*A, *m*B are proportional to the forces, and may be taken to represent them. The forces *m*A, *m*B are called component or partial forces, and *mo* is called the resulting force. The resulting force being that which, taken in a contrary direction, will keep the component forces in equilibrio.

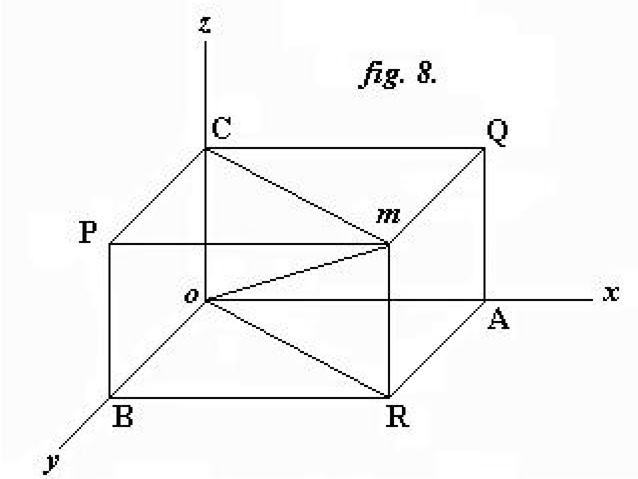


27. Thus the resulting force is represented in magnitude and direction by the diagonal of a parallelogram, whose sides are *m*A, *m*B the partial ones.

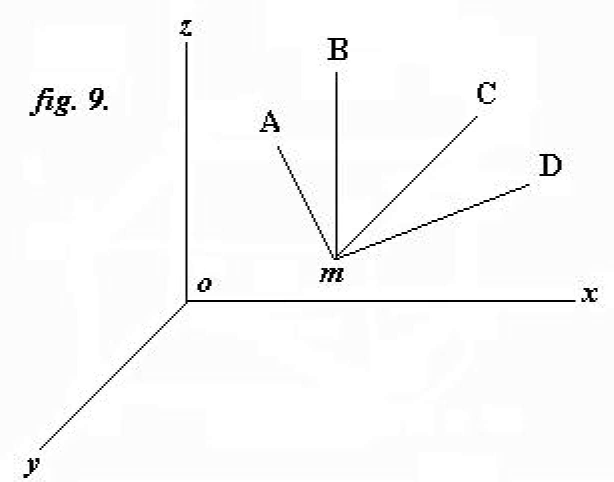
28. Since the diagonal *cm*, fig. 6, is the resultant of the two forces *m*A, *m*B, whatever may be the angle they make with each other, so, conversely these two forces may be used in place of the single force *mc*. But *mc* may be resolved into any two forces whatever which form the sides of a parallelogram of which it is the diagonal; it may, therefore, be resolved into two forces *ma*, *mb*, which are at right angles to each other. Hence it is always possible to resolve a force *mc* into two others which are parallel to two rectangular axes *ox*, *oy*, situate in the same plane with the force; by drawing through *m* the lines *ma*, *mb*, respectively, parallel to *ox*, *oy*, and completing the parallelogram *macb*.



29. If from any point C, fig. 7, of the direction of a resulting force *m*C, perpendiculars CD, CE, be drawn on the directions of the component forces *m*A, *m*B, these perpendiculars are reciprocally as the component forces. That is, CD is to CE as CA to CB, or as their equals *m*B to *m*A.



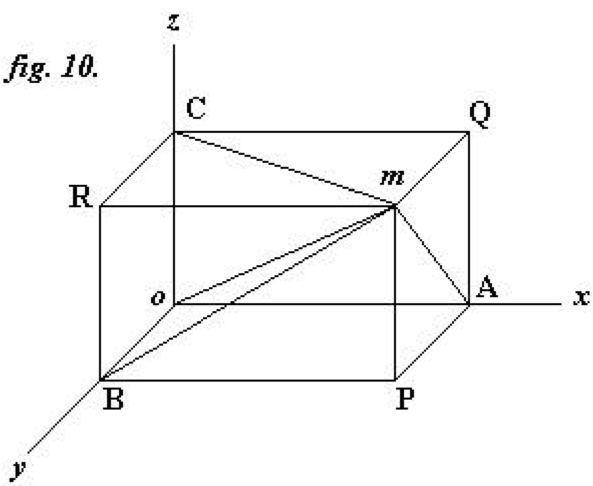
30. Let BQ, fig. 8, be a figure formed by parallel planes seen in perspective, of which *mo* is the diagonal. If *mo* represent any force both in direction and intensity, acting on a material point *m*, it is evident from what has been said, that this force may be resolved into two other forces, *m*C, *m*R, because *mo* is the diagonal of the parallelogram *m*C*o*R. Again *m*C is the diagonal of the parallelogram *m*QCP, therefore it may be resolved into the two forces *m*Q, *m*P; and thus the force *mo* may be resolved into three forces, *m*P, *m*Q, and *m*R; and is this is independent of the angles of the figure, the force *mo* may be resolved into three forces at right angles to each other. It appears then, that any force *mo* may be resolved into three other forces parallel to three rectangular axes given in position: and conversely, three forces *m*P, *m*Q, *m*R, acting on a material point *m*, the resulting force *mo* may be obtained by constructing the figure BQ with sides proportional to these forces, and drawing the diagonal *mo*.



31. Therefore, if the directions and intensities with which any number of forces urge a material point be given, they may be reduced to one single force whose direction and intensity is known. For example, if there were four forces, *m*A, *m*B, *m*C, *m*D, fig. 9, acting on *m*, if the resulting force of *m*A and *m*B be found, and then that of *m*C and *m*D; these four forces would be reduced to two, and by finding the resulting force of these two, the four forces would be reduced to one.

32. Again, this single resulting force may be resolved into three forces parallel to three rectangular axes *ox*, *oy*, *oz*, fig. 10, which would represent the action of the forces *m*A, *m*B, etc., estimated in the direction of the axes; or, which is the same thing, each of the forces *m*A, *m*B, etc. acting on *m*, may be resolved into three other forces parallel to the axes.

fig. 10.



33. It is evident that when the partial forces act in the same direction, their sum is the force in that axis; and when some act in one direction, and others in an opposite direction, it is their difference that is to be estimated.

34. Thus any number of forces of my kind are capable of being resolved into other forces, in the direction of two or of three rectangular axes, according as the forces act in the same or in different planes.

35. If a particle of matter remain in a state of equilibrium, though acted upon by any number of forces, and free to move in every direction, the resulting force must be zero.

36. If the material point be in equilibrio on a curved surface, or on a curved line, the resulting force must be perpendicular to the line or surface, otherwise the particle would slide. The line or surface resists the resulting force with an equal and contrary pressure.

37. Let *o*A=X, *o*B=Y, *o*C=Z, fig. 10, be three rectangular component forces, of which *om*=F is their resulting force. Then, if *m*A, *m*B, *m*C be joined, *om*=F will be the hypotenuse common to three rectangular triangles, *o*A*m*, *o*B*m*, and *o*C*m*. Let the angles *mo*A=*a*, *mo*B=*b*, *mo*C=*c*; then

$$\mathrm{X} = \mathrm{F}\cos a, \quad \mathrm{Y} = \mathrm{F}\cos b, \quad \mathrm{Z} = \mathrm{F}\cos c. \qquad \mathbf{(1)}$$

Thus the partial forces are proportional to the cosines of the angles which their directions make with their resultant. But BQ being a rectangular parallelopiped

$$\mathrm{F}^2 = \mathrm{X}^2 + \mathrm{Y}^2 + \mathrm{Z}^2. \qquad \mathbf{(2)}$$

Hence

$$\frac{\mathrm{X}^2 + \mathrm{Y}^2 + \mathrm{Z}^2}{\mathrm{F}^2} = \cos^2 a + \cos^2 b + \cos^2 c = 1.$$

When the component forces are known, equation (2) will give a value of the resulting force, and equations (1) will determine its direction by the angles *a*, *b*, and *c*; but if the resulting force be given, its resolution into the three component forces X, Y, Z, making with it the angles *a*, *b*, *c*, will be given by (1). If one of the component forces as Z be zero, then

$$c = 90^0, \ \ \mathrm{F}=\sqrt{\mathrm{X}^2+\mathrm{Y}^2}, \ \ \mathrm{X}=\mathrm{F}\cos a, \ \ \mathrm{Y}=\mathrm{F}\cos b\,.$$

38. Velocity and force being each represented by the same space, whatever has been explained with regard to the resolution and composition of the one applies equally to the other.

The General Principles of Equilibrium

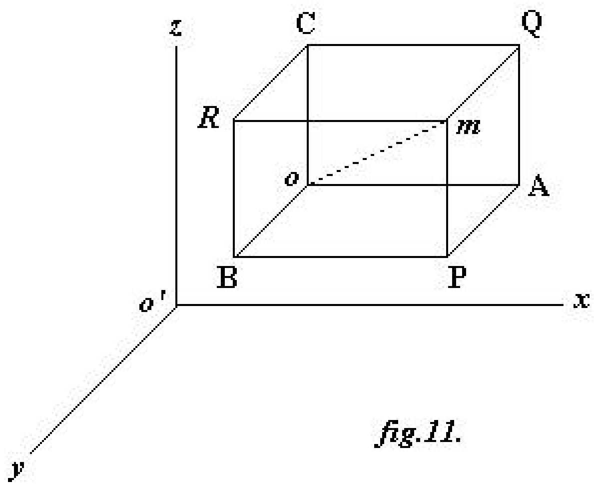


fig.11.

39. The general principles of equilibrium may be expressed analytically, by supposing *o* to be the origin of a force F, acting on a particle of matter at *m*, fig. 11, in the direction *om*. If o' be the origin of the co-ordinates; *a*, *b*, *c*, the co-ordinates of *o*, and *x*, *y*, *z* those of *m* ; the diagonal *om*, which may he represented by *r*, will be

$$r = \sqrt{(x-a)^2+(y-b)^2+(z-c)^2}$$

But F, the whole force in *om*, is to its component force in[2] $o\mathrm{A} :: r : a - x$, hence the component force parallel to the axis *ox* is

$$\mathrm{F}\frac{(x-a)}{r}.$$

In the same manner it may be shown, that

$$\mathrm{F}\frac{(y-b)}{r}; \ \ \mathrm{F}\frac{(z-c)}{r}$$

are the component forces parallel to *oy* and *oz*. Now the equation of the diagonal gives

$$\frac{dr}{dx}=\frac{(x-a)}{r}; \ \ \frac{dr}{dy}\frac{(x-b)}{r}; \ \ \frac{dr}{dz}\frac{(x-c)}{r};$$

hence the component forces of F are

$$\mathrm{F}\left(\frac{dr}{dx}\right);\;\mathrm{F}\left(\frac{dr}{dy}\right);\;\mathrm{F}\left(\frac{dr}{dz}\right).$$

Again, if F' be another force acting on the particle at *m* in another direction r', its component forces parallel to the co-ordinates will be,

$$\mathrm{F}'\left(\frac{dr'}{dx}\right);\;\mathrm{F}'\left(\frac{dr'}{dy}\right);\;\mathrm{F}'\left(\frac{dr'}{dz}\right).$$

And any number of forces acting on the particle *m* may be resolved in the same manner, whatever their directions may be. If Σ be employed to denote the sum of any number of finite quantities, represented by the same general symbol

$$\Sigma.\mathrm{F}.\left(\frac{dr}{dx}\right)=\mathrm{F}\left(\frac{dr}{dx}\right)+\mathrm{F}'\left(\frac{dr'}{dx}\right)+\mathrm{F}''\left(\frac{dr''}{dx}\right)+\&\text{c.}$$

is the sum of the partial forces urging the particle parallel to the axis *ox*. Likewise

$$\Sigma.\mathrm{F}.\left(\frac{dr}{dy}\right);\;\Sigma.\mathrm{F}.\left(\frac{dr}{dz}\right);$$

are the sums of the partial forces that urge the particle parallel to the axis *oy* and *oz*. Now if $\mathrm{F}_{/}$ be the resulting force of all the forces F, F', F'', etc. that act on the particle *m*, and if *u* be the straight line drawn from the origin of the resulting force to *m*, by what precedes

$$\mathrm{F}_{/}\left(\frac{du}{dx}\right);\;\mathrm{F}_{/}\left(\frac{du}{dy}\right);\;\mathrm{F}_{/}\left(\frac{du}{dz}\right),$$

are the expressions of the resulting force $\mathrm{F}_{/}$, resolved in directions parallel to the three co-ordinates; hence

$$\mathrm{F}_{/}\left(\frac{du}{dx}\right)=\Sigma.\mathrm{F}\left(\frac{dr}{dx}\right);\;\mathrm{F}_{/}\left(\frac{du}{dy}\right)=\Sigma.\mathrm{F}\left(\frac{dr}{dy}\right);\;\mathrm{F}_{/}\left(\frac{du}{dz}\right)=\Sigma.\mathrm{F}\left(\frac{dr}{dz}\right),$$

or if the sums of the component forces parallel to the axis *x*, *y*, *z*, be represented by X, Y, Z, we shall have

$$\mathrm{F}_{/}\left(\frac{du}{dx}\right)=\mathrm{X};\;\mathrm{F}_{/}\left(\frac{du}{dy}\right)=\mathrm{Y};\;\mathrm{F}_{/}\left(\frac{du}{dz}\right)=\mathrm{Z}.$$

If the first of these be multiplied by dx, the second by dy, and the third by dz, their sum will be

$$F_{/}du = Xdx + Ydy + Zdz.$$

40. If the intensity of the force can be expressed in terms of the distance of its point of application from its origin, X, Y, and Z may be eliminated from this equation, and the resulting force will then be given in functions of the distance only. All the forces in nature are functions of the distance, gravity for example, which varies inversely as the square of the distance of its origin from the point of its application. Were that not the case, the preceding equation could be of no use.

41. When the particle is in equilibrio, the resulting force is zero; consequently

$$Xdx + Ydy + Zdz = 0, \quad \textbf{(3)}$$

which is the general equation of the equilibrium of a free particle.

42. Thus, when a particle of matter urged by any forces whatever remains in equilibrio, the sum of the products of each force by the element of its direction is zero. As the equation is true, whatever be the values of dx, dy, dz, it is equivalent to the three partial equations in the direction of the axes of the co-ordinates, that is to

$$X = 0, \quad Y = 0, \quad Z = 0,$$

for it is evident that if the resulting force be zero, its component forces must also be zero.

On Pressure

43. A pressure is a force opposed by another force, so that no motion takes place.

44. Equal and proportionate pressures are such as are produced by forces which would generate equal and proportionate motions in equal times.

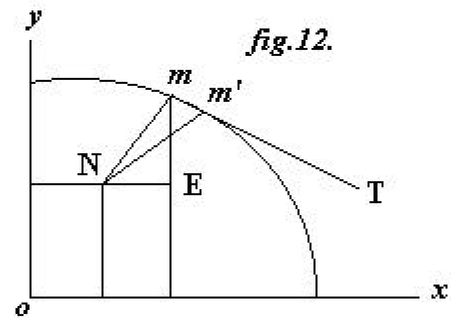
45. Two contrary pressures will balance each other, when the motions which the forces would separately produce in contrary directions are equal; and one pressure will counterbalance two others, when it would produce a motion equal and contrary to the resultant of the motions which would be produced by the other forces.

46. It results from the comparison of motions, that if a body remain at rest, by means of three pressures, they must have the same ratio to one another, as the sides of a triangle parallel to the directions.

On the Normal

47. The normal to a curve, or surface in any point m, fig. 12, is the straight line mN perpendicular to the tangent mT. If mm' be a plane curve

$$m\text{N} = \sqrt{(x-a)^2 + (y-b)^2}$$



x and y being the co-ordinates of m, a and b those of N. If the point m be on a surface, or curve of double curvature, in which no two of its elements are in the same plane, then,

$$m\text{N} = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$$

x, y, z being the co-ordinates of m, and a, b, c those of N. The centre of curvature N, which is the intersection of two consecutive normals mN, m'N, never varies in the circle and sphere, because the curvature is every where the same; but in all other curves and surfaces the position of N changes with every point in the curve or surface, and a, b, c, are only constant from one point to another. By this property, the equation of the radius of curvature is formed from the equation of the curve, or surface. If r be the radius of curvature, it is evident, that though it may vary from one point to another, it is constant for any one point m where $dr = 0$.

Equilibrium of a Particle on a curved Surface

48. The equation (3) is sufficient for the equilibrium of a particle of matter, if it be free to move in my direction; but if it be constrained to remain on a curved surface, the resulting force of all the forces acting upon it must be perpendicular to the surface, otherwise it would slide along it; but as by experience it is found that re-action is equal and contrary to action, the perpendicular force will be resisted by the re-action of the surface, so that the re-action is equal, and contrary to the force destroyed; hence if $\text{R}_/$, be the resistance of the surface, the equation of equilibrium will be

$$\text{X}dx + \text{Y}dy + \text{Z}dz = -\text{R}_/ d\text{r}\,.$$

dx, dy, dz are arbitrary; these variations may therefore be assumed to take place in the direction of the curved surface on which the particle moves: then by the property of the normal, $dr = 0$; which reduces the preceding equation to

$$\text{X}dx + \text{Y}dy + \text{Z}dz = 0\,.$$

But this equation is no longer equivalent to three equations, but to two only, since one of the elements dx, dy, dz, must be eliminated by the equation of the surface.

49. The same result may be obtained in another way. For if $u = 0$ be the equation of the surface, then $du=0$; but as the equation of the normal is derived from that of the surface, the equation $dr=0$ is connected with the preceding, so that $dr=\mathrm{N}du$. But

$$r=\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}$$

whence

$$\frac{dr}{dx}=\frac{x-a}{r};\quad \frac{dr}{dy}=\frac{y-b}{r};\quad \frac{dr}{dz}=\frac{z-c}{r};$$

consequently,

$$\left\{\left(\frac{dr}{dx}\right)^2+\left(\frac{dr}{dy}\right)^2+\left(\frac{dr}{dz}\right)^2\right\}=1,$$

on account of which, the equation

$$dr=\mathrm{N}du \text{ gives } \mathrm{N}^2\left\{\left(\frac{du}{dx}\right)^2+\left(\frac{du}{dy}\right)^2+\left(\frac{du}{dz}\right)^2\right\}=1,$$

or

$$\mathrm{N}=\frac{1}{\sqrt{\left(\frac{du}{dx}\right)^2+\left(\frac{du}{dy}\right)^2+\left(\frac{du}{dz}\right)^2}},$$

for u is a function of x, y, z; hence,

$$\mathrm{R}_{/}dr=\frac{\mathrm{R}_{/}du}{\sqrt{\left(\frac{du}{dx}\right)^2+\left(\frac{du}{dy}\right)^2+\left(\frac{du}{dz}\right)^2}};$$

and if

$$l=\frac{\mathrm{R}_{/}}{\sqrt{\left(\frac{du}{dx}\right)^2+\left(\frac{du}{dy}\right)^2+\left(\frac{du}{dz}\right)^2}},$$

then $\mathrm{R}_{/}dr$ becomes $l\,du$, and the equation of the equilibrium of a particle m, on a curved line or surface, is

$$\mathrm{X}dx+\mathrm{Y}dy+\mathrm{Z}dz+l\,du=0, \qquad \textbf{(4)}$$

where du is a function of the elements dx, dy, dz: and as this equation exists whatever these elements may be, each of them may be made zero, which will divide it into three equations; but they will be reduced to two by the elimination of l. And these two, with the equation of the surface $u = 0$, will suffice to determine x, y, z, the co-ordinates of m in its position of equilibrium. These found, N and consequently l become known. And since $R_{/}$, is the resistance

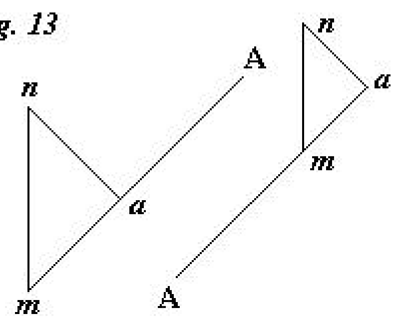
$$R_{/} = l\sqrt{\left(\frac{du}{dx}\right)^2+\left(\frac{du}{dy}\right)^2+\left(\frac{du}{dz}\right)^2}$$

is the pressure, which is equal and contrary to the resistance, and is therefore determined.

50. Thus if a particle of matter, either free or obliged to remain on a curved line or surface, be urged by any number of forces, it will continue in equilibrio, if the sum of the products of each force by the element of its direction be zero.

Virtual Velocities

fig. 13



51. This principle, discovered by John Bernoulli,[3] and called the principle of virtual velocities, is perfectly general, and may be expressed thus: –

If a particle of matter be arbitrarily moved from its position through an indefinitely small space, so that it always remains on the curve or surface, which it ought to follow, if not entirely free, the sum of the forces which urge it, each multiplied by the element of its direction, will be zero in the case of equilibrium.

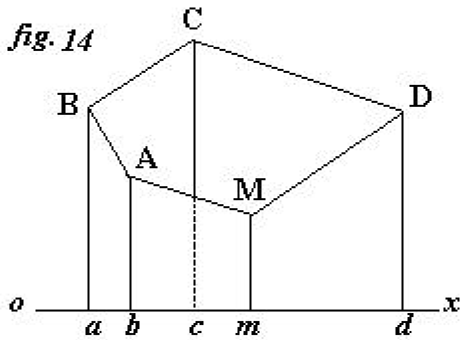
On this general law of equilibrium, the whole theory of statics depends.

52. An idea of what virtual velocity is, may be formed by supposing that a particle of matter m is urged in the direction mA by a force applied to m. If m be arbitrarily moved to any place n indefinitely near to m, then mn will be the virtual velocity of m.

53. Let na be drawn at right angles to mA, then mA is the virtual velocity of m resolved in the direction of the force mA: it is also the projection of mn on mA*;* for

$$mn : ma :: 1 : \cos nma \text{ and } ma = \cos nma\,.$$

54. Again, imagine a polygon ABCDM of any number of sides, either in the same plane or not, and suppose the sides MA, AB, etc., to represent, both in magnitude and direction, any forces applied to a particle at M. Let these forces be resolved in the direction of the axis ox, so that ma, ab, bc, etc. may be the projections of the sides of the polygon, or the cosines of the angles made by the sides of the polygon with ox to the several radii MA, AB, etc., then will the



segments *ma*, *ab*, *bc*, etc. of the axis represent the resolved portions of the forces estimated in that single direction, and calling a, b, g, &c., the angles above mentioned,

$$ma = \text{MA}\cos a;\ ab = \text{AB}\cos b;\ \text{and } bc = \text{BC}\cos g,$$

etc. and the sum of these partial forces will be

$$\text{MA}\cos a + \text{AB}\cos b + \text{BC}\cos g + \&c. = 0$$

by the general property of polygons, as will also be evident if we consider that *dm*, *ma*, *ab* lying towards *o* are to be taken positively, and *bc, cd* lying towards *x* negatively; and the latter making up the same whole *bd* as the former, their sums must be zero. Thus it is evident, that if any number of forces urge a particle of matter, the sum of these forces when estimated in any given direction, must be zero when the particle is in equilibrio; and *viceversâ,* when this condition holds, the equilibrium will take place. Hence, we see that a point will rest, if urged by forces represented by the sides of a polygon, taken in order.

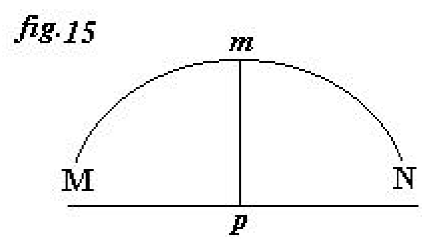
In this case also, the sum of the virtual velocities is zero; for, if M be removed from its place through an infinitely small space in any direction, since the position of *ox* is arbitrary, it may represent that direction, and *ma, ab, bc, cd, dm,* will therefore represent the virtual velocities of M in directions of the several forces, whose sum, as above shown, is zero.

55. The principle of virtual velocities is the same, whether we consider a material particle, a body, or a system of bodies.

Variations

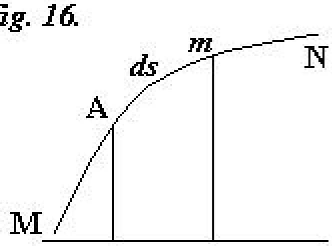
56. The symbol d is appropriated to the calculus of variations,[4] whose general object is to subject to analytical investigation the changes which quantities undergo when the relations which connect them are altered, and when the functions which are the objects of discussion undergo a change of form, and pass into other functions by the gradual variation of some of their elements, which had previously been regarded as constant. In this point of view, variations are only differentials on another hypothesis of constancy and variability, and are therefore subject to all the laws of the differential calculus.

57. The variation of a function may be illustrated by problems of maxima and minima, of which there are two kinds, one not subject to the law of variations, and another that is. In the former case, the quantity whose maximum or minimum is required depends by known relations on some arbitrary independent variable; for example, in a *given* curve MN, fig. 15, it is required to determine the point in which the ordinate *pm* is the greatest possible. In this case, the curve, or



function expressing the curve, remains the same; but in the other case, the form of the function whose maximum or minimum is required, is variable; for, let M, N, fig. 16, be any two given points in space, and suppose it were required, among the infinite number of curves that can be drawn between these two points, to determine that whose length is a minimum. If ds be the element of the curve, $\int ds$ is the curve itself; now as the required curve must be a minimum, the variation of $\int ds$ when made equal to zero, will give that curve, for when quantities are at their maxima or minima, their increments are zero. Thus the form of the function $\int ds$ varies so as to fulfil the conditions of the problem, that is to say, in place of retaining its general form, it takes the form of that particular curve, subject to the conditions required.

fig. 16.



58. It is evident from the nature of variations, that the variation of a quantity is independent of its differential, so that we may take the differential of a variation as $d\,.d\,y$, or the variation of a differential as $d\;.\,dy$, and that $d\,.d\,y = d\;.\,dy$.

59. From what has been said, it appears that virtual velocities are real variations; for if a body be moving on a curve, the virtual velocity may be assumed either to be on the curve or not on the curve; it is consequently independent of the law by which the co-ordinates of the curve vary, unless when we choose to subject it to that law.

Notes

[1] The proportionality implies that $\dfrac{o\mathrm{A}}{\mathrm{A}m} = \dfrac{1}{\tan \mathrm{A}om}$.

[2] $\mathrm{F} : o\mathrm{A} :: r : (a-x)$ so that $\dfrac{\mathrm{F}}{o\mathrm{A}} = \dfrac{r}{(a-x)}$.

[3] Bernoulli, Johann or Jean, 1667-1748, mathematician, born in Basel, Switzerland. He wrote on differential equations, isochronous curves, and curves of quickest descent. He was a pioneer in being one of the first to adopt the calculus recently developed by Leibniz and he quickly applied the calculus to differential equations and mechanical problems. His works are published in the four volume series *Opera Johannis Bernoullii* (1742).

[4] *Calculus of variations.* An important unifying principle involving the determination of a minimum or maximum value from an infinite number of possible solutions. The calculus of variations was pioneered by Bernoulli (see previous note), Euler (see note 6, *Bk. II*, *Chap, II*) who gave the method its first general rule, and Lagrange (see note 16, *Preliminary Dissertation*) who gave the method much of its terminology. Variational problems have been used in the determination of new rules in physics. For example, in classical mechanics, the calculation of trajectories in dynamical systems involve the application of a variational principle (Hamilton's principle). The development of Einstein's general theory of relativity depended heavily on the application of the calculus of variations.

BOOK I

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CHAPTER II

VARIABLE MOTION

60. WHEN the velocity of a moving body changes, the cause of that change is called an accelerating or retarding force; and when the increase or diminution of the velocity is uniform, its cause is called a continued, or uniformly accelerating or retarding force, the increments of space which would be described in a given time with the initial velocities being always equally increased or diminished.

Gravitation is a uniformly accelerating force, for at the earth's surface a stone falls $16\frac{1}{11}$ feet nearly, during the first second of its motion, $48\frac{3}{11}$ feet during the second, $80\frac{5}{11}$ feet during the third, &c., falling every second $32\frac{2}{11}$ feet more than during the preceding second.

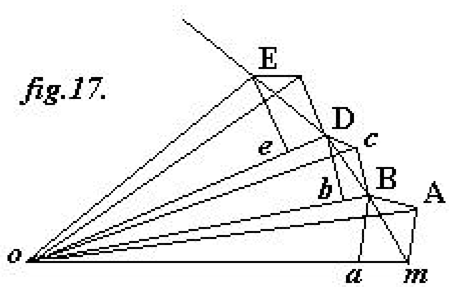
61. The action of a continued force is uninterrupted, so that the velocity is either gradually increased or diminished; but to facilitate mathematical investigation it is assumed to act by repeated impulses, separated by indefinitely small intervals of time, so that a particle of matter moving by the action of a continued force is assumed to describe indefinitely small but unequal spaces with a uniform motion, in indefinitely small and equal intervals of time.

62. In this hypothesis, whatever has been demonstrated regarding uniform motion is equally applicable to motion uniformly varied; and X, Y, Z, which have hitherto represented the components of an impulsive force, may now represent the components of a force acting uniformly.

Central Force

63. If the direction of the force be always the same, the motion will be in it straight line; but where the direction of a continued force is perpetually varying it will cause the particle to describe a curved line.

fig.17.



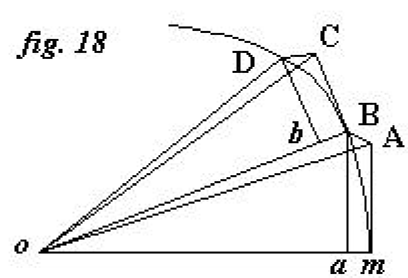
Demonstration. Suppose a particle impelled in the direction mA, fig. 17, and at the same time attracted by a continued force whose origin is in o, the force being supposed to act impulsively at equal successive infinitely small times. By the first impulse alone, in any given time the particle would move equably to A : but in the same time the action of the continued, or as it must now be considered the impulsive force alone, would cause it to move uniformly through *ma*; hence at the end of that time the particle would be found in B, having described the diagonal *m*B. Were the particle now left to itself, it would move uniformly to C in the next equal interval of time; but the action of the second impulse of the attractive force would bring it equably to *b* in the same time. Thus at the end of the second interval it would be found in D, having described the diagonal BD, and so on. In this manner the particle would describe the polygon *m*BDE; but if the intervals between the successive impulses of the attractive force be indefinitely small, the diagonals *m*B, BD, DE, &c., will also be indefinitely small, and will coincide with the curve passing through the points *m*, B, D, E, &c.

64. In this hypothesis, no error can arise from assuming that the particle describes the sides of a polygon with a uniform motion; for the polygon, when the number of its sides is indefinitely multiplied, coincides entirely with the curve.

65. The lines *m*A, BC, &c., fig. 17, are tangents to the curve in the points, *m*, B, &c.; it therefore follows that when a particle is moving in a curved line in consequence of any continued force, if the force should cease to act at any instant, the particle would move on in the tangent with an equable motion, and with a velocity equal to what it had acquired when the force ceased to act.

66. The spaces *ma*, B*b*, CD, fig. 18, &c., are the sagittae of the indefinitely small arcs *m*B, BD, DE, &c. Hence the effect of the central force is measured by *ma*, the sagitta of the arc *m*B described in an indefinitely small given time, or by

fig. 18



$$\frac{(\text{arc } m\text{B})^2}{2.om} = ma,$$

om being the radius of the circle coinciding with the curve in *m*.

67. We shall consider the element or differential of time to be a constant quantity; the element of space to be the indefinitely small space moved over in an element of time, and the element of velocity to be the velocity that a particle would acquire, if acted on by a constant force during an element of time. Thus, if *t*, *s* and *v* be the time, space, and velocity, the elements of these quantities are *dt*, *ds*, and *dv*; and as each element is supposed to express an arbitrary unit of its kind, these heterogeneous quantities become capable of comparison. As a decrement only differs from an increment by its sign, any expressions regarding increasing quantities will apply to those that decrease by changing the signs of the differentials; and thus the theory of retarded motion is included in that of accelerated motion.

68. In uniformly accelerated motion, the force at any instant is directly proportional to the second element of the space, and inversely as the square of the element of the time.

Demonstration. Because in uniformly accelerated motion, the velocity is only assumed to be constant for an indefinitely small time, $v = \frac{ds}{dt}$, and as the element of the time is constant, the differential of the velocity is $dv = \frac{d^2s}{dt}$; but since a constant force, acting for an indefinitely small time, produces an indefinitely small velocity, $\mathrm{F}dt = dv$; hence $\mathrm{F} = \frac{d^2s}{dt^2}$.

General Equations of the Motions of a Particle of Matter

69. The general equation of the motion of a particle of matter, when acted on by any forces whatever, may be reduced to depend on the law of equilibrium.

Demonstration. Let *m* be a particle of matter perfectly free to obey any forces X, Y, Z, urging it in the direction of three rectangular co-ordinates *x, y, z.* Then regarding velocity as an effect of force, and as its measure, by the laws of motion these forces will produce in the instant *dt*, the velocities X*dt*, Y*dt*, Z*dt*, proportional to the intensities of these forces, and in their directions. Hence when *m* is free, by article 68,

$$d\cdot\frac{dx}{dt} = \mathrm{X}dt; \quad d\cdot\frac{dy}{dt} = \mathrm{Y}dt; \quad d\cdot\frac{dz}{dt} = \mathrm{Z}dt; \qquad \textbf{(5)}$$

for the forces X, Y, Z, being perpendicular to each other, each one is independent of the action of the other two, and may be regarded as if it acted alone. If the first of these equations be multiplied by dx, the second by dy, and the third by dz, their sum will be

$$0 = \left(\mathrm{X} - \frac{d^2x}{dt^2}\right)dx + \left(\mathrm{Y} - \frac{d^2y}{dt^2}\right)dy + \left(\mathrm{Z} - \frac{d^2z}{dt^2}\right)dz, \qquad \textbf{(6)}$$

And since $\mathrm{X} - \frac{d^2x}{dt^2}$; $\mathrm{Y} - \frac{d^2y}{dt^2}$; $\mathrm{Z} - \frac{d^2z}{dt^2}$; are separately zero, dx, dy, dz, are absolutely arbitrary and independent; and *viceversâ*, *if* they *are* so, this one equation will be equivalent to the three separate ones.

This is the general equation of the motion of a particle of matter, when free to move in every direction.

2nd case. But if the particle *m* be not free, it must either be constrained to move on a curve, or on a surface, or be subject to a resistance, or otherwise subject to some condition. But matter is not moved otherwise than by force; therefore, whatever constrains it, or subjects it to conditions, is a force. If a curve, or surface, or a string constrains it, the force is called reaction: if a fluid medium, the force is called resistance: if a condition however abstract, (as for example that it move in a tautochrone,[1]) still this condition, by obliging it to move out of its free course, or with an unnatural velocity, must ultimately resolve itself into *force*; only that in this case it is an implicit and not an explicit function of the co-ordinates. This new force may therefore be considered first, as involved in X, Y, Z; or secondly, as added to them when it is resolved into $\mathrm{X}', \mathrm{Y}', \mathrm{Z}'$.

In the first case, if it be regarded as included in X, Y, Z, these really contain an indeterminate function: but the equations[2]

$$\mathrm{X}dt = \frac{d^2x}{dt};\ \ \mathrm{Y}dt = \frac{d^2y}{dt};\ \ \mathrm{Z}dt = \frac{d^2z}{dt};$$

still subsist; and therefore also equation (6).

Now however, there are not enough of equations to determine x, y, z, in functions of t, because of the unknown forms of $\mathrm{X}', \mathrm{Y}', \mathrm{Z}'$; but if the equation $u = 0$, which expresses the condition of restraint, with all its consequences $du = 0, du = 0$, &c., be superadded to these, there will then be enough to determine the problem. Thus the equations are

$$u = 0;\ \mathrm{X} - \frac{d^2x}{dt^2} = 0;\ \mathrm{Y} - \frac{d^2y}{dt^2} = 0;\ \mathrm{Z} - \frac{d^2z}{dt^2} = 0.$$

u is a function of $x, y, z,$ X, Y, Z, and t. Therefore the equation $u = 0$ establishes the existence of a relation

$$du = pdx + qdy + rdz = 0$$

between the variations $dx, dy, dz,$ which can no longer be regarded as arbitrary; but the equation (6) subsists whether they be so or not, and may therefore be used simultaneously with $du = 0$ to eliminate one; after which the other two being *really* arbitrary, their coefficients *must* be separately zero.

In the second case; if we do not regard the forces arising from the conditions of constraint as involved in X, Y, Z, let $du = 0$ be that condition, and let $\mathrm{X}', \mathrm{Y}', \mathrm{Z}',$ be the unknown forces brought into action by that condition, by which the action of X, Y, Z, is modified; then will the whole forces acting on m be $\mathrm{X}+\mathrm{X}', \mathrm{Y}+\mathrm{Y}', \mathrm{Z}+\mathrm{Z}',$ and under the influence of these the particle will move as a *free particle*; and therefore $dx, dy, dz,$ being any variations

$$0 = \left(\mathrm{X} + \mathrm{X}' - \frac{d^2x}{dt^2}\right)dx + \left(\mathrm{Y} + \mathrm{Y}' - \frac{d^2y}{dt^2}\right)dy + \left(\mathrm{Z} + \mathrm{Z}' - \frac{d^2z}{dt^2}\right)dz$$

or,

$$0 = \left(\mathrm{X} - \frac{d^2x}{dt^2}\right)dx + \left(\mathrm{Y} - \frac{d^2y}{dt^2}\right)dy + \left(\mathrm{Z} - \frac{d^2z}{dt^2}\right)dz + \mathrm{X}'dx + \mathrm{Y}'dy + \mathrm{Z}'dz; \qquad \textbf{(7)}$$

and this equation is independent of any particular relation between $dx, dy, dz,$ and holds good whether they subsist or not. But the condition $du = 0$ establishes a relation of the form $pdx + qdy + rdz = 0,$ where

$$p = \left(\frac{du}{dx}\right),\ \ q = \left(\frac{du}{dy}\right),\ \ r = \left(\frac{du}{dz}\right);$$

and since this is true, it is so when multiplied by any arbitrary quantity l; therefore,

$$l\,(p d x + q d y + r d z) = 0, \text{ or } l d u = 0;$$

because

$$d u = p d x + q d y + r d z = 0.$$

If this be added to equation (7), it becomes

$$0 = \left(X - \frac{d^2x}{dt^2}\right)dx + \left(Y - \frac{d^2y}{dt^2}\right)dy + \left(Z - \frac{d^2z}{dt^2}\right)dz + X'dx + Y'dy + Z'dz - l\,du,$$

which is true whatever l may be.

Now since X', Y', Z', are forces acting in the direction x, y, z, (though unknown) they may be compounded into one resultant $R_{/}$, which must have one direction, whose element may be represented by ds. And since the single force $R_{/}$ is resolved into X', Y', Z', we must have

$$X'dx + Y'dy + Z'dz = R_{/}ds;$$

So that the preceding equation becomes

$$0 = \left(X - \frac{d^2x}{dt^2}\right)dx + \left(Y - \frac{d^2y}{dt^2}\right)dy + \left(Z - \frac{d^2z}{dt^2}\right)dz + R_{/}ds - l\,du \qquad \textbf{(8)}$$

and this is true whatever X may be.

But l being thus left arbitrary, we are at liberty to determine it by any convenient condition. Let this condition be $R_{/}ds - l\,du = 0$, or $l = R_{/} \cdot \frac{ds}{du}$, which reduces equation (8) to equation (6). So when X, Y, Z, are the only acting forces explicitly given, this equation still suffices to resolve the problem, provided it be taken in conjunction with the equation $du = 0$, or, which is the same thing,

$$p dx + q dy + r dz = 0,$$

which establishes a relation between $dx,\ dy,\ dz$.

Now let the condition $l = s \cdot \frac{ds}{du}$ be considered which determines l.

Since $R_{/}$ is the resultant of the forces X', Y', Z', its magnitude must be represented by $\sqrt{X'^2 + Y'^2 + Z'^2}$ by article 37, and since $R_{/}ds = l\,du$, or

$$X'dx + Y'dy + Z'dz = l \cdot \frac{du}{dx}dx + l \cdot \frac{du}{dy}dy + l \cdot \frac{du}{dz}dz ,$$

therefore, in order that *dx, dy, dz,* may remain arbitrary, we must have

$$X' = l\,\frac{du}{dx},\ Y' = l\,\frac{du}{dy},\ Z' = l\,\frac{du}{dz};$$

and consequently

$$R_{/} = \sqrt{X'^2 + Y'^2 + Z'^2} = l \cdot \sqrt{\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 + \left(\frac{du}{dz}\right)^2} \qquad \textbf{(9)}$$

and

$$l = \frac{R_{/}}{\sqrt{\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 + \left(\frac{du}{dz}\right)^2}}$$

and if to abridge $\frac{1}{\sqrt{\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 + \left(\frac{du}{dz}\right)^2}} = K$; then if $a,\ b,\ g,$ be angles that the normal to the curve or surface makes with the co-ordinates,

$$K\frac{du}{dx} = \cos\ a,\ K\frac{du}{dy} = \cos\ b,\ K\frac{du}{dz} = \cos\ g,$$

and[3]

$$X' = R_{/}\,.\cos a,\ \ Y' = R_{/}\,.\cos b,\ \ Z' = R_{/}\,.\cos g.$$

Thus if u be given in terms of[4] *x, y, z,* the four quantities l, X', Y' and Z', will be determined. If the condition of constraint expressed by $u = 0$ be pressure against a surface, $R_{/}$ is the re-action.

Thus the general equation of a particle of matter moving on a curved surface, or subject to any given condition of constraint, is proved to be

$$0 = \left(X - \frac{d^2x}{dt^2}\right)dx + \left(Y - \frac{d^2y}{dt^2}\right)dy + \left(Z - \frac{d^2z}{dt^2}\right)dz + l\,du \qquad \textbf{(10)}$$

70. The whole theory of the motion of a particle of matter is contained in equations (6) and (10); but the finite values of these equations can only be found when the variations of the forces are expressed at least implicitly in functions of the distance of the moving particle from their origin.

71. When the particle is free, if the forces X, Y, Z, be eliminated from

$$\text{X}-\frac{d^2x}{dt^2}=0;\ \ \text{Y}-\frac{d^2y}{dt^2}=0;\ \ \text{Z}-\frac{d^2z}{dt^2}=0$$

by functions of the distance, these equations which then may be integrated at least by approximation, will only contain space and time; and by the elimination of the latter, two equations will remain, both functions of the co-ordinates which will determine the curve in which the particle moves.

72. Because the force which urges a particle of matter in motion, is given in functions of the indefinitely small increments of the coordinates, the path or trajectory of the particle depends on the nature of the force. Hence if the force be given, the curve in which the particle moves may be found; and if the curve be given, the law of the force may be determined.

73. Since one constant quantity may vanish from an equation at each differentiation, so one must be added at each integration; hence the integral of the three equations of the motion of a particle being of the second order, will contain six arbitrary constant quantities, which are the data of the problem, and are determined in each case either by observation, or by some known circumstances peculiar to each problem.

74. In most cases finite values of the general equation of the motion of a particle cannot be obtained, unless the law according to which the force varies with the distance be known; but by assuming from experience, that the intensity of the forces in nature varies according to some law of the distance and leaving them otherwise indeterminate, it is possible to deduce certain properties of a moving particle, so general that they would exist whatever the forces might in other respects be. Though the variations differ materially, and must be carefully distinguished from the differentials dx, dy, dz, which are the spaces moved over by the particle parallel to the co-ordinates in the instant dt; yet being arbitrary, we may assume them to be equal to these, or to any other quantities consistent with the nature of the problem under consideration. Therefore let $d x$, $d y$, $d z$, be assumed equal to dx, dy, dz, in the general equation of motion (6), which becomes in consequence

$$\text{X}dx+\text{Y}dy+\text{Z}dz=\frac{dxd^2x+dyd^2y+dzd^2z}{dt^2}.$$

75. The integral of this equation can only be obtained when the first member is a complete differential, which it will be if all the forces acting on the particle, in whatever directions, be functions of its distance from their origin.

Demonstration. If F be a force acting on the particle, and s the distance of the particle from its origin, $\text{F}\frac{x}{s}$ is the resolved portion parallel to the axis x; and if F', F'', &c., be the other forces acting on the particle, then, $\text{X}=\Sigma.\text{F}\frac{x}{s}$ will be the sum of all these forces resolved in a direction parallel to the axis x. In the same manner, $\mathrm{Y}=\Sigma.\mathrm{F}\frac{y}{s}$; $\mathrm{Z}=\Sigma.\mathrm{F}\frac{z}{s}$ are the sums of the forces resolved in a direction parallel to the axes y and z, so that

$$\mathrm{X}dx+\mathrm{Y}dy+\mathrm{Z}dz=\Sigma.\mathrm{F}\frac{xdx+ydy+zdz}{s}=\Sigma.\mathrm{F}\frac{sds}{s}=\Sigma.\mathrm{F}ds,$$

which is a complete differential when $\mathrm{F}', \mathrm{F}''$, &c., are functions of s.

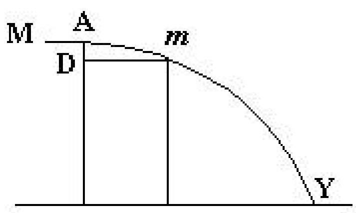


fig. 19

76. In this case, the integral of the first member of the equation is $\int(\mathrm{X}dx+\mathrm{Y}dy+\mathrm{Z}dz)$, or[5] $f(x, y, z)$ a function of x, y, z; and by integration the second is $\frac{1}{2}\frac{dx^2+dy^2+dz^2}{dt^2}$ which is evidently the half of the square of the velocity; for if any curve MN, fig. 19, be represented by s, its first differential ds or Am is

$$\sqrt{\mathrm{AD}^2+\mathrm{D}m^2}=\sqrt{dx^2+dy^2};$$

hence, $ds^2 = dx^2 + dy^2$ when the curve is in one plane, but when in space it is $ds^2=dx^2+dy^2+dz^2$: and as $\frac{ds}{dt}$, the element of the space divided by the element of the time is the velocity: therefore

$$\tfrac{1}{2}\frac{dx^2+dy^2+dz^2}{dt^2}=\tfrac{1}{2}v^2;$$

consequently,

$$2f(x,y,z)+c=v^2,$$

c being an arbitrary constant quantity introduced by integration.

77. This equation will give the velocity of the particle in any point of its path, provided its velocity in any other point be known: for if A be its velocity in that point of its trajectory whose co-ordinates are a, b, c, then

$$\mathrm{A}^2=c+2f(a,b,c),$$

and

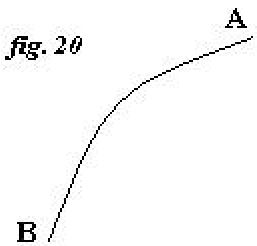
$$v^2-\mathrm{A}^2=2f(x,y,z)-2f(a,b,c);$$

whence v will be found when A is given, and the co-ordinates a, b, c, x, y, z, are known.

It is evident, from the equation being independent of any particular curve, that if the particle begins to move from any given point with a given velocity, it will arrive at another given point with the same velocity, whatever the curve may be that it has described.

78. When the particle is not acted on by any forces, then X, Y, and Z are zero, and the equation becomes $v^2 = c$. The velocity in this case, being occasioned by a primitive impulse, will be constant; and the particle, in moving from one given point to another, will always take the shortest path that can be traced between these points, which is a particular case of a more general law, called the principle of Least Action.

Principle of Least Action



79. Suppose a particle beginning to move from a given point A, fig. 20, to arrive at another given point B, and that its velocity at the point A is given in magnitude but not in direction. Suppose also that it is urged by accelerating forces X, Y, Z, such, that the finite value of $\mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz$ can be obtained. We may then determine v the velocity of the particle in terms of $x, y, z,$ without knowing the curve described by the particle in moving from A to B. If ds be the element of the curve, the finite value of vds between A and B will depend on the nature of the path or curve in which the body moves. The principle of Least Action consists in this, that if the particle be free to move in every direction between these two points, except in so far as it obeys the action of the forces X, Y, Z, it will in virtue of this action, choose the path in which the integral $\int vds$ is a minimum; and if it be constrained to move on a given surface, it will still move in the curve in which $\int vds$ is a minimum among all those that can be traced on the surface between the given points.

To demonstrate this principle, it is required to prove the variation of $\int vds$ to be zero, when A and B, the extreme points of the curve are fixed.

By the method of variations $d\int vds = \int d\,.\,vds$: for $\int$ the mark of integration being relative to the differentials, is independent of the variations. Now

$$d\,.\,vds = dv\,.\,ds + v\,d\,ds, \text{ but } v = \frac{ds}{dt} \text{ or } ds = vdt\,;$$

hence

$$d\,v \cdot ds = vd\,vdt = dt\,\tfrac{1}{2}d\,\cdot v^2,$$

and therefore

$$d\,\cdot vds = dt\cdot\tfrac{1}{2}d\,\cdot v^2 + v\cdot d\,\cdot ds\,.$$

The values of the two last terms of this equation must be found separately. To find $dt\cdot\frac{1}{2}d\cdot v^2$. It has been shown that

$$v^2 = c + 2\int(\mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz),$$

its differential is

$$vdv = (\mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz)\,,$$

and changing the differentials into variations,

$$\tfrac{1}{2}\delta . v^2 = \mathrm{X}\delta x + \mathrm{Y}\delta y + \mathrm{Z}\delta z .$$

If $\tfrac{1}{2}\delta . v^2$ be substituted in the general equation of the motion of a particle on its surface, it becomes

$$\tfrac{1}{2}\delta . v^2 = \frac{d^2x}{dt^2}\delta x + \frac{d^2y}{dt^2}\delta y + \frac{d^2z}{dt^2}\delta z + l\delta u = 0 .$$

But $l\delta u$ does not enter into this equation when the particle is free; and when it must move on the surface whose equation is $u = 0$, δu is also zero; hence in every case the term $l\delta u$ vanishes; therefore

$$dt . \tfrac{1}{2}\delta . v^2 = \frac{d^2x}{dt^2}\delta x + \frac{d^2y}{dt^2}\delta y + \frac{d^2z}{dt^2}\delta z$$

is the value of the first term required.

A value of the second term $v . \delta . ds$ must now be found. Since

$$ds^2 = dx^2 + dy^2 + dz^2 ,$$

its variation is $ds . \delta ds = dx . \delta dx + dy . \delta dy + dz . \delta dz$, but $ds = vdt$, hence

$$v . \delta ds = \frac{dx}{dt}\delta dx + \frac{dy}{dt}\delta dy + \frac{dz}{dt}\delta dz ,$$

which is the value of the second term, and if the two be added, their sum is

$$\delta . vds = d\left\{\frac{dx}{dt}\delta x + \frac{dy}{dt}\delta y + \frac{dz}{dt}\delta z\right\},$$

as may easily be seen by taking the differential of the last member of this equation. Its integral is

$$\delta\int vds = \frac{dx}{dt}\delta x + \frac{dy}{dt}\delta y + \frac{dz}{dt}\delta z .$$

If the given points A and B be moveable in space, the last member of this equation will determine their motion; but if they be fixed points, the last member which is the variation of the co-ordinates of these points is zero: hence also $\delta\int vds = 0$, which indicates either a maximum or minimum, but it is evident from the nature of the problem that it can only be a minimum. If the particle be not urged by accelerating forces, the velocity is constant, and the integral is *vs*. Then the curve s described by the particle between the points A and B is a minimum; and since the velocity is uniform, the particle will describe that curve in a shorter time than it would have done any other curve that could be drawn between these two points.

80. The principle of least action was first discovered by Euler:[6] it has been very elegantly applied to the reflection and refraction of light. If a ray of light IS, fig. 21, falls on any surface CD, it will be turned back or reflected in the direction S*r*, so that ISA = *r*SA . But if the medium whose surface is CD be diaphanous,[7] as glass or water, it will be broken or refracted at S, and will enter the denser medium in the direction SR, so that the sine of the angle of incidence ISA will be to the sine of the angle of refraction RSB, in a constant ratio for any one medium; Ptolemy[8] discovered that light, when reflected from any surface, passed from one given point to another by the shortest path, and in the shortest time possible, its velocity being uniform.

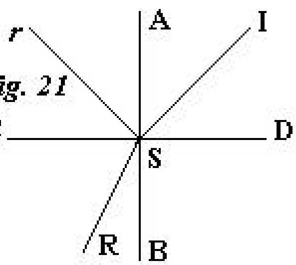


fig. 21

Fermat[9] extended the same principle to the refraction of light; and supposing the velocity of a ray of light to be less in the denser medium, he found that the ratio of the sine of the angle of incidence to that of the angle of refraction, is constant and greater than unity. Newton however proved by the attraction of the denser medium on the ray of light, that in the corpuscular hypothesis its velocity is greater in that medium than in the rarer, which induced Maupertuis[10] to apply the theory of maxima and minima to this problem. If IS, a ray of light moving in a rare medium, fall obliquely on CD the surface of a medium that is more dense, it moves uniformly from I to S; but at the point S both its direction and velocity are changed, so that at the instant of its passage from one to the other, it describes, an indefinitely small curve, which may be omitted without sensible error: hence the whole trajectory of the light is ISR; but IS and SR are described with different velocities; and if these velocities be v and v', then the variation of $\mathrm{IS} \times v + \mathrm{SR} \times v'$ must be zero, in order that the trajectory may be a minimum: hence the general expression $d\int vds = 0$ becomes in this case $d\,.(\mathrm{IS} \times v + \mathrm{SR} \times v') = 0$, when applied to the refraction of light; from whence it is easily found, by the ordinary analysis of maxima and minima, that $v \sin(\mathrm{ISA}) = v' \sin(\mathrm{RBS})$. As the ratio of these sines depends on the ratio of the velocities, it is constant for the transition out of any one medium into another, but varies with the media, on account of the velocity of light being different in different media. If the denser medium be a crystallized diaphanous substance, the velocity of light in it will depend on the direction of the luminous ray; it is constant for any one ray, but variable from one ray to another. Double refraction, as in Iceland spar[11] and in crystallized bodies, arises from the different velocities of the rays; in these substances two images are seen instead of one. Huygens[12] first gave a distinct account of this phenomenon, which has since been investigated by others.

Motion of a Particle on a curved Surface

81. The motion of a particle, when constrained to move on a curve or surface, is easily determined from equation (7); for if the variations be changed into differentials, and if X', Y', Z' be eliminated by their values in the end of article 69, that equation becomes

$$\frac{dx \cdot d^2x + dy \cdot d^2y + dz \cdot d^2z}{dt^2} = \mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz + \mathrm{R}_/ \{dx \,.\, \cos a + dy \,.\, \cos b + dz \,.\, \cos g\},$$

$\mathrm{R}_/$ being the reaction in the normal, and a, b, g the angles made by the normal with the co-ordinates. But the equation of the surface being $u = 0$,

$$du = \frac{du}{dx} \cdot dx + \frac{du}{dy} \cdot dy + \frac{du}{dz} \cdot dz = 0\,;$$

consequently, by article 69,

$$l\, du = dx \,.\, \cos a + dy \,.\, \cos b + dz \,.\, \cos g = 0\,;$$

so that the pressure vanishes from the preceding equation; and when the forces are functions of the distance, the integral is

$$2\, f\,(x, y, z) + c = v^2,$$

and

$$\mathrm{A}^2 - v^2 = 2\, f\,(x, y, z) - 2\, f\,(a, b, c),$$

as before. Hence, if the particle be urged by accelerating forces, the velocity is independent of the curve or surface on which the particle moves; and if it be not urged by accelerating forces, the velocity is constant. Thus the principle of Least Action not only holds with regard to the curves which a particle describes in space, but also for those it traces when constrained to move on a surface.

82. It is easy to see that the velocity must be constant, because a particle moving on a curve or surface[13] only loses an indefinitely small part of its velocity of the second order in passing from one indefinitely small plane of a surface or side of a curve to the consecutive; for if the particle be moving on *ab* with the velocity *v*; then if the angle $abe = b$, the velocity on *bc* will be $v \cos b$; but $\cos b = 1 - \frac{1}{2} b^2 - \&\mathrm{c.}$; therefore the velocity on *bc* differs from the velocity on *ab* by the indefinitely small quantity $\frac{1}{2} v \cdot b^2$. In order to determine the pressure of the particle on the surface, the analytical expression of the radius of curvature must be found.

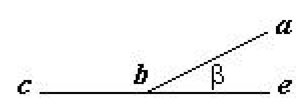
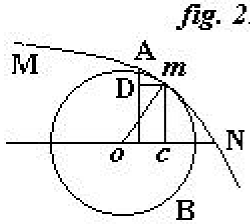


fig. 22



Radius of Curvature

83. The circle A*m*B, fig. 22, which coincides with a curve or curved surface through an indefinitely small space on each side of *m* the point of contact, is called the curve of equal curvature, or the osculating[14] circle of the curve MN, and *om* is the radius of curvature.

In a plane curve the radius of curvature r is expressed by

$$r = \frac{ds^2}{\sqrt{\left(d^2x\right)^2 + \left(d^2y\right)^2}}$$

and in a curve of double curvature it is

$$r = \frac{ds^2}{\sqrt{\left(d^2x\right)^2 + \left(d^2y\right)^2 + \left(d^2z\right)^2}},$$

ds being the constant element of the curve.

Let the angle com be represented by q, then if Am be the indefinitely small but constant element of the curve MN, the triangles com and ADm are similar; hence $m\text{A} : m\text{D} :: om : mc$, or $ds : dx :: 1 : \sin q$, and $\sin q = \frac{dx}{ds}$. In the same manner $\cos q = \frac{dy}{ds}$. But $d\,.\cos q = -dq \sin q$, and $dq = -\frac{d\,.\cos q}{\sin q}$; also $d\,.\sin q = dq \cos q$, and $dq = \frac{d\,.\sin q}{\cos q}$; but these evidently become

$$dq = +\frac{ds}{dy}\cdot d\frac{dx}{ds} \text{ and } dq = -\frac{ds}{dx}\cdot d\frac{dy}{ds};$$

or

$$dq = +\frac{d^2x}{dy}, \text{ and } dq = -\frac{d^2y}{dx}.$$

Now if om the radius of curvature be represented by r, then moA being the indefinitely small increment dq of the angle com, we have $r : ds :: 1 : dq$; for the sine of the infinitely small angle is to be considered as coinciding with the arc: hence $dq = \frac{ds}{r}$, whence $r = -\frac{ds\,.dy}{d^2x} = \frac{ds\,.dx}{d^2y}$. But $dx^2 + dy^2 = ds^2$, and as ds is constant[15] $dx\,.d^2x + dy\,.d^2y = 0$. Whence

$$\frac{d^2x}{d^2y} = -\frac{dy}{dx}, \text{ or } \left(\frac{d^2x}{d^2y}\right)^2 = \frac{dy^2}{dx^2},$$

and adding one to each side of the last equation, it becomes

$$\frac{dx^2 + dy^2}{dx^2} = \frac{ds^2}{dx^2} = \frac{\left(d^2x\right)^2 + \left(d^2y\right)^2}{(d^2y)^2}.$$

Whence

$$\frac{dx}{d^2y} = \frac{ds}{\sqrt{(d^2x)^2 + (d^2y)^2}}.$$

But it has been shown that $r = \frac{dxds}{d^2y}$; hence in a plane curve the radius of curvature is

$$r = \frac{ds^2}{\sqrt{(d^2x)^2 + (d^2y)^2}}.$$

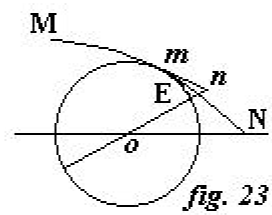


fig. 23

We may imagine MN to be the projection of a curve of double curvature on the plane *xoy*, fig. 23, then $r = \frac{ds^2}{\sqrt{(d^2x)^2 + (d^2y)^2}}$ will be the projection of the radius of curvature on *xoy*, and it is evident that a similar expression will be found for the projection of the radius of curvature on each of the other co-ordinate planes. In fact $\frac{1}{2}\sqrt{(d^2x)^2 + (d^2y)^2}$ is the sagitta of curvature nE; for $(nm)^2 = 2r \cdot n\mathrm{E}$, or

$$r = \frac{(nm)^2}{2n\mathrm{E}} = \frac{ds^2}{2n\mathrm{E}} = \frac{ds^2}{\sqrt{(d^2x)^2 + (d^2y)^2}}$$

for the arc being indefinitely small, the tangent may be considered as coinciding with it. Thus the three projections of the sagitta of curvature of the surface, or curve of double curvature, are

$$\tfrac{1}{2}\sqrt{(d^2x)^2 + (d^2y)^2}; \quad \tfrac{1}{2}\sqrt{(d^2x)^2 + (d^2z)^2}; \quad \tfrac{1}{2}\sqrt{(d^2y)^2 + (d^2z)^2};$$

hence the sum of their squares is

$$\tfrac{1}{2}\sqrt{(d^2x)^2 + (d^2y)^2 + (d^2z)^2};$$

and the radius of curvature of a surface, or curve of double curvature, is

$$r = \frac{ds^2}{\sqrt{(d^2x)^2 + (d^2y)^2 + (d^2z)^2}}.$$

Pressure of a Particle moving on a curved Surface

84. If the particle be moving on a curved surface, it exerts a pressure which the surface opposes with an equal and contrary pressure.

Demonstration. For if F be the resulting force of the partial accelerating forces X, Y, Z, acting on the particle at *m*, it may be resolved into two forces, one in the direction of the tangent *m*T, and the other in the normal *m*N, fig 12. The forces in the tangent have their full effect, and produce a change in the velocity of the particle, but those in the normal are destroyed by the resistance of the surface. If the particle were in equilibrio, the whole pressure would be that in the normal; but when the particle is in motion, the velocity in the tangent produces another pressure on the surface, in consequence of the continual effort the particle makes to fly off the in the tangent. Hence when the particle is in motion, its whole pressure on the surface is the difference of these two pressures, which are both in the direction of the normal, but one tends to the interior of the surface and the other from it. The velocity in the tangent is variable in consequence of the accelerating forces X, Y, Z, and becomes uniform if we suppose them to cease.

Centrifugal Force

85. When the particle is not urged by accelerating forces, its motion is owing to a primitive impulse, and is therefore uniform. In this case X, Y, Z, are zero, the pressure then arising from the velocity only, tends to the exterior of the surface.

And as v the velocity is constant, if ds be the element of the curve described in the time dt, then

$$ds = v dt, \text{ whence } dt = \frac{ds}{v},$$

therefore ds is constant; and when this value of dt is substituted in equation (7), in consequence of the values of X', Y', Z', in the end of article 69, it gives

$$v^2 \cdot \frac{d^2x}{ds^2} = \mathrm{R}_{/} \cos a$$
$$v^2 \cdot \frac{d^2y}{ds^2} = \mathrm{R}_{/} \cos b$$
$$v^2 \cdot \frac{d^2z}{ds^2} = \mathrm{R}_{/} \cos g$$

for by article 81 the particle may be considered as free, whence

$$\mathrm{R}_{/} = \frac{v^2\sqrt{\left(d^2x\right)^2 + \left(d^2y\right)^2 + \left(d^2z\right)^2}}{ds^2};$$

and as the osculating radius is

$$r = \frac{ds^2}{\sqrt{\left(d^2x\right)^2 + \left(d^2y\right)^2 + \left(d^2z\right)^2}},$$

so

$$\mathrm{R}_{/} = \frac{v^2}{r}.$$

The first member of this equation was shown to be the pressure of the particle on the surface, which thus appears to be equal to the square of the velocity, divided by the radius of curvature.

86. It is evident that when the particle moves on a surface of unequal curvature, the pressure must vary with the radius of curvature.

87. When the surface is a sphere, the particle will describe that great circle which passes through the primitive direction of its motion. In this case the circle A*m*B is itself the path of the particle; and in every part of its motion, its pressure on the sphere is equal to the square of the velocity divided by the radius of the circle in which it moves; hence its pressure is constant.

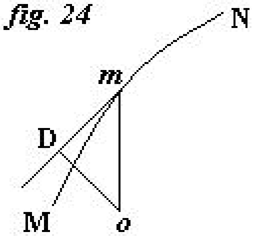
88. Imagine the particle attached to the extremity of a thread assumed to be without mass, whereof the other extremity is fixed to the centre of the surface; it is clear that the pressure which the particle exerts against the circumference is equal to the tension of the thread, provided the particle be restrained in its motion by the thread alone. The effort made by the particle to stretch the thread, in order to get away from the centre, is the centrifugal force. Hence the centrifugal force of a particle revolving about a centre, is equal to the square of its velocity divided by the radius.

89. The plane of the osculating circle, or the plane that passes through two consecutive and indefinitely small sides of the curve described by the particle, is perpendicular to the surface on which the particle moves. And the curve described by the particle is the shortest line that can be drawn between any two points of the surface, consequently this singular law in the motion of a particle on a surface depends on the principle of least action. With regard to the Earth, this curve drawn from point to point on its surface is called a perpendicular to the meridian; such are the lines which have been measured both in France and England, in order to ascertain the true figure of the globe.

90. It appears that when there are no constant or accelerating forces, the pressure of a particle on any point of a curved surface is equal to the square of the velocity divided by the radius of curvature at that point. If to this the pressure due to the accelerating forces be added, the whole pressure of the particle on the surface will be obtained, when the velocity is variable.

91. If the particle moves on a surface, the pressure due to the centrifugal force will be equal to what it would exert against the curve it describes resolved in the direction of the normal to the surface in that point; that is, it will be equal to the square of the velocity divided by the radius of the osculating circle, and multiplied by the sine of the angle that the plane of that circle makes with the tangent plane to the surface. Let MN, fig. 24, be the path of a particle on the surface; *mo* the radius of the osculating circle at *m*, and *m*D a tangent to the surface at *m*; then *om* being radius, *o*D is the sine of the inclination of the plane of the osculating circle on the plane that is tangent to the surface at *m*, the centrifugal force is equal to

fig. 24



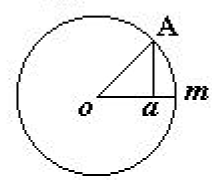
$$\frac{v^2 \times o\mathrm{D}}{om}.$$

If to this, the part of the pressure which is due to the accelerating forces be added, the sum will be the whole pressure on the surface.

92. It appears that the centrifugal force is that part of the pressure which depends on velocity alone; and when there are no accelerating forces it is the pressure itself.

93. It is very easy to show that in a circle, the centrifugal force is equal and contrary to the central force.

fig. 25



Demonstration. By article 63 a central force F combined with an impulse, causes a particle to describe an indefinitely small arc mA, fig. 25, in the time dt. As the sine may be taken for the tangent, the space described from the impulse alone is

$$a\mathrm{A} = vdt\,;$$

but

$$(a\mathrm{A})^2 = 2r \cdot am,$$

so

$$am = \frac{v^2 dt^2}{2r},$$

r being radius. But as the central force causes the particle to move through the space

$$am = \tfrac{1}{2}\mathrm{F} \cdot dt^2,$$

in the same time,

$$\frac{v^2}{r} = \mathrm{F}.$$

94. If v and v' be the velocities of two bodies, moving in circles whose radii are r and r', their velocities are as the circumferences divided by the times of their revolutions; that is, directly as the space, and inversely as the time, since circular motion is uniform. But the radii are as their circumferences, hence

$$v^2 : v'^2 :: \frac{r^2}{t^2} : \frac{r'^2}{t'^2},$$

t and t' being the times of revolution. If c and c' be the centrifugal forces of the two bodies, then

$$c : c' :: \frac{v^2}{r} : \frac{v'^2}{r'},$$

or, substituting for v^2 and v'^2, we have

$$c : c' :: \frac{r}{t^2} : \frac{r'}{t'^2}.$$

Thus the centrifugal forces are as the radii divided by the squares of the times of revolution.

95. With regard to the Earth the times of rotation are everywhere the same; hence the centrifugal forces, in different latitudes, are as the radii of these parallels. These elegant theorems discovered by Huygens, led Newton to the general theory of motion in curves, and to the law of universal gravitation.

Motion of Projectiles

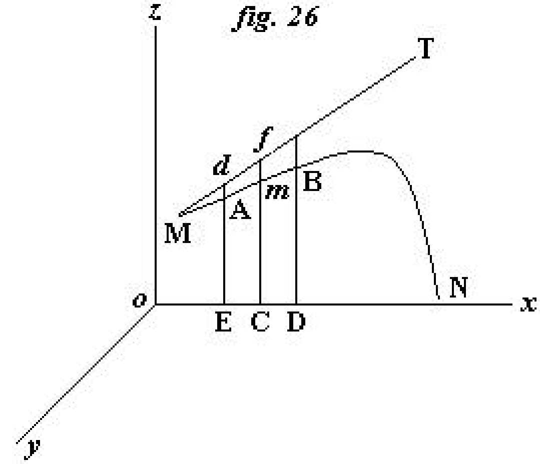
96. From the general equation of motion is also derived the motion of projectiles.

Gravitation affords a perpetual example of a continued force; its influence on matter is the same whether at rest or in motion; it penetrates its most intimate recesses, and were it not for the resistance of the air, it would cause all bodies to fall with the same velocity: it is exerted at the greatest heights to which man has been able to ascend, and in the most profound depths to which he has penetrated. Its direction is perpendicular to the horizon, and therefore varies for every point on the earth's surface; but in the motion of projectiles it may be assumed to act in parallel straight lines; for, any curves that projectiles could describe on the earth may be esteemed as nothing in comparison of its circumference.

The mean radius of the earth is about 4,000 miles,[16] and MM. Biot and Gay-Lussac ascended in a balloon to the height of about four miles,[17] which is the greatest elevation that has been attained, but even that is only the 1,000th part of the radius.

The power of gravitation at or near the earth's surface way, without sensible error, be considered as a uniform force; for the decrease of gravitation, inversely as the square of the distance, is barely perceptible at any height within our reach.

fig. 26



97. *Demonstration.* If a particle be projected in a straight line MT, fig. 26, forming any angle whatever with the horizon, it will constantly deviate from the direction MT by the action of the gravitating force, and will describe a curve MN, which is concave towards the horizon, and to which MT is tangent at M. On this particle there are two forces acting at every instant of its motion: the resistance of the air, which is always in a direction contrary to the motion of the particle, and the force of gravitation, which urges it with an accelerated motion, according to the perpendiculars E*d*, C*f*, &c. The resistance of the air may be resolved into three partial forces, in the direction of the three axes *ox*, *oy*, *oz*, but gravitation acts on the particle in the direction of *oz* alone. If A represents the resistance of the air, its component force in the axis *ox* is evidently $-\text{A}\frac{dx}{ds}$; for if A*m* or *ds* be the space proportional to the resistance, then

$$\text{A}m : \text{E}c :: \text{A} : \text{A}\frac{\text{E}c}{\text{A}m} = \text{A}\frac{dx}{ds};$$

but as this force acts in a direction contrary to the motion of the particle, it must be taken with a negative sign. The resistance in the axes *oy* and *oz* are $-\text{A}\frac{dy}{ds}$, $-\text{A}\frac{dz}{ds}$; hence if *g* be the force of gravitation, the forces acting on the particle are

$$\text{X} = -\text{A}\frac{dx}{ds};\ \text{Y} = -\text{A}\frac{dy}{ds};\ \text{Z} = g - \text{A}\frac{dz}{ds}.$$

As the particle is free, each of the virtual velocities is zero; hence we have

$$\frac{d^2x}{dt^2} = -\text{A}\frac{dx}{ds};\ \frac{d^2y}{dt^2} = -\text{A}\frac{dy}{ds};\ \frac{d^2z}{dt^2} = g - \text{A}\frac{dz}{ds};$$

for the determination of the motion of the projectile. If A be eliminated between the two first, it appears that

$$\frac{d^2x}{dt}\cdot\frac{dy}{dt} = \frac{d^2y}{dt}\cdot\frac{dx}{dt},\ \text{or } d\log\frac{dx}{dt} = d\log\frac{dy}{dt};$$

and integrating

$$\log\frac{dx}{dt} = \log\text{C} + \log\frac{dy}{dt}.$$

Whence $\frac{dx}{dt} = \text{C}\frac{dy}{dt}$, or $dx = \text{C}dy$, and if we integrate a second time,

$$x = \text{C}y + D,$$

in which C and D are the constant quantities introduced by double integration. As this is the equation to a straight line, it follows that the projection of the curve in which the body moves on the plane *xoy* is a straight line, consequently the curve MN is in the plane *zox*, that is at right angles to *xoy;* thus MN is a plane curve, and the motion of the projectile is in a plane at right angles to the horizon. Since the projection of MN on *xoy* is the straight line ED, therefore $y = 0$, and the equation $\frac{d^2y}{dt^2} = -\text{A}\frac{dy}{dt}$ is of no use in the solution of the problem, there being no motion in the direction *oy*. Theoretical reasons, confirmed to a certain extent by experience, show that the resistance of the air supposed of uniform density is proportional to the square of the velocity;[18] hence

$$\text{A}=hv^2 = h\frac{ds^2}{dt^2},$$

h being a quantity that varies with the density, and is constant when it is uniform; thus the general equations become

$$(a)\ \frac{d^2x}{dt^2} = -h\cdot\frac{ds}{dt}\cdot\frac{dx}{dt};\quad \frac{d^2z}{dt^2} = g - h\cdot\frac{ds}{dt}\cdot\frac{dz}{dt};$$

the integral of the first is

$$\frac{dx}{dt} = \text{C}\,.\,c^{-hs},$$

C being an arbitrary constant quantity, and c the number whose hyperbolic logarithm is unity.[19]

In order to integrate the second, let $dz = udx$, u being a function of z; then the differential according to t gives

$$\frac{d^2z}{dt^2} = \frac{du}{dt}\cdot\frac{dx}{dt} + u\cdot\frac{d^2x}{dt^2}.$$

If this be put in the second of equations (a), it becomes, in consequence of the first,

$$\frac{du}{dt}\cdot\frac{dx}{dt} = -g;$$

or, eliminating dt by means of the preceding integral, and making

$$-\frac{g}{2\text{C}^2} = a,$$

it becomes

$$\frac{du}{dx} = 2ac^{2h}.$$

The integral of this equation will give u in functions of x, and when substituted in

$$dz = udx,$$

it will furnish a new equation of the first order between z, x, and t, which will be the differential equation of the trajectory.

If the resistance of the medium be zero, $h = 0$, and the preceding equation gives

$$u = 2ax + b,$$

and substituting $\frac{dz}{dx}$ for u, and integrating again

$$z = ax^2 + bx + b'$$

b and b' being arbitrary constant quantities. This is the equation to a parabola whose axis is vertical, which is the curve a projectile would describe in vacuo. When

$$h = 0,\ d^2z = gdt^2;$$

and as the second differential of the preceding integral gives

$$d^2z = 2adx^2;\ dt = dx\sqrt{\frac{2a}{g}},$$

therefore

$$t = x\sqrt{\frac{2a}{g}} + a'.$$

If the particle begins to move from the origin of the co-ordinates, the time as well as x, y, z, are estimated from that point; hence b' and a' are zero, and the two equations of motion become

$$z = ax^2 + bx;\ \text{and}\ t = x\sqrt{\frac{2a}{g}};$$

whence

$$z = g\frac{t^2}{2} + tb\sqrt{\frac{g}{2a}}.$$

These three equations contain the whole theory of projectiles in vacuo; The second equation shows that the horizontal motion is uniform, being proportional to the time; the third expresses that the motion in the perpendicular is uniformly accelerated, being as the square of the time.

Theory of Falling Bodies

99[20]**.** If the particle begins to move from a state of rest, $b = 0$, and the equations of motion are

$$\frac{dz}{dt} = gt,\ \text{and}\ z = \tfrac{1}{2}gt^2.$$

The first shows that the velocity increases as the time; the second shows that the space increases as the square of the time, and that the particle moving uniformly with the velocity it has acquired in the time t, would describe the space $2z$, that is, double the space it has moved through. Since gt expresses the velocity v, the last of the preceding equations gives

$$2gz = g^2t^2 = v^2 ,$$

where z is the height through which the particle must have descended from rest, in order to acquire the velocity v. In fact, were the particle projected perpendicularly upwards, the parabola would then coincide with the vertical: thus the laws of parabolic motion include those of falling bodies; for the force of gravitation overcomes the force of projection, so that the initial velocity is at length destroyed, and the body then begins to fall from the highest point of its ascent by the force of gravitation, as from a state of rest. By experience it is found to acquire a velocity of nearly 32.19 feet in the first second of its descent at London, and in two seconds it acquires a velocity of 64.38, having fallen through 16.095 feet in the first second, and in the next 32.19 + 16.095 = 48.285 feet, &c. The spaces described are as the odd numbers 1, 3, 5, 7, &c.

These laws, on which the whole theory of motion depends, were discovered by Galileo.

Comparison of the Centrifugal Force with Gravity

100. The centrifugal force may now be compared with gravity, for if v be the velocity of a particle moving in the circumference of a circle of which r is the radius, its centrifugal force is $f = \frac{v^2}{r}$. Let h be the space or height through which a body must fall in order to acquire a velocity equal to v; then by what was shown in article 99, $v^2 = 2hg$, for the accelerating force in the present case is gravity; hence $f = \frac{2 . h . g}{r}$. If we suppose $h = \frac{1}{2} r$, the centrifugal force becomes equal to gravity.

101. Thus, if a heavy body be attached to one extremity of a thread, and if it be made to revolve in a horizontal plane round the other extremity of the thread fixed to a point in the plane; if the velocity of revolution be equal to what the body would acquire by falling through a space equal to half the length of the thread, the body will stretch the thread with the same force as if it hung vertically.

102. Suppose the body to employ the time T to describe the circumference whose radius is r; then p being the ratio of the circumference to the diameter, $v = \frac{2p r}{\mathrm{T}}$, whence

$$f = \frac{4p^2 r}{\mathrm{T}^2} .$$

Thus the centrifugal force is directly proportional to the radius, and in the inverse ratio of the square of the time employed to describe the circumference. Therefore, with regard to the earth, the centrifugal force increases from the poles to the equator, and gradually diminishes the force of gravity. The equatorial radius, computed from the mensuration of degrees of the meridian, is 20,920,600[21] feet, T = 365.2564 days,[22] and as it appears, by experiments with the pendulum, that bodies fall at the equator 16.0436 feet in a second, the preceding formulae give the ratio of the centrifugal force to gravity at the equator equal to $\frac{1}{289}$. Therefore if the rotation of the earth were 17 times more rapid, the centrifugal force would be equal to gravity, and at the equator bodies would be in equilibrio from the action of these two forces.

Simple Pendulum

103. A particle of matter suspended at the extremity of a thread, supposed to be without weight, and fixed at its other extremity, forms the simple pendulum.

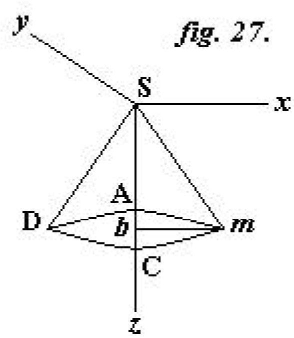
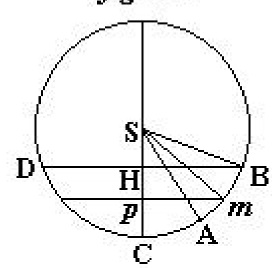


fig. 27.

104. Let m, fig. 27, be the particle of matter, Sm the thread, and S the point of suspension. If an impulse be given to the particle, it will move in a curve mADC, as if it were on the surface of the sphere of which S is the centre; and the greatest deviation from the vertical Sz would be measured by the sine of the angle CSm. This motion arises from the combined action of gravitation and the impulse.

105. The impulse may be such as to make the particle describe a curve of double curvature; or if it be given in the plane xSz, the particle will describe the arc of a circle DCm, fig. 28; but it is evident that the extent of the arc will be in proportion to the intensity of the impulse, and it may be so great as to cause the particle to describe an indefinite number of circumferences. But if the impulse be small, or if the particle be drawn from the vertical to a point B and then left to itself, it will be urged in the vertical by gravitation, which will cause it to describe the arc mC with an accelerated velocity; when at C it will have acquired so much velocity that it will overcome the force of gravitation, and having passed that point, it will proceed to D; but in this half of the arc its motion will be as much retarded by gravitation as it was accelerated in the other half; so that on arriving at D it will have lost all its velocity, and it will descend through DC with an accelerated motion which will carry it to B again. In this manner it would continue to move for ever, were it not for the resistance of the air. This kind of motion is called oscillation. The time of an oscillation is the time the particle employs to move through the arc BCD.

fig. 28.


106. *Demonstration.* Whatever may be the nature of the curve, it has already been shown in article 99, that at any point m, $v^2 = 2gz$, g being the force of gravitation, and $z = \text{H}p$, the height through which the particle must have descended in order to acquire the velocity v. If the particle has been impelled instead of falling from rest, and if I be the velocity generated by the impulse, the equation becomes $v^2 = \text{I}+2gz$. The velocity at m is directly as the element of the space, and inversely as the element of the time; hence

$$v^2 = \frac{(\mathrm{A}m)^2}{dt^2} = \frac{ds^2}{dt^2} = \mathrm{I} + 2g\,.\,z\,;$$

whence[23]

$$dt = \frac{-ds}{\sqrt{1+2g\,.\,z}}\,.$$

The sign is made negative, because z diminishes as t augments. If the equation of the trajectory or curve mCD be given, the value of $ds = \mathrm{A}m$ may be obtained from it in terms of $z = \mathrm{H}p$, and then the finite value of the preceding equation will give the time of an oscillation in that curve.

107. The case of greatest importance is that in which the trajectory is a circle of which Sm is the radius; then if an impulse be given to the pendulum at the point B perpendicular to SB, and in the plane *xoz*, it will oscillate in that plane. Let h be the height through which the particle must fall in order to acquire the velocity given by the impulse, the initial velocity I will then be 2gh; and if $\mathrm{BSC} = a$ be the greatest amplitude, or greatest deviation of the pendulum from the vertical, it will be a constant quantity. Let the variable angle $m\mathrm{SC} = q$, and if the radius be r, then

$$\mathrm{S}p = r\cos q;\; \mathrm{SH} = r\cos a;\quad \mathrm{H}p = \mathrm{S}p - \mathrm{SH} = r(\cos q - \cos a);$$

and the elementary[24] arc($m\mathrm{A} = rdq$); hence the expression for the time becomes

$$dt = \frac{-rdq}{\sqrt{2g\,(h + r\cos q - r\cos a)}}\,.$$

This expression will take a more convenient form, if $x = \mathrm{C}p = (1 - \cos q)$ be the versed sine of mSC, and $b = (1 - \cos a)$ the versed sine of BSC; then

$$dq = \frac{dx}{\sqrt{2x - x^2}}\,,$$

and

$$dt = \frac{-rdx}{\sqrt{2x - x^2}\,.\sqrt{2g\,(h + rb - rx)}}$$

$$v = \sqrt{2g\,(h + rb - rx)}.$$

Since the versed sine can never surpass 2, if $h + rb > 2r$, the velocity will never be zero, and the pendulum will describe an indefinite number of circumferences; but if $h + rb < 2r$, the velocity v will be zero at that point of the trajectory where $x = \dfrac{h + rb}{r}$, and the pendulum will oscillate on each side of the vertical.

If the origin of motion be at the commencement of an oscillation, $h = 0$, and

$$dt = -\tfrac{1}{2}\sqrt{\frac{r}{g}} \cdot \frac{dx}{\sqrt{b\,x - x^2}\sqrt{1 - \frac{x}{2}}}.$$

Now[25]

$$\left(1 - \frac{x}{2}\right)^{-\frac{1}{2}} = 1 + \frac{1}{2} \cdot \frac{x}{2} + \frac{1.3}{2.4} \cdot \frac{x^2}{4} + \frac{1.3.5}{2.4.6} \cdot \frac{x^3}{8} + \&c.$$

therefore,

$$dt = -\frac{1}{2}\sqrt{\frac{r}{g}} \cdot \frac{dx}{\sqrt{b\,x - x^2}}\left\{1 + \frac{1}{2} \cdot \frac{x}{2} + \frac{1.3}{2.4} \cdot \frac{x^2}{4} + \&c.\right\}$$

By Lacroix' *Integral Calculus*[26]

$$\int \frac{-dx}{\sqrt{b\,x - x^2}} = \text{arc}\left(\cos = \frac{2x - b}{b}\right) + \text{constant.}$$

But the integral must be taken between the limits $x = b$ and $x = 0$, that is, from the greatest amplitude to the point C. Hence

$$\int \frac{-dx}{\sqrt{b\,x - x^2}} = p\ ;$$

p being the ratio of the circumference to the diameter. From the same author[27] it will be found that

$$\int \frac{-xdx}{\sqrt{b\,x - x^2}} = \tfrac{1}{2}bp; \quad \int \frac{-x^2dx}{\sqrt{b\,x - x^2}} = \tfrac{1}{2} \cdot \tfrac{3}{4}\, b^2 p, \ \&\text{c. }\&\text{c.}$$

between the same limits. Hence, if $\tfrac{1}{2}$T be the time of half an oscillation,

$$\text{T} = p\sqrt{\frac{r}{g}}\left\{1 + \left(\frac{1}{2}\right)^2 \frac{b}{2} + \left(\frac{1.3}{2.4}\right)^2 \frac{b^2}{4} + \left(\frac{1.3.5}{2.4.6}\right)^2 \frac{b^3}{8} + \&c.\right\}$$

This series gives the time whatever may be the extent of the oscillations; but if they be very small, $\frac{b}{2}$ may be omitted in most cases; then

$$T=p\sqrt{\frac{r}{g}}. \quad \textbf{(11)}$$

As this equation does not contain the arcs, the time is independent of their amplitude, and only depends on the length of the thread and the intensity of gravitation; and as the intensity of gravitation is invariable for any one place on the earth, the time is constant at that place. It follows, that the small oscillations of a pendulum are performed in equal times, whatever their comparative extent may be.

The series in which the time of an oscillation is given however, shows that it is not altogether independent of the amplitude of the arc. In very delicate observations the two first terms are retained; so that

$$T=p\sqrt{\frac{r}{g}}\left\{1+\left(\frac{1}{2}\right)^2\frac{b}{2}\right\}, \text{ or } T=p\sqrt{\frac{r}{g}}\left\{1+\left(\frac{1}{2}\right)^2\frac{a^2}{2}\right\} \quad \textbf{(12)}$$

for as b is the versed sine of the arc a, when the arc is very small, $b=\frac{a^2}{2}$ nearly. The term $p\sqrt{\frac{r}{g}}\left(\frac{1}{2}\right)^2\frac{a^2}{4}$, which is very small, is the correction due to the magnitude of the arc described, and is the equation alluded to in article 9, which must be applied to make the times equal. This correction varies with the arc when the pendulum oscillates in air, therefore the resistance of the medium has an influence on the duration of the oscillation.

108. The intensity of gravitation at any place on the earth may be determined from the time and the corresponding length of the pendulum. If the earth were a sphere, and at rest, the intensity of gravity[28] would be the same in every point of its surface; because every point in its surface would then be equally distant from its centre. But as the earth is flattened at the poles, the intensity of gravitation increases from the equator to the poles; therefore the pendulum that would oscillate in a second at the equator, must be lengthened in moving towards the poles.

If h be the space a body would describe by its gravitation during the time T, then $2h=gT^2$, and because $T^2=p^2\cdot\frac{r}{g}$; therefore

$$h=\tfrac{1}{2}p^2\cdot r. \quad \textbf{(13)}$$

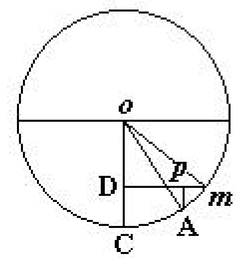
If r be the length of a pendulum beating seconds in any latitude, this expression will give h, the height described by a heavy body during the first second of its fall.

The length of the seconds pendulum at London is 39. 1387 inches; consequently in that latitude gravitation causes a heavy body to fall through 16. 0951 feet during the first second of its descent.

Huygens had the merit of discovering that the rectilinear motion of heavy bodies might be determined by the oscillations of the pendulum. It is found by experiments first made by Sir Isaac Newton, that the length of a pendulum vibrating in a given time is the same, whatever the substance may be of which it is composed; hence gravitation acts equally on all bodies, producing the same velocity in the same time, when there is no resistance from the air.

Isochronous Curve

fig. 29.



109. The oscillations of a pendulum in circular arcs being isochronous[29] only when the arc is very small, it is now proposed to investigate the nature of the curve in which a particle must move, so as to oscillate in equal times, whatever the amplitude of the arcs may be.

The forces acting on the pendulum at any point of the curve are the force of gravitation resolved in the direction of the arc, and the resistance of the air which retards the motion. The first is $-g\frac{\mathrm{A}p}{\mathrm{A}m}$, or $-g\cdot\frac{dz}{ds}$, the arc Am being indefinitely small; and the second, which is proportional to the square of the velocity, is expressed by $-n\left(\frac{ds}{dt}\right)^2$, in which n is any number, for the velocity is directly as the element of the space, and inversely as the element of the time. Thus $-g\cdot\frac{dz}{ds}-n\frac{ds^2}{dt^2}$ is the whole force acting on the pendulum, hence the equation $\mathrm{F}=\frac{d^2s}{dt^2}$ article 68, becomes $-g\frac{dz}{ds}-n\frac{ds^2}{dt^2}=\frac{d^2s}{dt^2}$. The integral of which will give the isochronous curve in air; but the most interesting results are obtained when the particle is assumed to move in vacuo; then $n=0$, and the equation becomes $\frac{d^2s}{dt^2}=-g\frac{dz}{ds}$, which, multiplied by $2ds$ and integrated, gives $\frac{ds^2}{dt^2}=c-2gz$, c being an arbitrary constant quantity.

Let $z=h$ at m, fig. 29, where the motion begins, the velocity being zero at that point, then will $c=2gh$, and therefore

$$\frac{ds^2}{dt^2}=2g\left(h-z\right);$$

whence

$$dt=-\frac{ds}{\sqrt{2g\left(h-z\right)}}:$$

the sign is negative, because the arc diminishes as the time increases. When the radical is developed,[30]

$$dt = -\frac{ds}{\sqrt{2gh}}\left\{1+\frac{1}{2}\cdot\frac{z}{h}+\frac{1.3}{2.4}\cdot\frac{z^2}{h^2}+\&c.\right\}$$

Whatever the nature of the required curve may be, *s* is a function of *z*; and supposing this function developed according to the powers of *z*, its differential will have the form,

$$\frac{ds}{dz} = az^{i} + bz^{i'} + \&c.$$

Substituting this value of *ds* in the preceding equation, it becomes

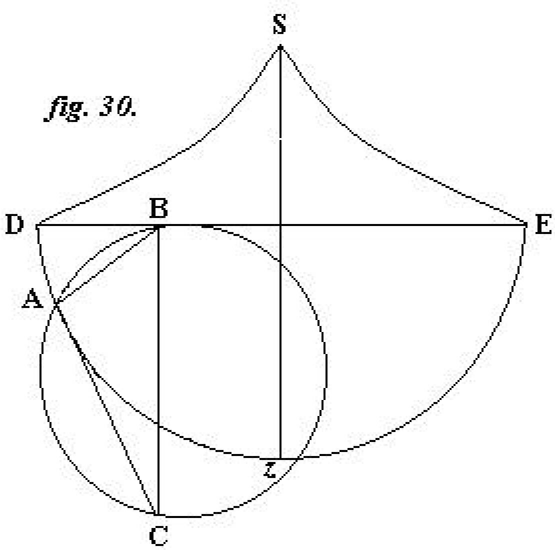
$$dt = -\frac{a}{\sqrt{2g}}\frac{z^{i}}{h^{\frac{1}{2}}}\left\{1+\frac{1}{2}\cdot\frac{z}{h}+\frac{1.3}{2.4}\cdot\frac{z^2}{h^2}+\&c.\right\}dz - \frac{b}{\sqrt{2g}}\frac{z^{i'}}{h^{\frac{1}{2}}}\left\{1+\frac{1}{2}\cdot\frac{z}{h}+\frac{1.3}{2.4}\cdot\frac{z^2}{h^2}+\&c.\right\}dz.$$

The integral of this equation, taken from $z = h$ to $z = 0$, will give the time employed by the particle in descending to C, the lowest point of the curve. But according to the conditions of the problem, the time must be independent of *h*, the height whence the particle has descended; consequently to fulfil that condition, all the terms of the value of *dt* must be zero, except the first; therefore *b* must be zero, and $i+1=\frac{1}{2}$ or $i=-\frac{1}{2}$; thus $ds = az^{-\frac{1}{2}}dz$; the integral of which is $s = 2az^{1/2}$, the equation to a cycloid D*z*E, fig. 30, with a horizontal base, the only curve in vacuo having the property required. Hence the oscillations of a pendulum moving in a cycloid are rigorously isochronous in vacuo. If $r = 2\text{BC}$, by the properties of the cycloid $r = 2a^2$, and if the preceding value of *ds* be put in

$$dt = -\frac{ds}{\sqrt{2g(h-z)}}$$

its integral is

$$t = \tfrac{1}{2}\sqrt{\frac{r}{g}}\cdot\text{arc}\left(\cos = \frac{2z-h}{h}\right).$$



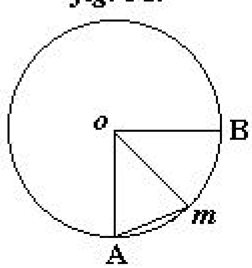
It is unnecessary to add a constant quantity if $z = h$ when $t = 0$. If $\tfrac{1}{2}\text{T}$ be the time that the particle takes to descend to the lowest point in the curve where $z = 0$, then

$$\text{T}=\sqrt{\frac{r}{g}}\cdot\ \text{arc}(\cos = -1) = p\cdot\sqrt{\frac{r}{g}}.$$

Thus the time of descent through the cycloidal arc is equal to a semi-oscillation of the pendulum whose length is r, and whose oscillations are very small, because at the lowest point of the curve the cycloidal arc ds coincides with the indefinitely small arc of the osculating circle whose vertical diameter is $2r$.

110. The cycloid in question is formed by supposing a circle ABC, fig. 30, to roll along a straight line ED. The curve EAD traced by a point A in its circumference is a cycloid. In the same manner the cycloidal arcs SD, SE, may be traced by a point in a circle rolling on the other side of DE. These arcs are such, that if we imagine a thread fixed at S to be applied to SD, and then unrolled so that it may always be tangent to SD, its extremity D will trace the cycloid DzE; and the tangent zS is equal to the corresponding arc DS. It is evident also, that the line DE is equal to the circumference of the circle ABC. The curve SD is called the involute, and the curve Dz the evolute. In applying this principle to the construction of clocks, it is so difficult to make the cycloidal arcs SE, SD, round which the thread of the pendulum winds at each vibration, that the motion in small circular arcs is preferred. The properties of the isochronous curve were discovered by Huygens, who first applied the pendulum to clocks.

fig. 31.



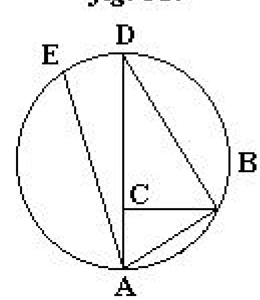
111. The time of the very small oscillation of a circular pendulum is expressed by $\mathrm{T}=\sqrt{\frac{\mathrm{r}}{\mathrm{g}}}$, r being the length of the pendulum, and consequently the radius of the circle AmB, fig. 31. Also $t=\sqrt{\frac{2z}{g}}$ is the time employed by a heavy body to fall by the force of gravitation through a height equal to z. Now the time employed by a heavy body to fall through a space equal to twice the length of the pendulum will be $t=\sqrt{\frac{4r}{g}}$ hence

$$\tfrac{1}{2}\mathrm{T} : t \;::\; \tfrac{1}{2}p\sqrt{\frac{r}{g}} : \sqrt{\frac{4r}{g}},$$

or

$$\frac{\mathrm{T}}{2} : t :: \frac{p}{2} : 2$$

fig. 32.



that is, the time employed to move through the arc Am, which is half an oscillation, is to the time of falling through twice the length of the pendulum, as a fourth of the circumference of the circle AmB to its diameter. But the times of falling through all chords drawn to the lowest point A, fig. 32, of a circle are equal: for the accelerating force F in any chord AB, is to that of gravitation as $\mathrm{AC} : \mathrm{AB}$, or as AB to AD, since the triangles are similar. But the forces being as the spaces, the times are equal: for as

$$\mathrm{F} : g :: \mathrm{AB} : \mathrm{AD} \text{ and } \mathrm{T} : t :: \frac{\mathrm{AB}}{\mathrm{F}} : \frac{\mathrm{AD}}{g},$$

it follows that $\mathrm{T} = t$.

112. Hence the time of falling through the chord AB, is the same with that of falling through the diameter; and thus the time of falling through the arc AB is to the time of falling through the chord AB as $\frac{p}{2} : 2$, that is, as one-fourth of the circumference to the diameter, or as 1.57079 to 2. Thus the straight line AB, through the shortest that can be drawn between the points B and A, is not the line of quickest descent.

Curve of quickest Descent

113. In order to find the curve in which a heavy body will descend from one given point to another in the shortest time possible, let CP=z, PM=y, and CM=s, fig. 33. The velocity of a body moving in the curve at M will be $\sqrt{2gz}$, g being the force of gravitation. Therefore $\sqrt{2gz} = \frac{ds}{dt}$ or $dt = \frac{ds}{\sqrt{2gz}}$ the time employed in moving from M to m. Now let

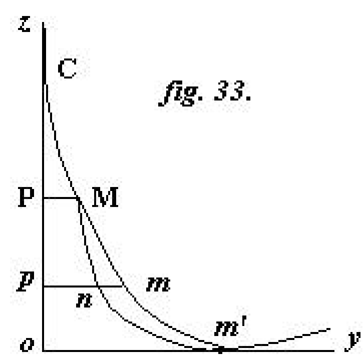


fig. 33.

$$\mathrm{C}p = z + dz = z', \quad pm = y + dy = y' \text{ and } \mathrm{C}m = ds + s = s'.$$

Then the time of moving through mm' is $\frac{ds'}{\sqrt{2gz'}}$. Therefore the time of moving from M to m' is $\frac{ds}{\sqrt{2gz}} + \frac{ds'}{\sqrt{2gz'}}$, which by hypothesis must be a minimum, or, by the method of variations,

$$d\,\frac{ds}{\sqrt{z}} + d\,\frac{ds'}{\sqrt{z'}} = 0.$$

The values of z and z' are the same for any curves that can be drawn between the points M and m': hence $d\,dz = 0$ and $d\,dz' = 0$. Besides, whatever the curves may be, the ordinate om' is the same for all; hence $dy + dy'$ is constant, therefore $d\,(dy + dy') = 0$: whence

$$d\,dy = -d\,dy'; \text{ and } d\,\frac{ds}{\sqrt{z}} + d\,\frac{ds'}{\sqrt{z'}} = 0,$$

from these considerations, becomes

$$\frac{dy}{ds\sqrt{z}} - \frac{dy'}{ds\sqrt{z'}} = 0\,.$$

Now it is evident, that the second term of this equation is only the first term in which each variable quantity is augmented by its increment, so that

$$\frac{dy}{ds\sqrt{z}} - \frac{dy'}{ds\sqrt{z'}} = d\cdot\frac{dy}{ds\sqrt{z}} = 0\,,$$

whence

$$\frac{dy}{ds\sqrt{z}} = \mathrm{A}\,.$$

But $\frac{dy}{ds}$ is the sine of the angle that the tangent to the curve makes with the line of the abscissae, and at the point where the tangent is horizontal this angle is a right angle, so that $\frac{dy}{ds} = 1$: hence if a be the value of z at that point, $\mathrm{A} = \frac{1}{\sqrt{a}}$, and $\frac{dy}{ds} = \sqrt{\frac{z}{a}}$, but, $ds^2 = dy^2 + dz^2$, therefore

$$\frac{dy}{dz} = \sqrt{\frac{z}{a-z}}\,,$$

the equation to the cycloid, which is the curve of quickest descent.

Notes

[1] A tautochrone is a curved line, such that a heavy body, descending along it by the action of gravity, will always arrive at the lowest point in the same time, wherever in the curve it may begin to fall; an inverted cycloid with its base horizontal is a tautochrone. *Webster's Dictionary, 1913.*

[2] A comma is used after the third equation in the 1st edition.

[3] Y' and Z' *read* Y and Z in the 1st edition.

[4] The series *reads x, y, z*; in the 1st edition.

[5] This *reads f(x, y, z,)* in the 1st edition.

[6] See note 4, *Foreword to the Second Edition.*

[7] *diaphanus*. Transparent or nearly so.

[8] See note 15, *Preliminary Dissertation*.

[9] Fermat, Pierre de, 1601-1665, mathematician, born in Beaumont-de-Lomagne, France. He founded the theory of probability but is best known for his work in number theory. Fermat's "last theorem" was the most famous unsolved problem in mathematics until proved in 1994. In optics, Fermat's principle of least time was the first statement of a

variational principle in physics (see note 4, *Bk. I, Chap. I*). Fermat's work in finding tangents to curves was instrumental in the emergence of differential calculus.

[10] Maupertuis, Pierre Louis Moreau de, 1698-1759, mathematician and French astronomer who popularized Newton's theory of gravitation, born in St Malo, France. In 1736 he accurately measured the length of a degree of the meridian, a verification of Newton's prediction that the earth was an oblate spheroid. Maupertuis is best known for his "principle of least action" published in his *Essai de cosmologie* (1750).

[11] *Iceland spar.* A colourless and transparent form of calcite known for its property of double refraction.

[12] Huygens, Christiaan, 1629-1695, physicist and astronomer, born in The Hague, The Netherlands. In optics Huygens propounded the wave theory of light, and discovered polarization. He also discovered the ring and fourth satellite of Saturn (1655), and made the first pendulum clock based on his theories in *Horologium Oscillatorium sive de motu pendulorum* (1673). Huygens derived the law of centrifugal force in the case of uniform circular motion. Huygens also experimentally demonstrated the principle of linear momentum conservation for elastic collisions. He later applied this principle to rotating bodies together with the application of the principle that the centre of gravity would remain fixed. This work led to the formulation of the inverse-square law of gravitational attraction by Robert Hooke (1635-1703) and Christopher Wren (1632-1723). This inverse-square law combined with the direct attraction of masses formed the basis for Newton's theory of universal gravitation.

[13] This figure is also unnumbered in the 1st edition.

[14] *osculating.* To be tangent; touch. *The Wordsmyth Educational Dictionary-Thesaurus.*

[15] Original text *reads* $dx.d^2x + dyd^2y = 0$.

[16] *mean radius of the earth.* The actual value is 3,964 miles.

[17] Biot, Jean Baptiste, 1774-1862, physicist and astronomer, born in Paris. Biot taught physics at the Collège de France. In 1804 he ascended in a balloon with chemist Joseph Louis Gay-Lussac (1778-1850). The flight demonstrated that the earth's magnetic field did not vary with altitude. Biot established fundamental laws of light polarization in optically active materials. The Biot-Savart law resulted from a collaboration with Félix Savart (1791-1841) in the demonstration of the relationship between an electric current and magnetic field. Biot's most important work is his *Traité élémentaire d'astronomie physique* (1805). Gay-Lussac was born in St Léonard, France. His balloon ascents led to discovery of the law of combining volumes of gases named after him (1808).

[18] An empirical relation.

[19] *the number whose hyperbolic logarithm is unity.* Although the letter *e* has been used for this number (2.71828…) since Euler's published work *Mechanica* (1736), other mathematicians used *b* and *c*. The mathematician Jean le Rond d'Alembert (1717-1783), frequently referenced by Somerville, used the letter *c*.

[20] There is no article 98 in the original text.

[21] Here as elsewhere in the text we have added comma separators not used in the 1st edition.

[22] This *reads* $365^{d}.2564$ in the 1st edition.

[23] This *reads* $dt = \frac{-ds}{\sqrt{\mathrm{I} + 2g\,.\,z}}$ in the 1st edition.

[24] This *reads* arc mA$=rdq$ in the 1st edition.

[25] The third term *reads* $\frac{1.3.5}{2.4.6} \cdot \frac{x^2}{8}$ in the 1st edition.

[26] Lacroix, Silvestre Francois, 1765-1843, *An elementary treatise on the differential and integral calculus*, 1816

[27] ibid.

[28] This *reads* 'gravitation' in the 1st edition (published erratum).

[29] *isochronous.* Motions or oscillations of equal duration.

[30] The first term inside bracket in next expression *reads* $\frac{1}{2}\frac{z}{h}$ in 1st edition.

BOOK I

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CHAPTER III

ON THE EQUILIBRIUM OF A SYSTEM OF BODIES

Definitions and Axioms

114. ANY number of bodies which can in any way mutually affect each other's motion or rest, is a system of bodies.

115. Momentum is the product of the mass and the velocity of a body.

116. Force is proportional to velocity, and momentum is proportional to the product of the velocity and the mass; hence the only difference between the equilibrium of a particle and that of a solid body is, that a particle is balanced by equal and contrary forces, whereas a body is balanced by equal and contrary momenta.

117. For the same reason, the motion of a solid body differs from the motion of a particle by the mass alone, and thus the equation of the equilibrium or motion of a particle will determine the equilibrium or motion of a solid body, if they be multiplied by its mass.

118. A moving force is proportional to the quantity of momentum generated by it.

Reaction equal and contrary to Action

119. The law of reaction being equal and contrary to action, is a general induction from observations made on the motions of bodies when placed within certain distances of one another; the law is, that the sum of the momenta generated and estimated in a given direction is zero. It is found by experiment, that if two spheres A and B of the same dimensions and of homogeneous matter as of gold, be suspended by two threads so as to touch one another when at rest, then if they be drawn aside from the perpendicular to equal heights and let fall at the same instant, they will strike one another centrically,[1] and will destroy each other's motion, so as to remain at rest in the perpendicular. The experiment being repeated with spheres of homogenous matter, but of different dimensions, if the velocities be inversely as the quantities of matter, the bodies after impinging will remain at rest. It is evident, that in this case, the smaller sphere must descend through a greater space than the larger, in order to acquire the necessary velocity. If the spheres move in the same or in opposite directions, with different momenta, and one strike the other, the body that impinges will lose exactly the quantity of momentum that the other acquires. Thus, in all cases, it is known by experience that reaction is equal and contrary to action, or that equal momenta in opposite directions destroy one another. Daily experience shows that one body cannot acquire motion by the action of another, without depriving the latter body of the same quantity of motion. Iron attracts the magnet with the same force that it is attracted by it; the same thing is seen in electrical attractions and repulsions, and also in animal forces; for whatever may be the moving principle of man and animals, it is found they receive by the reaction of matter, a force equal and contrary to that which they communicate, and in this respect they are subject to the same laws as inanimate beings.

Mass proportional to Weight

120. In order to show that the mass of bodies is proportional to their weight, a mode of defining their mass without weighing them must be employed; the experiments that have been described afford the means of doing so, for having arrived at the preceding results, with spheres formed of matter of the same kind, it is found that one of the bodies may be replaced by matter of another kind, but of different dimensions from that replaced. That which produces the same effects as the mass replaced, is considered as containing the same mass or quantity of matter. Thus the mass is defined independent of weight, and as in any one point of the earth's surface every particle of matter tends to move with the same velocity by the action of gravitation, the sum of their tendencies constitutes the weight of a body; hence the mass of a body is proportional to its weight, at one and the same place.

Density

121. Suppose two masses of different kinds of matter, A, of hammered gold, and B of cast copper. If A in motion will destroy the motion of a third mass of matter C, and twice B is required to produce the same effect, then the density of A is said to be double the density of B.

Mass proportional to the Volume into the Density

122. The masses of bodies are proportional to their volumes multiplied by their densities; for if the quantity of matter in a given cubical magnitude of a given kind of matter, as water, be arbitrarily assumed as the unit, the quantity of matter in another body of the same magnitude of the density r, will be represented by r; and if the magnitude of the second body to that of the first be as m to 1, the quantity of matter in the second body will be represented by $m \times r$.

Specific Gravity

123. The densities of bodies of equal volumes are in the ratio of their weights, since the weights are proportional to their masses; therefore, by assuming for the unit of density the maximum density of distilled water at constant temperature, the density of a body will be the ratio of its weight to that of a like volume of water reduced to this maximum.

This ratio is the specific gravity of a body.

Equilibrium of two Bodies

124. If two heavy bodies be attached to the extremities of an inflexible line without mass, which may turn freely on one of its points; when in equilibrio, their masses are reciprocally as their distances from the point of motion.

Demonstration. For, let two heavy bodies, m and m', fig. 34, be attached to the extremities of an inflexible line, free to turn round one of its points n, and suppose the line to be bent in n, but so little, that $m'nm$ only differs from two right angles by an indefinitely small angle[2] $amn + am'n$, which may be represented by w. If g be the force of gravitation, gm, gm' will be the gravitation of the two bodies. But the gravitation gm acting in the direction na may be resolved into two forces, one in the direction mn, which is destroyed by the fixed point n, and another acting on m' in the direction $m'm$. Let $mn = f$, $m'n = f'$; then $m'm = f + f'$ very nearly. Hence the whole force gm is to the part acting on m' :: $na : mm'$, and the action of m on m', is $\frac{gm(f + f')}{na}$; but $m'n : na :: 1 : w$, for the arc is so small that it may be taken for its sine. Hence $na = w \cdot f'$, and the action of m on m' is $\frac{gm \cdot (f + f')}{wf'}$.

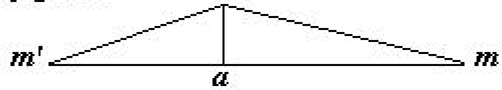


fig. 34.

In the same manner it may be shown that the action of m' on m is $\frac{gm'(f + f')}{wf}$; but when the bodies are in equilibrio, these forces must be equal: therefore

$$\frac{gm(f + f')}{wf'} = \frac{gm'(f + f')}{wf},$$

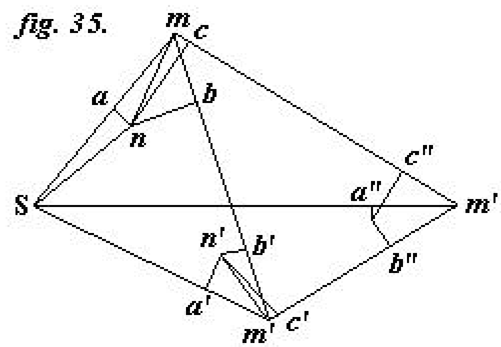
whence $gm \cdot f = gm' \cdot f'$, or $gm : gm' :: f' : f$, which is the law of equilibrium in the lever, and shows the reciprocal action of parallel forces.

Equilibrium of a System of Bodies

125. The equilibrium of a system of bodies way be found, when the system is acted on by any forces whatever, and when the bodies also mutually act on, or attract each other.

Demonstration. Let, m, m', m'', &c., be a system of bodies attracted by a force whose origin is in S, fig. 35;[3] and suppose each body to act on all the other bodies, and also to be itself subject to the action of each,–the action of all these forces on the bodies m, m', m'', &c., are as the masses of these bodies and the intensities of the forces conjointly.

Let the action of the forces on one body, as m, be first considered; and, for simplicity, suppose the number of bodies to be only three– m, m', and m''. It is evident that m is attracted by the force at S, and also urged by the reciprocal action of the bodies m' and m''.



Suppose m' and m'' to remain fixed, and that m is arbitrarily moved to n: then mn is the virtual velocity of m; and if the perpendiculars na, nb, nc be drawn, the lines ma, mb, mc, are the virtual velocities of m resolved in the direction of the forces which act on m. Hence, by the principle of virtual velocities, if the action of the force at S on m be multiplied by ma, the mutual action of m and m' by mb, and the mutual action of m and m'' by mc, the sum of these products must be zero when the point m is in equilibrio; or, m being the mass, if the action of S on m be $\mathrm{F}.m$, and the reciprocal actions of m on m' and m'' be p, p', then

$$m\mathrm{F}\times ma + p\times mb + p'\times mc = 0\,.$$

Now, if m and m'' remain fixed, and that m' is moved to n', then

$$m'\mathrm{F}'\times m'a' + p\times m'b' + p''\times m'c' = 0\,.$$

And a similar equation may be found for each body in the system. Hence the sum of all these equations must be zero when the system is in equilibrio. If, then, the distances $\mathrm{S}m$, $\mathrm{S}m'$, $\mathrm{S}m''$, be represented by s, s', s'', and the distances mm', mm'', $m'm''$, by f, f', f'', we shall have

$$\sum .m\mathrm{F}d\mathrm{s} + \sum .\,pd\,f + \sum .\,pd\,f' \pm, \text{\&c.} = 0\,,$$

$\sum$ being the sum of finite quantities; for it is evident that

$$d\,f = mb + m'b',\ d\,f' = mc + m''c'',\ \text{and so on.}$$

If the bodies move on surfaces, it is only necessary to add the terms $\mathrm{R}dr$, $\mathrm{R}'dr'$, &c., in which R and R' are the pressures or resistances of the surfaces, and dr, dr' the elements of their directions or the variations of the normals. Hence in equilibrio[4]

$$\sum .m\mathrm{F}d\mathrm{s} + \sum .\,pd\,f + \text{\&c.} + \mathrm{R}dr + \mathrm{R}'dr' + \text{\&c.} = 0\,.$$

Now, the variation of the normal is zero; consequently the pressures vanish from this equation: and if the bodies be united at fixed distances from each other, the lines mm', $m'm''$, &c., or f, f', &c., are constant:–consequently $d\,f = 0$, $d\,f' = 0$, &c.

The distance f of two points m and m' in space is

$$f = \sqrt{(x'-x)^2 + (y'-y)^2 + (z'-z)^2} \ ,$$

$x, y, z,$ being the co-ordinates of m, and $x', y', z',$ those of m'; so that the variations may be expressed in terms of these quantities: and if they be taken such that $d f = 0, \ d f' = 0,$ &c., the mutual action of the bodies will also vanish from the equation, which is reduced to

$$\sum \cdot m\mathrm{F} d s = 0 \ . \qquad \textbf{(14)}$$

126. Thus in every case the sum of the products of the forces into the elementary variations of their directions is zero when the system is in equilibrio, provided the conditions of the connexion of the system be observed in their variations or virtual velocities, which are the only indications of the mutual dependence of the different parts of the system on each other.

127. The converse of this law is also true−that when the principle of virtual velocities exists, the system is held in equilibrio by the forces at S alone.

Demonstration. For if it be not, each of the bodies would acquire a velocity $v, \ v',$ &c., in consequence of the forces $m\mathrm{F}, \ m'\mathrm{F}',$ &c. If $d n, \ d n',$ &c., be the elements of their direction, then

$$\sum \cdot m\mathrm{F} d s - \sum \cdot m v d n = 0 \ .$$

The virtual velocities $d n, \ d n'$, &c., being arbitrary, may be assumed equal to $vdt, \ v'dt$, &c., the elements of the space moved over by the bodies; or to $v, \ v',$ &c., if the element of the time be unity. Hence

$$\sum \cdot m\mathrm{F} d s - \sum \cdot m v^2 = 0 \ .$$

It has been shown that in all cases $\sum \cdot m\mathrm{F} d s = 0$, if the virtual velocities be subject to the conditions of the system. Hence, also, $\sum \cdot m v^2 = 0$; but as all squares are positive, the sum of these squares can only be zero if $v = 0, \ v' = 0,$ &c. Therefore the system must remain at rest, in consequence of the forces Fm, &c., alone.

Rotatory Pressure

128. Rotation is the motion of a body, or system of bodies, about a line or point. Thus the earth revolves about its axis, and [a] billiard-ball about its centre.

129. A rotatory pressure or moment is a force that causes a system of bodies, or a solid body, to rotate about any point or line. It is expressed by the intensity of the motive force or momentum, multiplied by the distance of its direction from the point or line about which the system or solid body rotates.

On the Lever

130. The lever first gave the idea of rotatory pressure or moments, for it revolves about the point of support or fulcrum.

When the lever mm', fig. 36, is in equilibrio, in consequence of forces applied to two heavy bodies at its extremities, the rotatory pressure of these forces, with regard to N, the point of support, must be equal and contrary.

Demonstration. Let $ma,\ m'a'$ [,] fig. 36, which are proportional to the velocities, represent the forces acting on m and m' during the indefinitely small time in which the bodies m and m' describe the indefinitely small spaces $ma,\ m'a'$. The distance of the direction of the forces $ma,\ m'a',$ from the fixed point N, are Nm, Nm'; and the momentum of m into Nm, must be equal to the momentum of m' into Nm'; that is, the product of ma by Nm and the mass m, must be equal to the product of $m'a'$ by Nm' and the mass m' when the lever is in equilibrio; or,

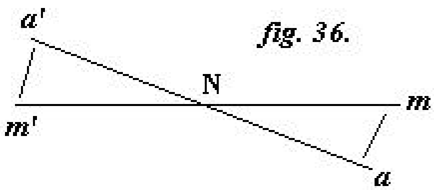


fig. 36.

$$ma \times \mathrm{N}m \times m = m'a' \times \mathrm{N}m' \times m'.$$

But

$$ma \times \mathrm{N}m \text{ is twice the triangle } \mathrm{N}ma\,,$$

and

$$m'a' \times \mathrm{N}m' \text{ is twice the triangle } \mathrm{N}m'a'\,;$$

hence twice the triangle Nma into the mass m, is equal to twice the triangle N$m'a'$ into the mass m', and these are the rotatory pressures which cause the lever to rotate about the fulcrum; thus, in equilibrio, the rotatory pressures are equal and contrary, and the moments are inversely as the distances from the point of support.

Projection of Lines and Surfaces

131. Surfaces and areas may be projected on the co-ordinate planes by letting fall perpendiculars from every point of them on these planes. For let oMN, fig. 37, be a surface meeting in a plane xoy in o, the origin of the co-ordinates, but rising above it towards MN. If perpendiculars be drawn from every point of the area oMN on the plane xoy, they will trace the area omn, which is the projection of oMN.

Since, by hypothesis, xoy is a right angle, if the lines mD, nC, be drawn parallel to oy, DC is the projection of mn on the axis ox. In the same manner AB is the projection of the same line on oy.

Equilibrium of a System of Bodies invariably united

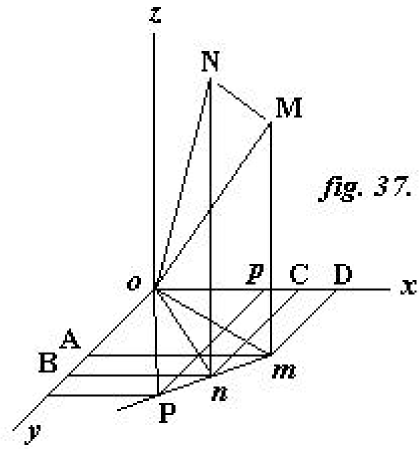


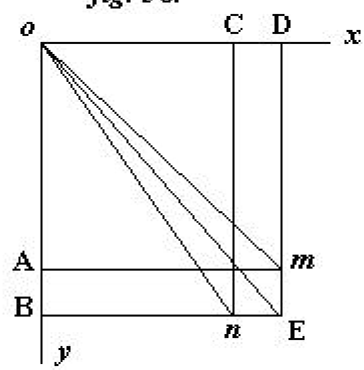
fig. 37.

132. A system of bodies invariably united will be in equilibrio upon a point, if the sum of the moments of rotation of all the forces that act upon it vanish, when estimated parallel to three rectangular co-ordinates.

Demonstration. Suppose a system of bodies invariably united, moving about a fixed point *o* in consequence of an impulse and a force of attraction; *o* being the origin of the attractive force and of the co-ordinates.

Let one body be considered at a time, and suppose it to describe the indefinitely small arc MN, fig. 37, in an indefinitely small time, and let *mn* be the projection of this arc on the plane *xoy*. If *m* be the mass of the body, then $m \times mn$ is its momentum, estimated in the plane *xoy*; and if *o*P be perpendicular to *mn*, it is evident that $m \times mn \times o\text{P}$ is its rotatory pressure. But $mn \times o\text{P}$ is twice the triangle *mon*; hence the rotatory pressure is equal to the mass *m* into twice the triangle *mon* that the body could describe in an element of time. But when *m* is at rest, the rotatory pressure must be zero; hence in equilibrio, $m \times mn \times o\text{P} = 0$.

fig. 38.



Let *omn*, fig. 38, be the projected area, and complete the parallelogram oDEB; then if *o*D, *o*A, the co-ordinates of *m*, be represented by *x* and *y*, it is evident that *y* increases, while *x* diminishes; hence

$$\text{CD} = -dx, \text{ and AB} = dy.$$

Join OE, then

$$no\text{E} = \tfrac{1}{2} n\text{D},$$

because the triangle and parallelogram are on the same base and between the same parallels; also $mo\text{E} = \tfrac{1}{2}\text{AE}$: hence the triangle[5]

$$mon = \tfrac{1}{2}\{n\text{D} + \text{AE}\}.$$

Now

$$n\text{D} = -dx(y + dy)$$

and

$$\text{AE} = xdy,$$

therefore

$$mon = \tfrac{1}{2}(xdy - ydx) - \tfrac{1}{2}dxdy;$$

but when the arc *mn* is indefinitely small, $\frac{1}{2}dxdy = \frac{1}{2}n\text{E} \cdot m\text{E}$ may be omitted in comparison of the first powers of these quantities, hence the triangle

$$mon = \tfrac{1}{2}\left(xdy - ydx\right),$$

therefore $m\left(xdy - ydx\right) = 0$ is the rotatory pressure in the plane *xoy* when *m* in is in equilibrio. A similar equation must exist for each co-ordinate plane when *m* is in a state of equilibrium with regard to each axis, therefore also

$$m\left(xdz - zdx\right) = 0,\ m\left(ydz - zdy\right) = 0.$$

The same may be proved for every body in the system, consequently when the whole is in equilibrio on the point *o*

$$\sum m\left(xdy - ydx\right) = 0 \quad \sum m\left(xdz - zdx\right) = 0 \quad \sum m\left(ydz - zdy\right) = 0. \qquad \textbf{(15)}$$

133. This property may be expressed by means of virtual velocities, namely, that a system of bodies will be at rest, if the sum of the products of their momenta by the elements of their directions be zero, or by article 125

$$\sum m\text{F}d\,s = 0.$$

Since the mutual distances of the parts of the system are invariable, if the whole system be supposed to be turned by an indefinitely small angle about the axis *oz*, all the co-ordinates z', z'', &c., will be invariable. If dv be any arbitrary variation, and if

$$\begin{aligned} d\,x &= yd\,v & d\,y &= -xd\,v \\ d\,x' &= y'd\,v & d\,y' &= -x'd\,v \ ; \end{aligned}$$

then f being the mutual distance of the bodies *m* and m' whose co-ordinates are *x, y, z*; x', y', z', there will arise

$$d\,f = d\sqrt{\left(x'-x\right)^2 + \left(y'-y\right)^2 + \left(z'-z\right)^2} =$$
$$\frac{x'-x}{f}\left(d\,x' - d\,x\right) + \frac{y'-y}{f}\left(d\,y' - d\,y\right) =$$
$$\tfrac{1}{2}\left\{\left(x'-x\right)\left(y'-y\right)d\,v - \left(y'-y\right)\left(x'-x\right)d\,v\right\} = 0.$$

So that the values assumed for $d\,x$, $d\,y$, $d\,x'$, $d\,y'$ are not incompatible with the invariability of the system. It is therefore a permissible assumption.

Now if *s* be the direction of the force acting on *m*, its variation is

$$ds = \frac{ds}{dx}dx + \frac{ds}{dy}dy\,,$$

since z is constant; and substituting the preceding values of dx, dy, the result is[6]

$$ds = \frac{ds}{dx}\cdot y dv - \frac{ds}{dy}\cdot x dv = dv\left\{\frac{ds}{dx}\cdot y - \frac{ds}{dy}\cdot x\right\}$$

or, multiplying by the momentum Fm,

$$\mathrm{F}m ds = \mathrm{F}m\left\{y\frac{ds}{dx} - x\frac{ds}{dy}\right\}dv\ .$$

In the same manner with regard to the body m'

$$\mathrm{F}'m'ds' = \mathrm{F}'m'\left\{y'\frac{ds'}{dx'} - x'\frac{ds'}{dy'}\right\}dv\ ,$$

and so on; and thus the equation $\sum m\mathrm{F}ds = 0$ becomes

$$\sum m\mathrm{F}\left\{y\frac{ds}{dx} - x\frac{ds}{dy}\right\} = 0\,.$$

It follows, from the same reasoning, that

$$\sum m\mathrm{F}\left\{z\frac{ds}{dx} - x\frac{ds}{dz}\right\} = 0,$$
$$\sum m\mathrm{F}\left\{z\frac{ds}{dy} - y\frac{ds}{dz}\right\} = 0.$$

In fact, if X, Y, Z be the components of the force F in the direction of the three axes, it is evident that

$$\mathrm{X}=\mathrm{F}\frac{ds}{dx};\quad \mathrm{Y}=\mathrm{F}\frac{ds}{dy};\quad \mathrm{Z}=\mathrm{F}\frac{ds}{dz};$$

and these equations become

$$\begin{aligned}&\sum my\,.\,\mathrm{X} - \sum mx\,.\,\mathrm{Y}=0\\ &\sum mz\,.\,\mathrm{X} - \sum mx\,.\,\mathrm{Z}=0\\ &\sum mz\,.\,\mathrm{Y} - \sum my\,.\,\mathrm{Z}=0.\end{aligned} \qquad \textbf{(16)}$$

But $\sum m\mathrm{F}y\frac{ds}{dx}$ expresses the sum of the moments of the forces parallel to the axis of x to turn the system round that of z, and $\sum m\mathrm{F}x\frac{ds}{dy}$ that of the forces parallel to the axis of y to do the same, but estimated in the contrary direction;–and it is evident that the forces parallel to z have no effect to turn the system round z. Therefore the equation $\sum m\mathrm{F}\left\{y\frac{ds}{dx}-x\frac{ds}{dy}\right\}=0$, expresses that the sum of the moments of rotation of the whole system relative to the axis of z must vanish, that the equilibrium of the system may subsist. And the same being true for the other rectangular axes (whose positions are arbitrary), there results this general theorem, viz., that in order that a system of bodies may be in equilibrio[7] upon a point, the sum of the moments of rotation of all the forces that act on it must vanish when estimated parallel to any three rectangular co-ordinates.

134. These equations are sufficient to ensure the equilibrium of the system when o is a fixed point; but if o, the point about which it rotates, be not fixed, the system, as well as the origin o, may be carried forward in space by a motion of translation at the same time that the system rotates about o, like the earth, which revolves about the sun at the same time that it turns on its axis. In this case it is not only necessary for the equilibrium of the system that its rotatory pressure should be zero, but also that the forces which cause the translation when resolved in the direction of the axes[8] ox, oy, oz, should be zero for each axis separately.

On the Centre of Gravity

135. If the bodies m, m', m'', &c., be only acted on by gravity, its effect would be the same on all of them, and its direction may be considered the same also; hence

$$\mathrm{F}=\mathrm{F}'=\mathrm{F}''=\&c.,$$

and also the directions

$$\frac{ds}{dx}=\frac{ds}{dx'}=\&c. \qquad \frac{ds}{dy}=\frac{ds}{dy'}=\&c. \qquad \frac{ds}{dz}=\frac{ds}{dz'}=\&c.,$$

are the same in this case for all the bodies, so that the equations of rotatory pressure become

$$\mathrm{F}\left\{\frac{ds}{dx}\cdot\sum my-\frac{ds}{dy}\sum mx\right\}=0$$

$$\mathrm{F}\left\{\frac{ds}{dz}\cdot\sum my-\frac{ds}{dy}\sum mz\right\}=0$$

$$\mathrm{F}\left\{\frac{ds}{dx}\cdot\sum mz-\frac{ds}{dz}\sum mx\right\}=0$$

or, if X, Y, Z, be considered as the components of gravity in the three co-ordinate axes by article 133

$$\begin{aligned} \mathrm{X}\cdot\sum my - \mathrm{Y}\cdot\sum mx &= 0 \\ \mathrm{Z}\cdot\sum my - \mathrm{Y}\cdot\sum mz &= 0 \\ \mathrm{X}\cdot\sum mz - \mathrm{Z}\cdot\sum mx &= 0. \end{aligned} \qquad \textbf{(17)}$$

It is evident that these equations will be zero whatever the direction of gravity may be, if

$$\sum mx = 0, \quad \sum my = 0, \quad \sum mz = 0\,. \qquad \textbf{(18)}$$

Now since $\mathrm{F}\frac{ds}{dx},\ \mathrm{F}\frac{ds}{dy},\ \mathrm{F}\frac{ds}{dz},$ are the components of the force of gravity of the force of gravity in the three co-ordinates *ox*, *oy*, *oz*,

$$\mathrm{F}\cdot\frac{ds}{dx}\cdot\sum m; \quad \mathrm{F}\cdot\frac{ds}{dy}\cdot\sum m; \quad \mathrm{F}\cdot\frac{ds}{dz}\cdot\sum m\,;$$

are the forces which translate the system parallel to these axes. But if *o* be a fixed point, its reaction would destroy these forces. By article 49,

$$\left(\frac{ds}{dx}\right)^2 + \left(\frac{ds}{dy}\right)^2 + \left(\frac{ds}{dz}\right)^2 = 1$$

is the diagonal of a parallelopiped, of which

$$\frac{ds}{dx},\ \frac{ds}{dy},\ \frac{ds}{dz},$$

are the sides; therefore these three compose one resulting force equal to $\mathrm{F}\cdot\sum m$. This resulting force is the weight of the system which is thus resisted or supported by the reaction of the fixed point *o*.

136. The point *o* round which the system is in equilibrio, is the centre of gravity of the system, and if that point be supported, the whole will be in equilibrio.

On the Position and Properties of the Centre of Gravity

137. It appears from the equations (18), that if any plane passes through the centre of gravity of a system of bodies, the sum of the products of the mass of each body by its distance from that plane is zero. For, since the axes of the co-ordinates are arbitrary, any one of them, as

xox', fig. 39,[9] may be assumed to be the section of the plane in question, the centre of gravity of the system of bodies m, m', &c., being in o. If the perpendiculars ma, $m'b$, &c., be drawn from each body on the plane xox', the product of the *mass* m by the distance ma plus the product of m' by $m'b$ plus, &C., must be zero; or, representing the distances by z, z', z'', &c., then[10]

$$mz + m'z' - m''z'' + m'''z''' + \&\text{c.} = 0\,;$$

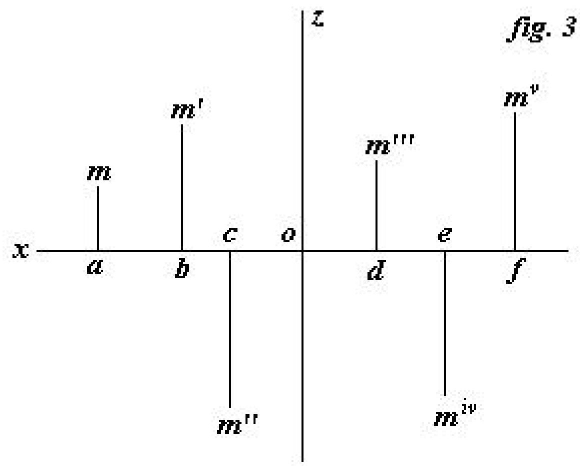


fig. 39.

or, according to the usual notation,

$$\sum \cdot mz = 0\,.$$

And the same property exists for the other two co-ordinate planes. Since the position of the co-ordinate planes is arbitrary, the property obtains for every set of co-ordinate planes having their origin in o. It is clear that if the distances ma, $m'b$, &c., be positive on one side of the plane, those on the other side must be negative, otherwise the sum of the products could not be zero.

138. When the centre of gravity is not in the origin of the coordinates, it may be found if the distances of the bodies m, m', m'', &c., from the origin and from each other be known.

Demonstration. For let o, fig. 40, be the origin, and c the centre of gravity of the system m, m', &c. Let MN be the section of a plane passing through c; then by the property of the centre of gravity just explained,

$$m\,.\,ma + m'\,.\,m'b - m''\,.\,m''d + \&c. = 0\,;$$

but

$$ma = o\text{A} - op;\quad m'b = o\text{A} - op', \ \&\text{c. } \&\text{c.},$$

hence

$$m\left(o\text{A} - op\right) + m'\left(o\text{A} - op'\right) + \&c. = 0\,;$$

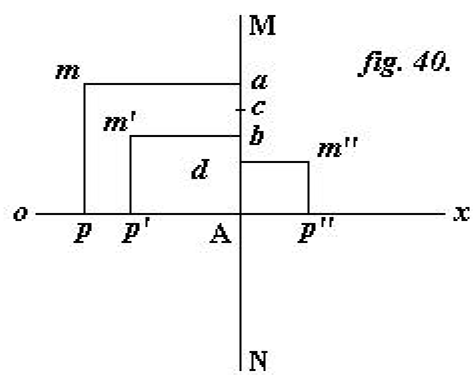


fig. 40.

or if Ao be represented by $\overline{x}$, and op, op', op'', &c., by x, x', x'', &c., then will

$$m\left(\overline{x} - x\right) + m'\left(\overline{x} - x'\right) - m''\left(\overline{x} - x''\right) + \&c. = 0\,.$$

Whence

$$\overline{x}\left(m + m' - m'' + \&\text{c.}\right) = mx + m'x' - m''x'' + \&\text{c.},$$

and

$$\bar{x} = \left(\frac{mx + m'x' + \&c.}{m + m' - m'' + \&c.} \right) = \frac{\Sigma . mx}{\Sigma m}. \tag{19}$$

Thus, if the masses of the bodies and their respective distances from the origin of the co-ordinates be known, this equation will give the distance of the centre of gravity from the plane *yoz*. In the same manner its distances from the other two co-ordinate planes are found to be

$$\bar{y} = \frac{\Sigma . my}{\Sigma m} \qquad \bar{z} = \frac{\Sigma . mz}{\Sigma m}. \tag{20}$$

139. Thus, because the centre of gravity is determined by its three co-ordinates $\bar{x}$, $\bar{y}$, $\bar{z}$, it is a single point.

140. But these three equations give

$$\bar{x}^2 + \bar{y}^2 + \bar{z}^2 = \frac{(\Sigma mx)^2 + (\Sigma my)^2 + (\Sigma mz)^2}{(\Sigma m)^2},$$

or

$$\bar{x}^2 + \bar{y}^2 + \bar{z}^2 = \frac{\Sigma m(x^2 + y^2 + z^2)}{\Sigma m} - \frac{\Sigma mm'\left\{(x'-x)^2 + (y'-y)^2 + (z'-z)^2\right\}}{(\Sigma m)^2}$$

The last term of the second member is the sum of all the products similar to those under Σ when all the bodies of the system are taken in pairs.

141. It is easy to show that the two preceding values of $\bar{x}^2 + \bar{y}^2 + \bar{z}^2$ are identical, or that

$$\frac{(\Sigma mx)^2}{(\Sigma m)^2} = \frac{\Sigma mx^2}{\Sigma m} - \frac{\Sigma mm'(x'-x)^2}{(\Sigma m)^2}$$

or

$$(\Sigma mx)^2 = \Sigma m . \Sigma mx^2 - \Sigma mm'(x'-x)^2.$$

Where[11] there are only two planets, then

$$\Sigma m = m + m', \quad \Sigma mx = mx + m'x', \quad \Sigma mm' = mm';$$

consequently

$$(\Sigma mx)^2 = (mx + m'x')^2 = m^2x^2 + 2m'^2x'^2 + mm'xx'.$$

With regard to the second member[12]

$$\sum m \,.\, \sum mx^2 = (m+m')(mx^2+m'x'^2) = m^2x^2 + m'^2x'^2 + mm'x^2 + mm'x'^2,$$

and

$$\sum mm'(x'-x)^2 = mm'x'^2 + mm'x^2 - 2mm'xx';$$

consequently

$$\sum m \cdot \sum mx^2 - \sum mm'(x'-x)^2 = m^2x^2 + m'^2x'^2 + 2mm'xx' = \left(\sum mx\right)^2.$$

This will be the case whatever the number of planets may be; and as the equations in question are symmetrical with regard to x, y, and z, their second members are identical.

Thus the distance of the centre of gravity from a given point may be found by means of the distances of the different points of the system from this point, and of their mutual distances.

142. By estimating the distance of the centre of gravity from any three fixed points, its position in space will be determined.

Equilibrium of a Solid Body

143. If the bodies m, m', m'', &c., be indefinitely small, infinite in number, and permanently united together, they will form a solid mass, whose equilibrium may be determined by the preceding equations.

For if x, y, z, be the co-ordinates of any one of its indefinitely small particles dm, and X, Y, Z, the forces urging it in the direction of these axes, the equations of its equilibrium will be

$$\int \mathrm{X}dm = 0 \quad \int \mathrm{Y}dm = 0 \quad \int \mathrm{Z}dm = 0$$

$$\int (\mathrm{X}y - \mathrm{Y}x)\,dm = 0; \quad \int (\mathrm{Y}z - \mathrm{Z}x)\,dm = 0; \quad \int (\mathrm{Z}y - \mathrm{Y}z)\,dm = 0.$$

The three first are the equations of translation, which are destroyed when the centre of gravity is a fixed point; and the last three are the sums of the rotatory pressures.

Notes

[1] *centrically*. At or near the center.

[2] This *reads* "indefinitely small angle *amn*." in the 1st edition (published erratum).

[3] In fig. 35 a'' *reads* a' in the 1st edition.

[4] The last two terms *read* $\mathrm{R}'dr'$, & c.=0 in the 1st edition.

[5] The right hand side *reads* $\frac{1}{2}\{n\mathrm{D}+\mathrm{AE}.\}$ in the 1st edition.

[6] The last term in the following expression reads $\frac{ds}{dy}x$ in the 1st edition.

[7] This *reads* "equilibro" in the 1st edition.

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[8] This *reads* "axis" in the 1st edition.
[9] Fig. 39 is mislabeled in the 1st edition: o is at c, c is omitted, m^{iv} is mislabeled m^{v}, and m^{v} is mislabeled m^{vi}.
[10] The second term in this equation *reads* $m'z$ in the 1st edition.
[11] This *reads* "Were" in the 1st edition.
[12] The last term in next equation reads $m\,m'x^{2\prime}$ in the 1st edition.

BOOK I

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CHAPTER IV

MOTION OF A SYSTEM OF BODIES

144. IT is known by observation, that the relative motions of a system of bodies, are entirely independent of any motion common to the whole; hence it is impossible to judge from appearances alone, of the absolute motions of a system of bodies of which we form a part; the knowledge of the true system of the world was retarded, from the difficulty of comprehending the relative motions of projectiles on the earth, which has the double motion of rotation and revolution. But all the motions of the solar system, determined according to this law, are verified by observation.

By article 117, the equation of the motion of a body only differs from that of a particle, by the mass; hence, if only one body be considered, of which m is the mass, the motion of its centre of gravity will be determined from equation (6), which in this case becomes

$$m\left\{\mathrm{X}-\frac{d^2x}{dt^2}\right\}dx+m\left\{\mathrm{Y}-\frac{d^2y}{dt^2}\right\}dy+m\left\{\mathrm{Z}-\frac{d^2z}{dt^2}\right\}dz=0\,.$$

A similar equation may be found for each body in the system, and one condition to be fulfilled is, that the sum of all such equations must be zero;–hence the general equation of a system of bodies is

$$0=\sum m\left\{\mathrm{X}-\frac{d^2x}{dt^2}\right\}dx+\sum m\left\{\mathrm{Y}-\frac{d^2y}{dt^2}\right\}dy+\sum m\left\{\mathrm{Z}-\frac{d^2z}{dt^2}\right\}dz\,, \qquad \textbf{(21)}$$

in which

$$\sum m\mathrm{F}\,ds=0$$

are the sums of the products of each mass by its corresponding component force, for

$$\sum m\mathrm{X}=m\mathrm{X}+m'\mathrm{X}'+m''\mathrm{X}''+\&c.;$$

and so for the other two. Also

$$\sum m\frac{d^2x}{dt^2},\quad \sum m\frac{d^2y}{dt^2},\quad \sum m\frac{d^2z}{dt^2},$$

are the sums of the products of each mass, by the second increments of the space respectively described by them, in an element of time in the direction of each axis, since

$$\sum m\frac{d^2x}{dt^2}=m\frac{d^2x}{dt^2}+m'\frac{d^2x'}{dt^2}+\&c.$$

the expressions

$$\sum m\frac{d^2y}{dt^2},\quad \sum m\frac{d^2z}{dt^2}$$

have a similar signification.

From this equation all the motions of the solar system are directly obtained.

145. If the forces be invariably supposed to have the same intensity at equal distances from the points to which they are directed, and to vary in some ratio of that distance, all the principles of motion that have been derived from the general equation (6), may be obtained from this, provided the sum of the masses be employed instead of the particle.

146. For example, if the equation, in article 74, be multiplied by $\sum m$ its finite value is found to be[1]

$$\sum m\mathrm{V}^2=\mathrm{C}+2\sum\int m\left(\mathrm{X}dx+\mathrm{Y}dy+\mathrm{Z}dz\right).$$

This is the Living Force or Impetus of a system, which is the sum of the masses into the square of their respective velocities, and is analogous to the equation[2]

$$\mathrm{V}^2=\mathrm{C}+2v^2,$$

relating to a particle.

147. When the motion of the system changes by insensible degrees, and is subject to the action of accelerating forces, the sum of the indefinitely small increments of the impetus is the same, whatever be the path of the bodies, provided that the points of departure and arrival be the same.

148. When there is a primitive impulse without accelerating forces, the impetus is constant.

149. Impetus is the true measure of labour; for if a weight be raised ten feet, it will require four times the labour to raise an equal weight forty feet. If both these weights be allowed to descend freely by their gravitation, at the end of their fall their velocities will be as 1 to 2; that is, as the square roots of their heights. But the effects produced will be as their masses into the heights from whence they fell, or as their masses into 1 and 4; but these are the squares of the velocities, hence the impetus is the mass into the square of the velocity. Thus the impetus is the true measure of the labour employed to raise the weights, and of the effects of their descent, and is entirely independent of time.

150. The principle of least action for a particle was shown, in article 80, to be expressed by $d\int vds = 0$, when the extreme points of its path are fixed; hence, for a system of bodies, it is

$$\Sigma d\int mvds = 0, \text{ or } \Sigma d\int mv^2 dt = 0.$$

Thus the sum of the living forces of a system of bodies is a minimum, during the time that it takes to pass from one position to another.

If the bodies be not urged by accelerating forces, the impetus of the system during a given time, is proportional to that time, therefore the system moves from one given position to another, in the shortest time possible: which is the principle of least action in a system of bodies.

On the Motion of the Centre of Gravity of a System of Bodies

151. In a system of bodies the common centre of gravity of the whole either remains at rest or moves uniformly in a straight line, as if all the bodies of the system were united in that point, and the concentrated forces of the system applied to it.

Demonstration. These properties are derived from the general equation (21) by considering that, if the centre of gravity of the system be moved, each body will have a corresponding and equal motion independent of any motions the bodies may have among themselves: hence each of the virtual velocities $d\,x$, $d\,y$, $d\,z$, will be increased by the virtual velocity of the centre of gravity resolved in the direction of the axes; so that they become

$$d\,x + d\,\overline{x},\ d\,y + d\,\overline{y},\ d\,z + d\,\overline{z}:$$

thus the equation of the motion of a system of bodies is increased by the term,[3]

$$\Sigma.m\left\{X - \frac{d^2x}{dt^2}\right\}d\,\overline{x} + \Sigma.m\left\{Y - \frac{d^2y}{dt^2}\right\}d\,\overline{y} + \Sigma.m\left\{Z - \frac{d^2z}{dt^2}\right\}d\,\overline{z}$$

arising from the consideration of the centre of gravity. If the system be free and unconnected with bodies foreign to it, the virtual velocity of the centre of gravity, is independent of the connexion of the bodies of the system with each other; therefore $d\,\overline{x}$, $d\,\overline{y}$, $d\,\overline{z}$ may each be zero, whatever the virtual velocity of the bodies themselves may be; hence

$$\Sigma.m\left\{X - \frac{d^2x}{dt^2}\right\} = 0,\quad \Sigma.m\left\{Y - \frac{d^2y}{dt^2}\right\} = 0,\quad \Sigma.m\left\{Z - \frac{d^2z}{dt^2}\right\} = 0.$$

But it has been shewn[4] that the co-ordinates of the centre of gravity are,

$$\overline{x} = \frac{\Sigma.mx}{\Sigma.m};\ \ \overline{y} = \frac{\Sigma.my}{\Sigma.m};\ \ \overline{z} = \frac{\Sigma.mz}{\Sigma.m}.$$

Consequently,

$$d^2\overline{x} = \frac{\sum .md^2x}{\sum .m}; \quad d^2\overline{y} = \frac{\sum .md^2y}{\sum .m}; \quad d^2\overline{z} = \frac{\sum .md^2z}{\sum .m}.$$

Now[5]

$$\sum .md^2x = dt^2 . \sum .m\mathrm{X}; \quad \sum .md^2y = dt^2 . \sum .m\mathrm{Y}; \quad \sum .md^2z = dt^2 . \sum .m\mathrm{Z};$$

hence

$$\frac{d^2\overline{x}}{dt^2} = \frac{\sum .m\mathrm{X}}{\sum .m}; \quad \frac{d^2\overline{y}}{dt^2} = \frac{\sum .m\mathrm{Y}}{\sum .m}; \quad \frac{d^2\overline{z}}{dt^2} = \frac{\sum .m\mathrm{Z}}{\sum .m}. \qquad \textbf{(22)}$$

These three equations determine the motion of the centre of gravity.

152. Thus the centre of gravity moves as if all the bodies of the system were united in that point, and as if all the forces which act on the system were applied to it.

153. If the mutual attraction of the bodies of the system be the only accelerating force acting on these bodies, the three quantities $\sum m\mathrm{X}$, $\sum m\mathrm{Y}$, $\sum m\mathrm{Z}$, are zero.

Demonstration. This evidently arises from the law of reaction being equal and contrary to action; for if F be the action that an element of the mass m exercises on an element of the mass m', whatever may be the nature of this action, $m'F$ will be the accelerating force with which m is urged by the action of m'; then if f be the mutual distance of m and m', by this action only

$$\mathrm{X} = \frac{m'F\left(x'-x\right)}{f}; \quad \mathrm{Y} = \frac{m'F\left(y'-y\right)}{f}; \quad \mathrm{Z} = \frac{m'F\left(z'-z\right)}{f}. \qquad \textbf{(23)}$$

For the same reasons, the action of m' on m will give

$$\mathrm{X}' = \frac{m\mathrm{F}\left(x-x'\right)}{f}; \quad \mathrm{Y}' = \frac{m\mathrm{F}\left(y-y'\right)}{f}; \quad \mathrm{Z}' = \frac{m\mathrm{F}\left(z-z'\right)}{f};$$

hence

$$m\mathrm{X}+m'\mathrm{X}'=0; \quad m\mathrm{Y}+m'\mathrm{Y}'=0; \quad m\mathrm{Z}+m'\mathrm{Z}'=0;$$

and as all the bodies of the system, taken two and two, give the same results, therefore generally

$$\sum .m\mathrm{X}=0; \quad \sum .m\mathrm{Y}=0; \quad \sum .m\mathrm{Z}=0.$$

154. Consequently

$$\frac{d^2\overline{x}}{dt^2} = 0; \quad \frac{d^2\overline{y}}{dt^2} = 0; \quad \frac{d^2\overline{z}}{dt^2} = 0;$$

and integrating,

$$\overline{x} = at+b; \quad \overline{y} = a't+b'; \quad \overline{z} = a''t+b'';$$

in which a, a', m'; b, b', y', are the arbitrary constant quantities introduced by the double integration. These are equations to straight lines; for, suppose the centre of gravity to begin to move at A, fig. 41, in the direction ox, the distance oA is invariable, and is represented by b; and as at increases with the time t, it represents the straight line AB.

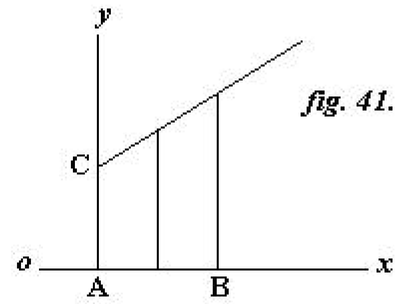


fig. 41.

155. Thus the motion of the centre of gravity in the direction of each axis is a straight line, and by the composition of motions it describes a straight line in space; and as the space it moves over increases with the time, its velocity is uniform; for the velocity, being directly as the element of the space, and inversely as the element of the time, is

$$\sqrt{\left(\frac{d\bar{x}}{dt}\right)^2+\left(\frac{d\bar{y}}{dt}\right)^2+\left(\frac{d\bar{z}}{dt}\right)^2}\;;$$

or

$$\sqrt{a^2+a'^2+a''^2}\,.$$

Thus the velocity is constant, and therefore the motion uniform.

156. These equations are true, even if some of the bodies, by their mutual action, lose a finite quantity of motion in an instant.

157. Thus, it is possible that the whole solar system may be moving in space; a circumstance which can only be ascertained by a comparison of its position with regard to the fixed stars at very distant periods. In consequence of the proportionality of force to velocity, the bodies of the solar system would maintain their relative motions, whether the system were in motion or at rest.

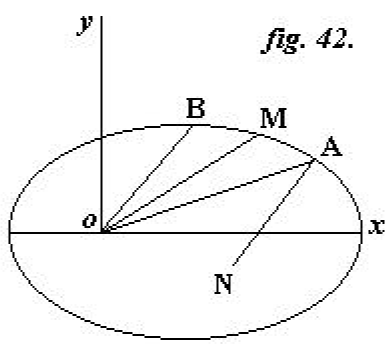


fig. 42.

On the Constancy of Areas

158. If a body propelled by an impulse describe a curve AMB, fig. 42, in consequence of a force of attraction in the point o, that force may be resolved into two, one in the direction of the normal AN, and the other in the direction of the curve or tangent: the first is balanced by the centrifugal force, the second augments or diminishes the velocity of the body; but the velocity is always such that the areas AoM, MoB, described by the radius vector Ao, are proportional to the time; that is, if the body moves from A to M in the same time that it would move from M to B, the area AoM will be equal to the area MoB.

If a system of bodies revolve about any point in consequence of an impulse and a force of attraction directed to that point, the sums of their masses respectively multiplied by the areas described by their radii vectores,[6] when projected on the three co-ordinate planes, are proportional to the time.

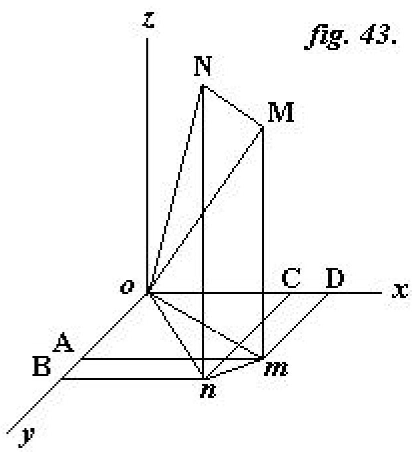


fig. 43.

Demonstration. For if we only consider the areas that are projected on the plane *xoy*, fig. 43, the forces in the direction *oz*, which are perpendicular to that plane, must be zero; hence Z=0, Z′=0, &c.; and the general equation of the motion of a system of bodies becomes[7]

$$\Sigma . m\left\{\mathrm{X}-\frac{d^2x}{dt^2}\right\}dx+\Sigma . m\left\{\mathrm{Y}-\frac{d^2y}{dt^2}\right\}dy=0 .$$

If the same assumptions be made here as in article 133, namely,

$$\begin{aligned} dx &= ydv \qquad dy=-xdv \\ dx' &= y'dv \qquad dy'=-x'dv \text{ , \&c. \&c.,} \end{aligned}$$

and if these be substituted in the preceding equation, it becomes, with regard to the plane *xoy*,

$$\Sigma . m\left(\frac{xd^2y-yd^2x}{dt^2}\right)=\Sigma m.\left(x\mathrm{Y}-y\mathrm{X}\right).$$

In the same manner

$$\begin{aligned} \Sigma m.\left(\frac{zd^2x-xd^2z}{dt^2}\right)&=\Sigma m.\left(z\mathrm{X}-x\mathrm{Z}\right) \\ \Sigma m.\left(\frac{yd^2z-zd^2y}{dt^2}\right)&=\Sigma m.\left(y\mathrm{Z}-z\mathrm{Y}\right) \end{aligned} \qquad \textbf{(24)}$$

are obtained for the motions of the system with regard to the planes *xoz*, *yoz*. These three equations, together with

$$\Sigma m\cdot\frac{d^2x}{dt^2}=\Sigma m\mathrm{X},\ \ \Sigma m\cdot\frac{d^2y}{dt^2}=\Sigma m\mathrm{Y},\ \ \Sigma m\cdot\frac{d^2z}{dt^2}=\Sigma m\mathrm{Z}, \qquad \textbf{(25)}$$

are the general equations of the motions of a system of bodies which does not contain a fixed point.

159. When the bodies are independent of foreign forces, and only subject to their reciprocal attraction and to the force at *o*, the sum of the terms

$$m\left\{\mathrm{X}y-\mathrm{Y}x\right\}+m'\left\{\mathrm{X}'y'-\mathrm{Y}'x'\right\},$$

arising from the mutual action of any two bodies in the system, m, m', is zero, by reason of the equality and opposition of action and reaction; and this is true for every such pair as m and m'', m' and m'', &c. If f be the distance of m from o, F the force which urges the body m towards that origin, then

$$\text{X}=-\text{F}\frac{x}{f},\ \ \text{Y}=-\text{F}\frac{y}{f},\ \ \text{Z}=-\text{F}\frac{z}{f}$$

are its component forces; and when substituted in the preceding equations, F vanishes; the same may be shown with regard to m', m'', &c. Hence the equations of areas are reduced to

$$\sum m\left\{\frac{yd^2x-xd^2y}{dt^2}\right\}=0,$$
$$\sum m\left\{\frac{zd^2x-xd^2z}{dt^2}\right\}=0,$$
$$\sum m\left\{\frac{yd^2z-zd^2y}{dt^2}\right\}=0,$$

and their integrals are

$$\begin{aligned}\sum m\{xdy-ydx\}&=cdt\\ \sum m\{zdx-xdz\}&=c'dt\\ \sum m\{ydz-zdy\}&=c''dt\end{aligned}\qquad\textbf{(26)}$$

As the first members or these equations are the sum of the masses of all the bodies of the system, respectively multiplied by the projections of double the areas they describe on the co-ordinate planes, this sum is proportional to the time.

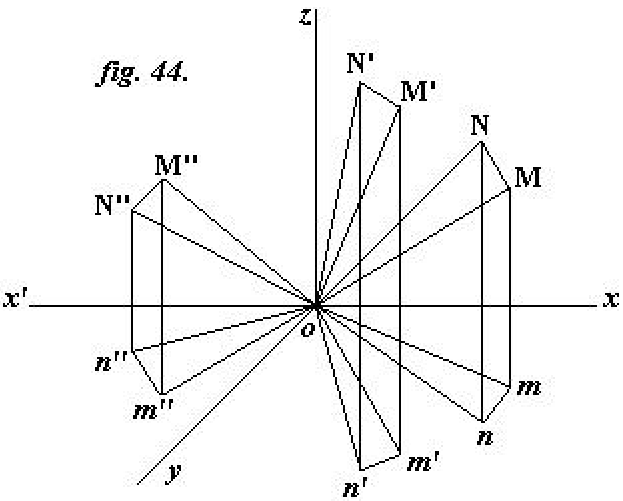
If the centre of gravity be the origin of the co-ordinates, the preceding equations may be expressed thus,

$$cdt=\frac{\sum mm'\{(x'-x)(dy'-dy)-(y'-y)(dx'-dx)\}}{\sum m},$$
$$c'dt=\frac{\sum mm'\{(z'-z)(dx'-dx)-(x'-x)(dz'-dz)\}}{\sum m},$$
$$c''dt=\frac{\sum mm'\{(y'-y)(dz'-dz)-(z'-z)(dy'-dy)\}}{\sum m}.$$

So that the principle of areas is reduced to depend on the co-ordinates of the mutual distances of the bodies of the system.

160. These results way be expressed by a diagram. Let m, m', m'', fig. 44, &c., be a system of bodies revolving about o, the origin of the co-ordinates, in consequence of a central force and a primitive impulse.–Suppose that each of the radii vectores, om, om', om'', &c., describes the indefinitely small areas, MoN, M′oN′, &c., in an indefinitely small time, represented by dt; and let mon, $m'on'$, &c., be the projections of these areas on the plane xoy. Then the equation

fig. 44.



$$\sum m\{xdy - ydx\} = cdt,$$

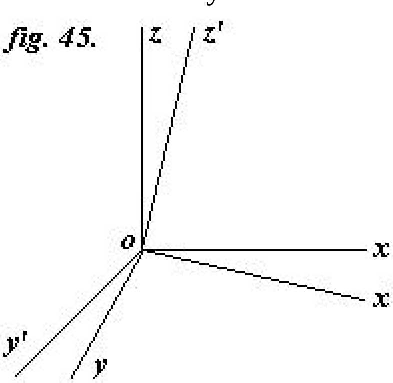
shows that the sum of the products of twice the area mon, by the mass m, twice the area $m'on'$ by the mass m', twice $m''on''$ by the mass m'', &c., is proportional to the element of the time in which they are described: whence it follows that the sum of the projections of the areas, each multiplied by the corresponding mass, is proportional to the finite time in which they are described. The other two equations express similar results for the areas projected on the planes xoz, yoz.

161. The constancy of areas is evidently true for any plane whatever, since the position of the co-ordinate planes is arbitrary. The three equations of areas give the space described by the bodies on each co-ordinate plane in value of the time: hence, if the time be known or assumed, the corresponding places of the bodies will be found on the three planes, and from thence their true positions in space may be determined, since that of the co-ordinate planes is supposed to be known. It was shown, in article 132, that

$$\sum m\{xdy - ydx\},$$
$$\sum m\{zdx - xdz\},$$
$$\sum m\{zdy - ydz\},$$

are the pressures of the system, tending to make it turn round each of the axes of the co-ordinates: hence the principle of areas consists in this–that the sum of the rotatory pressures which cause a system of bodies to revolve about a given point, is zero when the system is in equilibrio, and proportional to the time when the system is in motion.

fig. 45.



162. Let us endeavour to ascertain whether any planes exist on which the sums of the areas are zero when the system is in motion. To solve this problem it is necessary to determine one set of coordinates in values of another.

163. If ox, oy, oz, fig. 45, be the co-ordinates of a point m, it is required to determine the position of m by means of

ox', oy', oz', three new rectangular co-ordinates, having the same origin as the former.

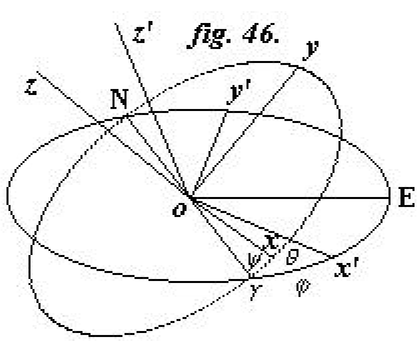
We shall find a value of ox or x first. Now,

$$ox : ox' :: 1 : \cos xox' \quad \text{or} \quad x' = x\cos xox'.$$
$$ox : oy' :: 1 : \cos xoy' \quad \text{or} \quad y' = x\cos xoy'.$$
$$ox : oz' :: 1 : \cos xoz' \quad \text{or} \quad z' = x\cos xoz'.$$

If the sum of these quantities be taken, after multiplying the first by $\cos xox'$, the second by $\cos xoy'$, and the third by $\cos xoz'$, we shall have

$$x'\cos xox' + y'\cos xoy' + z'\cos xoz' = x\left\{\cos^2 xox' + \cos^2 xoy' + \cos^2 xoz'\right\} = x.$$

Let og, fig. 46, be the intersection of the old plane xoy with the new $x'oy'$; and let q be the inclination of these two planes; also let $g\,ox$, $g\,ox'$, be represented by y and f. Values of the cosines of xox', xoy', xoz', must be found in terms of q, y, and f. In the right-angled triangle $g\,xx'$, the sides $g\,x$, $g\,x'$, are y and f, and the angle opposite the side xx' is q:–hence, by spherical trigonometry,



$$\cos xox' = \cos q \sin f \sin y + \cos y \cos f.$$

This equation exists, whatever the values of f and y may be; hence, if $f + 90^\circ$ be put for f, the line ox' will take the place oy', the angle xox' will become xoy', and the preceding equation will give

$$\cos xoy' = \cos q \sin y \cos f - \cos y \sin f.$$

$\cos xoz'$ is found from the triangle whose three sides are the arcs intercepted by the angles $g\,oz'$, $g\,ox$, and xoz'. The angle opposite to the last side is

$$90^\circ - q,\ g\,oz' = 90^\circ,\ g\,ox = y,$$

then the general equation becomes

$$\cos xoz' = \sin q \sin y.$$

If these expressions for the cosines be substituted in the value of x, it becomes

$$x = x'\left\{\cos q \sin f \sin y + \cos y \cos f\right\} + y'\left\{\cos q \cos f \sin y - \cos y \sin f\right\} + z'\sin q \sin y.$$

In the same manner, the values of y and z are found to be

$$y = x'\{\cos q \sin y \sin f - \sin y \cos f\} + y'\{\cos q \cos y \cos f + \sin y \sin f\} + z'\{\sin q \cos y\}$$
$$z = -x'\{\sin q \sin f\} - y'\{\sin q \cos f\} + z' \cos q \ .$$

By substituting these values of $x, y, z,$ in any equation, it will be transformed from the planes xoy, xoz, yoz, to the new planes $x'oy'$, $x'oz'$, $y'oz'$.

164. We have now the means of ascertaining whether, among the infinite number of co-ordinate planes whose origin is in o, the centre of gravity of a system of bodies, there be any on which the sums of the areas are zero. This may be known by substituting the preceding values of $x, y, z,$ and their differentials in the equations of areas: for the angles q, y, and f being arbitrary, such values may be assumed for two of them as will make the sums of the projected areas on two of the co-ordinate planes zero; and if there be any incongruity in this assumption, it will appear in the determination of the third angle, which in that case would involve some absurdity in the areas on the third plane. That, however, is by no means the case, for the sum of the areas on the third plane is then found to be a maximum. If the substitution be made, and the angles y and q so assumed that

$$\sin q \sin y = \frac{c''}{\sqrt{c^2 + c'^2 + c''^2}}, \quad \sin q \cos y = \frac{-c'}{\sqrt{c^2 + c'^2 + c''^2}},$$

it follows that

$$\cos q = \frac{c}{\sqrt{c^2 + c'^2 + c''^2}},$$

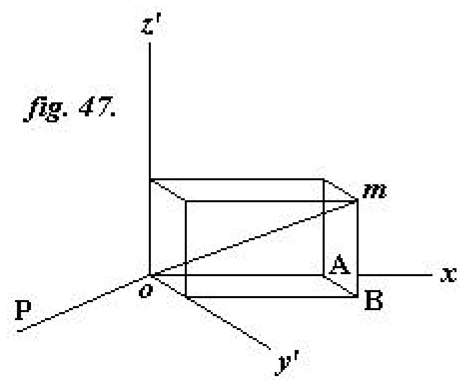
whence

$$\sum m \frac{x'dy' - y'dx'}{dt} = \sqrt{c^2 + c'^2 + c''^2}, \quad \sum m \frac{x'dz' - z'dx'}{dt} = 0, \quad \sum m \frac{y'dz' - z'dy'}{dt} = 0. \qquad \textbf{(27)}$$

Thus, in every system of revolving bodies, there does exist a plane, on which the sum of the projected areas is a maximum; and on every plane at right angles to it, they are zero. One plane alone possesses that property.

165. If the attractive force at o were to cease, the bodies would move by the primitive impulse alone, and the principle of areas would be also true in this case; it even exists independently of any abrupt changes of motion or velocity, among the bodies; and also when the centre of gravity has a rectilinear motion in space. Indeed it follows as a matter of course, that all the properties which have been proved to exist in the motions of a system of bodies, whose centre of gravity is at rest, must equally exist, if that point has a uniform and rectilinear motion in space, since experience shows that the relative motions of a system of bodies, are[8] independent of any motion common to them all.

fig. 47.



Demonstration. However, that will readily appear, if $\bar{x}$, $\bar{y}$, $\bar{z}$, be assumed, as the co-ordinates of o, the moveable centre of gravity estimated from a fixed point P, fig. 47, and if oA; AB, Bm, or x', y', z', be the co-ordinates of m, one of the bodies of the system with regard to the moveable point o. Then the co-ordinates of m relatively to P will be $\overline{x}+x'$, $\overline{y}+y'$, $\overline{z}+z'$. If these be put instead of x, y, z, in the different equations relative to the motions of a system, by attending to the properties of the centre of gravity, $\overline{x}$, $\overline{y}$, $\overline{z}$, vanish from these equations, which then become independent of them. If $\overline{x}+x'$, $\overline{y}+y'$, $\overline{z}+z'$ be put for x, y, z, in equations (25), they become[9]

$$\sum .m\left\{d^2\overline{x}+d^2x'\right\}-\sum m\mathrm{X}dt^2=0.$$
$$\sum .m\left\{d^2\overline{y}+d^2y'\right\}-\sum m\mathrm{Y}dt^2=0.$$
$$\sum .m\left\{d^2\overline{z}+d^2z'\right\}-\sum m\mathrm{Z}dt^2=0.$$

But when the centre of gravity has a rectilinear and uniform motion in space, it has been shown, that

$$\frac{d^2\overline{x}}{dt^2}=0;\quad \frac{d^2\overline{y}}{dt^2}=0;\quad \frac{d^2\overline{z}}{dt^2}=0;$$

which reduces the preceding equations to their original form, namely,[10]

$$\sum m\cdot\frac{d^2x'}{dt^2}=\sum m\mathrm{X},\quad \sum m\cdot\frac{d^2y'}{dt^2}=\sum m\mathrm{Y},\quad \sum m\cdot\frac{d^2z'}{dt^2}=\sum m\mathrm{Z}.$$

If the same substitution be made in[11]

$$\sum m\left(\frac{xd^2y-yd^2x}{dt^2}\right)=\sum m\left(x\mathrm{Y}-y\mathrm{X}\right)$$

it becomes

$$\frac{\overline{x}\sum .md^2y'-\overline{y}\sum md^2x'}{dt^2}+\sum m\cdot\frac{x'd^2y'-y'd^2x'}{dt^2}=\sum m\,.\left(\mathrm{Y}x'-\mathrm{X}y'\right)+\overline{x}\,.\sum m\mathrm{Y}-\overline{y}\,.\sum m\mathrm{X}\,.$$

But in consequence of the preceding equations it is reduced to[12]

$$\sum .m\,.\left\{\frac{x'd^2y'-y'd^2x'}{dt^2}\right\}=\sum .m\cdot\left(x'\mathrm{Y}-y'\mathrm{X}\right).$$

In the same manner it may be shown that

$$\sum .m\,.\left\{\frac{z'd^2x'-x'd^2z'}{dt^2}\right\}=\sum .m\,.\left(z'\mathrm{X}-x'\mathrm{Z}\right),$$

$$\Sigma \cdot m \cdot \left\{ \frac{y' d^2 z' - z' d^2 y'}{dt^2} \right\} = \Sigma \cdot m \cdot \left(y'\mathrm{Z} - z'\mathrm{Y} \right).$$

Thus the equations that determine the motions of a system of bodies are the same, whether the centre of gravity be at rest, or moving uniformly in a straight line; consequently the principles of Impetus, of Least Action, and of the Conservation of Areas, exist in either case.

166. Let the effect produced by the motion of the centre of gravity on the position of the plane for which the areas are a maximum, be now determined.

If $\overline{x} + x$, $\overline{y} + y$, $\overline{z} + z$, be put for x, y, z, in equations (26), they will retain the same form, namely,

$$\Sigma m \{ x' dy' - y' dx' \} = cdt,$$
$$\Sigma m \{ z' dx' - x' dz' \} = c' dt,$$
$$\Sigma m \{ y' dz' - z' dy' \} = c'' dt;$$

for, in consequence of the rectilinear motion of the origin,

$$\overline{x} d\overline{y} - \overline{y} d\overline{x} = 0, \quad \overline{z} d\overline{x} - \overline{x} d\overline{z} = 0, \quad \overline{y} d\overline{z} - \overline{z} d\overline{y} = 0 .$$

And as the position of the plane in question is determined by the constant quantities c, c', and c'', it will always remain parallel to itself during the motion of the system; on that account it is called the Invariable Plane.

167. Thus, when there are no foreign forces acting on the system, the centre of gravity either remains at rest, or moves uniformly in a straight line; and if that point be assumed as the origin of the co-ordinates; the principles of the conservation of areas and living forces will exist with regard to it; and the invariable plane, always passing through that point, will remain parallel to itself, and will be carried along with the centre of gravity in the general motion of the system.

On the Motion of a System of Bodies in all possible Mathematical relations between Force and Velocity

168. In nature, force is proportional to velocity; but as a matter of speculation, Laplace[13] has investigated the motions of a system of bodies in every possible relation between these two quantities. It is rather singular that such an hypothesis should involve no contradiction; on the contrary, principles similar to the preservation of impetus, the constancy of areas, the motion of the centre of gravity, and the least action, actually exist.

Notes

1 V^2 *reads* V_2 in the 1st edition.
2 This *reads* $V_2 = C + 2v$ in the 1st edition.
3 $d\bar{x}$ *reads* $d\,x$ in the 1st edition (published erratum).
4 *shewn*. Archaic for *shown*.
5 This *reads* $\sum \cdot md^2x = dt^2 \cdot \sum \cdot m\mathrm{X}; \;\; \sum \cdot md^2y = dt^2 \cdot \sum m\mathrm{Y}; \;\; \sum \cdot md^2z = dt^2 \sum \cdot m\mathrm{Z}$ in 1st edition.
6 *vectores*. Archaic plural for *vector*.
7 First term below *reads* $\sum m \left\{ \mathrm{X} - \frac{d^2x}{dt^2} \right\} dx$ in the 1st edition.
8 This *reads* "is" in the 1st edition.
9 Second member of third equation below *reads* $-\sum \cdot m\mathrm{Z}dt^2$ in the 1st edition.
10 The first term in first equation below *reads* $\sum \cdot m \frac{d^2x'}{dt^2} = \sum m\mathrm{X}$ in the 1st edition.
11 This is the first of equations (24) which we reproduce as formulated earlier.
12 The closing bracket on the right hand side of the following equation is omitted in the 1st edition.
13 See note 4, *Introduction*.

BOOK I

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CHAPTER V

THE MOTION OF A SOLID BODY OF ANY FORM WHATEVER

169. IF a solid body receives an impulse in a direction passing through its centre of gravity, all its parts will move with an equal velocity; but if the direction of the impulse passes on one side of that centre, the different parts of the body will have unequal velocities, and from this inequality results a motion of rotation in the body round its centre of gravity, at the same time that the centre is moved forward, or translated with the same velocity it would have taken, had the impulse passed through it. Thus the double motions of rotation and translation are produced by one impulse.

170. If a body rotates about its centre of gravity, or about an axis, and is at the same time carried forward in space; and if an equal and contrary impulse be given to the centre of gravity, so as to stop its progressive motion, the rotation will go on as before it received the impulse.

171. If a body revolves about a fixed axis, each of its particles will describe a circle, whose plane is perpendicular to that axis, and its radius is the distance of the particle from the axis. It is evident, that every point of the solid will describe an arc of the same number of degrees in the same time; hence, if the velocity of each particle be divided by its radius or distance from the axis, the quotient will be the same for every particle of the body. This is called the angular velocity of the solid.

172. The axis of rotation may change at every instant, the angular velocity is therefore the same for every particle of the solid for any one instant, but it may vary from one instant to another.

173. The general equations of the motion of a solid body are the same with those of a system of bodies, provided we assume the bodies m, m', m'', &c. to be a system of particles, infinite in number, and united into a solid mass by their mutual attraction.

Let x, y, z, be the co-ordinates of dm, a particle of a solid body urged by the forces X, Y, Z, parallel to the axes of the co-ordinates; then if **S** the sign of ordinary integrals[1] be put for $\sum$, and dm for m, the general equations of the motion of a system of bodies in article 158 become

$$\begin{aligned} \mathbf{S}\cdot\frac{d^2x}{dt^2}dm &= \mathbf{S}\,.\,\mathrm{X}dm, \\ \mathbf{S}\cdot\frac{d^2y}{dt^2}dm &= \mathbf{S}\,.\,\mathrm{Y}dm, \\ \mathbf{S}\cdot\frac{d^2z}{dt^2}dm &= \mathbf{S}\,.\,\mathrm{Z}dm, \end{aligned} \qquad \textbf{(28)}$$

[and][2]

$$\mathbf{S}\cdot\left(\frac{xd^2y-yd^2x}{dt^2}\right)dm=\mathbf{S}\cdot\left(x\mathrm{Y}-y\mathrm{X}\right)dm,$$
$$\mathbf{S}\cdot\left(\frac{zd^2x-xd^2z}{dt^2}\right)dm=\mathbf{S}\cdot\left(\mathrm{z}\mathrm{X}-x\mathrm{Z}\right)dm, \qquad \textbf{(29)}$$
$$\mathbf{S}\cdot\left(\frac{yd^2z-zd^2y}{dt^2}\right)dm=\mathbf{S}\cdot\left(y\mathrm{Z}-z\mathrm{Y}\right)dm,$$

which are the general equations of the motion of a solid, of which m is the mass.

Determination of the general Equations of the Motion of the Centre of Gravity of a Solid in Space

174. Let $\overline{x}+x'$, $\overline{y}+y'$, $\overline{z}+z'$ be put for *x*, *y*, *z*, in equations (28) then

$$\mathbf{S}\cdot dm\left\{\frac{d^2\overline{x}+d^2x'}{dt^2}\right\}=\mathbf{S}\cdot\mathrm{X}dm$$
$$\mathbf{S}\cdot dm\left\{\frac{d^2\overline{y}+d^2y'}{dt^2}\right\}=\mathbf{S}\cdot\mathrm{Y}dm \qquad \textbf{(30)}$$
$$\mathbf{S}\cdot dm\left\{\frac{d^2\overline{z}+d^2z'}{dt^2}\right\}=\mathbf{S}\cdot\mathrm{Z}dm$$

in which $\overline{x}$, $\overline{y}$, $\overline{z}$, are the co-ordinates of *o* the moveable centre of gravity of the solid referred to P a fixed point, and x', y', z', are the co-ordinates of *dm* referred to *o*, fig. 47. Now the co-ordinates of the centre of gravity being the same for all the particles of the solid,

$$\mathbf{S}\,.\,dm\frac{d^2\overline{x}}{dt^2}=m\frac{d^2\overline{x}}{dt^2}$$
$$\mathbf{S}\,.\,dm\frac{d^2\overline{y}}{dt^2}=m\frac{d^2\overline{y}}{dt^2}$$
$$\mathbf{S}\,.\,dm\frac{d^2\overline{z}}{dt^2}=m\frac{d^2\overline{z}}{dt^2}\,.$$

And, with regard to the centre of gravity,

$$\mathbf{S}\,.\,x'dm=0$$
$$\mathbf{S}\,.\,y'dm=0$$
$$\mathbf{S}\,.\,z'dm=0$$

which denote the sum of the particles of the body into their respective distances from the origin; therefore their differentials are

$$\mathbf{S}\,.\,dm\frac{d^2x'}{dt^2}=0$$
$$\mathbf{S}\,.\,dm\frac{d^2y'}{dt^2}=0$$
$$\mathbf{S}\,.\,dm\frac{d^2z'}{dt^2}=0\,.$$

This reduces the equations (30) to

$$m\frac{d^2\overline{x}}{dt^2}=\mathbf{S}\,.\,\mathrm{X}dm$$
$$m\frac{d^2\overline{y}}{dt^2}=\mathbf{S}\,.\,\mathrm{Y}dm \qquad \textbf{(31)}$$
$$m\frac{d^2\overline{z}}{dt^2}=\mathbf{S}\,.\,\mathrm{Z}dm.$$

These three equations determine the motion of the centre of gravity of the body in space, and are similar to those which give the motion of the centre of gravity of a system of bodies.

The solid therefore moves in space as if its mass were united in its centre of gravity, and all the forces that urge the body applied to that point.

175. If the same substitution be made in the first of equations (29), and if it be observed that $\overline{x}$, $\overline{y}$, $\overline{z}$, are the same for all the particles

$$\mathbf{S}\left(\overline{x}d^2\overline{y}-\overline{y}d^2\overline{x}\right)dm=m\left(\overline{x}d^2y-\overline{y}d^2\overline{x}\right)$$
$$\mathbf{S}\left(\overline{x}\mathrm{Y}-\overline{y}\mathrm{X}\right)dm=\overline{x}\cdot\mathbf{S}\cdot\mathrm{Y}dm-\overline{y}\cdot\mathbf{S}\cdot\mathrm{X}dm\,;$$

also

$$\mathbf{S}\left(x'd^2\overline{y}-y'd^2\,\overline{x}+\overline{x}d^{\,2}y'-\overline{y}d^{\,2}x'\right)dm=$$
$$d^{\,2}\overline{y}\cdot\mathbf{S}\cdot x'dm-d^{\,2}\overline{x}\cdot\mathbf{S}\cdot y'dm+\overline{x}\cdot\mathbf{S}\cdot d^2y'dm-\overline{y}\cdot\mathbf{S}\cdot d^2x'dm=0,$$

because x', y', z', are referred to the centre or gravity as the origin of the co-ordinates; consequently the co-ordinates $\overline{x}$, $\overline{y}$, $\overline{z}$, and their differentials vanish from the equation, which therefore retains its original form. Similar results will be obtained for the areas on the other two co-ordinate planes, and thus, equations (29) retain the same forms, whether the centre of gravity be in motion or at rest, proving the motions of rotation and translation to be independent of one another.

Rotation of a Solid

176. If to abridge

$$\mathbf{S}(y\mathrm{Z}-z\mathrm{Y})\,dm = M,$$
$$\mathbf{S}(z\mathrm{X}-x\mathrm{Z})\,dm = M',$$
$$\mathbf{S}(x\mathrm{Y}-y\mathrm{X})\,dm = M''.$$

The integrals of equations (29), with regard to the time, will be

$$\mathbf{S}\left(\frac{ydz-zdy}{dt}\right)dm = \int Mdt,$$
$$\mathbf{S}\left(\frac{zdx-xdz}{dt}\right)dm = \int M'dt, \qquad \textbf{(32)}$$
$$\mathbf{S}\left(\frac{xdy-ydx}{dt}\right)dm = \int M''dt.$$

These equations, which express the properties of areas, determine the rotation of the solid;–equations (31) give the motion of its centre of gravity in space. **S** expresses the sum of the particles of the body, and $\int$ relates to the time alone.

177. Impetus is the mass into the square of the velocity, but the velocity of rotation depends on the distance from the axis, the angle being the same; hence the impetus of a revolving body is the sum of the products of each particle, multiplied by the square of its distance from the axis of rotation. Suppose oA, oB, oC, fig. 10, to be the co-ordinates of a particle dm, situate in m, and let them be represented by x, y, z; then because $m\mathrm{A}=\mathrm{R}o$, $m\mathrm{B}=\mathrm{Q}o$, $m\mathrm{C}=\mathrm{P}o$, the squares of the distances of dm from the three axes ox, oy, oz, are respectively

$$(m\mathrm{A})^2 = y^2+z^2, \quad (m\mathrm{B})^2 = x^2+z^2, \quad (m\mathrm{C})^2 = x^2+y^2.$$

Hence if A′, B′, C′, be the impetus or moments of inertia of a solid with regard to the axes ox, oy, oz, then

$$\mathrm{A}' = \mathbf{S}.dm(y^2+z^2)$$
$$\mathrm{B}' = \mathbf{S}.dm(x^2+z^2) \qquad \textbf{(33)}$$
$$\mathrm{C}' = \mathbf{S}.dm(x^2+y^2).$$

178. If an impulse be given to a sphere of uniform density, in a direction which does not pass through its centre of gravity, it will revolve about an axis perpendicular to the plane passing through the centre of the sphere and the direction of the force; and it will continue to rotate about the same axis even if new forces act on the sphere, provided they act equally on all its particles; and the areas which each of its particles describes will be constant.

179. If the solid be not a sphere, it may change its axis of rotation at every instant; it is therefore of importance, to ascertain if any axes exist in the solid, about which it would rotate permanently.

180. If a body rotates permanently about an axis, the rotatory pressures arising from the centrifugal forces of the solid are equal and contrary in each point of the axis, so that their sum is zero, and the areas described by every particle in the solid are proportional to the time; but if foreign forces disturb the rotation, the rotatory pressures on the axis of rotation are unequal, which causes a perpetual change of axis, and a variation in the areas described by the particles of the body, so that they are no longer proportional to the time. Thus the inconstancy of areas becomes a test of disturbing forces. In this disturbed rotation the body may be considered to have a permanent rotation during an instant only.

181. When three axes of a solid body are permanent axes of rotation, the rotatory pressures on them are zero; this is expressed by the equations

$$\mathbf{S}\,.\,xydm = 0;\;\; \mathbf{S}\,.\,xzdm = 0;\;\; \mathbf{S}\,.\,yzdm = 0;$$

which characterize such axes. To show this, it is necessary to prove that when these equations hold, the rotation of the body round any one axis causes no twisting effort to displace that axis; for example, that the centrifugal forces developed by rotation round z, produce no rotatory pressure round y and x; and so for the other, and *vice versâ*.

Demonstration. Let $r=\sqrt{x^2+y^2}$ be the distance of a particle dm from z the axis of rotation, and let w be the angular velocity of the particle. By article 171 $w=\frac{v}{r}$, therefore $w^2\,.\,r=\frac{v^2}{r}$ is the œntrifugal force arising from rotation round z, and acting in the direction r. When resolved in the direction x, and multiplied by dm it gives

$$w^2 rdm\cdot\frac{x}{r}=w^2 xdm,$$

which, regarded as a force tending to turn the system round y, gives rotatory pressure = $w^2 xzdm$, because it acts at the distance z from the axis y. Therefore when $\mathbf{S}\,.\,xzdm=0$, the whole effect is zero. Similarly, when $\mathbf{S}\,.\,yzdm=0$, the whole effect of the revolving system to turn round x vanishes. Therefore, in order that z should be [the] permanent axis of rotation,

$$\mathbf{S}\,.\,xzdm=0,\;\; \mathbf{S}\,.\,yzdm=0.$$

In like manner, in order that y should be so,

$$\mathbf{S}\,.\,xydm = 0,\;\; \mathbf{S}\,.\,zydm = 0$$

must exist; and in order that x should be so,

$$\mathbf{S}\,.\,yxdm = 0,\;\; \mathbf{S}\,.\,zxdm = 0$$

must exist, all of which are in fact only three different equations, namely

$$\mathbf{S}\,.\,xydm = 0,\;\; \mathbf{S}\,.\,xzdm = 0,\;\; \mathbf{S}\,.\,yzdm = 0; \qquad \textbf{(34)}$$

and if these hold at once, x, y, z, will all be permanent axes of rotation.

Thus the impetus is as the square of the distance from the axis of rotation, and the rotatory pressures are simply as the distance from the same axis.

182. In order to ascertain whether a solid possesses any permanent axes of rotation, let the origin be a fixed point, and let x', y', z', be the co-ordinates of a particle dm, fixed in the solid, but revolving with it about its centre of gravity. The whole theory of rotation is derived from the equations (32) containing the principle of areas. These are the areas projected on the fixed co-ordinate planes xoy, xoz, yoz, fig. 48; but if ox', oy', oz', be the new axes that revolve with the solid, and if the values of x, y, z, given in article 163, be substituted, they will be the same sums, when projected on the new co-ordinate planes $x'oy'$, $x'oz'$, $y'oz'$. The angles q, y, and f, introduced by this change are arbitrary, so that the position of the new axes ox', oy', oz', in the solid, remains indeterminate; and these three angles may be made to fulfil any conditions of the problem.

183. The equations of rotation will take the most simple form if we suppose x', y', z', to be the principal axes of rotation, which they will become if the values of q, y, and f, can be so assumed as to make the rotatory pressures $\mathbf{S}\,.\,x'z'dm$, $\mathbf{S}\,.\,x'y'dm$, $\mathbf{S}\,.\,y'z'dm$, zero at once, then the equations (32) of the areas, when transformed to functions of x', y', z', and deprived of these terms, will determine the rotation of the body about its principal, or permanent axes of rotation, x', y', z'.

184. If the body has no principal axes of rotation, it will be impossible to obtain such values of q, f, and y, as will make the rotatory pressures zero; it must therefore be demonstrated whether or not it be possible to determine the angles in question, so as to fulfil the requisite condition.

185. To determine the existence and position of the principal axes of the body, or the angles q, f, and y, so that

$$\mathbf{S}\,.\,x'y'dm = 0;\;\; \mathbf{S}\,.\,x'z'dm = 0;\;\; \mathbf{S}\,.\,y'z'dm = 0\,.$$

Let values of x', y', z', in functions of x, y, z, determined from the equations in article 163 be substituted in the preceding expressions, then if to abridge,

$$\mathbf{S}.x^2dm = l^2;\quad \mathbf{S}.y^2dm = n^2;\quad \mathbf{S}.z^2dm = s^2$$
$$\mathbf{S}.xydm = f;\quad \mathbf{S}.xzdm = g;\quad \mathbf{S}.yzdm = h,$$

there will result

$$\begin{aligned}
&\cos f\,.\mathbf{S}.x'z'dm - \sin f\,.\mathbf{S}.y'z'dm = \\
&\left(l^2 - n^2\right)\sin q \sin y \cos y + f\sin q\left(\cos^2 y - \sin^2 y\right) \\
&+\cos q\left(g\cos y - h\sin y\right); \\
&\sin f\,.\mathbf{S}.x'z'dm + \cos f\,.\mathbf{S}.y'z'dm = \\
&\sin q\cos q\left\{l^2\sin^2 y + n^2\cos^2 y - s^2 + 2f\sin y\cos y\right\} \\
&+\left(\cos^2 q - \sin^2 q\right)\cdot\left(g\sin y + h\cos y\right).
\end{aligned}\qquad \mathbf{(35)}$$

If the second members of these be made equal to zero, there will be

$$\tan q = \frac{h\sin y - g\cos y}{\left(l^2 - n^2\right)\sin y\cos y + f\left(\cos^2 y - \sin^2 y\right)},$$

and

$$\tfrac{1}{2}\tan 2q = \frac{g\sin y + h\cos y}{s^2 - l^2\sin^2 y - n^2\cos^2 y - 2f\sin y\cos y},$$

but

$$\tfrac{1}{2}\tan 2q = \frac{\tan q}{1 - \tan^2 q},$$

by the arithmetic of sines; hence, equating these two values of $\tfrac{1}{2}\tan 2q$, and substituting for $\tan q$ its value in y; then if to abridge, $u = \tan y$, after some reduction it will be found that

$$\begin{aligned}
0 &= \left(gu + h\right)\left(hu - g\right)^2 + \\
&\left\{\left(l^2 - n^2\right)u + f\left(1 - u^2\right)\right\}\cdot\left\{\left(hs^2 - hl^2 + fg\right)u + gn^2 - gs^2 - hf\right\};
\end{aligned}$$

where u is of the third degree. This equation having at least one real root, it is always possible to render the first members of the two equations (35) zero at the same time, and consequently

$$\left(\mathbf{S}.x'z'dm\right)^2 + \left(\mathbf{S}.y'z'dm\right)^2 = 0.$$

But that can only be the case when $\mathbf{S}.x'z'dm = 0$, $\mathbf{S}.y'z'dm = 0$. The value of $u = \tan y$, gives y, consequently $\tan q$ and q become known.

It yet remains to determine the condition $\mathbf{S}.x'y'dm = 0$, and the angle f. If substitution be made in $\mathbf{S}.x'y'dm = 0$, for x' and y' from article 163, it will take the form $H\sin 2f + L\cos 2f$, H and L being functions of the known quantities q and y; as it must be zero, it gives

$$\tan 2q = -\frac{L}{H};$$

and thus the three axes ox', oy', oz', determined by the preceding values of q, y, and f, satisfy the equations

$$\mathbf{S}.x'z'dm = 0,\ \mathbf{S}.y'z'dm = 0,\ \mathbf{S}.x'y'dm = 0.$$

186. The equation of the third degree in u seems to give three systems of principal axes, one for each value of u; but u is the tangent of the angle formed by the axis x with the line of intersection of the plane xy with that of $x'y'$; and as any one of the three axes, x', y', z', may be changed into any other of them, since the preceding equations will still be satisfied, therefore the equation in u will determine the tangent of the angle formed by the axis x with the line of intersection of xy and $x'y'$, with that of xy and $x'z'$, or with that of xy and $y'z'$. Consequently the three roots of the equation in u are real, and belong to the same system of axes.

187. Whence every body has at least one system of principal and rectangular axes, round any one of which if the body rotates, the opposite centrifugal forces balance each other. This theorem was first proposed by Segner[3] in the year 1755, and was demonstrated by Albert Euler[4] in 1760.

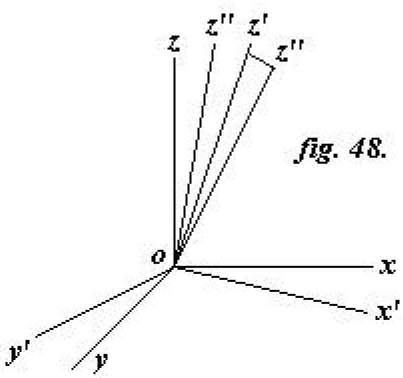


fig. 48.

188. The position of the principle axes ox', oy', oz', in the interior of the solid, is now completely fixed; and if there be no disturbing forces, the body will rotate permanently about any one of them, as oz', fig. 48; but if the rotation be disturbed by foreign forces, the solid will only rotate for in instant about oz', and in the next element of time it will rotate about oz'', and so on, perpetually changing. Six equations are therefore required to fix the position of the instantaneous axis oz''; three will determine its place with regard to the principal axes ox', oy', oz', and three more are necessary to determine the position of the principal axes themselves in space, that is, with regard to the fixed co-ordinates *ox*, *oy*, *oz*. The permanency of rotation is not the same for all the three axes, as will now be shown.

189. The principal axes possess this property—that the moment of inertia of the solid is a maximum for one of these, and a minimum for another. Let x', y', z', be the co-ordinates of dm, relative to the three principal axes, and let x, y, z, be the co-ordinates of the same element referred to any axes whatever having the same origin. Now if

$$C' = \mathbf{S}\left(x^2 + y^2\right)dm$$

be the moment of inertia relatively to one of these new axes, as *z*, then substituting for *x* and *y* their values from article 163, and making

$$A = \mathbf{S}\left(y'^2 + z'^2\right).dm;\ \ B = \mathbf{S}\left(x'^2 + z'^2\right).dm;\ \ C = \mathbf{S}\left(x'^2 + y'^2\right).dm;$$

the value of C' will become

$$C' = A\sin^2 q\sin^2 f + B\sin^2 q\cos^2 f + C\cos^2 q,$$

in which

$$\sin^2 q\sin^2 f,\ \sin^2 q\cos^2 f,\ \cos^2 q,$$

are the squares of the cosines of the angles made by ox', oy', oz', with *oz*; and *A*, *B*, *C*, are the moments of inertia of the solid with regard to the axes x', y', and z', respectively. The quantity C' is less than the greatest of the three quantities *A*, *B*, *C*, and exceeds the least of them; the greatest and the least moments of inertia belong therefore, to the principal axes. In fact, C' must be less than the greatest of the three quantities *A*, *B*, *C*, because their joint coefficients are always equal to unity; and for a similar reason it is always greater than the least.

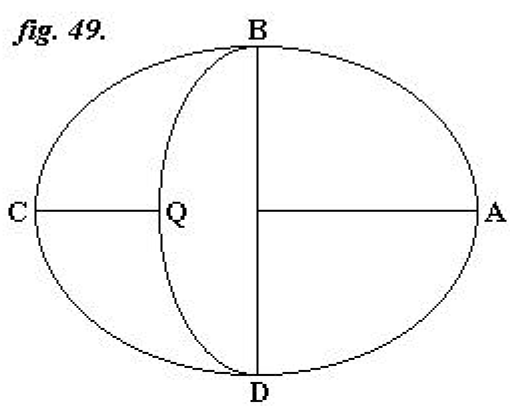
190. When $A = B = C$, then all the axes of the solid are principal axes, and it will rotate permanently about any one of them. The sphere of uniform density is a solid of this kind, but there are many others.

191. When two of the moments of inertia are equal, as $A=B$, then

$$C' = A\sin^2 q + C\cos^2 q\ ;$$

and all the moments of inertia in the same plane with these are equal: hence all the axes situate in that plane are principal axes. The ellipsoid of revolution of uniform density is of this kind; all the axes in the plane of its equator being principal axes.

fig. 49.



192. An ellipsoid of revolution is formed by the rotation of an ellipse ABCD about its minor axis BD. Then AC is its equator. When the moments of inertia are unequal, the rotation round the axes which have their moment of inertia a maximum or minimum is stable, that is, round the least or greatest axis; but the rotation is unstable round the third, and may be destroyed by the slightest cause. If stable rotation be slightly deranged, the body will never deviate far from its equilibrium; whereas in unstable rotation, if it be disturbed, it will deviate more and more, and will never return to its former state.

193. This theorem is chiefly of importance with regard to the rotation of the earth. If xoy (fig. 46) be the plane of the ecliptic, and z its pole; $x'oy'$ the plane of the equator, and z' its pole: then oz' is the axis of the earth's rotation, $zoz' = q$ is the obliquity of the ecliptic, g N the line of the equinoxes, and g the first point of Aries: hence $xog = y$ is the longitude of ox, and $x'og = f$ is the longitude of the principal revolving axis ox', or the measure of the earth's rotation: oz' is therefore one of the permanent axes of rotation.

The earth is flattened at the poles, therefore oz' is the least of the permanent axes of rotation, and the moment of inertia with regard to it, is a maximum. Were there no disturbing forces, the earth would rotate permanently about it; but the sun and moon, acting unequally on the different particles, disturb its rotation. These disturbing forces do not sensibly alter the velocity of rotation, in which neither theory nor observation have detected any appreciable variation; nor do they sensibly displace the poles of rotation on the surface of the earth; that is to say, the axis of rotation, and the plane of the equator which is perpendicular to it, always meet the surface in the same points; but these forces alter the direction of the polar axis in space, and produce the phenomena of precession and nutation;[5] for the earth rotates about oz'', fig. 50, while oz'' revolves about its mean place oz', and at the same time oz' describes a cone about oz; so that the motion of the axis of rotation is very complicated. That axis of rotation, of which all the points remain at rest during the time dt, is called an instantaneous axis of rotation, for the solid revolves about it during that short interval, as it would do about a fixed axis.

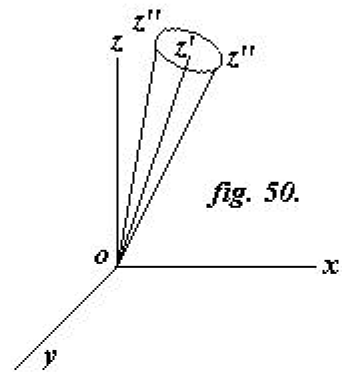


fig. 50.

The equations (32) must now be so transformed as to give all the circumstances of rotatory motion.

194. The equations in article 163, for changing the co-ordinates, will become

$$\begin{aligned} x &= ax' + by' + cz' \\ y &= a'x' + b'y' + c'z' \\ z &= a''x' + b''y' + c''z' \end{aligned} \qquad \textbf{(36)}$$

If to abridge

$$\begin{aligned} a &= \cos q \sin y \sin f + \cos y \cos f \\ b &= \cos q \sin y \cos f - \cos y \sin f \\ c &= \sin q \sin y \\ a' &= \cos q \cos y \sin f - \sin y \cos f \\ b' &= \cos q \cos y \cos f + \sin y \sin f \\ c' &= \sin q \cos y \\ a'' &= -\sin q \sin f \\ b'' &= -\sin q \cos f \\ c'' &= \cos q, \end{aligned}$$

where a, b, c are the cosines of the angles made by x with x', y', z'; a', b', c' are the cosines of the angles made by y with x', y', z'; and a'', b'', c'' are the cosines of the angles made by z with the same axes.

Whatever the co-ordinates of dm may be, since they have the same origin,

$$x^2 + y^2 + z^2 = x'^2 + y'^2 + z'^2 .$$

By means of these, it may be found that

$$\begin{aligned} a^2 + a'^2 + a''^2 &= 1 \qquad & ab + a'b' + a''b'' &= 0 \\ b^2 + b'^2 + b''^2 &= 1 \qquad & ac + a'c' + a''c'' &= 0 \\ c^2 + c'^2 + c''^2 &= 1 \qquad & bc + b'c' + b''c'' &= 0 . \end{aligned}$$

In the same manner, to obtain x', y', z', in functions of x, y, z,

$$\begin{aligned} x' &= ax + a'y + a''z \\ y' &= bx + b'y + b''z \\ z' &= cx + c'y + c''z , \end{aligned} \qquad \textbf{(37)}$$

whence the equations of condition,

$$\begin{aligned} a^2 + b^2 + c^2 &= 1 \qquad & aa' + bb' + cc' &= 0 \\ a'^2 + b'^2 + c'^2 &= 1 \qquad & aa'' + bb'' + cc'' &= 0 \\ a''^2 + b''^2 + c''^2 &= 1 \qquad & a'a'' + b'b'' + c'c'' &= 0 \end{aligned}$$

six of the quantities a, b, c, a', b', c', a'', b'', c'', are determined by the preceding equations, and three remain arbitrary.

If values of x', y', z', found from equations (36) be compared with their values in equations (37), there will result

$$\begin{aligned} a &= b'c'' - b''c' \qquad & a' &= b''c - bc'' \qquad & a'' &= bc' - b'c \\ b &= a''c' - a'c'' \qquad & b' &= ac'' - a''c \qquad & b'' &= a'c - ac' \\ c &= a'b'' - a''b' \qquad & c' &= a''b - ab'' \qquad & c'' &= ab' - a'b. \end{aligned} \qquad \textbf{(38)}$$

195. The axes x', y', z', retain the same position in the interior of the body during its rotation, and are therefore independent of the time; but the angles a, b, c, a', b', c', a'', b'', c'', vary with the time; hence, if values of y, z, $\frac{dy}{dt}$, $\frac{dz}{dt}$, from equations (36), be substituted in the first of equations (32), it will become

$$\mathbf{S}\left\{\begin{array}{l}\left(\frac{a'da''-a''da'}{dt}\right)x'^2+\left(\frac{b'db''-b''db'}{dt}\right)y'^2+\left(\frac{c'dc''-c''dc'}{dt}\right)z'^2\\ \qquad+\left(\frac{a'db''-b''da'+b'da''-a''db'}{dt}\right)x'y'\\ \qquad+\left(\frac{a'dc''-c''da'+c'da''-a''dc'}{dt}\right)x'z'\\ \qquad+\left(\frac{b'dc''-c''db'+c'db''-b''dc'}{dt}\right)y'z'\end{array}\right\}dm=\int M\cdot dt.$$

If a', a'', b', &c. be eliminated from this equation by their values in (38), and if to abridge

$$\begin{aligned}cdb+c'db'+c''db''&=-bdc-b'dc'-b''dc''=pdt\\ adc+a'dc'+a''dc''&=-cda-c'da'-c''da''=qdt\\ bda+b'da'+b''da''&=-adb-a'db'-a''db''=rdt\end{aligned}\qquad\textbf{(39)}$$

$$A=\mathbf{S}\left(y'^2+z'^2\right)dm;\quad B=\mathbf{S}\left(x'^2+z'^2\right)dm;\quad C=\mathbf{S}\left(x'^2+y'^2\right)dm.$$

And if

$$\mathbf{S}\,.\,x'y'dm=0\quad \mathbf{S}\,.\,x'zdm=0\quad \mathbf{S}\,.\,y'z'dm=0\,,$$

it will be found that

$$aAp+bBq+cCr=\int Mdt\,;$$

by the same process it may be found that

$$a'Ap+b'Bq+c'Cr=\int M'dt\,,$$
$$a''Ap+b''Bq+c''Cr=\int M''dt\,.$$

196. If the differentials of these three equations be taken, making all the quantities vary except A, B, and C, then the sum of the first differential multiplied by a, plus the second multiplied by a', plus the third multiplied by a'', will be

$$A\frac{dp}{dt}+(C-B)\cdot qr=aM+a'M'+a''M''\,,$$

in consequence of the preceding relations between $a\ a'\ a''$, $b\ b'\ b''$, $c\ c'\ c''$, and their differentials. By a similar process the coefficient $b\ b'\ b''$, &c., may be made to vanish, and then if

$$\begin{aligned}aM+a'M'+a''M''&=N\\ bM+b'M'+b''M''&=N'\\ cM+c'M'+c''M''&=N''\end{aligned}$$

the equations in question are transformed to

$$\begin{aligned} A\frac{dp}{dt}+(C-B).qr&=N \\ B\frac{dq}{dt}+(A-C).rp&=N' \\ C\frac{dr}{dt}+(B-A).pq&=N''. \end{aligned} \qquad \textbf{(40)}$$

And if $a,\ a',\ a'',\ b,\ b'$, &c., and their differentials, be replaced by their functions in $f,\ y$, and y, given in article 194, the equations (39) become

$$\begin{aligned} pdt &= \sin f \sin q\ .dy - \cos f\ .dq \\ qdt &= \cos f \sin q\ .dy + \sin f\ .dq \\ rdt &= df - \cos q\ .dy\ . \end{aligned} \qquad \textbf{(41)}$$

197. These six equations contain the whole theory of the rotation of the planets and their satellites, and as they have been determined in the hypothesis of the rotatory pressures being zero, they will give their rotation nearly about their principal axes.

198. The quantities p, q, r, determine oz'', the position of the real and instantaneous axis of rotation, with regard to its principal axis oz'; when a body has no motion but that of rotation, all the points in a permanent axis of rotation remain at rest; but in an instantaneous axis of rotation the axis can only be regarded as at rest from one instant to another.

If the equations (36) for changing the co-ordinates, be resumed, then with regard to the axis of rotation,

$$dx = 0,\ dy = 0,\ dz = 0,$$

since all its points are at rest; therefore the indefinitely small spaces moved over by that axis in the direction of these co-ordinates being zero, the equations in question become,

$$\begin{aligned} x'da + y'db + z'dc &= 0, \\ x'da' + y'db' + z'dc' &= 0, \\ x'da'' + y'db'' + z'dc'' &= 0, \end{aligned}$$

which will determine $x',\ y',\ z'$, and consequently oz'' the axis in question.

For if the first of these equations be multiplied by c, the second by c', and the third by c'', their sum is

$$py' - qz' = 0\ . \qquad \textbf{(42)}$$

Again, if the first be multiplied by b, the second by b', and the third by b'', their sum is[6]

$$rx' - pz' = 0. \qquad \textbf{(43)}$$

Lastly, if the first equation be multiplied by a, the second by a', and the third by a'', their sum is

$$qz' - ry' = 0.$$

The last of these is contained in the two first, which are the equations to a straight line oz'', which forms, with the principle axes x', y', z', angles whose cosines are

$$\frac{p}{\sqrt{p^2+q^2+r^2}};\quad \frac{q}{\sqrt{p^2+q^2+r^2}};\quad \frac{r}{\sqrt{p^2+q^2+r^2}}; \qquad \textbf{(44)}$$

for the two last give

$$x'^2 = z'^2\frac{p^2}{r^2};\quad y'^2 = z'^2\frac{q^2}{r^2};$$

whence

$$x'^2 + y'^2 + z'^2 = z'^2\left\{\frac{p^2+q^2+r^2}{r^2}\right\};$$

and therefore

$$\frac{z'}{\sqrt{x'^2+y'^2+z'^2}} = \frac{r}{\sqrt{p^2+q^2+r^2}}.$$

But

$$oz'' = \sqrt{x'^2+y'^2+z'^2};$$

and

$$oz'' : oc :: 1 : \cos z''oc;$$

then if x', y', z', be the co-ordinates of the point z'',

$$\cos z''oc = \frac{z'}{\sqrt{x'^2+y'^2+z'^2}} = \frac{r}{\sqrt{p^2+q^2+r^2}}.$$

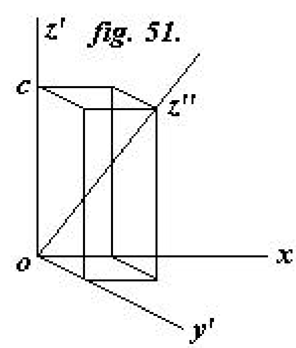


fig. 51.

In the same manner

$$\cos z''ox' = \frac{p}{\sqrt{p^2+q^2+r^2}}$$

and

$$\cos z''oy' = \frac{q}{\sqrt{p^2+q^2+r^2}}.$$

Consequently oz'' is the instantaneous axis of rotation.

201.[7] The angular velocity of rotation is also given by these quantities. If the object be to determine it for a point in the axis, as for example where $oc = 1$, then

$$x' = 0,\ y' = 0,$$

and the values of dx, dy, dz give, when divided by dt,

$$\frac{dy}{dt}\sin q,\ \frac{dq}{dt}\cos q,\ -\frac{dq}{dt}\sin q,$$

for the components of the velocity of a particle; hence the resulting velocity is

$$\frac{\sqrt{dq^2 + dy^2 \sin^2 q}}{dt} = \sqrt{q^2 + r^2}\ ,$$

which is the sum of the squares of the two last of equations (41).

199. But in order to obtain the angular velocity of the body, this quantity must be divided by the distance of the particle at c' from the axis oz''; but this distance is evidently equal to the sine of $z''oc$, the angle between oz' and oz'', the principal and instantaneous axes of rotation; but

$$\frac{r}{\sqrt{p^2 + q^2 + r^2}}$$

is the cosine of this angle; hence

$$\sqrt{1 - \frac{r^2}{p^2 + q^2 + r^2}}\ ,$$

or

$$\frac{\sqrt{q^2 + p^2}}{\sqrt{p^2 + q^2 + r^2}}\ ,$$

is the sine; and therefore

$$\sqrt{p^2 + q^2 + r^2}$$

is the angular velocity of rotation. Thus, whatever may be the rotation of a body about a point that is fixed, or one considered to be fixed, the motion can only be rotation about an axis that is fixed during an instant, but may vary from one instant to another.

200. The position of the instantaneous axis with regard to the three principal axes, and the angular velocity of rotation, depend on p, q, r, whose determination is very important in these researches; and as they express quantities independent of the situation of the fixed plane xoy, they are themselves independent of it.

201.[8] Equations (40) determine the rotation of a solid troubled by the action of foreign forces, as for example, that of the earth when disturbed by the sun and moon. But the same equations will also determine the rotation of a solid, when not disturbed in its rotation.

Rotation of a Solid not subject to the action of Disturbing Forces, and at liberty to revolve freely about a Fixed Point, being its Centre of Gravity, or not

202. Values of *p*, *q*, *r*, in terms of the time must be obtained, in order to ascertain all the circumstances of rotation at every instant.

If we suppose that there are no disturbing forces, the areas are constant: hence the equations (40) become

$$A\cdot dp+(C-B)\cdot q\cdot r\cdot dt=0;\quad B\cdot dq+(A-C)\cdot r\cdot p\cdot dt=0;\quad C\cdot dr+(B-A)\cdot p\cdot q\cdot dt=0. \qquad \textbf{(45)}$$

If the first be multiplied by *p*, the second by *q*, and the third by *r*, their sum is

$$Apdp+Bqdq+Crdr=0,$$

and its integral is

$$Ap^2+Bq^2+Cr^2=k^2, \qquad \textbf{(46)}$$

k^2 being a constant quantity. Again, if the three equations be multiplied respectively by *Ap*, *Bq*, *Cr*, and integrated, they give

$$A^2p^2+B^2q^2+C^2r^2=h^2, \qquad \textbf{(47)}$$

a constant quantity. Equation (46)[9] contains the principle of the preservation of impetus or living force which is constant in conformity with article 148. From these two integrals are obtained:

$$p^2=\frac{h^2-Bk+(B-C).Cr^2}{A(A-B)}\quad q^2=\frac{h^2-Ak+(A-C).Cr^2}{B(B-A)}. \qquad \textbf{(48)}$$

By the substitution of these values of *p* and *q*, the last of equations (45) when resolved according to *dt*, gives

$$dt=\frac{Cdr\,.\sqrt{AB}}{\sqrt{\left\{\left(h^2-Bk+(B-C)Cr^2\right).\left(-h^2+Ak+(C-A)\right).Cr^2\right\}}} \qquad \textbf{(49)}$$

This equation will give by quadratures the value of t in r, and reciprocally the value of r in t; and thus by the substitution of this value of r in equations (48) the three quantities p, q and r become known in functions of the time. This equation can only be integrated when any two of the moments of inertia are equal, either when

$$A = B, \quad B = C, \quad A = C;$$

in each of these cases the solid is a spheroid of revolution.

203. Thus p, q, r, being known functions of the time, the angular velocity of the solid, and its rotation with regard to the principal axes, are known at every instant.

204. This however is not sufficient. To become acquainted with all the circumstances of rotation, it is requisite to know the position of the principle axes themselves with regard to quiescent space, that is, their position relatively to the fixed axes x, y, z. But for that purpose the angles f, y, and q, must be determined in functions of the time, or, which is the same thing, in functions of p, q, r, which may now be regarded as known quantities.

If the first of equations (45) be multiplied by a, the second by b, and the third by c, their sum when integrated, in consequence of the relations between the angles in article 194, is

$$aAp + bBq + cCr = l,$$

by a similar process

$$\begin{aligned} a'Ap + b'Bq + c'Cr &= l', \\ a''Ap + b''Bq + c''Cr &= l'', \end{aligned} \qquad \textbf{(50)}$$

l, l', l'', being arbitrary constant quantities. These equations coincide with those in article 195, and contain the principle of areas. They are not however three distinct integrals, for the sum of their squares is

$$A^2p^2 + B^2q^2 + C^2r^2 = l^2 + l'^2 + l''^2,$$

in consequence of the equations in article 194. But this is the same with (47); hence

$$l^2 + l'^2 + l''^2 = h^2$$

being an equation of condition, equations (50) will only give values of two of the angles f, y, and q.

The constant quantities l, l', l'', correspond with c, c', c'', in article 164, therefore

$$\tfrac{1}{2}t\sqrt{l^2 + l'^2 + l''^2}$$

is the sum of the areas described in the time t by the projection of each particle of the body on the plane on which that sum is a maximum. If xoy be that plane, l and l' are zero: therefore, in every solid body in rotation about an axis, there exists a plane, on which the sum of the areas described by the projections of the particles of the solid during a finite time is a maximum. It is called the Invariable Plane, because it remains parallel to itself during the motion of the body: it is also named the plane of the Greatest Rotatory Pressure.

Since

$$l = 0, \quad l' = 0, \quad l'' = h,$$

if the first of equations (50) be multiplied by a, the second by a', and the third by a'', in consequence of the equations in article 194, their sum is

$$a'' = \frac{Ap}{h};$$

in the same manner it will be found that

$$b'' = \frac{Bq}{h}, \quad c'' = \frac{Cr}{h};$$

or, substituting the values of a'', b'', c'', from article 194,

$$\sin q' \sin f' = -\frac{Ap}{h}, \quad \sin q' \cos f' = -\frac{Bq}{h}, \quad \cos q' = \frac{Cr}{h}. \qquad \textbf{(51)}$$

The accented angles q', f', y', relate to the invariable plane, and angles q, f, y, to the fixed plane.

Because p, q, r, are known functions of the time, f' and q' are determined, and if dq be eliminated between the two first of equation (41), the result will be

$$\sin^2 q' . dy' = \sin q' . \sin f' . pdt + \sin q' . \cos f' . qdt .$$

But in consequence of equations (51), and because

$$Ap^2 + Bq^2 = k - Cr^2,$$
$$dy' = \frac{Cr^2 - k}{h^2 - C^2 r^2} \cdot hdt ;$$

and as r is given in functions of the time by equation (49), y' is determined.

Thus, p, q, r, y', q', and f', are given in terms of the time: so that the position of the three principal axes with regard to the fixed axes, ox, oy, oz; and the angular velocity of the body, are known at every instant.

205. As there are six integrations, there must be six arbitrary constant quantities for the complete solution of the problem. Besides h and k, two more will be introduced by the integration of dt and dy'. Hence two are still required, because by the assumption of *xoy* for the invariable plane, l and l' become zero.

Now the three angles, y', f', q', are given in terms of p, q, r, and these last are known in terms of the time; hence y', f', q', (fig. 49), are known with regard to the invariable plane *xoy*: and by trigonometry it will be easy to determine values of y, f, q, with regard to any fixed plane whatever, which will introduce two new arbitrary quantities, making in all six, which are requisite for the complete solution of the problem.

206. These two new arbitrary quantities are the inclination of the invariable plane on the fixed plane in question, and the angular distance of the line of intersection of these two planes from a line arbitrarily assumed on the fixed plane; and as the initial position of the fixed plane is supposed to be given, the two arbitrary quantities are known.

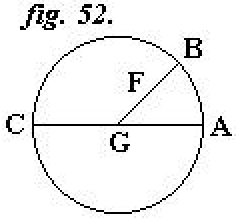
If the position of the three principal axes with regard to the invariable plane be known at the origin of the motion, f', q', will be given, and therefore p, q, r, will be known at that time; and then equation (46) will give the value of k.

The constant quantity arising from the integration of dt depends on the arbitrary origin or instant whence the time is estimated, and that introduced by the integration of dy' depends on the origin of the angle y', which may be assumed at pleasure on the invariable plane.

207. The determination of the sixth constant quantity h is very interesting, as it affords the means of ascertaining the point in which the sun and planets may be supposed to have received a primitive impulse, capable of communicating to them at once their rectilinear and rotatory motions.

The sum of the areas described round the centre of gravity of the solid by the radius of each particle projected on a fixed plane, and respectively multiplied by the particles, is proportional to the moment of the primitive force projected on the same plane; but this moment is a maximum relatively to the plane which passes through the point of primitive impulse and centre of gravity, hence it is the invariable plane.

fig. 52.



208. Let G, fig. 52, be the centre of gravity of a body of which ABC is a section, and suppose that it has received an impulse in the plane ABC at the distance GF, from its centre of gravity; it will move forward in space at the same time that it will rotate about an axis perpendicular to the plane ABC. Let v be the velocity generated in the centre of gravity by the primitive impulse; then if m be the mass of the body, $m \cdot v \cdot \mathrm{GF}$ will be the moment of this impulse, and multiplying it by $\frac{1}{2}t$, the product will be equal to the sum of the areas described during the time t; but this sum was shown to be

$$\tfrac{1}{2}t\sqrt{l^2 + l'^2 + l''^2}\ ;$$

hence

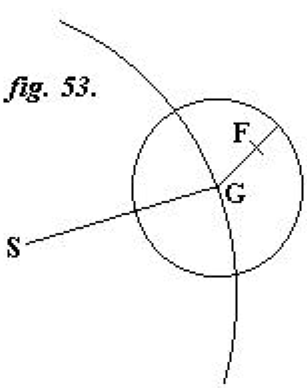
$$\sqrt{l^2 + l'^2 + l''^2} = m \cdot v \cdot \mathrm{GF} = h\ ;$$

which determines the sixth arbitrary constant quantity h. Were the angular velocity of rotation, the mass of the body and the velocity of its centre of gravity known, the distance GF, the point of primitive impulse, might be determined.

209. It is not probable that the primitive impulse of the planets and other bodies of the system passed exactly through their centres of gravity; most of them are observed to have a rotatory motion, though in others it has not been ascertained, on account of their immense distances, and the smallness of their volumes. As the sun rotates about an axis, he must have received a primitive impulse not passing through his centre of gravity, and therefore it would cause him to move forward in space accompanied by the planetary system, unless an impulse in the contrary direction had destroyed that motion, which is by no means likely. Thus the sun's rotation leads us to presume that the solar system may be in motion.

210. Suppose a planet of uniform density, whose radius is R, to be a sphere revolving round the sun in S, at the distance SG or $\bar{r}$, with an angular velocity represented by u, then the velocity of the centre of gravity will be $v = u\bar{r}$.

fig. 53.



Imagine the planet to be put in motion by a primitive impulse, passing through the point F, fig. 53, then the sphere will rotate about an axis perpendicular to the invariable plane, with an angular velocity equal to r, for the components q and p at right angles to that plane will be zero; hence, the equation[10]

$$\sqrt{l^2 + l'^2 + l''^2} = m \cdot v \cdot \mathrm{GF}$$

becomes

$$l'' = mu\bar{r} \, . \, GF \, ;$$

and

$$l'' = rC \, .$$

The centre of gyration is that point of a body in rotation, into which, if all the particles were condensed, it would retain the same degree of rotatory power. It is found that the square of the radius of gyration in a sphere, is equal to $\frac{2}{5}$ of the square of its semi-diameter; hence the rotatory inertia C becomes $\frac{2}{5}m\mathrm{R}^2$, thus

$$l'' = r \times \frac{2}{5} m\mathrm{R}^2, \text{ and } \mathrm{GF} = \frac{2}{5} \cdot \frac{\mathrm{R}^2}{\bar{r}} \cdot \frac{r}{u}.$$

211. Hence, if the ratio of the mean radius of a planet to its mean distance from the sun, and the ratio of its angular velocity of rotation to its angular velocity in its orbit, could be ascertained, the point in which the primitive impulse was given, producing its motion in space, might be determined.

212. Were the earth a sphere of uniform density, the ratio $\frac{\mathrm{R}}{r}$ would be 0.000042665; and the ratio of its rotatory velocity to that in its orbit is known by observation to be 366.25638, whence $\text{GF}=\frac{\text{R}}{160}$; and as the mean radius of the earth is about 4,000 miles, the primitive impulse must have been given at the distance of 25 miles from the centre. However, as the density of the earth is not uniform, but decreases from the centre to the surface, the distance of the primitive impulse from its centre of gravity must have been something less.

213. The rotation of the earth has established a relation between time and the arcs of a circle. Every point in the surface of the earth passes through 360^0 in 24 hours; and as the rotation is uniform, the arcs described are proportional to the time, so that one of these quantities may represent the other. Thus, if a be an arc of any number of degrees, and t the time employed to describe it, $360^0 : a :: 24 : t$: hence $a=\frac{360}{24}t$; or, if the constant co-efficient $\frac{360}{24}$ be represented by n, $a=nt$, and $\sin a=\sin nt$, $\cos a=\cos nt$.

In the same manner the periodic time of the moon being 27.3 days nearly, an arc of the moon's orbit would be $\frac{360}{27.3}t$, and may also be expressed by nt. Thus, n may have all values, so that nt is a general expression for any arc that increases uniformly with the time.

214. The motions of the planets are determined by equations of these forms,

$$\frac{d^2u}{dt^2}+n^2u=\text{R}$$
$$\frac{d^2u}{dt^2}+n^2u=0,$$

which are only transformations of the general equation of the motions of a system of bodies. The integrals of both give a value of u in terms of the sines and cosines of circular arcs increasing with the time; the first by approximation, but the integral of the second will be obtained by making $u=c^x$, c being the number whose Napierian[11] logarithm is unity. Whence

$$d^2u=c^x\left(d^2x+dx^2\right),$$

and the equation in question becomes

$$d^2x+dx^2+n^2dt^2=0.$$

Let

$$dx=ydt\text{, then } d^2x=dydt\,,$$

since the element of the time is constant, which changes the equation to

$$dy+dt\left(n^2+y^2\right)=0\,.$$

If $y = m$ a constant quantity, $dm = dy = 0$, hence

$$n^2 + m^2 = 0;$$

whence

$$m = \pm n\sqrt{-1},$$

but

$$dx = ydt = \pm ndt\sqrt{-1},$$

the integral of which is

$$x = \pm nt\sqrt{-1}.$$

As x has two values, $u = c^x$ gives

$$u = bc^{nt\sqrt{-1}}, \text{ and } u = b'c^{-nt\sqrt{-1}},$$

and because either of these satisfies the conditions of the problem, their sum

$$u = bc^{nt\sqrt{-1}} + u = b'c^{-nt\sqrt{-1}},$$

also satisfies the conditions and is the general solution, b and b' being arbitrary constant quantities. But

$$c^{nt\sqrt{-1}} = \cos nt + \sqrt{-1}\sin nt,$$
$$c^{-nt\sqrt{-1}} = \cos nt - \sqrt{-1}\sin nt.$$

Hence

$$u = (b + b')\cos nt + (b - b')\sqrt{-1}\sin nt\,.$$

Let

$$b + b' = M\,\sin e; \quad (b - b')\sqrt{-1} = M\,\cos e;$$

and then[12]

$$u = M\,\{\sin e\cos nt + \cos e\,\sin nt\}$$

or

$$u = M\sin(nt + e),$$

which is the integral required, because M and e are two arbitrary constant quantities.

215. Since a sine or cosine never can exceed the radius, $\sin.(nt + e)$ never can exceed unity, however much the time may increase; therefore u is a periodic quantity whose value oscillates between fixed limits which it never can surpass. But that would not be the case were n an imaginary quantity; for let

$$n = a \pm b\sqrt{-1};$$

then the two values of x become

$$x = bt + at\sqrt{-1} \qquad x = bt - at\sqrt{-1}\,,$$

consequently,

$$c^{bt+at\sqrt{-1}} = c^{bt}\,.\,c^{at\sqrt{-1}} = c^{bt}\left\{\cos at + \sqrt{-1}\sin at\right\}$$
$$c^{bt-at\sqrt{-1}} = c^{bt}\,.\,c^{-at\sqrt{-1}} = c^{bt}\left\{\cos at - \sqrt{-1}\sin at\right\}$$

whence

$$u = c^{bt}\left\{(b+b')\cos at + (b-b')\sqrt{-1}\sin at\right\}$$

or substituting for

$$b+b'; \quad (b-b')\sqrt{-1};$$

[then][13]

$$u = c^{bt}\cdot M\cdot\sin(at+e)\,.$$

But[14]

$$c^{bt} = 1 + bt + \tfrac{1}{2}b^2t^2 + \frac{1}{2.3}b^3t^3 + \&c.,$$

therefore c^{bt} increases indefinitely with the time, and u is no longer a periodic function, but would increase to infinity.

Were the roots of n^2 equal, then $x = bt$, and $u = \mathrm{C}.c^{bt}$, C being constant.

Thus it appears that if the roots of n^2 be imaginary or equal, the function u would increase without limit.

These circumstances are of the highest importance, because the stability of the solar system depends upon them.

Rotation of a Solid which turns nearly round one of its principal Axes, as the Earth and the Planets, but not subject to the action of accelerating Forces

216. Since the axis of rotation oz'' is very near oz', fig. 50, the angle $z'oz''$ is so small, that its cosine $\dfrac{r}{\sqrt{p^2+q^2+r^2}}$ differs but little from unity; hence p and q are so minute that their product may be omitted, which reduces equations (45) to

$$Cdr = 0,$$
$$Adp + (C-B)qrdt = 0,$$
$$Bdq + (A-C)prdt = 0;$$

the first shows the angular velocity to be uniform, and the two last give

$$\frac{d^2q}{dt^2}+\frac{(A-C)}{B}r\frac{dp}{dt}=0; \quad \frac{dp}{dt}=\frac{(B-C)}{A}qr=0;$$

hence if the constant quantity

$$\frac{(A-C)(B-C)}{AB}r^2=n^2,$$

the result will be

$$\frac{d^2q}{dt^2}+n^2q=0;$$

and by article 214,

$$q=M'\cos(nt+g).$$

In the same manner

$$p=M\sin(nt+g);$$

whence

$$M'=M\cdot\sqrt{\frac{A(A-C)}{B(B-C)}}.$$

217. If oz'' the real axis of rotation coincides with oz', the principal axis in the beginning of the motion, then q and p are zero; hence also, $M=0$, and $M'=0$. It follows therefore, that in this case q and p will always be zero, and the axis oz'' will always coincide with oz'; whence, if the body begins to turn round one of its principal axes, it will continue to rotate uniformly about that axis for ever. On account of this remarkable property these are called the natural axes of rotation; it belongs to them exclusively, for if the position of the real axis of rotation oz'' be invariable on the surface of the body, the angular velocity will be constant; hence

$$dp=0, \quad dq=0, \quad dr=0,$$

and

$$(C-B)qrdt=0, \quad (A-C)rpdt=0, \quad (B-A)pqdt=0.$$

218. If A, B, C, be unequal, these equations will only be zero in every case when two of the quantities p, q, r, are zero; but then, the real axis coincides with one of the principal axes.

If two of the moments of inertia be equal, as $A=B$, the three equations are reduced to $rp=0$, $qr=0$; both of which will be satisfied, that is, they will both be zero for every value of q and p, if $r=0$. The axis of rotation is, therefore, in a plane at right angles to the third principal axis; but as the body is then a solid of revolution, every axis in that plane is a principal axis.

219. When $A=B=C$, the three preceding equations are zero, whatever may be the values of p, q, r, then all the axes of the body will be principal axes. Thus the principal axes alone have the property of permanent rotation, though they do not possess that property in the same degree.

220. Suppose the real axis of rotation oz'', fig. 50, to deviate by an indefinitely small quantity from oz', the third principal axis, the coefficients M and M' will then be indefinitely small, since $q = M' \times \cos(nt+g)$, and $p = M \times \sin(nt+g)$ are indefinitely small. Now if n be a real quantity, $\sin(nt+g)$, $\cos(nt+g)$, will never exceed very narrow limits, therefore q and p will remain indefinitely small; so that the real axis oz'' will make indefinitely small oscillations about the third principal axis. But if n be imaginary, by article 215, $\sin(nt+g)$, $\cos(nt+g)$, will be changed into quantities which increase with the time, and the real axis of rotation will deviate more and more from the third principal axis, so that the motion will have no stability. The value of n will decide that important point.

Since[15]

$$n = r\sqrt{\frac{(A-C)(B-C)}{AB}},$$

it will be a real quantity when C the moment of inertia with regard to oz', is either the greatest or the least of the three moments of inertia A, B, C, for then the product[16] $(A-C)(B-C)$ will be positive; but if C have a value that is between those of A and B, that product will be negative, and n imaginary. Hence the rotation will be stable about the greatest and least of the principal axes, but unstable about the third.

221. Having determined the rotation of the solid, it only remains to ascertain the position of the principal axis with regard to quiescent space, that is, with regard to the fixed axes ox, oy, oz. That evidently depends on the angles f, y, and q.

If the third principal axis oz', fig. 50, be assumed to be nearly at right angles to the plane xoy, the angle zoz', or q, will be so very small that its square may be omitted, and its cosine assumed equal to unity; then the equations (41) give $df - dy = rdt$; or if $r = a$, be a constant quantity, the integral is,

$$y = f - at + e.$$

If $\sin q \cos f = s$, $\sin q \sin f = u$, the two first of equations (41), after the elimination of dy, give

$$\frac{ds}{dt} + a\,u = -p, \quad \frac{du}{dt} - a\,s = q.$$

The integrals of these two quantities are obtained by the method in article 214, and are

$$s = x\cos(at + l) - \frac{BM'}{Ca}\cos(nt+g),$$
$$u = x\sin(at + l) - \frac{AM}{Ca}\sin(nt+g),$$

x and l being new arbitrary quantities introduced by integration. The problem is completely solved, since s and u give q and f in values of the time, and y is given in values of f and the time.

Compound Pendulums

222. Hitherto the rotation of a solid about its centre of gravity has been considered either when the body is free, or when the centre of gravity is fixed; but imagine a solid OP, fig. 54, to revolve about a fixed axis in o which does not pass through its centre of gravity. If the body be drawn aside from the vertical oz, and then left to itself, it will oscillate about that axis by the action of gravitation alone. This solid body of any form whatever is the compound pendulum, and its motion is perfectly similar to that of the simple pendulum already described, depending on the property of areas.

fig. 54.

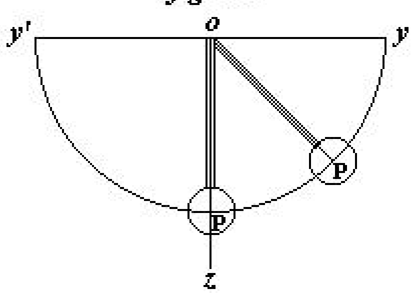
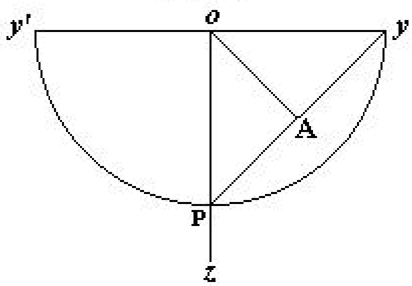


fig. 55.



The motion being in the plane zoy, the sums of the areas in the other two planes are zero; so that the motion of the pendulum is derived from the equation

$$\mathbf{S}\left(\frac{yd^2z-zd^2y}{dt^2}\right)dm=\mathbf{S}\left(y\mathrm{Z}-z\mathrm{Y}\right)dm\,.$$

In order to adapt that equation to the motion of the pendulum, let $oy=y$, $o\mathrm{P}=z$, $\mathrm{A}o=z'$, $\mathrm{A}y=y'$, hence $\mathrm{PA}=-y'$, fig.55; and let the angle PoA be represented by q. P is the centre of gravity of the pendulum, which is supposed to rotate about the axis ox, passing through o at right angles to the plane zoy, and therefore it cannot be represented in the diagram.

Now

$$-y'=z\sin q$$

$$z'=z\cos q$$

$$z'=y\cos q$$

$$y'=y\cos q$$

If the first of these four equations be multiplied by $\sin q$, and the second by $\cos q$, their sum is

$$z=z'\cos q-y'\sin q\ ;$$

in the same way

$$y=z'\sin q+y'\cos q\ .$$

If these values be substituted in the equation of areas it becomes

$$A\frac{d^2q}{dt^2} = -\mathbf{S}\left(y\mathrm{Z} - z\mathrm{Y}\right)dm\,,$$

for

$$A = \mathbf{S}\left(y'^2 + z'^2\right)dm.$$

If the pendulum moves by the force of gravitation alone in the direction *oz*,

$$\mathrm{Y} = 0 \qquad \mathrm{Z} = g\,.$$

Hence

$$A\frac{d^2q}{dt^2} = -\mathbf{S}gydm\,.$$

If the value of *y* be substituted in this it becomes,[17]

$$A\frac{d^2q}{dt^2} = -g\sin q\,.\,\mathbf{S}\,.\,z'dm - g\cos q\,.\,\mathbf{S}\,.\,y'dm\,.$$

Because z' passes through the centre of gravity of the pendulum, the rotatory pressure $\mathbf{S}.y'dm$ is zero; hence

$$A\frac{d^2q}{dt^2} = -g\sin q\,.\,\mathbf{S}\,.\,z'dm\,.$$

If L be the distance of the centre of gravity of the pendulum from the axis of rotation ox, the rotatory pressure $\mathbf{S}\cdot z'dm$ becomes *Lm*, in which *m* is the whole mass of the pendulum; hence

$$A\frac{d^2q}{dt^2} = -Lmg\sin q\,,$$

or

$$\frac{dq^2}{dt^2} = \frac{2Lmg}{A}\cdot\cos q + \mathrm{C}\,,$$

C being an arbitrary constant quantity.

223. If a simple pendulum be considered, of which all the atoms are united in a point at the distance of l from the axis of rotation ox, its rotatory inertia will be $A = ml^2$, m being the mass of the body, and $l^2 = z^2 + y^2$. In this case $l = L$. Substituting this value for *A*, we find

$$\frac{dq^2}{dt^2} = \frac{2g}{l}\cos q + \mathrm{C}\,.$$

224. Thus it appears, that if the angular velocities of the compound and simple pendulums be equal when their centres of gravity are in the vertical, their oscillations will be exactly the same, provided also that the length of the simple pendulum be equal to the rotatory inertia of the solid body with regard to the axis of motion, divided by the product of the mass by the distance of its centre of gravity from the axis, or $l = \frac{A}{mL}$.

Thus such a relation is established between the lengths of the two pendulums, that the length of a simple pendulum may be found, whose oscillations are performed in the same time with those of a compound pendulum.

In this manner the length of the simple pendulum beating seconds has been determined from observations on the oscillations of the compound pendulum.

Notes

[1] We use boldface for **S** in this edition. Somerville uses the plain face S.

[2] The left hand side of this expression is *reads* $\left(\frac{yd^2z - zd^2y}{dt^2}\right)dm$ in the 1st edition.

[3] Segner, Johann or Jan Andreas, (1704-1777), mathematician and physicist, born in Pressburg, Hungary. Segner discovered that solid bodies have three axes of symmetry. His publications include *Elements of Arithmetic and Geometry* and *Nature of Liquid Surfaces*.

[4] See note 6, *Book I, Chapter III.*

[5] *nutation*. Oscillatory movement of the axis of a rotating body (as the earth). *Merriam-Webster's Collegiate Dictionary*.

[6] The prime in the second term in (43) is interchanged in error as $rx' - p'z = 0$ in the 1st edition.

[7] This article (201) is out of sequence. It ought to be article 199. The ordering sequence resumes at 199 in the next article. As a consequence there are two articles numbered 201. We retain the ordering followed in 1st edition.

[8] There are two articles numbered 201 as noted above.

[9] "Equation (46)" *reads* "This equation" in the 1st edition (published erratum).

[10] The 1st edition has a period after this expression.

[11] The principle of logarithms was devised by John Napier (1550-1617) of Merchiston, Scotland, to abridge arithmetical calculations, by the use of addition and subtraction in place of multiplication and division.

[12] Written with a rounded right-hand bracket in 1st edition.

[13] Punctuation in equation changed to period from semicolon in 1st edition.

[14] Comma added.

[15] This *reads* $n = r\sqrt{\frac{(A-B)(B-C)}{AB}}$ in the 1st edition. But from the analysis in article 216, the first factor in the numerator under the root should be $(A-C)$ not $(A-B)$.

[16] As in the 1st edition (see preceding note.)

[17] The first right hand term in this expression *reads* $-g \sin q\ \mathrm{S} \cdot z'dm$ in 1st edition.

BOOK I

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CHAPTER VI

ON THE EQUILIBRIUM OF FLUIDS

Definitions, &c.

225. A FLUID is a mass of particles, which yield to the slightest pressure, and transmit that pressure in every direction.

226. Mobility of the particles constitutes the difference between fluids and solids.

227. There are, indeed, fluids in nature whose particles adhere more or less to each other, called viscous fluids; but those only whose particles do not adhere in any degree, but possess perfect mobility, are the subject of this investigation.

228. Strictly speaking, all fluids are compressible, for even liquids under very great pressure change their volume; but as the compression is insensible in ordinary circumstances, fluids of perfect mobility are divided into compressible or elastic fluids, and incompressible.

229. The elastic and compressible fluids are atmospheric air, the gases, and steam. When compressed, these fluids change both form and volume, and regain their primitive state as soon as the pressure is removed. Some of the gases are found to differ from atmospheric air in losing their elastic form, and becoming liquid when compressed to a certain degree, as lately proved by Mr. Faraday,[1] [2] and steam is reduced to water when its temperature is diminished; but atmospheric air, and others of the gases, always retain their gaseous form, whatever the degree of pressure may be.

230. It is impossible to ascertain the forms of the particles of fluids, but as all of them, considered in mass, afford the same phenomena, it can have no influence on the laws of their motions.

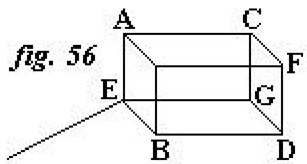
Equilibrium of Fluids

231. When a fluid is in equilibrio, each particle must itself be held in equilibrio by the forces acting upon it, together with the pressures of the surrounding particles.

232. It is evident, that whatever the accelerating forces or pressures may be, they can all be resolved into component forces parallel to three rectangular co-ordinates, *ox*, *oy*, *oz*.

Equation of Equilibrium

233. Imagine a system of fluid particles, forming a rectangular parallelopipedon[3] A B C D, fig. 56, and suppose its sides parallel to the co-ordinate axes. Suppose also, that it is pressed on all sides by the surrounding fluid, at the same time that it is urged by accelerating forces.



234. It is evident, that the pressure on the face A B, must be in a contrary direction to the pressure on the face C D; hence the mass will be urged by the difference of these pressures: but this difference may be considered as a single force acting either on the face A B, or C D; consequently the difference of the pressures multiplied by the very small area A B will be the whole pressure, urging the mass parallel to the side E G. In the same manner, the pressures urging the mass in a direction parallel to E B and E A, are the area E C into the difference of the pressures on the faces E C and B F; and the area E D into the difference of the pressures on E D and A F.

235. Because the mass is indefinitely small, if x, y, z, be the coordinates of E, the edges E G, E B, E A, may be represented by dx, dy, dz. Then p being the pressure on a unit of surface, $pdydz$ will be the pressure on the face A B, in the direction E G. At G, x becomes $x+dx$, y and z remaining the same; hence as p is considered a function of x, y, z, it becomes

$$p' = p + \left(\frac{dp}{dx}\right)dx \text{ at the point G};$$

hence

$$p - p' = -\left(\frac{dp}{dx}\right)dx,$$

and

$$pdydz - p'dydz = -\left(\frac{dp}{dx}\right)dx \cdot dydz.$$

Now $pdydz$ is the pressure on A B, and $p'dydz$ is the pressure on C D; hence

$$-\left(\frac{dp}{dx}\right)dx \cdot dydz = (p - p')\,dydz$$

is the difference of the pressures on the faces A B and C D. In the same manner it may be proved that

$$-\left(\frac{dp}{dy}\right)dy \,.\, dxdz, \text{ and } -\left(\frac{dp}{dz}\right)dz \cdot dydx$$

are the differences of the pressures on the faces B F, A G, and on E D, A F.

236. But if X, Y, Z, be the accelerating forces in the direction of the axes, when multiplied by the volume $dx\,dy\,dz$, and by r its density, they become the momenta

$$r\,.\,\mathrm{X}dxdydz,$$
$$r\,.\,\mathrm{Y}dxdydz,$$
$$r\,.\,\mathrm{Z}dxdydz.$$

But these momenta must balance the pressures in the same directions when the fluid mass is in equilibrio; hence, by the principle of virtual velocities

$$\left\{r\mathrm{X}-\frac{dp}{dx}\right\}dx+\left\{r\mathrm{Y}-\frac{dp}{dy}\right\}dy+\left\{r\mathrm{Z}-\frac{dp}{dz}\right\}dz=0,$$

or

$$\frac{dp}{dx}dx+\frac{dp}{dy}dy+\frac{dp}{dz}dz=r\left\{\mathrm{X}dx+\mathrm{Y}dy+\mathrm{Z}dz\right\}.$$

As the variations are arbitrary, they may be made equal to the differentials, and then

$$dp=r\left\{\mathrm{X}dx+\mathrm{Y}dy+\mathrm{Z}dz\right\} \qquad \textbf{(52)}$$

is the general equation of the equilibrium of fluids, whether elastic or incompressible. It shows, that the indefinitely small increment of the pressure is equal to the density of the fluid mass multiplied by the sum of the products of each force by the element of its direction.

237. This equation will not give the equilibrium of a fluid under all circumstances, for it is evident that in many cases equilibrium is impossible; but when the accelerating forces are attractive forces directed to fixed centres, it furnishes another equation, which shows the relation that must exist among the component forces, in order that equilibrium may be possible at all. It is called an equation of condition, because it expresses the general condition requisite for the existence of equilibrium.

Equations of Condition

238. Assuming the forces X, Y, Z, to be functions of the distance, by article 75[4] the second member or the preceding equation is an exact differential; and as p is a function of x, y, z, it gives the partial equations

$$\frac{dp}{dx}=r\mathrm{X};\ \frac{dp}{dy}=r\mathrm{Y};\ \frac{dp}{dz}=r\mathrm{Z};$$

but the differential of the first, according to y, is

$$\frac{d^2 p}{dxdy} = \frac{d\,.\,r\text{X}}{dy}$$

and the differential of the second, according to x, is

$$\frac{d^2 p}{dydx} = \frac{d\,.\,r\text{Y}}{dx};$$

hence

$$\frac{d\,.\,r\text{X}}{dy} = \frac{d\,.\,r\text{Y}}{dx}.$$

By a similar process, it will be found that

$$\frac{d\,.\,r\text{Y}}{dz} = \frac{d\,.\,r\text{Z}}{dy}; \quad \frac{d\,.\,r\text{X}}{dz} = \frac{d\,.\,r\text{Z}}{dx}.$$

These three equations of condition are necessary, in order that the equation (52) may be an exact differential, and consequently integrable. If the differentials of these three equations be taken, the sum of the first multiplied by Z, of the second multiplied by X, and of the third multiplied by $-\text{Y}$, will be

$$0 = \text{X}\cdot\frac{d\text{Y}}{dz} - \text{Y}\cdot\frac{d\text{X}}{dz} + \text{Z}\cdot\frac{d\text{X}}{dy} - \text{X}\cdot\frac{d\text{Z}}{dy} + \text{Y}\cdot\frac{d\text{Z}}{dx} - \text{Z}\cdot\frac{d\text{Y}}{dx}$$

an equation expressing the relation that must exist among the forces X, Y, Z, in order that equilibrium may be possible.

Equilibrium will always be possible when these conditions are fulfilled; but the exterior figure of the mass must also be determined.

Equilibrium of homogeneous Fluids

239. If the fluid be free at its surface, the pressure must be zero in every point of the surface when the mass is in equilibrio; so that $p = 0$, and

$$r\left\{\text{X}dx + \text{Y}dy + \text{Z}dz\right\} = 0,$$

whence

$$\int\left(\text{X}dx + \text{Y}dy + \text{Z}dz\right) = \text{constant},$$

supposing it an exact differential, the density being constant.

The resulting force on each particle must be directed to the interior of the fluid mass, and must be perpendicular to the surface; for were it not, it might be resolved into two others, one perpendicular, and one horizontal; and in consequence of the latter, the particle would slide along the surface.

If $u = 0$ be the equation of the surface, by article 69 the equation of equilibrium at the surface will be

$$\mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz = l\,du\,,$$

l being a function x, y, z; and by the same article, the resultant of the forces X, Y, Z, must be perpendicular to those parts of the surface where the fluid is free, and the first member must be an exact differential.

Equilibrium of heterogeneous Fluids

240. When the fluid mass is heterogeneous, and when the forces are attractive, and their intensities functions of the distances of the points of application from their origin, then the density depends on the pressure; and all the strata or layers of a fluid mass in which the pressure is the same, have the same density throughout their whole extent.

Demonstration. Let the function

$$\mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz$$

be an exact difference, which by article 75 will always be the case when the forces X, Y, Z, are attractive, and their intensities functions of the mutual distances of the particles. Assume

$$f = \int\left(\mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz\right), \qquad \textbf{(53)}$$

f being a function of x, y, z; then equation (52) becomes

$$dp = r\,.df\,. \qquad \textbf{(54)}$$

The first member of this equation is an exact differential, and in order that the second member may also be an exact differential, the density r must be a function of f. The pressure p will then be a function of f also; and the equation of the free surface of the fluid will be $f = \text{constant quantity},$ as in the case of homogeneity. Thus the pressure and the density are the same for all the points of the same layer. The law of the variation of the density in passing from one layer to another depends on the function in f which expresses it. And when that function is given, the pressure will be obtained by integrating the equation $dp = r\,df$.

241. It appears from the preceding investigation, that a homogeneous liquid will remain in equilibrio, if all its particles act on each other, and are attracted towards any number of fixed centres; but in that case, the resulting force must be perpendicular to the surface of the liquid, and must tend to its interior. If there be but one force or attraction directed to a fixed point, the mass would become a sphere, having that point in its centre, whatever the law of the force might be.

242. When the centre of the attractive force is at an infinite distance, its direction becomes parallel throughout the whole extent of the fluid mass; and the surface, when in equilibrio, is a plane perpendicular to the direction of the force. The surface of a small extent of stagnant water may be estimated plane, but when it is of great extent, its surface exhibits the curvature of the earth.

243. A fluid mass that is not homogeneous but free at its surface will be in equilibrio, if the density be uniform throughout each indefinitely small layer or stratum of the mass, and if the resultant of all the accelerating forces acting on the surface be perpendicular to it, and tending towards the interior. If the upper strata of the fluid be most dense, the equilibrium will be unstable; if the heaviest is undermost, it will be stable.

244. If a fixed solid of any form be covered by fluid as the earth is by the atmosphere, it is requisite for the equilibrium of the fluid that the intensity of the attractive forces should depend on their distances from fixed centres, and that the resulting force of all the forces which act at the exterior surface should be perpendicular to it, and directed towards the interior.

245. If the surface of an elastic fluid be free, the pressure cannot be zero till the density be zero; hence an elastic fluid cannot be in equilibrio unless it be either shut up in a close vessel, or, like the atmosphere, it extend in space till its density becomes insensible.

Equilibrium of Fluids in Rotation

246. Hitherto the fluid mass has been considered to be at rest; but suppose it to have a uniform motion of rotation about a fixed axis, as for example the axis *oz*. Let w be the velocity of rotation common to all the particles of the fluid, and r the distance of a particle *dm* from the axis of rotation, the co-ordinates of *dm* being x, y, z. Then wr will be the velocity of *dm*, and its centrifugal force resulting from rotation, will be w^2r, which must therefore be added to the accelerating forces which urge the particle; hence equation (53) will be

$$df = \mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz + w^2 rdr\,.$$

And the differential equation of the strata, and of the free surface of the fluid, will be

$$\mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz + w^2 .\, rdr = 0\,. \qquad \textbf{(55)}$$

The centrifugal force, therefore, does not prevent the function f from being an exact differential, consequently equilibrium will be possible, provided the condition of article 238 be fulfilled.

247. The regularity of gravitation at the surface of the earth; the increase of density towards its centre; and, above all, the correspondence of the form of the earth and planets with that of a fluid mass in rotation, have led to the supposition that these bodies may have been originally fluid, and that their parts, in consolidating, have retained nearly the form they would have acquired from their mutual attractions, together with the centrifugal force induced by rotation when fluid. In this case, the laws expressed by the preceding equations must have regulated their formation.

Notes

[1] Faraday, Michael, 1791-1867, chemist, experimental physicist, and natural philosopher, born in Newington Butts, England. Faraday's research included work on the condensation of gases, the conservation of force, and studies on benzene and steel. His major work is the series of *Experimental Researches on Electricity* (1839-55), in which he reports discoveries about electricity, electrolysis, and relationships between electricity and magnetism. Faraday and Somerville corresponded frequently. In the first edition of her *On the Connexion of the Physical Sciences* (see note 39, *Foreword to the Second Edition*) Mary Somerville was one of the first to report on Faraday's recent researches on the voltaic pile. Faraday reviewed the sheets of Somerville's draft manuscript and sent the following response: (see also next note.)

"*…I cannot resist saying too what pleasure I feel in your approbation of my later Experimental Researches. The approval of one judge is to me more stimulating than the applause of thousands that* <u>*cannot*</u> *understand the subject*." ." Dep c.370, 20, MSF-1: Michael Faraday to Mary Somerville, 1 March 1834, Mary Somerville Collection, Bodleian Library, Oxford University.

In her first edition Somerville had adopted Faraday's ideas about electricity but continued to use an older vocabulary. Somerville revised the vocabulary in her second edition in 1835 and incorporated materials on Faraday's research in five topic areas: definite proportions, atomic weights, definite proportions of electricity (Faraday's laws of electricity), crystallization, and the density of the atmosphere. (Patterson, Elizabeth, *Mary Somerville and the Cultivation of Science, 1815-1840*, Mattinus Nijhoff Publishers, p.144, 1983)

[2] Mary Somerville's experimental work included a series of studies on the permeability of various bodies to the chemical rays of the sun. She carried out this work in her garden at Chelsea. The work involved an application of a type of primitive photography whereby light in passing through different media produced discolorations on paper coated with silver chloride. The media Somerville used included various coloured glasses, clear and coloured mica, emeralds, garnets, beryls, tourmalines, and rock crystal . On 12 October, 1835 Michael Faraday wrote to Somerville:

"*I have been making some experiments with the papers but do not succeed in obtaining so good & regular a result as I wished & believed I might obtain.*

In the first place the precipitates made upon the paper are not so sensible or regular as that first formed & washed & applied in the usual way, the excess of the muriate or nitrate used & the resulting salt formed interfering with action of light by retarding more or less the change and that in an irregular manner. Chloride produced on the paper is therefore nothing like so regular in its change as chloride previously precipitated & well washed.

In the next place I do not find that I can lay a more regular coat of the substance in the method I mentioned than by using the moist precipitated chloride & a camel hair pencil.

I suspect your chloride is a good deal [illegible]. I will therefore precipitate & wash some and send it to you in the moist state. Allow me to suggest that when you refer to and apply it to paper for your experiments you do so in a dark place, or by candlelight only & thus you may keep it for a long time in good condition.

I send also Biot's report for your inspection." Dep c.370, 20, MSF-1: Michael Faraday to Mary Somerville, 12 Oct. 1835, Mary Somerville Collection, Bodleian Library, Oxford University.

Somerville was investigating whether the "chemical" solar rays, which were known to blacken silver and fade vegetable colours, displayed analogous activity in passing through various media. The then accepted solar spectrum consisted of five overlapping spectra: calorific (i.e. infrared), red, yellow, blue, and chemical (i.e. ultraviolet). Somerville's experimental work always exhibited careful design, economy, attention to the control of various parameters and the maintenance of standard controls (Patterson, Elizabeth, *Mary Somerville and the Cultivation of Science, 1815-1840*, Mattinus Nijhoff Publishers, p.173, 1983.)

[3] *parallelopipedon.* A parallelopiped, *Webster's Revised Unabridged, 1913.*

[4] Corrected from 1st edition which *reads* "…by article 75. The second member…"

BOOK I

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CHAPTER VII

MOTION OF FLUIDS

General Equation of the Motion of Fluids

248. THE MASS of a fluid particle being $r\,dxdydz$, its momentum in the axis x arising from the accelerating forces is, by article 144,

$$\left\{\mathrm{X}-\frac{d^2x}{dt^2}\right\}r\,dxdydz\,.$$

But the pressure resolved in the same direction is

$$\left(\frac{dp}{dx}\right)dxdydz\,.$$

Consequently the equation of the motion of a fluid mass in the axis ox, when free, is

$$\left\{\mathrm{X}-\frac{d^2x}{dt^2}\right\}r-\frac{dp}{dx}=0\,. \qquad \textbf{(56)}$$

In the same manner its motions in the axes y and z are[1]

$$\left\{\mathrm{X}-\frac{d^2y}{dt^2}\right\}r-\frac{dp}{dy}=0\,,$$
$$\left\{\mathrm{X}-\frac{d^2z}{dt^2}\right\}r-\frac{dp}{dz}=0\,. \qquad \textbf{(56)}$$

And by the principle of virtual velocities the general equation of fluids in motion is

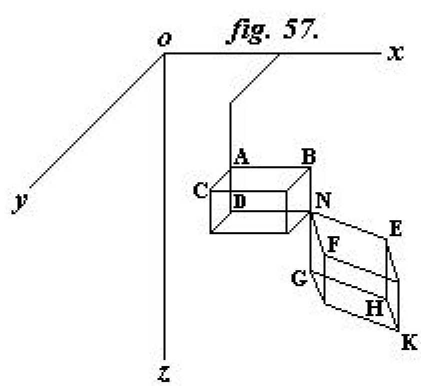
$$\left\{\mathrm{X}dx+\mathrm{Y}dy+\mathrm{Z}dz\right\}-\frac{dp}{r}=\frac{d^2x}{dt^2}dx+\frac{d^2y}{dt^2}dy+\frac{d^2z}{dt^2}dz\,. \qquad \textbf{(57)}$$

This equation is not rigorously true, because it is formed in the hypothesis of the pressures being equal on all sides of a particle in motion, which Poisson[2] has proved not to be the case; but, as far as concerns the following analysis, the effect of the inequality of pressure is insensible.

249. The preceding equation, however, does not express all the circumstances of the motion of a fluid. Another equation is requisite.

A solid always preserves the same form whatever its motion may be, which is by no means the case with fluids; for a mass ABCD, fig. 57, formed of particles possessing perfect mobility, changes its form by the action of the forces, so that it always continues to fit into the intervals of the surrounding molecules without leaving any void. In this consists the continuity of fluids, a property which furnishes the other equation necessary for the determination of their motions.

Equation of Continuity



250. Suppose at any given time the form of a very small fluid mass to be that of a rectangular parallelopiped ABCD, fig. 57. The action of the forces will change it into an oblique figure NEFK, during the indefinitely small time that it moves from its first to its second position. Now NEFG may differ from ABCD both in form and density, but not in mass; for if the density depends on the pressure, the same forces that change the form may also produce a change in the pressure, and, consequently, in the density; but it is evident that the mass must always remain the same, for the number of molecules in ABCD can neither be increased nor diminished by the action of the forces; hence the volume of ABCD into its primitive density must still be equal to [the] volume of NEFG into the new density; hence, if

$$r\, dxdydz\ ,$$

be the mass of ABCD, the equation of continuity will be

$$d\ .\ r\ dx\ dy\ dz = 0\ . \qquad \textbf{(58)}$$

251. This equation, together with equations (56), will determine the four unknown quantities x, y, z, and p, in functions of the time, and consequently the motion of the fluid.

Development of the Equation of Continuity

252. The sides of the small parallelopiped, after the time dt, become

$$dx + d\ .\ dx,\ dy + d\ .\ dy,\ dz + d\ .\ dz\ ;$$

or, observing that the variation of dx only arises from the increase of x, the co-ordinates y and z remaining the same, and that the variations of dy, dz, arise only from the similar increments of y and z; hence the edges of the new mass are

$$\text{NE} = dx\left(1+\frac{d^2x}{dx}\right)$$
$$\text{NG} = dy\left(1+\frac{d^2y}{dy}\right)$$
$$\text{NF} = dz\left(1+\frac{d^2z}{dz}\right)$$

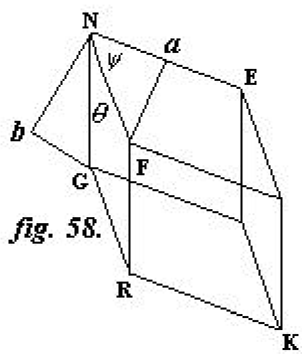


fig. 58.

If the angles GNF and FNE, fig. 58, be represented by q and y, the volume of the parallelopiped NK will be

$$\text{NE}\,.\,\text{NG}\sin q\,.\,\text{NF}\sin y\,;$$

for

$$\text{F}a = \text{NF}.\sin y$$
$$\text{N}b = \text{NG}.\sin q\,,$$

Fa, Nb being at right angles to NE and RG; but as q and y were right angles in the primitive volume, they could only vary by indefinitely small arcs in the time dt; hence in the new volume

$$q = 90^\circ \pm dq\,,\; y = 90^\circ \pm dy\,,$$

consequently,

$$\sin q = \sin\left(90^\circ \pm dq\right) = \cos dq = 1-\tfrac{1}{2}dq^2 + \&c.$$
$$\sin y = \sin\left(90^\circ \pm dy\right) = \cos dy = 1-\tfrac{1}{2}dy^2 + \&c.$$

and omitting

$$dq^2,\;\; dy^2,\;\; \sin q = \sin y = 1\,,$$

and the volume becomes $\text{NE}\,.\,\text{NG}\,.\,\text{NF}$; substituting for the three edges their preceding values, and omitting indefinitely small quantities of the fifth order, the volume after the time dt is

$$dxdydz\left\{1+\frac{d^2x}{dx}+\frac{d^2y}{dy}+\frac{d^2z}{dz}\right\}.$$

The density varies both with the time and space; hence r, the primitive density, is a function of t, x, y and z, and after the time dt, it is

$$r + \frac{dr}{dt}dt + \frac{dr}{dx}dx + \frac{dr}{dy}dy + \frac{dr}{dz}dz\ ;$$

consequently, the mass, being the product of the volume and density, is, after the time dt, equal to 3

$$dm = r \cdot dxdydz\left(1 + \frac{dr}{dt}dt + \frac{dr}{dx}dx + \frac{dr}{dy}dy + \frac{dr}{dz}dz + r\frac{d^2x}{dx} + r\frac{d^2y}{dy} + r\frac{d^2z}{dz}\right).$$

And the equation

$$d \,.\, r \,.\, dxdydz = 0$$

becomes

$$\frac{dr}{dt} + \frac{d \cdot r\dfrac{dx}{dt}}{dx} + \frac{d \cdot r\dfrac{dy}{dt}}{dy} + \frac{d \cdot r\dfrac{dz}{dt}}{dz} = 0 \qquad \textbf{(59)}$$

as will readily appear by developing this quantity, which is the general equation of continuity.

253. The equations (56) and (59) determine the motions both of incompressible and elastic fluids.

254. When the fluid is incompressible, both the volume and density remain the same during the whole motion; therefore the increments of these quantities are zero; hence, with regard to the volume

$$\frac{d^2x}{dx} + \frac{d^2x}{dx} + \frac{d^2x}{dx} = 0\ ; \qquad \textbf{(60)}$$

and with regard to the density,

$$\frac{dr}{dt} + \frac{dr}{dx}dx + \frac{dr}{dy}d + \frac{dr}{dz}dz = 0\,. \qquad \textbf{(61)}$$

255. By means of these two equations and the three equations (56), the five unknown quantities p, r, x, y, and z, may be determined in functions of t, which remains arbitrary; and therefore all the circumstances of the motion of the fluid mass may be known for any assumed time.

256. If the fluid be both incompressible and homogeneous, the density is constant, therefore $dr = 0$, and as the last equation becomes identical, the motion of the fluid is obtained from the other four.

Second form of the Equation of the Motions of Fluids

257. It is occasionally more convenient to regard x, y, z, the co-ordinates of the fluid particle dm, as known quantities, and

$$\frac{dx}{dt},\ \frac{dy}{dt},\ \frac{dz}{dt},$$

its velocities in the direction of the co-ordinates, as unknown. In order to transform the equations (56) and (59) to suit this case, let

$$s=\frac{dx}{dt},\ u=\frac{dy}{dt},\ v=\frac{dz}{dt};$$

these quantities being functions of x, y, z, and t. The differentials of these equations when x, y, z, and t vary all at once; and when sdt, udt, vdt, are put for dx, dy, dz, become

$$\begin{aligned} ds &= \frac{ds}{dt}dt+\frac{ds}{dx}\cdot sdt+\frac{ds}{dy}\cdot udt+\frac{ds}{dz}\cdot vdt\,,\\ du &= \frac{du}{dt}dt+\frac{du}{dx}\cdot sdt+\frac{du}{dy}\cdot udt+\frac{du}{dz}\cdot vdt\,,\\ dv &= \frac{dv}{dt}dt+\frac{dv}{dx}\cdot sdt+\frac{dv}{dy}\cdot udt+\frac{dv}{dz}\cdot vdt\,. \end{aligned} \tag{62}$$

And as

$$ds=\frac{d^2x}{dt},\ du=\frac{d^2y}{dt},\ dv=\frac{d^2z}{dt},$$

the equations (56) become, by the substitution of the preceding quantities,

$$\begin{aligned} \frac{dp}{dx} &= r\left\{\mathrm{X}-\frac{ds}{dt}-\frac{ds}{dx}\cdot s-\frac{ds}{dy}\cdot u-\frac{ds}{dz}\cdot v\right\}\\ \frac{dp}{dy} &= r\left\{\mathrm{Y}-\frac{du}{dt}-\frac{du}{dx}\cdot s-\frac{du}{dy}\cdot u-\frac{du}{dz}\cdot v\right\}\\ \frac{dp}{dz} &= r\left\{\mathrm{Z}-\frac{dv}{dt}-\frac{dv}{dx}\cdot s-\frac{dv}{dy}\cdot u-\frac{dv}{dz}\cdot v\right\} \end{aligned} \tag{63}$$

and by the same substitution, the equation (59) of continuity becomes

$$\frac{dr}{dt}+\frac{d\,.\,rs}{dx}+\frac{d\,.\,ru}{dy}+\frac{d\,.\,rv}{dz}=0\,, \tag{64}$$

which, for incompressible and homogeneous fluids, is

$$\frac{ds}{dx}+\frac{du}{dy}+\frac{dv}{dz}=0. \qquad \textbf{(65)}$$

The equations (63) and (64) will determine s, u, and v, in functions of x, y, z, t, and then the equations

$$dx = sdt \quad dy = udt \quad dz = vdt$$

will give x, y, z, in functions of the time. The whole circumstances of the fluid mass will therefore be known.

Integration of the Equations of the Motions of Fluids

258. The great difficulty in the theory of the motion of fluids, consists in the integration of the partial equations (63) and (64), which has not yet been surmounted, even in the most simple problems. It way, however, be effected in a very extensive case, in which

$$sdx + udy + vdx$$

is a complete differential of a function f, of the three variable quantities x, y, z; so that

$$sdx + udy + vdz = df\ .$$

259. If in the equation (57) the variations which are arbitrary, be made equal to the differentials of the same quantities; and if, as in nature, the accelerating forces X, Y, Z be functions of the distance, then $\mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz$ will be a complete differential, and may be expressed by dV, so that the equation in question becomes

$$\frac{dp}{r} = d\mathrm{V} - dx\cdot\frac{d^2x}{dt^2} - dy\cdot\frac{d^2y}{dt^2} - dz\cdot\frac{d^2z}{dt^2} \qquad \textbf{(66)}$$

But the function f gives the velocities of the fluid mass in the directions of the axes, *viz*.

$$s=\frac{df}{dx},\ u=\frac{df}{dy},\ v=\frac{df}{dz}.$$

By the substitution of these values in equation (62), ds, du, dv, and consequently

$$\frac{d^2x}{dt},\quad \frac{d^2y}{dt},\quad \frac{d^2z}{dt},$$

will be obtained in functions of f, by which the preceding equation becomes[4]

$$\frac{dp}{r} = d\mathrm{V} - \frac{ds}{dt}\cdot dx - \frac{du}{dt}\cdot dy - \frac{dv}{dt}\cdot dz - \tfrac{1}{2}d\left(\frac{df^2}{dx^2} + \frac{df^2}{dy^2} + \frac{df^2}{dz^2}\right).$$

Now

$$\frac{ds}{dt}\cdot dx + \frac{du}{dt}\cdot dy + \frac{dv}{dt}\cdot dz = d\cdot\frac{df}{dt};$$

consequently,[5]

$$\int\frac{dp}{r} = \mathrm{V} - \frac{df}{dt} - \tfrac{1}{2}\left(\frac{df^2}{dx^2} + \frac{df^2}{dy^2} + \frac{df^2}{dz^2}\right). \qquad \textbf{(67)}$$

The constant quantity introduced by integration is included in the function f. By the same substitution, the equation of continuity becomes

$$\frac{dr}{dt} + \frac{d\,.\,r\,\frac{df}{dx}}{dx} + \frac{d\cdot r\,\frac{df}{dy}}{dy} + \frac{d\,.\,r\,\frac{df}{dz}}{dz} = 0\,. \qquad \textbf{(68)}$$

The two last equations determine the motion of the fluid mass in the case under consideration.

260. It is impossible to know all the cases in which the function $sdx+udy+vdz$ is an exact differential, but it may be proved that if it be so at any one instant, it will be an exact differential during the whole motion of a fluid.

Demonstration. Suppose that at any one instant it is a complete differential, it will then be integrable, and may be expressed by df; in the following instant it will become[6]

$$df + \frac{ds}{dt}\cdot dx + \frac{du}{dt}\cdot dy + \frac{dv}{dt}\cdot dz\,.$$

It will still be an exact differential, if

$$\frac{ds}{dt}dx + \frac{du}{dt}dy + \frac{dv}{dt}dz \text{ be one.}$$

Now the latter quantity being equal to $d\cdot\frac{df}{dt}$, equation (67) gives[7]

$$\frac{ds}{dt}dx + \frac{du}{dt}dy + \frac{dv}{dt}dz = d\mathrm{V} - \frac{dp}{r} - \tfrac{1}{2}d\left(\frac{df^2}{dx^2} + \frac{df^2}{dy^2} + \frac{df^2}{dz^2}\right).$$

And if the density r be a function of p the pressure, the second member of this equation will be an exact differential, consequently the first member will be one also, and thus the function

$sdx+udy+vdz$ is a complete differential in the second instant, if it be one in the first; it will therefore be a complete differential during the whole motion of the fluid.

Theory of small Undulations of Fluids

261. If the oscillations of a fluid be very small, the squares and products of the velocities s, u, v, may be neglected: then the preceding equation becomes

$$d\mathrm{V} - \frac{dp}{r} = \frac{ds}{dt}dx + \frac{du}{dt}dy + \frac{dv}{dt}dz .$$

If r be a function of p, the first member will be a complete differential, therefore the second member, and consequently $sdx+udy+vdz$ is one also, so that the equation is capable of integration; and as its last member is equal to $d \cdot \frac{df}{dt}$, the integral is

$$\mathrm{V} - \int \frac{dp}{r} = \frac{df}{dt} . \qquad \textbf{(69)}$$

This equation, together with equation (68) of continuity, contain the whole theory of the small undulations of fluids.

262. An idea may be formed of these undulations by the effect of a stone dropped into still water; a series of small concentric circular waves will appear, extending from the point where the stone fell. If another stone be let fall very near the point where the first fell, a second series of concentric circular waves will be produced; but when the two series of undulations meet, they will cross, each series continuing its course independently of the other, the circles cutting each other in opposite points. An infinite number of such undulations may exist without disturbing the progress of one another. In sound, which is occasioned by undulations in the air, a similar effect is produced: in a chorus, the melody of one voice may be distinguished from the general harmony. Coexisting vibrations may also be excited in solid bodies, each undulation having its perfect effect, independently of the others. If the directions of the undulations coincide, their joint motions will be the sum or the difference of the separate motions, according as similar or dissimilar parts of the undulations are coincident. In undulations of equal frequency, when two series exactly coincide in point of time, the united velocity of the particular motions will be the greatest or least;–and if the undulations are of equal strength, they will totally destroy each other, when the time of the greatest direct motion of one undulation coincides with that of the greatest retrograde motion of the other.

The general principle of Interferences was first shown by Dr. Young[8] to be applicable to all vibratory motions, which he illustrated beautifully by the remarkable phenomena of two rays of light producing darkness, and the concurrence of two musical sounds producing silence. The first may be seen by looking at the flame of a candle through two extremely narrow parallel slits in a card; and the latter is rendered evident by what are termed beats in music.

The same principle serves to explain why neither flood nor ebb tides take place at Batsham in Tonquin on the day following the moon's passage across the equator; the flood tide arrives by one channel at the same instant that the ebb arrives by another, so that the interfering waves destroy each other.

Co-existing vibrations show the comprehensive nature and elegance of analytical formulae. The general equation of small undulations is the sum of an infinite number of equations, each of which gives a single series of undulations, like the surface of water in a shower, which at once contains an infinite number of undulations, and yet exhibits each independently of the rest.

Rotation of a Homogeneous Fluid

263. If a fluid mass rotates uniformly about an axis, its component velocity in the axis of rotation is zero; the velocities in the other two axes are angular velocities−independent of the time, the motion being uniform: indeed, the motion is the same with that of a solid body revolving about a fixed axis. If the mass revolves about the axis z, and if w be the angular velocity at the distance of unity from that axis, the component velocities will be

$$s=-wy,\quad u=wx,\quad v=0\,;$$

and from equations (63) it will be easily found that

$$\frac{dp}{r}=d\mathrm{V}+w^2\left(xdx+ydy\right);$$

and if r be constant, the integral is

$$\frac{p}{r}=\mathrm{V}+\frac{w^2}{2}\left(x^2+y^2\right).$$

The equation (65) of continuity will be satisfied, since

$$\frac{ds}{dx}=0,\quad \frac{du}{dy}=0,\quad \frac{dv}{dz}=0\,.$$

264. This motion of a fluid mass is therefore possible, although it is a case in which the function $sdx+udy+vdz$ is not an exact differential; for by the substitution of the preceding values of the velocities, it becomes

$$sdx+udy+vdz=w\left(xdy-ydx\right),$$

an expression that cannot be integrated. Therefore, in the theory of the tides caused by the disturbing action of the sun and moon on the ocean, the function $sdx+udy+vdz$ must not be regarded as an exact differential, since it cannot be integrated even when there is no disturbance in its rotatory motion.

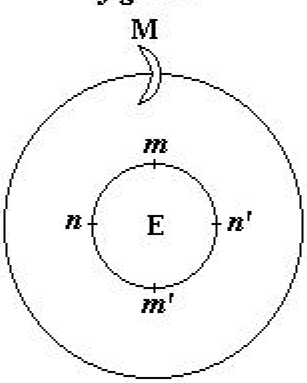
265. Thus a fluid mass or a fluid covering a solid of any form whatever, will rotate about an axis without alteration in the relative position of its particles. This would be the state of the ocean were the earth a solitary body, moving in space; but the attractions of the sun and moon not only trouble the ocean, but also cause commotions in the atmosphere, indicated by the periodic variations in the heights of the mercury in the barometer. From the vast distance of the sun and moon, their action upon the fluid particles of the ocean and atmosphere, is very small in comparison of that produced by the velocity of the earth's rotation, and by its attraction.

Determination of the Oscillations of a Homogeneous Fluid covering a Spheroid, the whole in rotation about an axis; supposing the fluid to be slightly deranged from its state of equilibrium by the action of very small forces

266. If the earth be supposed to rotate about its axis, uninfluenced by foreign forces, the fluids on its surface would assume a spheroidal form, from the centrifugal force induced by rotation; and a particle in the interior of the fluid would be subject to the action of gravitation and the pressure of the surrounding fluid only. But although the fluids would be moving with great velocity, yet to us they would seem at rest. When in this state the atmosphere and ocean are said to be in equilibrio.

Action of the Sun and Moon

fig. 59.



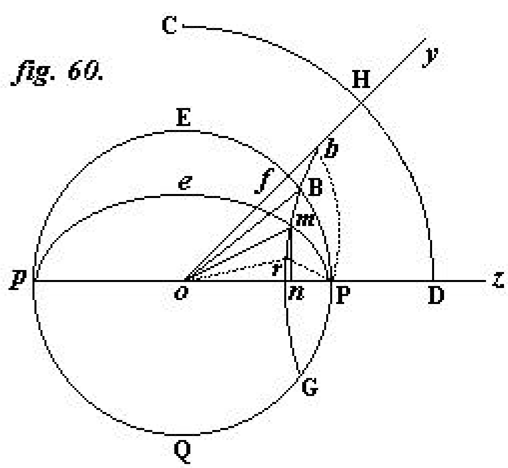
267. The action of the sun and moon troubles this equilibrium, and occasions tides in both fluids. The whole of this theory is perfectly general, but for the sake of illustration it will be considered with regard to the ocean. If the moon attracted the centre of gravity of the earth and all its particles with equal and parallel forces, the whole system of the earth and the waters that cover it, would yield to these forces with a common motion, and the equilibrium of the seas would remain undisturbed. The difference of the intensity and direction of the forces alone, trouble the equilibrium; for, since the attraction of the moon is inversely as the square of the distance, a molecule at m, under the moon M, is as much more attracted than the centre of gravity of the earth, as the square of EM is greater than the square of mM: hence the particle has a tendency to leave the earth, but is retained by gravitation, which this tendency diminishes. Twelve hours after, the particle is brought to m' by the rotation of the earth, and is then in opposition to the moon, which attracts it more feebly than it attracts the centre of the earth, in the ratio of the square of EM to the square of $m'\text{M}$. The surface of the earth has then a tendency to leave the particle, but the gravitation of the particle retains it; and gravitation is also in this case diminished by the action of the moon. Hence, when the particle is at m, the moon draws the particle from the earth; and when it is at m', it draws the earth from the particle: in both instances producing an elevation of the particle above the surface of equilibrium of nearly the same height, for the diminution of the gravitation in each position is almost the same on account of the distance of the moon being great in comparison of the radius of the earth. The action of the moon on a particle at *n*, 90° distant from *m*, may be resolved into two forces–one in the direction of the radius *n*E, and the other tangent to the surface. The latter force alone attracts the particle towards the moon, and makes it slide along the surface; so that there is a depression of the water in *n* and n', at the same time that it is high water at *m* and m'. It is evident that, after half a day, the particle, when at n', will be acted on by the same force it experienced at *n*.

268. Were the earth entirely covered by the sea, the water thus attracted by the moon would assume the form of an oblong spheroid, whose greater axis would point towards the moon; since the column of water under the moon, and the direction diametrically opposite to her, would be rendered lighter in consequence of the diminution of their gravitation: and in order to preserve the equilibrium, the axis 90° distant would be shortened. The elevation, on account of the smaller space to which it is confined, is twice as great as the depression, because the contents of the spheroid always remain the same. If the waters were capable of instantly assuming the form of a spheroid, its summit would always be directed towards the moon, notwithstanding the earth's rotation; but on account of their resistance, the rapid motion of rotation prevents them from assuming at every instant the form which the equilibrium of the forces acting on them requires, so that they are constantly approaching to, and receding from that figure, which is therefore called the *momentary equilibrium* of the fluid. It is evident that the action, and consequently the position of the sun modifies these circumstances, but the action of that body is incomparably less than that of the moon.

Determination of the general Equation of the Oscillations of all parts of the Fluids covering the Earth

269. Let *p*EPQ, fig. 60, be the terrestrial spheroid, E*o* the equatorial radius, P*p* the axis of rotation. Suppose the spheroid to be entirely covered with the fluid–the ocean, for example; and let *pe*P, or *p*EP, represent the bottom of the sea, CD its surface, PD its depth; also let *o* be the centre of the spheroid and origin of the co-ordinates, and *om* the radius. Imagine *m* to be a fluid particle at any point below the surface of the fluid–at the bottom, for example. It is evident that this particle, moved by rotation alone, would be carried to B without changing its distance from the centre of the spheroid, or from the axis of rotation; so that the arcs P*m*, PB, are equal to each other, as also the radii *om*, *o*B. If *pe*P be assumed as a given meridian, the origin of the time, and *g* the first point in Aries,[9] then *g*PB is the longitude of the particle when arrived at B, and E*o*B is its latitude.

fig. 60.



Now, if the disturbing forces were to act on the particle during its rotation from *m* to B, they would cause it to move to *b*, some point not far from B. By the disturbing forces alone, the longitude of the particle at B would be increased by the very small angle BP*b*; the latitude would be diminished by the very small angle B*ob*, and its distance from the centre of the spheroid increased by *fb*. The angle $g\,\mathrm{PB}$ is the rotation of the earth, and any may be represented by $nt+v$, since it is proportional to the time, (by Article 213); but in the time *t*, the disturbing forces bring the particle to *b*: therefore the angle $nt+v$ must be increased by BP*b* or v. Hence

$$g\,\mathrm{P}b = nt + v \;+ v\,.$$

Again, if q be the component of the latitude E*o*B, and *u*, its very small increment, B*ob*, the angle

$$\mathrm{P}o\mathrm{B} = q + u\,.$$

In the same manner, if *s* be the increment of the radius *r*, then

$$ob = r + s\,.$$

Hence the co-ordinates of the particle at *b* are,

$$x = (r+s)\cos(q+u)\,,$$
$$y = (r+s)\sin(q+u)\cos(nt+v\;+v)\,,$$
$$z = (r+s)\sin(q+u)\sin(nt+v\;+v)\,.$$

270. *v* and *u* very nearly represent the motion of the particle in longitude and latitude estimated from the terrestrial meridian P*ep*. These are so small, compared with *nt* the rotatory motion of the earth, that their squares may be omitted. But although the lateral motions *v*, *u* of the particle be very small, they are much greater than *s*, the increase in the length of the radius.

271. If these values of *x*, *y*, *z*, be substituted in (57) the general equation of the motion of fluids; and if to abridge

$$\mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz = dV\,,$$

then[10]

$$\begin{aligned} & r^2 dq \left\{ \frac{d^2u}{dt^2} - 2n\sin q\,\cos q\left(\frac{dv}{dt}\right)\right\} \\ & + r^2 dv\,\left\{\sin^2 q\left(\frac{d^2v}{dt^2}\right) + 2n\sin q\,\cos q\left(\frac{du}{dt}\right) + \frac{2n\sin^2 q}{r}\left(\frac{ds}{dt}\right)\right\} \\ & + dr\left\{\left(\frac{d^2s}{dt^2}\right) - 2nr\sin^2 q\left(\frac{dv}{dt}\right)\right\} \\ & = \frac{n^2}{2}d\left\{(r+s)\sin(q+u)\right\}^2 + dV - \frac{dp}{r}\,, \end{aligned} \qquad \textbf{(70)}$$

will determine the oscillations of a particle in the interior of the fluid when troubled by the action of the sun and moon. This equation, however, requires modification for a particle at the surface.

Equation at the Surface

272. If DH, fig. 60, be the surface of the sea undisturbed in its rotation, the particle at B will only be affected by gravitation and the pressure of the surrounding fluid; but when by the action of the sun and moon the tide rises to *y*, and the particle under consideration is brought to *b*, the forces which there act upon it will be gravitation, the pressure of the surrounding fluid, the action of the sun and moon, and the pressure of the small column of water between H and *y*. But the gravitation acting on the particle at *b* is also different from that which affects it when at B, for *b* being farther from the centre of gravity of the system of the earth and its fluids, the gravity of the particle at *b* must be less than at B, consequently the centrifugal force must be greater: the direction of gravitation is also different at the points B and *b*.

273. In order to obtain an equation for the motion of a particle at the surface of the fluid, suppose it to be in a state of momentary equilibrium, then as the differentials of v, u, and s, express the oscillations of the fluid about that state, they must be zero, which reduces the preceding equation to

$$\frac{n^2}{2} d\left\{(r+s)\sin(q+u)\right\}^2 + (dV) = 0; \qquad \textbf{(71)}$$

for as the pressure at the surface is zero, $dp = 0$, and (dV) represents the value of dV corresponding to that state. Thus in a state of momentary equilibrium, the forces (dV), and the centrifugal force balance each other.

274. Now dV is the sum of all the forces acting on the particle when troubled in its rotation into the elements of their directions, it must therefore be equal to (dV), the same sum suited to a state of momentary equilibrium, together with those forces which urge the particle when it oscillates about that state, into the elements of their directions. But these are evidently the variation in the weight of the little column of water H*y*, and a quantity which may be represented by dV', depending on the difference in the direction and intensity of gravity at the two points B and *b*, caused by the change in the situation of the attracting mass in the state of motion, and by the attraction of the sun and moon.

275. The force of gravity at *y* is so nearly the same with that at the surface of the earth, that the difference may be neglected; and if *y* be the height of the little column of fluid H*y*, its weight will be *gy* when the sea is in a state of momentary equilibrium; when it oscillates about that state; the variation in its weight will be gdy, g being the force of gravity; but as the pressure of this small column is directed towards the origin of the co-ordinates and tends to lessen them, it must have a negative sign. Hence in a state of motion,

$$dV = (dV) + dV' - gdy,$$

whence

$$(dV) = dV - dV' + gd\,y\,.$$

276. When the fluid is in momentary equilibrio, the centrifugal force is

$$\frac{n^2}{2}\{(r+s)\sin(q+u)\}^2;$$

but it must vary with $d\,y$, the elevation of the particle above the surface of momentary equilibrium. The direction Hy does not coincide with that of the terrestrial radius, except at the equator and pole, on account of the spheroidal form of the earth; but as the compression of the earth is very small, these directions may be esteemed the same in the present case without sensible error; therefore $r+s-y$ may be regarded as the value of the radius at y. Consequently

$$-d\,y\,.\,rn^2\sin^2 q$$

is the variation of the centrifugal force corresponding to the increased height of the particle; and when compared with $-gd\,y$ the gravity of this little column, it is of the order $\frac{n^2 r}{g}$, the same with the ratio of the centrifugal force to gravity at the equator, or to $\frac{1}{288}$, and therefore may be omitted; hence equation (71) becomes

$$dV - dV' + gd\,y + \frac{n^2}{2}d\{(r+s)\sin(q+u)\}^2 = 0\,.$$

277. As the surface of the sea differs very little from that of a sphere, $d\,r$ may be omitted; consequently if

$$\frac{n^2}{2}d\{(r+s)\sin(q+u)\}^2$$

be eliminated from equation (70), the result will be

$$r^2 dq\left\{\left(\frac{d^2u}{dt^2}\right) - 2n\sin q\cos q\left(\frac{dv}{dt}\right)\right\}$$
$$+r^2 dv\left\{\sin^2 q\left(\frac{d^2v}{dt^2}\right) + 2n\sin q\cos q\left(\frac{du}{dt}\right) + 2n\sin^2 q\left(\frac{ds}{dt}\right)\right\} = -gd\,y + dV', \qquad \textbf{(72)}$$

which is the equation of the motion of a particle at the surface of the sea. The variations $d\,y$, dV' correspond to the two variables q and v.

278. To complete the theory of the motions of the atmosphere and ocean, the equation of the continuity of the fluid must now be found.

Continuity of Fluids

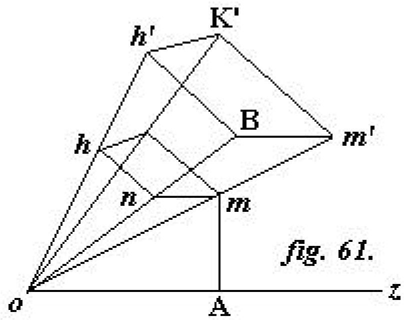


fig. 61.

Suppose $m'h$, fig. 61, to be an indefinitely small rectangular portion of the fluid mass, situate at B, fig. 60, and suppose the solid to be formed by the imaginary rotation of the area $\mathrm{B}nhh'$ about the axis *oz*; the centre of gravity of $\mathrm{B}nhh'$ will describe an arc, which on account of the smallness of the solid, may without sensible error be represented by mn, its radius being $m\mathrm{A}$; hence the arc mn is $m\mathrm{A}\times mn$. Now the area $\mathrm{B}nhh'$ multiplied by $m\mathrm{A}\times mn$, is equal to the solid $m'h$, supposing it indefinitely small and rectangular.

The colatitude of the point B or $\mathrm{A}om = q$, the longitude of B is $nt + v$, then the indefinitely small increments of these angles are $m'o\mathrm{K}' = dq$, $m'o\mathrm{B} = dv$, for as the figure is independent of the time, *nt* is constant. Hence if the radii *o*B, *on*, be represented by r' and r, the sectors $\mathrm{B}oh'$, noh, are $r'^2 dq$ and $r^2 dq$; hence the area

$$\mathrm{B}nhh' = \frac{\left(r'^2 - r^2\right)}{2} dq = \frac{\left(r' + r\right)\left(r' - r\right)}{2} dq\ .$$

But as the thickness is indefinitely small,

$$r' + r = 2r,\ r' - r = dr\ ;$$

therefore the area

$$\mathrm{B}nhh' = rdr\,.\,dq\ .$$

Again,

$$\mathrm{A}m = r\sin q\ ,$$

consequently,

$$\mathrm{A}m\,.\,mn = rdv\ \sin q\ ,$$

and thus the volume

$$m'h = r^2 drdq\,dv\ \sin q\ ;$$

and if r be the density,

$$dm = r\,r^2 drdq\,dv\ \sin q\ .$$

But in consequence of the disturbing forces, r, q, and v, become $r + s$, $q + u$, $v + v$, after the time *t*, and dr, dq, dv, become[11]

$$dr + \frac{ds}{dr}\cdot dr,\ dq + \frac{du}{dq}\cdot dq,\ dv + \frac{dv}{dv}\cdot dv\ ;$$

also the density is changed to $r + r'$. If these values be put in the preceding expression for the solid *dm*, it becomes after the time *t* equal to

$$(r + r')(r + s^2)\left(1 + \frac{ds}{dr}\right)\left(1 + \frac{du}{dq}\right)\left(1 + \frac{dv}{dv}\right) dr dq dv \ \sin(q + u),$$

but this must be equal to the original mass; hence

$$(r + r')(r + s^2)\left(1 + \frac{ds}{dr}\right)\left(1 + \frac{du}{dq}\right)\left(1 + \frac{dv}{dv}\right)\sin(q + u) = r r^2 \sin q\ .$$

If the squares and products of

$$s,\ \frac{ds}{dr},\ \frac{du}{dq},\ \frac{dv}{dv}$$

be omitted, and observing that

$$2rs + r^2\frac{ds}{dr} = \frac{d\cdot r^2 s}{dr},$$

and

$$\sin(q + u) = \sin q + u\cos q\ ;$$

for as *u* is very small, the arc may be put for the sine, and unity for the cosine, the equation of the continuity of the fluid is[12]

$$0 = r^2(r + r')\left\{\left(\frac{du}{dq}\right) + \left(\frac{dv}{dv}\right) + \frac{u\cos q}{\sin q}\right\} + r\left(\frac{d\cdot r^2 s}{dr}\right), \qquad \textbf{(73)}$$

expressed in polar co-ordinates.

279. The equations (70), (72), and (73), are perfectly general; and therefore will answer either for the oscillations of the ocean or atmosphere.

Oscillations of the Ocean

280. The density of the sea is constant, therefore $r' = 0$; hence the equation of continuity becomes

$$0=\left(\frac{d\,.r^2 s}{dr}\right)+r^2\left\{\left(\frac{du}{dq}\right)+\left(\frac{dv}{dv}\right)+\frac{u\cos q}{\sin q}\right\}.$$

In order to find the integral of this equation with regard to r only, it may be assumed, that all the particles that are on any one radius at the origin of the time, will remain on the same radius during the motion; therefore r, v, and u will be nearly the same on the small part of the terrestrial radius between the bottom and surface of the sea; hence, the integral will be

$$0=r^2s-\left(r^2s\right)+r^2g\left\{\left(\frac{du}{dq}\right)+\frac{dv}{dv}+\frac{u\cos q}{\sin q}\right\}$$

(r^2s) is the value of r^2s at the surface of the sea, but the change in the radius of the earth between the bottom and surface of the sea is so small, that $r^2(s)$ may be put for (r^2s); then dividing the whole by r^2, and neglecting the terms $\frac{2g(s)}{r}$, which is the ratio of the depth of the sea to the terrestrial radius, and therefore very small, the mean depth even of the Pacific ocean being only about four miles, whereas the mean radius of the earth is nearly 4,000 miles; the preceding equation becomes

$$0=s-(s)+g\left\{\left(\frac{du}{dq}\right)+\left(\frac{dv}{dv}\right)+\frac{u\cos q}{\sin q}\right\}. \qquad \textbf{(74)}$$

Now $g+s-(s)$ is the whole depth of the sea from the bottom to the highest point to which the tides rise at its surface of momentary equilibrium; and g varies with the angles v and q; hence at the surface of equilibrium, it becomes

$$g+u\frac{du}{dq}+v\frac{dg}{dv};$$

and as y is the height of a particle above the surface of equilibrium, it follows that

$$g+s-(s)=-y+g+u\frac{dg}{dq}+v\frac{dg}{dv},$$

or

$$s-(s)=-y+u\frac{dg}{dq}+v\frac{dg}{dv}.$$

Whence the equation of continuity becomes

$$y=-\frac{d\left(gu\right)}{dq}-\frac{d\left(gv\right)}{dv}-\frac{gu\cos q}{\sin q}. \qquad \textbf{(75)}$$

281. In order to apply the other equations to the motion of the sea, it must be observed that a fluid particle at the bottom of the sea would in its rotation from m to B always touch the spheroid, which is nearly a sphere; therefore the value of s would be very small for that particle, and would be to v, u, of the order of the eccentricity of the spheroid, to its mean radius taken as unity; consequently at the bottom of the sea, s may be omitted in comparison of u, v. But it appears from equations (74), that $s-(s)$ is a function of u and v, independent of r, when the very small quantity $\frac{2g(s)}{r}$ is omitted: hence s is the same throughout every part of the radius r, as it is at the bottom, and may therefore be omitted throughout the whole depth, when compared with u and v, so that equation (72) of the surface of the fluid is reduced to[13]

$$r^2 dq\left\{\left(\frac{d^2u}{dt^2}\right)-2n\sin q\cos q\left(\frac{dv}{dt}\right)\right\} \tag{76}$$

$$+r^2 dv\left\{\sin^2 q\left(\frac{d^2v}{dt^2}\right)+2n\sin q\cos q\left(\frac{du}{dt}\right)\right\}=-gdy+dV',$$

282. When the fluid mass is in momentary equilibrium, the equation for the motion of a particle in the interior of the fluid becomes

$$0=\tfrac{1}{2}n^2 d\left\{(r+s)\sin(q+u)\right\}^2+(dV)-\frac{(dp)}{r},$$

where (dV), (dp), are the values of dV and dp suited to that state. But we may suppose that in a state of motion,

$$dV=(dV)+dV', \text{ and } dp=(dp)+dp';$$

whence

$$(dV)=dV-dV',\ (dp)=dp-dp',$$

and

$$\tfrac{1}{2}n^2 d\left\{(r+s)\sin(q+u)\right\}^2=dV'-dV+\frac{dp}{r}-\frac{dp'}{r}.$$

283. If the first member of this expression be eliminated from equation (70), with regard to the independent variation of r alone, it gives

$$\frac{d\left(V'-\frac{p'}{r}\right)}{dr}=\left(\frac{d^2s}{dt^2}\right)-2nr\sin^2 q\left(\frac{dv}{dt}\right). \tag{77}$$

284. Now $n\left(\frac{dv}{dt}\right)$ is of the order y, s, or $\frac{gs}{r}$; for if the coefficients of dq, dv, be each made zero in equation (76), it will give

$$r^2\left(\frac{d^2u}{dt^2}\right) - 2nr^2 \sin q \cos q \left(\frac{dv}{dt}\right) = -g\left(\frac{dy}{dq}\right) + \left(\frac{dV'}{dq}\right),$$

$$r^2 \sin^2 q \left(\frac{d^2v}{dt^2}\right) + 2nr^2 \sin q \cos q \left(\frac{du}{dt}\right) = -g\left(\frac{dy}{dv}\right) + \left(\frac{dV'}{dv}\right);$$

add the differential of the last equation relative to t, to the first equation multiplied by

$$-2n \sin q \cos q$$

and let the second member of this equation be represented by

$$y' . r^2 \sin^2 q \ ,$$

then divide by

$$r^2 \sin^2 q \ ,$$

and put

$$2n \cos q = a,$$

and there will be found the linear equation

$$\left(\frac{d^2v}{dt^2}\right) + a^2\left(\frac{dv}{dt}\right) = y' .$$

The value of $\frac{dv}{dt}$ obtained from the integral of this equation will be a function of y', and as y' is a function of y and V', each of which is of the order s or $\frac{gs}{r}$, $\frac{dv}{dt}$; consequently

$$\frac{d\left(V' - \frac{p'}{r}\right)}{dr}$$

is of the same order. If then equation (77), be multiplied by dr its integral will be

$$V' - \frac{p'}{r} = \int dr \left\{ \left(\frac{d^2s}{dt^2}\right) - 2nr \sin^2 q \left(\frac{dv}{dt}\right) \right\} + l \ .$$

285. Since this equation has been integrated with regard to *r* only, l must be a function of q, v, and *t*, independent of *r*, according to the theory of partial equations. And as the function in *r* is of the order $\frac{gs}{r}$ it may be omitted; and then

$$dV' - \frac{dp'}{r} = dl\ ,$$

by which equation (70) becomes[14]

$$r^2 dq\left\{\left(\frac{d^2u}{dt^2}\right) - 2n\sin q\,\cos q\left(\frac{dV}{dt}\right)\right\}$$
$$+ r^2 dv\left\{\sin^2 q\left(\frac{d^2v}{dt^2}\right) + 2n\sin q\,\cos q\left(\frac{du}{dt}\right)\right\} = dl\ .$$

286. But as dl does not contain *r*, *s*, or *y*, it is independent of the depth of the particle; hence this equation is the same for a particle at the surface, or in its neighbourhood, consequently it must coincide with equation (76); and therefore

$$dl = dV' - gdy\ .$$

287. Thus it appears, that the whole theory of the tides would be determined if integrals of the equations

$$r^2 dq\left\{\left(\frac{d^2u}{dt^2}\right) - 2n\sin q\,\cos q\left(\frac{dv}{dt}\right)\right\}$$
$$+ r^2 dv\left\{\sin^2 q\left(\frac{d^2v}{dt^2}\right) + 2n\sin q\,\cos q\left(\frac{du}{dt}\right)\right\} = -gdy + dV'$$
$$y = -\frac{d(gu)}{dq} - \frac{d(gv)}{dv} - \frac{gu\cos q}{\sin q}$$

could be found, for the horizontal flow might be obtained from the first, by making the coefficients of the independent quantities dq, dv, separately zero, then the height to which they rise would be found from the second. This has not yet been done, as none of the known methods of analysis have hitherto succeeded.

288. These equations have been formed on the hypothesis of the earth being entirely covered by the sea; hence the integrals, if they could be found, would be inadequate to determine the oscillations of the ocean retarded or accelerated by the continents, islands, and innumerable other causes, beyond the reach of analysis. No attempt is therefore made to integrate the equations; but the theory of the tides is determined by comparing the general relations which subsist between the observed phenomena and the causes which produce them.

289. In order to integrate the equation of continuity, it was assumed that if the angles P*ob*, *m*P*b*, [fig. 60] or rather

$$u,\ \frac{du}{dt},\ v,\ \frac{dv}{dt},$$

be the same for every particle situate on the same radius throughout the whole depth of the sea at the beginning of the motion, they will always continue to be the same for that set of particles during their motion, therefore all the fluid particles that are at the same instant on any one radius, will continue very nearly on that radius during the oscillations of the fluid. Were this rigorously true, the horizontal flow of the tides would be isochronous, like the oscillations of a pendulum, and their velocity would be inversely as their depth, provided the particles had no motion in latitude; and it may be nearly so in the Pacific, whose mean depth is about four miles, and where the tides only rise to about five feet; but it is very far from being the case in shallow seas, and on the coasts where the tides are high; because the condition of isochronism depends on the omission of quantities of the order of the ratio of the height of the tides to the depth of the sea.

290. The reaction of the sea on the terrestrial spheroid is so small that it is omitted. The common centre of gravity of the spheroid and sea is not changed by this reaction, and therefore the ratio of the action of the sea on the spheroid, is to the reaction of the spheroid on the sea, as the mass of the sea to the solid mass; that is, as the depth of the sea to the radius of the earth, or at most as 1 to 1000, assuming the mean depth of the sea to be four miles. For that reason u, v, express the true velocity of the tides in longitude and latitude, as they were assumed to be.

On the Atmosphere

291. Experience shows the atmosphere to be an elastic fluid, whose density increases in proportion to the pressure. It is subject to changes of density from the variation of temperature in different latitudes, at different heights, and from various other causes; but in this investigation the temperature is assumed to be constant.

292. Since the air resists compression equally in all directions, the height of the atmosphere must be unlimited if its atoms be infinitely divisible. Some considerations, however, induced Dr. Wollaston[15] to suppose that the earth's atmosphere is of finite extent, limited by the weight of ultimate atoms of definite magnitude, no longer divisible by repulsion of their parts. But whether the particles of the atmosphere be infinitely divisible or not, all phenomena concur in proving its density to be quite insensible at the height of about fifty miles.

Density of the Atmosphere

293. The law by which the density of the air diminishes as the height above the surface of the sea increases, will appear by considering ρ, ρ', ρ'', to be the densities of three contiguous strata of air, the thickness of each being so small that the density may be assumed uniform throughout each stratum. Let p be the pressure of the superincumbent air on the lowest stratum, p' the pressure on the next, and p'' the pressure on the third; and let m be a coefficient, such that $r = a\,p$. Then, because the densities are as the pressures,

$$r' = a\,p', \text{ and } r'' = a\,p''.$$

Hence,[16]

$$r - r' = a\,(p - p') \text{ and } r' - r'' = a\,(p' - p'').$$

But $p - p'$ is equal to the weight of the first of these strata, and $p' - p''$ is equal to that of the second: hence

$$r - r' : r' - r'' :: r : r';$$

consequently

$$r\,r'' = r'^2.$$

The density of the middle stratum is therefore a mean proportional between the densities of the other two; and whatever be the number of equidistant strata, their densities are in continual proportion.

294. If the heights therefore, from the surface of the sea, be taken in an increasing arithmetical progression, the densities of the strata of air will increase in geometrical progression, a property that logarithms possess relatively to their numbers.

295. All the circumstances both of the equilibrium and motion of the atmosphere may be determined from equation (70), if the quantities it contains be supposed relative to that compressible fluid instead of to the ocean.

Equilibrium of the Atmosphere

296. When the atmosphere is in equilibrio v, u, and s are zero, which reduces equation (70) to

$$\frac{n^2}{2} \cdot r^2 \,.\, \sin^2 q + V - \int \frac{d\,p}{r} = \text{constant}.$$

Suppose the atmosphere to be every where of the same density as at the surface of the sea, let h be the height of that atmosphere which is very small, not exceeding $5\frac{1}{2}$ miles, and let g be the force of gravity at the equator; then as the pressure is proportional to the density, $p = h \,.\, g \,.\, r$, and[17]

$$\int \frac{dp}{r} = hg\,.\log r \ ,$$

consequently the preceding equation becomes

$$hg\,.\log r = \text{constant}+V + \frac{n^2}{2}\cdot r^2\ .\sin^2 q\ .$$

At the surface of the sea, V is the same for a particle of air, and for the particle of the ocean adjacent to it; but when the sea is in equilibrio

$$V + \frac{n^2}{2}\cdot r^2\ .\sin^2 q = \text{constant}\ ,$$

therefore r is constant, and consequently the stratum of air contiguous to the sea is every where of the same density.

297. Since the earth is very nearly spherical, it may be assumed that r the distance of a particle of air from its centre is equal to $R+r'$, R being the terrestrial radius extending to the surface of the sea, and r' the height of the particle above that surface. V, which relates to the surface of the sea, becomes at the height r';

$$V' = V + r'\left(\frac{dV}{dr}\right) + \&c.$$

by Taylor's theorem,[18] consequently the substitution of $R+r'$ for r in the value of[19] $hg\,.\log r$ gives

$$hg\cdot\log r = \text{constant}+V + r'\left(\frac{dV}{dr}\right) + \frac{r'^2}{2}\left(\frac{d^2V}{dr^2}\right) + \frac{n^2}{2}\cdot R^2\cdot\sin^2 q + n^2\cdot Rr'\cdot\sin^2 q$$

V, $\left(\frac{dV}{dr}\right)$, &c. relate to the surface of the sea where

$$V + \frac{n^2}{2}\cdot R^2\,.\sin^2 q = \text{constant}\,,$$

and as

$$-\left(\frac{dV}{dr}\right) - n^2\cdot R\cdot\sin^2 q\ ,$$

is the effect of gravitation at that surface, it may be represented by g', whence

$$hg \cdot \log r = \text{constant} - r'g' + \frac{r'^2}{2} \cdot \left(\frac{d^2V}{dr^2}\right).$$

298. Since $\left(\frac{d^2V}{dr^2}\right)$ is multiplied by the very small quantity r'^2, it may be integrated in the hypothesis of the earth being a sphere; but in that case

$$-\left(\frac{dV}{dr}\right) = g' - \frac{m}{R^2}$$

m being the mass of the earth; hence

$$\left(\frac{d^2V}{dr^2}\right) = -\frac{2m}{R^2} = -\frac{2g'}{R};$$

consequently the preceding equation becomes

$$\log r = -\frac{r'}{h} \cdot \frac{g'}{g}\left(1 + \frac{r'}{R}\right);$$

whence

$$r = r' . c^{-\frac{r'g'}{hg} \cdot \left(1 + \frac{r'}{R}\right)};$$

an equation which determines the density of the atmosphere at any given height above the level of the sea; c is the number whose logarithm is unity, and r' a constant quantity equal to the density of the atmosphere at the surface of the sea.

299. If g' and g be the force of gravity at the equator and in any other latitude, they will be proportional to l' and l, the lengths of the pendulum beating seconds at the level of the sea in these two places; hence l' and l, which are known by experiment, may be substituted for g' and g, and the formula becomes[20]

$$r = r' . c^{-\frac{r'l'}{hl} \cdot \left(1 + \frac{r'}{R}\right)}. \qquad \textbf{(78)}$$

Whence it appears that strata of the same density are every where very nearly equally elevated above the surface of the sea.

300. By this expression the density of the air at any height may be found, say at fifty-five miles. $\frac{r'}{R}$ is very small and may be neglected; and l may be made equal to l' without sensible error; hence

$$r = r'c^{-\frac{r'}{h}}.$$

Now the height of an atmosphere of uniform density is only about $h = 5\frac{1}{2}$ English miles; hence if

$$r' = 10h = 55,\ r = r'c^{-10},$$

and as[21]

$$c = 2.71828,\ r = \frac{r'}{22{,}026},$$

so that the density at the height of 55 English miles is extremely small, which corresponds with what was said in article 292.

301. From the same formula the height of any place above the level of the sea may be found; for the densities r' and r, and consequently h, are given by the height of the barometer, l' and l, the lengths of the seconds' pendulum for any latitude are known by experiment; and R, the radius of the earth is also a given quantity; hence r' may be found. But in estimating the heights of mountains by the barometer, the variation of gravity at the height r' above the level of the sea cannot be omitted, therefore $\frac{l'-l}{l'}r'$ must be included in the preceding formula.

Oscillations of the Atmosphere

302. The atmosphere has the form of an ellipsoid flattened at the poles, in consequence of its rotation with the earth, and its strata by article 299, are everywhere of the same density at the same elevation above the surface of the sea. The attraction of the sun and moon occasions tides in the atmosphere perfectly similar to those of the ocean; however, they are probably affected by the rise and the fall of the sea.

303. The motion of the atmosphere is determined by equations (70), (73), which give the tides of the ocean, with the exception of a small change owing to the elasticity of the air; hence the term $\frac{dp}{r}$, expressing the ratio of the pressure to the density cannot be omitted as it was in the case of the sea.

Let $r = (r) + r'$; (r) being the density of the stratum in equilibrio, and r' the change suited to a state of motion; hence

$$p = hg\left((r) + r'\right),$$

and

$$\frac{dp}{r} = hg\frac{d(r)}{(r)} + g\frac{d(hr')}{(r)}.$$

Let

$$\frac{hr'}{(r)} = y',$$

then

$$\frac{dp}{r} = hg\frac{d(r)}{(r)} + gdy' .$$

304. The part $hg\frac{d(r)}{(r)}$ vanishes, because in equilibrio

$$\frac{n^2}{2}d\{(r+s)\sin(q+u)\}^2 + (dV) - hg\frac{d(r)}{(r)} = 0,$$

therefore

$$\frac{dp}{r} = gdy' .$$

Let f be the elevation of a particle of air above the surface of equilibrium of the atmosphere which corresponds with y, the elevation of a particle of water above the surface of equilibrium of the sea. Now at the sea $f = y,$ for the adjacent particles of air and water are subject to the same forces; but it is necessary to examine whether the supposition of $f = y,$ and of y being constant for all the particles of air situate on the same radius are consistent with the equation of continuity (73), which for the atmosphere is

$$0 = r^2\left\{r' + (r)\left\{\left(\frac{du}{dq}\right) + \left(\frac{dv}{dv}\right) + \frac{u\cdot\cos q}{\sin q}\right\}\right\} + (r)\cdot\left(\frac{d\cdot r^2 s}{dr}\right).$$

If the value of $\frac{r'}{(r)}$ from this equation be substituted in $\frac{hr'}{(r)} = y',$ it becomes[22]

$$y' = -h\cdot\left\{\left(\frac{d\,.r^2 s}{r^2\,.dr}\right) + \left(\frac{du}{dq}\right) + \left(\frac{dv}{dv}\right) + \frac{u\,.\cos q}{\sin q}\right\}.$$

The part of s that depends on the variation of the angles q and v is so small, that it may be neglected, consequently $s = f;$ and if $f = y$ then $\left(\frac{df}{dr}\right) = 0$, since[23] the value of f is the same for all the particles situate on the same radius. Also y is of the order h or $\frac{n^2}{g};$ consequently[24]

$$y' = -h\cdot\left\{\left(\frac{du}{dq}\right) + \left(\frac{dv}{dv}\right) + \frac{u\cdot\cos q}{\sin q}\right\}, \qquad \textbf{(79)}$$

then u and v being the same for all the particles situate primitively on the same radius, the value of y' will be the same for all these particles, and as quantities of the order ds are omitted, equation (70) becomes

$$r^2 dq \left\{ \left(\frac{d^2u}{dt^2} \right) - 2n \sin q \cos q \left(\frac{dv}{dt} \right) \right\}$$
$$+ r^2 dv \left\{ \sin^2 q \left(\frac{d^2v}{dt^2} \right) + 2n \sin q \cos q \left(\frac{du}{dt} \right) \right\} \qquad \textbf{(80)}$$
$$= dV - g\,dy' - g\,dy\,.$$

Thus the equations that determine the oscillations of the atmosphere only differ from those that give the tides by the small quantity $g\,dy'$, depending on the elasticity of the air.

305. Finite values of the equations of the motion of the atmosphere cannot be obtained; therefore the ebb and flow of the atmosphere may be determined in the same manner as the tides of the ocean, by estimating the effects of the sun and moon separately. This can only be effected by a comparison of numerous observations.

Oscillations of the Mercury in the Barometer

306. Oscillations in the atmosphere cause analogous oscillations in the barometer. For suppose a barometer to be fixed at any height above the surface of the sea, the height of the mercury is proportional to the pressure on that part of its surface that is exposed to the action of the air. As the atmosphere rises and falls by the action of the disturbing forces like the waves of the sea, the surface of the mercury is alternately more or less pressed by the variable mass of the atmosphere above it. Hence the density of the air at the surface of the mercury varies for two reasons; first, because it belonged to a stratum which was less elevated in a state of equilibrium by the quantity *y*, and secondly, because the density of a stratum is augmented when in motion by the quantity $\frac{(r)}{h} \cdot y$. Now if *h* be the height of the atmosphere in equilibrio when its density is uniform, and equal to (r), then

$$h : y :: (r) : y \cdot \frac{(r)}{h},$$

the increase of density in a state of motion from the first cause. In the same manner, $y' \cdot \frac{(r)}{h}$ is the increase of density from the second cause. Thus the whole increase is

$$(r) \frac{(y' + y)}{h}.$$

And if *H* be the height of the mercury in the barometer when the atmosphere is in equilibrio, its oscillations when in motion will be expressed by

$$H\frac{(y'+y)}{h}. \qquad \textbf{(81)}$$

The oscillations of the mercury are therefore similar at all heights above the level of the sea, and proportional in their extent to the height of' the barometer.

Conclusion

307. The account of the first book of the *Mécanique Céleste*[25] is thus brought to a conclusion. Arduous as the study of it may seem, the approach in every science, necessarily consisting in elementary principles, must be tedious: but let it not be forgotten, that many important truths, coeval[26] with the existence of matter itself, have already been developed; and that the subsequent application of the principles which have been established, will lead to the contemplation of the most sublime works of the Creator. The general equation of motion has been formed according to the primordial laws of matter; and the universal application of this one equation, to the motion of matter in every form of which it is susceptible, whether solid or fluid, to a single particle, or to a system of bodies, displays the essential nature of analysis, which comprehends every case that can result from a given law. It is not, indeed, surprising that our limited faculties do not enable us to derive general values of the unknown quantities from this equation: it has been accomplished, it is true, in a few cases, but we must be satisfied with approximate values in by much the greater number of instances. Several circumstances in the solar system materially facilitate the approximations; these Laplace[27] has selected with profound judgment, and employed with the greatest dexterity.

Notes

[1] Two equations numbered 56 occur in the 1st edition. We retain this numbering.

[2] Poisson, Siméon Denis, 1781-1840, mathematician, born in Pithiviers, France. From 1806 Poisson was a professor at the École Polytechnique. His contributions included work in electromagnetism and probability and a law governing the distribution of randomly occurring events (the Poisson distribution). He is known best for his work in celestial mechanics through his *Traité de mécanique* (1811 & 1833). Poisson extended the work of Lagrange (see note 16, *Preliminary Dissertation*) and Laplace (see note 4, *Foreword to the Second Edition*) on the stability of planetary orbits and calculated the attraction of both spheroids and ellipsoids. He was also known for his work on the expression of planetary gravitational force as a function of mass distribution within the planet.

[3] Punctuation changed from a semicolon in the 1st edition.

[4] Period added.

[5] Period added.

[6] Period added.

[7] Middle left hand term in 1st edition contains an error written $\frac{ds}{dt}dy$.

[8] See note 35, *Preliminary Dissertation.*

[9] *Aries.* In astronomy, an autumn zodiacal constellation located between Pisces and Taurus. The first point of Aries, or vernal equinox, is an intersection of the celestial equator with the apparent annual pathway of the Sun and the point in the sky from which celestial longitude and right ascension are measured.

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[10] The first term contains an unmatched bracket and *reads* $r^2 dq \left\{ \left(\frac{d^2u}{dt^2} - 2n\sin q \cos\left(\frac{dv}{dt}\right)\right\} \right.$ in the 1st edition.

[11] The first expression below *reads* $dr + \frac{ds}{dr}dr$ in the 1st edition.

[12] Equation (73) *reads* with an unmatched bracket as $0 = r^2 (r + r'\left(\left(\left(\frac{du}{dq}\right)+\left(\frac{dv}{dv}\right)+\frac{u\cos q}{\sin q}\right)+r\left(\frac{d\cdot r^2 s}{dr}\right)\right.$ in the 1st edition. The form used here is in conformity with the presentation in article 280.
[13] A comma after the first term below in the 1st edition is not used here.
[14] Comma after first term in original text is not used here.
[15] See note 62, *Preliminary Dissertation.*
[16] Error in 1st edition *reads* $r - r = a\,(p' - p'')$ for the second expression.

[17] Equation below *reads* $\int \frac{dp}{r} = hg \cdot \log\cdot r$ in the 1st edition.

[18] *Taylor series.* If f possesses derivatives of all orders at $x = c$ and is represented by $\sum_{n=0}^{\infty} a_n (x-c)^n$ in an interval of convergence of positive radius, then we must have $a_0 = f(c)$, and $a_n = \frac{f^{(n)}(c)}{n!}(n = 1,2,3,\ldots)$, so that

$$f(x) = f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \ldots$$

This is called the *formal Taylor series* for f in powers of $(x-c)$. Fobes and Smyth, *Calculus and Analytic Geometry*, V.2, Prentice Hall, 1963.
[19] This *reads* $hg \log.r$ in the 1st edition. The form we use is consistent with the use in the next equation.
[20] Punctuation added at end of equation.

[21] Comma separator added in denominator. The 1st edition reads $r = \frac{r'}{22026}$.

[22] Punctuation added at end of next equation.

[23] Punctuation changed here. The 1st edition reads: "…then $\left(\frac{df}{dr}\right) = 0$. Since …"

[24] Comma added at end of next equation.
[25] See note 4, *Introduction.*
[26] Coinciding in time of origin or existence; contemporary. *The Wordsmyth Educational Dictionary-Thesaurus.*
[27] See note 25 above.

Mercury

This photomosaic of Mercury is composed of images taken by the Mariner 10 as it flew by the planet after the first encounter in March 1974. The image shows the Caloris basin at the left of the terminator surrounded and filled by younger smooth plains deposits. This 1,300 km diameter impact basin formed about 4 billion years ago when a large asteroid or comet struck Mercury. The smooth plains resemble the lunar maria, the smooth, dark lava plains that are concentrated on the Moon's near side. However, the Mercurian plains display less contrast in reflectivity with heavily cratered terrain shown on the right, top and bottom than is seen between the lunar maria (dark) and the lunar highland (light). (Courtesy of NASA)

BOOK II - UNIVERSAL GRAVITATION

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FOREWORD[1]

Gravitation

IT has been proved by Newton, that a particle of matter placed without the surface of a hollow sphere is attracted by it in the same manner as if the mass of the hollow sphere, or the whole matter it contains, were collected into one dense particle in its centre. The same is therefore true of a solid sphere, which may be supposed to consist of an infinite number of concentric hollow spheres, or shells, arranged around the same centre, like the coats of an onion. This, however, is not the case with a spheroid;[2] but the celestial bodies are so nearly spherical, and at such remote distances from one another, that they attract and are attracted as if each were condensed into a single particle situate in its centre of gravity[3]–a circumstance which greatly facilitates the investigation of their motions.

Newton has shown that the force which retains the moon in her orbit is the same with that which causes heavy substances to fall at the surface of the earth. If the earth were a sphere, and at rest, a body would be equally attracted, that is, it would have the same weight at every point of its surface, because the surface of a sphere is everywhere equally distant from its centre. But, as our planet is flattened at the poles, and bulges at the equator, the weight of the same body gradually decreases from the poles, where it is greatest, to the equator, where it is least. There is, however, a certain mean[4] latitude, or part of the earth intermediate between the pole and the equator, where the attraction of the earth on bodies at its surface is the same as if it were a sphere;[5] and experience shows that bodies there fall through 16.0697 feet in a second. The mean distance[6] of the moon from the earth is about sixty times the mean radius[7] of the earth. When the number 16.0697 is diminished in the ratio (or relation) of 1 to 3600, which is the square of the moon's distance[8] from the earth's centre, estimated in terrestrial radii, it is found to be exactly

[1] The material in this and the forewords to Books I, III and IV is extracted from the 10th and last edition of Mary Somerville's *On the Connexion of the Physical Sciences*, (corrected and revised by Arabella B. Buckley), p. 4-106, London : John Murray, 1877.

[2] *spheroid.* A solid body which sometimes has the shape of an orange; it is then called an oblate spheroid, because it is flattened at the poles. Such is the form of the earth and the planets. When, on the contrary, it is drawn out at the poles like an egg, it is called a prolate spheroid. (Somerville's note.)

[3] *centre of gravity.* A point in every body, which if supported, the body will remain at rest in whatever position it may be placed. About that point all the parts exactly balance one another. (Somerville's note.)

[4] *Mean* quantities are such as are intermediate between others that are greater or less. (Somerville's note.)

[5] The attraction of a sphere on an external body is the same as if its mass were collected into one heavy particle at its centre of gravity, and the intensity of its attraction diminishes as the square of its distance from the external body increases. (Somerville's note.)

[6] The *mean distance* of a planet from the centre of the sun, or of a satellite from the centre of its planet, is equal to half the sum of its greatest and least distances, and consequently, is equal to half the major axis of its orbit. (Somerville's note.)

[7] The *mean radius of the earth* is the mean distance from the centre to the surface of the earth. It is intermediate between the distances measured from the pole to the centre and from the equator to the centre. (Somerville's note).

[8] In order to avoid large numbers, the mean radius of the earth is taken for unity: then the mean distance of the moon is expressed by 60; and the square of that number is 3600, or 60 times 60. (Somerville's note.)

the space the moon would fall through in the first second of her descent to the earth were she not prevented by the centrifugal force[9] arising from the velocity with which she moves in her orbit. The moon is thus retained in her orbit by a force having the same origin, and regulated by the same law, with that which causes a stone to fall at the earth's surface. The earth may, therefore, be regarded as the centre of a force which extends to the moon; and, as experience shows that the action and reaction[10] of matter are equal and contrary, the moon must attract the earth with an equal and contrary force. Nevertheless, since the earth's mass is so much greater than the mass of the moon, this equal force will only draw it over a comparatively small space.

Newton also ascertained that a body set in motion or projected in space, if attracted by a force proceeding from a fixed point, will move in a conic section[11] with an intensity inversely as the square of the distance;[12] but that any deviation from that law will cause it to move in a curve of a different nature. Kepler[13] found, by direct observation, that the planets describe ellipses, or oval paths, round the sun.[14] Later observations show that some comets also move in ellipses, but the greater part seem to move in parabolas, while others move in hyperbolas. All these are conic sections. It consequently follows that the sun attracts all the planets and comets inversely as the square of their distances from its centre; the sun, therefore, is the centre of a force extending indefinitely in space, and including all the bodies of the system in its action.

Kepler also deduced from observation that the squares of the periodic times of the planets, or the times of their revolutions round the sun, are proportional to the cubes of their mean distances from its centre. Hence the intensity of gravitation of all the bodies towards the sun is the same at equal distances; and if the planets and comets were at equal distances from the sun, and left to the effects of gravity, they would arrive at his surface at the same time.[15] The

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[9] *Centrifugal force.* A term formerly, but erroneously used to express the tendency of a revolving body to fly from the centre of attraction round which it revolves. The real explanation of this fact is found in Newton's law that *a body continues in its state of rest or uniform motion in a straight line, except in so far as it is compelled by impressed forces to change that state.* The true tendency of any body moving in space is therefore to proceed in a straight line, and it will continue to do so unless the direction of its motion is altered by the attraction of any other body. (Somerville's note.)

[10] *Action and reaction.* When motion is communicated by collision or pressure, the action of the body which strikes is returned with equal force by the body which receives the blow. (Somerville's note.)

[11] *Conic sections.* Lines formed by any plane cutting a cone. When the axis is perpendicular to the base, the solid is a right cone. If a right cone with a circular base be cut at right angles to the base by a plane passing through the apex, the section will be a triangle. If the cone be cut through both sides by a plane parallel to the base, the section will be a circle. If the cone be cut slanting quite through both sides the section will be an ellipse. If the plane be cut parallel to one of the sloping sides the section will be a parabola. And if the plane cut only one side of the cone, and be not parallel to the other, the section will be a hyperbola. Thus there are five conic sections. (Somerville's note.)

[12] *Inversely as the square of the distance.* The attraction of one body for another at the distance of two miles is four times less than at the distance of one mile; at three miles it is nine times less than at one; at four miles it is sixteen times less, and so on. That is, the gravitating force decreases in intensity as the squares of the distances increase. (Somerville's note.)

[13] See note 3, *Preliminary Dissertation.*

[14] This was the second of Kepler's three celebrated laws. The first law is, *That the radii vectores of the planets and comets describe areas proportional to the time.* The second law is, *That the orbits or paths of the planets and comets are conic sections, having the sun in one of their foci.* The third law is, *That the squares of the periodic times of the planets are proportional to the cubes of their mean distances from the sun.* (Somerville's note.)

[15] But for the resistance of the air, all bodies would fall to the ground in equal times. In fact, a hundred equal particles of matter at equal distances from the surface of the earth would fall to the ground in parallel straight lines with equal rapidity, and no change whatever would take place in the circumstances of their descent, if 99 of them were united in one solid mass; for the solid mass and the single particle would touch the ground at the same instant, were it not for the resistance of the air. (Somerville's note.)

satellites, such as our moon, also gravitate to their primaries, or the planets about which they revolve, according to the same law that their primaries do to the sun. Thus, by the law of action and reaction, each body is itself the centre of an attractive force extending indefinitely in space, causing all the mutual disturbances which render the celestial motions so complicated, and their investigation so difficult.

The gravitation of matter directed to a centre, and attracting directly as the mass (or the quantity of matter in a given bulk), and inversely as the square of the distance, does not belong to it when considered in mass only; particle acts on particle according to the same law when at sensible distances from each other. If the sun acted on the centre of the earth, without attracting each of its particles, the tides would be very much greater than they now are and would also, in other respects, be very different. The gravitation of the earth to the sun results from the gravitation of all its particles, which, in their turn, attract the sun in the ratio of their respective masses. There is a reciprocal action likewise between the earth and every particle at its surface. The earth and a feather mutually attract each other in the proportion of the mass of the earth to the mass of the feather. Were this not the case, and were any portion of the earth, however small, to attract another portion, and not be itself attracted, the centre of gravity of the earth would be moved in space by this action, which is impossible.

The forms of the planets result from the reciprocal attraction of their component particles. A detached fluid mass, if at rest, would assume the form of a sphere, from the reciprocal attraction of its particles. But if the mass revolve about an axis, it becomes flattened at the poles and bulges at the equator, in consequence of the motion imparted to each particle by the velocity of rotation; for this tendency to direct onward motion diminishes the gravity of the particles especially at the equator where the movement of rotation is most rapid, and equilibrium can only exist where it is balanced by an increase of gravity. Therefore, as the attractive force is the same on all particles at equal distances from the centre of a sphere, the equatorial particles would recede from the centre, till their increase in number by matter brought down from the poles produces a counterbalancing amount of attraction. Consequently, the sphere would become an oblate or flattened spheroid, and a fluid, partially or entirely covering a solid, as the ocean and atmosphere cover the earth, must assume that form in order to remain in equilibrio. The surface of the sea is, therefore, spheroidal, and the surface of the earth only deviates from that figure where it rises above or sinks below the level of the sea. But the deviation is so small, that it is unimportant when compared with the magnitude of the earth; for the mighty chain of the Andes, and the yet more lofty Himalaya, bear about the same proportion to the earth that a grain of sand does to a globe three feet in diameter. Such is the form of the earth and planets. The compression or flattening at their poles is, however, so small, that even Jupiter, whose rotation is the most rapid, and which is therefore the most elliptical of the planets, may, from his great distance, be regarded as spherical. Although the planets attract each other as if they were spheres, on account of their distances, yet the satellites are near enough to be sensibly affected in their motions by the forms of their primaries. The moon, for example, is so near the earth, that the reciprocal attraction between each of her particles, and each of the particles in the prominent mass at the terrestrial equator, occasions considerable disturbances in the motions of both bodies; for the action of the moon on the protuberant matter at the earth's equator produces a nutation, or nodding motion, in the earth's axis of rotation like that of a spinning-top when it is about to fall, and the reaction of that matter on the moon is the cause of a corresponding nutation in the lunar orbit.[16]

If a sphere at rest in space receive an impulse passing through its centre of gravity, all its parts will move with an equal velocity in a straight line; but, if the impulse does not pass through the centre of gravity, its particles, having unequal velocities, will have a rotatory or revolving motion, at the same time that it is translated or carried forward in space. These motions are independent of one another; so that a contrary impulse, passing through its centre of gravity, will impede its progress, without interfering with its rotation. The sun rotates about an axis, and modern observations show that an impulse in a contrary direction has not been given to its centre of gravity, for he moves in space accompanied by all those bodies which compose the solar system–a circumstance which in no way interferes with their relative motions; for, in consequence of the principle that force is proportional to velocity, the reciprocal attractions of a system remain the same whether its centre of gravity be at rest, or moving uniformly in space. It is computed that, had the earth received its motion from a single impulse, that impulse must have passed through a point about twenty-five miles from its centre.

Since the motions of rotation and translation of the planets are independent of each other, though probably communicated by the same impulse, they form separate subjects of investigation.

Elliptical Orbits

A planet moves in its elliptical orbit with a velocity varying every instant, in consequence of two forces, one tending to the centre of the sun, and the other in the direction of a tangent (*mT*, Fig. 78, Article 407) to its orbit, arising from the primitive impulse given at the time when it was launched into space. Should the onward motion of the planet cease, it would fall to the sun by its gravity. Were the sun not to attract it, the planet would fly off in the tangent. Thus, when the planet is at the point of its orbit furthest from the sun, its velocity gradually decreases till it is overcome by the attraction of the sun which brings it back with such an accelerated motion, that when nearest to the sun it overcomes his attraction, and, shooting past him, gradually decreases in velocity until it arrives at the most distant point, where the sun's attraction again prevails.[17] In this motion the *radii vectores* or imaginary lines joining the centres of the sun and the planets, pass over equal areas or spaces in equal times, as stated in Kepler's first law (see note 14).

The mean distance of a planet from the sun is equal to half the major axis of its orbit (PA, fig. 76, article 392), therefore the periodic time, or time of revolution of the planet round the sun, would be the same whether it moved in a circular or elliptical orbit, since the curves coincide at the extremities of the major axis. If the planet described a circle round the sun, its velocity or speed would be the same at all points in its orbit; whereas when moving in an ellipse, its elliptic or true motion will be continually varying, being either faster or slower than the circular or mean

[16] *Nutation of lunar orbit*. The action of the bulging matter at the earth's equator on the moon occasions a variation in the inclination of the lunar orbit to the plane of the ecliptic. (Somerville's note.)

[17] *Motion in an elliptical orbit*. A planet *m*, (fig. 76, article 392) moves round the sun at *S* in an ellipse P*m*AP, in consequence of two forces, one urging it in the direction of the tangent at *m*, and another pulling it towards the sun in the direction *mS*. Its velocity, which is greatest at P, decreases throughout the arc to A, where it is least, and increases continually as it moves along the arc till it comes to P again. The whole force producing the elliptical motion varies inversely as the square of the distance. (Somerville's note.)

motion at all points except the extremities of the major axis where the curves coincide.[18] As it is necessary to have some fixed point in the heavens from whence to estimate these motions, the vernal equinox (article 360) at a given epoch has been chosen. The equinoctial, which is a great circle traced in the starry heavens by the imaginary extension of the plane of the terrestrial equator, is intersected by the ecliptic, or apparent path of the sun, in two points diametrically opposite to one another, called the vernal and autumnal equinoxes. The vernal equinox is the point through which the sun passes in going from the southern to the northern hemisphere; and the autumnal, that in which he crosses from the northern to the southern. The mean or circular motion of a body, estimated from the vernal equinox, is its mean longitude (article 392); and its elliptical, or true motion, reckoned from that point, is its true longitude (article 392), both being estimated from west to east, the direction in which the bodies move. The difference between the two is called the equation of the centre (article 382), which consequently vanishes at the apsides,[19] or extremities of the major axis, and is at its maximum ninety degrees distant from these points, or in quadratures,[20] where it measures the eccentricity[21] of the orbit; so that the place of the planet in its elliptical orbit is obtained by adding or subtracting the equation of the centre to or from its mean longitude.

The orbits of the principal planets have a very small obliquity or inclination to the plane of the ecliptic in which the earth moves;[22] and, on that account, astronomers refer their motions to this plane at a given epoch as a known and fixed position. The angular distance of a planet from the plane of the ecliptic is its latitude,[23] which is south or north according as the planet is south or north of that plane. When the planet is in the plane of the ecliptic, its latitude is zero; it is then said to be in its nodes.[24] The ascending node is that point in the ecliptic through which the planet passes in going from the southern to the northern hemisphere. The descending node is a corresponding point in the plane of the ecliptic diametrically opposite to the other, through which the planet descends in going from the northern to the southern hemisphere. The longitude and latitude of a planet cannot be obtained by direct observation, but are deduced from observations made at the surface of the earth by a very simple computation. These two quantities, however, will not give the place of a planet in space. Its distance from the sun[25] must also be known; and, for the complete determination of its elliptical motion, the nature and position of its orbit must be ascertained by observation. This depends upon seven quantities, called the elements of the orbit (article 378). These are, the length of the major axis, and the eccentricity, which determine the

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[18] Let AMP, fig. 76., article 392, be the upper half of the circular orbit moving round the sun in the centre, C, and A*m*P the elliptic orbit of a body moving round the sun situated at *S*, one of the foci of the ellipse. In both cases the periodic time of the planet would be the same, but whereas when describing the circle, its motion would be uniform in all parts, its speed at different points of the ellipse would be continually varying as explained in the previous note.

[19] *Apsides*. The points P and A, fig. 76, article 392, at the extremities of the major axis of an orbit. P is called the *perihelion*; and the point A the *aphelion*. (Somerville's note.)

[20] *Quadratures*. A celestial body is said to be in quadrature when it is 90 degrees distant from the sun. (Somerville note.)

[21] *Eccentricity*. Deviation from circular form. In fig. 76, article 392, C*S* is the eccentricity of the orbit P*m*AP. (Somerville note.)

[22] Angle PN*p*, fig. 77, article 397.

[23] Angle *mSp*, fig. 77, article 397.

[24] *Nodes*. The two points *n* and N, fig. 77, article 397, in which the orbit of a planet or comet intersects the plane of the ecliptic. The ascending node N is the point through which the body passes in rising above the plane of the ecliptic, and the descending node *n* is the point in which the body sinks below it. (Somerville's note.)

[25] *Distance from the sun*. *Sm* in fig. 77, article 397. When the three quantities: the latitude, the longitude, and the distance from the sun are known, the place of the planet is determined in space. (Somerville's note.)

form of the orbit; the longitude of the planet when at its least distance from the sun, called the longitude of the perihelion; the inclination of the orbit to the plane of the ecliptic, and the longitude of its ascending node: these give the position of the orbit in space; but the periodic time, and the longitude of the planet at a given instant, called the longitude of the epoch, are necessary for finding the place of the body in its orbit at all times. A perfect knowledge of these seven elements is requisite for ascertaining all the circumstances of undisturbed elliptical motion. By such means it is found that the path of the planets, when their mutual disturbances are omitted, are ellipses nearly approaching to circles, whose planes, slightly inclined to the ecliptic, cut it in straight lines, passing through the centre of the sun. The orbits of the recently discovered planets[26] deviate more from the ecliptic than those of the ancient planets: that of Pallas, for instance, has an inclination of $34^{\circ}\ 1'\ 31''$ to it; on which account it is more difficult to determine their motions.

Were the planets attracted by the sun only, they would always move in ellipses, invariable in form and position and because his action is proportional to his mass, which is much larger than that of all the planets put together, the elliptical is the nearest approximation to their true motions. The true motions of the planets are extremely complicated, in consequence of their mutual attraction, so that they do not move in any known or symmetrical curve, but in paths now approaching to, now receding from, the elliptical form; and their radii vectores do not describe areas or spaces exactly proportional to the time, so that the areas become a test of the disturbing forces.

To determine the motion of each body, when disturbed by all the rest, is beyond the power of analysis. It is therefore necessary to estimate the disturbing action of one planet at a time, whence the celebrated problem of the three bodies, originally applied to the moon, the earth, and the sun–namely, the masses being given of three bodies projected from three given points, with velocities given both in quantity and direction; and supposing the bodies to gravitate to one another with forces that are directly as their masses, and inversely as the square of the distances, to find the lines described by these bodies, and their positions at any given instant; or, in other words, to determine the path of a celestial body when attracted by a second body, and disturbed in its motion round the second body by a third–a problem equally applicable to planets, satellites, and comets.

By this problem the motions of translation of the celestial bodies are determined. It is an extremely difficult one, and would be infinitely more so if the disturbing action were not very small when compared with the central force; that is, if the action of the planets on one another were not very small when compared with that of the sun. As the disturbing influence of each body may be found separately, it is assumed that the action of the whole system, in disturbing any one planet, is equal to the sum of all the particular disturbances it experiences, on the general mechanical principle, that the sum of any number of small oscillations is nearly equal to their simultaneous and joint effect.

Perturbations of the Planets

The planets are subject to disturbances of two kinds, both resulting from the constant operation of their reciprocal attraction: one kind depending upon their positions with regard to

[26] See note 9, *Preliminary Dissertation.*

each other, begins from zero, increases to a maximum, decreases, and becomes zero when the planets return to the same relative positions. In consequence of these, the disturbed planet is sometimes drawn away from the sun, sometimes brought nearer to him: sometimes it is accelerated in its motion, and sometimes retarded. At one time it is drawn above the plane of its orbit, at another time below it, according to the position of the disturbing body. All such changes, being accomplished in short periods, some in a few months, others in years, or in hundreds of years, are denominated *periodic inequalities*. The inequalities of the other kind, though occasioned likewise by the disturbing energy of the planets, are entirely independent of their relative positions. They depend upon the relative positions of the orbits alone, whose forms and places in space are altered by very minute quantities, in immense periods of time, and are therefore called *secular inequalities*.

The *periodical perturbations* are compensated when the bodies return to the same relative positions with regard to one another and to the sun: the *secular inequalities* are compensated when the orbits return to the same positions relatively to one another and to the plane of the ecliptic.

Planetary motion, including both these kinds of disturbance, may be represented by a body revolving in an ellipse, and making small and transient deviations, now on one side of its path, and now on the other, whilst the ellipse itself is slowly, but perpetually. changing both in form and position.

The periodic inequalities are merely transient deviations of a planet from its path, the most remarkable of which only lasts about 918 years; but in consequence of the secular disturbances, the apsides, or extremities of the major axes of all the orbits, have a direct but variable motion in space, excepting those of the orbit of Venus, which are retrograde[27] and the lines of the nodes move with a variable velocity in a contrary direction. Besides these, the inclination and eccentricity of every orbit are in a state of perpetual but slow change. These effects result from the disturbing action of all the planets on each. But, as it is only necessary to estimate the disturbing influence of one body at a time, what follows may convey some idea of the manner in which one planet disturbs the elliptical motion of another.

Suppose two planets moving in ellipses round the sun; if one of them attracted the other and the sun with equal intensity, and in parallel directions, or such as could never meet however much prolonged, it would have no effect in disturbing the elliptical motion. The inequality of this attraction is the sole cause of perturbation, and the difference between the disturbing planet's action on the sun and on the disturbed planet constitutes the disturbing force, which consequently varies in intensity and direction with every change in the relative positions of the three bodies. Although both the sun and planet are under the influence of the disturbing force, the motion of the disturbed planet is referred to the centre of the sun as a fixed point, for convenience. The whole force which disturbs a planet is equivalent to three partial forces. One of these acts on the disturbed planet, in the direction of a tangent to its orbit, and is called the *tangential force*: it occasions secular inequalities in the form and position of the orbit in its own plane, and is the sole cause of the periodical perturbations in the planet's longitude. Another acts upon the same body in the direction of its *radius vector*, that is, in the line joining the centres of the sun and planet, and is called the *radial force*: it produces periodical changes in the distance of the planet

[27] *Retrograde*. Going backwards, as from east to west, contrary to the motion of the planets. This apparent anomaly in the case of Venus is caused by the combined action of the earth and Mercury, by which the apsides of Venus are made to recede more rapidly than the joint action of all the other planets can cause them to advance. (Somerville's note.)

from the sun, and affects the form and position of the orbit in its own plane. The third, which may be called the *perpendicular force*, acts at right angles to the plane of the orbit, occasions the periodic inequalities in the planet's latitude, and affects the position of the orbit with regard to the plane of the ecliptic.

It has been observed, that the radius vector of a planet, moving in a perfectly elliptical orbit, passes over equal spaces or areas in equal times; a circumstance which is independent of the law of the force, and would be the same whether it varied inversely as the square of the distance, or not, provided only that it be directed to the centre of the sun. Hence the tangential force, not being directed to the centre, occasions an unequable description of areas, or, what is the same thing, it disturbs the motion of the planet in longitude. The tangential force sometimes accelerates the planet's motion, sometimes retards it, and occasionally has no effect at all. Were the orbits of both planets circular, a complete compensation would take place at each revolution of the two planets, because the arcs in which the accelerations and retardations take place would be symmetrical on each side of the disturbing force. For it is clear, that if the motion be accelerated through a certain space, and then retarded through as much, the motion at the end of the time will be the same as if no change had taken place. But, as the orbits of the planets are ellipses, this symmetry does not hold: for, as the planet moves unequably in its orbit, it is in some positions more directly, and for a longer time, under the influence of the disturbing force than in others. And, although multitudes of variations do compensate each other in short periods, there are others, depending on peculiar relations among the periodic times of the planets, which do not compensate each other till after one, or even till after many revolutions of both bodies. A periodical inequality of this kind in the motions of Jupiter and Saturn has a period of no less than 918 years.

The radial force, or that part of the disturbing force which acts in the direction of the line joining the centres of the sun and disturbed planet, has no effect on the areas, but is the cause of periodical changes of small extent in the distance of the planet from the sun. It has already been shown, that the force producing perfectly elliptical motion varies inversely as the square of the distance, and that a force following any other law would cause the body to move in a curve of a very different kind. Now, the radial disturbing force varies directly as the distance; and, as it sometimes combines with and increases the intensity of the sun's attraction for the disturbed body, and at other times opposes and consequently diminishes it, in both cases it causes the sun's attraction to deviate from the exact law of gravity, and the whole action of this compound central force on the disturbed body is either greater or less than what is requisite for perfectly elliptical motion. When greater, the curvature of the disturbed planet's path, on leaving its perihelion (article 316), or point nearest the sun, is greater than it would be in the ellipse, which brings the planet to its aphelion (article 316), or point farthest from the sun, before it has passed through 180°, as it would do if undisturbed. So that in this ease the apsides, or extremities of the major axis, advance in space. When the central force is less than the law of gravity requires, the curvature of the planet's path is less than the curvature of the ellipse. So that the planet, on leaving its perihelion, would pass through more than 180° before arriving at its aphelion, which causes the apsides to recede in space. Cases both of advance and recess occur during a revolution of the two planets; but those in which the apsides advance preponderate. This, however, is not the full amount of the motion of the apsides; part arises also from the tangential force, which alternately accelerates and retards the velocity of the disturbed planet. An increase in the planet's tangential velocity diminishes the curvature of its orbit, and is equivalent to a decrease of central force. On the contrary, a decrease of the tangential velocity, which increases the curvature of the orbit, is equivalent to an increase of central force. These fluctuations, owing to the tangential force, occasion an alternate recess and advance of the apsides, after the manner already explained. An uncompensated portion of the direct motion, arising from this cause, conspires with that already impressed by the radial force, and in some cases even nearly doubles the direct motion of these points. The motion of the apsides may be represented by supposing a planet to move in an ellipse, while the ellipse itself is slowly revolving about the sun in the same plane. This motion of the major axis, which is direct in all the orbits except that of the planet Venus is irregular, and so slow that it requires more than 109,880 years for the major axis of the earth's orbit to accomplish a sidereal revolution, that is, to return to its original position among the stars; and 20,984 years to complete its tropical revolution, or to return to the same equinox. The difference between these two periods arises from a retrograde motion in the equinoctial point (see article 360), which meets the advancing axis before it has completed its revolution with regard to the stars. The major axis of Jupiter's orbit requires no less than 200,610 years to perform its sidereal revolution, and 22,748 years to accomplish its tropical revolution from the disturbing action of Saturn alone.

A variation in the eccentricity of the disturbed planet's orbit is an immediate consequence of the deviation from elliptical curvature, caused by the action of the disturbing force. When the path of the body, in proceeding from its perihelion to its aphelion, is more curved than it ought to be from the effect of the disturbing forces, it falls within the elliptical orbit, the eccentricity is diminished, and the orbit becomes more nearly circular; when that curvature is less than it ought to be, the path of the planet falls without its elliptical orbit, and the eccentricity is increased; during these changes, the length of the major axis is not altered, the orbit only bulges out, or becomes more flat. Thus the variation in the eccentricity arises from the same cause that occasions the motion of the apsides. There is an inseparable connection between these two elements: they vary simultaneously, and have the same period; so that whilst the major axis revolves in an immense period of time, the eccentricity increases and decreases by very small quantities, and at length returns to its original magnitude at each revolution of the apsides. The terrestrial eccentricity is decreasing at the rate of about 40 miles annually; and if it were to decrease equably it would be 39,861 years before the earth's orbit became a circle. M. Leverrier[28] has, however, proved that the diminution will not continue beyond 23,980 years, from the year AD 1800; after that time the eccentricity will begin to increase. The mutual action of Jupiter and Saturn occasions variations in the eccentricity of the orbits of both these planets, the greatest eccentricity of Jupiter's orbit corresponding to the least of Saturn's. The period in which these vicissitudes are accomplished is 70,414 years, estimating the action of these two planets alone; but, if the action of all the planets were estimated, the cycle would extend to millions of years.

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[28] Leverrier, Urbain Jean Joseph, (1811-1877), astronomer, born in St. Lô, France. Leverrier predicted the existence and location in the heavens of an undiscovered planet based on the perturbations in the motions of the planets. This deduction occurred eight months after that of J. C. Adams (1819-1892) who in 1845 was inspired to calculate the location of the undiscovered planet based upon a prediction by Mary Somerville written in the 6th edition of her *On the Connexion of the Physical Sciences* published in 1842. Neptune was actually discovered at that location by Galle in Germany on September 23, 1846 within a few days of Leverrier's calculations and at Leverrier's request (see also note 39 below, note 48, *Bk. I, Foreword*, and note 38, *Bk. II, Chap. XIV*). Leverrier focussed much of his attention on the planet Mercury and compiled tables of its motions. He was also intrigued by unusual characteristics of Mercury's motion, phenomena that were finally adequately explained by Einstein's general theory of relativity in 1915.

That part of the disturbing force is now to be considered which acts perpendicularly to the plane of the orbit, causing periodic perturbations in latitude, secular variations in the inclination of the orbit, and a retrograde motion to its nodes on the true plane of the *ecliptic*, or apparent path of the sun round the earth. This force tends to pull the disturbed body above, or push it below, the plane of its orbit, according to the relative positions of the two planets with regard to the sun, considered to be fixed. By this action, it sometimes makes the plane of the orbit of the disturbed body tend to coincide with the plane of the ecliptic, and sometimes increases its inclination to that plane. In consequence of which, its nodes alternately recede or advance on the ecliptic. When the disturbing planet is in the line of the disturbed planet's nodes, it neither affects these points, the latitude, nor the inclination, because both planets are then in the same plane. When it is at right angles to the line of the nodes, and the orbit symmetrical on each side of the disturbing force, the average motion of these points, after a revolution of the disturbed body, is retrograde, and comparatively rapid: but, when the disturbing planet is so situated that the orbit of the disturbed planet is not symmetrical on each side of the disturbing force, which is most frequently the case, every possible variety of action takes place. Consequently, the nodes are perpetually advancing or receding with unequal velocity; but, as a compensation is not effected, their motion is, on the whole, retrograde.

With regard to the variations in the inclination, it is clear, that, when the orbit is symmetrical on each side of the disturbing force, all its variations are compensated after a revolution of the disturbed body, and are merely periodical perturbations in the planet's latitude; and no secular change is induced in the inclination of the orbit. When, on the contrary, that orbit is not symmetrical on each side of the disturbing force, although many of the variations in latitude are transient or periodical, still, after a complete revolution of the disturbed body, a portion remains uncompensated, which forms a secular change in the inclination of the orbit to the plane of the ecliptic. It is true, part of this secular change in the inclination is compensated by the revolution of the disturbing body, whose motion has not hitherto been taken into the account, so that perturbation compensates perturbation; but still a comparatively permanent change is effected in the inclination, which is not compensated till the nodes have accomplished a complete revolution.

The changes in the inclination are extremely minute, compared with the motion of the nodes, and there is the same kind of inseparable connection between their secular changes that there is between the variation of the eccentricity and the motion of the major axis. The nodes and inclinations vary simultaneously; their periods are the same, and very great. The nodes of Jupiter's orbit, from the action of Saturn alone, require 36,261 years to accomplish even a tropical revolution. In what precedes, the influence of only one disturbing body has been considered; but, when the action and reaction of the whole system are taken into account, every planet is acted upon, and does itself act, in this manner, on all the others; and the joint effect keeps the inclinations and eccentricities in a state of perpetual variation. It makes the major axis of all the orbits continually revolve, and causes, on an average, a retrograde motion of the nodes of each orbit upon every other. The ecliptic itself is in motion from the mutual action of the earth and planets, so that the whole is a compound phenomenon of great complexity, extending through unknown ages. At the present time the inclinations of all the orbits are decreasing but so slowly, that the inclination of Jupiter's orbit is only about six minutes less than it was in the age of Ptolemy.[29]

[29] See note 15, *Preliminary Dissertation.*

But, in the midst of all these vicissitudes, the length of the major axes and the mean motions of the planets remain permanently independent of secular changes. They are so connected by Kepler's law of the squares of the periodic times being proportional to the cubes of the mean distances of the planets from the sun, that one cannot vary without affecting the other. And it is proved, that any variations which do take place are transient, and depend only on the relative positions of the bodies.

It is true that, according to theory, the radial disturbing force should permanently alter the dimensions of all the orbits, and the periodic times of all the planets, to a certain degree. For example, the masses of all the planets revolving within the orbit of any one, such as Mars, by adding to the interior mass, increase the attracting force of the sun, which, therefore, must contract the dimensions of the orbit of that planet, and diminish its periodic time; whilst the planets exterior to Mars' orbit must have the contrary effect. But the mass of the whole of the planets and satellites taken together is so small, when compared with that of the sun, that these effects are quite insensible, and could only have been discovered by theory. And, as it is certain that the length of the major axes and the mean motions are not permanently changed by any other power whatever, it may be concluded that they are invariable.

With the exception of these two elements, it appears that all the bodies are in motion, and every orbit in a state of perpetual change. Minute as these changes are, they might be supposed to accumulate in the course of ages sufficiently to derange the whole order of nature, to alter the relative positions of the planets, to put an end to the vicissitudes of the seasons, and to bring about collisions which would involve our whole system, now so harmonious, in chaotic confusion. It is natural to inquire, what proof exists that nature will be preserved from such a catastrophe? Nothing can be known from observation, since the existence of the human race has occupied comparatively but a moment in duration, while these vicissitudes embrace myriads of ages. The proof is simple and conclusive. All the variations of the solar system, secular as well as periodic, are expressed analytically by the sines and cosines of circular arcs which increase with the time; and, as a sine or cosine can never exceed the radius, but must oscillate between zero and unity, however much the time may increase, it follows that when the variations have accumulated to a maximum by slow changes in however long a time, they decrease, by the same slow degrees, till they arrive at their smallest value, again to begin a new course; thus for ever oscillating about a mean value. This circumstance, however, would be insufficient, were it not for the small eccentricities of the planetary orbits, their minute inclinations to the plane of the ecliptic, and the revolutions of all the bodies, as well planets as satellites, in the same direction. These secure the perpetual stability of the solar system.[30] However, at the time that the stability was proved by Lagrange and Laplace, the telescopic planets between Mars and Jupiter had not been discovered; but Lagrange, having investigated the subject under a very general point of view, showed that, if a planetary system be composed of very unequal masses, the whole of the

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[30] The small eccentricities and inclinations of the planetary orbits, and the revolutions of all the bodies in the same direction, were proved by Euler (see note 6, *Book I, Chapter II*), Lagrange (see note 16, *Preliminary Dissertation*), and Laplace (see note 4, *Introduction*), to be conditions necessary for the stability of the solar system. Subsequently, however, the periodicity of the terms of the series expressing the perturbations was supposed to be sufficient *alone*, but M. Poisson (see note 1, *Book I, Chapter 6*) has shown that to be a mistake; that these three conditions are requisite for the necessary convergence of the series, and that therefore the stability of the system depends on them *conjointly* with the periodicity of the sines and cosines of each term. The author (i.e. Somerville) is aware that this note can only be intelligible to the analyst, but she is desirous of correcting an error, and the more so as the conditions of stability afford one of the most striking instances of design in the original construction of our system, and of the foresight and supreme wisdom of the Divine Architect. (Somerville's note.)

larger would maintain an unalterable stability with regard to the form and position of their orbits, while the orbits of the lesser might undergo unlimited changes. M. Leverrier has applied this to the solar system, and has found that the orbits of all the larger planets will for ever maintain an unalterable stability in form and position; for, though liable to mutations of very long periods, they return again exactly to what they originally were, oscillating between very narrow limits; but he found a zone of instability between the orbit of Mars, and twice the mean distance of the earth from the sun,[31] or between 1.5 and 2.00; therefore the position and form of the orbits of such of the telescopic planets as revolve within that zone will be subject to unlimited variations. But the orbits of those more remote from the sun than Flora,[32] or beyond 2.20, will be stable, so that their eccentricities and inclinations must always have been, and will always remain, very great, since they must have depended upon the primitive conditions that prevailed when these planetary atoms were launched into space. The telescopic planets,[33] numerous as they are, 153 having been discovered up to the date of Jan. 1876,[34] have been shown by Leverrier's calculations, completed in 1875, to have no influence upon the motions of Jupiter and scarcely any upon those of Mars. This result was to be expected, for Jupiter has a diameter of 84,846 miles,[35] while that of Pallas, his nearest neighbour, is not more than 171 miles.[36] The diameter of Mars, on the other side of the small planets, is 4,363 miles,[37] and that of the earth 7,920 miles,[38] so that the telescopic group is too minute to disturb the others. M. Leverrier found another zone of instability between Venus and the sun, on the border of which Mercury is revolving the inclination of whose orbit to the plane of the ecliptic is about 70°, which is more than that of any of the large planets. Neptune's orbit is, no doubt, as stable as that of any other of the large planets, as the inclination is very small, but he will have periodical variations of very long duration from the reciprocal attraction between him and Uranus, one especially of an enormous duration, similar to those of Jupiter and Saturn, and, like them, depending on the time of his revolution round the sun being nearly twice as long as that of Saturn. Mr. Adams[39] has computed that Neptune produces a periodical perturbation in the motion of Uranus, whose duration is about 6,800 years.

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[31] The mean distance of the earth from the sun is 91,600,000 miles, but to avoid the inconvenience of large numbers, it is assumed to be the unit of distance; hence the mean distance of Mars is 1.52369, or 1.5 nearly, that of the earth being = 1. (Somerville's note.)

[32] Flora is an asteroid discovered in 1847 by J. R. Hind in London. It has a diameter of about 141 km. Its distance from the Sun varies between 1.86 and 2.55 AU. The asteroid has a period of 3.27 years with an orbit inclined at 5.9° to the ecliptic. *Royal Astronomical Society of New Zealand.*

[33] *telescopic planets.* A term Somerville uses to refer to the asteroids.

[34] Several hundred thousand asteroids are now known with 26 of those larger than 200 km in diameter.

[35] *diameter of Jupiter.* The modern value is 88,850 mi. (142,984 km.)

[36] Stone's *Astronomical Monthly Notices*, vol. xxvii. p. 302. (Somerville's note.)

[37] *diameter of Mars.* Modern value is 4,222 mi. (6,794 km.)

[38] *diameter of Earth.* Modern value is 7,926 mi. (12,756 km.)

[39] Adams, John Couch, (1819-1892), astronomer, born in Laneast, England. Adams mathematically deduced the existence of the planet Neptune based upon the writings of Mary Somerville (see *Foreword to the Second Edition*). Adams gave the Director of the Cambridge Observatory precise data on the (still unseen) planet's location in September, 1845. Adams' calculations were done eight months before French astronomer Leverrier (1811-1877) who performed similar calculations independently. Leverrier then requested a search by the German astronomer Galle who located the planet at the Berlin Observatory a few days later on September 23, 1846. (see also note 28 above, note 48, *Bk. I, Foreword*, and note 38, *Bk. II, Chap. XIV*.)

BOOK II

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CHAPTER I

PROGRESS OF ASTRONOMY

308. THE science of astronomy was cultivated very early, and many important observations and discoveries were made, yet no accurate inferences leading to the true system of the world were drawn from them, until a much later period. It is not surprising, that men deceived by appearances, occasioned by the rotation of the earth, should have been slow to believe the diurnal motion of the heavens to be an illusion; but the absurd consequence which the contrary hypothesis involves, convinced minds of a higher order, that the apparent could not be the true system of nature.

Many of the ancients were aware of the double motion of the earth; a system which Copernicus[1] adopted, and confirmed by the comparison of a series of observations, that had been accumulating for ages; from these he inferred that the precession of the equinoxes might be attributed to a motion of the earth's axis. He ascertained the revolution of the planets round the sun, and determined the dimensions of their orbits, till then unknown. Although he proved these truths by evidence which had ultimately dissipated the erroneous theories resulting from the illusions of the senses, and overcame the objections which were opposed to them by ignorance of the laws of mechanics, this great philosopher, constrained by the prejudices of the times, only dared to publish the truths he had discovered, under the less objectionable name of hypothesis.

In the seventeenth century, Galileo,[2] assisted by the discovery of the telescope, was the first who saw the magnificent system of Jupiter's satellites, which furnished a new analogy between the planets and the earth: he discovered the phases of Venus, by which he removed all doubts of the revolution of that planet round the sun. The bright spots which he saw in the moon beyond the line which separates the enlightened from the obscure part, showed the existence and height of its mountains. He observed the spots and rotation of the sun, and the singular appearances exhibited by the rings of Saturn; by which discoveries the rotation of the earth was confirmed: but if the rapid progress of mathematical science had not concurred to establish this essential truth, it would have been overwhelmed and stifled by fanatical zeal. The opinions of Galileo were denounced as heretical by the Inquisition, and he was ordered by the Church of Rome to retract them. At a late period he ventured to promulgate his discoveries, but in a different form, vindicating the system of Copernicus; but such was the force of superstition and prejudice, that he, who was alike an honour to his country, and to the human race, was again subjected to the mortification of being obliged to disavow what his transcendent genius had proved to be true. He died at Arcetri[3] in the year 1642, the year in which Newton[4] was born, carrying with him, says Laplace,[5] the regret of Europe, enlightened by his labours, and indignant at the judgment pronounced against him by an odious tribunal.

The truths discovered by Galileo could not fail to mortify the vanity of those who saw the earth, which they conceived to be the centre and primary object of creation, reduced to the rate of but a small planet in a system, which, however vast it may seem, forms but a point in the scale of the universe.

The force of reason by degrees made its way, and persecution ceased to be the consequence of stating physical truths, though many difficulties remained to impede its progress, and no ordinary share of moral courage was required to declare it: '*prejudice,*' says an eminent author, '*bars up the gate of knowledge; but he who would learn, must despise the timidity that shrinks from wisdom, he must hate the tyranny of opinion that condemns its pursuit: wisdom is only to be obtained by the bold; prejudices must first be overcome, we must learn to scorn names, defy idle fears, and use the powers of nature to give us the mastery of nature. There are virtues in plants, in metals, even in woods, that to seek alarms the feeble, but to possess constitutes the mighty.*

About the end of the sixteenth, or the beginning of the seventeenth century, Tycho Brahe[6] made a series of correct and numerous observations on the motion of the planets, which laid the foundation of the laws discovered by his pupil and assistant, Kepler.[7]

Tycho Brahe, however, would not admit of the motion of the earth, because he could not conceive how a body detached from it could follow its motion: he was convinced that the earth was at rest, because a heavy body, falling from a great height, falls nearly at the foot of the vertical.

Kepler, one of those extraordinary men, who appear from time to time, to bring to light the great laws of nature, adopted sounder views. A lively imagination, which disposed him eagerly to search for first causes, tempered by a severity of judgment that made him dread being deceived, formed a character peculiarly fitted to investigate the unknown regions of science, and conducted him to the discovery of three of the most important laws in astronomy.

He directed his attention to the motions of Mars, whose orbit is one of the most eccentric in the planetary system, and as it approaches very near the earth in its oppositions, the inequalities of its motions are considerable; circumstances peculiarly favorable for the determination of their laws.

He found the orbit of Mars to be an ellipse, having the sun in one of its foci; and that the motion of the planet is such, that the radius vector drawn from its centre to the centre of the sun, describes equal areas in equal times. He extended these results to all the planets, and in the year 1626, published the Rudolphine Tables,[8] memorable in the annals of astronomy, from being the first that were formed on the true laws of nature.

Kepler imagined that something corresponding to certain mysterious analogies, supposed by the Pythagoreans[9] to exist in the laws of nature, might also be discovered between the mean distances of the planets, and their revolutions around the sun: after sixteen years spent in unavailing attempts, he at length found that the squares of the times of their sidereal revolutions are proportional to the cubes of the greater axes of their orbits; a very important law, which was afterwards found equally applicable to all the systems of the satellites. It was obvious to the comprehensive mind of Kepler, that motions so regular could only arise from some universal principle pervading the whole system. In his work *De Stella Martis,*[10] he observes, that[11] '*two insulated bodies would move towards one another like two magnets, describing spaces reciprocally as their masses. If the earth and moon were not held at the distance that separates them by some force, they would come in contact, the moon describing* $\frac{53}{54}$ *of the distance, and the earth the remainder, supposing them to be equally dense*.' '*If,*' he continues, '*the earth ceased to attract the waters of the ocean, they would go to the moon by the attractive force of that body. The attraction of the moon, which extends to the earth, is the cause of the ebb and flow of the sea.*' Thus Kepler's work, *De Stella Martis*, contains the first idea of a principle which Newton and his successors have fully developed.

The discoveries of Galileo on falling bodies, those of Huygens[12] on Evolutes,[13] and the centrifugal force, led to the theory of motion in curves. Kepler had determined the curves on which the planets move, and Hook[e][14] was aware that planetary motion is the result of a force of projection combined with the attractive force of the sun.

Such was the state of astronomy when Newton, by his grand and comprehensive views, combined the whole, and connected the most distant parts of the solar system by one universal principle.

Having observed that the force of gravitation on the summits of the highest mountains is nearly the same as on the surface of the earth, Newton inferred that its influence extended to the moon, and, combining with her force of projection, causes that satellite to describe an elliptical orbit round the earth. In order to verify this conjecture, it was necessary to know the law of the diminution of gravitation. Newton considered, that if terrestrial gravitation retained the moon in her orbit, the planets must be retained in theirs by their gravitation to the sun; and he proved this to be the case, by showing the areas to be proportional to the times: but it resulted from the constant ratio found by Kepler between the squares of the times of revolutions of the planets, and the cubes of the greater axes of their orbits, that their centrifugal force, and consequently their tendency to the sun, diminishes in the ratio of the squares of their distances from his centre. Thus the law of diminution was proved with regard to the planets, which led Newton to conjecture, that the same law of diminution takes place on terrestrial gravitation.

He extended the laws deduced by Galileo from his experiments on bodies falling at the surface of the earth, to the moon; and on these principles determined the space she would move through in a second of time, in her descent towards the earth, if acted upon by the earth's attraction alone. He had the satisfaction to find that that the action of the earth on the moon is inversely as the square of the distance, thus proving the force which causes a stone to fall at the earth's surface, to be identical with that retains the moon in her orbit.

Kepler having established the point that the planets move in ellipses, having the sun in one of their foci, Newton completed his theory, by showing that a projectile might move in any of the conic sections, if acted on by a force directed to the focus, and inversely as the square of the distance: he determined the conditions requisite to make the trajectory a circle, an ellipse, a parabola, or hyperbola. Hence he also concluded, that comets move round the sun by the same laws as the planets.

A comparison of the magnitude of the orbits of the satellites and the periods of their revolutions, with the same quantities relatively to the planets, made known to him the respective masses and densities of the sun and of planets accompanied by satellites, and the intensity of gravitation at their surfaces. He observed, that the satellites move round their planets nearly as they would have done, had the planets been at rest, whence he concluded that all these bodies obey the same law of gravitation towards the sun: he also concluded, from the equality of action and re-action, that the sun gravitates towards the planets, and the planets towards their satellites; and that the earth is attracted by all bodies which gravitate towards it. He afterwards extended this law to all the particles of matter, thus establishing the general principle, that each particle of matter attracts all other particles directly as its mass, and inversely as the square of its distance.

These splendid discoveries were published by Newton in his *Principia*,[15] a work which has been the admiration of mankind, and which will continue to be so while science is cultivated.

Referring to that stupendous effort of human genius, Laplace, who perhaps only yields to Newton in priority of time, thus expresses himself in a letter to the writer of these pages:[16]

'*Je publie successivement les divers livres du cinquième volume qui doit terminer mon traité de* ***Mécanique Céleste****, et dans lequel je donne l'analyse historique des recherches des géomètres sur cette matière. Cela m'a fait relire avec une attention particulière l'ouvrage incomparable des* ***Principes Mathématiques*** *de la philosophie naturelle de Newton, qui contient le germe de toutes ces recherches. Plus j'ai étudié cet ouvrage, plus il m'a paru admirable, en me transportant surtout à l'époque où il a été publié. Mais en même tems que j'ai senti l'élégance de la méthode synthétique suivant laquelle Newton a présenté ses découvertes, j'ai reconnu l'indispensable nécessité de l'analyse pour approfondir les questions très difficiles qu'il n'a pu qu'effleurer par la synthèse. Je vois avec un grand plaisir vos mathématiciens se livrer maintenant à l'analyse; et je ne doute point qu'en suivant cette méthode avec la sagacité propre à votre nation, ils ne soient conduits à d'importantes découvertes*.'

The reciprocal gravitation of the bodies of the solar system is a cause of great irregularities in their motions; many of which had been explained before the time of Laplace, but some of the most important had not been accounted for, and many were not even known to exist. The author of the *Mécanique Céleste* therefore undertook the arduous task of forming a complete system of physical astronomy, in which the various motions in nature should be deduced from the first principles of mechanics. It would have been impossible to accomplish this, had not the improvements in analysis kept pace with the rapid advance in astronomy, a pursuit in which many have acquired immortal fame; that Laplace is pre-eminent amongst these, will be most readily acknowledged by those who are best acquainted with his works.

Having endeavoured in the first book to explain the laws by which force acts upon matter, we shall now compare those laws with the actual motions of the heavenly bodies, in order to arrive by analytical reasoning, entirely independent of hypothesis, at the principle of that force which animates the solar system. The laws of mechanics may be traced with greater precision in celestial space than on earth, where the results are so complicated, that it is difficult to unravel, and still more so to subject them to calculation: whereas the bodies of the solar system, separated by vast distances, and acted upon by a force, the effects of which may be readily estimated, are only disturbed in their respective movements by such small forces, that the general equations comprehend all the changes which ages have produced, or may hereafter produce in the system; and in explaining the phenomena it is not necessary to have recourse to vague or imaginary causes, for the law of universal gravitation may be reduced to calculation, the results of which, confirmed by actual observation, afford the most substantial proof of its existence.

It will be seen that this great law of nature represents all the phenomena of the heavens, even to the most minute details; that there is not one of the inequalities which it does not account for; and that it has even anticipated observation, by unfolding the causes of several singular motions, suspected by astronomers, but so complicated in their nature, and so long in their periods, that observation alone could not have determined them but in many ages.

By the law of gravitation, therefore, astronomy is now become a great problem of mechanics, for the solution of which, the figure and masses of the planets, their places, and velocities at any given time, are the only data which observation is required to furnish. We proceed to give such an account of the solution of this problem, as the nature of the subject and the limits of this work admit of.

Notes

[1] Copernicus, Nicolas, 1473-1543, astronomer and founder of the heliocentric world system, born in Torun, Poland. The heliocentric system was advanced in his *De revolutionibus orbium coelestium* (1543). The view eliminated several problems in Ptolemy's (see note 15, *Preliminary Dissertation*) geostatic model, but was no more accurate in predicting celestial motions and really no simpler. Copernicus, for example, used 44 more epicycles that Ptolemy. Furthermore, the main counter-argument against the Copernican system, the lack of observed stellar parallax, was not adequately explained; nor was there an effective defense of the new model in terms of Aristotle's physics on which Copernicus still relied. In addition his work relied on scant observational evidence. The work nonetheless became the cornerstone for modern astronomical science.

[2] See note 1, *Introduction.*

[3] Arcetri is the small village near Florence where Galileo remained under house arrest until his death.

[4] Galileo died on Jan 4, 1642. Newton was bom on Christmas day of the same year.

[5] Somerville uses the spelling *La Place* throughout the text.

[6] Brahe, Tycho, 1546-1601 astronomer, born in Knudstrup, Sweden. Brahe was known for his unprecedented observational accuracy. He operated an observatory called Uraniborg on the island of Ven where most of his observations were conducted. He rejected the Copernican theory (see note 1) and proposed an independent geocentric system of the world (the Tyconic system) which is mathematically equivalent to the Copernican system. In his system the earth remains stationary, but the five remaining planets revolve about the sun, which in tun revolves about the earth. After his death Kepler used Brahe's precise data to demonstrate the elliptical orbit of Mars and to establish his three laws of planetary motion which later formed the foundation for Newtonian mechanics.

[7] See note 3, *Preliminary Dissertation.*

[8] The Rudolphine Tables were the first to make use of Kepler's newly formulated Laws on planetary motions, calibrated using Tycho Brahe's (see note 6) store of accurate planetary observations. They received a spectacular validation on November 7, 1631, when the French philosopher and sometimes astronomer Pierre Gassendi (1592-1655) observed a transit of Mercury across the solar disk, as predicted by Kepler. Kepler's prediction of this event was far more accurate than those based on the Copernican Tables. This success paved the way for the general acceptance not only of the Rudolphine Tables, but also by extension, of Kepler's three Laws of planetary motions. *Paul Charbonneau, High Altitude Observatory (NCAR)*.

[9] See note 10.

[10] Kepler, Johannes, 1571-1630, *Astronomia nova aitiologetos, seu physica coelestis, tradita commentariis de motibus stellae Martis, ex observationibus G.V. Tychonis Brahe ... elaborata ...*, Pragae, 1609. (see also note 2, *Preliminary Dissertation.*)

[11] Not italicized in the 1st edition.

[12] See note 12, *Book I, Chapter II* .

[13] *Evolutes.* The locus of the centers of curvature of a given curve, *The American Heritage® Dictionary, 1996.*

[14] Robert Hooke (1635-1703), chemist and physicist, born in Freshwater, Isle of Wight, England. Hooke formulated the law governing elasticity (Hooke's law), and invented the balance spring for watches. His most important work is his *Micrographia* (1665).

[15] Newton, Isaac, Sir, 1642-1727, *Isaac Newton's Philosophiae naturalis principia mathematica*, Cambridge: Cambridge University Press, 1972. (see also note 1, *Preliminary Dissertation.*)

[16] "I am publishing consecutively the various books of the fifth volume which will complete my treatise of Celestial Mechanics, and in which I give an historical analysis of the investigations of the geometers into this matter. This made me reread with particular attention the incomparable work of Newton, *The Mathematical Principles of Natural Philosophy* (1687), which contains the germ of all this research. The more I studied this work, the more it appeared to me admirable, transporting me, above all, to the period when it was published. But at the same time as I sensed the elegance of the synthetic method according to which Newton presented his discoveries, I recognized the indispensable necessity of analysis for plumbing the depths of the very difficult questions that one could only treat superficially by synthesis. I note with great pleasure that your mathematicians now devote themselves to analysis; and I have no doubt whatsoever that by following this method with the sagacity characteristic of your nation, they will be led to important discoveries." *Translation by John Black, Malaspina University-College.*

Venus

This mosaic of Venus was composed from Magellan images taken during radar investigations from 1990-1994, centered at 180° east longitude. Magellan spacecraft imaged more than 98% of Venus' surface at a resolution of about 100 meters. This image has an effective resolution of about 3 kilometers. Gaps in the Magellan coverage were filled with images from Earth-based Arecibo radar in a region roughly centered at 0° latitude and longitude and near the south pole. This mosaic was colour-coded to represent elevation. Missing elevation data from the Magellan radar altimeter were filled with altimetry from the Venera spacecraft and the U.S. Pioneer Venus missions. The Lighter areas denote rough terrain; the darker areas are smooth surfaces or possibly areas covered with dust. (Courtesy of NASA)

BOOK II

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CHAPTER II

ON THE LAW OF UNIVERSAL GRAVITATION, DEDUCED FROM OBSERVATION

309. THE three laws of Kepler furnish the data from which the principle of gravitation is established, namely:–

i. That the radii vectores of the planets and comets describe areas proportional to the time.

ii. That the orbits of the planets and comets are conic sections, having the sun in one of their foci.

iii. That the squares or the periodic times of the planets are proportional to the cubes of their mean distances from the sun.

310. It has been shown, that if the law of the force which acts on a moving body be known, the curve in which it moves may be found; or, if the curve in which the body moves be given, the law of the force may be ascertained. In the general equation of the motion of a body in article 144, both the force and the path of the body are indeterminate; therefore in applying that equation to the motion of the planets and comets, it is necessary to know the orbits in which they move, in order to ascertain the nature of the force that acts on them.

311. In the general equation of the motion of a body, the forces acting on it are resolved into three component forces, in the direction of three rectangular axes; but as the paths of the planets, satellites, and comets, are proved by the observations of Kepler to be conic sections, they always move in the same plane: therefore the component force in the direction perpendicular to that plane is zero, and the other two component forces are in the plane of the orbit.

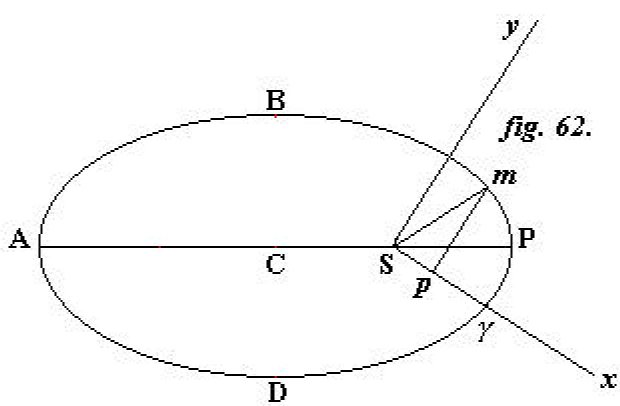


fig. 62.

312. Let *Am*P, fig. 62, be the elliptical orbit of a planet *m*, having the centre of the sun in the focus S, which is also assumed as the origin of the co-ordinates. The imaginary line S*m* joining the centre of the sun and the centre of the planet is the radius vector. Suppose the two component forces to be in the direction of the axes S*x*, S*y*, S*z*, then the component force Z, is zero; and as the body is free to move in every direction, the virtual velocities δx, δy are zero, which divides the general equation of motion in article 144 into

$$\frac{d^2x}{dt^2} = \text{X}; \ \frac{d^2y}{dt^2} = \text{Y};$$

giving a relation between each component force, the space that it causes the body to describe on ox, or oy, and the time. If the first of these two equations be multiplied by $-y$, and added to the second multiplied by x, their sum will be

$$\frac{d(xdy - ydx)}{dt^2} = \text{Y}x - \text{X}y.$$

But $xdy - ydx$ is double the area that the radius vector of the planet describes round the sun in the instant dt. According to the first law of Kepler, this area is proportional to the time, so that

$$xdy - ydx = cdt;$$

and as c is a constant quantity,

$$\frac{d(xdy - ydx)}{dt^2} = 0,$$

therefore

$$\text{Y}x - \text{X}y = 0,$$

whence

$$\text{X} : \text{Y} :: x : y;$$

so that the forces X and Y are in the ratio of x to y, that is as Sp to pm, and thus their resulting force mS passes through S, the centre of the sun. Besides, the curve described by the planet is concave towards the sun, whence the force that causes the planet to describe that curve, tends towards the sun. And thus the law of the areas being proportional to the time, leads to this important result,–that the force which retains the planets and comets in their orbits, is directed towards the centre of the sun.

313. The next step is to ascertain the law by which the force varies at different distances from the sun, which is accomplished by the consideration, that these bodies alternately approach and recede from him at each revolution; the nature of elliptical motion, then, ought to give that law. If the equation

$$\frac{d^2x}{dt^2} = \text{X}$$

be multiplied by dx, and

$$\frac{d^2y}{dt^2} = \text{Y},$$

by dy, their sum is

$$\frac{dxd^2x+dyd^2y}{dt^2}=\mathrm{X}dx+\mathrm{Y}dy\ ,$$

and its integral is

$$\frac{d^2x+d^2y}{dt^2}=2\int(\mathrm{X}dx+\mathrm{Y}dy)\,,$$

the constant quantity being indicated by the integral sign. Now the law of areas gives

$$dt=\frac{xdy-ydx}{c},$$

which changes the preceding equation to

$$\frac{c^2\left(dx^2+dy^2\right)}{\left(xdy-ydx\right)^2}=2\int(\mathrm{X}dx+\mathrm{Y}dy)\,. \qquad \textbf{(82)}$$

In order to transform this into a polar equation, let r represent the radius vector Sm, fig. 62, and v the angle $m\mathrm{S}g$, then

$$\mathrm{S}p=x=r\cos v;\ pm=y=r\sin v;\ \text{and } r=\sqrt{x^2+y^2}$$

whence

$$dx^2+dy^2=r^2dv^2+dr^2,\ xdy-ydx=r^2dv\,;$$

and if the resulting force of X and Y be represented by F, then

$$F:\mathrm{X}::\mathrm{S}m:\mathrm{S}p::1:\cos v\,;$$

hence

$$\mathrm{X}=-F\cos v\,;$$

the sign is negative, because the force F in the direction mS, tends to diminish the co-ordinates; in the same manner it is easy to see that

$$\mathrm{Y}=-F\sin v;\ F=\sqrt{\mathrm{X}^2+\mathrm{Y}^2};\ \text{and } \mathrm{X}dx+\mathrm{Y}dy=-Fdr\,;$$

so that the equation (82) becomes

$$0=\frac{c^2\left\{r^2dv^2+dr^2\right\}}{r^4dv^2}+2\int Fdr\,. \qquad \textbf{(83)}$$

Whence[1]

$$dv = \frac{cdr}{r\sqrt{-c^2 - 2r^2 \int Fdr}}.$$

314. If the force F be known in terms of the distance r, this equation will give the nature of the curve described by the body. But the differential of equation (83) gives

$$F = \frac{c^2}{r^3} - \frac{c^2}{2} d \frac{\left\{ \frac{dr^2}{r^4 dv^2} \right\}}{dr}. \qquad \textbf{(84)}$$

Thus a value of the resulting force F is obtained in terms of the variable radius vector Sm, and of the corresponding variable angle $m\text{S}g$; but in order to have a value of the force F in terms of mS alone, it is necessary to know the angle $g\text{S}m$ in terms of Sm. The planets move in ellipses, having the sun in one of their foci; therefore let v represent the angle $g\text{SP}$, which the greater axis AP makes with the axes of the co-ordinates Sx, and let v be the angle $g\text{S}m$. Then if $\frac{\text{CS}}{\text{CP}}$, the ratio of the eccentricity to the greater axis be e, and half[2] the greater axis CP=a, the polar equation of conic sections is

$$r = \frac{a\left(1-e^2\right)}{1+e\cos\left(v-v\right)},$$

which becomes a parabola when $e=1$, and a infinite; and a hyperbola when e is greater than unity and a negative. This equation gives a value of r in terms of the angle $g\text{S}m$ or v, and thence it may be found that

$$\frac{dr^2}{r^4 dv^2} = \frac{2}{ar\left(1-e^2\right)} - \frac{1}{r^2} - \frac{1}{a^2\left(1-e^2\right)}$$

which substituted in equation (84) gives

$$F = \frac{c^2}{a\left(1-e^2\right)} \cdot \frac{1}{r^2}.$$

The coefficient $\frac{c^2}{a\left(1-e^2\right)}$ is constant, therefore F varies inversely as the square of r or Sm. Wherefore the orbits of the planets and comets being conic sections, the force varies inversely as the square of the distance of these bodies from the sun.

Now as the force F varies inversely as the square of the distance, it may be represented by $\frac{h}{r^2}$, in which h is a constant coefficient, expressing the intensity of the force. The equation of conic sections will satisfy equation (84) when $\frac{h}{r^2}$ is put for F; whence as

$$h = \frac{c^2}{a\left(1 - e^2\right)}$$

forms an equation of condition between the constant quantities a and e, the three arbitrary quantities a, e, and v, are reduced to two; and as equation (83) is only of the second order, the finite equation of conic sections is its integral.

315. Thus, if the orbit be a conic section, the force is inversely as the square of the distance; and if the force varies inversely as the square of the distance, the orbit is a conic section. The planets and comets therefore describe conic sections in virtue of a primitive impulse and an accelerating force directed to the centre of the sun, and varying according to the preceding law, the least deviation from which would cause them to move in curves of a totally different nature.

316. In every orbit the point P, fig. 63, which is nearest the sun, is the perihelion, and in the ellipse the point A farthest from the sun is the aphelion. SP is the perihelion distance of the body from the sun.

317. A body moves in a conic section with a different velocity in every point of its orbit, and with a perpetual tendency to fly off in the direction of the tangent, but this tendency is counteracted by the attraction of the sun. At the perihelion, the velocity of a planet is greatest; therefore its tendency to leave the sun exceeds the force of attraction: but the continued action of the sun diminishes the velocity as the distance increases; at the aphelion the velocity of the planet is least: therefore its tendency to leave the sun is less than the force of attraction which increases the velocity as the distance diminishes, and brings the planet back towards the sun, accelerating its velocity so much as to overcome the force of attraction, and carry the planet again to the perihelion. This alternation is continually repeated.

318. When a planet is in the point B, or D, it is said to be in quadrature, or at its mean distance from the sun. In the ellipse, the mean distance, SB or SD, is equal to CP, half the greater axis; the eccentricity is CS.

319. The periodic time of a planet is the time in which it revolves round the sun, or the time of moving through 360^0. The periodic time of a satellite is the time in which it revolves about its primary.

320. From the equation[3]

$$F = \frac{c^2}{a\left(1-e^2\right)} \cdot \frac{1}{r^2},$$

it may be shown, that the force F varies, with regard to different planets, inversely as the square of their respective distances from the sun. The quantity $2a\left(1-e^2\right)$ is 2SV, the parameter of the orbit, which is invariable in any one curve, but is different in each conic section. The intensity of the force depends on

$$\frac{c^2}{a\left(1-e^2\right)} \text{ or } \frac{c^2}{\text{SV}},$$

which may be found by Kepler's laws. Let T represent the time of the revolution of a planet; the area described by its radius vector in this time is the whole area of the ellipse, or

$$p\,a^2 \cdot \sqrt{1-e^2}\,.$$

where $p = 3.14159$ the ratio of the circumference to the diameter. But the area described by the planet during the indefinitely small time dt, is $\tfrac{1}{2}cdt$; hence the law of Kepler gives

$$\tfrac{1}{2}cdt : p\,a^2\sqrt{1-e^2} :: dt : T\,;$$

whence

$$c = \frac{2p\,a^2\sqrt{1-e^2}}{T}. \qquad \textbf{(85)}$$

But, by Kepler's third law, the squares of the periodic times of the planets are proportional to the cubes of their mean distances from the sun; therefore

$$T^2 = k^2 a^3,$$

k being the same for all the planets. Hence

$$c = \frac{2p\sqrt{a\left(1-e^2\right)}}{k};$$

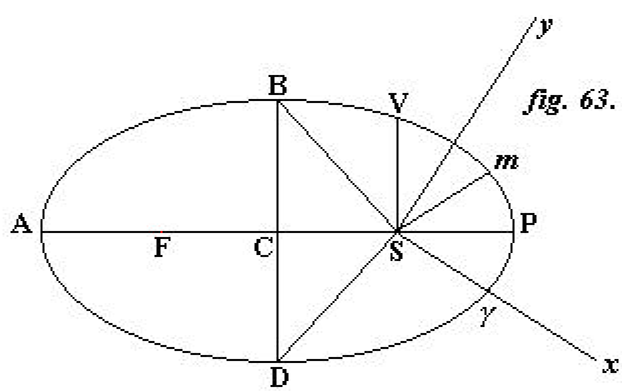


fig. 63.

but $2a\left(1-e^2\right)$ is 2SV, the parameter of the orbit.

Therefore, in different orbits compared together, the values of c are as the areas traced by the radii vectores in equal times; consequently those areas are proportional to the square roots of the parameters of the orbits, either of planets or comets. If this value of c be put in[4]

$$F = \frac{c^2}{a(1-e^2)} \cdot \frac{1}{r^2}$$

it becomes[5]

$$F = \frac{4p^2}{k^2} \cdot \frac{1}{r^2} = h \cdot \frac{1}{r^2};$$

in which $\frac{4p^2}{k^2}$ or *h*, is the same for all planets or comets; the force, therefore, varies inversely as the square of the distance of each from the centre of the sun: consequently, if all these bodies were placed at equal distances from the sun, and put in motion at the same instant from a state of rest, they would move through equal spaces in equal times; so that all would arrive at the sun at the same instant,–properties first demonstrated geometrically by Newton from the laws of Kepler.

321. That the areas described by comets are proportional to the square roots of the parameters of their orbits, is a result of theory more sensibly verified by observation than any other of its consequences. Comets are only visible for a short time, at most a few months, when they are near their perihelia; but it is difficult to determine in what curve they move, because a very eccentric ellipse, a parabola, and hyperbola of the same perihelion distance coincide through a small space on each side of the perihelion. The periodic time of a comet cannot be known from one appearance. Of more than a hundred comets, whose orbits have been computed, the return of only three has been ascertained. A few have been calculated in very elliptical orbits; but in general it has been found, that the places of comets computed in parabolic orbits agree with observation: on that account it is usual to assume, that comets move in parabolic curves.

322. In a parabola the parameter is equal to twice the perihelion distance, or

$$a(1-e^2) = 2\text{D};$$

hence, for comets,

$$c = \frac{2p}{k}\sqrt{2\text{D}}.$$

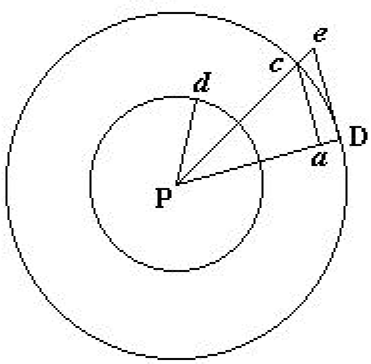
For, in this case, $e=1$ and *a*, is infinite; therefore, in different parabolae, the areas described in different times are proportional to the square roots of their perihelion distances. This affords the means of ascertaining how near a comet approaches to the sun. Five or six comets seem to have hyperbolic orbits; consequently they could only be once visible, in their transit through the system to which we belong, wandering in the immensity of space, perhaps to visit other suns and other systems.

It is probable that such bodies do exist in the infinite variety of creation, though their appearance is rare. Most of the comets that we have seen, however, are thought to move in extremely eccentric ellipses, returning to our system after very long intervals. Two hundred years have not elapsed since comets were observed with accuracy, a time which is probably greatly exceeded by the enormous periods of the revolutions of some of these bodies.

323. The discovery of the[6] laws of Kepler, deduced from the observations of Tycho Brahe,[7] and from his own observations of Mars, form an era of vast importance in the science of astronomy, being the bases on which Newton founded the universal principle of gravitation: they lead us to regard the centre of the sun as the focus of an attractive force, extending to an infinite distance in all directions, decreasing as the squares of the distance increase. Each law discloses a particular property of this force. The areas described by the radius vector of each planet or comet, being proportional to the time employed in describing them, shows that the principal force which urges these bodies, is always directed towards the centre of the sun. The ellipticity of the planetary orbits, and the nearly parabolic motion of the comets, prove that for each planet and comet this force is reciprocally as the square of the distance from the sun; and, lastly, the squares of the periodic times, being proportional to the cubes of the mean distances, proves that the areas described in equal times by the radius vector of each body in the different orbits, are proportional to the square roots of the parameters–a law which is equally applicable to planets and comets.

324. The satellites observe the laws of Kepler in moving round their primaries, and gravitate towards the planets inversely as the square of their distances from their centre; but they must also gravitate towards the sun, in order that their relative motions round their planets may be the same as if the planets were at rest. Hence the satellites must gravitate towards their planets and towards the sun inversely as the squares of the distances. The eccentricity of the orbits of the two first satellites of Jupiter is quite insensible; that of the third inconsiderable; that of the fourth is evident. The great distance of Saturn has hitherto prevented the eccentricity of the orbits of any of its satellites from being perceived, with the exception of the sixth. But the law of the gravitation of the satellites of Jupiter and Saturn is derived most clearly from this ratio,–that, for each system of satellites, the squares of their periodic times are as the cubes of their mean distances from the centres of their respective planets. For, imagine a satellite to describe a circular orbit, with a radius $PD = a$, fig. 64, its mean distance from the centre of the planet. Let T be the duration of a sidereal revolution of the satellite, then $3.14159 = p$, being the ratio of the circumference to the diameter, $a \cdot \frac{2p}{T}$ will be the very small arc D*c* that the satellite describes in a second. If the attractive force of the planet were to cease for an instant, the satellite would f1y off in the tangent D*e*, and would be farther from the centre of the planet by a quantity equal to *a*D, the versed sine of the arc D*c*. But the value of the versed sine is

fig. 64.



$$a \cdot \frac{2p^2}{T^2},$$

which is the distance that the attractive force of the planet causes the satellite to fall through in a second.

Now, if another satellite be considered, whose mean distance is $\text{PD} = a'$, and $T_{/}$, the duration of its sidereal revolution, its deflection will be[8] $a' \cdot \frac{2p^2}{T_{/}^2}$ in a second; but if F and F' be the attractive forces of the planet at the distances PD and P*d*, they will evidently be proportional to the quantities they make the two satellites fall through in a second; hence

$$F : F' :: a\frac{2p^2}{T^2} : a'\frac{2p^2}{T_{/}^2},$$

or

$$F : F' :: \frac{a}{T^2} : \frac{a'}{T_{/}^2};$$

but the squares of the periodic times are as the cubes of the mean distances; hence

$$T^2 : T_{/}^2 :: a^3 : a'^3;$$

Thus the satellites gravitate to their primaries inversely as the square of the distance.

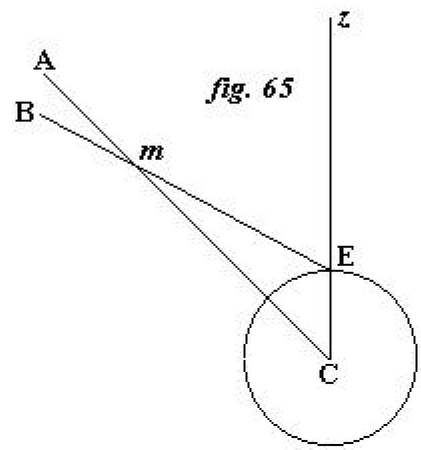
325. As the earth has but one satellite, this comparison cannot be made, and therefore the ellipticity of the lunar orbit is the only celestial phenomenon by which we can know the law of the moon's attractive force. If the earth and the moon were the only bodies in the system, the moon would describe a perfect ellipse about the earth; but, in consequence of the action of the sun, the path of the moon is sensibly disturbed, and therefore is not a perfect ellipse; on this account some doubts may arise as to the diminution of the attractive force of the earth as the inverse square of the distance.

The analogy, indeed, which exists between this force and the attractive force of the sun, Jupiter, and Saturn, would lead to the belief that it follows the same law, because the solar attraction acts equally on all bodies placed at the same distance from the sun, in the same manner that terrestrial gravitation causes all bodies in vacuo to fall from equal heights in equal times. A projectile thrown horizontally from a height, falls to the earth after having described a parabola. If the force of projection were greater, it would fall at a greater distance; and if it amounted to 30,772.4 feet in a second, and were not resisted by the air, it would revolve like a satellite about the earth, because its centrifugal force would then be equal to its gravitation. This body would move in all respects like the moon, if it were projected with the same force, at the same height.

It may be proved, that the force which causes the descent of heavy bodies at the surface of the earth, diminished in the inverse ratio of the square of the distance, is sufficient to retain the moon in her orbit, but this requires a knowledge of the lunar parallax.

On Parallax

326. Let *m*, fig. 65, be a body in its orbit, and C the centre of the earth, assumed to be spherical. A person on the surface of the earth, at E, would see the body *m* in the direction E*mB*; but the body would appear, in the direction C*mA*, to a person in C, the centre of the earth. The



angle CmE, which measures the difference of these directions, is the parallax of m. If z be the zenith of an observer at E, the angle zEm, called the zenith distance of the body, may be measured; hence mEC is known, and the difference between zEm and zCm is equal to CmE, the parallax, then if $\mathrm{CE} = R$, $\mathrm{C}m = r$, and $z\mathrm{E}m = z$, [then][9]

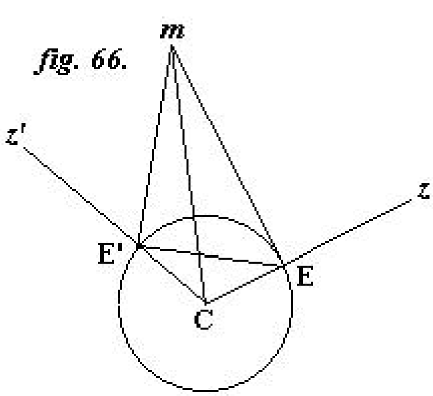
$$\sin \mathrm{C}m\mathrm{E} = \frac{R}{r}\sin z\,;$$

hence, if CE and Cm remain the same, the sine of the parallax, CmE, will vary as the sine of the zenith distance zEm; and when $z\mathrm{E}m = 90^\circ$, as in fig. 67,

$$\sin \mathrm{P} = \frac{R}{r};$$

P being the value of the angle CmE in this case; then the parallax is a maximum, for Em is tangent to the earth, and, as the body m is seen in the horizon, it is called the horizontal parallax; hence the sine of the horizontal parallax is equal to the terrestrial radius divided by the distance of the body from the centre of the earth.

327. The length of the mean terrestrial radius is known, the horizontal parallax may be determined by observation, therefore the distance of m from the centre of the earth is known. By this method the dimensions of the solar system have been ascertained with great accuracy. If the distance be very great compared with the diameter of the earth, the parallax will be insensible. if CmE were an angle of the fourth of a second, it would be inappreciable; an arc of $1'' = 0.000004848$ of the radius, the fourth of a second is therefore $0.000001212 = \frac{1}{825{,}082}$; and thus, if a body be distant from the earth by 825,082 of its semidiameters, or 3,265,660,000 miles, it will be seen in the same position from every point of the earth's surface. The parallax of all the celestial bodies is very small: even that of the moon at its maximum does not much exceed 1°.



328. P being the horizontal parallax, let p be the parallax EmC, fig. 66, at any height. When P is known, p may be found, and the contrary, for if $\frac{R}{r}$ be eliminated, then $\sin p = \sin P \sin z$, and when P is constant, $\sin p$ varies as $\sin z$.

329. The horizontal parallax is determined as follows: let E and E', fig. 66, be two places on the same meridian of the earth's surface; that is, which contemporaneously have the same noon. Suppose the latitudes of these two places to be perfectly known; when a body m is on the meridian, let its zenith distances $z\mathrm{E}m = z$, $z'E'm = z'$, be measured by two observers in E and E'. Then ECE', the sum of the latitudes, is known, and also the angles CEm, CE$'m$; hence EmE$'$, EmC, and E$'m$C may be determined; for P is so small, that it may be put for its sine; therefore

$$\sin p = P\sin z,\ \sin p' = P\sin z';$$

and as p and p' are also very small,[10]

$$p + p' = P\{\sin z + \sin z'\}.$$

Now, $p + p'$ is equal to the angle EmE$'$, under which the chord of the terrestrial arc EE$'$, which joins the two observers, would be seen from the centre of m, and it is the fourth angle of the quadrilateral CEmE$'$. But

$$\mathrm{CE}m = 180^\circ - z,\ \mathrm{CE'}m = 180^\circ - z',$$

and if

$$\mathrm{EC}m + \mathrm{E'C}m = f\ ,$$

then

$$180^\circ - z + 180^\circ - z' + p + p' + f = 360^\circ;$$

hence

$$p + p' = z + z' - f\ ;$$

therefore the two values of $p + p'$ give

$$P = \frac{z + z' - f}{\sin z + \sin z'},$$

which is the horizontal parallax of the body, when the observers are on different sides of Cm; but when they are on the same side,

$$P = \frac{z - z' - f}{\sin z - \sin z'}.$$

It requires a small correction, since the earth, being a spheroid, the lines[11] zE, z'E$'$ do not pass through C, the centre of the earth.

The parallax of the moon and of Mars were determined in this manner, from observations made by Lacaille[12] [13] at the Cape of Good Hope, in the southern hemisphere; and by Wargesten at Stockholm, which is nearly on the same meridian in the northern hemisphere.

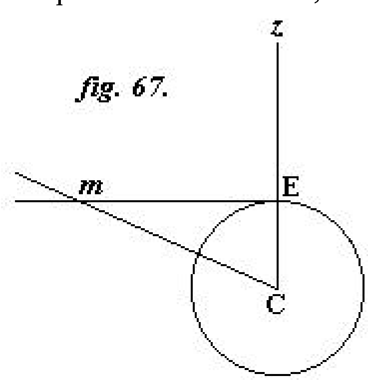
330. The horizontal parallax varies with the distance of the body from the earth; for it is evident that the greater the distance, the less the parallax. It varies also with the parallels of terrestrial latitude, the earth, being a spheroid, the length of the radius decreases from the equator to the poles. It is on this account that, at the mean distance of the moon, the horizontal parallax observed in different latitudes varies; proving the elliptical figure of the earth. The difference between the mean horizontal parallax at the equator and at the poles, from this cause, is $10''.3$.

331. In order to obtain a value of the moon's horizontal parallax, independent of these inequalities, the horizontal parallax is chosen at the mean distance of the moon from the earth, and on that parallel of terrestrial latitude, the square of whose sine is $\frac{1}{3}$, because the attraction of the earth upon the corresponding points of its surface is nearly equal to the mass of the earth, divided by the square of the mean distance of the moon from the earth. This is called the constant part of the horizontal parallax. The force which retains the moon in her orbit may now be determined.

Force of Gravitation at the Moon

332. If the force of gravity be assumed to decrease as the inverse square of the distance, it is clear that the force of gravity at E, fig. 67, would be, to the same force at *m*, the distance of the moon, as the square of C*m* to the square of CE; but CE divided by C*m* is the sine of the horizontal parallax of the moon, the constant part of which is found by observation to be $57'4.17''$ in the latitude in question; hence the force of gravity, reduced to the distance of the moon, is equal to the force of gravity at E on the earth's surface, multiplied by $\sin^2 57'4''.17$, the square of the sine of the constant part of the horizontal parallax.

fig. 67.



Since the earth is a spheroid, whose equatorial diameter is greater than its polar diameter, the force of gravity increases from the equator to the poles; but it has the same intensity in all points of the earth's surface in the same latitude.

Now the space through which a heavy body would fall during a second in the latitude the square of whose sine is $\frac{1}{3}$, has been ascertained by experiments with the pendulum to be 16.0697 feet; but the effect of the centrifugal force makes this quantity less than it would otherwise be, since that force has a tendency to make bodies fly off from the earth. At the equator it is equal to the 288th part of gravity; but as it decreases from the equator to the poles as the square of the sine of the latitude, the force of gravity in that latitude the square whose sine is $\frac{1}{3}$, is only diminished by two-thirds of $\frac{1}{288}$ or by its 432nd part. But the 432nd part of 16.0697 is 0.0372, and adding it to 16.0697, the whole effect of terrestrial gravity in the latitude in question is 16.1069 feet; and at the distance of the moon it is $16.1069 \cdot \sin^2 57'4''.17$ nearly. But in order to have this quantity more exactly it must be multiplied by $\frac{357}{358}$, because it is found by the theory of the moon's motion, that the action of the sun on the moon diminishes its gravity to the earth by a quantity, the constant part of which is equal to the 358th part of that gravity.

Again, it must be multiplied by $\frac{76}{75}$, because the moon in her relative motion round the earth, is urged by a force equal to the sum of the masses of the earth and moon divided by the square of C*m*, their mutual distance. It appears by the theory of the tides that the mass of the moon is only the $\frac{1}{75}$ of that of the earth which is taken as the unit of measure; hence the sum of the masses of the two bodies is

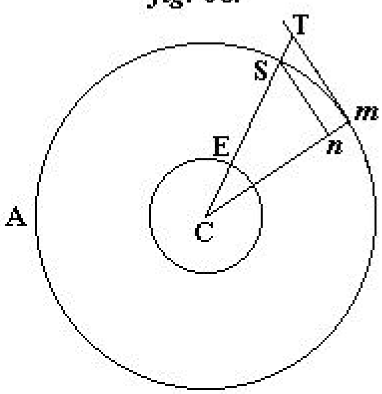
$$1+\frac{1}{75}=\frac{76}{75}.$$

Then if the terrestrial attraction be really the force that retains the moon in her orbit, she must fall through

$$16.1069\times\sin^2 57' \, 4''.17\times\frac{357}{358}\times\frac{76}{75}=0.00448474$$

of a foot in a second.

fig. 68.



333. Let mS, fig. 68, be the small arc which the moon would describe in her orbit in a second, and let C be the centre of the earth. If the attraction of the earth were suddenly to cease, the moon would go off in the tangent mT; and at the second she would be in T instead of S; hence the space that the attraction of the earth causes the moon to fall through in a second, is equal to mn the versed sine of the arc Sm.

The arc Sm is found by simple proportion, for the periodic time of the moon is 27.32166 days or $2{,}360{,}591''$, and since the lunar orbit without sensible error may be assumed equal to the circumference of a circle whose radius is the mean distance of the moon from the earth; it is

O Z ′ ,

therefore[14]

$$2{,}360{,}591'' : 1'' :: 2\mathrm{C}m\cdot\frac{355}{113} : \mathrm{S}m$$

and[15]

$$\mathrm{S}m=\frac{2(355)\,.\,\mathrm{C}m\cdot 1^{\text{sec.}}}{113(2{,}360{,}591^{\text{sec.}})}.$$

The arc Sm is so small that it may be taken for its chord, therefore $(m\mathrm{S})^2=\mathrm{C}m\,.\,mn$; hence[16]

$$\frac{4(355)^2(\mathrm{C}m)^2}{(113)^2(2{,}360{,}591'')^2}=2\mathrm{C}m\,.\,mn;$$

consequently

$$mn = \frac{2(355)^2 \,.\, \mathrm{C}m}{(113)^2 (2{,}360{,}591'')^2} \,.$$

Again, the radius CE of the earth in the latitude the square of whose sine is $\frac{1}{3}$, is computed to be 20,898,700 feet from the mensuration of the degrees of the meridian: and since

$$\frac{\mathrm{CE}}{\mathrm{C}m} = \sin 57' \, 4''.17 \,,$$

$$\mathrm{C}m = \frac{\mathrm{CE}}{\sin 57' \, 4''.17} = \frac{20{,}898{,}700}{\sin 57' \, 4''.17},$$

consequently,

$$mn = \frac{2(355)^2 (20{,}898{,}700)}{(113)^2 (2{,}360{,}591'')^2 \sin 57' \, 4''.17} = 0.00445983$$

of a foot, which is the measure of the deflecting force at the moon. But the space described by a body in one second from the earth's attraction at the distance of the moon was shown to be 0.00448474 of a foot in a second; the difference is therefore only the 0.00002491 of a foot, a quantity so small, that it may safely be ascribed to errors in observation.

334. Hence it appears that the force[17] that retains the moon in her orbit is terrestrial gravity, diminished in the ratio of the square of the distance. The same law then, which was proved to apply to a system of satellites, by a comparison of the squares of the times of their revolutions, with the cubes of their mean distances, has been demonstrated to apply equally to the moon, by comparing her motion with that of bodies falling at the surface of the earth.

335. In this demonstration, the distances were estimated from the centre of the earth, and since the attractive force of the earth is of the same nature with that of the other celestial bodies, it follows that the centre of gravity of the celestial bodies is the point from whence the distances must be estimated, in computing the effects of their attraction on substances at their surfaces, or on bodies in space.

336. Thus the sun possesses an attracting force, diminishing to infinity inversely as the squares of the distances, which includes all the bodies of the system in its action; and the planets which have satellites exert[18] a similar influence over them.

Analogy would lead us to suppose that the same force exists in all the planets and comets; but that this is really the case will appear, by considering that it is a fixed law of nature that one body cannot act upon another without experiencing an equal and contrary reaction from that body: hence the planets and comets, being attracted towards the sun, must reciprocally attract the sun towards them according to the same law; for the same reason, satellites attract their planets. This property of attraction being common to planets, comets, and satellites, the gravitation of the heavenly bodies towards one another may be considered as a general principle of the[19] universe; even the irregularities in the motions of these bodies are susceptible of being so well explained by this principle, that they concur in proving its existence.

337. Gravitation is proportional to the masses; for supposing the planets and comets to be at the same distance from the sun, and left to the action of gravity, they would fall through equal heights in equal times. The nearly circular orbits of the satellites prove that they gravitate like their planets towards the sun in the ratio of their masses: the smallest deviation from that ratio would be sensible in their motions, but none depending on that cause has been detected by observation.

338 Thus the planets, comets, and satellites, when at the same distance from the sun, gravitate as their masses; and as reaction is equal and contrary to action, they attract the sun in the same ratio; therefore their action on the sun is proportional to their masses divided by the square of their distances from his centre.

339. The same law obtains on earth; for very correct observations with the pendulum prove, that were it not for the resistance of the air, all bodies would fall towards its centre with the same velocity. Terrestrial bodies then gravitate towards the earth in the ratio of their masses, as the planets gravitate towards the sun, and the satellites towards their planets. This conformity of nature with itself upon the earth, and in the immensity of the heavens, shows, in a striking manner, that the gravitation we observe here on earth is only a particular case of a general law, extending throughout the system.

340. The attraction of the celestial bodies does not belong to their mass alone taken in its totality, but exists in each of their atoms, for if the sun acted on the centre of gravity of the earth without acting on each of its particles separately, the tides would be incomparably greater, and very different from what they now are. Thus the gravitation of the earth towards the sun is the sum of the gravitation of each of its particles; which in their turn attract the sun as their respective masses; besides, everything on earth gravitates towards the centre of the earth proportionally to its mass; the particle then reacts on the earth, and attracts it in the same ratio; were that not the case, and were any part of the earth however small not to attract the other part as it is itself attracted, the centre of gravity of the earth would be moved in space in virtue of this gravitation, which is impossible.

341. It appears then, that the celestial phenomena when compared with the laws of motion, lead to this great principle of nature, that all the particles of matter mutually attract each other as their masses directly, and as the squares of their distances inversely.

342. From the universal principle of gravitation, it may be foreseen, that the comets and planets will disturb each other's motion, so that their orbits will deviate a little from perfect ellipses; and the areas will no longer be exactly proportional to the time: that the satellites, troubled in their paths by their mutual attraction, and by that of the sun, will sensibly deviate from elliptical motion: that the particles of each celestial body, united by their mutual attraction, must form a mass nearly spherical; and that the resultant of their action at the surface of the body, ought to produce there all the phenomena of gravitation. It appears also, that centrifugal force arising from the rotation of the celestial bodies must alter their spherical form a little by flattening them at their poles; and that the resulting force of their mutual attractions not passing through their centres of gravity, will produce those motions that are observed in their axes of rotation. Lastly, it is clear that the particles of the ocean being unequally attracted by the sun and moon, and with a different intensity from the nucleus of the earth, must produce the ebb and flow of the sea.

343. Having thus proved from Kepler's laws, that the celestial bodies attract each other directly as their masses, and inversely as the square of the distance, Laplace inverts the problem, and assuming the law of gravitation to be that of nature, he determines the motions of the planets by the general theorem in article 144, and compares the results with observation.

Notes

[1] This word is not capitalized in 1st edition.
[2] The 1st edition omits the word "half" (published erratum).

[3] The equation *reads* $F = \frac{c}{a\left(1-e^2\right)} \cdot \frac{1}{r^2}$ in the 1st edition (published erratum).

[4] The expression *reads* $F = \frac{c}{a(1-e^2} \cdot \frac{1}{r^2}$ in the 1st edition (published erratum).

[5] Punctuation in this expression in the 1st edition is misplaced as $F = \frac{4p^2}{k^2} \cdot \frac{1}{r^2}; = h. \frac{1}{r^2}$.

[6] This *reads*, "The three laws of Kepler," *for* "The discovery of the laws of Kepler" in the 1st edition (published erratum).

[7] See note 6, *Book II, Chapter I.*

[8] This *reads* $a' \frac{2p^2}{T_/^2}$ in the 1st edition.

[9] This *reads* sine C*m*E = $\frac{R}{r} \sin z$ in the 1st edition.

[10] The punctuation in 1st edition is contained within the parenthesis as $p + p' = P\{\sin z + \sin z'.\}$

[11] This *reads* ZE, $Z'E'$ *for* zE, $z'E'$ in the 1st edition (published erratum).

[12] Somerville spells the name La Caille in the 1st edition.

[13] Lacaille, Nicolas Louis de, 1713-1762 astronomer, born in Rumigny, France. From 1750 to 1754 he compiled a catalogue of nearly 10,000 southern stars and was first to measure the arc of the meridian in South Africa. Lacaille's *Coelum Australe Stelliferum* was published in 1763.

[14] Expressed $2360591'' : 1'' :: 2Cm \frac{355}{113} : Sm$ in the 1st edition.

[15] Expressed $Sm = \frac{2(355) \cdot Cm \cdot 1^{\text{sec.}}}{113(2360591'')}$ in the 1st edition.

[16] Expressed $\frac{4(355)^2 (Cm)^2}{(113)^2 (2360591'')^2} = 2Cm \cdot mn$ in the 1st edition.

[17] This *reads* "principal force" *for* "force." in the 1st edition (published erratum).
[18] This *reads* "exact" *for* "exert" in the 1st edition (published erratum).
[19] This *reads* "this" *for* "the" in the 1st edition (published erratum).

BOOK II

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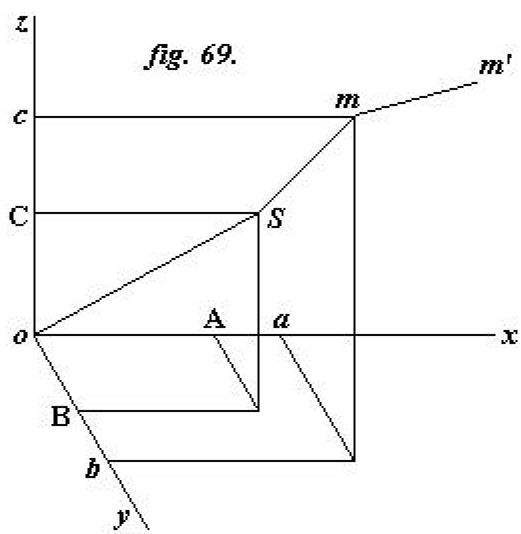
CHAPTER III

ON THE DIFFERENTIAL EQUATIONS OF THE MOTION OF A SYSTEM OF BODIES, SUBJECTED TO THEIR MUTUAL ATTRACTIONS

344. AS the earth which we inhabit is a part of the solar system, it is impossible for us to know any thing of its absolute motions; our observations must therefore be limited to its relative motions. In estimating the relative motion of planets, it is usual to refer them to the centre of the sun, and those of satellites to the centres of their primary planets. The sun and planets mutually attract each other; but in estimating the motions of a planet, the sun is supposed to be at rest, and all the motion is referred to the planet, which thus moves in consequence of the difference between its own action, and that of the sun. It is the same with regard to satellites and their primaries.

345. To determine the relative motions of a system of bodies m, m', m'', &c. fig. 69, considered as points revolving about one body S, which is the centre of their motions–



Let $\bar{x}$, $\bar{y}$, $\bar{z}$, be the co-ordinates of S referred to o as an origin, and x, y, z, x', y', z', &c. the co-ordinates of the bodies m, m', &c. referred to S as their origin. Then the co-ordinates of m when referred to o, are[1] $\bar{x}+x$, $\bar{y}+y$, $\bar{z}+z$, for it is easy to see that

$$\bar{x}+x=\text{OA}+\text{A}a,\quad \bar{y}+y=\text{OB}+\text{B}b,\quad \bar{z}+z=\text{OC}+\text{C}c\,.$$

In the same manner, the co-ordinates of m', when referred to o, are[2] $\bar{x}+x'$, $\bar{y}+y'$, $\bar{z}+z'$, and so for the other bodies. Let the distances of the bodies from S, or

$$Sm=\sqrt{x^2+y^2+z^2}\quad Sm'=\sqrt{x'^2+y'^2+z'^2}\,,\ \&\text{c.}$$

re represented by r, r', r'', &c. and the masses by m, m', &c. and S. The equations of the motion of m will be first determined.

346. The whole action of the system relative to m consists three parts:

1. Of the action of S on m.
2. Of the action of all the bodies m', m'', m''', &c. on m.
3. Of the action of all the bodies m, m', m'', &c. on S.

These will be determined separately [below].

i. The action of S on m is $-\frac{S}{r^2}$, that is directly as its mass, and inversely as the square of its distance. It has a negative sign, because the body S draws m towards the origin of the co-ordinates. This force when resolved in the direction ox is $-\frac{Sx}{r^3}$; for the force $-\frac{S}{r^2}$ is to its component force in ox, as Sm to Aa, that is as r to x.

ii. The distance of m' from m is

$$\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}$$

for x, y, z, x', y', z', being the co-ordinates of m and m' referred to S as their origin, the distance of these bodies from each other is the diagonal of a parallelopiped whose sides are $x'-x$, $y'-y$, $z'-z$. For the same reason, the distance of m'' from m is

$$\sqrt{(x''-x)^2+(y''-y)^2+(z''-z)^2}\text{, \&c.}$$

in order to abridge, let

$$l=\frac{m\cdot m'}{\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}}+\frac{m\cdot m''}{\sqrt{(x''-x)^2+(y''-y)^2+(z''-z)^2}}+\text{\&c.}$$

it is evident that[3]

$$\frac{1}{m}\left(\frac{d\,l}{dx}\right)=\frac{m'(x'-x)}{\left\{(x'-x)^2+(y'-y)^2+(z'-z)^2\right\}^{\frac{3}{2}}}+\frac{m''(x''-x)}{\left\{(x''-x)^2+(y''-y)^2+(z''-z)^2\right\}^{\frac{3}{2}}}+\&c.$$

is the sum of the actions of all the bodies m', m'', &c. on m when resolved in the direction ox. Hence the whole action of the system on m resolved in the axis[4] ox is

$$\frac{1}{m}\left(\frac{d\,l}{dx}\right)-\frac{Sx}{r^3};$$

but by the general theorem of motion

$$\frac{1}{m}\left(\frac{d\,l}{dx}\right)-\frac{Sx}{r^3}=\frac{d^2(\overline{x}+x)}{dt^2}, \qquad \textbf{(86)}$$

for $\bar{x}+x$ is the co-ordinate *oa*, or the distance of *m* from *o* in the direction *ox*.[5]

iii. The action of *m* on *S* is $\frac{m}{r^2}$, and its component force in *ox* is $\frac{mx}{r^3}$; likewise the actions of m', m'', &c. on *S*, when resolved in the same axes, are $\frac{m'x'}{r'^3}$, $\frac{m''x''}{r''^3}$, &c. hence the action of the system on *S* in the axis[6] *ox*, may be expressed by[7] $\Sigma\cdot\frac{mx}{r^3}$; but by the general theorem

$$\Sigma\cdot\frac{mx}{r^3}=\frac{d^2\bar{x}}{dt^2},$$

for the co-ordinates of *S* alone vary by this action. Now, if[8] $\Sigma\cdot\frac{mx}{r^3}$ be put for $\frac{d^2\bar{x}}{dt^2}$, in the equation (86) it becomes

$$0=\frac{d^2x}{dt^2}+\frac{Sx}{r^3}+\Sigma\cdot\frac{mx}{r^3}-\frac{1}{m}\left(\frac{d\,l}{dx}\right),$$

which is the whole action of the system relatively to *m*, when resolved in the direction *ox*, and because

$$\Sigma\cdot\frac{my}{r^3}=\frac{d^2\bar{y}}{dt^2},\qquad \Sigma\cdot\frac{mz}{r^3}=\frac{d^2\bar{z}}{dt^2};$$

the other two component forces are[9]

$$0=\frac{d^2y}{dt^2}+\frac{Sy}{r^3}+\Sigma\cdot\frac{my}{r^3}-\frac{1}{m}\left(\frac{d\,l}{dy}\right),$$
$$0=\frac{d^2z}{dt^2}+\frac{Sz}{r^3}+\Sigma\cdot\frac{mz}{r^3}-\frac{1}{m}\left(\frac{d\,l}{dz}\right).$$

The same equations will give the motions of m', m'', &c. round *S*, if m', x', y', z'; m'', x'', y'', z'', &c. be successively put for *m*, *x*, *y*, *z*, and *vice versâ*, and the equations

$$\frac{d^2\bar{x}}{dt^2}=\Sigma\cdot\frac{mx}{r^3},\quad \frac{d^2\bar{y}}{dt^2}=\Sigma\cdot\frac{my}{r^3},\quad \frac{d^2\bar{z}}{dt^2}=\Sigma\cdot\frac{mz}{r^3},$$

determine the motion of S.

347. These equations, however, may be put under a more convenient form for

$$\Sigma\cdot\frac{mx}{r^3}=\frac{mx}{r^3}+\frac{m'x'}{r'^3}+\&c.$$

and if $S+m$ the sum of the masses of the sun and of a planet, or of a planet and its satellite, be[10] represented by $m,$ the equation in x becomes

$$0=\frac{d^2x}{dt^2}+\frac{mx}{r^3}+\frac{m'x'}{r'^3}+\&\text{c.}-\frac{1}{m}\left(\frac{d\,l}{dx}\right).$$

The part $\frac{d^2x}{dt^2}+\frac{mx}{r^3}$ relates only to the undisturbed elliptical motion of m round S; it is much greater than the remaining part

$$\frac{m'x'}{r'^3}+\frac{m''x''}{r''^3}+\&\text{c.}-\frac{1}{m}\left(\frac{d\,l}{dx}\right),$$

which contains all the disturbances to which the body m is subject from the action of the other bodies of the system. [The term] $-\frac{1}{m}\left(\frac{d\,l}{dx}\right)$ contains the direct action of the bodies $m',$ $m'',$ &c. on m; but m is also troubled indirectly by the action of these bodies on[11] S; this part is contained in $\frac{m'x'}{r'^3}+\frac{m''x''}{r''^3}$+&c.

By the latter action S is drawn to or from m; and by the former, m is drawn to or from S; in both cases altering the relative positions of S and m. Let

$$R=\frac{m'}{\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}}-\frac{m'(x'x+y'y+z'z)}{r'^3}$$
$$+\frac{m''}{\sqrt{(x''-x)^2+(y''-y)^2+(z''-z)^2}}-\frac{m''(x''x+y''y+z''z)}{r''^3}+\&\text{c.}$$

where it is easy to see that

$$-\frac{dR}{dx}=\frac{m'x'}{r'^3}+\frac{m''x''}{r''^3}+\&\text{c.}-\frac{1}{m}\left(\frac{d\,l}{dx}\right),$$
$$-\frac{dR}{dy}=\frac{m'y'}{r'^3}+\frac{m''y''}{r''^3}+\&\text{c.}-\frac{1}{m}\left(\frac{d\,l}{dy}\right),$$

$$-\frac{dR}{dz}=\frac{m'z'}{r'^3}+\frac{m''z''}{r''^3}+\&\text{c.}-\frac{1}{m}\left(\frac{d\,l}{dz}\right),$$

and therefore the preceding equations become[12]

$$\begin{aligned}\frac{d^2x}{dt^2}+\frac{mx}{r^3}&=\left(\frac{dR}{dx}\right),\\ \frac{d^2y}{dt^2}+\frac{my}{r^3}&=\left(\frac{dR}{dy}\right),\\ \frac{d^2z}{dt^2}+\frac{mz}{r^3}&=\left(\frac{dR}{dz}\right).\end{aligned} \qquad \textbf{(87)}$$

The whole motions of the planets and satellites are derived from these equations, for S may either be considered to be the sun, and $m,\ m',$ &c. planets; or S may be taken for a planet, and $m,\ m',$ &c. for its satellites.

If one planet only moved round the sun, its orbit would be a perfect ellipse, but by the attraction of the other planets, its elliptical motion is very much altered, and rendered extremely complicated.

348. It appears then, that the problem of planetary motion, in its most general sense, is the determination of the motion of a body when attracted by one body, and disturbed by any number of others. The only results that can be obtained from the preceding equations, which express this general problem, are the principle of areas and living forces; and that the motion of the centre of gravity is uniform, rectilinear, and in no way affected by the mutual action of the bodies. As these properties have been already proved to exist in a system of bodies mutually attracting each other, whatever the law of the force might be, provided that it could be expressed in functions of the distance; it evidently follows, that they must exist in the solar system, where the force is inversely as the square of the distance, which is only a particular case of the more general theorem. As no other results can be obtained from these general equations in the present state of analysis, the effects of one disturbing body is estimated at a time, but as this can be repeated for each body in the system, the disturbing action of all the planets on any one may be found.

349. The problem of planetary motion when so limited is, to determine, at any given time, the place of a body when attracted by one body and disturbed by another, the masses, distances, and positions of the bodies being given. This is the celebrated problem of three bodies; it is extremely complicated, and the most refined and laborious analysis is requisite to select among the infinite number of inequalities to which the planets are liable, those that are perceptible, and to assign their values. Although this problem has employed the greatest mathematicians from Newton to the present day, it can only be solved by approximation.

350. The action of a planet on the sun, or of a satellite on its primary, shortens its periodic time, if the planet be very large when compared with the sun, or the satellite when compared with its primary; for, as the ratio of the cube of the greater axis of the orbit to the square of the periodic time is proportional to the sum of the masses of the sun and the planet, Kepler's law would vary in the different orbits, according to the masses if they were considerable. But as the law is nearly the same for all the planets, their masses must be very small in comparison to that of the sun; and it is the same with regard to the satellites and their primaries. The volumes of the sun and planets confirm this; if the centre of the sun were to coincide with the centre of the earth, his volume would not only include the orbit of the moon, but would extend as far again, whence we may form some idea of his magnitude; and even Jupiter, the largest planet of the solar system, is incomparably smaller than the sun.

351. Thus any modifications in the periodic times, that could be produced by the action of the planets on the sun, must be insensible. As the masses of the planets are so small, their disturbing forces are very much less than the force of the sun, and therefore their orbits, although not strictly elliptical, are nearly so; and the areas described so nearly proportional to the time, that the action of the disturbing force may at first be neglected; then the body may be estimated to move in a perfect ellipse. Hence the first approximation is, to find the place of a body revolving round the sun in a perfect ellipse at a given time. In the second approximation, the greatest effects of the disturbing forces are found; in the third, the next greatest, and so on progressively, till they become so small, that they may be omitted in computation without sensible error. By these approximations, the place of a body may be found with very great accuracy, and that accuracy is verified by comparing its computed place with its observed place. The same method applies to the satellites.

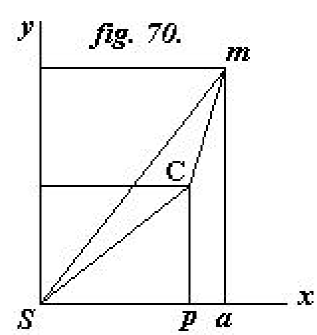
Fortunately, the formation of the planetary system affords singular facilities for accomplishing these approximations: one of the principal circumstances is the division of the system into partial systems, formed by the planets and their satellites. These systems are such, that the distances of the satellites from their primaries are very much less than the distances of their primaries from the sun. Whence, the action of the sun being very nearly the same on the planet and on its satellites, the satellites move very nearly as if they were only influenced by the attraction of the planet.

Motion of the Centre of Gravity

352. From this formation it also follows, that the motion of the centre of gravity of a planet and its satellites, is very nearly the same as if all these bodies were united in one mass at that point.

Let C be the centre of gravity of a system of bodies m, m', m'', &c., as, for example, of a planet and its satellites, and let[13] S be any body not belonging to the system, as the sun.

fig. 70.



It was shown, in the first book,[14] that the force which urges the centre of gravity of a system of bodies parallel to any straight line, Sx, is equal to the sum of the forces which urge the bodies m, m', &c. parallel to this straight line, multiplied respectively by their masses, the whole being divided by the sum of their masses.

It was also shown, that the mutual action and attraction of bodies united together in any manner whatever, has no effect on the centre of gravity of the system, whether at rest or in motion. It is, therefore, sufficient to determine the action of the body S, not belonging to the system, on its centre of gravity.

Let $\bar{x}$, $\bar{y}$, $\bar{z}$, be the co-ordinates of C, fig. 70, the centre of gravity of the system referred to *S*, the centre of the sun; and let x, y, z, x', y', z', &c., be the co-ordinates of the bodies m, m', m'', &c., referred to C, their common centre of gravity. Imagine also, that the distances Cm, Cm', &c., of the bodies from their centre of gravity, are very small in comparison of *S*C, the distance of the centre of gravity from the sun. The action of the body m on the sun at *S*, when resolved in the direction *Sx*, is

$$\frac{m\cdot(\bar{x}+x)}{r^3},$$

in which m is the mass of the body, and[15]

$$r=\sqrt{(\bar{x}+x)^2+(\bar{y}+y)^2+(\bar{z}+z)^2}\ .$$

But the action of the sun on m is to the action of m on the sun, as *S*, the mass of the sun, to m, the mass of the body; hence the action of these two bodies on C, the centre of gravity of the system, is

$$-S\cdot\frac{\frac{m(\bar{x}+x)}{r^3}}{m}.$$

The same relation exists for each of the bodies; if we therefore represent the sum of the actions in the axis[16] [17] *ox* by

$$\Sigma\cdot\frac{m(\bar{x}+x)}{r^3},$$

and the sum of the masses $\Sigma.m$, by the whole force that acts on the centre of gravity in the direction *Sx* will be

$$-S\cdot\frac{\Sigma\cdot\frac{m(\bar{x}+x)}{r^3}}{\Sigma.m}.$$

Now, $\bar{x}+x$, fig. 70, is equal to $Sp+pa$, but *Sp* and *pa*, are the distances of the sun and of the body m from C, estimated on *Sx*; as *pa* is incomparably less than *Sp*, the square of *pa* may be omitted without sensible error, and also the squares of y and z, together with the products of these small quantities; then if

$$\bar{r}=S\mathrm{C}=\sqrt{\bar{x}^2+\bar{y}^2+\bar{z}^2}\ ,$$

the quantity $\frac{\bar{x}+x}{r^3}$ becomes[18]

$$\frac{\bar{x}+x}{\left\{\bar{r}^2+2\left(\bar{x}x+\bar{y}y+\bar{z}z\right)\right\}^{\frac{3}{2}}},$$

or

$$\left(\bar{x}+x\right)\left\{\bar{r}^2+2\left(\bar{x}x+\bar{y}y+\bar{z}z\right)\right\}^{-\frac{3}{2}}.$$

And expanding this by the binomial theorem, it becomes[19]

$$\frac{\bar{x}}{\bar{r}^3}+\frac{x}{\bar{r}^3}-\frac{3\bar{x}\left\{\bar{x}x+\bar{y}y+\bar{z}z\right\}}{\bar{r}^5}.$$

Now, the same expression will be found for x', y', z', &c., the co-ordinates of the other bodies; and as by the nature of the centre of gravity $\sum .mx=0$, $\sum .my=0$, $\sum .mz=0$, the expression[20]

$$-S\cdot\frac{\sum\cdot\frac{m\left(\bar{x}+x\right)}{r^3}}{\sum .m} \text{ becomes } -\frac{S\cdot\bar{x}}{r^3},$$

that is, when the squares and products of the small quantities x, y, z, &c., are omitted; hence the centre of gravity of the system is urged by the action of the sun in the direction Sx, as if all the masses were united in C, their common centre of gravity. It is evident that

$$-\frac{S\,.\,\bar{y}}{\bar{r}^3},\qquad -\frac{S\,.\,\bar{z}}{\bar{r}^3},$$

are the forces urging the centre of gravity in the other two axes.

353. In considering the relative motion of the centre of gravity of the system round S, it will be found that the action of the system of bodies m, m', m'', &c., on S in the axes ox, oy, oz, are

$$\frac{\bar{x}\,.\sum m}{\bar{r}^3};\ \frac{\bar{y}\,.\sum m}{\bar{r}^3};\ \frac{\bar{z}\,.\sum m}{\bar{r}^3},$$

when the squares and products of the distances of the bodies from their common centre of gravity are omitted. These act in a direction contrary to the origin. Whence the action of the system on S is nearly the same as if all their masses were united in their common centre of gravity; and the centre of gravity is urged in the direction of the axes by the sum of the forces, or by

$$-\{S+\Sigma.m\}\frac{\bar{x}}{\bar{r}^3},$$
$$-\{S+\Sigma.m\}\frac{\bar{y}}{\bar{r}^3}, \qquad \textbf{(88)}$$
$$-\{S+\Sigma.m\}\frac{\bar{z}}{\bar{r}^3};$$

and thus the centre of gravity moves as if all the masses m, m', m'', &c., were united in their common centre of gravity; since the coordinates of the bodies m, m', m'', &c., have vanished from all the preceding results, leaving only $\bar{x}$, $\bar{y}$, $\bar{z}$, those of the centre of gravity.

From the preceding investigation, it appears that the system of a planet and its satellites, acts on the other bodies of the system, nearly as if the planet and its satellites were united in their common centre of gravity; and this centre of gravity is attracted by the different bodies of the system, according to the same law, owing to the distance between planets being comparatively so much greater than that of satellites from their primaries.

Attraction of Spheroids

354. The heavenly bodies consist of an infinite number of particles subject to the law of gravitation; and the magnitude of these bodies bears so small a proportion to the distances between them, that they act on one another as if the mass of each were condensed in its centre of gravity. The planets and satellites are therefore considered as heavy points, placed in their respective centres of gravity. This approximation is rendered more exact by their form being nearly spherical: these bodies may be regarded as formed of spherical layers or shells, of a density varying from the centre to the surface, whatever the law may be of that variation. If the attraction of one of these layers, on a point interior or exterior to itself, can be found, the attraction of the whole spheroid may be determined.

Let C, fig. 71, be the centre of a spherical shell of homogeneous matter, and $\mathrm{CP}=a$, the distance of the attracted point P from the centre of the shell. As everything is symmetrical round CP, the whole attraction of the spheroid on P must be in the direction of this line. If *dm* be an element of the shell at m, and $f=m\mathrm{P}$ be its distance from the point attracted, then, assuming the action to be in the inverse ratio of the distance, $\frac{dm}{f^2}$ is the attraction of the particle on P; and if $\mathrm{CP}m=g$, this action, resolved in the direction CP, will be $\frac{dm}{f^2}\cdot\cos g$, and the whole attraction A of the shell on P, will be

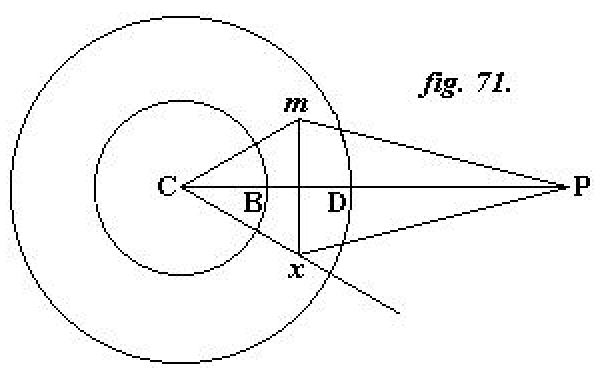


fig. 71.

$$A=\int\frac{dm\,.\cos g}{f^2}.$$

The position of the element dm, in space, will be determined by the angle mCP=q, C$m = r$, and by w, the inclination of the plane PCm on mCx. But, by article 278, $dm = r^2 \sin q \, dr dv \, dq$; and from the triangle CPm it appears that

$$f^2 = a^2 - 2ar\cos q + r^2; \;\; \cos g = \frac{a - r\cos q}{f};$$

hence

$$A = \int r^2 \sin q \, . dr dv \, dq \cdot \frac{a - r\cos q}{f^3},$$

is the attraction of the whole shell on P, for the integral must be taken from $r = \text{CB}$ to $r = \text{CD}$, and from $q = 0$, $w = 0$ to $q = p$, $w = 2p$, p being the semicircle whose radius is unity. The value of f, gives

$$\frac{df}{da} = \frac{a - r\cos q}{f};$$

hence

$$A = -\int r^2 \cdot \sin q \, dr dw dq \cdot \frac{d\frac{1}{f}}{da};$$

but as r, w, and q are independent of a,

$$A = -\frac{d\dfrac{\int r^2 \sin q \, . dr dw dq}{f}}{da}.$$

Thus the whole attraction of the spherical layer on the point P is obtained by taking the differential of

$$\int \frac{r^2 \sin q \, . dr dw dq}{f},$$

according to a, and dividing it by da. Let

$$\int \frac{r^2 \sin q \, . dr dw dq}{f} = V.$$

This integral from $w = 0$ to $w = 2p$, is

$$V = 2p \int \frac{r^2 dr \, . dq \sin q}{f}.$$

But from the value of *f*, it is easy to find

$$\frac{dq \sin q}{f} = \frac{1}{ar} df \;;$$

hence

$$V = \frac{2p}{a} \int rdr \,.\, df \;.$$

The integral with regard to q must be taken from $q = 0$ to $q = p$; but at these limits $f^2 = (a-r)^2$ and $f^2 = (a+r)^2$; and as f must always be positive, when the attracted point is within the spherical layer

$$f = r - a, \text{ and } f = r + a \;;$$

and when the attracted point P is without the spherical layer

$$f = a - r, \text{ and } f = a + r \;;$$

hence, in the first case,

$$V = 4p \int rdr$$

and in the second,

$$V = \frac{4p}{a} \int r^2 dr.$$

355. But the differential of V, according to a, and divided by da, when the sign is changed, is the whole attraction of the shell on P. Hence, from the first expression, $\frac{dV}{da} = 0$. Thus a particle of matter in the interior of a hollow sphere is equally attracted on all sides.

356. The second expression gives

$$-\frac{dV}{da} = \frac{4p}{a^2} \int r^2 dr.$$

The integral of this quantity from

$$r = \text{CB} = R' \text{ to } r = \text{CD} = R'',$$

is

$$-\frac{dV}{da} = \frac{4p}{3a^2} \left(R''^3 - R'^3 \right),$$

which is the action of a spherical layer on a point without it.

If M be the mass of the layer whose thickness is $R'' - R'$, it will be equal to the difference of two spheres whose radii are R'' and R'; hence

$$M = \frac{4p}{3}\left(R''^3 - R'^3\right);$$

and therefore

$$A = \frac{M}{a^2}.$$

Thus the attraction of a spherical layer on a point exterior to it, is the same as if its whole mass were united in its centre.

357. If R', the radius of the interior surface, be zero, the shell will be changed into a sphere whose radius is R''. Hence the attraction of a homogeneous sphere on a point at its surface, or beyond it, is the same as if its mass were united at its centre.

These results would be the same were the attracting solid composed of layers of a density varying, according to any law whatever, from the centre to the surface; for, as they have been proved with regard to each of its layers, they must be true for the whole.

358. The celestial bodies then attract very nearly as if the mass of each was united in its centre of gravity, not only because they are far from one another, but because their forms are nearly spherical.

Notes

[1] An error in the 1st edition lists these three co-ordinates as $\bar{x} + x'$, $\bar{y} + y$, $\bar{z} + z$.

[2] An error in the 1st edition lists these three co-ordinates as $\bar{x} + x'$, $\bar{y} + y'$, $\bar{z} + z$.

[3] The right hand terms *read* $\dfrac{m'(x'-x)}{\left\{(x'-x)^2 + (y'-y)^2 + (z'-z)^2\right\}^{\frac{3}{2}}}$ in the 1st edition.

[4] This *reads* “axes” *for* “axis” in the 1st edition (published erratum).

[5] Punctuation added.

[6] This *reads* “axes” in the 1st edition (published erratum).

[7] This *reads* $\Sigma \dfrac{mx}{r^3}$ in 1st edition.

[8] This *reads* $\Sigma \dfrac{mx}{r^3}$ in 1st edition.

[9] The third terms in the two expressions below *read* $\Sigma \dfrac{my}{r^3}$, $\Sigma \dfrac{mz}{r^3}$ in the 1st edition.

[10] Misprint in 1st edition *reads* “by”.

[11] Punctuation changed from a comma in 1st edition.

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[12] The second term in the last expression *reads* $\frac{mz}{\bar{r}^3}$ not $\frac{mx}{\bar{r}^3}$ as printed in the 1st edition (published erratum).

[13] The 1st edition is inconsistent in the use of italics for the body *S*. In this edition we use the italicized form throughout.

[14] See *Book I, Chapter IV*.

[15] The punctuation at the end of this expression is contained under the root in the 1st edition.

[16] This *reads* "axes", in the 1st edition.

[17] In fig. 70, the origin *o* must therefore coincide with *S*, although *o* is not labeled in the 1st edition figure.

[18] A bracket is missing in the following expression and *reads* $\frac{\bar{x} + x}{\left\{\bar{r}^2 + 2(\bar{x}x + \bar{y}y + \bar{z}z\right\}^{\frac{3}{2}}}$ in the 1st edition.

[19] The first term *reads* $\frac{}{\bar{r}^3}$ *for* $\frac{\bar{x}}{\bar{r}^3}$ as printed in the 1st edition (published erratum).

[20] Numerator below *reads* $\sum m$ in the 1st edition.

Mars

Twelve orbits a day provide the Mars Global Surveyor MOC wide angle cameras a global "snapshot" of weather patterns across the planet. Here, water ice clouds hang above the Tharsis volcanoes. The map is a mosaic of 24 images taken on a single northern summer day in April 1999. The center of the image is the point 15 degrees North, 90 degrees West. Photo courtesy of Malin Space Science Systems, San Diego, California. (Courtesy of NASA)

BOOK II

CHAPTER IV

ON THE ELLIPTICAL MOTION OF THE PLANETS

359. THE elliptical orbit of the earth is the plane of the ecliptic: the plane of the terrestrial equator cuts the plane of the ecliptic in a line passing through the vernal and autumnal equinoxes.

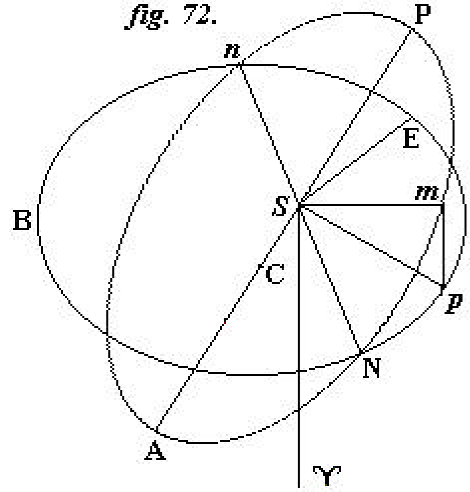
The vernal equinox is assumed as an origin from whence the angular distances of the heavenly bodies are estimated. Astronomers designate that point by the character ♈ the first point of Aries, although these points have not coincided for 2230 years, on account of the precession or retrograde motion of the equinoxes.

360. Angular distance from the vernal equinox, or first point of Aries, estimated on the plane of the ecliptic, is longitude, which is reckoned from west to east, the direction in which the bodies of the solar system revolve round the sun. For example, let E*n*BN, fig. 72, represent the ecliptic, *S* the sun, and ♈ the first point of Aries, or vernal equinox. If the earth be in E, its longitude is the angle ♈*S*E.

361. The earth alone moves in the plane of the ecliptic, the orbits of the other bodies of the system are inclined to it at small angles; so that the planets, in their revolutions, are sometimes seen above that plane, and sometimes below it. The angular distance of a planet above or below the plane of the ecliptic, is its latitude; when the planet is above that plane, it is said to have north latitude, and when below it, south latitude. Latitude is reckoned from zero to 180°.

362. Let E*n*BN represent the plane of the ecliptic, and let *m* be a planet moving round the sun *S* in the direction *m*P*n*, the orbit being inclined to the ecliptic at the angle PNE; the part of the orbit NP*n* is supposed to be above the plane of the ecliptic, and NA*n* below it. The line N*Sn*, which is the intersection of the plane of the orbit with the plane of the ecliptic, is the line of nodes; it always passes through the centre of the sun. When the planet is in N, it is in its ascending node; when in *n*, it is in its descending node. Let *mp* be a perpendicular from *m* on the plane of the ecliptic, *Sp* is the projection of the radius vector *Sm*, and is the curtate[1] distance of the planet from the sun. ♈*Sm* is the longitude of the ascending node; and it is clear that the longitude of *n*, the descending node, is 180° greater. The longitude of *m* is ♈*Sm*, or ♈*Sp*, according as it is estimated on the orbit, or on the ecliptic; and *mSp*, the angular height *m* above the plane of the ecliptic, is its latitude. As the position of the first point of Aries is known, it is evident that the place of a planet *m* in its orbit is found, when the angles ♈*Sm*, *mSp*, and *Sm*, its

fig. 72.



distance from the sun, are known at any given time, or ♈*Sp*, *pSm*, and *Sp*, which are more generally employed. But in order to ascertain the real place of a body, it is also requisite to know the nature of the orbit in which it moves, and the position of the orbit in space. This depends on six constant quantities, AP, the greater axis of the ellipse; $\frac{\mathrm{C}S}{\mathrm{CP}}$, the eccentricity; ♈*Sp*, the longitude of P, the perihelion; ♈*S*N, the longitude of N, the ascending node; ENP, the inclination of the orbit on the plane of the ecliptic; and on the longitude of the epoch, or position of the body at the origin of the time.

These six quantities, called the elements of the orbit, are determined by observation; therefore the object of analysis is to form equations between the longitude, latitude, and distance from the sun, in values of the time; and from them to compute tables which will give values of these three quantities, corresponding to any assumed time, for a planet or satellite; so that the situation of every body in the system may be ascertained by inspection alone, for any time past, present, or future.

363. The motion of the earth differs from that of any other planet, only in having no latitude, since it moves in the plane of the ecliptic, which passes through the centre of the sun. In consequence of the mutual attraction of the celestial bodies, the position of the ecliptic is variable to a very minute extent; but as the variation is known, its position can be ascertained.

364. The motions of the celestial bodies, and the positions of their orbits, will be referred to the known position of this plane at some assumed epoch, say 1750, unless the contrary be expressly mentioned. It will therefore be assumed to be the plane of the co-ordinates *x* and *y*, and will be called the FIXED PLANE.

Motion of one Body

365. If the undisturbed elliptical motion of one body round the sun be considered, the equations in article 346^2 become

$$\begin{aligned}&\frac{d^2x}{dt^2}+\frac{mx}{r^3}=0,\\&\frac{d^2y}{dt^2}+\frac{my}{r^3}=0,\\&\frac{d^2z}{dt^2}+\frac{mz}{r^3}=0,\end{aligned} \qquad \textbf{(89)}$$

where m is put for $S+m$, the sum of the masses of the sun and planet, and $r=\sqrt{x^2+y^2+z^2}$.

In these three equations, the force is inversely as the square of the distance; they ought therefore to give all the circumstances of elliptical motion. Their finite values will give *x*, *y*, *z*, in values of the time, which may be assumed at pleasure: thus the place of the body in its elliptical orbit will be known at any instant; and as the equations are of the second order, six arbitrary constant quantities will be introduced by their integration, which determine the six elements of the orbit.

366. These give the motion of the planet with regard to the sun; but the equations

$$0=\frac{d^2\overline{x}}{dt^2}-\frac{mx}{r^3};\quad 0=\frac{d^2\overline{y}}{dt^2}-\frac{my}{r^3};\quad 0=\frac{d^2\overline{z}}{dt^2}-\frac{mz}{r^3},$$

of article 346, give values of $\overline{x}$, $\overline{y}$, $\overline{z}$, in terms of the time which will determine the motion of the sun in space; for if the first of them be multiplied by $S+m$, and added to

$$\frac{d^2x}{dt^2}+\frac{(S+m)x}{r^3}=0,$$

multiplied by m, their sum will be

$$(S+m)\frac{d^2\overline{x}}{dt^2}+m\frac{d^2x}{dt^2}=0,$$

the integral of which is

$$\overline{x}=a+bt-\frac{mx}{S+m};$$

in the same manner,

$$\overline{y}=a'+b't-\frac{my}{S+m},$$

$$\overline{z}=a''+b''t-\frac{mz}{S+m}.$$

These equations give the motion of the sun in space accompanied by m; and as they are the same for each body, if $\sum m$ be substituted for m, they will determine the absolute motion of the sun attended by the whole system, when the relative motions of m, m', m'', &c., are known.

367. But in order to ascertain the values of x, y, z, the equations (89) must be integrated. Since these equations are linear and of the second order, their integrals must contain six constant quantities. They are also symmetrical and so connected, that any one of the variable quantities x, y, z, depends on the other two. M. Pontécoulant[3] has determined these integrals with great elegance and simplicity in the following manner.

368. If the first of the equations (89) of elliptical motion multiplied by y, be subtracted from the second multiplied by x, the result will be

$$\frac{xd^2y-yd^2x}{dt^2}=0;$$

consequently,

$$\frac{xdy-ydx}{dt}=c.$$

In the same way it is easy to find that

$$\frac{zdx - xdz}{dt} = c'; \quad \frac{ydz - zdy}{dt} = c'',$$

where c, c', c'', are arbitrary constant quantities introduced by integration. Again, if the first of the same equations be multiplied by $2dx$, the second by $2dy$, and the third by $2dz$, their sum will be

$$\frac{2dxd^2x + 2dyd^2y + 2dzd^2z}{dt^2} + \frac{2m\left(xdx + ydy + zdz\right)}{r^3} = 0.$$

But

$$r^2 = x^2 + y^2 + z^2;$$

whence

$$rdr = xdx + ydy + zdz;$$

and the integral of the preceding equation is

$$\frac{dx^2 + dy^2 + dz^2}{dt^2} - \frac{2m}{r} + \frac{m}{a} = 0, \tag{90}$$

$\frac{m}{a}$ being an arbitrary constant quantity. If

$$\frac{d^2y}{dt^2} = -\frac{my}{r^3}, \text{ multiplied by } c'' = \frac{ydz - zdy}{dt},$$

be subtracted from

$$\frac{d^2x}{dt^2} = -\frac{mx}{r^3}, \text{ multiplied by } c' = \frac{zdx - xdz}{dt},$$

the result will be

$$\frac{c'd^2x - c''d^2y}{dt} = \frac{mx}{r^3}\left(xdz - zdx\right) - \frac{my}{r^3}\left(zdy - ydz\right)$$

$$= \frac{m\left(rdz - zdr\right)}{r^2} = md \cdot \frac{z}{r}.$$

Whence

$$f + \frac{mz}{r} = \frac{c'dx - c''dy}{dt};$$

and by a similar process values of[4]

$$m \,.\, d\frac{y}{r}, \text{ and } m \,.\, d\frac{x}{r},$$

may be found, the integrals of which are

$$f' + \frac{my}{r} = \frac{c''dz - cdx}{dt}; \quad f'' + \frac{mx}{r} = \frac{cdy - c'dz}{dt}.$$

369. Thus the integrals of equations (89) are,

$$c = \frac{xdy - ydx}{dt}; \quad c' = \frac{zdx - xdz}{dt}; \quad c'' = \frac{ydz - zdy}{dt};$$
$$f + \frac{mz}{r} = \frac{c'dx - c''dy}{dt},$$
$$f' + \frac{my}{r} = \frac{c''dz - cdx}{dt}, \qquad \textbf{(91)}$$
$$f'' + \frac{mx}{r} = \frac{cdy - c'dz}{dt},$$
$$\frac{m}{a} - \frac{2m}{r} + \frac{dx^2 + dy^2 + dz^2}{dt^2} = 0,$$

containing the seven arbitrary constant quantities c, c', c'', f, f', f'', and a.

370. As two equations of condition exist among the constant quantities, they are reduced to five that are independent, consequently two of the seven integrals are included in the other five. For if the first of these equations be multiplied by z, the second by y, and the third by x, their sum is

$$cz + c'y + c''x = 0 \qquad \textbf{(92)}$$

Again, if the fourth integral multiplied by c, be added to the fifth multiplied by c',

$$fc + f'c' + m\frac{cz + c'y}{r} = c'' \cdot \frac{c'dz - cdy}{dt};$$

but

$$cz + c'y = -c''x;$$

hence

$$-\frac{fc + f'c'}{c''} + m\frac{x}{r} = \frac{cdy - c'dz}{dt};$$

but this coincides with the sixth integral, when

$$f'' = -\frac{fc + f'c'}{c''}, \text{ or } f''c'' + f'c' + fc = 0\,.$$

The six arbitrary quantities being connected by this equation of condition, the sixth integral results from the five preceding.

If the squares of f, f', and f'', from the fourth, fifth, and sixth integrals be added, and [letting] $f^2 + f'^2 + f''^2 = l^2$, they give

$$l^2 - m^2 = \left(c^2 + c'^2 + c''^2\right)\left\{\frac{dx^2 + dy^2 + dz^2}{dt^2} - \frac{2m}{r}\right\} - \left\{\frac{cdz + c'dy + c''x}{dt}\right\}^2$$

but

$$cz + c'y + c''x = 0; \text{ hence } cdz + c'dy + c''dx = 0\,;$$

consequently, if

$$c^2 + c'^2 + c''^2 = h^2\,,$$

[then]

$$0 = \frac{dx^2 + dy^2 + dz^2}{dt^2} - \frac{2m}{r} + \frac{m^2 - l^2}{h^2}\,,$$

and comparing this equation with the last of the integrals in article 369, it will appear that

$$\frac{m^2 - l^2}{h^2} = \frac{m}{a}\,;$$

thus, the last integral is contained in the others; so that the seven integrals and the seven constant quantities are in reality only equal to five distinct integrals and five constant quantities.

371. Although these are insufficient to determine x, y, z, in functions of the time, they give the curve in which the body m moves. For the equation

$$cz + c'y + c''x = 0$$

is that of a plane passing through the origin of the co-ordinates, whose position depends on the constant quantities c, c', c''. Thus the curve in which m moves is in one plane. Again, if the fourth of the integrals in article[5] 369 be multiplied by z, the fifth by y, and the sixth by x, their sum will be

$$fz + f'y + f''x + \frac{m\left(x^2 + y^2 + z^2\right)}{r} = c\,\frac{(xdy - ydx)}{dt} + c'\,\frac{(zdx - xdz)}{dt} + c''\,\frac{(ydz - zdy)}{dt}\,;$$

but in consequence of the three first integrals in article 369, it becomes[6]

$$0 = mr - \left(c^2 + c'^2 + c''^2\right) + fz + f'y + f''x\,,$$

or

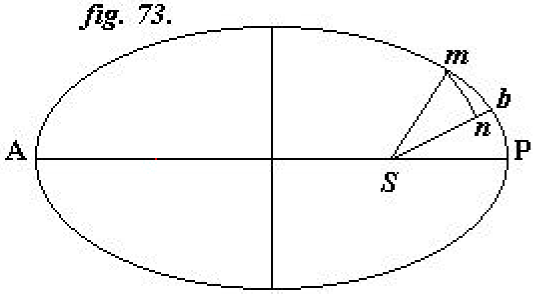
$$0 = mr - h^2 + fz + f'y + f''x.$$

This equation combined with

$$cz + c'y + c''x = 0, \quad \text{and} \quad r^2 = x^2 + y^2 + z^2,$$

gives the equation of conic sections, the origin of r being in the focus.

fig. 73.



372. Thus the planets and comets move in conic sections having the sun in one of their foci, and their radii vectores describe areas proportional to the time; for if dv represent the indefinitely small arc mb, fig. 73, contained between

$$Sm = r \text{ and } Sb = r + dr,$$

then

$$(mb)^2 = dx^2 + dy^2 + dz^2 = r^2dv^2 + dr^2,$$

but the sum of the squares of the three first of equations (91) is

$$\frac{(x^2 + y^2 + z^2)(dx^2 + dy^2 + dz^2)}{dt^2} - \frac{(xdx + ydy + zdz)}{dt^2} = h^2,$$

or

$$\frac{r^2(dx^2 + dy^2 + dz^2)}{dt^2} - \frac{r^2dr^2}{dt^2} = h^2;$$

hence

$$dv = \frac{hdt}{r^2}. \quad \textbf{(93)}$$

373. Thus the area $\frac{1}{2}r^2dv$ described by the radius vector r or Sm is proportional to the time dt, consequently the finite area described in a finite time is proportional to the time. It is evident also, that the angular motion of m round S is in each point of the orbit, inversely as the square of the radius vector, and as very small intervals of time may be taken instead of the indefinitely small instants dt, without sensible error, the preceding equation will give the horary[7] motion of the planets and comets in the different points of their orbits.

Determination of the Elements of Elliptical Motion

374. The elements of the orbit in which the body m moves depend on the constant quantities c, c', c'', f, f', f'', and $\frac{m}{a}$. In order to determine them, it must be observed that in the equations (89) the co-ordinates x, y, z, are SB, Bp, pm, fig. 74; but if they be referred to[8] ♈S the line of the equinoxes, so that $S\text{D} = x'$, $\text{D}p = y'$, $pm = z'$, and if ♈SN, ENP, the longitude of the node and inclination of the orbit on the fixed plane be represented by q and f ; it is evident, from the method of the co-ordinates in article 225, that

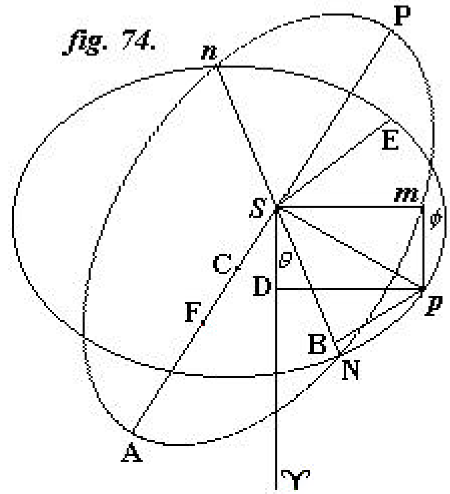


fig. 74.

$$x' = x\cos q + y\sin q\,,$$
$$y' = y\cos q - x\sin q\,,$$
$$z' = y\tan f\,,$$

consequently

$$z = y\cos q\,\tan f - x\sin q\,\tan f\ ;$$

but if this be compared with

$$0 = c''x + c'y + cz\ ,$$

it will be found that

$$c' = -c\cos q\,\tan f\ ,$$
$$c'' = c\sin q\,\tan f\ ,$$

whence

$$\left\{\begin{aligned} \tan q &= -\frac{c''}{c'} \\ \tan f &= \frac{\sqrt{c'^2 + c''^2}}{c} \end{aligned}\right\} \qquad \textbf{(94)}$$

Thus this position of the nodes and the inclination of the orbit are given in terms of the constant quantities c, c', c''.

375. Now $r^2 = x^2 + y^2 + z^2$, and $rdr = xdx + ydy + zdz$, but at the perihelion the radius vector r is a minimum; hence $dr = 0$, therefore

$$xdx + ydy + zdz = 0\,.$$

Let $x_{/}$, $y_{/}$, $z_{/}$, be the co-ordinates of the planet when in perihelio, then, substituting the values of c, c', c'', from [article] 369[9] in the equations in f' and f'' of the same number, and dividing the one by the other, the result in consequence of the preceding relation will be

$$\frac{y_{/}}{x_{/}} = \frac{f'}{f''}\,.$$

But if $v_{/}$ be the angle ♈SE, the projection of the longitude of the perihelion on the plane N*pn*, then $\frac{y_{/}}{x_{/}} = \tan v_{/}$; hence

$$\tan v_{/} = \frac{f'}{f''};$$

which determines the position of the greater axis of the conic section.

If $\frac{dx^2 + dy^2 + dz^2}{dt^2}$ be eliminated from the equation

$$r^2\left(\frac{dx^2 + dy^2 + dz^2}{dt^2}\right) - \frac{r^2 dr^2}{dt^2} = h^2,$$

by means of the last (91) the result will be

$$2mr - \frac{mr^2}{a} - \frac{r^2 dr^2}{dt^2} = h^2;$$

but at the extremities of the greater axis $dr = 0$, because the radius vector is either a maximum or minimum at these points, therefore at the aphelion and perihelion

$$0 = r^2 - 2ar + \frac{ah^2}{m};$$

whence

$$r = a \pm a\sqrt{1 - \frac{h^2}{ma}}.$$

The sum of these two values of *r* is the major axis of the conic section, and their difference is F*S* or double the eccentricity.

376. Thus *a* is half of AP, fig. 75, the major axis of the orbit, or it is the mean distance of *m* from *S*; and $\sqrt{1 - \frac{h^2}{ma}}$, is the ratio of the eccentricity to half the major axis. Let this ratio be represented by *e*, then as it was shown that $\frac{m^2 - l^2}{h^2} = \frac{m}{a}$; so also

$$me = l = \sqrt{f^2 + f'^2 + f''^2}.$$

Thus all the elements that determine the nature of the conic section and its position in space are known.

377. The three equations

$$r^2 = x^2 + y^2 + z^2, \quad mr - h^2 + fz + f'y + f''z = 0, \text{ and } c''x + c'y + cz = 0,$$

give x, y, z, in functions of r; but in order to have values of these co-ordinates in terms of the time, r must be found in terms of the same, which requires another integration. Resume the equation

$$2mr - \frac{mr^2}{a} - \frac{r^2dr^2}{dt^2} = h^2,$$

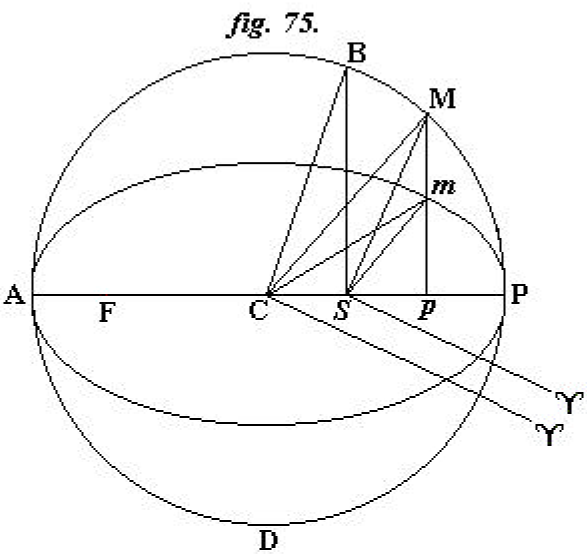
then

$$\sqrt{1 - \frac{h^2}{ma}} = e,$$

gives

$$h^2 = am\left(1 - e^2\right)$$

therefore



$$dt = \frac{rdr}{\sqrt{m}\sqrt{2r - \frac{r^2}{a} - a\left(1 - e^2\right)}}.$$

To integrate this equation, a value of r must be found from the conic sections. Let AmP, fig. 75, be an ellipse whose major axis is 2a, its minor axis 2b, the eccentricity $CS = e'$, and the radius vector $Sm = r$.

Let the circle PMA be described on the major axis, draw the perpendicular Mp through m, and join SM, CM, and Cm. Then

$$r^2 = Sp^2 + pm^2, \text{ and if MCP} = u,$$
$$Sp = Cp - CS = a\cos u - e',$$

or making

$$e = \frac{e'}{a}, \quad Sp^2 = a^2\left(\cos u - e\right)^2.$$

Again,

$$pm^2 = b^2 . \sin^2 u = b^2\left(1 - \cos^2 u\right);$$

but

$$b^2 = a^2 - e'^2 = a^2\left(1 - e^2\right);$$

hence

$$r^2 = a^2\left(1 - e^2\right)\left(1 - \cos^2 u\right) + a^2\left(\cos u - e\right)^2,$$

and

$$r = a\{1 - e\cos u\}.$$

This value of r and its differential being substituted in the value of dt it becomes

$$dt = \frac{a^{\frac{3}{2}}}{\sqrt{u}} \cdot du\left(1 - e\cos u\right)$$

the integral of which is

$$t + k = \frac{a^{\frac{3}{2}}}{\sqrt{u}}\{u - e\sin u\}; \qquad \textbf{(95)}$$

k being an arbitrary constant quantity.

This equation gives u and consequently r in terms of t, and as x, y, z, are given in functions of r, the values of these co-ordinates are known at any instant.

When $u = 1$ the values of dt and h^2 become

$$\frac{rdr}{\sqrt{2r - \frac{r^2}{a} - a\left(1 - e^2\right)}}, \text{ and } a\left(1 - e^2\right),$$

and when substituted in $dv = \frac{hdt}{r^2}$, the result is

$$dv = \frac{dr \cdot \sqrt{a\left(1 - e^2\right)}}{r\sqrt{2r - \frac{r^2}{a} - a\left(1 - e^2\right)}},$$

or

$$dv = \frac{\frac{a\left(1 - e^2\right)}{e} d \cdot \frac{1}{r}}{\sqrt{1 - \left\{\frac{a\left(1 - e^2\right)\frac{1}{r} - 1}{e}\right\}}};$$

the integral of which is

$$v = v + arc\left\{\cos = \frac{a\left(1 - e^2\right)\frac{1}{r} - 1}{e}\right\},$$

reciprocally[10]

$$r = \frac{a\left(1 - e^2\right)}{1 + e\cos\left(v - v\right)},$$

which is the general equation to the conic sections, when the origin of r the radius vector is in the focus; a is half the greater axis, and

$$\cos\left(v - v\right) = \cos\ (♈Sm - ♈SP),\ \text{fig. 77.}$$

Elements of the Orbit

378. Thus the finite values of the equations of elliptical motion are completely determined,

Six arbitrary constant quantities have been introduced, namely,

2a, the greater axis of the orbit.
e, the ratio of the eccentricity to half the greater axis.
$v_{/}$, the projection of the longitude of the perihelion.
q, the longitude of the ascending mode.
f, the inclination of the orbit on the plane of the ecliptic, and
$\in$, the longitude of the epoch.

The two first determine the nature of the orbit, the three following its position in space, and the last is relative to the position of the body at a given epoch; or, which is the same thing, it depends on the instant of its passage at the perihelion.

Equations of Elliptical Motion

379. It now becomes necessary to determine three equations which will give values of the longitude and latitude ♈*Sm*, *mSp*, and the distance *Sm*, fig. 72, in terms of the time from whence tables of the elliptical motions of the planets and satellites may be computed.

380. The motion of a body in an ellipse is not uniform, its velocity is greatest at the perihelion, and least at the aphelion, varying with the angle P*Sm*, which is the true angular motion of the planet; but if the circle PBAD, fig. 75, be described from the centre of the ellipse with the semigreater axis CP, or mean distance from S as radius, the motion of the planet in this circle would be uniform. This is called the mean motion of a body.

381. Were the motion of a planet uniform, the angle P*Sm* described by the planet in any interval of time after leaving perihelion might be found by simple proportion from knowing the periodic time, or time in which it describes 360°; but in order to preserve the equable description of areas, the true place of the planet will be before the mean place in going from perihelion to aphelion; and from aphelion to perihelion the true place will be behind the mean place. These angles are estimated from west to east, the direction in which the bodies of the system move, beginning at the perihelion. If, however, they are estimated from the aphelion, it is only necessary to add 180° to each.

382. The angular distance PCB between the perihelion and the mean place, is the mean anomaly, *PSm* the angular distance between the true place and the perihelion is the true anomaly; and *mS*B the angle at the sun, contained between the true and the mean place is called the equation of the centre. If then the mean anomaly be increased or diminished by the equation of the centre, the result will be the true place of the planet in its orbit. The equation of the centre is zero, both at the perihelion and aphelion, for if these points the true and mean places of the planets coincide; it is greatest when the planet is in quadratures, and at its maximum it is equal to an angle measured by twice the eccentricity of the orbit.

383. The mean place of a planet, at any given time may be found by simple proportion from its periodic time. The true place of the planet in its orbit, and its distance from the sun, may be found in terms of its mean place by help of the angle PCM, called the eccentric anomaly.

If the time be estimated from the perihelion,[11] $k=0$, which reduces equation (95) to[12]

$$t=\frac{a^{\frac{3}{2}}}{\sqrt{u}}\cdot\left(u-e.\sin u\right),\ \text{or}\ nt=u-e\sin u,\ \text{if}\ n=\frac{\sqrt{u}}{a^{\frac{3}{2}}}.$$

If the angles u and v be estimated from the perihelion, a comparison of the values of r in article 377 gives

$$1-e\cos u=\frac{1-e^2}{1+e\cos v},$$

whence

$$\cos v=\frac{\cos u-e}{1-e\cos u};\quad \sin v=\frac{\sin u\cdot\sqrt{1-e^2}}{1-e\cos u},$$

therefore

$$\tan\tfrac{1}{2}v=\sqrt{\frac{1+e}{1-e}}\cdot\tan\tfrac{1}{2}u.$$

384. The motions of the celestial bodies in elliptical orbits are therefore obtained from the three equations

$$\begin{aligned} nt &= u-e\sin u, \\ r &= a\left(1-e\cos u\right) \\ \tan\tfrac{1}{2}v &= \sqrt{\frac{1+e}{1-e}}\cdot\tan\tfrac{1}{2}u. \end{aligned} \qquad \textbf{(96)}$$

Where nt = PCB = mean anomaly, fig. 75,
v = P*Sm* = true anomaly,
u = PCM = eccentric anomaly,
r = *Sm* = radius vector,
a = CP = mean distance, and

$$e = \frac{\mathrm{C}S}{\mathrm{CP}} = \text{the ratio of the eccentricity to the mean distance}.$$

385. It appears from these expressions that when u becomes $u + 360^\circ$, r remains the same; and as v is then augmented by 360°, the planet returns to the same point of its orbit, having moved through four right angles, and the time becomes[13] $\mathrm{T} = \frac{a^{\frac{3}{2}}}{\sqrt{u}} \cdot 360^\circ$; so that the time of a complete revolution is independent of the eccentricity, and only depends on 2*a*, the major axis of the orbit; it is consequently the same as if the planet described a circle at its mean distance from the sun; for in this case $e = 0$, $r = a$, $u = nt$, $v = u$, consequently $v = nt$; the arcs described are therefore proportional to the time, and the planet moves uniformly in the circle whose radius is *a*. Generally *nt* represents the arc that a body would describe in the time *t*, if it set out from the perihelion at the same instant with a planet *m*, and moved with a uniform velocity represented by *n* in a circle described on the major axis of the orbit as diameter. This body would pass the perihelion and aphelion at the same instant with the planet *m*, but in one half of its revolution the planet would precede the body, and in the other half it would fall behind it. If $a = 1$, $m = 1$, then $n = 1$, and $v = t$, the time will therefore be expressed by the arcs described by the planet in the circle whose radius is unity.

Astronomers generally compare the motions of the solar system with those of the earth; they take the mean distance of the sun from the earth as the unit of distance, the sum of the masses of the sun and earth as the unit of mass; and supposing the time to be estimated in mean solar days, the unit of time will be represented by the arc that the earth describes round the sun in one day with its mean motion.

Determination of the Eccentric Anomaly in functions of the Mean Anomaly

386. If a value of *u* could be found in terms of *nt* from the first of these equations, both *r* and *v*, and consequently the place of the planet in its orbit at any instant, would be known from the two last.

Now an arc and its sine are incommensurate quantities, so that the one can only be obtained in functions of the other by an infinite series. Therefore a value of *u* in terms of *nt* must be found by an infinite series from the first of the preceding equations; but unless the terms of the series decrease rapidly in value u cannot be obtained, for a few of the first terms being computed, the value of the remaining part of the series must be so small that it may be neglected without sensible error. The small eccentricities of the orbits of the planets and satellites afford the means of approximation, for e the ratio of the eccentricity to half the greater axis is still smaller, consequently the powers of such quantities decrease rapidly, and therefore the second part of the equation $u = nt + e\sin u$ may be expanded into a series in functions of the time, and according to the powers of e, which will be sufficiently convergent. This may be accomplished by Maclaurin's *Theorem*,[14] for if u' be the value of *u* when $e = 0$,

$$u = u' + e \cdot \frac{du'}{de} + \frac{e^2}{1.2} \cdot \frac{d^2u'}{de^2} + \frac{e^3}{1.2.3} \cdot \frac{d^3u'}{de^3} + \&\text{c}.$$

But when $e=0$, $u=nt+e\sin u$, becomes $u'=nt$; and from the same equation

$$\frac{du}{de}=\frac{\sin u}{1-e\cos u};$$

or when

$$e=0,\quad \frac{du'}{de}=\sin nt.$$

Again,

$$\frac{d^2u}{de^2}=\frac{2\cos u\sin u}{(1-e\cos u)^2}-\frac{e\sin^3 u}{(1-e\cos u)^3};$$

or if[15]

$$e=0,\quad \frac{d^2u'}{de^2}=2\cos nt\sin nt$$

in the same manner, when $e=0$,

$$\frac{d^3u'}{de^3}=6\sin nt\cos^2 nt-3\sin^3 nt,$$

or

$$\frac{d^3u'}{de^3}=6\sin nt-9\sin^3 nt, \text{ \&c. \&c.}$$

But

$$2\cos nt\sin nt=\sin 2nt,$$

and

$$6\sin nt-9\sin^3 nt=-\tfrac{3}{4}\sin 3nt+\tfrac{9}{4}\sin 3nt;$$

hence

$$\frac{du'}{de}=\sin nt;\quad \frac{d^2u'}{de^2}=\frac{1}{2}\sin 2nt;\quad \frac{d^3u'}{de^3}=\frac{1}{2^2}\cdot\left\{3^2\sin 3nt-3\sin nt\right\}\text{ \&c.}$$

consequently,[16]

$$\begin{aligned}u=nt+e\sin nt&+\frac{e^2}{1.2.2}\cdot 2\sin 2nt+\frac{e^3}{2.3.2^2}\cdot\left\{3^2\sin 3nt-3\sin nt\right\}\\&+\frac{e^4}{2.3.4.2^3}\cdot\left\{4^3\sin 4nt-4.2^3\sin 2nt\right\}\\&+\frac{e^5}{2.3.4.5.2^4}\cdot\left\{5^4\sin 5nt-5.3^2\sin 3nt+\frac{5.4}{1.2}\sin nt\right\}\\&+\text{ \&c. \&c. \&c.}\end{aligned}$$

This series converges rapidly in most of the planetary orbits on account of the small value of the fraction which *e* expresses.

387. Having thus determined u for any instant, corresponding values of v and r may be obtained from the equations $r = a(1 - e\cos u)$ and

$$\tan\tfrac{1}{2}v = \sqrt{\frac{1+e}{1-e}} \cdot \tan\tfrac{1}{2}u\,;$$

but it is better to expand these also into series ascending according to the powers of e; and in functions of the sines or cosines of the mean anomaly.

Determination of the Radius Vector in functions of the Mean Anomaly

Let r' be the value of r when $e = 0$, then[17]

$$r = r' + e\frac{dr'}{de} + \frac{e^2}{1.2} \cdot \frac{d^2 r'}{de^2} + \&\text{c.}$$

but as r is a function of e by the equation $r = a(1 - e\cos u)$; and u is a function of e by $u = nt + e\sin u$, therefore,[18]

$$\frac{dr}{de} = \frac{dr'}{de} + \frac{dr'}{du} \cdot \frac{du}{de}.$$

Now when $e = 0$, $\frac{r}{a} = 1$; and[19] $u = nt$. But the differentials of the same equations, when $e = 0$, are[20]

$$\frac{dr}{de} = -a\cos nt; \text{ and } \frac{du}{de} = \sin nt\,;$$

consequently,

$$\frac{dr'}{de} = -a\cos nt + \sin nt \cdot \frac{dr}{ndt}, \text{ for } du = ndt\,;$$

or it may be written,

$$\frac{dr'}{de} = -a\cos nt + \frac{d\int \sin nt\,.\,dr}{ndt}.$$

Again,

$$\frac{d^2 r'}{de^2} = \frac{d^2\int \sin nt\,.\,dr}{ndt\cdot de}\,;$$

but if $\int \sin nt \cdot dr$ be put for r in

$$\frac{dr}{de} = \frac{d\int \sin nt\,.\,dr}{ndt},$$

then,

$$\frac{d\int \sin nt\,.\,dr}{de}=\frac{d\int \sin^2 nt\,.\,dr}{ndt}\,.$$

And if this be substituted in the value of $\frac{d^2r'}{de^2}$ it becomes[21]

$$\frac{d^2r'}{de^2}=\frac{d^2\int \sin^2 nt\,.\,dr}{(ndt)^2}=\frac{d\cdot\left(\sin^2 nt\cdot\frac{dr}{ndt}\right)}{ndt}\,.$$

The differential of the latter expression according to e is

$$\frac{d^3r'}{de^3}=\frac{d^3\int \sin^2 nt\,.\,dr}{(ndt)^2\cdot de};$$

and making the same substitution, it becomes[22]

$$\frac{d^3r'}{de^3}=\frac{d^3\int \sin^3 nt\,.\,dr}{(ndt)^3}=\frac{d^2\cdot\left(\sin^3 nt\cdot\frac{dr}{ndt}\right)}{(ndt)^2},$$

and so on. These coefficients being substituted,

$$r=a-ae\cos nt+e\sin nt\cdot\frac{dr}{ndt}+\frac{e^2}{2}\cdot\frac{d\left(\sin^2 nt\cdot\frac{dr}{ndt}\right)}{ndt}+\&c.$$

But

$$r=a(1-e\cos nt)\ \text{gives}\ \frac{dr}{ndt}=ae\,.\sin nt\,;$$

hence[23]

$$\frac{r}{a}=1-e\cos nt+e^2\sin^2 nt+\frac{e^3}{2}\cdot\frac{d\cdot\sin^3 nt}{ndt}+\frac{e^4}{2.3}\cdot\frac{d^2\cdot\sin^4 nt}{n^2dt^2}+\&\text{c.}$$

Now

$$\sin^2 nt=\tfrac{1}{2}-\tfrac{1}{2}\cos 2nt\,,$$

$$\frac{d\cdot\sin^3 nt}{ndt}=3\sin^2 nt\cos nt=\tfrac{3}{4}\{\cos nt-\cos 3nt\}$$

$$\frac{d^2\cdot\sin^4 nt}{(ndt)^2}=2\cos 2nt-2\cos 4nt,\ \&\text{c.}$$

thus[24]

$$\frac{r}{a}=1+\frac{e^2}{2}-e\cos nt-\frac{e^2}{2}\cos 2nt$$
$$-\frac{e^2}{1.2.2^2}\cdot\{3\cos 3nt-3\cos nt\}$$
$$-\frac{e^4}{1.2.3.2^3}\cdot\{4^2\cos 4nt-4.2^2\cos 2nt\}$$
$$-\frac{e^5}{1.2.3.4.2^4}\cdot\left\{5^3\cos 5nt-5.3^3\cos 3nt+\frac{5.4}{1.2}\cos nt\right\}$$
$$-\ \&c.\ \&c.\ \&c.$$

This gives a value of the radius vector in functions of the time.

Kepler's Problem. To find a Value of the true Anomaly in functions of the Mean Anomaly

388. The determination of v in terms of nt is Kepler's problem of finding the true anomaly in terms of the mean anomaly; or, to divide the area of a semicircle in a given ratio by a line drawn from a given point in the diameter–in order to accomplish this, a value of v in functions of u must be obtained from

$$\tan\tfrac{1}{2}v=\sqrt{\frac{1+e}{1-e}}\cdot\tan\tfrac{1}{2}u\,;$$

therefore let

$$l=\frac{e}{1+\sqrt{1-e^2}}\,,$$

then

$$e=\frac{2l}{1+l^2},\text{ and }\sqrt{\frac{1+e}{1-e}}=\frac{1+l}{1-l}\,.$$

Again,

$$\sin\tfrac{1}{2}v=c^{v\sqrt{-1}}-1,\ \cos\tfrac{1}{2}c^{v\sqrt{-1}}+1,$$

c being the number whose logarithm is unity; hence the equation in question becomes

$$\frac{c^{v\sqrt{-1}}-1}{c^{v\sqrt{-1}}+1}=\frac{1+l}{1-l}\cdot\frac{c^{u\sqrt{-1}}-1}{c^{u\sqrt{-1}}+1}\,,$$

whence

$$c^{v\sqrt{-1}}=\frac{1-lc^{-u\sqrt{-1}}}{1-lc^{u\sqrt{-1}}}\cdot c^{u\sqrt{-1}}\,;$$

or taking its logarithm,

$$v=u+\frac{\log\{1-l\,c^{-u\sqrt{-1}}\}-\log\{1-l\,c^{u\sqrt{-1}}\}}{\sqrt{-1}}.$$

Or

$$v=u+l\left\{\frac{c^{u\sqrt{-1}}-c^{-u\sqrt{-1}}}{\sqrt{-1}}\right\}+\frac{l^2}{2}\left\{\frac{c^{2u\sqrt{-1}}-c^{-2u\sqrt{-1}}}{\sqrt{-1}}\right\}+\&\text{c.}$$

but

$$\frac{c^{mu\sqrt{-1}}-c^{-mu\sqrt{-1}}}{2\sqrt{-1}}=\sin mu\,;$$

m being any whole positive number, therefore

$$v=u+2l\sin u+\frac{2l^2}{2}\sin 2u+\frac{2l^3}{3}\sin 3u+\&\text{c.}$$

The true anomaly may now be found in terms of the mean anomaly.

389. In order to have v in terms of the mean anomaly and of the powers of e, values of u, $\sin u$, $\sin 2u$, must be found in terms of the sines of nt and its multiples; and l, l^2, &c. must be developed into series according to the powers of e. Both may be accomplished by *Lagrange's*[25] Theorem,[26] for if

$$f=\frac{1}{a}=\frac{1}{1+\sqrt{1-e^2}}=\frac{l}{e}\,;$$

when

$$e=0,\ a=2,\ f'=\frac{1}{2},\ \frac{df'}{da}=-\frac{1}{2^2}$$

so that

$$f=\frac{l}{e}=\frac{1}{2}\left\{1+\left(\frac{e}{2}\right)^2+\frac{4}{2}\left(\frac{e}{2}\right)^4+\frac{5\,.\,6}{2\,.\,3}\left(\frac{e}{2}\right)^6+\&\text{c.}\right\}$$

or generally

$$f'=\frac{1}{2^i},\ \frac{df'}{da}=-\frac{1}{2^{i+1}},$$

consequently

$$l^i=\frac{e^i}{2^i}\left\{1+i\left(\frac{e}{2}\right)^2+\frac{i(i+3)}{2}\left(\frac{e}{2}\right)^4+\frac{i(i+3)(i+5)}{2\cdot 3}\left(\frac{e}{2}\right)^6+\&\text{c.}\right\}$$

If i be successively assumed to be 1, 2, 3, &c., this equation will give all the powers of l in series, ascending according to the powers of e.

Again. If we assume $f=u=nt+e\sin u$, f is a function of u which is a function of e; hence

$$\frac{df}{de} = \frac{df}{du} \cdot \frac{du}{de};$$

and as $f' = nt$, when $e = 0$, so $\frac{du}{de} = \sin nt$. And[27] $\frac{df'}{de} = \sin nt \cdot \frac{df}{du}$. Whence by the same process it will be found that[28]

$$u = f + e\sin nt \cdot \frac{df}{ndt} + \frac{e^2}{2} \cdot \frac{d \cdot \sin^2 nt \cdot df}{(ndt)^2} + \frac{e^3}{2 \cdot 3} \cdot \frac{d^2 \cdot \sin^2 nt \cdot df}{(ndt)^3} + \text{\&c. \&c.}$$

Values of u, $\sin u$, $\sin 2u$, &c., may be determined from this expression by making f successively equal to nt, $e \cdot \sin nt$, &c. The substitution of these, and of the powers of l, will complete the development of v, but the same may be effected very easily from the expression $dv = \frac{hdt}{r^2}$ of article 372, or rather from

$$dv = \sqrt{1-e^2} \cdot \frac{a^2}{r^2} \cdot ndt.$$

390. If $r^i = a^i(1-e\cos nt)^i$ be put for $r = a(1-e\cos nt)$, and $ia^i(1-e\cos nt)^{i-1} \cdot e\sin nt$ for $\frac{dr}{ndt}$ in the development of r in article 387, it becomes[29]

$$\begin{aligned}\frac{r^i}{a^i} &= (1-e\cos nt)^i + i \cdot e^2 \cdot \sin^2 . nt(1-e\cos nt)^{i-1} \\ &+ \frac{i \cdot e^3 \cdot \sin^3 . nt(1-e\cos . nt)^{i-1}}{2ndt} \\ &+ \frac{i \cdot e^4 d^2 \cdot \sin^4 . nt(1-e\cos . nt)^{i-1}}{2 \cdot 3 \cdot n^2dt^2} + \text{\&c.}\end{aligned}$$

whatever *i* may be. Let $i = -2$, then

$$\begin{aligned}\frac{a^2}{r^2} &= 1 + 2e . \cos . nt + \frac{e^2}{1 \cdot 2} \cdot (1 + 5 . \cos . nt) \\ &+ \frac{e^3}{1 \cdot 2^2}(13 . \cos . 3nt + 3 . \cos . nt) \\ &+ \frac{e^4}{1 \cdot 2^3 \cdot 3}(103 . \cos . 4nt + 8 . \cos . 2nt + 9) + \text{\&c.}\end{aligned}$$

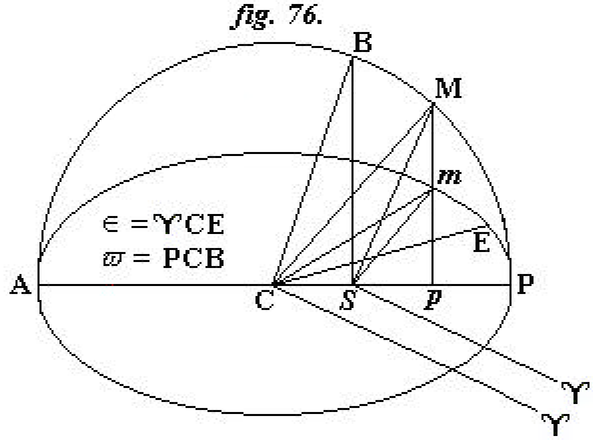
If this quantity be substituted in the preceding expression for *dv*, when the integration is accomplished, and the approximation only carried to the sixth powers of *e*, the result will be

$$v = nt + \left\{2e - \frac{1}{4}\cdot e^3 + \frac{5}{96}\cdot e^5\right\}\sin nt$$
$$+\left\{\frac{5}{4}\cdot e^2 - \frac{11}{24}\cdot e^4 + \frac{17}{192}\cdot e^6\right\}\sin 2nt$$
$$+\left\{\frac{13}{12}\cdot e^3 - \frac{45}{64}\cdot e^5\right\}\sin 3nt$$
$$+\left\{\frac{103}{96}\cdot e^4 - \frac{451}{480}\cdot e^6\right\}\sin 4nt, + \&c.\ \&c.$$

391. The angles v and nt which are the true and mean anomaly, begin at the perihelion; but if they be estimated from the aphelion, it will only be necessary to make e negative in the values of r and v, or to add 180° to each angle. This expression gives $v - nt$ the equation of the centre.

True Longitude and Radius Vector in functions of the Mean Longitude

fig. 76.



392. Instead of fixing the origin of the time at the instant of the planet's passage at the perihelion, let it be fixed at any point whatever, as E, fig. 76,[30] so that $nt = \text{ECB}$, then by adding the constant angle ♈CE represented by $\in$, the whole angle ♈$\text{CB} = nt + \in$ is the mean longitude of the planet, ♈ being the equinox of Spring; and if the constant angle ♈CP, which is the longitude of the perihelion, be represented by v, the angle $nt + \in - v = \text{PCB}$ must be put for nt, and if v be estimated from ♈, then $v - v$ must be put for v, and the preceding values of v and r become,

$$v = nt + \in + \left\{2e - \frac{1}{4}e^3\right\}\sin\left(nt + \in - v\right)$$
$$+\left\{\frac{5}{4}e^2 - \frac{11}{24}e^4\right\}\sin 2\left(nt + \in - v\right) + \&c.\ \&c. \qquad \textbf{(97)}$$

$$\frac{r}{a} = 1 + \frac{1}{2}e^2 - \left\{e - \frac{3}{8}e^3\right\}\cos\left(nt + \in - v\right)$$
$$-\left\{\frac{1}{2}e^2 - \frac{1}{3}e^4\right\}\cos 2\left(nt + \in - v\right) - \&c.\ \&c. \qquad \textbf{(98)}$$

393. v is the true longitude of the planet and $nt+\epsilon$ its mean longitude both being estimated on the plane of the orbit. The angle $\epsilon =$ ♈CE is the longitude of the point E, from whence the time is estimated, commonly called the longitude of the epoch.

394. In astronomical series, the quantities which multiply the sines and cosines are the coefficients; and the angles are called the arguments: for example in[31]

$$\left\{2e-\frac{1}{4}e^{3}\right\}\sin\left(nt+\epsilon-v\right)$$

the part $\left\{2e-\frac{1}{4}e^{3}\right\}$ is the coefficient, and $\left(nt+\epsilon-v\right)$ is the argument.

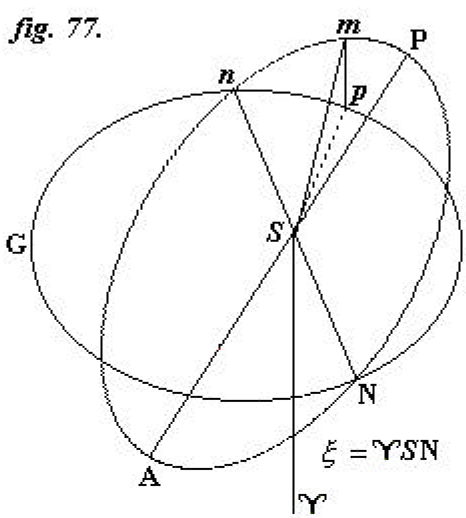
395. Although the time increases without limit, these series converge: for, as a sine or cosine never can exceed the radius, the values of the sines and cosines in these series never can be greater than unity, however much the time may increase, and as the powers of e soon become extremely small, they converge rapidly.

396. The values of v and r answer for all the planets and satellites, since they are independent of the masses, for the mass of a planet is so inconsiderable in comparison of that of the sun, that it may be omitted, and as the mass of the sun forms the standard of comparison for the masses of the other bodies of the system, it is assumed to be the unit of measure. The same holds with regard to a planet and its satellites.

Determination of the Position of the Orbit in space

397. The values of v and r give the place of a body in its orbit, but not its position in space; they however afford the means of ascertaining it. For let N*pn*G, fig. 77,[32] be the plane of the ecliptic, or fixed plane at the epoch, on which the plane of the orbit P*n*AN has a very small inclination; then N*n* is the line of the nodes; S the sun, and if mp be a perpendicular from the planet on the plane of the ecliptic, it will be the tangent of the latitude mSp. Let ♈*S*N the longitude of the node be represented by x when estimated on the plane of the orbit, and let q represent the same angle when projected on the plane of the ecliptic; also let $v_{/} =$ ♈*Sp* be the true longitude ♈*Sm* or v, when projected on the plane of the ecliptic. Then

fig. 77.



$$\mathrm{N}Sp = v_{/} - q\,, \quad \mathrm{N}Sm = v - z_{/}\,.$$

And if f be the inclination of the two planes, it appears from the right angled triangle *p*N*m*, that

$$\tan\left(v_{/}-q\right)=\cos f\,\tan\left(v-x\right). \tag{99}$$

Projected Longitude in Functions of true Longitude

398. This gives $v_{/}$ in terms of v, and the contrary. But these two angles may be obtained in terms of one another in very converging series by means of the expression,

$$\tfrac{1}{2}v=\tfrac{1}{2}u+l\,\sin u+\frac{l^{2}}{2}\sin 2u+\frac{l^{3}}{3}\sin 3u+\&c.$$

which was derived from $\tan\tfrac{1}{2}v=\sqrt{\dfrac{1+e}{1-e}}\cdot\tan\tfrac{1}{2}u$, by making $l=\dfrac{e}{1+\sqrt{1-e^{2}}}$. If $v_{/}-q$ be put for $\tfrac{1}{2}v$, $v-z$ for $\tfrac{1}{2}u$, and $\cos f$ for $\sqrt{\dfrac{1+e}{1-e}}$; then

$$l=\frac{\cos f-1}{\cos f+1}=-\tan^{2}\tfrac{1}{2}f\,,$$

and the series becomes

$$v_{/}-q=v-x-\tan^{2}\tfrac{1}{2}f\cdot\sin 2\left(v-x\right)+\tfrac{1}{2}\tan^{4}\tfrac{1}{2}f\cdot\sin 4\left(v-x\right)-\&c. \tag{100}$$

True Longitude in Functions of projected Longitude

On the contrary, if $v-x$ be put for $\tfrac{1}{2}v$, and $v_{/}-q$ for $\tfrac{1}{2}u$, the result will be

$$v-x=v_{/}-q+\tan^{2}\tfrac{1}{2}f\cdot\sin 2\left(v_{/}-q\right)+\tfrac{1}{2}\tan^{4}\tfrac{1}{2}f\cdot\sin 4\left(v_{/}-x\right)+\&c. \tag{101}$$

Projected Longitude in Functions of Mean Longitude

399. A value of $v_{/}-q$, or N*Sp*, may be found in terms of the sines and cosines of *nt*, and its multiple arcs, from the series

$$v=nt+\in+\{2e-\tfrac{1}{4}e^{3}\}\sin\left(nt+\in-v\right)+\left\{\frac{5}{4}e^{2}-\frac{11}{24}e^{4}\right\}\sin 2\left(nt+\in-v\right)+\&c.$$

which may be written

$$v=nt+\in+e\mathrm{Q}\,.$$

If x be subtracted from both sides of this equation, and the sines taken in place of the arcs, it becomes

$$\sin(v-x) = \sin(nt+\epsilon-x+e\text{Q}),$$

which may be expanded into a series, ascending, according to the powers of e, by the method already employed for the development of v and r; if

$$f = \sin(v-x) = \sin(nt+\epsilon-x+e\text{Q}).$$

Whence it may be found that,

$$\sin i(v-x) = \sin i(nt+\epsilon-x+e\text{Q}) = \left\{1-\frac{i^2e^2\text{Q}^2}{1.2}+\frac{i^4e^4\text{Q}^4}{1.2.3.4}-\&\text{c.}\right\}\times\sin i(nt+\epsilon-x)$$
$$+\left\{ie\text{Q}-\frac{i^3e^3\text{Q}^3}{1.2.3}+\frac{i^5e^5\text{Q}^5}{1.2.3.4.5}-\&\text{c.}\right\}\times\cos i(nt+\epsilon-x)+\&\text{c.}$$

Latitude

400. If mp, the tangent of the latitude, be represented by s, the right-angled triangle $m\text{N}p$ gives

$$s = \tan f \sin(v_{/}-q).$$

Curtate Distances

401. Let r, be the curtate distance Sp, then Spm, being a right angle,

$$Sp : Sm :: 1 : \sqrt{1+s^2};$$

hence

$$Sp = \frac{Sm}{\sqrt{1-s^2}};$$

or

$$r_{/} = r\left(1+s^2\right)^{-\frac{1}{2}} = r\left\{1-\tfrac{1}{2}s^2+\tfrac{3}{8}s^4-\&\text{c.}\right\} \qquad \textbf{(102)}$$

402. Thus $v_{/}$, s, and $r_{/}$, the longitude, latitude, and curtate distance[33] of the planet are determined in convergent series of the sines and cosines of nt and its multiples; if therefore the time be assumed, the place of the body will be known, and the means are thus furnished for computing tables of the motions of the planets and satellites, from which their elliptical places may be ascertained at any instant.

403. A particular period is chosen as an origin from whence the time is estimated, which is called the Epoch of the tables: the elements of the orbits are determined by observation; and the longitude, latitude, and distance of the body from the sun are computed for that period, and for every succeeding day, hour, and minute, if necessary, for any number of years; these are arranged in tables according to the time; so that by inspection alone the corresponding place of the body referred to the fixed plane, or position of the ecliptic at the epoch, may be found.

Fortunately for the facility of astronomical calculations, the orbits of the celestial bodies are either very nearly circular, as in the planets and satellites, or very eccentric, as in the comets. In both circumstances the series which determines the motion of the body may be made to converge rapidly, which would not be the case if the eccentricity bore a mean ratio to the greater axis.

Motion of Comets

404. If the ratio of the eccentricity to the greater axis be made very nearly equal to unity, instead of a very small fraction, the preceding series will then give the place of a comet in a very eccentric orbit, with this difference, that the terms have the increasing powers of the difference between unity and the ratio of the eccentricity to the greater axis, as coefficients, instead of the powers of that ratio itself. This difference is zero in the parabola; then the value of the radius vector becomes

$$r = \frac{\mathrm{D}}{\cos^2 . \frac{1}{2} v},$$

D being the perihelion distance: hence, in the parabola, the distance *Sm* is equal to the perihelion distance *S*P, divided by the square of the cosine of half the true anomaly P*Sm*. If, then, the true anomaly were known, the distance of the comet from the sun would be determined from this equation. When the body moves in a parabola, the equation between the mean and true anomaly is reduced to a cubic equation between the time and the tangent of half the true anomaly *PSm*.

Arbitrary Constant Quantities of Elliptical Motion, or Elements of the Orbits

405. There are six elements in the orbit of each celestial body: four of elliptical motion, namely, the mean distance of the planet from the sun; the eccentricity; the mean longitude of the planet at the epoch; and the longitude of the perihelion at the same epoch. The other two elements relate to the position of the orbit in space, namely, the longitude of the ascending mode at the epoch, and the inclination of the orbit on the plane of the ecliptic. The mean values of all these must be determined by observation, before the motion of the body call be ascertained, or tables computed. Hence there are forty-two elements to be determined for the seven principal planets, and twenty-four more for the four new planets, Ceres, Pallas, Juno, and Vesta,[34] besides those of the moon and satellites. Tables have been computed for most of these bodies; some of the satellites, however, are but little known, and the theory of the four new planets is still imperfect.

The same series that determine the motions of the planets answer equally well for the elliptical motion of the moon and satellites, only the mass of the planet is to be employed in place of that of the sun, omitting the mass of the satellite.

Co-ordinates of a Planet

406. The simplicity of analytical expressions very much depends on a skilful choice of co-ordinates, which are arbitrary and infinite in number, but so connected, that any one set may be expressed in values of any other. For example, the place of the planet *m* has been determined by the angles ♈*Sm*, *mSp*, and *Sm*, fig. 77, but these have been changed into ♈*Sp*, *pSm*, and *Sp*, which are the heliocentric longitude, latitude, and curtate distance of *m*. Again, from the latter, the geocentric longitude, latitude, and distance may be deduced, that is, the place of *m* as seen from the earth; and, lastly, the right ascension and declination of *m*, or its place referred to the equator, may be obtained from its geocentric longitude and latitude.

These quantities are given in terms of the mean longitude or time, since the first co-ordinates are given in series of the sines and cosines of that quantity. In the theory of the moon, the series are found to converge more rapidly, if the mean longitude, latitude, and distance are determined in functions of the true longitude. All these co-ordinates are connected by spherical triangles, so they are easily deduced from one another.

Determination of the Elements of Elliptical Motion

407. Were the primitive velocity with which the bodies of the solar system projected in space known, the values of the elements of their orbits might be determined; for if the equation (90) be resumed, and if the first member, which is the square of the velocity, be represented by V^2, then

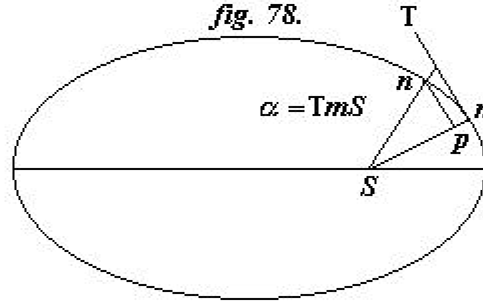
$$V^2 = m\left\{\frac{2}{r} - \frac{1}{a}\right\}$$

in which r is the radius vector, and a is half the greater axis of the conic section, m being the masses of the sun and planet. Thus the velocity is independent of the eccentricity of the orbit.

If u be the angular velocity which the planet would have if it described a circle at the distance of unity round the sun, then $r = a = 1$, and the preceding expression gives $u^2 = m$; hence

$$V^2 = u^2\left\{\frac{2}{r} - \frac{1}{a}\right\},$$

fig. 78.



V being the primitive velocity with which the body moved in a conic section. This equation will give a value of a by means of the primitive velocity of m, and its distance from *S*, fig. 78.[35] a is positive in the ellipse, infinite in the parabola, and negative in the hyperbola;

thus the orbit of m is an ellipse, a parabola, or hyperbola, according as V is less, equal to, or greater than $u=\sqrt{\frac{2}{r}}$. It is remarkable that the *direction* of the primitive impulse has no influence on the nature of the conic section in which the planet moves; the intensity alone has that effect.

To determine the eccentricity of the orbit, let a be the angle T*mS*, that the direction of the relative motion of *m* makes with the radius vector *r*; then

$$mn : mv :: ds : dr :: 1 : \cos a \; ;$$

then

$$\frac{ds}{dt}\cos a=\frac{dr}{dt}, \text{ but } \frac{ds}{dt}=V \; ,$$

hence

$$V^2\cos^2 a=\frac{dr^2}{dt^2}; \text{ or if } m\left\{\frac{2}{r}-\frac{1}{a}\right\}$$

be put for V,

$$\frac{dr^2}{dt^2}=m\left\{\frac{2}{r}-\frac{1}{a}\right\}\cos^2 a \; ;$$

but by article 377,

$$2mr-\frac{mr^2}{a}-\frac{r^2dr^2}{dt^2}=ma\left(1-e^2\right);$$

hence

$$a\left(1-e^2\right)^2=r^2\sin^2 a\left\{\frac{2}{r}-\frac{1}{a}\right\},$$

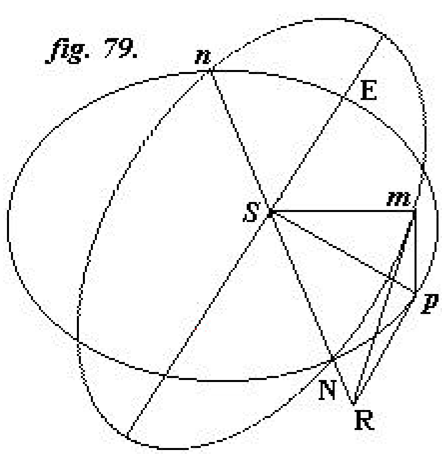
which gives the eccentricity of the orbit. The equation of conic sections,

$$r=\frac{a\left(1-e^2\right)}{1+e\cos v}$$

gives

$$\cos v=\frac{a\left(1-e^2\right)-r}{er}.$$

fig. 79.



Thus the angle v, that the radius vector makes with the perihelion distance, is found, and, consequently, the position of the perihelion. The equations (96) will then give the angle u, or eccentric anomaly, and, by means of it, the instant of the passage at the perihelion.

In order to have the position of the orbit, with regard to a fixed plane passing through the centre of *S*, fig. 77, supposed immovable,[36] let f be the inclination of the two planes, and $x = m$*S*N; also let $mp=z$ be the primitive elevation of the planet above the fixed plane, which is[37] supposed to be known;

then

$$r \sin x \sin f = z .$$

So that f, the inclination of the orbit, will be known when x shall be determined. For that purpose, let $l = m\mathrm{R}p$, fig. 79, be the angle made by mR, the primitive direction of the relative motion of m with the plane ENB; then the triangle $mS\mathrm{R}$, in which $Sm\mathrm{R} = a$, $NSm = x$, and $Sm = r$, gives

$$m\mathrm{R} = \frac{r \sin x}{\sin(x + a)};$$

then

$$\frac{z}{m\mathrm{R}} = \sin l ,$$

which is given, because l is supposed to be known; therefore

$$\tan x = \frac{z \sin a}{r \sin l - z \cos a}.$$

The elements of the orbit of the planet being determined by these formulae in terms of r, z, the velocity of the planet, and the direction of its motion, the variations of these elements, corresponding to the supposed variations in the velocity and its direction, may be obtained; and it will be easy, by means of methods that will be hereafter given, to have the differential variations of these elements, arising from the action of the disturbing forces.

Velocity of Bodies moving in Conic Sections

408. As the actual motions of the bodies of the solar system afford no information with regard to their primitive motions, the elements of their orbits can only be known by observation; but when these are determined, the velocities with which the bodies of the solar system were first projected in space, may be ascertained. If the equation

$$V^2 = u^2 \left\{ \frac{2}{r} - \frac{1}{a} \right\}$$

be resumed, then in the circle $r = a$, since the eccentricity is zero; hence[38]

$$V = u\sqrt{\frac{1}{r}};$$

therefore

$$V : u :: 1 : \sqrt{r} .$$

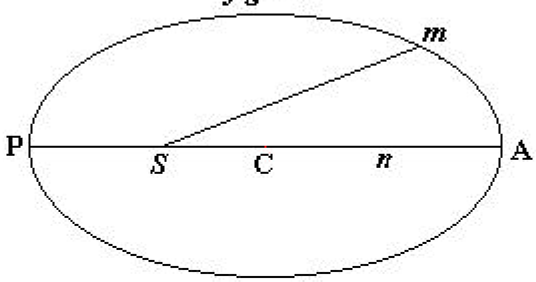
Thus the velocities of planets in different circles are as the square roots of their radii.

In the parabola, a is infinite; hence

$$\frac{1}{a} \text{ is zero, and } V = \sqrt{\frac{2}{r}}.$$

Thus the velocities in different points of a parabolic orbit are reciprocally as the square roots of the radii vectores, and the velocity in each point is to the velocity the planet would have if it moved in a circle with a radius equal to r, as $\sqrt{2}$ to 1.

fig. 80.



409. When an ellipse is infinitely flattened, it becomes a straight line; hence, in this case, V will express the velocity of m, if it were to descend in a straight line towards the sun; for then *Sm*, fig. 80, would coincide with *S*A. If m were to begin to fall from a state of rest at A, its velocity would be zero at that point; hence $\frac{2}{r} - \frac{1}{a} = 0$. Now, suppose that, in falling from A to n, the body had acquired the velocity V, then the equation would be

$$V^2 = u^2 \left\{ \frac{2}{r'} - \frac{1}{a} \right\},$$

and eliminating a, which is common to the two last equations,

$$V = u\sqrt{\frac{2(r - r')}{rr'}},$$

in which $r' = Sn$. This is the relative velocity the body m has acquired in falling from A through $r - r' = \mathrm{A}n$. Imagine the body m to have acquired, by its fall through An, the same velocity with a body moving in a conic section; the velocity of the latter body is

$$V' = u\sqrt{\frac{2}{r} - \frac{1}{u}}.$$

If these two be equated,

$$\mathrm{A}n = (r - r') = \frac{r(2a - r)}{4a - r}.$$

This expression gives the height through which a body moving in a conic section must fall, from the extremity A of the radius vector, in order to acquire the relative velocity which it had at A.

In the circle $a = r$, hence $An = \frac{1}{3} r$; in the ellipse, A*n* is less than $\frac{1}{2} r$;[39] in the parabola, *a* is infinite, which gives $An = \frac{1}{2} r$; and in the hyperbola *a* is negative, and therefore A*n* is greater than $\frac{1}{2} r$.

Notes

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[1] *curtate*. Shortened or reduced;–said of the distance of a planet from the sun or earth, as measured in the plane of the ecliptic, or the distance from the sun or earth to that point where a perpendicular, let fall from the planet upon the plane of the ecliptic, meets the ecliptic. *Webster's Revised Unabridged, 1913.*
[2] This *reads* "article 146" in the 1st edition (published erratum).
[3] Pontécoulant, Philippe Gustave le Doulcet, 1795-1874, French astronomer and graduate of the École Polytechnique.

[4] The second result below *reads* $md\frac{x}{r}$ in the 1st edition.
[5] This *reads* "article 269" in the 1st edition.
[6] The left hand side uses the letter *o* rather than the numeral 0 in the 1st edition.
[7] *horary*. Of or pertaining to an hour; noting the hours. *Webster's Revised Unabridged, 1913 Edition.*
[8] The 1st edition uses the symbol g here rather than symbol Υ introduced in the article 359.
[9] This *reads* "269" in the 1st edition (published erratum).
[10] The denominator *reads* $1 - e\cos(v - v\ )$ in the 1st edition (published erratum).
[11] This *reads* $l = 0$ in the 1st edition (published erratum).
[12] Note that the parameter *n* is used here for the first time.

[13] Or $T = \frac{360^\circ}{n}$, since $n = \frac{\sqrt{u}}{a^{\frac{3}{2}}}$ as defined in article 383.
[14] Colin Maclaurin (1698-1746). Maclaurin's *theorem* is a special case of Taylor's series (see note 18, *Book I, Chapter VII*) now named after him. The Taylor series for the function f at $x = c$ is $\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$. The Taylor series for the function f at $x = 0$ is also called a Maclaurin series for f.
[15] This *reads* $\frac{d^2u'}{de^2} = \cos nt \sin nt$ in the 1st edition (published erratum).
[16] A semicolon is omitted from second line below and the multiplier symbol added to several terms for consistency.
[17] $\frac{r}{a}$ is used for *r* in the 1st edition (published erratum).
[18] $\frac{dr'}{de}$ is used for $\frac{dr}{de}$ in the 1st edition (published erratum). However, the original equation reads $\frac{dr'}{de} = \frac{dr}{de} + \frac{dr}{du} \cdot \frac{du}{de}$; consequently, the corrected expression should read $\frac{dr}{de} = \frac{dr'}{de} + \frac{dr'}{du} \cdot \frac{du}{de}$.
[19] Punctuation added.
[20] $\cos nt$ is used for $a\cos nt$ in the next three equations in the 1st edition (published erratum).
[21] This *reads* "become" in the 1st edition.

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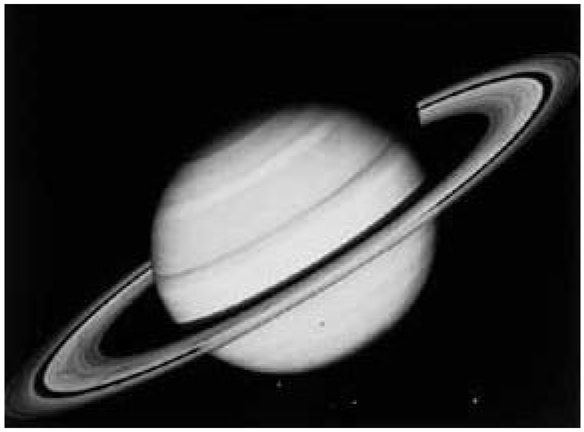
[22] Right hand side *reads* $\frac{d^2\left(\sin^3 nt \cdot \frac{dr}{ndt}\right)}{(ndt)^2}$ in the 1st edition.
[23] Equations modified to include multiplier symbol in relevant terms.
[24] Punctuation removed after first, second, third and fourth terms.
[25] see note 16, *Preliminary Dissertation*.
[26] *Lagrange's Theorem:* The theorem on the development of a function in series, for examples if y=x+ef(x), where e is small, then the theorem of Legrange gives the development of f(y) in series of y.
[27] This *reads* $\frac{df'}{de} = \sin nt \frac{df}{du}$ in the 1st edition.
[28] Equation modified to include multipliers.
[29] The remaining three equations in article 390 are modified slightly from the 1st edition text to reflect a consistent inclusion of the multiplier symbol where appropriate.
[30] Fig. 76 is modified from the 1st edition with inclusion of explicit definitions of $\in$ and v .
[31] This is the third term in equation (97). The presentation here is altered from the 1st edition with inclusion of the parentheses for clarity.
[32] Fig. 77 is modified from the 1st edition to include an explicit definition of x .
[33] See note 1.
[34] See note 9, *Preliminary Dissertation*.
[35] Figure 78 is modified to include an explicit definition of a .
[36] Spelled "immoveable" in the 1st edition.
[37] Typographical error in the 1st edition omits the letter "i" in "is".
[38] The symbol *V* reads *v* in the 1st edition (published erratum).
[39] Punctuation changed from a colon in the 1st edition to a semicolon.

Saturn



SATURN, the sixth planet from the Sun, is one of the five planets visible from Earth without a telescope. Since the 17th century, when Saturn's dazzling, complex ring system was first observed by the Italian astronomer Galileo Galilei, the planet has stood as a symbol of the majesty, mystery, and order of the physical universe. Over the past 20 years, we have discovered that Jupiter, Uranus, and Neptune also have rings; however, Saturn's ring system is the most extensive and brilliant. Although the origin of the rings is unknown, scientists hope to uncover clues by studying the planet's history.

Although Galileo was the first to see Saturn's rings (in 1610), it wasn't until 1659 that the Dutch astronomer Christiaan Huygens, using an improved telescope, observed that the are actually separate from the planet. In 1676, the French-Italian astronomer Jean Dominique Cassini first observed what appeared to be a division between the rings now known as the Cassini division. Improvements in telescopy over the next three centuries revealed much about the mysterious planet: the banded atmosphere, the storm "spots," and a very apparent "flattening" at the poles, three features Saturn was observed to share with Jupiter. (Courtesy of NASA)

BOOK II

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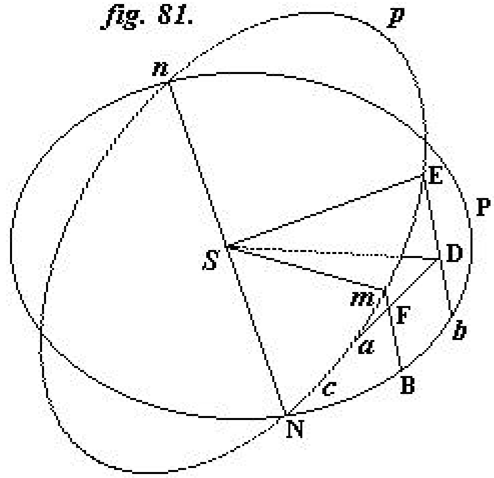
CHAPTER V

THEORY OF THE PERTURBATIONS OF THE PLANETS

410. THE tables computed on the theory of perfectly elliptical motion, are soon found inadequate to give the true place of a planet, on account of the reciprocal disturbances of the system. It is therefore necessary to investigate what these disturbances are, and to determine their effects.

In the first approximation to the celestial motions, the mutual action of the sun and one planet was considered: it then appeared that a planet, *m*, moves round the sun in an ellipse N*m*P*n*, fig. 81, inclined to the ecliptic NB*n*, at a very small angle P*np*. Now, if *m* be attracted by another planet m', which is much smaller than the sun, it will no longer go on in its elliptical orbit N*mn*, but will be drawn out of that orbit, and will move in some curved line, *ca*D, which may either be nearer to, or farther from, the plane of the ecliptic, according to the position of the disturbing body. In the first infinitesimal of time, the troubled orbit coincides with the ellipse through an indefinitely small space *ca*; in the second infinitely small interval of time, *am* will be the path of the planet in the ellipse, and *a*D will be its path in its troubled orbit: *am* is described in consequence of the action of the sun alone; *a*D by the combined action of the sun and of the disturbing body; *am* is the second increment of the space; *a*D is the second increment of the space, together with some very small space, FD, introduced by the action of the disturbing force. In consequence of the addition of FD, the longitude of *m* is increased by B*b*; its latitude is changed by the angle DSE, and the radius vector is increased by the difference between *S*D and *Sm*,–these three quantities are the perturbations of the planet in longitude, latitude, and distances.

fig. 81.



411. It is evident that the perturbations are true variations; and as the longitude, latitude, and radius vector of a planet moving in an elliptical orbit, have been represented by *v*, *s*, and *r*, the arcs $\text{B}b = dv$, $\text{ED} = ds$, and $S\text{D-}Sm - dr$, are the variations of these inequalities.

412. The perturbations in longitude, latitude, and distance, depend on the configuration of the bodies; that is, on the position of the bodies with regard to each other, to their perihelia and to their nodes. These inequalities, after going through a certain course of increase and decrease, are renewed as often as the bodies return to the same relative positions, and are therefore called Periodic Inequalities.

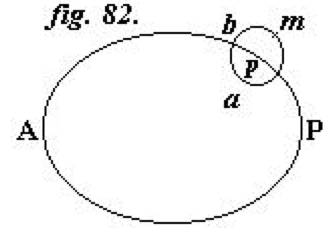
413. Thus the place of a planet, *m*, moving in its troubled orbit *ca*D, will be determined by the co-ordinates $v+dv$, $s+ds$, $r+dr$. These, however, are modified by a variation in the elements of the ellipse; for it is evident that, the path of the planet being changed from *a*E to *a*D, the elements of the ellipse N*m*E must vary. The variations of the elements are independent of the configuration or relative position of the bodies, and are only sensible in many revolutions; whereas those depending on the configuration, accomplish their changes in short periods. Thus $v+dv$, $s+ds$, $r+dr$, may be regarded as the co-ordinates of the planet in its true orbit, provided the elements contained in these functions be considered to vary by very slow degrees. This perfectly accords with observation, whence it appears that the perihelia of the orbits of the planets and satellites have a very slow direct motion in space; that the nodes have a slow retrograde motion; and that the eccentricities and inclinations are perpetually varying by very slow degrees. These very slow changes are really periodic, but many ages elapse before they accomplish their revolutions; on that account they are called Secular Inequalities, to distinguish them from the Periodic Inequalities, which pass rapidly from their maxima to their minima. Thus the Periodic Inequalities only depend on the configuration of the bodies, whereas the Secular Inequalities depend on the configuration of the perihelia and nodes alone.

414. Lagrange[1] took a new and very elegant view of the subject:–he considered the changes dv, ds, dr, to arise entirely from periodic and secular variations in the elements of elliptical motion, thus referring all the inequalities, to which a planet is liable, to changes in the elements of its orbit alone. In fact, as the curve *a*D very nearly coincides with the ellipse, it may be regarded as a portion of a new ellipse, having elements differing from those of the original one by infinitely minute variations. Of *these* a portion will be compensated in a whole revolution, or many revolutions of *m*, and of the disturbing planet constituting the Periodic Inequalities; but a portion will remain uncompensated, and entire1y independent of the position of the bodies with regard to each other. These uncompensated parts increase and diminish with extreme slowness; their effects on the motion of *m* partake of that character, and constitute what are called Secular Inequalities. Thus, in Lagrange's view, the co-ordinates of *m* in its elliptical orbit are modified, both by periodic and secular variations, in the elements of the ellipse.

415. The secular inequalities depend on the ratio of the disturbing mass to that of the sun, which, by article 350, is a very small fraction. Their arguments are not only different from those of the periodic inequalities, but, though also periodic, their periods are immensely longer.

416. Both periodic and secular inequalities may be represented by supposing a point *p* to revolve in an ellipse AP, fig. 82, where all the elements are perpetually varying by very slow degrees. Then, suppose a planet *m* to oscillate round the moveable point *p* in a curve *mab*, whose nature depends on the disturbing forces: this oscillating motion will represent the periodic inequalities, and the whole compound motion *m* represents the real motion of a planet in its troubled orbit.

fig. 82.



Demonstration of Lagrange's Theorem

417. The equations which determine the real motion of *m* in its troubled orbit are, by article 347,

$$\frac{d^2x}{dt^2}+\frac{mx}{r^3}=\left(\frac{dR}{dx}\right),$$
$$\frac{d^2y}{dt^2}+\frac{my}{r^3}=\left(\frac{dR}{dy}\right), \qquad \textbf{(87)}$$
$$\frac{d^2z}{dt^2}+\frac{mz}{r^3}=\left(\frac{dR}{dz}\right).$$

If $R=0,$ these equations would be the same with those in article 365, already integrated. Let a be one of the arbitrary constant quantities, or elements of the orbit of m, introduced by integration. When $R=0,$ then[2]

$$a=\text{Func.}\left(x,y,z,\frac{dx}{dt},\frac{dy}{dt},\frac{dz}{dt},t\right)$$

may represent any one whatever of the integrals (91); or, if to abridge

$$x_{/}=\frac{dx}{dt}\quad y_{/}=\frac{dy}{dt}\quad z_{/}=\frac{dz}{dt},$$

[then]

$$a=\text{Func.}\left(x,y,z,x_{/},y_{/},z_{/},t\right). \qquad \textbf{(103)}$$

During the instant dt, the ellipse and troubled orbit coincide; therefore x, y, z, $x_{/}$, $y_{/}$, $z_{/}$ have the same values in both, and a is constant. But at the end of the instant dt, the velocities $x_{/}$, $y_{/}$, $z_{/}$, are respectively augmented, from the action of the disturbing forces, by the indefinitely small quantities[3]

$$\frac{dR}{dx}dt,\quad \frac{dR}{dy}dt,\quad \frac{dR}{dz}dt;$$

then a is no longer constant; and when $x_{/}$, $y_{/}$, $z_{/}$ are increased by those quantities, the corresponding variation of a is

$$da=\left(\frac{da}{dx_{/}}\cdot\frac{dR}{dx}+\frac{da}{dy_{/}}\cdot\frac{dR}{dy}+\frac{da}{dz_{/}}\cdot\frac{dR}{dz}\right)dt. \qquad \textbf{(104)}$$

If equation (103) be regarded as the first integral of the equations (87), when $R=0,$ it will evidently satisfy the same equations when R is not zero, because the values of x, y, z, $x_{/}dt$, $y_{/}dt$, $z_{/}dt$, are supposed to be the same in each orbit, since these quantities only differ in the two curves by their second differentials.

Hence, if $(x_/)$, $(y_/)$, $(z_/)$ be the values of $x_/$, $y_/$, $z_/$, when $R=0$, then

$$x_/ = (x_/), \quad y_/ = (y_/), \quad z_/ = (z_/),$$

and

$$dx_/ = (dx_/) + d\,x_/, \quad dy_/ = (dy_/) + d\,y_/, \quad dz_/ = (dz_/) + d\,z_/.$$

Let func.$\left(x, y, z, x_/, y_/, z_/, t\right)$ be the differential of equation (103) when $R=0$, then will

$$0 = \text{func.}\left(x, y, z, x_/, y_/, z_/, t\right)$$

and the differential of the same equation, when R is not zero, will be

$$da = \text{func.}\left(x, y, z, x_/, y_/, z_/, t\right) + \left(\frac{da}{dx_/}d\,x_/ + \frac{da}{dy_/}d\,y_/ + \frac{da}{dz_/}d\,z_/\right),$$

because, in the latter case, all the quantities vary. If the first differential be subtracted from the second, the result will be

$$da = \left(\frac{da}{dx_/}d\,x_/ + \frac{da}{dy_/}d\,y_/ + \frac{da}{dz_/}d\,z_/\right). \qquad \textbf{(105)}$$

But if

$$(dx_/) + d\,x_/, \quad (dy_/) + d\,y_/, \quad (dz_/) + d\,z_/,$$

be put, in equations (87), in place of their equals,

$$\frac{d^2x}{dt^2}, \quad \frac{d^2y}{dt^2}, \quad \frac{d^2z}{dt^2},$$

they become

$$d\,x_/ = \frac{dR}{dx}dt, \quad d\,y_/ = \frac{dR}{dy}dt, \quad d\,z_/ = \frac{dR}{dz}dt.$$

Since $(dx_/)$, $(dy_/)$, $(dz_/)$, are supposed to satisfy these equations when $R=0$.

If the preceding values $d\,x_/$, $d\,y_/$, $d\,z_/$, be put in equation (105), it becomes identical with equation (104). Hence the integral (103) satisfies the equations (87), whether the disturbing forces be included or not, the only difference being that, in the first case, *a* must be regarded as a variable quantity, and in the last it is constant.

The same may be shown of all the first integrals of equations (87), when R is zero.

418. It appears, from what has been said, 1st, that as the motion is performed in the unvaried ellipse during the first element of time, *x*, *y*, *z*, *dx*, *dy*, *dz*, are alike in the varied and unvaried ellipse. 2nd, That as the motion is performed in the variable ellipse during the second element of time, if d^2x, d^2y, d^2z, be considered as belonging to the unvaried ellipse, d^2x+ddx, d^2y+ddy, d^2z+ddz will belong to the variable orbit of m. Hence the differential equation of the first order, which determines the motion of the body, answers for both orbits during the first instant of the time, the elements of the orbit being constant; in the second increment of time, the equations of elliptical motion have the form

$$\frac{d^2v}{dt^2}+n^2v=0,$$

the elements of the orbit being constant; but in the troubled orbit they have the form

$$\frac{d^2v}{dt^2}+n^2v+R=0,$$

where the elements of the orbit are variable, and R is the part containing the disturbing forces.

419. As the elements of the orbits only vary during the second increment of the time, their variation is of the first order; that is, the eccentricity e becomes $e+de$, the inclination f becomes

$$f+df, \text{ \&c. \&c.}$$

420. The elegant theory of the variation of the arbitrary constant quantities is due to Euler.[4] Lagrange first applies it to the celestial motions.

421. It is proposed, first, to determine the periodic and secular variations of the elements of orbits of any eccentricities and inclinations; in the second place, to find those of the planets and satellites, all of which have nearly circular orbits, slightly inclined to the plane of the ecliptic; and then to determine the periodic inequalities, dv, ds, dr, in longitude, latitude, and distance.

Variation of the Elements, whatever the Eccentricities and Inclinations may be

422. All the elements of the orbit have been determined from the seven arbitrary constant quantities, c, c', c'', f, f', f'', and a, introduced by the integration of the equations (87) of elliptical motion; but it was shown that the elements of the orbit, as well as the differentials dx, dy, dz, vary during the second element of time by the action of the disturbing forces, and then the differentials of the equations (91) will afford the means of finding the variations of the elements, whatever the eccentricities and inclinations of the orbits may be. Equations (87) give[5]

$$d^2x=dt^2\left(\frac{dR}{dx}\right);\quad d^2y=dt^2\left(\frac{dR}{dy}\right);\quad d^2z=dt^2\left(\frac{dR}{dz}\right);$$

which are the changes in dx, dy, dz, due to the disturbing forces alone, the elliptical part being omitted. If, therefore, the differentials of equations (91) be taken, considering c, c', c'', f, f', f'', a, dx, dy, dz, alone as variable, when the preceding values of d^2x, d^2y, d^2z, are substituted, they become[6]

$$dc = dt\left\{x\left(\frac{dR}{dy}\right)-y\left(\frac{dR}{dx}\right)\right\};$$
$$dc' = dt\left\{z\left(\frac{dR}{dx}\right)-x\left(\frac{dR}{dz}\right)\right\};$$
$$dc'' = dt\left\{y\left(\frac{dR}{dz}\right)-z\left(\frac{dR}{dy}\right)\right\};$$
$$df = dx\left\{z\left(\frac{dR}{dx}\right)-x\left(\frac{dR}{dz}\right)\right\}-dy\left\{y\left(\frac{dR}{dz}\right)-z\left(\frac{dR}{dy}\right)\right\}+c'dt\left(\frac{dR}{dx}\right)-c''dt\left(\frac{dR}{dy}\right); \qquad \textbf{(106)}$$
$$df' = dz\left\{y\left(\frac{dR}{dz}\right)-z\left(\frac{dR}{dy}\right)\right\}-dx\left\{x\left(\frac{dR}{dy}\right)-y\left(\frac{dR}{dx}\right)\right\}+c''dt\left(\frac{dR}{dz}\right)-cdt\left(\frac{dR}{dx}\right);$$
$$df'' = dy\left\{x\left(\frac{dR}{dy}\right)-y\left(\frac{dR}{dx}\right)\right\}-dz\left\{z\left(\frac{dR}{dx}\right)-x\left(\frac{dR}{dz}\right)\right\}+cdt\left(\frac{dR}{dy}\right)-c'dt\left(\frac{dR}{dz}\right);$$
$$d\cdot\frac{m}{a} = -2\mathrm{d}R\,.$$

423. If values of c, c', c'', f, f', f'', derived from these equations, be substituted instead of their constant values in equations

$$\tan f = \frac{\sqrt{c'^2+c''^2}}{c}, \quad \tan q = -\frac{c''}{c'},$$
$$h^2 = ma\left(1-e^2\right) = c^2 + c'^2 + c''^2,$$
$$\tan v_{/} = \frac{f'}{f''}, \text{ and } me = \sqrt{f^2 + f'^2 + f''^2},$$

given in article 374 and those following, they will determine the elements of the disturbed orbit.

The equations

$$c''x + c'y + cz = 0,$$
$$mr - h^2 + f''x + f'y + fz = 0;$$

and their differentials

$$c''dx + c'dy + cdz = 0,$$
$$mdr + f''dx + f'dy + fdx = 0,$$

will also answer in the disturbed orbit, provided the same substitution be made.

424. The mean distance a gives the mean motion of m, or more correctly that in the disturbed orbit, which corresponds with the mean motion in the elliptical orbit; for

$$n = a^{-\frac{3}{2}}\sqrt{m}\,.$$

If z be the mean motion of m, then in the elliptical orbit,

$$dz = ndt;$$

but this equation also answers for the disturbed orbit, since the two orbits coincide during the first instant of time. But

$$ddz = dndt, \quad dn = \frac{3an}{2m}\cdot d\frac{m}{a};$$

and as the last of equations (106) is[7]

$$d.\frac{m}{a} = -2\mathrm{d}R, \text{ so } dn = -\frac{3an}{m}\mathrm{d}R\,;$$

hence

$$ddz = -\frac{3andt\,.\,\mathrm{d}R}{m};$$

the integral of which is

$$z = -\frac{3}{m}\iint andt.\mathrm{d}R\,. \qquad \textbf{(107)}$$

425. The seven arbitrary constant quantities are only equivalent to five in consequence of the two equations

$$0 = fc + f'c' + f''c'',$$

$$0 = \frac{m}{a} + \frac{f^2 + f'^2 + f''^2 - m^2}{c^2 + c'^2 + c''^2}.$$

These also exist in the disturbed orbit, when the arbitrary quantities are replaced by their variable values.

426. Since R is given in article 347, all the elements of the disturbed orbit are determined with the exception of $\in$, the longitude of the planet at the epoch. From the equations

$$dv = \frac{hdt}{r^2}, \quad r^2 = \frac{a^2\left(1-e^2\right)^2}{\left(1+e\cos\left(v-v\ \right)\right)^2},$$

it is evident that,

$$dv \cdot \frac{\left(1-e^2\right)^2}{\left(1+e\cos\left(v-v\right)\right)^2} = \frac{h}{a^2}dt\,.$$

But

$$h = \sqrt{ma\left(1-e^2\right)}\,;$$

hence

$$\frac{h}{a^2} = a^{-\frac{3}{2}}\sqrt{m}\sqrt{\left(1-e^2\right)} = n\sqrt{1-e^2}\,;$$

therefore

$$ndt = dv \cdot \frac{\left(1-e^2\right)^{\frac{3}{2}}}{\left(1+e\cos\left(v-v\right)\right)^2}\,.$$

If

$$\frac{c^{(v-v)\sqrt{-1}} + c^{-(v-v)\sqrt{-1}}}{2}$$

be put for

$$\cos\left(v-v\right),$$

[then]

$$\frac{\sqrt{1-e^2}}{1+e\cos\left(v-v\right)} = \frac{2\sqrt{1-e^2}}{2+e\left\{c^{(v-v)\sqrt{-1}} + c^{-(v-v)\sqrt{-1}}\right\}}\,.$$

Again, if

$$l = \frac{e}{1+\sqrt{1-e^2}}\,;\text{ then } e = \frac{2l}{1+l^2}\,,$$

which, substituted in the second member of the last equation, gives

$$\frac{1}{1+e\cos\left(v-v\right)} = \frac{1}{\sqrt{1-e^2}}\left\{\frac{1-l^2}{1+l^2+l\left\{c^{(v-v)\sqrt{-1}} + c^{-(v-v)\sqrt{-1}}\right\}}\right\}.$$

The numerator of the last term is

$$1-l^2 = \left(1+l\,c^{-(v-v)\sqrt{-1}}\right) - l\,c^{-(v-v)\sqrt{-1}}\left(1+l\,c^{(v-v)\sqrt{-1}}\right)$$

And the denominator is equal to

$$\left(1+l\,c^{(v-v)\sqrt{-1}}\right)\left(1+l\,c^{-(v-v)\sqrt{-1}}\right)$$

hence

$$\frac{1}{1+e\cos(v-v)}=\frac{1}{\sqrt{1-e^2}}\left\{\frac{1}{1+lc^{(v-v)\sqrt{-1}}}-\frac{lc^{-(v-v)\sqrt{-1}}}{1+lc^{-(v-v)\sqrt{-1}}}\right\}.$$

By division,

$$\frac{1}{1+lc^{(v-v)\sqrt{-1}}}=1-lc^{(v-v)\sqrt{-1}}+l^2c^{2(v-v)\sqrt{-1}}-\&c.$$

[and]

$$\frac{lc^{-(v-v)\sqrt{-1}}}{1+lc^{-(v-v)\sqrt{-1}}}=lc^{-(v-v)\sqrt{-1}}-l^2c^{-2(v-v)\sqrt{-1}}+\&c.$$

And the difference of these is[8]

$$\frac{1}{1+e\cos(v-v)}=\frac{1}{\sqrt{1+e^2}}\left\{1-l\left(c^{(v-v)\sqrt{-1}}+c^{-(v-v)\sqrt{-1}}\right)+l^2\left(c^{2(v-v)\sqrt{-1}}+c^{-2(v-v)\sqrt{-1}}\right)-\&c.\right\};$$

but

$$c^{i(v-v)\sqrt{-1}}+c^{-i(v-v)\sqrt{-1}}=2\cos i(v-v);$$

hence

$$\frac{1}{1+e\cos(v-v)}=\frac{1}{\sqrt{1+e^2}}\left\{1-2l\cos(v-v)+2l^2\cdot\cos 2(v-v)-\&c.\right\}$$

or

$$\frac{1}{1+e\cos(v-v)}=\frac{1}{\sqrt{1+e^2}}\mp 2\cos i(v-v)\frac{l^i}{\sqrt{1-e^2}},$$

which is the general form of the series, i being any whole positive number.

Now,

$$\frac{1}{de}\cdot d\frac{e}{1+e\cos(v-v)}=\frac{1}{\left(1+e\cos(v-v)\right)^2}=\frac{1}{de}\cdot\left\{d\frac{e}{\sqrt{1-e^2}}\pm 2\cos(v-v)\cdot d\frac{el^i}{\sqrt{1-e^2}}\right\};$$

but

$$d\cdot\frac{e}{\sqrt{1-e^2}}=\frac{de}{\left(1-e^2\right)^{\frac{3}{2}}},\ \text{and}\ d\frac{el^i}{\sqrt{1-e^2}}=\pm\frac{e^i\left\{1+i\sqrt{1-e^2}\right\}de}{\left(1-e^2\right)^{\frac{3}{2}}\left(1+\sqrt{1-e^2}\right)^i}$$

the sign + is used here when i is even, and − when it is odd. Hence if to abridge

$$\mathrm{E}^{(i)}=\pm\frac{2e^i\cdot\left\{1+i\sqrt{1-e^2}\right\}}{\left(1+\sqrt{1-e^2}\right)^i},$$

the value of ndt becomes,

$$ndt = dv\left\{1 + \mathrm{E}^{(1)}\cos(v - v) + \mathrm{E}^{(2)}\cos 2(v - v) + \&\mathrm{c.}\right\}; \qquad \textbf{(108)}$$

The integral of which is[9]

$$\int ndt + \in = v + \mathrm{E}^{(1)}\sin(v - v) + \tfrac{1}{2}\mathrm{E}^{(2)}\sin 2(v - v) + \&\mathrm{c.},$$

$\in$ being arbitrary.

This equation is relative to the invariable ellipse; but in order that it may also suit the real orbit, every quantity in it must vary including e, v, and $\in$; and this differential must coincide with (108) since they are of the first order, and the two orbits coincide during the first element of time. Their difference is

$$d\in = de\left\{\left(\frac{d\mathrm{E}^{(1)}}{de}\right)\sin(v - v) + \tfrac{1}{2}\left(\frac{d\mathrm{E}^{(2)}}{de}\right)\sin 2(v - v) + \&\mathrm{c.}\right\}$$
$$-dv\left\{\mathrm{E}^{(1)}\cos(v - v) + \mathrm{E}^{(2)}\cos 2(v - v) + \&\mathrm{c.}\right\}$$

$v - v$ is the true anomaly of m estimated on the orbit, and v is the longitude of the perihelion on the orbit. Now equation (101) is[10]

$$v - x = v_{/} - q + \tan^2\tfrac{1}{2}f\cdot\sin 2(v_{/} - q) + \tfrac{1}{2}\tan^4\tfrac{1}{2}f\cdot\sin 4(v_{/} - x) + \&\mathrm{c.}$$

v being the longitude on the orbit, and $v_{/}$ its projection on the fixed plane. If v be put for v and $v_{/}$ for $v_{/}$; then

$$v - x = v_{/} - q + \tan^2\tfrac{1}{2}f\sin 2(v_{/} - q) + \&c.$$

Again, if we make v and $v_{/}$ zero in equation (101), it becomes

$$x = q + \tan^2\tfrac{1}{2}f\sin 2q + \tfrac{1}{2}\tan^4\tfrac{1}{2}f\sin 4q + \&\mathrm{c.}$$

hence

$$v = v_{/} + \tan^2\tfrac{1}{2}f\left\{\sin 2q + \sin 2(v_{/} - q)\right\} + \&\mathrm{c.}$$

therefore

$$dv = dv_{/}\left\{1 + 2\tan^2\tfrac{1}{2}f\cos 2(v_{/} - q) + \&\mathrm{c.}\right\}$$
$$+2dq\tan^2\tfrac{1}{2}f\left\{\cos 2q - \cos 2(v_{/} - q) + \&\mathrm{c.}\right\}$$
$$+\frac{df\tan\frac{1}{2}f}{\cos^2\frac{1}{2}f}\left\{\sin 2q + \sin 2(v_{/} - q) + \&\mathrm{c.}\right\}$$

Thus $dv_{/}$, dq, df, being determined, we shall have dv from this equation, and from thence $d\in$.

427. It appears from the preceding investigations, that the expressions in series given by the equations in article 392, and those following, of the radius vector, of its projection on the fixed plane, of the longitude, and its projection on the fixed plane, and of the latitude in the invariable orbit will answer for the disturbed orbit, provided *nt* be changed into $\int ndt$, and all the elements of the variable orbit be determined by the preceding equations; for the finite equations between *r, v, s, x, y, z,* and $\int ndt$, are the same in both cases, and all the equations in the articles alluded to are determined independently of the constancy or variation of the elements, consequently these expressions will still answer when the elements are variable.

These investigations relate to orbits of any inclination and eccentricity; but the orbits of the planetary system are nearly circular, and very little inclined either to one another, or to the plane of the ecliptic.

Variations of the Elliptical Elements of the Orbits of the Planets

428. The equation

$$n = a^{-\frac{3}{2}}\sqrt{m}$$

shows that the mean motions and greater axes of the orbits of the planets are so connected, that one cannot vary independently of the other; and as

$$\frac{m}{a} = -2\int \mathrm{d}R\,,$$

it is clear that the differential of R is taken only with regard to *nt* the mean motion of *m*. If the mass of the sun be assumed as the unit, and the mass of the planet omitted in comparison of it, $m = 1$, and

$$da = 2a^2\mathrm{d}R\,;$$

2*a* being the major axis.

429. The inequalities in the eccentricity and longitude of the perihelion are obtained from

$$\tan v_{/} = \frac{f'}{f''},\quad me = \sqrt{f^2 + f'^2 + f''^2}$$

$v_{/}$ being the longitude of the perihelion of *m* when projected on the fixed plane of the ecliptic. If the orbit of the planet *m* at a given epoch be assumed to be the fixed plane containing the axes *x* and *y*, any inclination the orbit may have at a subsequent period being entirely owing to the action of the disturbing forces must be so small, that the true longitude of the perihelion will only differ from its projection on that new fixed plane, by quantities of the order of the squares of the disturbing masses respectively multiplied by the squares of the inclinations of the orbits, therefore without sensible error it may be assumed that $v_{/} = v$; v being the longitude of the perihelion estimated on the orbit; thus

$$\tan v = \frac{f'}{f''},$$

whence

$$\sin v = \frac{f'}{\sqrt{f'^2 + f''^2}};$$

and

$$\cos v = \frac{f''}{\sqrt{f'^2 + f''^2}}.$$

But by article 370 $f = -\frac{f'c' + f''c''}{c}$. Now c, c', c'' are the areas described by the radius vector of m on its orbit, when projected on the co-ordinate planes; but as the orbit nearly coincides with the fixed plane of the orbit at the epoch containing the axes x and y, the other two co-ordinate planes are nearly at right angles to it; hence c', and c'' are extremely small, and as f is of the same order in consequence of the preceding equation it may be omitted, so that

$$e = \sqrt{f''^2 + f'^2}$$

whence

$$f'' = e\cos v\,; \quad f' = e\sin v\,,$$

and

$$ede = f''df'' + f'df'; \quad e^2dv = f''df' - f'df'',$$

making $m = 1$.

430. Since f is very small df is still smaller, therefore the fourth of the equations (91) may be omitted as well as $c'dt = zdx - xdz$, and $c''dt = ydz - zdy$, on account of the smallness of c' and c''. Also z, the height of the planet above the fixed plane of its orbit, is so small that its square may be neglected; therefore quantities having the factors zdz, or $dz\left(\frac{dR}{dz}\right)$ may be omitted, which reduces the values of the fifth and sixth of equations (106) to

$$df'' = dy\left\{x\left(\frac{dR}{dy}\right) - y\left(\frac{dR}{dx}\right)\right\} + cdt\left(\frac{dR}{dy}\right),$$

[and]

$$df' = -dx\left\{x\left(\frac{dR}{dy}\right) - y\left(\frac{dR}{dx}\right)\right\} - cdt\left(\frac{dR}{dx}\right).$$

431. If $r_/ = Sp$, fig. 77, be the radius vector of *m* projected on the fixed plane of the orbit of *m* containing the axes *x* and *y*; and if the angle N*Sp* be represented by $v_/$, and *pm* the tangent of the latitude of *m* above the fixed plane of its orbit by *s*, then

$$x = r_/ \cos v_/;\; y = r_/ \sin v_/;\; z = r_/ s\,.$$

Since *x* is a function of $r_/$ and $v_/$,

$$\frac{dR}{dx} = \frac{dR}{dr_/}\cdot\frac{dr_/}{dx},$$
$$\frac{dR}{dx} = \frac{dR}{dv_/}\cdot\frac{dv_/}{dx}\,.$$

But

$$\frac{dr_/}{dx} = \frac{1}{\cos v_/};\quad \frac{dv_/}{dx} = -\frac{1}{r_/\sin v_/};$$

hence

$$\frac{dR}{dx} = \frac{dR}{dr_/}\cdot\frac{1}{\cos v_/};\quad \frac{dR}{dx} = -\frac{dR}{dv_/}\cdot\frac{1}{r_/\sin v_/}\,.$$

If the first equation be multiplied by $\cos^2 v_/$, and the second by $\sin^2 v_/$, their sum will be,[11]

$$\frac{dR}{dx} = \left(\frac{dR}{dr_/}\right)\cos v_/ - \left(\frac{dR}{dv_/}\right)\frac{\sin v_/}{r_/}\,.$$

In like manner it may be found that

$$\frac{dR}{dy} = \left(\frac{dR}{dr_/}\right)\sin v_/ + \left(\frac{dR}{dv_/}\right)\frac{\cos v_/}{r_/};$$

whence

$$x\left(\frac{dR}{dy}\right) - y\left(\frac{dR}{dx}\right) = \frac{dR}{dv_/};$$

consequently,[12]

$$df'' = +dy\left(\frac{dR}{dv_/}\right) + cdt\left\{\left(\frac{dR}{dr_/}\right)\sin v_/ + \left(\frac{dR}{dv_/}\right)\frac{\cos v_/}{r_/}\right\},$$
$$df' = -dx\left(\frac{dR}{dv_/}\right) - cdt\left\{\left(\frac{dR}{dr_/}\right)\cos v_/ - \left(\frac{dR}{dv_/}\right)\frac{\sin v_/}{r_/}\right\};$$

but

$$dx = d(r_/ \cos v_/);\ \ dy = d(r_/ \sin v_/),$$

and

$$cdt = xdy - ydx = r_/^2 dv_/;$$

so that[13]

$$df'' = +\{dr_/ \sin v_/ + 2r_/ dv_/ \cos v_/\}\left(\frac{dR}{dv_/}\right) + r_/^2 dv_/ \sin v_/ \left(\frac{dR}{dr_/}\right),$$

$$df' = -\{dr_/ \cos v_/ - 2r_/ dv_/ \sin v_/\}\left(\frac{dR}{dv_/}\right) - r_/^2 dv_/ \cos v_/ \left(\frac{dR}{dr_/}\right).$$

432. The values of $r_/$, $dr_/$, $dv_/ \left(\frac{dR}{dr_/}\right)$, $\left(\frac{dR}{dr_/}\right)$, are the same from whatever point the longitudes may be estimated; but by diminishing the angle $v_/$ by a right angle, $\sin v_/$ becomes $-\cos v_/$; and $\cos v_/$ becomes $\sin v_/$, so that the expression of df'' is changed into that of df', whence it follows, that if the value of df'' be developed into a series of sines and cosines of angles increasing proportionally with the time, and if each of the angles $\in$, $\in'$, v, v', q, q', be diminished by 90°, the value of df' will be obtained.

433. By articles 398 and 401, the projection of the longitude on the fixed plane of the ecliptic, and the curtate distance are,[14]

$$v_/ - q = v - x - \tan^2 \tfrac{1}{2} f \sin 2(v - x) + \text{\&c.}$$
$$r_/ = r\{1 - \tfrac{1}{2}s^2 + \text{\&c.}\}.$$

But when the orbit of m at the epoch is assumed to be the fixed plane, any inclination it may have at a subsequent period, arises entirely from the action of the disturbing forces, and is so very small that the squares of the tangent of that inclination may be neglected, whence,

$$v_/ - q = v - x,\ \ r_/ = r,\ \ v_/ = v, \text{ and } q = x.$$

In the invariable orbit,[15]

$$r = \frac{a(1-e^2)}{1 + e\cos(v - v)},\ \ dr = \frac{r^2 dv \cdot e\sin(v - v)}{a(1-e^2)},\ \ r^2 dv = a^2 . n . dt\sqrt{1-e^2}.$$

But these equations answer also for the variable orbit, since the two ellipses coincide during the first element of time, and when substitution is made for r, dr, and $r^2 dv$ in the last values of df'' and df', they become

$$df'' = \frac{a\,.\,ndt}{\sqrt{1-e^2}}\left\{2\cos v + \tfrac{3}{2}e\cos v \ + \tfrac{1}{2}\cos(2v - v\ )\right\}\left(\frac{dR}{dv}\right) + a^2\,.\,ndt\sqrt{1-e^2}\,\sin v\left(\frac{dR}{dr}\right),$$
$$df' = \frac{a\,.\,ndt}{\sqrt{1-e^2}}\left\{2\sin v + \tfrac{3}{2}e\sin v \ + \tfrac{1}{2}\sin(2v - v\ )\right\}\left(\frac{dR}{dv}\right) - a^2\cdot ndt\sqrt{1-e^2}\,\cos v\left(\frac{dR}{dr}\right).$$

But

$$f'' = e\cos v\ ,\quad f' = e\sin v$$

and by means of these equations the expressions

$$ede = f''df'' + f'df'$$

and

$$e^2 dv\ = f''df' - f'df''$$

in consequence of

$$\cos(2v - 2v\ ) = 2\cos^2(v - v\ ) - 1,$$

become[16]

$$\begin{aligned} de = \frac{a\cdot ndt}{\sqrt{1-e^2}}&\left\{2\cos(v - v\ ) + e + e\cos^2(v - v\ )\right\}\cdot\left(\frac{dR}{dv}\right) \\ &+ a^2\cdot ndt\sqrt{1-e^2}\,\sin(v - v\ )\left(\frac{dR}{dr}\right), \end{aligned} \qquad \textbf{(109)}$$

$$\begin{aligned} edv\ = -\frac{a\cdot ndt}{\sqrt{1-e^2}}&\sin(v - v\ )\left\{2 + e\cos(v - v\ )\right\}\cdot\left(\frac{dR}{dv}\right) \\ &- a^2 ndt\sqrt{1-e^2}\,\cos(v - v\ )\left(\frac{dR}{dr}\right). \end{aligned} \qquad \textbf{(110)}$$

The variation of the eccentricity however may be obtained under a more simple form from the equation $c = \sqrt{ma(1-e^2)}$ article 422, c' and c'' being zero, for

$$de = \frac{da\sqrt{a(1-e^2)}}{2a} - \frac{ede\sqrt{a}}{\sqrt{1-e^2}};$$

but

$$\frac{de}{dt} = x\left(\frac{dR}{dy}\right) - y\left(\frac{dR}{dx}\right) = \left(\frac{dR}{dv}\right);$$

hence by comparing the two values of *de*, and observing that $\frac{da}{2a^2} = \mathrm{d}R$,

$$ede = -a \cdot ndt\sqrt{1-e^2}\left(\frac{dR}{dv}\right) + a\left(1-e^2\right)\mathrm{d}R. \qquad \textbf{(111)}$$

434. The variation in the longitude of the epoch may be found by the preceding equations (109) and (110). For it was shown in article 392, that if the mean anomaly be estimated from any other point than the perihelion, $nt+\in-v$ may be put for nt, or rather $\int ndt+\in-v$; hence the equations in article 385 are

$$\int ndt+\in-v = u - e\sin u\,,$$

$$r = a\left(1-e\cos u\right),$$

$$\tan\tfrac{1}{2}\left(v-v\right) = \sqrt{\frac{1+e}{1-e}}\tan\tfrac{1}{2}u\,,$$

and

$$r = \frac{a\left(1-e^2\right)}{1+e\cos\left(v-v\right)}.$$

In the invariable orbit,

$$ndt = du\left(1-e\cos u\right),$$

in which u varies with the time. But if we suppose the time constant, and u to vary only in consequence of the variation of e and v, then in the troubled orbit,

$$d\in - dv = du\left(1-e\cos u\right) - de\sin u\,.$$

From the third of the preceding equations,

$$-\frac{dv}{\cos^2\frac{1}{2}\left(v-v\right)} = \frac{du}{\cos^2\frac{1}{2}u}\cdot\sqrt{\frac{1+e}{1-e}} + \frac{2de\tan\frac{1}{2}u}{\left(1-e\right)\sqrt{1-e^2}}$$

and substituting for $\cos^2\frac{1}{2}\left(v-v\right)$, its value from the same equation, the result is

$$du = -\frac{dv\left(1-e\cos u\right)}{\sqrt{1-e^2}} - \frac{de\sin u}{1-e^2};$$

hence

$$d\in - dv = -\frac{dv\left(1-e\cos u\right)^2}{\sqrt{1-e^2}} - \frac{de\sin u\left(2-e^2-e\cos u\right)}{1-e^2};$$

or

$$d\in - dv\ + dv\ \sqrt{1-e^2} = \frac{edv}{\sqrt{1-e^2}}\left\{2\cos u - e - e\cos^2 u\right\} - \frac{de}{1-e^2}\sin u\left(2-e^2-e\cos u\right).$$

Now[17]

$$r = \frac{a\left(1-e^2\right)}{1+e\cos\left(v-v\ \right)} = a\left(1-e\cos u\right),$$

whence

$$\cos u = \frac{e+\cos\left(v-v\ \right)}{1+e\cos\left(v-v\ \right)},\quad \sin u = \frac{\sqrt{1-e^2}\sin\left(v-v\ \right)}{1+e\cos\left(v-v\ \right)}.$$

And substituting these,

$$d\in - dv\ \left(1-\sqrt{1-e^2}\right) = \sqrt{1-e^2}\,\frac{\left\{2\cos\left(v-v\ \right)+e+e\cos^2\left(v-v\ \right)\right\}}{\left(1+e\cos\left(v-v\ \right)\right)^2}\cdot edv$$
$$-\sqrt{1-e^2}\,\frac{\left\{2+e\cos\left(v-v\ \right)\right\}}{\left\{1+e\cos\left(v-v\ \right)\right\}^2}\,de\sin\left(v-v\ \right).$$

If the values of edv and de, given by equations (109) and (110), be substituted, the result will be

$$d\in = dv\ \left(1-\sqrt{1-e^2}\right) - 2a\cdot ndt\cdot r\left(\frac{dR}{dr}\right);$$

but

$$r\left(\frac{dR}{dr}\right) = a\left(\frac{dR}{da}\right);$$

hence

$$d\in = dv\ \left(1-\sqrt{1-e^2}\right) - 2a^2\left(\frac{dR}{da}\right)\cdot ndt,$$

which is the variation in the epoch.

435. The variations in the inclination of the orbits, and in the longitude of their nodes, are obtained from[18]

$$\tan f = \frac{\sqrt{c'^2+c''^2}}{c},\quad \tan q = \frac{-c''}{c'},$$
$$\text{for}\ \tan f\cos q = -\frac{c'}{c};\quad \tan f\sin q = \frac{c''}{c};$$

whence[19]

$$d\,.\tan f = \frac{1}{c}\left\{dc''\sin q - dc'\cos q - dc\tan f\right\},$$

$$dq\ .\tan f = \frac{1}{c}\left\{dc''\cos q + dc'\sin q\right\}.$$

If substitution be made for[20] $\frac{dc}{dt}$, $\frac{dc'}{dt}$, $\frac{dc''}{dt}$, of their values in article 422, and making

$$x = r\cos v,\quad y = r\sin v,$$
$$s = \tan f\sin(v-q),$$

there will result[21]

$$\begin{aligned} d\ .\tan f &= -\frac{dt\tan f\cos(v-q)}{c}\left\{r\left(\frac{dR}{dr}\right)\sin(v-q)+\left(\frac{dR}{dv}\right)\cos(v-q)\right\} \\ &\quad+\frac{(1+s^2)\,dt}{c}\cos(v-q)\left(\frac{dR}{ds}\right) \\ dq\ .\tan f &= -\frac{dt\tan f\sin(v-q)}{c}\left\{r\left(\frac{dR}{dr}\right)\sin(v-q)+\left(\frac{dR}{dv}\right)\cos(v-q)\right\} \\ &\quad+\frac{(1+s^2)\,dt}{c}\sin(v-q)\left(\frac{dR}{ds}\right). \end{aligned} \qquad \textbf{(112)}$$

These two equations determine the inclination of the orbit, and motion of the nodes. They give[22]

$$\sin(v-q)\cdot d\tan f - dq\cdot\cos(v-q)\tan f = 0,$$

which may also be obtained from

$$s = \tan f\sin(v-q).$$

436. If the orbit of m has so small an inclination on the fixed plane, that the squares of s and $\tan f$ may be omitted, then

$$d\cdot\tan f = \frac{dt}{c}\cos(v-q)\left(\frac{dR}{ds}\right),$$
$$dq\cdot\tan f = \frac{dt}{c}\sin(v-q)\left(\frac{dR}{ds}\right);$$

if to abridge

$$p = \tan f\sin q,\quad q = \tan f\cos q,$$

and as

$$c = \sqrt{a(1-e^2)};\quad a = \frac{1}{a^2n^2};$$

[then]

$$\frac{1}{c}=\frac{an}{\sqrt{1-e^2}}\,;$$

these become[23]

$$dp=\frac{andt}{\sqrt{1-e^2}}\sin v\left(\frac{dR}{ds}\right),$$
$$dq=\frac{andt}{\sqrt{1-e^2}}\cos v\left(\frac{dR}{ds}\right).$$

But

$$z=+qy-px\,;$$

and as the orbit is supposed to have a very small inclination on the fixed plane, $r\cos v$, $r\sin v$, and rs, may be put for x, y, and z, the last equation becomes

$$s=q\sin v-p\cos v\,,$$

whence

$$\frac{dR}{ds}=\frac{1}{\sin v}\left(\frac{dR}{dq}\right);\quad \frac{dR}{ds}=-\frac{1}{\cos v}\left(\frac{dR}{dp}\right);$$

consequently

$$dq=-\frac{andt}{\sqrt{1-e^2}}\left(\frac{dR}{dp}\right)$$
$$dp=\frac{andt}{\sqrt{1-e^2}}\left(\frac{dR}{dq}\right).$$

437. But when the inclination of the orbit is very small,

$$\frac{z}{a}=q\sin(nt+\in)-p\cos(nt+\in)$$

whence[24]

$$dp=-\frac{a^2ndt}{\sqrt{1-e^2}}\sin(nt+\in)\left(\frac{dR}{dz}\right),$$
$$dq=\frac{a^2ndt}{\sqrt{1-e^2}}\cos(nt+\in)\left(\frac{dR}{dz}\right);$$

for[25]

$$\frac{dR}{dp}=-\left(\frac{dR}{dz}\right)\cos(nt+\in)\,,$$
$$\frac{dR}{dq}=\left(\frac{dR}{dz}\right)\sin(nt+\in)\,;$$

and

$$x=a\cos(nt+\in),\quad y=a\sin(nt+\in)\,.$$

438. Since the elliptical and troubled orbits coincide during the first element of the time, the equations of motion are identical for that period, therefore the variation of the elements must be zero; consequently,

$$0=\left(\frac{dR}{da}\right)da+\left(\frac{dR}{de}\right)de+\left(\frac{dR}{dv}\right)dv+\left(\frac{dR}{d\in}\right)d\in+\left(\frac{dR}{dp}\right)dp+\left(\frac{dR}{dq}\right)dq \qquad \textbf{(113)}$$

Because nt is always accompanied by $-v$, therefore

$$\frac{dR}{dv}=\frac{dR}{ndt}+\frac{dR}{dv},$$

so that the differential *de* becomes

$$de=-andt\frac{\sqrt{1-e^2}}{e}\cdot\left(\frac{dR}{dv}\right)-a\frac{\sqrt{1-e^2}}{e}\left(1-\sqrt{1-e^2}\right)\mathrm{d}R.$$

If this value of de, and the preceding values of da, $d\in$, dp, dq, be substituted in equation (113), observing that $\frac{dR}{ndt}$ may be put for $\frac{dR}{d\in}$ and $\frac{dR}{dv}$, it will be reduced to

$$dv=\frac{andt\sqrt{1-e^2}}{e}\left(\frac{dR}{de}\right);$$

whence

$$d\in=\frac{andt\sqrt{1-e^2}}{e}\left(1-\sqrt{1-e^2}\right)\cdot\left(\frac{dR}{de}\right)-2a^2\left(\frac{dR}{da}\right)ndt.$$

By article 424,

$$dz=-3\int andt\,.\,\mathrm{d}R;$$

the integral of which is the periodic inequality in the mean motion.

439. The differential equations of the periodic variations of the elements of the orbit of *m* are therefore

$$da=2a^2\mathrm{d}R;$$
$$dz=-3\int andt\,.\,\mathrm{d}R;$$
$$d\in=\frac{andt\sqrt{1-e^2}}{e}\left(1-\sqrt{1-e^2}\right)\left(\frac{dR}{de}\right)-2a^2\left(\frac{dR}{da}\right)ndt;$$

$$de=-\frac{a\sqrt{1-e^2}}{e}\left(1-\sqrt{1-e^2}\right)\mathrm{d}R-\frac{andt\sqrt{1-e^2}}{e}\left(\frac{dR}{dv}\right); \qquad \textbf{(114)}$$

$$dv=\frac{andt\sqrt{1-e^2}}{e}\left(\frac{dR}{de}\right);$$

$$dp=\frac{andt}{\sqrt{1-e^2}}\left(\frac{dR}{dq}\right);$$

$$dq=-\frac{andt}{\sqrt{1-e^2}}\left(\frac{dR}{dp}\right).$$

Because $\in$ always accompanies nt,

$$\frac{dR}{d\in}=\frac{\mathrm{d}R}{ndt};\text{ whence } ndt\left(\frac{dR}{d\in}\right)=\mathrm{d}R;$$

so that da may also be expressed by

$$da=2a^2ndt\left(\frac{dR}{d\in}\right).$$

440. By article 347, R is a given function of x, y, z, x', y', z', &c., the co-ordinates m, m', m'', &c. and is of the first order with regard to the masses; and if the squares and products of the masses be omitted, the elliptical values of x, y, z, x', y', z', &c. may be substituted, and then R will be a function of the time, and of the elements of the orbits, and may therefore be developed in a series of sines and cosines containing the time. But the first part of this series is independent of the time, being a function of the elements of the orbits alone, as will be shown immediately, and may be represented by F.

441. As F does not contain the arc nt, its differential with regard to that quantity, is zero, consequently when F is put for R in the preceding equations they become[26]

$$da=0;\ dz=0;$$

$$d\in=\frac{andt\sqrt{1-e^2}}{e}\left(1-\sqrt{1-e^2}\right)\left(\frac{dF}{de}\right)-2a^2ndt\left(\frac{dF}{da}\right);$$

$$de=-\frac{andt\sqrt{1-e^2}}{e}\left(\frac{dF}{dv}\right);$$

$$dv=\frac{andt\sqrt{1-e^2}}{e}\left(\frac{dF}{de}\right); \qquad \textbf{(115)}$$

$$dp=\frac{andt}{\sqrt{1-e^2}}\left(\frac{dF}{dq}\right);$$

$$dq = -\frac{andt}{\sqrt{1-e^2}}\left(\frac{dF}{dp}\right).$$

The integrals of these equations are the secular variations the elements of the orbit of m.

442. In the determination of the periodic variations of the elements, all terms of the series *R*, that do not contain the time, must be omitted; and in the secular variations, all terms of that series that do contain the time must be rejected. Thus the periodic variations in the elements of the planetary orbits depend on the configuration, or relative position of the bodies, and their secular variations do not.

443. These periodic and secular variations, in the elements of elliptical motion, are sufficient for the determination of all the inequalities to which the bodies of the solar system are liable in their revolutions round the sun. On the same principle, the periodic and secular variations in the rotation of the earth and planets may be found from the variation of the six arbitrary constant quantities introduced by the integration of the equations of rotatory motion. The expressions of these variations are identical in the motions of translation and rotation; and as the perturbations in these two motions arise from the same cause, they are expressed by the same formulae. The analysis by which Lagrange[27] has united the two great problems of the solar system is the most refined and elegant in the science of astronomy.

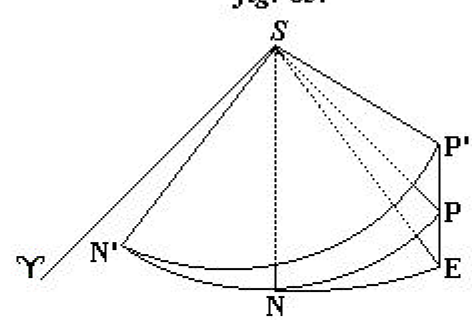
444. Observation shows the inclinations of the orbits of the planets on the plane of the ecliptic to be very small; hence if EN, Fig. 83, be the fixed plane of the ecliptic at a given epoch, PN the orbit of m, $\mathrm{P'N'}$ the orbit of m'[28]

$$\mathrm{ENP} = f, \quad \mathrm{EN'P'} = f',$$

the inclination of these orbits on the plane of the ecliptic; and

$$♈ S\mathrm{N} = q, \quad ♈ S\mathrm{N'} = q',$$

fig. 83.



the longitudes of their ascending nodes on the same plane, then if the planet m were moving on the orbit PN, the tangent of its latitude would be

$$z = \mathrm{EP} = \tan f \sin(nt + \epsilon - q).$$

And if it were moving on the orbit $\mathrm{P'N'}$, the tangent of its latitude would be

$$z' = \mathrm{EP'} = \tan f' \sin(nt + \epsilon - q').$$

Hence if g be the tangent of the inclination of the orbit P′N′ on the orbit PN, and Π the longitude of the line of common intersection of these two planes, or of the ascending node of the orbit of m' on that of m, then

$$\tan f' \sin(nt + \in - q') - \tan f \sin(nt + \in - q) = g \sin(nt + \in - \Pi) = z' - z = \text{PP}' \text{nearly}.$$

If then as before[29]

$$p = \tan f \sin q, \quad q = \tan f \cos q, \qquad \textbf{(116)}$$
$$p' = \tan f' \sin q', \quad q' = \tan f' \cos q';$$

there will be found[30 31]

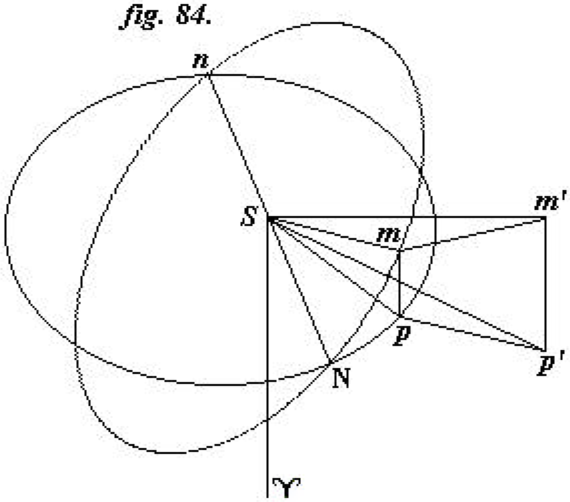
$$l \sin \Pi = p' - p, \quad g \cos \Pi = q' - q, \qquad \textbf{(117)}$$
$$g^2 = (p' - p)^2 + (q' - q)^2.$$

Now if EN be the primitive orbit of m at the epoch, and PN its orbit at any other period, $z = 0$, $f = 0$, and $g = \tan f'$; and it is evident that g, the tangent of the mutual inclination of these two planes, will be of the order of the disturbing forces; and therefore very small, since any inclination the orbit may acquire subsequently to the epoch is owing to the disturbing forces.

445. It is now requisite to develop R into a series of the sines and cosines of the mean angular distances of the bodies.

If the disturbing action of only one body be estimated at a time[32]

fig. 84.



$$R = \frac{m'}{\sqrt{(x'-x)^2 + (y'-y)^2 + (z'-z)^2}} - \frac{m'(x'x + y'y + z'z)}{(x'^2 + y'^2 + z'^2)^{\frac{3}{2}}},$$

in which

$$r = Sm = \sqrt{x^2 + y^2 + z^2}; \; r' = Sm' = \sqrt{x'^2 + y'^2 + z'^2},$$

$$mm' = \sqrt{(x'-x)^2 + (y'-y)^2 + (z'-z)^2}.$$

The orbits of the planets are nearly circular, and their greatest inclination on the plane of the ecliptic does not exceed 7°, R developed according to the powers and products of these quantities must necessarily be very convergent; but as R is independent of the position of the co-ordinate planes, the plane of projection N*pn*, fig. 84, may be so chosen as to make the inclination still less, consequently z and z' will be very small.

Let $v_{/} = \Upsilon Sp$, $v'_{/} = \Upsilon S\,p'$, be the projected longitudes of m and m' on the fixed plane, and let

$$r_{/} = Sp \quad r'_{/} = Sp'$$

be their curtate distances; then[33]

$$x = r_{/}\cdot\cos v_{/};\quad y = r_{/}\cdot\sin v_{/};$$
$$x' = r'_{/}.\cos v'_{/};\quad y' = r'_{/}.\sin v'_{/};$$

hence

$$R = \frac{m'}{\sqrt{r'^2_{/} - 2r_{/}r'_{/}.\cos(v'_{/} - v_{/}) + r_{/}^2 + (z'-z)^2}} - \frac{m'\left(r_{/}r'_{/}.\cos(v'_{/} - v_{/}) + z'z\right)}{\left(r'^2_{/} + z'^2\right)^{\frac{3}{2}}}$$

or z and z' being extremely small,[34]

$$R = \frac{m'}{\sqrt{r'^2_{/} - 2r_{/}r'_{/}.\cos(v'_{/} - v_{/}) + r_{/}^2}} - \frac{r_{/}\cdot\cos(v'_{/} - v_{/})\cdot m'}{r'^2_{/}} - \frac{m'.zz'}{r'^3_{/}}$$
$$+\frac{3m'r_{/}\cdot z'^2\cos(v'_{/} - v_{/})}{2r'^4_{/}} - \frac{m'(z'-z)^2}{2\left\{r'^2_{/} - 2r'r'_{/}\cos(v'_{/} - v_{/}) + r_{/}^2\right\}^{\frac{3}{2}}} + \&c$$

Because the eccentricities and inclinations of the orbits of the planets and satellites are very small, it appears from the values of the radius vector and true longitude in the elliptical orbit developed in article 398, and those following, that

$$r_{/} = a(1+u);\quad r'_{/} = a'(1+u');$$
$$v_{/} = nt + \in + v;\quad v'_{/} = n't + \in' + v';$$

u, u', v, v', being very small quantities depending on the eccentricities and inclinations, and a, a' the mean distances of m and m', or half the greater axes of their orbits.

If these quantities be substituted in R, and if to abridge

$$n't - nt + \in' - \in = b\ ,$$

observing also that,

$$\cos(b + v' - v) = \cos b\cdot\cos(v'-v) - \sin b\cdot\sin(v'-v) = \cos b - (v'-v)\sin b\ ,$$

because $v'-v$ is so small that it may be taken for its sine and unity for its cosine, thus

$$R = -m' \cdot \frac{a}{a'^2} \cdot \frac{1+u}{(1+u')^2} \cdot \cos b + m' \cdot \frac{a}{a'^2} \cdot \frac{1+u}{(1+u')^2} \cdot \sin b$$

$$+ \frac{m'}{\left\{a^2(1+u)^2 - 2aa'(1+u)(1+u').\cos b + a'^2(1+u')^2\right\}^{\frac{1}{2}}}$$

$$- \frac{m' \cdot zz'}{a'^3} + \frac{3m' \cdot az'^2}{2a'^4} \cdot \cos b$$

$$- \frac{m'(z-z')^2 - 3m' \cdot az'^2 (v'-v) \cdot \sin b}{2\left\{a^2(1+u)^2 - 2aa'(1+u)(1+u').\cos b + a'^2(1+u')^2\right\}^{\frac{3}{2}}} + \&c.$$

446. The expansion of this function into a series ascending, according to the powers and products of the very small quantities u, u', v, v', z, and z' is easily accomplished by the theorem for the development of a function of any number of variables, for if R' be the value of R when these small quantities are zero, that is, supposing the orbits to be circular and all in one plane, then

$$R = R' + au \cdot \frac{dR'}{da} + a'u' \cdot \frac{dR'}{da'} + (v'-v) \cdot \frac{dR'}{ndt} + \frac{a^2u^2}{2} \cdot \frac{d^2R'}{da^2} + \frac{a'^2u'^2}{2} \cdot \frac{d^2R'}{da'^2} + \&c.$$

because a is the only quantity that varies with u, a' with u', and t with $(v'-v)$. But

$$R' = m'\left\{(a^2 - 2aa'\cos b + a'^2)^{-\frac{1}{2}} - \frac{a}{a'^2}\cos b\right\};$$

and if [35] $(a'^2 - 2aa'\cos b + a^2)^{-\frac{1}{2}}$ be developed according to the cosines of the multiples of the arc b, it will have the form[36]

$$(a'^2 - 2aa'\cos b + a^2)^{-\frac{1}{2}} = \tfrac{1}{2}A_0 + A_1.\cos b + A_2.\cos 2b + A_3.\cos 3b + \&c.$$

in which A_0, A_1, &c., are functions of a and a' alone; in fact if to abridge $\frac{a}{a'} = \alpha$, the binomial theorem gives

$$A_0 = \frac{2}{a'}\left\{1 + \left(\frac{1}{2}\right)^2 \cdot \alpha^2 + \left(\frac{1.3}{2.4}\right)^2 \cdot \alpha^4 + \left(\frac{1.3.5}{2.4.6}\right) \cdot \alpha^6 + \&c.\right\},$$

the other coefficients are similar functions of the powers of α; but a general method of finding these coefficients in more convergent series will be given afterwards. Thus,

$$R' = m'\left\{\tfrac{1}{2}A_0 + \left(A_1 - \frac{a}{a'^2}\right)\cdot\cos b + A_2 . \cos 2b + \&c.\right\}$$

and if i represent every whole number either positive or negative including zero, the general term of this series is

$$R' = \frac{m'}{2}\cdot\Sigma . A_i . \cos ib \ ,$$

provided that when[37] $i=1$, $\left(A_1 - \frac{a}{a'^2}\right)$ be put for A_1.

Again, if

$$\left(a'^2 - 2aa'\cos b + a^2\right)^{-\frac{3}{2}} = \tfrac{1}{2}B_0 + B_1 . \cos b + B_2 . \cos 2b + B_3 . \cos 3b + \&c.$$

its general term is

$$\frac{m'}{2}\cdot\Sigma . B_i . \cos ib \ ;$$

and as

$$\frac{dR'}{da} = \frac{m'}{2}\cdot\Sigma\cdot\left(\frac{dA_i}{da}\right)\cdot\cos ib; \quad \frac{dR'}{da'} = \frac{m'}{2}\cdot\Sigma\cdot\left(\frac{dA_i}{da'}\right)\cdot\cos ib;$$

$$\frac{dR'}{ndt} = -\frac{m'}{2}\cdot\Sigma\cdot iA_i\cdot\sin ib; \quad \frac{d^2R'}{da^2} = -\frac{m'}{2}\cdot\Sigma\cdot\left(\frac{d^2A_i}{da^2}\right)\cdot\cos ib;$$

&c. &c.

The development of R is

$$\begin{aligned} R = {} & \frac{m'}{2}\cdot\Sigma . A_i . \cos i\left(n't - nt + \epsilon' - \epsilon\right) \\ & + \frac{m'}{2}\cdot u\cdot\Sigma\cdot a\cdot\left(\frac{dA_i}{da}\right)\cdot\cos i\left(n't - nt + \epsilon' - \epsilon\right) \\ & + \frac{m'}{2}\cdot u'\cdot\Sigma\cdot a'\cdot\left(\frac{dA_i}{da'}\right)\cdot\cos i\left(n't - nt + \epsilon' - \epsilon\right) \\ & - \frac{m'}{2}\cdot\left(v' - v\right)\cdot\Sigma\cdot i\cdot A_i\cdot\sin i\left(n't - nt + \epsilon' - \epsilon\right) \\ & + \frac{m'}{4}\cdot u^2\cdot\Sigma\cdot a^2\left(\frac{d^2A_i}{da^2}\right)\cdot\cos i\left(n't - nt + \epsilon' - \epsilon\right) \end{aligned}$$

$$+\frac{m'}{2}\cdot uu'\cdot\Sigma\cdot aa'\left(\frac{d^2A_i}{da\cdot da'}\right)\cdot\cos i\left(n't-nt+\in'-\in\right)$$
$$+\frac{m'}{4}\cdot u'^2\cdot\Sigma\cdot a'^2\left(\frac{d^2A_i}{da'^2}\right)\cdot\cos i\left(n't-nt+\in'-\in\right)$$
$$-\frac{m'}{2}\cdot u\cdot\left(v'-v\right)\cdot\Sigma\cdot ia\left(\frac{dA_i}{da}\right)\cdot\sin i\left(n't-nt+\in'-\in\right)$$
$$-\frac{m'}{2}\cdot u'\cdot\left(v'-v\right)\cdot\Sigma\cdot ia'\left(\frac{dA_i}{da'}\right)\cdot\sin i\left(n't-nt+\in'-\in\right)$$
$$-\frac{m'}{4}\cdot\left(v'-v\right)^2\cdot\Sigma\cdot i^2A_i\cdot\cos i\left(n't-nt+\in'-\in\right)$$
$$-\frac{m'\cdot zz'}{a'^3}+\frac{3m'\cdot a\cdot z'^2}{2\cdot a'^4}\cdot\cos i\left(n't-nt+\in'-\in\right)$$
$$-\frac{m'\left(z'-z\right)^2}{4}\cdot\Sigma\cdot B_i\cdot\cos i\left(n't-nt+\in'-\in\right)$$
$$+\frac{3m'\cdot a\cdot z'^2}{4}\cdot\left(v'-v\right)\cdot\Sigma\cdot B_i\cdot\cos i\left(n't-nt+\in'-\in\right)$$
$$+\&\text{c.}\quad\&\text{c.}$$

a series that may be extended indefinitely.

447. If $v_/$ be the projection of v, by articles 398 and 401, $v_/$ and the curtate distance are[38]

$$r_/=r\left(1-\tfrac{1}{2}s^2+\tfrac{3}{8}s^4-\&\text{c.}\right),$$
$$v_/=v-\tan^2\tfrac{1}{2}f\left\{\sin 2v+\tfrac{1}{2}\tan^2 f\,.\sin 4v+\&\text{c.}\right\}$$

or, if the values of r and v, in article 392, be substituted,

$$r_/=a\left(1+\tfrac{1}{2}e^2-e\cos\left(nt+\in-\varpi\right)+\&\text{c.}\right)\cdot\left(1-\tfrac{1}{2}s^2+\&\text{c.}\right),$$
$$v_/=nt+e+2e\sin\left(nt+\in-\varpi\right)+\&\text{c.}-\tan^2\tfrac{1}{2}f\left\{\sin 2v+\tfrac{1}{2}\tan^2 f\sin 4v+\&\text{c.}\right\}.$$

Where a is half the greater axis of the orbit of m, e the eccentricity, ϖ the longitude of the perihelion, θ the longitude of the ascending node, f the inclination of the orbit of m on the fixed ecliptic at the epoch, and $nt+\in$ the mean longitude of m.

But

$$r_/=a\left(1+u\right),\quad v_/=nt+\in+v;$$

hence

$$u=-e\cos\left(nt+\in-\varpi\right)+\tfrac{1}{2}e^2\left(1-\cos 2\left(nt+\in-\varpi\right)\right)-\tfrac{1}{2}\tan^2 f\,.\sin^2\left(nt+\in\right),$$
$$v=2e\,.\sin\left(nt+\in-\varpi\right)+\tfrac{5}{4}e^2\cdot\sin 2\left(nt+\in-\varpi\right)-\tan^2\tfrac{1}{2}f\cdot\sin 2\left(nt+\in\right)$$

when the approximation only extends to the squares and products of the eccentricities and inclinations.

In the same manner,

$$u' = -e'\cos\left(n't+\in'-v'\right)+\tfrac{1}{2}e'^2\left(1-\cos 2\left(n't+\in'-v'\right)\right)-\tfrac{1}{2}\tan^2 f'\cdot\sin^2\left(n't+\in'\right),$$
$$v' = 2e'.\sin\left(n't+\in'-v'\right)+\tfrac{5}{4}e'^2\cdot\sin 2\left(n't+\in'-v'\right)-\tan^2\tfrac{1}{2}f'\cdot\sin 2\left(n't+\in'\right).$$

448. The substitution of these quantities will give the value of R in series, if the products of the sines and cosines be replaced by the cosines of the sums and differences of the arcs, observing that cosines of the forms

$$\cos\left\{i\left(n't-nt+\in'-\in\right)+n't-nt+\in'-\in-v+v'\right\},$$
$$\cos\left\{i\left(nn't-nt+\in'-\in\right)+n't-nt+\in'=\in-v+v'\right\}$$
$$\cos\left\{i\left(n't-nt+\in'-\in\right)+n't+nt+\in'-\in-2v\right\}$$

become

$$\cos\left\{i\left(n't-nt+\in'-\in\right)-v+v'\right\},$$
$$\cos\left\{i\left(n't-nt+\in'-\in\right)+2nt+2e-2v\right\}$$

by the substitution of $i-1$ for *i*, and cosines of the form

$$\cos\left\{i\left(n't-nt+\in'-\in\right)-n't+nt-\in'+\in+v'-v\right\}$$

become

$$\cos\left\{i\left(n't-nt+\in'-\in\right)+v'-v\right\},$$

by the substitution of $i+1$ for *i*.

449. Attending to these circumstances, it will be found that

$$\begin{aligned}
R = {} & \frac{m'}{2}\cdot\sum A_i\cdot\cos i\left(n't-nt+\in'-\in\right) \\
& +\frac{m'}{2}\cdot M_0\cdot e\cdot\cos\left\{i\left(n't-nt+\in'-\in\right)+nt+\in-v\right\} \\
& +\frac{m'}{2}\cdot M_1\cdot e'\cdot\cos\left\{i\left(n't-nt+\in'-\in\right)+nt+\in-v'\right\} \\
& +\frac{m'}{2}\cdot N_0\cdot e^2\cdot\cos\left\{i\left(n't-nt+\in'-\in\right)+2nt+2\in-2v'\right\} \\
& +\frac{m'}{2}\cdot N_1\cdot ee'\cdot\cos\left\{i\left(n't-nt+\in'-\in\right)+2nt+2\in-v-v'\right\} \\
& +\frac{m'}{2}\cdot N_2\cdot e'^2\cdot\cos\left\{i\left(n't-nt+\in'-\in\right)+2nt+2\in-2v'\right\}
\end{aligned} \qquad \textbf{(118)}$$

$$+\frac{m'}{2}\cdot N_3\cdot\left(e^2+e'^2\right)\cdot\cos i\left(n't-nt+\epsilon'-\epsilon\right)$$
$$+\frac{m'}{2}\cdot N_4\cdot ee'\cdot\cos\left\{i\left(n't-nt+\epsilon'-\epsilon\right)+\nu-\nu'\right\}$$
$$+\frac{m'}{2}\cdot N_5\cdot ee'\cdot\cos\left\{i\left(n't-nt+\epsilon'-\epsilon\right)-\nu+\nu'\right\}$$
$$-\frac{m'\cdot zz'}{a'^3}+\frac{3m'\cdot a\cdot z'^2}{2a'^4}\cdot\cos\left(n't-nt+\epsilon'-\epsilon\right)$$
$$-\frac{m'\left(z-z'\right)^2}{4}\cdot\sum B_i\cdot\cos\left(n't-nt+\epsilon'-\epsilon\right)$$
$$+\frac{m'}{4}\cdot Q_0\cdot e'^3\cdot\cos\left\{i\left(n't-nt+\epsilon'-\epsilon\right)+3nt+3\epsilon-3\nu'\right\}$$
$$+\frac{m'}{4}\cdot Q_1\cdot e'^2\cdot e\cdot\cos\left\{i\left(n't-nt+\epsilon'-\epsilon\right)+3nt+3\epsilon-2\nu'-\nu\right\}$$
$$+\frac{m'}{4}\cdot Q_2\cdot e'\cdot e^2\cdot\cos\left\{i\left(n't-nt+\epsilon'-\epsilon\right)+3nt+3\epsilon-\nu'-2\nu\right\}$$
$$+\frac{m'}{4}\cdot Q_3\cdot e^3\cdot\cos\left\{i\left(n't-nt+\epsilon'-\epsilon\right)+3nt+3\epsilon-3\nu'\right\}$$
$$+\frac{m'}{4}\cdot z'^2\cdot e'\cdot\sum\cdot B_i\cdot\cos\left\{i\left(n't-nt+\epsilon'-\epsilon\right)+n't+\epsilon'-\nu'\right\}$$
$$+\frac{m'}{4}\cdot z'^2\cdot e\cdot\sum\cdot B_i\cdot\cos\left\{i\left(n't-nt+\epsilon'-\epsilon\right)+nt+\epsilon-\nu\right\}$$
$$+\&\text{c.}\quad\&\text{c.}$$

The coefficients being

$$M_0=-\left\{a\left(\frac{dA_i}{da}\right)+2iA_i\right\};$$
$$M_1=-a'\left(\frac{dA_{i-1}}{da'}\right)+2\left(i-1\right)A_{(i-1)};$$
$$N_0=\tfrac{1}{4}\left\{i\left(4i-5\right)\right\}A_i+2\left(2i-1\right)a\left(\frac{dA_i}{da}\right)+a^2\left(\frac{d^2A_i}{da^2}\right);$$
$$N_1=-\tfrac{1}{2}\left\{4\left(i-1\right)^2A_{(i-1)}+2\left(i-1\right)a\left(\frac{dA_{(i-1)}}{da}\right)-2\left(i-1\right)a'\left(\frac{dA_{(i-1)}}{da'}\right)-aa'\left(\frac{d^2A_{(i-1)}}{da\cdot da'}\right)\right\};$$
$$N_2=\tfrac{1}{4}\left\{\left(i-2\right)\left(4i-3\right)A_{(i-2)}-2\left(2i-3\right)a'\left(\frac{dA_{(i-2)}}{da'}\right)+a'^2\left(\frac{d^2A_{(i-2)}}{da'^2}\right)\right\};$$
$$N_3=-\tfrac{1}{2}\left\{4i^2A_i-2a\left(\frac{dA_i}{da}\right)-a^2\left(\frac{d^2A_i}{da^2}\right)\right\};$$

$$N_4 = \tfrac{1}{2}\left\{4(i-1)^2 A_{(i-1)} - 2(i-1)a\left(\frac{dA_{(i-1)}}{da}\right) - 2(i-1)a'\left(\frac{dA_{(i-1)}}{da'}\right) + aa'\left(\frac{d^2A_{(i-1)}}{da\cdot da'}\right)\right\};$$

$$N_5 = \tfrac{1}{2}\left\{4(i+1)^2 A_{(i+1)} + 2(i+1)a\left(\frac{dA_{(i+1)}}{da}\right) + 2(i+1)a'\left(\frac{dA_{(i+1)}}{da'}\right) + aa'\left(\frac{d^2A_{(i+1)}}{da\cdot da'}\right)\right\};$$

&c. &c.

450. But $z = r_/ s = r_/ \tan f \sin(nt + \epsilon - q)$, by article 435, or substituting the values of $r_/$ and $v_/$ in article 447, and rejecting the product $e\tan f$, it becomes

$$z = a\cdot\tan f \sin(nt+\epsilon - q);$$

also

$$z' = a'\cdot\tan f' \sin(n't+\epsilon' - q'),$$

f and f' being the inclinations of the orbits of m and m' on the ecliptic. These values of z and z' are referred to the ecliptic at the epoch; but if the orbit of m at the epoch be assumed to be the fixed plane, $f = 0$, $\tan f' = g$, the mutual inclination of the orbits of m and m', then $\prod$ being the longitude of the ascending node of the orbit of m' on that of m,

$$z = 0,\; z' = a'g\sin(n't+\epsilon' - \prod),$$

consequently the terms of R depending on z' with regard to g^2, eg^2, and $e'g^2$, become[39]

$$+\frac{m'}{2}\cdot N_6\cdot g^2\cdot\cos\{i(n't - nt + \epsilon' - \epsilon) + 2nt + 2\epsilon - 2\prod\},$$
$$+\frac{m'}{2}\cdot N_7\cdot g^2\cdot\cos\{i(n't - nt + \epsilon' - \epsilon)\},$$
$$+\frac{m'}{4}\cdot Q_4\cdot g^2 e'\cdot\cos\{i(n't - nt + \epsilon' - \epsilon) + 3nt + 3\epsilon - v' - 2\prod\},$$
$$+\frac{m'}{4}\cdot Q_5\cdot g^2 e\cdot\cos\{i(n't - nt + \epsilon' - \epsilon) + 3nt + 3\epsilon - v - 2\prod\}.$$

451. It appears from this series that the sum of the terms independent of the eccentricities and inclinations of the orbits, is

$$\frac{m'}{2}\cdot\sum A_i \cos i(n't - nt + \epsilon' - \epsilon),$$

which is the same as if the orbits were circular and in one plane.

The sum of the terms depending on the first powers of the eccentricities has the form

$$\frac{m'}{2}\sum M\cos\left\{i\left(n't-nt+\in'-\in\right)+nt+\in+K\right\}.$$

Those depending on the squares and products of the eccentricities and inclinations may be expressed by

$$\frac{m'}{2}\sum N\cdot\cos\left\{i\left(n't-nt+\in'-\in\right)+2nt+2\in+L\right\}$$
$$+\frac{m'}{2}\sum N'\cdot\cos\left\{i\left(n't-nt+\in'-\in\right)+L'\right\}.$$

Those depending on the third powers and products of these elements are

$$\frac{m'}{4}\sum Q\cdot\cos\left\{i\left(n't-nt+\in'-\in\right)+3nt+3\in+U\right\}$$
$$+\frac{m'}{4}\sum\cdot Q'\cdot\cos\left\{i\left(n't-nt+\in'-\in\right)+nt+\in+U'\right\},$$
&c. &c.

It may be observed that the coefficient of the sine or cosine of the angle v has always the eccentricity e for factor; the coefficient of the sine or cosine of $2v$ has e^2 for factor; the sine or cosine of $3v$ has e^3, and so on: also the coefficient of the sine or cosine of q has $\tan\cdot f$ for factor; the sine or cosine of $2q$ has $\tan^2.f$ for factor, &c. &c.

Determination of the Coefficients of the Series R

452. In order to complete the development of R, the coefficients A_i and B_i, and their differences, must be determined. Let

$$\left(a'^2-2aa'\cos b+a^2\right)^{-3}=A^{-3}=\tfrac{1}{2}A_0+A_1\cos b+A_2.\cos 2b+\&c.$$

The differential of which is

$$A^{-3-1}2saa'\sin b=A_1\sin b+2A_2\sin 2b+3A_3\sin 3b+\&\text{c.}$$

multiplying both sides of this equation by A, and substituting for A^{-3}, it becomes

$$2saa'\sin b\left\{\tfrac{1}{2}A_0+A_1\cos b+A_2\cos 2b+\&\text{c.}\right\}$$
$$=\left(a'^2-2aa'\cos b+a^2\right)\left\{A_1\sin b+2A_2\sin 2b+\&\text{c.}\right\}.$$

If it be observed that

$$\cos b \sin b = \tfrac{1}{2}\cos 2b\text{, \&c.}$$

when the multiplication is accomplished, and the sines and cosines of the multiple arcs put for the products of the sines and cosines, the comparison of the coefficients of like cosines gives

$$A_2 = \frac{\left(a^2 + a'^2\right)A_1 - saa'A_0}{aa'(2-s)};$$

$$A_3 = \frac{2\left(a^2 + a'^2\right)A_2 - (1+s)\,a\,a'A_1}{aa'(3-s)}$$

and generally

$$A_i = \frac{(i-1)\left(a^2 + a'^2\right)A_{(i-1)} - (i+s-2)\,a\,a'A_{(i-2)}}{(i-s)\,aa'}; \qquad \textbf{(119)}$$

in which i may be any whole number positive or negative, with the exception of 0 and 1. Hence A_i will be known, if A_0, A_1 can be found.

Let

$$A^{-3-1} = \tfrac{1}{2}B_0 + B_1\cos b + B_2\cos 2b + \text{\&c.}$$

multiplying this by

$$\left(a^2 - 2aa'\cos b + a'^2\right),$$

and substituting the value of A^{-3} in series

$$\tfrac{1}{2}A_0 + A_1\cos b + A_2\cos 2b + \text{\&c.}$$
$$= \left(a^2 - 2aa'\cos b + a'^2\right)\left(\tfrac{1}{2}B_0 + B_1\cos b + B_2\cos 2b + \text{\&c.}\right)$$

the comparison of the coefficients of like cosines gives

$$A_i = \left(a^2 + a'^2\right).B_i - aa'.B_{(i-1)} - aa'.B_{(i+1)}.$$

But as relations must exist among the coefficients $B_{(i-1)}$, B_i, $B_{(i+1)}$, similar to those existing among $A_{(i-1)}$, A_i, $A_{(i+1)}$, the equation (119) gives, when $s+1$ and $i+1$ are put for *s* and *i*,

$$B_{(i+1)} = \frac{i\left(a^2 + a'^2\right)B_i - (i+s).a\,a'B_{(i-1)}}{(i-s)aa'}. \qquad \textbf{(120)}$$

If this quantity be put in the preceding value of A_i, it becomes

$$A_i = \frac{2saa'B_{(i-1)} - s\left(a^2 + a'^2\right).B_i}{i-s}; \qquad \textbf{(121)}$$

or if $i+1$ be put for *i*,

$$A_{(i+1)} = \frac{2saa'B_i - s\left(a^2 + a'^2\right).\,B_{(i+1)}}{i-s+1}; \qquad \textbf{(122)}$$

whence may be obtained, by the substitution of the preceding value of $B_{(i+1)}$,

$$A_{(i+1)} = \frac{s(i+s)\cdot aa'\left(a^2 + a'^2\right)B_{(i-1)} + s\left\{2(i-s)a^2a'^2 - i\left(a^2 + a'^2\right)\right\}B_i}{(i-s)(i-s+1).aa'}.$$

If $B_{(i-1)}$ be eliminated between this equation and (121), there will result,

$$B_i = \frac{\frac{1}{s}(i+s)\left(a^2 + a'^2\right)\cdot A_i - \frac{2}{s}(i-s+1)\cdot aa'\cdot A_{(i+1)}}{\left(a'^2 - a^2\right)^2},$$

or substituting for $A_{(i+1)}$ its value given by equation (119),

$$B_i = \frac{\frac{1}{s}(s-1)\left(a^2 + a'^2\right)\cdot A_i + \frac{2}{s}(i+s-1)\cdot aa'\cdot A_{(i-1)}}{\left(a'^2 - a^2\right)^2}.$$

If to abridge $\frac{a}{a'} = \boldsymbol{a}$, the two last equations, as well as equation (119), when both the numerators and the denominators of their several members are divided by a'^2, take the form

$$A_i = \frac{(i-1)\left(1+\boldsymbol{a}^2\right).A_{(i-1)} - (i+s-2).\boldsymbol{a}\,.\,A_{(i-2)}}{(i-s)\boldsymbol{a}}, \qquad \textbf{(123)}$$

$$B_i = \frac{\frac{1}{s}(i+s)\left(1+\boldsymbol{a}^2\right)\cdot A_i - \frac{2}{s}(i-s+1)\cdot \boldsymbol{a}'\cdot A_{(i+1)}}{\left(1-\boldsymbol{a}^2\right)^2 a'^2}; \qquad \textbf{(124)}$$

$$B_i = \frac{\frac{1}{s}(s-i)(1+a^2)\cdot A_i + \frac{2}{s}(i+s-1)\cdot a'\cdot A_{(i-1)}}{(1-a^2)^2 a'^2}, \qquad \textbf{(125)}$$

which is very convenient for computation.

All the coefficients A_2, A_3, &c., B_0, B_1, &c., will be obtained from equations (123) and (125), when A_0, A_1, are known; it only remains, therefore, to determine these two quantities.

453.[40] Because

$$\cos b = \frac{c^{b\sqrt{-1}} + c^{-b\sqrt{-1}}}{2},$$

c being the number whose hyperbolic logarithm is unity; therefore

$$a'^2 - 2aa'\cos b + a^2 = \left\{a' - ac^{b\sqrt{-1}}\right\}\cdot\left\{a' - ac^{-b\sqrt{-1}}\right\}$$

consequently,

$$A^{-3} = \left\{a' - ac^{b\sqrt{-1}}\right\}^{-3}\cdot\left\{a' - ac^{-b\sqrt{-1}}\right\}^{-3}.$$

But[41]

$$\left(a' - ac^{b\sqrt{-1}}\right)^{-3} = \frac{1}{a'^3}\left\{1 + sa\,c^{b\sqrt{-1}} + \frac{s(s+1)}{2}a^2c^{2b\sqrt{-1}} + \text{\&c.}\right\},$$

$$\left(a' - ac^{-b\sqrt{-1}}\right)^{-3} = \frac{1}{a'^3}\left\{1 + sa\,c^{-b\sqrt{-1}} + \frac{s(s+1)}{2}a^2c^{-2b\sqrt{-1}} + \text{\&c.}\right\};$$

the product of which is

$$A^{-3} = \frac{1}{a'^{2s}}\left\{1 + s^2a^2 + \left(\frac{s(1+s)}{1.2}\right)^2 a^4 + \left(\frac{s(1+s)(2+s)}{1.2.3}\right)^2 a^6 + \text{\&c.}\right\}$$
$$+\frac{2}{a'^{2s}}\left\{sa + \frac{s^2(1+s)}{1.2}a^3 + \frac{s(s+1)}{1.2}\cdot\frac{s(1+s)(2+s)}{1.2.3}a^5 + \text{\&c.}\right\}\times\left(c^{b\sqrt{-1}} + c^{-b\sqrt{-1}}\right) + \text{\&c.}$$

whence it appears that $c^{ib\sqrt{-1}}$, and $c^{-ib\sqrt{-1}}$ have always the same coefficients; and as

$$c^{ib\sqrt{-1}} + c^{-ib\sqrt{-1}} = 2\cos ib,$$

it is to see that this series is the same with

$$A^{-3} = \left(a'^2 - 2aa'\cos b + a^2\right)^{-3} = \tfrac{1}{2}A_0 + A_1\cos b + \&c.$$

consequently,[42]

$$A_0 = \frac{2}{a'^{2s}}\left\{1 + s^2a^2 + \left(\frac{s(1+s)}{1.2}\right)^2 a^4 + \left(\frac{s(1+s)(2+s)}{1.2.3}\right)^2 a^6 + \&c.\right\},$$

$$A_1 = \frac{2}{a'^{2s}}\left\{sa + s\cdot\frac{s(1+s)}{1.2}\cdot a^3 + \frac{s(s+1)}{1.2}\cdot\frac{s(1+s)(2+s)}{1.2.3}a^5 + \&c.\right\}.$$

These series do not converge when $s = \tfrac{1}{2}$; but they converge rapidly when $s = -\tfrac{1}{2}$; then, however, A_0 and A_1 become the first and second coefficients of the development of

$$\left(a'^2 - 2aa'\cos b + a^2\right)^{\frac{1}{2}}.$$

Let[43] S and S' be the values of these two coefficients in this case, then

$$S = a'\left\{1 + \left(\frac{1}{2}\right)^2 a^2 + \left(\frac{1.1}{2.4}\right)^2 a^4 + \left(\frac{1.1.3}{2.4.6}\right)^2 a^6 + \&c.\right\}$$

$$S' = -a'\left\{a - \frac{1.1}{2.4}a^3 - \frac{1.1.1.3}{4.2.4.6}a^6 - \frac{1.3.5.1.1.3.5.7}{4.6.8.2.4.6.8.10}a^7 \text{-\&c.}\right\}$$

and as the values of A_0, A_1, may be obtained in functions of S and S', the two last series form the basis of the whole computation.

Because[44] A_0, A_1, become S and S' when $s = -\tfrac{1}{2}$, and that B_i becomes A_i; if $s = -\tfrac{1}{2}$, and $i = 0$, equation (124) gives[45]

$$A_0 = \frac{4aS + 3\left(1+a^2\right)S'}{\left(1-a^2\right)^2\cdot a'^2}.$$

and if $s = -\tfrac{1}{2}$, and $i = 1$, equation (125) gives

$$B_0 = \frac{\left(1+a^2\right)A_0 - 2aA_1}{a'^2\cdot\left(1-a^2\right)^2};$$

and substituting the preceding values of A_0 and A_1, it becomes

$$B_0 = \frac{2S}{a'^4 \cdot \left(1 - \boldsymbol{a}^2\right)^2}.$$

In the same manner it will be found that

$$B_1 = \frac{-3S'}{a'^4 \cdot \left(1 - \boldsymbol{a}^2\right)^2}.$$

454. It now remains to determine the differences of A_i and B_i with regard to a. Resume

$$A^{-3} = \tfrac{1}{2} A_0 + A_1 \cos \boldsymbol{b} + A_2 \cos 2\boldsymbol{b} + \&\text{c.}$$

and take its differential with regard to a, observing that

$$\frac{dA}{da} = 2\left(a - a'\right) \cos \boldsymbol{b} \;;$$

then

$$-2s \cdot \left(a - a'\right) \cos \boldsymbol{b} \cdot A^{-3-1} = \tfrac{1}{2} \cdot \frac{dA_0}{da} + \frac{dA_1}{da} \cdot \cos \boldsymbol{b} + \frac{dA_2}{da} \cdot \cos 2\boldsymbol{b} + \&\text{c.}$$

But

$$A = a'^2 - 2aa' \cdot \cos \boldsymbol{b} + a^2$$

gives

$$a - a' \cos \boldsymbol{b} = \frac{A + a^2 - a'^2}{2a};$$

therefore

$$A^{-3} + \left(a^2 - a'^2\right) A^{-3-1} = -\frac{1}{2} \cdot \frac{a}{s} \cdot \frac{dA_0}{da} - \frac{a}{s} \cdot \frac{dA_1}{da} \cos \boldsymbol{b} - \frac{a}{s} \cdot \frac{dA_2}{da} \cos 2\boldsymbol{b} - \&\text{c.}$$

or, substituting the values of A^{-3} and A^{-3-1} in series

$$\tfrac{1}{2} A_0 + A_1 \cos \boldsymbol{b} + A_2 \cos 2\boldsymbol{b} + \&\text{c.} + \left(a^2 - a'^2\right) \times \left\{\tfrac{1}{2} B_0 + B_1 \cos \boldsymbol{b} + B_2 \cos 2\boldsymbol{b} + \&c.\right\} =$$
$$-\frac{1}{2} \cdot \frac{a}{s} \cdot \frac{dA_0}{da} - \frac{a}{s} \cdot \frac{dA_1}{da} \cos \boldsymbol{b} - \frac{a}{s} \cdot \frac{dA_2}{da} \cos 2\boldsymbol{b} - \&\text{c.}$$

and the comparison of like cosines gives the general expression,

$$\frac{dA_i}{da} = \frac{s\left(a'^2 - a^2\right)}{a} \cdot B_i - \frac{s}{a} A_i; \qquad \textbf{(126)}$$

or substituting B_i its value in (124),[46] it becomes[47]

$$\frac{dA_i}{da} = \left(\frac{ia'^2 + (i+2s)\cdot a^2}{a(a'^2 - a^2)} \right) A_i - \left(\frac{2(i-s+1)\cdot a'}{(a'^2 - a^2)} \right) A_{(i+1)} .$$

If the differentials of this equation be taken with regard to a, and if, in the resulting equations, substitution be made for $\frac{dA_i}{da}$, $\frac{dA_{(i+1)}}{da}$ from the preceding formula, the successive differences of A_i, in functions of $A_{(i+1)}$, $A_{(i+2)}$, will be obtained.

Coefficients of the series R

455. If $\frac{1}{2}$ be put for s in the preceding equation, and in equation (123), and if it be observed that in the series R, article 446, $\frac{dA_i}{da}$ is always multiplied by a, $\frac{d^2A_i}{da^2}$ by a^2, and so on; then where i is successively made equal to 0, 1, 2, 3, &c. the coefficients and their differences are,

$$A_0 = \frac{2(1+a^2)S + 6aS'}{a'^2(1-a^2)^2}$$

$$A_1 = \frac{4aS + 3(1+a^2)S'}{a'^2(1-a^2)^2}$$

$$A_2 = \frac{1}{3a}\cdot\left\{2(1+a^2)A_1 - aA_0\right\}$$

$$A_3 = \frac{1}{5a}\cdot\left\{4(1+a^2)A_2 - 3aA_1\right\}$$

$$A_4 = \frac{1}{7a}\cdot\left\{6(1+a^2)A_3 - 5aA_2\right\}$$

$$A_5 = \frac{1}{9a}\cdot\left\{8(1+a^2)A_4 - 7aA_3\right\}$$

&c. &c.

$$a\left(\frac{d.A_0}{da}\right) = \frac{1}{1-a^2}\left\{a^2A_0 - aA_1\right\}$$

$$a\left(\frac{d.A_1}{da}\right) = \frac{1}{1-a^2}\left\{(1+2a^2)A_1 - 3aA_2\right\}$$

$$a\left(\frac{d.A_2}{da}\right) = \frac{1}{1-a^2}\left\{(2+3a^2)A_2 - 5aA_3\right\}$$

$$a\left(\frac{d.A_3}{da}\right) = \frac{1}{1-a^2}\left\{(3+4a^2)A_3 - 5aA_4\right\}$$

&c. &c.

$$a^2\left(\frac{d^2 . A_0}{da^2}\right)=\frac{1}{\left(1-a^2\right)^2}\left\{2a^4A_0+\left(a-3a^3\right)A_1\right\}$$

$$a^2\left(\frac{d^2 . A_1}{da^2}\right)=\frac{1}{\left(1-a^2\right)^2}\left\{\left(2-4a^2\right)A_1-\left(a-3a^3\right)A_0\right\}$$

$$a^2\left(\frac{d^2 . A_2}{da^2}\right)=\frac{1}{\left(1-a^2\right)^2}\left\{\begin{array}{l}\left\{\left(2+3a^2\right)^2+5a^2\left(1+a^2\right)-2\left(1-a^2\right)^2\right\}A_2\\-5a\left(5+9a^2\right)A_3+5.7a^2A_4\end{array}\right\}$$

$$a^2\left(\frac{d^2 . A_3}{da^2}\right)=\frac{1}{\left(1-a^2\right)^2}\left\{\begin{array}{l}\left\{\left(3+4a^2\right)^2+7a^2\left(1+a^2\right)-3\left(1-a^2\right)^2\right\}A_3\\-7a\left(7+11a^2\right)A_4+7.9\cdot a^2A_5\end{array}\right\}$$

$$a^2\left(\frac{d^2 . A_4}{da^2}\right)=\frac{1}{\left(1-a^2\right)^2}\left\{\begin{array}{l}\left\{\left(4+5a^2\right)^2+9a^2\left(1+a^2\right)-4\left(1-a^2\right)^2\right\}A_4\\-9a\left(9+13a^2\right)A_5+9.11\cdot a^2A_6\end{array}\right\}$$

&c. &c.

456. By the aid of equation (120), it is easy to see that

$$B_0=\frac{2S}{\left(a'^2-a^2\right)^2}$$

$$B_1=\frac{-3S'}{\left(a'^2-a^2\right)^2}$$

$$B_2=\frac{1}{a}\left\{2\left(1+a^2\right)B_1-3a\,B_0\right\}$$

$$B_3=\frac{1}{3a}\left\{4\left(1+a^2\right)B_2-5a\,B_1\right\}$$

$$B_4=\frac{1}{5a}\left\{6\left(1+a^2\right)B_3-7a\,B_2\right\}$$

&c. &c.

$$a\left(\frac{dB_0}{da}\right)=\frac{3a^2B_0+a\,B_1}{1-a^2}$$

$$a\left(\frac{dB_1}{da}\right)=\frac{3a\,B_0+\left(2a^2-1\right)B_1}{1-a^2}$$

&c. &c.

457. The coefficient A_i and its differences have a very simple form, when expressed in functions of B_i, for equations (121) and (126) give

$$A_0 = \left(a'^2 + a^2\right)B_0 - 2aa'B_1$$
$$A_1 = 2aa'B_0 - \left(a'^2 + a^2\right)B_1$$
$$A_2 = \frac{2aa'B_1 - \left(a'^2 + a^2\right)B_2}{3}$$
$$A_3 = \frac{2aa'B_2 - \left(a'^2 + a^2\right)B_3}{5}$$

&c. &c.

$$a\left(\frac{dA_0}{da}\right) = a'aB_1 - a^2B_0$$
$$a\left(\frac{dA_1}{da}\right) = a'^2B_1 - aa'B_0$$
$$a\left(\frac{dA_2}{da}\right) = \tfrac{1}{3}\left\{\left(2a'^2 - a^2\right)B_2 - aa'B_1\right\}$$
$$a\left(\frac{dA_3}{da}\right) = \tfrac{1}{5}\left\{\left(3a'^2 - 2a^2\right)B_3 - aa'B_2\right\}$$
$$a\left(\frac{dA_4}{da}\right) = \tfrac{1}{7}\left\{\left(4a'^2 - 3a^2\right)B_4 - aa'B_3\right\}$$

&c. &c.

$$a^2\left(\frac{d^2A_0}{da^2}\right) = 2a^2B_0 - a'aB_1$$
$$a^2\left(\frac{d^2A_1}{da^2}\right) = 3aa'B_0 - 2a'^2B_1$$

&c. &c.

458. The differences of A_i and B_i, with regard to a' are obtained from their differences with regard to a, for A_i being a homogeneous function of a and a' of the dimension -1,

$$a\left(\frac{dA_i}{da}\right) + a'\left(\frac{dA_i}{da'}\right) = -A_i\,;$$

as readily appears from

$$\left(a^2 - 2aa'\cos b + a'^2\right)^{-\frac{1}{2}},$$

therefore,

$$a'\left(\frac{dA_i}{da'}\right) = -A_i - a\left(\frac{dA_i}{da}\right)$$
$$a'\left(\frac{d^2A_i}{da\,.\,da'}\right) = -2\left(\frac{dA_i}{da}\right) - a\left(\frac{d^2A_i}{da^2}\right)$$

$$a'\left(\frac{d^2A_i}{da'^2}\right)=2A_i+4a\left(\frac{dA_i}{da}\right)+a^2\left(\frac{d^2A_i}{da^2}\right)$$

&c. &c.

Likewise B_i being a homogeneous function of the dimension -3,

$$a'\left(\frac{dB_i}{da'}\right)+a\left(\frac{dB_i}{da}\right)=-3B_i\,.$$

459. By means of these, all the differences of A_i, B_i, with regard to a', may be eliminated from the series R, so that the coefficients of article 449 become[48]

$$M_0=-a\left(\frac{dA_i}{da}\right)-2iA_i$$

$$M_1=a\left(\frac{dA_{(i-1)}}{da}\right)+2(i-1)A_{(i-1)}$$

$$N_0=\tfrac{1}{4}\left\{i(4i-5)A_i+2(2i-1)a\left(\frac{dA_i}{da}\right)+a^2\left(\frac{d^2A_i}{da^2}\right)\right\}$$

$$N_1=-\tfrac{1}{2}\left\{(2i-2)(2i-1)A_{(i-1)}+2(2i-1)a\left(\frac{dA_{(i-1)}}{da}\right)+a^2\left(\frac{d^2A_{(i-1)}}{da^2}\right)\right\}$$

$$N_2=\tfrac{1}{4}\left\{(4i^2-7i+2)A_{(i-2)}+2(2i-1)a\left(\frac{dA_{(i-2)}}{da}\right)+a^2\left(\frac{d^2A_{(i-2)}}{da^2}\right)\right\}$$

$$N_3=-\tfrac{1}{2}\left\{4i^2A_i-2a\left(\frac{dA_i}{da}\right)-a^2\left(\frac{d^2A_i}{da^2}\right)\right\}$$

$$N_4=\tfrac{1}{2}\left\{(2i-2)(2i-1)A_{(i-1)}-2a\left(\frac{dA_{(i-1)}}{da}\right)-a^2\left(\frac{d^2A_{(i-1)}}{da^2}\right)\right\}$$

$$N_5=\tfrac{1}{2}\left\{(2i+2)(2i+1)A_{(i+1)}-2a\left(\frac{dA_{(i+1)}}{da}\right)-a^2\left(\frac{d^2A_{(i+1)}}{da^2}\right)\right\}$$

$$N_6=\tfrac{1}{4}aa'\Sigma B_{(i-1)}$$

$$N_7=-\tfrac{1}{4}aa'\left(B_{(i-1)}+B_{(i+1)}\right)$$

&c. &c.

When $i=1$, $N_6=\tfrac{1}{4}aa'B_0-\tfrac{1}{2}\frac{a}{a'^2}$, and $\tfrac{1}{2}\frac{a}{a'^2}$ must be added to N_7.

460. The series represented by S and S' which are the bases of the computation, are numbers given by observation: for if the mean distance of the earth from the sun be assumed as the unit, the mean distances of the other planets determined by observation, may be expressed in functions of that unit, so that $a = \frac{a}{a'}$, the ratio of the mean distance of m to that of m' is a given number, and as the functions are symmetrical with regard to a and a', the denominator of $\frac{a}{a'}$ may always be so chosen is to make a less than unity, therefore if eleven or twelve of the first terms be taken and the rest omitted, the values of S and S' will be sufficiently exact; or, if their sum be found, considering them as geometrical series whose ratio is $1-a^2$, the values of S and S' will be exact to the sixth decimal, which is sufficient for all the planets and satellites. Thus A_i, B_i, their differences, and consequently the coefficients M_0, M_1, N_0, &c. of the series R are known numbers depending on the mean distances of the planets from the sun.

461. All the preceding quantities will answer for the perturbations of m' when troubled by m, with the exception of A_1, which becomes $A_1 - \frac{a}{a'}$; and when employed to determine the perturbations of Jupiter's satellites, the equatorial diameter of Jupiter, viewed at his mean distance from the sun, is assumed as the unit of distance, in functions of which the mean distances of the four satellites from the centre of Jupiter are expressed.

Notes

[1] See note 16, *Preliminary Dissertation*.

[2] This *reads* $a = \text{Func.}\left(x, y, z\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \cdot t \right)$ in the 1st edition.

[3] The differential element dt is missing in the middle term below in the 1st edition.

[4] See note 6, *Book I, Chapter II.*

[5] The third equation *reads* $d^2 z = dt\left(\frac{dR}{dz}\right)$ in the 1st edition.

[6] Punctuation after the frst three equations is changed from commas to semicolons, and from a period to a semicolon after the sixth.

[7] The 1st edition uses a non-italicized d rather than the italicized d in the differential element $\mathrm{d}R$.

[8] The right side *reads* $\frac{1}{\sqrt{1+e^2}}\{1 - l(c^{(v-v)\sqrt{-1}} + c^{-(v-v)\sqrt{-1}}\} + l^2\left(c^{2(v-v)\sqrt{-1}} + c^{-2(v-v)\sqrt{-1}} \right) - \text{\&c.}\}$ in 1st edition.

[9] The equation *reads* $\int ndt + e = v + \mathrm{E}^{(1)}\sin(v-v) + \frac{1}{2}\mathrm{E}^{(2)}\sin 2(v-v) + \text{\&c.}\}$ in the 1st edition.

[10] The 1st edition reproduces only the first three terms of equation (101).

[11] The second right hand term in this expression *reads* $\left(\frac{dR}{dv'}\right)\frac{\sin v_{/}}{r_{/}}$ in the 1st edition.

[12] Punctuation added after first equation.

[13] Punctuation added after first equation.

[14] Punctuation added after 2nd expression.

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[15] The 1st edition uses a comma after the third equation.

[16] The 2nd term in (110) contains a unbalanced bracket and *reads* $-a^2 ndt\sqrt{1-e^2}\cos(v-v\ )\}\left(\frac{dR}{dr}\right)$ in 1st edition.

[17] The middle term below contains a comma in the denominator and *reads* $1+e\cos(v-v\ )$, in the 1st edition.

[18] The first equation in the 1st edition uses the undefined angle symbol j rather than f.

[19] Multiplier symbol added to right hand side of second equation.

[20] Punctuation added after the next three terms.

[21] The closing bracket in the first term of the second equation is omitted in the 1st edition.

[22] A printing error in the 1st edition uses the symbol j for f in the second term below.

[23] These two equations are presented in the reverse order in following development.

[24] a^2 *reads* a in the next two equations (published erratum).

[25] A printing error in the 1st edition places the punctuation in the next two expressions inside the equations.

[26] Punctuation added after expression for $d\in$ in equations (115) below.

[27] See note 1.

[28] Punctuation added after first definition.

[29] Punctuation added after each expression.

[30] Punctuation added after each expression.

[31] The 2nd expression is obtained from the identity $\sin^2\Pi+\cos^2\Pi=1$.

[32] m' *reads* m in the numerator of 1st term in the 1st edition (published erratum).

[33] Product symbol inserted before sines in both expressions.

[34] Misprints in the arguments of the cosines in the numerator of the second term and denominator of the fifth term are printed $\cos(v'_{/}-v)$ and $\cos(v'-v_{/})$ respectively in the 1st edition.

[35] Note that the order of the terms a^2 and a'^2 is reversed in the previous equation.

[36] Product symbol is inserted into the second and third terms.

[37] Parentheses in 2nd term are omitted in the 1st edition.

[38] The 2nd equation *reads* $v_{/}=v-\tan^2\tfrac{1}{2}f\left\{\sin 2v+\tfrac{1}{2}\tan^2 f\cdot\sin 4v+\right\}$ &c. in the 1st edition.

[39] The 2nd term *reads* $+\frac{m'}{2}\cdot N_7\cdot g^2\cdot\cos i\,(n't-nt+\in'-\in)$ in the 1st edition.

[40] Article is numbered (454) in the 1st edition (published erratum).

[41] Not capitalized in the 1st edition.

[42] Punctuation added after second expression.

[43] We have used the symbol *S* (also italicized) in another context to represent the solar mass elsewhere in this text.

[44] A_1 *reads* $A_{/}$ in the 1st edition.

[45] The numerator *reads* $4aS+3(1+a^2)S'$ in the 1st edition (published erratum).

[46] Actually the value for B_i given before equation (123).

[47] The numerator in the second term in the 1st edition contains a misprint in which a' *reads* a^1.

[48] The third term in N_3 *reads* $-a^2\left(\frac{dA_i}{da^2}\right)$ in the 1st edition.

BOOK II

CHAPTER VI

SECULAR INEQUALITIES IN THE ELEMENTS OF THE ORBITS

Stability of the Solar System, with regard to the Mean Motions of
The Planets and the greater axes of their Orbits

462. WHEN the squares of the disturbing masses are omitted, however far the approximation may be carried with regard to the eccentricities and inclinations, the general form of the series represented by R, in article 449, is

$$m'k\cdot\cos\{i'n't-int+c\}=R,$$

k and c are quantities consisting entirely of the elements of the orbits, k being a function of the mean distances, eccentricities, and inclinations, and c a function of the longitudes of the epochs of the perihelia and nodes. The differential of this expression, with regard to nt the mean motion of m, is

$$\mathrm{d}R=m'kindt\sin\{i'n't-int+c\}\,.$$

The expression dR always relates to the mean motion of m alone; when substituted in

$$da=2a^2\mathrm{d}R\,,$$

it gives

$$da=2a^2m'ik\cdot ndt\cdot\sin\{i'n't-int+c\}\,,$$

the integral of which is

$$d\,a=\frac{2a^2im'nk}{i'n'-in}\cdot\cos\{i'n't-int+c\}\,.$$

It is evident that if the greater axes of the orbits of the planets be subject to secular inequalities, this value of $d\,a$ must contain terms independent of the sines and cosines of the angular distances of the bodies from each other. But a must be periodic unless $i'n'-in=0$; that is, unless the mean motions of the bodies m and m' be commensurable. Now the mean motions of no two bodies in the solar system are exactly commensurable, therefore $i'n'-in$ is in no case exactly zero; consequently the greater axes of the celestial bodies are not subject to secular inequalities; and on account of the equation $n=a^{-\frac{3}{2}}$, their mean motions are uniform.

Thus, when the squares and products of the masses m, m' are omitted, the differential $\mathrm{d}R$ does not contain any term proportional to the element of the time, however far the approximation may be carried with regard to the eccentricities and inclinations of the orbits, or which is the same thing, $\frac{\mathrm{d}R}{ndt}$ does not contain a constant term; for if it contained a term of the form $m'k$, then would $a = 2\int a^2 \,.\, \mathrm{d}R = 2a^2m'knt$, and $z = -3\iint andt \,.\, \mathrm{d}R$ would become

$$z = -3\iint an^2m'kdt^2 = -3an^2m'kt^2 ,$$

so that the greater axes would increase with the time, and the mean motion would increase with the square of the time, which would ultimately change the form of the orbits of the planets, and the periods of their revolutions. The stability of the system is so important, that it is necessary to inquire whether the greater axes and mean motions be subject to secular inequalities, when the approximation is carried to the squares and products of the masses.

463. The terms depending on the squares and products of the masses are introduced into the series R by the variation of the elements of the orbits, both of the disturbed and disturbing bodies. Hence, if $d\,a,\ d\,e,$ &c. be the integrals of the differential equations of the elements in article 439, the variable elements will be $a + d\,a,\ e + d\,e,$ &c. for m, and $a' + d\,a',\ e' + d\,e',$ &c. for m'; and when these are substituted for a, e, a', e', &c. in the series R, it takes the form

$$R_{/} = R + d\,R + d'R ;$$

and from what has been said, the greater axis and mean motion of m will not be affected by secular inequalities, unless the differential

$$\mathrm{d}R_{/} = \mathrm{d}R + \mathrm{d}\,.\,d\;R + \mathrm{d}\,.\,d'R$$

contains a term that is not periodic.

[The term] dR is of the first order relatively to the masses, and has been proved in the preceding article not to contain a term that is not periodic. [The terms] $\mathrm{d}\,.\,d\,R$ and $\mathrm{d}\,.\,d'R$ include the squares and products of the masses; the first is the differential of $d\,R$ with regard to the elements of the troubled planet m, and $\mathrm{d}\,.\,d'R$ is a similar function with regard to the disturbing body m'. It is proposed to examine whether either of these contains[1] a term that is not periodic, beginning with $\mathrm{d}\,.\,d\,R$.

464. The variation $d\,R$ regards the elements of m alone, and is[2]

$$d\,R = \frac{dR}{da}d\,a + \frac{dR}{d\epsilon}d\epsilon + \frac{dR}{de}d\,e + \frac{dR}{dv}d\,v + \frac{dR}{dp}d\,p + \frac{dR}{dq}d\,q .$$

If the values in article 439, be put for $d\,a,\ d\,e,$ &c. this expression becomes

$$d\,R = 2a^2\left\{\frac{dR}{da}\int\frac{dR}{d\epsilon}\cdot ndt - \frac{dR}{d\epsilon}\int\frac{dR}{da}\cdot ndt\right\}$$

$$+\frac{a\sqrt{1-e^2}}{e}\left(1-\sqrt{1-e^2}\right)\left\{\frac{dR}{d\in}\int\frac{dR}{de}\cdot ndt-\frac{dR}{de}\int\frac{dR}{d\in}\cdot ndt\right\}$$
$$+\frac{a\sqrt{1-e^2}}{e}\left\{\frac{dR}{dv}\int\frac{dR}{de}\cdot ndt-\frac{dR}{de}\int\frac{dR}{dv}\cdot ndt\right\}$$
$$+\frac{a}{\sqrt{1-e^2}}\left\{\frac{dR}{dp}\int\frac{dR}{dq}\cdot ndt-\frac{dR}{dq}\int\frac{dR}{dp}\cdot ndt\right\}.$$

And its differential, according to the elements of the orbit of *m* alone, is obtained by suppressing the signs $\int$ introduced by the integration of the differential equations of the elements in article 439, which reduces this expression to zero; therefore to obtain $\mathrm{d}.dR$, it is sufficient to take the differential according to *nt* of those terms in dR that are independent of the sign $\int$.

When the series in article 449 is substituted for *R*, dR will take the form

$$P.f.Qdt-Q.f.Pdt.$$

Where *P* and *Q* represent a series of terms of the form[3]

$$k.\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(i'n't-int+c),$$

i' and *i* being any whole numbers positive or negative. Let[4]

$$k\cos(i'n't-int+c)$$

belong to *P*, and let $k'\cos(i'n't-int+c')$ be the corresponding term of *Q*, *k*, k', *c*, c', being constant quantities.

A term that is not periodic could only arise in

$$\mathrm{d}dR=\mathrm{d}\left\{P\int Qdt-Q\int Pdt\right\},$$

if it contained such an expression as

$$kk'\cos\{i'n't-int+c\}\cos\{i'n't-int+c'\}=\tfrac{1}{2}kk'\cos(c-c')+\tfrac{1}{2}kk'\cos\{2i'n't-2int+c+c'\};$$

or a similar product of the sines of the same angles. But when $k\cos(i'n't-int+c)$ is put for *P*, and $k'\cos(i'n't-int+c')$ for *Q*, $\mathrm{d}.dR$ becomes

$$\mathrm{d}.dR=kindt.\sin(i'n't-int+c).\int k'dt.\cos(i'n't-int+c')$$
$$-k'indt.\sin(i'n't-int+c').\int kdt.\cos(i'n't-int+c'),$$

which is equal to zero when the integrations are accomplished. Whence it way be concluded that $\mathrm{d}.dR$ is altogether periodic.

465. It now remains to determine whether the variation of the elements of the orbit of m' produces terms that are not periodic in $\mathrm{d}.d'R$. This cannot be demonstrated by the same process, because the function R, not being symmetrical relatively to the co-ordinates of m and m', changes its value in considering the disturbance of m' by m. Let R' be what R becomes with regard to the planet m' troubled by m; then

$$R' = m\left\{\frac{1}{\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}} - \frac{xx'+yy'+zz'}{r^3}\right\}$$

hence

$$R = \frac{m'}{m}R' + m'(xx'+yy'+zz')\left(\frac{1}{r^3}-\frac{1}{r'^3}\right);$$

and[5]

$$d'R = \frac{m'}{m}d'R' + m'd'\left\{(xx'+yy'+zz')\left(\frac{1}{r^3}-\frac{1}{r'^3}\right)\right\}.$$

If the differential of this equation according to d be periodic, so will $\mathrm{d}.d'R$. Now in consequence of the variations of the elements of the orbit of m,[6]

$$d'R' = \frac{dR'}{da'}da' + \frac{dR'}{de'}de' + \frac{dR'}{d\in'}d\in' + \frac{dR'}{d\nu'}d\nu' + \frac{dR'}{dp'}dp' + \frac{dR'}{dq'}dq'.$$

And as this expression with regard to the planet m' is in all respects similar to that of dR in the preceding article with regard to m, by the same analysis it may be proved that $\mathrm{d}.dR'$ is altogether periodic. Thus the only terms that are not periodic, must arise from the differential[7 8] of,

$$m'd'\{xx'+yy'+zz'\}\left(\frac{1}{r^3}-\frac{1}{r'^3}\right).$$

Let,[9]

$$m'\{xx'+yy'+zz'\}\left(\frac{1}{r^3}-\frac{1}{r'^3}\right) = L.$$

Then by article 346,

$$\frac{m'x}{r^3} = -\frac{m'}{S}\cdot\frac{d^2x}{dt^2} - \frac{mm'}{S}\cdot\frac{x}{r^3} + \frac{m'}{S}\left(\frac{dR}{dx}\right);$$

likewise

$$\frac{m'x'}{r'^3} = -\frac{m'}{S}\cdot\frac{d^2x'}{dt^2} - \frac{m'^2}{S}\cdot\frac{x'}{r'^3} + \frac{m'}{S}\left(\frac{dR'}{dx'}\right).$$

The co-ordinates $y,\ z,\ y',\ z',$ furnish similar equations. Thus,

$$L=\frac{m'}{S}\left\{\frac{d\left(xdx'-x'dx+ydy'-y'dy+zdz'-z'dz\right)}{dt^2}\right\}+N,$$

where

$$N=\frac{m'^2}{S}\left(\frac{xx'+yy'+zz'}{r'^3}\right)-\frac{mm'}{S}\left(\frac{xx'+yy'+zz'}{r^3}\right)$$
$$+\frac{m'}{S}\left\{x'\left(\frac{dR}{dx}\right)-x\left(\frac{dR'}{dx'}\right)+y'\left(\frac{dR}{dy}\right)-y\left(\frac{dR'}{dy'}\right)+z'\left(\frac{dR}{dz}\right)-z\left(\frac{dR'}{dz'}\right)\right\}.$$

If N be omitted at first,

$$\mathrm{d}\cdot L=\frac{m'}{S}\cdot d\left\{\frac{d\left(x'dx-xdx'+y'dy-ydy'+z'dz-zdz'\right)}{dt^2}\right\}.$$

466. The elliptical values of the co-ordinates being substituted, every term must be periodic. For example, if

$$x=a\cdot\cos\left(nt+\epsilon-v\right)\qquad x'=a'\cdot\cos\left(n't+\epsilon'-v'\right)$$

[then]

$$\frac{x'dx-xdx'}{dt}=\tfrac{1}{2}aa'\left(n-n'\right).\sin\left\{n't-nt+\epsilon'-\epsilon-v'+v\right\};$$

a quantity that must be periodic unless $n't-nt=0,$ which never can happen, because the mean motions of no two bodies in the solar system are exactly commensurable; but even if a term that is not periodic were to occur, it would vanish in taking the second differential; and as the same thing may be shown with regard to the other products

$$y'dy-ydy'\ \text{[and]}\ z'dz-zdz',$$

$\mathrm{d}L$ is a periodic function. With regard to the term $\mathrm{d}L=\mathrm{d}N,$ if the elliptical values of the co-ordinates of m and m' be substituted, it will readily appear that this expression is periodic, for the equations of the elliptical motion of m and $m',$ in article 365, give

$$\frac{xx'+yy'+zz'}{r^3}=-\frac{x'd^2x+y'd^2y+z'd^2z}{\left(S+m\right)dt^2},$$
$$\frac{xx'+yy'+zz'}{r'^3}=-\frac{x'd^2x+y'd^2y+z'd^2z}{\left(S+m'\right)dt^2};$$

so that the function N becomes

$$N = -\frac{m'^2}{S(S+m)}\left\{\frac{xd^2x' + yd^2y' + z'd^2z}{dt^2}\right\} + \frac{mm'}{S(S+m)}\left(\frac{x'd^2x + y'd^2y + z'd^2z}{dt^2}\right)$$
$$+\frac{m'}{S}\left\{x'\left(\frac{dR}{dx}\right) - z\left(\frac{dR'}{dx'}\right) + y'\left(\frac{dR}{dy}\right) - y\left(\frac{dR'}{dy'}\right) + z'\left(\frac{dR}{dz}\right) - z\left(\frac{dR'}{dz'}\right)\right\}.$$

467. From what has been said, it will readily appear that the terms of this expression, consisting of the products $x'd^2x$, xd^2x', &c. &c., are periodic when the elliptical values are substituted for the co-ordinates, and their differentials.

468. The last term of the value of N is also periodic; for, if the elliptical values of the co-ordinates of m and m', be put in R, it may be developed into a series of cosines of the multiples of the arcs nt and $n't$, and the differential may be found by making R vary with regard to the quantities belonging to m alone; hence this differential may contain the sines and cosines of the multiples of nt, but no sine or cosine of $n't$ alone; and as[10]

$$x' = a'\cos(nt + \in' - v'),$$

the mean motions nt, $n't$, never vanish from $x'\left(\frac{dR}{dx}\right)$, which is consequently periodic; and as the same may be demonstrated for each of the products

$$x\left(\frac{dR'}{dx'}\right),\ y'\left(\frac{dR}{dy}\right), \text{ \&c. \&c.},$$

not only N but its differential are periodic, and consequently $\mathrm{d}.d'R$.

Thus it has been proved that when the approximation is carried to the squares and products of the masses, the expression

$$\mathrm{d}R_{/} = \mathrm{d}R + \mathrm{d}.d\ R + \mathrm{d}.d'R$$

relatively to the variations of the mean motions of the two planets m and m' is periodic.

469. These results would be the same whatever might be the number of disturbing bodies; for m'' being a second planet disturbing the motion of m, it would add to R the term

$$\frac{m''}{\sqrt{(x''-x)^2 + (y''-y)^2 + (z''-z)^2}} - \frac{m''(xx'' + yy'' + zz'')}{r''^3}.$$

The variation of the co-ordinates of[11] m and m'' resulting from the reciprocal action of these two planets, would produce terms multiplied by mm'' and m''^2 in the variation of R; and by the preceding analysis it follows that all the terms in $\mathrm{d}\,.d''R$ are periodic. $d''R$ relates to the variation of the elements of the orbit of m''.

The variations of the co-ordinates of m' arising from the action of m'' on m', will cause a variation in the part of R depending on the action of m' on m represented by

$$\frac{m'}{\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}}-\frac{m'(xx'+yy'+zz')}{r'^3}$$

There will arise terms in R, multiplied by $m'm''$, which will be functions of nt, $n't$, $n''t$, when substitution is made of the elliptical values of the co-ordinates; and as the mean motions cannot destroy each other, these terms will only produce periodic terms in dR. Should there be any terms independent of the mean motion nt in the development of R, they will vanish by taking the differential dR. And as terms depending on nt alone will have the form $m'm''.\mathrm{d}P$, P being a function of the elliptical co-ordinates of m,[12] there will arise in $\int \mathrm{d}\,.R$ terms of the form $m'm''\int \mathrm{d}P = m'm''.P$, since d$P$ is an exact differential. These terms will then be of the second order after integration, and such terms are omitted in the value of this function.

The variation of the co-ordinates x, y, z, produced by the action of m'' on m only introduce into the preceding part of R terms multiplied by $m'm''$ and functions of the three angles nt, $n't$, $n''t$; and as these three mean motions cannot destroy each other, there can only be periodic terms in dR. The terms depending on nt alone, only produce periodic terms of the order $m'm''$ in dR.

The same may be proved with regard to the part of R depending on the action of m'' on m.

470. Hence whatever may be the number of disturbing bodies, when the approximation includes the squares and products of the masses, the variation of the elliptical elements of the disturbed and disturbing planets only produce periodic terms in dR.

471. Now the variation of

$$z = -3\iint andt\,.\,\mathrm{d}R$$

is

$$dz = -3an\iint dt\,.\,\mathrm{d}\,.d\,R + 3a^2\iint\left(ndt\,.\,\mathrm{d}R\,.\int \mathrm{d}R\right).$$

It was proved in article 464 that $\mathrm{d}d\,R = 0$ in considering only secular quantities of the order of the squares of the masses. It is easy to see from the form of the series R that $\mathrm{d}R\int \mathrm{d}R = 0$ with regard to these quantities, consequently the variation of the mean motion of a planet cannot contain any secular inequality of the first or second order with regard to the disturbing forces that can become sensible in the course of ages, whatever the number of planets may be that trouble its motion. And as $da = 2a^2 \mathrm{d}R$ becomes

$$da = \left\{ 2a^2 \int \mathrm{d}R + 8a^3 \int \left(\mathrm{d}R \int \mathrm{d}R \right) \right\},$$

by the substitution of $\left(a + d\,a^2 \right)$ for a^2, $d\,a$ cannot contain a secular inequality if dz does not contain one.

472. It therefore follows, that when periodic inequalities are omitted as well as the quantities of the third order with regard to the disturbing forces, the *mean motions* of the planets, and the *greater axes* of their orbits, are *invariable*.

The whole of this analysis is given in the Supplement to the third volume of the *Mécanique Céleste*; but that part relating to the second powers of the disturbing forces is due to M. Poisson.[13]

Differential Equations of the Secular Inequalities in the Eccentricities, Inclinations, Longitudes of the Perihelia and Nodes, which are the annual and sidereal variations of these four elements

473. That part of the series *R*, in article 449, which is independent of periodic inequalities, is found by making $i = 0$, for then[14]

$$\sin i\left(n't - nt + \in' - \in \right) = 0,$$
$$\cos i\left(n't - nt + \in' - \in \right) = 1;$$

and if the differences of $A_0\ A_1$ with regard to a' be eliminated by their values in article 458, the series *R* will be reduced to

$$F = \frac{m'}{2} A_0 + \frac{m'}{4} \left\{ a\left(\frac{dA_0}{da} \right) + \tfrac{1}{2} a^2 \left(\frac{d^2 A_0}{da^2} \right) \right\} \left(e^2 + e'^2 \right)$$
$$+ \frac{m'}{2} \left\{ A_1 - a\left(\frac{dA_1}{da} \right) - \tfrac{1}{2} a^2 \left(\frac{d^2 A_1}{da^2} \right) \right\} ee' \cos\left(\nu' - \nu \right)$$
$$- \frac{m'}{8} a\, a' B_1 g^2$$

But the formulae in articles 456 and 457 give

$$a\left(\frac{dA_0}{da} \right) + \tfrac{1}{2} a^2 \left(\frac{d^2 A_0}{da^2} \right) = -\frac{3aa'.S'}{2\left(a'^2 - a^2 \right)^2}$$

$$A_1 - a\left(\frac{dA_1}{da}\right) - \tfrac{1}{2}a^2\left(\frac{d^2A_1}{da^2}\right) = \frac{3\left(aa'S + \left(a^2 + a'^2\right)S'\right)}{\left(a'^2 - a^2\right)}$$

$$aa'B_1 = -\frac{3aa'.S'}{\left(a'^2 - a^2\right)^2};$$

consequently

$$F = \frac{m'}{2}A_0 - \frac{3m'.aa'.S'}{2.4.\left(a'^2 - a^2\right)^2}\cdot\left\{e^2 + e'^2 - (p' - p)^2 - (q' - q)^2\right\}$$

$$+\frac{3m'\left(a'a.S + \left(a^2 + a'^2\right)S'\right)}{2\left(a'^2 - a^2\right)^2}\cdot ee'.\cos(v' - v);$$

for by article 444

$$g^2 = (p' - p)^2 + (q' - q)^2,$$

whence

$$\frac{dF}{dv} = \frac{3m'\left(a'a.S + \left(a^2 + a'^2\right)S'\right)}{2\left(a'^2 - a^2\right)^2}\cdot ee'.\sin(v' - v)$$

$$\frac{dF}{de} = -\frac{3m'aa'S'}{4\left(a'^2 - a^2\right)^2}\cdot e + \frac{3.m'\left(aa'S + \left(a^2 + a'^2\right)S'\right)}{2\left(a'^2 - a^2\right)^2}\cdot e'.\cos(v' - v)$$

$$\frac{dF}{dp} = -\frac{3m'.aa'S'}{4\left(a'^2 - a^2\right)^2}\cdot(p' - p)$$

$$\frac{dF}{dq} = -\frac{3m'.aa'S'}{4\left(a'^2 - a^2\right)^2}\cdot(q' - q).$$

474. When the squares of the eccentricities are omitted, the differential equations in article 441 become[15]

$$\frac{de}{dt} = -\frac{an}{e}\frac{dF}{dv}; \quad \frac{dv}{dt} = \frac{an}{e}\frac{dF}{de};$$

$$\frac{dp}{dt} = an\cdot\frac{dF}{dq}; \quad \frac{dq}{dt} = -an\cdot\frac{dF}{dp}.$$

If the differentials of F, according to the elements, be substituted in these, and if to abridge[16]

$$-\frac{3m'.na^2a'S'}{4\left(a'^2 - a^2\right)^2} = (0.1);$$

$$-\frac{3m' \cdot an \cdot \left(a a' S + \left(a^2 + a'^2\right) S'\right)}{2\left(a'^2 - a^2\right)} = \boxed{0.1}\,;$$

they become

$$\begin{aligned}
\frac{de}{dt} &= \boxed{0.1}\, e' \sin\left(v' - v\right) \\
\frac{dv}{dt} &= (0.1) - \boxed{0.1}\, \frac{e'}{e} \cos\left(v' - v\right) \\
\frac{dp}{dt} &= -(0.1)\left(q - q'\right) \\
\frac{dq}{dt} &= +(0.1)\left(p - p'\right).
\end{aligned} \qquad \textbf{(127)}$$

475. But $\tan f = \sqrt{p^2 + q^2}$ and $\tan q = \frac{p}{q}$, and when the squares of the inclinations are omitted $\cos q = 1$, hence

$$d f = dp \sin q + dq \cos q\,; \quad dq = \frac{dp \cos q - dq \sin q}{\tan f}\,;$$

and substituting the preceding values of *dp*, *dq*, the variations in the inclinations and longitude of the node are,[17]

$$\begin{aligned}
\frac{d f}{dt} &= (0.1) . \tan f . \sin\left(q - q'\right) \\
\frac{dq}{dt} &= -(0.1) + (0.1) \cdot \frac{\tan f'}{\tan f} \cdot \cos\left(q - q'\right).
\end{aligned}$$

476. The preceding quantities are the secular variations in the orbit of *m* when troubled by m' alone, but all the bodies in the system act simultaneously on the planet *m*, and whatever effect is produced in the elements of the orbit of *m* by the disturbing planet m', similar effects will be occasioned by the disturbing bodies[18] m'', m''', &c. Hence, as the change produced by m' in the elements of the orbit of *m* are expressed by the second terms of the preceding equations, it is only necessary to add to them a similar quantity for each disturbing body, in order to have the whole action of the system on *m*.

The expressions (0.1), $\boxed{0.1}$ have been employed to represent the coefficients relative to the action of m' on *m*; for quantities relative to *m* which has no accent, are represented by 0; and those relating to m' which has one accent, by 1; following the same notation, the coefficients relative to the action of m'' on *m* will be (0.2), $\boxed{0.2}$; those relating to m''' on *m* by (0.3), $\boxed{0.3}$; and so on. Therefore the secular action of m'' in disturbing the elements of the orbit of *m* will be

$$\boxed{0.2}\ e''\sin(v''-v); \quad (0.2)-\boxed{0.2}\frac{e''}{e}\cos(v''-v)$$
$$(0.2)\tan f\sin(q-q''); \quad -(0.2)+(0.2)\frac{\tan f''}{\tan f}\cos(q-q'').$$

477. Therefore the differential equations of the secular inequalities of the elements of the orbit of *m*, when troubled by the simultaneous action of all the bodies in the system, are

$$\begin{aligned}
\frac{de}{dt} &= \boxed{0.1}\ e'\sin(v'-v)+\boxed{0.2}\ e''\sin(v''-v)+\boxed{0.3}\ e'''\sin(v'''-v)+\&\text{c.}\\
\frac{dv}{dt} &= (0.1)+(0.2)+\&\text{c.}-\boxed{0.1}\ \frac{e'}{e}\cos(v'-v)-\boxed{0.2}\ \frac{e''}{e}\cos(v''-v)-\&\text{c.}\\
\frac{df}{dt} &= (0.1)\tan f'\sin(q-q')+(0.2)\tan f''\sin(q-q'')+\&\text{c.}\\
\frac{dq}{dt} &= -\{(0.1)+(0.2)+\&\text{c.}\}+(0.1)\frac{\tan f'}{\tan f}\cos(q-q')+(0.2)\frac{\tan f''}{\tan f}\cos(q-q'')+\&\text{c.}
\end{aligned} \qquad \textbf{(128)}$$

478. All the quantities in these equations are determined by observation for a given epoch assumed as the origin of the time, and when integrated, or (which is the same thing) multiplied by *t*, they give the annual variation in the elements of the orbit of a planet, on account of the immense periods of the secular inequalities, which admit of one year being regarded as an infinitely short time in which the elements e, v, &c., may be supposed to be constant.

479. It is evident that the secular variations in the elements of the orbits of m', m'', m''', &c., will be obtained from the preceding equations, if every thing relating to *m* be changed into the corresponding quantities relative to m', and the contrary, and so for the other bodies. Thus the variation in the elements of m', m'', &c., from the action of all the bodies in the system, will be

$$\begin{aligned}
\frac{de'}{dt} &= \boxed{1.0}\ .e.\sin(v-v')+\boxed{1.2}.\ e''.\sin(v''-v')+\&\text{c.}\\
\frac{de''}{dt} &= \boxed{2.0}\ .e.\sin(v-v'')+\boxed{2.1}.\ e'.\sin(v'-v'')+\&\text{c.}\\
&\qquad\qquad\qquad \&\text{c.} \qquad \&\text{c.}\\
\frac{dv'}{dt} &= (1.0)+(1.2)+\&\text{c.}-\boxed{1.0}\ .\frac{e}{e'}.\cos(v-v')-\boxed{1.2}\ .\frac{e''}{e'}\cos(v''-v').-\&\text{c.}\\
\frac{dv''}{dt} &= (2.0)+(2.1)+\&\text{c.}-\boxed{2.0}\ .\frac{e}{e''}\cos(v-v'')-\boxed{2.1}\ .\frac{e'}{e''}\cos(v'-v'').-\&\text{c.}\\
&\qquad\qquad\qquad \&\text{c.} \qquad \&\text{c.}\\
\frac{df'}{dt} &= (1.0).\tan f.\sin(q'-q)+(1.2).\tan f''.\sin(q'-q'')+\&\text{c.}\\
\frac{df''}{dt} &= (2.0).\tan f.\sin(q''-q)+(2.1).\tan f'.\sin(q''-q')+\&\text{c.}
\end{aligned} \qquad \textbf{(129)}$$

$$\frac{dq'}{dt} = -\{(1.0)+(1.2)+\text{\&c.}\}+(1.0)\cdot\frac{\tan f}{\tan f'}\cdot\cos(q'-q)+(1.2)\frac{\tan f''}{\tan f'}\cdot\cos(q'-q'')+\text{\&c.}$$

$$\frac{dq''}{dt} = -\{(2.0)+(2.1)+\text{\&c.}\}+(2.0)\cdot\frac{\tan f}{\tan f''}\cos(q''-q)+(2.1)\cdot\frac{\tan f}{\tan f''}\cos(q''-q')+\text{\&c.}$$

&c. &c.

As these quantities do not contain the mean longitude, nor its sines or cosines, they depend on the configuration of the orbits only.

Approximate Values of the Secular Variations in these four Elements in Series, ascending according to the powers of the Time

480. The annual variations in the elements are readily obtained from these formulae; but as the secular inequalities vary so slowly that they may be assumed to vary as the time for a great many centuries without sensible error, series may be formed, whence very accurate values of the elements may be computed for at least a thousand years before and after the epoch. Let the eccentricity be taken as an example. With the given values of the masses and mean longitudes of the perihelia determined by observation, let a value of $\frac{de}{dt}$, the variation in the eccentricity, be computed from the preceding equation for the epoch, say 1750, and another for 1950. If the latter be represented by $\left(\frac{de}{dt}\right)$, and the former by $\left(\frac{d\bar{e}}{dt}\right)$, then

$$\left(\frac{de}{dt}\right)-\frac{d\bar{e}}{dt}=200\cdot\frac{d^2\bar{e}}{dt^2};\text{ or, }\left(\frac{de}{dt}\right)=\left(\frac{d\bar{e}}{dt}\right)+200\cdot\frac{d^2\bar{e}}{dt^2}$$

the quantities $\frac{d\bar{e}}{dt}$, $\frac{d^2\bar{e}}{dt^2}$, being relative to the year 1750. Hence, $\bar{e}$ being the eccentricity of any orbit at that epoch, the eccentricity *e* at any other assumed time *t*, may be found from[19]

$$e=\bar{e}+\frac{d\bar{e}}{dt}\cdot t+\tfrac{1}{2}\cdot\frac{d^2\bar{e}}{dt^2}t^2+\text{\&c.}$$

with sufficient accuracy for 1,000 or 1,200 years before and after 1750.

In the same manner all the other elements may be computed from

$$
\begin{aligned}
v &= \bar{v} + \frac{d\bar{v}}{dt}\cdot t + \tfrac{1}{2}\frac{d^2\bar{v}}{dt^2}\cdot t^2 + \&\text{c.}\\
f &= \bar{f} + \frac{d\bar{f}}{dt}\cdot t + \tfrac{1}{2}\frac{d^2\bar{f}}{dt^2}\cdot t^2 + \&\text{c.}\\
q &= \bar{q} + \frac{d\bar{q}}{dt}\cdot t + \tfrac{1}{2}\frac{d^2\bar{q}}{dt^2}\cdot t^2 + \&\text{c.}\\
g &= \bar{g} + \frac{d\bar{g}}{dt}\cdot t + \tfrac{1}{2}\frac{d^2\bar{g}}{dt^2}\cdot t^2 + \&\text{c.}\\
\Pi &= \bar{\Pi} + \frac{d\bar{\Pi}}{dt}\cdot t + \tfrac{1}{2}\frac{d^2\bar{\Pi}}{dt^2}\cdot t^2 + \&\text{c.}
\end{aligned}
\qquad \textbf{(130)}
$$

For as and $\bar{f}$ and $\bar{q}$ are given by observation, $\bar{g}$ and $\bar{\Pi}$, which are functions of them, may be found. All the quantities in these equations are relative to the epoch.

These expressions are sufficient for astronomical purposes; but as very important results may be deduced from the finite values of the secular variations, the integrals of the preceding differential equations must be determined for any given time.

Finite Values of the Differential Equations relative to the eccentricities and longitudes of the Perihelia

481. Direct integration is impossible in the present state of analysis, but the differential equations in question may be changed into linear equations capable of being integrated by the following method of Lagrange. Let

$$
\begin{aligned}
h &= e\sin v & l &= e\cos v\\
h' &= e'\sin v' & l' &= e'\cos v',\\
&\&\text{c.} & &\&\text{c.}
\end{aligned}
$$

then

$$
\begin{aligned}
\frac{dh}{dt} &= \frac{de}{dt}\sin v + \frac{dv}{dt}\cdot e\cos v\,,\\
\frac{dl}{dt} &= \frac{de}{dt}\cos v - \frac{dv}{dt}\cdot e\sin v\,;
\end{aligned}
$$

and substituting the differentials in article 477, the result will be[20]

$$
\begin{aligned}
\frac{dh}{dt} &= \{(0.1)+(0.2)+\&\text{c.}\}l - \boxed{0.1}\,l' - \boxed{0.2}\,l'' - \boxed{0.3}\,l''' - \&\text{c.}\\
\frac{dl}{dt} &= -\{(0.1)+(0.2)+\&\text{c.}\}h + \boxed{0.1}\,h' + \boxed{0.2}\,h'' + \boxed{0.3}\,h''' + \&\text{c.}
\end{aligned}
\qquad \textbf{(131)}
$$

likewise
$$
\frac{dh'}{dt} = \{(0.1)+(0.2)+\&\text{c.}\}l' - \boxed{0.1}\,l - \boxed{1.2}\,l'' - \boxed{0.3}\,l''' - \&\text{c.}
$$

$$\frac{dl'}{dt} = -\{(0.1)+(1.2)+\&c.\}h' + \boxed{0.1}\,h + \boxed{1.2}\,h'' + \boxed{1.3}\,h''' + \&c.$$
$$\&c. \qquad \&c.$$

It is obvious that there must be twice as many such equations, and as many terms in each, as there are bodies in the system.

482. The integrals of these equations will be obtained by making

$$\begin{aligned} h &= N\sin(gt+x) \qquad l = N\cos(gt+x) \\ h' &= N'\sin(gt+x) \qquad l' = N'\cos(gt+x), \\ &\&c. \qquad \&c. \end{aligned}$$

It is easy to see why these quantities take this form, for if $h'=0$, $h''=0$, &c., $l=0$, $l'=0$, &c., then

$$\frac{dh}{dt} = (0.1)\,l; \;\; \frac{dl}{dt} = -(0.1)\,h.$$

Let[21]

$$\frac{dh}{dt} = gl; \;\; \frac{dl}{dt} = -gh,$$

but

$$\frac{d^2h}{dt^2} = g\,\frac{dl}{dt},$$

therefore

$$\frac{d^2h}{dt^2} + g^2h = 0.$$

And by article 214 $h = N\sin(gt+x)$, N and x being arbitrary constant quantities. In the same manner $l = N\cos(gt+x)$.

483. If the preceding values of h, h', h'', &c., l, l', l'', &c., and their differentials be substituted in equations (131), the sines and cosines vanish, and there will result a number of equations,

$$\begin{aligned} Ng &= \{(0.1)+(0.2)+(0.3)+\&c.\}N - \boxed{0.1}\,N' - \boxed{0.2}\,N'' - \&c. \\ N'g &= \{(1.0)+(1.2)+(1.3)+\&c.\}N' - \boxed{1.0}\,N - \boxed{1.2}\,N'' - \&c. \\ &\&c. \qquad \&c. \end{aligned} \qquad \textbf{(132)}$$

equal to the number of quantities N, N', N'', &c., consequently equal to the number of bodies in the system; hence, if N', N'', N''', &c., be eliminated, N will vanish, and will therefore remain indeterminate, and there will result an equation in g only, the degree of which will be equal to the number of bodies m, m', m'', &c. The roots of this equation way be represented by g, g_1, g_2, &c., which are the mean secular motions of the perihelia of the orbits of m, m', m'', &c., and are functions of the known quantities (0.1), $\boxed{0.1}$, (1.0), $\boxed{1.0}$, &c., only. When successively substituted in equations (132), these equations will only contain the indeterminate quantities N, N', N'', &c.; but it is clear, that for each root of g, quantities N, N', N'', &c., will have different values. Therefore let N, N', N'', &c., be their values corresponding to the root g; N_1, N_1', N_1'', &c., those corresponding to the root g_1;[22] N_2, N_2', N_2'', &c., those arising from the substitution of g_2, &c. &c.; and as the complete integral of a differential linear equation is the sum of the particular equations, the integrals of (131) are

$$\begin{aligned}
h &= N\sin\left(gt+x\right)+N_1\sin\left(g_1t+x_1\right)+N_2\sin\left(g_2t+x_2\right)+\text{\&c.}\\
h' &= N'\sin\left(gt+x\right)+N_1'\sin\left(g_1t+x_1\right)+N_2'\sin\left(g_2t+x_2\right)+\text{\&c.}\\
&\qquad\qquad \text{\&c.} \qquad \text{\&c.}\\
l &= N\cos\left(gt+x\right)+N_1\cos\left(g_1t+x_1\right)+N_2\cos\left(g_2t+x_2\right)+\text{\&c.}\\
l' &= N'\cos\left(gt+x\right)+N_1'\cos\left(g_1t+x_1\right)+N_2'\cos\left(g_2t+x_2\right)+\text{\&c.}\\
&\qquad\qquad \text{\&c.} \qquad \text{\&c.}
\end{aligned} \qquad \textbf{(133)}$$

for each term contains two arbitrary quantities N, x; N_1, x_1, &c.

484. Since each term of the equations (132) has one of the quantities N, N', &c., for coefficient, these equations will only give values of the ratios

$$\frac{N'}{N};\ \frac{N''}{N'};\ \text{\&c.},$$

so that for each of the roots by g, g_1, g_2, &c., one of the quantities N, N_1, N_2, &c., will remain indeterminate.

To show how these are determined, it must be observed that in the expression

$$\boxed{0.1} = -\frac{3m'.an\left(aa'S+\left(a^2+a'^2\right)S'\right)}{2\left(a'^2-a^2\right)^2}$$

of article 474, S and S' are the coefficients of the first and second terms of the development of[23]

$$\left(a^2-2aa'\cos b+a'^2\right)^{\frac{1}{2}},$$

which remain the same when a' is put for a; and the contrary, that is to say, whether the action of m' on m be considered, or that of m on m'. Hence if m, n', and a', be put for m', n, and a,

$$\boxed{0.1} = -\frac{3m.a'n'\left(aa'S + \left(a'^2 + a^2\right)S'\right)}{2\left(a'^2 - a^2\right)^2}$$

consequently[24]

$$\boxed{0.1}\ .m.n'a' = \boxed{1.0}\ .m'.na\,.$$

It is also evident that

$$(0.1)\,m.n'a' = (1.0)\,m'.na\,.$$

But if the mass of the planet be omitted in comparison of that of the sun considered as the unit,

$$n^2 = \frac{1}{a^3};\ n'^2 = \frac{1}{a'^3},\ \&c.;$$

therefore

$$\boxed{0.1}\ m\sqrt{a} - \boxed{1.0}\ m'\sqrt{a'} = 0\,,$$
$$\boxed{0.2}\ m\sqrt{a} - \boxed{2.0}\ m''\sqrt{a''} = 0\,,$$
$$\&c. \qquad \&c.$$
$$(0.1)\,m\sqrt{a} - (1.0)\,m'\sqrt{a'} = 0\,,$$
$$(0.2)\,m\sqrt{a} - (2.0)\,m''\sqrt{a''} = 0\,,$$
$$\&c. \qquad \&c.$$

485. Now let those of equations (131) that give

$$\frac{dh}{dt},\ \frac{dh'}{dt},\ \&c.,$$

be respectively multiplied by

$$Nm\sqrt{a},\ N'm'\sqrt{a'},\ N''m''\sqrt{a''},\ \&c.;$$

then, in consequence of equations (132), and the preceding relations, it will be found that

$$N\frac{dh}{dt}m\sqrt{a} + N'\frac{dh'}{dt}m'\sqrt{a'} + N''\frac{dh''}{dt}m''\sqrt{a''} + \&c.$$
$$= g\left\{Nlm\sqrt{a} + N'l'm'\sqrt{a'} + N''l''m''\sqrt{a''} + \&c.\right\};$$

if the preceding values of h, h', h'', &c., l, l', &c., be put in this, a comparison of the coefficients of like cosines gives[25]

$$0 = NN_1m\sqrt{a} + N'N_1'm'\sqrt{a'} + N''N_1''m''\sqrt{a''} + \&c.$$
$$0 = NN_2m\sqrt{a} + N'N_2'm'\sqrt{a'} + N''N_2''m''\sqrt{a''} + \&c.$$

Again, if the values of $h,\ h',\ h''$, &c., in equations (133) be respectively multiplied by

$$Nm\sqrt{a},\ N'm'\sqrt{a'},\ \text{\&c.}$$

They give

$$\begin{aligned}&Nmh\sqrt{a}+N'm'h'\sqrt{a'}+N''m''h''\sqrt{a''}+\text{\&c.}=\\&\left\{N^2m\sqrt{a}+N'^2m'\sqrt{a'}+N''^2m''\sqrt{a''}+\text{\&c.}\right\}\sin\left(gt+x\right),\end{aligned} \qquad \textbf{(134)}$$

in consequence of the preceding relations.

By the same analysis the values of $l,\ l',\ l''$, &c., give

$$\begin{aligned}&Nml\sqrt{a}+N'm'l'\sqrt{a'}+N''m''l''\sqrt{a''}+\text{\&c.}=\\&\left\{N^2m\sqrt{a}+N'^2m'\sqrt{a'}+N''^2m''\sqrt{a''}+\text{\&c.}\right\}\cos\left(gt+x\right).\end{aligned}$$

The eccentricities of the orbits of the planets, and the longitudes of their perihelia, are known by observation at the epoch, and if these be represented by $\bar{e},\ \bar{e}'$, &c., $\bar{\varpi},\ \bar{\varpi}'$, &c. by article 481,

$$\begin{aligned}h&=\bar{e}\sin\bar{\varpi},\ h'=\bar{e}'\sin\bar{\varpi}',\ \text{\&c.},\\ l&=\bar{e}\cos\bar{\varpi},\ l'=\bar{e}'\cos\bar{\varpi}',\ \text{\&c.};\end{aligned}$$

therefore $h,\ h'$, &c., $l,\ l'$, &c., are given at that period. And if it be taken as the origin of the time $t=0$, and the preceding equations give[26]

$$\tan x=\frac{N.\bar{e}\sin\bar{\varpi}.m\sqrt{a}+N'.\bar{e}'\sin\bar{\varpi}'.m'\sqrt{a'}+\text{\&c.}}{N.\bar{e}\cos\bar{\varpi}.m\sqrt{a}+N'.\bar{e}'\cos\bar{\varpi}'.m'\sqrt{a'}+\text{\&c.}}.$$

But, for the root g, the equations (132) give

$$N'=CN,\ N''=C'N,\ N'''=C''N,\ \text{\&c.},$$

$C,\ C',\ C''$ being constant and given quantities; therefore

$$\tan x=\frac{\bar{e}\sin\bar{\varpi}.m\sqrt{a}+C.\bar{e}'\sin\bar{\varpi}'.m'\sqrt{a'}+\text{\&c.}}{\bar{e}\cos\bar{\varpi}.m\sqrt{a}+C.\bar{e}'\cos\bar{\varpi}'.m'\sqrt{a'}+\text{\&c.}}.$$

If these values of $N',\ N''$, &c., be eliminated from equation (134), it gives

$$N = \frac{\bar{e}\sin\bar{v}\, m\sqrt{a} + C\bar{e}'\sin\bar{v}'m'\sqrt{a'} + \&c.}{\left\{m\sqrt{a} + C^2 m'\sqrt{a'} + C'^2 m''\sqrt{a''} + \&c.\right\}\sin x}.$$

Thus $\tan x$ and N are determined, all the remaining coefficients N', N'', &c., may be computed from equations (132), for the root g.

In this manner the indeterminate quantities belonging to the other roots g_1, g_2, &c., may be found. Thus the equations (133) are completely determined, whence the eccentricities of the orbits and the longitudes of their perihelia may be found for any instant $\pm t$, before or after the epoch.

486. The roots g, g_1, g_2, &c., express the mean secular motions of the perihelia, in the same manner that n represents the mean motion of a planet.

For example, the periodic time of the earth is about $365\frac{1}{4}$ days; hence $n = \frac{360^0}{365\frac{1}{4}}$, which is the mean motion of the earth for a day, and nt is its mean motion for any time t. The perihelion of the terrestrial orbit moves through 360^0 in 113,270 years nearly; hence, for the earth,[27]

$$g = \frac{360^0}{113,270} = 11''.44$$

in a century; and gt is the mean motion for any time t so that $nt + \in$ being the mean longitude of a planet, $gt + x$ is the mean longitude of its perihelion at any given time.

487. The equations (133), as well as observation, concur in proving that the perihelia have a motion in space, and that the eccentricities vary slowly. As, however, that variation might in process of time alter the nature of the orbits so much as to destroy the stability of the system, it is of the greatest importance to inquire whether these variations are unlimited, or if limited, what their extent is.

Stability of the Solar System with regard to the Form of the Orbits

488. Because

$$h = e\sin v\,,\quad l = e\cos v\,,\quad e^2 = h^2 + l^2;$$

and in consequence of the values of h and l in equations (133), the square of the eccentricity of the orbit of m becomes[28]

$$\begin{aligned} e^2 = N^2 + N_1^2 + N_2^2 + \&c. + 2NN_1\cos\left\{(g_1 - g)t + x_1 - x\right\} \\ +2NN_2\cos\left\{(g_2 - g)t + x_2 - x\right\} + \&c. \end{aligned} \qquad \textbf{(135)}$$

When the roots $g,\ g_1$, &c., are all real and unequal, the cosines in this expression will oscillate between fixed limits, and e^2 will always be less than[29]

$$\left(N+N_1+N_2+\&\text{c.}\right)^2=N^2+N_1^2+\&\text{c.}+2NN_1+2NN_2+\&\text{c.}$$

taken with the same sign, for it could only obtain that maximum if

$$\left(g_1-g\right)t+x_1-x=0,\ \left(g_2-g\right)t+x_2-x=0,+\&\text{c.}$$

which could never happen unless the time were to vanish; that is, unless

$$g_1-g=0,\ g_2-g=0,\ \&\text{c.};$$

thus, if $g,\ g_1,\ g_2$, &c., be real and unequal, the value of e^2 will be limited.

489. If however any of these roots be imaginary or equal, they will introduce circular arcs or exponentials into the values of $h,\ h'$, &c., $l,\ l'$, &c.; and as these quantities would then increase indefinitely with the time, the eccentricities would no longer be confined to fixed limits, but would increase till the orbits of the planets, which are now nearly circular, become very eccentric.

The stability of the system therefore depends on the nature of the roots $g,\ g_1,\ g_2$, &c.: however it is easy to prove that they will all be real and unequal, if all the bodies $m,\ m',\ m''$, &c., in the system revolve in the same direction.

490. For that purpose let the equations

$$\frac{de}{dt}=\boxed{0.1}\,e'\sin\left(v'-v\right)+\boxed{0.2}\,e''\sin\left(v''-v'\right)+\&\text{c.}$$
$$\frac{de'}{dt}=\boxed{1.0}\,e\sin\left(v-v'\right)+\boxed{1.2}\,e''\sin\left(v'-v''\right)+\&\text{c.}$$
$$\&\text{c.}\qquad\&\text{c.}$$

be respectively multiplied by

$$me\sqrt{a},\ m'e'\sqrt{a'},\ m''e''\sqrt{a''},\ \&\text{c.},$$

and added; then in consequence of the relations in article 484, and because

$$\sin\left(v-v'\right)=-\sin\left(v'-v\right)$$
$$\sin\left(v-v''\right)=-\sin\left(v''-v\right),\ \&c.\ \&c.,$$

the sum will be

$$0=ede\,.\,m\sqrt{a}+e'de'.m'\sqrt{a'}+e''de''\,.\,m''\sqrt{a''}+\&\text{c.}$$

and as the greater axes of the orbits are constant, its integral is

$$e^2 m\sqrt{a} + e'^2 m'\sqrt{a'} + e''^2 m''\sqrt{a''} + \&\text{c.} = C\,. \tag{136}$$

491. The radicals $\sqrt{a}$, $\sqrt{a'}$, &c., must all have the same sign if the planets revolve in the same direction; since by Kepler's law they depend on the periodic times; and in analysis motions in one direction have a different sign from those in a contrary direction: but as all the planets and satellites revolve from west to east, the radicals, and consequently all the terms of the preceding equations must have positive signs; therefore each term is less than the constant quantity C.

But observation shows that the orbits of the planets and satellites are nearly circular, hence each of the quantities

$$e^2 m\sqrt{a},\ e'^2 m'\sqrt{a'},\ \&\text{c.}$$

is very small; and C being a very small constant quantity given by observation, the first number of equation (136) is very small.

As C never could have changed since the system was constituted as it now is, so it never can change while the system remains the same; therefore equation (136) cannot contain any quantity that increases indefinitely with the time; so that none of the roots $g,\ g_1,\ g_2$, &c., are either equal or imaginary.

492. Since the greater axes and masses are invariable, and the eccentricities are perpetually changing, they have the singular property of compensating each other's variation, so that the sum of their squares, respectively multiplied by the coefficients $m\sqrt{a},\ m'\sqrt{a'}$, &c., remains constant and very small.

493. To remove all doubts on a point so important, suppose some of the roots, $g,\ g_1,\ g_2$, &c., to be imaginary, then some of the cosines or sines will be changed into exponentials; and, by article 215, the general value of h in (133) would contain the term Cc^{at}, c being the number whose hyperbolic logarithm is unity. If Dc^{at}, $C'c^{at}$, $D'c^{at}$, &c., be the corresponding terms introduced by these imaginary roots in $h,\ h',\ l'$, &c., the e^2 would contain a term $(C^2+D^2)c^{2at}$, e'^2 would contain $(C'^2+D'^2)c^{2at}$, and so on; hence the first number of equation (136) would contain [30]

$$c^{2at}\left\{m\sqrt{a}\left(C^2+D^2\right)+m'\sqrt{a'}\left(C'^2+D'^2\right)+\&\text{c.}\right\},$$

a quantity that increases indefinitely with the time.

If c^{at} be the greatest exponential that in $h,\ l,\ h',\ l'$, &c., contain, c^{2at} will be the greatest in the first member of equation (136); therefore the preceding term cannot be destroyed by any other term in that equation. In order, therefore, that its first member may be reduced to a constant quantity, the coefficient of c^{2at} must itself be zero; hence

$$m\sqrt{a}\left(C^2+D^2\right)+m'\sqrt{a'}\left(C'^2+D'^2\right)+\&\text{c.}=0\,.$$

But if the radicals $\sqrt{a}$, $\sqrt{a'}$, &c., have the same sign, that is, if all the bodies m, m', &c., move in the same direction, this coefficient can only be zero when each of the quantities C, D, C', D', &c., is zero separately; thus, h, l, h', l', &c., do not contain exponentials, and therefore the roots of [31] g, g_1, &c., are all real. If the roots g and g_1 be equal, then the preceding integral becomes[32]

$$h=(b+b')c^{at}=(b+b')\left(1+\frac{at}{2}+\frac{a^2t^2}{1.2}+\&\text{c.}\right).$$

Thus the general value of h will contain a finite number of terms of the form Ct^r, which increases indefinitely with the time; the same roots would introduce the terms Dt^r, $C't^r$, $D't^r$, &c., in the general value of l, h', l', &c.; therefore the first member of equation (136) would contain the term

$$t^2\left\{m\sqrt{a}\left(C^2+D^2\right)+m'\sqrt{a'}\left(C'^2+D'^2\right)\right\}+\&\text{c.}=0\,;$$

and if t^r be the highest power of t in h, l, h', l', &c.; t^{2r} will be the highest power of t in equation (136); consequently its first member can only be constant when

$$m\sqrt{a}\left(C^2+D^2\right)+m'\sqrt{a'}\left(C'^2+D'^2\right)+\&\text{c.}=0\,,$$

which cannot happen when all the planets revolve in the same direction, unless

$$C=0,\ D=0,\ C'=0,\ D'=0,\ \&\text{c.}$$

Thus, h, l, h', l', &c., neither contain exponentials nor circular arcs, when the bodies of the solar system revolve in the same direction, and as they really do so, the roots g, g_1, g_2, &c., are all real and unequal.

494. Because the equation (135) does not contain any quantity that increases with the time, on account of the roots[33] g, g_1, &c., being real and unequal, and that the eccentricities themselves and their variations are extremely small, the eccentricities increase and decrease with the cosines, between fixed but very narrow limits, in immense periods: for, considering only the mutual disturbances of Jupiter and Saturn, the eccentricities of their orbits would take no less than 70,414 years to accomplish their changes; but if more than two planets be taken, and compound periods established, they would evidently extend to millions of years.

495. The positions of the perihelia now remain to be considered.

$$e\sin v = h,\ e\cos v = l \text{ give } \tan v = \frac{h}{l},$$

and substituting the values of h and l in article 483,[34]

$$\tan v = \frac{N\sin(gt+x) + N_1\sin(g_1 t + x_1) + \text{\&c.}}{N\cos(gt+x) + N_1\cos(g_1 t + x_1) + \text{\&c.}};$$

or, if $gt+x$ be subtracted from v,

$$\tan(v - gt - x) = \frac{\tan v - \tan(gt+x)}{1+\tan v\ \tan(gt+x)};$$

and when substitution is made for $\tan v$,

$$\tan(v - gt - x) = \frac{N'\sin\{(g_1-g)t + x_1 - x\} + N_2\sin\{(g_2-g)t+x_2-x\}}{N + N_1\cos\{(g_1-g)t+x_1-x\} + N_2\cos\{(g_2-g)t+x_2-x\} + \text{\&c.}}.$$

This tangent never can be infinite, if the sum $N + N_1 + N_2 + \text{\&c.}$, of the coefficients in the denominator be less than N with a positive sign; for in this case the denominator never can be zero; so that the angle $v - gt - x$ never can attain to a quadrant, but will oscillate between $+90^\circ$ and -90°; hence the true motion of the perihelion is $gt+x$.

From this equation it appears that the motions of the perihelia are not uniform, and that they may experience variations in the course of ages, to which no limits can be assigned, though observation shows that the variations are very slow.

496. Because the equations which give the secular variations in the eccentricities and longitudes of the perihelia do not contain the mean longitudes nor the inclinations of the orbits, they are independent of the configuration of the planets, and would be the same if all the bodies revolved in one plane, at least when the approximation does not extend to the higher powers of the eccentricities, inclinations, or masses. These secular inequalities depend on the angular distances of the perihelia of all the planets taken two and two, that is, on the configuration of the orbits.

497. It may be concluded from the preceding analysis, that when periodic inequalities are omitted, the mean motions of the planets are uniform; and that the system is stable with regard to the species of the orbits, which, retaining the greater axis invariable, deviate but little from the circular form; the eccentricities being subject to the condition expressed by equations (136)–that the sum of their squares, multiplied by the masses of the bodies, and the square roots of the greater axes of their orbits is invariably the same. The perihelia alone are subject to unlimited variations.

Secular Variations in the Inclinations of the Orbits and Longitudes of their Nodes

498. In order to determine the secular inequalities in the inclinations of the orbits and longitudes of the nodes, let the equations in article 474 be resumed

$$\frac{dp}{dt}=(0.1)(q'-q)$$

and

$$\frac{dq}{dt}=-(0.1)(p'-p),$$

which express the variations in the position of the orbit of m, when troubled by m' alone. But as all the bodies in the system act simultaneously on m, each of them will produce a variation in the inclination of its orbit, and in the longitude of its nodes, similar to those caused by the action of m'; hence

$$\frac{dp}{dt}=(0.1)(q'-q)+(0.2)(q''-q)+\&\text{c.}$$
$$\frac{dq}{dt}=-(0.1)(p'-p)-(0.2)(p''-p)-\&\text{c.}$$

will express the whole action of the system on the position of the orbit of m. Similar equations must exist for every body in the system: there will consequently be the following series of equations,

$$\begin{aligned}\frac{dp}{dt}&=-\{(0.1)+(0.2)+\&\text{c.}\}q+(0.1)q'+(0.2)q''+\&c.\\ \frac{dq}{dt}&=\{(0.1)+(0.2)+\&\text{c.}\}p-(0.1)p'-(0.2)p''-\&c.\\ \frac{dp'}{dt}&=-\{(0.1)+(1.2)+\&\text{c.}\}q'+(1.0)q+(1.2)q''+\&c.\\ \frac{dq'}{dt}&=\{(1.0)+(1.2)+\&\text{c.}\}p'-(1.0)p-(1.2)p''-\&c.\\ &\quad\&\text{c.}\qquad\&\text{c.}\end{aligned}\qquad \textbf{(137)}$$

These equations are perfectly similar to those in article 481, and may be integrated on the same principle; whence

$$\begin{aligned}p&=N\sin(gt+x)+N_1\sin(g_1t+x_1)+\&\text{c.}\\ q&=N\cos(gt+x)+N_1\cos(g_1t+x_1)+\&\text{c.}\\ p'&=N'\sin(gt+x)+N_1'\sin(g_1t+x_1)+\&\text{c.}\\ q'&=N'\cos(gt+x)+N_1'\cos(g_1t+x_1)+\&\text{c.}\end{aligned}\qquad \textbf{(138)}$$

Stability of the Solar System with regard to the Inclination of the Orbits

499. The equation in g resulting from these, $g,\ g_1,\ g_2$, &c., for its roots, and the constant quantities $N,\ N_1$, &c. and $x,\ x_1$, &c. are determined. in a similar manner to what was employed for the eccentricities. For since $\bar{f},\ \bar{q}$, &c. are the values of $f,\ q$, &c. when $t=0$,[35]

$$\begin{array}{ll} p\ = \tan \bar{f} \sin \bar{q} & q = \tan \bar{f} \cos \bar{q}, \\ p' = \tan \bar{f}' \sin \bar{q}' & q' = \tan \bar{f}' \cos \bar{q}', \\ \quad\&c. & \quad\&c. \end{array}$$

hence, if all the inclinations of the orbits of the planets, and the longitudes of their nodes be known by observation at any given epoch, when $t=0$, there will be a sufficient number of equations to determine all the quantities $N,\ N_1$, &c. and $x,\ x_1$, &c.

500. Also the roots $g,\ g_1$, &c., are real and unequal, for if the equations (137) be respectively multiplied by[36]

$$m\sqrt{a}\,.\,p;\quad m\sqrt{a}\,.\,q;\quad m'\sqrt{a'}\,.\,p';\quad m'\sqrt{a'}\,.\,q';\ \&c.$$

and added, the integral of their sum will be

$$\left(p^2+q^2\right)m\sqrt{a}+\left(p'^2+q'^2\right)m'\sqrt{a'}+\&c.=C \qquad \textbf{(139)}$$

in consequence of the relations

$$\begin{array}{c} (0.1)\,m\sqrt{a}=(1.0)\,m'\sqrt{a'}, \\ (0.2)\,m\sqrt{a}=(2.0)\,m''\sqrt{a''}, \\ \&c. \qquad \&c. \end{array}$$

Whence we may be assured by the same reasoning employed with regard to the eccentricities, that this equation neither contains arcs of circles nor exponentials, when the bodies all revolve in the same direction, so that all the roots are real and unequal.

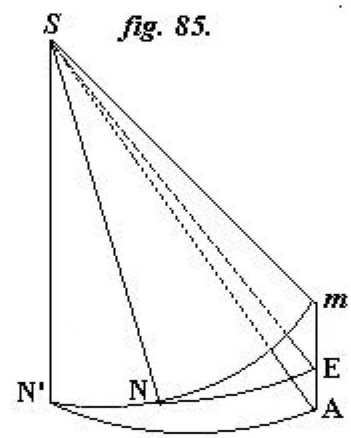
501. Now $\tan f=\sqrt{p^2+q^2}$, and if the values of p and q be substituted[37]

$$\tan f=\sqrt{p^2+q^2}=$$
$$\sqrt{\left\{N^2+N_1{}^2+\&c.+2NN_1\cos\left\{(g_1-g)t+x_1-x\right\}+2NN_2\cos\left\{(g_2-g)t+x_2-x\right\}+\&c.\right\}}.$$

The expression $\sqrt{p^2+q^2}$ is less than $N+N_1+N_2+$&c., on account of these coefficients being multiplied by cosines which diminish their values. The maximum of $\tan f$ would be $N+N_1+$&c., which it never can attain, since the differences of the roots $g_1-g,\ g_2-g$ are never zero; and as the inclinations of the orbits of the planets on the plane of the ecliptic are very small, the coefficients $N,\ N_1$, &c., which depend on the inclinations, are very small also, and will always remain so. And the inclinations of the orbits will oscillate between very narrow limits in periods depending on the roots $g,\ g_1$, &c.

502. The plane of the ecliptic in which the earth moves, changes its position in space from the action of the planets, each producing a retrograde motion in the intersection of the plane of the ecliptic, and that of its own orbit; whence it appears, that if EN be the orbit of the earth at a given epoch, AN′ will be its position at a subsequent period, and so on. The secular inequality in the position or the terrestrial orbit changes the obliquity of the ecliptic; but as it is determined from equations (138) it oscillates between narrow limits, never exceeding 3°, therefore the equator never has coincided, and never will coincide with the ecliptic, supposing the system constituted as it is at present, so that there never was, and there never will be eternal spring.

fig. 85.



503. Since $p^2+q^2=\tan^2 f,\ p'^2+q'^2=\tan^2 f'$, equation (139) becomes

$$m\sqrt{a}\tan^2 f+m'\sqrt{a'}\tan^2 f'+\text{\&c.}=C. \qquad \textbf{(140)}$$

Whence it may be concluded that the sum of the masses of all the bodies in the system multiplied by the square roots of half the greater axes of their orbits, and by the squares of the tangents of their inclinations on a fixed plane, will always be the same. If this sum be very small at any one period, and if all the radicals have the same sign, that is, if all the bodies revolve in the same direction, it will always remain so; and as in nature, the inclinations of all the orbits on the plane of the ecliptic are very small, and the bodies revolve in the same direction, the variations of the inclinations compensate each other, so that this expression will remain for ever constant, and very small.

504. Other two integrals may be obtained from the equations (137). For if the first be multiplied by $m\sqrt{a}$, the third by $m'\sqrt{a'}$, the fifth by $m''\sqrt{a''}$, &c., &c. their sum will be

$$m\sqrt{a}\frac{dp}{dt}+m'\sqrt{a'}\frac{dp'}{dt}+m''\sqrt{a''}\frac{dp''}{dt}+\text{\&c.}=0,$$

in consequence of the relations in article 484, the integral of which is

$$m\sqrt{a}\,.\,p+m'\sqrt{a'}\,.\,p'+m''\sqrt{a''}\,.\,p''+\text{\&c.}=\text{constant}.$$

In a similar manner the differential equations in $q,\ q',$ give

$$m\sqrt{a}\,.\,q + m'\sqrt{a'}\,.\,q' + m''\sqrt{a''}\,.\,q'' + \&\text{c.} = \text{constant}\,.$$

505. With regard to the nodes $\tan q = \dfrac{p}{q}$, and substituting for *p* and *q*,

$$\tan q = \frac{N\sin(gt+x) + N_1\sin(g_1t+x_1) + \&\text{c.}}{N\cos(gt+x) + N_1\cos(g_1t+x_1) + \&\text{c.}};$$

or subtracting $gt+x$ from q,

$$\tan(q-gt-x) = \frac{N_1\sin\{(g_1-g)t+x_1-x\} + N_2\sin\{(g_2-g)t+x_2-x\} + \&\text{c.}}{N + N_1\cos\{(g_1-g)t+x_1-x\} + N_2\cos\{(g_2-g)t+x_2-x\} + \&\text{c.}}$$

If the sum of the coefficients $N + N_1 + N_2 + \&\text{c.}$ of the cosines in the denominator taken positively be less than N, $\tan(q-gt-x)$ never can be infinite; hence the angle $q-gt-x$ will oscillate between $+90^\circ$ and -90°, so that $gt+x$ is the true motion of the nodes of the orbit of *m*, and $g = \dfrac{360^\circ}{\text{period of }\Omega\text{ of }m}$. As in general the periods of the motions of the nodes are great, the inequalities increase very slowly. From these equations it may be seen, that the motion of the nodes is indefinite and variable.

The method of computing the constant quantities will be given in the theory of Jupiter, whence the laws, periods, and limits of the secular variations in the elements of his orbit, will be determined.

506. The equations which give[38] $p,\ q,\ p',\ \in$ may be expressed by a diagram. Let A*n* be the orbit of the planet *m* at any assigned time, as the beginning of January, 1750, which is the epoch of many of the French tables. After a certain time, the action of the disturbing body m' alone on the planet *m*, changes the inclination of its orbit, and brings it to the position B*n*. But m'' acting simultaneously with m' brings the orbit into the position Cn: m'' acting along with the preceding bodies changes it to Dn'', and so on. It is evident that the last orbit will be that in which *m* moves. So the whole inclination of the orbit of *m* on the plane A*n*, after a certain time, will be the sum of the finite and simultaneous changes. Hence if N be the inclination of the circle B*n* on the fixed plane A*n*, and $g\,Sn = gt+x$ the longitude of its ascending node; N′ the inclination of the circle Cn' on B*n*, and

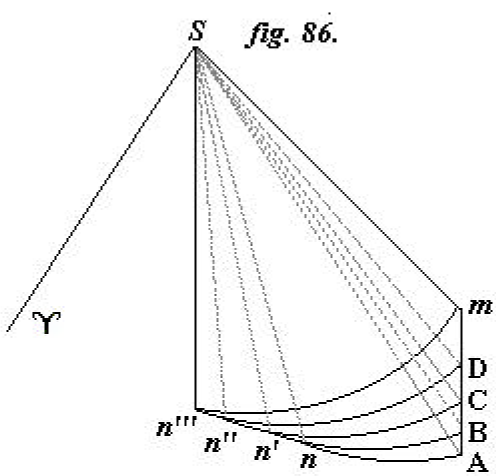


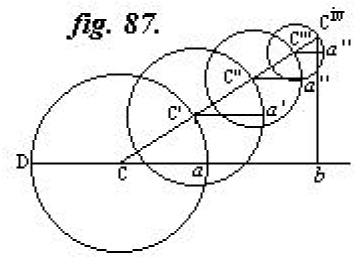
fig. 86.

$g\,Sn' = g't + x'$ the longitude of the node n'; N'' the inclination of the circle Dn'' on Cn', and $g\,Sn'' = g_2 t + x_2$ the longitude of the node n''; and so on for each disturbing body, the last circle will be the orbit of m.

507. Applying the same construction to h and l (133), it will be found that the tangent of the inclination of the last circle on the fixed plane is equal to the eccentricity of the orbit of m; and that the longitude of the intersection of this circle with the same plane is equal to that of the perihelion of the orbit of m.

508. The values of p and q in equations (138) may be determined by another construction; for let C, fig. 87,[39] be the centre of a circle whose radius is N; draw any diameter Da, and take the arc

fig. 87.



$$a\text{C}' = gt + x\,;$$

on C' as a centre with radius equal to[40] N_1, describe a circle, and having drawn $\text{C}'a'$ parallel to Ca, take[41] $a'\text{C}' = g_1 t + x_1$: on C'' as it centre with radius equal to N_2, describe a circle, and having drawn $\text{C}''a''$ parallel to Ca, take the $a''\text{C}'' = g_2 t + x_2$, and so on. Let $a^{\text{iv}}C^{\text{iv}}$ be the arc in the last circle, then if $\text{C}^{\text{iv}}b$ be perpendicular to Ca produced, it is evident that

$$\text{C}^{\text{iv}}b = p, \quad \text{C}b = q,$$

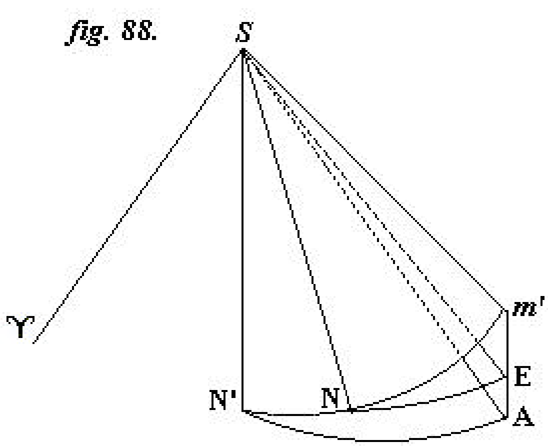
and if CC^{iv} be joined,

$$\tan f = \sqrt{p^2 + q^2}, \quad \tan q = \frac{p}{q},$$

q being the angle $\text{C}^{\text{iv}}\text{C}b$.

509. The equations which determine the secular variations in the inclinations and motions of the nodes being independent of the eccentricities, are the same as if the orbits of the planets were circular.

fig. 88.



Annual and Sidereal Variations in the Elements of the Orbits, with regard to the variable Plane of the Ecliptic

510. Equations (128) give the annual variations in the inclinations and longitudes of the nodes with regard to a fixed plane, but astronomers refer the celestial motions to the movable orbit of the earth whence observations are made; its motion occasioned by the action of the planets is indeed extremely minute, but it is important to know the secular variations in the position of the orbits with regard to it. Suppose AN fig. 88, to be the plane of the ecliptic or orbit of the earth, EN the variable plane of the ecliptic in which the earth is moving at a subsequent period, and $m'\mathrm{N}'$ the orbit of a planet m', whose position with regard to EN is to be determined.

By article 444,

$$\mathrm{EA} = q\sin(n't+\epsilon') - p\cos(n't+\epsilon')$$

is the latitude of *m* above AN; and the latitude of m' above AN' is

$$\mathrm{A}m' = q'\sin(n't+\epsilon') - p'\cos(n't+\epsilon').$$

As the inclinations are supposed to be very small, the difference of these two, or $m'\mathrm{A}-\mathrm{EA}$ is very nearly equal to $m'\mathrm{E}$ the latitude of m' above the variable plane of the ecliptic EN.

If $f`$ be the inclination of $m'\mathrm{N}'$ the orbit of $m'$ to EN the variable ecliptic, and $q`$ the longitude of its ascending node, then will

$$\tan f`.\sin q` = p'-p; \quad \tan f`.\cos q` = q'-q.$$

Whence[42]

$$\tan f` = \sqrt{(p'-p)^2+(q'-q)^2}; \quad \tan q` = \frac{p'-p}{q'-q}.$$

If EN be assumed to be the fixed plane at a given epoch, then $p=0$, $q=0$, but neither *dp* nor *dq* are zero; hence[43]

$$df` = (dp'-dp).\sin q' + (dq'-dq).\cos q',$$

$$dq` = \frac{(dp'-dp).\cos q' - (dq'-dq).\sin q'}{\tan f'},$$

and substituting the values in article 498 in place of the differentials *dp*, *dq*, &c. there will result

$$\frac{df`}{dt} = \{(1.2)-(0.2)\}\tan f''\sin(q'-q'') + \{(1.3)-(0.3)\}\times\tan f'''\sin(q'-q''') + \&c.$$

$$\begin{aligned}\frac{dq`}{dt} = &-\{(1.0)+(1.2)+(1.3)+\&c.\}-(0.1)\\ &+\{(1.2)-(0.2)\}\cdot\frac{\tan f''}{\tan f'}\cdot\cos(q'-q'')\\ &+\{(1.3)-(0.3)\}\ \frac{\tan f'''}{\tan f'}\ \cos(q'-q''')\\ &+\&c.\end{aligned} \qquad \textbf{(141)}$$

Motion of the Orbits of two Planets

511. Imagine two planets m and m' revolving round the sun so remotely from the rest of the system, that they are not sensibly disturbed by the other bodies.

Let $g=\sqrt{(p'-p)^2+(q'-q)^2}$ be the mutual inclination of the two orbits supposed to be very small. If the orbit of m at the epoch be assumed as the fixed plane

$$f=0,\ \ g=f',\ \ p=0,\ \ q=0,$$

and

$$\tan^2 f'=\tan^2 g=p'^2+q'^2 .$$

In this case, equations (140) and (128) become

$$m'\sqrt{a'}\tan^2 f'=C,\quad \frac{dq'}{dt}=-(0.1).$$

Since the greater axes of the orbits are constant, the first shows that the inclination is constant, and the second proves the motion of the node of the orbit of m' on that of m to be uniform and retrograde, and the motion of the intersection of the two orbits on the orbit of m, in consequence of their mutual attraction, will be $-(0.1)t$.

Secular Variations in the Longitude of the Epoch

512. The mean place of a planet in its orbit at a given instant, assumed to be the origin of the time, is the longitude of the epoch. It is one of the most important elements of the planetary orbits, being the origin whence the antecedent and subsequent longitudes are estimated. If the mean place of the planet at the origin of the time should vary from the action of the disturbing forces, the longitudes estimated from that point would be affected by it; to ascertain the secular inequalities of that element is therefore of the greatest consequence.

The differential equation of the longitude of the epoch in article 441, is

$$d\in=\frac{an\sqrt{1-e^2}}{e}\cdot\left(1-\sqrt{1-e^2}\right)\cdot\frac{dF}{de}dt-2a^2n\frac{dF}{da}dt .$$

By article 473,

$$\frac{dF}{da}=-\frac{m'}{2}\cdot\left(\frac{3a'S'+2aS}{\left(a'^2-a^2\right)^2}\right)$$

$$-\frac{m'}{4}\cdot\frac{a'^2}{a}\cdot\left(\frac{3S'\left(2a'^2-3a^2\right)+6a\,a'S}{\left(a'^2-a^2\right)^3}\right)ee'\cos\left(v'-v\right)$$

$$+\frac{m'}{2.4}aa'\left(\frac{6a'S-3aS'}{\left(a'^2-a^2\right)^3}\right)\left\{e^2+e'^2-\left(p'-p\right)^2-\left(q'-q\right)^2\right\}$$

$$\frac{dF}{de}=-\frac{3m'a\,a'.S'}{4\left(a'^2-a^2\right)^2}\cdot e$$

$$+\frac{3m'}{2\left(a'^2-a^2\right)^2}\left\{\left(a'^2+a^2\right)S'+a\,a'S\right\}e'\cos\left(v'-v\right).$$

If these be put in the value of $d\in$, rejecting the powers of e above the second, and if to abridge

$$C=\frac{m'.na^2.\left(2aS+3a'S'\right)}{\left(a'^2-a^2\right)^2},$$

$$C_1=-\frac{3m'.na^2a'\left(4a\,a'S-\left(3a^2-a'^2\right)S'\right)}{2.4.\left(a'^2-a^2\right)^3},$$

$$C_2=-\frac{3m'.na.\left\{\left(a^2-5a'^2\right)a\,a'S+\left(a^4+6a^2a'^2-5a'^4\right)S'\right\}}{4.\left(a'^2-a^2\right)^3},$$

$$C_3=\frac{3m'.na^3a'\left(2a'S-aS'\right)}{4\left(a'^2-a^2\right)^3},$$

$d\in$ becomes

$$\frac{d\in}{dt}=C+C_1e^2+C_2ee'\cos\left(v'-v\right)$$
$$+C_3\left\{\left(p'-p\right)^2+\left(q'-q\right)^2-e'^2\right\}.$$

But

$$h=e\sin v \quad l=e\cos v,$$
$$h'=e'\sin v' \quad l'=e'\cos v';$$

hence

$$\frac{d\in}{dt}=C+C_1\left(h^2+l^2\right)+C_2\left(hh'+ll'\right)+C_3\left\{\left(p'-p\right)^2+\left(q'-q\right)^2-h'^2-l'^2\right\}.$$

513. This equation only expresses the variation in the epoch of m when troubled by m'; but, in order to have the effect of the whole system in disturbing the epoch of m, a similar set of terms must be added for each of the planets; but if the two planets m and m' alone be considered, their mutual inclination will be constant by article 511, hence $g^2 = (p' - p)^2 + (q' - q)^2 = M^2$, a constant quantity.

Again by article 483,[44]

$$h^2 + l^2 = N^2 + N_/^2 + 2NN_/ \cos\{(g_/ - g)t + x_/ - x\}$$
$$h'^2 + l'^2 = N'^2 + N_/'^2 + 2N'N_/' \cos\{(g_/ - g)t + x_/ - x\}$$
$$hh' + ll' = NN' + N_/N_/' + (NN_/' + N'N_/)\cos\{(g_/ - g)t + x_/ - x\}.$$

Substituting these in $d\epsilon$, and to abridge, making

$$A`n = C + C_/(N^2 + N_/^2) + C^2(NN' + N'N_/') + C_3(M^2 - N'^2 - N_/'^2),$$
$$B` = 2C_/NN' - 2C_3N'N_/' + C_2(NN_/' + N_/N'),$$

it becomes[45]

$$d\epsilon = A`.ndt + B`\cos\{(g_/ - g)t + x_/ - x\}dt.$$

The integral of which is

$$d\epsilon = A`nt + \frac{B`}{g_/ - g}\sin\{(g_/ - g)t + x_/ - x\}.$$

514. The term $A`nt$ only augments the mean primitive motion of the planet $m$ in the ratio of 1 to $1 + A'$, so that the mean motion which should result from observation would be $(1 + A')nt$, corresponding to the mean distance $\frac{a}{(1 + A`)^{\frac{2}{3}}}$.

Knowing this distance, which is given by a comparison of the periodic times, the primitive distance a may be determined; but as $A`$ is an infinitely small fraction of the order of the masses $m$ and $m'$, this correction in the mean distance is insensible. The term $A`nt$ may therefore be omitted, so that the secular variation in the epoch is[46]

$$d\epsilon = \frac{B`}{g_/ - g}\sin\{(g_/ - g)t + x' - x\}. \quad \textbf{(142)}$$

The variation in the epoch, like the other secular inequalities in article 480, may be expressed in series ascending according to the powers of the time; but as the term depending on its first power is insensible, it will have the form

$$d\epsilon = Ht^2 + \&c.$$

This inequality is insensible for the planets; its greatest effect is produced in the theory of Jupiter and Saturn: but even then it is only $d\epsilon = -0.0000006501''.t^2$ for Jupiter, and for Saturn $d\epsilon' = +0''.0000015114.t^2$, t being any number of Julian years from 1750. This inequality is not the 60th part of a sexagesimal second in a century, a quantity altogether insensible. Like all other inequalities it is periodic; but its period, which depends on $g_{/}-g$, is for Jupiter and Saturn no less than 70,414 years. The variation $d\in$, though of the order of disturbing forces, may, in the course of many centuries, become sensible, on account of the small divisor $g_{/}-g$ introduced by integration; but although it is insensible with regard to the planets, it is of much importance in the theories of the Moon and of Jupiter's Satellites.

Stability of the System, whatever may be the powers of the Disturbing Masses

515. The stability of the system has been proved with regard to the greater axes of the orbits, even when the approximation extends to the squares of the disturbing forces, and to all powers of the eccentricities and inclinations. Its invariability with regard to the other elements has only been proved on the hypothesis of the orbits being nearly circular, and very little inclined to each other and to the plane of the ecliptic; but as the same results may be derived from the general equations of the motion of a system of bodies, they equally exist whatever the eccentricities and inclinations may be, and when the approximation includes the squares of the disturbing forces, and they remain the same whatever changes the secular inequalities may introduce in the lapse of ages.

516. If the equations of the motion of a system of bodies in article 346 be resumed, and the equations in x, x', &c., multiplied respectively by

$$my-m\cdot\frac{\sum m\,.\,y}{S+\sum m};\;\; m'y'-m'\cdot\frac{\sum m\,.\,y}{S+\sum m};\;\; \text{\&c.}$$

and those in y, y', &c. by

$$-mx+m\cdot\frac{\sum m\,.\,x}{S+\sum m};\;\; -m'x'+m'\cdot\frac{\sum m\,.\,x}{S+\sum m};\;\; \text{\&c.}$$

their sum will be

$$\sum m\cdot\left(\frac{xd^2y-yd^2x}{dt^2}\right)+\frac{\sum .my}{S+\sum m}\cdot\sum m\cdot\frac{d^2x}{dt^2}-\frac{\sum .mx}{S+\sum m}\cdot\sum m\cdot\frac{d^2y}{dt^2};$$

for the nature of the function l gives

$$y\cdot\frac{d\,l}{dx}+y'\cdot\frac{d\,l}{dx'}+\text{\&c.}=0;\;\; -x\cdot\frac{d\,l}{dy}-x'\cdot\frac{d\,l}{dy'}-\text{\&c.}=0,$$

$$\frac{d\,l}{dx}+\frac{d\,l}{dx'}+\text{\&c.}=0;\;\; \frac{d\,l}{dy}+\frac{d\,l}{dy'}+\text{\&c.}=0;$$

as may be seen by trial. The integral of the preceding equation is

$$\sum m\cdot\left(\frac{xdy-ydx}{dt}\right)+\frac{\sum .my}{S+\sum m}\cdot\sum m\cdot\frac{dx}{dt}-\frac{\sum .mx}{S+\sum m}\cdot\sum m\cdot\frac{dy}{dt}=C\,.$$

A similar equation may be found in x, z, and y, z; and when $S+m'=1$, it will be found that[47]

$$\begin{aligned}
&\sum m\cdot\frac{ydx-xdy}{dt}+\sum mm'\left(\frac{xdy'-y'dx+x'dy-ydx'}{dt}\right)=C\\
&\sum m\cdot\frac{xdz-zdx}{dt}+\sum mm'\left(\frac{zdx'-x'dz+z'dx-xdz'}{dt}\right)=C'\\
&\sum m\cdot\frac{zdy-ydz}{dt}+\sum mm'\left(\frac{y'dz-zdy'+y'dz-zdy'}{dt}\right)=C''
\end{aligned} \qquad \textbf{(143)}$$

C, C', C'', being constant quantities. Now $\frac{ydx-xdy}{dt}$ is double the area described in the time dt by the projection of the radius vector of m on the plane xy. This area on the orbit is $\sqrt{a\left(1-e^2\right)}$; and if f be the inclination of the orbit on the plane xy, $\cos f\sqrt{a\left(1-e^2\right)}$ is its projection. In the same manner

$$\frac{y'dx'-x'dy'}{dt}=\cos f'\sqrt{a'\left(1-e'^2\right)}$$

is the area described by the projection of the radius vector of m' on the same plane, and so on. In consequence of these the first of the preceding equations becomes

$$m\sqrt{a\left(1-e^2\right)}\cos f+m'\sqrt{a'\left(1-e'^2\right)}\cos f'+\text{\&c.}=mm'\left(\frac{ydx'-x'dy+y'dx-xdy'}{dt}\right)+\text{\&c.}+C\,.$$

If the elliptical values of x, y, x', y', be substituted, the first term of the second member of this equation must always be periodic; for, in consequence of the observations in article 466, the arcs nt, $n't$, never destroy one another in the expressions ydx', $x'dy$, &c. Hence, if periodic quantities and those of the fourth be neglected, the last number of the equation is constant. If the products ydx', $x'dy$, &c., contained constant terms, they would be of the first order with regard to the masses; and as they are functions of the elliptical elements, their variation is of the second order; consequently, the variation of the terms $mm'.y'dx$, &c. is of the fourth order. If the periodic part of the values of the elliptical elements be substituted in the first member of the preceding equation, any terms resulting from that substitution that are not periodic will be of the third order, and may be regarded as constant. The second member of the equation in question may therefore be esteemed constant. Hence,

$$m\sqrt{a\left(1-e^2\right)}\cos f+m'\sqrt{a'\left(1-e'^2\right)}\cos f'+\text{\&c.}=\text{C} \qquad \textbf{(144)}$$

517. Again, $\frac{xdz - zdx}{dt}$ and $\frac{zdy - ydz}{dt}$ are the areas described by the radius vector of m in the time dt, projected on the co-ordinate planes xz, and yz. But it is easy to see by trigonometry that the cosines of the inclination of the orbit on these planes are $\sin f \cos q$, and $\sin f \sin q$; hence

$$\frac{xdz - zdx}{dt} = \sqrt{a\left(1-e^2\right)}\sin f \cos q \;,$$

and

$$\frac{zdy - ydz}{dt} = \sqrt{a\left(1-e^2\right)}\sin f \sin q \;.$$

Similar expressions exist for all the bodies; and as the same reasoning applies to the two last equations (143), as to the first, they give[48]

$$\begin{aligned} m\sqrt{a\left(1-e^2\right)}\sin f \cos q + m'\sqrt{a'\left(1-e'^2\right)}\sin f' \cos q' + \&\text{c.} &= C', \\ m\sqrt{a\left(1-e^2\right)}\sin f \sin q + m'\sqrt{a'\left(1-e'^2\right)}\sin f' \sin q' + \&\text{c.} &= C''. \end{aligned} \quad \textbf{(145)}$$

518. These relations exist whatever the eccentricities and inclinations may be, and whatever may be the changes that they undergo in the course of ages from their secular inequalities, the approximation extending to the third order inclusively, and even to the squares of the disturbing forces.

519. A variety of results may be derived from them. Because

$$\cos f = \frac{1}{\sqrt{1+\tan^2 f}} \;,$$

equation (144) gives

$$m\sqrt{\frac{a\left(1-e^2\right)}{1+\tan^2 f}} + m'\sqrt{\frac{a'\left(1-e'^2\right)}{1+\tan^2 f'}} + \&\text{c.} = C \;.$$

If e^4 and $e^2 f^2$ be omitted,

$$m\sqrt{\frac{a\left(1-e^2\right)}{1+\tan^2 f}} = m\sqrt{a\left(1-e^2\right)}\left(1+\tan^2 f\right)^{-\frac{1}{2}} = m\sqrt{a} - \tfrac{1}{2} m\sqrt{a}\left(e^2 + \tan^2 f\right),$$

consequently

$$\tfrac{1}{2} m\sqrt{a}\left(e^2 + \tan^2 f\right) + \tfrac{1}{2} m'\sqrt{a'}\left(e'^2 + \tan^2 f'\right) + \&\text{c.} = 2m\sqrt{a} + 2m'\sqrt{a'} + \&\text{c.} + 2C \;.$$

But the last member is altogether constant: hence

$$m\sqrt{a}\left(e^2+\tan^2 f\right)+m'\sqrt{a'}\left(e'^2+\tan^2 f'\right)+\&\text{c.} = \text{constant}.$$

It was shown that when the squares and products of the eccentricities and inclinations are omitted, the variations in the eccentricities are the same as if all the planets moved in one plane; and that the variations in the inclinations are the same as if the orbits were circular, as these quantities vary independently of one another, e, e', &c., and f, f', &c., may be alternately zero in the last equation, consequently,

$$m\sqrt{a}\,.\,e^2+m'\sqrt{a'}\,.\,e'^2+m''\sqrt{a''}\,.\,e''^2+\&\text{c.}=\text{constant};$$
$$m\sqrt{a}\,.\tan^2 f+m'\sqrt{a'}\,.\tan^2 f'+m''\sqrt{a''}\,.\tan^2 f''+\&\text{c.}=\text{constant};$$

results that are the same with equations (136) and (140).

If quantities of the order of the squares of the eccentricities and inclinations be omitted, the tangents of the very small quantities f, f', may be taken in place of their sines, so that by the substitution of [49]

$$\begin{aligned} p &= \tan f \sin q, & q &= \tan f \cos q, \\ p' &= \tan f' \sin q', & q' &= \tan f' \cos q', \\ &\&\text{c.} & &\&\text{c.} \end{aligned}$$

in equations (145) they become

$$m\sqrt{a}\,.\,q+m'\sqrt{a'}\,.\,q'+m''\sqrt{a''}\,.\,q''+\&\text{c.}=\text{constant},$$
$$m\sqrt{a}\,.\,p+m'\sqrt{a'}\,.\,p'+m''\sqrt{a''}\,.\,p''+\&\text{c.}=\text{constant}.$$

520. Since the eccentricities and inclinations of all the orbits in the solar system are very small, the constant quantities in all the preceding equations of condition must be very small, provided the radicals $\sqrt{a}$, $\sqrt{a'}$, &c., have the same signs, that is, if the bodies all move in one direction, which is the case in nature; it may therefore be concluded that the elements vary within very narrow limits.

521. Let there be only two bodies m and m', the mutual inclination of their orbits being

$$\cos g = \cos f \cos f' + \sin f \sin f' \cos\left(q'-q\right);$$

then if the squares of the equations (144) and (145) be added, the result will be

$$m^2 a\left(1-e^2\right)+m'a'\left(1-e'^2\right)+2mm'\sqrt{a\left(1-e^2\right)}\,.\sqrt{a'\left(1-e'^2\right)}\times\cos g = \text{constant}. \qquad \textbf{(146)}$$

Neglecting quantities of the fourth order, and putting all the constant quantities in the second member, it becomes

$$m\sqrt{a}\,.\,e^2+m'\sqrt{a'}\,.\,e'^2+\frac{4mm'\sqrt{aa'}\sin^2\frac{1}{2}g}{m\sqrt{a}+m'\sqrt{a'}}=\text{constant}\,,$$

for

$$\cos g=1-2\sin^2\tfrac{1}{2}g\ .$$

The constant in the second part of this equation is equal to the first member at a given epoch, for at that epoch all the elements are supposed to be known by observation; it ought, therefore, to be independent of the variation of the elements $e,\ e'$, and g : its variation will be

$$m\sqrt{a}\,.\,ede+m'\sqrt{a'}\,.\,e'de'+\frac{2mm'\sqrt{aa'}\,.\,gdg}{m\sqrt{a}+m'\sqrt{a'}}=0\,, \qquad \textbf{(147)}$$

for a and a' are constant. This relation must always exist among the secular variations of the eccentricities of the two orbits and their mutual inclination.

If the constant part of equation (146) be included in the second member it becomes

$$m^2ae^2+m'^2a'e'^2-2mm'a^2a'^2nn'\sqrt{1-e^2}\sqrt{1-e'^2}\cos g=\text{constant}\,,$$

by the substitution of a^2n and a'^2n' for $\sqrt{a}$ and $\sqrt{a'}$; and if it be observed that

$$\sqrt{1-e^2}=1-\frac{e^2}{1+\sqrt{1-e^2}},\quad \sqrt{1-e'^2}=1-\frac{e'^2}{1+\sqrt{1-e'^2}},\quad \cos g=1-\frac{\sin^2 g}{1+\cos g}\,,$$

then will

$$\sqrt{1-e^2}\sqrt{1-e'^2}\cos g=1-\frac{e^2\sqrt{1-e'^2}\cos g}{1+\sqrt{1-e^2}}-\frac{e'^2\cos g}{1+\sqrt{1-e'^2}}-\frac{\sin^2 g}{1+\cos g}\ .$$

If this value be put in the preceding equation, and all constant quantities included in the second member, it becomes

$$m^2\,.\,ae^2+m'^2\,.\,a'e'^2+2mm'\,.\,a^2a'^2nn'\cdot\frac{e^2\sqrt{1-e'^2}\,.\cos g}{1+\sqrt{1-e^2}}$$
$$+2mm'\,.\,a^2a'^2nn'\,.\,\frac{e'^2\cos g}{1+\sqrt{1-e^2}}+2mm'\,.\,a^2a'^2nn'\,.\,\frac{\sin^2 g}{1+\cos g}=C_{/}\,;$$

$C_{/}$ being an arbitrary constant quantity.

$C_{/}$ is a very small quantity with regard to the squares and products of m and m', since they are multiplied by $e^2,\ e'^2,\ \sin g^2$; and that the mutual inclination of the two planes and their eccentricities are supposed to be very small, as is really the case in nature. Each term of the first member of this equation will therefore remain very small with regard to the squares and products of m and m'; if all the terms have the same sign, each term will then be less than $C_{/}$. But because all the planets revolve in the same direction round the sun, nt, $n't$, will have the same sign. Hence all the terms in the first member will be positive as long as g is less than 90°. But if $g = 90^\circ$, then $\sin g = 1$; $\cos g = 0$, which reduces the equation to

$$m^2ae^2 + m'^2a'e'^2 + 2mm'a^2a'^2nn' = C\ ,$$

and the last term is no longer very small with regard to mm', which is impossible, since $C_{/}$ is very small with regard to the product of m and m', and that the other terms of the first member are positive. Thus, because the angle g never can attain to 90°, it follows that g, the inclination, and the eccentricities e, e^2, of the two orbits, will always be small; for, as $\cos g$ never can become negative, every term in the first member of the equation under discussion will be positive, and will remain very small with regard to the squares and products of the masses m and m'. That is to say, the coefficients[50] e^2, e'^2, $\sin g^2$, will always remain very small, because they are small at present.

522. This reasoning would be the same whatever might be the number of planets, since each of them would only add terms to the first member of the equation under consideration, similar to those that compose it.

523. Thus it may be concluded that the planetary system is stable with regard to the eccentricities, the inclinations, and greater axes of the orbits, however far the approximation may be carried with regard to the elements of the orbits, even including the second powers of the disturbing forces.

524. Laplace[51] and Poisson[52] have proved the stability of the solar system when the approximation extends to the first and second powers of the disturbing force, on the hypothesis that all the planets revolve in nearly circular orbits, little inclined to each other; but in a very able paper read before the Royal Society on the 29th April, 1830, Mr. Lubbock[53] has shown that these conditions are not necessary in a system subject to the law of gravitation. He has obtained expressions for the variations of the elliptical constants, which are rigorously true, whatever the power of the disturbing force may be, whence it appears, that, however far the approximation may be carried, the eccentricities, the major axes, and the inclinations of the orbits to a fixed plane, contain no term that varies with the time, and that their secular variations oscillate between fixed limits in very long periods.

The Invariable Plane

525. It has been already mentioned that in the motion of a system of bodies there exists an invariable plane, which, always retaining a parallel position, is easily found, because the sum of the masses of the bodies of the system respectively multiplied by the projections of the areas described by their radii vectores in a given time, is a maximum on that plane, and the sum of the projections on any other planes at right angles to it is zero. It is principally in the solar system that this plane is of importance, on account of the proper motions of the stars, and of the plane of the ecliptic, which render it difficult to determine the celestial motions with precision, this difficulty indeed is already perceptible, and will increase when very accurate observations, separated by very long intervals of time, must be compared with each other.

If I be the inclination of the invariable plane on the fixed plane which contains the co-ordinates x and y, and if Ω be the longitude of its ascending node, by article 166

$$\tan I \sin\Omega = \frac{C''}{C}; \quad \tan I \cos\Omega = \frac{C'}{C};$$

and substituting the values of $C,\ C',\ C''$, given by equations (144) and (145),

$$\tan I \sin\Omega = \frac{m\sqrt{a\left(1-e^2\right)}\sin f \sin q + m'\sqrt{a'\left(1-e'^2\right)}\sin f' \sin q' + \&c.}{m\sqrt{a\left(1-e^2\right)}\cos f + m'\sqrt{a'\left(1-e'^2\right)}\cos f' + \&c.}$$

$$\tan I \cos\Omega = \frac{m\sqrt{a\left(1-e^2\right)}\sin f \cos q + m'\sqrt{a'\left(1-e'^2\right)}\sin f' \cos q' + \&c.}{m\sqrt{a\left(1-e^2\right)}\cos f + m'\sqrt{a'\left(1-e'^2\right)}\cos f' + \&c.}.$$

The second members of these two equations have been proved to be invariable, even in carrying out the approximation to the squares and products of the masses, whatever changes the secular variations may induce in the course of ages; and, by what Mr. Lubbock has shown, they must be constant, whatever the power of the disturbing force may be: hence it follows, that the invariable plane retains its position, notwithstanding the secular variations in the elliptical elements of the planetary system.

526. The determination of this plane requires a knowledge of the masses of all the bodies in the system, and of the elements of their orbits. Approximate values of these are only known with regard to the planets, but of the masses of the comets we are in total ignorance; however, as the mutual gravitation of the planets is sufficient to represent all their inequalities, it shows that, hitherto at least, the action of the comets on the planetary system is insensible. Besides, the comet of 1770 approached so near to the earth that its periodic time was increased by 2.046 days; and if its mass had been equal to that of the earth, it would have increased the length of the sidereal year by nearly one hour fifty-six minutes, according to the computation of Laplace; but he adds, that if an increase of only two seconds had taken place in the length of the year, it would have been detected by Delambre,[54] when he computed his astronomical tables from the observations of Dr. Maskelyne;[55] whence the mass of the comet must have been less than the $\frac{1}{3{,}000}$ part of the mass of the earth. The same comet passed through the satellites of Jupiter in the years 1767 and 1779, without producing the smallest effect. Thus, though comets are greatly disturbed by the action of the planets, they do not appear to produce any sensible effects by their reaction.

527. If the position of the ecliptic in the beginning of 1750 be assumed the fixed plane of the co-ordinates x and y, and if the line of the equinoxes be taken as the origin of the longitudes, it is found that at the epoch 1750 the longitude of the ascending node of the invariable plane was

$\Omega = 102^\circ\ 57'\ 30''$, and its inclination on the ecliptic $I = 1^\circ\ 35'\ 31''$; and if the values of the elements for 1950 be substituted in the preceding formulae, it will appear that in 1950

$$\Omega = 102^\circ\ 57'\ 15''; \quad I = 1^\circ\ 35'\ 31'';$$

which differ but little from the first.

528. The position of this plane is really approximate, since it has been determined in the hypothesis of the solar system being an assemblage of dense points mutually acting on one another, whereas the celestial bodies are neither homogeneous nor spherical; but as the quantities omitted have hitherto been insensible, the position of the plane as it is here given, will enable future astronomers to ascertain the real changes that may have taken place in the forms and positions of the planetary orbits.

Notes

1 This *reads* "contain" in the 1st edition (published erratum).

2 The third term *reads* $\frac{d}{de} d\,e$ in the 1st edition (published erratum).

3 The term $i'n't$ *reads* $i'nt$ in the 1st edition (published erratum).

4 The term $i'n't$ *reads* $i\,n't$ in the 1st edition (published erratum).

5 In the 1st edition $\frac{1}{r^3} - \frac{1}{r'^3}$ *reads* $\frac{1}{r^3} - \frac{1}{r'^2}$.

6 The last term in the 1st edition *reads* $\frac{dR'}{dq'} d\,q$.

7 An error in a published erratum would replace $\frac{1}{r^3} - \frac{1}{r^3}$ with $\frac{1}{r^3} - \frac{1}{r'^2}$. The replacement should be $\frac{1}{r^3} - \frac{1}{r'^3}$ (see next note).

8 A misplaced parenthesis *reads* $m'd'\left\{xx' + yy' + zz'\left(\frac{1}{r^3} - \frac{1}{r^3}\right)\right\}$ in the 1st edition.

9 The prime in $-\frac{1}{r'^3}$ is misplaced and *reads* $-\frac{1}{r^{3\prime}}$ in the 1st edition.

10 This *reads* r' *for* a' in the 1st edition (published erratum).

11 This *reads* m' in the 1st edition (published erratum).

12 This *reads* "elliptical co-ordinates of m.," in the 1st edition.

13 See note 1, *Book1, Chapter 6*.

14 These expressions *read* $\mathrm{Sin}'\,i(n't - nt + \epsilon' - \epsilon) = 0$ and $\mathrm{C}osi(n't - nt + \epsilon' - \epsilon) = 1$ in the 1st edition.

15 The third differential *reads* $\frac{dp}{dt} = .\,an \cdot \frac{dF}{dq}$ in the 1st edition.

16 The right hand side of the first expression *reads* (0,1) in the 1st edition.

17 The 1st edition contains the element $\frac{\tan f'}{\tan f'}$ in the second equation. This should be $\frac{\tan f'}{\tan f}$ as in equation (128).

18 This *reads* m', m'', &c. in the 1st edition.

19 The second term *reads* $\frac{1}{2}\cdot\frac{d^3\overline{e}}{dt^2}t^2$ in the 1st edition (published erratum).

20 The second term in the 2nd equation *reads* $-\boxed{0.1}\,h'$ in the 1st edition (published erratum).

21 Punctuation added after first term.

22 The subscript in g_1 is omitted in the 1st edition.

23 The expression is printed with mismatched parentheses as $(a^2 - 2aa'\cos b + a'^2\}^{\frac{1}{2}}$ in the 1st edition.

24 The 1st edition contains a missing multiplier and a misplaced comma written, $\boxed{0.1}\, m,n'a' = \boxed{1.0}\ .m'.\,na$.

25 The 3rd term in the 2nd equation *reads* $N''N_2''m'\sqrt{a''}$ in the 1st edition.

26 The first term in the numerator in the 1st edition contains an error, $\overline{e}$ *reads* *e*.

27 This *reads* $g = \frac{360^0}{113270} = 19'\,4''.7$ in the 1st edition (published erratum).

28 The 2nd, 3rd and 4th instances of the subscript 1 in N_1, g_1, and x_1 *read* as $N_/$, $g_/$, and $x_/$ in the 1st edition.

29 As in equation (135) the three instances of the subscript 1 in N_1, *read* as $N_/$ in the 1st edition.

30 Throughout this article the 1st edition text *reads* C^{at} and C^{2at} *for* c^{at} and c^{2at} .

31 The next two instances of g_1 *read* $g_/$ in the 1st edition.

32 The right hand parenthesis and punctuation are omitted in the 1st edition.

33 Again, the subscript on g_1 *reads* $g_/$ in the 1st edition.

34 See the previous note regarding the subscripts on N_1, g_1, and x_1 throughout the development in articles 495-500.

35 In the 1st edition the sines and cosines are reversed (published erratum).

36 The accent on q' in the fourth member of this series is omitted in the 1st edition.

37 See note 33 above regarding the subscripts on N_1, g_1, and x_1.

38 Punctuation added after third and fourth elements.

39 This is labeled fig. 90 in the 1st edition (published erratum).

40 The 1st edition text uses the radius $\mathrm{N}_/$ for N_1.

41 The 1st edition text expresses this as $a'\mathrm{C}' = g_/t + x_/$.

42 Punctuation added.

43 We have added the multiplier symbol in the second term of both expressions; also, there is a misplaced comma in the denominator of the second expression after $\tan f$` in the 1st edition.

44 A parenthesis is omitted in the third expression of $\cos\{g_/ - g)t + x_/ - x\}$ in the 1st edition.

45 An unbalanced parenthesis in the 1st edition *reads* $\cos\{(g_/ - g)\,t + x_/ - x)dt$.

46 Punctuation in the 1st edition is contained inside the parentheses.

47 The first term in the 1st edition *reads* $\sum m\cdot\frac{xdx - xdy}{dt}$ (published erratum).

48 A printing error on the right hand side of the second expression *reads* $+ =$ &c.C'' in the 1st edition.

49 The 4th expression *reads* $q = \tan f'\cos q'$ in the 1st edition.

50 Punctuation added.

51 See note 4, *Introduction.*

52 See note 1, *Book I, Chapter 6.*

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[53] Lubbock, John William, Sir, 1803-1865, *Account of the "Traite sur le flux et reflux de la mer," of Daniel Bernoulli; and a treatise on the attraction of ellipsoids*, London : C. Knight, 1830.
[54] See note 54, *Preliminary Dissertation.*
[55] Maskelyne, Nevil, 1732-1811, astronomer, born in London, England. Maskelyne produced the *British Mariner's Guide* (1763) and published the first volume of the *Nautical Almanac* in 1765 (see note 22, *Preliminary Dissertation*). His inventions include the prismatic micrometer. In 1774 he measured the earth's density using a plumb line. His measurements showed that the density was 4.5 times that of water. He was also the first to make measurements of time with a precision to one tenth of a second.

BOOK II

CHAPTER VII

PERIODIC VARIATIONS IN THE ELEMENTS OF THE PLANETARY ORBITS

Variations depending on the first Powers of the Eccentricities and Inclinations

529. THE differential d*R* relates to the arc *nt* alone, consequently the differential equation $da = 2a^2 \,.\, \text{d}R$ in article 439 becomes

$$da = +m'a^2 \,.\, in \,.\, \sum A_i \sin i\left(n't - nt + \in' - \in\right)$$
$$+m'a^2 en\left(i-1\right).\, M_0 \sin\left\{i\left(n't - nt + \in' - \in\right) + nt + \in - v\right\}$$
$$+m'a^2 e'n\left(i-1\right).\, M_1 \sin\left\{i\left(n't - nt + \in' - \in\right) + nt + \in - v'\right\}.$$

The integral of this equation is the periodic variation in the mean distance, and if represented by $d\,a$, then

$$d\,a = -m'a^2 \frac{n}{n'-n} \cdot \sum A_i \cos i\left(n't - nt + \in' - \in\right)$$
$$-m'a^2 e \frac{\left(i-1\right)n}{i\left(n'-n\right)+n} M_0 \cos\left\{i\left(n't - nt + \in' - \in\right) + nt + \in - v\right\}$$
$$-m'a^2 e' \frac{\left(i-1\right)n}{i\left(n'-n\right)+n} M_1 \cos\left\{i\left(n't - nt + \in' - \in\right) + nt + \in - v'\right\}.$$

In a similar manner it may be found that the periodic variation in the mean motion $dz = -3\int andt \text{d}R$ is,

$$dz = \tfrac{3}{2} \cdot m'a \cdot \frac{n^2}{i\left(n'-n\right)^2} \cdot A_i \sin i\left(n't - nt + \in' - \in\right)$$
$$+\tfrac{3}{2} \cdot m'ae \cdot \frac{\left(i-1\right)n^2}{\left\{i\left(n'-n\right)+n\right\}^2} \cdot M_0 \sin\left\{i\left(n't - nt + \in' - \in\right) + nt + \in - v\right\}$$
$$+\tfrac{3}{2} \cdot m'ae' \cdot \frac{\left(i-1\right)n^2}{\left\{i\left(n'-n\right)+n\right\}^2} \cdot M_1 \sin\left\{i\left(n't - nt + \in' - \in\right) + nt + \in - v'\right\}.$$

From the other differential equations in article 439 it may also be found that the periodic variation in the eccentricity is[1]

$$
\begin{aligned}
de = {} & \tfrac{1}{2} m'a \frac{n}{i(n'-n)+n} M_0 \cos\{i(n't - nt + \epsilon' - \epsilon) + nt + \epsilon - \varpi\} \\
& + \tfrac{1}{4} m'ae \frac{n}{n'-n} A_i \cos i(n't - nt + \epsilon' - \epsilon) \\
& + m'ae' \frac{n}{i(n'-n)+2n} N_0 \cos\{i(n't - nt + \epsilon' - \epsilon) + 2nt + 2\epsilon - 2\varpi\} \\
& + \tfrac{1}{2} m'a.e' \frac{n}{i(n'-n)+2n} N_1 \cos\{i(n't - nt + \epsilon' - \epsilon) + 2nt + 2\epsilon - \varpi - \varpi'\} \\
& - \tfrac{1}{2} m'a.e' \frac{n}{i(n'-n)} N_4 \cos\{i(n't - nt + \epsilon' - \epsilon) + \varpi - \varpi'\} \\
& + \tfrac{1}{2} m'a.e' \frac{n}{i(n'-n)} N_5 \cos\{i(n't - nt + \epsilon' - \epsilon) - \varpi + \varpi'\}.
\end{aligned}
$$

The variation of the epoch

$$
\begin{aligned}
d\epsilon = {} & -ma' \frac{n}{i(n'-n)} a \left(\frac{dA_i}{da}\right) \sin i(n't - nt + \epsilon' - \epsilon) \\
& + \tfrac{1}{4} m'ae \frac{n}{i(n'-n)+n} M_0 \sin\{i(n't - nt + \epsilon' - \epsilon) + nt + \epsilon - \varpi\} \\
& - m'a^2 e \frac{n}{i(n'-n)+n} \cdot \frac{dM_0}{da} \sin\{i(n't - nt + \epsilon' - \epsilon) + nt + \epsilon - \varpi\} \\
& - m'a^2 e' \frac{n}{i(n'-n)+n} \cdot \frac{dM_1}{da} \sin\{i(n't - nt + \epsilon' - \epsilon) + nt + \epsilon - \varpi'\}.
\end{aligned}
$$

The variation in the longitude of the perihelion

$$
\begin{aligned}
ed\varpi = {} & \tfrac{1}{2} m'a \frac{n}{i(n'-n)+n} M_0 \sin\{i(n't - nt + \epsilon' - \epsilon) + nt + \epsilon - \varpi\} \\
& + m'ae \frac{n}{i(n'-n)+2n} N_0 \sin\{i(n't - nt + \epsilon' - \epsilon) + 2nt + 2\epsilon - 2\varpi\} \\
& + mae \cdot \frac{n}{i(n'-n)} N_3 \sin i(n't - nt + \epsilon' - \epsilon) \\
& + \tfrac{1}{2} m'ae' \frac{n}{i(n'-n)+2n} N_1 \sin\{i(n't - nt + \epsilon' - \epsilon) + 2nt + 2\epsilon - \varpi - \varpi'\}
\end{aligned}
$$

$$+\tfrac{1}{2}m'ae'\frac{n}{i(n'-n)}N_4\sin\left\{i(n't-nt+\epsilon'-\epsilon)+\nu-\nu'\right\}$$
$$+\tfrac{1}{2}m'ae'\frac{n}{i(n'-n)}N_5\sin\left\{i(n't-nt+\epsilon'-\epsilon)-\nu+\nu'\right\}.$$

When e^2, eg, $e'g$, are omitted, the differentials of p and q in article 437 become

$$dp=a^2ndt\sin(nt+\epsilon)\frac{\mathrm{d}R}{dz}$$
$$dq=a^2ndt\cos(nt+\epsilon)\frac{\mathrm{d}R}{dz}.$$

When the orbit of m at the epoch is assumed to be the fixed plane,

$$z=0,\text{ and }z'=a'g\,\sin(n't+\epsilon'-\Pi).$$

the products of the inclination by the eccentricities being omitted.

Now although z be zero, its differential is not, therefore $\frac{\mathrm{d}R}{dz}$ must be determined from

$$R=\frac{m'zz'}{a'^3}+\frac{m'(z'-z)^2}{4}\Sigma B_i\cos i(n't-nt+\epsilon'-\epsilon);$$

whence

$$\frac{\mathrm{d}R}{dz}=\frac{m'z'}{a'^3}-\frac{m'z'}{2}\Sigma B_i\cos i(n't-nt+\epsilon'-\epsilon),$$

and

$$\frac{\mathrm{d}R}{dz}=\frac{-m'}{a'^2}g\,\sin\left\{(n't+\epsilon'-\Pi)+\frac{m'}{2}a'\Sigma B_{(i-1)}g\,\sin\left\{i(n't-nt+\epsilon'-\epsilon)+nt+\epsilon-\Pi\right\}\right\}$$

where i may be any whole number, positive or negative, except zero. When this quantity is substituted in dp, dq, their integrals are[2]

$$d\,p=-\frac{m'}{2}\cdot\frac{a^2n}{a'^2}g\left\{\frac{1}{n'-n}\sin(n't-nt+\epsilon'-\epsilon-\Pi)-\frac{1}{n'+n}\times\sin(n't+nt+\epsilon'+\epsilon-\Pi)\right\},$$
$$+\frac{m'}{4}a^2a'n\Sigma B_{(i-1)}g\left\{\begin{array}{l}\frac{1}{i(n'-n)}\sin(i(n't-nt+\epsilon'-\epsilon)-\Pi)\\-\frac{1}{i(n'-n)+2n}\sin\left\{i(n't-nt+\epsilon'-\epsilon)+2nt+2\epsilon-\Pi\right\}\end{array}\right\}$$

[and] [3] [4]

$$d\,q = \frac{m'}{2}\cdot\frac{a^2 n}{a'^2}g\left\{\frac{1}{n'+n}\cos\left(n't+nt+\in'+\in-\Pi\right)+\frac{1}{n'-n}\times\cos\left(n't-nt+\in'-\in-\Pi\right)\right\},$$

$$-\frac{m'}{4}a^2 a' n\Sigma B_{(i-1)}g\left\{\begin{array}{l}\dfrac{1}{i\left(n'-n\right)}\cos\left(i\left(n't-nt+\in'-\in\right)+\Pi\right)\\ +\dfrac{1}{i\left(n'-n\right)+2n}\cos\left\{i\left(n't-nt+\in'-\in\right)+2nt+2\in-\Pi\right\}\end{array}\right\}.$$

530. The equations which determine the variations in the greater axes and mean motion show that these two elements are subject to very considerable periodic variations, depending on the configurations of the bodies, when the divisor $i\left(n'-n\right)+n$ or $i'n'-in$ is very small.

There is no instance of the mean motions of any two of the celestial bodies being so exactly commensurable as to have $i'n'-in=0,$ therefore the greater axes and mean motions have no secular inequalities, but in several instances this divisor is a very small fraction, and as a quantity is increased in value when divided by a fraction, the divisor $i'n'-in,$ and still more its square, increases the values of these periodic variations very much. For this reason the periodic variation in the mean motion is much greater than that in the greater axis, evidently arising from the double integration in the former.

531. It is unnecessary to add constant quantities to the preceding integrals, for they may be included in the elements of elliptical motion, which then become

$$a+a_/,\ e+e_/,\ v+v_/,\ \in+\in_/,\ p+p_/,\ q+q_/;$$

and in the troubled orbit they are

$$a+a_/+d\,a,\ e+e_/+d\,e,\ v+v_/+d\,v,\ \in+\in_/+d\in,\ p+p_/+d\,p,\ q+q_/+d\,q.$$

Since $a_/,\ e_/,$ &c., $d\,a,\ d\,e,$ &c., are very small quantities of the order $m',\ a+a_/,\ e+e_/,$ &c., may be substituted in the latter quantities instead of *a*, *e*, &c., they will then be functions of the time and of the six constant quantities $a+a_/,\ e+e_/,$ &c.: so that the formulae of troubled motion in reality contain but six arbitrary constant quantities, as they ought to do. In order to determine $a_/,\ e_/,$ &c., suppose the perturbations of the planet m were required during a given interval of time. The quantities *a*, *e*, &c., are given by observation at the epoch when $t=0$ in the elliptical orbit, that is, assuming the disturbing force to be zero; but as $a_/+d\,a,\ e_/+d\,e,$ &c., arise entirely from the disturbing force, they must also be zero at the epoch; therefore, values of the arbitrary constant quantities $a_/,\ e_/,$ &c., are obtained from the equations[5]

$$a_/+d\bar{a}=0,\ e_/+d\bar{e}=0,\ v+d\bar{v}=0,\ \&\text{c.},$$

$d\,\bar{a},\ d\,\bar{e},$ &c. being the values of $d\,a,\ d\,e,$ &c., at the epoch.

The effect of the disturbing forces upon each of the elliptical elements will be completely expressed by $a_{/}+d\bar{a}$, $e_{/}+d\bar{e}$, &c. during the time under consideration. Thus both the periodic and secular variations of the elements of the orbits are determined.

Notes

[1] A misplaced extra parenthesis in the fifth term *reads* $N_4 \cos\{i(n't-nt)+\in'-\in)+v-v'\}$ in the 1st edition.

[2] The first line factor $\sin(n't+nt+\in'+\in-\Pi)$ *reads* $\sin(n't+n't+\in'+\in-\Pi)$ in the 1st edition.

[3] The factor $\frac{1}{n'-n}\times\cos(n't-nt+\in'-\in-\Pi)$ in the 1st line *reads* $\frac{1}{n'-n}\times\cos(n't-nt+\in'-\in)-\Pi)$ in 1st ed.

[4] The factor $\frac{1}{i(n'-n)}\cos(i(n't-nt+\in'-\in)-\Pi)$ in the second line contains two errors in the 1st edition. It *reads* $\frac{1}{i(n'-n}\cos(i(n't-nt+\in'-\in)+\Pi)$ (published erratum in sign before Π).

[5] The 1st equation *reads* $a_{/}+d\,a=0$ in the 1st edition.

BOOK II

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CHAPTER VIII

PERTURBATIONS OF THE PLANETS IN LONGITUDE, LATITUDE, AND DISTANCE

532. THE position of a planet in space is fixed when its curtate distance *Sp*, fig. 77, its projected longitude γSp, and its latitude *pm*, are known. The determination of these three co-ordinates in functions of the time is the principal object of Physical Astronomy; these quantities in series ascending according to the powers of the eccentricities and inclinations are given in article 399, and those following, supposing the planet to move in a perfect ellipse; but if values of the elements of the orbits corrected by their periodic and secular variations be substituted instead of their elliptical elements, the same series will determine the motion of the planet in its real perturbed orbit.

533. The projected longitude and curtate distance only differ from the true longitude and distance on the orbit by quantities of the second order with regard to the inclinations; and when the orbit at the epoch is assumed to be the fixed plane, these quantities as well as those of the latitude that depend on the product of the inclination by the eccentricity are so small that they are insensible, as will readily appear if it be considered that any inclination the orbit may have acquired subsequently to the epoch, can only have arisen from the small secular variation in the elements; besides the epoch may be chosen to make it so, being arbitrary. Hence the perturbations in the longitude and radius vector may be determined as if the orbits were in the same plane, and the latitude may be found in the hypothesis of the orbits being circular, provided the orbit at the epoch be taken as the fixed plane: circumstances which greatly facilitate the determination of the perturbations.

The following very elegant method of finding the perturbations, by considering the troubled orbit as an ellipse whose elements are varying every instant, was employed by Lagrange;[1] but Laplace's method,[2] which will be explained afterwards, has the advantage of greater simplicity, especially in the higher approximations.

534. In the elliptical hypothesis the radius vector and true longitude are expressed, in article 392, by

$$r = \text{functions}.\left(r, z, e, \in, v\right),$$
$$v = \text{functions}.\left(z, e, \in, v\right),$$

but in the true orbit these quantities become

$$a + da,\ z + dz,\ e + de,\ v + dv\ ;$$

therefore

$$dr = \frac{dr}{da} \cdot da + \frac{dr}{dz} \cdot dz + \frac{dr}{de} \cdot de + \frac{dr}{d\epsilon} \cdot d\epsilon + \frac{dr}{dv} \cdot dv,$$

$$dv = \frac{dv}{dz} \cdot dz + \frac{dv}{de} \cdot de + \frac{dv}{d\epsilon} \cdot d\epsilon + \frac{dv}{dv} \cdot dv;$$

and if the values of the periodic variations in the elements in article 529 be substituted instead of da, dz, &c., the perturbations in the radius vector and true longitude will be obtained; the approximation extending to the first powers of the eccentricities and inclinations inclusively.

535. The perturbations in longitude way be expressed under a more simple form; for by article 372,

$$dv = \frac{\sqrt{a(1-e^2)}}{r^2} \cdot dt,$$

an equation belonging both to the elliptical and to the real orbit, since it is a differential of the first order; on that account it ought not to change its form when the elements vary; hence

$$d\,.dv = \tfrac{1}{2}\sqrt{\frac{1-e^2}{a}} \cdot \frac{da}{r^2} \cdot dt - \sqrt{\frac{a}{1-e^2}} \cdot \frac{ede}{r^2} \cdot dt - 2\sqrt{a(1-e^2)}\,\frac{dr}{r^3} \cdot dt;$$

and neglecting the squares of the disturbing forces, the integral is

$$dv = \frac{1}{2a} \cdot \int da\,.dv - \frac{e}{1-e^2} \cdot \int de\,.\,dv - 2\int \frac{dr}{r} \cdot dv\,.$$

But

$$h = \sqrt{a\,.(1-e^2)},$$

then

$$\frac{dh}{h} = \frac{1}{2a} \cdot da - \frac{e}{1-e^2} de\,;$$

therefore

$$dv = \int \left(\frac{dh}{h} - \frac{2dr}{r} \right) \cdot dv \qquad \textbf{(148)}$$

will give the perturbations in longitude when those in the radius vector are known.

Perturbations in the Radius Vector

536. By article 392,

$$r = a\left(1+\tfrac{1}{2}e^2 - e\cos\left(nt+\in-v\right)-\tfrac{1}{2}e^2\cos 2\left(nt+\in-v\right)\right);$$

whence

$$dr = da - a\,de\cos\left(nt+\in-v\right) - ae\,dv\,\sin\left(nt+\in-v\right)\left(1+2e\cos\left(nt+\in-v\right)\right)$$
$$-3ed\,a\cos\left(nt+\in-v\right)+2aed\,e+ae\left(dz+d\in\right)\sin\left(nt+\in-v\right).$$

If the values of da, de, dv, dz, $d\in$, from article 529, be substituted in this expression, after the reduction of the products of the sines and cosines to the cosines of multiple arcs, and substitution for M_0, M_1, N_3, N_4, N_5, from article 459, it becomes

$$\begin{aligned}\frac{dr}{a} &= \frac{m'}{2}\cdot\Sigma.C_i\,.\cos i\left(n't-nt+\in'-\in\right)\\ &+m'.e.\Sigma.D_i\,.\cos\left\{i\left(n't-nt+\in'-\in\right)+nt+\in-v\right\}\\ &+m'.e'.\Sigma.E_i\,.\cos\left\{i\left(n't-nt+\in'-\in\right)+nt+\in-v'\right\},\end{aligned} \qquad \textbf{(149)}$$

where

$$C_i = \frac{n^2}{n^2-i^2\left(n'-n\right)^2}\left\{\frac{2n}{n-n'}aA_i + a^2\frac{dA_i}{da}\right\},$$

$$D_i = \frac{n^2}{\left\{i\left(n'-n\right)+n\right\}^2-n^2}\left\{\frac{3n}{n'-n}aA_i - \frac{i^2\left(n'-n\right)\left\{i\left(n'-n\right)-n\right\}-3n^2}{n^2}C_i+\tfrac{1}{2}a^3\left(\frac{d^2A_i}{da^2}\right)\right\},$$

$$E_i = \frac{n^2}{\left\{i\left(n'-n\right)+n\right\}^2-n^2}\left\{\frac{\left(i-1\right)\left(2i-1\right)n}{i\left(n'-n\right)+n}\cdot aA_{(i-1)} - \frac{i^2\left(n'-n\right)+n}{i\left(n'-n\right)+n}a^2\frac{dA_{(i-1)}}{da}-\tfrac{1}{2}a^3\frac{d^2A_{(i-1)}}{da^2}\right\}.$$

The Perturbations in Longitude

537. Having thus determined the perturbations in the radius vector, the term $\frac{2dr}{r}$ is known; and if substitution be made for da and de, from article 529, $\frac{dh}{h}$ will be obtained, and the integral of equation (148) will give[3]

$$\begin{aligned}dv &= \frac{m'}{2}\Sigma.F_i\,.\sin i\left(n't-nt+\in'-\in\right)\\ &+m'e.\Sigma.G_i\,.\sin\left\{i\left(n't-nt+\in'-\in\right)+nt+\in-v\right\}\\ &+m'e'.\Sigma\,H_i\,.\sin\left\{i\left(n't-nt+\in'-\in\right)+nt+\in-v'\right\}.\end{aligned}$$

Where

$$F_i = \frac{n}{i(n-n')}\left\{-\frac{n}{n-n'}\cdot aA_i + 2C_i\right\}$$

$$G_i = \frac{n}{i(n'-n)+n}\left\{\frac{(i-1)n}{n'-n}\cdot aA_i - 2D_i - \frac{i\{i(n'-n)-n\}+6n}{2n}\cdot C_i\right\}$$

$$H_i = \frac{n}{i(n'-n)+n}\times\left\{\frac{-(i-1)(2i-1)naA_{(i-1)} - (i-1)na^2\frac{dA_{(i-1)}}{da}}{2(i(n'-n)+n)} - 2E_i\right\}.$$

538. In these values of $d\,r$ and $d\,v,$ includes all whole numbers, either positive or negative, zero excepted: $d\,r$ and $d\,v$ will now be determined in the latter case, which is very important, because it gives the part of the perturbations that is not periodic.

539. If $i=0$ in the series R in article 449, the only constant term introduced by this value into $d\,r$ will be

$$\frac{m}{2}a^3\left(\frac{dA_0}{da}\right).$$

Again, in finding the integral $d\,a$ the arbitrary constant $a_/$ that ought to have been added, would produce a constant term in $d\,r$. In order to find it, let the origin of the time be at the instant of the conjunction of the two bodies m and $m',$ when

$$n't - nt + \in' - \in = 0\,;$$

whence $\cos 0 = 1,$ and the first term of $d\,a$ in article 529 becomes

$$d\,a = -2m'a^2\frac{n}{n'-n}\sum A_i\,,$$

whence

$$d\,r = \frac{m'}{2}a^3\frac{dA_0}{da} - 2m'a^2\frac{n}{n'-n}\sum A_i\,;$$

where $\sum$ extends to all positive values of i from $i=1$ to $i=\infty$.

540. If these values of $d\,r$ and $d\,a$ be put in equation (148), the result will be

$$d\,v = m'a\left\{\frac{3n}{n'-n}\sum A_i - a\left(\frac{dA_0}{da}\right)\right\}\cdot nt\,.$$

And as by article 392 the elliptical parts or r and v that are not periodic, or that do not depend on sines and cosines, are $r = a$, and $v = nt + \in$: those parts of the radius vector and true longitude that are not periodic are expressed by

$$r + dr = a - 2m'a^2 \frac{n}{n' - n} \cdot \sum A_i + \tfrac{1}{2} m'a^3 \left(\frac{dA_0}{da} \right)$$
$$v + dv = nt + \in + m'a \left\{ \frac{3n}{n' - n} \sum A_i - a \left(\frac{dA_0}{da} \right) \right\} nt \qquad \textbf{(150)}$$

in the real orbit.

Thus the perturbations in longitude seem to contain a term that increases indefinitely with the time; were that really the case, the stability of the solar system would soon be at an end. This term however is only introduced by integration, since the differential equations of the perturbations contain no such terms; it is therefore foreign to their nature, and may be made to vanish by a suitable determination of the arbitrary constant quantities. In fact the true longitude of a planet in its disturbed orbit consists of three parts,–of the mean motion, of the equation of the centre, and of the perturbations. The mean motion of the planet is the only quantity in the problem of three bodies that increases with the time: the equation of the centre is a periodic correction which is zero in the apsides and at its maximum in quadratures; and the perturbations being functions of the sines of the mean longitudes of the disturbed and disturbing bodies are consequently periodic, and are applied as corrections to the equation of the centre. All the coefficients of these quantities are functions of the elements of the orbits, which vary periodically but in immensely long periods. The arbitrary constant quantities introduced by integration, must therefore be determined so that the mean motion of the troubled planet may be entirely contained in that part of the longitude represented by v.

541. The values of a, n, e, $\in$, and v, in the preceding equations, are for the epoch $t = 0$, and would be the elliptical values of the elements of the orbit of m, if at that instant the disturbing forces were to cease. Let $n_/ t$ be the mean motion of m given by observation, then the second of the equations under consideration gives

$$n_/ = n \left\{ 1 + m'a \left(\frac{3n}{n' - n} \cdot \sum A_i - a \left(\frac{dA_0}{da} \right) \right) \right\},$$

and let $a_/$ be the mean distance corresponding to $n_/$ resulting from the equation,

$$n_/^{\,2} = \frac{S + m}{a_/^{\,3}}.$$

If in this last expression $n + n_/ - n$, and $a + a_/ - a$, be put for $n_/$ and $a_/$, and if $(n_/ - n)^2$, $(a_/ - a)^2$, which are very small be omitted, then

$$2n(n_{/}-n) = -\frac{3n^2}{a}(a_{/}-a);$$

and substituting for $n_{/}$ it becomes

$$a - a_{/} = \frac{2m'a^2}{3}\left\{\frac{3n}{n'-n}\sum A_i - a\left(\frac{dA_0}{da}\right)\right\};$$

and as a may be put for $a_{/}$ in the terms multiplied by m', the equations (150) become

$$r + d\,r = a_{/} - \tfrac{1}{6}m'a_{/}^{3}\left(\frac{dA_0}{da_{/}}\right)$$
$$v + d\,v = n_{/}t + \in .$$

Thus $d\,v$ no longer contains a term proportional to the time, and the mean motion of the disturbedplanet is altogether included in the part of the longitude expressed by v, in consequence of the introduction of the arbitrary constant quantities $n_{/}$ and $a_{/}$, instead of n and a.

The part of $d\,r$ depending on the first powers of the eccentricities may be found by making $i = 0$ in the values of $d\,a$, $d\,e$, &c., in article 529; after which their substitution in $d\,r$ of article 536, will give[4]

$$d\,r = -\frac{m'}{4}ae\left\{3a\left(\frac{dA_0}{da}\right)+\tfrac{1}{2}a^2\left(\frac{d^2A_0}{da^2}\right)\right\}\cos(nt+\in -v)$$
$$-\frac{m'}{4}ae'\left\{3A_1 - 3a\left(\frac{dA_1}{da}\right)-\tfrac{1}{2}a^2\left(\frac{d^2A_1}{da^2}\right)\right\}\cos(nt+\in -v').$$

The corresponding part of $d\,v$ from article 535 is

$$d\,v = \frac{m'}{2}ae\left\{3a\left(\frac{dA_0}{da}\right)+\tfrac{1}{2}a^2\left(\frac{d^2A_0}{da^2}\right)\right\}\sin(nt+\in -v)$$
$$+\frac{m'}{2}ae'\left\{2A_1 - 2a\left(\frac{dA_1}{da}\right)-\tfrac{1}{2}a^2\left(\frac{d^2A_1}{da^2}\right)\right\}\sin(nt+\in -v').$$

542. If the different parts of the value of $d\,r$ and $d\,v$ be added, and if

$$f = \tfrac{1}{4}\left\{3a^2\left(\frac{dA_0}{da}\right)+\tfrac{1}{2}a^3\left(\frac{d^2A_0}{da^2}\right)\right\}$$

$$f' = \tfrac{1}{4}\left\{3aA_1 - 3a^2\left(\frac{dA_1}{da}\right) - \tfrac{1}{2}a^3\left(\frac{d^2A_1}{da^2}\right)\right\}$$

$$f'' = \tfrac{1}{4}\left\{2A_1 - 2a\left(\frac{dA_1}{da}\right) - \tfrac{1}{2}a^2\left(\frac{d^2A_1}{da^2}\right)\right\}$$

The periodic inequalities in the radius vector and true longitude of m when troubled by m', are

$$\begin{aligned}\frac{dr}{a} = &-\frac{m'}{6}a_/^3\left(\frac{dA_0}{da_/}\right) + \frac{m'}{2}\cdot\sum .C_i\,.\cos i\left(n't - nt + \in' - \in\right)\\ &- m'.e\,.f\,.\cos\left(nt + \in - v\right) - m'e'f'.\cos\left(nt + \in - v'\right)\\ &+ m'.e.\sum .D_i\,.\cos\left\{i\left(n't - nt + \in' - \in\right) + nt + \in - v\right\}\\ &+ m'.e'.\sum .E_i\,.\cos\left\{i\left(n't - nt + \in' - \in\right) + nt + \in - v'\right\}\end{aligned}$$

[and]

$$\begin{aligned}dv = &+\frac{m'}{2}\cdot\sum .F_i\,.\sin i\left(n't - nt + \in' - \in\right)\\ &+ 2m'.e\,.f\,.\sin\left(nt + \in - v\right) + 2m'e'f''.\sin\left(nt + \in - v'\right)\\ &+ m'.e.\sum .G_i\,.\sin\left\{i\left(n't - nt + \in' - \in\right) + nt + \in - v\right\}\\ &+ m'.e'.\sum .H_i\,.\sin\left\{i\left(n't - nt + \in' - \in\right) + nt + \in - v'\right\}\end{aligned}$$

The action of each disturbing body will produce a similar effect on the radius vector and longitude of m, and the sum of all will be perturbations in these two co-ordinates arising from the disturbing action of the whole system on the planet m.

543. It has been already observed that each of the periodic variations $d\,a$, $d\,e$, &c., ought to contain an arbitrary constant quantity $a_/$, $e_/$, $v_/$, &c., introduced by their integrations, so that their true values are

$$a_/ + d\,a;\ e_/ + d\,e;\ v_/ + d\,w;\ \text{\&c. \&c.}$$

Now, if the values of $d\,r$, $d\,v$, are to express the effects of the disturbing forces on the radius vector and longitude during a given time, these constant quantities must be so determined, that when $t = 0$, they must give

$$e_/ + d\,e = 0;\ v_/ + d\,w = 0;\ \text{\&c. \&c.},$$

as was done with $d\,a$.

Substituting these values in place of $d\,e$, $d\,v$, &c., in equation (149), the resulting values will complete those of $d\,r$ and $d\,v$, which will no longer contain any arbitrary quantity, but will express the whole change in the longitude and distance arising from the action of the disturbing forces. Hence, if (r) (v) be the elliptical values of *r* and *v*, given in article 392, but corrected for the secular variation of the elements, the radius vector and longitude of m in its troubled orbit will be determined by

$$r = (r) + d\,r, \quad v = (v) + d\,v\,.$$

Perturbations in Latitude

544. If the second powers of the masses be omitted as well as the squares of the eccentricities, and the products of the eccentricities by the inclination, the orbit at the epoch being the fixed plane, then by article 437

$$d\,s = \frac{d\,z}{a}, \;\; d\,z = y d\,q - x d\,p,$$

and in this case

$$y = a \sin\left(nt + \in\right), \; x = a\cos\left(nt + \in\right),$$

then

$$\frac{d\,z}{a} = d\,q\,.\sin\left(nt + \in\right) - d\,p\,.\cos\left(nt + \in\right),$$

and substituting the values of $d\,q,\;\; d\,p,$ from article 529,

$$\begin{aligned} d\,s = & \frac{m'.n^2}{n'^2 - n^2}\cdot\frac{a^2}{a'^2} g \sin\left(n't - nt + \in' - \Pi\right) \\ & + \frac{m'^2 .n^2 .a^2 a'}{2} g \sum \frac{B_{(i-1)}}{n^2 - \left(n + i\left(n' - n\right)\right)^2} \sin\left\{i\left(n't - nt + \in' - \in\right) + nt + \in - \Pi\right\}. \end{aligned}$$

Now if a plane very little inclined to the orbit of *m* be assumed for the fixed plane instead of that of the orbit at the epoch, and if $f,\; f',\; q,\; q',$ be the inclinations and longitudes of the nodes of the orbits of m and $m',$ on this new plane; then as g is the tangent of the mutual inclination of the two orbits, and Π the longitude of their mutual intersection, by article 444,

$$g \sin\Pi = p' - p; \;\; g \cos\Pi = q' - q\,.$$

If these values be substituted in $d\,s,$ and if $(s) + d\,s = s$ be the whole latitude of m in its troubled orbit above the fixed plane, then will

$$s = q\sin\left(nt+\in\right) - p\cos\left(nt+\in\right)$$
$$+\frac{m'.n^2}{n'^2-n^2}\cdot\frac{a^2}{a'^2}\left\{\left(q'-q\right)\sin\left(n't+\in'\right)-\left(p'-p\right)\sin\left(n't+\in'\right)\right\}$$
$$-\frac{m'n^2\ .a^2a'}{2}\left(q'-q\right)\Sigma\frac{B_{(i-1)}}{\left\{i\left(n'-n\right)+n\right\}^2-n^2}\times\sin\left(i\left(n't-nt+\in-\in\right)+nt+\in\right)$$
$$+\frac{m'n^2\ .a^2a'^2}{2}\left(p'-p\right)\Sigma\frac{B_{(i-1)}}{\left\{i\left(n'-n\right)^2+n\right\}^2-n^2}\times\sin\left(i\left(nt-nt+\in-\in\right)+nt+\in\right).$$

The two terms independent of m' are the latitude of m above the fixed plane when m remains on the plane of its primitive orbit. If the exact latitude of m be substituted for these two terms, this expression will be more correct.

Each disturbing planet will add an expression to s similar to ds; the sum of the whole will be the true latitude of m when troubled by all the bodies in the system.

545. By a similar process, the perturbations depending on the other powers and products of the eccentricities way be obtained, but it would lead to long and intricate reductions, from which Laplace's method, deduced directly from the equations (87), is exempt.

Notes

1 See note 16, *Preliminary Dissertation.*
2 See note 4, *Introduction.*
3 The third term element $\in'$ *reads* $\in$\` in the 1st edition.
4 The term $-3a\left(\frac{dA_1}{da}\right)$ *reads* $-3a\left(\frac{dA_/}{da}\right)$ in the 1st edition.

BOOK II

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CHAPTER IX

SECOND METHOD OF FINDING THE PERTURBATIONS OF A PLANET IN LONGITUDE, LATITUDE, AND DISTANCE

Determination of the general Equations

546. TO determine the perturbations $\delta v,\ \delta r,\ \delta s,$ from the three general equations,[1]

$$\frac{d^2x}{dt^2}+\frac{mx}{r^3}=\frac{dR}{dx}$$
$$\frac{d^2y}{dt^2}+\frac{my}{r^3}=\frac{dR}{dy}$$
$$\frac{d^2z}{dt^2}+\frac{mz}{r^3}=\frac{dR}{dz}.$$

The sum of these equations respectively multiplied by dx, dy, dz is

$$\frac{dxd^2x+dyd^2y+dzd^2z}{dt^2}+\frac{m\left(xdx+ydy+zdz\right)}{r^3}=dx\left(\frac{dR}{dx}\right)+dy\left(\frac{dR}{dy}\right)+dz\left(\frac{dR}{dz}\right). \qquad \textbf{(151)}$$

The integral of which is evidently

$$\frac{dx^2+dy^2+dz^2}{dt^2}-\frac{2m}{r}+\frac{m}{a}=2\int \mathrm{d}R\,. \qquad \textbf{(152)}$$

The differential of R is only relative to the co-ordinates of m, because the motions of that body alone are under consideration; a is an arbitrary constant quantity introduced by integration; it is half the greater axis of the orbit of m when R is zero. Again, the same equations, respectively multiplied by x, y, and z, and added to the preceding integral, give

$$\frac{xd^2x+yd^2y+zd^2z}{dt^2}+\frac{dx^2+dy^2+dz^2}{dt^2}+\frac{m\left(x^2+y^2+z^2\right)}{r^3}$$
$$-\frac{2m}{r}+\frac{m}{a}=x\left(\frac{dR}{dx}\right)+y\left(\frac{dR}{dy}\right)+z\left(\frac{dR}{dz}\right)+2\int \mathrm{d}R\,.$$

The two first members of this equation are equal to $\frac{1}{2}\frac{d^2r^2}{dt^2}$, the third is $\frac{m}{r}$ and if to abridge rR' be put for

$$x\left(\frac{dR}{dx}\right)+y\left(\frac{dR}{dy}\right)+z\left(\frac{dR}{dz}\right),$$

the equation becomes

$$\frac{1}{2}\frac{d^2r^2}{dt^2}-\frac{m}{r}+\frac{m}{a}=2\int \mathrm{d}\,.R+rR'.$$

Let dv be the indefinitely small angle mSh, fig. 89, contained between

$$Sm=r, \text{ and } Sh=r+\mathbf{d}r, \text{ then } mh^2=ma^2+ah^2;$$

but

$$ma=rdv, \text{ and } ah=dr,$$

hence

$$mh^2=dr^2+r^2dv^2=dx^2+dy^2+dz^2.$$

But

$$xd^2x+yd^2y+zd^2z=d\left(xdx+ydy+zdz\right)-\left(dx^2+dy^2+dz^2\right)=rd^2r-r^2dv^2;$$

so that the equation in question becomes,

$$\frac{rd^2r-r^2\,.dv^2}{dt^2}+\frac{m}{r}=rR',$$

whence[2]

$$\frac{dv^2}{dt^2}-\frac{d^2r}{rdt^2}-\frac{m}{r^3}=-\frac{1}{r}R'.$$

547. In solving equations by approximation, a value of the unknown quantity is found by omitting some of the smaller terms, then the value so found is substituted in the equation, and a new value is sought, including the terms that were at first omitted.

Now values of r and v have been determined in the elliptical orbit by omitting the parts containing the disturbing forces, but if $r+\mathbf{d}r$ [and] $v+\mathbf{d}v$ be put for r and v in the preceding equations, the parts containing the elliptical motion may be omitted, and the remaining terms will give the perturbations. It is evident, however, that this substitution must not be made in R, since it contains the first powers of the disturbing action already. Consequently, the equations in question become

$$\frac{d^2\,.r\mathbf{d}r}{dt^2}+\frac{mr\mathbf{d}r}{r^3}=2\int \mathrm{d}R+rR'$$

$$\frac{2r^2dv\,.\,d\mathbf{d}v}{dt^2}=\frac{rd^2\mathbf{d}r-\mathbf{d}r\,.\,d^2r}{dt^2}-\frac{3mr\mathbf{d}r}{r^3}-rR'$$

and eliminating $\frac{mrdr}{r^3}$ from the second by means of the first

$$ddv = \frac{d\left\{dr\,.\,d\,r + 2r\,.\,ddr\right\} - dt^2\left(3\int \mathrm{d}R + 2rR'\right)}{r^2dv}. \qquad \textbf{(153)}$$

548. The integral of the angle $dv = mSh$ does not lie all in one plane, but it is easy to obtain a value of that indefinitely small angle in functions of its projection $pS\mathrm{B} = dv'$. For let pB be the projection of the arc mh on the fixed plane of the ecliptic, then $Sm = r$, $mp = s$, the tangent of the latitude, and the curtate distance $Sp = \frac{r}{\sqrt{1+s^2}}$.

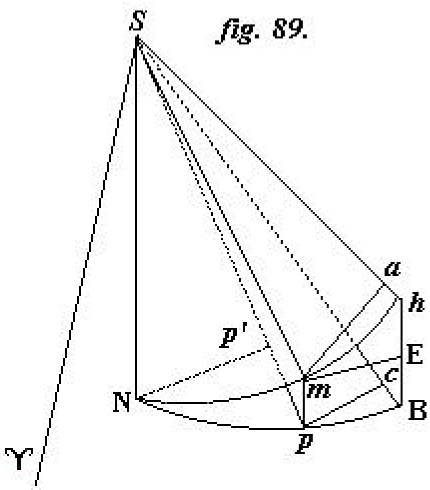

fig. 89.

Draw mE at right angles to Bh; and describe the arc pc with the radius Sp. Now the arc mh being indefinitely small, its projection pB is indefinitely small, therefore both may be taken in place of their sines; also[3]

$$(mh)^2 = (m\mathrm{E})^2 + (\mathrm{E}h)^2$$

and as

$$m\mathrm{E} = p\mathrm{B},$$

therefore[4]

$$(pc)^2 + (c\mathrm{B})^2 + (\mathrm{E}h)^2 = (mh)^2.$$

But

$$pc = \frac{rdv'}{\sqrt{1+s^2}};\; c\mathrm{B} = d\,.\,\frac{r}{\sqrt{1+s^2}} = \frac{dr\left(1+s^2\right) - rsds}{\left(1+s^2\right)^{\frac{3}{2}}}.$$

Again

$$mp = \frac{sr}{\sqrt{1+s^2}}.$$

And[5]

$$h\mathrm{E} = d\,.\,(mp) = \frac{rds + sdr\left(1+s^2\right)}{\left(1+s^2\right)^{\frac{3}{2}}};$$

hence

$$(mh)^2 = \frac{r^2dv'^2}{1+s^2} + dr^2 + \frac{r^2ds^2}{\left(1+s^2\right)^2};$$

but[6]

$$(mh)^2 = r^2dv^2 + dr^2,$$

and lastly,

$$dv' = dv \cdot \frac{\sqrt{\left(1+s^2\right)^2 - \frac{ds^2}{dv^2}}}{\sqrt{1+s^2}}. \tag{154}$$

Thus dv' is known when ds is determined; however, if the latitude be estimated from a known position of the orbit of the planet itself at any given epoch, instead of from the fixed plane of the ecliptic, it will be zero at that instant; and any latitude the planet may have at a subsequent period, can only arise from the disturbing forces, and must on that account be very small.

549. Assuming therefore *NpB*, fig. 89, to have been the orbit of the planet *m* at any given time, *s* and *ds* will be so small, that their squares may be omitted, and then $dv = dv'$. Also the radius vector *r*, only differs from the curtate distance $\frac{r}{\sqrt{1+s^2}}$ by the extremely small quantity $\frac{1}{2}rs^2$ which may be omitted, and then the true longitude *v* may be estimated on the plane *NpB* without sensible error; so that

$$SN = x = r\cos v;\quad Np' = y = r\sin v;$$

and as $z\left(\frac{dR}{dz}\right)$ is so small that it may be omitted,

$$rR' = x\left(\frac{dR}{dx}\right) + y\left(\frac{dR}{dy}\right) = r\left(\frac{dR}{dr}\right);$$

hence, the equation which determines the perturbations in the radius vector becomes[7]

$$\frac{d^2 . rdr}{dt^2} + \frac{mrdr}{r^3} = 2\int \mathrm{d}R + r\left(\frac{dR}{dr}\right). \tag{155}$$

550. It was shown in article 372, that r^2dv is the area described by the body in the indefinitely small time *dt*, therefore

$$r^2dv = \sqrt{ma\left(1-e^2\right)}dt = na^2\sqrt{1-e^2}\,.\,dt$$

hence the value of ddv becomes

$$ddv = \frac{1}{\sqrt{1-e^2}}\left\{\frac{d\left(2rd\,.dr + dr\,.dr\right)}{a^2ndt^2} - \frac{an}{m}\left(3\int \mathrm{d}R + 2r\left(\frac{dR}{dr}\right)\right)\right\}$$

and its integral is

$$d v = \frac{1}{\sqrt{1-e^2}} \left\{ \frac{2rd\,.d\,r + dr\,.d\,r}{a^2 ndt} - \frac{an}{m} \left(3\iint dt\,.\,\mathrm{d}R + 2\int r \left(\frac{dR}{dr} \right) dt \right) \right\} \qquad \textbf{(156)}$$

which determines the perturbations of m in longitude.

551. Since the orbit of m at the epoch is taken as the fixed plane, the only latitude the planet will have at a subsequent period must arise from the perturbations, and may therefore be represented by $d\,s$; hence $z = rd\,s$. And substituting this value of z in the third of the equations of motion in article 546, it becomes

$$\frac{d^2\,.\,rd\,s}{dt^2} + \frac{m\,.\,rd\,s}{r^3} - \frac{dR}{dz} = 0\,. \qquad \textbf{(157)}$$

A value of $d\,s$ may easily be found from this; and if it be then desired to refer the position of the planet to a plane which is but little inclined to that of its primitive orbit, it will only be necessary to add to this value of $d\,s$ the latitude of the planet, supposing it not to quit the plane of its primitive orbit.

Perturbations in the Radius Vector

552. These are obtained by successive approximations from the equation

$$\frac{d^2 rdr}{dt^2} + \frac{mrd\,r}{r^3} = 2\int \mathrm{d}R + r\left(\frac{dR}{dr}\right).$$

A value of $d\,r$ is first determined by omitting the eccentricities; that value is substituted in the same equation, and a new value of $d\,r$ is found, including the first powers of the eccentricities; that is again substituted, and a third value of $d\,r$ is obtained, containing the squares and products of the eccentricities and inclinations, and so on, till the remaining or rejected quantities are less than the errors of observation.

553. Supposing the orbits to be circular, then $r^{-3} = a^{-3}$; and by article 383, $\frac{m}{a^3} = n^2$. And if the mass of the planet be omitted when compared with that of the sun taken as the unit, the preceding equation, after these substitutions, becomes

$$\frac{d^2\,.\,rd\,r}{dt^2} + n^2 rdr = 2\int \mathrm{d}R + r\left(\frac{dR}{dr}\right).$$

But

$$r\left(\frac{dR}{dr}\right) = a\left(\frac{dR}{du}\right);$$

and in this case

$$R=\frac{m'}{2}\sum A_i \cos i\left(n't-nt+\in'-\in\right).$$

When

$$i=0,\ \cos i\left(n't-nt+\in'-\in\right)=1,$$

$$R=\frac{m}{2}A_0+\frac{m}{2}\cdot\sum.A_i\cos i\left(n't-nt+\in'-\in\right),$$

and as dR is the differential of R with regard to nt alone, therefore

$$2\int \mathrm{d}R+r\left(\frac{dR}{dr}\right)=2m'g+\frac{m'}{2}a\left(\frac{dA_0}{da}\right)+\frac{m'}{2}\sum\left\{\frac{2n}{n-n'}A_i+a\left(\frac{dA_i}{da}\right)\right\}\times\cos i\left(n't-nt+\in'-\in\right);$$

whence

$$\frac{d^2 rd\,r}{dt^2}+n^2 rd\,r=2m'g+\frac{m'}{2}a.\left(\frac{dA_0}{da}\right)+\frac{m'}{2}\sum\left\{\frac{2n}{n-n'}A_i+a\left(\frac{dA_i}{da}\right)\times\cos i\left(n't-nt+\in'-\in\right)\right\}.$$

The integral of this equation is

$$\frac{rd\,r}{a^2}=B+B'.\cos i\left(n't-nt+\in'-\in\right),$$

B and B' being indeterminate coefficients, then

$$\frac{d^2 rd\,r}{dt^2}+n^2rd\,r=Bn^2a^2+B'a^2\left(n^2-i^2\left(n'-n\right)^2\right)\cos i\left(n't-nt+\in'-\in\right).$$

And comparing the coefficients of like cosines,

$$B=2m'ag+\frac{m'}{2}a^2\left(\frac{dA_0}{da}\right),$$

$$B'=\frac{m'}{2}\frac{an^2}{n^2-i^2\left(n-n'\right)^2}\cdot\sum\cdot\left\{\frac{2n}{n-n'}A_i+a\left(\frac{dA_i}{da}\right)\right\},$$

and so

$$\frac{rd\,r}{a^2}=2m'ag+\frac{m'}{2}a^2\left(\frac{dA_0}{da}\right)+\frac{m'n^2}{2}\cdot\sum\cdot\frac{\left\{\frac{2n}{n-n'}aA_i+a^2\left(\frac{dA_i}{da}\right)\right\}}{n^2-i^2\left(n-n'\right)^2}\times\cos i\left(n't-nt+\in'-\in\right);$$

or if a be put for r in the first member, and because by article 536,

$$C_i = \frac{n^2\left\{\frac{2n}{n-n'}aA_i + a^2\left(\frac{dA_i}{da}\right)\right\}}{n^2 - i^2(n-n')^2}$$

[then]

$$\frac{d\,r}{a} = 2m'a\,g + \frac{m'}{2}a^2\left(\frac{dA_0}{da}\right) + \frac{m'}{2}\Sigma C_i \cos i(n't - nt + \epsilon' - \epsilon);$$

which is the first approximation.

554. When the first powers of the eccentricities are included,

$$r^{-3} = \frac{1}{a^3}\{1 + 3e\cos(nt + \epsilon - \varpi)\};$$

and therefore

$$\frac{d^2 r d\,r}{dt^2} + n^2 r d\,r + 3n^2 a\,.\,d\,r\,.\,e\cos(nt + \epsilon - \varpi) = 2\int \mathrm{d}R + r\left(\frac{dR}{dr}\right),$$

but

$$R = \frac{m'}{2}\Sigma M_0 e\cos\{i(n't - nt + \epsilon' - \epsilon) + nt + \epsilon - \varpi\} + \frac{m'}{2}\Sigma M_1 e'\cos\{i(n't - nt + \epsilon' - \epsilon) + nt + \epsilon - \varpi'\}$$

therefore

$$2\int \mathrm{d}R + r\left(\frac{dR}{dr}\right) = \frac{m'}{2}\Sigma\left\{\frac{2(i-1)n}{i(n-n')-n}M_0 + a\left(\frac{dM_0}{da}\right)\right\}e\cos\{i(n't - nt + \epsilon' - \epsilon) + nt + \epsilon - \varpi\}$$
$$+ \frac{m'}{2}\Sigma\left\{\frac{2(i-1)n}{i(n-n')-n}M_1 + a\left(\frac{dM_1}{da}\right)\right\}e'\cos\{i(n't - nt + \epsilon' - \epsilon) + nt + \epsilon - \varpi'\}.$$

By the substitution of this quantity, and of the preceding value of $\frac{d\,r}{a}$,

$$\frac{d^2 r d\,r}{dt^2} + n^2 r d\,r = -\frac{m'}{2}\Sigma\left\{3a^2n^2C_i - \frac{2(i-1)n}{i(n-n')-n}M_0 - a\frac{dM_0}{da}\right\}e\cos\{i(n't - nt + \epsilon' - \epsilon) + nt + \epsilon - \varpi\}$$
$$+ \frac{m'}{2}\Sigma\left\{\frac{2(i-1)n}{i(n-n')-n}M_1 + a\frac{dM_1}{da}\right\}e'\cos\{i(n't - nt + \epsilon' - \epsilon) + nt + \epsilon - \varpi'\}.$$

Let

$$\frac{rd\,r}{a^2} = \frac{m'}{2}Be\cos\{i(n't - nt + \epsilon' - \epsilon) + nt + \epsilon - \varpi\} + \frac{m'}{2}B'e'\cos\{i(n't - nt + \epsilon' - \epsilon) + nt + \epsilon - \varpi'\}$$

B and B' being indeterminate coefficients, then

$$\frac{d^2 r\delta r}{dt^2} + n^2 r\delta r = +\frac{m'}{2}\cdot Ba^2\left\{n^2 - \left(i\left(n-n'\right)+n\right)^2\right\}.e.\cos\left\{i\left(n't-nt+\epsilon'-\epsilon\right)+nt+\epsilon-\nu\right\}$$
$$+\frac{m'}{2}\cdot B'a^2\left\{n^2 - \left(i\left(n-n'\right)+n\right)^2\right\}.e'.\cos\left\{i\left(n't-nt+\epsilon'-\epsilon\right)+nt+\epsilon-\nu'\right\}.$$

If to abridge

$$K_i = 3C_i - \frac{2(i-1)n}{i(n-n')-n} aM_0 - a^2\cdot\frac{dM_0}{da}$$

$$L_i = -\frac{2(i-1)}{i(n-n')-n} aM_1 - a^2\frac{dM_1}{da},$$

$$B = \frac{-n^2.K_i}{n^2-\left\{i\left(n-n'\right)+n\right\}^2};\quad B' = \frac{-n^2.L_i}{n^2-\left\{i\left(n-n'\right)+n\right\}^2};$$

and because $a^3n^2 = 1$,

$$\frac{r\delta r}{a^2} = \Sigma\frac{m'n^2}{2\left\{i\left(n-n'\right)+n\right\}^2-2n^2}\left\{\begin{array}{l}+K_i.e\cos\left\{i\left(n't-nt+\epsilon'-\epsilon\right)+nt+\epsilon-\nu\right\}\\ +L_i.e'\cos\left\{i\left(n't-nt+\epsilon'-\epsilon\right)+nt+\epsilon-\nu'\right\}\end{array}\right\};$$

where i may have any whole value, positive or negative, except zero. But in order to have the complete value of $\frac{r\delta r}{a^2}$, according to the theory of linear equations the integral of

$$\frac{d^2.r\delta r}{dt^2} + n^2 r\delta r = 0$$

must be added. The true integral of this equation is

$$\frac{r\delta r}{a^2} = m'fe.\cos\left(nt\left(1+c\right)+\epsilon-\nu\right) + m'f'e'.\cos\left(nt\left(1+c'\right)+\epsilon-\nu'\right)$$

where c and c' are given functions of the elements; but if it be assumed as is generally done, that the elliptical elements have already been corrected by their secular variations c and c' may be omitted, and then

$$\frac{r\delta r}{a^2} = m'fe.\cos\left(nt+\epsilon-\nu\right) + m'f'e'.\cos\left(nt+\epsilon-\nu'\right).$$

If all the parts of $\frac{r\delta r}{a^2}$ that have been determined in this and in the first approximation be collected, and if [8] $a\left(1-\cos\left(nt+\epsilon-\nu\right)\right)$ be put for r, then will

$$\frac{dr}{a} = +2m'ag + \frac{m'}{2}a^2\left(\frac{dA_0}{da}\right)$$
$$+m'f.e.\cos(nt+\in-v)+m'f'.e'.\cos(nt+\in-v')$$
$$+\frac{m'}{2}\cdot\frac{n^2}{\left(n^2-i^2(n'-n)^2\right)}\cdot\Sigma\left\{\frac{2n}{n-n'}aA_i+a^2\left(\frac{dA_i}{da}\right)\right\}\cos(n't-nt+\in'-\in)$$
$$+\frac{m'}{2}\cdot\Sigma\cdot\left\{C_i+\frac{n^2K_i}{\{i(n'-n)+n\}^2-n^2}\right\}\cdot e\cdot\cos\{i(n't-nt+\in'-\in)+nt+\in-v\}$$
$$+\frac{m'}{2}\cdot\Sigma\cdot\left\{\frac{nL_i}{\{i(n'-n)+n\}^2-n^2}\right\}\cdot e'\cdot\cos\{i(n't-nt+\in'-\in)+nt+\in-v\}.$$

If substitution be made for K_i and L_i, it will be found that the coefficients in this expression are identical with those in article 536, so that

$$\frac{n^2\Sigma\left\{\frac{2n}{n-n'}aA_i+a^2\left(\frac{dA_i}{da}\right)\right\}}{n^2-i^2(n'-n)^2}=C_i,$$
$$C_i+\frac{n^2K_i}{\{i(n'-n)+n\}-n^2}=D_i,$$
$$\frac{n^2L_i}{\{i(n'-n)+n\}^2-n^2}=F_i,$$

consequently

$$\frac{dr}{a} = +2m'ag + \frac{m'}{2}\cdot a^2\left(\frac{dA_0}{da}\right)+\frac{m'}{2}\Sigma C_i\cos i(n't-nt+\in'-\in)$$
$$+m'.fe.\cos(nt+\in-v)+m'.f'.e'.\cos(nt+\in-v)$$
$$+m'.\Sigma.D_i.e.\cos\{i(n't-nt+\in'-\in)+nt+\in-v\}$$
$$+m'.\Sigma.F_i.e'.\cos\{i(n't-nt+\in'-\in)+nt+\in-v'\}.$$

Perturbations in Longitude

555. The perturbations in longitude may now he found from equation (156); which becomes, when e^2 is omitted, and $m=a^3n^2=1$,

$$dv=\frac{2rd.dr+dr.dr}{a^2.ndt}-3an\iint dt\mathrm{d}R-2an\int r\left(\frac{dR}{dr}\right)dt.$$

By the substitution of the preceding values of R and $d\,r$ it will be found that the perturbations in longitude are[9]

$$d\,v = -m'a\left(3g + a\left(\frac{dA_0}{da}\right)\right).\,nt$$
$$+\frac{m'}{2}\sum\left\{-\frac{n^2}{i\left(n-n'\right)^2}aA_i + \frac{2n^3\left\{\frac{2n}{n-n'}aA_i + a^2\left(\frac{dA_i}{da}\right)\right\}}{i\left(n-n'\right)n^2 - i^2\left(n-n'\right)^2}\right\}\times\sin i\left(n't - nt + \epsilon' - \epsilon\right)$$
$$+m'f_/ e\sin\left(nt+\epsilon - v\right) + m'f_/'e'.\sin\left(nt+\epsilon - v'\right)$$
$$+m'e\sum G_i\sin\left\{i\left(nt - nt + \epsilon - v\right) + nt + \epsilon - v\right\}$$
$$+m'e'\sum H_i\sin\left\{i\left(nt - nt + \epsilon - v\right) + nt + \epsilon - v'\right\} + C,$$

where

$$f_/ = 3a^2\frac{dA_0}{da} + a^3\frac{d^2A_0}{da^2} + 2ag$$

[and]

$$f_/' = \tfrac{3}{2}aA_1 - \tfrac{3}{2}a^2\frac{dA_1}{da} - a^3\frac{d^2A_1}{da^2} - 2f'.$$

556. If all the periodic terms be omitted in the expressions $r + d\,r$ and $v + d\,v$, they become[10]

$$r + d\,r = a + 2m'a^2g + \tfrac{1}{2}m'a^2\left(\frac{dA_0}{da}\right)$$
$$v + d\,v = nt + \epsilon - m'\left(3ag + a^2\left(\frac{dA_0}{da}\right)\right).\,nt;$$

$v + d\,v$ is the mean longitude of the planet at the end of the time t; and if it be assumed that this longitude is the same as the elliptical orbit of the planet, and in the orbit it really describes, this condition will determine g. Whence

$$g = -\tfrac{1}{3}a\left(\frac{dA_0}{da}\right);$$

and, as before

$$r + d\,r = a - \frac{m'}{6}a^3\left(\frac{dA_0}{da}\right),$$

which is the constant part of the radius vector in the troubled orbit.

Thus a is not the mean distance of the planet from the sun in the troubled orbit, as it is in the elliptical orbit. In the latter case a is deduced from the mean motion by the equation

$$n^2 = \frac{1}{a^3},$$

whereas in the troubled orbit it is

$$a - \frac{m'}{6} a^3 \left(\frac{dA_0}{da} \right).$$

Therefore the mean motion and periodic time are different from what they would have been had there been no disturbance; but when they are produced they are permanent, and unchangeable in their quantity by the subsequent action of the other bodies of the system.

The perturbations in the co-ordinates of a planet depend on the angular distances of the disturbed and disturbing bodies, that is, on the differences of their mean longitudes; but terms of the form

$$f_{/} e \sin\left(nt + \in - \nu'\right),\ f'_{/} e' \sin\left(nt + \in - \nu'\right)$$

belong to elliptical motion; they form a part of the equation of the centre, but they will vanish from $d\nu$ if $f_{/}$ and $f'_{/}$, which are perfectly arbitrary, be made zero; in that case

$$f = \tfrac{1}{2}\left(\tfrac{7}{3} a^2 \frac{dA_0}{da} + a^3 \frac{d^2 A_0}{da^2} \right);$$

$$f' = \tfrac{1}{2}\left\{ \tfrac{3}{2} a A_1 - \tfrac{3}{2} a^2 \frac{dA_1}{da} - a^3 \frac{d^2 A_1}{da} \right\}.$$

and as the arbitrary constant quantity C may be made zero, the perturbations in the radius vector and longitude of m are

$$\begin{aligned} dr = &-\frac{m'}{6} a^2 \left(\frac{dA_0}{da} \right) + \frac{m'}{2} \sum C_i \cos i\left(n't - nt + \in' - \in\right) \\ &+ m' f e \cos\left(nt + \in - \nu\right) + m' f' e' \cos\left(nt + \in - \nu'\right) \\ &+ m' e \sum D_i \cos\left\{ i\left(n't - nt + \in' - \in\right) + nt + \in - \nu \right\} \\ &+ m' e' \sum E_i \cos\left\{ i\left(n't - nt + \in' - \in\right) + nt + \in - \nu' \right\}; \end{aligned} \qquad \textbf{(158)}$$

[and]

$$\begin{aligned} d\nu = &+\frac{m'}{2} \sum F_i \sin i\left(n't - nt + \in' - \in\right) \\ &+ m' e \sum G_i \sin\left\{ i\left(n't - nt + \in' - \in\right) + nt + \in - \nu \right\} \\ &+ m' e' \sum H_i \sin\left\{ i\left(n't - nt + \in' - \in\right) + nt + \in - \nu' \right\}. \end{aligned} \qquad \textbf{(159)}$$

The coefficients being the same as in article 537, and i may have any whole value, except zero.

557. The integral

$$\frac{rdr}{a^2} = m'.f\ .e.\cos\left(nt(1+c)+\in-v\ \right)+m'.f'\ .e'.\cos\left(nt(1+c')+\in-v\ \right),$$

by the resolution of the cosines becomes

$$\frac{rdr}{a^2} = +m'.f\ .e.\cos\left(nt+\in-v\ \right)+m'.f'\ .e'.\cos\left(nt+\in-v\ '\right)$$
$$-m'.f\ .e.cnt.\sin\left(nt+\in-v\ \right)-m'.f'.e'\ .c'nt.\sin\left(nt+\in-v\ '\right);$$

and as it is given under this form by direct integration, it was very[11] embarrassing to mathematicians, because the terms containing the arcs cnt, $c'nt$, as coefficients, increase indefinitely with the time, and if such inequalities really had existence in our system, its stability would soon be at end. The expression (98) for the radius vector does not contain a term that increases with the time, neither does the series R; consequently the arc nt could not be introduced into the differential equation (155), unless R contained terms of the form

$$A.\frac{\sin}{\cos}\left(a+b\,t+g\,t^2+\&c.\right),$$

the differential of which would produce them.

Now, the powers and products of sines and cosines introduce the sines and cosines of multiple arcs, but never the sines or cosines of the powers of arcs; consequently R does not contain terms of the preceding form, and therefore the differential equation (155) does not contain any term that increases with the time. Terms in the finite equations, that have the arc nt as coefficient, really arise from the imperfection of analysis, by which, in the course of integration, periodic terms, such as $A.\cos\left(nt+\in-v\ \right)$, are introduced under their developed form $a+b\,t+g\,t^2+\&c.$; and, as Mr. Herschel[12] observes, that is not done at once, but by degrees; a first approximation giving only a, the next bt, and so on. In stopping here, it is obvious that we should mistake the nature of this inequality, and that a really periodical function, from the effect of an imperfect approximation, would appear under the form of one not periodical, and would lead to erroneous conclusions as to the stability of the system and the general laws of its perturbations.

When, by this manner of integration, terms that increase with the time are introduced, the method of reducing the integrals to the periodic form will be found in a Memoir by Laplace, in the *Mem. Acad. Sci.*, 1772, and in the fifth chapter, second book, of the *Mécanique Céleste.*[13]

Perturbations in Latitude

558. Those are found by substituting

$$\frac{dR}{dz}=-\frac{m'}{a'^2}g\ \sin\left(n't+\epsilon'-\Pi\right)+\frac{m'}{2}a'\sum B_{(i-1)}g\ \sin\left\{i\left(n't-nt+\epsilon'-\epsilon\right)+nt+\epsilon-\Pi\right\}$$

in

$$\frac{d^2 r\delta s}{dt^2}+n^2 . r\delta s=\frac{dR}{dz};$$

where the primitive orbit of m is assumed to be the fixed plane, and the product of the eccentricity by the inclination neglected. Making $a^2n^2=1,$ and integrating, the result is

$$\begin{aligned}\frac{\delta s}{a}=&+\frac{m'n^2}{n'^2-n^2}\cdot\frac{a^2}{a'^2}g\ \sin\left(n't+\epsilon'-\Pi\right)\\&+\frac{m'}{2}\cdot\frac{a'}{a}g\ \frac{\sum B_{(i-1)}}{n^2-\left(i\left(n'-n\right)+n\right)^2}\sin\left(i\left(n't-nt+\epsilon'-\epsilon\right)+nt+\epsilon-\Pi\right).\end{aligned} \quad \textbf{(160)}$$

This expression is the same with that in article 544. No constant quantities are added, because, being arbitrary, they are assumed to be zero, which does not interfere with the generality of the problem, and is more convenient for use: i may have any whole value, positive or negative, zero excepted.

Perturbations, including the Squares of the Eccentricities and Inclinations

559. When the approximation extends to the squares and products of the eccentricities and inclinations

$$r^{-3}=\frac{1}{a^3}\left\{1+3e\cos\left(nt+\epsilon-v\right)+3e^2\cos 2\left(nt+\epsilon-v\right)\right\};$$

and by article 451,

$$\begin{aligned}R=&+\frac{m'}{2}\cdot\sum N\ .\cos\left\{i\left(n't-nt+\epsilon'-\epsilon\right)+2nt+2\epsilon+L\right\}\\&+\frac{m'}{2}\cdot\sum N'.\cos\left\{i\left(n't-nt+\epsilon'-\epsilon\right)+L'\right\};\end{aligned}$$

whence

$$\begin{aligned}&2\int \mathrm{d}R+r\left(\frac{dR}{dr}\right)=\\&+\frac{m'}{2}\cdot\sum\left\{a\frac{dN}{da}+N\cdot\frac{2\left(2-i\right)n}{i\left(n'-n\right)+2n}\right\}\cos\left\{i\left(n't-nt+\epsilon'-\epsilon\right)+2nt+2\epsilon+L\right\}\\&+\frac{m'}{2}\cdot\sum\left\{a\frac{dN'}{da}-N'\cdot\frac{2n}{\left(n'-n\right)}\right\}\cos\left\{i\left(n't-nt+\epsilon'-\epsilon\right)+L'\right\}\end{aligned}$$

hence

$$\frac{d^2 r\delta r}{dt^2}+n^2 r\delta r+3n^2 a\,.\delta r\,.\left\{e\cos\left(nt+\in-\varpi\right)+e^2\cos 2\left(nt+\in-\varpi\right)\right\}=$$
$$+\frac{m'}{2}\cdot\Sigma\left\{a\frac{dN}{da}+N\cdot\frac{2\left(2-i\right)n}{i\left(n'-n\right)+2n}\right\}\cos\left\{i\left(n't-nt+\in'-\in\right)+2n+L\right\}$$
$$+\frac{m'}{2}\cdot\Sigma\left\{a\frac{dN'}{da}-N'\cdot\frac{2n}{\left(n'-n\right)}\right\}\cos\left\{i\left(n't-nt+\in'-\in\right)+L'\right\}.$$

The value of $\frac{\delta r}{a}$, given in article 558, must be substituted in the last term of the first member; but as all terms are rejected that do not contain the squares or products of the eccentricities and inclinations, the only part of $\frac{\delta r}{a}$ that is requisite is

$$\frac{\delta r}{a}=+\frac{m'}{2}\Sigma.C_i\cos i\left(n't-nt+\in'-\in\right)$$
$$+m'e\Sigma D_i\cos\left\{i\left(n't-nt+\in'-\in\right)+nt+\in-\varpi\right\}$$
$$+m'e'\Sigma E_i\cos\left\{i\left(n't-nt+\in'-\in\right)+nt+\in-\varpi'\right\};$$

where *i* may have every value, positive or negative, zero excepted. But if *i* be made negative in the two last terms, and if D_i', E_i', be the two coefficients in this case, then

$$\frac{\delta r}{a}=+\frac{m'}{2}\Sigma C_i\cos i\left(n't-nt+\in'-\in\right)$$
$$+m'e\Sigma D_i\cos\left\{i\left(n't-nt+\in'-\in\right)+nt+\in-\varpi\right\}$$
$$+m'e\Sigma D_i'\cos\left\{-i\left(n't-nt+\in'-\in\right)+nt+\in-\varpi\right\}\qquad\textbf{(161)}$$
$$+m'e'\Sigma E_i\cos\left\{i\left(n't-nt+\in'-\in\right)+nt+\in-\varpi'\right\}$$
$$+m'e'\Sigma E_i'\cos\left\{-i\left(n't-nt+\in'-\in\right)+nt+\in-\varpi'\right\};$$

but now *i* can only be a *positive* whole number.

When this quantity is substituted in the last term of the first member for δr, and terms of the second order alone retained, the differential equation becomes, when integrated by the method of indeterminate coefficients, or otherwise,

$$\frac{rdr}{a^2} = +\frac{m'.n^2}{\{i(n'-n)+3n\}\cdot\{i(n'-n)+n\}}\times \left\{\begin{array}{l} \frac{3}{2}e^2.\sum.\{\frac{1}{2}C_i+D_i\}.\cos\{i(n't-nt+\epsilon'-\epsilon)+2nt+2\epsilon-2v\} \\ +\frac{3}{2}ee'.\sum.E_i.\cos\{i(n't-nt+\epsilon'-\epsilon)+2nt+2\epsilon-v-v'\} \\ -\frac{1}{2}\sum\left\{\frac{2(2-i)n}{in'+(2-i)n}aN+a^2\frac{dN}{da}\right\}\cdot\cos\{i(n't-nt+\epsilon'-\epsilon)+2nt+L\} \end{array}\right\} \quad \textbf{(162)}$$

$$+\frac{m'.n^2}{\{i(n'-n)-n\}\cdot\{i(n'-n)+n\}}\times \left\{\begin{array}{l} \frac{3}{2}ee'.\sum.E_i.\cos\{i(n't-nt+\epsilon'-\epsilon)-v'+v\} \\ +\frac{3}{2}ee'.\sum.E'_i.\cos\{i(n't-nt+\epsilon'-\epsilon)+v'-v\} \\ +\frac{3}{2}\sum\{D_i+D'_i\}.e^2.\cos i(n't-nt+\epsilon'-\epsilon) \\ -\frac{1}{2}\sum\left\{a^2\frac{dN'}{da}-\frac{2n}{n'-n}aN'\right\}\cdot\cos\{i(n't-nt+\epsilon'-\epsilon+L')\} \end{array}\right\}.$$

Now in order to obtain the value of $\frac{dr}{a}$ from this expression it must be observed that

$$\frac{rdr}{a^2} = \frac{r}{a}\cdot\frac{dr}{a};$$

and when the elliptical value of r is substituted it becomes

$$\frac{rdr}{a^2} = \frac{dr}{a}\left\{1+\tfrac{1}{2}e^2 - e.\cos(nt+\epsilon-v) - \tfrac{1}{2}e^2\cos 2(nt+\epsilon-v)\right\}$$

whence

$$\frac{dr}{a} = \frac{rdr}{a^2} - \frac{dr}{a}\left\{\tfrac{1}{2}e^2 - e\cos(nt+\epsilon-v) - \tfrac{1}{2}e^2\cos 2(nt+\epsilon-v)\right\}.$$

If the value of $\frac{dr}{a}$ from equation (161) be substituted in the second member, it will be found, after the reduction of the products of the cosines, that the perturbations in the radius vector depending on the second powers of the eccentricities and inclinations are expressed by

$$
\begin{aligned}
\frac{dr}{a} = &+\frac{rd\,r}{a^2} + \frac{m'}{4}\cdot\Sigma\cdot\left\{\tfrac{1}{2}C_i + 2D_i\right\}\times e^2\,.\cos\left\{i\left(n't - nt + \in' - \in\right) + 2nt + 2\in - 2v\right\} \\
&+\frac{m'}{2}\cdot\Sigma\cdot E_i ee'\,.\cos\left\{i\left(n't - nt + \in' - \in\right) + 2nt + 2\in - v - v'\right\} \\
&+\frac{m'}{2}\cdot\Sigma\left\{D_i + D_i' - \tfrac{1}{2}C_i\right\}e^2\cos i\left(n't - nt + \in' - \in\right) \\
&+\frac{m'}{2}\cdot\Sigma\cdot E_i ee'\,.\cos\left\{i\left(n't - nt + \in' - \in\right) + v - v'\right\} \\
&+\frac{m'}{2}\cdot\Sigma\cdot E_i\,.\,ee'\,.\cos\left\{i\left(n't - nt + \in' - \in\right) - v + v'\right\};
\end{aligned}
\qquad \textbf{(163)}
$$

where $\frac{rd\,r}{a^2}$ represents equation (162).

560. With the values of $\frac{d\,r}{a}$ in (163) and (161), together with those terms of R that depend on the second powers and products of the eccentricities and inclinations, equation (156) gives the perturbations in longitude equal to[14]

$$
d\,v = \frac{1}{\sqrt{1-e^2}}\left\{
\begin{aligned}
&\frac{2d\left(rd\,r\right)}{a^2\,.\,ndt} + \frac{m'}{2}\cdot\Sigma\cdot\left\{D_i - D_i'\right\}e^2\,.\sin i\left(n't - nt + \in' - \in\right) \\
&+\frac{m'}{2}\cdot\Sigma\cdot E_i\,.\,ee'\,.\sin\left\{i\left(n't - nt + \in' - \in\right) + v - v'\right\} \\
&-\frac{m'}{2}\cdot\Sigma\cdot E_i'\,.\,ee'\,.\sin\left\{i\left(n't - nt + \in' - \in\right) - v - v'\right\} \\
&-\frac{m'}{2}\cdot\Sigma\cdot\left\{\tfrac{1}{2}C_i + D_i\right\}.\,e^2\,.\sin\left\{i\left(n't - nt + \in' - \in\right) + 2nt + 2\in - 2v\right\} \\
&-\frac{m'}{2}\cdot\Sigma\cdot E_i\,.\,ee'\,.\sin\left\{i\left(n't - nt + \in' - \in\right) + 2nt + 2\in - v - v'\right\} \\
&-\frac{m'}{2}\Sigma\left\{\frac{\left(6-3i\right)n^2}{\left(i\left(n'-n\right)+2n\right)^2}\cdot aN + a^2\cdot\frac{dN}{da}\cdot\frac{2n}{i\left(n'-n\right)+2n}\right\}\times \\
&\qquad\sin\left\{i\left(n't - nt + \in' - \in\right) + 2nt + L\right\} \\
&-\frac{m'}{2}\Sigma\left\{\frac{2n}{i\left(n'-n\right)}\cdot a^2\cdot\frac{dN'}{da} - \frac{3n^2}{i\left(n'-n\right)^2}\cdot aN'\right\}\times \\
&\qquad\sin\left\{i\left(n't - nt + \in' - \in\right) + 2nt + L'\right\}
\end{aligned}
\right\}. \qquad \textbf{(164)}
$$

561. The inequalities of this order are very numerous, it is therefore necessary to select those that have the greatest values and to reject the rest, which can only be done in each particular case from the values of the divisors

$$i(n'-n)+3n,\ i(n'-n)+2n,\ i(n'-n)+n,\ i(n'-n).$$

For if the mean motions of the bodies m and m' be so nearly commensurable as to make any of these a small fraction, the inequality to which it is divisor will in general be of sufficient magnitude to be computed.

562. The inequalities in latitude will be determined afterwards.

Perturbations depending on the Cubes and Products of three Dimensions of the Eccentricities and Inclinations

563. These perturbations are only sensible when the divisor $i(n'-n)+3n,$ is a very small fraction, that is, when the mean motions of the two bodies are nearly commensurable; but as this divisor arises from the angle $i(n't-nt+\in'-\in)+3nt+3\in$ alone, the only part of the series R that is requisite by article 451, is[15]

$$\begin{aligned}R=&+\frac{m'}{4}Q_0e'^3\cos\left\{i\left(n't-nt+\in'-\in\right)+3nt+3\in-3v'\right\}\\&+\frac{m'}{4}Q_1e'^2e\cos\left\{i\left(n't-nt+\in'-\in\right)+3nt+3\in-2v'-v\right\}\\&+\frac{m'}{4}Q_2e'e^2\cos\left\{i\left(n't-nt+\in'-\in\right)+3nt+3\in-v'-2v\right\}\\&+\frac{m'}{4}Q_3e^3\cos\left\{i\left(n't-nt+\in'-\in\right)+3nt+3\in-3v\right\}\\&+\frac{m'}{4}Q_4e'g^2\cos\left\{i\left(n't-nt+\in'-\in\right)+3nt+3\in-v-2\Pi\right\}\\&+\frac{m'}{4}Q_5eg^2\cos\left\{i\left(n't-nt+\in'-\in\right)+3nt+3\in-v-2\Pi\right\}.\end{aligned}$$

But

$$\begin{aligned}&\cos\left\{i\left(n't-nt+\in'-\in\right)+3nt+3\in-3v\right\}\\&=+\cos 3v\ .\cos\left\{i\left(n't-nt+\in'-\in\right)+3nt+3\in\right\}\\&\quad+\sin 3v\ .\sin\left\{i\left(n't-nt+\in'-\in\right)+3nt+3\in\right\};\end{aligned}$$

each cosine may be resolved in the same manner; and if

$$P = \tfrac{1}{4}\left\{\begin{array}{l} +Q_0\,.\,e'^3\,.\sin 3v' + Q_1 e'^2 e \sin(2v' + v) \\ +Q_2 e e'^2 \sin(v' + 2v) + Q_3 e^3 \sin 3v \\ +Q_4 e' g^2 \sin(2\Pi + v') + Q_5 e g^2 \sin(2\Pi + v) \end{array}\right\}, \qquad \textbf{(165)}$$

[and]

$$P' = \tfrac{1}{4}\left\{\begin{array}{l} +Q_0\,.\,e'^3\,.\cos 3v' + Q_1 e'^2 e \cos(2v' + v) \\ +Q_2 e e'^2 \cos(v' + 2v) + Q_3 e^3 \cos 3v \\ +Q_4 e' g^2 \cos(2\Pi + v') + Q_5 e g^2 \cos(2\Pi + v) \end{array}\right\}. \qquad \textbf{(166)}$$

This part of R becomes

$$\begin{aligned} R = &+ m'P\,.\sin\{i(n't - nt + \epsilon' - \epsilon) + 3nt + 3\epsilon\} \\ &+ m'P'\,.\cos\{i(n't - nt + \epsilon' - \epsilon) + 3nt + 3\epsilon\} \end{aligned} \qquad \textbf{(167)}$$

564. Let[16] $\frac{r\,dr}{a^2} = m'K\cos\{i(n't - nt + \epsilon' - \epsilon) + 2nt + 2\epsilon + B\}$ be the part of the equation (162) that has the divisor $i(n' - n) + 3n$; by the substitution of this, and of the preceding value of R, equation (155) gives, when integrated,[17]

$$\begin{aligned} \frac{r\,dr}{a^2} = &-\frac{2(i-3)m'n}{i(n'-n)+3n}\left\{\begin{array}{l} +aP\sin\{i(n't - nt + \epsilon' - \epsilon) + 3nt + 3\epsilon\} \\ +aP'\cos\{i(n't - nt + \epsilon' - \epsilon) + 3nt + 3\epsilon\} \end{array}\right\} \\ &-\tfrac{3}{2}m'eK\cos\{i(n't - nt + \epsilon' - \epsilon) + 3nt + 2\epsilon + B - v\} \\ &+\tfrac{1}{2}m'eK\cos\{i(n't - nt + \epsilon' - \epsilon) + nt + \epsilon + B + v\}; \end{aligned}$$

and because

$$\frac{r\,dr}{a^2} = \frac{r}{a}\cdot\frac{dr}{a} = \frac{r}{a}m'K\cos\{i(n't - nt + \epsilon' - \epsilon) + 2nt + 2\epsilon + B\}$$

the whole perturbations in the radius vector having the divisor $i(n' - n) + 3n$, are

$$\begin{aligned} \frac{dr}{a} = &+m'K\cos\{i(n't - nt + \epsilon' - \epsilon) + 2nt + 2\epsilon + B\} \\ &-m'Ke\cos\{i(n't - nt + \epsilon' - \epsilon) + 3nt + 3\epsilon - v + B\} \\ &+m'Ke\cos\{i(n't - nt + \epsilon' - \epsilon) + nt + \epsilon + v + B\} \\ &-\frac{2(i-3)nm'}{i(n'-n)+3n}\left\{\begin{array}{l} +aP\,.\sin\{i(n't - nt + \epsilon' - \epsilon) + 3nt + 3\epsilon\} \\ +aP'\,.\cos\{i(n't - nt + \epsilon' - \epsilon) + 3nt + 3\epsilon\} \end{array}\right\}. \end{aligned} \qquad \textbf{(168)}$$

565. If this quantity and the preceding value of R be substituted in equation (156) the result will be,

$$\begin{aligned} dv = &-\frac{3(3-i)m'n^2}{\{i(n'-n)+3n\}^2}\left\{\begin{array}{l} +aP'.\sin\{i(n't-nt+\epsilon'-\epsilon)+3nt+3\epsilon\} \\ -aP.\cos\{i(n't-nt+\epsilon'-\epsilon)+3nt+3\epsilon\} \end{array}\right\} \\ &+\frac{2m'n}{i(n'-n)+3n}\left\{\begin{array}{l} +a^2\left(\frac{dP}{da}\right)\cos\{i(n't-nt+\epsilon'-\epsilon)+3nt+3\epsilon\} \\ -a^2\left(\frac{dP'}{da}\right)\sin\{i(n't-nt+\epsilon'-\epsilon)+3nt+3\epsilon\} \end{array}\right\} \\ &-\frac{m'e}{2}K\sin\{i(n't-nt+\epsilon'-\epsilon)+3nt+3\epsilon-v+B\} \\ &+\frac{5}{2}m'eK\sin\{i(n't-nt+\epsilon'-\epsilon)+nt+\epsilon+v+B\}. \end{aligned} \qquad \textbf{(169)}$$

And as that part of dv article 560, having the divisor $i(n'-n)+3n$ is nearly[18]

$$dz = 2m'K\sin\{i(n't-nt+\epsilon'-\epsilon)+2nt+2\epsilon+B\}$$

if [19] $2K = K'$ the term $\frac{5}{4}m'eK'.\sin\{i(n't-nt+\epsilon'-\epsilon)+nt+\epsilon+v+B\}$ must replace the last term in the preceding value of dv, &c.

Secular Variation of the Elliptical Elements during the periods of the Inequalities

566. An inequality

$$\frac{C}{\{5n'-2n\}}\begin{array}{c}\sin\\ \cos\end{array}\{(5n'-2n)t+B\}$$

is at its maximum when the sine or cosine is unity; and if $5n'-2n$ be a small fraction, the coefficient

$$\frac{C}{\{5n'-2n\}^2}$$

will be very great. The period of an inequality is the time the argument or angle $(5n'-2n)t+B$ takes to increase from zero to 360°; it is evident that the period will be the greater, the less the difference $5n'-2n$.

Thus, the perturbations in longitude expressed by

$$d\,v = -\frac{3(3-i)\,m'n^2}{\{i(n'-n)+3n\}^2}\cdot\left\{\begin{array}{l} +aP'\sin\{i(n't-nt+\epsilon'-\epsilon)+3nt+3\epsilon\} \\ -aP\cos\{i(n't-nt+\epsilon'-\epsilon)+3nt+3\epsilon\}\end{array}\right\}$$

are very great, and of long periods, when $i(n'-n)+3n$ is a small fraction.

567. The square of the divisor could only be introduced by a double integration, consequently the preceding value of $d\,v$ is the integral of the part

$$d\,v = -3a\iint ndt\,.\,\mathrm{d}R$$

of equation (156), which is the periodic inequality in the mean motion of m, when troubled by m', in article 439. Thus, when the mean motions are nearly commensurable, all terms having the small divisors in question, must be applied as corrections to the mean motion of the troubled planet.

568. In some cases the periods of these inequalities extend to many centuries; in so long a time the secular variations of the elements of the orbits have a very sensible influence on these perturbations and in order to include this effect, the expression

$$d\,v = -3\iint andt\,.\,\mathrm{d}R$$

must be integrated *by parts* in the hypothesis of P and P' being variable functions of the elements. Now

$$R = +m'P\sin\{i(n't-nt+\epsilon'-\epsilon)+3nt+3\epsilon\}$$
$$+m'P'\cos\{i(n't-nt+\epsilon'-\epsilon)+3nt+3\epsilon\};$$

whence

$$\mathrm{d}R = +m'P\,.\,(3-i)\,ndt\,.\cos\{i(n't-nt+\epsilon'-\epsilon)+3nt+3\epsilon\}$$
$$-m'P'\,.\,(3-i)\,ndt\,.\sin\{i(n't-nt+\epsilon'-\epsilon)+3nt+3\epsilon\}$$

and

$$-3a\iint ndt\,.\,\mathrm{d}R = +3a(3-i)\,.\,m'\iint P'\,.\,n^2dt^2\,.\sin\{i(n't-nt+\epsilon'-\epsilon)+3nt+3\epsilon\}$$
$$-3a(3-i)\,.\,m'\iint P\,.\,n^2dt^2\,.\cos\{i(n't-nt+\epsilon'-\epsilon)+3nt+3\epsilon\}.$$

From the integration of this equation it will be found that the periodic inequality in the mean motion, depending on the third dimensions of the eccentricities and inclinations, and affected by the secular variations during its period, is

$$dv = dz = \frac{3(3-i)m'n^2}{\{i(n'-n)+3n\}^2} \times \left\{ \begin{array}{l} +\left\{a'P - \dfrac{2a\,.\,dP'}{\{i(n'-n)+3n\}dt} - \dfrac{3a\,.\,d^2P}{\{i(n'-n)+3n\}^2\,dt^2} - \&c.\right\}\times \\ \qquad \cos\{i(n't-nt+\epsilon'-\epsilon)+3nt+3\epsilon\} \\ -\left\{a'P + \dfrac{2a\,.\,dP}{\{i(n'-n)+3n\}dt} - \dfrac{3a\,.\,d^2P'}{\{i(n'-n)+3n\}^2\,dt^2} + \&c.\right\}\times \\ \qquad \sin\{i(n't-nt+\epsilon'-\epsilon)+3nt+3\epsilon\} \end{array} \right\}. \qquad \textbf{(170)}$$

This correction must be applied to the mean motions in the elliptical part of such planets as have their motions nearly commensurable.

569. The same method of integration may be employed for the term in equation (164), that has the divisor $n^2-\{i(n'-n)+2n\}^2$ when the quantity $i(n'-n)+3n$ is a small fraction, and in general to all inequalities of long periods having small divisors.

The variation of the elements during the periods of the inequalities may be estimated by the following approximate method, which will answer for several centuries before and after the epoch. By the method employed in article 563 the sum of the terms in equations (164) depending on the angle $i(n't-nt+\epsilon'-\epsilon)+2nt+2\epsilon$ may be put under the form

$$dv = m'\overline{P}\sin\{i(n't-nt+\epsilon'-\epsilon)+2nt+2\epsilon\}+m'\overline{P}'\cos\{i(n't-nt+\epsilon'-\epsilon)+2nt+2\epsilon\};$$

$\overline{P}$ and $\overline{P}'$ being functions of the elements of the orbits of m and m' determined by observation for a given epoch, say 1750. Since $\overline{P}$ and $\overline{P}'$ are known quantities, let

$$\frac{\overline{P}'}{\overline{P}} = \tan\overline{E}, \text{ and } \sqrt{\overline{P}^2+\overline{P}'^2} = \overline{F}$$

$\sin\overline{E}$ having the same sign with $\overline{P}'$ and $\overline{P}'$ with $\overline{P}$; hence

$$dv = m'\overline{F}\sin\{i(n't-nt+\epsilon'-\epsilon)+2nt+2\epsilon+\overline{E}\}$$

are the perturbations in question for the epoch 1750. Now if the time t be made equal to 500 in the expressions for the elements in article 480, values of P and P' will be found for the year 2250, with which new values of F and E may be computed for that era. Again, values of P and P' may be obtained from the same formulae for the year 2750, and by the method employed in article 480, the series

$$F = \overline{F} + \frac{d\overline{F}}{dt}t + \tfrac{1}{2}\frac{d^2\overline{F}}{dt^2}t^2 + \&c.$$
$$E = \overline{E} + \frac{d\overline{E}}{dt}t + \tfrac{1}{2}\frac{d^2\overline{E}}{dt^2}t^2 + \&c.$$

will give values of the variable coefficients for any time t during many centuries, consequently

$$d\,v = m'\left\{\overline{F} + \frac{d\overline{F}}{dt}t + \&c.\right\}\sin\left\{i\left(n't - nt + \epsilon' - \epsilon\right) + 2nt + 2\epsilon\ + \overline{E} + \frac{d\overline{E}}{dt}t^2 + \&c.\right\} \qquad \textbf{(171)}$$

will give the perturbations, including the secular variations in the elements of the orbits during their periods, $\overline{F}$, $\overline{E}$ and their differences being relative to the epoch 1750.

570. The formulae that have been obtained will give the places of all the planets at any instant with great accuracy, except those of Jupiter and Saturn, which are so remote from the rest, as to be almost beyond the sphere of their disturbing influence; but their proximity to one another, and their immense magnitude, render their mutual disturbances greater than those of any of the other planets. They may be regarded as forming with the sun a system by themselves; and as there are some circumstances in their motions peculiar to them alone, their theory will form a separate subject of consideration.

Notes

[1] The right hand side of the second equation *reads* $\frac{dR}{dx}$ in the 1st edition (published erratum).

[2] The second term *reads* $-\frac{d^2r}{rdt_2}$ in the 1st edition.

[3] The 1st edition expression *reads* $mh^2 = m\mathrm{E}^2 + \mathrm{E}h^2$.

[4] The 1st edition expression *reads* $pc^2 + c\mathrm{B}^2 + \mathrm{E}h^2 = mh^2$.

[5] The term $d\,.(mp)$ *reads*: " $.\,dmp$ " in the 1st edition.

[6] The expression *reads* $(mh)^2 = r^2dv + dr^2$ in the 1st edition (published erratum).

[7] Punctuation added.

[8] Right hand parenthesis is omitted in the 1st edition.

[9] First term *reads* $-m'a(3g + a\left(\frac{dA_0}{da}\right).nt$ in the 1st edition.

[10] The second and third terms in the first equation *read* $2m'ag + \tfrac{1}{2}m'a^2\left(\frac{dA_0}{da}\right)$ in the 1st edition (published erratum).

[11] This *reads* "very very embarrassing" in the 1st edition (published erratum).

[12] See note 63, *Preliminary Dissertation.*

[13] See note 4, *Introduction.*

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[14] A missing parenthesis in line 6 and missing accent on $\in$ in line 7 of the 1st edition *read* $i\left(n'-n\right)+2n)^2$ and $\sin\left\{i\left(n't-nt+\in-\in\right)+2nt+L'\right\}$.

[15] Punctuation added.

[16] This *reads* $\frac{rdr}{a^2}=m'K\cos\left\{i\left(n't-n't+\in-\in\right)+2nt+2\in+B\right\}$ in the 1st edition.

[17] Right hand parenthesis is omitted in the 1st edition.

[18] This *reads* $m'He$ *for* $2m'K$ in the 1st edition (published erratum).

[19] In the 1st edition, the remaining two lines in this article *read*:

"the terms $\frac{5}{4}m'eK\,.\sin\left\{i\left(n't-nt+\in'-\in\right)+nt+\in+v+B\right\}$ must be added to the preceding value of dv, which will then be the whole perturbations in longitude having the divisor in question." (published erratum).

Jupiter

This processed image of Jupiter was produced in 1990 by the U.S. Geological Survey from a Voyager image captured in 1979. Zones of ascending clouds alternate with bands of dark, descending clouds. The clouds travel around the planet in alternating eastward and westward belts at speeds of up to 540 kilometers per hour. Tremendous storms as big as earthly continents surge around the planet. The Great Red Spot (oval shape toward the lower-left) is an enormous anti-cyclonic storm that drifts along its belt, eventually circling the entire planet. (Courtesy of NASA)

BOOK II

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CHAPTER X

THE THEORY OF JUPITER AND SATURN

571. BY comparing ancient with modern observations, Halley[1] discovered that the mean motion of Jupiter had been accelerated, and that of Saturn retarded. Halley, Euler, Lagrange, Laplace, and other eminent mathematicians, were led by their researches to the certain conclusion that these inequalities do not depend on the configuration of the orbits; and as Laplace proved that they are not occasioned by the action of comets, or bodies foreign to the system, he could only suppose them to belong to the class of periodic inequalities.

Observation had already shown that five times the mean motion of Saturn is so nearly equal to twice the mean motion of Jupiter, that the difference of these two quantities, or $5n' - 2n$, is an extremely small fraction, being about the 74th part of the mean motion of Jupiter. Laplace perceived that the square of this minute quantity is divisor to some of the perturbations in the longitude of Jupiter and Saturn, which led him to conjecture that the nearly commensurable ratio in the mean motions might be the cause of this anomaly in the theory of these two planets; a conjecture which computation amply confirmed, showing that a great inequality of $48'2''.207$ at its maximum exists in the theory of Saturn, which at the present time increases the mean motion of the planet, and accomplishes its changes in about 929 years; and that the mean motion of Jupiter is also affected by a corresponding and contrary inequality of nearly the same period, only amounting to $19'46''.62$ at its maximum, which diminishes the mean motion of Jupiter.

These two inequalities attained their maximum in the year 1560; from that period, the apparent mean motion of the two planets approached to their true motions, and became equal to them in 1790, which accounts for Halley finding the mean motion of Saturn slower, and that of Jupiter faster, by a comparison of ancient with modern observations, than modern observations alone allowed them to be: whilst on the other hand, modern observations indicated to Lambert[2] an acceleration in Saturn's motion, and a retardation in that of Jupiter; and the quantities of the inequalities found by these astronomers are nearly the same with those determined by Laplace.

Recorded observations of these mean motions at very remote periods enable us to ascertain the chronology of the nations in which science had made early advances. Thus the Indians determined the mean motions of Jupiter and Saturn, when the mean motion of Jupiter was at its maximum of acceleration, and that of Saturn at its greatest retardation; the two periods at which that was the case, were 3102 years before the Christian era, and 1491 years after it.

The formulae of the motions of Jupiter and Saturn determined by Laplace, agree with their oppositions, the error not amounting to $12''.96$, when it is to be recollected that only twenty years ago[3] the errors in the best tables exceeded $1296''$. These formulae also represent with great precision the observations of Flamstead,[4] of the Arabian astronomers,[5] and of Ptolemy,[6] leaving no grounds to doubt that Laplace has succeeded in solving this difficulty, by assigning the true cause of these inequalities, which had for so many ages baffled the acuteness of astronomers; so that anomalies which seemed at variance with the law of gravitation, do in fact furnish the strongest corroboration of the universal influence it exerts throughout the solar system. Such, says Laplace, has been the fate of that brilliant discovery of Newton, that every difficulty which has been raised against it, has formed a new subject of triumph, the sure characteristic of a law of nature.

The precision with which these two greatest planets of our system have obeyed the laws of mutual gravitation from the earliest periods at which we have records of their motions, proves the stability of the system, since Saturn has experienced no sensible action of foreign bodies from the time of Hipparchus,[7] although the sun's attraction on Saturn is about a hundred times less than that exerted on the earth.

Periodic Variations in the Elements of the Orbits of Jupiter and Saturn, depending on the First Powers of the Disturbing Forces

572. If i be made equal to 5 in equation (169), the great inequality of Jupiter, including the secular variations of the elements of both orbits during its period of 929 years, is [8]

$$\begin{aligned} dv = dz = &+\frac{6m'n^2}{(5n'-2n)^2}\left\{\begin{array}{l}\left\{+aP'+\dfrac{2a\,.\,dP}{(5n'-2n)dt}-\text{\&c.}\right\}\cdot\sin(5n't-2nt+5\in'-2\in)\\ \left\{-aP+\dfrac{2a\,.\,dP'}{(5n'-2n)dt}+\text{\&c.}\right\}\cdot\cos(5n't-2nt+5\in'-2\in)\end{array}\right\}\\ &+\frac{2m'n}{5n'-2n}\left\{\begin{array}{l}a^2\cdot\dfrac{dP}{da}\cdot\cos(5n't-2nt+5\in'-2\in)\\ -a^2\cdot\dfrac{dP'}{da}\cdot\sin(5n't-2nt+5\in'-2\in)\end{array}\right\}\\ &-\frac{m'}{2}\cdot eK\,.\sin(5n't-2nt+5\in'-2\in-v+B)\\ &+\frac{5m'}{4}\cdot eK'.\sin(5n't-4nt+5\in'-2\in+v+B), \end{aligned} \qquad \textbf{(172)}$$

which must be applied as a correction to the mean motion of Jupiter.

573. Because of the equality and opposition of action and reaction, the great inequality in the mean motion of Saturn may be determined when that of Jupiter is known, and *vice versâ*; for by article 546,

$$\frac{dx^2+dy^2+dz^2}{dt^2}-2\frac{(S+m)}{r}=2\int \mathrm{d}R$$

may be assumed to belong to Jupiter, and

$$\frac{dx'^2+dy'^2+dz'^2}{dt^2}-2\frac{(S+m')}{r'}=2\int \mathrm{d}R'$$

to Saturn, dR and dR' relate to the co-ordinates of m and m'. Their sum, when the first equation is multiplied by m, and the second by m', is

$$2m\int \mathrm{d}R + 2m'\int \mathrm{d}R' = -2m\frac{(S+m)}{r} + m\frac{dx^2+dy^2+dz^2}{dt^2} - 2m'\frac{(S+m')}{r'} + m'\frac{dx'^2+dy'^2+dz'^2}{dt^2}.$$

The second member of this equation does not contain any term of the order of the squares of the disturbing masses having the divisor $5n't-2nt$, which can only arise from the integration of the sines or cosines of the angle $5n't-2nt$; because, when the elliptical values are substituted instead of x, y, z, the part [9]

$$-2m\frac{(S+m)}{r} + m\frac{dx^2+dy^2+dz^2}{dt^2},$$

will only contain the sines or cosines of the angle nt, and the remaining part of the second member is a function of $n't$ only; and as such terms as have the square of the divisor $5n't-2nt$ are alone under consideration, the second member may be omitted, then

$$m\int \mathrm{d}R + m'\int \mathrm{d}R' = 0. \qquad \textbf{(173)}$$

574. When $S+m=m$ is restored, which has hitherto been assumed equal to unity, the general expression for the periodic inequality in the mean motion of Jupiter is

$$dz = -3\iint \cdot \frac{andt\,.\,\mathrm{d}R}{S+m}.$$

The corresponding inequality in the mean motion of Saturn is

$$dz' = -3\iint \cdot \frac{a'n'dt\,.\,\mathrm{d}R'}{S+m'}.$$

From these two it is easy to find [10]

$$m(S+m).a'n'.dz + m'(S+m').an.dz' + 3m.a'n'\iint andt.\mathrm{d}R + 3m'.an\iint a'n'dt.\mathrm{d}R' = 0.$$

And in consequence of equation (173)[11]

$$m(S+m).a'n'.dz + m'(S+m').an.dz' = 0.$$

But

$$n = \frac{\sqrt{S+m}}{a^{\frac{3}{2}}} \quad n' = \frac{\sqrt{S+m'}}{a'^{\frac{3}{2}}};$$

and if the masses m and m' be omitted in $(S+m)$, $(S+m')$; in comparison of the mass of the sun taken as the unit, the preceding equation becomes

$$m\sqrt{a}\,.dz = -m'\sqrt{a'}\,.dz'.$$

Thus the periodic inequality in the mean motion of Jupiter is contrary to that in the mean motion of Saturn when n and n' have the same sines, which must always be the case, because both planets revolve about the sun in the same direction, so that one body is accelerated when the other is retarded, which corresponds with observation. These inequalities are in the ratio of $m\sqrt{a}$ to $m'\sqrt{a'}$; hence, if the inequality in the mean motion of Jupiter be known, that in the mean motion of Saturn will be found from

$$dz' = -\frac{m\sqrt{a}}{m'\sqrt{a'}}dz\,. \qquad \textbf{(174)}$$

575. As the whole of the following analyses depends[12] on the angle $5n't - 2nt + 5\epsilon' - 2\epsilon$, it will be represented by l for the sake of abridgement. If i be made equal to 5 in equation (167), it becomes

$$R = m'P\,.\sin l + m'P'.\sin l\,.$$

From this, values of dR, $\frac{dR}{de}$, $\frac{dR}{dv}$, may be found; but equations (165) and (166), show that

$$\left(\frac{dP}{dv}\right) = e\left(\frac{dP'}{de}\right); \quad \left(\frac{dP'}{dv}\right) = -e\left(\frac{dP}{de}\right);$$

consequently, by the substitution of dR, $\frac{dR}{de}$, $\frac{dR}{dv}$ in equations (114), the periodic variations in the eccentricity, longitude of the perihelion, and semigreater axis of Jupiter’s orbit, depending on the third powers of the eccentricities and inclinations, are easily found to be

$$de_{/} = +\frac{m'.an}{5n'-2n}\left\{\frac{dP}{de}\cdot\sin l + \frac{dP'}{de}\cdot\cos l\right\} \qquad \textbf{(175)}$$

$$edv_{/} = -\frac{m'.an}{5n'-2n}\left\{\frac{dP}{de}\cdot\cos l - \frac{dP'}{de}\cdot\sin l\right\}. \qquad \textbf{(176)}$$

576. The periodic inequalities in g and Π, the mutual inclination of the orbits of Jupiter and Saturn, and the longitude of the ascending node of the orbit of Saturn on that of Jupiter, are obtained from

$$R=\frac{m'}{4}\cdot Q_4 e' g^2 .\cos(l-2\Pi-v')+\frac{m'}{4}\cdot Q_5 eg^2 .\cos(l-2\Pi-v);$$

or

$$R=+\frac{m'}{4}\cdot g^2\cos 2\Pi\left\{Q_4 .e'\cos(l-v')+Q_5 .e\cos(l-v)\right\}$$
$$+\frac{m'}{4}\cdot g^2\sin 2\Pi\left\{Q_4 .e'\sin(l-v')+Q_5 .e\sin(l-v)\right\};$$

or to abridge

$$R=\frac{m'}{4}\cdot g^2\cos 2\Pi .A+\frac{m'}{4}\cdot g^2\sin 2\Pi .B.$$

But from article 444 it appears that

$$g^2 .\cos 2\Pi=(q'-q)^2-(p'-p)^2;\quad g^2 .\sin 2\Pi=2(q'-q)^2(p'-p)^2;$$

whence

$$R=\frac{m'}{4}\left\{(q'-q)^2-(p'-p)^2\right\}.A+\frac{m'}{4}\cdot 2(q'-q)(p'-p).B,$$

and

$$\frac{dR}{dp}=\frac{m'}{2}(p'-p).A-\frac{m'}{2}(q'-q).B,$$

or

$$\frac{dR}{dp}=\frac{m'}{2}.g\sin\Pi .A-\frac{m'}{2}g\cos\Pi .B;$$

restoring the values of A and B, and reducing the products of the sines and cosines,

$$\frac{dR}{dp}=-\frac{m'}{2}\cdot Q_4 .e'g .\sin(l-v-\Pi)-\frac{m'}{2}\cdot Q_5 .eg\cos(l-v-\Pi).\sin(l-v-\Pi),$$

But

$$\sin(l-v'-\Pi)=\sin(l+\Pi).\cos(v+2\Pi)-\cos(l+\Pi).\sin(v+2\Pi),$$

hence[13]

$$\frac{dR}{dp}=+\frac{m'}{2}\left\{Q_4 .e'g .\sin(v'+2\Pi)+Q_5 .eg .\sin(v+2\Pi)\right\}.\cos(l+\Pi)$$
$$+\frac{m'}{2}\left\{Q_4 .e'g .\cos(v'+2\Pi)+Q_5 .eg .\cos(v+2\Pi)\right\}.\sin(l+\Pi);$$

and from equations (165) and (166) it is clear that

$$\frac{dR}{dp} = m' \frac{dP}{dg} \cdot \cos\left(l + \Pi \right) - m' \frac{dP'}{dg} \cdot \sin\left(l + \Pi \right).$$

In the same manner it may be found that

$$\frac{dR}{dq} = -m' \cdot \frac{dP}{dg} \cdot \sin\left(l + \Pi \right) - m' \cdot \frac{dP'}{dg} \cdot \cos\left(l + \Pi \right);$$

with these values the two last of equations (114) become, when integrated,

$$d\,p = + \frac{m'.an}{5n' - 2n} \cdot \left\{ \frac{dP}{dg} \cdot \cos\left(l + \Pi \right) - \frac{dP'}{dg} \cdot \sin\left(l + \Pi \right) \right\};$$
$$d\,q = - \frac{m'.an}{5n' - 2n} \cdot \left\{ \frac{dP}{dg} \cdot \sin\left(l + \Pi \right) + \frac{dP'}{dg} \cdot \cos\left(l + \Pi \right) \right\}.$$

If s be the latitude of Jupiter, by article 436

$$s = q \sin v - p \cos v;$$

hence

$$d\,s = d\,q\,.\sin v - d\,p\,.\cos v,$$

and substituting for $d\,p,\ d\,q,$

$$d\,s = - \frac{m'.an}{5n' - 2n} \left\{ \frac{dP}{d\,l} \cdot \cos\left(l - v + \Pi \right) - \frac{dP'}{d\,l} \cdot \sin\left(l - v + \Pi \right) \right\} \qquad \textbf{(177)}$$

which is the only sensible inequality in the latitude of Jupiter in this approximation.

The latitude of Jupiter above the primitive orbit of Saturn is

$$s = -g \sin\left(v - \Pi \right)$$

whence

$$-d\,s = dg \sin\left(v - \Pi \right) - g d\Pi \cos\left(v - \Pi \right)$$

and a comparison of the two values of $d\,s,$ gives

$$dg' = \frac{m'.an}{5n' - 2n} \left\{ \frac{dP'}{dg} \cos\, l + \frac{dP}{dg} \sin\, l \right\}$$
$$g d\Pi' = - \frac{m'.an}{5n' - 2n} \left\{ \frac{dP}{dg} \cdot \cos\, l - \frac{dP'}{dg} \cdot \sin\, l \right\}.$$

These are the variations occasioned by the action of Saturn in the mutual inclination of the two orbits, and in the ascending node of their common intersection; but Jupiter produces a corresponding effect in these two quantities, and if it be expressed by dg'', $gd\Pi''$; then the whole variations will be

$$dg = dl' + dg'', \quad d\Pi = d\Pi' + d\Pi'';$$

but by article

$$dg'' = \frac{m.a'n'}{m'.an} \cdot dg'; \quad d\Pi'' = \frac{m.a'n'}{m'.an} \cdot d\Pi';$$

or, substituting for n and n', the whole variations in the two elements in question are

$$\begin{aligned} dg &= +\frac{m'.an}{5n'-2n} \cdot \frac{m\sqrt{a} + m'\sqrt{a'}}{m'\sqrt{a'}} \cdot \left\{ \frac{dP'}{dg} \cos l + \frac{dP}{dg} \sin l \right\} \\ gd\Pi &= +\frac{m'.an}{5n'-2n} \cdot \frac{m\sqrt{a} + m'\sqrt{a'}}{m'\sqrt{a'}} \cdot \left\{ \frac{dP'}{dg} \sin l - \frac{dP}{dg} \cos l \right\}. \end{aligned} \qquad \textbf{(178)}$$

577. The corresponding periodic inequalities in the latitude and elements of the orbit of Saturn are[14]

$$ds = -\frac{2a'n'.m}{(5n'-2n)m'} \begin{Bmatrix} +\dfrac{dP'}{dg} \sin\{4n't - 2nt + 4\epsilon' - 2\epsilon - v + \Pi\} \\ -\dfrac{dP}{dg} \cos\{4n't - 2nt + 4\epsilon' - 2\epsilon - v + \Pi\} \end{Bmatrix},$$

$$dz' = -\frac{m\sqrt{a}}{m'\sqrt{a'}} \cdot dz, \qquad \textbf{(179)}$$

$$de' = +\frac{m.a'n'}{5n'-2n} \left\{ \frac{dP'}{de'} \cos l + \frac{dP}{de'} \sin l \right\},$$

$$e'dv' = -\frac{m.a'n'}{5n'-2n} \left\{ \frac{dP}{de'} \cos l - \frac{dP'}{de'} \sin l \right\}.$$

It is evident that the variations in the mean motions are by much the greatest, on account on account of the divisor $(5n'-2n)^2$.

Periodic Variations in the Elements of the Orbits of Jupiter and Saturn, depending on the Squares of the Disturbing Forces

578. The equations in the preceding articles, which determine the periodic inequalities in the elements of the orbits of Jupiter and Saturn, are functions of the sines and cosines of their mean motions; and when the mean motions are corrected by the application of their great inequalities, the equations in question give secular as well as periodic inequalities in the elements of both orbits; depending on the squares and products of the disturbing masses.

The great inequalities may be put under a convenient form for this analysis, if the value of *R*, in article 563, be expressed by

$$R = m' . \Sigma . Q . \cos\{5n't - 2nt + 5\epsilon' - 2\epsilon - b\},$$

where b is a function of the longitudes of the perihelia and node of the common intersection of the two orbits. The substitution of this in

$$dz = -3 \iint . andt . dR,$$

gives

$$dz = -6m' \iint . an^2 dt^2 . \Sigma Q . \sin(5n't - 2nt + 5\epsilon' - 2\epsilon - b). \quad \textbf{(180)}$$

Since dz and dz' represent the great inequalities of Jupiter and Saturn, their corrected mean motions are $nt + dz$, and $n't + dz'$; and, by the substitution of these in the preceding equation, it becomes

$$(dz) = -6m' \iint . an^2 dt^2 . \Sigma Q . \sin\{5n't - 2nt + 5\epsilon' - 2\epsilon - b + 5dz' - 2dz\} \quad \textbf{(181)}$$

(dz) being the great inequality of Jupiter when the mean motions are corrected. In order to abridge, let

$$5n't - 2nt + 5\epsilon' - 2\epsilon = l,$$

then

$$\sin(l - b + 5dz' - 2dz) = \sin(l - b)\cos(5dz' - 2dz) + \cos(l - b)\sin(5dz' - 2dz).$$

But $5dz' - 2dz$ is so small, that it may be taken for its sine, and unity for its cosine; and as quantities of the order of the square of the disturbing forces are alone to be retained, $\sin(l - b)$ may be omitted; hence

$$\sin(l - b + 5dz' - 2dz) = \{5dz' - 2dz\}\cos(l - b);$$

or, as

$$dz' = -\frac{m\sqrt{a}}{m'\sqrt{a'}} \cdot dz$$

therefore

$$\sin(l - b + 5dz' - 2dz) = -\left\{\frac{5m\sqrt{a} + 2m'\sqrt{a'}}{m'\sqrt{a'}}\right\} \cdot dz \cdot \cos(l - b)$$

but the integral of equation (180) is

$$dz = \frac{6m'.an^2.\Sigma.Q}{(5n'-2n)^2}\cdot\sin(l-b),$$

consequently [15]

$$\sin(l-b+5dz'-2dz) = -\frac{(3m'.an^2.\Sigma.Q)}{(5n'-2n)^2}\cdot\left\{\frac{5m\sqrt{a}+2m'\sqrt{a'}}{m'\sqrt{a'}}\right\}\cdot\sin(2l-2b).$$

When this quantity is substituted in equation (181), instead of the sine, its integral

$$(dz) = -\frac{(3m'.an^2.\Sigma.Q)^2}{2(5n'-2n)^4}\left\{\frac{5m\sqrt{a}+2m'\sqrt{a'}}{m'\sqrt{a'}}\right\}\sin 2(5n't-2nt+5\epsilon'-2\epsilon-b) \qquad \textbf{(182)}$$

is the variation in the mean motion of Jupiter, and on account of the relation in article 574, the corresponding inequality in the mean motion of Saturn is

$$(dz') = \frac{(3m.an^2.\Sigma.Q)^2}{2(5n'-2n)^4}\left\{\frac{5m'\sqrt{a'}+2m\sqrt{a}}{m'\sqrt{a'}}\right\}\frac{m\sqrt{a}}{m'\sqrt{a'}}\sin 2(5n't-2nt+5\epsilon'-2\epsilon). \qquad \textbf{(183)}$$

These inequalities have a sensible effect, on account of the minute divisor $(5n'-2n)^4$.

579. The great inequalities in the mean motions also occasion variations in the eccentricities and longitudes of the perihelia, depending on the squares of the disturbing forces.

The principal term of the great inequality is sufficient for this purpose; and if the secular variations in the elements of the orbits during the period of the inequalities be omitted, the first term of the great inequality in the mean motion of Jupiter (172), when l is put for $5n't-2nt+5\epsilon'-2\epsilon$, is,

$$-\frac{6m'.an^2}{(5n'-2n)^2}\{P\cos l - P'\sin l\}.$$

The corresponding inequality in the mean motion of Saturn is

$$+\frac{6m'.an^2}{(5n'-2n)^2}\cdot\frac{m\sqrt{a}}{m'\sqrt{a'}}\{P\cos l - P'\sin l\}.$$

If these be applied as corrections to nt and $n't$, in the differential of equation (175), or

$$dde = +m' \cdot andt \cdot \left\{ \frac{dP}{de} \cdot \cos l - \frac{dP'}{de} \cdot \sin l \right\},$$

it will be found, by the same analysis that was employed in the last article, that

$$\begin{aligned} d\,.\,de &= +m'\,.\,andt \left\{ \frac{dP}{de} \cdot \cos l - \frac{dP'}{de} \cdot \sin l \right\} \\ &\quad - m'\,.\,andt \frac{dP}{de} \left\{ \frac{6m'\,.\,an^2}{(5n'-2n)^2} \frac{5\,m'\sqrt{a'} + 2\,m\sqrt{a}}{m'\sqrt{a'}} \right\} \left\{ P\,.\cos l \sin l - P' \sin^2 l \right\} \\ &\quad - m'\,.\,andt \frac{dP'}{de} \left\{ \frac{6m'\,.\,an^2}{(5n'-2n)^2} \frac{5m\sqrt{a'} + 2m\sqrt{a}}{m'\sqrt{a'}} \right\} \left\{ P\,.\cos^2 l - P' \cos l \sin l \right\}. \end{aligned} \qquad \textbf{(184)}$$

But

$$P \cos l \sin l - P' \sin^2 l = \tfrac{1}{2} P \sin 2l + \tfrac{1}{2} P' \cos 2l - \tfrac{1}{2} P'$$

$$P \cos^2 l - P' \cos l \sin l = \tfrac{1}{2} P \cos 2l - \tfrac{1}{2} P' \sin 2l + \tfrac{1}{2} P;$$

and, as terms depending on the first powers of the masses are to be rejected, the periodic part of the preceding equation is

$$\begin{aligned} de_2 &= -\frac{3m'^2\,.\,a^2 n^3}{2(5n'-2n)^3} \cdot \frac{5m\sqrt{a} + 2m'\sqrt{a'}}{m'\sqrt{a'}} \cdot \left\{ P' \cdot \frac{dP}{de} + P \cdot \frac{dP'}{de} \right\} \times \\ &\qquad \sin 2(5n't - 2\,nt + 5\epsilon' - 2\epsilon) \\ &\quad - \frac{3m'^2\,.\,a^2 n^3}{2(5n'-2n)^3} \cdot \frac{5m\sqrt{a} + 2m'\sqrt{a'}}{m'\sqrt{a'}} \cdot \left\{ P' \cdot \frac{dP'}{de} - P \cdot \frac{dP}{de} \right\} \times \\ &\qquad \cos 2(5n't - 2\,nt + 5\epsilon' - 2\epsilon). \end{aligned} \qquad \textbf{(185)}$$

By the same process it may be found that the periodic variations of *nt*, and $n't$, produce the periodic variation

$$\begin{aligned} dv_2 &= +\frac{3m'^2\,.\,a^2 n^3}{2e(5n'-2n)^3} \cdot \frac{5m\sqrt{a} + 2m'\sqrt{a'}}{m'\sqrt{a'}} \cdot \left\{ P \cdot \frac{dP}{de} - P' \cdot \frac{dP'}{de} \right\} \times \\ &\qquad \sin 2(5n't - 2\,nt + 5\epsilon' - 2\epsilon) \\ &\quad + \frac{3m'^2\,.\,a^2 n^3}{2e(5n'-2n)^3} \cdot \frac{5m\sqrt{a} + 2m'\sqrt{a'}}{m'\sqrt{a'}} \cdot \left\{ P' \cdot \frac{dP}{de} - P \cdot \frac{dP'}{de} \right\} \times \\ &\qquad \cos 2(5n't - 2\,nt + 5\epsilon' - 2\epsilon), \end{aligned} \qquad \textbf{(186)}$$

in the longitude of the perihelion of Jupiter. These are the only sensible periodic inequalities in the elements of Jupiter's orbit of this order. Corresponding variations obtain in those of the orbit of Saturn.

Secular Variations in the Elements of the Orbits of Jupiter and Saturn, depending on the Squares of the Disturbing Forces

580. The secular variations in the elements of the orbits of Jupiter and Saturn depending on the first powers of the disturbing forces, are determined by the formulae (130), in common with the other planets; but to these must be added their variations depending on the squares of the masses, quantities only sensible in the motions of Jupiter and Saturn.
The secular part of equation (184), arising from the corrected values of $nt,\ n't,$ is

$$(de) = -\frac{3m'^2\,.\,a^2n^3}{(5n'-2n)^2}\cdot t\cdot\frac{5m\sqrt{a}+2m'\sqrt{a'}}{m'\sqrt{a'}}\cdot\left\{P\cdot\frac{dP'}{de}-P'\cdot\frac{dP}{de}\right\}. \qquad \textbf{(187)}$$

and the corresponding variation in the longitude of the perihelion of Jupiter's orbit, depending on the squares of the disturbing forces, is

$$(dv) = \frac{3m'^2\,.\,a^2n^3}{e(5n'-2n)^2}\cdot t\cdot\frac{5m\sqrt{a}+2m'\sqrt{a'}}{m'\sqrt{a'}}\cdot\left\{P\cdot\frac{dP}{de}+P'\cdot\frac{dP'}{de}\right\}. \qquad \textbf{(188)}$$

The corresponding inequalities for Saturn are,

$$(de') = -\frac{3m^2\,.\,a^2n^3}{a'(5n'-2n)^2}\cdot t\cdot\frac{5m\sqrt{a}+2m'\sqrt{a'}}{m\sqrt{a}}\cdot\left\{P\cdot\frac{dP'}{de'}-P'\cdot\frac{dP}{de'}\right\}$$
$$(dv') = \frac{3m^2\,.\,a^3n^3}{a'e'(5n'-2n)}\cdot t\cdot\frac{5m\sqrt{a}+2m'\sqrt{a'}}{m'\sqrt{a'}}\cdot\left\{P\cdot\frac{dP}{de'}+P'\cdot\frac{dP'}{de'}\right\}. \qquad \textbf{(189)}$$

581. Thus the periodic inequalities in the mean motions cause both periodic and secular variations in the elements of the two orbits of the order of the squares of the disturbing forces; but the periodic variations in the other elements have the same effect; for, making

$$5n't-2nt+5\in'-2\in = l,$$

the differential of equation (175) is

$$d\in = +m'\cdot andt\left\{\frac{dP}{de}\cos l-\frac{dP'}{de}\sin l\right\};$$

and when all the elements are variable except the mean motion, the effects of which have already been determined,

$$d\,.\,de = +m'\,.\,andt\,.\left\{\begin{array}{l} -de\left(\dfrac{d^2P'}{de^2}\right)\cdot\sin\,l - \left(\dfrac{d^2P}{de^2}\right)\cdot\cos\,l \\ -dv\left(\dfrac{d^2P'}{dedv}\right)\cdot\sin\,l - \left(\dfrac{d^2P}{dedv}\right)\cdot\cos\,l \\ -de'\left(\dfrac{d^2P'}{dede'}\right)\cdot\sin\,l - \left(\dfrac{d^2P}{dede'}\right)\cdot\cos\,l \\ -dv'\left(\dfrac{d^2P'}{dedv'}\right)\cdot\sin\,l - \left(\dfrac{d^2P}{dedv'}\right)\cdot\cos\,l \\ -dg\left(\dfrac{d^2P'}{dedg}\right)\cdot\sin\,l - \left(\dfrac{d^2P}{dedl}\right)\cdot\cos\,l \\ -d\Pi\left(\dfrac{d^2P'}{ded\Pi}\right)\cdot\sin\,l - \left(\dfrac{d^2P}{ded\Pi}\right)\cdot\cos\,l \end{array}\right\}$$

If the values of dv, de, de', dv', dg, and $d\Pi$, from article 575, and those that follow, be substituted, observing at the same time, that equations (165) and (166) give

$$\begin{array}{ll} \dfrac{d^2P}{de\,.\,dv} = e\cdot\dfrac{d^2P'}{de^2}; & \dfrac{d^2P'}{de\,.\,dv} = -e\cdot\dfrac{d^2P'}{de^2}; \\ \dfrac{d^2P}{de\,.\,dv'} = e'\cdot\dfrac{d^2P'}{de\,.\,de'}; & \dfrac{d^2P'}{de\,.\,dv'} = -e'\cdot\dfrac{d^2P'}{de\,.\,de'}; \\ \dfrac{d^2P}{de\,.\,d\Pi} = g\cdot\dfrac{d^2P'}{de\,.\,dg}; & \dfrac{d^2P'}{de\,.\,d\Pi} = -g\cdot\dfrac{d^2P}{de\,.\,dg}; \end{array} \qquad \mathbf{(190)}$$

It will be found, when the periodic terms are omitted, and equation (187) added, that the whole secular variation in the eccentricity of Jupiter's orbit, depending on the squares of the disturbing forces, is

$$(de) = -\frac{3m'^2\,.\,a^2n^3}{(5n'-2n)^2}\cdot t\cdot\frac{2m'\sqrt{a'}+5m\sqrt{a}}{m'\sqrt{a'}}\cdot\left\{P\cdot\left(\frac{dP'}{de}\right) - P'\cdot\left(\frac{dP}{de}\right)\right\}$$

$$+\frac{m'^2\,.a^2\,.n^2}{5n'-2n}\cdot t\cdot\left\{\begin{array}{l} +\left(\dfrac{dP'}{de}\right)\cdot\left(\dfrac{d^2P}{de^2}\right) - \left(\dfrac{dP}{de}\right)\cdot\left(\dfrac{d^2P'}{de^2}\right) \\ +\left(\dfrac{dP'}{dg}\right)\cdot\left(\dfrac{d^2P}{dedg}\right) - \left(\dfrac{dP}{dg}\right)\cdot\left(\dfrac{d^2P'}{dedg}\right) \end{array}\right\}$$

$$+\frac{mm'.aa'.nn'}{5n'-2n}\cdot t\cdot\left\{\begin{array}{l}+\left(\frac{dP'}{de'}\right)\cdot\left(\frac{d^2P}{de.de'}\right)-\left(\frac{dP}{de'}\right)\cdot\left(\frac{d^2P'}{de.de'}\right)\\+\left(\frac{dP'}{dg}\right)\cdot\left(\frac{d^2P}{de.dg}\right)-\left(\frac{dP}{dg}\right)\cdot\left(\frac{d^2P'}{de.dg}\right)\end{array}\right\}. \qquad \textbf{(191)}$$

By the same process it may be found, that where the periodic terms which are quite insensible are omitted, the secular variation in the longitude of the perihelion of Jupiter's orbit, depending on the squares of the disturbing force, including the equation (188), is[16]

$$(dv)=+\frac{3m'^2.a^2n^3}{e(5n'-2n)^2}\cdot t\cdot\frac{5m\sqrt{a}+2m'\sqrt{a'}}{m'\sqrt{a'}}\cdot\left\{P\cdot\left(\frac{dP}{de}\right)+P'\cdot\left(\frac{dP'}{de}\right)\right\}$$
$$+\frac{m'^2.a^2.n^2}{e(5n'-2n)}\cdot t\cdot\left\{\begin{array}{l}+\left(\frac{dP}{de}\right)\cdot\left(\frac{d^2P}{de^2}\right)+\left(\frac{dP'}{de}\right)\cdot\left(\frac{d^2P'}{de^2}\right)\\+\left(\frac{dP}{dg}\right)\cdot\left(\frac{d^2P}{dedg}\right)+\left(\frac{dP'}{dg}\right)\cdot\left(\frac{d^2P'}{dedg}\right)\end{array}\right\} \qquad \textbf{(192)}$$
$$+\frac{mm'.aa'.nn'}{e(5n'-2n)}\cdot t\cdot\left\{\begin{array}{l}+\left(\frac{dP}{de'}\right)\cdot\left(\frac{d^2P}{de.de'}\right)+\left(\frac{dP'}{de'}\right)\cdot\left(\frac{d^2P'}{de.de'}\right)\\+\left(\frac{dP}{dg}\right)\cdot\left(\frac{d^2P}{de.dg}\right)+\left(\frac{dP'}{dg}\right)\cdot\left(\frac{d^2P}{de.dg}\right)\end{array}\right\}.$$

582. The corresponding variations for Saturn, including equations (189), are,

$$(de')=-\frac{3m^2.a^2n^3}{a'(5n'-2n)^2}\cdot t\cdot\frac{5m\sqrt{a}+2m'\sqrt{a'}}{m\sqrt{a}}\cdot\left\{P\cdot\left(\frac{dP'}{de'}\right)-P'\cdot\left(\frac{dP}{de'}\right)\right\}$$
$$+\frac{m^2.a'^2.n'^2}{5n'-2n}\cdot t\cdot\left\{\begin{array}{l}+\left(\frac{dP'}{de'}\right)\cdot\left(\frac{d^2P}{de'^2}\right)-\left(\frac{dP}{de'}\right)\cdot\left(\frac{d^2P'}{de'^2}\right)\\+\left(\frac{dP'}{dg}\right)\cdot\left(\frac{d^2P}{de'dg}\right)-\left(\frac{dP}{dg}\right)\cdot\left(\frac{d^2P'}{de'dg}\right)\end{array}\right\} \qquad \textbf{(193)}$$
$$+\frac{mm'.aa'.nn'}{5n'-2n}\cdot t\cdot\left\{\begin{array}{l}+\left(\frac{dP'}{de}\right)\cdot\left(\frac{d^2P}{de.de'}\right)-\left(\frac{dP'}{de}\right)\cdot\left(\frac{d^2P'}{de.de'}\right)\\-\left(\frac{dP'}{dg}\right)\cdot\left(\frac{d^2P}{de'.dg}\right)-\left(\frac{dP}{dg}\right)\cdot\left(\frac{d^2P'}{de'.dg}\right)\end{array}\right\};$$

$$(dv') = +\frac{3m^2 \,.\, a^3 n^3}{a'e'(5n'-2n)^2}\cdot t\cdot\frac{5m\sqrt{a}+2m'\sqrt{a'}}{m\sqrt{a}}\cdot\left\{P\cdot\left(\frac{dP}{de'}\right)+P'\cdot\left(\frac{dP'}{de'}\right)\right\}$$
$$+\frac{m^2 \,.\, a'^2 \,.\, n'^2}{e'(5n'-2n)}\cdot t\cdot\left\{\begin{array}{l}+\left(\frac{dP}{de'}\right)\cdot\left(\frac{d^2P}{de'^2}\right)+\left(\frac{dP'}{de'}\right)\cdot\left(\frac{d^2P'}{de'^2}\right)\\ +\left(\frac{dP}{dg}\right)\cdot\left(\frac{d^2P}{de'dg}\right)+\left(\frac{dP'}{dg}\right)\cdot\left(\frac{d^2P'}{de'dg}\right)\end{array}\right\} \qquad \textbf{(194)}$$
$$+\frac{mm'\,.\,aa'\,.\,nn'}{e'(5n'-2n)}\cdot t\cdot\left\{\begin{array}{l}+\left(\frac{dP}{de}\right)\cdot\left(\frac{d^2P}{de\,.\,de'}\right)+\left(\frac{dP'}{de}\right)\cdot\left(\frac{d^2P'}{de\,.\,de'}\right)\\ +\left(\frac{dP}{dg}\right)\cdot\left(\frac{d^2P}{de'\,.\,dg}\right)+\left(\frac{dP}{dg}\right)\cdot\left(\frac{d^2P'}{de'\,.\,dg}\right)\end{array}\right\}.$$

583. Secular variations, depending on the squares of the disturbing forces, arise from the same cause in the mutual inclination of the orbits, and in the longitude of the ascending mode of the orbit of Saturn on that of Jupiter. These are obtained from equations (178), considering the elements to be variable; then the substitution of their periodic variations will give, in consequence of

$$\left(\frac{dP'}{dg}\right)\cdot\left(\frac{d^2P}{dg^2}\right)-\left(\frac{dP}{dg}\right)\cdot\left(\frac{d^2P'}{dg^2}\right)=0.$$

$$(dg) = -\frac{3m'^2 \,.\, a^2 n^3}{(5n'-2n)^2}\cdot t\cdot\frac{m\sqrt{a}+m'\sqrt{a'}}{m'\sqrt{a'}}\cdot\frac{5m\sqrt{a}+2m'\sqrt{a'}}{m'\sqrt{a'}}\left\{P\cdot\left(\frac{dP'}{dg}\right)-P'\cdot\left(\frac{dP}{dg}\right)\right\}$$
$$+\frac{m'^2 \,.\, a^2 \,.\, n^2}{5n'-2n}\cdot t\cdot\frac{m\sqrt{a}+m'\sqrt{a'}}{m'\sqrt{a'}}\cdot\left\{\left(\frac{dP'}{de}\right)\cdot\left(\frac{d^2P}{de\,.\,dg}\right)-\left(\frac{dP}{de}\right)\cdot\left(\frac{d^2P'}{de\,.\,dg}\right)\right\} \qquad \textbf{(195)}$$
$$+\frac{mm'\,.\,aa'\,.\,nn'}{5n'-2n}\cdot t\cdot\frac{m\sqrt{a}+m'\sqrt{a'}}{m'\sqrt{a'}}\cdot\left\{\left(\frac{dP'}{de'}\right)\cdot\left(\frac{d^2P}{de'\,.\,dg}\right)-\left(\frac{dP}{de'}\right)\cdot\left(\frac{d^2P'}{de'\,.\,dg}\right)\right\};$$

$$(d\Pi) = +\frac{3m'^2 \,.\, a^2 n^3}{g\,(5n'-2n)^2}\cdot t\cdot\frac{m\sqrt{a}+m'\sqrt{a'}}{m'\sqrt{a'}}\cdot\frac{5m\sqrt{a}+2m'\sqrt{a'}}{m'\sqrt{a'}}\left\{P\left(\frac{dP}{dg}\right)+P'\left(\frac{dP'}{dg}\right)\right\}$$
$$+\frac{m'^2 \,.\, a^2 \,.\, n^2}{g\,(5n'-2n)}\cdot t\cdot\frac{m\sqrt{a}+m'\sqrt{a'}}{m'\sqrt{a'}}\left\{\left(\frac{dP}{de}\right)\left(\frac{d^2P}{de\,.\,dg}\right)+\left(\frac{dP'}{de}\right)\left(\frac{d^2P'}{de\,.\,dg}\right)\right\} \qquad \textbf{(196)}$$
$$+\frac{mm'\,.\,aa'\,.\,nn'}{5n'-2n}\cdot t\cdot\frac{m\sqrt{a}+m'\sqrt{a'}}{m'\sqrt{a'}}\left\{\begin{array}{l}+\left(\frac{dP'}{de'}\right)\left(\frac{d^2P}{de'\,.\,dg}\right)-\left(\frac{dP}{de'}\right)\left(\frac{d^2P'}{de'\,.\,dg}\right)\\ +\left(\frac{dP}{dg}\right)\left(\frac{d^2P}{dg^2}\right)+\left(\frac{dP'}{dg}\right)\left(\frac{d^2P'}{dg^2}\right)\end{array}\right\};$$

584. These are the variations with regard to the plane of Jupiter's orbit at a given time, but the variations in the position of the orbits of Jupiter and Saturn with regard to the ecliptic may easily be found, for $f,\ f'$, being the inclinations of the orbits of m and m' on the fixed ecliptic at the epoch, and $q,\ q'$, the longitudes of the ascending nodes estimated on that plane, by article 444,

$$p' - p = g \sin\Pi; \qquad q' - q = g \cos\Pi;$$

or

$$f'\sin q' - f\sin q = g \sin\Pi,$$
$$f'\cos q' - f\cos q = g \cos\Pi.$$

and on account of the action and reaction of Jupiter and Saturn,

$$d\left(f'.\sin q'\right) = -\frac{m\sqrt{a}}{m'\sqrt{a'}}\cdot d\left(f\sin q\right),$$
$$d\left(f'.\cos q'\right) = -\frac{m\sqrt{a}}{m'\sqrt{a'}}\cdot d\left(f\cos q\right).$$

And from these four equations, it will readily be found, that

$$(df) = -\frac{m'\sqrt{a'}}{m\sqrt{a}+m'\sqrt{a'}}\left\{dg\,.\cos\left(\Pi-q\right)-gd\Pi\sin\left(\Pi-q\right)\right\}$$
$$(fdq) = -\frac{m'\sqrt{a'}}{m\sqrt{a}+m'\sqrt{a'}}\left\{dg\,.\sin\left(\Pi-q\right)+gd\Pi\cos\left(\Pi-q\right)\right\}$$
$$(df') = -\frac{m\sqrt{a}}{m\sqrt{a}+m'\sqrt{a'}}\left\{dg\,.\cos\left(\Pi-q'\right)-gd\Pi\sin\left(\Pi-q\right)\right\} \qquad \textbf{(197)}$$
$$(f'dq') = -\frac{m\sqrt{a}}{m\sqrt{a}+m'\sqrt{a'}}\left\{dg\,.\cos\left(\Pi-q'\right)+gd\Pi\cos\left(\Pi-q'\right)\right\}.$$

Thus when dg and $gd\Pi$ are computed, the variations in the inclinations and longitude of the nodes when referred to the fixed plane of the ecliptic may be found.

585. The periodic variations in the eccentricities, inclinations, longitudes of the perihelia, and nodes, do not affect the mean motion with any sensible inequalities depending on the squares and product of the masses; for if the variation of

$$dz = +\frac{6m'.an^2}{\left(5n'-2n\right)^2}\left\{P'.\sin l - P.\cos l\right\}$$

be taken, considering all the elements as variable, the substitution of their periodic variations will make the whole vanish in consequence of the relations between the partial differences.

586. The longitude of the epoch is not affected by any variations of this order that are sensible in the planets, but they are of much importance in the theories of the moon and Jupiter's satellites.

587. The variations in the elements depending on the squares of the disturbing forces, are insensible in the theories of all the planets, except those of Jupiter and Saturn; they are only perceptible in the motions of these two planets, on account of the nearly commensurable ratio in their mean motions introducing the minute divisor $5n' - 2n$; therefore, if

$$(d\bar{e}),\ (d\bar{v}),\ (d\bar{g}),\ (d\bar{\Pi}),\ (d\bar{f}),\ (d\bar{q}),$$

be the secular variations in the elements depending on the second powers of the disturbing forces, and computed for the epoch from the equations in articles 580, and the two following, the equations (130) become, with regard to Jupiter and Saturn only,[17]

$$\begin{aligned}
e &= \bar{e} + \left\{ \frac{d\bar{e}}{dt} + (d\bar{e}) \right\} t + \&\text{c.} \\
v &= \bar{v} + \left\{ \frac{d\bar{v}}{dt} + (d\bar{v}) \right\} t + \&\text{c.} \\
g &= \bar{g} + \left\{ \frac{d\bar{g}}{dt} + (d\bar{g}) \right\} t + \&\text{c.} \\
\Pi &= \bar{\Pi} + \left\{ \frac{d\bar{\Pi}}{dt} + (d\bar{\Pi}) \right\} t + \&\text{c.} \\
f &= \bar{f} + \left\{ \frac{d\bar{f}}{dt} + (d\bar{f}) \right\} t + \&\text{c.} \\
q &= \bar{q} + \left\{ \frac{d\bar{q}}{dt} + (d\bar{q}) \right\} t + \&\text{c.}
\end{aligned} \qquad \textbf{(198)}$$

Whence the elements of the orbits of these two planets may be determined with greater accuracy for 1,000 or 1,200 years before and after the time assumed as the epoch.

Periodic Perturbations in Jupiter's Longitude depending on the Squares of the disturbing Forces

588. Where e^2 is omitted, equation (97) becomes

$$v = 2e\sin\left(nt + \epsilon - v\right).$$

The eccentricity and longitude of the perihelion, when corrected for their periodic inequalities (175), and (185) (176) and (186), become,

$$e+de_1+de_2 \text{ and } v+dv_1+dv_2,$$

and the longitude of the epoch when corrected by its periodic variation, is $\in + d\in_1$; by the substitution of these v becomes

$$dv = (2e+2de_1+2de_2)\sin\{nt+\in -v+d\in_1 - dv_1 - dv_2\}:$$

when the quantities that do not contain the squares of the disturbing forces are rejected, the development of this expression is

$$dv = \{2de_2 + 2edv_1.d\in_1 - edv^2\}\sin(nt+\in -v) - \{2dv_1 + 2ede_1.dv_1 - ede.d\in\}\cos(nt+\in -v);$$

when the values of the periodic variations are substituted, the result will be the inequality[18]

$$\begin{aligned} dv = &-\frac{3m'^2.a^2n^3}{(5n'-2n)^3}\cdot\frac{5m\sqrt{a}+4m'\sqrt{a'}}{m'\sqrt{a'}}\cdot\left\{P\cdot\frac{dP'}{de}+P'\cdot\frac{dP}{de}\right\}\times \\ &\cos\{5nt-10n't+5\in -10\in' -v\} \\ &-\frac{3m'^2.a^2n^3}{(5n'-2n)^3}\cdot\frac{5m\sqrt{a}+4m'\sqrt{a'}}{m'\sqrt{a'}}\cdot\left\{P'\cdot\frac{dP'}{de}-P\cdot\frac{dP}{de}\right\}\times \\ &\sin\{5nt-10n't+5\in -10\in' -v\}. \end{aligned} \qquad \textbf{(199)}$$

The corresponding inequality for Saturn is found from

$$v' = 2e'\sin(n't+\in -v').$$

589. The radii vectores and true longitudes of m and m' in their elliptical orbits have been represented by $r,\ r',\ v,\ v',$ but as

$$dr,\ dr',\ dv,\ dv'$$

are the periodic perturbations of these quantities, these two co-ordinates of m and m' in their troubled orbits, are

$$r+dr,\ r'+dr',\ v+dv,\ v'+dv'.$$

When these quantities are substituted in

$$R = \frac{m'\left(rr'\cos\left(v'-v\right)\right)+zz'}{\left(r'^2+z'^2\right)^{\frac{3}{2}}} - \frac{m'}{\sqrt[3]{r^2-2rr'\cos\left(v'-v\right)+r'^2}},$$

R becomes a function of the squares and products of the masses, it consequently produces terms of that order in the mean motion

$$z = -3\iint . andt . \mathrm{d}R$$

having the factor $\left(5n'-2n\right)^2$; they therefore form a part of the great inequalities in the mean motions of Jupiter and Saturn. A mistake has been observed in Laplace's determination of those inequalities,[19] which has been, and still is, a subject of controversy between three of the greatest mathematicians of the present age, MM. Plana,[20] Poisson,[21] and Pontécoulant,[22] to whose very learned papers the reader is referred for a full investigation of this difficult subject.

590. The numerical values of the perturbations of Jupiter in longitude are computed from equations (159), (164), (172), (182), and (199), together with some terms depending on the fifth powers of the eccentricities and inclinations which may be determined by the same process as in the other approximations; his perturbations in latitude are computed from equations (160) and (177), and those in his radius vector from (158) and (163).

591. Hitherto the mass of the planet has been omitted when compared with that of the sun taken as the unit; so that half the greater axes has been determined by the equation $a^3 = \frac{1}{n^2}$, whereas its real value is found from

$$\frac{1+m}{a^3} = n^2, \text{ or } a = n^{-\frac{3}{2}}\left(1+\tfrac{1}{3}m\right);$$

the semigreater axes of the orbits of Jupiter and Saturn ought therefore to be augmented by $\tfrac{1}{3}ma$, $\tfrac{1}{3}m'a'$, quantities that are only sensible in these two planets.

Notes

[1] See note 55, *Preliminary Dissertation.*
[2] Lambert, Johann Heinrich, 1728-1777, self-educated mathematician, bom in Mülhausen, Germany. One of the first to appreciate the nature of the Milky Way. He became editor of the astronomical almanac *Astronomisches Jahrbuch oder Ephemeriden* in 1774. In his *Theorie der Parallellinien* (1766) he developed several pioneering but inconclusive theorems in non-Euclidean geometry. In 1768 he demonstrated the irrational nature of the number pi (p). Lambert also developed the first methods for the measurement of light intensity (1760). A unit of light intensity is named after him.
[3] That is around 1810.

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[4] Flamstead or Flamsteed, John, 1646-1719, the first Astronomer Royal (1675-1719). He was born in Denby, England. Flamsteed instituted reliable observations at Greenwich in 1676 (see note 22, *Preliminary Dissertation*). Isaac Newton (see note 1, *Preliminary Dissertation*) used Flamsteed's data to verify his gravitational theory. His star catalog *Historia Coelestis Britannica* (1725) lists over 3000 stars with great precision.

[5] *Arabian astronomers.* Somerville is likely referring to the Muslin astronomers al-Farghani (c. 860), al-Battani (868-929), and Thabit ibn Qurrah (c. 836-901). Al-Farghani's *Kitab fi al-Harakat al-Samawiya wajawami Ilm al-Nujum* (Elements of Astronomy) was translated into Latin in the 12th century and exerted great influence upon European astronomy. Al-Battani made a remarkably accurate determination of the solar year as being 365 days, 5 hours, 46 minutes and 24 seconds, which is very close to modern values. He found that the longitude of the sun's apogee had increased by 16° 47' since Ptolemy. This implied the important discovery of the motion of the solar apsides and of a slow variation in the equation of time. He also measured the obliquity of the ecliptic, the length of the seasons and the true and mean orbit of the sun. His measurements of lunar and solar eclipses were used in 1749 to determine the secular acceleration of the moon. Thabit ibn Qurrah was one of the early reformers of Ptolemaic views. He analysed several problems related to the movements of sun and moon and wrote treatises on sun-dials.

[6] See note 15, *Preliminary Dissertation.*

[7] See note 32, *Preliminary Dissertation.*

[8] In the 1st edition term 4 *reads* $+\frac{5m_{/}}{2}\cdot eK\,.\sin\left(5n't-2nt+5\epsilon'-2\epsilon+v+b\right)$ and B in term 3 *reads* b In addition a final term $+m'.He.\sin\left(5n't-2nt+5\epsilon'-2\epsilon+v+b\right)$ is deleted (published errata). In the 1st edition the first term is expressed as two terms with identical coefficients but opposite signs.

[9] The 1st term *reads* $-2m\frac{(s+m)}{r}$ in the 1st edition.

[10] In the 1st edition dz' in the second term has no accent (published erratum).

[11] As above, in the 1st edition dz' in the second term has no accent (published erratum).

[12] This *reads* "depend" in the 1st edition (published erratum).

[13] The coefficient of second term *reads* $+\frac{m'}{2}$ in the 1st edition (published erratum).

[14] The elements $\frac{dP'}{dl}$ and $\frac{dP}{dl}$ in the first equation are reversed in the 1st edition (published erratum).

[15] $\left(3m'.an^2.\Sigma.Q\right)$ *reads* as its square in the 1st edition (published erratum).

[16] The original pagination in this area of the 1st edition text is out of sequence. The numbering *reads* 337, 338, 337 (repeated), 338 (repeated), 339.

[17] The 4th equation *reads* $p=\overline{p}+\left\{\frac{d\overline{\Pi}}{dt}+(d\overline{\Pi})\right\}t+\&c.$ in the 1st edition (published erratum).

[18] The arguments in the 3rd and 4th lines *read* $5n't-10nt+5\epsilon'-10\epsilon-v$ in the 1st edition (published erratum).

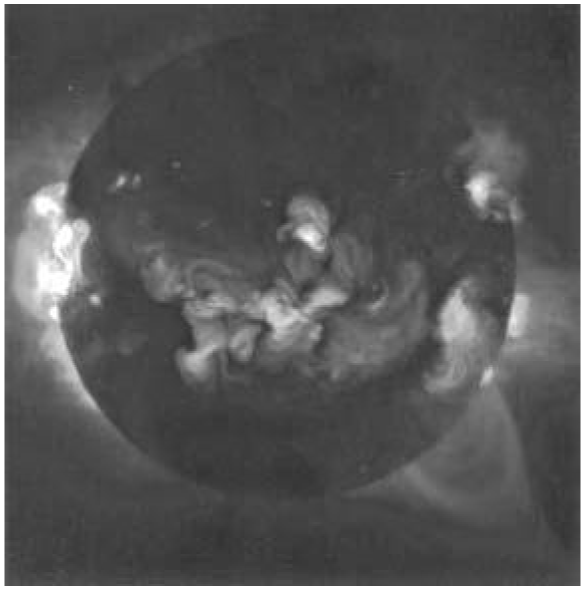
[19] See note 4, *Introduction.*

[20] Plana, Giovanni Antonio Amedeo, barone, 1781-1864, *Integration des formules propres a determiner les equations seculaires des elements des planetes et des cometes : produites par la resistance d'un milieu tres-rare*, Genes : Imprimerie Carniglia, 1825.

[21] See note 1, *Book I, Chapter VI.*

[22] See note 3, *Book II, Chapter IV.*

The Sun



This image of the Sun, taken January 24, 1992, is viewed from space at x-ray wavelengths. The image, as seen by the Soft X-ray Telescope on the Japan/US/UK Yohkoh Mission (orbiting solar observatory), reveals the hot, three-dimensional geometry of the corona across the full disk of the Sun. The large bright areas are regions where the Sun's magnetic field is so strong that it can trap hot gasses even though the temperature of the region is over 1 million degrees K. The dark areas are coronal holes, which are the origin of streams of particles, called the high speed solar wind, that flows past Earth and through the solar system at about 700 kilometers per second. (Courtesy of NASA)

BOOK II

CHAPTER XI

INEQUALITIES OCCASIONED BY THE ELLIPTICITY OF THE SUN

592. As the sun has hitherto been considered a sphere, his action was assumed to be the same as if his mass were united in his centre of gravity; but from his rotatory motion, his form must be spheroidal on account of his centrifugal force, therefore the excess of matter at his equator may have an influence on the motions of the planets.

In the theory of spheroids it is found that the attraction of the redundant matter at the equator is expressed by

$$\left(r - \tfrac{1}{2}y\right)\cdot \frac{R'^2}{r^3}\left(h^2 - \tfrac{1}{3}\right).$$

Where r is the ellipticity of the sun, y the ratio of the centrifugal force to gravity at the solar equator, h the declination of a planet *m* relative to this equator, R' the semidiameter of the sun, his mass being unity. Therefore, the attraction of the elliptical part of the sun's mass adds the term

$$\left(r - \tfrac{1}{2}y\right)\cdot \frac{R'^2}{r^3}\left(h^2 - \tfrac{1}{3}\right)$$

to the disturbing action expressed by the series *R* in article 449. If this disturbing action of the sun's spheroidal form be alone considered, omitting h^2, and substituting

$$\frac{1}{a^3}\left(1 - \tfrac{3}{2}e^2\right), \text{ for } r^{-3},$$

it gives, with regard to secular quantities alone,

$$F = -\tfrac{1}{3}\left(r - \tfrac{1}{2}y\right).\frac{R'^2}{a^3}\left(1 - \tfrac{3}{2}e^2\right),$$

and

$$\frac{dF}{de} = e\left(r - \tfrac{1}{2}y\right)\cdot\frac{R'^2}{a^3}.$$

The substitution of which in

$$dv = \frac{andt}{e}\cdot\frac{dF}{de},$$

gives by integration,

$$dv = \left(r - \tfrac{1}{2}y\right) \cdot \frac{R'^2}{a^2} \cdot nt\,.$$

Thus the action of the excess of matter at the sun's equator produces a direct motion in the perihelia of the planetary orbits.

593. The effect of the sun's ellipticity on the position of the orbits may be ascertained from the last of equations (115), or

$$dp = andt \cdot \frac{dF}{dq}\,.$$

Since h is the declination of the planet m on the plane of the sun's equator, if the equator be taken as the fixed plane, then will

$$h^2 = \frac{z^2}{r^2}\,.$$

And if the eccentricity be omitted,

$$F = \left(r - \tfrac{1}{2}y\right) \cdot \frac{R'^2}{a^5}\left(z^2 - a^2\right),$$

therefore

$$\frac{dF}{dz} = 2\,.\left(r - \tfrac{1}{2}y\right) \cdot \frac{R'^2}{a^5} \cdot z\,.$$

But

$$\frac{dF}{dq} = \frac{dF}{dz} \cdot \frac{dz}{dq} = \frac{dF}{dz} \cdot a\sin\left(nt + \in\right)$$

On account of equation,

$$\frac{z}{a} = q\,.\sin\left(nt + \in\right) - p\cos\left(nt + \in\right)$$

consequently,

$$\frac{dF}{dq} = 2 \cdot \left(r - \tfrac{1}{2}y\right) \cdot \frac{R'^2}{a^5} \cdot z\,.\sin\left(nt + \in\right)$$

or substituting $a\,.\tan f\,.\sin\left(nt + \in - q\right)$ for *z*,

$$\frac{dF}{dq} = -\left(r - \tfrac{1}{2}y\right) \cdot \frac{R'^2}{a^3} \cdot \cos q\,.\tan f,$$

whence

$$dp = -ndt\,.\left(r - \tfrac{1}{2}y\right) \cdot \frac{R'^2}{a^2} \cdot \cos q\,.\tan f\,.$$

But

$$p = \tan f \,.\sin q \;;$$

whence

$$dp = dq \,.\tan f \,.\cos q \;.$$

Therefore

$$dq = -ndt \,.\left(r - \tfrac{1}{2} y \right) \cdot \frac{R'^2}{a^2} ,$$

and

$$dq = -nt \,.\left(r - \tfrac{1}{2} y \right) \cdot \frac{R'^2}{a^2} .$$

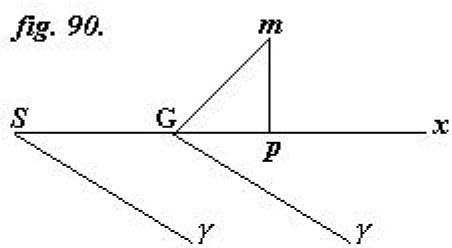
Thus the nodes of the planetary orbits have a retrograde motion on the plane of the solar equator equal to the direct motion of their perihelia on the same plane, both so small that they are scarcely perceptible even in Mercury. As neither the eccentricities nor the inclinations are affected by this disturbance, it has no influence on the stability of the system.

BOOK II

CHAPTER XII

PERTURBATIONS IN THE MOTIONS OF THE PLANETS OCCASIONED BY THE ACTION OF THEIR SATELLITES

594. THE common centre of gravity of a planet and its satellites very nearly describes an ellipse round the sun. If that orbit be considered to be the orbit of the planet itself, the respective positions of the satellites with regard to each other, and to the sun, will give that of the planet with regard to their common centre of gravity, and consequently the perturbations produced by the satellites on their primary.



Let G, fig. 90, be the common centre of gravity of a planet, and of its satellites,[1] S the sun, g the equinoctial point, and $\overline{x}$, $\overline{y}$, $\overline{z}$, the co-ordinates of G, so that $S\mathrm{G} = \overline{x}$, and $\overline{z}$ perpendicular to the plane of the orbit. Then if x, y, z, be the co-ordinates of a satellite m, and $v = gS\mathrm{G}$, $U = g\mathrm{G}m$, the longitudes of G and m; it is evident that $\mathrm{G}p = x - \overline{x}$, and r being the radius Gm,

$$\mathrm{G}p = x - \overline{x} = r \,.\, \cos(U - v);$$

hence, if $\sum m$ be the sum of the masses of the satellites, and P that of their primary,

$$\sum m \,.\, x = \overline{x} \,.\, P + \sum m \,.\, r \cos(U - v),$$

or

$$\sum mx = \overline{x} \,.\, P + mr \,.\, \cos(U - v) + m'r' \,.\, \cos(U - v') + \&c.$$

In the same manner

$$\sum my = \overline{y} \,.\, P + mr \,.\, \sin(U - v) + m'r' \,.\, \sin(U - v') + \&c.$$

[and]

$$\sum mz = \overline{z} P + m \,.\, rs + m' \,.\, r's' + \&c.$$

s, s', s'', &c., being the latitudes of the satellites above the orbit of their common centre of gravity. But by the property of the centre of gravity,

$$\sum m \,.\, x = 0, \ \sum m \,.\, y = 0, \ \sum m \,.\, z = 0;$$

consequently,

$$0 = \overline{x} \,.\, P + mr \,.\, \cos(U - v) + \&c.$$
$$0 = \overline{y} \,.\, P + mr \,.\, \sin(U - v) + \&c.$$
$$0 = \overline{z} \,.\, P + mr \,.\, s + m'r's' + \&c.$$

By article 353 the centre of gravity is urged in a direction parallel to the co-ordinates, by the forces

$$-(P + \Sigma m)\overline{x}; \quad \frac{-(P + \Sigma m)\overline{y}}{\overline{r}}; \quad \frac{-(P + \Sigma m)\overline{z}}{\overline{r}}.$$

[where] $\overline{r} = S\mathrm{G}$, the radius vector of the centre of gravity. These forces vary very nearly as $\overline{x}$, $\frac{\overline{y}}{\overline{r}}$, and $\frac{\overline{z}}{\overline{r}}$; therefore the perturbations in the radius vector *S*G are very nearly proportional to $\overline{x}$, that is, to

$$-\frac{m}{P} \cdot r \cos(U - v) - \frac{m'}{P} \cdot r' \cos(U - v') - \&c.$$

The perturbations in longitude are nearly proportional to

$$-\frac{m}{P} \cdot r \sin(U - v) - \frac{m'}{P} \cdot \frac{r'}{\overline{r}} \sin(U - v') - \&c.;$$

and those in latitude to[2]

$$-\frac{m}{P} \cdot \frac{rs}{\overline{r}} - \frac{m'r's'}{\overline{r}P} - \&c.$$

The masses of Jupiter's satellites compared with the mass of that planet are so small, and their elongations seen from the sun subtend so small an angle, that the perturbations produced by them in Jupiter's motions are insensible; and there is reason to believe this to be the case also with regard to Saturn and Uranus.

595.[3] But the Earth is sensibly troubled in its motions by the Moon, her action produces the inequalities[4]

$$d\,r = -\frac{m}{E} \cdot r \cos(U - v)$$
$$d\,v = -\frac{m}{E} \cdot \frac{r}{\overline{r}} \sin(U - v)$$
$$d\,s = -\frac{m}{E} \cdot \frac{r}{\overline{r}} \cdot s;$$

or, more correctly,

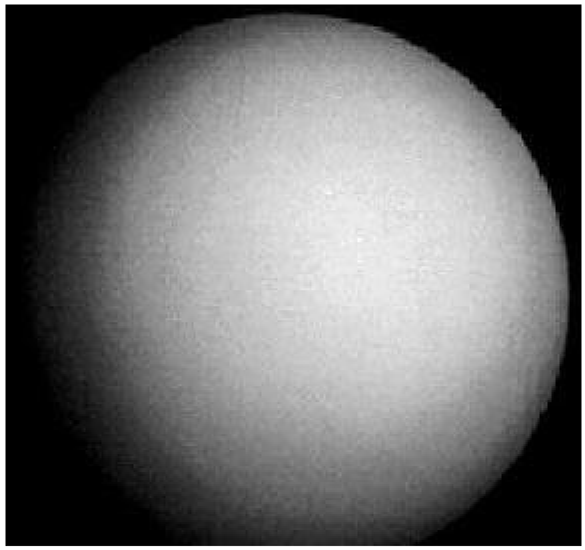
$$\begin{aligned} dr &= -\frac{m}{E+m}\cdot r\cos(U-v) \\ dv &= -\frac{m}{E+m}\cdot\frac{r}{\bar{r}}\sin(U-v) \\ ds &= -\frac{m}{E+m}\cdot\frac{r}{\bar{r}}\cdot s; \end{aligned} \qquad \textbf{(200)}$$

in the radius vector, longitude and latitude of the Earth, *E* and *m* being the masses of the Earth and Moon.

Notes

[1] In keeping with earlier use we italicize *S* in this edition. The 1st edition text *reads* S.
[2] The denominator in the third term *reads* $\bar{r}$ in the 1st edition (published erratum).
[3] This article is numbered 495 in the 1st edition.
[4] The third equation *reads* $ds = -\frac{m}{E}\cdot\frac{r}{r}\cdot s$ in the 1st edition.

Uranus



Uranus' greenish atmosphere is due to methane and high-altitude photochemical smog. Voyager 2 acquired this view of the seventh planet while departing the Uranian system in late January 1986. This image looks at the planet approximately along its rotational pole. (Courtesy of NASA)

BOOK II

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CHAPTER XIII

DATA FOR COMPUTING THE CELESTIAL MOTIONS

596. THE data requisite for computing the motions of the planets determined by observation for any instant arbitrarily assumed as the epoch or origin of the time, are

The masses of the planets;
Their mean sidereal motions for a Julian year of 365.25 days;
The mean distances of the planets from the sun;
The ratios of the eccentricities to the mean distances;
The inclinations of the orbits on the plane of the ecliptic;
The longitudes of the perihelia;
The longitudes of the ascending nodes on the ecliptic;
The longitudes of the planets.

Masses of the Planets

597. Satellites afford the means of ascertaining the masses of their primaries; the masses of such planets as have no satellites are found from a comparison of their inequalities determined by analysis, with values of the same obtained from numerous observations. The secular inequalities will give the most accurate values of the masses, but till they are perfectly known the periodic variations must be employed. On this account there is still some uncertainty as to the masses of several bodies. It is only necessary to know the ratio of the mass of each planet to that of the sun taken as the unit; the masses are consequently expressed by very small fractions.

598. If the time of a sidereal revolution of a planet m, whose mean distance from the sun is a, p the ratio of the circumference to the diameter, and $m = \sqrt{m+S}$ the sum of the sun and planet, by article 383,

$$T = \frac{2p \,.\, a^{\frac{3}{2}}}{\sqrt{m}}.$$

From this expression the masses of such planets as have satellites may be obtained.

Suppose this equation relative to the earth, and that the mass of the earth is omitted when compared with that of the sun, it then becomes

$$T = \frac{2p \cdot a^{\frac{3}{2}}}{\sqrt{S}}.$$

Again, let $m = m + m'$ the sum of the masses of a planet and of its satellite m', T' being the time of a sidereal revolution of the planet at the mean distance a' from the sun, then

$$T' = \frac{2p \cdot a'^{\frac{3}{2}}}{\sqrt{m + m'}};$$

and dividing the one by the other the result is,

$$\frac{m + m'}{S} = \frac{a'^3}{a^3} \cdot \frac{T^2}{T'^2}.$$

If the values of T, T', a, and a', determined from observation, be substituted in this expression, the ratio of the sum of the masses of the planet and of its satellite to the mass of the sun will be obtained; and if the mass of the satellite be neglected when compared with that of its primary, or if the ratio of these masses be known, the preceding equation will give the ratio of the mass of the planet to that of the sun. For example,

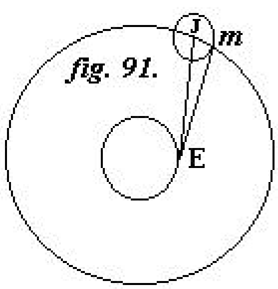

fig. 91.

599. Let m be the mass of Jupiter [fig. 91], that of his satellite being omitted, and let the mass of the sun be taken as the unit, then

$$m = \frac{a'^3}{a^3} \cdot \frac{T^2}{T'^2}.$$

J*m* the mean radius of the orbit of the fourth satellite at the mean distance of the earth from the sun taken as the unit, is seen under the angle JE*m*=2580″.579. The radius of the circle reduced to seconds is 206264″.8; hence the mean radii of the orbit of the fourth satellite and of the terrestrial orbit are in the ratio of these two numbers. The time of a sidereal revolution of the fourth satellite is 16.6890 days, and the sidereal year is 365.2564 days, hence

$$a = 206264.8$$
$$a' = 2580.58$$
$$T = 365.2564$$
$$T' = 16.6890.$$

With these data it is easy to find that the mass of Jupiter is

$$m = \frac{1}{1066.09}.$$

The sixth satellite of Saturn accomplishes a sidereal revolution in 15.9453 days; the mean radius of its orbit, at the mean distance of the planet, is seen from the sun under an angle of 179″; whence the mass of Saturn is

$$\frac{1}{3359.40}.$$

By the observations of Sir William Herschel[1] the sidereal revolutions of the fourth satellite of Uranus are performed in 13.4559 days, and the mean radius of its orbit seen from the sun at the mean distance of the planet is 44”.23. With these data the mass of Uranus is found to be

$$\frac{1}{19{,}504}.$$

600. This method is not sufficiently accurate for finding the mass of the Earth, on account of the numerous inequalities of the Moon. It has already been observed, that the attraction of the Earth on bodies at its surface in the parallel where the square of the sine of the latitude is $\frac{1}{3}$, is nearly the same as if its mass were united at its centre of gravity. If R be the radius of the terrestrial spheroid drawn to that parallel, and m its mass, this attraction will be

$$g = \frac{m}{R^2}; \text{ whence } m = g\,.\,R^2.$$

Then, if a be the mean distance of the Sun from the Earth, T the duration of the sidereal year,

$$T = \frac{2\pi\,.\,a^{\frac{3}{2}}}{\sqrt{S}};$$

and, by division,

$$\frac{m}{S} = \frac{g\,.\,R^2T^2}{4\pi\,.\,a^3}.$$

R, g, T, and a, are known by observation, therefore the ratio of the mass of the Earth to that of the Sun[2] may be found from this expression.

The sine of the solar parallax at the mean distance of the sun from the earth,[3] and in the latitude in question, is

$$\sin P = \frac{R}{a} = \sin 8''.75;$$

the attraction of the Earth, and the terrestrial radius in the same parallel, are

$$g = 2\times 16.1069 = 32.2138$$
$$R = 2089870,$$

and the sidereal year is

$$T = 31558152''.9$$

with these data the mass of the earth is computed to be[4]

$$\frac{1}{337{,}103},$$

the mass of the sun being unity. This value varies as the cube of the solar parallax compared with that adopted.

601. The compression of the three larger planets, and the ring of Saturn, probably affect the values of the masses computed from the elongations of their satellites; but the comparison of numerous well chosen observations, with the disturbances determined from theory, will ultimately give the masses of all the planets with great accuracy.

The action of each disturbing body adds a term of the form $m'dv'$ to the longitude, so that the longitude of m at any given instant in its troubled orbit, is

$$v + m'dv' + m''dv'' + \&c.$$

[where][5] v, dv', dv'', are susceptible of computation from theory; and as they are given by the Tables of the Motions of the Planets, the true longitude of m is

$$v + m'dv' + m''dv'' + \&c. = L\,.$$

When this formula is compared[6] with a great number of observations, a series of equations,

$$m'dv' + m''dv'' + \&c. = L - v,$$
$$m'dv_2' + m''dv_2'' + \&c. = L' - v_2,$$
$$+ \&c. = \&c.$$

are obtained, where m', m'', &c., are unknown quantities, and by the resolution of these the masses of the planets may be estimated by the perturbations they produce,

602. As there are ten planets,[7] ten equations would be sufficient to give their masses, were the observed longitudes and the computed quantities v, dv', dv'', &c., mathematically exact; but as that is far from being the case, many hundreds of observations made on all the planets must be employed to compensate the errors. The method of combining a series of equations more numerous than the unknown quantities they contain, so as to determine these quantities with all possible accuracy, depends on the theory of probabilities, which will be explained afterwards. The powerful energy exercised by Jupiter on the four new planets in his immediate vicinity occasions very great inequalities in the motions of these small bodies, whence that highly distinguished mathematician, M. Gauss,[8] has obtained a value for the mass of Jupiter, differing considerably from that deduced from the elongation of his satellites, it cannot however be regarded as conclusive till the perturbations of these small planets are perfectly known.

603. The mass of Venus is obtained from the secular diminution in the obliquity of the Ecliptic. The plane of the terrestrial equator is inclined to the plane of the ecliptic at an angle of 23º 28′ 47″ nearly, but this angle varies in consequence of the action of the planets. A series of tolerably correct observations of the Sun's altitude at the solstices chiefly by the Chinese and Arabs, have been handed down to us from the year 1100 before Christ, to the year 1473 of the Christian era; by a comparison of these, it appears that the obliquity was then diminishing, and it is still decreasing it the rate of 50″.2 in a century. From numerous observations on the obliquity of the ecliptic made by Bradley[9] about a hundred years ago, and from later observations by Dr. Maskelyne,[10] Delambre[11] determined the maximum of the inequalities produced by the action of Venus, Mars, and the Moon, on the Earth, and by comparing these observations with the analytical formulae, he obtained nearly the same value of the mass of Venus, whether he deduced it from the joint observations of Bradley and Maskelyne, or from the observations of each separately. From this correspondence in the values of the mass of Venus, obtained from these different sets of observations, there can be little doubt that the secular diminution in the obliquity of the ecliptic is very nearly 50″.2, and the probability of accuracy is greater as it agrees with the observations made by the Chinese and Arabs so many centuries ago. Notwithstanding doubts still exist as to the mass of Venus.

604. The mass of Mars has been determined by the same method, though with less precision than that of Venus, because its action occasions less disturbance in the Earth's motions, for it is evident that the masses of those bodies that cause the greatest disturbance will be best known. The action of the new planets is insensible, and that of Mercury has a very small influence on the motions of the rest. An ingenious method of finding the mass of that planet has been adopted by Laplace, although liable to error.

605. Because mass is proportional to the product of the density and the volume, if m, m', be the masses of any two planets of which r, r', are the densities, and V, V', the volumes, then

$$m : m' :: r.V : r'.V'.$$

But as the planets differ very little from spheres, their volumes may be assumed proportional to the cubes of their diameters; hence if D, D', be the diameters of m, and m',

$$m : m' :: r.D^3 : r'.D'^3;$$

whence

$$\frac{r}{r'} = \frac{D'^3}{D^3} \cdot \frac{m}{m'}. \qquad \textbf{(201)}$$

The apparent diameters of the planets have been measured so that D and D' are known; this equation will therefore give the densities if the masses be known, and *vice versâ*.

By comparing the masses of the Earth, Jupiter, and Saturn, with their volumes, Laplace found that the densities of these three planets are nearly in the inverse ratio of their mean distances from the sun, and adopting the same hypothesis with regard to Mercury, Mars, and Jupiter, he obtained the preceding values of the masses of Mars and Mercury, which are found nearly to agree with those determined from other data. Irradiation, or the spreading of the light round the disc of a planet, and other difficulties in measuring the apparent diameters, together with the uncertainty of the hypothesis of the law of the densities, makes the values of the masses obtained in this way the more uncertain, as the hypothesis does not give a true result for the masses of Venus and Saturn. Fortunately the influence of Mercury on the solar System is very small.

606. The mass of the Sun being unity, the masses of the planets are,[12]

Mercury. . . . . $\dfrac{1}{2,025,810}$

Venus . . . . $\dfrac{1}{405,871}$

The Earth . . . . $\dfrac{1}{354,936}$

Mars . . . . $\dfrac{1}{2,546,320}$

Jupiter. . . . . $\dfrac{1}{1,070.5}$

Saturn . . . . $\dfrac{1}{3,512}$

Uranus. . . . . $\dfrac{1}{17,918}$

Densities of the Planets

607. The densities of bodies are proportional to the masses divided by the volumes, and when the masses are spherical, their volumes are as the cubes of their radii; as the sun and planets are nearly spherical, their densities are therefore as their masses divided by the cubes of their radii; but the radii must be taken in those parallels of latitude, the squares of whose sines are $\frac{1}{3}$.

The mean apparent semidiameters of the Sun and Earth at their mean distance are,

Sun	961″
The Earth	8″.6

The radius of Jupiter's spheroid in the latitude in question, when viewed at the mean distance of the earth from the sun, is 94″.344; and the corresponding radius of Saturn at his mean distance from the sun is 8″.1. Whence the densities are,

Sun	1
The Earth	3.9326
Jupiter	.99239
Saturn	.59496

Thus the densities decrease with the distance from the sun; however that of Uranus does not follow this law, being greater than that of Saturn, but the uncertainty of the value of its apparent diameter may possibly account for this deviation.

Intensity of Gravitation at the Surfaces of the Sun and Planets

608. Let g and g' represent the force of gravity at the surfaces of two bodies m and m', whose apparent diameters are D and D'. If the bodies be spherical and without rotation, the force of gravity at their equators will be as their masses divided by the squares of their diameters; hence

$$g = g' \cdot \frac{m}{m'} \cdot \frac{D'^2}{D^2}.$$

Because the masses, apparent diameters, and the intensity of gravity at the terrestrial equator are known, g, the intensity of the gravitating force at the equator of any other body may be found; and as the rotation of the sun and planets is determined by observation, their centrifugal forces, and consequently the intensity of gravitation at their surfaces may be computed. With the preceding values of the masses and apparent diameters it will be found, that if the weight of a body at the terrestrial equator be the unit, the same body transported to the equator of Jupiter, would weigh 2.716; but this would be diminished by about a ninth, on account of the centrifugal force. The same body would weigh 27.9 at the sun's equator, and a body at the sun's equator would fall through 448.39 feet in the first second of its descent, that would only fall through 16.0436 feet at the earth's equator.

To determine the fall of bodies at the surfaces of the sun and planets was hopeless till Newton's immortal discovery connected us with remote worlds.

609. The mean sidereal motions of the planets in a Julian year of 365.25 days are the second data.

When the sun is in the tropics his declination is a maximum, and equal to the obliquity of the ecliptic; the time at which that happens is found by observing his declination at noon for several days before and after the instant of a solstice, so that an equation can be formed between the time and the declination, which is sufficiently exact for a few days. If the differential of the declination in this equation be made zero, the instant of the solstice and the obliquity of the ecliptic will be obtained. The instant of the equinoxes is determined in the same manner, only that in the equation between the time and the declination, the declination is made zero, for in these points the sun is in the plane of the equator.[13] The length of the year is determined by comparing together the time of the sun's being in either equinox, or in either tropic, with the time of his being in the same point for another year distant from the former by a long period; the interval reckoned in days and parts of a day, divided by the number of years elapsed, will give the true length of the year; and the greater the interval, the more correct will it be. The length of the year however, like all astronomical data, was determined by successive approximations, but it was very early known to be 365.25 days.

The Julian year being known, if the synodic revolutions of the planets be known, their mean motion for any given interval may be found.

610. The longitude of an inferior planet in superior[14] conjunction, or of a superior planet in opposition, is the same as if viewed from the centre of the sun. The synodical revolution of the planet, which is the interval between two conjunctions, or two oppositions, may be ascertained by observation, and from thence its periodic time. Let T be the synodic revolution of a planet, P its periodic time, then

$$P:365.25::360^0:360^0 \pm a,$$

the angle described by the planet in 365.25 days. If it be an inferior planet, its angular motion will be greater than that of the earth; hence the angle described in 365.25 days is equal to 360^0 plus the angle gained by the planet on the earth, or $360^0 + a$. But if it be a superior planet, its angular velocity being less than that of the earth, the angle described in a Julian year is $360^0 - a$. But these angles are as the times in which they are described, therefore

$$360^0:360^0 \pm a::T:365.25 \pm T;$$

hence

$$P:365.25::T:365.25 \pm T,$$

and

$$P = \frac{365.25 \times T}{365.25 \pm T}.$$

As the synodic revolutions are known, the sidereal revolutions of the planets are as follow.

	Days
Mercury	87.9705
Venus	224.7
The Earth	365.2564

Mars	686.99
Vesta	1,592.69
Juno	1,331.
Ceres	1,681.42
Pallas	1,686.56
Jupiter	4,332.65
Saturn	10,759.4
Uranus	30,687.5

Whence it will be found by simple proportion that the mean sidereal motions of the planets in a Julian year of 365.2564 days, or the values of n, n', &c., are

Mercury	53,831,034″.99
Venus	2,106,644″.82
The Earth	12,995,977″.74
Mars	689,051″.63
Vesta	355,681″.17
Juno	297,216″.21
Ceres	281,531″.00
Pallas	280,672″.32
Jupiter	109,256″.78
Saturn	43,996″.13
Uranus	15,425″.64

These have been determined by approximation, continually corrected by a long series of observations on the oppositions and conjunctions of the planets.

Mean Distances of the Planets, or Values of a, a', a'', &c.

611. The mean distances are obtained from the mean motions of the planets: for, assuming the mean distance of the earth from the sun as the unit, Kepler's law of the squares of the periodic times being as the cubes of the mean distances, gives the following values of the mean distances of the planets from the sun.[15]

	[A.U.][16]
Mercury	0.3870981
Venus	0.7233316
The Earth	1.0000000
Mars	1.5236923
Vesta	2.3678700
Juno	2.6690090
Ceres	2.7672450
Pallas	2.7728860
Jupiter	5.2011524

Saturn	9.5379564
Uranus	19.1823927

Ratio of the Eccentricities to the Mean Distances, or Values of e, e′, &c. for 1801

612. The eccentricity of an orbit is found by ascertaining that heliocentric longitude of the planet at which it is moving with its mean angular velocity, for there the increments of the true and mean anomaly are equal to one another, and the equation of the œntre, or difference between the mean and true anomaly is a maximum, and equal to half the eccentricity. By repeating this process for a series of years, the effects of the secular variations will become sensible, and may be determined; and when they are known, the eccentricity may be determined for any given period. The values of e, e', e'', &c., for 1801, are

Mercury	0.20551494
Venus	0.00686074
The Earth	0.01685318
Mars	0.09330700
Vesta	0.08913000
Juno	0.25784800
Ceres	0.07843900
Pallas	0.24164800
Jupiter	0.04816210
Saturn	0.05615050
Uranus	0.04661080

Inclinations of the Orbits on the Plane of the Ecliptic, in 1801

613. When the earth is in the line of a planet’s nodes, if the planet’s elongation from the sun and its geocentric latitude be observed, the inclination of the orbit may be found; for the sine of the elongation is to the radius, as the tangent of the geocentric latitude to the tangent of the inclination. If the planet be 90° distant from the sun, the latitude observed is just equal to the inclination. By this method Kepler determined the inclination of the orbit of Mars. The secular inequalities become sensible after a course of years. The values of f, f', f'', &c. were in 1801

		°	′	″
	Mercury	7	0	9.1
	Venus	3	23	28.5
	Mars	1	51	6.2
1820	Vesta	7	8	9.0
1820	Juno	13	4	9.7
1820	Ceres	10	37	26.2
1820	Pallas	34	34	55.0
	Jupiter	1	18	51.3
	Saturn	2	29	35.7
	Uranus	0	46	28.4

Longitudes of the Perihelia

614. The angular velocity of a body is least in aphelion, and greatest in perihelion; consequently, if its longitude be observed when the increments of the angular velocity are greatest or least, these points will be in the extremities of the major axis: if these be really the two observed longitudes, the interval between them will be exactly half the time of a revolution, a property belonging to no other diameter in the ellipse. As it is very improbable that the observations should differ by 180^{o}, they require a small correction to reduce them to the true times and longitudes. On this principle the longitudes of the perihelia may be determined, and if the observations be continued for a series of years, their secular motions will be obtained, whence their places may be computed for any epoch. The longitude of the perihelion is the distance of the perihelion from the ascending node estimated on the orbit, plus the longitude of the node. In the beginning of 1801, the values of v, v', v'', &c., were,

		°	′	″
	Mercury	74	21	46.8
	Venus	128	43	53.0
	The Earth	99	30	4.8
	Mars	332	23	56.4
1820	Vesta	249	33	24.2
1820	Juno	53	33	46.0
1820	Ceres	147	7	31.1
1820	Pallas	121	7	4.3
	Jupiter	11	8	34.4
	Saturn	89	9	29.5
	Uranus	167	30	23.7

Longitudes of the Ascending Nodes

615. When a planet is in its nodes, it is in the plane of the ecliptic; its longitude is then the same with the longitude of its node, and its latitude is zero. The place of the nodes may therefore be found by a series of observations, and if they be continued long enough, their secular motions will be obtained; whence their positions at any time may be computed. In the beginning of 1801 the values of q, q', q'', &c., were,

		°	′	″
	Mercury	45	57	30.9
	Venus	74	54	12.9
	Mars	48	0	3.5
1820	Vesta	103	13	18.2
1820	Juno	171	7	40.4
1820	Ceres	80	41	24.0
1820	Pallas	172	39	26.8

Jupiter	98	26	18.9
Saturn	111	56	37.3
Uranus	72	59	35.4

616. Mean longitudes of the planets on the 1st January, 1801, at midnight, or values of $\in$, $\in'$, $\in''$, &c.

		°	′	″
	Mercury	163	56	26.9
	Venus	10	44	21.6
	The Earth	100	9	12.9
	Mars	64	6	59.9
1820 1st Jan at noon	Vesta	278	30	0.4
1820 1st Jan at noon	Juno	200	16	19.1
1820 1st Jan at noon	Ceres	123	16	11.9
1820 1st Jan at noon	Pallas	108	24	57.9
	Jupiter	112	12	51.3
	Saturn	135	19	5.5
	Uranus	177	48	1.1

All the longitudes are estimated from the mean equinox of spring, the epoch being the 1st January, 1801.

617. With these data the motions of the planets are computed; they are, however, only approximate, since each element is determined independently of the rest; whereas they are so connected, that their values ought to be determined simultaneously by equations of condition formed from thousands of observations.

618. Elements of the orbits of the three comets belonging to the solar system.

Halley's Comet of 1682[17]

Period of revolution 76 years, nearly. Instant of passage at perihelion 1835, October 31st 2.

Half the greater axis	17.98355
Eccentricity	0.967453
Longitude of perihelion on orbit	304° 34′ 19″
Longitude of ascending node	55 6 59
Inclination	17 46 50

Motion Retrograde

Encke's Comet of 1819[18][19]

Period of Revolution 1205.55 days.[20] Passage at perihelion 1829, January 10th, 573.

Mean diurnal motion	1069″.557

Half the greater axis	2.224346
Eccentricity	0.8446862
Longitude of perihelion	157° 18′ 35″
Longitude of ascending node	334 24 15
Inclination	13 22 34

Claussen and Gambart's Comet of 1825

Period of revolution 6.7 years.[21] Passage at perihelion 1832, November 27th, 4808.

Half the greater axis	3.53683
Eccentricity	0.7517481
Longitude of perihelion	109° 56′ 45″
Longitude of ascending node	248 12 24
Inclination	13 13 13

The computation, in the next Chapter, of the perturbations of Jupiter and Saturn will be sufficient to show the method of finding their numerical values, especially as there are many peculiar to these two planets.

Notes

[1] See note 52, *Preliminary Dissertation.*
[2] The capitalization of planetary names is not consistent in the text. We retain the 1st edition assignments.
[3] See preceding note.
[4] The modern value is closer to $\frac{1}{333{,}000}$ or 3.33×10^{-5}.
[5] This *reads* $v_{/}$, in the 1st edition.
[6] This *reads* "composed" in the 1st edition (published erratum).
[7] Including Earth, at the time of writing there are actually eleven "planets." These are Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus and the recently discovered "telescopic planets," the asteroids Ceres, Pallas, Juno, and Vesta.
[8] Gauss, (Johann) Carl Friedrich, 1777-1855, mathematician, born in Brunswick, Germany. He was professor of mathematics and director of the observatory at Göttingen and considered one of the greatest mathematicians of all time. Gauss wrote the first modern book on number theory and pioneered a number of mathematical applications in gravitation, magnetism, and electricity. His astronomical work included careful calculations of the orbits of the asteroids Ceres and Pallas using his method of least squares (see also note 60, *Bk. II, Chap. XIV*). Gauss was also interested in the shape of the earth and invented a heliotrope to increase the accuracy of surveying measurements. His *Theoria motus corporum coelestium in sectionibus conicis Solem ambientium* (1809) is a two volume treatise on celestial motion. It treats conic sections, elliptical orbits, and methods for refining planetary orbits. The unit of magnetic induction is named after him.
[9] See note 38, *Preliminary Dissertation.*
[10] See note 55, *Bk. II, Chap. VI.*
[11] See note 54, *Preliminary Dissertation.*
[12] The modern values of the reciprocals of these ratios to 4 significant figures (with Somerville's values also rounded to 4 significant figures in parentheses for comparison) are: Mercury 602,700 (2,026,000), Venus 408,500 (405,900), Earth 333,000 (355,000), Mars 3,097,000 (255,000), Jupiter 1,047 (1,071), Saturn 3,502 (3,512), Uranus

22,910 (17,920). Note that Somerville does not use comma separators in the presentation of her data in the 1st edition.

[13] This *reads* "ecliptic" in the 1st edition (published erratum).

[14] This modifier is omitted in the 1st edition (published erratum).

[15] The modern values of these distances in AU (with Somerville's values in parentheses for comparison) are: Mercury 0.38 (0.39), Venus 0.72 (0.72), Earth 1.00 (1.00), Mars 1.52 (1.52), Vesta 2.36 (2.37), Juno 2.67 (2.67), Ceres 2.77 (2.77), Pallas 2.77 (2.77), Jupiter 5.20 (5.20), Saturn 9.54 (9.54), Uranus 19.218 (19.182).

[16] The 1st edition text does not use the AU designation. An AU (Astronomical Unit) is the mean distance of the Earth

[17] See note 55, *Preliminary Dissertation*.

[18] *Encke*. Somerville spells the name "Enke" in the 1st edition.

[19] Encke, Johann Franz, 1791-1865, astronomer, born in Hamburg, Germany. Encke made the first accurate calculation of solar distance and calculated the solar parallax with a precision close to the modern value. He directed the Seeberg Observatory (1822-5) and later became director of the Observatory at Berlin University. In 1819 Encke measured the period of the comet that bears his name. He also discovered the outer ring of Saturn, known now as Encke's Division. Encke also developed new methods for the determination of the orbits of the newly discovered small planets or asteroids.

[20] This *reads* 1203.687 days in the 1st edition (published erratum).

[21] This *reads* 6,7 years in the 1st edition.

BOOK II

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CHAPTER XIV

NUMERICAL VALUES OF THE PERTURBATIONS[1]

619. THE epoch assumed for this computation is that of the French Tables, namely, the 31st of December, at midnight, 1749, mean time at Paris. The data for that epoch are as follow:–

Values of e, e', e'', &c.

Mercury	0.20551320
Venus	0.00688405
The Earth	0.01681395
Mars	0.09305767
Jupiter	0.04807670
Saturn	0.05622460
Uranus	0.04669950

Values of v, v', v'', &c.

Mercury	73°.5661
Venus	127°.9117
The Earth	98°.6211
Mars	331°.473
Jupiter	10°.3511
Saturn	88°.1519
Uranus	166°.614

Values of f, f', f'', &c.

Mercury	7°
Venus	3°. 3931
Mars	1°.8499
Jupiter	1° .3172
Saturn	2°.4986
Uranus	0°.7736

Values of q, q', q'', &c.

Mercury	45°.3452
Venus	74°.4384
Mars	47°.6438
Jupiter	97°.906
Saturn	111°.5064
Uranus	72°.6314

The longitudes are estimated from the mean equinox of spring.

620. The series represented by S and S' in article 453 form the basis of the whole computation, but twelve or fourteen of the first terms of each will be sufficiently correct for all the planets.

The numerical values of the coefficients, A_0, A_1, &c., B_0, B_1, &c., and their differences, for Jupiter and Saturn, are obtained from the formulae in article 455, and those that follow. The mean distances of these two planets are, according to Laplace,

$$a = 5.20116636, \quad a' = 9.5378709,$$

whence

$$a = 0.54531726.$$

$A_0 = 0.228576$	$A_1 = 0.065071$	$A_2 = 0.027012$
$A_3 = 0.012369$	$A_4 = 0.005929$	$A_5 = 0.002918$
$A_6 = 0.001458$	$A_7 = 0.000738$	$A_8 = 0.000376$
$A_9 = 0.000189$	$A_{10} = 0.000091$	$A_{11} = 0.000034$
$\frac{dA_0}{da} = 0.008891$	$\frac{dA_1}{da} = 0.016305$	$\frac{dA_2}{da} = 0.012149$
$\frac{dA_3}{da} = 0.007987$	$\frac{dA_4}{da} = 0.004983$	$\frac{dA_5}{da} = 0.00302$
$\frac{dA_6}{da} = 0.001789$	$\frac{dA_7}{da} = 0.001056$	$\frac{dA_8}{da} = 0.000617$
$\frac{dA_9}{da} = 0.000364$	$\frac{dA_{10}}{da} = 0.000223$	
$\frac{d^2A_0}{da^2} = 0.003314$	$\frac{d^2A_1}{da^2} = 0.002942$	$\frac{d^2A_2}{da^2} = 0.004058$
$\frac{d^2A_3}{da^2} = 0.004070$	$\frac{d^2A_4}{da^2} = 0.003453$	$\frac{d^2A_5}{da^2} = 0.002654$
$\frac{d^2A_6}{da^2} = 0.001919$	$\frac{d^2A_7}{da^2} = 0.001319$	$\frac{d^2A_8}{da^2} = 0.000877$
$\frac{d^2A_9}{da^2} = 0.000559$		

$$\frac{d^3A_0}{da^3} = 0.001466 \qquad \frac{d^3A_1}{da^3} = 0.001556 \qquad \frac{d^3A_2}{da^3} = 0.001551$$

$$\frac{d^3A_3}{da^3} = 0.001868 \qquad \frac{d^3A_4}{da^3} = 0.002061 \qquad \frac{d^3A_5}{da^3} = 0.002013$$

$$\frac{d^3A_6}{da^3} = 0.001808 \qquad \frac{d^3A_7}{da^3} = 0.001478 \qquad \frac{d^3A_8}{da^3} = 0.001156$$

$$\frac{d^4A_0}{da^4} = 0.001069 \qquad \frac{d^4A_1}{da^4} = 0.001064 \qquad \frac{d^4A_2}{da^4} = 0.001107$$

$$\frac{d^4A_3}{da^4} = 0.001138 \qquad \frac{d^4A_4}{da^4} = 0.001284 \qquad \frac{d^4A_5}{da^4} = 0.001808$$

$$\frac{d^4A_6}{da^4} = 0.001503 \qquad \frac{d^4A_7}{da^4} = 0.001469$$

$$\frac{d^5A_0}{da^5} = 0.000993 \qquad \frac{d^5A_1}{da^5} = 0.001001 \qquad \frac{d^5A_2}{da^5} = 0.001011$$

$$\frac{d^5A_3}{da^5} = 0.001044 \qquad \frac{d^5A_4}{da^5} = 0.001088 \qquad \frac{d^5A_5}{da^5} = 0.001175$$

$$\frac{d^5A_6}{da^5} = 0.001212$$

$$B_0 = 0.005026 \qquad B_1 = 0.003674 \qquad B_2 = 0.0024$$

$$B_3 = 0.001493 \qquad B_4 = 0.000904 \qquad B_5 = 0.000537$$

$$B_6 = 0.000315 \qquad B_7 = 0.000183 \qquad B_8 = 0.000107$$

$$B_9 = 0.000062$$

$$\frac{dB_0}{da} = 0.001774 \qquad \frac{dB_1}{da} = 0.000184 \qquad \frac{dB_2}{da} = 0.000162$$

$$\frac{dB_3}{da} = 0.000128 \qquad \frac{dB_4}{da} = 0.000943 \qquad \frac{dB_5}{da} = 0.000661$$

$$\frac{dB_6}{da} = 0.000448 \qquad \frac{dB_7}{da} = 0.000448 \qquad \frac{dB_8}{da} = 0.000293$$

$$\frac{dB_9}{da} = 0.000189$$

$$\frac{d^2B_0}{da^2} = 0.001225 \qquad \frac{d^2B_1}{da^2} = 0.001203 \qquad \frac{d^2B_2}{da^2} = 0.001181$$

$$\frac{d^2B_3}{da^2} = 0.001101 \qquad \frac{d^2B_4}{da^2} = 0.000951 \qquad \frac{d^2B_5}{da^2} = 0.000774$$

$$\frac{d^2B_6}{da^2} = 0.000602 \qquad \frac{d^2B_7}{da^2} = 0.000453$$

$$\frac{d^3B_0}{da^3} = 0.001102 \qquad \frac{d^3B_1}{da^3} = 0.001102 \qquad \frac{d^3B_2}{da^3} = 0.001076$$

$$\frac{d^3B_3}{da^3} = 0.001043 \qquad \frac{d^3B_4}{da^3} = 0.000984 \qquad \frac{d^3B_5}{da^3} = 0.000885$$

$$\frac{d^3B_6}{da^3} = 0.000764$$

Jupiter and Mercury

$$a' = 0.38709812 \qquad a = 5.20116636$$
$$\alpha = 0.0744256$$
$$S = 5.20887 \qquad S' = -0.38683.$$

Jupiter and Venus

$$a' = 0.7233323 \qquad \alpha = 0.13907116$$
$$S = 5.22634 \qquad S' = -0.721579.$$

Jupiter and the Earth

$$a' = 1. \qquad \alpha = 0.19226461$$
$$S = 5.24933 \qquad S' = -0.995358.$$

Jupiter and Mars

$$a' = 1.52369352 \qquad \alpha = 0.29295212$$
$$S = 5.31338 \qquad S' = -1.50717.$$

Jupiter and Uranus

$$a' = 19.183305 \qquad \alpha = 0.2711298$$
$$S = 19.5375 \qquad S' = -5.1528.$$

Secular Variations of Jupiter and Saturn

621. These are given by the numerical values of equations (198), which are computed from the formulae

$$\boxed{4.0} = -\frac{3m'.an\left\{a a' S + \left(a^2 + a'^2\right)S'\right\}}{2\left(a'^2 - a^2\right)^2}$$

$$(4.0) = -\frac{3m'.a^2 a' n.S'}{4\left(a'^2 - a^2\right)^2},$$

as the numerical values of all the quantities in these expressions are given, it is easy to find by their substitution, that

$$\begin{aligned}
(4.0) &= 0''.000226 \qquad \boxed{4.0} = 0''.000021,\\
(4.1) &= 0''.004291 \qquad \boxed{4.1} = 0''.00744,\\
(4.2) &= 0''.009862 \qquad \boxed{4.2} = 0''.002359,\\
(4.3) &= 0''.22451 \qquad \boxed{4.3} = 0''.001633,\\
(4.5) &= 7''.702 \qquad \boxed{4.5} = 5''.0342,\\
(4.6) &= 0''.09665 \qquad \boxed{4.6} = 0''.03247,
\end{aligned} \qquad \textbf{(202)}$$

where the digits 0, 1, 2, 3, &c. refer to Mercury, Venus, the Earth, Mars, Jupiter, Saturn, and Uranus.

622. By the substitution of the preceding data, equations (128) and (141), give the following results, when multiplied by the radius reduced to seconds, or, by 206,264″.8, where $\frac{dv}{dt}$ is the sidereal motion of the perihelion of Jupiter in longitude at the epoch 1750, during a period of $365\frac{1}{4}$ days: $2\frac{de}{dt}$ is the annual variation of the equation of the centre: $\frac{df}{dt}$ is the annual inclination[2] of the orbit of Jupiter on the fixed ecliptic of 1750; $\frac{df`}{dt}$ is the annual variation of the inclination on the true ecliptic: $\frac{dq}{dt}$ is the annual and sidereal motion of the ascending node of the orbit of Jupiter on the fixed ecliptic of 1750; and $\frac{dq`}{dt}$ is the same variation with regard to the true ecliptic.

$$\frac{d\bar{v}}{dt} = 6''.5998 \qquad \frac{d\bar{e}}{dt} = 0''.27721 \qquad \frac{d\bar{f}}{dt} = -0''.07814$$
$$\frac{d\bar{f}`}{dt} = -0''.223178 \qquad \frac{d\bar{q}}{dt} = 6''.4562 \qquad \frac{d\bar{q}`}{dt} = -14''.6634.$$

By article 484,

$$(4.0) = \frac{m\sqrt{a}}{m'\sqrt{a'}}(0.4); \qquad \boxed{4.0} = \frac{m\sqrt{a}}{m'\sqrt{a'}}\boxed{0.4};$$

if then, the quantities (202) relating to Jupiter, be multiplied by $\frac{m\sqrt{a}}{m'\sqrt{a'}}$, those corresponding to Saturn will be found, and the formulae (128) give for Saturn

$$\frac{d\bar{v}'}{dt} = 16''.1127 \qquad \frac{d\bar{e}'}{dt} = 0''.54021$$
$$\frac{d\bar{f}'}{dt} = 0''.099741 \qquad \frac{d\bar{q}'}{dt} = -9''.0053.$$

By article 444,

$$f'\sin q' - f\sin q \;= g\;\sin\bar{\Pi},$$
$$f'\cos q' - f\cos q = g\cos\bar{\Pi};$$

and by the substitution of the numerical values of article 613 and 615, it will readily be found, that in 1750

$$\bar{g} = 1^{\circ}\ 15'\ 30'' \qquad \bar{\Pi} = 125^{\circ}\ 44'\ 34'',$$

g being the mutual inclination of the orbits of Jupiter and Saturn, and Π the longitude of the ascending node of the orbit of Saturn on that of Jupiter. If the differential of these equations be taken and the numerical values of

$$\frac{dq'}{dt},\ \frac{dq}{dt},\ \frac{df}{dt},\ \frac{df'}{dt}$$

substituted, it will be found, that

$$\frac{d\bar{g}}{dt} = -0''.000105, \qquad \frac{d\bar{\Pi}}{dt} = -26''.094.$$

623. The variations in the elements that depend on the squares of the disturbing forces must now be computed, and for that purpose the numerical values of $P,\ P',$ and their differences, must be found from equations (165) and (166).

The coefficients $Q_0,\ Q_1,$ &c., are given by the expansion of R, article 446; so that[3]

$$Q_0 = -\frac{1}{12}\left\{389A_2 + 201a\cdot\frac{dA_2}{da} + 27a^2\cdot\frac{d^2A_2}{da^2} + a^3\cdot\frac{d^3A_2}{da^3}\right\}$$
$$Q_1 = +\frac{1}{4}\left\{402A_3 + 193a\cdot\frac{dA_3}{da} + 26a^2\cdot\frac{d^2A_3}{da^2} + a^3\cdot\frac{d^3A_3}{da^3}\right\}$$
$$Q_2 = -\frac{1}{4}\left\{396A_4 + 184a\cdot\frac{dA_4}{da} + 25a^2\cdot\frac{d^2A_4}{da^2} + a^3\cdot\frac{d^3A_4}{da^3}\right\}$$
$$Q_3 = +\frac{1}{12}\left\{380A_5 + 174a\cdot\frac{dA_5}{da} + 24a^2\cdot\frac{d^2A_5}{da^2} + a^3\cdot\frac{d^3A_5}{da^3}\right\}$$

$$Q_4 = -\frac{1}{4}aa'\left\{10B_3 + a\frac{dB_3}{da}\right\}$$
$$Q_5 = +\frac{1}{4}aa'\left\{7B_4 + a\frac{dB_4}{da}\right\}.$$

If the values of A_2, A_3, &c. and their differences in article 620, be substituted, then will

$$Q_0 = -2.199192,$$
$$Q_1 = +4.0292,$$
$$Q_2 = -2.43538,$$
$$Q_3 = +0.487332,$$
$$Q_4 = -0.267808,$$
$$Q_5 = +0.139264.$$

With the preceding data, equations (165) and (166) give

$$P = 0.0000114596, \quad P' = -0.000107267;$$

and as the differences of these equations are

$$\frac{dP}{de} = \frac{1}{4}\left\{Q_1 e'^2 \sin(2v' + v) + 2Q_2 ee' \sin(v' + 2v) + 3Q_3 e^2 \sin 3v + Q_5 g^2 \sin(2\Pi + v)\right\},$$
$$\frac{dP'}{de} = \frac{1}{4}\left\{Q_1 e'^2 \sin(2v' + v) + 2Q_2 ee' \cos(v' + 2v) + 3Q_3 e^2 \cos 3v + Q_5 g^2 \cos(2\Pi + v)\right\},$$

&c. &c.

all the quantities in equations (191), (192), (195), and (196), are known; whence for Jupiter at the epoch,[4]

$$(d\bar{v}) = +0''.352941, \qquad (d\bar{e}) = +0''.052278,$$
$$(d\bar{g}) = +0''.000184, \qquad (d\bar{\Pi}) = -0''.007631,$$

and from equations (193) and (194), the corresponding variations in the elements of the orbit of Saturn are

$$(d\bar{v}') = +3''.242722, \qquad (d\bar{e}') = -0''.102763;$$

so that

$$\frac{d\bar{v}}{dt} = 6''.95281; \quad \frac{d\bar{v}'}{dt} = 19''.355448; \quad \frac{d\bar{e}}{dt} = 0''.329487;$$
$$\frac{d\bar{e}'}{dt} = -0''.642968; \quad \frac{d\bar{g}}{dt} = -0''.000079; \quad \frac{d\bar{\Pi}}{dt} = -26''.10163.$$

But at the epoch,

$$\bar{e} = 9916''.53; \quad \bar{e}' = 11597''.1; \quad v = 10^\circ.35108;$$
$$\bar{v}' = 88^\circ 15194; \quad \bar{g} = 1^\circ.25838; \quad \bar{\Pi} = 125^\circ.74278.$$

Consequently the elements of the two orbits at any time t are

$$\begin{aligned} e &= 9916''.53 + 0''.329487\,.\,t, \\ v &= 10^\circ.35108 + 6''.95281\,.\,t, \\ e' &= 11597''.1 - 0''.642968\,.\,t, \\ v' &= 88^\circ.15194 + 19''.3555448\,.\,t, \\ g &= 1^\circ.25838 + 0''.000079\,.\,t, \\ \Pi &= 125^\circ.74278 - 26''.10163\,.\,t. \end{aligned} \quad \textbf{(203)}$$

If $t = 0$ these expressions will give the elements in 1950, and if the computation be repeated with them it will be found that in 1950

$$\frac{dv}{dt} = 7''.053178; \quad \frac{dv'}{dt} = 19''.424739; \quad \frac{de}{dt} = 0''.326172;$$
$$\frac{de'}{dt} = -0''.648499; \quad \frac{dg}{dt} = -0''.001487; \quad \frac{d\Pi}{dt} = -26''.402056.$$

The differences between these and their values for 1750, divided by 200, will be their second differences, therefore the formulae (198), with regard to Jupiter and Saturn, are

$$\begin{aligned} e &= 9916''.53 + 0''.329487\,.\,t - 0''0000082871\,.\,t^2, \\ v &= 10^\circ 21'4'' + 6''.952808\,.\,t + 0''.0002509259\,.\,t^2, \\ e' &= 11597''.1 - 0''.642968\,.\,t - 0''.0000138275\,.\,t^2, \\ v' &= 88^\circ 9'6'' + 19''.3555440\,.\,t + 0''.0001732274\,.\,t^2, \\ g &= 1^\circ 15'30''.2 + 0''.000078\,.\,t - 0''.0000391311\,.\,t^2, \\ \Pi &= 125^\circ 44'33'' - 26''.1028\,.\,t - 0''.0007507307\,.\,t^2, \end{aligned} \quad \textbf{(204)}$$

which will give the elements of the orbits of these two planets for 1000 or 1200 years before or after 1750.

Periodic Inequalities of Jupiter

624. The inequalities in the radius vector and longitude, which are independent of the eccentricities and inclinations, are computed from

$$\frac{dr}{a} = -\frac{m'}{6} a^2 \cdot \frac{dA_0}{da} + \frac{m'}{2} \cdot \Sigma . C_i \cdot \cos i\left(n't - nt + \epsilon' - \epsilon\right),$$
$$dv = \frac{m'}{2} \cdot \Sigma . F_i \cdot \sin i\left(n't - nt + \epsilon' - \epsilon\right);$$

If $i = 0$, then by articles 536 and 537

$$C_1 = \frac{n^2}{n'(2n - n')}\left\{\frac{2n}{n - n'} \cdot aA_1 + a^2 \cdot \frac{dA_1}{da}\right\}$$
$$F_1 = \frac{n}{n - n'}\left\{-\frac{2n}{n - n'} \cdot aA_1 + 2C_1\right\}.$$

But

$$n = 109256''; \quad n' = 43996''.7; \quad a = 5.20116626;$$
$$m' = \frac{1}{3359.4}; \quad A_1 = 0.0078973; \quad \frac{dA_1}{da} = 0.00531108;$$
$$\frac{2n}{n - n'} \cdot aA_1 = 0.1375352; \quad a^2 \cdot \frac{dA_1}{da} = 0.143676;$$
$$\frac{2n}{n - n'} aA_1 + a^2 \frac{dA_1}{da} = 0.281209;$$
$$\log 0.281209 = 9.4490293$$
$$\log \frac{2n^2}{n'(2n - n')} = 0.4926697$$
$$\log 2C_1 = \overline{9.9416990} = \log 0.874378$$

hence

$$-\frac{n}{n - n'} aA_1 + 2C_1 = 0.8056104.$$

$$\log 0.8056104 = 9.9061248$$
$$\log \frac{n}{n - n'} = 0.2238068$$
$$\text{log of radius in seconds} = \underline{5.3144256}$$
$$\text{the sum is. . . . . } 5.4443572$$
$$\log 3359.4 = 3.5262617$$
$$\log 82''.812 = 1.9180955$$

Consequently, when $i = 1$, $dv = 82''.821 \sin\left(n't - nt + \epsilon' - \epsilon\right)$. Hence if i be made successively equal to all the positive numbers from 1 to 9, and the corresponding quantities substituted in the preceding formulae, it will be found that the inequalities of this order in the longitude and radius vector of Jupiter arising from the action of Saturn, are

$$d v = \begin{Bmatrix} +82''.811711 \sin(n't - nt + \epsilon' - \epsilon) \\ -204''.406384 \sin 2(n't - nt + \epsilon' - \epsilon) \\ -17''.071564 \sin 3(n't - nt + \epsilon' - \epsilon) \\ -3''.926319 \sin 4(n't - nt + \epsilon' - \epsilon) \\ -1''.210573 \sin 5(n't - nt + \epsilon' - \epsilon) \\ -0''.42843 \sin 6(n't - nt + \epsilon' - \epsilon) \\ -0''.170923 \sin 7(n't - nt + \epsilon' - \epsilon) \\ -0''.076086 \sin 8(n't - nt + \epsilon' - \epsilon) \\ -0''.041273 \sin 9(n't - nt + \epsilon' - \epsilon) \end{Bmatrix}$$

$$d r = \begin{Bmatrix} -0.0000620586 \\ +0.000676876 \cos(n't - nt + \epsilon' - \epsilon) \\ -0.00289662 \cos 2(n't - nt + \epsilon' - \epsilon) \\ -0.0003021367 \cos 3(n't - nt + \epsilon' - \epsilon) \\ -0.0000782514 \cos 4(n't - nt + \epsilon' - \epsilon) \\ -0.0000258952 \cos 5(n't - nt + \epsilon' - \epsilon) \\ -0.0000094779 \cos 6(n't - nt + \epsilon' - \epsilon) \\ -0.000003756 \cos 7(n't - nt + \epsilon' - \epsilon) \\ -0.0000014781 \cos 8(n't - nt + \epsilon' - \epsilon) \\ 0.0000004799 \cos 9(n't - nt + \epsilon' - \epsilon) \end{Bmatrix}.$$

625. The inequalities depending on the first powers of the eccentricities are obtained from

$$\begin{aligned} d r &= m'fe \cos(nt + \epsilon - v) + m'f'e' . \cos(nt + \epsilon - v') \\ &\quad + m'e . \Sigma . D_i . \cos\{i(n't - nt + \epsilon' - \epsilon) + nt + \epsilon - v\} \\ &\quad + m'e' . \Sigma . E_i . \cos\{i(n't - nt + \epsilon' - \epsilon) + nt + \epsilon - v'\}, \\ d v &= m'e . \Sigma . G_i . \sin\{i(n't - nt + \epsilon' - \epsilon) + nt + \epsilon - v\} \\ &\quad + m'e' . \Sigma . H_i . \sin\{i(n't - nt + \epsilon' - \epsilon) + nt + \epsilon - v'\}. \end{aligned}$$

by making successively equal to the whole positive numbers, from 1 to 7, and to the whole negative numbers, from -1 to -5, and substituting the numerical data corresponding to each in the coefficients D_i, E_i, &c., which are given in articles 536 and 537. The values of e and e' at the epoch are sufficiently exact for all the terms of this order, except those having the arguments $2n't - nt + 2\epsilon' - \epsilon$, and $3n' - 2nt + 3\epsilon' - 2\epsilon$, whose periods are so long, that

$$9916''.53 + 0''.329487\,.\,t, \text{ and } 11597''.1 - 0''.642968\,.\,t$$

must be employed instead of e and e'. It will then be found that the perturbations of Jupiter are

$$d\,r = \left\{\begin{array}{l} +0.0000206111\cos\left(nt + \epsilon - v\right) \\ -0.0000795246\cos\left(n't + \epsilon' - v\right) \\ +0.0000492096\cos\left(n't + \epsilon' - v'\right) \\ -0.000292213\cos\left\{2n't - nt + 2\epsilon' - \epsilon - v\right\} \\ +0.0001688085\cos\left\{2n't - nt + 2\epsilon' - \epsilon - v'\right\} \\ -0.0004584483\cos\left\{3n't - 2nt + 3\epsilon' - 2\epsilon - v\right\} \\ +0.0009047822\cos\left\{3n't - 2nt + 3\epsilon' - 2\epsilon - v'\right\} \\ +0.0001259429\cos\left\{4n't - 3nt + 4\epsilon' - 3\epsilon - v\right\} \\ -0.0002424413\cos\left\{4n't - 3nt + 4\epsilon' - 2\epsilon - v'\right\} \\ +0.0000268383\cos\left\{5n't - 4nt + 5\epsilon' - 4\epsilon - v\right\} \\ -0.0000516048\cos\left\{5n't - 4nt + 5\epsilon' - 4\epsilon - v'\right\} \\ +0.0000579151\cos\left\{2nt - n't + 2\epsilon - \epsilon' - v\right\} \\ -0.000134653\cos\left\{3nt - 2n't + 3\epsilon - 2\epsilon' - v\right\} \end{array}\right\}.$$

[and] [5]

$$d\,v = \left\{\begin{array}{l} +8''.608489\sin\left(n't + \epsilon' - v\right) \\ -9''.692386\sin\left(n't + \epsilon' - v'\right) \\ -\left\{138''.373337 + t\,.\,0''.0045985\right\}\sin\left\{2n't - nt + 2\epsilon' - \epsilon - v\right\} \\ +\left\{56''.634099 - t\,.\,0''.0031398\right\}\sin\left\{2n't - nt + 2\epsilon' - \epsilon - v'\right\} \\ -\left\{44''.460822 + t\,.\,0''.0014775\right\}\sin\left\{3n't - 2nt + 2\epsilon' - 2\epsilon - v\right\} \\ +\left\{84''.942569 - t\,.\,0''.004794\right\}\sin\left\{3n't - 2nt + 3\epsilon' - 2\epsilon - v'\right\} \\ +7''.925312\sin\left\{4n't - 3nt + 4\epsilon' - 3\epsilon - v\right\} \\ -15''.629621\sin\left\{4n't - 3nt + 4\epsilon' - 3\epsilon - v'\right\} \\ +1''.047717\sin\left\{5n't - 4nt + 5\epsilon' - 4\epsilon - v\right\} \\ -2''.781664\sin\left\{5n't - 4nt + 5\epsilon' - 4\epsilon - v'\right\} \\ +0''.407251\sin\left\{6n't - 5nt + 6\epsilon' - 5\epsilon - v\right\} \\ \quad \text{Continued on next page} \end{array}\right\}.$$

$$\left\{\begin{array}{l}\text{Continued from previous page}\\ -0''.913302\sin\{6n't-5nt+6\in'-5\in-v'\}\\ +0''.149277\sin\{7n't-6nt+7\in'-6\in-v\}\\ -0''.325592\sin\{7n't-6nt+7\in'-6\in-v'\}\\ -5''.208122\sin\{2n't-nt+2\in-\in'-v\}\\ -0''.569738\sin\{2n't-nt+2\in-\in'-v'\}\\ +12''.87665\sin\{3n't-2nt+3\in-2\in'-v\}\\ -0''.352399\sin\{3n't-2nt+3\in-2\in'-v'\}\\ +1''.287482\sin\{4n't-3nt+4\in-3\in'-v\}\\ -0''.172892\sin\{4n't-3nt+4\in-3\in'-v'\}\\ +0''.356627\sin\{5n't-4nt+5\in-4\in'-v\}\\ -0''.083189\sin\{5n't-4nt+5\in-4\in'-v'\}\end{array}\right\}.$$

Inequalities depending on the Squares of the Eccentricities and Inclinations

626. These are computed by making i successively equal to 1, 2, 3, &c., in formulae (163) and (164).

If $i=1$, that part of the perturbations in longitude, depending on the argument $n't+nt+\in'+\in$, is[6]

$$dv=\frac{1}{\sqrt{1-e^2}}\left\{\begin{array}{l}\dfrac{2d.(rdr)}{a^2.ndt}-\dfrac{m'}{2}\left\{\begin{array}{l}\left(\frac{1}{2}C_1+D_1\right)e^2.\sin\left(n't+nt+\in'+\in-2v\right)\\+E_1ee'.\sin\left(n't+nt+\in'+\in-v-v'\right)\end{array}\right\}\\ \qquad -\dfrac{m'}{2}\left\{\dfrac{3n^2}{(n'+n)^2}\cdot\sum.aN+\sum a^2\cdot\dfrac{dN}{da}\cdot\dfrac{2n}{n'+n}\right\}\sin\left(n't+nt+\in'+\in+L\right)\end{array}\right\};$$

where[7]

$$\frac{2d(rdr)}{a^2.ndt}=-\frac{m'.n^2}{n'^2+2nn'}\left\{\begin{array}{l}3\left(\frac{1}{2}C_1+D_1\right)e^2.\sin\left(n't+nt+\in'+\in-2v\right)\\+3E_1.ee'.\sin\left(n't+nt+\in'+\in-v-v'\right)\\+\left\{\dfrac{2n}{n'+n}\cdot\sum.dN+\sum.a^2\dfrac{dN}{da}\right\}\cdot\sin\left(n't+nt+\in'+\in+L\right)\end{array}\right\}.$$

$$C_1=\frac{n^2}{n^2-(n'-n)^2}\cdot\left\{\frac{2n}{n-n'}\cdot aA_1+a^2\frac{dA_1}{da}\right\}$$

$$D_1=\frac{n^2}{n'^2-n^2}\left\{\frac{3n}{n'-n}\cdot aA_1-\frac{(n'-n)(n'-2n)-3n^2}{n^2}\cdot C_1+\tfrac{1}{2}a^3\frac{d^2A_1}{da^2}\right\}$$

$$E_1 = -\frac{n^2}{n'^2 - n^2} \cdot \left\{ a^2 \frac{dA_0}{da} + \tfrac{1}{2} a^3 \frac{d^2 A_0}{da^2} \right\}.$$

$$\begin{aligned}
&\Sigma . N . \sin\left(n't + nt + \in' + \in - L\right) = \\
&+ N_0 . e^2 . \sin\left(n't + nt + \in' + \in - 2v\right) \\
&+ N_1 . ee' . \sin\left(n't + nt + \in' + \in - v - v'\right) \\
&+ N_2 . e'^2 . \sin\left(n't + nt + \in' + \in - 2v'\right) \\
&+ N_6 . g^2 . \sin\left(n't + nt + \in' + \in - 2\Pi\right).
\end{aligned}$$

The coefficients N_0, N_1, &c., are given in article 459, and if the numerical values of A_0, A_1, their differences, and also[8] $n = 109256''$, $n' = 43996''.6$, be substituted, it will be found that $d v$ takes the form

$$\begin{aligned}
d v = {} & b . e^2 . \sin\left(n't + nt + \in' + \in - 2v\right) \\
& + b_1 . ee' . \sin\left(n't + nt + \in' + \in - v - v'\right) \\
& + b_2 . e'^2 . \sin\left(n't + nt + \in' + \in - 2v'\right) \\
& + b_3 . g^2 . \sin\left(n't + nt + \in' + \in - 2\Pi\right),
\end{aligned}$$

where b, b_1, b_2 and b_3 are given numbers. But $d v$ may be expressed by

$$d v = P . \sin\left(n't + nt + \in' + \in\right) - P' . \cos\left(n't + nt + \in' + \in\right),$$

where,[9]

$$P' = be^2 . \sin 2v + b_1 ee' . \sin\left(v + v'\right) + b_2 . e'^2 \sin 2v' + b_3 . g^2 \sin 2\Pi$$

[and]

$$P = be^2 . \sin 2v + b_1 ee' . \cos\left(v + v'\right) + b_2 . e'^2 \cos 2v' + b_3 . g^2 \cos 2\Pi;$$

substituting the values of the elements given in article 619, it will be found by the method in article 569, that

$$\sqrt{P^2 + P'^2} = 1''.004; \qquad \frac{P'}{P} = -\tan 45^\circ.4894 = -\frac{\sin 45^\circ.4894}{\cos 45^\circ.4894}.$$

Consequently the inequality depending on $i = 0$, becomes

$$d v = 1''.004 . \sin\left(n't + nt + \in' + \in + 45^\circ.4894\right).$$

627. It will be found by this method of computation that all the sensible inequalities in longitude and in the radius vector depending on the squares and products of the eccentricities and inclinations, are included in the following expressions; observing that the inequality having the argument $3n't - 5nt + 3\epsilon' - 5\epsilon$, must be computed with the formulae (204), on account of the great length of its period.[10 11]

$$d\,v = \left\{ \begin{array}{l} +1''.004\sin\left(n't + nt + \epsilon' + \epsilon + 45^\circ.4894\right) \\ -5''.57871\sin\left(2n't + 2\epsilon' + \epsilon + 15^\circ.93999\right) \\ +11''.72425\sin\left(3n't - nt + 3\epsilon' - \epsilon + 79^\circ.6633\right) \\ -18''.07528\sin\left(4n't - 2nt + 4\epsilon' - 2\epsilon - 57^\circ.2072\right) \\ +\{169''.2659 - t\,.0''.004277\}\sin\left(3n't - 5nt + 3\epsilon' - 5\epsilon + 55^\circ.6802 + t\,.50''.5084\right) \\ +1''.64714\sin\left(6n't - 4nt + 6\epsilon' - 4\epsilon - 54^\circ.43\right) \\ +2''.4764\sin\left(n't - nt + \epsilon - \epsilon' + 43^\circ.2836\right) \\ -5''.288\sin\left(2n't - 2nt + 2\epsilon' - 2\epsilon + 42^\circ.6789\right) \\ +0.000082242\cos\left(2n't + 2\epsilon + 11^\circ.0153\right) \\ +0.000022625\cos\left(3n't - nt + 3\epsilon' - 2\epsilon - 21^\circ.7884\right) \\ -0.0001010533\cos\left(4n't - 2nt + 4\epsilon' - 2\epsilon - 51^\circ.0677\right) \\ -\{0.00211145 - t\,.\,0.00000005323\}\cos\left(3n't - 5nt + 3\epsilon' - 5\epsilon + 55^\circ.597 + 50''.4144\,.\,t\right) \\ -0.0000652204\cos\left(2n't - 2nt + 2\epsilon' - 3\epsilon + 54^\circ.1477\right) \end{array} \right\}.$$

Perturbations depending on the Third Powers and Products of the Eccentricities and Inclinations

628. These are contained in equation (172). But, in order to find the numerical value of the principal term, the differences of P and P' must be computed. By article 623,

$$P = 0.0000114596, \qquad P' = -0.000107267$$

are the values of these quantities in 1750; but their values in the years 2250, and 2750, will be obtained by making t successively equal to 500 and 1000, in equations (204); whence the elements of the orbits of Jupiter and Saturn at these two periods will be known; and if the same computation that was employed for the determination of P and P' be repeated with them, the results in 2250, and 2750, will be

$$\begin{aligned} P &= -0.000008407 \text{ [in 2250]} \\ P' &= -0.00010552 \text{ [in 2250]} \\ P &= -0.000027365 \text{ [in 2750]} \\ P' &= -0.00010009 \text{ [in 2750]}; \end{aligned}$$

and, by the method of article 480

$$\frac{dP}{dt} = -0.000000040645 \text{ [in 2250]};$$
$$\frac{dP'}{dt} = -0.0000000002249 \text{ [in 2250]};$$
$$\frac{dP}{dt} = -0.000000000003642 \text{ [in 2750]};$$
$$\frac{dP'}{dt} = -0.000000000014865 \text{ [in 2750]};$$

with these data the principal term of the great inequality put under the form of equation (171) becomes

$$d\,v = \left\{ \begin{array}{l} +1263''.79967 - 0''.008418\,.\,t - 0''.00001925\,.\,t^2 \sin\left(5n't - 2\,nt + 5\in' - 2\in\right) \\ +119''.52695 - 0''.473686\,.\,t - 0''.000078562\,.\,t^2 \cos\left(5n't - 2\,nt + 5\in' - 2\in\right) \end{array} \right\}.$$

In order to compute the inequality[12]

$$d\,v = -\frac{2m'n}{5n' - 2n}\left\{ a^2 \cdot \frac{dP}{da} \cdot \cos\left(5n't - 2\,nt + 5\in' - 2\in\right) - a^2 \cdot \frac{dP'}{da} \cdot \sin\left(5n't - 2\,nt + 5\in' - 2\in\right) \right\}$$

equation (165), gives[13]

$$\begin{aligned} 4\frac{dP}{da} = &+\frac{dQ_0}{da} \cdot e^3 \sin 3v' + \frac{dQ_1}{da} \cdot e'^2 \sin\left(v' + v\right) \\ &+ \frac{dQ_2}{da} \cdot e'e^2 \sin\left(v + 2v'\right) + \frac{dQ_3}{da} \cdot e^3 \sin 3v \\ &+ \frac{dQ_4}{da} \cdot e'g^2 \sin\left(2\Pi + v'\right) + \frac{dQ_5}{da} \cdot eg \sin\left(2\Pi + v\right). \end{aligned}$$

The quantities $\frac{dQ_0}{da}$, &c. are obtained from the values of Q_0, Q_1, &c. in article 623.[14]

With which and the numerical values of the elements at the epoch 1750, the preceding value of $\frac{dP}{da}$ gives

$$-\frac{2m'.n}{5n' - 2n} a^2 \cdot \frac{dP}{da} = -17''.22886;$$

and, by changing the sines into cosines, the same expression gives

$$\frac{2m'n}{5n'-2n}a^2 \cdot \frac{dP'}{da} = 5''.360016 .$$

If t be made equal to 200 in the equations (204), and the computation repeated with the resulting values of the elements, it will be found that in 1950

$$-\frac{2m'.n}{5n'-2n}a^2 \cdot \frac{dP}{da} = -16''.836801$$
$$\frac{2m'n}{5n'-2n}a^2 \cdot \frac{dP'}{da} = 6''.449839;$$

but

$$\frac{-17''.22886+16''.83680}{200} = -0''.0019603,$$

and[15]

$$\frac{6''.449839-5''.360016}{200} = 0''.0004491;$$

hence[16]

$$d\,v = -\{17''.228862 - 0''.0019603\,.\,t\}\,.\sin(5n't - 2\,nt + 5\in' - 2\in)$$
$$+\{5''.360016 + 0''.0004491\,.\,t\}\,.\cos(5n't - 2\,nt + 5\in' - 2\in).$$

The only remaining inequalities of this order are,[17]

$$-m'Ke\,.\sin(5n't - 2\,nt + 5\in' - 2\in - v + B)$$
$$+\frac{5m'}{4}\cdot K'e\,.\sin(5n't - 4\,nt + 5\in' - 4\in + v + B)$$

the numerical values of which may easily be found equal to

$$d\,v = (0''.8203 - 0''.00059324\,.\,t)\,.\sin(5n't - 2\,nt + 5\in' - 2\in)$$
$$-(1''.83796 - 0''.00000149\,.\,t)\,.\cos(5n't - 2\,nt + 5\in' - 2\in)$$
$$+10''.0847\,.\sin(4nt - 5n't + 4\in' - 5\in - 45^\circ.36225).$$

The great inequality of Jupiter also contains the terms

$$d\,v = (12''.5365 - 0''.001755t).\sin(5n't - 2\,nt + 5\in' - 2\in)$$
$$-(8''.1211 + 0''.004885t).\cos(5n't - 2\,nt + 5\in' - 2\in);$$

depending on the fifth powers and products of the eccentricities and inclinations, the computation of these is exactly the same with the examples given, but very tedious on account of the form of the coefficients of the series *R.* If all the terms depending on the argument $5n't - 2nt + 5\epsilon' - 2\epsilon$ be collected, it will be found that the great inequality of Jupiter is

$$d v = \left\{ \begin{array}{l} + \left\{ 1261''.56 - 0''.013495 \,.\, t - 0''.00001925 \,.\, t^2 \right\} . \sin\left(5n't - 2nt + 5\epsilon' - 2\epsilon\right) \\ + \left\{ 96''.4661 - 0''.47466 \,.\, t - 0''.00007856 \,.\, t^2 \right\} . \cos\left(5n't - 2nt + 5\epsilon' - 2\epsilon\right) \end{array} \right\}.$$

Inequalities depending on the Squares of the Disturbing Force

629. These are given by equations (182) and (199): their numerical values are

$$\begin{aligned} d v = {} & +4''.0248 \,.\, \sin\left(5nt - 10n't + 5\epsilon - 10\epsilon' + 51^\circ.3653\right) \\ & -13''.2389 \sin\left(\text{twice the argument of the great inequality of Jupiter}\right). \end{aligned}$$

The inequality mentioned in article 589, according to Pontécoulant,[18] is [for Jupiter],

$$dz = 2''.16304 \,.\, \sin\left(5n't - 2nt + 5\epsilon' - 2\epsilon\right) + 16''.9712 \times \cos\left(5n't - 2nt + 5\epsilon' - 2\epsilon\right);$$

and [for Saturn]

$$dz' = 3''.4645 \,.\, \sin\left(5n't - 2nt + 5\epsilon' - 2\epsilon\right) - 40''.3437 \times \cos\left(5n't - 2nt + 5\epsilon' - 2\epsilon\right).$$

Periodic Inequalities in the Radius Vector, depending on the Third Powers and Products of the Eccentricities and Inclinations

630. These are occasioned by Saturn, and are easily found from equation (168) to be [19]

$$d r = \left\{ \begin{array}{l} -0.0003042733 \,.\, \cos\left(5n't - 2nt + 5\epsilon' - 2\epsilon - 12^\circ.14694\right) \\ +0.0001001860 \,.\, \cos\left(5n't - 2nt + 5\epsilon' - 2\epsilon + 45^\circ.27972\right) \end{array} \right\}.$$

Periodic Inequalities in Latitude

631. These are obtained from equations (160) and (177).

$$f = 1^\circ.3172,$$

is the inclination of Jupiter's orbit on the fixed ecliptic of 1750,

$$\frac{df}{dt} = -0''.07821 \text{ is its secular variation,}$$

and

$$\frac{df`}{dt} = -0''.22325,$$

is the same, with regard to the variable ecliptic; also

$$q = 97^\circ.906,$$

is the longitude of the ascending node of Jupiter's orbit on the fixed ecliptic; $\frac{dq}{dt} = 6''.4571$, is its secular variation with regard to that plane, and $\frac{df`}{dt} = -14''.6626$ is its secular variation with regard to the variable ecliptic. Equations (197) give

$$(df) = -0.0000726, \text{ and } (dq) = 0.0008113,$$

for the variations depending on the squares of the disturbing forces; hence

$$\frac{df}{dt} = -0''.078283, \ \frac{dq}{dt} = 6''.457,$$

with regard to the fixed ecliptic, and

$$\frac{df`}{dt} = -0''.22325, \ \frac{dq`}{dt} = -14''.6626.$$

With these it will be found that

$$ds = \left\{ \begin{array}{l} +0''.564458 . \sin\left(n't + \in' - \Pi\right) \\ +0''.663927 . \sin\left(2n't - nt + 2\in' - \in - \Pi\right) \\ +1''.119782 . \sin\left(3n't - 2nt + 3\in' - 2\in - \Pi\right) \\ -0''.279382 . \sin\left(4n't - 3nt + 4\in' - 3\in - \Pi\right) \\ -0''.26913 . \sin\left(2nt - n't + 2\in - \in' - \Pi\right) \\ +3''.94168 . \sin\left(3nt - 5n't + 3\in - 5\in' + 59^\circ.50097\right) \end{array} \right\};$$

which are the only sensible inequalities in the latitude of Jupiter.

632. The action of the earth occasions the inequalities

$$dv = \begin{Bmatrix} +0''.120833 . \sin(n't - nt + \epsilon' - \epsilon) \\ -0''.000086 . \sin 2(n't - nt + \epsilon' - \epsilon) \end{Bmatrix}$$

in the longitude of Jupiter, n' being the mean motion of the earth, and the action of Uranus is the cause of the following perturbations in the longitude of Jupiter,[20]

$$dv = \begin{Bmatrix} +1''.051737 . \sin 2(n't - nt + \epsilon' - \epsilon) \\ -0''.427296 . \sin 2(n't - nt + \epsilon' - \epsilon) \\ -0''.044085 . \sin 3(n't - nt + \epsilon' - \epsilon) \\ -0''.005977 . \sin 4(n't - nt + \epsilon' - \epsilon) \\ +0''.123506 . \sin(nt + \epsilon - v) \\ -0''.23524 \ . \sin(nt + \epsilon - v') \\ -0''.53308 \ . \sin(2n't - nt + 2\epsilon' - \epsilon - v) \\ +0''.102673 . \sin(2n't - nt + 2\epsilon' - \epsilon - v') \\ -0''.127963 . \sin(3n't - 2nt + 3\epsilon' - \epsilon - v') \end{Bmatrix}$$

where n' is the mean motion of Uranus.

These are all the inequalities that are sensible in the motions of Jupiter; those of Saturn may be computed in the same manner.

On the Laws, Periods, and Limits of the Variations in the Orbits of Jupiter and Saturn

633. When the values of $p, \ p', \ q, \ q',$ are substituted in equations (137) they give

$$gN = (4.5)(N' - N); \quad gN' = (5.4)(N - N');$$

and as

$$(5.4) = (4.5)\frac{m\sqrt{a}}{m'\sqrt{a'}},$$

$$g^2 + g\left\{\frac{m'\sqrt{a'} + m\sqrt{a}}{m'\sqrt{a'}}\right\}(4.5) = 0 .$$

The roots of which are,

$$g_1 = 0; \quad g = -\frac{m'\sqrt{a'} + m\sqrt{a}}{m'\sqrt{a'}}(4.5)$$

so that equations (138) become

$$\begin{aligned} p &= N\,.\sin\left(gt+x\right)+N_{/}\,.\sin x_{/} \\ q &= N\,.\cos\left(gt+x\right)+N_{/}\,.\cos x_{/} \\ p' &= N'\,.\sin\left(gt+x\right)+N_{/}\,.\sin x_{/} \\ q' &= N'\,.\cos\left(gt+x\right)+N_{/}\,.\cos x_{/}. \end{aligned} \qquad \textbf{(205)}$$

Whence,

$$p'-p=\left(N'-N\right)\sin\left(gt+x\right);\quad q'-q=\left(N'-N\right)\cos\left(gt+x\right),$$

and at the epoch when $t=0$

$$\tan x=\frac{p'-p}{q'-q}.$$

But as

$$N'=-\frac{m\sqrt{a}}{m'\sqrt{a'}}\cdot N;$$

and

$$p'-p=\left(N'-N\right)\sin x,$$

so

$$N=-\frac{m'\sqrt{a'}\left(p'-p\right)}{\left(m\sqrt{a}+m'\sqrt{a'}\right)\sin x}.$$

Again, by article 504

$$\begin{aligned} m\sqrt{a}\,.\,p+m'\sqrt{a'}\,.\,p' &= \text{constant}, \\ m\sqrt{a}\,.\,q+m'\sqrt{a'}\,.\,q' &= \text{constant}; \end{aligned}$$

or in consequence of

$$Nm\sqrt{a}+N'm'\sqrt{a'}=0$$

$$\begin{aligned} \left(m\sqrt{a}+m'\sqrt{a'}\right)N_{/}\,.\sin x_{/} &= \text{constant}, \\ \left(m\sqrt{a}+m'\sqrt{a'}\right)N_{/}\,.\cos x_{/} &= \text{constant}. \end{aligned}$$

Whence[21]

$$\tan x_{/}=\frac{m\sqrt{a}\,.\,p+m'\sqrt{a'}\,.\,p'}{m\sqrt{a}\,.\,q+m'\sqrt{a'}\,.\,q'},$$

and

$$N_{/}=\frac{m\sqrt{a}\,.\,p+m'\sqrt{a'}\,.\,p'}{\left(m\sqrt{a}+m'\sqrt{a'}\right)\sin x_{/}}$$

and as at the epoch

$$\begin{aligned} p &= \tan\bar{f}\,.\sin\bar{q} \qquad & q &= \tan\bar{f}\,.\cos\bar{q} \\ p' &= \tan\bar{f}'\,.\sin\bar{q}' \qquad & q' &= \tan\bar{f}'\,.\cos\bar{q}' \end{aligned}$$

are given, all the constant quantities g, $g_/$, x, $x_/$, N, N', and $N_/$, are obtained from the preceding equations.

The variations in the inclinations are at their maxima and minima when $gt + x - x_/$ is either zero or 180°; hence if $x_/$ be substituted for $gt + x$, equations (205) give

$$\tan f = N + N_/; \qquad \tan f' = N' + N_/$$

for the maxima of the inclinations; and when $x_/ + 180^\circ$ is put for $gt + x$, they give for the minima,[22]

$$\tan f = N - N_/; \qquad \tan f' = N' - N_/ .$$

The maxima and minima of the longitude of the nodes are given by the equations $dq = 0$, $dq' = 0$, or $d\,.\tan q = 0$, whence

$$q\frac{dp}{dt} - p\frac{dq}{dt} = 0,$$

and therefore $pp' + qq' = p^2 + q^2$, and by the substitution of the quantities in equations (205), it becomes

$$N + N_/\,.\cos\left(gt + x - z_/\right) = 0,$$

or

$$\cos\left(gt + x - x_/\right) = -\frac{N}{N_/}.$$

If $N_/$ be greater than N independently of the signs, the nodes will have a libratory[23] motion; but if $N_/$ be less than N, they will circulate in one direction.

Tan $f = \sqrt{N_/^2 - N^2}$ corresponds to the preceding value of

$$\cos\left(gt + x - x_/\right);$$

it gives the inclination corresponding to the stationary points of the node.

These points are attained when

$$\cos\left(gt + x - x_/\right) = -\frac{N}{N_/},$$

whereas the maxima and minima of the inclinations happen when

$$\cos\left(gt + x - x_{/}\right) = \pm 1.$$

The stationary positions of the nodes therefore do not correspond either to the maxima or minima of the inclination, or to the semi-intervals between them.

In 1700, by Halley's Tables,[24 25]

$$f = 1^{\circ}\,19'\,10'' \qquad q = 97^{\circ}\,34'\,9''$$
$$f' = 2^{\circ}\,30'\,10'' \qquad q' = 101^{\circ}\,5'\,6''$$

hence at that time,

$$p = 0.02283 \qquad q = -0.00303$$
$$p' = 0.04078 \qquad q' = -0.01573,$$

with these values, Mr. Herschel[26] found

$$N_{/} = 0.02905 \qquad N' = 0.01537 \qquad N = -0.00661$$
$$x = 125^{\circ}\,15'\,40'' \quad x_{/} = 103^{\circ}\,38'\,40'' \quad g = -25''.5756,$$

consequently for Jupiter

$$\tan f = 0.029880\sqrt{1 - 0.43290\cos\left\{21^{\circ}\,37' - t \times 25''.5756\right\}}$$

and for Saturn

$$\tan f' = 0.03287\sqrt{1 + 0.82665\cos\left\{21^{\circ}\,37' - t \times 25''.5756\right\}}.$$

Also

$$N_{/} + N' = 0.04442 \qquad N_{/} - N' = 0.01368;$$

so that the maxima and minima of the inclinations of Saturn's orbit are $2^{\circ}\,32'\,40''$ and $0^{\circ}\,47'$, and its greatest deviation from its mean state does not exceed $52^{\circ}\,50''$. In Jupiter's orbit, the maximum is $2^{\circ}\,2'\,30''$, and the minimum $1^{\circ}\,17'\,10''$, and the greatest deviation from a mean state is $0^{\circ}\,22'\,40''$.

The longitude of the node q has a maximum and minimum in both orbits, because $N_{/} \rangle N'$. The extent of its librations in Jupiter's orbit will be $13^{\circ}\,9'\,40''$, and in Saturn's $31^{\circ}\,56'\,20''$, on either side of its mean station on the plane of the ecliptic supposed immovable.[27] The period in which the inclinations vary from their greatest to their least values, and the nodes from their greatest to their least longitudes, is by article 486

$$= \frac{360^{\circ}}{g} = \frac{360^{\circ}}{25''.5756} = 50{,}673 \text{ Julian years}.$$

634. The limits and periods of the variations in the eccentricities and longitudes of the perihelia are obtained by a similar process, from equations (133), and those in article 485. The quantities

$$h = e\sin v\,,\quad l = e\cos v\,,\quad h' = e'\sin v',\quad l' = e'\cos v',$$

are known at the epoch, and equations (132) give

$$g^2 - g\left\{\frac{m'\sqrt{a'} + m\sqrt{a}}{m'\sqrt{a'}}\right\}(4.5) = \frac{m\sqrt{a}}{m'\sqrt{a'}}\left\{\boxed{4.5}^2 - (4.5)^2\right\};$$

whence[28]

$$g_{/} = 3''.5851;\qquad g = 21''.9905;$$

$$N = -0.01715;\qquad N_{/} = 0.04321;\qquad N' = 0.04877;$$

$$N_{/}' = 0.03532;\qquad x_{/} = 210^\circ 16' \, 40'';\qquad x = 306^\circ \, 34' \, 40'';$$

and equation (135) gives

$$e = \sqrt{h^2 + l^2},$$

or

$$e = 0.04649\sqrt{1 + 0.68592\cos\left(83^\circ\ 42' - t\times 18''.4054\right)}$$

for the eccentricity of Jupiter's orbit; and

$$e' = 0.06021\sqrt{1 - 0.95009\cos\left(83^\circ\ 42' - t\times 18''.4054\right)}$$

for that of Saturn for any number t of Julian years after the epoch.

The longitudes of the perihelia are found from the value of $\tan v$ in article 495. The greatest deviation of these from their mean place will happen when

$$\cos\left\{\left(g - g_{/}\right)t + x - x_{/}\right\} = -\frac{gN'^2 + g_{/}N_{/}'^2}{\left(g + g_{/}\right)N'.N_{/}'}.$$

If this fraction be less than unity, the perihelia will librate like the nodes about a mean position, if not, they will move continually in one direction. In the case of Jupiter and Saturn $gN'^2 + g_{/}N_{/}'^2$ is greater than $\left(g + g_{/}\right)N'.N_{/}'$; so that the perihelia go on for ever in one direction.

The period in which the eccentricities accomplish their changes is

$$\frac{360^\circ}{g - g_{/}} = \frac{360^\circ}{18''.4054} = 70{,}414 \text{ Julian years}.$$

The greatest and least values of the eccentricities are expressed by

$$N' \pm N'_{/} \text{ and } N \pm N_{/}.$$

For Saturn these are

$$0.08409 \text{ and } 0.01345,$$

and for Jupiter

$$0.06036 \text{ and } 0.02606;$$

the maximum of one planet corresponding to the minimum of the other.

The numerical values of the perturbations of the other planets will be found in the *Mécanique Céleste*; it is therefore only necessary to observe the circumstances that are peculiar to each planet.

Mercury

635. The motions of Mercury are less disturbed than those of any other body, on account of his proximity to the sun, his greatest elongation not exceeding $28^\circ.8$. His periodic inequalities are caused by Venus, the Earth, Jupiter, and Saturn; those from Saturn are very small, and Mars only affects the elements of his orbit.

The secular variations in the elements of Mercury's orbit were in the beginning of the year 1801, in the eccentricity

$$0.000003867;$$

secular and sidereal variation in the longitude of the perihelion,

$$9'43''.5;$$

secular and sidereal variation in the longitude of the node,

$$-13'2'';$$

secular variation of the inclination of the orbit on the true ecliptic,

$$19''.8.$$

636. Mercury sometimes appears as a morning and sometimes as an evening star, and exhibits phases like the moon. He occasionally is seen to pass over the disc of the sun like a black spot: these transits are true annular eclipses of the sun, proving that Mercury is an opaque body shining only by reflected light. The recurrence of the transits of Mercury depends on his periodic time being nearly equal to four times that of the earth. This ratio can be expressed by several pairs of small whole numbers, so that if the planet be in conjunction with the sun while in one of his nodes, he will be in conjunction again at the same node, after the Earth and he have completed a certain number of revolutions. The periodic revolutions of the earth have the following ratios to those of Mercury:

Periods of the Earth	Periods of Mercury
7	29
13	54
33	137
&c.	&c.

Consequently transits of Mercury will happen at intervals of 7, 13, 33, &c. years.

Had the orbit of Mercury coincided with the plane of the ecliptic, there would have been a transit at each revolution; but in consequence of the inclination of his orbit, transits do not happen often; for when a transit takes place, the latitude of Mercury must be less than the apparent semi-diameter of the sun. The return of the transits are also irregular from the great eccentricity of the orbit, which makes the motion of Mercury very unequal; the retrograde motion of the nodes also prevents the planet from returning to the same latitude when it returns to the same conjunction. A transit of Mercury took place at the descending node in 1799, the next that will happen at that node will be in 1832.

Transits happened at the ascending node in the years 1802, 1815, and 1822.

The mean apparent diameter of Mercury is $6''.9$.

Venus

637. 'The Morning Star' is the only planet mentioned in the sacred writings, and has been the theme of the poet's song, from Hesiod[29] and Homer,[30] to the days of Milton.[31]

Venus is next to Mercury, and exhibits similar phenomena. Like him she is alternately an evening and a morning star, has phases, and when in her nodes, occasionally appears to pass over the sun's disc, though her transits are not so frequent as those of Mercury. The returns of the transits of Venus depend on five times the mean motion of the earth being nearly equal to three times that of Venus: this however cannot be expressed by pairs of small whole numbers as in the case of Mercury; therefore the transits of Venus do not happen so often. It appears from the ratio of the periodic time of Venus to that of the earth, that eight periods of the earth's revolution are nearly equal to thirteen periods of the revolution of Venus, and 235 periods of the earth are nearly equal to 382 of Venus; hence a transit of Venus may happen at the same node after an interval of eight years, but if it does not happen, it cannot take place again at the same node for 235 years. At present, the heliocentric longitude of Venus's ascending node is something less than 75°, and that of her descending node is about 164°. The earth, as seen from the sun, has nearly the former longitude in the beginning of December, and the latter in the beginning of June; hence the transits of Venus for ages to come will happen in December and June. Those of Mercury will take place in May and November.

Table of the Transits of Venus

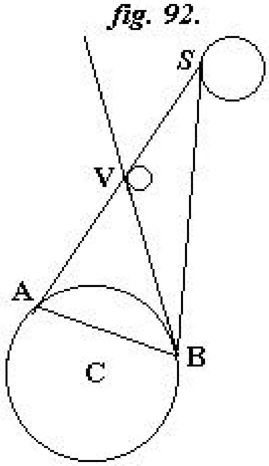
Year	
1631	6th December, ascending node.
1639	4th December, same.

1761	5th June, descending node.
1769	3rd June, same.
1874	8th December, ascending node.
1882	6th December, same.
2004	7th June, descending node.

The transits of Venus afford the most accurate method of finding the sun's parallax, and consequently his distance from the earth, from whence the true magnitude of the whole system is determined; for unless the actual distance of the sun were known, only the ratios of the magnitudes could have been ascertained.

638. The sun's parallax $\mathrm{E}m\mathrm{E}'$, fig. 65, which is the angle subtended at the sun by the earth's radius, can be found, if another angle $\mathrm{E}m\mathrm{E}'$, fig. 66, subtended by a chord EE' lying between two known places on the earth's surface be known; that is, if the sun's parallax at any one altitude be known, his horizontal parallax may be determined, as it has been shown in article 329. However, the method employed in that number is not sufficiently accurate when applied to the sun, because in measuring the zenith distances, an error of three or four seconds might happen, which is immaterial in the case of the moon, whose parallax is nearly a degree, but an error of that magnitude in finding[32] the parallax of the sun, which is less than nine seconds, would render the results useless; hence, astronomers have endeavoured to compute the angle $\mathrm{E}m\mathrm{E}'$ instead of measuring it. Let AB, fig. 92, represent the equator, *S* and V the discs of the sun and Venus perpendicular to it: suppose them both to be moving in the equator, the motion of Venus retrograde, that of the sun direct. To a person at A, the internal contact, or total ingress of Venus on the sun commences, when to a spectator at B, the edge of Venus's disc is distant from the sun by the angle VB*S*. The difference between the times of total ingress as seen from B and A is the time of describing VB*S* by the approach of the sun and Venus to each other. Hence from the difference of the times, and the rate at which Venus and the sun approach each other, the angle VB*S* may be found, because the motions of both the sun and Venus are known. And sine VB*S* is to sine V*S*B, as Venus's distance from the sun to Venus's distance from the earth. But the ratio of Venus's distance from the sun to her distance from the earth is known, therefore the angle A*S*B is found, and C*S*B, the parallax of the sun may be computed, and from that his horizontal parallax; whence the distance of the sun from the earth may be determined in multiples of the terrestrial radius, or even in miles since the length of the radius is known. The computation of the transit is complicated chiefly on account of the inclination of Venus's orbit to the ecliptic, and the situations of the places of observation A and B being always at different distances from the equator. The investigation of this problem, and the computation of the parallax, will be found in Biot's and Woodhouse's *Astronomy*.[33]

fig. 92.

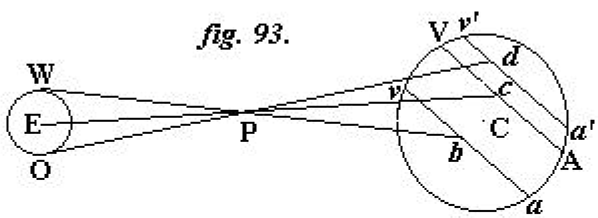


The times of internal contact can be observed with much greater accuracy than any angular distance can be measured, and on this depends the superiority of the preceding method of finding the parallax.

At inferior conjunction, the sun and Venus approach each other at the rate of $4''$ in a minute; hence, if the time of contact be erroneous at each place of observation $4''$ of time, the angle VB*S*, fig. 92, may be erroneous $\frac{4\times 8}{60}=\frac{8}{15}$ of a second, therefore the limit of the error in A*S*B is about $\frac{4}{15}$ of a second, and thus by the transit of Venus, an angle only $\frac{4}{15}$ of a second can be measured, a less quantity than can be determined by any other method.

639. The preceding method requires the difference of longitudes of the two places A and B to be accurately known, in order to compare the actual times of contact. In 1761 a transit of Venus was observed at the Cape of Good Hope, and at many places in Europe, the longitudes of all being well known: by comparing the observations the mean result determined the parallax to be $8''.47$; this is only an approximate value, but it was useful in obtaining the true value from the transit of 1769, which was observed at Wardhus in Lapland, and in Otaheite[34] in the southern hemisphere;[35] but as the longitude of the latter was unknown, astronomers avoided the difficulty by changing their method of calculation. In place of observing the ingress only, they observed the duration of the transit, and from the difference of duration at different places, they deduced the parallax.

Let P be Venus; E the earth, W Wardhus towards the north pole; O Otaheite towards the south; and VA the disc of the sun: then the true line of transit seen from E, the centre of the earth, would be VA, at W the transit would appear to be in the line va, and from O it would be seen in $v'a'$.



If T be the true duration of the transit, or the time of describing VA, then the time of describing va nearer to the sun's centre, and therefore greater than VA, would be $\mathrm{T}+t$; whilst that of describing $v'a'$, which is farther from the centre, and therefore less than VA, would be $\mathrm{T}-t'$. The difference of the durations of the transits seen from O and W is $\mathrm{T}+t-(\mathrm{T}-t')=t+t'$, which is entirely the effect of parallax. With an approximate value of the parallax, t and t', the differences in the durations at W and O from what they would have been if observed at[36] E, the centre of the earth may be computed; then comparing the computed value of $t+t'$ with its observed value, the error in the assumed parallax will be found. With the parallax $8''.83$ it has been calculated that at Wardhus the duration was lengthened by $11'.16''.9$

And diminished at Otaheite by	$12'.10''$
Sum $t+t'$	$23'.26''.9$
But by observation	$23'.10''$
Difference	$16''.9$

Consequently the parallax $8''.45$ is less than that assumed; therefore to make the observed and computed differences of durations agree, the parallax must be $8''.72$. This does not differ much from what is given by the lunar theory $8''.6$, but an error recently detected by M. Bessel,[37] reduces it to $8''.575$. The transit commenced at Otaheite at half-past nine in the morning, and ended at half-past three in the afternoon.

640. Venus is by far the most brilliant and beautiful of the planets, but her splendour is variable. Her phases increase with her distance from the earth, and therefore she ought to become brighter as her disc enlarges; but the increase of the distance diminishes her lustre, since the intensity of light decreases proportionally to the square of the distance: there is, however, a mean position in which Venus is more brilliant than in any other; the interval of her returns to that position is about eight years, depending on the ratio of her periodic time to that of the earth. She is then visible to the naked eye during the day, but she is also visible in daylight every eighteen months though less distinctly.

The variations in the apparent diameter of Venus are very great; she is nearest the earth in her transit; her apparent diameter is then $16''.904$. M. Arago[38] has found its mean value to be $16^{\circ}.904$.

Shröeter, by observing the horns of Venus, determined her rotation about an axis, considerably inclined to the plane of the ecliptic, to be performed in $23^{h}\ 21'$; he discovered also very high mountains on her surface.

641. Venus is too near the sun to be very irregular in her motions, her greatest elongation not exceeding $47^{\circ}\ 7'$. In 1801, the secular variation in the eccentricity of her orbit was 0.000062711.

In the longitude of the perihelion,	$4'28''$
In the longitude of the ascending node,	$-31'10''$
In the inclination on the true ecliptic,	$4''.5$

The Earth

642. Uranus is too distant to have a sensible influence on the earth. Besides the disturbances occasioned by the other planets, there are some inequalities produced by the moon which are to be found in article 498.

It will be shown in the theory of the moon, that if $U-\Omega$ be her distance from her ascending node, the greatest inequality in her latitude is

$$18542''.8\sin\left(U-\Omega\right),$$

and if $S=18542''.8,$ the inequality (195) in the earth's latitude is

$$-\frac{m}{E}\cdot\frac{r}{\bar{r}}\cdot 18542''.8\sin\left(U-\Omega\right).$$

In order to compute the inequalities occasioned by the moon, it is requisite to know the ratio of the mass of the moon to that of the earth. The theory of the tides shows that the action of the moon in raising the waters of the ocean is 2.35333 times greater than that of the sun. The action of the moon on the earth, resolved in the direction r, is $\frac{E+m}{r^3}$; and the action of the sun, according to his radius vector $\bar{r}$, is $\frac{S}{\bar{r}^3}$: S and m being the masses of the sun and moon; hence

$$\frac{E+m}{r^3} = 2.35333 \cdot \frac{S}{r^3}.$$

By the theory of central forces,

$$\frac{E+m}{r^3} = n_/^2, \text{ and } \frac{S}{r^3} = n^2;$$

n and $n_/$ being the mean motions of the earth and moon; whence

$$\frac{m}{E+m} = 2.35333 \cdot \frac{n^2}{n_/^2}.$$

By observation,

$$\frac{n}{n_/} = 0.0748301;$$

hence

$$\frac{m}{E+m} = \frac{1}{75.928};$$

and if the mass of the earth be taken as the unit, the mass of the moon is

$$\frac{m}{E} = m = \frac{1}{75} \text{ nearly}.$$

Again, the ratio of the earth's distance from the sun to its distance from the moon is equal to the horizontal parallax of the sun, divided by the mean horizontal parallax of the moon, as will appear by considering that, as the parallax of both the sun and moon is very small, the arc may be taken for the sine, and the mean horizontal parallax of the moon is then the mean terrestrial radius divided by the mean distance of the moon from the earth; and the solar parallax is equal to the same terrestrial radius divided by the mean distance of the earth from the sun. The parallax of the sun is known, by observation, to be $8''.575$, that of the moon is $3454''.16$; hence the ratio of the distances is

$$\frac{8''.575}{3454''.16}.$$

With these data, the coefficients are

$$dv = -6''.8274 \sin(U - v),$$
$$dr = -0.0000331 \cos(U - v),$$
$$ds = -0''.61377 \sin(U - \Omega).$$

643. The inequality caused by the moon in the earth's radius vector is small; the mass of the moon being only $\frac{1}{75}$ part of that of the earth, the distance of the common centre of gravity of the earth and moon from the centre of the former must be less than the semidiameter, that is, it must be within the mass of the earth, and therefore the inequality in the earth's place must be less than $8''.575$, the sun's horizontal parallax.

644. The inequality produced by the moon in the earth's longitude is the lunar equation of the tables of the sun; it is of much importance for correcting the value of the mass of the moon. Its coefficient being computed with a value of the mass of the moon determined from the theory of the tides, compared with the coefficient of the same inequality determined by observation, will give the error in the mass of the moon, supposing the parallax of the sun and moon to be correct.

645. The irregularities communicated to the earth by the moon and planets are referred to the sun by observers on the earth's surface; therefore the sun appears to have a motion in longitude, by which he alternately advances before, and falls behind the point that describes the elliptical orbit in the heavens. In like manner be seems alternately to ascend above the plane of the ecliptic, and to descend below it by the disturbance in latitude. The perturbations in latitude, by the action of the planets, are computed from (160), and are[39]

$$ds'' = \begin{Bmatrix} +0''.991803\sin\left(2n''t - n't + 2\epsilon'' - \epsilon' - q'\right) \\ +0''.234256\sin\left(4n''t - 3n't + 4\epsilon'' - 3\epsilon' - q'\right) \\ +0''.164703\sin\left(2n^{\mathrm{iv}}t - n''t + 2\epsilon^{iv} - \epsilon'' - q^{\mathrm{iv}}\right) \end{Bmatrix};$$

this, added to

$$-0.61377\sin\left(U - \Omega\right),$$

is the whole periodic disturbance in the earth's motion in latitude, taken with a different sign. It affects the obliquity of the ecliptic, determined from the observations of the altitude of the sun in the solstices; it also has an influence on the time of the equinoxes, determined from observations of the sun at that period, as well as on the right ascensions and declinations of the fixed stars, determined by comparison with the sun; for it is clear that any inequalities in the motion of the earth will be referred to the observations made at its surface.

Considering the great accuracy of modern observations, these circumstances must be attended to. It is easy to see that this variation in the sun's latitude will increase his apparent declination by

$$-\frac{ds''.\cos\{\text{obliquity of ecliptic}\}}{\cos\{\text{declination of sun}\}};$$

and his apparent right ascension by

$$\frac{ds''.\sin\{\text{obliquity of ecliptic}\}.\cos\{\text{sun's R.A.}\}}{\cos\{\text{declination of sun}\}}$$

The observed right ascensions and declinations of the sun must therefore be diminished by these quantities, in order to have those that would be observed if the sun never left the plane of the ecliptic.

Secular Inequalities in the Terrestrial Orbit

646. The eccentricity and place of the perihelion of the terrestrial orbit may be determined with sufficient accuracy for 1000 or 1200 years before and after the epoch 1750, from

$$e = 2\bar{e} - 0''.187638 \times t - 0''000006721 \times t^2, \text{ and}$$
$$v = \bar{v} + 11''.949588 \times t + 0''.000079522 \times t^2,$$

$\bar{e}$ and $\bar{v}$ are the eccentricity and longitude of the perihelion at the epoch.

The secular diminution of the eccentricity is $18''.79$, about 3,914 miles, in reality an exceedingly small fraction in astronomy, though it appears so great in terrestrial measures. Were the diminution uniform, which there is no reason to believe, the earth's orbit would become a circle in 36,300 years; its variation has a great influence on the motions of the moon.

The longitude of the perihelion increases annually at the rate of $11''.9496$, so that it accomplishes a sidereal revolution in 109,758 years.

647. A remarkable period in astronomy was that in which the greater axis of the terrestrial orbit coincided with the line of the equinoxes, then the true equinox coincided with the mean. This occurred 4,084 years before the epoch in which chronologists place the creation of man;[40] at that time the solar perigee coincided with the equinox of spring. This however is but an approximate value, on account of the masses of the planets and the doubts as to the exact value of precession; the error may therefore be 80 years, which is not much in such a quantity.

Another remarkable astronomical period was, when the greater axis of the terrestrial orbit was perpendicular to the line of equinoxes; it was then that the true and mean solstice were united; this coincidence took place in the year 1248 of the Christian era. It is evident that these two periods depend on the direct motion of the perihelion and precession of the equinoxes conjointly.

648. The position of the ecliptic is changed by the reciprocal action of the planets on one another, and on the earth, each of them producing a retrograde motion in the intersection of the plane of its own orbit with the plane of the ecliptic. This action also changes the position of the plane of the ecliptic, with regard to itself, a change that may be determined from the values of p and q by formulae (138), or rather from

$$p = +0''.0767209 \times t + 0''.000021555 \times t^2,$$
$$q = -0''.5009545 \times t + 0''.000067473 \times t^2.$$

These will give the variation of the ecliptic, with regard to its fixed position in 1750, for 1000 or 1200 years, before and after that epoch.

This change in the ecliptic alters its position with regard to the earth's equator; but as the formulae in article 498 are periodic, these two planes never have and never will coincide. It occasions also a small motion in the equinoxes of about $0''.0846$ annually. Both of these variations are entirely independent of the form of the earth, and would be the same were it a sphere. However, the action of the sun and moon on the protuberant matter at the earth's equator is the cause of the precession of the equinoxes, or of that slow angular motion by which the intersection of the equator and ecliptic goes backward at the rate of $50''.34$ annually, so that the pole of the equator describes a circle round the pole of the ecliptic in the space of 25,748 years. This motion is diminished by the very small secular inequality $0''.0846$, arising from the action of the planets on the ecliptic. The formulae for computing the obliquity of the ecliptic and precession of the equinoxes depend on the rotation of the earth.

Mars

649. Mars is troubled by all the planets except Mercury. Jupiter alone affects the latitude of Mars. The secular variations in the elements of his orbit were, in 1801, as follow:

In the eccentricity.	0.000090176
In the longitude of the perihelion	$26''.22$
In the inclination on the true ecliptic	$1''.5$
In the longitude of the ascending node	$-38'\,48''$

The eccentricity is diminishing.

The greatest elongation of Mars is[41] $126^\circ.8$. By spots on his surface it appears that he rotates in one day about an axis that is inclined to the plane of the ecliptic at an angle of $59^\circ.697$. His equatorial is to his polar diameter in the ratio of 194 to 189; his apparent diameter subtends an angle of $6''.29$, at his mean distance, and of $18''.28$ at his greatest distance, when his parallax is nearly twice that of the sun. The disc of Mars is occasionally gibbous. Spots near his poles that augment or diminish according as they are exposed to the sun, give the idea of masses of ice.

The New Planets

650. The orbits of Vesta, Juno, Ceres and Pallas are situate between those of Mars and Jupiter. Ceres was discovered by Piazzi, at Palermo,[42] on the first day of the present century; Pallas was discovered by Olbers,[43] in 1802; Juno in 1803, by Harding; and Vesta in 1807, by Olbers. These bodies are nearly at equal distances from the sun, their periodic times are therefore nearly the same. The eccentricities of the orbits of Juno and Vesta, and the position of their nodes are nearly the same.

These small planets are much disturbed by the proximity and vast magnitude of Jupiter and Saturn, and the series which determine their perturbations converge slowly, on account of the greatness of the eccentricities and inclinations of their orbits. The inclination of the old planets is so small, that they are all contained within the zodiac, which extends 8° on each side of the ecliptic, but those of the new planets very much exceed these limits. They are invisible to the naked eye, and so minute that their apparent diameters have not yet been measured. Sir William Herschel[44] estimated that they cannot amount to the fourth of a second, which would make the real diameter less than 65 miles. However, Juno, the largest of these asteroids, is supposed to have a real diameter of about 200 miles.

Jupiter

651. Jupiter is the largest planet in the system, and with his four moons exhibits one of the most splendid spectacles in the heavens. His form is that of an oblate spheroid whose polar diameter is $35''.65$, and his equatorial $= 38''.44$; he rotates in 9 hours 56 minutes about an axis nearly perpendicular to the plane of the ecliptic. The circumference of Jupiter's equator is about eleven times greater than that of the earth, and as the time of his rotation is to that of the earth as 1 to 0.414, it follows that during the time a point of the terrestrial equator describes 1°, a point in the equator of Jupiter moves through $2^\circ.41$; but these degrees are longer than the terrestrial degrees in the ratio of 11 to 1, consequently each point in Jupiter's equator moves 26 times faster than a point in the equator of the earth. In the beginning of 1801 the secular variations of his orbit were,

In the eccentricity.	0.00015935
In the longitude of the perihelion	$11'.4''$
In the inclination on the true ecliptic	$23''$
In the longitude of the ascending node . . .	$-26'\,17''$

Saturn

652. Viewed through a telescope Saturn is even more interesting than Jupiter: he is surrounded by a ring concentric with himself, and of the same or even greater brilliancy; the ring exhibits a variety of appearances according to the position of the planet with regard to the sun and earth, but is generally of an elliptical form: at times it is invisible to common observation, and can only be seen with superior instruments; this happens when the plane of the ring either passes through the centre of the sun or of the earth, for its edge, which is very thin, is then directed to the eye. On the 29th September, 1832, the plane of the ring will pass through the centre of the earth, and will be seen with a very high magnifying power like a line across the disc of the planet. On the 1st December of the same year, the plane of the ring will pass through the sun. Professor Struve[45] has discovered that the rings are not concentric with the planet. The interval between the outer edge of the globe and the outer edge of the ring on one side is $11''.037$, and on the other side the interval is $11''.288$, consequently there is an eccentricity of the globe in the ring of $0''.215$. In 1825 the ring of Saturn attained its greatest ellipticity; the proportion of the major to the minor axis was then as 1000 to 498, the minor being nearly half the major. Stars have been observed between the planet and his ring. It is divided into two parts by a dark concentric band, so that there are really two rings, perhaps more. These revolve about the planet on an axis perpendicular to their plane in about $10^{h}\ 29^{m}\ 17^{s}$, the same time with the planet.

The form of Saturn is very peculiar. He has four points of greatest curvature, the diameters passing through these are the greatest; the equatorial diameter is the next in size, and the polar the least; these are in the ratio of 36, 35, and 32. Besides the rings, Saturn is attended by seven satellites which reciprocally reflect the sun's rays on each other and on the planet. The rings and moons illuminate the nights of Saturn; the moons and Saturn enlighten the rings, and the planet and rings reflect the sun's beams on the satellites when they are deprived of them in their conjunctions. The rings reflect more light than the planet. Sir William Herschel observed, that with a magnifying power of 570, the colour of Saturn was yellowish, whilst that of the rings was pure white. Saturn has several belts parallel to his equator: changes have been observed in the colour of these and in the brightness of the poles, according as they are turned to or from the sun, probably occasioned by the melting of the snows. Saturn's motions are disturbed by Jupiter and Uranus alone; the secular variations in the elements of his orbit were as follows, in the beginning of 1801.

In the eccentricity.	0.000312402
In the longitude of the perihelion	$32'.17''$
In the inclination on the true ecliptic	$15'\ 5''$
In the longitude of the ascending node . . .	$-37'\ 54''$

Uranus, or the Georgium Sidus

653. This planet was discovered by Sir William Hershel, in 1781. The period of his sidereal revolution is 30,687 days. If we judge of the distance of the planet by the slowness of its motion, it must be on the very confines of the solar system; its greatest elongation is $103^{\circ}.5$, and its apparent diameter $4''$: it is accompanied by six satellites, only visible with the best telescopes. The only sensible perturbations in the motions of this planet arise from the action of Jupiter and Saturn; the secular variations in the elements of its orbit were, in 1801, as follow:

In the eccentricity.	0.000025072
In the longitude of the perihelion	$4'$
In the inclination on the true ecliptic	$3''.7$
In the longitude of the ascending node . . .	$-59'\ 57''$

The rotation of Uranus[46] has not been determined.

654. It is remarkable that the rotation of the celestial bodies is from west to east, like their revolutions; and that Mercury, Venus, the Earth, and Mars, accomplish their rotations in about twenty-four hours, while Jupiter and Saturn perform theirs in $\frac{4}{10}$ of a day.

On the Atmosphere of the Planets

655. Spots and belts are observed on the discs of some of the planets varying irregularly in their position, which shows that they are surrounded by an atmosphere; these spots appear like clouds driven by the winds, especially in Jupiter. The existence of an atmosphere round Venus is indicated by the progressive diffusion of the sun's rays over her disc. Schroëter measured the extension of light beyond the semicircle when she appeared like a thin crescent, and found the zone that was illuminated by twilight to be at least four degrees in breadth, whence be inferred that her atmosphere, must be much more dense than that of the earth. A small star hid by Mars was observed to become fainter before its appulse[47] to the body of the planet, which must have been occasioned by his atmosphere. Saturn and his rings are surrounded by a dense atmosphere, the refraction of which may account for the irregularity apparent in his form: his seventh satellite has been observed to hang on his disc more than $20'$ before its occultation, giving by computation a refraction of two seconds, a result confirmed by observation of the other satellites. An atmosphere so dense must have the effect of preventing the radiation of the heat from the surface of the planet, and consequently of mitigating the intensity of cold that would otherwise prevail, owing to his vast distance from the sun. Schroëter observed a small twilight in the moon, such as would be occasioned by an atmosphere capable of reflecting the sun's rays at the height of about a mile. Had a dense atmosphere surrounded that satellite, it would have been discovered by the duration of the occultations of the fixed stars being less than it ought to be, because its refraction would have rendered the stars visible for a short time after they were actually behind the moon, in the same manner as the refraction of the earth's atmosphere enables us to see celestial objects for some minutes after they have sunk below our horizon, and after they have risen above it, or distant objects hid by the curvature of the earth. A friend of the author's was astonished one day on the plain of Hindostan,[48] to behold the chain of the Himala[49] mountains suddenly start into view after a heavy shower of rain in hot weather.

The Bishop of Cloyne[50] says, that the duration of the occultations of stars by the moon is never lessened $8''$ of time, so that the horizontal refraction at the moon must be less than $2''$: if therefore a lunar atmosphere exists, it must be 1000 times rarer than the atmosphere at the surface of the earth, where the horizontal refraction is nearly $2000''$. Possibly the moon's atmosphere may have been withdrawn from it by the attraction of the earth. The radiation of the heat occasioned by the sun's rays must be rapid and constant, and must cause intense cold and sterility in that cheerless satellite.

The Sun

656. The sun viewed with a telescope, presents the appearance of an enormous globe of fire, frequently in a state of violent agitation or ebullition; black spots of irregular form rarely visible to the naked eye sometimes pass over his disc, moving from east to west, in the space of nearly fourteen days: one was measured by Sir W. Herschel in the year 1779, of the breadth of 30,000 miles. A spot is surrounded by a penumbra, and that by a margin of light, more brilliant than that of the sun. A spot when first seen on the eastern edge, appears like a line, progressively extending in breadth till it reaches the middle, when it begins to contract, and ultimately disappears at the western edge: in some rare instances, spots re-appear on the east side; and are even permanent for two or three revolutions, but they generally change their aspect in a few days, and disappear: sometimes several small spots unite into a large one, as a large one separates into smaller ones which soon vanish.

The paths of the spots are observed to be rectilinear in the beginning of June and December, and to cut the ecliptic at in angle of $7^\circ\ 20'$. Between the first and second of these periods, the lines described by the spots are convex towards the north, and acquire their maximum curvature about the middle of that time. In the other half year the paths of the spots are convex towards the south, and go through the same changes. From these appearances it has been concluded, that the spots are opaque bodies attached to the surface of the sun, and that the sun rotates about an axis, inclined at an angle of $7^\circ\ 20'$ to the axis of the ecliptic. The apparent revolution of a spot is accomplished in twenty-seven days; but during that time, the spot has done more, having gone through a revolution, together with an arc equal to that described by the sun in his orbit in the same time, which reduces the time of the sun's rotation to $25^d\ 9^m\ 35^s$.

These phenomena induced Sir W. Herschel to suppose the sun to be a solid dark nucleus, surrounded by a vast atmosphere, almost always filled with luminous clouds, occasionally opening and discovering the dark mass within. The speculations of Laplace were different: he imagined the solar orb to be a mass of fire, and that the violent effervescences and explosions seen on its surface are occasioned by the eruption of elastic fluids formed in its interior, and that the spots are enormous caverns, like the craters of our volcanoes.

Light is more intense in the centre of the sun's disc than at the edges, although, from his spheroidal form, the edges exhibit a greater surface under the sane angle than the centre does, and therefore might be expected to be more luminous. The fact may be accounted for, by supposing the existence of a dense atmosphere absorbing the rays which have to penetrate a greater extent of it at the edges than at the centre; and accordingly, it appears by Bouguer's[51] observations on the moon, which has little or no atmosphere, that it is more brilliant at the edges than in the centre.

657. A phenomenon denominated the zodiacal light, from its being seen only in that zone, is somehow connected with the rotation of the sun. It is observed before sunrise and after sunset, and is a luminous appearance, in some degree similar to the milky way, though not so bright, in the form of an inverted cone with the base towards the sun, its axis inclined to the horizon, and only inclined to the plane of the ecliptic at an angle of 7°; so that it is perpendicular to the axis of the sun's rotation. Its length from the sun to its vertex varies from 45° to 120°. It is seen under the most favorable circumstances after sunset in the beginning of March: its apex extends towards Aldebaran,[52] making an angle of 64° with the horizon. The zodiacal light varies in brilliancy in different years.

It was discovered by Cassini[53] in 1682, but had probably been seen before that time. It was observed in great splendour at Paris on the 16th of February, 1769.

658. The elliptical motion of the planets is occasioned by the action of the sun; but by the law of reaction, the planets must disturb the sun, for the invariable point to which they gravitate is not the centre of the sun, but the centre of gravity of the system; the quantity of motion in the sun in one direction must therefore be equal to that of all the planets in a contrary direction. The sun thus describes an orbit about the centre of gravity of the system, which is a very complicated curve, because it results from the action of a system of bodies, perpetually changing their relative positions; it is such however as to furnish a centrifugal force with regard to each planet, sufficient to counteract the gravitation towards it.

Newton[54] has shown that the diameter of the sun is nearly equal to 0.009 of the radius of the earth's orbit. If all the great planets of the system were in a straight line with the sun, and on the same side of him, the centre of the sun would be nearly the farthest possible from the common centre of gravity of the whole; yet it is found by computation, that the distance is not more than 0.0085 of the radius vector of the earth; so that the centre of the sun is never distant from the centre of gravity of the system by as much as his own diameter.

Influence of the Fixed Stars in disturbing the Solar System

659. It is impossible to estimate the effects of comets in disturbing the solar system, on account of our ignorance of the elements of their orbits, and even of the existence of such as have a great perihelion distance, which nevertheless may trouble the planetary motions; but there is every reason to believe that their masses are too small to produce a sensible influence; the effect of the fixed stars may, however, be determined.

Let m' be the mass of it fixed star, x', y', z', its co-ordinates referred to the centre of gravity of the sun, and r' its distance from that point. Also let x, y, z, be the co-ordinates of a planet m, and r its radius vector; then the disturbing influence of the star is[55]

$$R=\frac{m'}{\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}}-\frac{m'(xx'+yy'+zz')}{r'^3};$$

or [56]

$$R=+\frac{m'}{r'}-\frac{m'r^2}{2r'^3}+\tfrac{3}{2}m'\frac{\left((xx'+yy'+zz')-\tfrac{1}{2}r^2\right)^2}{r'^5}+\&\text{c.}$$

when developed according to the powers of r'. The fixed plane being the orbit of m at the epoch, then

$$x=r\cos v,\quad y=r\sin v,\quad z=rs,$$

let l be the latitude of the fixed star, and u its longitude, then

$$x'=r'\cos l\cos u,\quad y'=r'\cos l\sin u,\quad z'=r'\sin l;$$

and if all the powers of r' above the cube be omitted, it will be found that

$$R=+\frac{m'}{r'}-\frac{m'r^2}{4r'^3}\left\{2-3\cos^2 l-3\cos^2 l\,.\cos(2v-2u)-6s\,.\sin 2l\,.\cos(v-u)\right\}.$$

But neglecting s, the substitution of this in equation (155) gives

$$\frac{dr}{a} = -\frac{m' a^3 nt}{r'^3}\left\{\left(1-\tfrac{3}{2}\cos^2 l\right)e\sin\left(v-v\right) - \tfrac{3}{4}\cos^2 l\,.\,e\,.\sin\left(v+v-2u\right)\right\}.$$

But

$$r = a\left(1+e\cos\left(v-v\right)\right);$$

whence

$$\frac{dr}{a} = de\cos\left(v-v\right) + edv\,.\sin\left(v-v\right);$$

and comparing the two values of $\frac{dr}{a}$, there will be found

$$dv = -\frac{m'a^3}{r'^3 e}\cdot nt\left\{1-\tfrac{3}{2}\cos^2 l - \tfrac{3}{4}\cos^2 l\cos\left(2v-2u\right)\right\}$$

$$de = \frac{3m'a^3}{4r'^3}\cdot\cos^2 l nt\,.\,e\,.\sin\left(2v-2u\right).$$

Whence it appears, that the star occasions secular variations in the eccentricity and longitude of the perihelion of m, but these variations are incomparably less than those caused by the planets. For if m be the earth, the distance of the star from the centre of the sun cannot be less than 100,000 times the mean distance of the earth from the sun, because the annual parallax of the nearest fixed star is less than $1''$; therefore assuming $r' = 100{,}000\,.\,a$ the coefficient $\frac{m'a^3}{r'^3}nt$ does not exceed $0''.0000000013\,.\,m't$, t being any number of Julian years. This quantity is incomparably less than the corresponding variation in the eccentricity of the earth's orbit, arising from the action of the planets, which is

$$-0''.0938191\,.\,t,$$

unless the mass m' of the fixed stars be much greater than what is probable. Whence it may be concluded that, the attraction of the fixed stars has no sensible influence on the form of the planetary orbits; and it may be easily proved, that the positions of the orbits are also uninfluenced.

Disturbing Effect of the Fixed Stars on the Mean Motions of the Planets

660. The part of equation (156) that depends on R, when $m = 1$, is

$$d\,.\,dz = -3a\int ndt\cdot \mathrm{d}R - 2a\,.\,ndt\cdot r\left(\frac{dR}{dr}\right).$$

The preceding value of R gives

$$d\,.dz = +\frac{m'a^3}{r'^3}ndt\left(2-3\cos^2 l\right)-\frac{6m'a^3}{r'^3}\cdot s\,.\sin 2l\,.\cos\left(v-u\right)$$
$$-\frac{9}{2}\cdot m'\,.\,a^3\,.\,ndt\int d\cdot\frac{s\,.\sin 2l}{r^3}\cdot\cos\left(v-u\right),$$

which is the whole variation in the mean motion of *m* from the action of the fixed stars. The parts will be examined separately.

Let r'' and l' be the distance and latitude of the star at the epoch 1750, and let it be assumed, that these quantities diminish annually by a and b, then *t* being any indefinite time, r' and *l* become

$$r' = r''\left(1-at\right),\quad l = l'\left(1-bt\right)$$

whence the first term of $d\,.dz$ becomes

$$d\,.dz = \frac{3m'\,.\,a^3}{r''^3}\left(1-\tfrac{3}{2}\cos^2 l'\right)a\,nt^2-\frac{3m'\,.\,a^3}{r''^3}\cdot\sin 2l'.\,b\,.\,nt^2\,.$$

We know nothing of the changes in the distance of the fixed stars; but with regard to the earth, they may be assumed to vary $0''.324$ annually in latitude; hence[57]

$$b = 0''.324,\quad r'' = 100{,}000a,$$

so that $\frac{m'a^3}{r'^3}\cdot b\,.\,nt^2$ becomes

$$\frac{m't^2\,.\,2''.0357}{10^{15}}.$$

a quantity inappreciable from the earliest observations.

With regard to the terms in *s*,

$$s = t\cdot\frac{dp}{dt}\sin v - t\cdot\frac{dq}{dt}\cos v;$$

consequently, rejecting the periodic part,

$$d\cdot\frac{s\,.\sin 2l}{r'^3}\cdot\cos\left(v-u\right)=\frac{\sin 2l}{2r'^3}\left\{\frac{dp}{dt}\cdot\sin u-\frac{dq}{dt}\cdot\cos u\right\},$$

so that

$$d\,.dz = -\frac{21}{4}\cdot\frac{m'a^3}{r'^3}\cdot n\,.\,tdt\,.\sin 2l\left\{\frac{dp}{dt}\sin u - \frac{dq}{dt}\cos u\right\};$$

the integral of which is

$$dz = -\frac{21}{8}\cdot\frac{m'a^3}{r'^3}\cdot nt^2\,.\sin 2l\left\{\frac{dp}{dt}\sin u - \frac{dq}{dt}\cos u\right\}.$$

But with regard to the earth

$$p = +0''.076721\times t + 0''.000021555\times t^2$$
$$q = -0''.50096\times t + 0''.0000067474\times t^2.$$

If these quantities be substituted, it will be found that the secular inequalities in the mean motion of the earth are quite insensible; the earliest records also prove them to be so. The same results will be obtained for the most distant planets, whence it may be concluded that the fixed stars are too remote to affect the solar system.

Construction of Astronomical Tables

661. The motion of a planet in longitude consists of three parts, of the mean or circular motion; of a correction depending on the eccentricity, which is the equation of the centre; and of the periodic inequalities.

In the construction of tables, the mean longitude of the body, and the mean longitude of the aphelion, or perihelion, are determined in degrees, minutes, seconds, and tenths, at the instant assumed as the origin of the tables. These initial values are generally computed for the beginning of each year, and are called the epoch of the tables; from them subsequent values are deduced at convenient intervals, by adding the daily increments. These intervals are longer or shorter according to the motion of the body, or its importance, and the intermediate values are found by simple proportion, or by tables of proportional parts. The mean anomaly is given by the tables, since it is the difference between the mean longitudes of the body and of the aphelion.

The tables of the equation of the centre, and of the mean longitude of the aphelion, give these quantities for each degree of mean anomaly. To these are added tables of the periodic inequalities in longitude, and of the secular inequalities in the eccentricity and longitude of the aphelion. From these tables the true longitude of the body may be known at any instant, by applying the corrections to the mean longitude.

The radius vector consists of three parts,–of a mean value, which is equal to half the greater axis of the orbit; of the elliptical variations, and of its periodic inequalities. The two latter are given in the tables for every degree of mean anomaly. The latitude is computed in terms of the mean anomaly at stated intervals: besides these, the mean longitude of the ascending node and the inclination of the orbit at the beginning of each year, and the secular inequalities of these two quantities are given. Thus the mean motions are given, and the true motions are found by applying the inequalities, the numerical values of which are called equations: for, in astronomy, an equation signifies the quantities that must be added or taken from the mean results, to make them equal to the true results.

The mean motion and equation of the centre are computed from Kepler's problem; the motions of the nodes and perihelia, the secular inequalities of the elements, and the periodic inequalities, are computed from the formulae determined by the problem of three bodies.

Method of correcting Errors in the Tables

662. As astronomical tables are computed from analytical formulae, determined on the principles of universal gravitation, no error can arise from that source; but the elements of the orbit, though determined with great accuracy by numerous observations, will lead to errors, because each element is found separately; whereas these quantities are so connected with each other, that a perfectly correct value of one, cannot be determined independently of the others. For example, the expressions in articles[58] 477-479 show, that the eccentricity depends on the longitudes of the perihelia, and the longitude of the perihelion is given in terms of the eccentricities. A reciprocal connexion exists also between the inclination of the orbit and the longitude of the nodes. Hence, in an accurate determination of the elements, it is necessary to attend to this reciprocal connexion.

The tables are computed with the observed values of the elements; an error in one of the elements will affect every part of the tables, and will be perceived in the comparison [of][59] the place of the body derived from them, with its place determined by observation. Were the observation exact, the difference would be the true error of the tables; but as no observation is perfectly accurate, the comparison is made with 1000, or even many thousands of observations, so that their errors are compensated by their numbers.

The simultaneous correction is accomplished, by comparing a longitude of the body derived from observation, with the longitude corresponding to the same instant in the tables.

Suppose the tables of the sun to require correction, and let E represent the error of the tables, or the difference between the longitude of the tables and that deduced from observation, at that point of the orbit where his mean anomaly is 198°. There are three sources from whence this error may arise, namely, the mean longitude of the perigee, the greatest equation of the centre, and the epoch of the tables; for, if an error has been made in computing the initial longitude, it will affect every subsequent longitude. Now, as we do not know to which of these quantities to attribute the discrepancy, part of it is assumed to arise from each. Let P be the unknown error in the longitude of the perigee, e that in the greatest equation of the centre, and $\in$ that in the epoch. In order to determine these three errors, let us ascertain what effect would be produced on the place of the sun, where his mean anomaly is 198°, by an error of $60''$ in the longitude of the perigee. As the mean anomaly is estimated from perigee, a minute of change in the perigee will produce the change of one minute in the mean anomaly corresponding to each longitude; but the table of the equation of the centre shows that the change of $60''$ in the mean anomaly at that part of the orbit which corresponds to 198° produces an increment of $1''.88$ in the equation of the centre; and as that quantity is subtractive at that part of the orbit, the true longitude of the sun is diminished by $1''.88$; hence, if $60''$ produce a change of $1''.88$ in the true longitude, the error P will produce a change of

$$\frac{1''.88}{60''}P = 0''.3133P \, .$$

Again, if we suppose the greatest equation of the centre to be augmented by any arbitrary quantity as $17''.18$, it is easy to see by the tables that the equation of the centre at that point of the orbit where the mean anomaly is 198° is increased by $5''.1$; whence the true longitude is diminished by $5''.1$. Thus, if $17''.18$ produce a change of $5''.1$ in the true longitude, the error e will produce the change

$$-\frac{5''.1}{17''.18}e = -0''.2969e \, .$$

Hence the sum of the three errors is equal to E, the error of the tables

$$\in +0''.3133P - 0''.2969e = E \, .$$

This is called an equation of condition between the errors, because it expresses the condition that the sum of the errors must fulfil.

As there are three unknown quantities, three equations would be sufficient for their determination, if the observations were accurate; but as that is not the case, a great number of equations of condition must be formed from an equal number of observed longitudes, and they must be so combined by addition or subtraction, as to form others that are as favorable as possible for the determination of each element. For example, in finding the value of P before the other two, the numerous equations must be so combined, as to render the coefficient of P as great as possible; and the coefficients of e and $\in$ as small is may be; this may always be accomplished by changing the signs of all the equations, so as to have the terms containing P positive, and then adding them; for some of the other terms will be positive, and some negative, as they may chance to be; therefore the sum of their coefficients will be less than that of P.

Having determined this equation, in which P has the greatest coefficient possible, two others must be formed on the same principle, in which the coefficients of the other two errors must be respectively as great as possible, and from these three equations values of the three errors will be easily obtained, and their accuracy will be in proportion to the number of observations employed. These values are referred to the mean interval between the first and last observations, supposing them not to be separated by any great length of time, and that the mean motion is perfectly known. Were it not, as might happen in the case of the new planets, an additional error may be assumed to arise from this source, which may be determined in the same manner as the others. This method of correcting errors in astronomical tables was employed by Mayer, in computing tables of the moon, and is applicable to a variety of subjects.

663. The numerous equations of condition of the form

$$E = \in +0''.3133P + 0''.2969e,$$

may be combined in a different manner, used by Legendre,[60] called the principle of the least squares.

If the position of a point in space, is to be determined, and if a series of observations had given it the positions $n,\ n',\ n''$, &c., not differing much from each other, a mean place M must be found, which differs as little as possible from the observed positions $n,\ n',\ n''$, &c.: hence it must be so chosen that the sum of the squares of its distances from the points $n,\ n',\ n''$, &c., may be a minimum; that is,

$$(Mn)^2 + (Mn')^2 + (Mn'')^2 + \&\text{c.} = \text{minimum}\,.$$

A demonstration of this is given in Biot's Astronomy,[61] vol. ii.; but the rule for forming the equation of the minimum, with regard to one of the unknown errors, as P, is to multiply every term of all the equations of conditions by $0''.3133$, the coefficient of P, taken with its sign, and to add the products into one sum, which will be the equation required. If a similar equation be formed for each of the other errors, there will be as many equations of the first degree as errors; whence their numerical values may be found by elimination.

It is demonstrated by the Theory of Probabilities, that the greatest possible chance of correctness is to be obtained from the method of least squares; on that account it is to be preferred to the method of combination employed by Mayer, though it has the disadvantage of requiring more laborious computations.

The principle of least squares is a corollary that follows from a proposition of the Loci Plani, that the sum of the squares of the distances of any number of points from their centre of gravity is a minimum.

664. Three centuries have not elapsed since Copernicus[62] introduced the motions of the planets round the sun, into astronomical tables: about a century later Kepler[63] introduced the laws of elliptical motion, deduced from the observations of Tycho Brahe[64], which led Newton to the theory of universal gravitation. Since these brilliant discoveries, analytical science has enabled us to calculate the numerous inequalities of the planets, arising from their mutual attraction, and to construct tables with a degree of precision till then unknown. Errors existed formerly, amounting to many minutes; which are now reduced to a few seconds, a quantity so small, that a considerable part of it may perhaps be ascribed to inaccuracy in observation.

Notes

1 *Numerical Values of the Perturbations.* This chapter title in the 1st edition *reads* "Numerical Values of the Perturbations of Jupiter."

2 This *reads* "variation" in the 1st edition (published erratum).

3 The first parenthesis in the last equation is rounded in the 1st edition.

4 The fourth element $(d\overline{\Pi})$ *reads* $(d\Pi)$ in the 1st edition.

5 The argument in the 17th term (next page) in the series for dv *reads* $\{3nt - 2nt + 3\in' - 2\in' - v\}$ in the 1st edition.

6 The closing parenthesis in this equation is omitted in the 1st edition.

7 The left hand side of the fifth equation *reads* $\sum .N\,.\sin(n't + nt + \in' + \in - L =$ in the 1st edition.

8 In the 1st edition $n' = 43996''.6$ *reads* $n'' = 43996''.6$.

9 This is incorrectly capitalized in the 1st edition.

10 This expression is expressed in two parts containing eight and five terms respectively in the 1st edition.

11 The + sign before the 9th term is omitted in the 1st edition.

[12] The accent is omitted on the first instance of $\in'$ in the 1st edition.

[13] The left hand side term $4\frac{dP}{da}$ *reads* $\frac{dP}{da}$ in the 1st edition (published erratum).

[14] A comma is used at the end of this sentence in the 1st edition.

[15] The value $0''.0004491$ *reads* $0''.0054491$ in the 1st edition (published erratum).

[16] The value $0''.0004491$ *reads* $0''.0054491$ in the 1st edition (published erratum).

[17] This expression contains two published errata in the 1st edition: The last term *reads*:

$$+\frac{5m'}{2}\cdot Ke\,.\sin\left(5n't-2nt+5\in'-2\in-\nu+B\right).$$

and an erroneous third term *reads*:

$$+m'He\,.\sin\left(5n't-2nt+5\in'-2\in-\nu+B\right).$$

[18] See note 3, *Bk. II, Chap. IV*.

[19] Parentheses are omitted in the arguments of both cosines in the 1st edition.

[20] The first term *reads* $+0''.051737\,.\sin\left(n't-nt+\in'-\in\right)$ in the 1st edition (published erratum).

[21] Not capitalized in the 1st edition.

[22] $\tan f = N - N_{/}$ *reads* $\tan f = N - N'$ in the 1st edition (published erratum).

[23] *libratory*. An oscillation in the apparent aspect of a secondary body (as a planet or a satellite) as seen from the primary object around which it revolves. *Merriam-Webster Collegiate Dictionary*.

[24] See note 55, *Preliminary Dissertation*.

[25] In the 1st edition q and q' *read* 0 and $0'$ (published erratum).

[26] See note 64, *Preliminary Dissertation*.

[27] Spelled "immoveable" in the 1st edition.

[28] In the 1st edition g and $g_{/}$ are reversed (published erratum).

[29] Hesiod, c. 750 BC, poet, born in Ascra, Greece. One of the earliest known Greek poets, he is best known for two works, *Works and Days* and *Theogony*.

[30] Homer, c. 850 BC, Greek poet, author of the *Odyssey* and the *Iliad*.

[31] Milton, John, 1608-1674, poet, born in London, England. His major works were *L'Allegro* and *Il Penseroso* (1632), *Comus* (1633), *Lycidas* (1637), *Areopagitica* (1644),), *Paradise Lost* (1663), *Paradise Regained* (1671) and *Samson Agonistes* (1671).

[32] This word is omitted in the 1st edition (published erratum).

[33] Biot, Jean Baptiste, 1774-1862, *Traite elementaire d'astronomie physique par J.B. Biot; avec des additions relatives a l'astronomie nautique par M. de Rossel*, Paris: J. Klostermann, 1810-11 (see also note 17, *Bk. I*, *Chap. II*)

[34] *Otaheite*. Known now as Tahiti.

[35] Measurements made by Nevil Maskelyne, 1732-1811 (see note 55, *Preliminary Dissertation* and note 5, *Bk. II*, *Chap. IX*).

[36] This *reads* C, in the 1st edition (published erratum).

[37] Bessel, Friedrich Wilhelm, 1784-1846, mathematician and astronomer, born in Minden, Germany. He was appointed director of the observatory and professor at Königsberg in 1819. He catalogued stars, and predicted a planet beyond Uranus as well as the existence of dark stars. He was also the first person to measure stellar parallax (of the star 61 Cygni) in 1838. Bessel's measured value of 0.314 seconds compares favorably with the modern value of 0.292 seconds. Struve (see note 29, *Preliminary Dissertation*) independently measured the parallax of the star Vega in 1839. Bessel later investigated the so-called Kepler's problem of heliocentricity. A series of mathematical functions bear Bessel's name.

[38] Arago, (Dominique) François (Jean), 1786-1853, scientist, born in Estagel, France. In 1830 he became secretary to the Polytechnic Observatory where he had worked since the age of 17. His was active in areas of astronomy, magnetism, and optics. Arago is known also for his involvement in a dispute between U. Leverrier (see note 28, *Bk. II Foreword*) and John Adams (see note 39, *Bk. II Foreword*) over priority in the discovery of the planet Neptune. (see also note 48, *Bk. I Foreword*, and *Foreword to the second edition*.)

[39] The + sign on the second term is missing in the 1st edition, as is the closing parenthesis.

[40] A determination based on a chronology of biblical events placed the creation of man around 4000 BCE.

[41] This *reads* $126^\circ.8$ in the first edition.

[42] Piazzi, Giuseppe, 1746-1826, astronomer and theologian, born in Ponte di Valtellina, Italy. Piazzi was a Theatine monk and professor of theology and mathematics in Rome (1779) and Palermo (1780) where he established an observatory in 1789. He also measured the proper motion of the star 61 Cygni, the same star Bessel later used for his first measurement of stellar parallax (see note 37 above). Piazzi is known best for his discovery and naming of the first minor planet or asteroid, Ceres (see note 9, *Preliminary Dissertation*).

[43] Olbers, (Heinrich) Wilhelm (Matthäus), 1758-1840, physician and astronomer, born in Arbergen, Germany. He invented a method for calculating the velocity of falling stars. He also discovered the minor planets or asteroids Pallas (1802) and Vesta in 1807 (see note 9, *Preliminary Dissertation*) as well as five new comets (one bears his name). He theorized that a comet's tail was due to radiation pressure. Light pressure was demonstrated experimentally in the 20th century.

[44] See note 26.

[45] See note 29, *Preliminary Dissertation.*

[46] The 1st edition text *reads* Saturn *for* Uranus.

[47] *appulse.* In astronomy, the approach of any planet to a conjunction with the sun, or a star. *Webster's 1828 Dictionary.*

[48] *Hindostan.* 'The country of the Hindus,' India. In modern native parlance this word indicates distinctively India north of the Nerbudda, and exclusive of Bengal and Behar. *The Anglo-Indian Dictionary*.

[49] *Himala.* Himalayas *The Anglo-Indian Dictionary*.

[50] Berkeley, George, 1685-1753, Anglican bishop and philosopher, born at Dysert Castle, Kilkenny, Ireland. He studied at Trinity College, Dublin, where he remained, as fellow and tutor, until 1713. His most important works include: *Essay towards a New Theory of Vision* (1709), *A Treatise concerning the Principles of Human Knowledge* (1710), and *Three Dialogues between Hylas and Philonous* (1713). He became Bishop of Cloyne in 1734.

[51] Bouguer, Pierre, 1698-1758, physicist, born in Le Croisie, France. In 1735 he was sent with others to Peru to measure a degree of the meridian at the equator. His views on the intensity of light laid the foundation of photometry. In 1748 he invented the heliometer.

[52] *Aldebaran.* An orange binary star in the constellation Taurus, having a combined magnitude of 0.9.

[53] Cassini, Giovanni (Gian) Domenico, 1625-1712, astronomer, born in Perinaldo, Italy. He was a professor of astronomy at Bologna, and the first director of the observatory at Paris (1669). Cassini measured solar parallax, as well as the periods of Mars, Venus and Jupiter (by measuring the shadows of Jupiter's shadows as they passed between the planet and the sun). He was also first to document the zodiacal light and first to observe Saturn's four moons (Iapetus, Rhea, Tethys, and Dione) as well as the gap between two of Saturn's rings (Cassini's Division). Cassini's laws on lunar rotation were formulated in 1693.

[54] See note 1, *Preliminary Dissertation.*

[55] The numerator in the second term *reads* $m'(xx+yy+zz)$ in the 1st edition.

[56] A parenthesis in the numerator in the second term is missing and *reads* $(xx'+yy' +zz') - \frac{1}{2}r^2)^2$ in the 1st edition.

[57] The second value *reads* $r' = 100.000\ a$ in the 1st edition.

[58] This *reads* "in page 261," in the 1st edition (original pagination).

[59] *[of].* This word is corrupted in the 1st edition.

[60] Legendre, Adrien-Marie, 1752-1833, mathematician, born in Paris. Legendre was professor of mathematics at the École Militaire (1775-80), and professor at the Ecole Normale (1795). The method of least squares mentioned by Somerville was indeed devised by Legendre and appeared in an appendix to his *Nouvelles méthodes pour la détermination des orbites des comètes* in 1806. However Carl Gauss (see note 8, *Bk. II, Chap XIII*) claimed priority for the method in his *Theoria motus corporum coelestium in sectionibus conicis Solem ambientium* in 1809 while acknowledging that it had appeared earlier in Legendre's work. At École Militaire Legendre taught with Pierre-Simon Laplace (see note 4, *Foreword to the Second Edition*) who is said to have appropriated some of Legendre's work with little credit to him. Legendre made important contributions to number theory and elliptic integrals (*Traité des fonctions elliptiques* in 1825-37). He studied the attraction of spheroids and ellipsoids. Using what are known now as Legendre functions he was able to determine the attraction of an ellipsoid at any exterior point.

[61] Op. Cit.

[62] See note 1, *Bk. II, Chap. I.*

[63] See note 3, *Preliminary Dissertation.*

[64] See note 6, *Bk. II, Chap. I.*

Moon

This photograph of the Moon was taken in December 1972 by the Apollo 17 mission--shortly after the spacecraft left the Moon to return to Earth. The view shows the full Moon. The region at the right (about two-thirds of the total) is part of the Moon's far side, the side never seen from Earth. The dark regions are the maria, which are covered with dark-colored basalt lava flows. The dark, nearly circular mare region at the upper left is called Mare Crisium. Below it and to the left is Mare Fecunditatis, with the large white crater Langrenus. The light-colored regions are the lunar highlands, which are made of older rocks and contain extensive large craters made by large projectiles that struck the Moon more than 4 billion years ago. The bright, rayed crater near the upper-right rim is Giordano Bruno, a fresh crater formed by a much younger impact event. (Courtesy of NASA)

BOOK III - LUNAR THEORY

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FOREWORD[1]

Rotation of the Moon

THE periods of rotation of the moon and the other satellites are equal to the times of their revolutions, consequently these bodies always turn the same face to their primaries. However, as the mean motion of the moon is subject to a secular inequality, which will ultimately amount to many circumferences if the rotation of the moon were perfectly uniform and not affected by the same inequalities, it would cease exactly to counterbalance the motion of revolution; and the moon, in the course of ages, would successively and gradually discover every point of her surface to the earth. But Laplace showed that this never can happen; for the rotation of the moon, though it does not partake of the periodic inequalities of her revolution, is affected by the same secular variations, so that her motions of rotation and revolution round the earth will always balance each other, and remain equal. This circumstance can only be accounted for by the form of the moon herself. She has three principal axes at right angles to each other. The first of these, the polar axis, is supposed to be the least, because, according to theory, the moon is flattened at her poles from her centrifugal force. The other two axes are in the plane of her equator, and of these the one which is directed towards the earth is the greatest. The attraction of the earth, as if it had drawn out that part of the moon's equator, constantly brings the greatest axis, and consequently the same hemisphere, towards us, which makes her rotation participate in the secular variations of her mean motion of revolution. Even if the angular velocities of rotation and revolution had not been nicely balanced in the beginning of the moon's motion, the attraction of the earth would have recalled the greatest axis to the direction of the line joining the centres of the moon and earth; so that it would have vibrated on each side of that line in the same manner as a pendulum oscillates on each side of the vertical from the influence of gravitation. No such libration[2] is perceptible; but another movement very similar, which was theoretically proved by Newton, has been detected by some observers. This movement, which is called *physical libration*, depends upon the fact, that as the moon does not move with equal velocity in all parts of her orbit, her elongated axis is not always directed exactly to the earth, but oscillates perpetually on each side of its mean place.

Besides these almost imperceptible movements, the moon is subject to very important librations which cause us to see, from time to time, slightly round the edge of her globe. These, which are called *libration in latitude* and *libration in longitude*, are due to two different causes, namely, the inclination of the moon's axis of rotation to the plane of her orbit, and the balance between her motions of rotation and revolution. The moon's equator-plane is inclined $1^\circ 30' 10''.8$ to the ecliptic, and in consequence of this inclination her northern and southern poles lean alternately in a slight degree to and from the earth, so that the middle of her visible disc lies at one time $6^\circ\ 39'$ north of the equator, and at another time as far south. This is the

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[1] The material in this and the forewords to Books I, II and IV is extracted from the 10th and last edition of Mary Somerville's *On the Connexion of the Physical Sciences*, (corrected and revised by Arabella B. Buckley), p. 4-106, London : John Murray, 1877.

[2] *libration.* A balancing movement to and fro (Somerville's note).

libration in latitude. The second libration, or the *libration in longitude*, is caused by the fact, that while the moon rotates with perfect uniformity on her axis, she does not move with equal velocity in all parts of her orbit on account of its eccentricity. When in perigee her movement is accelerated, and her revolution outstrips her rotation; when in apogee her movement is retarded, and she has turned round on her axis more than she has advanced in her orbit. The consequence of this is, that a meridian drawn across the centre of her true disc lies sometimes to the west and sometimes to the east of her visible disk, according as her rotation or revolution are in advance, and we see more of the eastern or western halves of her surface. As the eccentricity of the moon's orbit is variable, the amount of her libration in longitude is also variable. There is also another libration of very slight importance, called *diurnal libration*, which depends upon the fact that we are on the surface of a rotating globe, which causes the moon to appear to oscillate about her radius vector. This libration never exceeds $1^\circ 1' 28''.8$.

For the same reason that one hemisphere of the moon must remain eternally concealed from us, the earth, which must be so splendid an object to one lunar hemisphere, will be for ever veiled from the other. On account of these circumstances, the remoter hemisphere of the moon has its day a fortnight long, and a night of the same duration, not even enlightened by a moon, while the favoured side is illuminated by the reflection of the earth during its long night. A planet exhibiting a surface thirteen time's larger than that of the moon, with all the varieties of clouds, land, and water coming successively into view, would be a splendid object if seen from the moon. The great height of the lunar mountains probably has a considerable influence on the phenomena of her motion, the more so as her compression is small and her mass considerable. In the curve passing through the poles, and that diameter of the moon which always points to the earth, nature has furnished a permanent meridian to which the different spots on her surface have been referred, and their positions are determined with as much accuracy as those of many of the most remarkable places on the surface of our globe. The surface of the moon has been of late years most extensively studied, and some of the results are very remarkable. Not only is that face of the planet which is turned towards us covered with large extinct volcanic craters, some of these having a diameter of from 40 to 50 miles, and a depth of 10,000 feet; but large circular formations, with no cones in the centre, and measuring from 100 to 300 miles across, also occur. Their origin is not easy to explain, though Professor Dana[3] has suggested that they may once have been boiling lakes of lava, like the pit-crater of Kilauea in the Hawaiian Islands. Besides these evidently volcanic productions, there are the peaks and mountain ranges which have been generally supposed to be caused by the shrinking of the moon's surface, but which Messrs. Nasmyth and Carpenter[4] suggest may have been also formed by successive flows of lava oozing out from the crust and solidifying in slopes which would grow gradually steeper and steeper. One of the most remarkable appearances, however, is that of systems of streaks of light and shade which radiate from the borders of some of the largest ring mountains, and spread for hundreds of miles, having the appearance of glass or ice which has been starred by a blow. It is now generally believed that these streaks are cracks caused by the rending of the solid crust of the moon, and afterwards filled with molten matter from beneath.

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[3] Dana, James Dwight, (1813-1895), Geologist, born in Utica, N.Y., USA. Dana was an editor of the American Journal of Science (1846-95). He wrote a text on Hawaiian volcanoes *(Characteristics of volcanoes*, New York, Dodd, Mead and company, 1891) and several books on geology including *Manual of Geology* (1862) and a highly successful book on mineralogy *A System of Mineralogy* (1837). Dana was a personal acquaintance of Mary Somerville, and corresponded with botanist Asa Gray, naturalist Louis Agassiz, and Charles Darwin.

[4] Nasmyth and Carpenter, *The Moon*, 1874. (Somerville's note.)

Lunar Perturbations

Several circumstances concur to render the moon's motions the most interesting, and at the same time the most difficult to investigate, of all the bodies of our system. In the solar system, planet troubles planet; but in the lunar theory, the sun is the great disturbing cause, his vast distance being compensated by his enormous magnitude, so that the motions of the moon are more irregular than most of the planets; and, on account of the great ellipticity of her orbit, and the size of the sun, the approximations of her motions are tedious and difficult, beyond what those unaccustomed to such investigations could imagine. The average distance of the moon from the centre of the earth is only 238,818 miles,[5] so that her motion among the stars is perceptible in a few hours. She moves in an orbit whose eccentricity is about 12,985 miles, and completes a circuit of the heavens in $27^d\ 7^h\ 43^m\ 11^s.5$, although to the fact that the earth is also moving onward, the exact time from one new moon to another is $29^d\ 12^h\ 44^m\ 2^s.87$. The moon is about four hundred times nearer to the earth than the sun. The proximity of the moon to the earth keeps them together; for so great is the attraction of the sun, that, if the moon were farther from the earth, she would leave it altogether, and would revolve as an independent planet about the sun.

The disturbing action of the sun on the moon is equivalent to three forces. The first, or radial force, acting in the direction of the line joining the moon and earth, increases or diminishes her gravity to the earth. The second, or tangential force acting in the direction of a tangent to her orbit, disturbs her motion in longitude. And the third, or perpendicular force, acting perpendicularly to the plane of her orbit, disturbs her motion in latitude; that is, it brings her nearer to, or removes her farther from, the plane of the ecliptic than he would otherwise be. The periodic perturbations in the moon, arising from these forces, are perfectly similar to the periodic perturbations of the planets. But they are much greater and more numerous; because the sun is so large, that many inequalities which are quite insensible in the motions of the planets are of great magnitude in those of the moon. Among the innumerable periodic inequalities to which the moon's motion in longitude is liable, the most remarkable are, the Equation of the Centre (see Article 382, *Bk. II*, *Chap. IV*), which is the difference between the moon's mean and true longitude, the Evection, the Variation, and the Annual Equation. The disturbing force which acts in the line, joining the moon and earth produces the Evection:[6] it diminishes the eccentricity of the lunar orbit in conjunction and opposition, thereby making it more circular, and augments it in quadrature, which consequently renders it more elliptical. The period of this inequality is less than thirty-two days. Were the increase and diminution always the same, the Evection would only depend upon the distance of the moon from the sun; but its absolute value also varies with her distance from the perigee,[7] of her orbit. Ancient astronomers who observed the moon solely with a view to the prediction of eclipses, which can only happen in conjunction and opposition, where no eccentricity is diminished by the Evection, assigned too small a value to the ellipticity

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[5] Modern value is 238,866 miles (384,400 km).

[6] *Evection.* The evection is produced by the action of the radial force acting along the line joining the earth and the moon. It sometimes increases and sometimes diminishes the earth's attraction o the moon. It produces a corresponding temporary change in the eccentricity, which varies with the position of the major axis of the lunar orbit in respect of the line joining the centres of the earth and the sun. (Somerville's note.)

[7] *Perigee.* A Greek word signifying round the earth. The perigee of the lunar orbit is the point where the moon is nearest to the earth. It corresponds to the perihelion of a planet (see art. 316, *Bk. II, Chap. II*). Sometimes the word is used to denote the point where the sun is nearest the earth. (Somerville's note.)

of her orbit. The Evection was discovered by Ptolemy,[8] from observation, about AD 140. The Variation produced by the tangential disturbing force, which is at its maximum when the moon is 45° distant from the sun, vanishes when that distance amounts to a quadrant, and also when the moon is in conjunction and opposition; consequently, that inequality never could have been discovered from the eclipses: its period is half a lunar month.[9] The Annual Equation depends upon the sun's distance from the earth: it arises from the moon's motion being accelerated when that of the earth is retarded, and *vice versâ*—for, when the earth is in its perihelion, the lunar orbit is enlarged by the action of the sun; therefore, the moon requires more time to perform her revolution. But, as the earth approaches its aphelion, the moon's orbit contracts, and less time is necessary to accomplish her motion—its period, consequently, depends upon the time of the year. In eclipses the Annual Equation combines with the Equation of the Centre of the terrestrial orbit, so that ancient astronomers imagined the earth's orbit to have a greater eccentricity than modern astronomers assign to it.

The planets disturb the motion of the moon both directly and indirectly; their action on the earth alters its relative position with regard to the sun and moon, and occasions inequalities in the moon's motion, which are more considerable than those arising from their direct action; for the same reason the moon, by disturbing the earth, indirectly disturbs her own motion. Neither the eccentricity of the lunar orbit, nor its mean inclination to the plane of the ecliptic, have experienced any changes from secular inequalities; for, although the mean action of the sun on the moon depends upon the inclination of the lunar orbit to the ecliptic, and the position of the ecliptic is subject to a secular inequality, yet analysis shows that it does not occasion a secular variation in the inclination of the lunar orbit, because the action of the sun constantly brings the moon's orbit to the same inclination to the ecliptic. The mean motion, the nodes, and the perigee, however, are subject to very remarkable variations.

From the eclipse observed at Babylon, on March 19, seven and twenty-one years before the Christian era, the place of the moon is known from that of the sun at the instant of opposition, whence her mean longitude may be found. But the comparison of this mean longitude with another mean longitude, computed back for the instant of the eclipse from modern observations, shows that the moon performs her revolution round the earth more rapidly and in a shorter time now than she did formerly, and that the acceleration in her mean motion has been increasing from age to age as the square of the time.[10] All ancient and intermediate eclipses confirm this result. As the mean motions of the planets have no secular inequalities, this seemed to be an unaccountable anomaly. It was at one time attributed to the resistance of an ethereal medium pervading space, and at another to the successive transmission of the gravitating force. But, as Laplace proved that neither of these causes, even if they exist, have any influence on the motions of the lunar perigee or nodes, they could not affect the mean motion; a variation in the mean motion from such causes being inseparably connected with variations in the motions of the perigee and nodes. That great mathematician, in studying the theory of Jupiter's satellites, perceived that the secular variation in the elements of Jupiter's orbit, from the action of the

[8] See note 15, *Preliminary Dissertation*.
[9] *Variation*. The lunar perturbation called the variation is the alternate acceleration and retardation of the moon in longitude, from the action of the tangential force. She is accelerated when approaching the two points located 90 degrees from the perigee and apogee (called the syzygies), and retarded when returning from the syzygies to the perigee or apogee. (Somerville's note.)
[10] *Square of the time*. If the times increase at the rate of 1, 2, 3, 4, &c., years or hundreds of years, the squares of the times will be 1, 4, 9, 16, &c., years or hundreds of years (Somerville's note).

planets, occasions corresponding changes in the motions of the satellites, which led him to suspect that the acceleration in the mean motion of the moon might be connected with the secular variation in the eccentricity of the terrestrial orbit. Professor Adams[11] has now shown that the whole amount cannot be explained by secular variation, and it is probable that a part of it is only apparent, the real cause being a retardation in the earth's motion of rotation, in consequence of the friction of the tides, as suggested by Hansen and Delaunay.[12]

Nevertheless, Laplace was right in attributing the acceleration, so far as it is a fact, to the variation of the eccentricity of the earth's orbit. It is proved that the greater the eccentricity of the terrestrial orbit, the greater is the disturbing action of the sun on the moon. Now, as the eccentricity has been decreasing for ages, the effect of the sun in disturbing the moon has been diminishing during that time. Consequently the attraction of the earth has had a more and more powerful effect on the moon, and has been continually diminishing the size of the lunar orbit. So that the moon's velocity has been gradually augmenting for many centuries to balance the increase of the earth's attraction. This secular increase in the moon's velocity is called the Acceleration, a name peculiarly appropriate at present, and which will continue to be so for a vast number of ages; because, as long as the earth's eccentricity diminishes, the moon's mean motion will be accelerated; but when the eccentricity has passed its minimum, and begins to increase, the mean motion will be retarded from age to age. The secular acceleration is now about $11''.9$, but its effect on the moon's place increases as the square of the time.[13] It is remarkable that the action of the planets, thus reflected by the sun to the moon, is much more sensible than their direct action either on the earth or moon. The secular diminution in the eccentricity, which has not altered the equation of the centre of the sun by eight minutes since the earliest recorded eclipses, has produced a variation of about $1^\circ\ 48'$ in the moon's longitude, and of $7^\circ\ 12'$ in her mean anomaly.[14]

The action of the sun occasions a rapid but variable motion in the nodes and perigee of the lunar orbit. Though the nodes recede during the greater part of the moon's revolution, and advance during the smaller, they perform their sidereal revolution in $6793^d\ 9^h\ 23^m\ 9^s.3$, or about 18.6 years; and the perigee accomplishes a revolution, called of the moon's apsides, in

[11] See note 39, *Foreword, Bk. II.*

[12] *Ast. Soc. Monthly Notices*, vol. xxviii. p. 117. (Somerville's note.)

[13] In all investigations hitherto made with regard to the acceleration, it was tacitly assumed that the areas described by the radius vector of the moon were not permanently altered; that is to say, that the tangential disturbing force produced no disturbing effect. But Mr. Adams has discovered that, in consequence of the constant decrease in the eccentricity of the earth's orbit, there is a gradual change in the central disturbing force which affects the areal velocity, and consequently it alters the amount of the acceleration by a very small quantity, as well as the variation and the other periodic inequalities of the moon. On the latter, however, it has no permanent effect, because it affects them in opposite directions in very moderate intervals of time, whereas a very small error in the amount of the acceleration goes on increasing as long as the eccentricity of the earth's orbit diminishes, so that it would ultimately vitiate calculations of the moon's place for distant periods of time. This shows how complicated the moon's motions are, and what rigorous accuracy is required in their determination.

To give an idea of the labour requisite *merely* to *perfect* or *correct* the lunar tables, the moon's place was determined by observation at the Greenwich Observatory (see note 22, *Preliminary Dissertation*) in 6,000 different points of her orbit, each of which was compared with the same points calculated from Baron Plana's (see note 19, *Bk. II, Chap. X*) formulae, and to do that *sixteen computers* were constantly employed for *eight years*. Since the longitude is determined by the motions of the moon, the lunar tables are of the greatest importance. (Somerville's note.)

[14] *Mean anomaly*. The mean anomaly of a planet is its angular distance from the perihelion, supposing it to move in a circle. The true anomaly is the angular distance from the perihelion in its elliptical orbit. (Somerville note.)

$3232^d\ 13^h\ 48^m\ 29^s.6$, or a little more than nine years, notwithstanding its motion is sometimes retrograde and sometimes direct: but such is the difference between the disturbing energy of the sun and that of all the planets put together, that it requires no less than 109,830 years for the greater axis of the terrestrial orbit to do the same, moving at the rate of $11''.8$ annually. The form of the earth has no sensible effect either on the lunar nodes or apsides. It is evident that the same secular variation which changes the sun's distance from the earth, and occasions the acceleration in the moon's mean motion, must affect the nodes and perigee. It consequently appears, from theory as well as observation, that both these elements are subject to a secular inequality, arising from the variation in the eccentricity of the earth's orbit, which connects them with the acceleration, so that both are retarded when the mean motion is anticipated. The secular variations in these three elements are in the ratio of the numbers 3, 0.735, and 1; whence the three motions of the moon, with regard to the sun, to her perigee, and to her nodes, are continually accelerated, and their secular equations are as the numbers 1, 4.702, and 0.612. A comparison of ancient eclipses observed by the Arabs, Greeks, and Chaldeans, imperfect as they are, with modern observations, confirms these results of analysis. Future ages will develop these great inequalities, which at some most distant period will amount to many circumferences.[15] They are, indeed, periodic; but who shall tell their period? Millions of years must elapse before that great cycle is accomplished.

The moon is so near, that the excess of matter at the earth's equator occasions periodic variations in her longitude, and also that remarkable inequality in her latitude, already mentioned as a nutation in the lunar orbit, which diminishes its inclination to the ecliptic when the moon's ascending node coincides with the equinox of spring, and augments it when that node coincides with the equinox of autumn. As the cause must be proportional to the effect, a comparison of these inequalities, computed from theory, with the same given by observation, shows that the compression of the terrestrial spheroid, or the ratio of the difference between the polar and the equatorial diameters, to the diameter of the equator, is $\frac{1}{305}$. It is proved analytically, that, if a fluid mass of homogeneous matter, whose particles attract each other inversely as the squares of the distance, were to revolve about an axis as the earth does, it would assume the form of a spheroid whose compression is $\frac{1}{305}$. Since that is not the case, the earth cannot be homogeneous, but must decrease in density from its centre to its circumference. Thus the moon's eclipses show the earth to be round; and her inequalities not only determine the form, but even the internal structure of our planet; results of analysis which could not have been anticipated. Similar inequalities in the motions of Jupiter's satellites prove that his mass is not homogeneous. His equatorial diameter exceeds his polar diameter by about 6000 miles.

The phases of the moon, which vary from a slender silvery crescent soon after conjunction, to a complete circular disc of light in opposition, decrease by the same degrees till the moon is again enveloped in the morning beams of the sun. These changes regulate the returns of the eclipses. Those of the sun can only happen in conjunction, when the moon, coming between the earth and the sun, intercepts his light. Those of the moon are occasioned by the earth intervening between the sun and moon when in opposition. As the earth is opaque and nearly spherical, it throws a conical shadow on the side of the moon opposite to the sun, the axis of which passes through the centres of the sun and earth. The length of the shadow terminates at the

[15] *Many circumferences.* There are 360 degrees, or 1,296,000 seconds in a circumference; and, as the acceleration of the moon only increases at the rate of eleven seconds in a century, it must be a prodigious number of ages before it accumulates to many circumferences. (Somerville's note.)

point where the apparent diameters[16] of the sun and earth would be the same. When the moon is in opposition, and at her mean distance, the diameter of the sun would be seen from her centre under an angle of $1918''.1$. That of the earth would appear under an angle of $6908''.3$. So that the length of the shadow is at least three times and a half greater than the distance of the moon from the earth, and the breadth of the shadow, where it is traversed by the moon, is about eight-thirds of the lunar diameter. Hence the moon would be eclipsed every time she is in opposition, were it not for the inclination of her orbit to the plane of the ecliptic, in consequence of which the moon, when in opposition, is either above or below the cone of the earth's shadow, except when in or near her nodes. Her position with regard to them occasions all the varieties in the lunar eclipses. Every point of the moon's surface successively loses the light of different parts of the sun's disc before being eclipsed. Her brightness therefore gradually diminishes before she plunges into the earth's shadow. The breadth of the space occupied by the penumbra[17] is equal to the apparent diameter of the sun, as seen from the centre of the moon. The mean duration of a revolution of the sun, with regard to the node of the lunar orbit, is to the duration of a synodic revolution[18] of the moon as 223 to 19. So that, after a period of 223 lunar months, the sun and moon would return to the same relative position with regard to the node of the moon's orbit, and therefore the eclipses would recur in the same order were not the periods altered by irregularities in the motions of the sun and moon. In lunar eclipses, our atmosphere, bends the sun's rays which pass through it all round into the cone of the earth's shadow. And as the horizontal refraction[19] or bending of the rays surpasses half the sum of the semidiameters of the sun and moon, divided by their mutual distance, the centre of the lunar disc, supposed to be in the axis of the shadow, would receive the rays from the same point of the sun, round all sides of the earth; so that it would be more illuminated than in full moon, if the greater portion of the light were not stopped or absorbed by the atmosphere. Instances are recorded where this feeble light has been entirely absorbed, so that the moon has altogether disappeared in her eclipses.

The sun is eclipsed when the moon intercepts his rays. The moon though incomparably smaller than the sun is so much nearer the earth, that her apparent diameter differs but little from his, but both are liable to such variations that they alternately surpass one another. Were the eye of a spectator in the same straight line with the centres of the sun and moon, he would see the sun eclipsed. If the apparent diameter of the moon surpassed that of the sun, the eclipse would be total. If it were less, the observer would see a ring of light round the disc of the moon, and the eclipse would be annular, as it was on the 17th of May, 1836, on the 15th of March, 1858, and on the 6th of March, 1867. If the centre of the moon should not be in a straight line joining the centres of the sun and the eye of the observer, the moon might only eclipse a part of the sun. The variation, therefore, in the distances of the sun and moon from the centre of the earth, and of the moon from her node at the instant of conjunction, occasions great varieties in the solar eclipses. Besides, the height of the moon above the horizon changes her apparent diameter, and may augment or diminish the apparent distances of the centres of the sun and moon, so that an eclipse of the sun may occur to the inhabitants of one country, and not to those of another. In this respect the solar eclipses differ from the lunar, which are the same for every part of the earth where the moon is above the horizon. In solar eclipses, the light reflected by the atmosphere diminishes the

[16] *Apparent diameter.* The diameter of a celestial body as seen from the earth. (Somerville's note.)
[17] *Penumbra.* The shadow or imperfect darkness which precedes and follows an eclipse. (Somerville's note.)
[18] *Synodic revolution of the moon.* The time between two consecutive new or full moons. (Somerville's note.)
[19] *Horizontal refraction.* The light, in coming from a celestial object, is bent into a curve as soon as it enters our atmosphere; and that bending is greatest when the object is in the horizon. (Somerville's note.)

obscurity they produce. Even in total eclipses the higher part of the atmosphere is enlightened by a part of the sun's disc, and reflects its rays to the earth. The whole disc of the new moon is frequently visible from atmospheric reflection. In total solar eclipses the slender luminous arc that is visible for a few seconds before the sun vanishes and also before he reappears, resembles a string of pearls surrounding the dark edge of the moon; it is occasioned by the sun's rays passing between the tops of the lunar mountains; it occurs likewise in annular eclipses.

A phenomenon altogether unprecedented was seen during the total eclipse of the sun which happened on the 8th of July, 1842. The moon was like a black patch on the sky surrounded by the well known faint whitish light or corona about the eighth of the moon's diameter in breadth, which is supposed to be the solar atmosphere rendered visible by the intervention of the moon. In this whitish corona there appeared three rose-coloured flames like the teeth of a saw. Similar flames were also seen in the white corona of the total eclipse which took place in 1851, and a long rose-coloured chain of what appeared to be jagged mountains or sierras united at the base by a red band seemed to be raised into the corona by mirage; but there is no doubt that the corona and red phenomena belong to the sun.

Planets sometimes eclipse one another. On the l7th of May, 1737, Mercury was eclipsed by Venus near their inferior conjunction; Mars passed over Jupiter on the 9th of January, 1591; and on the 30th of October, 1825, the moon eclipsed Saturn. These phenomena, however, happen very seldom, because all the planets, or even a part of them, are very rarely seen in conjunction at once; that is, in the same part of the heavens at the same time. More than 2500 years before our era the five great planets were in conjunction. On the 15th of September, 1186, a similar assemblage took place between the constellations of Virgo and Libra; and in 1801 the Moon, Jupiter, Saturn, and Venus were united in the heart of the Lion. These conjunctions are so rare, that Lalande has computed that more than seventeen millions of millions of years separate the epochs of the contemporaneous conjunctions of the six great planets.

The motions of the moon have now become of more importance to the navigator and geographer than those of any other heavenly body, from the precision with which terrestrial longitude is determined by occultations of stars, and by lunar distances. In consequence of the retrograde motion of the nodes of the lunar orbit, at the rate of $3'10''.64$ daily, these points make a tour of the heavens in a little more than eighteen years and a half. This causes the moon to move round the earth in a kind of spiral, so that her disc at different times passes over every point in a zone of the heavens extending rather more than $5^\circ 9'$ on each side of the ecliptic. It is therefore evident that at one time or other she must eclipse every star and planet she meets within this space. Therefore the occultation of a star by the moon is a phenomenon of frequent occurrence. The moon seems to pass over the star, which almost instantaneously vanishes at one side of her disc, and after a short time as suddenly reappears on the other. A lunar distance is the observed distance of the moon from the sun, or from a particular star or planet, at any instant. The lunar theory is brought to such perfection, that the times of these phenomena, observed under any meridian, when compared with those computed for that of Greenwich, and given in the *Nautical Almanac*, furnish the longitude of the observer within a few miles.

From the lunar theory, the mean distance of the sun from the earth, and thence the whole dimensions of the solar system, are known; for the forces which retain the earth and moon in their orbits are respectively proportional to the radii vectores of the earth and moon, each being divided by the square of its periodic time. And, as the lunar theory gives the ratio of the forces, the ratio of the distances of the sun and moon from the earth is obtained. Hence it appears that the sun's mean distance from the earth is 384.26 times greater than that of the moon. The method of finding the absolute distances of the celestial bodies, in miles, is in fact the same with that employed in measuring the distances of terrestrial objects. From the extremities of a known base, the angles which the visual rays from the object form with it are measured; their sum subtracted from two right angles gives the angle opposite the base, therefore, by trigonometry, all the angles and sides of the triangle may be computed–consequently the distance of the object is found. The angle under which the base of the triangle is seen from the object is the parallax of that object. It evidently increases and decreases with the distance. Therefore the base must be very great indeed to be visible from the celestial bodies. The globe itself, whose dimensions are obtained by actual measurement, furnishes a standard of measures with which we compare the distances, masses, densities, and volumes of the sun and planets.

BOOK III

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CHAPTER I

LUNAR THEORY

665. THERE is no object within the scope of astronomical observation which affords greater variety of interesting investigation to the inhabitant of the earth, than the various motions of the moon: from these we ascertain the form of the earth, the vicissitudes of the tides, the distance of the sun, and consequently the magnitude of the solar system. These motions which are so obvious, served as a measure of time to all nations, until the advancement of science taught them the advantages of solar time; to these motions the navigator owes that precision of knowledge which guides him with well-grounded confidence through the deep.

Phases of the Moon

666. The phases of the moon depend upon her synodic motion, that is to say, on the excess of her motion above that of the sun. The moon moves round the earth from west to east; in conjunction she is between the sun and the earth; but as her motion is more rapid than that of the sun, she soon separates from him, and is first seen in the evening like a faint crescent, which increases with her distance till in quadrature, or 90° from him, when half of her disc is enlightened: as her elongation increases, her enlightened disc augments till she is in opposition, when it is full moon, the earth being between her and the sun. In describing the other half of her orbit, she decreases by the same degrees, till she comes into conjunction with the sun again. Though the moon receives no light from the sun when in conjunction, she is visible for a few days before and after it, on account of the light reflected from the earth.

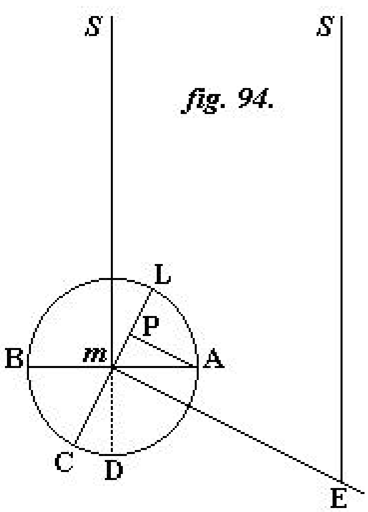


fig. 94.

The law of the variation of the phases of the moon proves her form to be spherical, since they vary as the versed sine of her angular distance from the sun.

If E be the earth, fig. 94, *m* the centre of the moon, supposed to be spherical, and *Sm*, *S*E parallel rays from the sun. Then, if AB be at right angles to the ray *mS*, BLA is the part of the disc that is enlightened by the sun; and CL, being at right angles to *m*E, the part of the moon that is turned to the earth will be CNL; hence the only part of the enlightened disc seen from the earth is LA; or, if it be projected on CL, it is PL, the versed sine of AL. But A*m*L is complement to A*m*N, and is therefore equal to D*m*N, or to *m*ES, the elongation or angular distance of the moon from the sun. When the moon is in quadrature, that is, either 90° or 180° from the sun, a little more than half her disc is enlightened; for when the exact half is visible, the moon is a little nearer to the sun than 90°;

at that instant, which is known by the division between the light and the dark half being a straight line; the lunar radius E*m* fig. 95, is perpendicular to *mS*, the line joining the centres of the sun and moon; hence, in the right-angled triangle E*mS*, the angle E, at the observer, may be measured, and therefore we can determine *S*E, the distance of the sun from the earth, by the solution of a right-angled triangle, when the moon's distance from the earth is known. The difficulty of ascertaining the exact time at which the moon is bisected, renders this method of ascertaining the distance of the sun incorrect. It was employed by Aristarchus of Samos[1] at Alexandria, about two hundred and eighty years before the Christian era,[2] and was the first circumstance that gave any notion of the vast distance and magnitude of the sun.

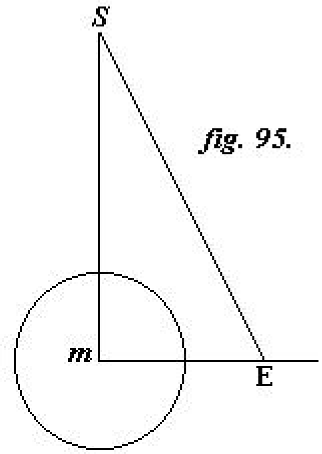


fig. 95.

Mean or Circular Motion of the Moon

667. The mean motion of the moon may be determined by comparing ancient with modern observations. The moon when eclipsed is in opposition, and her place is known from the sun's place, which can be accurately computed back to the earliest ages of antiquity. Three eclipses of the moon observed at Babylon in the years 720 and 719 before the Christian era, are the oldest observations recorded with sufficient precision to be relied on. By comparing these with modern observations, it is found that the mean are described by the moon in one hundred Julian years, or the difference of the mean longitudes of the moon in a century, was $481,267^{\circ}.8793$ in the year 1800; it is called the moon's tropical motion, which, omitting 1336 entire circumferences, is $307^{\circ}.8793$; and dividing it by 365.25, the number of days in the Julian year, her diurnal tropical motion is $13^{\circ}.17636$, about thirteen times greater than that of the sun.

668. From the tropical motion of the moon, her periodic revolution, or the time she employs in returning to the same longitude, may be found by simple proportion; for

$$481,267^{\circ}.8793 : 360^{\circ} :: 365.25 : 27.321582,$$

the periodic revolution of the moon, or a periodic lunar month.

669. By subtracting $5010''$, or the precession of the equinoxes for a century, from the secular tropical motion of the moon, her sidereal motion in a century is $481,266^{\circ}.48763$; or, omitting the whole circumferences, it is $306^{\circ}.48763$; whence, by simple proportion, her sidereal revolution is $27^{d}\ 7^{h}\ 43'\ 11''.5$. These two motions of the moon only differ by the precession of the equinoxes: her sidereal daily motion is, therefore, $13^{\circ}\ 10'\ 35''.034$.

670. The synodic revolution of the moon is her mean motion from conjunction to conjunction, or from opposition to opposition. The mean motion of the moon in a century being $481,267^{\circ}.8793$, and that of the sun being $36,000^{\circ}.7625$, their difference, $445,267^{\circ}.1168$, is the excess of the moon's motion above the sun's in one hundred Julian years; hence her motion through 360° is accomplished in $29^d\ 12^h\ 44'\ 2''.8$, a lunar month. The lunar month is to the tropical as 19 to 235 nearly, so that 19 solar years are equal to 235 lunar months. The mean motion of the moon is variable, which affects all the preceding results.

671. The apparent diameter of the moon is either measured by a micrometer, or computed from the duration of the occultations of the fixed stars. Its greatest value is thus found to be $2{,}011''.1$, and the least $1{,}761''.91$. The analogous values in the apparent diameter of the sun are $1{,}955''.6$ and $1{,}890''.96$; whence the variations in the moon's distance from the earth are much greater than those of the sun; consequently the eccentricity of the lunar orbit is much greater than that of the terrestrial orbit.

672. It appears from observation, that the horizontal parallax of the moon takes all possible values between the limits $1^\circ.0248$ and $0^\circ.8975$ which give 55.9164 and 63.8419 for the least and greatest distances of the moon from the earth; consequently, her mean distance is nearly sixty times the terrestrial radius. The solar parallax shows, that the sun is immensely more distant. Because the lunar parallax is equal to the radius of the terrestrial spheroid divided by the moon's distance from the earth, it is evident that, at the same distance of the moon, the parallax varies with the terrestrial radii; consequently, the variations in the parallax not only prove that the moon moves in an ellipse, having the earth in one of its foci, but that the earth is a spheroid.

Elliptical Motion of the Moon

673. The greatest inequality in the moon's motion is the equation of the centre, which was discovered at a very early period: it is by this quantity alone that the undisturbed elliptical motion of a body differs from its mean or circular motion; it therefore arises entirely from the eccentricity of the orbit, being zero in the apsides, where the elliptical motion is the same with the mean motion, and greatest at the mean distance, or in quadratures, where the two motions differ most. Its maximum is found, by observation, to be $6^\circ\ 17'28''$. This quantity which appears to be invariable, is equal to twice the eccentricity; and if the radius be unity, an arc of

$$3^\circ\ 8'\ 44'' = 0.0549003 = e,$$

the eccentricity of the lunar orbit when the mean distance of the moon from the earth is one.

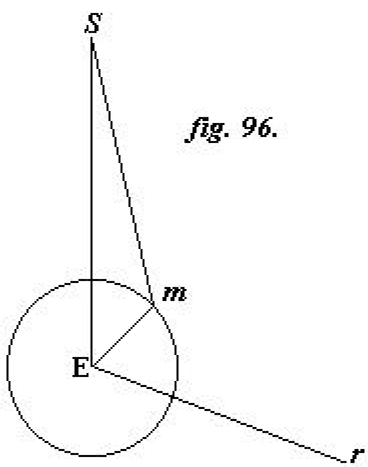
674. In consequence of the action of the sun, the perigee of the lunar orbit has a direct motion in space. Its mean motion in one hundred Julian years, deduced from a comparison of ancient with modern observations, was $4{,}069^\circ.0395$ in 1800, with regard to the equinoxes, which by simple proportion gives $3{,}231^d.4751$ for its tropical revolution, and $3{,}232^d.5807$, or a little more than nine years for its sidereal revolution; hence its daily mean motion is $6'41''$. These motions change on account of the secular variation in the motion of the perigee.

675. The anomalistic[3] revolution of the moon is her revolution with regard to her apsides, because the moon moves in the same direction with her perigee; after separating from that point, she only comes to it again by the excess of her velocity. That excess is $477{,}198^{\circ}.69184$ in one hundred Julian years; therefore by simple proportion, the moon's anomalistic[4] year is $27^{d}.5546$.

676. The nodes of the lunar orbit have a retrograde motion, which may be computed from observation, in the same manner with the motion of the perigee. The mean tropical motion of the nodes in 1800 was $1{,}936^{\circ}.940733$, which gives $6{,}788^{d}.54019$ for their tropical revolution, and $6{,}793^{d}.42118$ for their sidereal revolution, or $3'\,10''.64$ in a day; hence the moon's daily motion, with regard to her node, is $13^{\circ}\;13'\;45''.534$. The motion of the perigee and nodes arises from the disturbing action of the sun, and depends on the ratio of his mass to that of the earth; this being very great, is the reason why the greater axis and nodes of the lunar orbit move so much more rapidly than those of any other body in the system.

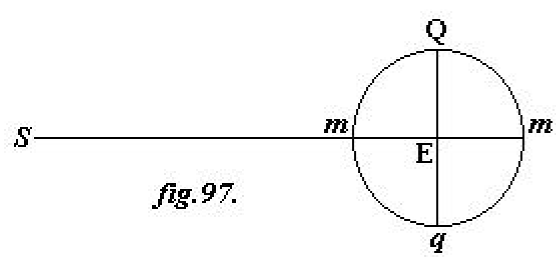
Lunar Inequalities

677. The moon is troubled in her motion by the sun; by her own action on the earth, which changes the relative positions of the bodies, and thus affects her motions; by the direct action of the planets; by their disturbing action on the earth, and by the form of the terrestrial spheroid.



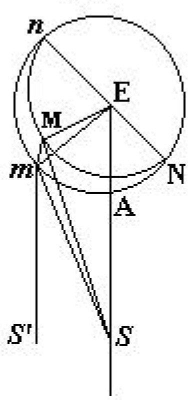
678. Previous to the analytical investigations, it may perhaps be of use to give some idea of the action of the sun, which is the principle cause of the lunar inequalities.

The moon is attracted by the sun and by the earth at the same time, but her elliptical motion is only troubled by the difference of the actions of the sun on the earth and on herself. Were the sun at an infinite distance, he would act equally and in parallel straight lines, on the earth and moon, and their relative motions would not be troubled by an action common to both; but the distance of the sun although very great, is not infinite. The moon is alternately nearer to the sun and farther from him than the earth; and the straight line *Sm*, fig. 96, which joins the centres of the sun and moon, makes angles more or less acute with *S*E, the radius vector of the earth. Thus the sun acts unequally, and in different directions, on the earth and moon; whence inequalities result in the lunar motions, depending on *m*E*S*, the elongation of the sun and moon, on their distances and the moon's latitude.



When the moon is in conjunction at *m*, fig. 97, she is nearer the sun than the earth is; his action is therefore greater on the moon than it is on the earth; the difference of their actions tends to diminish the moon's gravitation to the earth. In opposition at m', the earth is nearer to the sun than the moon is, and therefore the sun attracts the earth more powerfully than he attracts the moon. The difference of these actions tends also to diminish the moon's gravitation to the earth. In quadratures, at Q and *q*, the action of the sun on the moon resolved in the direction of the radius vector QE, tends to augment the gravitation of the moon to the earth; but this increment of gravitation in quadratures is only half of the diminution of gravitation in syzigies; and thus, from the whole action of the sun on the moon in the course of a synodic revolution, there results a mean force directed according to the radius vector of the moon, which diminishes her gravity to the earth, and may be determined as follows:–

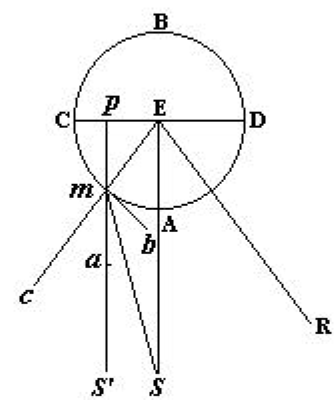
fig. 98.



679. Let M, fig. 98, be the moon in her nearly circular orbit *n*MN; E and *S* the earth and sun in the plane of the ecliptic; *nm*N the moon's orbit projected on the same. Then M*m* is the tangent of the moon's latitude, and E*m* her curtate distance. Let *S*E, E*m*, be represented by r' and *r*, and the angle AE*m* by x, m' being the mass of the sun.

The attraction of the sun on the moon at M is $\frac{m'}{(S\text{M})^2}$. This force may be resolved into three; one in the direction M*m*, which troubles the moon in latitude; another in *m*E, which, being directed towards the centre E, increases the gravity of the moon to the earth, and does not disturb the equable description of areas; and into a third in the direction mS', the excess of which above that by which the sun attracts the earth disturbs the relative position of the moon and earth. The inclination of the lunar orbit is so small that it may be omitted at first, and then the force $\frac{m'}{(Sm)^2}$, fig. 99, is resolved into two, one in the direction mE, which only increases the gravity of the moon, and the other in $N_0, N_1, \ldots$ which disturbs her motion. Let *ma* represent this last force, and suppose it resolved into *mb* and *mc*. The force *mb* accelerates the moon in the quadrants CA and DB, and retards her in the other two; the force *mc* lessens the gravity of the moon.

fig. 99.



680. The analytical expression of these forces is readily found. For the action of the sun on the moon in the direction *Sm*, is $\frac{m'}{(Sm)^2}$, but on account of the great distance of the sun,

$$Sm = \text{SE} - mp = r' - r\cos x, \text{ nearly,}$$

hence the action of the sun on the moon in *Sm* is

$$\frac{m'}{\left(r' - r\cos x\right)^2},$$

which, resolved in the direction SE, is

$$\frac{m'}{r'^2} + \frac{m'}{r'^3} 3r\cos x, \text{ nearly}.$$

But the action of the sun on the earth is $\frac{m'}{r'^2}$, and their difference

$$\frac{3m'}{r'^3}\cdot r\cos x$$

is the force *ma*.

Now $\frac{m}{r'^3}\cdot 3r\cos^2 x,$ is the force *ma* resolved in *mc*, and

$$\frac{m'}{r'^3}\cdot 3r\sin x\cos x = \frac{m'}{r'^3}\cdot \tfrac{3}{2} r\sin 2x,$$

is the same resolved in *mb*. But the force in *m*E which increases the moon's gravity to the earth, is evidently $\frac{m'r}{r'^3}$; hence the whole force by which the sun increases or diminishes the gravity of the moon to the earth is,

$$\text{force in } m\text{E} - \text{force in } mc, \text{ or } \frac{m'r}{r'^3}\left(1-3\cos^2 . x\right).$$

In syzigy[5] $x = 0^\circ$, or 180°, and $\cos^2 x = +1$; thus the action of the sun in conjunction and opposition is $-\frac{2m'r}{r'^3}$. In quadratures $x = 90^\circ$, or 270°; hence $\cos x = 0$, and the sun's action at these points is $\frac{m'r}{r'^3}$. The mean value of the force $\frac{m'r}{r'^3}\left(1-3\cos^2 x\right)$ for an entire revolution, is the integral of

$$\frac{m'r}{r'^3}\left(1-3\cos^2 x\right)dx = \frac{m'r}{r'^3}\left(1-\tfrac{3}{2}-\tfrac{3}{2}\cos 2x\right)dx,$$

or

$$-\frac{m'r}{r'^3}\left(\tfrac{1}{2}x+\tfrac{3}{4}\sin 2x\right);$$

and when $x = 360^\circ$, it becomes $-\frac{m'r}{2r'^3}$, which is the mean disturbing force acting on the moon in the direction of the radius vector.

681. In order to have the ratio of this mean force to the gravity of the moon, we must observe that if E and *m* be the masses of the earth and moon, $\frac{\text{E}+m}{r^2}$ is the force that retains the moon in its orbit, and $\frac{m'}{r'^2}$ is the force that retains the earth in its orbit. But these forces are as

$$\frac{r}{(27.321661)^2} \text{ to } \frac{r'}{(365.25)^2},$$

which are the radii vectores of the moon and earth divided by the squares of their periodic times, whence

$$\frac{mr'}{r'^3} = \frac{1}{179} \cdot \frac{m+\mathrm{E}}{r^2};$$

and thus it appears that the mean action of the sun diminishes the gravity of the moon to the earth by its 358th part, for

$$\frac{mr'}{2r'^3} = \frac{1}{358} \cdot \frac{m+\mathrm{E}}{r^2}.$$

682. In consequence of this diminution of the moon's gravity by its 358th part,[6] she describes her orbit at a greater distance from the earth with a less angular velocity, and in a longer time than if she were urged to the earth by her gravity alone; but as the force is in the direction of the radius vector, the areas are not affected by it; hence, if her radius vector be increased by its 358th part, and her angular velocity diminished by its 179th part, the areas described will be the same as they would have been without that action. The force in the tangent mb disturbs the equable description of areas, and that in mM troubles the moon in latitude. The true investigation of these forces can only be conducted by an analytical process, which will now be given, without carrying the approximation so far as may be necessary, referring for the complete development of the series, to Damoiseau's profound analysis in the *Memoirs* of the *French Institute* for 1827.[7]

683. The peculiar disturbances to which the moon is liable, and the variety of inferences that may be drawn from them, render her motions better adapted to prove the universal prevalence of the law of gravitation, than those of any other body. The perfect coincidence of theory with observation, shows that analytical formulae not only express all the observed phenomena, but that they may be employed as a means of discovery not less certain than observation itself.

684. Although the motions of the moon be similar to those of a planet, they cannot be determined by the same analysis, on account of the great eccentricity of the lunar orbit, and the immense magnitude of the sun, which make it necessary to carry the approximation at least to the fourth powers of the eccentricities, and to the square of the disturbing force; and although the smallness of the mass of the moon compared with that of the earth, enables us to obtain her perturbations by successive approximations, yet the series converge slowly when the disturbing action of the sun is expressed in functions of the mean longitudes of the sun and moon; and as the facility of analytical investigations, and the fitness of formulae for computation, depend on a skilful choice of co-ordinates, the motions of the moon are first determined in functions of the true longitudes, and then her co-ordinates are obtained by reversion of series in functions of the mean longitudes of the two bodies.

685. The successive approximations are determined by the magnitude of the coefficients. Those terms belong to the first approximation which have for coefficients, either the ratio of the mean motion of the sun to that of the moon, or the eccentricities of the earth and moon, or the inclination of the lunar orbit on the ecliptic. Those terms belong to the second approximation, which have the squares or these quantities as coefficients; those which have their cubes belong to the third, and so on.

The terms having the constant ratio $\frac{a}{a'} = \frac{1}{400}$ of the parallax of the sun to that of the moon for coefficients, are included in the second approximation, and also those depending on the disturbing force of the sun, which is of the order

$$\frac{m'a^3}{a'^3}, \text{ or } m^2;$$

for it has been observed that a permanent change is produced by the disturbing forces in the mean distance: hence if

$$a',\ a,\ n',\ n,\ m',\ m,$$

be the mean distances, mean motions and masses of the sun and moon, and $\bar{a}$ the value of a in the troubled orbit, so that $a = \bar{a}$ when there is no disturbing force, then will

$$\frac{a^2}{\sqrt{\bar{a}}} = \frac{1}{n}, \text{ and as } \frac{m'}{a^3} = n'^2,$$

therefore[8]

$$\frac{m'a^3}{a'^3} \cdot \frac{a}{\bar{a}} = \frac{n'^2}{n^2} = \left(\frac{1}{13.368}\right)^2 = 0.005595;$$

but the mass of the moon is $m = 0.0748013,$ consequently

$$m^2 = 0.005595,$$

so that

$$\frac{m'a^3}{a'^3} \cdot \frac{a}{\bar{a}} = m^2; \text{ or if } \frac{m'a^3}{a'^3} = \bar{m}^2,$$

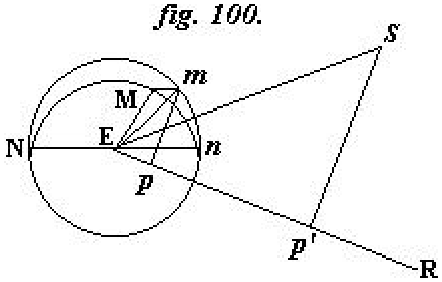
then

$$\frac{\bar{m}^2}{\bar{a}} = \frac{m^2}{a}.$$

686. By arranging the series according to the magnitudes of their terms, each approximation may be had separately by taking a certain part and rejecting the rest. This process must be continued till the value of the remainder is so small as to be insensible to observation; but even then it is necessary to ascertain not only that it is so at present, but that it will remain so after the lapse of ages. Besides selecting from the innumerable terms of the series those that have considerable coefficients, it is requisite to examine what values the different terms acquire in the determination of the finite values of the perturbations from their indefinitely small changes, for it has been shown that by integration some of the terms acquire divisors, which increase their values so much that great errors would ensue from omitting them.

Analytical Investigation of the Lunar Inequalities

fig. 100.



687. Suppose the motion of the earth to be referred to the sun, and that both sun and moon revolve round the earth assumed to be at rest in E, fig. 100. Let M be the moon in her orbit, *m* her place projected on the plane of the ecliptic, so that E*m* is her curtate distance; and let E*p*, *pm*, M*m*, or x, y, z, be the co-ordinates of the moon, and x', y', z', those of the sun in S, both referred to the centre of the earth, and to the fixed ecliptic at a given epoch.

If m', E, m, be the masses of the sun, the earth, and the moon, the equations of article 347 are

$$0=\frac{d^2x}{dt^2}+\frac{\mathrm{E}+m}{r^3}-\frac{1}{m}\left(\frac{dl}{dx}\right);$$
$$0=\frac{d^2y}{dt^2}+\frac{\mathrm{E}+m}{r^3}-\frac{1}{m}\left(\frac{dl}{dy}\right);$$
$$0=\frac{d^2z}{dt^2}+\frac{\mathrm{E}+m}{r^3}-\frac{1}{m}\left(\frac{dl}{dz}\right).$$

In which $r=\sqrt{x^2+y^2+z^2}$ is the radius vector of the moon,

$$l=\frac{mm'}{\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}};$$

and the element of the time is assumed to be constant in taking the differentials; but if that element be variable, and if [9]

$$R=\frac{(\mathrm{E}+m)}{r}-\frac{m'(x'x+y'y+z'z)}{(x'^2+y'^2+z'^2)^{\frac{3}{2}}}+\frac{m'}{\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}}$$

the relative motion of the moon and earth will be determined by the following equations,

$$\frac{d^2x}{dt^2} - \frac{dxd^2t}{dt^3} = \left(\frac{dR}{dx}\right)$$
$$\frac{d^2y}{dt^2} - \frac{dyd^2t}{dt^3} = \left(\frac{dR}{dy}\right)$$
$$\frac{d^2z}{dt^2} - \frac{dzd^2t}{dt^3} = \left(\frac{dR}{dz}\right).$$

688. In very small angles the arc may be taken for its sine; hence the lunar parallax is the radius of the terrestrial spheroid divided by the moon's distance from the earth, and thus the parallax varies inversely as the radius vector. Then if $R`$ be the radius of the earth, and $r$ the radius vector of the moon, the lunar parallax will be $\frac{R`}{r}$, which thus becomes the third co-ordinate of the moon. But if the earth be assumed to be spherical, its radius may be taken equal to unity, and then the lunar parallax will be $\frac{1}{r}$. Therefore let $u = \frac{1}{r}$, [the angle] $\mathrm{RE}m = v$ [fig. 100]; and $m\mathrm{M} = s$, the tangent of the moon's latitude; then

$$r = \sqrt{x^2 + y^2 + z^2} = \frac{\sqrt{1+ss}}{u},$$

[where]

$$x = \frac{\cos v}{u}, \quad y = \frac{\sin v}{u}, \quad z = \frac{s}{u}.$$

But in taking the differentials of these, dv must be constant, since dt is assumed to be variable.

689. Let the first of the preceding equations multiplied by $-\sin v$ be added to the second multiplied by $\cos v$; and let the first multiplied by $\cos v$ be added to the second multiplied by $\sin v$; then, if the foregoing values of $x, y, z,$ be substituted, and if to abridge

$$\left(\frac{dR}{dx}\right)\sin v - \left(\frac{dR}{dy}\right)\cos v = p$$
$$\left(\frac{dR}{dx}\right)\cos v + \left(\frac{dR}{dy}\right)\sin v = \Pi,$$

the result will be

$$\frac{d^2v}{dt^2} - \frac{2dvdu}{udt^2} - \frac{dvd^2t}{dt^3} = -pu;$$
$$\frac{d^2u}{u\,.\,dt^2} + \frac{dv^2}{dt^2} - \frac{2du^2}{u^2dt^2} - \frac{dud^2t}{udt^3} = -\Pi u; \qquad \textbf{(206)}$$
$$\frac{d^2s}{dt^2} + \frac{sdv^2}{dt^2} - \frac{dsd^2v}{dvdt^2} = pu\frac{ds}{dv} - \Pi su + u\left(\frac{dR}{dz}\right).$$

The first of these equations multiplied by $\frac{2dv}{u^3}$, and integrated, is

$$\left(\frac{dv}{u^2dt}\right)^2 = h^2 - \int 2p \cdot \frac{dv}{u^3},$$

h^2 being a constant quantity; whence

$$dt = \frac{dv}{u^2\sqrt{h^2 - 2\int \frac{p\,dv}{u^3}}}.$$

The elimination of d^2t between the first and second of equations (206), gives

$$\frac{dud^2v}{u^2dvdt^2} - \frac{d^2u}{u^2dt^2} - \frac{dv^2}{udt^2} = \Pi - \frac{p\,du}{udv};$$

and if dv be assumed to be constant, and substituting for dt its preceding value, it becomes

$$\frac{d^2u}{dv^2} + u = -\frac{\Pi - p\,\frac{du}{udv}}{u^2\left(h^2 - 2\int \frac{p\,dv}{u^3}\right)}.$$

In the same manner the third of equations (206) gives

$$\frac{d^2s}{dv^2} + s = -\frac{\left(\frac{dR}{dz}\right) - \Pi s + \frac{p\,ds}{dv}}{u^2\left(h^2 - 2\int \frac{p\,dv}{u^3}\right)}.$$

Now

$$dR = dx\left(\frac{dR}{dx}\right) + dy\left(\frac{dR}{dy}\right) + dz\left(\frac{dR}{dz}\right),$$

and when substitution is made for dx, dy, dz,

$$dR = -\frac{du}{u^2}\left\{\left(\frac{dR}{dx}\right)\cos v + \left(\frac{dR}{dy}\right)\sin v + \left(\frac{dR}{dz}\right)s\right\}$$
$$-\frac{dv}{u}\left\{\left(\frac{dR}{dx}\right)\sin v - \left(\frac{dR}{dy}\right)\cos v\right\} + \frac{ds}{u}\left(\frac{dR}{dz}\right).$$

But

$$dR = du\left(\frac{dR}{du}\right) + dv\left(\frac{dR}{dv}\right) + ds\left(\frac{dR}{ds}\right);$$

hence, by comparison,

$$\frac{dR}{du} = -\frac{1}{u^2}\left\{\left(\frac{dR}{dx}\right)\cos v + \left(\frac{dR}{dy}\right)\sin v + \left(\frac{dR}{ds}\right)s\right\}$$

$$\frac{dR}{dv} = -\frac{1}{u}\left\{\left(\frac{dR}{dx}\right)\sin v - \left(\frac{dR}{dy}\right)\cos v\right\}$$

$$\frac{dR}{ds} = \frac{1}{u}\left(\frac{dR}{dz}\right).$$

Whence

$$\Pi = -u^2\left(\frac{dR}{du}\right) - su\left(\frac{dR}{ds}\right)$$

$$p = -u\left(\frac{dR}{dv}\right), \text{ and } \frac{dR}{dz} = u\left(\frac{dR}{ds}\right).$$

690. Thus the differential equations which determine the motions of the moon become

$$dt = \frac{dv}{hu^2\left\{1 + \frac{2}{h^2}\int\left(\frac{dR}{dv}\right)\cdot\frac{dv}{u^2}\right\}^{\frac{1}{2}}};$$

$$0 = \left(\frac{d^2u}{dv^2} + u\right)\left\{1 + \frac{2}{h^2}\int\left(\frac{dR}{dv}\right)\cdot\frac{dv}{u^2}\right\} + \frac{du}{h^2u^2dv}\left(\frac{dR}{dv}\right) - \frac{1}{h^2}\left(\frac{dR}{du}\right) - \frac{s}{h^2u}\left(\frac{dR}{ds}\right); \quad \textbf{(207)}$$

$$0 = \left(\frac{d^2s}{dv^2} + s\right)\left\{1 + \frac{2}{h^2}\int\left(\frac{dR}{dv}\right)\cdot\frac{dv}{u^2}\right\} + \frac{1}{h^2u^2}\cdot\frac{ds}{dv}\left(\frac{dR}{dv}\right) - \frac{s}{h^2u}\left(\frac{dR}{du}\right) - \frac{(1+s^2)}{h^2u^2}\left(\frac{dR}{ds}\right).$$

In the determination of these equations no quantities have been neglected, therefore the influence of such terms as may be omitted in the final result can be fully appreciated.

691. In order to develop[10] the disturbing forces represented by R, the action of the sun alone will be first considered, assuming the masses of the three bodies to be spherical, and $m + \mathrm{E} = 1$. If x', y', z', be the co-ordinates of the sun, and r' its radius vector, then

$$\frac{1}{\sqrt{(x'-x)^2 + (y'-y)^2 + (z'-z)^2}} = \frac{1}{\sqrt{r^2 + r'^2 - 2xx' - 2yy' - 2zz'}}$$

and the second member developed according to the powers of $\frac{1}{r'}$ is

$$\frac{1}{r'}+\frac{xx'+yy'+zz'-\frac{1}{2}r^2}{r'^3}+\frac{3}{2}\frac{\left(xx'+yy'+zz'-\frac{1}{2}r^2\right)^2}{r'^5}+\text{\&c.}$$

692. Since the earth is assumed to be a sphere, its radius may be taken equal to unity, and therefore the solar parallax will be $\frac{1}{r'}$; and if $u'=\frac{1}{r'}$, then $r'=\frac{\sqrt{1+s'^2}}{u'}$. But the ecliptic may be taken for the plane of projection, although it be not fixed; for in its secular motion it carries the orbit of the moon with it, as will be shown afterwards, so that the mean inclination of this orbit on the ecliptic remains constant, and the phenomenon depending on the relative inclination of the two planes are always the same; hence $s'=0$, and therefore $z'=0$, and the co-ordinates of the sun are

$$x'=\frac{\cos v'}{u'}\quad y'=\frac{\sin v'}{u'}.$$

693. Now the distance of the sun from the earth being nearly 400 times greater than that of the moon, $u'=\frac{1}{r'}$ is very small in comparison of $u=\frac{1}{r}$, consequently in the lunar theory u'^5 may be omitted; and if the preceding values of x, y, z; x', y', $\frac{1}{r}$, and $\frac{1}{r'}$, be substituted in R, it becomes

$$\begin{aligned}R=&+\frac{u}{\sqrt{1+s^2}}+m'u'+\frac{m'u'^3}{4u^2}\left\{1+3\cos\left(2v-2v'\right)-2s^2\right\}\\&+\frac{m'u'^4}{8u^3}\left\{3\left(1-4s^2\right)\cos\left(v-v'\right)+5\cos\left(3v-3v'\right)\right\}.\end{aligned}\qquad\textbf{(208)}$$

694. But the quantities u, u', v', and s, in the elliptical hypothesis, become

$$u+du,\ u'+du',\ v'+dv',\ s+ds,$$

in the troubled orbit; and as the mass of the sun is so great that the second powers of the disturbing forces must be taken into account, the co-ordinates of the moon must not only contain R but dR.

695. Now

$$\frac{dR}{du}=\frac{1}{\sqrt{1+s^2}}-\frac{m'u'^3}{2u^3}\left\{1+3\cos\left(2v-2v'\right)-2s^2\right\}-\frac{3m'u'^4}{8u^4}\left\{3\left(1-4s^2\right)\cos\left(v-v'\right)+5\cos\left(3v-3v'\right)\right\}$$

$$\frac{dR}{dv}=-\frac{3m'u'^3}{2u^2}\sin\left(2v-2v'\right)-\frac{3m'u'^4}{8u^3}\left\{\left(1-4s^2\right)\sin\left(v-v'\right)+5\sin\left(3v-3v'\right)\right\}$$

$$\frac{dR}{ds} = -\frac{su}{\left(1+s^2\right)^{\frac{3}{2}}} - \frac{m'u'^3}{u^2}\cdot s - \frac{3m'u'^4}{u^3}\cdot s\cdot\cos\left(v-v'\right);$$

and if the approximation be only carried to terms of the third order inclusively, the co-ordinates of the moon in her troubled orbit will be

$$\begin{aligned}
\frac{d^2u}{dv^2}+u = &+\frac{1}{h^2\left(1+s^2\right)^{\frac{3}{2}}} - \frac{m'u'^3}{2h^2u^3} - \frac{3m'u'^3}{2h^2u^3}\cdot\cos\left(2v-2v'\right) + \frac{3m'u'^3}{2h^2u^4}\cdot\frac{du}{dv}\sin\left(2v-2v'\right)\\
&-\frac{9}{8}\frac{m'u'^4}{h^2u^4}\cos\left(v-v'\right) + \left(\frac{d^2u}{dv^2}+u\right)\frac{3m'}{h^2}\int\frac{u'^3dv}{u^4}\sin\left(2v-2v'\right)\\
&+\left(\frac{d^2u}{dv^2}+u\right)\frac{3m'}{4h^2}\int\frac{u'^4dv}{u^5}\sin\left(v-v'\right)\\
&+d\left\{\begin{array}{l}
\frac{1}{h^2\left(1+s^2\right)^{\frac{3}{2}}} - \frac{m'u'^3}{2h^2u^3}\\
-\frac{3m'u'^3}{2h^2u^3}\cos\left(2v-2v'\right)\\
+\frac{3m'u'^3}{2h^2u^4}\cdot\frac{du}{dv}\sin\left(2v-2v'\right)\\
-\frac{3m'u'^4}{8h^2u^4}\left\{3\cos\left(v-v'\right)+5\cos\left(3v-3v'\right)\right\}\\
+\frac{m'u'^4}{8h^2u^5}\cdot\frac{du}{dv}\left\{3\sin\left(v-v'\right)+15\sin\left(3v-3v'\right)\right\}\\
+\frac{3m'}{h^2}\left(\frac{d^2u}{dv^2}+u\right)\left\{\int\frac{u'^3}{u^4}dv\sin\left(2v-2v'\right)+\tfrac{1}{4}\int\frac{u'^4}{u^5}dv\sin\left(v-v'\right)\right\}\\
+\frac{3m'}{h^2}\left\{\frac{d^2u}{dv^2}+u\right\}\left\{\int\frac{u'^4}{4u^5}dv\,.5\sin\left(3v-3v'\right)\right\}.
\end{array}\right\}
\end{aligned} \qquad \textbf{(209)}$$

$$\begin{aligned}
\frac{d^2s}{dv^2}+s = &-\frac{3m'u'^3}{2h^2u^4}s - \frac{3m'u'^3}{2h^2u^4}s\,.\cos\left(2v-2v'\right) + \frac{3m'u'^3}{2h^2u^4}\frac{ds}{dv}\cdot\sin\left(2v-2v'\right)\\
&+\frac{3m'u'^4}{2h^2u^5}\left\{11us\cos\left(v-v'\right)+\tfrac{1}{4}\frac{ds}{dv}\sin\left(v-v'\right)\right\}\\
&+\frac{3m'}{h^2}\left(\frac{d^2s}{dv^2}+s\right)\cdot\int\frac{u'^3dv}{u^4}\sin\left(2v-2v'\right)
\end{aligned} \qquad \textbf{(210)}$$

$$-d\left\{\begin{array}{l}\frac{3m'u'^3}{2h^2u^4}s+\frac{3m'u'^3}{2h^2u^4}\cos(2v-2v') \\ -\frac{3m'u'^3}{2h^2u^4}\cdot\frac{ds}{dv}\cdot\sin(2v-2v') \\ -\frac{3m'}{h^2}\left(\frac{d^2s}{dv^2}+s\right)\int\frac{u'^3dv}{u^4}\sin(2v-2v')\end{array}\right\}$$

[and]

$$dt=\frac{dv}{h^2(u+du)^2\sqrt{1-\frac{3m'}{h^2}\int\frac{u'^3dv}{u^4}\left\{\sin(2v-2v')+\frac{u'}{4u}\sin(v-v')\right\}}}. \qquad \textbf{(211)}$$

696. These equations contain five unknown quantities, u, v, u', v', and s; but h^2 and v' may be eliminated by their functions in v by integrating the equations (207) when $R=0$, that is, assuming the moon to move without disturbance. By the method already employed, the two last of these equations give

$$s=g\ \sin(v-q)$$
$$u=\frac{1}{h^2(1+g^2)}\left\{\sqrt{1+s^2}+e\cos(v-v\ )\right\}$$

g being the tangent of the inclination of the lunar orbit on the ecliptic, q the longitude of the ascending node, e the eccentricity, and v the longitude of the perigee.

697. In these equations the lunar orbit is assumed to be immovable,[11] but observation shows that the nodes and perigee have a rapid motion in space from the action of the sun; the latter accomplish a revolution in a little more than nine years, so that the lunar ellipse revolves in its own plane in the same direction with the moon's motion; hence if c be such that $1:1-c::v$, the moon's motion in longitude, is to the motion of the apsides, then $v(1-c)$ will be the angle described by the apsis,[12] while the moon describes v. Assuming the instant when the apsis coincided with the axis of x as the origin of the time, then $v-v(1-c)=cv$ will be the moon's true anomaly. In the same manner $(g-1)v$ will represent the retrograde motion of the node, while the moon moves through v. Hence if gv and cv be put for v in the preceding values of s and u, they become

$$s=g\ .\sin(gv-q)$$
$$u=\frac{1}{h^2(1+g^2)}\left\{1+\tfrac{1}{4}g^2+e\cos(cv-v\ )-\tfrac{1}{4}g^2\cos 2(gv-q)\right\} \qquad \textbf{(212)}$$

which are the latitude and parallax of the moon in her orbit considered as a revolving ellipse.

This value of u put in $dt=\frac{dv}{h^2u^2}$, which is the first of equations (207), when $R=0$, gives[13]

$$dt=h^3dv\begin{Bmatrix}1+\tfrac{3}{2}\left(e^2+g^2\right)-2e\left(1+\tfrac{3}{2}e^2+\tfrac{5}{4}g^2\right)\cos\left(cv-v\right)\\+\tfrac{3}{2}e^2\cos\left(2cv-2v\right)-e^3\cos\left(3cv-3v\right)+\tfrac{1}{2}g^2\cos\left(2gv-2q\right)\\-\tfrac{3}{4}eg^2\left\{\cos\left(2gv+cv-v-2q\right)+\cos\left(2gv-cv+v-2q\right)\right\}\end{Bmatrix}$$

its integral is

$$t=\text{constant}+h^3\begin{Bmatrix}v\left(1+\tfrac{3}{2}e^2+\tfrac{3}{2}g^2\right)-\dfrac{2e}{c}\left(1+\tfrac{3}{2}e^2+\tfrac{5}{4}g^2\right)\sin\left(cv-v\right)\\+\dfrac{3e^2}{4c}\sin\left(2cv-2v\right)-\dfrac{e^3}{3c}\sin\left(3cv-3v\right)+\dfrac{g^2}{4g}\sin\left(2gv-2q\right)\\-\dfrac{3eg^2}{4\left(2g+c\right)}\sin\left(2gv+cv-v-2q\right)-\dfrac{3eg^2}{4\left(2g-c\right)}\sin\left(2gv-cv+v-2q\right)\end{Bmatrix}.$$

698. The coefficients are somewhat modified by the action of the sun. In elliptical motion the coefficient of dv is $a^{\frac{3}{2}}$; a being half the greater axis of the lunar orbit, hence

$$h^3\left(1+\tfrac{3}{2}e^2+\tfrac{3}{2}g^2\right)=a^{\frac{3}{2}}.$$

699. Again, because $m=0.0748013$, $c=1-\tfrac{3}{2}m^2=0.991548$, $g=1+\tfrac{3}{4}m^2=1.00402175$, nearly, therefore c and g may be taken equal to unity in the coefficients of the preceding integral, which becomes, when quantities of the order e^3 are rejected and n put for $a^{-\frac{3}{2}}$,

$$\begin{aligned}nt+\in=v-2e\sin\left(cv-v\right)+\tfrac{3}{4}e^2\sin\left(2cv-2v\right)\\+\tfrac{1}{4}g^2\sin\left(2gv-2q\right)-\tfrac{3}{4}eg^2\sin\left(2gv-cv-2q+v\right),\end{aligned}\qquad \textbf{(213)}$$

$\in$ being the arbitrary constant quantity.

700. Now, when quantities of the order g^4 are omitted, the coefficient of the second of equations (212) becomes

$$\frac{1}{h^2\left(1+g^2\right)}=h^{-2}\left(1-g^2\right);$$

but,

$$h^{-2}=\frac{1}{a}\left(1+e^2+g^2+x\right),$$

x being the remaining part of the development of h^{-2}, and therefore of the fourth order in e and g, consequently

$$\frac{1}{h^2\left(1+g^2\right)}=\frac{1}{a}\left(1+e^2+x\right),$$

and the parallax becomes

$$u=\frac{1}{a}\left\{1+e^2+\tfrac{1}{4}g^2+x+e\left(1+e^2\right)\cos\left(cv-v\right)-\tfrac{1}{4}g^2\cos\left(2gv-2q\right)\right\}$$

the constant part of which is

$$\frac{1}{a}\left(1+e^2+\tfrac{1}{4}g^2+x\right);$$

but as this is modified by the action of the sun, it will be expressed by

$$\frac{1}{\bar{a}}\left(1+e^2+\tfrac{1}{4}g^2+x'\right),$$

so that without that action

$$\frac{1}{a}=\frac{1}{\bar{a}};$$

and when quantities of the fourth order are omitted,

$$u=\frac{1}{\bar{a}}\left\{1+e^2+\tfrac{1}{4}g^2+e\left(1+e^2\right)\cos\left(cv-v\right)-\tfrac{1}{4}g^2\cos\left(2gv-2q\right)\right\}. \qquad \textbf{(214)}$$

701. If accented letters are employed for the sun, his parallax and mean longitude will be,

$$u'=\frac{1}{a'}\left\{1+e'^2+e'\left(1+e'^2\right)\cos\left(c'v'-v'\right)\right\} \qquad \textbf{(215)}$$

$$n't+\in'=v'-2e'\sin\left(c'v'-v'\right)+\tfrac{3}{4}e'^2\sin\left(2c'v'-2v'\right). \qquad \textbf{(216)}$$

For $g'=0$ since the sun moves in the plane of the ecliptic, and $g'=1$, $c'=1$, without error in the coefficients.

In order to abridge, let $n't+\in'=v'+f'$, and for the same reason, equation (213) may be expressed by $nt+\in=v+f$. For the elimination of v', suppose the sun and moon to have the same epoch; hence $\in=0$, $\in'=0$, and comparing their mean motions

$$v' = m(v+f) - f', \text{ since } \frac{n'}{n} = m.$$

By the substitution of this in

$$f' = -2e' \sin(c'v' - v') + \tfrac{3}{4}e'^2 \sin(2c'v' - v'),$$

it becomes

$$f' = -2e' \sin\{c'mv - v' + (c'mf - c'f')\} + \tfrac{3}{4}e'^2 \sin\{2c'mv - 2v' + 2(c'mf - c'f')\};$$

or [14]

$$f' = -2e' \sin\{c'mv - v' + c'mf\} \cos c'f' + 2e' \cos\{c'mv - v' + c'mf\} \sin c'f'$$
$$+ \tfrac{3}{4}e'^2 \sin\{2c'mv - 2v' + 2c'mf\} \cos 2c'f' - \tfrac{3}{4}e'^2 \cos\{2c'mv - 2v' + 2c'mf\} \sin 2c'f'.$$

But if

$$c' = 1 \text{ and } \cos c'f' = 1 - \tfrac{1}{2}f'^2 + \&\text{c.}$$
$$\sin c'f' = f' - \tfrac{1}{6}f'^3 + \&\text{c.},$$

then omitting f'^3 the result will be,

$$\begin{aligned} f' = &-2e' \sin(c'mv - v' + c'mf) \\ &+ 2e'f' \cos(c'mv - v' + c'mf) \\ &+ e'f'^2 \sin(c'mv - v' + c'mf) \\ &+ \tfrac{3}{4}e'^2 \sin(2c'mv - 2v' + 2c'mf) \\ &- \tfrac{3}{2}e'^2 \cos(2c'mv - 2v' + 2c'mf) \\ &\quad \&\text{c.} \qquad \&\text{c.} \end{aligned}$$

If substitution be again made for f', and the same process repeated, it will be found, that

$$f' = -e'\left(2 - \tfrac{1}{4}e'^2\right) \sin(c'mv - v') - e'\left(2 - \tfrac{1}{4}e'^2\right) mf \cos(c'mv - v')$$
$$- \tfrac{5}{4}e'^2 \sin(2c'mv - 2v') - \tfrac{5}{2}m'e^2 f \cos(2c'mv - 2v').$$

If this value of f' be put in $v' = m(v+f) - f'$ the value of f restored, and the products of the sines and cosines reduced to the sines of the sums and differences of the arcs, when e^3 is rejected, the result will be

$$\begin{aligned} v' = &+mv - 2me\sin(cv - v) \\ &+\tfrac{3}{4}e^2 m\sin(2cv - 2v) \\ &+\tfrac{1}{4}mg^2\sin(2gv - 2q) \\ &-\tfrac{3}{4}meg^2\sin(2gv - cv + v - 2q) \\ &+2e'\left(1 - \tfrac{1}{8}e'^2\right)\sin(c'mv - v') \\ &-2mee'\sin(cv + c'mv - v - v') \\ &-2mee'\sin(cv - c'mv - v + v') \\ &+\tfrac{5}{4}e'^2\sin(2c'mv - 2v'). \end{aligned} \qquad \textbf{(217)}$$

702. If this value of v' be expressed by $v' = mv + y$, and substituted in equation (215) it becomes

$$u' = \frac{1}{a'}\left\{1 + e'^2 + e'\left(1 + e'^2\right)\cos(c'mv - v' + c'y)\right\}.$$

It will readily appear by the same process, when all powers of the eccentricities above the second are rejected, that

$$u' = \frac{1}{a'}\left\{\begin{aligned} &1 + e'\left(1 - \tfrac{1}{8}e'^2\right)\cos(c'mv - v') + e'^2\cos(2c'mv - 2v') \\ &+mee'\cos(cv - c'mv - v + v') - mee'\cos(cv + cmv - v - v') \end{aligned}\right\}. \qquad \textbf{(218)}$$

703. By the same substitution,

$$\cos(v - v') = \cos(v - m'v)\cos y + \sin(v - mv)\sin y;$$

but

$$\cos y = 1 - \tfrac{1}{2}y^2 + \&\text{c.}, \quad \sin y = y - \tfrac{1}{6}y^3 + \&\text{c.}$$

hence

$$\cos(v - v') = \cos(v - mv) + y\sin(v - mv) - \tfrac{1}{2}y^2\cos(v - mv) - \tfrac{1}{6}y^3\sin(v - mv) + \&\text{c.};$$

and

$$\begin{aligned} \cos(v - v') = &+\cos(v - mv) \\ &-me\cos(v - mv - cv + v) \\ &+me\cos(v - mv + cv - v) \\ &+\tfrac{3}{8}me^2\cos(2cv - v + mv - 2v) \\ &-\tfrac{3}{8}me^2\cos(2cv + v - mv - 2v) \\ &+\tfrac{1}{8}mg^2\cos(2gv - v + mv - 2q) \end{aligned} \qquad \textbf{(219)}$$

$$
\begin{aligned}
&-\tfrac{1}{8}mg^2\cos\left(2gv+v-mv-2q\right)\\
&-\tfrac{3}{8}meg^2\cos\left(v-mv-2gv+cv-v+2q\right)\\
&+\tfrac{3}{8}meg^2\cos\left(v-mv+2gv-cv+v-2q\right)\\
&+e'\left(1-\tfrac{1}{8}e'^2\right)\cos\left(v-mv-c'mv+v'\right)\\
&-e'\left(1-\tfrac{1}{8}e'^2\right)\cos\left(v-mv+c'mv-v'\right)\\
&+\qquad \&c. \qquad \&c.
\end{aligned}
$$

Thus the series expressing $\cos\left(v-v'\right)$ may extend to any powers of the disturbing force and eccentricities.

704. Now

$$
\begin{aligned}
\cos\left(2v-2v'\right)=&+\cos\left(2v-2mv\right)\\
&+2y\sin\left(2v-2mv\right)\\
&-2y^2\cos\left(2v-2mv\right)\\
&-\tfrac{4}{3}y^3\sin\left(2v-2mv\right)\\
&\quad \&c. \qquad \&c.
\end{aligned}
$$

which shows that $\cos\left(2v-2v'\right)$ may be readily obtained from the development of $\cos\left(v-v'\right)$ by putting $2v$ for v, and $2y$ for y; and the same for any cosine, as $\cos i\left(v-v'\right)$.

705. Again, if $90^\circ+v$ be put for v, $\cos\left(v-mv\right)$ becomes

$$\cos\left\{\left(v+90^\circ\right)\left(1-m\right)\right\}=-\sin\left(v-mv\right);$$

hence also the expansion of $\sin\left(v-v'\right)$ may be obtained from the expression (219), and generally the development of $\sin i\left(v-v'\right)$ may be derived from that of $\cos i\left(v-v'\right)$.

Thus all the quantities in the equations of the moon's motions in article 695 are determined, except the variation δu, $\delta u'$, $\delta v'$, and δs.

706. It is evident from the value of $\dfrac{d^2u}{dv^2}+u$ in equation (209), that u is a function of the cosines of all the angles contained in the products of the developments[15] of u, u', $\cos\left(v-v'\right)$, $\cos\left(2v-2v'\right)$, &c.; and δu, being the part of u arising from the disturbing action of the sun, must be a function of the same quantities: hence if A_0, A_1, A_2, &c. be indeterminate coefficients, it may be assumed, that

$$
\begin{aligned}
a\,d\,u = &+A_0 \cdot \cos\left(2v-2mv\right) \\
&+A_1 e \cdot \cos\left(2v-2mv-cv+v\right) \\
&+A_2 e \cdot \cos\left(2v-2mv+cv-v\right) \\
&+A_3 e' . \cos\left(2v-2mv+c'mv-v'\right) \\
&+A_4 e' . \cos\left(2v-2mv-c'mv+v'\right) \\
&+A_5 e' . \cos\left(c'mv-v'\right) \\
&+A_6 ee' . \cos\left(2v-2mv-cv+c'mv+v-v'\right) \\
&+A_7 ee' . \cos\left(2v-2mv-cv-c'mv+v+v'\right) \\
&+A_8 ee' \cdot \cos\left(cv+c'mv-v-v'\right) \\
&+A_9 ee' \cdot \cos\left(cv-c'mv-v+v'\right) \\
&+A_{10} e^2 \cdot \cos\left(2cv-2v\right) \\
&+A_{11} e^2 . \cos\left(2cv-2v+2mv-2v\right) \\
&+A_{12} g^2 \cdot \cos\left(2gv-2q\right) \\
&+A_{13} g^2 \cdot \cos\left(2gv-2v+2mv-2q\right) \\
&+A_{14} e'^2 . \cos\left(2c'mv-2v'\right) \\
&+A_{15} eg^2 \cdot \cos\left(2gv-cv-2q+v\right) \\
&+A_{16} eg^2 . \cos\left(2v-2mv-2gv+cv+2q-v\right) \\
&+A_{17} \frac{a}{a'} . \cos\left(v-mv\right) \\
&+A_{18} \frac{a}{a'} e' . \cos\left(v-mv+c'mv-v'\right) \\
&+A_{19} \frac{a}{a'} e' . \cos\left(v-mv-c'mv+v'\right) \\
&+A_{20} \frac{a}{a'} . \cos\left(3v-3mv\right).
\end{aligned}
\qquad \textbf{(220)}
$$

The term depending on $\cos\left(cv-v\right)$ which arises from the disturbing action of the sun is omitted, because it has already been included in the value of u.

707. It is evident from equation (210) that $d\,s$, the variation of the tangent of the latitude, can only have the form

$$
\begin{aligned}
ds = &+B_0 . g \sin(2v - 2mv - gv + q) \\
&+ B_1 . g \sin(2v - 2mv + gv - q) \\
&+ B_2 . eg \sin(gv + cv - q - v) \\
&+ B_3 . eg \sin(gv - cv - q + v) \\
&+ B_4 . eg \sin(2v - 2mv - gv + cv + q - v) \\
&+ B_5 . eg \sin(2v - 2mv + gv - cv - q + v) \\
&+ B_6 . eg \sin(2v - 2mv - gv - cv + q + v) \\
&+ B_7 . e'g \sin(gv + c'mv - q - v') \\
&+ B_8 . e'g \sin(gv - c'mv - q + v') \\
&+ B_9 . e'g \sin(2v - 2mv - gv + c'mv + q - v') \\
&+ B_{10} . e'g \sin(2v - 2mv + gv - c'mv - q + v') \\
&+ B_{11} . e^2 g \sin(2cv - gv - 2v + q) \\
&+ B_{12} . e^2 g \sin(2v - 2mv - 2cv + gv + 2v - q) \\
&+ B_{13} . e^2 g \sin(2cv + gv - 2v + 2mv - 2v - q) \\
&+ B_{14} . \frac{a}{a'} g \sin(gv - v + mv - q) \\
&+ B_{15} . \frac{a}{a'} g \sin(gv + v - mv - q),
\end{aligned}
\tag{221}
$$

B_0, B_1, &c. being indeterminate coefficients.

708. The variation in the longitude of the earth from the action of the planets troubles the motion of the moon. Equation (216), when $d(nt + \in)$ is put for dv, gives[16]

$$
dv' = md(nt + \in)\left\{1 + 2e' \cos(c'mv - v') - \tfrac{5}{2} e'^2 \cos(2c'mv - 2v')\right\}. \tag{222}
$$

But dv or $d(nt + \in)$, arising from the disturbing force, is entirely independent of equation (213), which belongs to the elliptical motion only; and from equation (211) it appears that if C_6, C_9, &c. be indeterminate coefficients,

$$
\begin{aligned}
d(nt + \in) = &+C_6 \sin(2v - 2mv) \\
&+ C_9 e' \sin(2v - 2mv + c'mv - v') \\
&+ C_{10} e' \sin(2v - 2mv - c'mv + v') \\
&+ \quad \&c. \qquad \&c.
\end{aligned}
\tag{223}
$$

By this value, equation (222) becomes

$$d\,v' = +m\left\{C_6 + C_9 e'^2 + C_{10} e'^2\right\}\sin\left(2v - 2mv\right) \qquad \textbf{(224)}$$
$$+ \qquad \&\text{c.} \qquad \&\text{c.}$$

709. But the longitude of the earth is troubled by the action of the moon as well as by that of the planets, and thus the moon indirectly troubles her own motions. In the theory of the earth it is found that the action of the moon occasions the inequality

$$d\,v' = m\,\frac{r}{r'}\sin\left(v - v'\right)$$

in the earth's longitude, and thus the whole variation of v' is

$$d\,v' = +m\left\{C_6 + C_9 e'^2 + C_{10} e'^2\right\}\sin\left(2v - 2mv\right) + m\,\frac{u'}{u}\sin\left(v - v'\right); \qquad \textbf{(225)}$$

where m is the ratio of the mass of the moon to the sum of the masses of the earth and moon.

710. The parallax of the moon is troubled by both these causes, but that arising from the action of the planets may be omitted at present. The moon's attraction produces the inequality

$$d\,r' = m\,r\cos\left(v - v'\right)$$

in the radius vector of the earth, and consequently the variation

$$d\,u' = -\frac{mu'^2}{u}\cos\left(v - v'\right) \qquad \textbf{(226)}$$

in the solar parallax.

711. Lastly, $\frac{du}{dv}$ is obtained from equation (214).

712. Thus every quantity in the equation of article 695 are determined, and by their substitution, the co-ordinates of the moon will be obtained in her troubled orbit in functions of her true longitude.

The Parallax

713. The substitution of the given quantities in the differential equation (209) of the parallax is extremely simple, though tedious. The first term

$$-\frac{1}{h^2\left(1+s^2\right)^{\frac{3}{2}}}=-\frac{1}{h^2}\left(1-\tfrac{3}{2}s^2\right)$$

when the higher powers of s^2 are omitted; putting

$$\frac{1}{\bar{a}}\left(1+e^2+g^2+x\right)\text{ for }h^{-2}$$

and

$$\tfrac{1}{2}g^2-\tfrac{1}{2}g^2\cos\left(2gv-2q\right)\text{ for }s^2$$

becomes

$$-\frac{1}{h^2\left(1+s^2\right)^{\frac{3}{2}}}=-\frac{1}{\bar{a}}\left\{1+e^2+\tfrac{1}{4}g^2+x+\tfrac{3}{4}g^2\left(1+e^2-\tfrac{1}{4}g^2\right)\cos\left(2gv-2q\right)\right\}.$$

Again,

$$u'^3=\frac{1}{a'^3}\left\{1+\tfrac{3}{2}e'^2+3e'\cos\left(c'mv-v'\right)+\&\text{c.}\right\}$$

$$u^{-3}=a^3\left\{1-\tfrac{3}{4}g^2-3e\cos\left(cv-v\right)+\&\text{c.}\right\};$$

and as by article 685,

$$\frac{m'a^3}{a'^3}=\bar{m}^2$$

$$\frac{m'u^3}{2h^2u^3}=\frac{\bar{m}^2}{2\bar{a}}\left\{1+e^2+\tfrac{1}{4}g^2+\tfrac{3}{2}e'^2-3e\left(1+\tfrac{1}{2}e^2+\tfrac{3}{2}e'^2\right)\cos\left(cv-v\right)+\&\text{c.}\right\}$$

In this and all the other terms, x is omitted, being of the fourth order in e and g .

714. Terms of the form $\frac{9m'u'^4}{8h^2u^4}\cos\left(v-v'\right)$ become

$$+\frac{9\bar{m}^2}{8\bar{a}}\cdot\frac{a}{a'}\left(1+2e^2+2e'^2\right)\cos\left(v-mv\right)$$

$$+\frac{9\bar{m}^2}{8\bar{a}}\cdot\frac{a}{a'}e'\cos\left(v-mv+c'mv-v'\right)$$

$$+\frac{27\bar{m}^2}{8\bar{a}}\cdot\frac{a}{a'}e'\cos\left(v-mv-c'mv+v'\right);$$

and, by comparing their coefficients with observation, serve for the determination of $\frac{a}{a'}$, the ratio of the parallax of the sun to that of the moon; but as it is a very small quantity, any error would be sensible, and on that account the approximation must extend to quantities of the fifth order inclusively with regard to the angle $v - v'$; but in every other case, it will only be carried to quantities of the third order.

715. Attending to these circumstances, and observing that in the variation of $\frac{1}{h^2\left(1+s^2\right)^{\frac{3}{2}}}$ the square of ds must be included, so that

$$d\frac{1}{h^2\left(1+s^2\right)^{\frac{3}{2}}} = -\frac{3s\,ds}{h^2} + \frac{3}{2\bar{a}}ds^2$$

and as

$$\frac{\bar{m}^2}{\bar{a}} = \frac{m^2}{a},$$

it will readily be found, that

$$\begin{aligned}
0 = \frac{d^2u}{dv^2} &+ u \\
&- b_0 \\
&- b_1 e\cos\left(cv - v\ \right) \\
&+ b_2 \cos\left(2v - 2mv\right) \\
&+ b_3 e\cos\left(2v - 2mv - cv + v\ \right) \\
&- b_4 e\cos\left(2v - 2mv + cv - v\ \right) \\
&- b_5 e'\cos\left(2v - 2mv + c'mv - v'\right) \\
&+ b_6 e'\cos\left(2v - 2mv - c'mv + v'\right) \\
&+ b_7 e'\cos\left(c'mv - v'\right) \\
&+ b_8 ee'\cos\left(2v - 2mv - cv + c'mv + v\ - v'\right) \\
&- b_9 ee'\cos\left(2v - 2mv - cv - c'mv + v\ + v'\right) \\
&- b_{10} ee'\cos\left(cv + c'mv - v\ - v'\right) \\
&- b_{11} ee'\cos\left(cv - c'mv - v\ + v'\right) \\
&+ b_{12} e^2\cos\left(2cv - 2v\ \right) \\
&+ b_{13} e^2\cos\left(2cv - 2v + 2mv - 2v\ \right) \\
&- b_{14} g^2\cos\left(2gv - 2q\ \right) \\
&+ b_{15} g^2\cos\left(2gv - 2v + 2mv - 2q\ \right) \\
&+ b_{16} e'^2\cos\left(2c'mv - 2v'\right)
\end{aligned} \qquad \textbf{(227)}$$

$$
\begin{aligned}
&-b_{17}eg^{2}\cos\left(2gv-cv-2q+v\right)\\
&-b_{18}eg^{2}\cos\left(2v-2mv-2gv+cv+2q-v\right)\\
&+b_{19}\frac{a}{a'}\cos\left(v-mv\right)\\
&+b_{20}e'\frac{a}{a'}\cos\left(v-mv+c'mv-v'\right)\\
&+b_{21}e'\frac{a}{a'}\cos\left(v-mv-c'mv+v'\right).
\end{aligned}
$$

716. The coefficients being[17]

$$
\begin{aligned}
b_0 &= +\frac{1}{\bar{a}}\left\{1+e^2+\tfrac{1}{4}g^2+x\right\}-\frac{\bar{m}^2}{2\bar{a}}\left\{1+e^2+\tfrac{1}{4}g^2+\tfrac{3}{2}e'^2\right\}\\
&\quad+\frac{3\bar{m}^2}{4\bar{a}}\left(4-3m-m^2\right)A_0\left(1-\tfrac{5}{2}e'^2\right)-\frac{3}{4\bar{a}}B_0^{2}g^2
\end{aligned}
$$

$$
b_1=\frac{3m^2}{4a}\left\{\begin{aligned}
&+2+e^2+3e'^2-2\left(B_2+B_3\right)\frac{g^2}{m^2}+\left(1+2m-c\right)A_0\left(1-\tfrac{5}{2}e'^2\right)\\
&-4\left\{1+2m+\left(4\left(1-m^2\right)-1\right)\left(\frac{1+m}{2-2m-c}+\frac{1-m}{2-2m+c}\right)\right\}\times A_0\left(1-\tfrac{5}{2}e'^2\right)\\
&+\frac{1}{1-m}\left\{\left(1+6m+c\right)\left(1+m\right)+7+\left(2-2m-c\right)^2\right\}A_1\left(1-\tfrac{5}{2}e'^2\right)\\
&-\tfrac{1}{2}\left(9+m+c\right)A_6\,.\,e'^2+\tfrac{7}{2}\left(9+3m+c\right)A_7e'^2+3\left(A_8+A_9\right).e'^2
\end{aligned}\right\}
$$

$$
b_2=\frac{3m^2}{4a}\left\{1+\left(1+2m\right)e^2+\tfrac{1}{4}g^2-\tfrac{5}{2}e'^2+\frac{1}{1-m}\left(1+3e^2+\tfrac{1}{4}g^2-\tfrac{5}{2}e'^2\right)-A_0-\left(B_0-B_1\right)\frac{g^2}{m^2}\right\}
$$

$$
b_3=\frac{3m^2}{a}\left\{\begin{aligned}
&-\tfrac{1}{4}\left(3+4m\right)\left(1+\tfrac{1}{2}e^2-\tfrac{5}{2}e'^2\right)+\frac{1-c^2}{4\left(1-m\right)}\\
&-\frac{2\left(1+m\right)}{2-2m-c}\left(1+\tfrac{7}{4}e^2-\tfrac{5}{2}e'^2\right)-\tfrac{1}{2}\left(A_1-2A_0\right)+\tfrac{1}{2}\left(B_5-B_6\right)\frac{g^2}{m^2}
\end{aligned}\right\}
$$

$$
b_4=\frac{3m^2}{4a}\left\{3+c-4m+\frac{8\left(1-m\right)}{2-2m+c}+2A_2\right\}
$$

$$
b_5=\frac{3m^2}{4a}\left\{\frac{4-m}{2-m}+2B_9\frac{g^2}{m^2}+2A_3\right\}
$$

$$
b_6=\frac{3m^2}{4a}\left\{\frac{7\left(4-3m\right)}{2-3m}-2B_{10}\frac{g^2}{m^2}-2A_4\right\}
$$

$$b_7 = +\frac{3m^2}{4a}\left\{\begin{array}{l} +1+e^2+\tfrac{1}{4}g^2+\tfrac{9}{8}e'^2+\left(B_7+B_8\right)\dfrac{g^2}{m^2}-\tfrac{3}{2}(1+2m)A_0 \\ -\dfrac{2(1-2m)(3-2m)(3-m)}{(2-3m)(2-m)}A_0-2A_3-(2-3m)A_4 \\ +\left(B_9+B_{10}\right)B_0\dfrac{g^2}{m^2}-A_5-11C_6-2C_9+2C_{10} \end{array}\right\}$$

$$+\frac{6m'}{a}\left\{4A_0+A_3-A_4-10A_1e^2+\tfrac{5}{2}(A_7-A_6)e^2\right\}$$

$$b_8 = \frac{3m^2}{4a}\left\{\frac{3+2m-c}{4}+\frac{(2+m)}{2-m-c}-\tfrac{3}{2}A_1-A_6-\left(\frac{3+m-c}{2}+\frac{4}{2-m-c}\right)A_9\right\}$$

$$b_9 = \frac{3m^2}{4a}\left\{\frac{7(3+6m-c)}{4}+\frac{7(2+3m)}{2-3m-c}+\tfrac{3}{2}A_1+A_7+\left(\frac{3-m-c}{2}+\frac{4}{2-3m-c}\right)A_8\right\}$$

$$b_{10} = \frac{3m^2}{4a}\left\{\frac{3+2m}{2}-\left(\frac{1+2m+c}{4}+\frac{2}{c+m}\right)A_1+A_8+\left(\frac{1+3m+c}{2}+\frac{4}{c+m}\right)A_7\right\}$$

$$b_{11} = \frac{3m^2}{4a}\left\{\frac{3-2m}{2}+A_9+7\left(\frac{1+2m+c}{4}+\frac{2}{c-m}\right)A_1+\left(\frac{1+m+c}{2}+\frac{4}{c-m}\right)A_6\right\}$$

$$b_{12} = \frac{3m^2}{4a}\left\{1-B_{11}\cdot\frac{g^2}{m^2}-A_{10}\right\}$$

$$b_{13} = \frac{3m^2}{4a}\left\{\frac{2+11m+8m^2}{2}-\frac{(10+19m+8m^2)}{2c-2+2m}+4A_1+\frac{\left\{8A_{10}+10A_1^2\right\}}{2c-2+2m}-2A_{11}\right\}$$

$$b_{14} = \frac{3}{4a}\left\{1+e^2-\tfrac{1}{4}g^2-\tfrac{1}{2}m^2+2m^2A_{12}\right\}$$

$$b_{15} = \frac{3m^2}{4a}\left\{\frac{1+2m-2g}{4}+\frac{(4g^2-1)}{4(1-m)}-\frac{(2+m)}{2g-2+2m}+\frac{2B_0}{m^2}-2A_{13}+\frac{8A_{12}}{2g-2+2m}\right\}$$

$$b_{16} = \frac{3m^2}{4a}\left(\tfrac{3}{2}-A_{14}\right)$$

$$b_{17} = \frac{3m^2}{4a}\left\{\tfrac{1}{2}+\frac{B_3}{m^2}+\frac{(1+c-2g-10m)}{4}A_1-(10+5m)A_{13}+(5+m)A_{16}-\frac{1}{m^2}B_0B_5+A_{15}\right\}$$

$$b_{18} = \frac{3m^2}{4a}\left\{1+2m+\frac{(5+m)}{1-2m}-\frac{3(1-m)}{3-2m}+2A_{16}-\frac{2}{m^2}B_4+\frac{10}{1-2m}A_{15}\right\}$$

$$b_{19} = \frac{m^2}{a}\left\{\begin{array}{l} +\left\{\tfrac{9}{8}(1-2m)\left(1+2e^2+2e'^2\right)+\dfrac{3(1-2m)\left(1+\tfrac{9}{2}e^2+2e'^2\right)}{4(1-m)}+\dfrac{3(1+m)}{2(1-m)}\right\}\times A_{18}e'^2 \\ -\dfrac{(36+21m-15m^2)}{4(1-m)}A_{17}-\dfrac{(57-38m)}{4(1-m)}A_0+\tfrac{3}{2}\left(B_{14}+B_{15}\right)\dfrac{g^2}{m^2} \end{array}\right\}$$

$$b_{20} = \frac{3m^2}{4a}\left\{\frac{5(1-2m)}{4} - A_{18} + \frac{(4+m)}{4}A_{17} - (5+m)A_{19}\right\}$$

$$b_{21} = \frac{3m^2}{2a(1-2m)}\left\{\frac{15-18m}{4}(1-2m) - \frac{76-33m}{m}A_{17} - 5A_{18} - (1-2m)A_{19}\right\}.$$

717. The integral of the preceding equation is evidently

$$u = \frac{1}{a}\left\{\begin{matrix} +1+e^2+\tfrac{1}{4}g^2+x+e(1+e^2)\cos(cv-v) \\ -\tfrac{1}{4}g^2\left(1+e^2-\tfrac{1}{4}g^2\right)\cos(2gv-2q) \end{matrix}\right\} + du. \qquad \textbf{(228)}$$

Where du is given by equation (220).

718. In order to find values of the indeterminate coefficients A_0, A_1, &c., this value of u must be substituted in equation (227); but to determine the unknown quantity c, both e and v must vary in the term $e(1+e^2)\cos(cv-v)$, which expresses the motion of the perigee. Hence, when $\frac{d\frac{e}{a}}{dv^2}$ is omitted, a comparison of the coefficients of corresponding sines and cosines gives

$$\textbf{(229)}$$

$$0 = 1+e^2+\tfrac{1}{4}g^2+x-ab_0$$

$$0 = 1-\left(c-\frac{dv}{dv}\right)^2 - \frac{ab_1}{1+e^2}$$

$$0 = \frac{e(1+e^2)}{a}\cdot\frac{d^2v}{dv^2} - 2\left(c-\frac{dv}{dv}\right)\frac{d\cdot e\frac{(1+e^2)}{a}}{dv}$$

$$0 = A_0\left(1-4(1-m)^2\right)+ab_2$$

$$0 = A_1\left(1-(2-2m-c)^2\right)+ab_3$$

$$0 = A_2\left(1-(2-2m+c)^2\right)-ab_4$$

$$0 = A_3\left(1-(2-m)^2\right)-ab_5$$

$$0 = A_4\left(1-(2-3m)^2\right)+ab_6$$

$$0 = A_5\left(1-m^2\right)+ab_7$$
$$0 = A_6\left(1-\left(2-m-c\right)^2\right)+ab_8$$
$$0 = A_7\left(1-\left(2-3m-c\right)^2\right)-ab_9$$
$$0 = A_8\left(1-\left(c+m\right)^2\right)-ab_{10}$$
$$0 = A_9\left(1-\left(c-m\right)^2\right)-ab_{11}$$
$$0 = A_{10}\left(1-4c^2\right)+ab_{12}$$
$$0 = A_{11}\left(1-\left(2c-2+2m\right)^2\right)+ab_{13}$$
$$0 = A_{12}\left(1-4g^2\right)+ab_{14}$$
$$0 = A_{13}\left(1-\left(2g-2+2m\right)^2\right)+ab_{15}$$
$$0 = A_{14}\left(1-4m^2\right)+ab_{16}$$
$$0 = A_{15}\left(1-\left(2g-c\right)^2\right)-ab_{17}$$
$$0 = A_{16}\left(1-\left(2-2m-2g+c\right)^2\right)-ab_{18}$$
$$0 = A_{17}\left(1-\left(1-m\right)^2\right)+ab_{19}$$
$$0 = ab_{20}$$
$$0 = A_{19}\left(1-\left(1-2m\right)^2\right)+ab_{21}$$
$$0 = A_{20}\left(1-\left(3-3m\right)^2\right).$$

719. The secular inequalities in the form of the lunar orbit are derived from the three first of these equations; from the rest are obtained values of the indeterminate coefficients A_0, A_1, &c. &c. It is evident that these coefficients will be more correct, the farther the approximation is carried in the development of equation (209).

Secular Inequalities in the Lunar Orbit

720. When the action of the sun is omitted, by article 685, $\frac{1}{a}=\frac{1}{\bar{a}}$; and x, being of the fourth order, may be omitted: hence $1+e^2+\frac{1}{4}g^2-ab_0=0$ becomes

$$\frac{1}{a}=\frac{1}{\bar{a}}-\frac{\bar{m}^2}{2\bar{a}}\left(1+\tfrac{3}{2}e'^2\right)+\frac{3\bar{m}^2}{4\bar{a}}\left(1-\tfrac{5}{2}e'^2\right)\left(4-3m-m^2\right)A_0-\frac{3}{4\bar{a}}B_0^2g^2. \qquad \textbf{(230)}$$

Since a is the mean distance of the moon from the earth, or half the greater axis of the lunar orbit, the constant part of the moon's parallax is proportional to $\frac{1}{a}$. But the action of the planets produces a secular variation in e', the eccentricity of the terrestrial orbit, without affecting $2a'$, the greater axis. The preceding value of $\frac{1}{a}$ must therefore be subject to a secular inequality, in consequence of the variation of the term $-\frac{3\bar{m}^2}{4\bar{a}}e'^2$; but this variation will always be insensible.

721. The motion of the perigee may be obtained from the second of equations (229), put under the form

$$1-\left(c-\frac{d\varpi}{dv}\right)^2-p-p'e'^2=0\,;$$

for since b_1 is a function of e'^2, the quantity $\frac{ab_1}{1+e^2}$ may be expressed by $p+p'e'^2$.

If $\frac{d\varpi}{dv}$ be omitted, $c=\sqrt{1-p-p'.e'^2}$, so that c varies in consequence of e'^2. Now

$$\frac{d\varpi}{dv}=c-\sqrt{1-p}+\frac{p'e'^2}{2\sqrt{1-p}}\,,$$

the integral of which is

$$\varpi=cv-v\sqrt{1-p}+\frac{p'}{2\sqrt{1-p}}\int e'^2dv+\in\,;$$

for e'^2 is variable, and p, p' may be regarded as constant, without sensible error, as appears from the value of b_1, and $\in$ is a constant quantity, introduced by integration; hence

$$\cos\left(cv-\varpi\right)=\cos\left\{v\sqrt{1-p}-\frac{p'}{2\sqrt{1-p}}\int e'^2dv-\in\right\}. \qquad \textbf{(231)}$$

722. Thus, from theory, we learn that the perigee has a motion equal to

$$\left(1-\sqrt{1-p}\right)v+\frac{p'}{2\sqrt{1-p}}\int e'^2dv\,,$$

which is confirmed by observation; but this motion is subject to a secular inequality, expressed by

$$\frac{p'}{2\sqrt{1-p}}\int e'^2 dv\,, \qquad \textbf{(232)}$$

on account of the variation in e'^2, the eccentricity of the earth's orbit.

In consequence of the preceding value of c, v is equal to the constant quantity $\in$, together with the secular equation of the motion of the perigee.

723. The eccentricity of the moon's orbit is affected by a secular variation similar to that in the parallax, and proportional to $\frac{d\,\mathrm{v}}{dv}$, but as the variations of the latter are only sensible in the integral $\int \frac{d\,\mathrm{v}}{dv}dv$, the eccentricity of the lunar orbit may be regarded as constant.

Latitude of the Moon

724. The development[18] of the parallax will greatly assist in that of the latitude, as most of the terms differ only in being multiplied by s, its variation, or its differentials; and by substitution of the requisite quantities in equation (210), it will readily be found, when all the powers of the eccentricities and inclination above the cubes are omitted, that

$$\begin{aligned}
0 = &+\frac{d^2s}{dv^2}+s \qquad \textbf{(233)}\\
&+a_0 g\sin(gv-q)\\
&-a_1 g\sin(2v-2mv-gv+q)\\
&+a_2 g\sin(2v-2mv+gv-q)\\
&+a_3 eg\sin(gv+cv-q-\varpi)\\
&+a_4 eg\sin(gv-cv-q+\varpi)\\
&+a_5 eg\sin(2v-2mv-gv+cv+q-\varpi)\\
&+a_6 eg\sin(2v-2mv+gv-cv-q+\varpi)\\
&+a_7 eg\sin(2v-2mv-gv-cv+q+\varpi)\\
&+a_8 e'g\sin(gv+c'mv-q-\varpi')\\
&+a_9 e'g\sin(gv-c'mv-q+\varpi')\\
&+a_{10} e'g\sin(2v-2mv-gv+c'mv+q-\varpi')\\
&+a_{11} e'g\sin(2v-2mv-gv-c'mv+q+\varpi')\\
&+a_{12} e^2g\sin(2cv-gv-2q+\varpi)\\
&+a_{13} e^2g\sin(2v-2mv-2cv+gv+2\varpi-q)
\end{aligned}$$

$$+a_{14}e^2 g\sin\left(2cv+gv-2v+2mv-2v-q\right)$$

$$+a_{15}\frac{a}{a'}g\sin\left(gv-v+mv-q\right)$$

$$+a_{16}\frac{a}{a'}g\sin\left(gv+v-mv-q\right).$$

725. The coefficients in consequence of $\dfrac{\overline{m}^2}{\overline{a}}=\dfrac{m^2}{a}$ being

$$a_0=\frac{3m^2}{2}\left\{\begin{array}{l}+1+2e^2-\tfrac{1}{4}g^2+\tfrac{3}{2}e'^2\\ -\tfrac{1}{2}\left(1-\tfrac{5}{2}e'^2\right)\left(\dfrac{(3-2m-g)(g+m)}{1-m}B_0+4A_0\right)\\ -\tfrac{7}{2}(3-3m-g)B_{10}e'^2+\tfrac{1}{4}(3-m-g)B_9e'^2+\tfrac{3}{2}(B_7+B_8)e'^2\end{array}\right\}$$

$$a_1=\frac{3m^2}{4}\left\{\begin{array}{l}+(1+g)\left(1+2e^2-\tfrac{1}{4}(2+m)g^2-\tfrac{5}{2}e'^2\right)\\ +\dfrac{(1-g^2)}{1-m}-4A_0+10A_1e^2-2B_0\end{array}\right\}$$

$$a_2=\frac{3m^2}{2}\left\{\frac{1-g}{2}+B_1\right\}$$

$$a_3=\frac{3m^2}{2}\left\{B_2-2+(1-m)(3-2m-g)B_0\right\}$$

$$a_4=\frac{3m^2}{2}\left\{B_3-2-2A_1+(1+m)(3-2m-g)B_0\right\}$$

$$a_5=\frac{3m^2}{2}\left\{(1+g)(1-m)-2B_0+B_4\right\}$$

$$a_6=\frac{3m^2}{2}\left\{(g-1)(1+m)+B_5-2A_1\right\}$$

$$a_7=\frac{3m^2}{2}\left\{(1+g)(1+m)+B_6+2A_1-2B_0\right\}$$

$$a_8=\frac{3m^2}{4}\left\{3+2B_7+\tfrac{1}{2}(3-2m-g)B_0-(3-3m-g)B_{10}\right\}$$

$$a_9=\frac{3m^2}{4}\left\{3+2B_8-\tfrac{7}{2}(3-2m-g)B_0-(3-m-g)B_9\right\}$$

$$a_{10}=\frac{3m^2}{4}\left\{\frac{1+g}{2}+2B_9+3B_0-(1+g-m)B_8\right\}$$

$$a_{11}=\frac{3m^2}{4}\left\{2B_{10}-\tfrac{7}{2}(1+g)+3B_0-(1+g+m)B_7\right\}$$

$$a_{12}=\frac{3m^2}{4}\left\{\begin{array}{l}+2B_{11}-5-10A_1+4A_{11}-\left(3-2m-2c+2g\right)B_{12}\\ +\left(10+19m+8m^2\right)B_0\left(\dfrac{3-3m-g}{4}+\dfrac{\left(2-2m-g\right)^2-1}{2\left(2c+2m-2\right)}\right)\end{array}\right\}$$

$$a_{13}=\frac{3m^2}{4}\left\{2B_{12}+\left(1-g\right)\tfrac{1}{4}\left(10+19m+8m^2\right)+10A_1-4A_{11}-2B_{11}\right\}$$

$$a_{14}=\frac{3m^2}{4}\left\{\tfrac{1}{2}\left(10+19m+8m^2\right)+2B_{13}+10A_1-4A_{11}-5B_0\right\}$$

$$a_{15}=\frac{3m^2}{4}\left\{3+2B_{14}\right\}$$

$$a_{16}=\frac{3m^2}{4}\left\{\tfrac{5}{2}+2B_{15}\right\}.$$

726. The integral of the differential equation of the latitude is

$$s=g\sin\left(gv-q\right)+\delta s;\ \delta s \text{ is given in (221).} \qquad \textbf{(234)}$$

If this quantity be substituted in equation (233) instead of *s*, a comparison of the coefficients of like sines and cosines will furnish a sufficient number of equations; whence the indeterminate coefficients B_0, B_1, &c. will be known, but in order to find a value of the unknown quantity *g*, both q and g must vary in the terms $g\sin\left(gv-q\right)$ in taking the differentials of *s*. Attending to these circumstances, it will readily be found that,[19]

$$2\frac{dg}{dv}\left(g-\frac{dq}{dv}\right)-g\frac{d^2q}{dv^2}=0 \qquad \textbf{(234)}$$

$$g-g\left(g-\frac{dq}{dv}\right)^2+\frac{d^2g}{dv^2}+g\,a_0=0$$

$$B_0\left(1-\left(2-2m-g\right)^2\right)-a_1=0$$

$$B_1\left(1-\left(2-2m+g\right)^2\right)+a_2=0$$

$$B_2\left(1-\left(g+c\right)^2\right)+a_3=0$$

$$B_3\left(1-\left(g-c\right)^2\right)+a_4=0$$

$$B_4\left(1-\left(2-2m+c-g\right)^2\right)+a_5=0$$

$$B_5\left(1-\left(2-2m-c+g\right)^2\right)+a_6=0$$

$$B_6\left(1-\left(2-2m-c-g\right)^2\right)+a_7=0$$

$$B_7\left(1-(g+m)^2\right)+a_8=0$$
$$B_8\left(1-(g-m)^2\right)+a_9=0$$
$$B_9\left(1-(2-m-g)^2\right)+a_{10}=0$$
$$B_{10}\left(1-(2-3m-g)^2\right)+a_{11}=0$$
$$B_{11}\left(1-(2c-g)^2\right)+a_{12}=0$$
$$B_{12}\left(1-(2-2m-2c+g)^2\right)+a_{13}=0$$
$$B_{13}\left(1-(2c+g-2+2m)^2\right)+a_{14}=0$$
$$B_{14}\left(1-(g-1+m)^2\right)+a_{15}=0$$
$$B_{15}\left(1-(g+1-m)^2\right)+a_{16}=0.$$

The two first of these equations will give the secular variations in the nodes and inclination of the orbit, the rest serve for the determination of the coefficients B_0, B_1, &c.

Secular Inequalities in the Position of the Lunar Orbit

727. The coefficient a_0 may be represented by $q+q'e'^2$, then the second of the equations in the last article becomes

$$\frac{d^2g}{dv^2}+g\left(1-\left(g-\frac{dq}{dv}\right)^2\right)+g\left(q+q'e'^2\right)=0\,;$$

q' is a function of A_0 and B_0; and as these are functions of $1-\frac{5}{2}e'^2$, therefore $q'e'^2$ may be omitted, as well as $\frac{d^2g}{dv^2}$, which is insensible, and neglecting $\frac{dq}{dv}$ in the first instance,

$$g=\sqrt{1-q-q'e'^2},$$

so that g varies with e'^2. But

$$\frac{dq}{dv}=g-\sqrt{1+q}-\frac{q'}{2\sqrt{1+q}}e'^2\,;$$

and as q and q' may be regarded as constant, the integral is

$$q = gv - v\sqrt{1+q} - \frac{q'}{2\sqrt{1+q}}\int e'^2 dv + a \ ,$$

a being a constant quantity introduced by integration; hence

$$\sin\left(gv - q\right) = \sin\left\{v\sqrt{1+q} + \frac{q'}{2\sqrt{1+q}}\int e'^2 dv - a\right\}, \qquad \textbf{(235)}$$

which shows the nodes of the lunar orbit to have a retrograde motion on the true ecliptic equal to

$$\left(\sqrt{1+q} - 1\right)v + \frac{q'}{2\sqrt{1+q}}\int e'^2 dv \, ,$$

which accords with observation. This motion is not uniform, but is affected by a secular inequality expressed by

$$\frac{q'}{2\sqrt{1+q}}\int e'^2 dv \, , \qquad \textbf{(236)}$$

corresponding to the secular variation of e', the eccentricity of the terrestrial orbit.

728. The first of the equations (234) determines the inclination of the lunar orbit on the plane of the ecliptic. Its integral is

$$g = \left\{H\left(g - \frac{dq}{dv}\right)\right\}^{-\frac{1}{2}} .$$

H being an arbitrary constant quantity. Hence it appears that the inclination is subject to a secular inequality; but as it is quite insensible, the inclination g may be regarded as constant, which is the reason why the most ancient observations do not indicate any change in the inclination of the lunar orbit on the plane of the ecliptic, although the position of the ecliptic has varied sensibly during that interval.

The Mean Longitude of the Moon

729. When the square root is extracted, equation (211) becomes,

$$dt = \frac{dv}{h^2\left(u^2 + 2udu + du^2\right)}\left\{1 - \frac{3m}{h^2}\int\frac{u'^3 dv}{u^4}\sin\left(2v - 2v'\right) + \frac{3m}{2h^4}\left(\int\frac{u^3 dv}{u^4}\sin\left(2v - 2v'\right)\right)^2 - \&\text{c.}\right\};$$

and, making the necessary substitutions there will result [20]

$$\begin{aligned} dt = \frac{a^2 dv}{\sqrt{\bar{a}}}\{ & x_0 + x_1 e\cos(cv - v) \\ & + x_2 e^2 \cos(2cv - 2v) \\ & + x_3 e^3 \cos(3cv - 3v) \\ & + x_4 g^2 \cos(2gv - 2q) \\ & + x_5 eg^2 \cos(2gv - cv - 2q + v) \\ & + x_6 eg^2 \cos(2gv + cv - 2q - v) \\ & + x_7 \cos(2v - 2mv) \\ & + x_8 e\cos(2v - 2mv - cv + v) \\ & + x_9 e\cos(2v - 2mv + cv - v) \\ & + x_{10} e' \cos(2v - 2mv + c'mv - v') \\ & + x_{11} e' \cos(2v - 2mv - c'mv + v') \\ & + x_{12} e' \cos(c'mv - v') \\ & + x_{13} ee' \cos(2v - 2mv - cv + c'mv + v - v') \\ & + x_{14} ee' \cos(2v - 2mv - cv - c'mv + v + v') \\ & + x_{15} ee' \cos(cv + c'mv - v - v') \\ & + x_{16} ee' \cos(cv - c'mv - v + v') \\ & + x_{17} e^2 \cos(2cv - 2v + 2mv - 2v) \\ & + x_{18} g^2 \cos(2gv - 2v + 2mv - 2q) \\ & + x_{19} e'^2 \cos(2c'mv - 2v') \\ & + x_{20} \frac{a}{a'} \cos(v - mv) \\ & + x_{21} \frac{a}{a'} e' \cos(v - mv + c'mv - v')\}. \end{aligned} \tag{237}$$

730. The coefficients of which are[21]

$$x_0 = 1 + \frac{27m^4}{64(1-m)^2} + \frac{3m^2 \cdot A_0}{4(1-m)} + \tfrac{3}{2}\{A_2^2 + A_1^2 e^2\}$$

$$x_1 = -2\left(1 - \tfrac{1}{4}g^2\right) + \frac{15m^2}{4(1-m)} A_1 + 3A_0 \cdot A_1$$

$$x_2 = \tfrac{3}{2} + \tfrac{1}{4}e^2 - \tfrac{3}{2}g^2 - 2A_{10}$$

$$x_3 = -1$$

$$x_4 = \tfrac{1}{2}\left(1+\tfrac{3}{2}e^2 - \tfrac{1}{2}g^2 - 2A_{12} + 3A_{15}e^2\right)$$

$$x_5 = -\tfrac{3}{4} - 2A_{15}$$

$$x_6 = -\tfrac{3}{4}$$

$$x_7 = -\frac{3m^2\left(1+2e^2+\tfrac{5}{2}e'^2\right)}{4(1-m)} - 3m^2e^2\left\{\frac{1+m}{2-2m-c} + \frac{1-m}{2-2m+c}\right\}$$

$$-2A_0\left(1+\tfrac{1}{2}e^2 - \tfrac{1}{4}g^2\right) + 3e^2A_1 + 3e^2A_2$$

$$x_8 = +\frac{3m^2\left(1+2e^2 - \tfrac{1}{4}g^2 - \tfrac{5}{2}e'^2\right)}{4(1-m)} + \frac{3m^2(1+m)\left(1+\tfrac{3}{4}e^2 - \tfrac{1}{4}g^2 - \tfrac{5}{2}e'^2\right)}{2-2m-c}$$

$$-\frac{3m^2e^2\left(10+19m+8m^2\right)}{8(2c-2+2m)} - 2A_1\left(1+\tfrac{1}{2}e^2 - \tfrac{1}{4}g^2\right) + 3A_0 + 3e^2A_{11}$$

$$x_9 = \frac{3m^2}{4(1-m)} + \frac{3m^2(1-m)}{2-2m+c} - 2A_2 + 3A_0 - 3A_1e^2$$

$$x_{10} = \frac{3m^2}{4(2-m)} - 2A_3 + 3A_6e^2$$

$$x_{11} = -\frac{21m^2}{4(2-3m)} - 2A_4 + 3A_7e^2$$

$$x_{12} = -3m\left\{4A_0 + A_3 - A_4 - 10A_1e^2 + \tfrac{5}{2}(A_7 - A_6)e^2\right\}$$

$$+\left\{\frac{3m^2A_0}{4} + \frac{27m^4}{32(1-m)}\right\}\left\{\frac{7}{2-3m} - \frac{1}{2-m}\right\}$$

$$+\left\{\frac{3m^2}{4(1-m)} + 3A_0\right\}\left\{A_3 + A_4\right\} - 2A_5\left(1+\tfrac{1}{2}e^2 - \tfrac{1}{4}g^2\right)$$

$$+3(A_8 + A_9)e^2 + 3A_1(A_6 + A_7)e^2$$

$$+\frac{3m^2}{4}\left\{11C_6 + 2C_9 - 2C_{10}\right\}$$

$$x_{13} = -\frac{3m^2(2+m)}{4(2-m-c)} - \frac{3m^2}{4(2-m)} - 2A_6 + 3A_3$$

$$x_{14} = \frac{21m^2(2+3m)}{4(2-3m-c)} + \frac{21m^2}{4(2-3m)} - 2A_7 + 3A_4$$

$$x_{15} = -2A_8 + 3A_5$$

$$x_{16} = -2A_9 + 3A_5$$

$$x_{17} = +\frac{3m^2\left(10+19m+8m^2\right)}{8\left(2c-2+2m\right)} - \frac{3m^2\left(1+m\right)}{2-2m-c} - \frac{9m^2}{16\left(1-m\right)}$$

$$-3A_0 + 3A_1 - 2A_{11} - \frac{3m^2A_{10} + \frac{15}{4}m^2A_1^2}{2c-2+2m}$$

$$x_{18} = \frac{3m^2\left(2+m\right)}{8\left(2g-2+2m\right)} - \frac{3m^2}{16\left(1-m\right)} - 2A_{13} - \tfrac{3}{4}A_0 - \frac{3m^2A_{12}}{2g-2+2m}$$

$$x_{19} = -A_{14}$$

$$x_{20} = -\frac{3m^2}{8\left(1-m\right)} + \frac{3m^2\left(5+3m\right)}{4\left(1-m\right)}A_{17} - 2A_{17}\left(1+\tfrac{1}{2}e^2 - \tfrac{1}{4}g^2\right) + 3A_0 \cdot A_{17}$$

$$x_{21} = -A_{18}.$$

731. Now if quantities of the order m^4 be omitted,

$$\frac{a^2dv}{\sqrt{\overline{a}}}x_0 \text{ becomes } \frac{a^2dv}{\sqrt{\overline{a}}};$$

but in this case equation (230) is reduced to

$$\frac{1}{a} = \frac{1}{\overline{a}}\left\{1 - \frac{m^2}{2} - \frac{3m^2}{4}e'^2\right\},$$

because m^2 differs very little from $\overline{m}^2$, whence

$$\left(\frac{a}{\overline{a}}\right)^2 = 1 + m^2 + \tfrac{3}{2}m^2e',$$

and

$$\frac{a^2dv}{\sqrt{\overline{a}}} = \left(\overline{a}\right)^{\frac{3}{2}}\left\{\left(1+m^2\right) + \tfrac{3}{2}m^2e'^2\right\}dv, \qquad \textbf{(238)}$$

so that $\frac{a^2dv}{\sqrt{\overline{a}}}$ varies with e', the eccentricity of the terrestrial orbit; but if that variation be omitted, the part that is not periodic of

$$\frac{a^2dv}{\sqrt{\overline{a}}} = \left(\overline{a}\right)^{\frac{3}{2}}\left(1+m^2\right).dv.$$

If the action of the sun be omitted $a = \bar{a}$, and if $\frac{1}{n}$ be put for $a^{\frac{3}{2}}$, then the part that is not periodic becomes

$$\frac{a^2 dv}{\sqrt{\bar{a}}} = \frac{dv}{n} = a^{\frac{3}{2}}\left(1 + m^2\right).dv,$$

and equation (238) is transformed to

$$\frac{a^2 dv}{\sqrt{\bar{a}}} = \frac{dv}{n} + \frac{3m^2}{2n} e'^2 dv,$$

and

$$dt = \frac{a^2 dv}{\sqrt{\bar{a}}} x_0$$

becomes

$$ndt = dv + \tfrac{3}{2} m^2 e'^2 dv,$$

the integral of which is

$$nt + \in = v + \tfrac{3}{2} m^2 \int \left(e'^2 - \bar{e}^2\right) dv,$$

$\bar{e}^2$ being a constant quantity equal to the eccentricity of the earth's orbit at the epoch.

732. Thus the mean longitude of the moon is affected by a secular inequality, occasioned by the variation of the eccentricity of the earth's orbit, and the true longitude of the moon in functions of her mean longitude contains the secular inequality

$$-\tfrac{3}{2} m^2 \int \left(e'^2 - \bar{e}^2\right) dv, \text{ or } -\tfrac{3}{2} m^2 \int \left(e'^2 - \bar{e}^2\right) ndt,$$

called the acceleration; hence the secular inequalities in the mean longitude of the moon, in the longitude of her perigee and nodes, are as the three quantities

$$3\bar{m}^2, \quad -\frac{p'}{\sqrt{1 - p}}, \quad \frac{q'}{\sqrt{1 + q}}.$$

It is true that the terms depending on the squares of the disturbing force alter the value of the secular equations in the mean longitude a little; but the terms of this order that have a considerable influence on the secular equation of the perigee have but little effect on that of the mean motion.

733. Thus the integral of equation (237) is

$$
\begin{aligned}
nt+\in=+v+\tfrac{3}{2}m^2\int\left(e'^2-\bar{e}^2\right)dv \qquad \textbf{(239)}\\
&+C_0 e\sin\left(cv-v\right)\\
&+C_1 e^2\sin\left(2cv-2v\right)\\
&+C_2 e^3\sin\left(3cv-3v\right)\\
&+C_3 g^2\sin\left(2gv-2q\right)\\
&+C_4 eg^2\sin\left(2gv-cv-2q+v\right)\\
&+C_5 eg^2\sin\left(2gv+cv-2q-v\right)\\
&+C_6\sin\left(2v-2mv\right)\\
&+C_7 e\sin\left(2v-2mv-cv+v\right)\\
&+C_8 e\sin\left(2v-2mv+cv-v\right)\\
&+C_9 e'\sin\left(2v-2mv+c'mv-v'\right)\\
&+C_{10} e'\sin\left(2v-2mv-c'mv+v'\right)\\
&+C_{11} e'\sin\left(c'mv-v'\right)\\
&+C_{12} ee'\sin\left(2v-2mv-cv+c'mv+v-v'\right)\\
&+C_{13} ee'\sin\left(2v-2mv-cv-c'mv+v+v'\right)\\
&+C_{14} ee'\sin\left(cv+cmv-v-v'\right)\\
&+C_{15} ee'\sin\left(cv-cmv-v+v'\right)\\
&+C_{16} e^2\sin\left(2cv-2v+2mv-2v\right)\\
&+C_{17} g^2\sin\left(2gv-2v+2mv-2q\right)\\
&+C_{18} e'^2\sin\left(2c'mv-2v'\right)\\
&+C_{29}\frac{a}{a'}\sin\left(v-mv\right)\\
&+C_{20}\frac{a}{a'}e'\sin\left(v-mv+c'mv-v'\right).
\end{aligned}
$$

734. If the differential of this equation be compared with equation (237), the following values will be obtained for the indeterminate coefficients–

$$C_0=\frac{x_1}{c} \qquad C_{10}=\frac{x_{11}}{2-3m}$$

$$C_1=\frac{x_2}{2c} \qquad C_{11}=\frac{x_{12}}{m}$$

$$C_2=\frac{x_3}{3c} \qquad C_{12}=\frac{x_{13}}{2-m-c}$$

$$C_3 = \frac{x_4}{2g} \qquad C_{13} = \frac{x_{14}}{2-3m-c}$$

$$C_4 = \frac{x_5}{2g-c} \qquad C_{14} = \frac{x_{15}}{c+m}$$

$$C_5 = \frac{x_6}{2g+c} \qquad C_{15} = \frac{x_{16}}{c-m}$$

$$C_6 = \frac{x_7}{2-2m} \qquad C_{16} = \frac{x_{17}}{2c-2+2m}$$

$$C_7 = \frac{x_8}{2-2m-c} \qquad C_{17} = \frac{x_{18}}{2g-2+2m}$$

$$C_8 = \frac{x_9}{2-2m+c} \qquad C_{18} = \frac{x_{19}}{m}$$

$$C_9 = \frac{x_{10}}{2-m} \qquad C_{19} = \frac{x_{20}}{1-m}$$

$$C_{20} = -2A_{18}\,.$$

Notes

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1 Aristarchus of Samos, c. 310-230 BC, Alexandrian astronomer called the "ancient Copernicus" (see note 1, *Bk. II, Chap. I*), maintained that the Earth moves round the Sun. Only one of his works *On the Sizes and Distances of the Sun and Moon* survives. In it Aristarchus calculates the sun's diameter at about 19 times the moon's diameter, and the sun's distance at 19 times the moon's distance. Although Aristarchus' heliocentric writings did not survive, Archimedes wrote: "*Aristarchus of Samos has published in outline certain hypotheses…He supposes that the fixed stars and the sun are immovable, but that the earth is carried round the sun in a circle…*" (Dreyer, J. E. L., *A History of Astronomy from Thales to Kepler*, 2nd ed., Dover Publications, p. 152, 1953). The lack of stellar parallax counter-argument against the heliocentric model is answered by Aristarchus by assuming the sphere of the fixed stars as infinitely distant with respect to the earth-sun distance.

2 i.e. 280 BCE

3 This *reads* "anomalastic" in the 1st edition (published erratum).

4 See note 3.

5 *syzigy*. Also spelled "syzygy." The nearly straight-line configuration of three celestial bodies (as the sun, moon, and earth during a solar or lunar eclipse) in a gravitational system. *Merriam-Webster's Collegiate Dictionary*.

6 This *reads* "385th part" in the 1st edition.

7 Damoiseau, Théodore, baron de, 1768-1846, *Tables de la lune, formées par la seule théorie de l'attraction, et suivant la division de la circonférence en 360 degrés; par m. le baron de Damoiseau ... Publiées par le Bureau des longitudes*, Paris, Bachelier (successeur de mme. Ve Courcier) 1828.

8 Left hand side *reads* $\frac{m'a^3}{a'^3}\cdot\frac{a_/}{\bar{a}}$ in 1st edition.

9 The numerator in the second term *reads* $m'(xx+yy+zz)$ in the 1st edition.

10 Spelled "develope" in the 1st edition.

11 The 1st edition spelling is, "immoveable".

12 *apsis*. The point in an astronomical orbit at which the distance of the body from the center of attraction is either greatest or least. *Merriam-Webster's Collegiate Dictionary*.

13 The right-hand factor h^3dv *reads* h^3dt in the 1st edition.

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[14] The argument of the 3[rd] term *reads* $2'cmv - 2v' + 2c'mf$ in the 1[st] edition.

[15] This *reads* "developments" in the 1[st] edition.

[16] Period added after equation (222).

[17] A parenthesis is omitted in the 1[st] term in b_{19} and in the 1[st] edition *reads* as follows:

$$+\{\tfrac{9}{8}(1-2m)\left(1+2e^2+2e'^2\right)+\frac{3(1-2m)\left(1+\frac{9}{2}e^2+2e'^2\right)}{4(1-m)}+\frac{3(1+m)}{2(1-m)}\times A_{18}e'^2$$

[18] This is spelled "developement" in the 1[st] edition.

[19] In the 1[st] edition the previous equation is also numbered (234). We retain the original numbering.

[20] The 4[th] term in the argument of x_{14} *reads* $-cmv$ in the 1[st] edition.

[21] The closing parenthesis is missing in the equation for x_4 in the 1[st] edition.

BOOK III

CHAPTER II

NUMERICAL VALUES OF THE COEFFICIENTS

735. THE following data are obtained by observation.

$$m = 0.0748013$$
$$e = 0.05486281$$
$$g = 0.0900807$$
$$c = 0.99154801$$
$$g = 1.00402175$$
$$e' = 0.016814, \text{ at the epoch 1750,}$$
$$m = \frac{1}{75}.$$

e and g result from the comparison of the coefficients of the sines of the angles $cv - v$ and $gv - q$, computed from observation with those from theory. With these data equation (230) gives

$$\frac{1}{a} = \frac{1}{a_1} \cdot 0.9973020; \quad \frac{a^2}{\sqrt{a_1}} = 1.0003084 = \frac{1}{n};$$

whence

$$\frac{1}{a} = \sqrt[3]{\frac{n^2\,(1.0003084)^2}{0.9973020}} .$$

With these the formulae of articles 718 and 726 and 734 give

$A_0 =$	$+0.0070962$	$A_{11} =$	$+0.349187$
$A_1 =$	$+0.201816$	$A_{12} =$	$+0.0026507$
$A_2 =$	-0.00372953	$A_{13} =$	$+0.0077734$
$A_3 =$	-0.00300427	$A_{14} =$	-0.012989
$A_4 =$	$+0.0284957$	$A_{15} =$	-0.742373
$A_5 =$	-0.00591628	$A_{16} =$	-0.041378
$A_6 =$	-0.0698493	$A_{17} =$	-0.113197
$A_7 =$	$+0.516751$	$A_{18} =$	$+1.08469$
$A_8 =$	-0.20751	$A_{19} =$	$+0.001601$

$A_9 = +0.274122$
$A_{10} = +0.0008107$
$B_2 = -0.0055075$
$B_3 = +0.019553$
$B_4 = +0.0063661$
$B_5 = -0.0013668$
$B_6 = -0.021272$
$B_7 = +0.07824$
$B_8 = -0.0833684$
$B_9 = -0.0327678$
$B_{10} = +0.0720448$
$B_{11} = +0.491954$
$B_{12} = +0.0061023$
$B_{13} = +0.0920621$
$B_{14} = -0.0125619$
$B_{15} = +0.0038663$
$C_0 = -2.003974$
$C_1 = +0.752886$
$C_2 = -0.336175$
$C_3 = +0.243118$

$B_0 = +0.0282636$
$B_1 = -0.0000024$
$C_4 = +0.722823$
$C_5 = -0.250034$
$C_6 = -0.0091988$
$C_7 = -0.414046$
$C_8 = +0.0129865$
$C_9 = +0.0039255$
$C_{10} = -0.0387853$
$C_{11} = +0.196755$
$C_{12} = +0.12765$
$C_{13} = -1.081734$
$C_{14} = +0.373115$
$C_{15} = -0.616738$
$C_{16} = +0.272377$
$C_{17} = +0.033825$
$C_{18} = +0.173647$
$C_{19} = -0.236616$
$C_{20} = -2.16938$

736. If these coefficients be reduced to sexagesimal seconds, the mean longitude of the moon will become[1]

$$
\begin{aligned}
nt+\epsilon = {} & +v+\tfrac{3}{2}m^2\int\left(e^2-\bar{e}^2\right)dv \\
& -22{,}677''.5\cdot\sin(cv-v) \\
& +\quad 467''.42\cdot\sin(2cv-2v) \\
& -\quad 11''.45\cdot\sin(3cv-3v) \\
& +\quad 406''.92\cdot\sin(2gv-2q) \\
& +\quad 66''.37\cdot\sin(2gv-cv-2q+v) \\
& -\quad 22''.96\cdot\sin(2gv+cv-2q-v) \\
& -\quad 1{,}906''.93\cdot\sin(2v-2mv) \\
& -\quad 4{,}685''.46\cdot\sin(2v-2mv-cv+v) \\
& +\quad 147''.68\cdot\sin(2v-2mv+cv-v)
\end{aligned}
\qquad \textbf{(240)}
$$

$$
\begin{aligned}
&+ \quad 13''.61.\sin\left(2v-2mv+c'mv-v'\right)\\
&- \quad 134''.51.\sin\left(2v-2mv-c'mv+v'\right)\\
&+ \quad 682''.37\cdot\sin\left(c'mv-v'\right)\\
&+ \quad 24.29\cdot\sin\left(2v-2mv-cv+c'mv+v-v'\right)\\
&- \quad 205.82\cdot\sin\left(2v-2mv-cv-c'mv+v+v'\right)\\
&+ \quad 70.99\cdot\sin\left(cv+c'mv-v-v'\right)\\
&- \quad 117.35\cdot\sin\left(cv-c'mv-v+v'\right)\\
&+ \quad 169.09\cdot\sin\left(2cv-2v+2mv-2v\right)\\
&+ \quad 56.62\cdot\sin\left(2gv-2v+2mv-2q\right)\\
&+ \quad 10.13\cdot\sin\left(2c'mv-2v'\right)\\
&+ \quad 122.014\cdot(1+i)\sin\left(v-mv\right)\\
&- \quad 18.81\cdot(1+i)\sin\left(v-mv+c'mv-v'\right).
\end{aligned}
$$

737. The two last terms have been determined in supposing

$$\frac{a}{a'}=\frac{(1+i)}{400}.$$

This fraction is the ratio of the parallax of the sun to that of the moon; it differs very little from $\frac{1}{400}$, but for greater generality it is multiplied by the indeterminate coefficient $1+i$; and by comparing the coefficient of $\sin(v-mv)$ with the result of observations the solar parallax is obtained, as will be shown afterwards.

738. It has been shown that the action of the moon produces the inequality

$$m\cdot\frac{a}{a'}\sin(v-mv)$$

in the earth's longitude. This action of the moon changes the earth's place, and, consequently, the moon's place with regard to the sun, so that the moon indirectly troubles her own motion, producing in her mean longitude the inequality

$$0.54139\,.\,m\,.\,\frac{a}{a'}\sin(v-nv).$$

Thus the direct action of the moon is weakened by reflection in the ratio of 0.54139 to unity.

739. Equation (233) gives the tangent of the latitude, but the expression of the arc by the tangent s is

$$s - \tfrac{1}{3}s^3 + \tfrac{1}{5}s^5 - \&c.$$

Thus the latitude is nearly

$$\begin{aligned}&g\left(1-\tfrac{1}{4}g^2\right)\sin(gv-q)+ds\times\\&\left\{1-\tfrac{1}{2}g^2+\tfrac{1}{2}g^2\cos(2gv-2q)+\tfrac{1}{12}g^3\sin(3gv-3q)\right\}.\end{aligned}$$

And from the preceding data the latitude of the moon is easily found to be

$$\begin{aligned}s = +18{,}542''.0.\ &\sin(gv-q)\\+\ 12''.57.\ &\sin(3gv-3q)\\+\ 525''.23.\ &\sin(2v-2mv-gv+q)\\+\ 1''.14.\ &\sin(2v-2mv+gv-q)\\-\ 5''.53.\ &\sin(gv+cv-q-v)\\+\ 19''.85.\ &\sin(gv-cv-q+v)\\+\ 6''.46.\ &\sin(2v-2mv-gv+cv+q-v)\\-\ 1''.39.\ &\sin(2v-2mv+gv-cv-q+v)\\-\ 21''.6\ .\ &\sin(2v-2mv-gv-cv+q+v)\\+\ 24''.34.\ &\sin(gv+c'mv-q-v')\\-\ 25''.94.\ &\sin(gv-c'mv-q+v')\\-\ 10''.2\ .\ &\sin(2v-2mv-gv+c'mv+q-v')\\+\ 22''.42.\ &\sin(2v-2mv-gv-cmv+q+v')\\+\ 27''.41.\ &\sin(2cv-gv-2v+q)\\+\ 5''.29.\ &\sin(2cv+gv-2v+2mv-2v-q).\end{aligned} \qquad \textbf{(241)}$$

740. The sine of the horizontal parallax of the moon is

$$\frac{R`}{r} = \frac{R`u}{\sqrt{1+ss}},$$

$R`$ being the terrestrial radius, but as this arc is extremely small, it may be taken for its sine; hence, if

$$\frac{1}{a}\left\{1+e^2+\tfrac{1}{4}g^2+e\left(1+e^2\right)\cos\left(cv-v\right)-\tfrac{1}{4}g^2\cos\left(2gv-2q\right)\right\}+du$$

be put for m, and quantities of the order $\frac{R\grave{}}{a}e^4$ rejected, the parallax will be

$$\frac{R\grave{}}{r}=\frac{R\grave{}u}{\sqrt{1+s^2}}=\frac{R\grave{}}{a}\left(1+e^2\right)\left\{1+e\left[1-\tfrac{1}{4}g^2+\tfrac{1}{4}g^2\cos\left(2gv-2q\right)\right]\cos\left(cv-v\right)+adu-sds\right\}.$$

In the untroubled orbit of the moon the radius vector, and, consequently, the parallax, varies according to a fixed law through every point of the ellipse. Its mean value, or the constant part of the horizontal parallax, is $\frac{R\grave{}}{a}$, to which the rest of the series is applied as corrections arising both from the ellipticity of the orbit and the periodic inequalities to which it is subject.

741. In order to compute the constant part of the parallax, let s be the space described by falling bodies in a second in the latitude, the square of whose sine is $\frac{1}{3}$, l and $R\grave{}$ the corresponding lengths of the pendulum and terrestrial radius, p the ratio of the semicircumference to the radius, E and m the masses of the earth and moon; then, supposing

$$E+m=1,$$

$$\frac{E}{\left(E+m\right)R\grave{}^2}=2s=p^2l,\text{ also } n=\frac{2p}{T},$$

T being the number of seconds a sidereal revolution of the moon; and by article 735

$$\frac{1}{a}=\sqrt[3]{\frac{n^2\left(1.0003084\right)^2}{0.9973020}},$$

therefore

$$\frac{R\grave{}}{a}=\sqrt[3]{\frac{E}{E+m}\cdot\frac{R\grave{}}{l}\cdot\frac{4\left(1.0003084\right)^2}{T^2 0.997320}}.$$

Now the length of the pendulum, independent of the centrifugal force, is

$$l=32.648\text{ feet},$$

also

$$R\grave{}=20{,}898{,}500\text{ feet},$$
$$T=2{,}360{,}591''.8;$$

and if

$$m=\frac{E}{58.6}$$

it will be found that

$$\frac{R\grave{}}{a} = 0.01655101\text{, and therefore } \frac{R\grave{}}{a}\left(1+e^{2}\right) = 3{,}424''.16\,;$$

this value augmented by $3''.74$, to reduce it to the equator, is $3{,}427''.9$; hence the equatorial parallax of the moon in functions of its true longitude is

$$\begin{aligned}
\frac{1}{r} = {} & +3{,}427''.9 \\
& + 187''.48\cos\left(cv - v\right) \\
& + 24''.68\cos\left(2v - 2mv\right) \\
& + 47''.92\cos\left(2v - 2mv - cv + v\right) \\
& - 0''.7\ \cos\left(2v - 2mv + cv - v\right) \\
& - 0''.17\cos\left(2v - 2mv + c'mv - v'\right) \\
& + 1''.64\cos\left(2v - 2mv - c'mv + v'\right) \\
& - 0''.33\cos\left(c'mv - v'\right) \\
& - 0''.22\cos\left(2v - 2mv - cv + c'mv + v - v'\right) \\
& + 1''.63\cos\left(2v - 2mv - cv - cmv + v + v'\right) \\
& - 0''.45\cos\left(cv + c'mv - v - v'\right) \\
& + 0''.86\cos\left(cv - c'mv - v + v'\right) \\
& + 0''.01\cos\left(2cv - 2v\right) \\
& + 3''.6\cos\left(2cv - 2v + 2mv - 2v\right) \\
& + 0''.07\cos\left(2gv - 2q\right) \\
& - 0''.18\cos\left(2gv - 2v + 2mv - 2q\right) \\
& - 0''.01\cos\left(2c'mv - 2v'\right) \\
& - 0''.95\cos\left(2gv - cv - 2q + v\right) \\
& - 0''.06\cos\left(2v - 2mv - 2gv + cv + 2q - v\right) \\
& - 0''.97\left(1+i\right)\cos\left(v - mv\right) \\
& + 0''.16\left(1+i\right)\cos\left(v - mv + c'mv - v'\right) \\
& - 0''.04\cos\left(2v - 2mv + 2cv - 2c'mv - v + v'\right) \\
& - 0''.15\cos\left(4v - 4mv - cv + v\right) \\
& + 0''.05\cos\left(4v - 4mv - cv + 2v\right) \\
& + 0''.13\cos\left(2cv - 2v + 2mv + c'mv - 2v - v'\right)
\end{aligned} \tag{242}$$

$$+ \quad 0''.02\cos\left(2cv + 2v - 2mv - 2\nu\right)$$
$$- \quad 0''.12\left(1+i\right)\cos\left(cv - v + mv - \nu\right)$$

The greatest value of the parallax is $1^\circ 1' 29''.32$, which happens when the moon is in perigee and opposition; the least, $58' 29''.93$, happens when the moon is in apogee and conjunction.

742. With $m = \frac{E}{74}$, Mr. Damoiseau[2] finds the constant part of the equatorial parallax equal to $3{,}431''.73$.

743. The lunar parallax being known, that of the sun may be determined by comparing the coefficients of the inequality

$$122''.014\left(1+i\right)\sin\left(v - mv\right)$$

in the moon's mean longitude with the same derived from observation. In the tables of Burg,[3] reduced from the true to the mean longitude, this coefficient is $122''.378$; hence

$$i + 1 = \frac{122''.378}{122''.014} = 1''.00298, \text{ and } \frac{a}{a_{/}} = \frac{1''.00298}{400}.$$

But the solar parallax is

$$\frac{R`}{a'} = \frac{R`}{a}\frac{a}{a'} = \frac{R`}{a}\cdot\frac{1''.00298}{400},$$

but

$$\frac{R`}{a} = 0.01655101,$$

hence

$$\frac{R`}{a'} = \frac{1''.00298 \times 0.01655101}{400} = 8''.5602,$$

which is the mean parallax of the sun in the parallel of latitude, the square of whose sine is $\frac{1}{3}$.

Burckhardt's tables[4] give $122''.97$ for the value of the coefficient, whence the solar parallax is $8''.637$, differing very little from the value deduced from the transit of Venus. This remarkable coincidence proves that the action of the sun upon the moon is very nearly equal to his action on the earth, not differing more than the three millionth part.

744. The constant part of the lunar parallax is $3{,}432''.04$, by the observations of Mr. Maskelyne,[5] consequently the equation

$$3{,}432''.04 = \sqrt[3]{\frac{E}{E+m} \cdot \frac{R`}{l} \cdot \frac{4\left(1.0003084\right)^2}{T^2\left(0.9973020\right)}}$$

gives the mass of the moon equal to

$$\frac{1}{74.2}$$

of that of the earth.

Since by article 646, $\frac{R`}{a} = 0.01655101,$ in the latitude the square of whose sine is $\frac{1}{3}$; if $R`,$ the mean radius of the earth, be assumed as unity, the mean distance of the moon from the earth is 60.4193 terrestrial radii, or about 247,583 English miles.

745. As theory combined with observations with the pendulum, and the mensuration of the degrees of the meridian, give a value of the lunar parallax nearly corresponding with that derived from astronomical observations, we may reciprocally determine the magnitude of the earth from these observations; for if the radius of the earth be assumed as the unknown quantity in the expression in article 646, it will give its value equal to 20,897,500 English feet.

'*Thus,*' says Laplace,[6] '*an astronomer, without going out of his observatory, can now determine with precision the magnitude and distance of the earth from the sun and moon, by a comparison of observations with analysis alone; which in former times it required long voyages in both hemispheres to accomplish*.'

746. The apparent diameter of the moon varies with its parallax, for if P be the horizontal parallax, $R`$ the terrestrial radius, r the radius vector of the moon, D her real, and A her apparent diameters; then

$$P = \frac{R`}{r},\ A = \frac{D}{r};\ \text{whence}\ \frac{P}{A} = \frac{R`}{D}$$

a ratio that is constant if the earth be a sphere. It is also constant at the same point of the earth's surface, whatever the figure of the earth may be.

$$\text{If } P = 56'4''.168 \text{ and } \tfrac{1}{2}A = 31'7''.73;$$

then

$$\frac{A}{2P} = 0.27293 = \tfrac{3}{11} \text{ nearly;}$$

thus if $\frac{11}{3}$ be multiplied by the moon's apparent semidiameter, the corresponding horizontal parallax will be obtained.

Secular Inequalities in the Moon's Motions

747. It has been shown, that the action of the planets is the cause of a secular variation in the eccentricity of the earth's orbit, which variation produces analogous inequalities in the mean motion of the moon, in the motion of her perigee and in that of her nodes.

The Acceleration

748. The secular variation in the mean motion of the moon denominated the Acceleration, was discovered by Halley;[7] but Laplace[8] first showed that it was occasioned by the variation in the eccentricity in the earth's orbit. The acceleration in the mean motion of the moon is ascertained by comparing ancient with modern observations; for if the ancient observations be assumed as observed longitudes of the moon, a calculation of her place for the same epoch from the lunar tables will render the acceleration manifest, since these tables may be regarded as data derived from modern observations.

An eclipse of the moon observed by the Chaldeans at Babylon, on the 19th of March, 721 years before the Christian era, which began about an hour after the rising of the moon, as recorded by Ptolemy,[9] has been employed. As an eclipse can only happen when the moon is in opposition, the instant of opposition may be computed from the solar tables, which will give the true longitude of the moon at the time, and the mean longitude may be ascertained from the tables. Now, if we compare this result with another mean longitude of the moon computed from modern observations, the difference of the longitudes augmented by the requisite number of circumferences will give the arc described by the moon parallel to the ecliptic during the interval between the observations, and the mean motion of the moon during 100 Julian years may be ascertained by dividing this arc by the number of centuries elapsed. But the mean motion thus computed by Delambre,[10] Bouvard,[11] and Burg, is more than 200″ less than that which is derived from a comparison of modern observations with one another. The same results are obtained from two eclipses observed by the Chaldeans in the years 719 and 720 before the Christian era. This acceleration was confirmed by comparing less ancient eclipses with those that happened recently; for the epoch of intermediate observations being nearer modern times, the differences of the mean longitudes ought to be less than in the first case, which is perfectly confirmed, by the eclipses observed by Ibn-Junis,[12] an Arabian astronomer of the eleventh century. It is therefore proved beyond a doubt, that the mean motion of the moon is accelerated, and her periodic time consequently diminished from the time of the Chaldeans.

Were the eccentricity of the terrestrial orbit constant, the term

$$\tfrac{3}{2} m^2 \int \left(e'^2 - \overline{e}^2 \right) dv$$

would be united with the mean angular velocity of the moon; but the variation of the eccentricity, though small, has in the course of time a very great influence on the lunar motions. The mean motion of the moon is accelerated, when the eccentricity of the earth's orbit diminishes, which it has continued to do from the most ancient observations down to our times; and it will continue to be accelerated until the eccentricity begins to increase, when it will be retarded. In the interval between 1750 and 1850, the square of the eccentricity of the terrestrial orbit has diminished by 0.00000140595. The corresponding increment in the angular velocity of the moon is the

0.000000011782th part of this velocity. As this increment takes place gradually and proportionally to the time, its effect on the motion of the moon is less by one half than if it had been uniformly the same in the whole course of the century as at the end of it. In order, therefore, to determine the secular equation of the moon at the end of a century estimated from 1801, we must multiply the secular motion of the moon by half the very small increment of the angular velocity; but in a century the motion of the moon is $1{,}732{,}559{,}351''.514$, which gives $10''.2065508$ for her secular equation. Assuming that for 2000 years before and after the epoch 1750, the square of the eccentricity of the earth's orbit diminishes as the time, the secular equation of the mean motion will increase as the square of the time: it is sufficient then during that period to multiply $10''.2065508$ by the square of the number of centuries elapsed between the time for which we compute and the beginning of the nineteenth century; but in computing back to the time of the Chaldeans, it is necessary to carry the approximation to the cube of the time. The numerical formula for the acceleration is easily found, for since[13]

$$\tfrac{3}{2}m^2\int\left(e'^2-\bar{e}^2\right)dv$$

is the acceleration in the mean longitude of the moon, the true longitude of the moon in functions of her mean longitude will contain the term

$$-\tfrac{3}{2}m^2\int\left(e'^2-\bar{e}^2\right)ndt,$$

$\bar{e}$ being the eccentricity of the terrestrial orbit at the epoch 1750. If then, t be any number of Julian years from 1750, by article 480,

$$2e'=2\bar{e}-0''.171793t-0''.000068194t^2$$

is the eccentricity of the earth's orbit at any time t, whence the acceleration is

$$10''.1816213.T^2+0''.018538444.T^3,$$

T being any number of centuries before or after 1801.

In consequence of the acceleration, the mean motion of the moon is $7'\,30''$ greater in a century now than it was 2,548 years ago.

Motion of the Moon's Perigee

749. In the first determination of the motion of the lunar perigee, the approximation had not been carried far enough, by which the motion deduced from theory was only one half of that obtained by observation; this led Clairaut[14] to suppose that the law of gravitation was more complicated than the inverse ratio of the squares of the distance; but Buffon[15] opposed him on the principle that, the primordial laws of nature being the most simple, could only depend on one principle, and therefore their expression could only consist of one term. Although such reasoning is not always conclusive, Buffon was right in this instance, for, upon carrying the approximation to the squares of the disturbing force, the law of gravitation gives the motion of the lunar perigee exactly conformable to observation, for $\bar{e}'^2$ being the eccentricity of the terrestrial orbit at the epoch, the equation $c = \sqrt{1 - p - p'\bar{e}^2}$ when reduced to numbers is $c = 0.991567$, consequently $(1-c)v$ the motion of the lunar perigee is $0.008433.v$; and with the value of c in article 735 given by observation, it is $0.008452.v$, which only differs from the preceding by 0.000019. In Damoiseau's theory[16] it is $0.008453.v$, which does not differ much from that of Laplace. The terms depending on the squares of the disturbing force have a very great influence on the secular variation in the motion of the lunar perigee; they make its value three times as great as that of the acceleration: for the secular inequality in the lunar perigee is

$$\frac{p'}{2\sqrt{1+p}}\int\left(e'^2 - \bar{e}^2\right)ndt,$$

or, when the coefficient is computed, it is

$$3.00052\tfrac{3}{2}m^2\int\left(e'^2 - \bar{e}^2\right)ndt,$$

and has a contrary sign to the secular equation in the mean motion.

The motion of the perigee becomes slower from century to century, and is now $8'.2$ slower than in the time of Hipparchus.[17]

Motion of the Nodes of the Lunar Orbit

750. The sidereal motion of the node on the true ecliptic as determined by theory, does not differ from that given by observation by a 350th part; for the expression in article 727 gives the retrograde motion of the node equal to $0.00400105v$, and by observation

$$(g-1)v = 0.00402175v,$$

the difference being 0.00001125. Mr. Damoiseau makes it

$$g - 1 = 0.0040215.$$

The secular inequality in the motion of the node depends on the variation in the eccentricity of the terrestrial orbit, and has a contrary sign to the acceleration. Its analytical expression gives

$$\frac{q'}{2\sqrt{1+q}}\int\left(e'^2 - \bar{e}^2\right)dv = 0.735452\tfrac{3}{2}m^3\int\left(e'^2 - \bar{e}^2\right)dv.$$

As the motion of the nodes is retrograde, this inequality tends to augment their longitudes posterior to the epoch.

751. It appears from the signs of these three secular inequalities, as well as from observation, that the motion of the perigee and nodes become slower, whilst that of the moon is accelerated; and that their inequalities are always in the ratio of the numbers 0.735452, 3.00052, and 1.

752. The mean longitude of the moon estimated from the first point of Aries is only affected by its own secular inequality; but the mean anomaly estimated from the perigee, is affected both by the secular variation of the mean longitude, and by that of the perigee; it is therefore subject to the secular inequality $-4.00052\frac{3}{2}m^2\int\left(e'^2-\bar{e}'^2\right)dv$ more than four times that of the mean longitude. From the preceding values it is evident that the secular motion of the moon with regard to the sun, her nodes, and her perigee, are as the numbers 1; 0.265; and 4; nearly.

753. At some future time, these inequalities will produce variations equal to a fortieth part of the circumference in the secular motion of the moon; and in the motion of the perigee, they will amount to no less than a thirteenth part of the circumference. They will not always increase: depending on the variation of the eccentricity of the terrestrial orbit they are periodic, but they will not run through their periods for millions of years. In process of time, they will alter all those periods which depend on the position of the moon with regard to the sun, to her perigee, and nodes; hence the tropical, synodic, and sidereal revolutions of the moon will differ in different centuries, which renders it vain to attempt to attain correct values of them for any length of time.

Imperfect as the early observations of the moon may be, they serve to confirm the results that have been detailed, which is surprising, when it is considered that the variation of the eccentricity of the earth's orbit is still in some degree uncertain, because the values of the masses of Venus and Mars are not ascertained with precision; and it is worthy of remark, that in process of time the development of the secular inequalities of the moon will furnish the most accurate data for the determination of the masses of these two planets.

754. The diminution of the eccentricity of the earth's orbit has a greater effect on the moon's motions than on those of the earth. This diminution, which has not altered the equation of the centre of the sun by more than $8'.1$ from the time of the most ancient eclipse on record, has produced a variation of $1^\circ\,8'$ in the longitude of the moon, and of $7^\circ.2$ in her mean anomaly.

Thus the action of the sun, by transmitting to the moon the inequalities produced by the planets on the earth's orbit, renders this indirect action of the planets on the moon more considerable than their direct action.

755. The mean action of the sun on the moon contains the inclination of the lunar orbit on the plane of the ecliptic; and as the position of the ecliptic is subject to a secular variation, from the action of the planets, it might be expected to produce a secular variation in the inclination of the moon's orbit. This, however, is not the case, for the action of the sun retains the lunar orbit at the same inclination on the orbit of the earth; and thus in the secular motion of the ecliptic, the orbit of the earth carries the orbit of the moon along with it, as it will be demonstrated, the change in the ecliptic affecting only the declination of the moon. No perceptible change has been observed in the inclination of the lunar orbit since the time of Ptolemy, which confirms the result of theory.

756. Although the inclination of the orbit does not vary from the change in the plane of the ecliptic; yet, as the expressions which determine the inclination and eccentricity of the lunar orbit, the parallax of the moon, and generally the coefficients of all the moon's inequalities, contain the eccentricity of the terrestrial orbit, they are all subject to secular inequalities corresponding to the secular variation of that quantity. Hitherto they have been insensible, but in the course of time will increase to an estimable quantity. Even now, it is necessary to include the effects of this variation in the inequality called the annual equation, when computing ancient eclipses.

757. The three co-ordinates of the moon have been determined in functions of the true longitudes, because the series converge better, but these quantities may be found in functions of the mean longitudes by reversion of series. For if nt, ϖ, q, and $\in$, represent the mean motion of the moon, the longitudes of her perigee, ascending node and epoch, at the origin of the time, together with their secular equations for any time t, equation (240) becomes[18]

$$v-(nt+\in)=-\left\{C_0 . e . \sin(cv-\varpi)+C_1 . e^2 \sin 2(cv-\varpi)+C_2 . e^3 . \sin 3(cv-\varpi)+\&c.\right\}$$

or to abridge

$$v-(nt+\in)=S .$$

The general term of the series is

$$Q \cdot \sin(xv+y) .$$

And if Q' be the sum of the coefficients arising from the square of the series S, and depending on the angle $xv+y'$; Q'' the sum of the coefficients arising from the cube of S, and depending on the angle $xv+y$, &c. &c., the general term of the new series, which gives the true longitude of the moon in functions of her mean longitude, is

$$-\left\{Q+\tfrac{1}{2}x . Q'-\tfrac{1}{6}x^2 . Q''-\tfrac{1}{24}x^3 . Q'''+\&c.\right\} . \sin\left(x(nt+\in)+y\right)$$

Laplace does not give this transformation, but Damoiseau[19] has computed the coefficients for the epoch of January 1st, 1801, and has found that the true longitude of the moon in functions of its mean longitude $nt+\in=l$ is

$$\begin{aligned} v = nt+\in &+22{,}639''.7 . \sin\{cl-\varpi\} \\ &+ \quad 768''.72 . \sin(2cl-2\varpi) \\ &+ \quad 36''.94 . \sin(3cl-3\varpi) \\ &- \quad 411''.67 . \sin(2gl-2q) \end{aligned}$$

$$
\begin{aligned}
&+ \quad 39''.51 . \sin\left(cl - 2gl - v + 2q\right) \\
&- \quad 45''.12 . \sin\left(cl + 2gl - v - 2q\right) \\
&+ \quad 2{,}370''.00 . \sin\left(2l - 2ml\right) \\
&+ \quad 4{,}589''.61 . \sin\left(2l - 2ml - cl + v\right) \\
&+ \quad 192''.22 . \sin\left(2l - 2ml + cl - v\right) \\
&- \quad 24''.82 . \sin\left(2l - 2ml + c'ml - v'\right) \\
&+ \quad 165''.56 . \sin\left(2l - 2ml - c'ml + v'\right) \\
&- \quad 673''.70 . \sin\left(c'ml - v'\right) \\
&- \quad 28''.67 . \sin\left(2l - 2ml - cl + c'ml + v - v'\right) \\
&+ \quad 207''.09 . \sin\left(2l - 2ml - cl - c'ml + v + v'\right) \\
&- \quad 109''.27 . \sin\left(cl + cml - v - v'\right) \\
&+ \quad 147''.74 . \sin\left(cl - cml - v + v'\right) \\
&+ \quad 211''.57 . \sin\left(2l - 2ml - 2cl + 2v\right) \\
&+ \quad 54''.83 . \sin\left(2l - 2ml - 2gl + 2q\right) \\
&- \quad 7''.34 . \sin\left(2c'ml - 2v'\right) \\
&- \quad 122''.48 . \sin\left(l - ml\right) \\
&- \quad 17''.56 . \sin\left(l - ml + c'ml - v'\right).
\end{aligned}
$$

This is only the transformation of Laplace's equation (240), but Damoiseau[20] carries the approximation much farther.

758. The first term of this series is the mean longitude of the moon, including its secular variation.

The second term[21]

$$22{,}639''.7 \sin\left(cl - v\right)$$

is the equation of the centre, which is a maximum when

$$\sin\left(cl - v\right) = \pm 1,$$

that is, when the mean anomaly of the moon is either 90° or 270°. Thus, when the moon is in quadrature, the equation of the centre is[22] $\pm 6^\circ\ 17'\ 19''.7$, double the eccentricity of the orbit. In syzigies it is zero.

759. The most remarkable of the periodic inequalities next to the equation of the centre, is the evection[23]

$$4{,}589''.61\sin\left(2l-2ml-cl+v\right),$$

which is at its maximum and $=\pm 4{,}589''.61$, when $2l-2ml-cl+v$ is either 90° or 270°, and it is zero when that angle is either 0° or 180°. Its period is found by computing the value of its argument in a given time, and then finding by proportion the time required to describe 360°, or a whole circumference. The synodic motion of the moon in 100 Julian years is

$$445{,}267^\circ.1167992 = l-ml$$

and

$$890{,}534^\circ.2335984 = 2\{l-ml\}$$

is double the distance of the sun from the moon in 100 Julian years. If $477{,}198^\circ.839799$ the anomalistic motion of the moon in the same period be subtracted, the difference $413{,}335^\circ.3937994$ will be the angle $2l-2ml-cl+v$, or the argument of the evection in 100 Julian years: whence

$$413{,}335^\circ.3937994 : 360^\circ :: 365^{d}.25 : 31^{d}.811939 =$$

the period of the evection. If t be any time elapsed from a given period, as for example, when the evection is zero, the evection may be represented for a short time by

$$4{,}589''.61\cdot\sin\left\{\frac{360^\circ.t}{31.811939}\right\}.$$

This inequality is a variation in the equation of the centre, depending on the position of the apsides of the lunar orbit. When the apsides are in syzigies, as in figure 101, the action of the sun increases the eccentricity of the moon's orbit or the equation of the centre. For if the moon be in conjunction at m, the sun draws her from the earth; and if she be in opposition in m', the sun draws the earth from her; in both cases increasing the moon's distance from the earth, and thereby the eccentricity or equation of the centre. When the moon is in any other point of her orbit, the action of the sun may be resolved into two, one in the direction of the tangent, and the other according to the radius vector. The latter increases the moon's gravitation to the earth, and is at its maximum when the moon is in quadratures; as it tends to diminish the distance QE, it makes the ellipse still more eccentric, which increases the equation of the centre. This increase is the evection. Again, if the line of apsides be at right angles to *S*E, the line joining the centres of the sun and the earth, the action of the sun on the moon at m or m', figure 102, by

fig. 101.

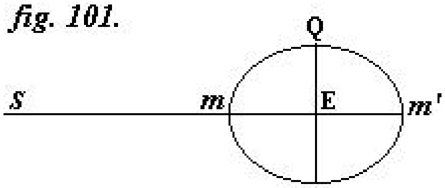
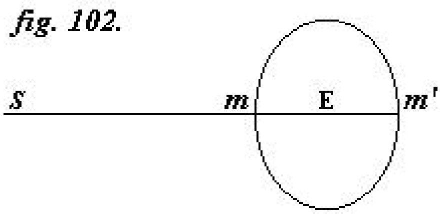


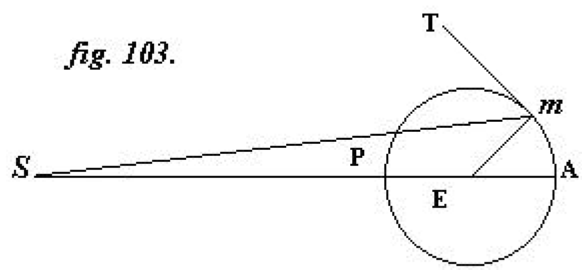
fig. 102.



increasing the distance from the earth, augments the breadth of the orbit, thereby making it approach the circular form, which diminishes the eccentricity. If the moon be in quadratures, the increase in the moon's gravitation diminishes her distance from the earth, which also diminishes the eccentricity, and consequently the equation of the centre. This diminution is the evection. Were the changes in the evection always the same, it would depend on the angular distances of the sun and moon, but its true value varies with the distance of the moon from the perigee of her orbit. The evection was discovered by Ptolemy,[24] in the first century after Christ, but Newton[25] showed on what it depends.

760. The variation is an inequality in the moon's longitude, which increases her velocity before conjunction, and retards her velocity after it. For the sun's force, acting on the moon according to *Sm*, fig. 103, may be resolved into two other forces, one in the direction of *m*E, which produces the evection, and the other in the direction of *m*T, tangent to the lunar orbit. The latter produces the variation which is expressed by

fig. 103.



$$2{,}370''\sin 2\{l - ml\}.$$

This inequality depends on the angular distance of the sun from the moon, and as she runs through her period whilst that distance increases 90°, it must be proportional to the sine of twice the angular distance. Its maximum happens in the octants when $l - ml = 45^\circ$, it is zero when the angular distance of the moon fom the sun is either zero, or when the moon is in quadratures. Thus the variation vanishes in syzigies and quadratures, and is a maximum in the octants.

The angular distance of the moon from the sun depends on its synodic motion: it varies

$$\frac{360^\circ}{29^d.530588}\ \text{daily},$$

and

$$2(l - ml) = \frac{2.360^\circ}{29^d.530588},$$

hence its period is

$$\frac{29^d.530588}{2} = 14^d.765294.$$

Thus the period of the variation is equal to half the moon's synodic revolution. The variation was discovered by Tycho Brahe[26], and was first determined by Newton.[27]

761. The annual equation

$$673''.70\sin\{c'ml - v'\}$$

is another remarkable periodic inequality in the moon's longitude. The action of the sun which produces this inequality is similar to that which causes the acceleration of the moon's mean motion. The annual equation is occasioned by a variation in the sun's distance from the earth, it consequently arises from the eccentricity of the terrestrial orbit. When the sun is in perigee his action is greatest, and he dilates the lunar orbit, so that the angular motion of the moon is diminished; but as the sun approaches the apogee the orbit contracts, and the moon's angular motion is accelerated. This change in the moon's angular velocity is the annual equation. It is a periodic inequality similar to the equation of the centre in the sun's orbit, which retards the motion of the moon when that of the sun increases, and accelerates the motion of the moon when the motion of the sun diminishes, so that the two inequalities have contrary signs.

The period of the annual equation is an anomalistic year. It was discovered by Tycho Brahe[28] by computing the places of the moon for various seasons of the year, and comparing them with observation. He found the observed motion to be slower than the mean motion in the six months employed by the sun in going from perigee to apogee, and the contrary in the other six months. It is evident that as the action of the sun on the moon varies with his distance, and therefore depends on the eccentricity of the earth's orbit, whatever affects the eccentricity must influence all the motions of the moon.

762. The variation has been ascribed to the effect of that part of the sun's force that acts in the direction of the tangent; and the evection to the effect of the part which acts in the direction of the radius vector, and alters the ratio of the perigean and apogean gravities of the moon from that of the inverse squares of the distance. The annual equation does not arise from the direct effect of either, but from an alteration in the mean effect of the sun's disturbing force in the direction of the radius vector which lessens the gravity of the moon to the earth.

763. Although the causes of the lesser inequalities are not so easily traced as those of the four that have been analysed, yet some idea of the sources from whence they arise may be formed by considering that when the moon is in her nodes, she is in the plane of the ecliptic, and the action of the sun being in that plane is resolved into two forces only; one in the direction of the moon's radius vector, and the other in that of the tangent to her orbit. When the moon is in any other part of her orbit, she is either above or below the plane of the ecliptic, and the line joining the sun and moon, which is the direction of the sun's disturbing force, being out of that plane, the sun's force is[29] resolved into three component forces; one in the direction of the moon's radius vector, another in the tangent to her orbit, and the third perpendicular to the plane of her orbit, which affects her latitude. If then the absolute action of the sun be the same in these two positions of the moon, the component forces in the radius vector and tangent must be less than when the moon is in her nodes by the whole action in latitude. Hence any inequality like the evection, whose argument does not depend on the place of the nodes, will be different in these two positions of the moon, and will require a correction, the argument of which should depend on the position of the nodes. This circumstance introduces the inequality

$$54''.83 \cdot \sin\left(2gl - 2l + 2ml - 2q\right)$$

in the moon's longitude. The same cause introduces other inequalities in the moon's longitude, which are the corrections of the variation and annual equation. But the annual equation requires a correction from another cause which will introduce other terms in the perturbations of the moon in longitude; for since it arises from a change in the mean effect of the sun's disturbing force, which diminishes the moon's gravity, its coefficient is computed for a certain value of the moon's gravity, consequently for a given distance of the moon from the earth; hence, when she has a different distance, the annual equation must be corrected to suit that distance.

764. In general, the numerical coefficients of the principal inequalities are computed for particular values of the sun's disturbing force, and of the moon's gravitation; as these are perpetually changing, new inequalities are introduced, which are corrections to the inequalities computed in the first hypothesis. Thus the perturbations are a series of corrections. How far that system is to be carried, depends on the perfection of astronomical instruments, since it is needless to compute quantities that fall within the limits of the errors of observation.

765. When Laplace[30] had determined all the inequalities in the moon's longitude of any magnitude arising from every source of disturbance, he was surprised to find that the mean longitude computed from the tables in Lalande's astronomy[31] for different epochs did not correspond with the mean longitudes computed for the same epochs from the tables of Lahere and Bradley,[32] the difference being as follows:–

Epochs.	Errors.[33]
1766	$-3''$
1779	$9''.3$
1789	$17''.6$
1801	$28''.5$

Whence it was to be presumed that some inequality of a very long period affected the moon's mean motion, which induced him to revise the whole theory of the moon. At last he found that the series which determines the mean longitude contains the term

$$g^2 e'^3 e \cdot \frac{a}{a'} \cdot \frac{\sin\{3v - 3mv + 3c'mv - 2gv - cv + 2q + v - 3v'\}}{\{3 - 3m + 3c'm - 2g - c\}^2} = g^2 e'^3 . e \cdot \frac{a}{a'} \cdot \frac{\sin\{2q + v - 3v'\}}{\{3 - 3m + 3c'm - 2g - c\}^2}$$

depending on the disturbing action of the sun, that appeared to be the cause of these errors.

The coefficient of this inequality is so small that its effect only becomes sensible in consequence of the divisor

$$\{3 - 3m + 3c'm - 2g - c\}^2$$

acquired from the double integration. Its maximum, deduced from the observations of more than a century, is $15''.4$. Its argument is twice the longitude of the ascending node of the lunar orbit, plus the longitude of the perigee, minus three times the longitude of the sun's perigee, whence its period may be found to be about 184 years.

The discovery of this inequality made it necessary to correct the whole lunar tables.

766. By reversion of series the moon's latitude in functions of her mean motion is found to be[34 35]

$$
\begin{aligned}
s = +18{,}539''.8 \cdot \sin(gl - q) \\
+ \quad 12''.6 \cdot \sin(3gl - 3q) \\
+ \quad 527''.7 \cdot \sin(2l - 2mv - gl + q) \\
+ \quad 1''.0 \cdot \sin(2l - 2ml + gl - q) \\
- \quad 1''.3 \cdot \sin(gl + cl - v - q) \\
- \quad 14''.4 \cdot \sin(cl - gl - v + q) \\
+ \quad 1''.8 \cdot \sin(2l - 2ml - gl + cl - v + q) \\
- \quad 0''.3 \cdot \sin(2l - 2ml + gl - cl + v - q) \\
- \quad 15''.8 \cdot \sin(2l - 2ml - gl - cl + v + q) \\
+ \quad 23''.8 \cdot \sin(gl + c'ml - v' - q) \\
- \quad 25''.1 \cdot \sin(gl - c'ml + v' - q) \\
- \quad 10''.3 \cdot \sin(2l - 2ml - gl + c'ml - v' + q) \\
+ \quad 22''.0 \cdot \sin(2l - 2ml - gl - cml + v' + q) \\
+ \quad 25''.7 \cdot \sin(2cl - gl - 2v + q) \\
- \quad 5''.4 \cdot \sin(2l - 2ml - 2cl - gl + 2v + q).
\end{aligned}
$$

767. The only inequality in the moon's latitude that was discovered by observation is

$$527''.7 \sin(2l - 2ml - gl + q).$$

Tycho Brahe[36] observed, in comparing the greatest latitude of the moon in different positions with regard to her nodes, that it was not always the same, but oscillated about its mean value of $5^\circ\, 9'$, and as the greatest latitude is the measure of the inclination of the orbit, it was evident that the inclination varied periodically. Its period is a semi-revolution of the sun with regard to the moon's nodes.

768. By reversion of series it will he found that the lunar parallax at the equator in terms of the mean motions is[37]

$$
\begin{aligned}
\frac{1}{r} = +3{,}420''.89 \\
+ \quad 186''.48 \cos(cl - v) \\
+ \quad 28''.54 \cos(2l - 2ml) \\
+ \quad 34''.43 \cos(2l - 2ml - cl + v) \\
+ \quad 3''.05 \cos(2l - 2ml + cl - v)
\end{aligned}
$$

$$
\begin{aligned}
&- \quad 0''.26\cos(2l-2ml+c'ml-v') \\
&+ \quad 1''.92\cos(2l-2ml-c'ml+v') \\
&- \quad 0''.32\cos(c'ml-v') \\
&- \quad 0''.24\cos(2l-2ml-cl+c'ml+v-v') \\
&+ \quad 1''.45\cos(2l-2ml-cl-c'ml+v+v') \\
&+ \quad 1''.20\cos(cl-c'ml-v+v') \\
&- \quad 0''.92\cos(cl+c'ml-v-v') \\
&+ \quad 10''.24\cos(2cl-2v) \\
&- \quad 0''.41\cos(2cl-2l+2ml-2v) \\
&+ \quad 0''.03\cos(2gl-2q) \\
&- \quad 0''.15\cos(2l-2ml-2gl+2q) \\
&- \quad 0''.70\cos(cl-2gl-v+2q) \\
&- \quad 0''.06\cos(2l-2ml-2gl+cl+2q-v) \\
&- \quad 0''.98\cos(l-ml) \\
&+ \quad 0''.14\cos(l-ml+c'ml-v') \\
&+ \quad 0''.18\cos(2l-2ml+cl-c'ml-v+v') \\
&+ \quad 0''.57\cos(4l-4ml-cl+v) \\
&+ \quad 0''.40\cos(4l-4ml-2cl+2v) \\
&- \quad 0''.03\cos(2l-2ml-2cl-c'ml+2v+v') \\
&+ \quad 0''.14\cos(2cl+2l-2ml-2v).
\end{aligned}
$$

769. The planets are at so great a distance from the sun, and from one another, that their form has no perceptible effect on their mutual motions; and, considered as spheres, their action is the same as if their mass were united in their centre of gravity: but the satellites are so near their respective planets that the ellipticity of the latter has a considerable influence on the motions of the former. This is particularly evident in the moon, whose motions are troubled by the spheroidal form of the earth.

Notes

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[1] The terms $+c'mv$ and $-c'mv$ in the 11th and 12th lines are unaccented in the 1st edition. However, the equivalent terms in equation (239) are accented.
[2] See note 6, *Bk. III, Chap. I.*
[3] France. Bureau des longitudes, *Tables astronomiques par M. Burg / publiees par le bureau des longitudes de France*, Paris : Chez Courcier, 1806.

4 Burckhardt, Johann Carl, *Tables astronomiques, publiees par le Bureau des Longitudes de France: tables de la lune, par M. Burckhardt*, Paris : Veuve Courcier, 1812.
5 See note 55, *Bk. II, Chap. VI.*
6 See note 4, *Introduction.*
7 See note 55, *Preliminary Introduction.*
8 See note 5.
9 See note 15, *Preliminary Dissertation.*
10 See note 54, *Preliminary Dissertation.*
11 Bouvard, Alexis, 1767-1843, astronomer, born in Contamines, France. Bouvard was Director of the Paris Observatory and discovered eight comets. He wrote the *Tables astronomiques* of Jupiter and Saturn in 1808 and of Uranus in 1821. The eventual failure of the Uranus tables led him to speculate later that the errors were due to disturbances from another celestial body. Mary Somerville offered a similar interpretation in the 6th edition of her *On the Connexion of the Physical Sciences* in 1842. Bouvard died three years before the discovery of Neptune in 1846. (see also note 48, *Bk. I Foreword*; notes 28 & 39, *Bk. II Foreword*; note 38, *Bk. II Foreword*; *note 38, Bk. II Chap. XIV*; and *Foreword to the Second Edition.*)
12 See note 49, *Preliminary Dissertation.*
13 The equation *reads* $\frac{3}{2}m\int\left(e'^2-\overline{e}^2\right)dv$ in the 1st edition (published erratum).
14 Clairaut, Alexis Claude, 1713-1765, mathematician, born in Paris, France. Clairaut translated Newton's *Principia* into French, but later when working on the three-body problem (1745), he concluded that Newton's inverse square law was incorrect. By 1748 he had retracted this conclusion and published his new results in *Théorie de la lune* in 1752. Clairaut later used his techniques to calculate the return of Halley's comet in 1759. When the comet returned to perihelion only a month before his predicted date Clairaut became a public hero.
15 Buffon, Georges-Louis Leclerc, comte de, 1707-1788, naturalist, born in Montbard, France. In 1739 he was made director of the Jardin du Roi, and wrote his *Histoire naturelle* (1749-67, Natural History). (see also note 56, *Preliminary Dissertation.*)
16 See note 6, *Bk. III, Chap. I.*
17 See note 32, *Preliminary Dissertation.*
18 The coefficient of the 2nd term *reads* $C_{/}$ in the 1st edition.
19 See note 6, *Bk. III, Chap. I.*
20 See note 6, *Bk. III, Chap. I.*
21 The parentheses in the argument of sine are mismatched in the 1st edition.
22 This value is punctuated with a period and *reads* $\pm 6^{\circ}\ 17'\ 19''.7$. in the 1st edition.
23 The parentheses in the argument of sine are mismatched in the 1st edition.
24 See note 15, *Preliminary Dissertation.*
25 See note 1, *Preliminary Dissertation.*
26 See note 6, *Bk. II, Chap. I.*
27 See note 1, *Preliminary Dissertation.*
28 See note 6, *Bk. II, Chap. I.*
29 The "i" is missing in "is" in the 1st edition.
30 See note 4, *Introduction.*
31 Lalande, Joseph Jérôme Le Français de, 1732-1807, astronomer, born in Bourg-en-Bresse, France. He was professor of Astronomy in the Collège de France, and later director of the Paris Observatory. He determined the lunar parallax in 1751. His major work is *Traité d'astronomie* (1764). He wrote his *Histoire céleste française* in 1801 and produced a comprehensive star catalogue in the same year.
32 See note 38, *Preliminary Dissertation.*
33 The 1801 error in the table *reads* $28''\ 5$ (missing decimal) in the 1st edition.
34 The parentheses in the arguments of the last 13 terms are miss-matched and *read* {argument) in the 1st edition.
35 The period is omitted at the end of this expression in the 1st edition.
36 See note 6, *Bk. II, Chap. I.*
37 The coefficient of the 23rd term *reads* $+0''.4$ in the 1st edition. If this is a misprint it perhaps ought to read $+0''.04$.

BOOK III

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CHAPTER III

INEQUALITIES FROM THE FORM OF THE EARTH

770. THE attraction of the disturbing matter is equal to the sum of all the molecules in the excess of the terrestrial spheroid above a sphere whose radius is half the axis of rotation, each molecule being divided by its distance from the moon; and the finite values of this action, after it has been resolved in the direction of the three co-ordinates of the moon, are the perturbations in longitude, latitude, and distance, caused by the non-sphericity of the earth. In the determination of these inequalities, therefore, results must be anticipated that can only be obtained from the theory of the attraction of spheroids. By that theory it is found that if r be the ellipticity of the earth, R` its mean radius, f the ratio of the centrifugal force at the equator to gravity, and n the sine of the moon's declination, the attraction of the redundant matter at the terrestrial equator is

$$\left(\tfrac{1}{2}f-r\right)\frac{R`}{r^3}\left(n-\tfrac{1}{3}\right)$$

the sum of the masses of the earth and moon being equal to unity. Hence the quantity R, which expresses the disturbing forces of the moon in equation (208) must be augmented by the preceding expression.

771. By spherical trigonometry n, the sine of the moon's declination in functions of her latitude and longitude, is

$$n=\sin w\sqrt{1-s^2}\sin fv+s\cos w\ ,$$

in which w is the obliquity of the ecliptic, s the tangent of the moon's latitude, and fv her true longitude, estimated from the equinox of spring. The part of the disturbing force R that depends on the action of the sun, has the form Qr^2 when the terms depending on the solar parallax are rejected. Hence

$$R=Qr^2-\left(r-\tfrac{1}{2}f\right)\cdot\frac{R`^2}{r^3}\left(\sin^2 w\,.\sin^2 fv+2s\sin w\,.\cos w\,.\sin fv\right)$$

very nearly; but $s=g\sin\left(gv-q\right)$ by article 696, and if

$$\frac{1}{a^3}\text{ be put for }\frac{1}{r^3}$$

[then]

$$R = Qr^2 - \left(r - \tfrac{1}{2}f\right) \cdot \frac{R'^2}{a^3} \sin w \, . \cos w \, . g \cos\left(gv - fv - q\right);$$

when all terms are rejected except those depending on the angle $gv - fv - q$, which alone have a sensible effect in troubling the motion of the moon.

772. If this force be resolved in the direction of the three co-ordinates of the moon, and the resulting values of

$$\frac{dR}{du} \quad \frac{dR}{dv} \quad \frac{dR}{ds}$$

substituted in the equations in article 695, they will determine the effect which the form of the earth has in troubling the motions of that body. But the same inequalities are obtained directly and with more simplicity from the differential of the periodic variation of the epoch in article 439, which, in neglecting the eccentricity of the lunar orbit, becomes

$$d\in = -2a^2 \left(\frac{dR}{da}\right) ndt \, .$$

Now

$$2a^2 \left(\frac{dR}{da}\right) = 4ar^2Q + 6\left(r - \tfrac{1}{2}f\right) \cdot \frac{R'^2}{a^2} \cdot \sin w \, . \cos w \, . g \cos\left(gv - fv - q\right).$$

But by article 438 the variation of dR is zero, consequently the coefficient of $\cos\left(gv - fv - q\right)$ must be zero in R. Then if $d \, . r^2Q$ be the part of r^2Q that depends on the compression of the earth,

$$0 = d \, . r^2Q - \left(ar - \tfrac{1}{2}af\right) \cdot \frac{R'^2}{a^3} \cdot \sin w \, . \cos w \, . g \cos\left(gv - fv - q\right),$$

and eliminating $d\, r^2Q$,

$$2da^2 \left(\frac{dR}{da}\right) = 10\left(r - \tfrac{1}{2}f\right) \frac{R'^2}{a^3} \cdot \sin w \, . \cos w \, . g \cos\left(gv - fv - q\right).$$

And if dv be put for ndt,

$$d\in = -10\left(r - \tfrac{1}{2}f\right) \cdot \frac{R'^2}{a^3} \cdot \sin w \, . \cos w \, . g \, dv \, . \cos\left(gv - fv - q\right).$$

This equation is referred to the plane of the lunar orbit, but in order to reduce it to the plane of the ecliptic the equation (154) must be resumed, which is

$$dv_{/} = dv\left(1 + \tfrac{1}{2}s^2 - \tfrac{1}{2}\frac{ds^2}{dv^2}\right)$$

$dv_{/}$ being the arc dv projected on the plane of the ecliptic, or fixed plane. By article 436

$$s = q\sin fv - p\cos fv,$$

whence

$$\frac{ds}{dv} = \left(fq - \frac{dp}{dv}\right)\cos fv + \left(fp + \frac{dq}{dv}\right)\sin fv + \&c.$$

and neglecting periodic quantities depending on fv,

$$dv_{/} = dv + \frac{qdp - pdq}{2}, \text{ very nearly}.$$

Hence, in order to have $d.dv_{/}$ it is only necessary to add

$$\frac{qdp - pdq}{2} \text{ to } d.dv.$$

For the same reason the angle $d\in$ will be projected on the plane of the ecliptic if $\frac{qdp - pdq}{2}$ be added to it, so that

$$d\in_{/} = d\in + \frac{qdp - pdq}{2}.$$

Now $s = g\sin(gv - q)$ may be put under the form

$$s = g\cos(gv - fv - q)\sin fv + g\sin(gv - fv - q)\cos fv,$$

and comparing it with

$$s = q\sin fv - p\cos fv,$$

the result is

$$p = -g\sin(gv - fv - q) \quad q = g\cos(gv - fv - q),$$

whence

$$dp = -(g - f)qdv$$
$$dq = +(g - f)pdv$$

$$R = r^2 Q - \left(r - \tfrac{1}{2} f \right) \frac{R'^2}{a^2} \sin w \cos w \,.\, q,$$

and

$$\frac{dR}{dq} = -\left(r - \tfrac{1}{2} f \right) \frac{R'^2}{a^2} \sin w \cos w;$$

in consequence of this the values of *dp*, *dq*, in article 439, become

$$dp = -\left(g - f \right) q dv + \left(r - \tfrac{1}{2} f \right) \frac{R'^2}{a^2} \sin w \cos w \cdot dv$$
$$dq = +\left(g - f \right) p dv;$$

therefore *dp* contains the term

$$\frac{\left(r - \tfrac{1}{2} f \right)}{g - f} \cdot \frac{R'^2}{a^2} \sin w \cos w \,.\, dv;$$

and as $ds = dq \,.\sin fv - dp \cos fv,$ the latitude of the moon is subject to the inequality

$$-\frac{\left(r - \tfrac{1}{2} f \right)}{g - f} \cdot \frac{R'^2}{a^2} \sin w \cos w g \sin fv. \qquad \textbf{(243)}$$

773. The constant part of q produces in $\dfrac{qdp - pdq}{2}$ the term[1]

$$+\tfrac{1}{2} \left(r - \tfrac{1}{2} f \right) \frac{R'^2}{a^2} \cdot \sin w \cos w \,.\, g \cos\left(gv - fv - q \right);$$

whence $d\in_{/}$, which is the value of $d\in$ when referred to the plane of the ecliptic, becomes

$$d\in = -\frac{19}{2} \cdot \left(r - \tfrac{1}{2} f \right) \frac{R'^2}{a^2} \sin w \cos w \,.\, g \cos\left(gv - fv - q \right) . dv,$$

which gives in $\in$, and consequently in the true longitude of the moon the inequality

$$-\frac{19}{2} \cdot \frac{\left(r - \tfrac{1}{2} f \right)}{g - f} \cdot \frac{R'^2}{a^2} \cdot \sin w \cos w \,.\, g \sin\left(gv - fv - q \right). \qquad \textbf{(244)}$$

774. Now,

$$\frac{R'}{a} = 0.0165695, \qquad w = 23^\circ\ 28', \text{ \&c.}$$
$$f = \frac{1}{289}, \quad g = 0.0900684, \quad g = 1.00402175,$$

and the argument $gv - fv - q$ is the mean longitude of the moon. Thus every quantity is known, except r, the compression, which may therefore be determined by comparing the coefficient, computed with these data, with the coefficient of the same inequality given by observation. By Burg's Tables,[2] it is $-6''.8$, and by Burckhardt's,[3] $-7''.0$. The mean of these $-6''.9$ gives the compression

$$r = \frac{1}{303.22}.$$

By the theory of the rotation of spheroids, it is found that if the earth be homogeneous, the compression is $\frac{1}{230}$. Consequently the earth is of variable density.

That the inequalities of the moon should disclose the interior structure of the earth, is a singular instance of the power of analysis.

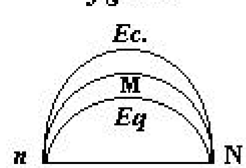
775. The inequality in the moon's latitude, depending on the same cause, confirms these results. Its coefficient, determined by Burg[4] and Burckhardt[5] from the combined observations of Maskelyne[6] and Bradley,[7] is $-8''.0$, which, compared with the coefficient of

$$-\frac{\left(r - \frac{1}{2}f\right)}{g - f} \cdot \frac{R^2}{a^2} \sin w \cos w \,.\, g \sin fv,$$

computed with the preceding data, gives $\frac{1}{305.26}$ for the compression, which also proves that the earth is not homogeneous.

776. Since the coefficients of both inequalities are greater in supposing the earth to be homogeneous, it affords another proof that the gravitation of the moon to the earth is composed of the attraction of all its particles. Thus the eclipses of the moon in the early ages of astronomy showed the earth to be spherical, and her motions, when perfectly known, determine its deviation from that figure. The ellipticity of the earth, obtained from the motions of the moon, being independent of the irregularities of its form, has an advantage over that deduced from observations with the pendulum, and from the arcs of the meridian.

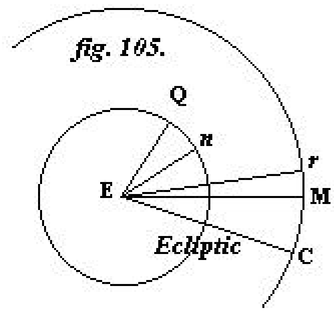
fig. 104.



777. The inequality in the moon's latitude, arising from the ellipticity of the earth, may be represented by supposing that the orbit of the moon, in place of moving with the earth on the plane of the ecliptic, and preserving the same inclination of $5^\circ\ 9'$ to that plane, moves with a constant inclination of $8''$ on a plane NM*n* passing between the ecliptic and the equator, and through *n*N, the line of the equinoxes. The inequality in question diminishes the inclination of the lunar orbit to the ecliptic, when its ascending node coincides with the equinox of spring; it augments it when this node coincides with the autumnal equinox.

778. This inequality is the re-action of the nutation in the terrestrial axis, discovered by Bradley;[8] hence there would be equilibrium round the centre of gravity of the earth, in consequence of the forces which produce the terrestrial nutation and this inequality in the moon's latitude, if all the molecules of the earth and moon were fixedly united by means of a lever; the moon compensating the smallness of the force which acts on her by the length of the lever to which she is attached, for the distance of the common centre of gravity of the earth and moon from the centre of the earth is less than the earth's semidiameter.

fig. 105.



The proof of this depends on the rotation of the earth; but some idea may be formed of this re-action from the annexed diagram. Let EC be the plane of the ecliptic, seen edgewise; Q the earth's equator; E its centre, and M the moon. Then QEC is the obliquity of the ecliptic, and MEC the latitude of the moon.

The moon, by her action on the redundant matter at Q, draws the equator to some point *n* nearer to the ecliptic, producing the nutation QE*n*; but as re-action is equal and contrary to action, the matter at Q draws the moon from M to some point *r*, thereby producing the inequality ME*r* in her latitude, that has been determined. Laplace[9] finds the analytical expressions of the areas ME*r* and QE*n*, and thence their moments; the one from the preceding inequality in the moon's latitude, the other from the formulae of nutation in the axis of the earth's rotation from the direct action of the moon. These two expressions are identical, but with contrary signs, proving them, as he supposed, to be the effects of the direct and reflected action of the moon.

779. The form of the earth increases the motion of the lunar nodes and perigee by $0.000000085484v$, an insensible quantity. The ellipticity of the lunar spheroid has no perceptible effect on her motion.

Notes

[1] The factor $\frac{R^{-2}}{a^2}$ *reads* $\frac{R^2}{a^2}$ in the 1st edition.
[2] See note 3, *Bk. III, Chap. II.*
[3] See note 4, *Bk. III, Chap. II.*
[4] See note 2.
[5] See note 3.
[6] See note 55, *Bk. II, Chap. VI.*
[7] See note 38, *Preliminary Dissertation.*
[8] See note 7.
[9] See note 4, *Introduction.*

BOOK III

CHAPTER IV

INEQUALITIES FROM THE ACTION OF THE PLANETS

780. THE action of the planets produces three different kinds of inequalities in the motions of the moon. The first, and by far the greatest, arising from their influence on the eccentricity of the earth's orbit, which is the cause of the secular inequalities in the mean motion, in the perigee, and nodes of the lunar orbit. The other two are periodic inequalities in the moon's longitude; one from the direct action of the planets on the moon, the other from the perturbations they occasion in the longitude and radius vector of the earth, which are reflected back to the moon by means of the sun.

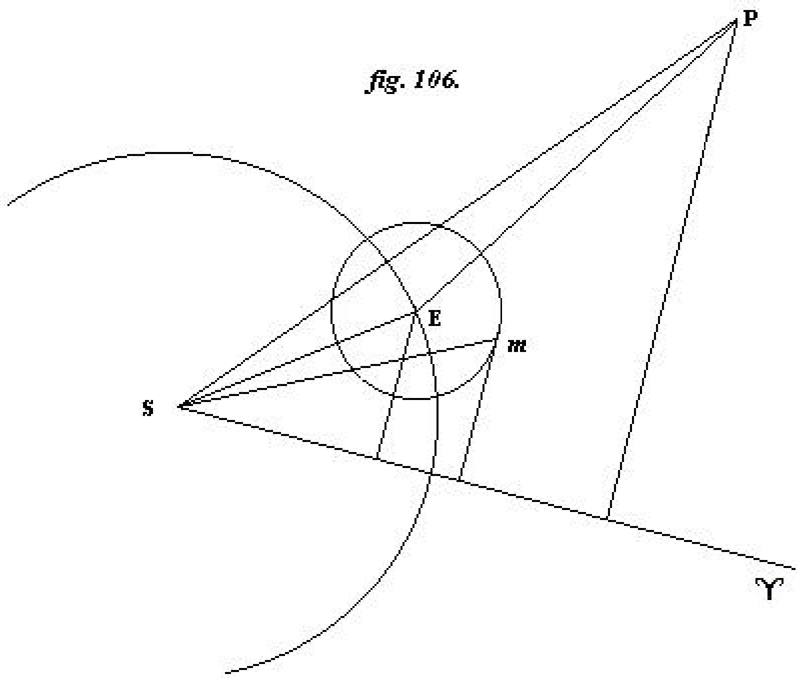


fig. 106.

For, let S be the sun, E and m the earth and moon, P a planet, and *g* the first point of Aries: then, if P be the mass of the planet, its direct action on the moon is $\frac{P}{(Pm)^2}$, which alters the position of the moon with regard to the earth. Again, the disturbing action of the planet on the earth is $\frac{P}{(PE)^2}$, which changes the position of the earth with regard to the moon, in each case producing inequalities of the same order. The latter become sensible from the very small divisors they acquire by integration.

The direct action will be determined first.

If $X`$, $Y`$, $Z`$, x, y, z, be the co-ordinates of the planet and moon, referred to the centre of the earth, and f the distance of the planet from this centre, then

$$f=\sqrt{(X`-x)^2+(Y`-y)^2+(Z`-z)^2}\,.$$

But if X', Y', Z', x', y', z', be the co-ordinates of the planet and the earth referred to the centre of the sun,

$$X` = X' - x', \quad Y` = Y' - y', \quad Z` = Z' - z';$$

and

$$f = \sqrt{\left(X' - (x' + x)\right)^2 + \left(Y' - (y' + y)\right)^2 + \left(Z' - (z' + z)\right)^2};$$

and the attraction of the planet on the moon is

$$\frac{P}{f} - \frac{\tfrac{1}{2}Pr^2}{f^3} + \tfrac{3}{2}P\frac{\left(X'x + Y'y + Z'z - xx' - yy' - zz'\right)^2}{f^5} + \&\text{c.}$$

The ecliptic being the fixed plane,

$$z' = 0, \quad r' = \frac{1}{u'}, \quad v' = g\,SE.$$

Then, if $R_{/} = SP$, $U = g\,SP$, and S, be the radius vector, longitude, and heliocentric latitude of the planet, it is evident that

$$x' = \frac{\cos v'}{u'}, \qquad y' = \frac{\sin v'}{u'}, \qquad r' = \frac{\sqrt{1+s^2}}{u'},$$
$$X' = R_{/}\cos U, \quad Y' = R_{/}\sin U, \quad Z' = R_{/}S;$$

hence

$$f = \sqrt{R^2\left(1+S^2\right) + r'^2 - 2Rr'\cos\left(U - v'\right)};$$

therefore the action of the planet on the moon is

$$\frac{P}{f} - \frac{\tfrac{1}{2}P\left(1+s^2\right)}{u^2 f^3} + \tfrac{3}{2}P\frac{\left(R_{/}\cos(v-U) - r\cos(v-v') + R_{/}sS\right)^2}{u^2 f^5} + \&\text{c.}$$

or, omitting S^2, it is

$$\frac{P}{f} + \frac{P\left(1-2s^2\right)}{4u^2 f^3} + \&\text{c. }\&\text{c.}$$

The first term does not contain the co-ordinates of the moon, and therefore does not affect her motion; and the only term of the remainder of the series that has a sensible influence is $\frac{P}{4u^2 f^3}$, which, therefore, forms a part of R in (208); and, with regard to the action of the planets alone, $R = \frac{P}{4u^2 f^3}$. But, by article 446, the development of f is

$$\tfrac{1}{2}A^{(0)}+A^{(1)}\cos(U-v')+A^{(2)}\cos 2(U-v')+\&c.$$

If i be the ratio of the mean motion of the planet to that of the moon, by equation (212)

$$U = iv - 2ie\sin(cv - v\ ) + \&c.$$

Hence, if iv be put for U, and mv for v', it is evident that

$$R = \frac{P}{4u^2}\left\{\tfrac{1}{2}A^{(0)}+A^{(1)}\cos(i-m)v+A^{(2)}\cos 2(i-m)v+\&c.\right\}$$

The only term of the parallax in which this value of R is sensible is

$$-\frac{1}{h^2}\left(\frac{dR}{du}\right)$$

which becomes

$$\frac{PA^{(0)}}{4h^2u^3}+\frac{P}{2h^2u^3}\left\{A^{(1)}\cos(i-m)v+A^{(2)}\cos 2(i-m)v+\&c.\right\};$$

or, if e^2 and g^2 be neglected, $u^{-3}=a^3$, and the periodic part of $-\frac{1}{h^2}\left(\frac{dR}{du}\right)$ is

$$\frac{Pa^3}{2h^2}\left\{A^{(1)}\cos(i-m)v+A^{(2)}\cos 2(i-m)v+\&c.\right\}$$

But, by the second of equations (209), $-\frac{1}{h^2}\left(\frac{dR}{du}\right)$ contains the variation of $\frac{m'u'^3}{2h^2u^3}$ which is

$$-\frac{3m'u'^3}{2h^2u^4}\delta u = -\frac{3m^2}{2}\cdot\delta u\ .$$

Let

$$\delta u = G_1\cos(i-m)v+G_2\cos 2(i-m)v+G_3\cos 3(i-m)v+\&c.$$

Therefore the direct disturbance of the planets gives

$$\frac{d^2u}{dv^2}+u = -\frac{Pa^3}{2}\left\{A_1\cos(i-m)v+A_2\cos 2(i-m)v+\&c.\right\}$$
$$+\frac{3m^2}{2}\left\{G_1\cos(i-m)v+G_2\cos 2(i-m)v+\&c.\right\} =$$
$$G_1\left(1-(i-m)^2\right)\cos(i-m)v+G_2\left(1-4(i-m)^2\right)\cos 2(i-m)v+\&c.$$

And comparing similar cosines,[1]

$$G_1 = -\frac{\tfrac{1}{2}P.\,A_1.a^3}{1-\tfrac{3}{2}m^2-(1-m)^2}$$
$$G_2 = -\frac{\tfrac{1}{2}P.\,A_2.a^3}{1-\tfrac{3}{2}m^2-4(1-m)^2}$$
$$G_3 = -\frac{\tfrac{1}{2}P.\,A_3.a^3}{1-\tfrac{3}{2}m^2-9(1-m)^2}$$
$$\&c. \qquad \&c.$$

and thus the integral u or (228) acquires the term

$$-\tfrac{1}{2}Pa^3\left\{\frac{A_1\cos(i-m)v}{1-\tfrac{3}{2}m^2-(i-m)^2}+\frac{A_2\cos 2(i-m)v}{1-\tfrac{3}{2}m^2-4(i-m)^2}+\&c.\right\}$$

consequently, the mean longitude $nt+\epsilon$ contains the term

$$\frac{Pa^3}{i-m}\left\{\frac{A_1\sin(i-m)v}{1-\tfrac{3}{2}m^2-(i-m)^2}+\frac{\tfrac{1}{2}A_2\sin 2(i-m)v}{1-\tfrac{3}{2}m^2-4(i-m)^2}+\&c.\right\}$$

or if a^3 be eliminated by $\dfrac{m'a^3}{a'^3}=m^2$

$$\frac{\dfrac{P}{m'}m^2a'^3}{i-m}\left\{\frac{A_1\sin(i-m)v}{1-\tfrac{3}{2}m^2-(i-m)^2}+\frac{\tfrac{1}{2}A_2\sin 2(i-m)v}{1-\tfrac{3}{2}m^2-4(i-m)^2}+\&c.\right\} \qquad \textbf{(245)}$$

m' being the mass of the sun.

If B_1, B_2, &c., be put for A_1, A_2, &c., it becomes

$$\frac{\dfrac{P}{m'}m^2a^3}{i-m}\left\{\frac{B_1\sin(i-m)v}{1-\tfrac{3}{2}m^2-(i-m)^2}+\frac{\tfrac{1}{2}B_2\sin 2(i-m)v}{1-\tfrac{3}{2}m^2-4(i-m)^2}+\&c.\right\} \qquad \textbf{(246)}$$

which is the inequality in the moon's mean longitude, arising from the action of a planet inferior to the earth.

And if a be the ratio of the mean distance of the planet from the sun to that of the sun from the earth, the substitution of a^3B_1, a^3B_2, &c., for A_1, A_2, &c., in equation (245), gives[2]

$$\frac{\frac{P}{m'}m^2 \cdot a'^3a^3}{i-m}\left\{\frac{B_1\sin(i-m)v}{1-\frac{3}{2}m^2-(i-m)^2}+\frac{\frac{1}{2}B_2\sin 2(i-m)v}{1-\frac{3}{2}m^2-4(i-m)^2}+\text{\&c.}\right\} \qquad \textbf{(247)}$$

for the action of a superior planet on the mean longitude of the moon.

781. Besides these disturbances, which are occasioned by the direct action of the planets on the moon, there are others of the same order caused by the perturbations in the radius vector of the earth. The variation of u' was omitted in the development of the coordinates of the moon, but

$$d\cdot\frac{m'u'^3}{2h^2u^3}=\frac{3m'u'^2}{2h^2u^3}du'$$

and when the eccentricities are omitted,

$$h^2=a, \text{ and } \frac{m'a^3}{a'^2}=a'm^2.$$

So

$$d\,\frac{m'u'^3}{2h^2u^3}=\frac{3a'm^2}{2a}du';$$

since $du'=\frac{dr'}{a'}$ are the periodic inequalities in the radius vector of the earth produced by the action of a planet, they are given in (158), and may be represented by

$$a'du'=-\frac{P}{m'}\{K_1\cos(i-m)v+K_2\cos 2(i-m)v+\text{\&c.}\}$$

where the coefficients K_1, K_2, &c. are known, and $(i-m)v$ is the mean longitude of the planet minus that of the earth. Thus

$$\frac{3a'm^2}{2a}du'=-\frac{3m^2}{2}\cdot\frac{P}{m'}\{K_1\cos(i-m)v+K_2\cos 2(i-m)v+\text{\&c.}\}$$

By the method of indeterminate coefficients, it will be found that adu contains the function

$$\frac{3m^2}{2}\cdot\frac{P}{m'}\left\{\frac{K_1\cos(i-m)v}{1-\frac{3}{2}m^2-(i-m)^2}+\frac{K_2\cos 2(i-m)v}{1-\frac{3}{2}m^2-4(i-m)^2}+\text{\&c.}\right\}$$

and the mean longitude of the moon is subject to the inequality

$$-\frac{3m^2}{i-m}\cdot\frac{P}{m'}\left\{\frac{K_1\cos(i-m)v}{1-\frac{3}{2}m^2-(i-m)^2}+\frac{K_2\cos 2(i-m)v}{1-\frac{3}{2}m^2-4(i-m)^2}+\&c.\right\} \qquad \textbf{(248)}$$

Numerical Values of the Lunar Inequalities occasioned by the Action of the Planets

782. With regard to the action of Venus, the data in articles 611 and 610 give

$$a=0.7233325; \quad i-m=0.04679$$

and

$$\frac{P}{m'}=\frac{1}{356{,}632};$$

hence because

$$a'^3B_1=8.872894,$$
$$a'^3B_2=7.386580,$$
$$a'^3B_3=5.953940,$$

function (246) becomes

$$+0''.62015\sin(i-m)v+0''.25990\sin 2(i-m)v+0''.14125\sin 3(i-m)v$$

which is the direct action of Venus on the moon. Now $dr'=-\frac{du'}{u'^2}$, and when the eccentricity is omitted, $u'^2=\frac{1}{a'^2}$; hence $\frac{dr'}{a'}=-a'du'$. But if the action of Venus on the radius vector of the earth be computed by the formula (158), it will be found that

$$\begin{aligned}a'du'=&+0.0000064475\cos(i-m)v\\&-0.0000184164\cos 2(i-m)v\\&+0.000002908\cos 3(i-m)v.\end{aligned}$$

This gives the numerical values of the coefficients K^0, K^1, &c.; hence formula (248) becomes

$$\begin{aligned}&+0''.482200\sin(i-m)v\\&-0''.69336\sin 2(i-m)v\\&-0''.07380\sin 3(i-v)v,\end{aligned}$$

which is the indirect action of the planets on the moon's longitude. Added to the preceding the sum is

$$\begin{aligned}&+1''.10235.\sin(i-m)v\\&-0''.43336.\sin 2(i-m)v\\&+0''.06745.\sin 3(i-m)v,\end{aligned}$$

the whole action of Venus on the moon's mean longitude.

783. Relative to Mars:

$$\begin{aligned}a &= 0.65630030\\a'^3B_1 &= 5.727893\\a'^3B_2 &= 4.404530\\a'^3B_3 &= 3.255964\\i-m &= -0.0350306\\\frac{P}{m} &= \frac{1}{1,846,082}.\end{aligned}$$

and by formula (158) with regard to Mars,

$$\begin{aligned}a'du' = &+0''.00000017778\ \cos(i-m)v\\&+0''.0000026121\quad \cos 2(i-m)v\\&+0''.000000111\quad \cos 3(i-m)v;\end{aligned}$$

whence the action of Mars on the moon's mean longitude, both direct and indirect, is

$$\begin{aligned}&+0''.025583.\sin(i-m)v\\&+0''.389283.\sin 2(i-m)v\\&-0''.027337.\sin 3(i-m)v.\end{aligned}$$

784. With regard to Jupiter,

$$\begin{aligned}a &= 0.192205\\a'^3B_1 &= 0.618817\\a'^3B_2 &= 0.147980\\a'^3B_3 &= 0.0331045\\i-m &= -0.06849523\end{aligned}$$

$$\frac{P}{m} = \frac{1}{1{,}067.09}.$$

And formula (158) gives for the action of Jupiter on the radius vector of the earth,

$$\begin{aligned} a'du' = &- 0.0000159055 \quad \cos(i-m)v \\ &- 0.0000090791 \quad \cos 2(i-m)v \\ &- 0.00000064764 \cos 3(i-m)v. \end{aligned}$$

Whence it is easy to see that the whole action of Jupiter on the mean longitude of the moon, both direct and indirect, is

$$\begin{aligned} &+0''.74435 . \sin(i-m)v \\ &-0''.24440 . \sin 2(i-m)v \\ &-0''.01282 . \sin 3(i-m)v. \end{aligned}$$

If all these inequalities, resulting from the action of the planets on the moon, be taken with a contrary sign, we shall have the inequalities that this action produces in the expression of the true longitude of the moon, $(i-m)v$ being supposed equal to the mean motion of the planet minus that of the earth.

785. The secular action of the planets on the moon, and the elements of her orbit, may be determined from the term $\frac{PA^0}{4h^2u^3}$; but as it is insensible, the investigation may be omitted.

Notes

[1] The left hand side of the 2nd equation *reads* G^2 in the 1st edition.

[2] The numerator in the 2nd term of equation (247) *reads* $\frac{1}{2}B_2 \sin(i-m)v$ in the 1st edition.

BOOK III

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CHAPTER V

EFFECTS OF THE SECULAR VARIATION IN THE PLANE OF THE ECLIPTIC

780. HAVING developed all the inequalities to which the moon is subject, we shall now show that the secular variation in the plane of the ecliptic has no effect on the inclination of the lunar orbit.

The latitude of the earth s', being extremely small, was omitted in the values of R, No. (208): it can only arise from disturbances either secular or periodic: both oscillate between fixed limits; but we shall suppose s' to relate only to the secular variations in the plane of the ecliptic, and according to equations (138) shall only assume it to be equal to a series of terms of and form,

$$\sum K\cdot\sin\left(v'+it+\in\right),$$

i being a very small coefficient. Then omitting quantities of the order s^3, the tangent of the moon's latitude is

$$s=g\ \sin\left(gv-q\right)+\sum K\sin\left(v+it+\in\right)+d\,s;$$

equation (205), which determines the latitude, is

$$0=\frac{d^2s}{dv^2}+s+\frac{3m'u'^3s}{2h^2u^4}-\frac{3mu'^3s}{2h^2u^4}-\frac{3mu'^3s}{h^2u^4}\cos\left(v-v'\right)+\frac{3mu'^3s}{2h^2u^4}d\,s.$$

Now

$$\frac{3m'u'^3s}{2h^2u^4}=\frac{3m^2}{2}\cdot\frac{a'}{a}\sum K\sin\left(v+it+\in\right).$$

The following term gives the same quantity with a contrary sign. And if

$$d\,s=\sum bK\sin\left(v+it+\in\right),$$

the last term gives

$$\frac{3m^2}{2}\frac{a'}{a}\sum bK\sin\left(v+it+\in\right),$$

so that the differential equation of the moon's latitude becomes

$$0=\frac{d^2s}{dv^2}+s+\frac{3m^2}{2}\cdot\frac{a'}{a}\sum bK\sin\left(v+it+\in\right)$$

and if $\sum bK \sin(v+it+\in)$ be put for $\frac{d^2 s}{dv^2}+s,$ the equation becomes

$$0=\sum(1+b)K\left\{1+(1+i)^2\right\}\sin(v+iv+\in)+\frac{3m^2}{2}\frac{a'}{a}\sum bK\sin(v+iv+\in)$$

for *iv* may be put for it, whence

$$b=-\frac{1-(1+i)^2}{1-(1+i)^2+\frac{3m^2}{2}\cdot\frac{a'}{a}}=\frac{2i+i^2}{\frac{3m^2}{2}\cdot\frac{a'}{a}-2i-i^2}.$$

Hence the variation of *s*, the moon's latitude, with regard to the secular motion of the ecliptic is [1]

$$\frac{\sum\left(2i+i^2\right)K\sin(v+iv+\in)}{\frac{3m^2}{2}\cdot\frac{a'}{a}-2i-i^2}$$

This quantity is insensible, for *iv* is only about $16''$ a year, and

$$\frac{3m^2}{3}\cdot\frac{a'}{a}$$

being nearly $40^\circ\ 37'$, the value of the factor

$$\frac{2i+i^2}{\frac{3m^2}{2}\cdot\frac{a'}{a}-2i-i^2}$$

is only $0''.00022$.

So that the ecliptic in its motion carries the orbit of the moon along with it.

787. The coincidence of theory with observation, in explaining the inequalities in the motions of the moon, affords the most conclusive proof of the universality of the law of gravitation. Having deduced all these inequalities from that one cause, Laplace[2] established the correctness of the results obtained by analysis by comparing them with the lunar tables computed by Mason[3] from 1137 observations made by Bradley[4] between the years 1750 and 1760, and corrected by Burg[5] by means of upwards of 3000 observations made by Maskelyne[6] between the years 1765 and 1793. He had the satisfaction to find that the greatest difference did not exceed $8''$ in the longitude, while the difference in latitude was only $1''.94$, a degree of accuracy sufficient to warrant the tables of latitude being regarded as equivalent to the result of theory: the approximations in latitude, indeed, are more simple and convergent than those in longitude. The inequalities in the lunar parallax are so small, that theory will determine them more correctly than observation. Accurate as these results are, it is still possible that the motions of the moon may be affected by the resistance of an ethereal medium surrounding the sun.

Notes

[1] The term i^2 in the denominator *reads* $i2$ in the 1st edition.
[2] See note 4, *Introduction*.
[3] Mason, Charles, 1730-1787, astronomer, known for the "Mason–Dixon Line" in the USA. He was an assistant at Greenwich Observatory, and observed the 1761 transit of Venus at the Cape of Good Hope with the English surveyor Jeremiah Dixon.
[4] See note 38, *Preliminary Dissertation*.
[5] See note 3, *Bk. III, Chap. II.*
[6] See note 55, *Bk. II, Chap. VI.*

BOOK III

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CHAPTER VI

EFFECTS OF AN ETHEREAL MEDIUM ON THE MOTIONS OF THE MOON

788. IN order to determine its effects in the hypothesis of its existence, let x, y, z be the co-ordinates of the moon referred to the centre of gravity of the earth, and x', y', z' those of the earth referred to the centre of the sun. The absolute velocity of the moon round the sun will be

$$\frac{\sqrt{(dx+dx')^2+(dy+dy')^2+(dz+dz')^2}}{dt}.$$

If K be a coefficient depending on the density of the ether, on the surface of the moon, and on her density; and if the resistance of the ether be assumed proportional to the square of the velocity, it will be

$$\frac{K\left\{(dx+dx')^2+(dy+dy')^2+(dz+dz')^2\right\}}{dt^2}.$$

In the same manner[1]

$$\frac{K'\left(dx'^2+dy'^2+dz'^2\right)}{dt^2}$$

is the resistance the earth experiences from the ether, K' being a coefficient for the earth similar to, but different from K. In the theory of the moon the earth is assumed to be at rest, therefore this resistance must be in a contrary direction from that acting on the moon, consequently the whole action of the ether in disturbing the moon will be the difference of these forces: so with regard to the action of the ether alone, (208) becomes

$$R=\frac{K'\left(dx'^2+dy'^2+dz'^2\right)}{dt^2}-\frac{K\left\{(dx+dx')^2+(dy+dy')^2+(dz+dz')^2\right\}}{dt^2}$$

and because the resistance is in the plane of the orbit, its component forces are parallel to the axes x and y only; hence

$$\frac{dR}{dx}=K'\frac{dx'}{dt^2}\cdot\sqrt{dx'^2+dy'^2+dz'^2}-K\frac{(dx+dx')}{dt^2}\cdot\sqrt{(dx+dx')^2+(dy+dy')^2+(dz+dz')^2}$$
$$\frac{dR}{dy}=K'\frac{dy'}{dt^2}\cdot\sqrt{dx'^2+dy'^2+dz'^2}-K\frac{(dy+dy')}{dt^2}\cdot\sqrt{(dx+dx')^2+(dy+dy')^2+(dz+dz')^2}$$

But in the theory of the moon

$$x=\frac{\cos v}{u},\ \ y=\frac{\sin v}{u},\ \ z=\frac{s}{u},\ \ x'=\frac{\cos v'}{u'},\ \ y'=\frac{\sin v'}{u'},$$

and if the ecliptic of 1750 be assumed as the fixed plane $z'=0:v'$ is the heliocentric longitude of the earth.

Let $\sqrt{dx'^2+dy'^2+dz'^2}$, the little arc described by the earth in the time dt be represented by $r'ds'$. This arc is to that described by the moon in her relative motion round the earth as $\frac{a'm}{a}$ to unity, consequently at least thirty times as great. If the eccentricity of the terrestrial orbit be omitted, $ds'=mdt$. If these quantities be substituted for the co-ordinates

$$\sqrt{\left(dx+dx'\right)^2+\left(dy+dy'\right)^2+\left(dz+dz'\right)^2}=ma'dt-dx\,.\sin v'+dy\,.\cos v';$$

and if quantities depending on the arc $2v'$ be rejected,

$$\begin{aligned}\frac{dR}{dx}&=\frac{\left(K-K'\right)m^2}{u'^2}\cdot\sin v'-\frac{3Km}{2u'}\cdot\frac{dx}{dt}\\ \frac{dR}{dy}&=\frac{\left(K-K'\right)m^2}{u'^2}\cdot\cos v'-\frac{3Km}{2u'}\cdot\frac{dy}{dt}.\end{aligned} \qquad \textbf{(249)}$$

But

$$d\frac{1}{a}=-2dR=-2dx\left(\frac{dR}{dx}\right)-2dy\left(\frac{dR}{dy}\right). \qquad \textbf{(250)}$$

and[2]

$$d\frac{1}{a}=-\frac{2\left(K-K'\right)m^2}{u'^2}\cdot\left\{dx\cdot\sin v'-dy\,.\cos v'\right\}+\frac{3\left(K-K'\right)m^2}{u'}\cdot\left\{\frac{dx^2+dy^2}{dt}\right\}. \qquad \textbf{(251)}$$

The different quantities contained in this equation must now be determined.

789. The distance of the moon from the earth is $Em=\frac{1}{u}$, that of the earth from the sun is $ES=\frac{1}{u'}$, and that of the moon from the sun is

$$mS=u'\sqrt{1+\frac{u'^2}{u^2}-2\frac{u'}{u}\cos\left(v-v'\right)}$$

but $\frac{u'^2}{u^2}$ is a very small fraction that may be omitted; consequently, when the square root is extracted, the distance of the moon from the sun is

$$mS = u' - \frac{u'^2}{u} \cdot \cos(v - v').$$

If we assume the density of the ether to be proportional to a function of the distance from the sun, and represent that function by $f(u')$, with regard to the moon, it will be

$$f(u') - \frac{u'^2}{u} \cdot f'(u') . \cos(v - v')$$

$f'(u')$ being the differential of $f(u')$ divided by du'. As K is a quantity depending on the density of the ether it is variable, hence it may be assumed that

$$K = H . f(u') - \frac{Hu'^2}{u} \cdot f'(u') . \cos(v - v').$$

But as

$$x = \frac{\cos v}{u}, \;\; y = \frac{\sin v}{u}, \;\; u = \frac{1}{a}\left(1 + e\cos(cv - v)\right),$$

therefore

$$dx = -a^2\left(udv . \sin v + du . \cos v\right) . \left(1 - 2e\cos(cv - v)\right),$$
$$dy = +a^2\left(udv \cdot \cos v - du \cdot \sin v\right) \cdot \left(1 - 2e\cos(cv - v)\right),$$

also

$$dt = dv\left(1 - 2e \cdot \cos(cv - v)\right).$$

790. By the substitution of these quantities in equation[3] (251) it will be found, after rejecting periodic quantities, and integrating, that[4]

$$\frac{1}{a} = -Hma^3\left\{\frac{3f(u')}{u'} - m \cdot f'(u')\right\} \cdot v + Hma^3\left\{\frac{6f(u')}{u'} - \frac{9}{2}m \cdot f'(u')\right\} \cdot e\sin(cv - v),$$

which is the secular variation in the mean parallax of the moon in consequence of the resistance of the ether.

In order to abridge, let[5 6]

$$\alpha = Hma^3\left\{\frac{3f(u')}{u'} - m \cdot f'(u')\right\},$$
$$\chi = Hma^3\left\{\frac{6f(u')}{u'} - \frac{9}{2}m \cdot f'(u')\right\},$$

then[7]

$$\frac{\bar{a}}{a} = -a\,v + x\ .e\sin\left(cv - v\ \right).$$

The value of $\frac{1}{a}$ in equation (225) will be augmented by $a\,v$, therefore a will be diminished by $a\,v$. Since

$$d\,\frac{1}{a} = -2\mathrm{d}R\,,$$

therefore

$$\mathrm{d}R = \frac{a}{2\bar{a}}dv - \frac{x}{2\bar{a}}dv\,.\,e\,.\cos\left(cv - v\ \right).$$

Consequently, when periodic quantities are omitted, $z\ = -3\int a dv\,.\,\mathrm{d}R$ gives

$$z\ = -\frac{3a}{4\bar{a}}av^2$$

or, omitting the action of the sun,

$$z\ = -\tfrac{3}{4}a\,v^2.$$

Thus the mean motion is affected by a secular variation from the resistance of the ethereal medium; but it may easily be shown, from the value of R in article 788, that this medium has no effect whatever on the motion of the lunar nodes or perigee. However, in consequence of that action the second of equations (224), which is the coefficient of $\sin\left(cv - v\ \right)$, ought to be augmented by $x\ .\,e$; hence, rejecting c^2, dv , and making $c = 1$ it gives[8]

$$\frac{x\ .\,edv}{\bar{a}} = 2\,.\,d\,\frac{e}{a},$$

or

$$\frac{e}{a} = \text{constant}\ \left(1 + \tfrac{1}{2}xv\right);$$

but as $\frac{1}{a}$ must be augmented by $a\,v$, if the square of v be omitted,

$$e = \text{constant}\ \left(1 - \left(a - \tfrac{1}{2}x\right)v\right).$$

Thus the eccentricity of the lunar orbit is affected by a secular inequality from the resistance of ether, but it is insensible when compared with the corresponding inequality in the mean motion.

It appears then that the mean motion of the moon is subject to a secular variation in consequence of the resistance of ether, which neither affects the motion of the perigee nor the position of the orbit; and, as the secular inequalities of the moon deduced theoretically from the variation of the eccentricity of the earth's orbit are perfectly confirmed by the concurrence of ancient and modern observations, they cannot be ascribed to the resistance of an ethereal medium.

791. The action of the ether on the motions of the earth may be found by the preceding formulae to be

$$\frac{dR}{dx} = +K'm^2a'^2 \cdot \sin v'$$
$$\frac{dR}{dy} = -K'm^2a'^2 \cdot \cos v';$$

when the eccentricity of the earth's orbit is omitted, so that

$$u' = \frac{1}{a'}.$$

Consequently the general equation (250) gives

$$\mathrm{d}R = -K'.a'^3.m^3.dt, \text{ and therefore}$$
$$dv = -\frac{3a'}{m'} \cdot \iint dt \,.\, \mathrm{d}R = \tfrac{3}{2}\frac{K'.a'^4m^4.t^2}{m'},$$

m' being the mass of the sun.

If $f(u')$ be a function of the distance of the earth from the moon, then must

$$K' = H' \cdot f(u'),$$

H' being a constant quantity depending on the mass and surface of the earth. Whence it may be found by the same method with that employed, that the resistance of ether in the mean motion of the earth would be

$$z = \tfrac{3}{2}\frac{H'a'^4m^4t^2 \cdot f(u')}{m'}.$$

Whence it appears that the acceleration in the mean motion of the moon is to that in the mean motion of the earth as unity to

$$\frac{2H' \cdot m \cdot f(u')}{H\left\{3f(u') - \frac{m}{a'}f'(u')\right\}},$$

or as unity to

$$\tfrac{2}{3}m.\frac{H'}{H},\text{ if }-\frac{m}{a'}f'(u')$$

be omitted, and because

$$\frac{m'a^3}{a^3}=m^2\,.$$

Now H' and H depend on the masses and surfaces of the earth and moon; and as the resistance is directly as the surface, and inversely as the mass, therefore

$$H=\frac{\text{surface}}{\text{mass}}.$$

But by article 652, if the radius of the earth be unity, the moon's true diameter =

$$\frac{\tfrac{1}{2}\text{ moon's apparent diameter}}{\text{moon's horizontal parallax}};$$

hence surface of moon[9] =

$$\frac{\left(\tfrac{1}{2}\text{ apparent diameter}\right)^2}{(\text{lunar parallax})^2}$$

and

$$H=\frac{\left\{\tfrac{1}{2}\text{ apparent diameter of moon}\right\}^2}{\text{mass of moon }\{\text{lunar parallax}\}^2}.$$

But as the terrestrial radius is assumed = 1, the earth's surface is unity; so

$$H'=\frac{1}{\text{mass of earth}};$$

hence

$$\frac{H'}{H}=\frac{\text{mass of moon}}{\text{mass of earth}}\cdot\frac{\text{square horizontal parallax of moon}}{\text{square of }\tfrac{1}{2}\text{ moon's apparent diameter}}.$$

From observation half the moon's apparent diameter is $943''.164$, her horizontal parallax is $3{,}454.16$, and her mass is $\frac{1}{75}$ of that of the earth, so $\frac{H'}{H}=0.17883$; and as $m=\frac{1}{13.3}$, it follows that the acceleration in the mean motion of the earth from the resistance of ether is equal to the corresponding acceleration in the mean motion of the moon multiplied by 0.008942, or about a hundred times less than the acceleration of the moon from the resistance of ether. No such acceleration has been detected in the earth's motion, nor could it be expected, since it is insensible with regard to the moon.

In the preceding investigation, the resistance was assumed to be as the square of the velocity, but Mr. Lubbock[10] has obtained general formulae, which will give the variations in the elements, whatever the law of this resistance may be.

792. Although we have no reason to conclude that the sun is surrounded by ether, from any effects that can be ascribed to it in the motions of the moon and planets, the question of the existence of such a fluid has lately derived additional interest from the retardation that has been observed in the returns of Encke's comet[11] at each revolution, which it is difficult to account for by any other supposition than this existence of such a medium.

Mr. Enke has proved that this retardation does not arise from the disturbing action of the planets. But on computing the effects of the resistance of an ether diffused through space, he found that the diminution in the periodic time, and on the eccentricity arising from the ether, supposing it to exist, corresponds exactly with observation. This coincidence is very remarkable, because ignorance of the nature of the medium in question imposes the necessity of forming an hypothesis of the law of its resistance. Future returns of this comet will furnish the best proof of the existence of an ether, which, by the computation of Mazotti, must be 360,000 millions of times more rare than atmospheric air, in order to produce the observed retardation. The existence of an ethereal medium, if established, would not only be highly important in astronomy, but also from the confirmation it would afford of the undulating theory of light; among whose chief supporters we have to number Huygens,[12] Descartes,[13] Hooke,[14] Euler,[15] and, in later times, the illustrious names of Young[16] and Fresnel,[17] who have applied it with singular success and ingenuity to the explanation of those classes of phenomena which present the greatest difficulties to the corpuscular doctrine.

793. Laplace[18] employs the same analysis to determine the effects that the resistance of light has on the motions of the bodies of the solar system, whether considered as propagated by the undulations of a very rare medium as ether, or emanating from the sun. He finds that it has no effect whatever on the motion of the perigee, either of the sun or moon; that its action on the mean motions of the earth and moon is quite insensible; but that the action of light, on the mean motion of the moon, in the corpuscular hypothesis, is to that in the undulating system as -1 to 0.01345.

794. If gravitation be produced by the impulse of a fluid towards the centre of the attracting body, the same analysis will give the secular equation due to the successive transmission of the attractive force. The result is, that if g be the attraction of any body as the earth; G the ratio of the velocity of the fluid which causes gravitation to that of the moon, at her mean distance, and t any finite time, the secular equation of the mean motion of the moon from the transmission of the attractive force is $\frac{3}{2}\frac{gt^2}{aG}$.

The gravity of a body moving in its orbit is equal to its centrifugal force; and the latter is equal to the nature of the velocity divided by the radius vector; and as the square of the moon's velocity is[19] $a\left(27.32166\right)^2$ its centrifugal force is $\left(27.32166\right)^2$, whence[20]

$$g = a\left(27.32166\right)^2;$$

and the secular equation becomes

$$\tfrac{3}{2}\left(\frac{(27.32166)^2}{G}\right)\cdot t^2\,.$$

Since G is the ratio of the velocity of the fluid in question to the velocity of the moon

$$G=\frac{\text{velocity of the fluid}}{a(27.32155)};$$

hence the velocity of the fluid is $(27.32166)\,aG$.

If

$$L=\frac{\text{velocity of the fluid}}{\text{velocity of light}},$$

then the velocity of the gravitating fluid is equal to L [times the] velocity of light; whence L [times the] velocity of light = (27.32166)aG; but by Bradley's theory,[21] the velocity of light is[22]

$$\frac{(365.25)\,a'}{\tan 20''.25},$$

a' being the mean distance of the earth from the sun; whence

$$L\cdot\frac{(365.25)\,a'}{\tan 20''.25}=(27.32166)\,aG,$$

$$G=\frac{L(365.25)\,a'}{(27.32166)\,a\cdot\tan 20''.25}.$$

And the secular equation of the moon from the successive transmission of gravity becomes[23]

$$\tfrac{3}{2}\frac{(27.32166)^3}{L(365.25)}\cdot\frac{a}{a'}\cdot t^2\cdot\tan 20''.25.$$

Now, if the acceleration in the moon's mean motion arises from the successive transmission of gravity, and not from the secular variation in the earth's eccentricity, the preceding expression would be equal to $10''.1816213$, the acceleration in 100 Julian years. Therefore, making $t=100$,

$$L=\tfrac{3}{2}\frac{a}{a'}\frac{(27.32166)^3}{365.25}\cdot\frac{10{,}000\tan 20''.25}{10''.1816213};$$

but

$$\frac{a}{a'} = \frac{1}{400};$$

whence[24]

$$L = 50{,}464{,}700;$$

thus the velocity with which gravity is transmitted must be more than fifty[25] million times greater than the velocity of light:[26] hence we must suppose the velocity of the moon to be many a hundred million times greater than that of light to preserve her from being drawn to the earth, if her acceleration be owing to the successive transmission of gravity. The action of gravity may therefore be regarded as instantaneous.

795. These investigations are general, though they have only been applied to the earth and moon; and, as the influence of the ethereal media and of the transmission of gravity on the moon is quite insensible, though greater than on the earth, it may be concluded, that they have no sensible effect on the motions of the solar system; but as they do not affect the motions of the lunar perigee and the perihelia of the earth and planets at all, these motions afford a more conclusive proof of the law of gravitation, than any other circumstance in the system of the world. The length of the day is proved to be constant by the secular equation of the moon. For if the day were longer now than in the time of Hipparchus[27] by the 0.00324th of a second, the century would be $118''.341$ longer than at that period. In this interval, the moon would describe an arc of $173''.2$, and her actual mean secular motion would appear to be augmented by that quantity; so that her acceleration, which is $10''.206$ for the first century, beginning from 1801, would be increased by $4''.377$; but observations do not admit of so great an increase. It is therefore certain, that the length of the day has not varied the 0.00324th of a second since the time of Hipparchus.

796. It is evident then, that the lunar motions can be attributed to no other cause than the gravitation of matter: of which the concurring proofs are the motion of the lunar perigee and nodes; the mass of the moon; the magnitude and compression of the earth; the parallax of the sun and moon, and consequently the magnitude of the system; the ratio of the sun's action to that of the moon, and the various secular and periodic inequalities in the moon's motions, every one of which is determined by analysis on the hypothesis of matter attracting inversely as the square of the distance; and the results thus obtained, corroborated by observation, leave not a doubt that the whole obey the law of gravitation. Thus the moon is, of all the heavenly bodies, the best adapted to establish the universal influence of this law of nature; and, from the intricacy of her motions, we may form some idea of the powers of analysis, that marvelous instrument, by the aid of which so complicated a theory has been unraveled.

797. Before we leave the subject, it may be interesting to show that the differential equations of the lunar co-ordinates, given in (207), may be derived from Newton's theory.

If the inclination of the lunar orbit be omitted, the whole force which disturbs the moon may be resolved into two; one perpendicular to the radius vector, and another, according to the radius vector, and directed towards the centre of the earth. Now, $\frac{1}{r}\left(\frac{dR}{dv}\right)$ is the first of these forces, and $-\left(\frac{dR}{dr}\right)$ is the other. The force $\frac{1}{r}\left(\frac{dR}{dv}\right)$, multiplied by dt, gives the increment of the velocity of the moon perpendicular to the radius during the instant dt; and when multiplied by $\frac{1}{2}rdt$, it becomes $\frac{1}{2}\left(\frac{dR}{dv}\right)dt$ = the increment of the area described by the radius vector in the time dt. It must therefore be equal to $\frac{1}{2}\cdot\frac{d\,.r^2dv}{dt}$; hence

$$\frac{d\,.r^2dv}{dt}=\left(\frac{dR}{dv}\right)dt\,.$$

If this equation be multiplied by $\frac{r^2dv}{dt}$ and integrated, the result will be

$$\left(r^2dv\right)^2=h^2dt^2\left(1+\frac{2}{h^2}\int\left(\frac{dR}{dv}\right)r^2dv\right);$$

and as $r=\frac{1}{u}$, it becomes

$$dt=\frac{dv}{hu^2\sqrt{1+\frac{2}{h^2}\int\left(\frac{dR}{dv}\right)\frac{dv}{u^2}}},$$

which is the first of equations (207).

Again, if ds be the element of the curve described by the moon, $\frac{ds^2}{dt^2}$ will be the square of her velocity; and, substituting the preceding value of dt, the square of the moon's velocity will be

$$h^2u^4\cdot\frac{ds^2}{dv^2}\cdot\left\{1+\frac{2}{h^2}\int\left(\frac{dR}{dv}\right)\cdot\frac{dv}{u^2}\right\}.$$

If $r`$ be the oscillating radius of the orbit, the expression of the radius of curvature, in article 83, will give, when substitution is made for $x, y, z,$ in supposing dv constant,

$$\frac{1}{r`}=dv^3\,\frac{\left(\frac{d^2u}{dv^2}+u\right)}{u^3ds^3}.$$

Hence the square of the moon's velocity, divided by the radius of curvature, is

$$u \cdot \frac{dv}{ds} \cdot h^2 \left\{ \frac{d^2u}{dv^2} + u \right\} \cdot \left\{ 1 + \frac{2}{h^2} \int \left(\frac{dR}{dv} \right) \frac{dv}{u^2} \right\}. \qquad \textbf{(252)}$$

By the theorems of Huygens, this expression must be equal to the lunar force resolved in the radius of curvature, and directed towards the centre of curvature. Now, if the force $-\left(\frac{dR}{dr}\right)$ be resolved into two, one parallel to the element of the curve, and the other directed to the centre of curvature, the latter will be $u\left(\frac{dR}{du}\right)\cdot\frac{dv}{ds}$. Also the force $\frac{1}{r}\left(\frac{dR}{dv}\right)$, resolved according to the radius of curvature, will be $-\frac{du}{uds}\left(\frac{dR}{dv}\right)$. The sum of these two forces directed towards the centre of curvature is

$$u \cdot \frac{dv}{ds}\left(\frac{dR}{du}\right) - \frac{du}{uds}\left(\frac{dR}{dv}\right).$$

If the square of this expression be made equal to that of (252), then

$$0 = \left(\frac{d^2u}{dv^2} + u\right)\left\{1 + \frac{2}{h^2}\int\left(\frac{dR}{dv}\right)\frac{dv}{u^2}\right\} - \frac{1}{h^2}\left(\frac{dR}{du}\right) + \frac{du}{h^2u^2dv}\left(\frac{dR}{dv}\right);$$

which is the same with the second of equations (202), when the inclination of the orbit is omitted.

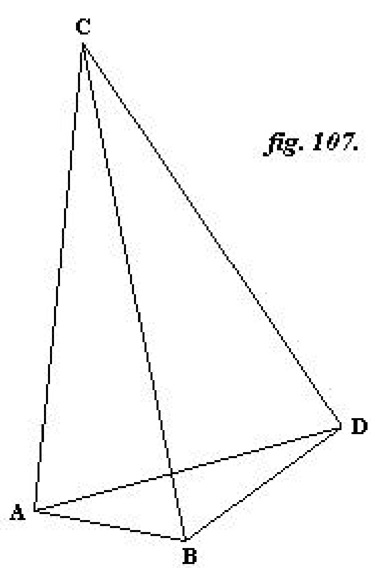

fig. 107.

The equation in latitude is not so easily found as the other two; but the method followed by Newton was to resolve the action of the sun on the moon into two, one in the direction of the radius vector of the lunar orbit, the other parallel to a line joining the centres of the sun and earth. The difference between the last force and the action of the sun on the earth, he saw to be the only force that could change the position of the lunar orbit, since it is not in that plane. He determined the effect of this force, by supposing AB, fig. 107, to be the arc described by the moon in an instant; then ABC is the plane of the orbit during that time; in the next instant, the difference of the two forces causes the moon to describe the small arc BD in a different plane; then if BD represent the difference of the forces, and if AB be the velocity of the moon in the first instant, the diagonal BD will be the direction of the velocity in the second instant; and ACD will be the position of the orbit. Newton deduced the horary and mean motion of the nodes, their principal variation, and the inequalities in latitude, from these considerations. Laplace considered the theory of the moon as the most profound and ingenious part of the Principia.

Notes

[1] The closing parenthesis in the denominator is missing in the 1st edition.

[2] $(K - K')$ *reads* K in equation (251) in the 1st edition (published erratum).

[3] This *reads* "equation (241)" in the 1st edition (published erratum).

[4] The multipliers a^3 *read* a^2 in the 1st edition (published erratum).

[5] The multipliers a^3 *read* a^2 in the 1st edition (published erratum).

[6] The term $\frac{3f(u')}{u'}$ *reads* $\frac{3f(n')}{u'}$ in the 1st edition.

[7] $\frac{\overline{a}}{a}$ *reads* $\frac{1}{a}$ in the 1st edition (published erratum).

[8] The left hand side *reads* $x\ .edv$ in the 1st edition (published erratum).

[9] The factor $\frac{1}{2}$ in the numerator is omitted in the 1st edition (published erratum).

[10] Lubbock, John William, Sir, 1803-1865, *On the theory of the moon, and on the perturbations of the planets*, London : C. Knight, 1833.

[11] Encke, Johann Franz, 1791-1865, *On Encke's comet, Encke's dissertation contained in no. CCX and CCXI of the Astonomische Nachrichten ; translated from German by G. B. Airy*, Cambridge : Printed by. J. Smith, 1832.

[12] See note 12, *Bk. II, Chap. II.*

[13] Descartes, René, 1596-1650, philosopher and mathematician, born in La Haye, France. His principle philosophic work is his *Meditationes de prima philosophia* (1641, Meditations on First Philosophy). His *Discours de la méthode pour bien conduire sa raison, et chercher la vérité dans les sciences* (1637, Discourse on the Method for Rightly Conducting One's Reason and Searching for Truth in the Sciences) contains an appendix in which he establishes the foundational principles of analytic geometry. Descartes also formulated a mechanical "vortex" model of planetary motion in which the earth and other planets in contact with air whirled about the sun.

[14] See note 14, *Bk. II, Chap. I.*

[15] See note 6, *Bk. I, Chap. II.*

[16] See note 35, *Preliminary Dissertation.*

[17] See note 51, *Preliminary Dissertation.*

[18] See note 4, *Introduction.*

[19] This *reads* $a^2(27.32166)^2$ in the 1st edition (published erratum).

[20] This *reads* $g = (27.32166)^2$ in the 1st edition (published erratum).

[21] See note 38, *Preliminary Dissertation.*

[22] The numerator *reads* $(365.25)a$ in the 1st edition (published erratum).

[23] L is omitted from the denominator in the 1st edition (published erratum).

[24] This *reads* L = 42,145,000 in the 1st edition (published erratum).

[25] This *reads* "forty-two" in the 1st edition. This is the final published erratum in the 1st edition.

[26] The phrase "the velocity of light" is repeated twice in the 1st edition.

[27] See note 32, *Preliminary Dissertation.*

BOOK IV - THE SATELLITES

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FOREWORD[1]

Rotation of the Planets

THE oblate form of several of the planets indicates rotatory motion. This has been confirmed in most cases by tracing spots on their surface, by which their poles and times of rotation have been determined. The rotation of Mercury is still doubtful, but Schröter believes that by examining daily the cusps of the crescent he has discovered a rotation of $24^h\ 5^m\ 28^s$; that of the new planets has not yet been ascertained. The sun revolves in twenty-five days seven hours and forty-eight minutes about an axis which is directed towards a point half-way between the pole-star and of alpha of Lyra,[2] the plane of rotation being inclined by $7^\circ\ 30'$, or a little more than seven degrees, to the plane of the ecliptic: it may therefore be concluded that the sun's mass is a spheroid, flattened at the poles. From the rotation of the sun, there was every reason to believe that he has a progressive motion in space, a circumstance which is confirmed by observation. But, in consequence of the reaction of the planets, he describes a small irregular orbit about the centre of gravity of the system, never deviating from his position by more than twice his own diameter, or a little more than seven times the distance of the moon from the earth. The sun and all his attendants rotate from west to east, on axes that remain nearly parallel to themselves in every point of their orbit, and with angular velocities that are sensibly uniform. Although the uniformity in the direction of their rotation is a circumstance hitherto unaccounted for in the economy of nature, yet, from the design and adaptation of every other part to the perfection of the whole, a coincidence so remarkable cannot be accidental. And, as the revolutions of the planets and satellites are also from west to east, it is evident that both must have arisen from the primitive cause which determined the planetary motions. Indeed, Laplace has computed the probability to be as four millions to one that all the motions of the planets, both of rotation and revolution, were at once imparted by an original common cause, but of which we know neither the nature nor the epoch.

The larger planets rotate in shorter periods than the smaller planets and the earth. Their compression is consequently greater, and the action of the sun and of their satellites occasions a nutation in their axes and a precession of their equinoxes similar to that which obtains in the terrestrial spheroid, from the attraction of the sun and moon on the prominent matter at the equator. Jupiter revolves in less than ten hours round an axis at right angles to certain dark belts or bands, which always cross his equator. This rapid rotation occasions a very great compression in his form. His equatorial axis exceeds his polar axis by 6,000 miles, whereas the difference in the axes of the earth is only about twenty-six and a half [miles]. It is an evident consequence of Kepler's law of the squares of the periodic times of the planets being as the cubes of the major axes of their orbits, that the heavenly bodies move slower the farther they are from the sun. In comparing the periods of the revolutions of Jupiter and Saturn with the times of their rotation, it

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[1] The material in this and the forewords to Books I, II and III is extracted from the 10th and last edition of Mary Somerville's *On the Connexion of the Physical Sciences*, (corrected and revised by Arabella B. Buckley), p. 4-106, London : John Murray, 1877.

[2] Alpha Lyra or alpha Lyr is the fifth brightest star, called Vega, situated in the bright northern constellation of Lyra.

appears that a year of Jupiter contains 10,484 of his days, and that of Saturn 24,620 Saturnian days.

The appearance of Saturn is unparalleled in the system of the world. He is a spheroid nearly 700 times larger than the earth, surrounded by a ring even brighter than himself, which always remains suspended in the plane of his equator: and, viewed with a very good telescope, it is found to consist of two concentric rings, divided by a dark band. The exterior ring, as seen through Mr. Lassell's[3] great equatorial at Malta, has a dark-striped band through the centre, which may possibly be another division: it is altogether less bright than the interior ring, one half of which is extremely brilliant; while the interior half is shaded in rings like the seats in an amphitheatre. In 1850 Mr. Dawes in England, and Professor Bond[4] in America, made the remarkable discovery of a dark transparent ring, whose edge coincides with the inner edge of the interior rim, and which occupies about half the space between it and Saturn. The transparency of this ring was better ascertained in 1852 by Mr. Lassell, who compares it to a band of dark-coloured crape drawn across a portion of the disc of the planet, and the part projected upon the blue sky is also transparent. At the time these observations were made at Malta, Captain Jacob also discovered the transparent ring at Madras. None of these rings can be very dense, since the density of Saturn himself is known to be less than the eighth part of that of the earth and there are strong reasons given by Professor Clerk-Maxwell[5] for believing that they are composed of myriads of minute satellites revolving around the globe of the planet, as was originally suggested by Cassini[6] in the eighteenth century. A transit of the sun across a star might reveal something concerning this wonderful object. The ball of Saturn is striped by belts of different colours. At the time of these observations, the part above the ring was bright white; at his equator there was a ruddy belt divided in two, above which were belts of a bluish green, alternately dark and light, while at the pole there was a circular space of a pale colour. The mean distance of the interior part of the double ring from the surface of the planet is about 18,846 miles, it is no less than 28,384 miles broad, but by the estimation of Sir John Herschel[7], its thickness does not much exceed 250 miles, so that it appears like a plane. By the laws of mechanics, it is impossible that this body can retain its position by the adhesion of its particles alone. It must necessarily revolve with a velocity that will generate a centrifugal force sufficient to balance the attraction of Saturn. Observation confirms the truth of these principles, showing that the rings rotate from west to east about the planet in ten hours and a half, which is nearly the time a satellite would take to revolve about Saturn at the same distance. Their plane is inclined to the ecliptic, at an angle of $28^\circ\ 10'\ 44''.7$; in consequence of this obliquity of position, they always appear elliptical to us, but with an eccentricity so variable as even to be occasionally like a straight line drawn across

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[3] Lassell, William, (1799-1880), astronomer, born in Bolton, England. Lassell discovered several planetary satellites including Triton, Ariel, Umbriel and Hyperion (independently of W. C. Bond [see next note] who made his discovery the same night).

[4] Bond, William Cranch, (1789-1859), astronomer, born in Portland, Maine, USA. Bond was a pioneer in celestial photography (he made the first daguerreotype of the moon in 1850) and discovered Hyperion the seventh satellite of Saturn (independently of W. Lassell [see previous note] who made his discovery the same night).

[5] Maxwell, James Clerk, (1831-1879), physicist, born in Edinburgh, Scotland. His greatest work was the theory of electromagnetic radiation outlined in his *Treatise on Electricity and Magnetism* (1873). He is ranked with Isaac Newton and Albert Einstein for the importance of his fundamental contributions to science which paved the way for both the quantum theory of Planck and Einstein's theory of relativity. Maxwell read and publicly admired the works of Mary Somerville (see *Foreword to the Second Edition*).

[6] see note 53, *Bk. II, Chap. XIV*.

[7] see note 64, *Preliminary Dissertation*.

the planet. In the beginning of October 1832, the plane of the ring passed through the centre of the earth; in that position they are only visible with very superior instruments, and appear like a fine line across the disc of Saturn. About the middle of December, in the same year, the rings became invisible, with ordinary instruments, on account of their plane passing through the sun. In the end of April, 1883, the rings vanished a second time, and reappeared in June of that year. Similar phenomena will occur as often as Saturn has the same longitude with either node of his rings. Each side of these rings has alternately fifteen years of sunshine and fifteen years of darkness.

It has been proved theoretically, that the rings could not maintain their stability of rotation if they were everywhere of uniform thickness; for the smallest disturbance would destroy the equilibrium, which would become more and more deranged, till, at last, they would be precipitated on the surface of the planet. But if some parts of the rings were enormously thicker than others, it could not escape observation, therefore Professor Clerk-Maxwell concludes that they cannot be irregular solids, and the only other theory which will account for their stability is, that they are composed of an immense number of unconnected particles revolving round the planet with different velocities. Professor Struve has shown that the centre of the rings is not concentric with the centre of Saturn.[8] The interval between the outer edge of the globe of the planet and the outer edge of the rings on one side is $11''.390$, consequently there is an eccentricity of the globe in the rings of $0''.215$. If the rings obeyed different forces, they would not remain in the same plane, but the powerful attraction of Saturn always maintains them and his satellites in the plane of his equator. The rings, by their mutual action, and that of the sun and satellites, must oscillate about the centre of Saturn, and produce phenomena of light and shadow whose periods extend to many years. According to M. Bessel[9] the mass of Saturn's ring is equal to the $\frac{1}{118}$ part of that of the planet.

The distance and minuteness of Jupiter's satellites render it extremely difficult to ascertain their rotation. It was, however, attempted by Sir William Herschel[10] by ascertaining their relative brightness. He observed that they alternately exceed each other in brilliancy, and, by comparing the maxima and minima of their illumination with their positions relatively to the sun and to their primary, he found reason to believe that, like the moon, the time of their rotation was equal to the period of their revolution about Jupiter. Later observations, however, render this conclusion doubtful. The eighth satellite of Saturn, Iapetus, is the only one whose period of rotation has been fairly ascertained. This was done first by Cassini and afterwards by Sir W. Herschel, who concluded that the time of its rotation on its axis must agree very nearly with the period of its revolution round Saturn.

Jupiter's Satellites

The changes which take place in the planetary system are exhibited on a smaller scale by Jupiter and his satellites; and, as the period requisite for the development of the inequalities of these moons only extends to a few centuries, it may be regarded as an epitome of that grand cycle which will be accomplished by the planets in myriads of ages. The revolutions of the

[8] see note 29, *Preliminary Dissertation.*
[9] see note 37, *Bk. II, Chap. XIV*
[10] See note 52, *Preliminary Dissertation.*

satellites about Jupiter are precisely similar to those of the planets about the sun; it is true they are disturbed by the sun, but his distance is so great, that their motions are nearly the same as if they were not under his influence. The satellites, like the planets, were probably projected in elliptical orbits: but, as the masses of the satellites are nearly 100,000 times less than that of Jupiter and as the compression of Jupiter's spheroid is so great, in consequence of his rapid rotation, that his equatorial diameter exceeds his polar diameter by no less than 6000 miles; the immense quantity of prominent matter at his equator must soon have given the circular form observed in the orbits of the first and second satellites, which its superior attraction will always maintain. The third and fourth satellites, being farther removed from its influence, revolve in orbits with a very small eccentricity. And, although the first two sensibly move in circles, their origins acquire a small ellipticity, from the disturbances they experience.

It has been stated, that the attraction of a sphere on an exterior body is the same as if its mass were united in one particle in its centre of gravity, and therefore inversely as the square of the distance. In a spheroid, however, there is an additional force arising from the bulging mass at its equator, which acts as a disturbing force. One effect of this disturbing force in the spheroid of Jupiter is to occasion a direct motion in the greater axes of the orbits of all his satellites, which is more rapid the nearer the satellite is to the planet, and very much greater than that part of their motion which arises from the disturbing action of the sun. The same cause occasions the orbits of the satellites to remain nearly in the plane of Jupiter's equator,[11] on account of which the satellites are always seen nearly in the same line; and the powerful action of that quantity of prominent matter is the reason why the motions of the nodes of these small bodies are so much more rapid than those of the planet. The nodes of the fourth satellite accomplish a tropical revolution in 531 years, while those of Jupiter's orbit require no less than 36,261 years;–a proof of the reciprocal attraction between each particle of Jupiter's equator and of the satellites. In fact, if the satellites moved exactly in the plane of Jupiter's equator, they would not be pulled out of that plane, because his attraction would be equal on both sides of it. But, as their orbits have a small inclination to the plane of the planet's equator, there is a want of symmetry, and the action of the protuberant matter tends to make the nodes regress by pulling the satellites above or below the planes of their orbits; an action which is so great on the interior satellites, that the motions of their nodes are nearly the same as if no other disturbing force existed.

The orbits of the satellites do not retain a permanent inclination either to the plane of Jupiter's equator, or to that of his orbit, but to certain planes passing between the two, and through their intersection. These have a greater inclination to his equator the farther the satellite is removed, owing to the influence of Jupiter's compression; and they have a slow motion corresponding to secular variations in the planes of Jupiter's orbit and equator.

The satellites are not only subject to periodic and secular inequalities from their mutual attraction, similar to those which affect the motions and orbits of the planets, but also to others peculiar to themselves. Of the periodic inequalities arising from their mutual attraction the most remarkable take place in the angular motions[12] of the three nearest to Jupiter, the second of which receives from the first a perturbation similar to that which it produces in the third and it experiences from the third a perturbation similar to that which it communicates to the first. In the eclipses these two inequalities are combined into one, whose period is 437.659 days. The

[11] The plane of Jupiter's equator is the imaginary plane passing through his centre at right angles to his axis of rotation. (Somerville's note.)

[12] *Angular motion or velocity* is the swiftness with which a body revolves–a sling, for example; or the speed with which the surface of the earth performs its daily motion about its axis. (Somerville's note.)

variations peculiar to the satellites arise from the secular inequalities occasioned by the action of the planets in the form and position of Jupiter's orbit, and from the displacement of his equator. It is obvious that whatever alters the relative positions of the sun, Jupiter, and his satellites, must occasion a change in the directions and intensities of the forces, which will affect the motions and orbits of the satellites. For this reason the secular variations in the eccentricity of Jupiter's orbit occasion secular inequalities in the mean motions of the satellites, and in the motions of the nodes and apsides of their orbits. The displacement of the orbit of Jupiter, and the variation in the position of his equator, also affect these small bodies. The plane of Jupiter's equator is inclined to the plane of his orbit at an angle of $3^\circ\, 5'\, 30''$, so that the action of the sun and of the satellites themselves produces a nutation and precession in his equator, precisely similar to that which takes place in the rotation of the earth, from the action of the sun and moon. Hence the protuberant matter at Jupiter's equator is continually changing its position with regard to the satellites, and produces corresponding nutations in their motions. And, as the cause must be proportional to the effect, these inequalities afford the means, not only of ascertaining the compression of Jupiter's spheroid, but they prove that his mass is not homogeneous. Although the apparent diameters of the satellites are so small that they can scarcely be measured, yet their perturbations give the values of their masses with considerable accuracy—a striking proof of the power of analysis.

A singular law obtains among the mean motions and mean longitudes of the first three satellites. It appears from observation that the mean motion of the first satellite, plus twice that of the third, is equal to three times that of the second; and that the mean longitude of the first satellite, minus three times that of the second, plus twice that of the third, is always equal to two right angles. It is proved by theory, that, if these relations had only been approximate when the satellites were first launched into space, their mutual attractions would have established and maintained them, notwithstanding the secular inequalities to which they are liable. These extend to the synodic motions[13] of the satellites: consequently they affect their eclipses, and have a very great influence on their whole theory. The satellites move so nearly in the plane of Jupiter's equator, which has a very small inclination to his orbit, that the first three are eclipsed at each revolution by the shadow of the planet, which is much larger than the shadow of the moon: the fourth satellite is not eclipsed so frequently as the others. The eclipses take place close to the disc of Jupiter when he is near opposition;[14] but at times his shadow is so projected with regard to the earth, that the third and fourth satellites vanish and reappear on the same side of the disc. These eclipses are in all respects similar to those of the moon: but, occasionally, the satellites eclipse Jupiter, sometimes passing like obscure spots across his surface, resembling annular eclipses of the sun, sometimes like a bright spot traversing one of his dark belts, and even sometimes as a dark spot upon the belt. This last fact, observed by Schröeter and Harding, has led to the conclusion that some of the satellites have occasionally obscure spots on their own bodies, or in their atmospheres. Before opposition, the shadow of the satellite, like a round black spot, precedes its passage over the disc of the planet; and, after opposition, the shadow follows the satellite.

In consequence of the relations already mentioned in the mean motions and mean longitudes of the first three satellites, they never can be all eclipsed at the same time: for, when

[13] *Synodic motion of a satellite.* Its motion during the interval between two of its consecutive eclipses. (Somerville's note.)

[14] *Opposition.* A body is said to be in opposition when its longitude differs from that of the sun by 180°. (Somerville's note.)

the second and third are in one direction, the first is in the opposite direction; consequently, when the first is eclipsed, the other two must be between the sun and Jupiter. The instant of the beginning or end of an eclipse of a satellite marks the same instant of absolute time to all the inhabitants of the earth; therefore, the time of these eclipses observed by a traveler, when compared with the time of the eclipse computed for Greenwich, or any other fixed meridian, gives the difference of the meridians in time, and, consequently, the longitude of the place of observation.[15] The longitude is determined with extreme precision wherever it is possible to convey the time instantaneously by means of electricity from one place to another, since it obviates the errors of clocks and chronometers.

Satellites of Saturn, Uranus and Neptune

The little that is known of the theories of the satellites of Saturn and Uranus is, in all respects, similar to that of Jupiter. Saturn is accompanied by eight satellites. There has been much confusion about their nomenclature owing to some astronomers numbering them in the order of their discovery, and others according to their distances from Saturn. At the suggestion of Sir J. Herschel they are now called after the heathen deities Iapetus, Hyperion, Titan, Rhea, Dione, Tethys, Enceladus, and Mimas; Iapetus being the most distant and Mimas the nearest to Saturn. The size of these satellites is not accurately known. Titan, which is the largest, is about the size of Mars. Hyperion was simultaneously discovered in 1848 by Mr. Bond[16] in America, and the distinguished astronomer Mr. Lassell,[17] of Liverpool. The orbit of the most distant satellite is inclined about $12^\circ\, 14'$ to the plane of the ring; but the great compression of Saturn occasions the other satellites to move nearly in the plane of his equator. So many circumstances must concur to render the two interior satellites visible, that they are only seen with difficulty. They move exactly at the edge of the ring and their orbits never deviate from its plane. When Sir William Herschel discovered them in 1789, he saw them like beads, threading the slender line of light which the ring is reduced to when seen edgewise from the earth. And for a short time he perceived them advancing off it at each end, when turning round in their orbits. The eclipses of the exterior satellites only take place when the ring is in this position, and even then, owing to the great distance of Saturn, these eclipses cannot be used, like those of Jupiter, for the determination of longitudes. Of the situation of the equator of Uranus we know nothing, nor of his compression; but the orbits of his satellites are nearly perpendicular to the plane of the ecliptic; and by analogy, they ought to be in the plane of his equator. Uranus is so remote that he has more the appearance of a planetary nebula than a planet, which renders it extremely difficult to distinguish the satellites at all; and quite hopeless without such a telescope as is rarely to be met with even in observatories. Sir William Herschel discovered the two that are farthest from the planet, and ascertained their approximate periods, which his son [Sir John Herschel]

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[15] The meridian passing through the Observatory of Greenwich (see note 22, *Preliminary Dissertation*) is assumed by the British as a fixed origin from whence terrestrial longitudes are measured. And as each point on the surface of the earth passes through 360°, or a complete circle in twenty-four hours, at the rate of 15° in an hour, time becomes a representative of angular motion. Hence, if the eclipse of a satellite happens at any place at eight o'clock in the evening, and the *Nautical Almanac* shows that the same phenomena will take place at Greenwich at nine, the place of observation will be in the 15th degree of west longitude. (Somerville's note.)

[16] See note 4.

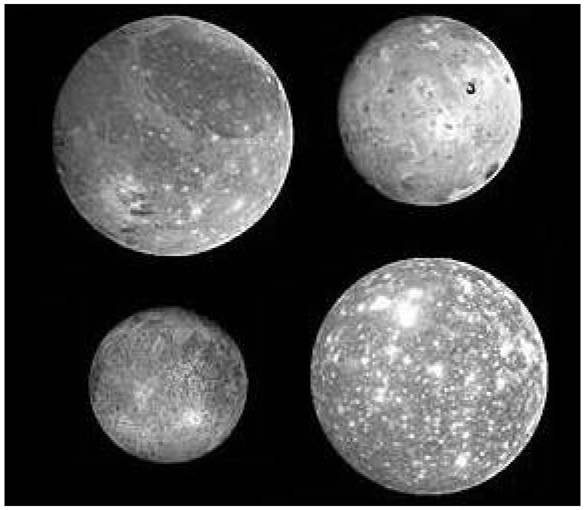
[17] See note 3.

afterwards determined to be $13^d\ 11^h\ 7^m\ 12^s.6$ and $8^d\ 16^h\ 56^m\ 31^s.30$ respectively. The orbits of both have an inclination of $78^\circ\ 58'$ to the plane of the ecliptic. The two interior satellites are so faint and small, and so near the edge of the planet, that they can with difficulty be seen even under the most favourable circumstances: however, Mr. Lassell has ascertained that the most distant of the two revolves about Uranus in $4^d\ 3^h\ 28^m\ 8^s.0$ days and that nearest to the planet is $2^d\ 12^h\ 29^m\ 20^s.7$ days, and from a long and minute examination he is convinced that the system only consists of four satellites. Soon after Neptune was seen Mr. Lassell (see note 6) discovered the only satellite known certainly to belong to that planet, although he believes he has discovered a second. The elements of the first satellite have been determined by M. Otto Struve,[18] and Mr. Lassell has determined its period to be $5^d\ 21^h\ 2^m\ 7^s$. The satellites of Uranus offer the singular and only instance of a revolution from east to west, while all the planets and all the other satellites revolve from west to east. Retrograde motion is occasionally met with in the comets and double stars, and the known satellite of Neptune may possibly also have a retrograde motion,[19] but this is not yet clearly ascertained.

[18] See note 29, *Preliminary Dissertation.*

[19] *retrograde motion.* Somerville was correct. Triton, Neptune's largest satellite (six others have since been discovered) is indeed retrograde and the only "large" moon in the solar system to orbit "backwards." The other moons with retrograde orbits are Jupiter's Ananke, Carme, Pasiphae and Sinope and Saturn's Phoebe, all of which are less than one tenth the diameter of Triton. *Bill Arnett's "nine planets" web site*.

Satellites of Jupiter



In this composite of images, Jupiter's four largest moons are shown to scale, Ganymede, Io, Europa and Callisto. All the images were taken by the Galileo spacecraft in 1996 and 1997 during its orbital tour of the Jovian system, except for the globe of Callisto, which was taken in 1979 by the Voyager spacecraft. (Courtesy of NASA)

BOOK IV

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CHAPTER I

THEORY OF JUPITER'S SATELLITES

798. JUPITER is attended by four satellites, which were discovered by Galileo[1] on the 1st of June, 1610; their orbits are nearly in the plane of Jupiter's equator, and they exhibit all the phenomena of the solar system, on a small scale and in short periods. The eclipses of these satellites afford the easiest method of ascertaining terrestrial longitudes; and the frequency of the occurrence of an eclipse renders the theory of their motions nearly as important to the geographer as that of the moon.

799. The orbits of the two first satellites are circular, subject only to such eccentricities as arise from the disturbing forces; the third and fourth satellites have elliptical orbits; the eccentricity however is so small, that their elliptical motion is determined along with those perturbations that depend on the eccentricities of the orbits.

800. Although Jupiter's satellites might be regarded as an epitome of the solar system, they nevertheless require a new investigation, on account of the nearly commensurable ratios in the mean motions of the three first satellites, the action of the sun, the ellipticity of Jupiter's spheroid, and the displacement of his orbit by the action of the planets.

801. It appears, from observation, that the mean motion of the first satellite is nearly equal to twice that of the second; and that the mean motion of the second is nearly equal to twice that of the third; whence the mean motion of the first, minus three times that of the second, plus twice that of the third, is zero; but the last ratio is so exact, that from the earliest observations it has always been zero. It is also found that, from the time of the discovery of the satellites, the mean longitude of the first, minus three times that of the second, plus twice that of the third, is equal to 180°: and it will be shown, in the theory of these bodies, that even if these ratios had not been exact in the origin of their motions, their mutual attractions would have made them so. They are the cause of the principal inequalities in the longitude of the satellites; and as they exist also in their synodic motions, they have a great influence on the times of their eclipses, and indeed on their whole theory.

802. The prominent matter at Jupiter's equator, together with the action of the satellites themselves, causes a direct motion in the apsides, which changes the relative position of the orbits, and alters the attractive force of the satellites; consequently each satellite has virtually four equations of the centre, or rather, that part of the longitude of each satellite that depends on the eccentricity, consists of four principal terms; one that arises from the true ellipticity of its own orbit, and three others, depending on the positions of the apsides of the other three orbits. Inequalities perfectly similar to these are produced in the radii vectores by the same cause, consisting of the same number of terms, and depending on the same quantities.

803. Astronomers imagined that the orbits of the satellites had a constant inclination to the plane of Jupiter's equator; however, they have not always the same inclination, either to the plane of his equator or orbit, but to certain imaginary fixed planes passing between these, and also through their intersection.

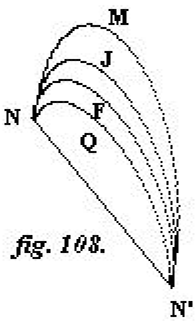


fig. 108.

Let NJN′ be the orbit of Jupiter, NQN′ the plane of his equator extended so as to cut his orbit in NN′; then, if NMN′ be the orbit of a satellite, it will always preserve very nearly the same inclination to a fixed plane NFN′, passing between the planes NQN′ and NJN′, and through the line of their nodes. But although the orbit of the satellite preserves nearly the same inclination to NFN′, its nodes have a retrograde motion on that plane. The plane FN itself is not absolutely fixed, but moves slowly with the equator and orbit of Jupiter. Each satellite has a different fixed plane, which is less inclined to the plane of Jupiter's equator the nearer the satellite is to the planet, evidently arising from the attraction of the protuberance at Jupiter's equator, which retains the satellites nearly in the plane of the equator; furnishing another proof of the mutual attraction of the particles of matter.

804. The equatorial matter of Jupiter's spheroid causes a retrograde motion in the nodes of the orbits of the satellites; which alters their mutual attraction, by changing the relative position of their planes, so that the latitude of any one satellite not only depends on the position of the node of its own orbit, but on the nodes of the other three; and as the position of Jupiter's equator is perpetually varying, in consequence of the action of the sun and satellites, the latitude of these bodies varies also with the inclination of Jupiter's equator on his orbit, and the position of its nodes. Thus, the principal inequalities of the satellites arise from the compression of Jupiter's spheroid, and from the direct and indirect action of the sun and satellites themselves.

805. The secular variation in the form and position of Jupiter's orbit is the cause also of secular variations in the motions of the satellites, similar to those in the motions of the moon occasioned by the variation in the eccentricity and position of the earth's orbit.

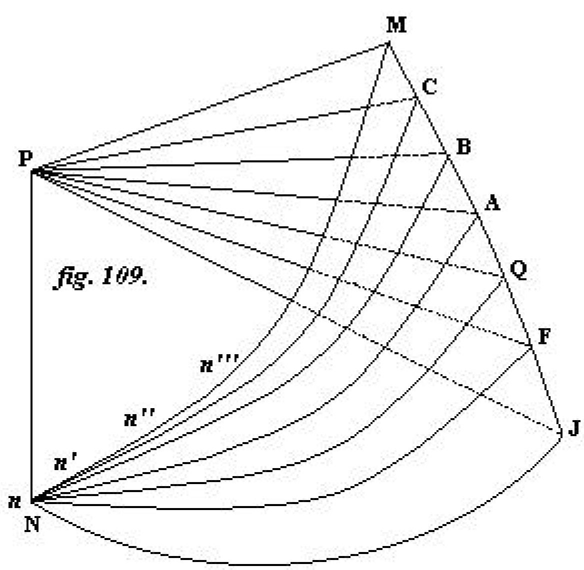


fig. 109.

806. The position of the orbit of a satellite may be known by supposing five planes, of which FN, passing between JN and QN, the planes of Jupiter's orbit and equator, always retains very nearly the same inclination to them. The second plane A*n* moves uniformly on FN, retaining nearly the same inclination on it. The third B*n*′ moves in the same manner on A*n*; the fourth C*n*″ moves similarly on B*n*′; and the fifth M*n*‴, which has the same kind of motion on C*n*″, is the orbit of the satellite. The motion of the nodes are retrograde, and each satellite has set of planes peculiar to itself. In conformity with this, the latitude of a satellite above the variable orbit of Jupiter, is expressed by five terms; the first of which is relative to the displacement of the orbit and equator of Jupiter; the second is relative to the inclination of the orbit of the satellite on its fixed plane; and the other three terms depend on the position and motion of the nodes of the other three orbits. The inequalities which have small divisions, arising from the configuration of the bodies, are insensible in latitude, with the exception of those produced by the sun, which modify the preceding quantities.

807. For the solution of the problem of the satellites, the data that must be determined by observation for a given epoch, are, the compression of Jupiter's spheroid, the inclination of his equator on his orbit, the longitude of its nodes, the eccentricity of his orbit, its position, and its secular variations; the masses of the four satellites, their mean distances, periodic times, the eccentricities and inclinations of their orbits, together with the longitude of their apsides and nodes. The masses of the satellites and the compression of Jupiter are determined from the inequalities of the satellites themselves.

808. The orbits of the four satellites may be regarded as circular, because the eccentricity of the third, and even the fourth, is so small, that their equations of the centre will be determined with the perturbations depending on the eccentricities and inclinations. Thus, with regard to the two first, and nearly for the other two, the true longitude is the sum of the mean longitude and perturbations; and the radius vector will be found by adding the perturbations to the mean distance.

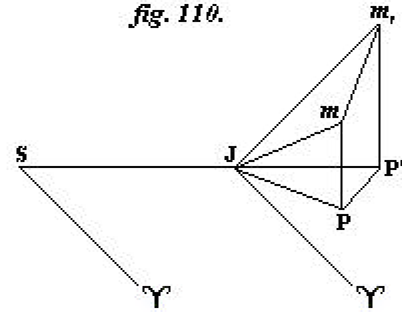
809. A satellite *m* is troubled by the other three, by the sun, and by the excess of matter at Jupiter's equator. The problem however will be limited to the action of the sun, of Jupiter's spheroid, and of one satellite; the resulting equations will be general, from whence the action of each body may be computed separately, and the sum will be the effect of the whole.

810. Let m and $m_{/}$ be the masses of any two satellites, $x, y, z, x', y', z',$ their rectangular co-ordinates referred to the centre of gravity of Jupiter, supposed to be at rest; r, r' their radii vectores; then the disturbing action of $m_{/}$ on m is

$$\frac{m_{/}\left(xx'+yy'+zz'\right)}{r_{/}^{3}}-\frac{m_{/}}{\sqrt{\left(x'-x\right)^{2}+\left(y'-y\right)+\left(z'-z\right)^{2}}}=R\,;$$

Consequently the sign of R must be changed in equations (155) and (156), since it is assumed to be negative in this case.

fig. 110.



The satellites move nearly in the plane of Jupiter's equator, which in 1750 was inclined to the plane of his orbit at an angle of $3^{\circ}\,5'\,30''$; and as the fixed planes pass between these two, the inclinations of the orbits of the satellites to them are very small; consequently $s = mP,\ s_{/} = m_{/}P,$ fig. 110, the tangents of the latitude of the two satellites on $PJP',$ the fixed plane of m, are very small.

If g be the vernal equinox of Jupiter, the longitudes of the two satellites are $g\,JP = v,\ g\,JP' = v_{/},$ and therefore

$$x = \frac{r\cos v}{\sqrt{1+s^2}},\;\; y = \frac{r\sin v}{\sqrt{1+s^2}},\;\; x = \frac{rs}{\sqrt{1+s^2}}.$$

If x', y', z', the co-ordinates of $m_/$, be equal to the same quantities accented, the action of $m_/$ on m, expressed in polar co-ordinates, will be

$$R = +\frac{m_/ r}{r_/^{\,2}}\left\{ss_/ + \left(1-\tfrac{1}{2}s^2 - \tfrac{1}{2}s_/^{\,2}\right)\cos(v_/ - v)\right\} - \frac{m_/}{\sqrt{r^2 - 2rr_/\cos(v_/ - v) + r_/^{\,2}}}$$
$$-\frac{m_/ \cdot rr_/ \cdot \left\{ss_/ - \tfrac{1}{2}\left(s^2 + s_/^{\,2}\right)\cos(v_/ - v)\right\}}{\left\{r^2 - 2rr_/ \cdot \cos(v_/ - v) + r_/^2\right\}^{\frac{3}{2}}},$$

when s^4, $s_/^{\,4}$ are neglected.

811. If S' be the mass of the sun, and X', Y', Z', his co-ordinates, his action upon m will be expressed by

$$R = \frac{S'(X'x + Y'y + Z'z)}{D^3} - \frac{S'}{\sqrt{(X'-x)^2 + (Y'-y)^2 + (Z'-z)^2}},$$

D being his distance from the centre of Jupiter.

Let Jupiter and his orbit be assumed to be at rest, and let his motion be referred to the sun, which is the same as supposing the sun to move in the orbit of Jupiter with the velocity of that planet; if S be the tangent of the sun's latitude above the fixed plane PJP', and $U = g\,SJ$, his longitude seen from the centre of Jupiter when at rest, then

$$X' = \frac{D\cos U}{\sqrt{1+S^2}},\;\; Y' = \frac{D\sin U}{\sqrt{1+S^2}},\;\; Z' = \frac{D\,.\,S}{\sqrt{1+S^2}},$$

and[2]

$$R = -\frac{S'}{D} - \frac{S'r^2}{4D^3}\left\{1 - 3s^2 - 3S^2 + 12sS\left(\cos(U-v) + 3\cos 2(U-v)\right)\right\},$$

which is the action of the sun on the satellite when terms divided by D^4 are omitted, for the distance of the satellite from Jupiter is incomparably less than the distance of Jupiter from the sun.

812. The attraction of the excess of matter at Jupiter's equator is expressed by

$$R = -\left(\rho - \tfrac{1}{2}\phi\right)\left(\tfrac{1}{3} - \nu^2\right) \cdot \frac{J \cdot R^{`2}}{r^3},$$

in which ν is the sine of the declination of the satellite on the plane of Jupiter's equator; J the mass of Jupiter; $2R^{`}$ his equatorial diameter; $\rho$ his ellipticity, and $\phi$ the ratio of the centrifugal force to gravity at his equator. Now it may be assumed that $J = 1$, $R^{`} = 1$; and if s' be the tangent of the latitude that the satellite would have above the fixed plane if it moved in the plane of Jupiter's equator, and as s is its latitude above that plane, when moving in its own orbit, $\nu = s - s'$ nearly; hence

$$R = -\frac{\left(\rho - \tfrac{1}{2}\phi\right)}{r^3}\left\{\tfrac{1}{3} - \left(s - s'\right)^2\right\}.$$

813. Thus the whole force that troubles the motion of m is

$$R = +\frac{m_{/}r}{r_{/}^{2}}\left\{ss_{/} + \left(1 - \tfrac{1}{2}s^2 - \tfrac{1}{2}s_{/}^{2}\right)\cos\left(v_{/} - v\right)\right\} - \frac{m_{/}}{\sqrt{r^2 - 2rr_{/}\cos\left(v_{/} - v\right) + r_{/}^{2}}}$$

$$- \frac{m_{/}rr_{/}\left\{ss_{/} - \tfrac{1}{2}\left(s^2 + s_{/}^{2}\right)\cos\left(v_{/} - v\right)\right\}}{\left\{r^2 - 2rr_{/}\cos\left(v_{/} - v\right) + r_{/}^{2}\right\}^{\frac{3}{2}}} - \frac{S'}{D} - \frac{S'r^2}{4D^3}\left\{1 - 3s^2 - 3S^2 + 12sS\cos\left(U - v\right) + 3\cos 2\left(U - v\right)\right\}$$

$$- \frac{\left(\rho - \tfrac{1}{2}\phi\right)}{r^3}\left\{\tfrac{1}{3} - \left(s - s'\right)^2\right\}.$$

814. If the squares of S, s, and s' be omitted, the only force that troubles the satellites in longitude and distance is

$$R = +\frac{m_{/}r}{r_{/}^{2}}\cos\left(v_{/} - v\right) - \frac{m_{/}}{\sqrt{r^2 - 2rr_{/}\cos\left(v_{/} - v\right) + r_{/}^{2}}} - \frac{S'}{D} - \frac{S'r^2}{4D^3}\left\{1 + 3\cos 2\left(U - v\right)\right\} - \frac{\left(\rho - \tfrac{1}{2}\phi\right)}{3r^3}.$$

When the eccentricities are omitted, the radii vectores, r and r', become a, $a_{/}$, half the greater arcs of the orbits, and that part of R that depends on the mutual attraction of the satellites, is

$$R' = \frac{m_{/}a}{a_{/}^{2}}\cos\left(n_{/}t - nt + \epsilon_{/} - \epsilon\right) - \frac{m_{/}}{\sqrt{a^2 - 2aa_{/}\cos\left(n_{/}t - nt + \epsilon_{/} - \epsilon\right) + a_{/}^{2}}}$$

$nt + \epsilon$, $n_{/}t + \epsilon_{/}$, being the mean longitudes of m and $m_{/}$. This expression may be developed into the series

$$R' = m_/ \left\{ \tfrac{1}{2} A_0 + A_1 \cdot \cos\left(n_/ t - nt + \in_/ - \in \right) + A_2 \cos 2\left(n_/ t - nt + \in' - \in \right) + \&c. \right\}$$

This is the part of R that is independent of the eccentricities, and is identical with the series in article 446; therefore the coefficients A_0, A_1, &c., and their differences, may be computed by the same formulae as for the planets, observing to substitute $A_/ - \frac{a}{a_/^2}$ for $A_/$.

But, by article 445,

$$r = a(1+u) \qquad r_/ = a_/(1+u_/)$$
$$v = nt + \in + v' \qquad v_/ = n_/ t + \in_/ + v_/',$$

where u, $u_/$, v', $v_/'$, are the elliptical parts of the radii vectores, and of the longitudes of m and $m_/$. By the same article, the general formula for the development of R, according to the powers and products of these minute quantities, is

$$R = R' + au_/ \cdot \frac{dR'}{da} + a_/ u_/ \cdot \frac{dR'}{da_/} + \left(v_/' - v' \right) \frac{dR'}{ndt} + \&c.$$

From the preceding value of R' the quantities $\frac{dR'}{da}$, $\frac{dR'}{da_/}$, &c., may be found; and, when substituted, it will be seen afterwards that the only requisite part of R is

$$\begin{aligned} R = &+ m_/ \left\{ \tfrac{1}{2} A_0 + A_1 \cdot \cos\left(n_/ t - nt + \in_/ - \in \right) + A_2 \cos 2\left(n_/ t - nt + \in' - \in \right) + \&c. \right\} \\ &+ \frac{m_/}{2} \cdot au \cdot \frac{dA_0}{da} + m_/ a u \frac{dA_2}{da} \cdot \cos 2\left(n_/ t - nt + \in_/ - \in \right) \\ &+ m_/ a_/ u' \frac{dA_1}{da_/} \cdot \cos\left(n_/ t - nt + \in_/ - \in \right) \\ &- m_/ v_/' A_1 \cdot \sin\left(n_/ t - nt + \in_/ - \in \right) \\ &+ 2 m_/ v' A_2 \cdot \sin 2\left(n_/ t - nt + \in_/ - \in \right). \end{aligned}$$

Because the satellites move in nearly circular orbits, u, $u_/$, v', and $v_/'$, may be regarded as variations arising either entirely from the disturbing forces, as in the first and second satellites, or from that force conjointly with a real but very small ellipticity, as in the third and fourth; therefore

$$r = a(1 + du), \qquad r_/ = a_/(1 + du_/)$$
$$v = nt + \in + dv, \qquad v_/ = n_/ t + \in_/ + dv_/$$

Now, $r = a(1+u)$ gives $r^2 = a^2(1+2u)$; for u is so small, that its square may be omitted; hence $du = \frac{rdr}{a^2}$: consequently $du_/ = \frac{r_/ dr_/}{a_/^2}$; and when R = 0, equation (156) gives, for the elliptical part of rdr only,

$$dv = \frac{2d(rdr)}{a^2 . ndt}, \text{ and } dv_/ = \frac{2d(r_/ dr_/)}{a_/^2 . n_/ dt},$$

when the squares of the eccentricities are omitted.

815. If these quantities be substituted in R instead of u, $u_/$, v', and $v'_/$, it becomes

$$\begin{aligned} R = &+m_/\left\{\tfrac{1}{2}A_0 + A_1 \cdot \cos(n_/ t - nt + \in_/ - \in) + A_2 \cos 2(n_/ t - nt + \in' - \in) + \&c.\right\} \\ &+ \frac{m_/}{2} \cdot \frac{rdr}{a^2} \cdot a\left(\frac{dA_0}{da}\right) \\ &+ m_/ \cdot \frac{rdr}{a^2} \cdot a\left(\frac{dA_2}{da}\right) \cdot \cos 2(n_/ t - nt + \in_/ - \in) \\ &+ m_/ \cdot \frac{r_/ dr_/}{a_/^2} \cdot a_/\left(\frac{dA_1}{da_/}\right) \cdot \cos(n_/ t - nt + \in_/ - \in) \\ &+ 4m_/ \cdot \frac{d(rdr)}{a^2 \cdot ndt} \cdot A_2 . \sin 2(n_/ t - nt + \in_/ - \in) \\ &- 2m_/ \cdot \frac{d(r_/ dr_/)}{a_/^2 . n_/ dt} \cdot A_1 . \sin(n_/ t - nt + \in_/ - \in) \\ &+ \&c. \end{aligned} \qquad \textbf{(253)}$$

816. If $\frac{S'}{D}$ and the eccentricity be omitted, the action of the sun on m is

$$R = -\frac{S'a^2}{4D'^3}\left\{1 + 3\cos 2(Mt - nt + E - \in)\right\};$$

where D' is the mean distance of Jupiter from the sun, and $Mt+E$ his mean longitude referred to the sun. In the troubled orbit a, $nt+\in$, and D' become

$$a\left(1 + \frac{rdr}{a^2}\right), \quad nt + \in - \frac{2d(rdr)}{a^2 \cdot ndt}, \quad \text{and } D'\left(1 - \frac{DdD}{D'^2}\right);$$

and as, by article 383, $\frac{S'}{D'^3} = M^2$, when the mass of Jupiter is omitted in comparison of that of the sun, the whole disturbing action of the sun is

$$\begin{aligned} R = &-\frac{M^2a^2}{4} - \frac{M^2}{2}\cdot rd\,r - \frac{3M^2a^2}{4}\cdot\cos 2(nt - Mt + \in - E) \\ &- \tfrac{3}{4}M^2a^2\cdot\frac{Dd\,D}{D'^2} - M^2\cdot\frac{6rd\,r}{4}\cdot\cos 2(nt - Mt + \in - E) \\ &+ 3M^2\cdot\frac{d(rd\,r)}{ndt}\cdot\sin 2(nt - Mt + \in - E) \end{aligned} \quad \textbf{(254)}$$

when the squares of the eccentricities are omitted.

817. In the same manner it is easy to see that the effect of Jupiter's compression is

$$R = -\frac{(r - \frac{1}{2}f)}{3a^3} + \frac{(r - \frac{1}{2}f)}{a^5}\cdot rd\,r\,.$$

The three last values of R contain all the forces that trouble the longitude and radius vector of m.

FIRST APPROXIMATION

Perturbations in the Radius Vector and Longitude of m that are independent of the Eccentricities

818. Since R has been taken with a negative sign, equation (155) becomes

$$\frac{d^2.rd\,r}{dt^2} + m\cdot\frac{rd\,r}{r^3} + 2\int \mathrm{d}R + r\left(\frac{dR}{dr}\right) = 0\,. \quad \textbf{(255)}$$

The mass of each satellite is about ten thousand times less than the mass of Jupiter, and may therefore be omitted in the comparison, and if Jupiter be taken as the unit of mass $m = 1$.

When the eccentricity is omitted $r = a$; but by article 556 the action of the disturbing forces produces a permanent increase in a, which may be expressed by $d\,a$, therefore if $(a + d\,a)^{-3}$ be put for r^{-3},

$$\frac{d^2.rd\,r}{dt^2} + \frac{rd\,r}{a^3}\left(1 - 3\frac{d\,a}{a}\right) + 2\int \mathrm{d}R + r\left(\frac{dR}{dv}\right) = 0\,. \quad \textbf{(256)}$$

819. When the eccentricities are omitted,

$$R = +m_/\left\{\tfrac{1}{2}A_0 + A_1\cos(n_/t - nt + \in_/ - \in) + A_2\cos 2(n_/t - nt + \in_/ - \in)\right\}$$

$$+\frac{m_{/}}{2}\cdot\frac{rd\,r}{a^2}\cdot a\left(\frac{dA_0}{da}\right)$$
$$-\tfrac{1}{4}M^2a^2-\tfrac{1}{2}M^2\cdot\frac{rd\,r}{a^2}-\tfrac{3}{4}M^2a^2\cos 2(nt-Mt+\epsilon-E) \qquad \textbf{(257)}$$
$$-\frac{\left(r-\frac{1}{2}f\right)}{3a^3}+\frac{\left(r-\frac{1}{2}f\right)}{a^5}\cdot rd\,r.$$

Since dR relates to the mean motion of m, the term

$$\frac{m_{/}}{2}\cdot\frac{rd\,r}{a^2}\cdot a\left(\frac{dA_0}{da}\right)$$

gives

$$2\int \mathrm{d}R=m_{/}\cdot\frac{rd\,r}{a}\cdot\left(\frac{dA_0}{da}\right);$$

moreover the same term gives

$$r\left(\frac{dR}{dr}\right)=\frac{m_{/}}{2}\cdot\frac{rd\,r}{a}\left\{\frac{dA_0}{da}+a\frac{d^2A_0}{da^2}\right\}.$$

With regard to Jupiter's compression

$$\int \mathrm{d}R=R,\quad r\left(\frac{dR}{dr}\right)=-3R,$$

consequently

$$2\int \mathrm{d}R+r\left(\frac{dR}{dr}\right)=\frac{\left(r-\frac{1}{2}f\right)}{3a^3}-\frac{\left(r-\frac{1}{2}f\right)}{a^5}\cdot rd\,r.$$

Attending to these circumstances, and observing that

$$\frac{1}{a^3}=n^2=\frac{1+m}{a^3},$$

it will be found, when the eccentricities are omitted and the whole divided by a^2, that

$$+\frac{d^2.rd\,r}{a^2dt^2}+N^2\cdot\frac{rd\,r}{a^2}+2n^2K+n^2\cdot\frac{r-\frac{1}{2}f}{3a^2}-M^2$$
$$+\sum.\frac{m_{/}n^2}{2}\cdot a^2\left(\frac{dA_0}{da}\right)-3M^2\cdot\frac{2n-M}{2n-2M}\cdot\cos 2(nt-Mt+\epsilon-E)$$
$$+\sum.m_{/}n^2\cdot\left\{a^2\left(\frac{dA_1}{da}\right)+\frac{2n}{n-n_{/}}\cdot aA_1\right\}.\cos(n_{/}t-nt+\epsilon_{/}-\epsilon) \qquad \textbf{(258)}$$

$$+\sum . m_{/} n^{2} \cdot \left\{ a^{2} \left(\frac{dA_{2}}{da} \right) + \frac{2n}{n-n_{/}} \cdot aA_{2} \right\} . \cos 2 \left(n_{/} t - nt + \in_{/} - \in \right)$$
$$+ \&c. \&c. = 0.$$

Where to abridge

$$N^{2} = n^{2} \left\{ 1 - \frac{3da}{a} - \frac{\left(r - \frac{1}{2} f \right)}{a^{2}} - \frac{2M^{2}}{n^{2}} + \sum \frac{m_{/} a^{2}}{2} \left\{ 3 \left(\frac{dA_{0}}{da} \right) + a \left(\frac{d^{2} A_{0}}{da^{2}} \right) \right\} \right\};$$

a quantity that differs little from n^2, for the last term is extremely small in consequence of the factor $m_{/}$: the variation of the mean distance a is very small, and so are the other two parts depending on the compression of Jupiter and the action of the sun. Indeed M and $N - n$ may be omitted, in comparison of n in the terms arising from the action of the sun after integration, for the motion of Jupiter is much slower than the motion of his satellites.

820. The preceding equation may be integrated by the method of indeterminate coefficients, if it be assumed that

$$\frac{rdr}{a^{2}} = B + m_{/} b \cos \left(n_{/} t - nt + \in_{/} - \in \right) + m_{/} b_{(1)} \cos 2 \left(n_{/} t - nt + \in_{/} - \in \right) + \&c. + Gm_{/} \cos 2 \left(nt - Mt + \in - E \right).$$

For a comparison of the coefficients of similar cosines after the substitution of this quantity and its differential gives

$$B = -\frac{n^{2}}{N^{2}} \left\{ 2K + \frac{\left(r - \frac{1}{2} f \right)}{3a^{2}} - \frac{M^{2}}{n^{2}} + \sum \frac{m_{/}}{2} a^{2} \left(\frac{dA_{0}}{da} \right) \right\}$$

$$b = \frac{\left\{ a^{2} \left(\frac{dA_{1}}{da} \right) + \frac{2n}{n-n_{/}} \cdot aA_{1} \right\} n^{2}}{\left(n - n_{/} \right)^{2} - N^{2}}$$

$$b_{(1)} = \frac{\left\{ a^{2} \left(\frac{dA_{2}}{da} \right) + \frac{2n}{n-n_{/}} \cdot aA_{2} \right\} n^{2}}{4 \left(n - n_{/} \right)^{2} - N^{2}}$$

$$b_{(2)} = \frac{\left\{ a^{2} \left(\frac{dA_{3}}{da} \right) + \frac{2n}{n-n_{/}} \cdot aA_{3} \right\} n^{2}}{9 \left(n - n_{/} \right)^{2} - N^{2}} \&c.$$

$$G = -\frac{M^{2}}{n^{2}},$$

and the integral of (258) is

$$\frac{rdr}{a^2} = -\frac{n^2}{N^2}\left\{2K + \frac{\left(r - \frac{1}{2}f\right)}{3a^2} - \frac{M^2}{n^2} + \Sigma\frac{m_/}{2}a^2\left(\frac{dA_0}{da}\right)\right\}$$

$$-\frac{M^2}{n^2}\cos 2(nt - Mt + \in - E)$$

$$+\Sigma m'\left\{\begin{array}{l} \dfrac{n^2}{(n-n_/)^2 - N^2}\left\{a^2\left(\dfrac{dA_1}{da}\right) + \dfrac{2n}{n-n_/}aA_1\right\}\cos(n_/t - nt + \in_/ - \in) \\ \dfrac{n^2}{4(n-n_/)^2 - N^2}\left\{a^2\left(\dfrac{dA_2}{da}\right) + \dfrac{2n}{n-n_/}aA_2\right\}\cos 2(n_/t - nt + \in_/ - \in) \\ \dfrac{n^2}{9(n-n_/)^2 - N^2}\left\{a^2\left(\dfrac{dA_3}{da}\right) + \dfrac{2n}{n-n_/}aA_3\right\}\cos 3(n_/t - nt + \in_/ - \in) \end{array}\right\}$$

$$+\&\text{c.} \qquad \&\text{c.}$$

The first term of this equation is what was expressed by $\frac{da}{a}$, for if all the periodic quantities be omitted $r = a$, and this equation becomes

$$\frac{da}{a} = -2K - \frac{\left(r - \frac{1}{2}f\right)}{3a^2} + \frac{M^2}{n^2} - \Sigma\frac{m_/}{2}a^2\left(\frac{dA_0}{da}\right);$$

for N differs so little from n that $\frac{n^2}{N^2} = 1$, without sensible error: this is the permanent change in the radius vector from the disturbing influence.

These are the principal perturbations in the radii vectores of the satellites.

821. Since the squares of the eccentricity are omitted $\sqrt{1-e^2} = 1$, and as $m = 1$, equation (156) of the longitude becomes

$$dv = \frac{2d(rdr)}{a^2 \cdot ndt} - \frac{dr \,.\, dr}{a^2 \cdot ndt} + 3a\iint ndt \,.\, dR + 2a\int ndt \,.\, r\left(\frac{dR}{dr}\right) \qquad \textbf{(259)}$$

since the sign of R is changed.

If the preceding value of $\frac{rdr}{a^2}$ be put in this equation, and also if substitution be made for dR and $r\left(\frac{dR}{dr}\right)$ derived from equation (257), observing that $\frac{1+m_/}{a^3} = \frac{1}{a^3} = n^2$, and that M and N differ but little from n, the result will be

$$
\begin{aligned}
d\,v = +nt&\left\{3K + \frac{\left(r - \tfrac{1}{2}f\right)}{a^2} - \tfrac{7}{4}\frac{M^2}{n^2} + \sum \cdot m_{/} a^2\left(\frac{dA_0}{da}\right)\right\} \\
&+ \tfrac{11}{8}\cdot\frac{M^2}{n^2}\cdot \sin 2\left(nt - Mt + \in - E\right) \\
&+ \sum \frac{m_{/} n}{n - n_{/}}\left\{\begin{aligned}
&+\left\{\frac{n}{n-n_{/}} aA_1 + \frac{2N^2}{\left(n-n_{/}\right)^2 - N^2}\left(a^2\left(\frac{dA_1}{da}\right) + \frac{2n}{n-n_{/}} aA_1\right)\right\} \\
&\times \sin\left(n_{/}t - nt + \in_{/} - \in\right) \\
&+\tfrac{1}{2}\left\{\frac{n}{n-n_{/}} aA_2 + \frac{2N^2}{4\left(n-n_{/}\right)^2 - N^2}\left(a^2\left(\frac{dA_2}{da}\right) + \frac{2n}{n-n_{/}} aA_2\right)\right\} \\
&\times \sin 2\left(n_{/}t - nt + \in_{/} - \in\right) \\
&+\tfrac{1}{3}\left\{\frac{n}{n-n_{/}} aA_3 + \frac{2N^2}{9\left(n-n_{/}\right)^2 - N^2}\left(a^2\left(\frac{dA_3}{da}\right) + \frac{2n}{n-n_{/}} aA_3\right)\right\} \\
&\times \sin 3\left(n_{/}t - nt + \in_{/} - \in\right)
\end{aligned}\right. \\
&+ \&c. \qquad \&c.
\end{aligned}
$$

By article 540 $d\,v$ ought not to contain the mean motion nt, so the first term must be zero, by which the arbitrary constant quantity is determined to be

$$K = -\frac{\left(r - \tfrac{1}{2}f\right)}{3a^2} + \tfrac{7}{12}\frac{M^2}{n^2} - \tfrac{1}{3}\sum m_{/} a^2\left(\frac{dA_0}{da}\right),$$

whence

$$\frac{d\,a}{a} = \frac{\left(r - \tfrac{1}{2}f\right)}{3a^2} - \tfrac{1}{6}\frac{M^2}{n^2} + \tfrac{1}{6}\sum m_{/} a^2\left(\frac{dA_0}{da}\right)$$

and

$$N^2 = n^2\left\{1 - 2\frac{\left(r - \tfrac{1}{2}f\right)}{a^2} - \tfrac{3}{2}\frac{M^2}{n^2} + \sum m_{/} a^2\left\{\left(\frac{dA_0}{da}\right) + \tfrac{1}{2}a\left(\frac{d^2A_0}{da}\right)\right\}\right\}$$

The preceding value of $d\,v$, deprived of its first term, contains all the perturbations in longitude that are independent of the eccentricities; and as the square of s, the tangent of the latitude, is omitted, by article 548 the very small angle $d\,v$ may either be estimated on the orbit of the satellite, or on the fixed plane, since it coincides with its projection. The term depending on the action of the sun corresponds with the variation[3] in the motion of the moon.

822. If the masses of the four satellites be m, m_1, m_2, m_3, the perturbations that m experiences by the action of the other two will be found by changing successively the quantities relative to m_1 into those belonging to m_2 and m_3, and the sum of these will be the action of the three satellites m_1, m_2, and m_3 on m. The perturbations of the others are found by making similar changes.

823. Hereafter the four satellites will be represented m, m_1, m_2, m_3. Where m is the first, or that nearest Jupiter, and m_3 is the fourth and the most distant, all quantities related to them will be accented in the same manner, except it be stated to the contrary.

824. Because $2n_/ = n = N$ nearly,

$$\frac{2m_/ n \cdot N^2}{n - n_/} = m_/ n^2,$$

and the perturbations expressed by

$$\frac{rdr}{a} = +\frac{m_/ n^2}{(n-n_/)^2 - N^2}\left\{a^2\left(\frac{dA_1}{da}\right) + \frac{2n}{n-n_/}\cdot aA_1\right\}.\cos(n_/ t - nt + \epsilon_/ - \epsilon)$$
$$+\frac{m_/ n^2}{(n-n_/)^2 - N^2}\left\{a^2\left(\frac{dA_2}{da}\right) + \frac{2n}{n-n_/}\cdot aA_2\right\}.\cos 2(n_/ t - nt + \epsilon' - \epsilon)$$

[and]

$$dv = +\frac{2m_/ n^2}{(n-n_/)^2 - N^2}\left\{a^2\left(\frac{dA_1}{da}\right) + \frac{2n}{n-n_/}\cdot aA_1\right\}.\sin(n_/ t - nt + \epsilon_/ - \epsilon)$$
$$+\frac{2m_/ n^2}{4(n-n_/)^2 - N^2}\left\{a^2\left(\frac{dA_2}{da}\right) + \frac{2n}{n-n_/}\cdot aA_2\right\}.\sin 2(n_/ t - nt + \epsilon_/ - \epsilon)$$

are the greatest to which the three first satellites are liable, on account of the very small divisors arising from the nearly commensurable ratios in the mean motions of these three bodies.

825. The greatest inequality in the first satellite is occasioned by the action of the second, and expressed by

$$\frac{rdr}{a} = +\frac{m_/ n^2}{(n-n_/)^2 - N^2}\left\{a^2\left(\frac{dA_2}{da}\right) + \frac{2n}{n-n_/}\cdot aA_2\right\}.\cos 2(n_/ t - nt + \epsilon_/ - \epsilon)$$

[and]

$$dv = +\frac{2m_/ n^2}{4(n-n_/)^2 - N^2}\left\{a^2\left(\frac{dA_2}{da}\right) + \frac{2n}{n-n_/}\cdot aA_2\right\}.\sin 2(n_/ t - nt + \epsilon_/ - \epsilon)$$

Because the mean motion of the first satellite is nearly double that of the second, $n = 2n_/$, and as $N = n = 2n_/$ nearly, the divisor

$$4\left(n-n_{/}\right)^{2}-N^{2}=\left\{\left(2n-2n_{/}\right)-N\right\}\left\{\left(2n-2n_{/}\right)+N\right\}=2n\cdot\left(2n-2n_{/}-N\right);$$

and if to abridge

$$F=-a^{2}\left(\frac{dA_{2}}{da}\right)-\frac{2n}{n-n_{/}}\cdot aA_{2},$$

the greatest inequalities in the motion of the first satellite are

$$\begin{aligned}\frac{rdr}{a^{2}}&=-\frac{m_{/}n\,.F}{2\left(2n-2n_{/}-N\right)}.\cos 2\left(n_{/}t-nt+\epsilon_{/}-\epsilon\right)\\ dv&=+\frac{m_{/}n\,.F}{2n-2n_{/}-N}.\sin 2\left(n_{/}t-nt+\epsilon'-\epsilon\right).\end{aligned}\qquad\textbf{(260)}$$

826. The principal inequalities in the second satellite arise from the action of the first and third. Those occasioned by the first depend on the terms that have the divisor $\left(n-n_{/}\right)^{2}-N_{/}^{2}$; the quantities having one accent belong to $m_{/}$, the second satellite. Let $A_{1}^{(1.2)}$ be the value of A_{1} when the second satellite is troubled by the first; then if

$$G=-a_{/}^{2}\left(\frac{dA_{1}^{(1.2)}}{da_{/}}\right)+\frac{2n_{/}}{n-n_{/}}\cdot a_{/}A_{1}^{(1.2)},$$

the principal inequalities in the second satellite occasioned by the first are

$$\begin{aligned}\frac{r_{/}dr_{/}}{d_{/}^{2}}&=-\frac{mn_{/}\,.G}{2\left(n-n_{/}-N_{/}\right)}.\cos\left(nt-n_{/}t+\epsilon-\epsilon_{/}\right)\\ dv_{/}&=+\frac{mn_{/}\,.G}{n-n_{/}-N_{/}}.\sin\left(nt-n_{/}t+\epsilon-\epsilon_{/}\right)\end{aligned}\qquad\textbf{(261)}$$

for

$$n=2n_{/},\quad N_{/}=n_{/},$$

and

$$\left(n-n_{/}\right)^{2}-N^{2}=\left\{n_{/}-n-N_{/}\right\}\cdot\left\{n_{/}-n+N_{/}\right\}=2n_{/}\left(n-n_{/}-N_{/}\right)$$

The action of the third satellite on the second is perfectly similar to the action of the second on the first, on account of the ratios $n=2n_{/}$ and $n_{/}=2n_{2}$ in their mean motions; therefore, the inequalities in the motion of the second, occasioned by the action of the third, will be obtained from equations (260), by changing what relates to the first and second into the quantities relative to the second and third. In this case let $A_{2}^{(3.2)}$ be the value of A_{2} and let

$$F' = -a_/^{\,2}\left(\frac{dA_2^{(3.2)}}{da_/}\right) - \frac{2n_/}{n_/ - n_2}\cdot a_/ A_2^{(3.2)}$$

be the value of *F*, then

$$\begin{aligned} \frac{r_/ d\,r_/}{a_1^2} &= -\frac{m_2 n_/ \,.\, F'}{2\left(n_1 - n_2 - N_/\right)}.\cos 2\left(n_/ t - n_2 t + \epsilon_/ - \epsilon_2\right) \\ d\,v_/ &= +\frac{m_2 n_1 \,.\, F'}{2n_1 - 2n_2 - N_/}.\sin 2\left(n_/ t - n_2 t + \epsilon_/ - \epsilon_2\right). \end{aligned} \qquad \textbf{(262)}$$

By observation,

$$nt - 3n_/ t + 2n_2 t + \epsilon - 3\epsilon_/ + 2\epsilon_2 = 180^\circ,$$

consequently,[4]

$$2\left(n_/ t - n_2 t + \epsilon_/ - \epsilon_2\right) = nt - n_/ t + \epsilon - \epsilon_/ - 180^\circ;$$

for

$$n = 2n_/ \qquad n_/ = 2n_2 \text{ nearly,}$$

the two last inequalities may be added to the preceding, since they depend on the same angle; the principal inequalities in the motion of the second satellite from the action of the first and third are therefore[5]

$$\begin{aligned} \frac{r_/ d\,r_/}{a_1^2} &= -\frac{n_/}{2\left(n - n_/ - N_/\right)}\left\{mG - m_2 F'\right\}.\cos\left(nt - n_/ t + \epsilon - \epsilon_/\right) \\ d\,v_/ &= +\frac{n_/}{n - n_/ - N_/}\left\{mG - m_2 F'\right\}.\sin\left(nt - n_/ t + \epsilon - \epsilon_/\right). \end{aligned} \qquad \textbf{(263)}$$

In consequence of the ratios in the mean motions these inequalities will never be separated.

827. The action of the second satellite produces inequalities in the theory of the third, analogous to those occasioned by the action of the first on the second; hence, if all the quantities in equations (261) relating to the second and first be changed into those belonging to the third and second, and if $A_/^{(2.3)}$ and G' be the values of $A_/^{(1.3)}$ and G in this case, so that

$$A_/^{(2.3)} = A_/^{(1.2)} + \frac{a_2}{a_/^2} - \frac{a_1}{a_2^2},$$

and

$$G' = -a_2^2\left(\frac{dA_/^{(2.3)}}{da_2}\right) + \frac{2n_2}{n_1 - n_2}\cdot a_2 A_/^{(2.3)},$$

the resulting equations for m_2 will be[6]

$$\frac{r_2 d r_2}{a_2{}^2} = -\frac{m_1 n_2 G'}{2(n_1 - n_2 - N_2)} . \cos(n_1 t - n_2 t + \in_1 - \in_2)$$
$$d v_2 = +\frac{m_1 n_2 G'}{n_1 - n_2 - N_2} \cdot \sin(n_1 t - n_2 t + \in_1 - \in_2). \qquad \textbf{(264)}$$

These inequalities have only been detected by observation in the motion of the first satellite.

828. G, which is a function of $A_/^{(1.2)}$ may be expressed by a function of $A_/$, for

$$A_/^{(1.2)} = \frac{a_/}{a^2} - \frac{a}{a_/^2} + A_/,$$

whence on account of

$$n = \frac{1}{a^3}, \quad n_/ = \frac{1}{a_/^3};$$

and that $n = 2n_/$ it may be found that

$$G = 2a_/ A_/ - a_/^2 \left(\frac{dA_/}{da_/} \right).$$

SECOND APPROXIMATION

Inequalities depending on the First Powers of the Eccentricities

829. If $a + \frac{rd\,r}{a}$ be put for r, equation (255) becomes

$$0 = \frac{d^2 rd\,r}{dt^2} + \frac{rd\,r}{a^3}\left\{1 - \frac{3rd\,r}{a^2}\right\} + 2\int dR + r\left(\frac{dR}{dr}\right)$$

or as

$$\frac{1}{a^3} = n^2 = N^2, \text{ very nearly,}$$

$$0 = \frac{d^2 rd\,r}{dt^2} + N^2 rd\,r\left\{1 - \frac{3rd\,r}{a^2}\right\} + 2\int dR + r\left(\frac{dR}{dr}\right). \qquad \textbf{(265)}$$

If the action of the sun be omitted, the only part of the preceding value of R, that has a sensible effect on the radius vector is

$$R = m_{/}\left\{\begin{array}{l} A_1 \cos\left(n_{/}t - nt + \in_{/} - \in\right) + \dfrac{r_{/}d\,r_{/}}{a_{/}^2}\cdot a_{/}\dfrac{dA_1}{da_{/}}\cos\left(n_{/}t - nt + \in_{/} - \in\right) \\ -2A_1\dfrac{d\left(r_{/}d\,r_{/}\right)}{a_{/}^2\cdot n_{/}dt}\sin\left(n_{/}t - nt + \in_{/} - \in\right) \end{array}\right\};$$

but these terms are very important, because they serve for the determination of the secular inequalities in the eccentricities and motions of the apsides. With regard to the terms depending on nt, $\int dR = R$, substituting for R, and dividing the whole equation (265) by a^2, it becomes, when $\left(\dfrac{rdr}{a^2}\right)^2$ is omitted,

$$0 = +\frac{d^2\,rd\,r}{a^2dt^2} + N^2\cdot\frac{rd\,r}{a^2}$$

$$+\Sigma\left\{\begin{array}{l} +m_{/}n^2\,\dfrac{r_{/}d\,r_{/}}{a_{/}^2}\left\{2aa_{/}\left(\dfrac{dA_1}{da_{/}}\right) + a^2a_{/}\left(\dfrac{d^2A_1}{dada_{/}}\right)\right\}\cos\left(n_{/}t - nt + \in_{/} - \in\right) \\ -\dfrac{2m_{/}n^2\,.d\ .r_{/}d\,r_{/}}{a_{/}^2\cdot n_{/}d\,t}\left\{2aA_1 + a^2\left(\dfrac{dA_1}{da}\right)\right\}\sin\left(n_{/}t - nt + \in_{/} - \in\right) \end{array}\right\}$$

830. In order to integrate this equation, it may be assumed that

$$\frac{rd\,r}{a^2} = h\cos\left(nt + \in - gt - \Gamma\right); \quad \frac{r_{/}d\,r_{/}}{a_{/}^2} = h_{/}\cos\left(n_{/}t + \in_{/} - gt - \Gamma\right), \text{ \&c.};$$

h and $h_{/}$ are indeterminate coefficients, and $gt + \Gamma$ is the motion of the apsides of the orbits of the satellites.

When these quantities and their differentials are substituted, the square of g neglected, and those terms alone retained that depend on the angle $nt + \in - gt - \Gamma$, a comparison of the coefficients of similar cosines gives

$$0 = h\left\{N^2 + 2ng - n^2\right\} + \Sigma\frac{m_{/}n^2}{2}h_{/}\left\{2aa_{/}\left(\frac{dA_1}{da_{/}}\right) + a^2a_{/}\left(\frac{d^2A_1}{dada_{/}}\right) + 4aA_1 + 2a^2\left(\frac{dA_1}{da}\right)\right\}$$

but by article 458,

$$a\left(\frac{dA_1}{da}\right) + a_{/}\left(\frac{dA_1}{da_{/}}\right) = -A_1;$$

and if the value of N^2 in article 819 be substituted, this coefficient becomes[7]

$$0=+h\left\{\frac{g}{n}-\frac{\left(r-\tfrac{1}{2}f\right)}{a^2}-\tfrac{3}{4}\frac{M^2}{n^2}+\tfrac{1}{2}\Sigma m_{/}\left\{a^2\left(\frac{dA_0}{da}\right)+\tfrac{1}{2}a^3\left(\frac{d^2A_0}{da^2}\right)\right\}\right\}$$

$$+\tfrac{1}{2}\Sigma m_{/}h_{/}\left\{aA_1-a^2\left(\frac{dA_1}{da}\right)-\tfrac{1}{2}a^3\left(\frac{d^2A_1}{da^2}\right)\right\}.$$

And is in article 474, if

$$(0.1)=-\frac{m_{/}n}{2}\left\{a^2\left(\frac{dA_0}{da}\right)+\tfrac{1}{2}a^3\left(\frac{d^2A_0}{da^2}\right)\right\};$$

$$\boxed{0.1}=\frac{m_{/}n}{2}\left\{aA_1-a^2\left(\frac{dA_1}{da}\right)-\tfrac{1}{2}a^3\left(\frac{d^2A_1}{da^2}\right)\right\};$$

and if

$$(0)=\frac{\left(r-\tfrac{1}{2}f\right)}{a^2}n;\quad \boxed{0}=\tfrac{3}{4}\frac{M^2}{n},$$

this equation becomes

$$0=h\left\{g-(0)-\boxed{0}-(0.1)\right\}+\boxed{0.1}\,h_{/}$$

with regard to the first satellite troubled by the second; but the action of m_2 and m_3 produces terms similar to those caused by m_1; and if the same notation be used that was employed for the planets, this equation, when m is troubled by the other three satellites, by the sun, and by the compression of Jupiter, becomes

$$0=h\left\{g-(0)-\boxed{0}-(0.1)-(0.2)-(0.3)\right\}+\boxed{0.1}\,h_1+\boxed{0.2}\,h_2+\boxed{0.3}\,h_3 \qquad \textbf{(266)}$$

A similar equation exists for each satellite, and may be determined from this by changing the quantities relative to m into those relating to m_1 m_2 m_3, and reciprocally; hence, for the others,

$$\begin{aligned}
0&=h_1\left\{g-(1)-\boxed{1}-(1.0)-(1.2)-(1.3)\right\}+\boxed{1.0}\,h+\boxed{1.2}\,h_2+\boxed{1.3}\,h_3,\\
0&=h_2\left\{g-(2)-\boxed{2}-(2.0)-(2.1)-(2.3)\right\}+\boxed{2.0}\,h+\boxed{2.1}\,h_1+\boxed{2.3}\,h_3,\\
0&=h_3\left\{g-(3)-\boxed{3}-(3.0)-(3.1)-(3.2)\right\}+\boxed{3.0}\,h+\boxed{3.1}\,h_1+\boxed{3.2}\,h_3.
\end{aligned} \qquad \textbf{(267)}$$

By (484)

$$(0.1)m\sqrt{a}=(1.0)m_{/}\sqrt{a_{/}},\ \&\text{c}.$$

and also

$$\boxed{0.1}\,m\sqrt{a}=\boxed{1.0}\,m_{/}\sqrt{a_{/}},\ \&\text{c}.$$

for any two satellites, so these functions are easily deduced from one another, which saves computation.

These results are perfectly similar to those obtained for the planets, $h\ h_{/}$, &c., correspond to $N\ N'$, &c.

831. It has already been mentioned that the part of the longitude of each satellite depending on the eccentricity consists of four terms, of one that is really the equation of the centre, and of three others arising from the variations in the orbits, chiefly induced by the action of the excess of matter at Jupiter's equator. The coefficients of these sixteen terms are obtained by the aid of the preceding equations, and also of the annual and sidereal motions of the apsides of the orbits. The variations in the radii vectores depend on the same cause, contain the same values of g, and have the same coefficients. $h,\ h_1,\ h_2,\ h_3,$ are the real eccentricities of the four orbits, and if they be eliminated there will result an equation of the fourth degree in g. These four values of g, which will be represented by $g,\ g_1,\ g_2,\ g_3,$ are the annual and sidereal motions of the apsides of the orbits of the four satellites.

832. Let g, the annual and sidereal motion of the first satellite, belong to the first of the preceding equations, and assume $h_1 = x_1 h;\ \ h_2 = x_2 h;\ \ h_3 = x_3 h;$ then the substitution of these in equation (266) will make h vanish, and $x_1,\ x_2,\ x_3,$ will be given in functions of g. Thus h, which may be regarded as the real eccentricity of the orbit of m, is an arbitrary quantity, known by observation. Again, if $g_{/}$ be the value of g in the second of the preceding equations, and if

$$h = x_1^{(1)} h_1, \quad h_2 = x_2^{(1)} h_1, \quad h_3 = x_3^{(1)} h_1,$$

by the substitution of these, h_1 will vanish from the equation in question, $x_1^{(1)},\ x_2^{(1)},\ x_3^{(1)},$ will be given in functions of $g_{/}$; and $h_{/}$, the real eccentricity of the orbit of $m_{/}$, is determined by observation. In the same manner, if $x_1^{(2)}, x_2^{(2)}, x_3^{(2)}, x_1^{(3)}, x_2^{(3)}, x_3^{(3)},$ be the quantities corresponding to g_2 and g_3, h_2 and h_3 will be arbitrary constant quantities, which vanish from the two last of equations (267); whence $x_1^{(2)}, x_2^{(2)}, x_3^{(2)},$ and $x_1^{(3)}, x_2^{(3)}, x_3^{(3)},$ will be given in functions of g_2 and g_3.

Thus the coefficients of the sixteen terms of the equations of the centre, corresponding to the four values of g, are $h,\ h_1,\ h_2,\ h_3,\ x_1 h_1,\ x_2 h_1,\ x_3 h_1,\ x_1^{(1)} h_2,$ &c. &c., of which $h,\ h_1,\ h_2,\ h_3,$ are the real eccentricities of the orbits of the four satellites, and are determined by observation: by means of these, and the equations (266) and (267), values of x, x_1, &c. will be obtained; and also the four roots of g. Observation shows, however, that h and h_1 are insensible.

833. It was assumed, that

$$\frac{rdr}{a^2} = h\cos\left(nt + \epsilon - gt - \Gamma\right);$$

and as g has four roots, to each of which there are four corresponding values of h, this expression becomes

$$\frac{rdr}{a^2} = +h\cos(nt+\epsilon-gt-\Gamma) + h_1\cos(nt+\epsilon-g_1t-\Gamma_1) + h_2\cos(nt+\epsilon-g_2t-\Gamma_2) + h_3\cos(nt+\epsilon-g_3t-\Gamma_3):$$

thus the whole variation in the radius vector of the first satellite depends on h, the eccentricity of its own orbit, on g the motion of its own nodes, and on those of the other three. The corresponding inequalities in the radii vectores of the other three satellites are,

$$\frac{r_1dr_1}{a_1^2} = +x_1\cdot h\cdot\cos(n_1t+\epsilon_1-gt-\Gamma) + x_1^{(1)}h_1\cos(n_1t+\epsilon_1-g_1t-\Gamma_1) + x_1^{(2)}h_2\cdot\cos(n_1t+\epsilon_1-g_2t-\Gamma_2) + x_1^{(3)}h_3\cos(n_1t+\epsilon_1-g_3t-\Gamma_3)$$

$$\frac{r_2dr_2}{a_2^2} = +x_2\cdot h\cdot\cos(n_2t+\epsilon_2-gt-\Gamma) + x_2^{(1)}h_1\cos(n_2t+\epsilon_2-g_1t-\Gamma_1) + x_2^{(2)}h_2\cdot\cos(n_2t+\epsilon_2-g_2t-\Gamma_2) + x_2^{(3)}h_3\cos(n_2t+\epsilon_2-g_3t-\Gamma_3)$$

$$\frac{r_3dr_3}{a_3^2} = +x_3\cdot h\cdot\cos(n_3t+\epsilon_3-gt-\Gamma) + x_3^{(1)}h_1\cos(n_3t+\epsilon_3-g_1t-\Gamma_1) + x_3^{(2)}h_2\cdot\cos(n_3t+\epsilon_3-g_2t-\Gamma_2) + x_3^{(3)}h_3\cos(n_3t+\epsilon_3-g_3t-\Gamma_3).$$

These equations contain the perturbations in the radii vectores of the four satellites, depending on the first powers of the eccentricities, and are the complete integrals of the differential equation (265), when applied to each satellite, since they contain the eight arbitrary constant quantities h, h_1, h_2, h_3, Γ, Γ_1, Γ_2, Γ_3, all of which are known by observation. The four last are the mean longitudes of the lower apsides of the orbits of the satellites at the epoch.

834. If the orbits be considered as variable ellipses, ae being the eccentricity of the orbit of the first satellite, and v the longitude of its lower apsis, estimated from the origin of the angles,

$$\frac{rdr}{a^2} = -e\cos(nt+\epsilon-v);$$

comparing this with the preceding value of $\frac{rdr}{a^2}$ the result is

$$e\cos v = -h\cos(gt+\Gamma) - h_1\cos(g_1t+\Gamma_1) - \&c.$$
$$e\sin v = -h\sin(gt+\Gamma) - h_1\sin(g_1t+\Gamma_1) - \&c.$$

whence e and v may be obtained; and for the same reasons, e_1, v_1, e_2, v_2, and e_3, v_3.

835. When the squares of the eccentricity are omitted, the elliptical part of the longitude is $v = 2e\sin(nt + \in - v)$ by 392; or representing it by dv for the satellites, where it chiefly arises from the disturbing forces, it gives

$$dv = 2e\cos v\ \sin(nt + \in) - 2e\sin v\ \cos(nt + \in);$$

and substituting for $e\cos v$, and $e\sin v$,

$$\begin{aligned} dv = &-2h\ \sin(nt + \in - gt - \Gamma) \quad - 2h_1 \sin(nt + \in - g_1 t - \Gamma_1) \\ &- 2h_2 \sin(nt + \in - g_2 t - \Gamma_2) - 2h_3 \sin(nt + \in - g_3 t - \Gamma_3), \end{aligned}$$

which is the equation of the centre of the first satellite. It appears, that the first term depends on the eccentricity and apsis of its own orbit, the second term arises from the action of the second satellite, and depends on the eccentricity and apsis of the orbit of that body; the other two inequalities arise from the attraction of the third and fourth satellites, and depend on the eccentricities and apsides of their orbits.

The corresponding inequalities in the longitude of the other three satellites are,

$$\begin{aligned} dv_1 = &-2x_1 h\sin(n_1 t + \in_1 - gt - \Gamma) - 2x_1^{(1)} h_1 \sin(n_1 t + \in_1 - g_1 t - \Gamma_1) \\ &- 2x_1^{(2)} h_2 \sin(n_1 t + \in_1 - g_2 t - \Gamma_2) - 2x_1^{(3)} h_3 \sin(n_1 t + \in_1 - g_3 t - \Gamma_3) \\ dv_2 = &-2x_2 h\sin(n_2 t + \in_2 - gt - \Gamma) - 2x_2^{(1)} h_1 \sin(n_2 t + \in_2 - g_1 t - \Gamma_1) \\ &- 2x_2^{(2)} h_2 \sin(n_2 t + \in_2 - g_2 t - \Gamma_2) - 2x_2^{(3)} h_3 \sin(n_2 t + \in_2 - g_3 t - \Gamma_3) \\ dv_3 = &-2x_3 h\sin(n_3 t + \in_3 - gt - \Gamma) - 2x_3^{(1)} h_1 \sin(n_3 t + \in_3 - g_1 t - \Gamma_1) \\ &- 2x_3^{(2)} h_2 \sin(n_3 t + \in_3 - g_2 t - \Gamma_2) - 2x_3^{(3)} h_3 \sin(n_3 t + \in_3 - g_3 t - \Gamma_3). \end{aligned}$$

These inequalities are very considerable in the motions of the satellites in longitude.

The whole then depends on the resolution of the equations (266) and (267); these, however, are not complete, as several terms arise from the perturbations depending on the squares and products of the disturbing forces.

Action of the Sun depending on the Eccentricities

836. The part of R depending on the action of the sun in the elliptical hypothesis is

$$\begin{aligned} R = &-\tfrac{3}{4} M^2 a^2 \cdot \frac{DdD}{D'^2} - \frac{6rdr}{4} M^2 \cos(2nt - 2Mt + 2\in - 2E) \\ &+ \tfrac{12}{4} M^2 \cdot \frac{d(rdr)}{ndt} \sin(2nt - 2Mt + 2\in - 2E). \end{aligned}$$

But

$$\frac{rdr}{a^2} = h\cos\left(nt + \in - gt - \Gamma\right);$$

and

$$\frac{DdD}{D'^2} = H\cos\left(Mt + E - \Pi\right),$$

H being the eccentricity of Jupiter's orbit, and Π the longitude the perihelion; hence

$$R = -\tfrac{3}{4}M^2a^2 \cdot H \cdot \cos\left(Mt + E - \Pi\right) - \tfrac{9}{4}M^2 \cdot a^2 \cdot h\cos\left(nt - 2Mt + \in - 2E + gt + \Gamma\right);$$

and therefore, equation (265) becomes

$$0 = +\frac{d^2 rdr}{dt^2} + N^2\frac{rdr}{a^2}\left\{1 - 3h\cos\left(nt + \in - gt - \Gamma\right)\right\}$$
$$-\tfrac{3}{2}M^2 . H . \cos\left(Mt + E - \Pi\right) - 9M^2 . h . \cos\left(nt - 2Mt + \in - 2E + gt + \Gamma\right).$$

By article 820, it appears that $\frac{rdr}{a^2}$ contains the terms

$$-\frac{M^2}{n^2} \cdot \cos\left(2nt - 2Mt + 2\in - 2E\right);$$

hence

$$-3N^2 \cdot \frac{rdr}{a^2} \cdot h . \cos\left(nt + \in - gt + \Gamma\right)$$

contains

$$\tfrac{3}{2}M^2 \cdot h \cdot \cos\left(nt - 2Mt + \in - 2E + gt + \Gamma\right),$$

N^2 being very nearly equal to n^2, so that $\frac{N^2}{n^2} = 1$: thus,

$$0 = \frac{d^2 rdr}{a^2dt^2} + N^2 \cdot \frac{rdr}{a^2} - \tfrac{15}{2} \cdot M^2 . h . \cos\left(nt - 2Mt + \in - 2E + gt + \Gamma\right) - \tfrac{3}{2}M^2 . H . \cos\left(Mt + E - \Pi\right),$$

whence by the method of indeterminate coefficients, the integral is

$$\frac{rdr}{a^2} = \frac{15M^2 \cdot h}{4n\left(2M + N - n - g\right)}\cos\left(nt - 2Mt + \in - 2E + gt + \Gamma\right) + \frac{3M^2 \cdot H}{2n^2} \cdot \cos\left(Mt + E - \Pi\right),$$

which is the effect of the sun's action on the radius vector; and if it be substituted in equation (259), the perturbations in longitude depending on the same cause will be

$$dv = -\frac{15M^2 \cdot h}{4n(2M+N-n-g)} \cdot \sin(nt - 2Mt + \epsilon - 2E + gt + \Gamma) - \frac{3M}{n} \cdot H \cdot \sin(Mt + E - \Pi).$$

837. The first term of the second number of this expression corresponds to the evection in the lunar theory, and is only sensible in the motions of the third and fourth satellites; but it is not the only inequality of this kind, for each of the roots g_1, g_2, g_3, furnishes another. The perturbations corresponding to these for the other satellites are found, by reciprocally changing the quantities relative to one into those relating to the others.

Inequalities depending on the Eccentricities which become sensible in consequence of the Divisors they acquire by double integration

838. It is found by observation, that the mean motion of the first satellite is nearly equal to twice that of the second; and that the mean motion of the second is nearly equal to twice that of the third; or

$$n = 2n_1, \quad \mathrm{n}_1 = 2n_2.$$

In consequence of the squares of these nearly commensurable quantities becoming divisors to the inequalities by a double integration, they have a very sensible effect on the preceding equations in longitude.

839. The only part of equation (259) that has a double integral is $3a\iint ndt \,.\, \mathrm{d}R$; and as the divisors in question arise from the angles $nt - 2n_1t$, $n_1t - 2n_2t$ alone, it is easy to see that the part of R containing these angles is,

$$\begin{aligned} R = &+m_1 \frac{r_1 d r_1}{a_1^2} \cdot a_1 \left(\frac{dA_1}{da_1}\right) \cdot \cos(n_1 t - nt + \epsilon_1 + \epsilon) \\ &-2m_1' \cdot \frac{d \cdot (r_1 d r_1)}{a_1^2 \cdot n_1 dt} \cdot A_1 \,.\, \sin(n_1 t - nt + \epsilon_1 - \epsilon) \\ &+m_/ \cdot \frac{rd r}{a^2} a \cdot \left(\frac{dA_2}{da}\right) \cdot \cos 2(n_/ t - nt + \epsilon_/ - \epsilon) \\ &+4m_/ \frac{d \cdot (rd r)}{a^2 \cdot ndt} \cdot A_2 \,.\, \sin e(n_/ t - nt + \epsilon_/ - \epsilon). \end{aligned}$$

With regard to the action of $m_/$ on m, if $h_/ \cos(n_/ t + \epsilon_/ - gt - \Gamma)$, be put instead of $\frac{r_/ d r_/}{a_/^2}$, and $h\cos(nt + \epsilon - gt - \Gamma)$ instead of $\frac{rd r}{a^2}$; and as by articles 828 and 826

$$G = -a_{/}^{2}\left(\frac{dA_1}{da_{/}}\right) + 2a_{/}A_1$$
$$F = -a^{2}\left(\frac{dA_2}{da}\right) - 2aA_2,$$

observing that $n = 2n_{/}$ nearly, the result will be

$$R = -\frac{m_{/}}{2a}\cdot\left\{Fh + \frac{a}{a_{/}}Gh_{/}\right\}\cdot\cos\left(nt - 2n_{/}t + \in - 2\in_{/} + gt + \Gamma\right),$$

which substituted in $3a\iint ndt\,.\,dR,$ and integrated, gives for the first satellite,

$$d\,v = \frac{-3m_{/}\,.\,n^{2}}{2\left(n - 2n_{/} + g\right)^{2}}\cdot\left\{F'h + \frac{a}{a_{/}}Gh_{/}\right\}\cdot\sin\left(nt - 2n_{/}t + \in - 2\in_{/} + gt + \Gamma\right).$$

Again, since $n_1 = 2n_2$ nearly, the action of m_2 on $m_{/}$ produces in $d\,v_{/}$ an inequality similar to the preceding, which is

$$d\,v_{/} = \frac{-3m_2\,.\,n_{/}^{2}}{2\left(n_{/} - 2n_2 + g\right)^{2}}\cdot\left\{Fh_{/} + \frac{a_1}{a_2}G'h_2\right\}\cdot\sin\left(n_1t - 2n_2t + \in_1 - 2\in_2 + gt + \Gamma\right).$$

An inequality of the same kind, and from the same cause, is produced also in the equation of the centre of $m_{/}$ by the action of m, for with regard to the inequalities we are now considering, article 574 shows that

$$d\,v_{/} = -\frac{m\sqrt{a}}{m_{/}\sqrt{a_{/}}}d\,v$$

whence the inequality produced by the action of m on $m_{/}$ is

$$d\,v_{/} = \frac{-3m\,.\,n^{2}\sqrt{a}}{2\left(n - 2n_{/} + g\right)^{2}\sqrt{a_{/}}}\cdot\left\{Fh + \frac{a}{a_{/}}Gh_{/}\right\}\cdot\sin\left(nt - 2n_{/}t + \in - 2\in_{/} + gt + \Gamma\right).$$

This inequality may be added to the preceding, for

$$nt - 2n_{/}t + \in - 2\in_{/} = n_1t - 2\,n_2t + \in_1 - 2\in_2 + 180^{\circ},$$

and as $n = 2n_{/}$ nearly, and $\left(\frac{a}{a_{/}}\right)^2 = \left(\frac{n_{/}}{n}\right)^2$; therefore

$$\frac{n^2\sqrt{a}}{\sqrt{a_{/}}} = 2n_{/}^2 \cdot \frac{a_{/}}{a},$$

and thus the two terms become

$$d\,v_{/} = \frac{3n_{/}^2}{(n-2n_{/}+g)^2}\cdot\left\{m\left\{Gh_{/}+\frac{a_{/}}{a}Fh\right\}+\frac{m_2}{2}\left\{F'h_{/}+\frac{a_{/}}{a_2}G'h_2\right\}\right\}\cdot\sin\left(nt-2n_{/}t+\in-2\in_{/}+gt+\Gamma\right).$$

Lastly, the action of m_1 on m_2 produces an inequality in m_2, analogous to that produced by the action of m on $m_{/}$, which is therefore

$$d\,v_2 = \frac{-3m_{/}\,.\,n_2^2}{(n_1-2n_2+g)^2}\cdot\left\{G'h_2+\frac{a_2}{a_1}F'h_{/}\right\}\cdot\sin\left(nt-2n_{/}t+\in-2\in_{/}+gt+\Gamma\right).$$

We shall represent the preceding inequalities by

$$d\,v = -Q\sin\left(nt-2n_{/}t+\in-2\in_{/}+gt+\Gamma\right) \qquad \textbf{(268)}$$

$$d\,v_1 = +Q_1\sin\left(nt-2n_{/}t+\in-2\in_{/}+gt+\Gamma\right) \qquad \textbf{(269)}$$

$$d\,v_2 = -Q_2\sin\left(nt-2n_{/}t+\in-2\in_{/}+gt+\Gamma\right) \qquad \textbf{(270)}$$

These inequalities are relative to the root g, but each of the roots g_1, g_2, g_3, give similar inequalities in the motions of the three first satellites.

No such inequality exists in the motion of the fourth satellite, since its mean motion is not nearly commensurable with that of any of the others.

Inequalities depending on the Square of the Disturbing Force

840. On account of the nearly commensurable ratios in the mean motions of the three first satellites the preceding equations must be added as periodic variations to the mean motions, as in the case of Jupiter and Saturn, by means of them several terms are added to equations (266) and (267), which determine the secular variations in the eccentricities and longitudes of the apsides. For if the eccentricities be omitted, and $m = 1$, the equations df, df' in article 433 relative to the planets, become

$$d\left(e\cos v\right) = -andt\left\{2\cos v\left(\frac{dR}{dv}\right)+a\sin v\left(\frac{dR}{dr}\right)\right\},$$

$$d\,(e\sin v\,) = -andt\left\{2\sin v\left(\frac{dR}{dv}\right) - a\cos v\left(\frac{dR}{dr}\right)\right\}.$$

The secular variations with regard to the first satellite will be found by substituting

$$R = -\frac{\left(r - \tfrac{1}{2}f\right)}{3r^3} + m_1 A_2 \cos 2(v_/ - v)$$

in the first of the preceding equations, and putting $nt + \in + d\,v$ for v, and $a^2 + 2rd\,r$ for r^2; whence

$$\begin{aligned} d\,(e\cos v\,) = {} & +4andt \cdot m_/ A_2 \sin(2v - 2v_/)\cos v \\ & -a^2 ndt \cdot m_/\left(\frac{dA_2}{da}\right)\cos(2v - 2v_/)\sin v \\ & -ndt \cdot \frac{\left(r - \tfrac{1}{2}f\right)}{a^2}\cdot \sin(nt + \in) \\ & -ndt \cdot \frac{\left(r - \tfrac{1}{2}f\right)}{a^2} d\,v \cos(nt + \in) \\ & +4ndt \cdot \frac{\left(r - \tfrac{1}{2}f\right)}{a^2}\cdot \frac{rd\,r}{a^2}\sin(nt + \in)\,. \end{aligned}$$

Then on1y attending to the terms depending on $nt - 2n_/ t + \in - 2\in_/$, if the values of $\frac{rd\,r}{a^2}$ and $d\,v$ given by (260) be substituted; and as

$$F = -4aA_2 - a^2\left(\frac{dA_2}{da}\right),$$

the result will be

$$d\,(e\cos v\,) = -\frac{m_/ F\,.\,ndt}{2}\cdot\left\{1 - \frac{(0)}{2n - 2n_/ - N}\right\}\cdot \sin\left(nt - 2n_/ t + \in - 2\in_/\right)$$

in which

$$(0) = \frac{\left(r - \tfrac{1}{2}f\right)}{a^2} n\,.$$

Since the mean longitudes $nt + \in$ and $n_/ t + \in_/$ are variable, these angles must be augmented by the values of $d\,v$, $d\,v_/$, in equations (268) and (269), so that

$$nt + \epsilon + Q\sin\left(nt - 2n_{/}t + \epsilon - 2\epsilon_{/} + gt + \Gamma\right)$$
$$n_{/}t + \epsilon_{/} + Q_{/}\sin\left(nt - 2n_{/}t + \epsilon - 2\epsilon_{/} + gt + \Gamma\right)$$

must be substituted in the sine of the preceding equation, which becomes, in consequence,

$$d\left(e\cos v\right) = \frac{m_{/}F\,.\,ndt}{4}\cdot\left\{1 - \frac{(0)}{2n - 2n_{/} - N}\right\}\cdot\left(2Q_{/} - Q\right)\cdot\sin\left(gt + \Gamma\right)$$

when the periodic part is omitted. But by article 834,

$$e\cos v = -h\cos\left(gt + \Gamma\right);$$

hence

$$d\left(e\cos v\right) = hg\cdot dt\cdot\sin\left(gt + \Gamma\right),$$

and thus

$$\frac{m_{/}\,.\,Fn}{4}\cdot\left\{1 - \frac{(0)}{2n - 2n_{/} - N}\right\}\cdot\left(2Q_{/} - Q\right)$$

must be subtracted from equation (266).

841. The same analysis applied to $d\left(e_{/}\cos v_{/}\right)$ will determine the increment of the first of equations (267), with regard to the second satellite. But, in this case,

$$R = -\frac{\left(r - \tfrac{1}{2}f\right)}{3r^{3}} + m_{/}A_{1}^{(1.2)}\cos\left(v - v_{/}\right) + m_{2}A_{2}^{(3.2)}\cos 2\left(v_{/} - v_{2}\right),$$

and equations (269) and (270) must be employed. The result is, that

$$\frac{m_{2}n_{/}}{4}\left\{1 - \frac{(1)}{n - n_{/} - N_{/}}\right\}F'\cdot\left(2Q_{2} - Q_{1}\right) - \frac{mn_{/}}{4}\cdot\left\{\frac{(1)}{n - n_{/} - N_{/}}\right\}G\cdot\left(2Q_{/} - Q\right)$$

must be added to the first of equations (267).

For the same reason

$$\frac{m_{/}n_{2}}{4}\cdot G'\cdot\left(2Q_{2} - Q_{1}\right)\cdot\left\{1 - \frac{(2)}{n_{/} - n_{2} - N_{2}}\right\}$$

must be added to the second of equation (267).

As these quantities only arise from the ratios among the mean motions of the three first satellites, the secular variations of the fourth are not affected by them. In consequence of these additions, equations (266) and (267) become

$$
\begin{aligned}
0 &= +h\left\{g-(0)-\boxed{0}-(0.1)-(0.2)-(0.3)\right\}+\boxed{0.1}h_1+\boxed{0.2}h_2+\boxed{0.3}h_3 \\
&\quad -\frac{m_{/}n}{4}\left\{1-\frac{(0)}{2n-2n_{/}-N_{/}}\right\}F\left(2Q_{/}-Q\right); \\
0 &= +h_1\left\{g-(1)-\boxed{1}-(1.0)-(1.2)-(1.3)\right\}+\boxed{1.0}h+\boxed{1.2}h_2+\boxed{1.3}h_3 \\
&\quad -\frac{mn_{/}}{4}\left\{1-\frac{(1)}{n-n_{/}-N_{/}}\right\}G\left(2Q_{/}-Q\right)+\frac{m_2n_{/}}{4}\left\{1-\frac{(1)}{n-n_{/}-N_{/}}\right\}F'\left(Q_2-Q_1\right); \\
0 &= +h_2\left\{g-(2)-\boxed{2}-(2.0)-(2.1)-(2.3)\right\}+\boxed{2.0}h+\boxed{2.1}h_1+\boxed{2.3}h_2 \\
&\quad +\frac{m_{/}n_2}{4}\left\{1-\frac{(2)}{n_{/}-2n_2-N_2}\right\}G'\left(2Q_2-Q_1\right); \\
0 &= +h_3\left\{g-(3)-\boxed{3}-(3.0)-(3.1)-(3.2)\right\}+\boxed{3.0}h+\boxed{3.1}h_1+\boxed{3.2}h_2\,.
\end{aligned}
\qquad \textbf{(271)}
$$

842. An inequality which is only sensible in the theory of the second satellite may now be determined; for, by (260),

$$d\,v=\frac{m_{/}nF}{2n-2n_{/}-N}\sin\left(2nt-2n_{/}t+2\in-2\in_{/}\right);$$

or[8]

$$d\,v=\frac{m_{/}nF}{2n-2n_{/}-N_{/}}\left\{\cos\left(nt-2n_{/}t+\in-2\in_{/}\right).\sin\left(nt+\in\right)+\sin\left(nt-2n_{/}t+\in-2\in_{/}\right).\cos\left(nt+\in\right)\right\};$$

but as $v=2e\sin\left(nt+\in-v\right)$, and for the variable ellipse which we are now considering,

$$d\,v=2d\cdot\left(e\cos v\right)\cdot\sin\left(nt+\in\right)-2d\cdot\left(e\sin v\right)\cdot\cos\left(nt+\in\right).$$

By comparing these two values,

$$
\begin{aligned}
2d\left(e\sin v\right)&=-\frac{m_{/}nF}{2n-2n_{/}-N_{/}}\sin\left(nt-2n_{/}t+\in-2\in_{/}\right),\\
2d\left(e\sin v\right)&=+\frac{m_{/}nF}{2n-2n_{/}-N_{/}}\cos\left(nt-2n_{/}t+\in-2\in_{/}\right).
\end{aligned}
$$

But the elliptical expression of v contains the term

$$\tfrac{5}{4}e^2\sin\left(2nt+2\in-2v\right),$$

or

$$\tfrac{5}{4}\left(e^2\cos^2 v-e^2\sin^2 v\right).\sin2\left(nt+\in\right)-\tfrac{5}{4}.e^2\sin v\,.\cos v\,.\cos2\left(nt+\in\right).$$

If $e\sin v + d(e\sin v)$, and $e\cos v + d(e\cos v)$ be put for $e\sin v$, and $e\cos v$, it becomes[9]

$$dv = \tfrac{5}{4}\left\{(d\cdot e\cos v)^2 - (d\cdot e\sin v)^2\right\}\cdot\sin 2(nt+\epsilon) - \tfrac{5}{4}d\cdot e\cos v\cdot d\cdot e\sin v\cdot\cos 2(nt+\epsilon);$$

and in consequence of the preceding values of $d(e\cos v)$, $d(e\sin v)$, there is the following inequality in the longitude of the first satellite,

$$dv = \tfrac{5}{16}\left(\frac{m_{\prime}nF}{2n-2n_{\prime}-N}\right)^2\sin 4(nt-n_{\prime}t+\epsilon-\epsilon_{\prime}).$$

By the same process the corresponding inequalities in the second and third satellites are found to be

$$dv_{\prime} = \tfrac{5}{16}\frac{n_{\prime}^2}{(n-n_{\prime}-N_{\prime})^2}\{mG-m_2F'\}^2\sin 2(nt-n_{\prime}t+\epsilon-\epsilon_{\prime})$$

$$dv_2 = \tfrac{5}{16}\left(\frac{m_{\prime}n_2G'}{n_{\prime}-n_2-N_2}\right)^2\sin 2(n_1t-n_2t+\epsilon_{\prime}-\epsilon_2).$$

Librations of the three first Satellites

843. Some very interesting inequalities arising from the equation

$$nt-3n_1t+2n_2t+\epsilon-3\epsilon_1+2\epsilon_2=180^\circ,$$

are found among the terms depending on the squares of the disturbing forces, that affect the whole theory of the satellites, in consequence of the very small divisor $(n-3n_1+2n_2)^2$ which they acquire by double integration. If the orbits be considered as variable ellipses, and if z, z_1, z_2, be the mean longitudes of the three first satellites, it is clear that the terms having the square of $n-3n_1+2n_2$ for divisor, can only be found from

$$d^2z = 3andt\,.\,\mathrm{d}R$$
$$d^2z_1 = 3a_1n_1dt\,.\,\mathrm{d}R_1$$
$$d^2z_2 = 3a_2n_2dt\,.\,\mathrm{d}R_2$$

which are variations in the mean motions by article 439.

844. With regard to the action of m_1 on m, the series R in article 815 only contains the angle $n_/ t - nt + \in_/ - \in$ and its multiples, it is evident therefore, that the angle $nt - 3n_1 t + 2n_2 t$ can only arise from the substitution of the perturbations (262) which depend on the angle $2n_1 t - 2n_2 t$. By article 814, $d v_1$ contains both the elliptical part of the longitude and the perturbations, and if the latter be expressed by[10] $d\overline{v}_/$ then[11]

$$d v_/ = \frac{2d\left(r_/ d v_/\right)}{a_/^2 \cdot n_/ dt} + d\overline{v}_/$$

and when the square of the eccentricity is omitted $\frac{r_/ d r_/}{a_/^2}$ becomes $\frac{d r_/}{a_/}$. If then $d\overline{v}_/$ and $\frac{d r_/}{a_/}$ be put for $\frac{2d\left(r_/ d r_/\right)}{a_/^2 \cdot n_/ dt}$ and $\frac{r_/ d r_/}{a_/^2}$ the part of R required is

$$R = m_/ \cdot \left(\frac{dA_/}{da_/}\right) \cdot d r_/ \cdot \cos\left(n_/ t - nt + \in_/ - \in\right) - m_/ \cdot d\overline{v}_/ \cdot A_/ \cdot \sin\left(n_/ t - nt + \in_/ - \in\right) \cdot ndt,$$

or

$$\mathrm{d}R = m_/ \cdot A_/ d\overline{v}_/ \cdot \cos\left(n_/ t - nt + \in_/ - \in\right) \cdot ndt - m_/ \cdot \left(\frac{dA_/}{da_/}\right) \cdot d r_/ \cdot \sin\left(n_/ t - nt + \in_/ - \in\right) \cdot ndt.$$

for in this case $dd r_/$ and $dd\overline{v}_/$ are zero, since equations (262), or

$$d r_/ = -\frac{m_2 n_/ a_/ F'}{2\left(n_1 - n_2 - N_/\right)} . \cos\left(2n_/ t - 2n_2 t + 2\in_1 - 2\in_2\right)$$
$$d\overline{v}_/ = +\frac{m_2 n_1 F'}{2n_/ - 2n_2 - N_/} . \sin\left(2n_1 t - 2n_2 t + 2\in_1 - 2\in_2\right)$$

do not contain the arc nt. If these quantities be substituted in dR, it will be found, in consequence of

$$G = 2a_/ A_1 - a_/^2 \left(\frac{dA_1}{da_/}\right), \text{ and } n = 2n_/,$$

that

$$\frac{d^2 z}{dt^2} = -\frac{3n^2 m_/ m_2 F' G}{8\left(n - n_/ - N_/\right)} \frac{a}{a_/} \sin\left(nt - 3n_1 t + 2n_2 t + \in - 3\in_/ + 2\in_2\right);$$

for as

$$2n_/ - 2n_2 = n - n_/ \text{ nearly,}$$

the divisor

$$2n_/ - 2n_2 - N_/ = n - n_/ - N_/ .$$

The variation in the mean motion of the second satellite consists of two parts; one arising from the action of m, and the other from that of m_2.

The value of R for the first is

$$R = m \cdot A^{(1.2)} \cdot d\overline{v}_{/} \cdot \sin\left(nt - n_{/}t + \epsilon - \epsilon_{/}\right) + m \cdot \left(\frac{dA^{(1.2)}}{da}\right) \cdot dr_{/} \cdot \cos\left(nt - n_{/}t + \epsilon - \epsilon_{/}\right).$$

If the differential of R be taken with regard to $n_{/}t$, making $d\overline{v}_{/}$ and $dr_{/}$ vary, by the substitution of the preceding values of $d\overline{v}_{/}$, $dr_{/}$, and their differentials, it will be found, in consequence of

$$G = 2a_{/}A_{/}^{(1.2)} - a_{/}^{2}\left(\frac{dA_{/}^{(1.2)}}{da_{/}}\right),$$

and $n_2 = \frac{1}{2}n_1$, that the variation in the mean motion of the second satellite from the action of the first must be

$$\frac{3n^3 m . m_2 F' G}{16\left(n - n_{/} - N_{/}\right)} \sin\left(nt - 3n_{/}t + 2n_2 t + \epsilon - 3\epsilon_{/} - 2\epsilon_2\right).$$

Again, if

$$\frac{dr_{/}}{a_{/}} = -\frac{mn_{/}G}{2\left(n - n_{/} - N_{/}\right)} \cos\left(nt - n_{/}t + \epsilon - \epsilon_{/}\right),$$

and

$$dv_{/} = +\frac{mn_{/}G}{n - n_{/} - N_{/}} \sin\left(nt - n_{/}t + \epsilon - \epsilon_{/}\right),$$

from article 826, be substituted in the differential of

$$R = m_2 \left\{ \left(\frac{dA_2^{(3.2)}}{da}\right) dr_{/} \cos\left(2n_{/}t - 2n_2 t + 2\epsilon_1 - \epsilon_2\right) - 2A_2^{(3.2)} . d\overline{v}_{/} . \sin\left(2n_{/}t - 2n_2 t + 2\epsilon_1 - 2\epsilon_2\right) \right\},$$

which is the value of R with regard to m_2 and m_1, observing that $n = 2n_{/}$; and, by article 826,

$$F' = -4a_{/}A_2^{(3.2)} - a_{/}^{2}\left(\frac{dA_2^{(3.2)}}{da_{/}}\right),$$

the part of $\frac{d^2z_{/}}{dt^2}$, arising from the action of m_2 on $m_{/}$, will be found equal to

$$\frac{3m \cdot m_2 n^3}{32\left(n - n_{/} - N_{/}\right)} F'G \sin\left(nt - 3n_{/}t + 2n_2t + \in - 3\in_1 + 2\in_2\right);$$

and the whole variation in the mean motion of $m_{/}$, from the combined action of m and m_2, is

$$\frac{d^2z_{/}}{dt^2} = \frac{9mm_2n^3F'G}{32\left(n - n_{/} - N_{/}\right)} \sin\left(nt - 3n_{/}t + 2n_2t + \in - 3\in_1 + 2\in_2\right).$$

With regard to the action of m_1 on m_2

$$R = m_1\left\{-2A_2^{(3.2)}.d\bar{v}_{/}\sin 2\left(n_1t - n_2t + \in_1 - \in_2\right) + \left(\frac{dA_2^{(3.2)}}{da_{/}}\right)\cdot dr_{/}.\cos 2\left(n_1t - n_2t + \in_1 - \in_2\right)\right\}.$$

If the same values of $d\bar{v}_{/}$ and $dr_{/}$ be substituted in the differential of this with regard to n_2t, it will be found that the action of m_1 and m_2 produces the inequality

$$\frac{d^2z_2}{dt^2} = -\frac{3n^3mm_{/}F'G}{64\left(n - n_{/} - N_{/}\right)} \cdot \frac{a_2}{a} \sin\left(nt - 3n_{/}t + 2n_2t + \in - 3\in_1 + 2\in_2\right).$$

845. As

$$\frac{d^2z}{dt^2} = 3andt\,.\mathrm{d}R; \quad \frac{d^2z_{/}}{dt^2} = 3a_{/}n_{/}dt\,.\mathrm{d}R_{/}; \quad \frac{d^2z_2}{dt^2} = 3a_2n_2dt\,.\mathrm{d}R_2;$$

by comparing the values of these three quantities in the last article the result is

$$mdR + m_{/}dR_{/} = 0, \text{ and } m_{/}dR_{/} + m_2dR_2 = 0,$$

which is conformable with what was shown in article 573, with regard to the planets.

846. As the three first satellites move in orbits, they are nearly circular, the error would be very small, in assuming[12]

$$nt + \in, \quad n_{/}t + \in_{/}, \quad n_2t + \in_2,$$

to be their true longitudes.

The preceding inequalities in the mean motions of the three first satellites are therefore

$$\frac{d^2v}{dt^2} = -\frac{3n^3 m_1 m_2 \frac{a}{a_/} F'G}{8\left(n - n_/ - N_/\right)} \sin\left(v - 3v_/ + 2v_2\right)$$
$$\frac{d^2v_/}{dt^2} = +\frac{9n^3 m m_2 F'G}{32\left(n - n_/ - N_/\right)} \sin\left(v - 3v_/ + 2v_2\right) \qquad \textbf{(272)}$$
$$\frac{d^2v_2}{dt^2} = -\frac{3n^3 m m_/ F'G}{64\left(n - n_/ - N_/\right)} \frac{a_2}{a_/} \sin\left(v - 3v_/ + 2v_2\right).$$

847. In order to abridge, let $f = v - 3v_1 + 2v_2$; whence

$$\frac{d^2f}{dt^2} = \frac{d^2v}{dt^2} - 3\frac{d^2v_/}{dt^2} + 2\frac{d^2v_2}{dt^2}.$$

If the preceding values be put in this, and if to abridge,

$$K = -\frac{3nF'G}{8\left(n - n_/ - N_/\right)}\left\{\frac{a}{a_/} m_1 m_2 + \tfrac{9}{4} m m_2 + \frac{a_2}{4a_/} m m_2\right\},$$

the result will be[13]

$$\frac{d^2f}{dt^2} = K \,.\, n^2 \,.\sin f \,.$$

K and n^2 may be assumed to be constant quantities, their variations are so small; hence the integral of this equation is

$$dt = \frac{\pm df}{\sqrt{c - 2Kn^2 \cos f}};$$

c is a constant quantity introduced by integration, the different values of which give rise to the three following cases.

848. 1st. If c be greater than $2Kn^2$, without regard to the sign, it must be positive; and the angle $\pm f$ will increase indefinitely, and will become equal to one, two, three, &c., circumferences.

2nd. If K be positive, and c less than $2n^2K$, abstracting from the sign, the radical will be imaginary when $\pm f$ is equal to zero, or to one, two, three, &c. circumferences. The angle f must therefore oscillate about the semi-circumference,[14] since it never can he zero, or equal to a whole circumference, which would make the time an imaginary quantity. Its mean value must consequently be 180°.

3rd. If c be less than $2Kn^2$, and K negative, the radical would be imaginary when the angle $\pm f$ is equal to any odd number of semi-circumferences; the angle f must therefore oscillate about zero, its mean value, since the time cannot be imaginary. However, as it will be shown that K is a positive quantity, the latter case does not exist, so that f must either increase indefinitely, or oscillate about 180°. In order to ascertain which of these is the law of nature, let

$$f = p \pm v ,$$

p being 180° and v any angle whatever; hence

$$dt = \frac{dv}{\sqrt{c + 2Kn^2 \cos v}} . \qquad \textbf{(273)}$$

If the angles $\pm f$ and v increase indefinitely, c is positive, and greater than $2Kn^2$; hence, in the interval between $v = 0$, and its increase to 90°, dt is less than

$$\frac{dv}{n\sqrt{2K}}; \text{ and } t < \frac{v}{n\sqrt{2K}} .$$

Thus the time t that the angle v employs in increasing till it be equal to 90°, will be less than

$$\frac{v}{2n\sqrt{2K}} .$$

This time is less than two years: but from the discovery of the satellites the libration or angle v has always been zero, or extremely small; therefore this angle does not increase indefinitely, it can only oscillate about its mean value of zero.

The second case, then, is what really exists, and the angle

$$v - 3v_1 + 2v_2 ,$$

must oscillate about 180°, which is its mean value.

849. Several important results are given by the equation

$$v - 3v_1 + 2v_2 = p + v .$$

If the insensible part v be omitted,

$$nt - 3n_1 t + 2n_2 t + \epsilon - 3\epsilon_1 + 2\epsilon_2 = p .$$

Whence

$$n - 3n_1 + 2n_2 = 0$$
$$\epsilon - 3\epsilon_1 + 2\epsilon_2 = 180^\circ .$$

These two equations are perfectly confirmed by observation, for Delambre[15] found, from the comparison of a great number of eclipses of the three first satellites, that their mean motions in a hundred Julian years, with regard to the equinox, are

1st Satellite	$7{,}432{,}435^\circ.46982$
2nd Satellite	$3{,}702{,}713^\circ.231493$
3rd Satellite	$1{,}837{,}852^\circ.113582$

whence it appears, that the mean motion of the first, minus three times that of the second, plus twice that of the third, is equal to $9''.0072$, so small a quantity, that it affords an astonishing proof of the accuracy both of the theory and observation. Delambre determined also, from a great number of eclipses, that the epochs of the mean motions of the three first satellites, at midnight, on the first of January 1750, were

$$\epsilon = 15^\circ.02626$$
$$\epsilon' = 310^\circ.44689$$
$$\epsilon'' = 10^\circ.27219 ,$$

whence

$$\epsilon - 3\epsilon_1 + 2\epsilon_2 = 180^\circ\ 1'\ 3'',$$

a result that is less accurate than the preceding; but it will be shown, in treating of the eclipses of the satellites, that it probably arises from errors of observation, depending on the discs of the satellites, which vanish to us before they are quite immersed in the shadow.

850. The same laws exist in the synodic motions of the satellites; for in the equation

$$nt - 3n_1 t + 2n_2 t + \epsilon - 3\epsilon_1 + 2\epsilon_2 = 180^\circ,$$

the angles may be estimated from a moveable axis, since the position of the axis would vanish in this equation: we may therefore suppose that

$$nt + \epsilon, \quad n_1 t + \epsilon_1, \quad n_2 t + \epsilon_2,$$

are the mean synodic longitudes. This has a great influence on the eclipses of the three first satellites, as will appear afterwards.

851. On account of these laws the actions of the first and third satellites on the second are united in one term, given in article 826, which is the great inequality in that body indicated by observations. These inequalities will never be separated.

852. Without the mutual attraction of the satellites the two equations

$$n - 3n_1 + 2n_2 = 0$$
$$\epsilon - 3\epsilon_1 + 2\epsilon_2 = 0$$

would be unconnected. It would have been necessary in the beginning of their motions that their epochs and mean motions had been so arranged as to suit these equations, which is most improbable; and in this case the slightest action from any foreign cause, as the attraction of the planets and comets, would have changed the ratios. But the mutual action of the satellites gives perfect stability to these relations, for, at the origin of the motion, when $t = 0$,

$$\frac{dv}{ndt} - 3\frac{dv_1}{n_1dt} + 2\frac{dv_2}{n_2dt} = \pm\sqrt{\frac{c}{n^2} - 2K\cos\left(\epsilon - 2\epsilon_1 + 3\epsilon_2\right)}$$

c being less than $2Kn^2$. It would be sufficient for the accuracy of the preceding results that the first member of this equation had been comprised between the limits

$$+2K\sin\left(\tfrac{1}{2}\epsilon - \tfrac{3}{2}\epsilon_1 + \epsilon_2\right)$$
$$-2K\sin\left(\tfrac{1}{2}\epsilon - \tfrac{3}{2}\epsilon_1 + \epsilon_2\right)$$

at the origin of their motions, and it is sufficient for their stability that no foreign force disturbs it.

853. It appears then, that if the preceding laws among the mean motions of the three first satellites had only been approximate at their origin, their mutual attraction would ultimately have rendered them exact.

854. The angle v is so small, that we may make

$$\cos v = 1 - \tfrac{1}{2}v^2;$$

and if to abridge

$$x^2 = \frac{c + 2Kn^2}{n^2K},$$

x being arbitrary, on account of the arbitrary constant quantity c that it contains, equation (273) becomes

$$v = x\sin\left(nt\sqrt{K} + A\right),$$

A being a new arbitrary quantity.

855. As the motions of the four satellites in longitude, latitude, and distance, are determined by twelve differential equations of the second order, their integrals must contain twenty-four arbitrary quantities, which are the data of the problem, and are given by observation. Two of these are determined by the equations

$$n - 3n_1 + 2n_2 = 0$$
$$\in - 3\in_1 + 2\in_2 = 180^\circ \;;$$

they are, however, replaced by x and A, the first determines the extent of the libration, and A marks the time when it is zero: neither are determined, since the inequality v has as yet been insensible.

856. The integrals of the three equations (272) may now be found, for as

$$v - 3v_1 + 2v_2 = p + v = p + x\sin\left(nt\sqrt{K} + A\right),$$
$$\sin\left(v - 3v_1 + 2v_2\right) = \sin\left\{p + x\sin\left(nt\sqrt{K} + A\right)\right\}$$
$$= -x\sin\left(nt\sqrt{K} + A\right);$$

hence the first of equations (272) becomes

$$\frac{d^2v}{dt^2} = \frac{3n^2m_1m_2F'G}{8\left(n - n_1 - N_1\right)}\frac{a}{a_1}x\sin\left(nt\sqrt{K} + A\right),$$

the integral of which is

$$v = \frac{x\sin\left(nt\sqrt{K} + A\right)}{1 + \frac{9a_1m}{4am_1} + \frac{a_2m}{4am_2}}.$$

In the same way

$$v_1 = -\frac{x\sin\left(nt\sqrt{K} + A\right)\cdot\frac{3a_/m}{4am_/}}{1 + \frac{9a_1m}{4am_1} + \frac{a_2m}{4am_2}}$$

$$v_2 = \frac{\frac{a_2m}{8am_2}x\sin\left(nt\sqrt{K} + A\right)}{1 + \frac{9a_1m}{4am_1} + \frac{a_2m}{4am_2}}$$

which are the three equations of the libration. They have hitherto been insensible, but they modify all the inequalities of long periods in the theory of the three first satellites.

857. For example, the inequality

$$v = -\frac{3M}{n} H \sin\left(Mt + E - \Pi\right),$$

gives

$$\frac{d^2v}{dt^2} = +\frac{3M^2}{n} H \sin\left(Mt + E - \Pi\right);$$

But the differential of the first of the equations of libration is

$$\frac{d^2v}{dt^2} = -\frac{Kn^2 \sin\left(v - 3v_1 + 2v_2\right)}{1 + \frac{9a_1m}{4am_1} + \frac{a_2m}{4am_2}};$$

or, if to abridge,

$$b = 1 + \frac{9a_1m}{4am_1} + \frac{a_2m}{4am_2}$$

$$\frac{d^2v}{dt^2} = -\frac{Kn^2}{b} \cdot \sin\left(v - 3v_1 + 2v_2\right),$$

and adding the two values of $\frac{d^2v}{dt^2}$

$$\frac{d^2v}{dt^2} = -\frac{Kn^2}{b} \sin\left(v - 3v_1 + 2v_2\right) + \frac{3M^3}{n} \cdot H \sin\left(Mt + E - \Pi\right). \tag{274}$$

To integrate this equation let

$$v = l \sin\left(Mt + E - \Pi\right), \quad v_1 = l_1 \sin\left(Mt + E - \Pi\right), \quad v_2 = l_2 \sin\left(Mt + E - \Pi\right),$$

hence,

$$v - 3v_1 + 2v_2 = \left(l - 3l_1 + 2l_2\right) \cdot \sin\left(Mt + E - \Pi\right),$$

and

$$\frac{d^2v}{dt^2} = \left\{\frac{3M^3 . H}{n} - \frac{Kn^2}{b}\left(l - 3l_1 + 2l_2\right)\right\} \sin\left(Mt + E - \Pi\right);$$

and if

$$l \sin\left(Mt + E - \Pi\right)$$

be put for v,

$$l = -\frac{3M \cdot H}{n} + \frac{Kn^2}{bM^2}\left(l - 3l_1 + 2l_2\right).$$

In the same manner it may be found that

$$l_1 = -\frac{6M \cdot H}{n} - \frac{3a_1 m}{4am_1} \cdot \frac{Kn^2}{bM^2}\left(l - 3l_2 + 2l_2 \right),$$

$$l_2 = -\frac{12M \cdot H}{n} + \frac{3a_2 m}{8am_2} \cdot \frac{Kn^2}{bM^2}\left(l - 3l_1 + 2\, l_2 \right),$$

whence

$$l - 3l_1 + 2l_2 = \frac{9M^3 \cdot H}{n\left(Kn^2 - M^2\right)};$$

so that equation (274) becomes

$$\frac{d^2v}{dt^2} = +\frac{3M^3}{n}\left\{1 + \frac{3K \,.\, n^2}{b\left(Kn^2 - M^2\right)}\right\} H \cdot \sin\left(Mt + E - \Pi\right)$$

and

$$d\,v = -\frac{3M}{n}\left\{1 + \frac{3K \,.\, n^2}{b\left(Kn^2 - M^2\right)}\right\} H \cdot \sin\left(Mt + E - \Pi\right).$$

The inequalities in the longitude of m_1 and m_2 are found by the same analysis, consequently

$$d\,v = -\frac{3M}{n}\left\{1 + \frac{3K \,.\, n^2}{b\left(M^2 - Kn^2\right)}\right\} H \cdot \sin\left(Mt + E - \Pi\right)$$

$$d\,v_1 = -\frac{6M}{n}\left\{1 - \frac{9a_1 mK \,.\, n^2}{8am_1 b\left(M^2 - Kn^2\right)}\right\} H \cdot \sin\left(Mt + E - \Pi\right)$$

$$d\,v_2 = -\frac{12M}{n}\left\{1 + \frac{3a_2 mK \,.\, n^2}{32am_2 b\left(M^2 - Kn^2\right)}\right\} H \cdot \sin\left(Mt + E - \Pi\right).$$

This inequality replaces the term depending on the same angle in article 836. It corresponds with the annual equation in the lunar theory, and its period is very great.

858. The variation in the form and position of Jupiter's orbit is the cause of secular inequalities in the mean motions of the satellites, similar to those produced by the variation of the earth's orbit on the moon; hitherto, however, they have been insensible, and will remain so for a long time, with the exception of one depending on the displacement of Jupiter's equator, and that is only perceptible in the motions of the fourth satellite; but these cannot be determined till the equations in latitude have been found.

Notes

[1] Galilei, Galileo, *Sidereus Nuncius or the Sidereal Messenger*, University of Chicago Press, 1989 (see also note 1, *Introduction*).

[2] The closing parenthesis in the argument $(\cos(U-v)+3\cos 2(U-v))$ is omitted in the 1st edition.

[3] This word is capitalized in the 1st edition.

[4] The 1st left hand term inside the parenthesis *reads* $nt_{/}$ in the 1st edition.

[5] The right hand side argument in the 1st equation *reads* $(\cos nt - n_{/}t + \in - \in_{/})$ in the 1st edition.

[6] The right hand side argument in the 1st equation *reads* $(n_1 t - n_2 + \in_1 - \in_2)$ in the 1st edition.

[7] The closing parenthesis of the 1st term is omitted in the 1st edition.

[8] The term $\sin(nt - 2n_{/}t + \in - 2\in_{/})$ *reads* $\sin(nt - 2nt + \in - 2\in)$ in the 1st edition.

[9] The left hand parenthesis is omitted in the term $(d\,.\,e\cos v)^2$ and *reads* $d\,.\,e\cos v)^2$ in the 1st edition.

[10] This *reads* $\overline{d}v_{/}$ in the 1st edition.

[11] The right hand parenthesis is omitted in the numerator of the 1st term and *reads* $2d(r_{/}d v_{/}$ in the 1st edition.

[12] The third term *reads* $nt_2 + \in_2$ in the 1st edition.

[13] $\sin f$ *reads* $\sin j$ in the 1st edition.

[14] This *reads* semicircumference in the 1st edition.

[15] See note 54, *Preliminary Dissertation*.

BOOK IV

CHAPTER II[1]

PERTURBATIONS OF THE SATELLITES IN LATITUDE

859. THE perturbations in latitude are found with most facility from

$$0=\frac{d^2s}{dv^2}\left\{1-\frac{2}{h^2}\int\left(\frac{dR}{dv}\right)\cdot\frac{dv}{u^2}\right\}-\frac{1}{h^2u^2}\cdot\frac{ds}{dv}\cdot\left(\frac{dR}{dt}\right)+\frac{s}{h^2u^2}\left(\frac{dR}{du}\right)+\frac{\left(1+s^2\right)}{h^2u^2}\left(\frac{dR}{ds}\right),$$

which was employed for the moon, but in that case R was a function of u, v, and s, and the differential $\frac{dR}{ds}$ was taken in that hypothesis, which we shall represent by $\frac{dR'}{ds}$, but now R is a function of r, v, and s, hence

$$du\left(\frac{dR}{du}\right)+ds\left(\frac{d'R}{ds}\right)=dr\left(\frac{dR}{dr}\right)+ds\left(\frac{dR}{ds}\right)$$

and as

$$r=\frac{\sqrt{1+s^2}}{u},\quad du=-\frac{dr}{r^2}\sqrt{1+s^2}+\frac{sds}{r\sqrt{1+s^2}};$$

and comparing the coefficients of ds in these two equations

$$\frac{us}{1+s^2}\left(\frac{dR}{du}\right)+\frac{d'R}{ds}=\left(\frac{dR}{ds}\right),$$

so the preceding equation of latitude, when $\frac{\sqrt{1+s^2}}{r}$ is put for u, and the product of the disturbing force by $s^2\cdot\frac{ds}{dv}$ omitted, becomes

$$0=\frac{d^2s}{dv^2}+s+\frac{r^2}{h^2}\left(\frac{dR}{ds}\right)-\frac{r^2}{h^2}\cdot\frac{ds}{dv}\left(\frac{dR}{dv}\right).$$

860. The only part of the disturbing force that affects the latitude is

$$R=-\frac{m_/ r_/ r\left\{ss_/-\frac{1}{2}s^2\cos\left(v_/-v\right)\right\}}{\left\{r^2-rr_/\cos\left(v_/-v\right)+r_/^2\right\}^{\frac{3}{2}}}+\frac{3m_/}{4n^2}\left\{s_/^2+s^2\cos\left(2v-2u\right)-4ss_/\cos\left(v-u\right)\right\}+\frac{\left(r-\frac{1}{2}f\right)}{a^2}\left(s-s'\right)^2.$$

If the eccentricities be omitted, $r=a$, $r_/=a_/$, and

$$\frac{1}{\left(r^2-2rr_/\cos\left(v_/-v\right)+r_/^2\right)^{\frac{3}{2}}}=\left\{a^2-2aa_/\cos\left(v_/-v\right)+a_/^2\right\}^{-\frac{3}{2}}=\tfrac{1}{2}B_0+B_1\cos\left(v_/-v\right)+\&c.$$

as for the planets; hence

$$R=-\sum m_/a^2a_/\left\{ss_/-\tfrac{1}{2}s^2\cos\left(v_/-v\right)\right\}B_1\cos\left(v_/-v\right)+\frac{3M^2}{4n^2}\left\{s^2-4sS\cos\left(v-U\right)\right\}+\frac{\left(r-\frac{1}{2}f\right)}{a^2}\left(s-s'\right)^2.$$

Whence

$$0=+\frac{d^2s}{dt^2}+s\left\{1+2\frac{\left(r-\frac{1}{2}f\right)}{a^2}+\frac{3}{2}\frac{M^2}{n^2}+\tfrac{1}{2}\sum m_/a^2a_/\,.\,B_1\right\}$$
$$-\frac{2\left(r-\frac{1}{2}f\right)}{a^2}s'-\frac{3M^2}{n^2}S\cos\left(U-v\right)-\sum m_/a^2\cdot a_/B_1s_/\cos\left(v_/-v\right).$$

861. In order to integrate this equation, let

$$s=l\,.\sin\left(v+pt+\Lambda\right);\quad s_/=l_/\sin\left(v_1+pt+\Lambda\right)$$
$$s_2=l_2\,.\sin\left(v_2+pt+\Lambda\right);\quad s_3=l_3\sin\left(v_3+pt+\Lambda\right)$$
$$S=L'\cdot\sin\left(U+pt+\Lambda\right);\quad s'=L\sin\left(v+pt+\Lambda\right),$$

l, l_1, l_2, l_3, L' and L being the inclination of the orbits of the four satellites, of Jupiter's orbit and equator on the fixed plane, p and Λ, quantities on which the sidereal motions and longitudes of the nodes depend.

If the motion of only one satellite be considered at a time, then substituting for s, $s_/$ and S, also putting $\frac{v}{n}$ for t, and omitting p^2, the comparison of the coefficients of $\sin\left(v+pt+\Lambda\right)$ gives

$$0 = +l\left\{\frac{p}{n} - \frac{\left(r - \frac{1}{2}f\right)}{a^2} - \frac{3}{4}\frac{M^2}{n^2} - \frac{1}{4}\sum m_{/}a^2 \,.\, a_{/}B_1\right\} + \frac{\left(r - \frac{1}{2}f\right)}{a^2}L + \frac{3}{4}\frac{M^2}{n^2}\cdot L' + \frac{1}{4}\sum m_{/}a^2 \cdot a_{/}B_1 l_{/}. \qquad \textbf{(275)}$$

If $\dfrac{a}{a'_{/}} = \boldsymbol{a}$, and $v_{/} - v = n_{/}t - nt + \in_{/} - \in = \boldsymbol{b}$

$$\left\{1 - 2\boldsymbol{a}\cos\boldsymbol{b} + \boldsymbol{a}^2\right\}^{-\frac{3}{2}} = a_{/}^3\left(\tfrac{1}{2}B_0 + B_1\cos\boldsymbol{b} + B_2\cos 2\boldsymbol{b} + \&\text{c.}\right),$$

which is identical with the series in article 446, and therefore the formulae for the planets give by article

$$(0.1) = \frac{m_1 n \cdot a^2 a_1 B_1}{4},$$

consequently equation (275) becomes

$$0 = l\left\{p - (0) - \boxed{0} - (0.1)\right\} + L(0) + L'\boxed{0} + (0.1)\,l_{/},$$

but the action of the satellites m_2 and m_3 produce terms analogous to those produced by m_1; so the preceding equation, including the disturbing action of all the bodies, and the compression of Jupiter, is

$$0 = +l\left\{p - (0) - \boxed{0} - (0.1) - (0.2) - (0.3)\right\} + (0)L + \boxed{0}L' + (0.1)\,l_1 + (0.2)\,l_2 + (0.3)\,l_3. \qquad \textbf{(276)}$$

By the same process the corresponding equations for the other satellites are

$$\begin{aligned}
0 &= +l_1\left\{p - (1) - \boxed{1} - (1.0) - (1.2) - (1.3)\right\} \\
&\quad + (1)L + \boxed{1}L' + (1.0)\,l + (1.2)\,l_2 + (1.3)\,l_3; \\
0 &= +l_2\left\{p - (2) - \boxed{2} - (2.0) - (2.1) - (2.3)\right\} \\
&\quad + (2)L + \boxed{2}L' + (2.0)\,l + (2.1)\,l_1 + (2.3)\,l_3; \\
0 &= +l_3\left\{p - (3) - \boxed{3} - (3.0) - (3.1) - (3.2)\right\} \\
&\quad + (3)L + \boxed{3}L' + (3.0)\,l + (3.1)\,l_1 + (3.2)\,l_2.
\end{aligned} \qquad \textbf{(277)}$$

862. These four equations determine the coefficients of the latitude; they include the reciprocal action of the satellites, together with that of the sun, and the direct action of Jupiter considered as a spheroid, but in the hypothesis that the plane of his equator retains a permanent inclination on the fixed plane: that, however, is not the case, for as neither the sun nor the orbits of all the satellites are in the plane of Jupiter's equator, their action on the protuberant matter causes a nutation in the equator, and a precession of its equinoxes, in all respects similar to those occasioned by the action of the moon on the earth, which produce sensible inequalities in the motions of the satellites. Thus the satellites, by troubling Jupiter, indirectly disturb their own motions.

The Effect of the Nutation and Precession of Jupiter on the Motion of his Satellites

863. The reciprocal action of the bodies of the solar system renders it impossible to determine the motion of any one part independently of the rest; this creates a difficulty of arrangement, and makes it indispensable to anticipate results which can only be obtained by a complete investigation of the theory on which they depend. The nutation and precession of Jupiter's spheroid can only be known by the theory of the rotation of the planets, from whence it is found that if q and g be the inclinations of Jupiter's equator and orbit on the fixed plane, y the retrograde motion of the descending node of his equator on that plane, and estimated from the vernal equinox of Jupiter, t the longitude of the ascending node of his orbit, *it* the rotation of Jupiter, and *A*, *B*, *C*, the moments of inertia of his spheroid with regard to the principal axes of rotation, as in article 177, the precession of Jupiter's equinoxes is

$$\frac{dq}{dt}=\frac{3(2C-A-B)}{4iC}\left\{M^2 g\sin(t+y)+\sum mn^2 g'\sin(t'+y)\right\}$$

whence $M^2 g\sin(t+y)$ is the action of the sun, and $\sum mn^2 g'\sin(t'+y)$ that of the satellites. The nutation is

$$q\cdot\frac{dy}{dt}=\frac{3(2C-A-B)}{4iC}\left\{q\left(M^2+\sum mn^2\right)+M^2 g\sin(t+y)+\sum mn^2 g'\sin(t'+y)\right\}.$$

The first of these equations, multiplied by $\sin y$, added to the second multiplied by $\cos y$, gives

$$\frac{d(q\sin y)}{dt}=\frac{3(2C-A-B)}{4iC}\left\{\left(M^2+\sum mn^2\right)q\sin y+M^2 g\cos t+\sum mn^2 g'\cos t'\right\} \qquad \textbf{(278)}$$

likewise

$$\frac{d(q\cos y)}{dt}=\frac{3(2C-A-B)}{4iC}\left\{-\left(M^2+\sum mn^2\right)q\sin y+M^2 g\sin t+\sum mn^2 g'\sin t'\right\} \qquad \textbf{(279)}$$

864. Now, in order to have some idea of the positions of the different planes, let JN be the orbit of Jupiter, QN the plane of his equator, FN the fixed plane, and *m*N the orbit of a

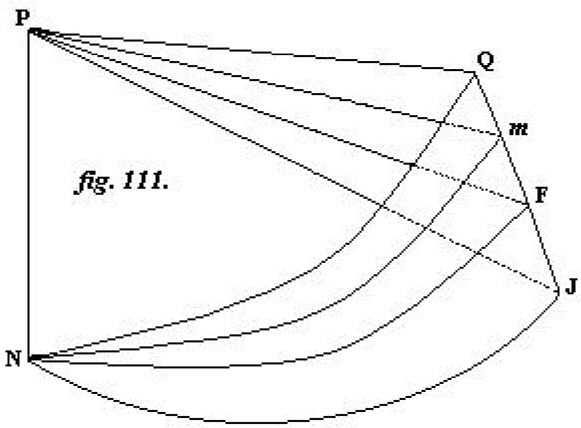


fig. 111.

satellite. Then the integrals of these equations may be found by considering that as $q = \text{QNF}$ is the inclination of Jupiter's equator on the fixed plane, $-q\sin(v+y)$ would be the latitude of a satellite if it moved on the plane of Jupiter's equator, for the latitudes are all referred to the fixed plane FN; and if they are positive on the side FJ, they must be negative on the side FQ; but by the value assumed for s, in article 861, that latitude is equal to a series of terms of the form

$$L\sin(v+pt+\Lambda),$$

hence

$$q\sin y = -\Sigma'.L\sin(pt+\Lambda) \qquad \textbf{(280)}$$
$$q\cos y = -\Sigma'.L\cos(pt+\Lambda).$$

865. Likewise, $g = \text{JNF}$, being the inclination of Jupiter's orbit on the fixed plane, $g\sin(U-t)$ is the latitude of the sun above the fixed plane, by article 863; but by the value assumed for S, in article 861, it is easy to see that

$$\begin{aligned} g\sin t &= -\Sigma'.L'\sin(pt+\Lambda) \\ g\cos t &= -\Sigma'.L'\cos(pt+\Lambda). \end{aligned} \qquad \textbf{(281)}$$

In the same manner $g' = m\text{NF}$ being the inclination of the orbit of a satellite on the fixed plane, its latitude is $g'\sin(v+l_{/})$, and by article 861

$$\begin{aligned} g'\sin t' &= -\Sigma' l\,.\sin(pt+\Lambda) \\ g'\cos t' &= +\Sigma' l\cdot\cos(pt+\Lambda). \end{aligned} \qquad \textbf{(282)}$$

Σ' denotes the sum of a series, but Σ is the sum of the terms relating to the different satellites.

When these quantities are put in equations (279) and (278), a comparison of the coefficients of similar sines and cosines gives

$$0 = pL + \frac{3(2C-A-B)}{4iC}\left\{M^2(L'-L) + \Sigma mn^2(l-L)\right\},$$

which equation determines the effect of the displacement of Jupiter's equator.

866. If Jupiter be an elliptical spheroid, theory gives

$$\frac{2C-A-B}{C}=\frac{2\left(r-\tfrac{1}{2}f\right)\int P.\bar{R}^2 d\bar{R}}{\int P\bar{R}^4\cdot d\bar{R}}.$$

As the celestial bodies decrease in density from the centre to the surface, P represents the density of a shell or layer of Jupiter's spheroid at the distance of $\bar{R}$ from his centre, the integral being between $\bar{R}=0,$ the value of the radius at the center to $\bar{R}=1,$ its value at the surface.

867. The equations[2] (277) may be put under the form

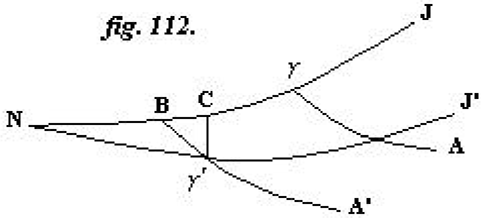
$$\begin{aligned}
0=&+\left\{p-(0)-\boxed{0}-(0.1)-(0.2)-(0.3)\right\}(L-l)\\
&+(0.1)(L-l_1)+(0.2)(L-l_2)+(0.3)(L-l_3)+\boxed{0}(L-L')-pL;\\
0=&+\left\{p-(1)-\boxed{1}-(1.0)-(1.2)-(1.3)\right\}(L-l_1)\\
&+(1.0)(L-l)+(1.2)(L-l_2)+(1.3)(L-l_3)+\boxed{1}(L-L')-pL;\\
0=&+\left\{p-(2)-\boxed{2}-(2.0)-(2.1)-(2.3)\right\}(L-l_2)\\
&+(2.0)(L-l)+(2.1)(L-l_1)+(2.3)(L-l_3)+\boxed{2}(L-L')-pL;\\
0=&+\left\{p-(3)-\boxed{3}-(3.0)-(3.1)-(3.2)\right\}(L-l_3)\\
&+(3.0)(L-l)+(3.1)(L-l_1)+(3.2)(L-l_2)+\boxed{3}(L-L')-pL;\\
0=&+pL-\frac{3(2C-A-B)}{4iC}\left\{\begin{array}{l}+M^2(L-L')+mn^2(L-l)+m_1n_1^2(L-l_1)\\+m_2n_2^2(L-l_2)+m_3n_3^2(L-l_3)\end{array}\right\};
\end{aligned}\tag{283}$$

which determine the positions of the orbits of the satellites, including the effects of Jupiter's nutation and precession.

Inequalities occasioned by the Displacement of Jupiter's Orbit

868. The position of Jupiter's orbit is perpetually changing by very slow degrees with regard to the ecliptic, from the action of the planets. In consequence of this displacement the inclination of the plane of Jupiter's equator on his orbit is changed, and a corresponding change takes place in the precession of the nodes of the equator on the orbit, which may be represented by considering JN to be the orbit of Jupiter, and gA the plane of his equator at the epoch, g being the ascending node of his equator at that period. After a time the action of the planets brings the orbit into the position J′N, which increases the inclination by JNJ′, and the node, which by its retrograde motion alone would come to B, is brought to g', so that the motion of the node is lessened by BC. Thus the inclination of the plane of Jupiter's equator on his orbit is affected by two totally different and unconnected causes,

fig. 112.



the one arising from the direct action of the sun and satellites on the protuberant matter at the equator producing nutation and precession, and the other from the displacement of his orbit by the action of the planets, which diminishes the precession: both disturb the motions of the satellites; but in order to determine the effects of the displacement of the orbit, it must be observed that if the inclinations of the orbits were eliminated from the equations (283) the result would be an equation in *p*, the roots of which $p,\ p_1,\ p_2,\ p_3,$ would be the annual and sidereal motions of the nodes of the satellites depending on their mutual attraction and that of the sun, but there would also be very small values of *p* of the order BC, fig. 112, depending on the displacement of the orbit and equator of Jupiter. Now if we regard the equations (283) as relative to the displacement of the orbit and equator of Jupiter alone, omitting, for the present, the mutual action of the sun and satellites, these very small values of *p* may be omitted in comparison of $(0),\ (0.1),$ &c.; and if it be assumed that

$$\begin{aligned} L-l &= l\ (L-L') \\ L-l_1 &= l_1(L-L') \\ L-l_2 &= l_2(L-L') \\ L-l_3 &= l_3(L-L'), \end{aligned} \tag{284}$$

the four first of equations (283) become

$$\begin{aligned} 0 &= +\left\{(0)+\boxed{0}+(0.1)+(0.2)+(0.3)\right\} l \\ &\quad -(0.1)\,l_1-(0.2)\,l_2-(0.3)\,l_3-\boxed{0} \\ 0 &= +\left\{(1)+\boxed{1}+(1.0)+(1.2)+(1.3)\right\} l_1 \\ &\quad -(1.0)\,l-(1.2)\,l_2-(1.3)\,l_3-\boxed{1} \\ 0 &= +\left\{(2)+\boxed{2}+(2.0)+(2.1)+(2.3)\right\} l_2 \\ &\quad -(2.0)\,l-(2.1)\,l_2-(2.3)\,l_3-\boxed{2} \\ 0 &= +\left\{(3)+\boxed{3}+(3.0)+(3.1)+(3.2)\right\} l_3 \\ &\quad -(3.0)\,l-(3.1)\,l_1-(3.2)\,l_2-\boxed{3}, \end{aligned} \tag{285}$$

which are relative to the displacement of the orbit and equator of Jupiter; by these, $l,\ l_1,\ l_2,\ l_3,$ may be computed, whence the relations among the inclinations will be known.

On the Constant Planes

869. The preceding equations afford the means of finding the permanent planes mentioned in article 803, for $l = m\text{NF}$ and $L' = \text{JNF},$ fig. 111, are the inclinations of the satellite and Jupiter on the fixed plane; $l - L' = m\text{NJ}$ is the inclination of the orbit of the satellite on that of Jupiter, by article 864; hence, the latitude of the satellite *m* above the orbit of Jupiter is equal to a series of terms of the form

$$(l - L')\sin(v + pt + \Lambda).$$

But the first of equations (284) gives

$$l - L' = (1 - l)(L - L');$$

thus, with regard to the displacement of Jupiter's orbit and equator,

$$\Sigma'(l - L')\sin(v + pt + \Lambda) = (1 - l)\Sigma'(L - L')\sin(v + pt + \Lambda).$$

Again, $L = \text{QNF}$, $L_{/} = \text{JNF}$ being the inclinations of Jupiter's equator and orbit on the fixed plane; $L - L' = \text{QNJ}$ is the inclination of his equator on his orbit, for $L = \text{QNF}$ is a negative quantity by article 864, therefore

$$\Sigma'(L - L')\sin(v + pt + \Lambda)$$

would be the latitude of the satellite *m* above the orbit of Jupiter, if it moved on the plane of his equator. But the inclination $(1 - l)\times(L - L')$ is less than $L - L' = \text{QNJ}$, both being referred to the plane of Jupiter's orbit; hence, $(1 - l)(L - L') = l - L' < L - L'$; therefore the plane having the inclination $l - L'$, or $(1 - l)(L - L')$ must come between the equator and orbit of Jupiter; and as Λ and *p*, the longitude of the node and its annual and sidereal precession, are the same in both, this plane passes through NP, the line of the nodes. But

$$L - L' : (1 - l)(L - L') :: 1 : 1 - l :: \text{QNJ} : \text{FNJ},$$

and the plane FN always retains the same inclination to the equator and orbit of Jupiter, since l is a constant quantity: each of the other satellites has its own permanent plane depending on l_1, l_2, l_3. It is hardly possible that these planes could have been discovered by observation alone.

870. If $q' = \text{QNJ} = L - L'$ be the inclination of Jupiter's equator on his orbit, and $-y' = pt + \Lambda$ the longitude of its descending node on the orbit estimated from the vernal equinox of Jupiter, the preceding expression, with regard to that part of the latitude of *m* above the orbit of Jupiter which is relative to the displacement of his orbit and equator, is

$$(l - 1)q'\sin(v + y'),$$

for $lq' = \text{FNQ}$, the inclination of the constant plane FN on Jupiter's equator, therefore

$$(l-1)q' = \text{FNJ},$$

is the inclination of the same constant plane on Jupiter's orbit, and

$$(l-1)q' \cdot \sin(v+y')$$

is the latitude the satellite would have if it moved on its constant plane.

To determine the Effects of the Displacements of the Equator and Orbit of Jupiter on the quantities $q = \text{QNF}$, $q' = \text{QNJ}$, y, y', and Λ

871. The displacements of the equator and orbit of Jupiter affect the quantities q, y, q', y', and Λ. The general equations which determine this effect may easily be found; but if the values of these quantities be obtained in functions of the time, it will be sufficiently correct for astronomical purposes for several centuries, before or after any period that may be assumed as the epoch.

It will answer the same purpose, and facilitate the determination of these quantities, if Jupiter's orbit be assumed to coincide with the fixed plane FN; for the whole effect of its displacement will be referred to the equator, which will then vary both from nutation and the variation in the orbit of Jupiter. In this case $L' = 0$, and equations (284) give

$$l = (1-l)L; \;\; l_1 = (1-l_1)L; \;\; l_2 = (1-l_2)L; \;\; l_3 = (1-l_3)L.$$

In consequence of these, the four first of equations (283) vanish, and L remains indeterminate, and may be represented by[3] $-\grave{L}$, and the last of the same equation becomes

$$p = \frac{3(2C-A-B)}{4iC}\left\{M^2 + mn^2 l + m_1 n_1^2 l_1 + m_2 n_2^2 l_2 + m_3 n_3^2 l_3\right\},$$

which may be expressed by $\grave{p}$, and relates to the displacement of the equator of Jupiter.

872. Since JN coincides with FN, fig. 111, $-\grave{L} = \text{Q}n\text{J}$ is the inclination of the equator on the fixed orbit of Jupiter and

$$-\grave{L}\sin(v + \grave{p}t + \grave{\Lambda})$$

would be the latitude of the satellite if it were moving on the equator of Jupiter, $\grave{\Lambda}$ being an arbitrary quantity, or the longitude of the node of the equator corresponding to $\grave{p}$. But this latitude has also been expressed by $-q\sin(v+y)$. Whence[4]

$$
\begin{aligned}
q\sin y &= \grave{L}\sin\left(\grave{p}t+\grave{\Lambda}\right),\\
q\cos y &= \grave{L}\cos\left(\grave{p}t+\grave{\Lambda}\right),
\end{aligned}
\tag{286}
$$

$\grave{p}t$ being the mean precession of the equinoxes of Jupiter. Again, since $q = \mathrm{QNF}$, $g = \mathrm{JNF}$ are the inclinations of the equator and orbit of Jupiter on the fixed plane;

$$-q\sin\left(v+y\right)-g\sin\left(v-t\right)$$

is the latitude the satellite would have above the orbit of Jupiter, if it moved on the plane of his equator, but $-q'\sin\left(v+y'\right)$ is the same; so

$$q\sin\left(v+y\right)+g\sin\left(v-t\right)=q'\sin\left(v+y'\right),$$

v being indeterminate. If it be successively made equal to $-\grave{p}t$ and to $90^\circ-\grave{p}t,$ the preceding equation gives

$$
\begin{aligned}
q'\sin\left(y'-pt\right) &= q\sin\left(y-\grave{p}t\right)-g\sin\left(t+\grave{p}t\right)\\
q'\cos\left(y'-\grave{p}t\right) &= q\cos\left(y-\grave{p}t\right)+g\cos\left(t+\grave{p}t\right).
\end{aligned}
\tag{287}
$$

The sum of the squares of equations (286) gives $q=\grave{L},$ and as by this $\sin y=\sin\left(\grave{p}t+\grave{\Lambda}\right);$ therefore $y-\grave{p}t=\grave{\Lambda}.$

In consequence of this, the first of equations (287) becomes

$$q'\sin\left(y'-\grave{p}t\right)=\grave{L}\sin\grave{\Lambda}-g\sin\left(t+\grave{p}t\right),$$

or

$$q'\sin y'\cos\grave{p}t-q'\cos y'\sin\grave{p}t-\grave{L}\sin\grave{\Lambda}+g\sin t\cos\grave{p}t+g\cos t\sin\grave{p}t=0\,.$$

This expression must be zero, whatever the time may be, which can only happen when $\sin\grave{\Lambda}=0,$ for $\grave{L}=q;$ consequently,

$$\grave{\Lambda}=0,$$

and therefore

$$y=\grave{p}t\,.$$

873. In order to determine q' and $y',$ let the orbit of Jupiter in the beginning of 1750 be the fixed plane, let that period be the epoch, and the line of the vernal equinox of Jupiter the origin of the angles. Then at the epoch $t=0,$ whence equations (287) become

$$
\begin{aligned}
q'\sin y' &= q\sin y-g\sin t\\
q'\cos y' &= q\cos y+g\cos t\,.
\end{aligned}
$$

Now y' and y are so small, that the arc may be put for the sine, and unity for the cosine; also $g \cos t$, $g \sin t$ may be expressed by series increasing as the powers of the time for many centuries to come; therefore let

$$g \sin t = at \qquad g \cos t = bt$$

then, because

$$q = \grave{L}, \quad y = \grave{p}t, \quad \grave{\Lambda} = 0, \tag{288}$$
$$q'y' = \grave{L}\,.\,\grave{p}t - at; \qquad q' = \grave{L} + bt$$

whence

$$y' = \grave{p}t - \frac{at}{\grave{L}},$$

when the square of the time is neglected.

874. Since $g \sin t$, $g \cos t$, relate to the change in the position of Jupiter's orbit from the action of the planets, they are determined by equations (137); but as Jupiter's orbit is principally disturbed by the action of Saturn and Uranus, if f, f', be the inclinations of the orbits of Saturn and Uranus on the orbit of Jupiter in the beginning of 1750, and Ω, Ω', the longitudes of the ascending nodes of the two orbits on that of Jupiter at the same epoch, estimated from the equinox of spring of Jupiter; then will

$$a = +(4.5)f \cos\Omega + (4.6)f'.\cos\Omega'$$
$$b = -(4.5)f \sin\Omega - (4.6)f'\cdot\sin\Omega',$$

where (4.5), (4.6), are given by equations (202).

875. It only remains to determine the effects of the displacement on $g'\sin t$, $g'\sin t'$, the inclination and node of a satellite *m* with regard to its fixed plane.

By equations (248)

$$l = (1 - l)L + lL'.$$

If this value of *l* be put in the equations (282) they become

$$g'\sin t' = -\Sigma'.(1 - l).L.\sin(pt + \Lambda) - \Sigma'.\ lL'.\sin(pt + \Lambda)$$
$$g'\cos t' = +\Sigma'.(1 - l).L.\cos(pt + \Lambda) + \Sigma'.lL'.\cos(pt + \Lambda),$$

and in consequence of equations (280) and (281)

$$g'\sin t' = (1 - l).q\ \sin y + l.\sin t$$
$$g'\cos t' = (l - 1).q\cos y + l.\cos t,$$

but

$$q = \grave{}L, \quad y = \grave{}pt, \quad g \sin t = at, \quad g \cos t = bt;$$

and putting $\grave{}pt$ for the sine and unity for the cosine; with regard to the displacement of Jupiter's orbit and equator,

$$\begin{aligned} g' \sin t' &= (1 - l)\grave{}L . \grave{}pt + l . at \\ g' \cos t' &= (l - 1)\grave{}L + l \cdot bt \end{aligned} \qquad \textbf{(289)}$$

Thus the quantities relating to the displacement of the orbit and equator are completely determined.

876. With regard to the values of p, which depend on the mutual action of the satellites, L' is zero, since the action of the satellites has no sensible effect on the displacement of Jupiter's orbit. The values of L may be omitted relatively to $l, \ l_1, \ l_2, \ l_3;$ and since by the last of equations (283), pL is multiplied by $\dfrac{2C - A - B}{C}$, it is of the order of the product of the ellipticity of Jupiter by the masses of the satellites; and therefore it may be omitted also, which reduces equations (283) to

$$\begin{aligned} 0 &= +l\left\{p - (0) - \boxed{0} - (0.1) - (0.2) - (0.3)\right\} + (0.1)\,l_1 + (0.2)\,l_2 + (0.3)\,l_3 \\ 0 &= +l_1\left\{p - (1) - \boxed{1} - (1.0) - (1.2) - (1.3)\right\} + (1.0)\,l + (1.2)\,l_2 + (1.3)\,l_3 \\ 0 &= +l_2\left\{p - (2) - \boxed{2} - (2.0) - (2.1) - (2.3)\right\} + (2.0)\,l + (2.1)\,l_1 + (2.3)\,l_3 \\ 0 &= +l_3\left\{p - (3) - \boxed{3} - (3.0) - (3.1) - (3.2)\right\} + (3.0)\,l + (3.1)\,l_1 + (3.2)\,l_2 \end{aligned} \qquad \textbf{(290)}$$

877. These equations determine the annual and sidereal motion of the nodes and inclinations of the orbits, and are precisely similar to those which determine the eccentricities and motions of the apsides, for if the terms depending on the displacement of the orbit of Jupiter be omitted, each satellite has four terms in latitude similar to the four equations of the centre, and arising like them from the changes in the positions of the orbits by the action of the matter at Jupiter's equator and their mutual attraction, they therefore depend on the inclinations and motions of the nodes of their own orbits, and on those of the other three. Hence, with the values of $l, \ l_1, \ l_2, \ l_3,$ known by observation, these equations will give the four roots of p, the annual and sidereal motion of the nodes and the coefficients of the sixteen terms in the latitudes; for if it be assumed, that

$$l_1 = z_1 l; \quad l_2 = z_2 l; \quad l_3 = z_3 l,$$

these quantities will make l vanish from the preceding equations; the result will be four equations between $z_1, \ z_2, \ z_3,$ and p, whence p will be obtained by an equation of the fourth degree.

Let $p, \ p_1, \ p_2, \ p_3,$ be the roots of that equation, and let

$$z_1^{(1)},\ z_2^{(1)},\ z_3^{(1)};\quad z_1^{(2)},\ z_2^{(2)},\ z_3^{(2)};\quad z_1^{(3)},\ z_2^{(3)},\ z_3^{(3)};$$

be the values of $z_1,\ z_2,\ z_3,$ when p is successively changed to $p_1,\ p_2,\ p_3,$ they will give the coefficients required.

878. In article 861, it was assumed, that the latitudes of the satellites above the fixed plane were

$$s\ = l\ \sin(v\ + pt + \Lambda) \qquad s_/ = l_/ \sin(v_/ + pt + \Lambda)$$
$$s_2 = l_2 \sin(v_2 + pt + \Lambda) \qquad s_3 = l_3 \sin(v_3 + pt + \Lambda);$$

but if we refer them to the orbit of Jupiter, the term arising from the displacement of that orbit must be added to each, and if the different values of p be substituted, and the corresponding coefficients, the latitudes of the satellites above JN, the orbit of Jupiter at any time t, will be

$$\begin{aligned}
s\ &= +(l - 1)q'\sin(v + y') \\
&\quad + l\ \sin(v + p\ t + \Lambda) \\
&\quad + l_1 \sin(v + p_1 t + \Lambda_1) \\
&\quad + l_2 \sin(v + p_2 t + \Lambda_2) \\
&\quad + l_3 \sin(v + p_3 t + \Lambda_3) \\
s_1 &= +(l_1 - 1)q'\sin(v_1 + y') \\
&\quad + z_1\, l \sin(v_1 + pt + \Lambda) \\
&\quad + z_1^{(1)} l_1 \sin(v_1 + p_1 t + \Lambda_1) \\
&\quad + z_1^{(2)} l_2 \sin(v_1 + p_2 t + \Lambda_2) \\
&\quad + z_1^{(3)} l_3 \sin(v_1 + p_3 t + \Lambda_3) \\
s_2 &= +(l_2 - 1)q'\sin(v_2 + y') \\
&\quad + z_2\, l \sin(v_2 + pt + \Lambda) \\
&\quad + z_2^{(1)} l_1 \sin(v_2 + p_1 t + \Lambda_1) \\
&\quad + z_2^{(2)} l_2 \sin(v_2 + p_2 t + \Lambda_2) \\
&\quad + z_2^{(3)} l_3 \sin(v_2 + p_3 t + \Lambda_3) \\
s_3 &= +(l_3 - 1)q'\sin(v_3 + y') \\
&\quad + z_3\, l \sin(v_3 + pt + \Lambda) \\
&\quad + z_3^{(1)} l_1 \sin(v_3 + p_1 t + \Lambda_1) \\
&\quad + z_3^{(2)} l_2 \sin(v_3 + p_2 t + \Lambda_2) \\
&\quad + z_3^{(3)} l_3 \sin(v_3 + p_3 t + \Lambda_3).
\end{aligned} \qquad \textbf{(291)}$$

879. The first term of each depends on the displacement of Jupiter's orbit, and the eight quantities

$$l,\ l_1,\ l_2,\ l_3,\qquad \Lambda,\ \Lambda_1,\ \Lambda_2,\ \Lambda_3,$$

are determined by observation; the first are the respective inclinations of the four satellites on Jupiter's orbit, and the last four are the longitudes of the nodes at the epoch. If it be required to find the latitude of the satellites above the fixed plane, it will be necessary to add to these the values of the latitudes, supposing the satellites to move on the orbit of Jupiter.

880. The inequalities in latitude which depend on the configuration of the bodies that acquire small divisors by integration are insensible, with the exception of those arising from the action of the sun depending on the angle $2v - 2U$. The part of the disturbing force whence these come is

$$R = \frac{3M^2}{4n^2}\left\{s^2\cos 2(v-U) - 4sS\cos(v-U) - \cos 2(v-U)\right\}$$

omitting the squares and products of S and s,[5]

$$\frac{dR}{dv} = \frac{3M^2}{2n^2}\sin 2(v-U)$$
$$\frac{dR}{ds} = \frac{3M}{2n^2}\left\{s\cos 2(v-U) - S\cos 2(v-U)\right\}$$

but

$$S = L'\sin(v + pt + \Lambda);\qquad S = l\sin(v + pt + \Lambda)$$

and

$$\frac{ds}{dv} = l\cos(v + pt + \Lambda),$$

and if $\frac{v}{n}$ be put for t, observing that

$$U = Mt = \frac{M}{n}v,$$

the equation in article 859 becomes

$$0 = \frac{d^2s}{dv^2} + N_/^2 s + \frac{3M^2}{2n^2}(L' - l)\sin\left(v - \frac{2M}{n}v - \frac{p}{n}v - \Lambda\right),$$

in which

$$N_/^2 = 1 + 2\frac{\left(r - \tfrac{1}{2}f\right)}{a^2} + \tfrac{3}{2}\frac{M^2}{n^2} + \sum m_/ a^2 a_/ B_1;$$

but, without sensible error,

$$N_{/}^2 = 1 + \frac{\left(r - \frac{1}{2}f\right)}{a^2}.$$

In order to integrate this equation, let

$$s = K \cdot \sin\left(v - \frac{2M}{n}v - \frac{p}{n}v - \Lambda\right),$$

K being an indeterminate coefficient.

If that value of *s* be put in the equation, it will give

$$K = \frac{3M^2}{2n^2} \cdot \frac{L' - l}{\left(1 - 2\frac{M}{n} - \frac{p}{n}\right)^2 - N_{/}^2}.$$

But in the divisor,

$$\left\{1 - \frac{2M}{n} - \frac{p}{n} + N_{/}\right\} \cdot \left\{1 - \frac{2M}{n} - \frac{p}{n} - N_{/}\right\};$$

$\frac{p}{n}$ is very small, and $N_{/}$ differs but little from unity; hence

$$1 - 2\frac{M}{n} - \frac{p}{n} + N_{/} = 2, \text{ nearly};$$

therefore

$$s = -\frac{3M^2 (L' - l)}{4n^2\left(2\frac{M}{n} + \frac{p}{n} + N_{/} - 1\right)} \sin\left(v - 2\frac{M}{n}v - \frac{p}{n}v - \Lambda\right);$$

a similar inequality exists for each root of *p*, including $\grave{}p$, which is the value of *p* depending on the displacement of the equator and orbit of Jupiter.

Now,

$$\frac{p}{n} + N_{/} = 1, \text{ nearly};$$

consequently,

$$s = -\frac{3M}{8n}(L' - l) . \sin(v - 2U - pt - \Lambda). \qquad \textbf{(292)}$$

Secular Inequalities of the Satellites, depending on the Variations in the Elements of Jupiter's Orbit

881. The secular inequalities in the elements of Jupiter's orbit, occasioned by the action of the planets, produce corresponding variations in the mean motions of the satellites, which, in the course of ages, will have a considerable effect on the theory of these bodies. These are obtained from

$$R = -\frac{S'r^2}{4D^3}\left\{1 - 3s^2 - 3S^2 + 12Ss\cos(U - v)\right\}$$
$$- \frac{\left(r - \frac{1}{2}f\right)}{r^3}\left(\tfrac{1}{3} - (s - s')^2\right).$$

But, by articles 864 and 865,

$$s = g'\sin(v - t'), \quad S = g\sin(U - t), \quad s' = -q\sin(v + y);$$

and as the periodic inequalities are to be rejected,

$$\frac{S'}{D^3} = \frac{S'}{D'^3}\left\{1 + 3\left(\frac{DdD}{D'^2}\right)^2\right\} = M^2\left\{1 + 3H^2\sin^2(Mt + E - \Pi)\right\} = M^2\left(1 + \tfrac{3}{2}H^2\right).$$

For the same reason,

$$r^2 = a^2\left(1 + \tfrac{1}{2}e^2 - e\cos(nt + \in - v)\right)^2 = a^2\left(1 + \tfrac{3}{2}e^2\right);$$

and if it be observed that

$$\frac{3a^3nM^2}{4} = \boxed{0}; \qquad \frac{\left(r - \frac{1}{2}f\right)}{a^2}n = (0),$$

the value of *anR* is

$$anR = -\tfrac{1}{2}\boxed{0}\left\{e^2 + H^2 - g^2 + 2gg_{/}\cos(t' - t) - g_{/}^2\right\} - \tfrac{1}{2}(0)\left\{q^2 + 2qg_{/}\cos(t' + y) + g_{/}^2 - e^2\right\}.$$

If this quantity be put in equation (259), the result will be

$$\frac{d\,.dv'}{dt} = -2\boxed{0}\left\{e^2 + H^2 - g^2 + 2gg'\cos(t' - t) - g'^2\right\} + 3(0)\left\{q^2 + 2qg'\cos(t + y) + g'^2 - e^2\right\}.$$

This, however, only gives the inequalities on the orbit; but its projection on the fixed plane, by article 548, is[6]

$$d = dv\left\{1 + \tfrac{1}{2}s^2 - \tfrac{1}{2}\frac{ds^2}{dv^2}\right\}.$$

Now,

$$s = g'\sin(v - t') = g'\sin v\cos t' - g'\cos v\sin t'.$$

The substitution of this quantity, and of its differential, gives

$$dv' = dv + \tfrac{1}{2}\left\{g'.\sin t'\frac{d(g'\cos t')}{dt} - g'.\cos t'\frac{d(g'\sin t')}{dt}\right\},$$

the value of dv' projected on the fixed plane; therefore

$$\begin{aligned}\frac{d.dv}{dt} &= +\tfrac{1}{2}\left\{g'\sin t'\frac{d(g'\cos t')}{dt} - g'\cos t'\frac{d(g'\sin t')}{dt}\right\}\\ &\quad -2\boxed{0}\left\{e^2 + H^2 - g^2 + 2gg'\cos(t'-t) - g'^2\right\}\\ &\quad +\tfrac{1}{2}(0)\left\{e^2 - q^2 - 2qg'\cos(t'+y) - g'^2\right\}.\end{aligned} \qquad \textbf{(293)}$$

Since all the quantities $g\sin t$, $g\cos t$, $g_/\sin t_/$, $g_/\cos t'$, y and q, are given in the preceding articles, it may be found that

$$\begin{aligned}\frac{d.dv}{dt} &= +4.\boxed{0}.(1-l)^2.\grave{}Lbt - 6.(0).\ l^2.\grave{}L.bt\\ &\quad +(1-l).\ l.\left\{(0)+\boxed{0}+(0.1)+(0.2)+(0.3)\right\}.\grave{}L.bt\\ &\quad -\tfrac{1}{2}\,l\,(0).\grave{}L.bt - \tfrac{1}{2}(1-l).\boxed{0}.\grave{}L.bt\\ &\quad +\tfrac{1}{2}(0.1).\left\{(l-1)\,l' + (l_1-1)\,l\right\}.\grave{}L.bt\\ &\quad +\tfrac{1}{2}(0.2)\left\{(l-1)\,l_2 + (l_2-1)\,l\right\}.\grave{}L.bt\\ &\quad +\tfrac{1}{2}(0.3)\left\{(l-1)\,l_3 + (l_3-1)\,l\right\}\grave{}L.bt - 2\boxed{0}H^2.\end{aligned}$$

But in consequence of the relations between l, l_1, l_2, l_3, in article 859,

$$\frac{d.dv}{dt} = 4(1-l)^2.\boxed{0}.\grave{}Lbt - 6(0)\,l^2\grave{}Lbt - 2H^2.\boxed{0}\ .$$

In considering the action of Saturn only, equations (204) give the numerical value of H; to abridge, if $\bar{e}$ be the value of H at the epoch, then

$$H = \bar{e} + ct + \&c.;$$

and omitting the square of the time,

$$H^2 = 2\bar{e}ct,$$

and the integral becomes

$$dv = 2(1-l)^2\boxed{0}.\grave{}Lbt^2 - 3(0)\,l^2\grave{}Lbt^2 - 2\bar{e}ct^2\boxed{0}\ . \qquad \textbf{(294)}$$

This inequality in the mean motion of m varies with the eccentricity of the orbit of Jupiter, and is similar to the acceleration in the mean motion of the moon, but it will not be perceptible for many years, nor has it hitherto been perceived.

882. If there was but one satellite, the first of equations (285) would give

$$l = \frac{\boxed{0}}{\boxed{0} + (0)}.$$

In the theory of the moon, $\boxed{0}$ is vastly greater than (0), so that $l = 1 - \frac{(0)}{\boxed{0}}$ differs but little from unity, which reduces the equation (294) to $dv = -2\boxed{0}\bar{e}\,.ct^2$, where $\bar{e}$ is the eccentricity of the earth's orbit at the epoch; and substituting $\tfrac{3}{4}\frac{M}{n}$ for $\boxed{0}$, it becomes

$$dv = -\frac{3}{2}\frac{M^2}{n}\bar{e}ct^2;$$

which is the same with the acceleration of the moon.

883. One secular variation alone is sensible at present, and that only in the mean motion of the fourth satellite; it is derived from equation (293), each term of which must be determined separately. When e^2 and H^2 are omitted, its second term[7]

$$\tfrac{1}{2}\left\{g'^2 - 2gg'\cos(t' - t) + g^2\right\}$$

is the square of that part of the latitude of the satellite m above the orbit of Jupiter, which is independent of v; therefore the expression is equal to the square of s in article 861, where v is omitted; but as l, $l_{/}$, &c., are very small, their squares and products may be neglected, so that the quantity required, after the reduction of the products of the sines to the cosines of the differences of the arcs, is

$$g^2 - 2gg'\cos(t' - t) + g'^2 = (1 - l)^2 q'^2 + 2(l - 1)q'\left\{l\cos(pl + \Lambda - y') + l_{/}\cos(p_{/}l + \Lambda' - y') + \&c.\right\}$$

hence

$$\frac{d\,.dv}{dt} = 4(l - 1)q'\boxed{0}\left\{l\cos(pt + \Lambda - y') + \&c.\right\}.$$

Again,

$$q^2 + 2qg'\cos(t' + y) + g'^2,$$

the third term of equation (293), is the square of the latitude of m above the equator of Jupiter, when v is omitted, and is therefore equal to the square of

$$lq'\sin y' + l\sin(pt+\Lambda) + l_{/}\sin(p_{/}t+\Lambda_{/}) + \&c.$$

which is given by the first of equation (291). Whence

$$\frac{d\,.dv}{dt} = -6(0)\,lq'\{l\cos(pt+\Lambda-y') + \&c.\}.$$

In the third place the same expression of s gives

$$g'\sin t' = +(l-1)q'\cos y' + l\cos(pt+\Lambda) + \&c. - l\cos t$$
$$g'\cos t' = -(l-1)q'\sin y' - l\sin(pt+\Lambda) - \&c. - l\sin t.$$

By means of these values the first term of equation (293) becomes

$$\frac{ddv}{dt} = -\tfrac{1}{2}(l-1)\,.\,q'\{pl\cos(pt+\Lambda-y') + \&c.\}$$

When these three parts are added, they constitute the whole of equation (293), the integral of which is

$$dv = -\left\{6(0)\,l + 4(1-l)\boxed{0}\right\}q'\left\{\frac{l}{p}\sin(pt+\Lambda-y') + \frac{l_{/}}{p_{/}}\sin(p_{/}t+\Lambda_{/}-y') + \&c.\right\}$$
$$+\tfrac{1}{2}(1-l)q'\{l\sin(pt+\Lambda-y') + l_{/}\sin(p_{/}t+\Lambda_{/}-y') + \&c.\}$$

The only part that has a sensible effect is

$$dv = -\frac{\left\{4(1-l_3)\boxed{3} - \tfrac{1}{2}(1-l_3)\,p_3 + 6(3)\,l_3\right\}}{p_3}q'l_3 \times \sin(p_3 + \Lambda_3 - y'), \qquad \textbf{(295)}$$

and that in the motions of the fourth satellite only.

884. With regard to the moon, $l = 1 - \frac{(0)}{\boxed{0}}$ differs but little from unity, and $p = \boxed{0}$ nearly; hence, for that body,

$$dv = -\frac{19}{2}\cdot(0)\cdot\frac{q'l}{p}\sin(v + pt - y'),$$

which coincides with equation (244), supposing the obliquity of the ecliptic to be very small.

Notes

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1 The remaining three chapters (II, III, and IV) in Book IV are numbered VII, VIII, and IX in the 1st edition.
2 This *reads* "The equations in article 277" in the 1st edition.
3 This is the first instance of the use of the back-prime syntax in the text.
4 The arguments of both sine functions *read* $\left(p\grave{t} + \grave{}\Lambda \right)$ in the 1st edition.
5 The 2nd term in the 2nd equation *reads* $-2S\cos(\cos(v-U)$ in the 1st edition.
6 The parentheses are mismatched. The expression *reads* $d = dv(1+\frac{1}{2}s^2 - \frac{1}{2}\frac{ds^2}{dv^2}\}$ in the 1st edition.
7 The middle term *reads* $-2gg^2\cos\left(t' - t\right)$ in the 1st edition.

BOOK IV

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CHAPTER III[1]

NUMERICAL VALUES OF THE PERTURBATIONS

885. IT is known by observation that the sidereal revolutions of the satellites are accomplished in the following periods:–

	Days
1st Satellite in	1.769137787
2nd Satellite in	3.551181017
3rd Satellite in	7.154552808
4th Satellite in	16.689019396

The values of $n,\ n_1,\ n_2,\ n_3,$ being reciprocally as these periods,

$$n = n_3 \,.\, 9.433419$$
$$n_1 = n_3 \,.\, 4.699569$$
$$n_2 = n_3 \,.\, 2.332643\,.$$

And as the sidereal revolution of Jupiter is 4332.602208 days,

$$M = n_3 \,.\, 00385196.$$

886. The mean distances of the satellites from Jupiter are known from observation; with them, by a method to be shown afterwards, the equations (271) and (290) give the following approximate values of the masses of the satellites, and of the compression of Jupiter

$$m = 0.0000184113$$
$$m_1 = 0.0000258325$$
$$m_2 = 0.0000865185$$
$$m_3 = 0.00005590808$$
$$r - \tfrac{1}{2} f = 0.0217794,$$

the mass of Jupiter being the unit.

887. The mean distances of the three first satellites cannot be measured with sufficient accuracy for computing the inequalities; it is therefore necessary to determine them from the value of a_3 by Kepler's law.

At the mean distance of Jupiter from the sun, his equatorial diameter is seen under an angle of 38″.99; taking this diameter as the unit, the mean distance of the fourth satellite in functions of the diameter is

$$a_3 = 25.43590.$$

By article 818 the mean distance of a satellite is $a + da$, in consequence of the action of the disturbing forces; but as the variation da is principally owing to the compression of Jupiter, the only part of da in article 821 that has a sensible effect on the mean distances is $a\frac{\left(r - \frac{1}{2}f\right)}{3a^2}$, hence $a = n^{-\frac{2}{3}}$ becomes

$$a = n^{-\frac{2}{3}}\left(1 + \tfrac{1}{3}\left(\frac{r - \frac{1}{2}f}{a^2}\right)\right),$$

also

$$a_3 = n_3^{-\frac{2}{3}}\left(1 + \tfrac{1}{3}\left(\frac{r - \frac{1}{2}f}{a_3^2}\right)\right);$$

and thus, by Kepler's law,

$$a = \left\{1 + \tfrac{1}{3}\left(r - \tfrac{1}{2}f\right)\left(\frac{1}{a^2} - \frac{1}{a_3^2}\right)\right\} a_3 \sqrt[3]{\frac{n^2}{n_3^2}}$$

in which

$$\frac{1}{a^2} = \frac{1}{\left(a_3 \sqrt[3]{\frac{n^2}{n_3^2}}\right)^2};$$

whence, with the preceding values, it is easy to find that

$$a = 5.698491$$
$$a_1 = 9.066548$$
$$a_2 = 14.461893$$
$$a_3 = 25.43590;$$

with these the series S and S' in article 453 may be computed, and from them all the coefficients A_0, A_1, &c.; B_0, B_1, &c.; and their differences may be found by the same method of computation, and from the same formulae, as for the planets; and thence

$$N = n_3 . 9.4269167$$

$$N_1 = n_3 \,.\, 4.6979499$$
$$N_2 = n_3 \,.\, 2.332309$$
$$N_3 = n_3 \,.\, 0.9999070.$$

888. With these quantities, the perturbations in longitude and distance computed from the expressions in articles 820 and 821 are

$$d\,v = +m_1 \left\{ \begin{array}{l} + \;\; 60''.7333 \sin\{n_1 t - nt + \in_1 - \in\} \\ -7042''.63 \sin 2\{n_1 t - nt + \in_1 - \in\} \\ - \;\; 22''.949 \sin 3\{n_1 t - nt + \in_1 - \in\} \\ - \;\; 5''.2464 \sin 4\{n_1 t + nt + \in_1 - \in\} \\ - \;\; 1''.7518 \sin 5\{n_1 t - nt + \in_1 - \in\} \\ - \;\; 0''.69443 \sin 6\{n_1 t - nt + \in_1 - \in\} \end{array} \right\} + m_2 \left\{ \begin{array}{l} + \;\; 7''.1065 \sin\{n_2 t - nt + \in_2 - \in\} \\ - \;\; 6''.0005 \sin 2\{n_2 t - nt + \in_2 - \in\} \\ - \;\; 0''.6162 \sin 3\{n_2 t - nt + \in_2 - \in\} \\ - \;\; 0''.1156 \sin 4\{n_2 t - nt + \in_2 - \in\} \end{array} \right\} + 0''.04731 \sin\{2nt - Mt + 2\in - \in E\} \tag{296}$$

The inequalities depending on m_3 are insensible.

$$d\,r = +m_1 \left\{ \begin{array}{l} + \;\; 0.000084865 \\ + \;\; 0.00046652 \cos\{n_1 t - nt + \in_1 - \in\} \\ - \;\; 0.09764199 \cos 2\{n_1 t - nt + \in_1 - \in\} \\ - \;\; 0.00040917 \cos 3\{n_1 t - nt + \in_1 - \in\} \\ - \;\; 0.00010761 \cos 4\{n_1 t - nt + \in_1 - \in\} \\ - \;\; 0.00003824 \cos 5\{n_1 t - nt + \in_1 - \in\} \\ - \;\; 0.00001642 \cos 6\{n_1 t - nt + \in_1 - \in\} \end{array} \right\} + m_2 \left\{ \begin{array}{l} + \;\; 0.00000703 \\ + \;\; 0.00007780 \cos\{n_2 t - nt + \in_2 - \in\} \\ - \;\; 0.00010631 \cos 2\{n_2 t - nt + \in_2 - \in\} \\ - \;\; 0.00001310 \cos 3\{n_2 t - nt + \in_2 - \in\} \\ - \;\; 0.00000269 \cos 4\{n_2 t - nt + \in_2 - \in\} \end{array} \right\}$$

$$+m_3\left\{\begin{array}{ll}+ & 0.00000113 \\ + & 0.00001478\cos\{n_3t-nt+\in_3-\in\} \\ - & 0.00000968\cos2\{n_3t-nt+\in_3-\in\} \\ - & 0.00000078\cos3\{n_3t-nt+\in_3-\in\}\end{array}\right\}$$

$$\begin{array}{ll}+ & 0.00000095 \\ - & 0.00000095\cos\{2Mt-nt+2E-2\in\}\end{array}$$

(297)

$$d\,v_{\prime}=+m\left\{\begin{array}{ll}- & 2252''.28\sin\{nt-n_1t+\in-\in_1\} \\ - & 17''.053\sin2\{nt-n_1t+\in-\in_1\} \\ - & 3''.4102\sin3\{nt-n_1t+\in-\in_1\} \\ - & 1''.0837\sin4\{n_1t-nt+\in-\in_1\} \\ - & 0''.4202\sin5\{nt-n_1t+\in-\in_1\}\end{array}\right\}$$

$$+m_2\left\{\begin{array}{ll}+ & 59''.784\sin\{n_2t-n_1t+\in_2-\in_1\} \\ - & 3923''.3\sin2\{n_2t-n_1t+\in_2-\in_1\} \\ - & 22''.318\sin3\{n_2t-n_1t+\in_2-\in_1\} \\ - & 5''.1076\sin4\{n_2t-n_1t+\in_2-\in_1\} \\ - & 1''.7041\sin5\{n_2t-n_1t+\in_2-\in_1\} \\ - & 0''.6744\sin6\{n_2t-n_1t+\in_2-\in_1\}\end{array}\right\}$$

$$+m_3\left\{\begin{array}{ll}+ & 4''.0098\sin\{n_3t-n_1t+\in_3-\in_1\} \\ - & 3''.5108\sin2\{n_3t-n_1t+\in_3-\in_1\} \\ - & 0''.3449\sin3\{n_3t-n_1t+\in_3-\in_1\}\end{array}\right\}$$

$$+\quad 0''.1906\sin\{2n_1t-2Mt+2\in_1-2E\}$$

$$d\,r_1=+m\left\{\begin{array}{ll}- & 0.00044608 \\ + & 0.05069318\cos\{nt-n_1t+\in-\in_1\} \\ + & 0.00059197\cos2\{nt-n_1t+\in-\in_1\} \\ + & 0.00014002\cos3\{nt-n_1t+\in-\in_1\} \\ + & 0.00004784\cos4\{nt-n_1t+\in-\in_1\} \\ + & 0.00001928\cos5\{nt-n_1t+\in-\in_1\}\end{array}\right\}$$

$$+m_2\left\{\begin{array}{ll}+ & 0.00006497 \\ + & 0.00073255\cos\{n_2t-n_1t+\epsilon_2-\epsilon_1\} \\ - & 0.08670960\cos2\{n_2t-n_1t+\epsilon_2-\epsilon_1\} \\ - & 0.00063398\cos3\{n_2t-n_1t+\epsilon_2-\epsilon_1\} \\ - & 0.00016685\cos4\{n_2t-n_1t+\epsilon_2-\epsilon_1\} \\ - & 0.00006067\cos5\{n_2t-n_1t+\epsilon_2-\epsilon_1\}\end{array}\right\}$$

$$+m_3\left\{\begin{array}{ll}+ & 0.00000798 \\ + & 0.00007146\cos\{n_3t-n_1t+\epsilon_3-\epsilon_1\} \\ - & 0.00010133\cos2\{n_3t-n_1t+\epsilon_3-\epsilon_1\} \\ - & 0.00001189\cos3\{n_3t-n_1t+\epsilon_3-\epsilon_1\}\end{array}\right\}$$

$$\begin{array}{ll}+ & 0.00000609 \\ - & 0.00000609\cos\{2Mt-2n_1t+2E-2\epsilon_1\}\end{array}$$

$$d v_2=+m\left\{\begin{array}{ll}+ & 7''.862\sin\{nt-n_2t+\epsilon-\epsilon_2\} \\ - & 0''.228\sin2\{nt-n_2t+\epsilon-\epsilon_2\} \\ - & 0''.0414\sin3\{nt-n_2t+\epsilon-\epsilon_2\}\end{array}\right\} \qquad \textbf{(298)}$$

$$+m_1\left\{\begin{array}{ll}- & 1126''.96\sin\{n_1t-n_2t+\epsilon_1-\epsilon_2\} \\ - & 16''.504\sin2\{n_1t-n_2t+\epsilon_1-\epsilon_2\} \\ - & 3''.2995\sin3\{n_1t-n_2t+\epsilon_1-\epsilon_2\} \\ - & 1''.0467\sin4\{n_1t-n_2t+\epsilon_1-\epsilon_2\} \\ - & 0''.4067\sin5\{n_1t-n_2t+\epsilon_1-\epsilon_2\} \\ - & 0''.1767\sin6\{n_1t-n_2t+\epsilon_1-\epsilon_2\}\end{array}\right\}$$

$$+m_3\left\{\begin{array}{ll}+ & 34''.396\sin\{n_3t-n_2t+\epsilon_3-\epsilon_2\} \\ - & 117''.32\sin2\{n_3t-n_2t+\epsilon_3-\epsilon_2\} \\ - & 8''.251\sin3\{n_3t-n_2t+\epsilon_3-\epsilon_2\} \\ - & 1''.919\sin4\{n_3t-n_2t+\epsilon_3-\epsilon_2\} \\ - & 0''.609\sin5\{n_3t-n_2t+\epsilon_3-\epsilon_2\} \\ - & 0''.227\sin6\{n_3t-n_2t+\epsilon_3-\epsilon_2\}\end{array}\right\}$$

$$+ \quad 0''.7734\sin\{2n_2t-2Mt+2\epsilon_2-2E\}$$

$$d r_2 = +m \begin{Bmatrix} - & 0.00054798 \\ + & 0.00059147\cos\{nt - n_2t + \in - \in_2\} \\ + & 0.00001906\cos2\{nt - n_2t + \in - \in_2\} \\ + & 0.00000348\cos3\{nt - n_2t + \in - \in_2\} \end{Bmatrix}$$

$$+m_1 \begin{Bmatrix} - & 0.00070942 \\ + & 0.04137743\cos\{n_1t - n_2t + \in_1 - \in_2\} \\ + & 0.00091726\cos2\{n_1t - n_2t + \in_1 - \in_2\} \\ + & 0.00021712\cos3\{n_1t - n_2t + \in_1 - \in_2\} \\ + & 0.00007409\cos4\{n_1t - n_2t + \in_1 - \in_2\} \\ + & 0.00002980\cos5\{n_1t - n_2t + \in_1 - \in_2\} \\ + & 0.00001318\cos6\{n_1t - n_2t + \in_1 - \in_2\} \end{Bmatrix}$$

$$+m_3 \begin{Bmatrix} + & 0.00006850 \\ + & 0.00075191\cos\{n_3t - n_2t + \in_3 - \in_2\} \\ - & 0.0044961\cos2\{n_3t - n_2t + \in_3 - \in_2\} \\ - & 0.00039801\cos3\{n_3t - n_2t + \in_3 - \in_2\} \\ - & 0.00010474\cos4\{n_3t - n_2t + \in_3 - \in_2\} \\ - & 0.00003569\cos5\{n_3t - n_2t + \in_3 - \in_2\} \\ - & 0.00001379\cos6\{n_3t - n_2t + \in_3 - \in_2\} \end{Bmatrix}$$

$$\begin{matrix} + & 0.00003944 \\ - & 0.00003944\cos\{2Mt - 2n_3t + 2E - 2\in_3\} \end{matrix}$$

$$d v_3 = +m \begin{Bmatrix} + & 4''.6156\sin\{nt - n_3t + \in - \in_3\} \\ - & 0''.0067\sin2\{nt - n_3t + \in - \in_3\} \end{Bmatrix} \qquad \textbf{(299)}$$

$$+m_1 \begin{Bmatrix} - & 7''.2745\sin\{n_1t - n_3t + \in_1 - \in_3\} \\ - & 0''.09995\sin2\{n_1t - n_3t + \in_1 - \in_3\} \\ - & 0''.0175\sin3\{n_1t - n_3t + \in_1 - \in_3\} \end{Bmatrix}$$

$$+m_2 \begin{Bmatrix} - & 11''.482\sin\{n_2t - n_3t + \in_2 - \in_3\} \\ - & 5''.1701\sin2\{n_2t - n_3t + \in_2 - \in_3\} \\ - & 1''.0787\sin3\{n_2t - n_3t + \in_2 - \in_3\} \\ - & 0''.3304\sin4\{n_2t - n_3t + \in_2 - \in_3\} \\ - & 0''.1210\sin5\{n_2t - n_3t + \in_2 - \in_3\} \end{Bmatrix}$$

$$+ \quad 4''.2082\sin2\{n_3t - Mt + \in_3 - E\}$$

$$
\begin{aligned}
\delta r_3 = +m &\begin{Bmatrix} - & 0.00088152 \\ + & 0.00057018\cos\{nt - n_3t + \epsilon - \epsilon_3\} \\ + & 0.00000113\cos 2\{nt - n_3t + \epsilon - \epsilon_3\} \end{Bmatrix} \\
+m_1 &\begin{Bmatrix} - & 0.00093981 \\ + & 0.00091758\cos\{n_1t - n_3t + \epsilon_1 - \epsilon_3\} \\ + & 0.00001095\cos 2\{n_1t - n_3t + \epsilon_1 - \epsilon_3\} \\ + & 0.00000166\cos 3\{n_1t - n_3t + \epsilon_1 - \epsilon_3\} \end{Bmatrix} \\
+m_2 &\begin{Bmatrix} - & 0.00114443 \\ + & 0.00326071\cos\{n_2t - n_3t + \epsilon_2 - \epsilon_3\} \\ + & 0.00057836\cos 2\{n_2t - n_3t + \epsilon_2 - \epsilon_3\} \\ + & 0.00013614\cos 3\{n_2t - n_3t + \epsilon_2 - \epsilon_3\} \end{Bmatrix} \\
&+ \quad 0.00037741 \\
&- \quad 0.00037741\cos\{2Mt - 2n_3t + 2E - 2\epsilon_3\}.
\end{aligned}
$$

These inequalities in the circular orbits are independent of their positions.

Determination of the Masses of the Satellites and the Compression of Jupiter

889. Approximate values of the masses of the satellites, and of the compression of Jupiter, are sufficiently accurate for calculating the periodic inequalities in the circular orbit; but it is necessary to have more correct values of these quantities for computing the secular variations. The periodic and secular inequalities determined by theory, when compared with their observed values, furnish the means of finding the true values of these very minute quantities. The principle periodic inequality in the longitude of the first satellite is, by observation, $1636''.4$ at its maximum; but by article 888 this inequality is, by theory, $7042''.6m_1$, whence

$$m_1 = 0.232355.$$

The greatest periodic inequality in the longitude of the second satellite is, by observation, $3862''.3$ at its maximum; the same, by (298), is

$$m \,.\, 2252''.28 + m_2 \,.\, 3923''.3,$$

which arises from the combined action of the first and third satellites, hence

$$m = 1.714843 - m_2 \,.\, 1.741934. \qquad \textbf{(300)}$$

The other unknown quantities must be computed from equations (271) and (290). For that purpose let

$$r-\tfrac{1}{2}f=m\cdot 0.0217794,$$

m being an indeterminate quantity depending on the compression of Jupiter's spheroid. Then from the expressions

$$\frac{\left(r-\tfrac{1}{2}f\right)}{a^2}n=(0)\qquad \tfrac{3}{4}\cdot\frac{\mathrm{M}^2}{n}=\boxed{0}$$

and the formulae in article 474, it will be found that

$$\begin{array}{llll}
(0)=179{,}459''m & & \boxed{0}= & 33''.47 \\
(1)=35{,}317''m & & \boxed{1}= & 67''.16 \\
(2)=6{,}889''.6m & & \boxed{2}= & 135''.31 \\
(3)=954''.82m & & \boxed{3}= & 315''.64 \\
(0.1)=m_1 & 12{,}903''.6 & \boxed{0.1}=m_1 & 9{,}563''.2 \\
(0.2)=m_2 & 1{,}686''.44 & \boxed{0.2}=m_2 & 813''.69 \\
(0.3)=m_2 & 248''.57 & \boxed{0.3}=m_3 & 69''.16 \\
(1.0)=m & 10{,}229''.9 & \boxed{1.0}=m & 7{,}581''.6 \\
(1.2)=m_2 & 6{,}339''.61 & \boxed{1.2}=m_2 & 4{,}688''.2 \\
(1.3)=m_3 & 584''.554 & \boxed{1.3}=m_3 & 256''.12 \\
(2.0)=m & 1{,}058''.61 & \boxed{2.0}=m & 510''.77 \\
(2.1)=m_1 & 5{,}019''.6 & \boxed{2.1}=m_1 & 3{,}712''.1 \\
(2.3)=m_3 & 1{,}907''.34 & \boxed{2.3}=m_2 & 1{,}294''.4 \\
(3.0)=m & 117''.64 & \boxed{3.0}=m & 32''.74 \\
(3.1)=m_1 & 348''.99 & \boxed{3.1}=m_1 & 152''.93 \\
(3.2)=m_2 & 1{,}438''.2 & \boxed{3.2}=m_2 & 976''.01
\end{array}\qquad \textbf{(301)}$$

The numerical values of F, G; F', G', are determined from articles 825, 826, and 927, to be

$$\begin{array}{ll}
F=1.483732 & G=-0.857159 \\
F'=1.466380 & G'=-0.855370
\end{array}$$

and with the same quantities the coefficients Q, Q_1, Q_2, of the equations in article are found to be[2]

$$Q = -m_1\left\{\frac{16.850204h - 6.118274h_1}{\left(1+\frac{g}{972421''}\right)^2}\right\}$$

$$Q_1 = +m_2\left\{\frac{4.133080h_1 - 1.511476h_2}{\left(1+\frac{g}{972421''}\right)^2}\right\}$$

$$+m\left\{\frac{13.307450h - 4.831907h_1}{\left(1+\frac{g}{972421''}\right)^2}\right\} \qquad \textbf{(302)}$$

$$Q_2 = -m_1\left\{\frac{3.248934h_1 - 1.188133h_2}{\left(1+\frac{g}{972421''}\right)^2}\right\}$$

Not only these quantities, but several data from observation are requisite for the determination of the unknown quantities from equations (271) and (290).

890. The eclipses of the third satellite show it to have two distinct equations of the centre; the one depending on the apsides of the fourth satellite is $2h_2 = 245''.14.$ The other datum is the equation of the centre of the fourth satellite, which is, by observation, equal to $3002''.04 = 2h_3.$ Again, observation gives the annual and sidereal motion of the apsides of the fourth satellite equal to $2578''.75,$ which, by article 831, is one of the four roots of g in equation (271), so that $g_3 = 2578''.75.$ And, lastly, observation gives $43,374''$ for the annual and sidereal motion of the nodes of the orbit of the second satellite on the fixed plane, which is one of the roots of p in equation (290), so that

$$p_/ = 43,374''.$$

891. If the values of $m_/$ and $m,$ as well as all the quantities that precede, be substituted in equations (271) and (290), they become, when the first are divided by $h_3,$ and the last by $l_1,$[3]

$$0 = 2182'' - 954''.81m - 117''.64m + 32''.73m\frac{h}{h_3} - 1358''.5m_2 + 35''.533\frac{h_1}{h_2}. \qquad \textbf{(303)}$$

$$0=-\frac{h}{h_2}\{8040''.9+179457''m+51581''.5m+1686''.44m_2+248''.55m_3\}$$
$$+\{4977''.22+18729''m-16020''m_2\}\frac{h_1}{h_3}+544''.86m_2+69''.16m_3. \qquad \textbf{(304)}$$

$$0=+\{18305''.3m+72999''.2m^2-63180''.4mm_2\}\frac{h_1}{h_3}$$
$$+\left\{\begin{matrix}+2511''.6-35317''m-14128m-13455''m_2-584''.554m_3\\ -26505''.7m^2+45344''.8mm_2-19393''.4m_2^2\end{matrix}\right\}\frac{h_1}{h_3} \qquad \textbf{(305)}$$
$$+594''.41m+256''.12\,m_3-677''.04mm_2+592''.6m_2^2.$$

$$0=+4831''.9m\frac{h}{h_3}+\{1352''.8-1569''m+1342''m_2\}\frac{h_1}{h_3}$$
$$+89''.7-562''.6m-86''.44m-40''m_2+1138''.7m_2. \qquad \textbf{(306)}$$

$$0=+43306''.9-35317''m-10229''.9m\left(1-\frac{l}{l_/}\right)$$
$$-6339''.6m_2\left(1-\frac{l_2}{l_1}\right)-584''.554m_3\left(1-\frac{l_3}{l_1}\right). \qquad \textbf{(307)}$$

$$0=+2998''.23+(40342''.3-179457''m-1686''.44m_2-248''.57m_3)\frac{l}{l_1}$$
$$+1686''.44m_2\frac{l_2}{l_1}+248''.57m_3\frac{l_3}{l_1}. \qquad \textbf{(308)}$$

$$0=+1166''.5+1058''.6m\frac{l}{l_1}+1907''.34m_3\frac{l_3}{l_1}$$
$$+\{42072''.4-6889''.6m-1058''.6m-1907''.35m_3\}\frac{l_2}{l_1}. \qquad \textbf{(309)}$$

$$0=+81''.09+117''.64m\frac{l}{l_1}+1438''.2m_2\frac{l_2}{l_1}$$
$$+\{42976''.3-954''.82m-117''.64m-1438''.2m_2\}\frac{l_3}{l_1}. \qquad \textbf{(310)}$$

892. These are the particular values of equations (271) and (290) corresponding to the roots[4] $g_3 = 2578''.82$ and $p_1 = 43374''$ alone. By the following method of approximation, nine of the unknown quantities are obtained from these eight equations, together with equation (300).

The inclinations of the satellites are very small, and the two first move nearly in circular orbits, therefore the quantities

$$\frac{h}{h_3},\ \frac{h_1}{h_3},\ \frac{l}{l_1},\ \frac{l_2}{l_1},\ \frac{l_3}{l_1},$$

are so minute, that they may be made zero in the equations (303), (306), (307), in the first instance; and if m be eliminated by equation (300), these three equations will give approximate values of the masses m_2, m_3, and of m, and then m will be obtained from equation (300). But, in order to have these four quantities more accurately, their approximate values must be substituted in equations (304), (305), (308), (309), and (310), whence approximate values of

$$\frac{h}{h_3},\ \frac{h_1}{h_3},\ \frac{l}{l_1},\ \frac{l_2}{l_1},\ \frac{l_3}{l_1},$$

will be found. Again, if these approximate values of

$$\frac{h}{h_3},\ \frac{h_1}{h_3},\ \frac{l}{l_1},\ \frac{l_2}{l_1},\ \frac{l_3}{l_1},$$

be substituted in equations (303), (306), and (307), and if m be eliminated by means of equation (300), new and more accurate values of the masses and of m will be obtained. If with the last values of the masses and of m the same process be repeated, the unknown quantities will be determined with still more precision. This process must be continued till two consecutive values of each unknown quantity are nearly the same. In this manner it is found that

$$m = 1.0055974;$$
$$m = 0.173281; \qquad m_1 = 0.232355;$$
$$m_2 = 0.884972; \qquad m_3 = 0.426591;$$
$$h = 0.00206221 h_3; \qquad l = 0.0207938 l_{/};$$
$$h_1 = 0.0173350 h_3; \qquad l_2 = -0.0342530 l_{/};$$
$$h_2 = 0.0816578 h_3; \qquad l_3 = -0.000931164 l_{/}.$$

893. m determines the compression of Jupiter's spheroid, for

$$r - \tfrac{1}{2} f = m \cdot 0.0217794,$$

whence

$$r - \tfrac{1}{2} f = 0.0219012.$$

if t be the time of Jupiter's rotation, T the time of the sidereal revolution of the fourth satellite, then

$$f = \frac{T^2}{a_3^3 \cdot t^2}$$

is the ratio of the centrifugal force to gravity at Jupiter's equator. But

$$a_3 = 25.4359,\ T = 16.689019 \text{ days};$$

and, according to the observations of Cassini[5] $t = 0.413889$ of a day, hence

$$f = 0.0987990, \text{ and } r = 0.0713008.$$

As the equatorial radius of Jupiter's spheroid has been taken for unity, half his polar axis will be

$$1 - r = 0.9286992.$$

The ratio of the axis of the pole to that of his equator has often been measured: the mean of these is 0.929, which differs but little from the preceding value; but on account of the great influence of the matter at Jupiter's equator on the motions of the nodes and apsides of the orbits of the satellites, this ratio is determined with more precision by observation of the eclipses than by direct measurement, however accurate.

The agreement of theory with observation in the compression of Jupiter shows that his gravitation is composed of the gravitation of all his particles, since the variation in his attractive force, arising from his observed compression, exactly represents the motions of the nodes and apsides of his satellites.

894. If the preceding values of the masses of the satellites be divided by 10,000, the ratios of these bodies to that of Jupiter, taken as the unit, are

1st	0.0000173281
2nd	0.0000232355
3rd	0.0000884972
4th	0.0000426591.

895. Assuming the values of the masses of the earth and Jupiter in article 606, the mass of the third satellite will be 0.027337 of that of the earth, taken as a unit. But it was shown that the mass of the moon is

$$\frac{1}{75} = 0.013333, \text{\&c.}$$

of that of the earth. Thus the mass of the third satellite is more than twice as great as that of the moon, to which the mass of the fourth is nearly equal.

896. In the system of quantities,[6]

$$\begin{aligned} g_3 &= 2578''.82 \\ h &= 0.00206221 h_3 = x^{(3)} h_3 \\ h_1 &= 0.0173350 h_3 = x_1^{(3)} h_3 \\ h_2 &= 0.0816578 h_3 = x_2^{(3)} h_3 \end{aligned}$$

h_3 may be regarded as the true eccentricity of the orbit of the fourth satellite, arising from the elliptical form of the orbit, and given by observation. And the values of $h,\ h_1,\ h_2,$ are those parts of the eccentricities of the other three orbits, which arise from the indirect action of the matter at Jupiter's equator; for the attraction of that matter, by altering the position of the apsides of the fourth satellite, changes the relative position of the four orbits, and consequently alters the mutual attraction of the satellites, and is the cause of the changes in the form of the orbits expressed by the preceding values of $h,\ h_1,\ h_2.$ This is the reason why these quantities depend on the annual and sidereal motion of the apsides of the fourth satellite.

897. A similar system exists for each root of g, arising from the same cause, and depending on the annual and sidereal motions of the apsides of the other three satellites. These are readily obtained from the general equations (271), which become, when the values of the masses and of the quantities in equations (301) are substituted,

$$\begin{aligned} 0 = &+\left\{ g - 185091''.3 - \frac{16613''.78}{\left(1+\frac{g}{972421''}\right)^2} \right\} h + \left\{ 2222''.1 - \frac{8220''.4}{\left(1+\frac{g}{972421''}\right)^2} \right\} h_1 \\ &+ \left\{ 270''.1 + \frac{5212''.2}{\left(1+\frac{g}{972421''}\right)^2} \right\} h_2 + 29''.5 h_3; \end{aligned} \qquad \textbf{(311)}$$

$$0 = +\left\{ 1313''.7 - \frac{5668''.5}{\left(1+\frac{g}{972421''}\right)^2} \right\} h + \left\{ g - 43214'' - \frac{15936''.3}{\left(1+\frac{g}{972421''}\right)^2} \right\} h_1 \qquad \textbf{(312)}$$

$$+\left\{4148''.9+\frac{6740''.6}{\left(1+\frac{g}{972421''}\right)^2}\right\}h_2+109''.3h_3;$$

$$0=+\left\{89''.5+\frac{752''.6}{\left(1+\frac{g}{972421''}\right)^2}\right\}h+\left\{862''.5+\frac{1413''.5}{\left(1+\frac{g}{972421''}\right)^2}\right\}h_1$$

(313)

$$+\left\{g-9227''.1-\frac{616''.4}{\left(1+\frac{g}{972421''}\right)^2}\right\}h_2+552''.2h_3;$$

$$0=5''.7h+35''.53h_1+863''.74h_2+\left(g-2650''.1\right)h_3. \qquad \textbf{(314)}$$

898. As the motion of the apsides of the orbits of the satellites is almost entirely owing to the compression of Jupiter, in the first approximation the coefficient of h_2 may be made zero in equation (311); whence

$$g=9227''.1+\frac{616''.4}{\left(1+\frac{g}{972421''}\right)^2},$$

or, omitting g in the divisor,

$$g=9843''.5=10000'' \text{ nearly};$$

hence, if $10000''$ be put for g in equations (311), (312), (314), they will give values of

$$\frac{h}{h_2},\ \frac{h_1}{h_2},\ \frac{h_3}{h_2};$$

and, by the substitution of these in equation (311), a still more approximate value of g will be found. This process must be continued till two consecutive values of g are nearly the same. In this manner it may be found that

$$g_3=9399''.17$$
$$h\ =0.0238111h_2=x_1^{(2)}h_2$$

$$h_1 = 0.2152920 h_2 = x_2^{(2)} h_2$$
$$h_3 = 0.1291564 h_2 = x_3^{(2)} h_2$$

h_2 may be regarded as the true eccentricity of the orbit of the third satellite, and $h,\ h_1,\ h_3,$ are those parts of the eccentricities of the other three orbits, arising from the action of Jupiter's equator on the apsides of the third, and depending on $g_2 = 9399''.17,$ their annual and sidereal motion.

899. Again, if h and h_1 be made zero in equations (311) and (312), and g omitted in the divisor, then will

$$g = 35114''.7, \quad g_1 = 59152''.3,$$

and by the same method it will be found that

$$g = 196665'', \qquad g_/ = 0.57718''$$
$$h_1 = +0.0185238 h = x_1 h; \qquad h_1 = -0.0375392 h_1 = x_1^{(1)} h_1$$
$$h_2 = -0.0034337 h = x_2 h; \qquad h_2 = -0.0436686 h_1 = x_2^{(1)} h_1$$
$$h_3 = -0.00001735 h = x_3 h; \qquad h_3 = +0.00004357 h_1 = x_3^{(1)} h_1$$

In these h and h_1 are the real eccentricities of the orbits of the first and second satellites, and the other values, $h,\ h_1,\ h_2,\ h_3,$ &c., arise from the action of the other satellites corresponding to the roots g and $g_/$.

900. With regard to the inclinations of the orbits and the longitudes of the nodes, it appears, from article 892, that the system of inclinations for the root $p_/$ is

$$p_/ = 43374''.01$$
$$l = 0.0207938 l_/ = z_1^{(1)} l_/$$
$$l_2 = -0.0342530 l_/ = z_2^{(1)} l_/$$
$$l_3 = -0.00093116 l_/ = z_3^{(1)} l_/$$

$l_/$ is the real inclination of the orbit of the second satellite on its fixed plane, passing between the equator and orbit of Jupiter; and $l,\ l_2,\ l_3,$ are those parts of the inclination of the other three orbits depending on the root $p_/,$ and arising principally from the action of Jupiter's equator; for the attraction of that protuberant matter, by changing the place of the nodes of the second satellite, alters the relative position of the orbits, which changes the mutual attraction of the bodies, and produces the variations in the inclinations expressed by $l,\ l_2,\ l_3;$ and it is for this reason that these quantities depend on the annual and sidereal motion of the nodes of the second satellite.

901. A similar system depends on each root of p, that is, on the annual and sidereal motions of the nodes of the orbits of the other three satellites. These are obtained from equations (307), &c.; for when the values of the masses and of m are substituted, they become

$$\begin{aligned}
0 &= (p - 185091'')l + 2998''.23 l_1 + 1492''.5 l_2 + 106''.03 l_3 \\
0 &= 1772''.6 l + (p - 43214'') l_1 + 5610''.4 l_2 + 249''.4 l_3 \\
0 &= 183''.44 l + 1166''.3 l_1 + (p - 9227''.2) l_2 + 813''.7 l_3 \\
0 &= 20''.4 l + 81''.09 l_1 + 1272''.8 l_2 + (p - 2650'') l_3.
\end{aligned} \qquad \textbf{(315)}$$

902. The first approximate value of p is found by making the coefficient of l zero in the first of equations (315); whence $p = 185091''$; and if this value of p be put in the three last of these equations divided by l, values of $\frac{l_1}{l}, \frac{l_2}{l}, \frac{l_3}{l}$, will be found; and when these last quantities are put in the first of equations (315), a new and more correct value of p will be found: by repeating the process till two consecutive values of p nearly coincide, it will be found that

$$\begin{aligned}
p &= 185130''.14 \\
l_1 &= -0.0124527 l = z_1 l \\
l_2 &= -0.0009597 l = z_2 l \\
l_3 &= -0.0000995 l = z_3 l
\end{aligned}$$

l is the inclination of the first satellite on its fixed plane, arising chiefly from the attraction of Jupiter's equator, and given by observation; and l_1, l_2, l_3, are the parts of the inclination of the other three orbits depending on p, the annual and sidereal motion of the nodes of the first satellite.

903. The third and fourth roots of p will be obtained by making the coefficients of l_2 and l_3 respectively zero in the third and fourth of the preceding equations; and, by the same method of approximation, it will be found that

$$\begin{aligned}
p_2 &= 9193''.56, & p_3 &= 2489''.2 \\
l &= 0.0111626 l_2 = z_1^{(2)} l_2, & l &= 0.0019856 l_3 = z_1^{(3)} l_3 \\
l_1 &= 0.164053 l_2 = z_2^{(2)} l_2, & l_1 &= 0.0234108 l_3 = z_2^{(3)} l_3 \\
l_3 &= -0.196565 l_2 = z_3^{(2)} l_2, & l_2 &= 0.1248622 l_3 = z_3^{(3)} l_3,
\end{aligned}$$

where l_2 and l_3 are the real inclinations of the third and fourth satellites on their fixed planes, given by observation.

904. It now remains to compute the quantities depending on the displacement of Jupiter's equator and orbit, namely, the four values of l, $q' = `L + bt$, and $y' = `pt - \frac{at}{`L}$. The first are found by the substitution of the numerical values of the masses and of $r = \frac{1}{2}f$, in equations (285). Whence

$$\begin{aligned} l &= 0.00057879 \\ l_1 &= 0.00585888 \\ l_2 &= 0.02708801 \\ l_3 &= 0.13235804. \end{aligned}$$

Again,

$$`p = \frac{3}{4\,i}\left(\frac{2C - A - B}{C}\right)\left\{M^2 + mn^2\,l + m_1 n_1^2\,l_1 + m_2 n_2^2\,l_2 + m_3 n_3^2\,l_3\right\}.$$

As A, B, C, are the moments of inertia of Jupiter's spheroid, assumed to be elliptical, the theory of spheroids gives

$$\frac{2C - A - B}{C} = 0.14735;$$

and by observation, it is known that Jupiter's rotation is performed in 0. 41377 of a day; and that his sidereal revolution is 4332.6 days; therefore

$$\frac{M}{i} = \frac{0.41377}{4332.6};$$

then, by the substitution of the numerical values of the other quantities, all of which are given, it will appear that

$$`p = 3''.2007.$$

By observation, the inclination of Jupiter's equator on his orbit was, in 1750, $`L = 3^\circ.09996$, and as

$$a = \frac{dp}{dt}, \; b = \frac{dq}{dt}$$

are given by the theory of Jupiter at that epoch,

$$\frac{a}{`L} = 2''.93314, \; b = 0''.02279;$$

whence

$$q' = 3^\circ\!.09996 + 0''\!.02279\,t; \quad y' = 0''\!.2676,$$

which is nearly the annual precession of Jupiter's equinoxes on his orbit. $-y'$ expresses the longitude of the descending node of Jupiter's equator on his orbit, $180^\circ - y' = \Pi$ will be the longitude of his ascending node; consequently

$$\sin(v + y') = \sin(v - \Pi).$$

By observation, it is known that, in the beginning of 1750,

$$\Pi = 313^\circ\!.7592;$$

whence

$$y' = 46^\circ\!.241 + 0''\!.2676\,t\,;$$

and, with the preceding value of q', it will be found that

$$(1 - l\ \ )q' = 3^\circ\!.0899$$
$$(1 - l_1)q' = 3^\circ\!.0736$$
$$(1 - l_2)q' = 3^\circ\!.0079$$
$$(1 - l_3)q' = 2^\circ\!.6825.$$

905. It appears, from observation, that the two first satellites move in circular orbits, and that the first moves sensibly on its fixed plane, from the powerful attraction of Jupiter's equator; consequently *h* and $h_/$, corresponding to the roots *g* and $g_/$, are zero, as well as the inclination *l*, depending on the root *p*. Hence the systems of quantities in articles 899 and 902 are zero; and as, by observation, the real equations of the centre of the third and fourth satellite are

$$2h_2 = 245''\!.14, \quad 2h_2 = 553''\!.73, \quad 2h_3 = 3002''\!.04;$$

and the real inclinations of the second, third, and fourth on their fixed planes, are

$$l_1 = -1669''\!.31, \quad l_2 = -739''\!.98, \quad l_3 = -897''\!.998.$$

By the substitution of these quantities in the different systems,

$$x_1^{(2)}h_2,\ x_1^{(3)}h_3,\ \text{\&c. \&c.}$$

it will be found that the equations in articles 835 and 878,

$$\begin{aligned}
d v &= + \quad 13''.18 \sin\left(nt + \in - g_2 t - \Gamma_2\right) \\
&\quad - \quad 6''.19 \sin\left(nt + \in - g_3 t - \Gamma_3\right) \\
d v_1 &= + \ 119''.22 \sin\left(n_1 t + \in_1 - g_2 t - \Gamma_2\right) \\
&\quad - \quad 52''.04 \sin\left(n_1 t + \in_1 - g_3 t - \Gamma_3\right) \\
d v_2 &= - \ 552''.02 \sin\left(n_2 t + \in_2 - g_2 t - \Gamma_2\right) \\
&\quad - \ 244''.38 \sin\left(n_2 t + \in_2 - g_3 t - \Gamma_3\right) \\
d v_3 &= -3002''.04 \sin\left(n_3 t + \in_3 - g_3 t - \Gamma_3\right) \\
&\quad - \quad 71''.52 \sin\left(n_3 t + \in_3 - g_2 t - \Gamma_2\right).
\end{aligned} \tag{316}$$

$$\begin{aligned}
s &= +3^\circ.0899 \sin\left(v + 46^\circ.241 - 49''.8t\right) \\
&\quad - 34''.03 \sin\left(v + p_1 t + \Lambda_1\right) \\
&\quad + \ 8''.26 \sin\left(v + p_2 t + \Lambda_2\right) \\
s_1 &= +3^\circ.0736 \sin\left(v_1 + 46^\circ.241 - 49''.8t\right) \\
&\quad - 1669''.3 \sin\left(v_1 + p_2 t + \Lambda_2\right) \\
&\quad + \ 121''.4 \sin\left(v_1 + p_1 t + \Lambda_1\right) \\
&\quad + \quad 21''.02 \sin\left(v_1 + p_3 t + \Lambda_3\right) \\
s_2 &= +3^\circ.0079 \sin\left(v_2 + 46^\circ.241 - 49''.8t\right) \\
&\quad - 739''.98 \sin\left(v_2 + p_2 t + \Lambda_2\right) \\
&\quad + 112''.13 \sin\left(v_2 + p_3 t + \Lambda_3\right) \\
&\quad + \ 57''.18 \sin\left(v_2 + p_1 t + \Lambda_1\right) \\
s_3 &= +2^\circ.6825 \sin\left(v_3 + 46^\circ.241 - 49''.8t\right) \\
&\quad - 897''.998 \sin\left(v_3 + p_3 t + \Lambda_3\right) \\
&\quad + 145''.45 \sin\left(v_3 + p_3 t + \Lambda_3\right) \\
&\quad + \quad 1''.6 \sin\left(v_3 + p_1 t + \Lambda_1\right).
\end{aligned} \tag{317}$$

906. The following data are requisite for the complete determination of the motions of the satellites, all of them being estimated from the vernal equinox of the earth; the epoch being the instant of midnight, December 31st, 1749, mean time at Paris.

The secular mean motions of the four satellites.

$$\begin{aligned}
n &= 7432435^\circ.47 \\
n_1 &= 3702713^\circ.2215 \\
n_2 &= 1837852^\circ.112 \\
n_3 &= 787885^\circ.
\end{aligned}$$

The longitudes of the epochs of the satellites, estimated from the vernal equinox, were

$$\begin{aligned} \in &= 15^{\circ}.0128 \\ \in_1 &= 131^{\circ}.8404 \\ \in_2 &= 10^{\circ}.26083 \\ \in_3 &= 72^{\circ}.5513. \end{aligned}$$

Longitudes of the lower apsides of the third and fourth satellites.

$$\begin{aligned} \Gamma_2 &= 309^{\circ}.438603 \\ \Gamma_3 &= 180^{\circ}.343. \end{aligned}$$

Longitudes of ascending nodes.

$$\begin{aligned} \Lambda_1 &= 273^{\circ}.2889 \\ \Lambda_2 &= 187^{\circ}.4931 \\ \Lambda_3 &= 74^{\circ}.9687. \end{aligned}$$

The values, of g_2, g_3, &c.,p ,$p_/$, &c. are referred to the vernal equinox of Jupiter; but in order to refer them to the vernal equinox of the earth, the precession of the equinoxes, $= 50''$, must be added to the first and subtracted from the second; and as all the quantities in question have a1ready been given, it will be found that the annual and sidereal motions of the apsides were

$$\begin{aligned} g_2 &= 2628''.9 \\ g_3 &= 9449''.28. \end{aligned}$$

The annual and sidereal motions of the nodes were

$$\begin{aligned} p_1 &= 43324''.01 \\ p_2 &= 9143''.56 \\ p_3 &= 2439''.08. \end{aligned}$$

Also the annual and sidereal motion of Jupiter’s equinox, with regard to the vernal equinox of the earth, is

$$49''.8.$$

The longitude of Jupiter’s equinox at the epoch was $46^{\circ}.25$, consequently

$$y' = 46^\circ.25 + t\,.\,49''.8,$$

and the eccentricity of Jupiter's orbit at the epoch was

$$\bar{e} = 19831''.47.$$

In order to abridge $g_2 t + \Gamma_2$, $g_3 t + \Gamma_3$, $pt + \Lambda$, &c., will be represented by v_2, v_3, Ω, Ω_1, Ω_2, Ω_3.

Theory of the First Satellite

Longitude

907. Since h and $h_{/}$ are zero, equations (302) give only the two following values of Q;

$$Q = 0.208780 \cdot h_2 = 57''.8$$
$$Q = 0.016482\,.\,h_3 = 24''.7;$$

consequently equation (268) becomes

$$\begin{aligned} d\,v = &-57''.8\sin\left(nt - 2n_{/}t + \in - 2\in_{/} + v_2\right) \\ &- 24''.7\sin\left(nt - 2n_{/}t + \in - 2\in_{/} + v_3\right) \end{aligned}$$

If equation (296) and the first of equations (316) be added to this, observing that

$$2nt + 2\in - 2n_2 t - 2\in_2 = 180^\circ + 3nt + 3\in - 3n_1 t - 3\in_1,$$

it will be found that the true longitude of the first satellite in its eclipses, is

$$\begin{aligned} v = nt + \in\ &+ \ 13''.18\sin\left(nt + \in - v_2\right) \\ &+ \ 6''.19\sin\left(nt + \in - v_3\right) \\ &- \ 14''.11\sin\left(nt - n_{/}t + \in - \in_{/}\right) \\ &- \ 6''.29\sin\tfrac{3}{2}\left(nt - n_{/}t + \in - \in_{/}\right) \\ &+ 1636''.39\sin 2\left(nt - n_{/}t + \in - \in_{/}\right) \\ &+ \ 1''.22\sin 4\left(nt - n_{/}t + \in - \in_{/}\right) \\ &+ \ 0''.512\sin 5\left(nt - n_{/}t + \in - \in_{/}\right) \\ &- \ 57''.8\sin\left(nt - 2n_{/}t + \in - 2\in_{/} + v_2\right) \\ &- \ 24''.7\sin\left(nt - 2n_{/}t + \in - 2\in_{/} + v_3\right) \end{aligned} \qquad \textbf{(318)}$$

for in the eclipses of the satellites by Jupiter, or of Jupiter by the satellites, the longitudes of both bodies are the same; the Earth, Jupiter, and the satellites being then in the same straight line, consequently

$$Mt + E = nt + \in, \quad U = v,$$

consequently the term depending on the argument $2(nt - Mt + \in - E)$ vanishes.

Latitude

908. By article 880 the action of the sun occasions the inequality

$$s = -\frac{3M}{8n}(L' - l)\sin(v - 2U - pt - \Lambda)$$

but in the eclipses $U = v$, therefore

$$s = +\frac{3M}{8n}(L' - l)\sin(v + pt + \Lambda);$$

and as

$$l - L' = (1 - l)(L - L'),$$

and that

$$(1 - l)(L - L')\sin(v + pt + \Lambda)$$

is the latitude of the first satellite above its fixed plane, which was shown to be

$$3^{\circ}.0899\sin(v + 46^{\circ}.241 - 49''.8t),$$

therefore the preceding inequality is

$$-s = 1''.7\sin(v + 46^{\circ}.241 - 49''.8t).$$

When this quantity, which arises from the action of the sun, is added to the first of equations (310), it gives

$$\begin{aligned} s = &+3^{\circ}.0894\sin(v + 46^{\circ}.241 - 49''.8t) \\ &-34''.03\sin(v + \Omega_1) \\ &-\ 8''.26\sin(v + \Omega_2) \end{aligned}$$

for the latitude of the first satellite in its eclipses.

The inclination of the fixed plane on the equator of Jupiter is $6''.48$, which is insensible; and as the orbit has no perceptible inclination on the fixed plane, the first satellite moves nearly in a circular orbit in the plane of Jupiter's equator.

Theory of the Second Satellite

909. Because h and $h_/$ are insensible, equations (295) give

$$Q_/ = -0.662615h_2 \quad Q_/ = -0.055035h_3;$$

therefore equation (260) becomes

$$\begin{aligned} d v_/ &= +183''.46\sin\left(nt - 2n_/t + \epsilon - 2\epsilon_/ + \nu_2\right) \\ &\quad - 82''.6 \sin\left(nt - 2n_/t + \epsilon - 2\epsilon_/ + \nu_3\right). \end{aligned}$$

Again, equations

$$d v_/ = \frac{5}{16}\cdot\frac{n_/^2}{\left(n - n_/ - N_/\right)^2}\left\{mG - m_2F'\right\}^2 \sin 2\left(nt - n_/t + \epsilon - \epsilon_/\right)$$

$$d v_/ = -\frac{6M}{n}\left\{1 - \frac{9a_/mn^2K}{8am_/b\left(M^2 - Kn^2\right)}\right\}H\sin\left(Mt + E - \Pi\right)$$

in articles 766 and 752, have a sensible effect on the motions of the second satellite, and in consequence of

$$nt + \epsilon = 180^\circ - 2n_2t + 3n_1t - 2\epsilon_2 + 3\epsilon_1,$$

they become, by the substitution of the numerical values of the quantities,

$$\begin{aligned} d v_/ &= +22''.61\sin 4\left(n_1t - n_2t + \epsilon_1 - \epsilon_2\right) \\ &\quad -36''.07\sin\left(Mt + E - \Pi\right). \end{aligned}$$

If to these the second of equations (309) be added, together with equation (290), it will be found, in consequence of the relation,

$$nt - n_/t + \epsilon - \epsilon_/ = 180^\circ + 2n_1t - 2n_2t + 2\epsilon_1 - 2\epsilon_2,$$

that the true longitude of the second satellite is, in its eclipses,

$$\begin{aligned} d v_/ = n_/ + \epsilon_/ &+ 119''.22\sin\left(n_/t + \epsilon_1 - \nu_2\right) \\ &+ 52''.04\sin\left(n_1t + \epsilon_1 - \nu_3\right) \end{aligned}$$

$$\begin{aligned}
&- \quad 52''.91\sin\left(n_1t-n_2t+\in_1-\in_2\right)\\
&+3862''.3\sin 2\left(n_1t-n_2t+\in_1-\in_2\right)\\
&+ \quad 19''.75\sin 3\left(n_1t-n_2t+\in_2-\in_2\right)\\
&+ \quad 24''.18\sin 4\left(n_1t-n_2t+\in_1-\in_2\right)\\
&+ \quad 1''.51\sin 5\left(n_1t-n_2t+\in_1-\in_2\right)\\
&+ \quad 1''.19\sin 6\left(n_1t-n_2t+\in_1-\in_2\right)\\
&- \quad 1''.71\sin\left(n_1t-n_3t+\in_1-\in_3\right)\\
&+ \quad 1''.5\sin 2\left(n_1t-n_3t+\in_1-\in_3\right)\\
&+183''.46\sin\left(nt-2n_1t+\in-2\in_1+\nu_2\right)\\
&+ \quad 82''.6\sin\left(n_1t-2n_2t+\in_1-2\in_2+\nu_3\right)\\
&- \quad 36''.07\sin\left(Mt+E-\Pi\right),
\end{aligned} \qquad \textbf{(319)}$$

for the last term of equation (290) vanishes.

The Latitude

910. The equation (284),

$$s_/=-\frac{3M}{8n_/}\left\{L'-l\right\}\sin\left(v_/-2U-pt-\Lambda\right)$$

has a different value for each root of p, including $`p$ the root, that depends on the displacement of Jupiter's orbit and equator; but because

$$v_/=U,\quad \left(l_/-L'\right)=\left(1-l_/\right)\left(L-L'\right),$$

and that

$$\left(1-l_/\right)\left(L-L'\right)\sin\left(v_/+pt+\Lambda\right)$$

is the latitude of the second satellite above its fixed plane, which is

$$3^\circ.0736\sin\left(v_/+46^\circ 241-49''.8t\right)$$

the equation in question becomes

$$s_/=3''.4\sin\left(v_/+46^\circ 241-49''.8t\right).$$

The only remaining root of p that gives the preceding equation a sensible value in the theory of this satellite is $p_{/} = 43324''.9$; and by the substitution of the corresponding values

$$s_{/} = 0''.512 \sin\left(v_{/} + \Omega_{/}\right).$$

In consequence of these two inequalities the second of equations (310) becomes

$$\begin{aligned} s_{/} = &+3''.07262 \sin\left(v_{/} + 46^{\circ}241 - 49''.8t\right) \\ &-1669''.3 \sin\left(v_{/} + \Omega_2\right) \\ &-\ 121''.4 \sin\left(v_{/} + \Omega_1\right) \\ &-\ 21''.04 \sin\left(v_{/} + \Omega_3\right). \end{aligned} \qquad \textbf{(320)}$$

The inclination of the fixed plane on the equator of Jupiter is $63''.124$. The orbit of the satellite revolves on this plane, to which it is inclined at an angle of $27'48''.3$, its nodes completing a revolution in $29^{\text{yrs}}.914$.

Theory of the Third Satellite

911. The inequalities represented by

$$d v_2 = -Q_2 \sin\left(nt - 2n_{/}t + \in - 2\in_{/} + gt + T\right)$$

have a very sensible influence on the motions of the third satellite, because observation proves that body to have two distinct equations of the centre, one depending on the lower apsis of the orbit of the second satellite, and the other on that of the fourth. Consequently h_1 and h_2 in the coefficient

$$Q_2 = -m_1 \frac{\left(3.248934h_1 - 1.188133h_2\right)}{\left(1 + \dfrac{g}{972421''}\right)^2}$$

have respectively two values, namely,

$$h_1 = 0.2152920h_2, \text{ and } h_2 = -276''.865;$$

corresponding to g_2 and Γ_2, also

$$h_1 = 0.0173350h_3, \text{ and } h_2 = 0.0816578h_3,$$

corresponding to g_3 and Γ_3; therefore the preceding inequality, in consequence of the relations among the mean longitudes of the three first satellites, gives

$$\begin{aligned} d v_2 = & -30''.84 \sin\left(n_1 t - 2n_2 t + \epsilon_1 - 2\epsilon_2 + \varpi_2\right) \\ & +14''.12 \sin\left(n_1 t - 2n_2 t + \epsilon_1 - 2\epsilon_2 + \varpi_3\right) \end{aligned}$$

By articles 766 and 747 the action of the sun occasions the inequalities

$$\begin{aligned} d v_2 = & -\frac{12M}{n_2}\left\{1 + \frac{3a_2 mn^2 . K}{32am_2 . b\left(M^2 - Kn^2\right)}\right\}\overline{e}\sin\left(Mt + E - \Pi\right) \\ & -\frac{15Mh_2}{4n_2}\sin\left(n_2 t - 2Mt + \epsilon - 2E + gt + T\right) \end{aligned}$$

In consequence of the two values of h_2, and because

$$2Mt + 2E = 2n_2 t + 2\epsilon_2, \text{ in the eclipses}$$

these give

$$\begin{aligned} d v_2 = & +\ 1''.71 \sin\left(n_2 t + \epsilon_2 - \varpi_2\right) \\ & +\ 0''.76 \sin\left(n_2 t + \epsilon_2 - \varpi_3\right) \\ & -47''.76 \sin\left(Mt + E - \Pi\right). \end{aligned}$$

Adding the preceding inequalities to those in (291), and to the third of (309), it will be found that the longitude of the third satellite, in its eclipses, is

$$\begin{aligned} v_2 = n_2 t + \epsilon_2 & +552''.031 \sin\left(n_2 t + \epsilon_2 - \varpi_2\right) \\ & +244''.38 \sin\left(n_2 t + \epsilon_2 - \varpi_3\right) \\ & -261''.86 \sin\left(n_1 t - n_2 t + \epsilon_1 - \epsilon_2\right) \\ & -\ 3''.84 \sin 2\left(n_1 t - n_2 t + \epsilon_1 - \epsilon_2\right) \\ & -\ 2''.13 \sin 3\left(n_1 t - n_2 t + \epsilon_1 - \epsilon_2\right) \\ & -\ 14''.65 \sin\left(n_2 t - n_3 t + \epsilon_2 - \epsilon_3\right) \\ & +\ 50''.06 \sin 2\left(n_2 t - n_3 t + \epsilon_2 - \epsilon_3\right) \\ & +\ 3''.52 \sin 3\left(n_2 t - n_3 t + \epsilon_2 - \epsilon_3\right) \\ & +\ 0''.82 \sin 4\left(n_2 t - n_3 t + \epsilon_2 - \epsilon_3\right) \\ & +\ 30''.84 \sin\left(n_1 t - 2n_2 t + \epsilon_1 - 2\epsilon_2 + \varpi_2\right) \\ & +\ 14''.12 \sin\left(n_1 t - 2n_2 t + \epsilon_1 - 2\epsilon_2 + \varpi_3\right) \\ & -\ 47''.76 \sin\left(Mt + E - \Pi\right). \end{aligned} \qquad \textbf{(321)}$$

912. The double equation of the centre, occasions some peculiarities in the motion of the third satellite. By a comparison of

$$v_2 = 9449''.28\,t + 309^\circ.438603$$
$$v_2 = 2628''.9\,t + 180^\circ.343,$$

it appears that the lower apsides of the third and fourth satellites coincided in 1682, and then the coefficient of the equation of the centre was equal to the sum of the coefficients of the two partial equations. In 1777 the lower apsis of the third satellite was 180° before that of the fourth, and the coefficient of the equation of the centre was equal to the difference of the coefficients of the partial equations; results that were confirmed by observation.

Latitude

913. The only part of the equation

$$s_2 = -\frac{3M}{2n_2}\left(L' - l_2\right)\sin\left(v_2 - 2U - pt - \Lambda\right)$$

that is sensible in the motions of the third satellite is that relating to the equator of Jupiter, whence it is easy to see that

$$s_2 = -6''.7068\sin\left(v_2 + 46^\circ.241 - 49''.8t\right);$$

the same expression with regard to the third satellite, gives

$$0''.46\sin\left(v_2 + \Omega_2\right),$$

the first subtracted from the third of equations (310), gives the latitude of the third satellite equal to

$$\begin{aligned} s_2 = &+3^\circ.0061\sin\left(v_2 + 46^\circ.241 - 49''.8t\right) \\ &-739''.53\sin\left(v_2 + \Omega_2\right) \\ &-112''.13\sin\left(v_2 + \Omega_3\right) \\ &+\ 57''.18\sin\left(v_2 + \Omega_1\right) \end{aligned} \qquad \textbf{(322)}$$

in its eclipses.

The inclination of the fixed plane of the third satellite on the equator of Jupiter is $301''.49 = l_2 q_1$. Its orbit revolves on this plane, to which it is inclined at an angle of $12'\ 20''$, the nodes accomplishing their retrograde revolution in $141^{\text{yrs}}.739$.

Theory of the Fourth Satellite

914. By article 746 the action of the sun occasions the inequalities

$$d\,v_3 = +\frac{15}{4}\cdot\frac{Mh_3}{n_3}\cdot\left(n_3t+\in_3+v_3-2Mt-2E\right)$$
$$-\frac{3M}{n_3}\cdot\bar{e}\cdot\sin\left(Mt+E-\Pi\right),$$

and the secular variation in the inclination of the equator and orbit of Jupiter, by article 792, occasions the inequality

$$d\,v_3 = -\frac{\left\{4\left(1-l_3\right)\boxed{3}-\frac{1}{2}\left(1-l_3\right)p_3+6\left(3\right)l_3\right\}}{p_3}\cdot q'l_3\sin\left(p_3t+\Lambda_3-y'\right)$$

It is easy to see that the two first inequalities are,

$$d\,v_3 = +\;\;21''.69\sin\left(n_3t+\in_3+v_3-2Mt-2E\right)$$
$$-133''.33\sin\left(Mt+E-\Pi\right);$$

but in the eclipses $Mt+E=n_3t+\in_3$. So

$$d\,v_3 = -\;\;21''.69\sin\left(n_3t+\in_3-v_3\right)$$
$$-133''.33\sin\left(Mt+E-\Pi\right),$$

and the third inequality is[7]

$$d\,v_3 = -16''.04\sin\left(28^\circ\!.812+2488''.91t\right).$$

If these be added to equation (292), and the last of equations (309), the longitude of the fourth satellite in its eclipses is,

$$\begin{aligned} v_3 = n_3t+\in_3 &+2980''.35\sin\left(n_3t+\in_3-v_3\right)\\ &+\;\;13''.65\sin2\left(n_3t+\in_3-v_3\right)\\ &+\;\;\;\;0''.09\sin3\left(n_3t+\in_3-v_3\right)\\ &-\;\;71''.28\sin\left(n_3t+\in_3-v_2\right)\\ &-\;\;10''.16\sin\left(n_3t-n_2t+\in_3-\in_2\right)\\ &-\;\;\;\;4''.58\sin2\left(n_3t-n_2t+\in_3-\in_2\right)\\ &-\;\;\;\;0''.96\sin3\left(n_3t-n_2t+\in_3-\in_2\right)\\ &-\;\;\;\;0''.29\sin4\left(n_3t-n_2t+\in_3-\in_2\right) \end{aligned} \qquad \textbf{(323)}$$

$$\begin{aligned}&- \quad 0''.11\sin 5(n_3t - n_2t + \epsilon_3 - \epsilon_2)\\ &- 113''.33\sin(Mt + E - \Pi)\\ &- \quad 16''.04\sin(2488''.91t + 28^\circ.73).\end{aligned}$$

The terms having the coefficients $13''.65$ and $0''.09$ belong to the equation of the centre, which in this satellite extends to the squares and cubes of the eccentricity.

Latitude

915. The inequality of article 789

$$s_3 = \frac{3M}{8n_3}(l_3 - L')\sin(v_3 - 2U - pt - \Lambda),$$

arising from the action of the sun, has two sensible values, one arising from the displacement of Jupiter's orbit, and the other depending on the inclination of the orbit of the fourth satellite on its fixed plane. Because

$$l_3 - L' = (1 - l_3)(L - L') = 2^\circ.6825,$$

the first of these inequalities is

$$s_3 = 13''.98\sin(v_3 + 46^\circ.241 - 49''.8t),$$

in the eclipses when $U = v_3$, and the other depending on

$$p_3 = 2439''.08$$

is in the eclipses

$$s_3 = 1''.3\sin(v_3 + \Omega_3).$$

Adding these to the last of equations (310) the latitude of the fourth satellite in its eclipses is[8]

$$\begin{aligned}s_3 = &+2^\circ.6786\sin(v_3 + 46^\circ.241 - 49''.8t)\\ &-896''.702\sin(v_3 + \Omega)\\ &+145''.46\sin(v_3 + \Omega)\\ &+ \quad 1''.6\sin(v_3 + \Omega).\end{aligned} \qquad \textbf{(324)}$$

916. The inclination of the fixed plane of the fourth satellite on Jupiter's equator is

$$l_3 q' = 1473''.14.$$

The orbit of the satellite revolves on that plane to which it is inclined at an angle of[9] $14'\,58''$; its nodes accomplish a revolution in 531 years.

917. The preceding expression for the latitude explains a singular phenomenon observed in the motion of the fourth satellite. The inclination of its orbit on the orbit of Jupiter appeared to be constant, and equal to $2^\circ.43$ from the year 1680 to 1760; during that time the nodes had a direct motion of about $4'.32$ annually. From 1760 the inclination has increased. The latitude may be put under the form

$$A \sin v_3 - B \cos v_3;$$

A and B will be determined by making

$$v_3 = 90^\circ, \text{ and } v_3 = 180^\circ$$

successively in the expression s_3; $\dfrac{B}{A}$ will be the tangent of the longitude of the node and $\sqrt{A^2 + B^2}$, the inclination of the orbit. If then t be successively made equal to -70; -30; and 10 which corresponds to the years 1680, 1720, and 1760, estimated from the epoch of 1750, the result will be[10]

	Inclination	Longitude Ω
1680	$2^\circ.4764$	$311^\circ.4172$
1720	$2^\circ.4489$	$313^\circ.3067$
1760	$2^\circ.4411$	$317^\circ.0914$

If the inclination be represented by

$$2^\circ.4764 + Nt + Pt^2$$

t being the number of years elapsed since 1680. Comparing this formula with the preceding inclination

$$N = -0^\circ.0009315 \quad P = 0^\circ.000061313.$$

The minimum of the formula corresponds to t = 75.953, or to the year 1756. The mean of the three preceding values is $2^\circ.4555$, and the mean annual motion of the node from 1680 to 1760 is $4'.255$. These results are conformable to observation during this interval, but from 1760 the inclination has varied sensibly. The preceding value of s_3 gives an inclination of $2^\circ.5791$ in 1800, and the longitude of the node equal to $320^\circ.2935$; and as observation confirms these results, it must be concluded, that the inclination is a variable quantity, but the law of the variation could hardly have been determined independently of theory.

Notes

[1] This chapter is numbered VIII in the 1st edition.

[2] The subscript on the 2nd equation is omitted in the 1st edition; it *reads* $Q = -m_1$ &c. The third equation *reads* $Q_3 - m_1$ &c.

[3] The element $594''.41m$ in equation (305) *reads* $594''.41m'$ in the 1st edition. The element $89''.7$ in equation (306) *reads* $89'.7$ in the 1st edition. The punctuation after equation (307) is altered from a semicolon to a period.

[4] This *reads* $g_3 = 2578''.82_2$ in the 1st edition.

[5] See note 53, *Bk. II, Chap. XIV*.

[6] The right hand sides of the 2nd, 3rd, and 4th equations all *read* $x_1^{(3)}h_3$ in the 1st edition.

[7] The coefficient $-16''.04$ *reads* -16.04 in the 1st edition.

[8] The coefficient of the fourth term *reads* $1'6$ in the 1st edition.

[9] This *reads* $14'.58'$ in the 1st edition.

[10] The year "1720" in the following table *reads* "1620" in the 1st edition.

Asteroids

These are views of the three asteroids that have been imaged at close range by the Galileo and Near Earth Asteroid Rendezvous (NEAR) spacecraft. The image of Mathilde (left) was taken by the NEAR spacecraft on June 27, 1997. Images of the asteroids Gaspra (middle) and Ida (right) were taken by the Galileo spacecraft in 1991 and 1993, respectively. All three objects are presented at the same scale. The visible part of Mathilde is 59 km wide by 47 km high (37x29 miles). Mathilde has more large craters than the other two asteroids. The relative brightness has been made similar for easy viewing; Mathilde is actually much darker than either Ida or Gaspra. (Courtesy of NASA)

BOOK IV

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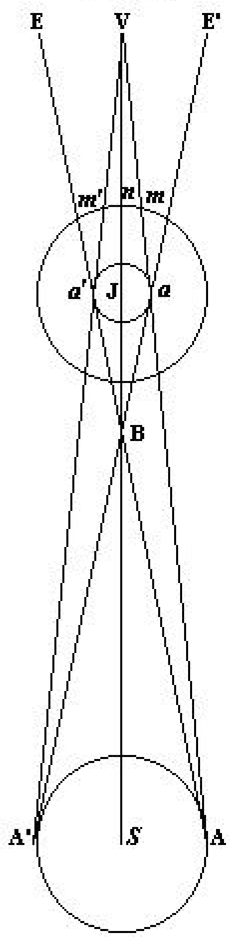
CHAPTER IV[1]

ECLIPSES OF JUPITER'S SATELLITES

918. JUPITER throws a shadow behind him relatively to the sun, in which the three first satellites are always immersed at their conjunctions, on account of their orbits being nearly in the plane of Jupiter's equator; but the greater inclination of the orbit of the fourth, together with its distance, render its eclipses less frequent.

fig. 113.



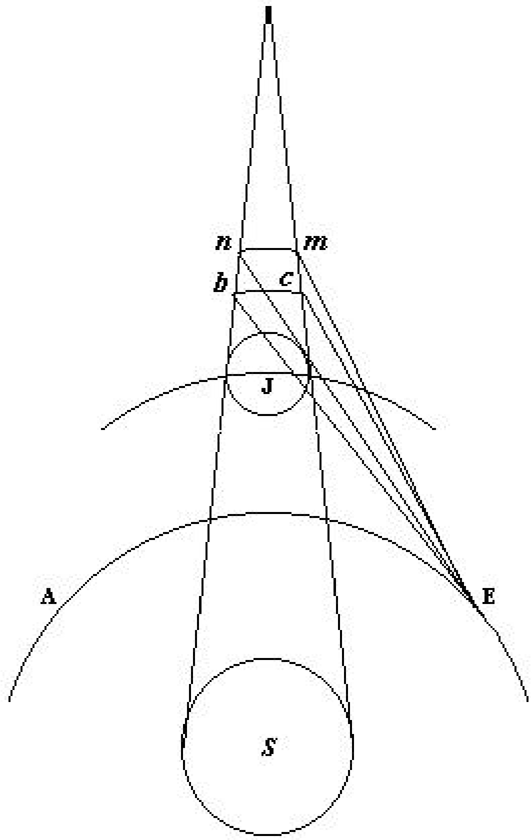
919. Let *S* and J, fig. 113, be sections of the sun and Jupiter, and *mn* the orbit of a satellite. Let AE, A′E′ touch the sections internally, and AV, A′V externally. If these lines be conceived to revolve about *S*JV they will form two cones, *a*V*a*′ and EBE′. The sun's light will be excluded from every part of the cone *a*V*a*′, and the spaces E*a*′V, E′*a*V will be the penumbra, from which the light of part of the sun will be excluded; less of it will be visible near *a*V, *a*′V, than near *a*E′, *a*′E.

920. As the satellites are only luminous by reflecting the sun's rays, they will suddenly disappear when they immerge into the shadow, and they will reappear on the other side of the shadow after a certain time. The duration of the eclipse will depend on the form and size of the cone, which itself depends on the figure of Jupiter, and his distance from the sun.

921. If the orbits of the satellites were in the plane of Jupiter's orbit, they would pass through the axis of the cone at each eclipse, and at the instant of heliocentric conjunction, the sun, Jupiter and the satellite would be on the axis of the cone, and the duration of the eclipses would always be the same, if the orbit were circular. But as all the orbits are more or less inclined to the plane of Jupiter's orbit, the duration of the eclipses varies. If the conjunction happened in the node, the eclipse would still be central; but at a certain distance from the node, the orbit of the satellite would no longer pass through the centre of the cone of the shadow, and the satellite would describe a chord more or less great, but always less than the diameter; hence the duration is variable. The longest eclipses will be those that happen in the nodes, whose position they will determine: the shortest will be observed in the limit or point farthest from the node at which an eclipse can take place, and will consequently determine the inclination of the orbit of that of Jupiter. With the inclination and the node, it will always be possible to compute the duration of the eclipse, its beginning and end.

922. The radius vector of Jupiter makes an angle *S*JE , fig. 114, with his distance from the earth, varying from 0° to 12°, which is the cause of great variations in the distances at which the eclipses take place, and the phenomena they exhibit.

fig. 114.



923. The third and fourth satellites always, and the second sometimes disappear and reappear on the same side of Jupiter, for if *S* be the sun, E the earth, and *m* the third or fourth satellite, the immersion and emersion, are seen in the directions E*m*, E*n*; only the immersions or emersions of the first satellite are visible according to the position of the earth; for if *ab* be the orbit of the first satellite, before the opposition of Jupiter, the immersion is seen in the direction E*a*, but the emersion in the direction E*b* is hid by Jupiter. On the contrary when the earth is in A, after the opposition of Jupiter, the emersion is seen, and not the immersion; it sometimes happens, that neither of the phases of the eclipses of the first satellite are seen. Before the opposition of Jupiter the eclipses happen on the west side of the planet, and after opposition on the east. The same satellite disappears at different distances from the primary according to the relative positions of the sun, the earth, and Jupiter, but they vanish close to the disc of Jupiter when he is near opposition. The eclipses only happen when the satellites are moving towards the east, the transits only when they are moving towards the west; their motion round Jupiter must therefore be from west to east, or according to the order of the signs. The transits are real eclipses of Jupiter by his moons, which appear like black spots passing over his disc.

924. It is important to determine with precision the time of the disappearance of a satellite, which is however rendered difficult by the concurrence of circumstances: a satellite disappears before it is entirely plunged in the shadow of Jupiter; its light is obscured by the penumbra: its disc, immerging into the shadow, becomes invisible to us before it is totally eclipsed, its edge being still at a little distance from the shadow of Jupiter, although we cease to see it. With regard to this circumstance, the different satellites vary, since it depends on their apparent distance from Jupiter, whose splendour weakens their light, and makes them more difficult to be seen at the instant of immersion. It also depends on the greater or less aptitude of their surfaces for reflecting light, and probably on the refraction and extinction of the solar rays in the atmosphere of Jupiter. By comparing the duration of the eclipses of all the satellites, an estimate may be formed of the influence of the causes enumerated. The variations in the distance of Jupiter and the sun from the earth, by changing the intensity of the light of the satellites, affects the apparent durations. The height of Jupiter above the horizon, the clearness of the air, and the power of the telescope employed in the observations, likewise affect their apparent duration; whence it not infrequently[2] happens that two observations of the same eclipse of the first satellite differ by half a minute: for the second satellite the error may be more than double; for the third, the difference may exceed 3′, and even 4′ in the fourth satellite. When the immersion and emersion are both observed, the mean is taken, but an error of some seconds may arise, for the phase nearest the disc of Jupiter is liable to the greatest uncertainty on account of the light of the planet; so that an eclipse may be computed with more certainty than it can be observed. Although the eclipses of Jupiter's satellites may not be the most accurate method of finding the longitude, it is by much the easiest, as it is only requisite to reduce the time of the observation into mean time, and compare it with the time of the same eclipse computed for Greenwich in the *Nautical Almanac*,[3] the difference of time is the longitude of the place of observation. The frequency of the eclipses renders this method very useful. The first satellite is eclipsed every forty-two hours; eclipses of the second recur in about four days, those of the third every seven days, and those of the fourth once in seventeen days. The latter is often a long time without being eclipsed, on account of the inclination of its orbit. Of course, the satellites are invisible all the time of Jupiter's immersion in the sun's rays.

925. Let *mn*, fig. 113, be the orbit of the satellite projected on the plane of Jupiter's orbit, then J*n* will be the curtate distance of the satellite at the instant of conjunction, and mm' the projection of the arc described by the satellite on its orbit in passing through the shadow. In order to know the whole circumstances of an eclipse, the form and length of the shadow must first be determined; then its breadth where it is traversed by the satellite, which must be resolved into the polar co-ordinates of the motion of the satellite; whence may be found the duration of the eclipse, its beginning and its end. These are functions of the actual path of the satellite through the shadow, and of its projection mm'. If Jupiter were a sphere, the shadow would be a cone, with a circular base tangent to his surface; but as he is a spheroid, the cone has an elliptical base; its shape and size may be perfectly ascertained by computation, since both the form and magnitude of Jupiter are known.

926. The whole theory of eclipses may be analytically determined, if, instead of supposing the cone of the shadow to be traced by the revolution of the tangent AV, we imagine it to be formed by the successive intersections of an infinite number of plane surfaces, all of which touch the surfaces of the sun, and Jupiter in straight lines A*a*V.

927. A plane tangent to a curved surface not only touches the surface in one point, but it coincides with it through an indefinitely small space; therefore the co-ordinates of that point must not only have the same value in the finite equations of the two surfaces, but also the first differentials of these co-ordinates must be the same in each equation. Let the origin of the co-ordinates be in the centre of the sun; then if his mass be assumed to be a sphere of which R' is the radius, the equation of his surface will be

$$x'^2 + y'^2 + z'^2 = R'^2 .$$

The general equation of a plane is

$$x = ay + bz + c,$$

a and *b* being the tangents of the angles this plane makes with the co-ordinate planes. In the point of tangency,[4] x, y, z, must not only be the same with x', y', z', but dx, dy, dz, must coincide with dx', dy', dz'; hence the equation of the plane and its differential become

$$x' = ay' + bz' + c$$
$$dx' = ady' + bdz'.$$

If this value of dx' be put in

$$x'dx' + y'dy' + z'dz' = 0,$$

which is the differential equation of the surface of the sun, it becomes

$$ax'dy' + bx'dz' + y'dy' + z'dz' = 0,$$

whatever the values of dy' and dz' may be. But this equation can only be zero under every circumstance when

$$ax' + y' = 0$$
$$bx' + z' = 0\,.$$

Thus the plane in question will touch the surface of the sun in a point A, when the following relations exist among the co-ordinates.

$$x'^2 + y'^2 + z'^2 = R'^2$$
$$ax' + y' = 0, \quad bx' + z' = 0 \qquad \textbf{(325)}$$
$$x' = ay' + bz' + c.$$

928. This plane only touches the surface of the sun, but it must also touch the surface of Jupiter, therefore the same relations must exist between the co-ordinates of the surface of Jupiter and those of the plane, as exist between the co-ordinates of the plane, and those of the surface of the sun. So the equations must be similar in both cases. Without sensible error it may be assumed that Jupiter's equator coincides with his orbit. Were he a sphere, there would be no error at all, consequently it can only be of the order of his ellipticity into the inclination of his equator on his orbit, which is $3^\circ\ 5'\ 27''$.

The centre of the sun being the origin of the co-ordinates, if *S*J, the radius vector of Jupiter, be represented by D, the equation of Jupiter's surface, considered as a spheroid of revolution, will be

$$\left(x_/ - D\right)^2 + y_/^{\,2} + \left(1 + r\right)^2\left(z_/^{\,2} - R_/^{\,2}\right) = 0, \qquad \textbf{(326)}$$

$R_/$ being half his polar axis, and r his ellipticity. The equations of contact are, therefore,

$$y_/ + a\left(x_/ - D\right) = 0$$

$$(1+r)^2 z_/ + b(x_/ - D) = 0 \qquad \textbf{(327)}$$
$$x_/ - D = ay_/ + bz_/ + c - D.$$

929. These eight equations determine the line A*a*V, according to which the plane touches the sun and Jupiter; but in order to form the cone of the shadow, a succession of such plane surfaces must touch both bodies. The equations

$$x = ay + bz + c, \text{ and } dx = ady + bdz,$$

both belong to the same plane, but because one plane surface only differs from another by position, which depends on the tangents *a* and *b*, and on *c*, the distance from the origin of the co-ordinates; these quantities being constant for any one plane, it is evident they must vary in passing to that which is adjacent, therefore

$$dx = ady + bdz + yda + zdb + dc;$$

and subtracting

$$dx = ady + bdz,$$

there results

$$0 = y + z\frac{db}{da} + \frac{dc}{da},$$

in which *b* and *c* are considered to be functions of *a*.

If values of $b, c, \frac{db}{da}, \frac{dc}{da}$, be determined from (325), (327), and substituted in this equation, and in that of the plane, they will only contain *a*, the elimination of which will give the equation of the shadow; hence, if to these be added

$$x = ay + bz + c \qquad \textbf{(328)}$$

$$0 = y + z\frac{db}{da} + \frac{dc}{da} \qquad \textbf{(329)}$$

they will determine the whole theory of eclipses. If the bodies be spheres, it is only necessary to make $r = 0$.

930. In order to determine the equation of the shadow, values of

$$b, \ c, \ \frac{db}{da}, \ \frac{dc}{da},$$

must be found. The three first of equations (325) give

$$x'^2(1 + a^2 + b^2) = R'^2,$$

and the three last give

$$x'\left(1+a^2+b^2\right)=c\,;$$

whence

$$c=R'\sqrt{1+a^2+b^2}\,,$$

and

$$c-D=R'\sqrt{1+a^2+b^2}-D;$$

but from equations (326) and (327)

$$c-D=\left(1-r\right)R_{/}\sqrt{1+a^2+b^2\,\frac{1}{\left(1+r\right)^2}}$$

the square of r being neglected.

If

$$\frac{\left(1+r\right)R_{/}}{R'}=l\,,$$

[then]

$$f^2=\frac{D^2}{R'^2\left(1-l\right)^2}-1, \qquad \textbf{(330)}$$

it may easily be found that

$$b=\left(1-\frac{l\,r}{1-l}\right)\sqrt{f^2-a^2}$$

$$c=\frac{D}{1-l}-l\,r\cdot\frac{R'^2}{D}\left(f^2-a^2\right);$$

whence

$$\frac{db}{da}=-\left(1-\frac{l\,r}{1-l}\right)\frac{a}{\sqrt{f^2-a^2}}\,;\quad \frac{dc}{da}=l\,r\,\frac{R'^2}{D}\cdot 2a;$$

and the equation

$$0=y+z\frac{db}{da}+\frac{dc}{da}$$

becomes

$$0=y-\left(1-\frac{l\,r}{1-l}\right)\frac{az}{\sqrt{f^2-a^2}}+l\,r\,\frac{R'^2}{D}\cdot 2a\,.$$

In order to have the equation of the shadow, a value of a must be found from this equation; which, with b and c, must be put in equation (328) of the plane. This will be accomplished with most ease by making $r=0$ in the preceding expression; whence

$$a=\frac{fy}{\sqrt{y^2+z^2}}$$

is the value of *a* in the spherical hypothesis; but as Jupiter is a spheroid,

$$a=\frac{fy}{\sqrt{y^2+z^2}}+qr;$$

consequently,

$$b=\left(1-\frac{lr}{1-l}\right)\sqrt{f^2-a^2}=\frac{fz}{\sqrt{y^2+z^2}}-\frac{qry}{z}-\frac{lfrz}{(1-l)\sqrt{y^2+z^2}}.$$

If this expression, together with the last value of a, and that of c be put in equation (328), it becomes

$$x=f\sqrt{y^2+z^2}-\frac{lfrz^2}{(1-l)\sqrt{y^2+z^2}}+\frac{D}{1-l}-\frac{lrR'^2\cdot f^2z^2}{D\left(y^2+z^2\right)};$$

whence[5]

$$\left(x-\frac{D}{(1-l)}\right)^2=f^2\left(y^2+z^2\right)-\frac{2f^2\,.\,lrz^2}{1-l}-\frac{2f^3\,.\,lr\,.R'^2\,.z^2}{D\sqrt{y^2+z^2}}.$$

931. At the summit of the cone *y* and *z* are zero, hence

$$x=\frac{D}{1-l}=S\text{V, fig. 113,}$$

but for every other value of y and z, x is less than $\frac{D}{1-l}$, consequently the square root of f^2 in (330) must have a negative sign; and as D is very much greater than R', $R'^2(1-l)^2$ may be neglected in comparison of D^2, hence equation (330) becomes

$$f=\frac{-D}{R'(1-l)}\text{, nearly;}$$

therefore the equation of the shadow of Jupiter is

$$\frac{R'^2(1-l)^2}{D^2}\left(\frac{D}{1-l}-x\right)^2=y^2+z^2+\frac{2l}{1-l}\cdot rz^2\left\{\frac{R'}{\sqrt{y^2+z^2}}-1\right\} \qquad \textbf{(331)}$$

and that of the penumbra is

$$\frac{R^2(1+l)^2}{D^2}\left(x-\frac{D}{1+l}\right)^2 = y^2+z^2+\frac{2l}{1+l}\cdot rz^2\left\{\frac{R}{\sqrt{y^2+z^2}}+1\right\} \qquad \textbf{(332)}$$

932. In order to know the breadth of the shadow through which the satellite passes, and thence to compute the duration of the eclipse, it is necessary to determine the section made by a plane perpendicular to *S*V, fig. 113, the axis of the cone, and at the distance *r* from Jupiter. In this case

$$x = Sn = D + r,$$

and the equation of the shadow is

$$\frac{R'^2}{D^2}\left\{Dl - r(1-l)\right\}^2 = y^2+z^2+\frac{2l}{1-l}\cdot rz^2\left\{\frac{R'}{\sqrt{y^2+z^2}}-1\right\}.$$

If at first $r=0$,

$$\sqrt{y^2+z^2} = R'l\left\{1-\frac{r(1-l)}{Dl}\right\}.$$

If this be put in the term which has r as a factor, and if to abridge

$$r' = \frac{r\left(1+\frac{r}{D}\right)}{1-\frac{r(1-l)}{Dl}},$$

the result will be

$$(1+r)^2 R_{/}^2\left\{1-\frac{r(1-l)}{Dl}\right\}^2 = y^2+z^2+2z^2r',$$

the equation to an ellipse whose eccentricity is r', and half the greater axis,

$$=(1+r)R_{/}\left\{1-\frac{r(1-l)}{Dl}\right\}=a \qquad \textbf{(333)}$$

$(1+r)R_{/}$ is the equatorial radius of Jupiter; hence the section of Jupiter's shadow at the distance of the satellite is

$$a^2 - y^2 = (1+r')^2 z^2,$$

and $2a$ is its greatest breadth. $2a$ is the actual path of the satellite through the shadow, and mm', fig. 113, is its projection on the orbit of Jupiter.

If l be made negative in the values of a and r', the preceding equation will be the section of the penumbra at the distance r from the centre of Jupiter, the difference of the two sections

$$\frac{2r}{Dl}(1+r)R_{/} = \frac{2R'r}{D} \text{ nearly,}$$

is the greatest breadth of the penumbra at that point, R' being the semidiameter of the sun.

933. To express the section of the shadow in polar co-ordinates of the motion of the satellite, let z be the height of a satellite above the orbit of Jupiter at the instant of its conjunction, r its radius vector, the projection of which on the orbit of Jupiter is $Jn = \sqrt{r^2 - z^2}$, fig. 113. Let v' be the angle described by the satellite from the instant of conjunction by its synodic motion, and projected on Jupiter's orbit, of which $\pm mn$ is the corresponding arc; and let SV be the axis of the co-ordinates x, then

$$y^2 = \left(r^2 - z^2\right)\sin^2 v'$$

which makes the equation of the section of the surface of the shadow

$$\left(r^2 - z^2\right)\sin^2 v' = a^2 - (1 + r')z^2,$$

or rejecting quantities of the order z^4, $z^2 \sin^2 v'$,

$$r^2 \sin^2 v' = a^2 - (1 + r')^2 z^2.$$

But as r is nearly constant, we have

$$z = r\left\{s + \sin v' \cdot \frac{ds}{dv'} + \tfrac{1}{2}\sin^2 v' \cdot \frac{d^2 s}{dv'} + \&\text{c.}\right\}, \qquad \textbf{(334)}$$

s being the tangent of the latitude of the satellite above the orbit of Jupiter at n at the instant of conjunction

$$z^2 = r^2\left\{s^2 + 2\sin v' \cdot \frac{sds}{dv'}\right\} \text{ nearly,}$$

hence

$$r^2 \sin^2 v' = a^2 - (1 + r')^2 r^2 s^2 - 2r^2 (1 + r')^2 \frac{sds}{dv'} \sin v'$$

from which[6]

$$\sin v' = -\left(1+r'\right)^2 \cdot \frac{sds}{dv'} \pm \sqrt{\left\{\frac{a^2}{r^2} - \left(1+r'\right)^2 s^2\right\}}.$$

With the positive sign of the radical this formula is the sine of the arc nm' described by the satellite in its synodic motion from conjunction to emersion on the orbit of Jupiter, and with the negative sign it is the arc *mn* from immersion to conjunction.

934. In order to find the duration of the eclipse, let *T* be the time employed by the satellite to describe a, half the breadth of the shadow on its orbit by its synodic motion, and let *t* be the time it takes to describe its projection v'. Then *nt* and *Mt* being the mean motions of the satellite and Jupiter, it is evident that dv' the arc described by the satellite during the time *dt*, must be equal to the difference of the mean motions of the satellite and Jupiter, or $dv' = dt\left(n-M\right)$, if the disturbing forces be omitted; but if w be the indefinitely small change in the equation of the centre during the time *dt*, then

$$dv' = dt\left(n-M\right)\left\{1+w\right\},$$

or

$$\frac{dv_{/}}{\left(n-M\right)dt} = 1+w.$$

Again, since a has been taken to represent the mean distance of the satellite m from Jupiter, $\frac{a}{a}$ is the sine of the angle under which a, half the breadth of the shadow, is seen from the centre of Jupiter. Let x be this angle, which is very small, and may be taken for its sine, then

$$t = \frac{Tv'\left(1-w\right)}{x}.$$

But $v_{/}$ is so small that

$$t = \frac{T\left(1-w\right)\sin v'}{x};$$

and if the preceding values of $\sin v'$ be substituted, putting also ax for a, the result will be[7]

$$t = T\left(1-w\right)\left\{-\left(1+r'\right)^2 \cdot \frac{sds}{x\,dv'} \pm \sqrt{\left\{\frac{a^2}{r^2} - \left(1+r'\right)^2 \frac{s^2}{x^2}\right\}}\right\}.$$

If all the inequalities be omitted, except the equations of the centre,

$$r = a\left(1-\tfrac{1}{2}w\right);$$

and as the same equation exists, even including the principal inequalities[8]

$$t = T(1-w)\left\{-(1+r')^2\cdot\frac{s}{x}\cdot\frac{ds}{dv'}\pm\sqrt{\left\{1+\tfrac{1}{2}w+(1+r')\frac{s}{x}\right\}\left\{1+\tfrac{1}{2}w-(1+r')\frac{s}{x}\right\}}\right\} \quad \textbf{(335)}$$

and if t' be the whole duration of the eclipse,

$$t' = 2T(1-w)\sqrt{\left\{1+\tfrac{1}{2}w+(1+r')\frac{s}{x}\right\}\left\{1+\tfrac{1}{2}w-(1+r')\frac{s}{x}\right\}}$$

whence may be derived

$$s = \frac{x\sqrt{4T^2(1-w)-t'^2}}{2T(1+r')(1-w)}.$$

Since s is given by the equations of latitude, this expression will serve for the determination of the arbitrary constant quantities that it contains, by choosing those observations of the eclipses on which the constant quantities have the greatest influence.

935. Both Jupiter and the satellite have been assumed to move in circular orbits, but a, half the breadth of the shadow, varies with their radii vectores. D' being the mean distance of Jupiter from the sun, $D'-dD$ may represent the true distance, so that equation (333) becomes

$$(1+r)R_{/}\left\{\tfrac{1}{2}w-\frac{dD}{D'}\right\}\frac{(1-l)a}{lD'}$$

$\tfrac{1}{2}w$ is always much less than

$$\frac{dD}{D'} = H\cos(Mt+E-\Pi) = H\cos V,$$

so the change in a is

$$-a\,\frac{(1-l)}{l}\cdot\frac{a}{D'}H\cos V;$$

and the value of $\dfrac{a}{x}$ becomes

$$\frac{a}{x}\left\{1-\frac{(1-l)}{l}\cdot\frac{a}{D}H\cos V\right\}.$$

In this function x is relative to the mean motions and mean distances of the satellite from Jupiter, and of Jupiter from the sun.

936. Since the breadth of the shadow is diminished by this cause, the time T of describing half of it will be diminished by

$$T\frac{(1-l)}{l}\frac{a}{D'}H\cos V;$$

but as the synodic motion in the time dt is nearly

$$(n-M)dt\left\{1+w-\frac{2M}{n-M}H\cos V\right\},$$

the time will be increased by

$$T\left\{\frac{2M}{n-M}H\cos V-w\right\}.$$

Omitting w, the time T on the whole will become from these two causes

$$T\left\{1+\left(\frac{2M}{n-M}-\frac{1-l}{l}\cdot\frac{a}{D'}\right)H\cos V\right\}; \quad \textbf{(336)}$$

but this is only sensible in the fourth satellite.

937. The arcs $v_{/}$ and x are so small, that no sensible error arises from taking them for their sine, and the contrary; indeed, the observations of the eclipses are liable to so many sources of error, that theory will determine these phenomena with most precision, notwithstanding these approximate values; should it be necessary, it is easy to include another term of the series in article 933.

938. The duration of the eclipses of each satellite may be determined from equation (335).

Delambre[9] found, from the mean of a vast number of observations, that half the mean duration of the eclipses of the fourth satellite in its nodes, is $T=3204''.4,$ which is the maximum; $x=7650''.6$ is the mean synodic motion of the satellite during the time T. In article 893, $r=0.0713008$. The semidiameter of Jupiter is by observation, $2(1+r)R_{/}=39''$. R' is the semidiameter of the sun seen from Jupiter. The semidiameter of the sun, at the mean distance of the earth, is $1923''.26$; it is therefore $\frac{1923''.26}{D'}$, when seen from Jupiter; $D'=5.20116636,$ is the mean distance of Jupiter from the sun, and as $a_3=25.4359,$ it is easy to find that[10]

$$r'=\frac{r\left(1+\frac{a_3}{D'}\right)}{1-\frac{(1-l)}{l}\cdot\frac{a_3}{D'}}$$

becomes $r' = 0.0729603$. $w = \frac{dv_3}{n_3 dt}$ is the indefinitely small variation in the equation of the equation of the centre during the time dt; and if the greatest term alone be taken,

$$w = 0.0145543 \cos\left(n_3 t + \epsilon_3 - v_3\right);$$

but the time T must be multiplied by

$$1 + \left\{\frac{2M}{n_3 - M} - \frac{(1-l)}{l} \cdot \frac{a_3}{D'}\right\} H \cos V,$$

H being the eccentricity of Jupiter's orbit; as the numerical values of all the quantities in this expression are given, this factor is $1 - 0.0006101 \cos V$; and if

$$z_3 = \frac{(1 + r')s_3}{x},$$

s_3 being the latitude of the fourth satellite, given in (324); then

$$\begin{aligned} z_3 = &+1.352380 \sin\left(v_3 + 46^\circ.241 - 49''.8t\right) \\ &-0.125759 \sin\left(v_3 + 74^\circ.969 + 2439''.07t\right) \\ &+0.020399 \sin\left(v_3 + 187^\circ.4931 + 9143''.6t\right) \\ &+0.000218 \sin\left(v_3 + 273^\circ.2889 + 43323''.9t\right). \end{aligned}$$

If the square of w be omitted, it reduces the quantity under the radical in equation (327) to $1 + w - z^2$; and if the products of w and H by $\frac{z_3 dz_3}{dv_3}$ be neglected, the expression (335) becomes

$$t = -118''.9 \frac{z_3 dz_3}{dv_3} \pm 3204''.4\left(1 - w - 0.0006101 \sin V\right)\sqrt{1 + w - z_3^2}.$$

From this expression it is easy to find the instants of immersion and emersion; for t was shown to be the time elapsed from the instant of the conjunction of the satellite projected on the orbit of Jupiter in n, which instant may be determined by the tables of Jupiter, and the expressions in (323) and (324) of v_3 and s_3, the longitude and latitude of the satellite.

The whole duration of the eclipses of the fourth satellite will be

$$6408''.7\left(1 - w - 0.0006101 \sin V\right) \cdot \sqrt{1 + w - z_3^2}.$$

939. With regard to the eclipses of the third satellite, $T = 2403''.8$, which is the maximum. The mean motion of the satellite, during the time *T*, is

$$x = 13416''.8, \quad a_2 = 14.461893;$$

whence

$$r' = 0.072236;$$

and if only the three greatest terms of v_2, in equation (321) be employed, $w = \frac{dv_2}{n_2 dt}$ becomes

$$\begin{aligned} w = &+0.00268457\cos(n_2 t + \epsilon_2 - \varpi_2) \\ &+0.00118848\cos(n_2 t + \epsilon_2 - \varpi_3) \\ &-0.00126952\cos(n_1 t - n_2 t + \epsilon_1 - \epsilon_2). \end{aligned}$$

The factor in (336) becomes, with regard to this satellite,

$$-0.00039871\sin V.$$

Then, if $z_2 = \frac{(1+r')s_2}{x}$, s_2 being the latitude of the third satellite,[11]

$$\begin{aligned} z_2 = &+0.864850\sin(v_2 + 46^\circ.241 - 49''.8t) \\ &-0.059101\sin(v_2 + 187^\circ.4931 + 9143''.6t) \\ &-0.008961\sin(v_2 + 74^\circ.969 + 2439''.08t) \\ &+0.004570\sin(v_2 + 273^\circ.2889 + 43323''.9t). \end{aligned}$$

Hence

$$t = -167''.64 \cdot \frac{z_2 dz_2}{dv_2} \pm 2403''.8(1 - w - 0.00039871\sin V)\sqrt{1 + w - z_2^2};$$

from whence the instants of immersion and emersion may be computed, by help of the tables of Jupiter, and of the longitude and latitude of the third satellite in (321) and (322).

The whole duration of the eclipses of the third satellite is

$$4807''.5(1 - w - 0.00039871\sin V)\sqrt{1 + w - z_2^2}.$$

940. The value of *T* from the eclipses of the second satellite, is $T = 1936''.13$; and x, the synodic mean motion of the second satellite during the time *T*, is $x = 21790''.4$; $a_{/} = 9.066548$, $r' = 0.0718862$. If we only take the greatest terms of $v_{/}$ in (319)

$$w = \frac{dv_{/}}{n_{/}dt} \text{ will be}$$

$$w = +0.00057797\cos\left(n_1 t + \epsilon_1 - v_2\right)$$
$$+0.0187249\cos 2\left(n_1 t - n_2 t + \epsilon_1 - \epsilon_2\right).$$

The factor (336) has no sensible effect on the eclipses, either of this satellite or the first, and may therefore be omitted.

If $z_{/} = \dfrac{(1+r')S_{/}}{x}$, $s_{/}$ being the latitude of the second satellite in (320); then

$$z_{/} = +0.507629\sin\left(v_{/} + 46^\circ.241 - 49''.8t\right)$$
$$-0.076569\sin\left(v_{/} + 273^\circ.2889 - 43323''.9t\right)$$
$$-0.005571\sin\left(v_{/} + 187^\circ.4931 - 9143''.6t\right)$$
$$-0.0009214\sin\left(v_{/} + 75^\circ.059 - 2439''.07t\right)$$

[and][12]

$$t = -204''.54\frac{z_{/}dz_{/}}{dv_{/}} \pm 1936''.13(1-w)\sqrt{1+w-z_{/}^2}$$

and the whole duration of the eclipses of the second satellite is

$$3872''.25(1-w)\sqrt{1+w-z_{/}^2}.$$

941. The value of T from the eclipses of the first satellite, is $T = 1527''$, and the mean synodic motion of the first satellite during the time T, is $x = 34511''.2$; and as $a = 5.698491$, $r' = 0.0716667$. If only the greatest term of v in (318) be taken

$$w = \frac{dv}{ndt} \text{ becomes}$$

$$w = 0.0079334\cos 2\left(nt - n_{/}t + \epsilon - \epsilon_{/}\right);$$

and if $z = \dfrac{(1+r')s}{x}$, s being the latitude of the first satellite in article 908, then

$$z = +0.345364\sin\left(v + 46^\circ.241 - 49''.8t\right)$$
$$-0.001057\sin\left(v + 273^\circ.2889 + 43323''.9t\right)$$
$$-0.000256\sin\left(v + 187^\circ.4931 + 9143''.6t\right);$$

also

$$t = -255''.49 \frac{z\,dz}{dv} \pm 1527''(1-w)\sqrt{1+w-z^2},$$

and the whole duration of the eclipses of the first satellite is

$$3054''(1-w)\sqrt{1+w-z^2}\ .$$

942. The errors to which the durations of the eclipses are liable, may be ascertained. Equation (333) divided by a, or which is the same thing $\frac{a}{a}$ is the sine of the angle described by each satellite during half the duration of its eclipses, supposing the satellite to be eclipsed the instant it enters the shadow. This angle, divided by the circumference, and multiplied by the time of a synodic revolution of the satellite, will give half the duration of the eclipse; and, comparing it with the observed semi-duration, the errors, arising from whatever cause, will be obtained. If $q,\ q,\ q_2,\ q_3,$ be this angle for each satellite, equation (333) gives

$$\frac{(1+r)R_{/}}{a_3}\left\{\frac{a_3}{a} - \frac{(1-l)}{l}\cdot\frac{a_3}{D'}\right\} = \sin q$$

$$\frac{(1+r)R_{/}}{a_3}\left\{\frac{a_3}{a_1} - \frac{(1-l)}{l}\cdot\frac{a_3}{D'}\right\} = \sin q_1$$

$$\frac{(1+r)R_{/}}{a_3}\left\{\frac{a_3}{a_2} - \frac{(1-l)}{l}\cdot\frac{a_3}{D'}\right\} = \sin q_2$$

$$\frac{(1+r)R_{/}}{a_3}\left\{1 - \frac{(1-l)}{l}\cdot\frac{a_3}{D'}\right\} = \sin q_3.$$

By what precedes, $l = 0.105469$,

$$\frac{(1+r)R_{/}}{D'} = 0.000094549\,;$$

whence

$$\frac{1}{a} - 0.000801823 = \sin q$$

$$\frac{1}{a_1} - 0.000801823 = \sin q_1$$

$$\frac{1}{a_2} - 0.000801823 = \sin q_2$$

$$\frac{1}{a_3} - 0.000801823 = \sin q_3;$$

and if the values of a_1, a_2, a_3, in article 87, be substituted,

$$q = 10^{\circ}.0602$$
$$q_1 = 6^{\circ}.2861$$
$$q_2 = 3^{\circ}.919$$
$$q_3 = 2^{\circ}.2072.$$

These are the angles described by the satellites during half the eclipse; and when divided by the circumference, and multiplied by the time of the synodic revolution of the satellites, they will give the duration of half the eclipse, whence half the duration of the eclipses are

1st Satellite	1602″.46
2nd Satellite	2010″.72
3rd Satellite	2527″.62
4th Satellite	3328″.01.

The semi-durations, from observation, are,

1st Satellite	1527″
2nd Satellite	1936″
3rd Satellite	2404″
4th Satellite	3204″.

943. The observed values are less than the computed, for they are diminished by the whole of the time that the discs of the satellites take to disappear after their centres have entered the shadow. The duration may be lessened by the refraction of the solar light on Jupiter's atmosphere, but it is augmented by the penumbra. These two last causes however are not sufficient to account for the difference between the computed and the observed semi-durations; therefore the time that half the discs of the satellites employ to pass into the shadow must be computed.

944. The effects of the penumbra, and of the reflected light of the sun on the atmosphere of Jupiter, are inconsiderable with regard to the first satellite. In order to have the breadth of the disc of the first satellite seen from Jupiter, let the density of this satellite be the same with that of Jupiter, and the mass and semidiameter of the planet be unity; then the apparent semidiameter of the satellite seen from the centre of Jupiter, is $\frac{\sqrt[3]{m}}{a}$; and substituting the values of *a* and $m_/$

$$\frac{\sqrt[3]{m}}{a} = 15' \, 10''.42.$$

This angle multiplied by $1^{\text{day}}.769138$, and divided by 360°, gives $41''.44$ for the time half the disc would take to pass into the shadow. Subtracting it from 1602″.46, the remainder

$1561''.02$ is the computed semi-duration, which is greater than the observed time; and yet there is reason to believe that the satellite disappears before it is quite immersed. It appears then, that the diameter of Jupiter must be diminished by at least a 50th part, which reduces it from $39''$ to $38''$. The most recent observations give $38''.44$ for the apparent equatorial diameter of Jupiter, and $35''.65$ for his polar diameter.

By this method it is computed that the discs of the satellites, seen from the centre of Jupiter, and the time they take to penetrate perpendicularly into the shadow, are

	Discs	Times
1st Satellite	$1820''.83$	$82''.888$
2nd Satellite	$1298''.37$	$115''.362$
3rd Satellite	$1271''.19$	$227''.744$
4th Satellite	$566''.7$	$237''.352$.

Whence the times of immersion and emersion of the satellites and of their shadows on the disc of Jupiter may be found, when they pass between him and the sun.

945. The observations of the eclipses of Jupiter by his satellites, may throw much light on their theory. The beginning and end of their transits may almost always be observed, which with the passage of the shadow afford four observations; whereas the ellipse of a satellite only gives two. Laplace[13] thinks these phenomena particularly worthy of the attention of practical astronomers.

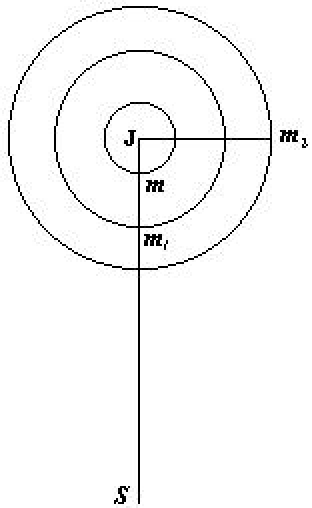
946. In the preceding investigations, the densities of the satellites were assumed to be the same with that of Jupiter. By comparing the computed times with the observed times of duration, the densities of the satellites will be found when their masses shall be accurately ascertained.

947. The perturbations of the three first satellites have a great influence on the times of their eclipses. The principal inequality of the first satellite retards, or advances its eclipses 72.41 seconds at its maximum. The principal inequality of the second satellite accelerates or retards its eclipses by $343^{\circ}.2$, at its maximum, and the principal inequality of the third satellite advances or retards its eclipses by $261^{\circ}.9$ at its maximum.

948. Since the perturbations of the satellites depend only on the differences of their mean longitudes, it makes no alteration in the value of these differences, whether the first point of Aries be assumed as the origin of the angles, or *S*J the radius vector of Jupiter supposed to move uniformly round the sun. If the angles be estimated from *S*J, nt, n_1t, n_2t, become the mean synodic motion of the three first satellites; and in both cases

$$nt-3n_1t+2n_2t+\in-3\in_1+3\in_2=180^{\circ}.$$

Suppose the longitudes of the epochs of the two first satellites to be zero or $\in=0$, $\in_1=0$, so that these two bodies are in conjunction with Jupiter when t = 0, then it follows that $\in_2=90^{\circ}$, and thus when the two first satellites are in conjunction, the third is a right angle in advance, as in fig. 115; and the principal inequalities of the three first satellites become fig. 115.



$$dv = +1636''.4\sin(2nt - 2n_1t)$$
$$dv_1 = -3862''.3\sin(nt - n_1t)$$
$$dv_2 = +\ 261''.86\sin(n_1t - n_2t).$$

In the eclipses of the first satellite at the instant of conjunction $nt = 0$, or it is equal to a multiple of 360°. Let

$$2n - 2n_1 = n + w,\ \text{or}\ n - 2n_1 = w$$

then

$$dv = 1636''.4\sin wt.$$

In the eclipses of the second satellite at the instant of conjunction $n_1t = 0$, or it is equal to a multiple of 360°; hence

$$dv_1 = -3862''.3\sin wt.$$

Lastly, in the eclipses of the third satellite, $n_2t + \epsilon_2 = 0$, or it is a multiple of 360° at the instant of conjunction, hence[14]

$$dv_2 = 261''.86\sin wt.$$

Thus it appears that the periods of these inequalities in the eclipses are the same, since they depend on the same angle. This period is equal to the product of $\dfrac{n}{n - 2n_1}$ by the duration of the synodic revolution of the first satellite, or to 437.659 days, which is perfectly conformable to observation.

949. On account of the ratio

$$nt - 3n_1t + 2n_2t + \epsilon - 3\epsilon_1 + 3\epsilon_2 = 180^\circ,$$

the three first satellites never can be eclipsed at once, neither can they be seen at once from Jupiter when in opposition or conjunction; for if

$$nt + \epsilon,\ n_1t + \epsilon_1,\ n_2t + \epsilon_2,$$

be the mean synodic longitudes, in the simultaneous eclipses of the first and second

$$nt + \epsilon = n_1t + \epsilon_1 = 180^\circ;$$

and from the law existing among the mean longitudes, it appears that

$$n_2t+\in_2=270^\circ.$$

In the simultaneous eclipses of the first and third satellites

$$nt+\in=n_2t+\in_2=180^\circ,$$

and on account of the preceding law, $n_1t+\in_1=120^\circ$.

Lastly in the simultaneous eclipses of the second and third satellites

$$n_1t+\in_1=n_2t+\in_2=180^\circ;$$

hence $nt+\in=0$, thus the first satellite in place of being eclipsed, may eclipse Jupiter.

Thus in the simultaneous eclipses of the second and third satellites, the first will always be in conjunction with Jupiter; it will always be in opposition in the simultaneous transits of the other two.

950. The comparative distances of the sun and Jupiter from the earth may be determined with tolerable accuracy from the eclipses of the satellites. In the middle of an eclipse, the sidereal position of the satellite, and the centre of Jupiter is the same when viewed from the centre of the sun, and may easily be computed from the tables of Jupiter. Direct observation, or the known motion of the sun gives the position of the earth as seen from the centre of the sun; hence, in the triangle formed by the sun, the earth, and Jupiter, the angle at the sun will be known; direct observation will give that at the earth, and thus at the instant of the middle of the eclipse, the relative distances of Jupiter from the earth and from the sun, may be computed in parts of the distance of the sun from the earth. By this method, it is found that Jupiter is at least five times as far from us as the sun is when his apparent diameter is $36''.742$. The diameter of the earth at the same distance, would only appear under an angle of $3''.37$. The volume of Jupiter is therefore at least a thousand times greater than that of the earth.

951. On account of Jupiter's distance, some minutes elapse from the instant at which an eclipse of a satellite begins or ends, before it is visible at the earth.

Roëmer observed, that the eclipses of the first satellite happened sooner, than they ought by computation when Jupiter was in opposition, and therefore nearer the earth; and later when Jupiter was in conjunction, and therefore farther from the earth. In 1675, he shewed[15] that this circumstance was owing to the time the light of the satellite employed in coming to the observer at the different distances of Jupiter. It was objected to this explanation, that the circumstance was not indicated by the eclipses of the other satellites, in which it was difficult to detect so small a quantity among their numerous inequalities then little known; but it was afterwards proved by Bradley's discovery of the aberration of light in the year 1725;[16] when he was endeavouring to determine the parallax of γ Draconis. He observed that the stars had a small annual motion. A star near the pole of the ecliptic appears to describe a small circle about it parallel to the ecliptic, whose diameter is $40''$, the pole being the true place of the star. Stars situate in the ecliptic appear to describe arcs of the ecliptic of $40''$ in length, and all stars between these two positions seem to describe ellipses whose greater axes are $40''$ in length, and are parallel to the ecliptic. The lesser axes vary as the sine of the star's latitude. This apparent motion of the stars arises from the velocity of light combined with the motion of the earth in its orbit. The sun is so very distant, that his rays are deemed parallel; therefore let S'A, SB, fig. 116, be two rays of light coming from the sun to the earth moving in its orbit in the direction AB. If a telescope be held in the direction AC, the ray S'A in place of going down the tube CA will impinge on its side, and be lost in consequence of the telescope being carried with the earth in the directions AB; but if the tube be in a position SEA, so that BA : BS as the velocity of the earth to the velocity of light; the ray will pass in the diagonal SA, which is the component of these two velocities, that is, it will pass through the axis of the telescope while carried parallel to itself with the earth. The star appears in the direction AS, when it really is in the direction AS'; hence $S'\text{A}S = \text{A}S\text{B}$ is the quantity or angle of aberration, which is always in the direction towards which the earth is moving.

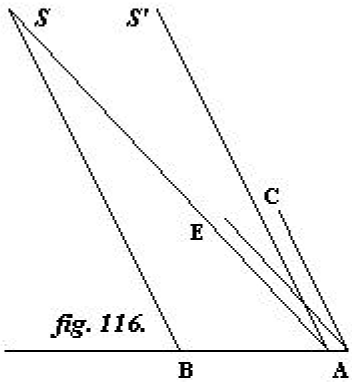


fig. 116.

Delambre[17] computed from 1000 eclipses of the first satellite, that light comes from the sun at his mean distance of about 95 millions of miles in $8'13''$;[18] therefore the velocity of light is more than ten thousand times greater than the velocity of the earth, which is nineteen miles in a second: hence BS is about 10,000 times greater than AB, consequently the angle ASB is very small. When EAB is a right angle, ASB is a maximum, and then

$$\sin \text{A}S\text{B} : 1 :: \text{AB} : \text{B}S :: \text{velocity of earth} : \text{velocity of light};$$

but ASB = the aberration; hence the sine of the greatest aberration is equal to

$$\frac{\text{rad. velocity of light}}{\text{velocity of light}} = \sin 20''.25$$

by the observation of Bradley[19] which perfectly correspond with the maximum of aberration computed by Delambre[20] from the mean of 6000 eclipses of the first satellite.

This coincidence shews the velocity of light to be uniform within the terrestrial orbit, since the one is derived from the velocity of light in the earth's orbit, and the other from the time it employs to traverse its diameter. Its velocity is also uniform in the space included in the orbit of Jupiter, for the variations of his radius vector are very sensible in the times of the eclipses of his satellites, and found to correspond exactly with the uniform motion of light.

If light be propagated in space by the vibrations of an elastic fluid, its velocity being uniform, the density of the fluid must be proportional to its elasticity.

952. The concurrent exertions of the most eminent practical and scientific astronomers have brought the theory of the satellites to such perfection, that calculation furnishes more accurate results than observation. Galileo[21] obtained approximate values of the mean distances and periodic times of the satellites from their configurations, and Kepler[22] was able to deduce from these imperfect data, proofs that the squares of their periodic times are proportional to the cubes of their mean distances, establishing an analogy between these bodies and the planetary systems, subsequently confirmed.

Bradley[23] found that the two first satellites return to the same relative positions in 437 days. Wangentin discovered a similar inequality in the third of the same period, which was concluded to be the cycle of their disturbances.

In the year 1766, the Academy of Sciences at Paris proposed the theory of the satellites of Jupiter as a prize question, which produced a masterly solution of the problem by Lagrange.[24] In the first approximation he obtained the inequalities depending on the elongations previously discovered by Bradley; in the second, he obtained four equations of the centre for each satellite, and by the same analysis shewed that each satellite has four principle equations in latitude, which he represented by four planes moving on each other at different but constant inclinations; however, his equations of the latitude were incomplete, from the error of assuming Jupiter's equator to be on the plane of his orbit. It was reserved for Laplace[25] to perfect this important theory, by including in these equations the inclination of Jupiter's equator, the effects of his nutation, precision, and the displacement of his orbit, and also by the discovery of the four fixed planes, of the libration, and of the law in the mean longitudes, discoveries that rank high among the many elegant monuments of genius displayed in his system of the world. The perfect harmony of these laws with observation, affords one of the numerous proofs, of the universal influence of gravitation. They are independent of secular inequalities, and of the resistance of a rare medium in space, since such resistance would only cause secular inequalities so modified by the mutual attraction of the satellites, that the secular equation of the first, minus three times that of the second, plus twice that of the third, would always be zero; therefore the inequalities in the return of the eclipses, whose period is 437 days, will always be the same.

953. The libration by which the three first satellites balance each other in space, is analogous to a pendulum performing an oscillation in 1135 days. It influences all the secular variations of the satellites, although only perceptible at the present time in the inequality depending on the equation of the centre of Jupiter; and as the observations of Sir William Herschel[26] shew that the periods of the rotation of the satellites are identical with the times of their revolutions, the attraction of Jupiter affects both with the same secular inequalities.

954. Thus Jupiter's three first satellites constitute a system of bodies mutually connected by the inequalities and relations mentioned, which their reciprocal action will ever maintain if the shock of some foreign cause does not derange their motion and relative position: as, for instance, if a comet passing through the system, as that of 1770 appears to have done, should come in collision with one of its bodies. That such collisions have occurred since the origin of the planetary system, is probable: the shock of a comet, whose mass only equaled the one hundred thousandth part of that of the earth, would suffice to render the libration of the satellites sensible; but since all the pains bestowed by Delambre upon the subject did not enable him to detect this, it may be concluded that the masses of any comets which may have impinged upon one of the three satellite's nearest to Jupiter must have been extremely small, which corresponds with what we have already had occasion to observe on the tenuity of the masses of the comets, and their hitherto imperceptible influence on the motions of the solar system.

955. To complete the theory, thirty-one unknown quantities remained to be derived from observation, all of which Delambre determined from 6000 eclipses, and with these data he computed tables of the motions of the satellites from Laplace's formulae, subsequently brought to great perfection by Mr. Bouvard.

The Satellites of Saturn

956. Saturn is surrounded by a ring, and seven satellites revolve from west to east round him, but their distance from the earth is so great that they are only discernible by the aid of very powerful telescopes, and consequently their eclipses have not been determined, their mean distances and periodic times alone have been ascertained with sufficient accuracy to prove that Kepler's third law extends to them. If $8''.1$ the apparent equatorial semidiameter of Saturn in his mean distance from the sun be assumed as unity, the mean distances and periodic times of the seven satellites are,

	Mean distance	Periodic times. days
1st	3.351	0.94271
2nd	4.300	1.37024
3rd	5.284	1.8878
4th	6.819	2.73948
5th	9.524	4.51749
6th	22.081	15.9453
7th	64.359	79.3296

The masses of the satellites and rings and the compression of Saturn being unknown, their perturbations cannot be determined. The orbits of the six interior satellites remain nearly in the plane of Saturn's equator, owing to his compression, and the reciprocal attraction of the bodies.

The orbit of the seventh satellite has a motion nearly uniform on a fixed plane passing between the orbit and equator of that planet, inclined to that plane at an angle of $15^\circ.264$. The nodes have a retrograde annual motion of $304''.6$; the fixed plane maintains a constant inclination of $21^\circ.6$ to Saturn's equator, but the approximation must be imperfect that results from data so uncertain.

957. The action of Saturn on account of his compression, retains the rings and the orbits of the six first satellites in the plane of his equator. The action of the sun constantly tends to make them deviate from it; but as this action increases very rapidly, and nearly as the 5th power of the radius of the orbit of the satellite, it is sensible in the seventh only. This is also the reason why the orbits of Jupiter's satellites are more inclined in proportion to their greater distance from their primary, because the attraction of his equatorial matter decreases rapidly, while that of the sun increases.

When the seventh satellite is east of the planet, it is scarcely perceptible from the faintness of its light, which must rise from spots on the hemisphere presented to us. Now, in order to exhibit always the same appearance like the moon and satellites of Jupiter, it must revolve on its axis in a time equal to that in which it revolves round its primary. Thus the equality of the time of rotation to that of revolution seems to be a general law in the motion of the satellites.

The compression of Saturn must be considerable, its revolution being performed in $11^{h}\ 42'\,43''$, nearly the same with that of Jupiter.

Satellites of Uranus

958. The slow motion of Uranus in its orbit shows it to be on the confines of the solar system. Its distance is so vast that its apparent diameter is but $3''.9$, its satellites are therefore only within the scope of instruments of very high powers; Sir William Herschel discovered six revolving in circular orbits nearly perpendicular to the plane of the ecliptic. Taking the semidiameter of the planet for unity, their mean distances and periodic times are

	Mean distance	Periodic times. days
1st	13.120	5.8926
2nd	17.022	8.7068
3rd	19.845	10.9611
4th	22.752	13.4559
5th	45.509	38.0750
6th	91.008	107.6944

subject, therefore, to the third law of Kepler. The compression of their primary and their reciprocal attraction retains their orbits in the plane of the planet's equator.

Notes

[1] This chapter is numbered "IX" in the 1st edition.
[2] This *reads* "unfrequently" in the 1st edition.
[3] See note 22, *Preliminary Dissertation*.
[4] This *reads* "tangence" in the 1st edition.
[5] The last two terms *read* $-\frac{2f.^{2}l\,r\,z^{2}}{1-l}-\frac{2f.^{3}l\,r\,.R'^{2}\,.z^{2}}{D\sqrt{y^{2}+z^{2}}}$ in the 1st edition.
[6] Punctuation added.
[7] The closing parenthesis is omitted in the 1st edition.
[8] There is a misplaced prime in $\frac{ds}{dv'}$ that *reads* $\frac{ds}{dv_{,}}$ in the 1st edition.
[9] See note 54, *Preliminary Dissertation*.
[10] The denominator *reads* $1-\frac{(1-l)}{l}\cdot\frac{a^{3}}{D'}$ in the 1st edition.
[11] The argument in the 4th term *reads* $v_{2}+273^{\circ}.2889+43323''9t$ in the 1st edition.

[12] The last term *reads* $\pm 1936''.13(1-w)\sqrt{1+w}-z_{/}^{2}$ in the 1st edition.

[13] See note 4, *Introduction.*

[14] This *reads* $dv_2 = 261''86 \sin y\,t$ in the 1st edition.

[15] *shewed.* Archaic use of "showed."

[16] See note 38, *Preliminary Dissertation.*

[17] See note 9.

[18] This *reads* $8'.13''$ in the 1st edition.

[19] See note 17.

[20] See note 9.

[21] See note 1, *Introduction.*

[22] See note 3, *Preliminary Dissertation.*

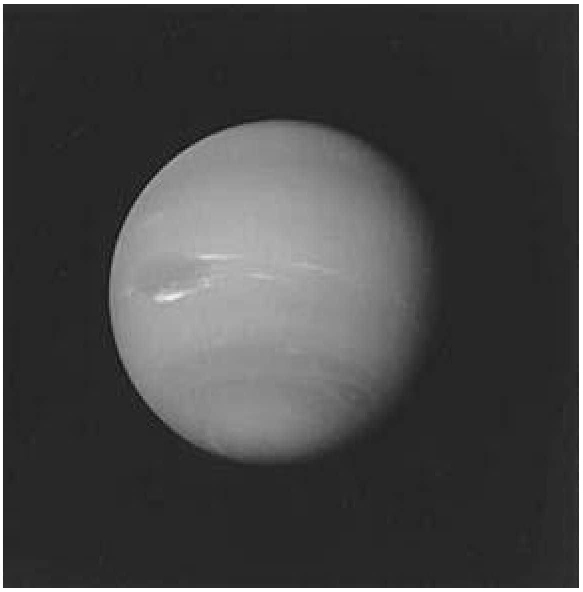
[23] See note 9.

[24] See note 16, *Preliminary Dissertation.*

[25] See note 4, *Introduction.*

[26] See note 52, *Preliminary Dissertation.*

Neptune



During 1989, the Voyager 2 narrow-angle camera was used to photograph Neptune almost continuously for two days, recording approximately two and a half rotations of the planet. This image shows two of four cloud features Voyager 2 tracked including the Great Dark Spot and a smaller dark spot. The image has been processed to enhance the visibility of features. The Great Dark Spot near the left side of the planet circuits Neptune every 18.3 hours. The bright clouds immediately to the south and east of this oval are seen to substantially change their appearance in periods as short as four hours. The smaller dark spot at the lower right of the planet circuits Neptune every 16.1 hours. (Courtesy of NASA)

CRITICAL REVIEWS OF *MECHANISM OF THE HEAVENS*

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THE LITERARY GAZETTE , AND JOURNAL OF THE BELLES LETTRES

778 (1831), p. 806-807, December, 1831

WE opened this book with no inconsiderable apprehensions for the reputation–we mean the scientific and literary reputation–of the fair author; for although Mrs. Somerville has long been considered, by persons acquainted with such subjects, as one of the most accomplished and most highly informed mathematicians of the day, no public evidence, antecedent to the appearance of this work, has been afforded of the correctness of this very high praise. We felt, therefore, the deepest interest in the result of the gigantic experiment our countrywoman undertook to perform, which was no less than to give to the world a succinct, profound, but, at the same time, as popular a view as possible of the great Laplace's *Mécanique Céleste*.[1]

A mere translation of that work would of itself have been a formidable task; and we may remark, by the way, that the distinguished American mathematician, Bowditch,[2] has already given to the world a portion of a translation, illustrated by copious notes, which cannot fail to be of the highest value to the student in those intricate pursuits. Mrs. Somerville's object, however, was of a different order, and one more consonant to the boldness and vigour of an original thinker, conscious of adequate powers to invest even the most abstruse topic with the virgin interest which true genius alone can create. She saw that Laplace's book was sealed to all who were not in familiar possession of that marvelous language in which his history of the heavens is exclusively written, and without which familiar acquaintance, the study became one of almost hopeless labour. Mrs. Somerville had fortunately obtained not only all the requisite knowledge to understand Laplace's exposition of the subject, but believed that she could facilitate its acquisition by others; and she conceived the energetic and public-spirited idea of acting as interpreter between the great continental successor of Newton and the less-instructed mathematicians, astronomers, and, we may add, general readers of her native country.

From the dedication, which we have much pleasure in stating is addressed, with singular taste and propriety, to Lord Brougham,[3] it appears that we are indebted to the sagacity of this extraordinary person for having first suggested the undertaking, and to his great influence afterwards in securing its accomplishment. His lordship, it seems, wished to embody Laplace's work in the publications of the Society for the Diffusion of Useful Knowledge; but, as may readily be supposed, Mrs. Somerville could not accomplish this purpose. Now, however, that the first grand step has been made towards giving a popular character to the highest flights of astronomical knowledge, we believe the object which the lord chancellor had in view no longer impossible, but only under one condition of the problem, namely, that the master-hand which originally gave the impulse shall undertake its completion.

It would be quite foreign to the purposes and habits of our Journal to give even the slightest sketch of the work before us; in fact, as the author says in her introduction, *"To*

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[1] See note 4, *Introduction*.
[2] See note 3, *Foreword to the Second Edition*.
[3] See note 35, *Foreword to the Second Edition*.

accomplish the task of giving an account of the 'Mécanique Céleste' without having recourse to the higher branches of mathematics is impossible;" and as we cannot pretend to have time (although, of course, we have the knowledge) to do the topic justice, we shall spare our readers the shock of a whole army of figures and symbols, with which we might cover our pages, if we were disposed to shew off our learning. As we despise such ostentation, for reasons best known to ourselves, we shall rest content with alluding to the most obvious distinction between this work and its great original, and then advert to the strictly popular branch of the undertaking.

"*Diagrams*," says Mrs. Somerville, "*are not used in Laplace's works, being unnecessary to those versed in analysis: some, however, will be occasionally introduced for the convenience of the reader*."

We do not know what meaning others may attach to the word occasionally, but to us it gave no idea of the extent of the assistance which this departure from the plan of Laplace is calculated to afford. Of course, this is not done throughout; for although, as our author says, "many subjects admit of geometrical demonstration, yet the object of the work being rather to give the spirit of Laplace's method, than to pursue a regular system of demonstration, it would be a deviation from the unity of his plan to adopt it in this."

For all this, we have good authority for saying, that the student who really wished to understand the mechanism of the heavens as developed in that splendid work, will here discover, by this and other aids furnished by Mrs. Somerville, the readiest means that are possible of acquiring that knowledge. Our author well characterizes Laplace's work as the resolution of a "a great problem in dynamics, wherein it is required to deduce all the phenomena of the solar system from the abstract laws of motion, and to confirm the truth of those laws by comparing theory with observation."

Still, however, had Mrs. Somerville confined herself to the mighty task just described, and which she has executed with a degree of address every way worthy of her principal, she would have fallen short of our wishes. We should have felt greatly disappointed, indeed, if she had not condescended to give the host of general readers some conception of the wonders concealed from the sight under the mystic garb of the differential calculus. With great good sense, therefore, and no small kindness, Mrs. Somerville has given all that we could have desired, in a preliminary dissertation, which, independently of its own intrinsic excellence, cannot fail to stimulate many readers to pursue for themselves the investigation of the phenomena it describes.

We possess already innumerable discourses on astronomy, in which the wonders of the heavens and their laws are treated of; but we can say most conscientiously, that we are acquainted with none—not even Laplace's own beautiful *exposé* in his *Système du Monde*—in which all that is essentially interesting in the motions and laws of the celestial bodies, or which is capable of popular enunciation, is so admirably, so graphically, or we may add, so unaffectedly and simply placed before us. The style is luminous and precise throughout, totally without ambition, either in thought or expression, and untouched by any depreciating apologies as to the execution, or marred by any feebleness in the design. We see nothing of the author, and think only of the subject; and it is quite clear, from the strongest possible internal evidence, that while she was penning this dissertation, a single thought of self never once crossed her mind. She felt—she must have felt—perfectly competent to treat the subject as it ought to be treated; and under this conviction, gave most spontaneous currency to her knowledge. That such perfection in style is the result of labour, at some time or other, we hold to be quite as certain as any proposition demonstrated in this book; but we are also quite sure, that the ease, vigour, and clearness, throughout such a dissertation as this, can spring only from the completest familiarity with the subject in all its bearings, chastened by the single-hearted purpose of telling what is to be told in the plainest and most acceptable language.

Is it asking too much of Mrs. Somerville to express a hope that she will allow this beautiful preliminary dissertation to be printed separately, for the delight and instruction of thousands of readers, young and old, who cannot understand, or who are too indolent to apply themselves to the elaborate parts of the work? If she will do this, we hereby promise to exert our best endeavours to make its merits known. As present we have left ourselves no space to enter into the analysis we shall be delighted to be again called upon to undertake; and we can only repeat, that we are not acquainted with any account of the celestial movements which is at once so complete in all its parts, and yet so judiciously condensed. Indeed, when we came to the conclusion, we felt only regret that our intellectual feast was so short: but on reading it again, we discovered much more matter for careful reflection than we had discovered when hurried along by the witchery of the style, or seduced into new curiosity by the evergreen freshness of this delicious subject. For of astronomy it may be more truly said, that almost of any other science, that the further we advance, the greater is our desire to proceed. In this pursuit every thing is pure, serene, certain. It is truly the "image of eternity, the throne of the invisible," that we are then contemplating; and the mind which is not raised by such contemplation above the selfish object and angry passions of this earth, must be gross indeed. But we must not forget that it involves still higher and more important considerations, by teaching us at once the wisdom, the power, and the beneficence of God, the Creator of all these things. And it must go hard indeed with our hearts if they be not touched by these important proofs of the Divine goodness to the creatures he has placed on one of the smallest of the countless myriads of orbs he has set in motion.

The following passage, with which we shall conclude this notice, is a good specimen of our fair author's style and is much in point.[4]

"The heavens afford the most sublime subject of study which can be derived from science: the magnitude and splendour of the objects, the inconceivable rapidity with which they move, and the enormous distances between them, impress the mind with some notion of the energy that maintains them in their motions with a durability to which we can see no limits. Equally conspicuous is the goodness of the great First Cause in having endowed man with faculties by which he can not only appreciate the magnificence of his works, but trace, with precision, the operation of his laws, use the globe he inhabits as a base wherewith to measure the magnitude and distance of the sun and planets, and make the diameter of the earth's orbit the first step of a scale by which he may ascend to the starry firmament. Such pursuits, while they ennoble the mind, at the same time inculcate humility, by showing that there is a barrier, which no energy, mental or physical, can ever enable us to pass: that however profoundly we may penetrate the depths of space, there still remain innumerable systems, compared with which those which seem so mighty to us must dwindle into insignificance, or even become invisible; and that not only man, but the globe he inhabits, nay the whole system of which it forms so small a part, might be annihilated, and its extinction be unperceived in the immensity of creation."

[4] See *Preliminary Dissertation*.

THE ATHENAEUM

[Charles Buller?][5]

221 (1832), p. 43-44, January, 1832

WE have universities, a considerable portion of whose vast revenues is annually paid for the support of *men* of science, and a further portion annually set apart for the printing of *books* on science. How is it that no English edition of the *Mécanique Céleste* has hitherto appeared under the sanction of a learned body and a respectable editor? If other evidence of the decline of science in this country were wanting, a strong case of suspicion might be grounded upon this one fact.[6]

It is recorded on the authority of the *Edinburgh Review*, that some fifteen years ago the British empire did not contain six individuals sufficiently learned in the exact sciences, to read this work; and here we have, at the hands of a lady, the very spirit and essence of its four quarto volumes and supplements, in a single octavo. In the preface to her book, Mrs. Somerville very properly gives us some account of its parentage. Lord Brougham,[7] it appears, was father to the thought,−having expressed a wish that its talented authoress would endeavour to introduce the working classes to a knowledge of the doctrines of the Mechanique−a wish which, conceived in the very spirit of that boundless philanthropy for which his Lordship is remarkable, and encouraged by the Society for the Diffusing [sic] of Useful Knowledge, is realized in the work before us.

We are convinced that the gratitude of the working classes would be unlimited, could they but appreciate the extent of the obligation. We are not, however, sanguine on this subject. With the very best wished for the general diffusion of knowledge, we do not expect, for many years, to find the work of Laplace much read among the labouring poor; and, indeed, looking at the splendour of the typography of the volume before us, and the patrician name of the bibliopole, we are disposed to think that Mrs. Somerville herself never seriously contemplated an early period

Contrectatus ubi manibus sordescere vulgi coeperit:

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[5] Buller, Charles, (1806-1848) an M.P. for West Looe in Cornwall and a frequent contributor to the *Athenaeum.* Buller had earlier attacked Somerville's book in the House of Commons. Although the style used in this review reflects Buller's proclivity for "*ready taunts and a prankish disposition*," the author is not identified in the review article. Elizabeth Patterson characterizes the review as "*mockingly and patronizingly derisive*" of both Somerville and in its accompanying "*ridicule of Brougham and the Society for the Diffusion of Useful Knowledge for their illusion that the working class wished to be introduced to Laplace and for their foolishness in choosing a woman to undertake such a task*." Patterson goes on to note that the "*words and tone demonstrate disdain for learned women, while its mathematical contents reveal its writer's inadequacies in that discipline*." In her autobiography Somerville recalls how "*a Mr. Buller member for someplace I have forgotten in the west of England spoke of…my book with sovereign contempt. I was much annoyed more so than I ought to have been for he showed that he was totally ignorant of the state of science*." (Patterson, Elizabeth Chambers, *Mary Somerville and the Cultivation of Science*, International Archives of the History of Science, Martinus Nijhoff Pub., 1983 , p. 83-4.)

[6] An English translation of Laplace is at present publishing at Boston, in North America, one volume of which has found its way to this country. The translator is Mr. Bowditch. The text is excellently printed, and accompanied by notes. (note in *Athenaeum* Review.)

[7] See note 35, *Foreword to the Second Edition.*

There is reserved for it a higher destiny than the hands of the unwashed. We behold it, in our critical imagination, reposing in graceful indolence on the table of every confirmed blue of the United Kingdom; its leaves will be cut; its pages turned over, by the fair hands of the very fairest of created things–and not more fair than wise. On the mysterious symbols which so mysteriously shadow forth its meaning, there will dwell (in beautiful wonder,) the brightest eyes that, since the days of our first mother, have shone, for evil or for good, upon the less fortunate portion of humanity. What a world of delightful prattle it will originate! And then, when the novelty of its youth has passed away, how dignified, how conspicuous a place will be assigned to it in the library!–how perfect, how uninterrupted will be its retirement! A more complete realization of the "*otium cum dignitate*" of a book cannot be imagined.

Although we have long considered an English translation of Laplace the great desideratum in our science, yet we confess that, when the rumour was brought to us that such a work had been undertaken by a lady, we found the information somewhat comfortless–all the chances appeared to us to be against her success. We foresaw, in the promised translation, an occasional echo of that understanding of his doctrines which had established itself in her own mind, and the prospect was discouraging. Our critical discomfort arrived, however, at a maximum, when, on opening the book, we found it blazoned in the preface that, instead of a translation, we had the spirit of Laplace, according to Mrs. Somerville, bottled up in an octavo. The gloomiest of our forebodings had never led us to dream that the sacrilege of remodeling the thoughts of Laplace would be otherwise than an occasional evil, insinuating itself, as it were, upon the task of the translator: we were utterly unprepared to find it thus openly avowed.

Laplace is perfectly competent to convey his meaning in his own words: his style is simple, and yet full of power; his words a fitting vehicle for the sublime truths which they convey; and his method strictly logical. He was far too great a man to deal in verbiage; and it is our religious belief–that any person capable of *understanding* (we use the word emphatically) the mechanism of the heavens at all, will understand it best with his own pages. We want his work as fresh from his intellect as it can be brought to us through the medium of a translation; and we like not the task which Mrs. Somerville has undertaken, of giving us his thoughts in language different from that which he thought best calculated to convey them. If her object was to simplify his reasonings, we cannot but applaud the intention; but we have every excuse for not having observed it, inasmuch as the work itself laughs all simplicity to scorn. The following instances of lucid explanation are from the first page: "*The activity of matter seems to be a law of the universe, as we know of no particle at rest.*"

Now this proposition is manifestly true, provided always, that if the particle were at rest, we should know it. But we do not know this;–as Mrs. Somerville proceeds immediately to inform us; for

"*were a body absolutely at rest, <u>we could not prove it to be so</u>, because there are no fixed points to which it could be referred.*"

The argument therefore stands thus: The activity of matter would seem to us to be a law of the universe, provided that, if any particle (of whose existence we are conscious) were at rest, we should know it, and that we know of no such particle at rest. But the particles of matter may be at rest, and we do not know it: therefore, the activity of matter does *not seem* to us to be a law of the universe.

This is the first proposition laid down in Mrs. Somerville's book; it is particularly unfortunate. We continue the quotation:–

"Consequently, if only one particle of matter were in existence, it would be impossible to determine whether it were at rest or in motion."

Now, we submit that the rest or unrest of this solitary particle of matter, would remain equally in doubt, were the world ever so thickly peopled with particles, provided there were no one point known to be at rest. Mrs. Somerville proceeds:–

"Thus, being totally ignorant of absolute motion, relative motion forms the subject of investigation: a body is therefore said to be in motion, when it changes its position, with regard to other bodies which are said to be at rest."

We, for our own parts, protest against Mrs. Somerville's comprehensive admission of ignorance. It seems to us pretty plain, that relative motion cannot exist without absolute motion. Now, of this relative motion we are allowed to know something; we are not therefore *totally* ignorant of absolute motion.

We have give the whole of the first sentence of *Mechanism of the Heavens*; we will now give that of the *Mécanique Céleste.*

"A body appears to us to move, when it changes its situation with reference to a system of bodies which we consider at rest; but, as all bodies, even those which appear to us to enjoy the most absolute repose, may be in motion, we imagine a space without limits, immovable, and penetrable to matter: it is to the parts of this space, real or imaginary, that we refer, in thought, the positions of bodies; and we conceive them in motion, when they occupy, successively different situations in space."

Our readers will perceive that Mrs. Somerville has framed her definition of motion according to that idea of it which Laplace has mentioned only to discard. Now it is to the discussion of this motion, with reference to which Mrs. Somerville and her author are thus at variance, that the whole work is devoted. It appears to us, from a careful consideration of the question, that in this first remarkable sentence of her book, Mrs. Somerville has endeavoured the whole universe to be in a state of unrest; in which she has failed, the proof being, *as she has shown*, impossible. She has then proceeded to establish the incontrovertible proposition, that there is no one point in the universe known to be at rest. From which proposition, laid down with a naiveté such as few could bring to so grave a discussion, she infers, that, if there were but one particle of matter in the universe, we should not know whether it were at rest or in motion–a useful conclusion, which leads her to terminate the discussion of absolute motion, by an admission of absolute ignorance.

On the subject of force, Mrs. Somerville is singularly unintelligible. We are not quite sure whether she admits the existence of a principle passing by that name or not. She talks of force *exerted by matter*–of matter *acting* upon matter–and much more in the same strain. At length, however, her mind grasps a definition; it is this:–"*analytically*

$$F = \frac{dv}{dt},$$

WHICH IS ALL WE KNOW ABOUT IT,"[8]

Spirit of the working classes, here is a boon! How admirable is the arrangement of symbols which thus concisely develops to us all that may be known of force. This is in the very spirit of that compression, by which an OCTAVO volume of mathematics is brought into the

[8] Emphasis in original review.

compass of a threepenny pamphlet, and, at the same time, simplified from the intellectual standard of the well read student in physics to the mind of a mechanic.

Having thus told us all that is known of force, Mrs. Somerville proceeds, in the most natural manner in the world, to tell us something more, and then this over again. She afterwards becomes quite diffuse on the subject, and that so plausibly, that had she not before defined *all* that was *known* of force, we should have believed that we were really adding to our knowledge of it. In the fundamental proposition of the parallelogram of forces, Mrs. Somerville has replaced the demonstration of Laplace by that of Paisson [sic][9] or Pontécoulant,[10] but by an old method now generally admitted to be no proof at all, and to be found in Dr. Wood's Mechanics.

We open the book casually at page 14, and we learn that the centre of curvature is the intersection of two normals–that, "*it never varies in the circle and sphere, <u>because</u> the curvature is everywhere the same.*" Now, it appears to us, that the term curvature, having no other than a conventional signification, dependent upon the position of the centre of curvature, it is beginning at the wrong end to argue a permanency of that position in any case from an equality of the curvature. The opposite is the true order of induction.

We find in the next sentence, that r being the radius of curvature, "*It is evident, that though it may vary from one point to another, it is constant for any one point, where* $dr = 0$*.*" Now, that for the *same point* the radius of curvature is the same, and for different points, different, we need not have been told, but how these facts involve the inference that $dr = 0$, escapes us.

The calculus of variations is dispatched *in a page*. In the theory of areas, the beautiful demonstration of Laplace is replaced by the method of the *Principia*. There appears to be few pages of the book which do not offer matter for similar animadversion: the subject will not, however, we fear, be interesting to the generality of our readers; we will therefore stop here.

Before we satisfy our critical conscience by recording an impartial opinion on the merits of a book, about which more than an unusual share of nonsense will, we foresee, be talked, we may be allowed to state, that we have risen from the perusal of it with the conviction, that Mrs. Somerville is a person of very extraordinary talents, and that we are possessed with an admiration, all but unlimited, for that what we understand to be the extent and variety of her attainments. Having said thus much, we feel ourselves compelled to add, that, in our belief, the work before us has been rashly undertaken, and very imperfectly completed; and that, remarkable as Mrs. Somerville's powers undoubtedly are, she has here assigned to herself a task considerably beyond them.

[9] Poisson, Siméon Dennis, see note 1, *Bk. I, Chap. VI.* Poisson actually praised Somerville's book and urged her to continue through Laplace's other books. (Patterson, Elizabeth Chambers, *Mary Somerville and the Cultivation of Science*, International Archives of the History of Science, Martinus Nijhoff Pub., 1983 , p. 87.)

[10] See note 3, *Bk. II, Chap. IV*

EDINBURGH REVIEW

[Thomas Galloway][11]

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THIS unquestionably is one of the most remarkable works that female intellect ever produced, in any age or country; and with respect to the present day, we hazard little in saying, that Mrs. Somerville is the only individual of her sex in the world who could have written it. The higher branches of the mathematics are not among the recognized objects of female accomplishment; and accordingly the education of women is so directed, that they have rarely the means afforded them of acquiring even the elements of scientific knowledge. Hence if, prompted by curiosity, or the consciousness of a capacity for such studies, they attempt to deviate into a path in which only a few men of exalted genius have been able to make great progress, they must possess no ordinary strength of purpose and powers of application, if they avoid being repulsed at the very entrance. But, notwithstanding the difficulties inseparable from the pursuit of abstract truth, and the obstacles interposed by fashion and prejudice to render the results of science inaccessible to females, examples occasionally occur of individuals of that sex raising themselves to the very highest eminence in mathematical learning; as if to prove that they are no less capable of excelling in those studies which require the patient exercise of profound thought, than they are of adorning the higher walks of literature. Our learned readers will call to mind the beautiful and unfortunate Hypatia,[12] the commentator of Apollonius and Diophantus, and president of the Alexandrian school, whose attainments in all the sciences of her age have been depicted in such glowing terms as to render her an object of admiration to posterity. A modern, and equally illustrious example, is afforded by Agnesi,[13] who, to a profound knowledge of mathematics, added an almost miraculous acquaintance with literature and philosophy, and gave the world, in her *Analytical Institutions*, a treatise which does honour not only to her sex, but to her age and country. The *Principia* of Newton, we may add, was translated into French by the celebrated *Marquise du Chastelet*,[14]–who thereby contributed, perhaps in no unimportant degree, to promote the knowledge of the Newtonian philosophy on the continent. With these illustrious names, that of Mrs. Somerville, already known in the annals of science, must henceforth be associated, on account of her great proficiency in the most sublime and difficult applications of mathematical analysis, evinced by this compend of the *Mécanique Céleste* of Laplace;–a work which, after the ample justice that has already been done to it in this Journal, and the unanimous decision of all who are capable of appreciating its merit, it would be superfluous, perhaps presumptuous, to undertake to criticise or to praise.

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[11] Galloway, Thomas, mathematics master at Sandhurst and a student of Somerville's mentor William Wallace (1768-1843). Like Somerville, Wallace was self taught. Wallace and Somerville maintained a mathematical correspondence by mail (Patterson, Elizabeth Chambers, *Mary Somerville and the Cultivation of Science*, International Archives of the History of Science, Martinus Nijhoff Pub., 1983 , p. 5.)
[12] Hypatia of Alexandria (355-415).
[13] Agnesi, Maria Gaetana, (1718-1799).
[14] Châtelet, Gabrielle-Émilie Le Tonnelier de Breteuil, Marquise du, (1706-1749), French mathematician and physicist and mistress of Voltaire.

The publication of the *Mécanique Céleste*, forms an important epoch in the history of Physical Astronomy. In the course of that century of brilliant discovery which had elapsed since the appearance of the *Principia*, the different branches of analysis had been assiduously cultivated, and successfully applied to the computation of the greater part of the celestial phenomena. Difficulty after difficulty had yielded to the successive efforts of the illustrious men who, with emulous rivalry, undertook to develop the theory of gravity, till the mechanism of the solar system was completely revealed, and the whole science of astronomy founded on a single law. In the *Mécanique Céleste*, which embodied the results of their united labours and discoveries, the long series of proofs which had been begun by Newton was completed. Every inequality of the planetary motions which the most refined observation had been able to detect, as well as numerous others too minute to be sensible to observation, was referred to its immediate cause, and subjected to rigorous computation. All the changes which can take place in the system were explained, and included in formulae, which represent not merely its present state, but its past and future condition, even to remote ages.

Such was the sublime picture exhibited in that extraordinary production; but into none of the productions of the human intellect does time bring greater ameliorations that onto those of the mathematician. Although the *Mécanique Céleste* must ever continue–what it was described by its author to be–a monument to the genius of the age in which it was composed, it is already in some respects behind the actual state of science. Embracing most of the principle questions connected with the constitution of the universe and the laws of matter, it has furnished themes for the speculations of all succeeding geometers; the investigations have been re-considered under every different point of view of which they were susceptible; and numerous and important simplifications have been made, which have superceded the original methods. In one respect, indeed, the analytical theory of the system of the world is susceptible of indefinite improvement. Many of the problems it presents are of so difficult a nature, that the most powerful analysis is unable to grasp the solutions in a finite expression; in such cases, recourse must be had to successive approximations, and however far these may be pushed, the solutions obtained in this manner necessarily fall short of absolute accuracy. In the finite integration of formulae that have hitherto been found intractable; in the investigation of series that converge more rapidly; in the reduction of difficulties to classes, and rendering the methods already known more simple and uniform, ample scope will always remain for the exercise of the most inventive talent. The future results of analysis cannot, indeed, have that imposing character which belongs to the discovery of a primordial law of the universe, or of those beautiful relations which 'bind and perpetuate the revolutions of nature,' but in reference to the simplification and more general diffusion of science, they may still be of very great importance. The analytical processes by which the more refined truths of astronomy are reached, are of so abstruse a nature, and so far removed from ordinary apprehension, that they who contribute to render them more easily understood, may justly claim to be regarded as benefactors of science.

The work of which we are about to give an account, was originally intended, as appears from the dedication, to form one of the series of treatises published under the superintendence of the Society for the Diffusion of Useful Knowledge; but by reason of the great variety and importance of the subjects that Physical Astronomy presents for discussion, it unavoidable exceeded the limits of the Society's publications. The very eminent nobleman,[15] however, at whose request it is stated to have been undertaken, and to whom it is dedicated, still thought that

[15] Lord Chancellor Brougham.

in its present form it might tend to promote the views of the Society: and under this high sanction it has been given to the world.

Mrs. Somerville has not very distinctly intimated the precise object she had in view in the composition of this treatise; and we are at some loss to discover whether an original work was contemplated, or merely an abridgement of that of Laplace. The only information given respecting its nature and purpose is contained in a sentence in the *Introduction*, in which it is said that '*in the following pages it is not intended to limit the account of the Mécanique Céleste to a detail of results, but rather to endeavour to explain the methods by which these results are deduced from one general equation of the motion of matter*.' From this we may infer, that while the main object was to demonstrate the results of the *Mécanique Céleste*, it was not intended to adhere strictly to the analysis of Laplace, but that the investigations would be rendered more simple and perspicuous where they admitted of improvement, and advantage taken of the recent discoveries in analysis to render the processes more comprehensive and uniform. This at least appears to have been the plan on which the work has been executed. In many cases the demonstrations of Laplace are given without alteration; in others they have been partially changed; and in a few instances they have been entirely supplanted by others drawn from other sources. Near the commencement, the explanations are full; as the work advances, and the difficulties increase, they become more rare; and in some of the most important problems the analysis of Laplace is transcribed without any explanation whatever. This, however, could hardly be otherwise. Indeed, when we consider the extent and abstruse nature of the various subjects that come under consideration, it will readily appear, that to give a clear and satisfactory explanation of the analytical methods of Laplace, without employing his own expressions, and exhibiting his own formulae, would be a task of no ordinary difficulty. The language of the calculus is the most concise by which human thought can be expressed; and when employed by so great a master, it receives a form which can rarely be altered without injury. The general style of Laplace is also remarkable for its perspicacity and precision; so that there is no hope of giving his meaning in different words with greater exactness, or more briefly.

With reference to the amount and species of explanation that the *Mécanique Céleste* may be thought to stand in need of, no general rule can be laid down; as all depends on the mathematical acquirements of the reader, and the direction his studies may have taken. Mrs. Somerville has evidently wished to render the theories of physical astronomy more accessible to those who have made only a moderate proficiency in analysis; but we fear, that in order to comprehend fully the *Mechanism of the Heavens*, little, very little, abatement can be made from the amount of mathematical knowledge which is indispensably required to enter with advantage or profit on the study of Laplace. In conformity with the practice of English writers, diagrams have been inserted '*for the convenience of the reader*;' and the analysis has been broken up into distinct propositions, by which means, without interrupting the process of investigation, the particular subject under discussion is set more prominently before the eye of the student. Such alterations, however, refer merely to the mode of representing the demonstrations, and do not at all touch the real difficulties. By some readers they will be probably be regarded as an impediment; and, in a mathematical investigation, it is obvious that whatever is not absolutely required to complete the chain of evidence, serves only to fatigue and distract attention. It may also be remarked, that the assistance which can be derived from the introduction of elementary illustrations into the higher problems of analysis, can only be partial and limited. From the first axioms of geometry to the sublime results of physical astronomy, the distance is immense; and if it were necessary to demonstrate every intermediate step, the bulk of a treatise containing these results, would exceed all reasonable limits. However numerous the explanations may be, they can never supercede the necessity of a very extensive acquaintance with the abstract theories of pure mathematics; nor will it be found possible, by any amount of explanation whatever, to convert the *Mécanique Céleste* into an elementary treatise of dynamics.

The entire assemblage of methods and researches comprehended in the *Mécanique Céleste*, may be divided into three principle classes. The first relates to the translation of the bodies of the solar system in space, or the motions of their centres of gravity, supposing their masses to be united at those points. The second embraces the theory of their figures, their motions of rotation, and the positions and oscillations of their axes. The third is devoted to the consideration of a number of particular phenomena, including the oscillations of the fluids at the surfaces of planets, or rather of the earth; the aberration and refraction of light, and molecular attraction. Mrs. Somerville's work extends only to the first of these classes, and does not even include the comets. The subjects which come under discussion are consequently the trajectories described by each of the planets about the sun, and of the satellites about their primary planets; the forms, positions, and magnitudes of the different orbits; the various changes which the elements of these orbits undergo; the periods and extent of the evagations of the bodies themselves from their mean places; and, lastly, the conservatory principles which ensure the stability of the system, and prevent any unlimited departure from its actual state. It is only with respect to this department of astronomy that the theory can be said to be perfect. The fundamental conditions are simple, and all supplied by observation; and the phenomena are in consequence accurately represented by the analytical equations. But at the surfaces of the planets the law of attraction is modified by various causes, of which the effect cannot be exactly appreciated; and hence the phenomena are less accessible to analysis. Their figures, for example, depend on their initial state, and the law of their density; with respect to which, we can only make arbitrary assumptions–and the motions of their axes of rotation are modified by their figures. For these reasons, the determination of the figures and rotation of the celestial bodies is attended with great and particular difficulties, to the solution of which the most illustrious analysts of the present age have devoted their efforts; and this branch of the theory of gravitation has in consequence received vast improvements since the publication of the *Mécanique Céleste*.

In the development of the planetary theory, Mrs. Somerville has derived great assistance from the *Théorie Analytique du Système du Monde* of Pontécoulant, a recent work of very considerable merit. Though grounded entirely on the *Mécanique Céleste*, the demonstrations in this treatise are occasionally original; while, by a better arrangement, and the adoption of a more uniform method of investigation, they are in numerous cases greatly simplified, without being rendered less general. The mathematical sciences must have undergone some considerable revolution before the theories of physical astronomy can be exhibited in a form much superior to that in which they appear in the work of Pontécoulant.

In a 'Preliminary Dissertation,' extending to seventy pages, Mrs. Somerville has collected and detailed, in a very interesting manner, most of the striking facts which theory and observation have made known respecting the constitution of the universe. This discourse is not indeed strictly confined to the subjects which are discussed in the subsequent part of the work; yet it is not too excursive, if designed as an introduction to the study of the *Mécanique Céleste*. It is calculated to give us a very high opinion of the industry and scientific attainments of its author; as it displays a correct and intimate acquaintance, not only with theoretical astronomy, but with the whole range of physical science, and the best and most recent works which treat of it. The diction, though occasionally deficient in accuracy and precision, is easy, flowing, and perspicuous; and the topics selected are among the most interesting that science offers to contemplation. The whole is eminently calculated to inspire a taste for the pleasures and pursuits of science; and to promote a desire to penetrate the recesses of that sublime geometry which presides over the motions, and determines the forms and distances, of the planetary bodies. We will quote a few sentences to give a specimen of the style, and the author's opinions on a subject of some moment–the degree of mathematical acquirement required to enter with advantage on the study of the analytical theory of the world. She will be admitted to be no incompetent judge.[16]

'*The heavens afford the most sublime subject of study which can be derived from science: the magnitude and splendour of the objects, the inconceivable rapidity with which they move, and the enormous distances between them, impress the mind with some notion of the energy that maintains them in their motions with a durability to which we can see no limits. Equally conspicuous is the goodness of the great First Cause in having endowed man with faculties by which he can not only appreciate the magnificence of his works, but trace, with precision, the operation of his laws, use the globe he inhabits as a base wherewith to measure the magnitude and distance of the sun and planets, and make the diameter of the earth's orbit the first step of a scale by which he may ascend to the starry firmament. Such pursuits, while they ennoble the mind, at the same time inculcate humility, by showing that there is a barrier, which no energy, mental or physical, can ever enable us to pass: that however profoundly we may penetrate the depths of space, there still remain innumerable systems, compared with which those which seem so mighty to us must dwindle into insignificance, or even become invisible; and that not only man, but the globe he inhabits, nay the whole system of which it forms so small a part, might be annihilated, and its extinction be unperceived in the immensity of creation.*

'*A complete acquaintance with Physical Astronomy can only be attained by those who are well versed in the higher branches of mathematical and mechanical science: such alone can appreciate the extreme beauty of the results, and of the means by which these results are obtained. Nevertheless a sufficient skill in analysis to follow the general outline, to see the mutual dependence of the different parts of the system, and to comprehend by what means some of the most extraordinary conclusions have been arrived at, is within the reach of many who shrink from the task, appalled by difficulties, which perhaps are not more formidable than those incident to the study of the elements of every branch of knowledge, and possibly overrating them by not making a sufficient distinction between the degree of mathematical acquirement necessary for making discoveries, and that which is requisite for understanding what others have done. That the study of mathematics and their application to astronomy are full of interest will be allowed by all who have devoted their time and attention to these pursuits, and they only can estimate the delight of arriving at truth, whether it be in the discovery of a world, or of a new property of numbers*.'

The more obvious consequences of the general laws of the universe have been so frequently noticed and illustrated, that it is often extremely difficult to discover by whom they were first remarked. Delambre,[17] in the preface to his *Abrégé d'Astronomie*, takes credit to himself for having always, in speaking of an instrument, a solution, or a formulae, endeavoured to name the author. It is a practice, he observes, too much neglected by the writers of elementary

[16] See *Preliminary Dissertation.*
[17] See note 54, *Preliminary Dissertation.*

works; and the consequence is, that the reader attributes to the author all that he finds in his book, in the same manner as we are led to ascribe to Euclid the theorems he has only transmitted to us. Though it may not be possible, and it is perhaps not necessary, unless where some general principle is involved, to adhere strictly to this practice, yet it is of great importance that the student receive no wrong impressions respecting the history of science; and therefore we cannot help regarding the following as a singularly unfortunate mode of introducing the name of an eminent individual.

'*A fluid, as Mr. Babbage observes, in falling from a higher to a lower level, carries with it the velocity due to its revolution with the earth at a greater distance from its centre. It will therefore accelerate, although to an almost infinitesimal extent, the earth's daily rotation.*'

As well might Mr. Babbage[18] have been quoted as remarking that the tides are caused by the attraction of the moon. The consequence here mentioned is one of those very obvious results of theory, that could not escape the slightest attention to the various circumstances that effect the rotation of the earth. It was stated with great clearness and detail, and without the slightest pretension to originality, by Professor Playfair,[19] in one of the notes to his *Illustrations of the Huttonian Theory*. (Works, vol. i. p. 419.)

Mrs. Somerville's work contains four books, of which the first, like the corresponding one of the *Mécanique Céleste*, forms a comprehensive and general treatise of Dynamics. On a subject which has been so often discussed by the most eminent mathematicians, we can expect to meet with little novelty or originality; and the principle merit of a new work must consist in the judicious selection and perspicuous arrangement of the materials. The necessary definitions and axioms are given very briefly and clearly in the first chapter; and among the deviations of the methods of Laplace, we cannot forbear noticing the demonstration of the formulae for the composition and resolution of forces; which Laplace, in order to avoid the assumption of force being proportional to velocity–a thing which cannot be known *a priori*–had deduced immediately from the theory of functions. This demonstration is remarkable; but forms perhaps too great a difficulty at the very commencement of the work. For this reason Mrs. Somerville has rejected it, and returned to the usual demonstration, which depends on the composition and resolution of motion. In this however, she has the countenance of the high authority of Lagrange, who admits that, in separating the principle of the composition of forces from that of the composition of motion, we deprive it of its principle advantages–evidence and simplicity–and reduce it to depend on a mere result of geometrical constructions, or of combinations of algebraical symbols. After the definitions comes the subject of the variable motion of a particle under different circumstances; then the equilibrium and motion of a system of bodies. In these preliminary chapters, the subjects of discussion are the same as those that occur in the *Mécanique Céleste*; and the changes that have been made are chiefly confined to the mode of illustration. The problem of the rotation of a solid body, which occupies the fifth chapter, is of great importance in astronomy, in consequence of its connexion with the theory of the nutation of the earth's axis, the precession of the equinoxes, and the libration of the moon. The analysis which Mrs. Somerville has given, is the same as that of Pontécoulant, and is sufficiently compact and symmetrical; but the subject is of so difficult a nature, that the general theory cannot be well understood without some special application. The same remarks apply to

[18] See note 14, *Foreword to the Second Edition.*
[19] See note 17, *Preliminary Dissertation.*

the two following chapters, which treat of the equilibrium and motion of fluids. As the theories of the rotation of the earth and of the tides are not comprehended in Mrs. Somerville's work, its unity would, perhaps, have been more perfect, if these last three chapters, which have no subsequent application, had been altogether omitted.

The second book, by far the longest of the four, is devoted to the development of the effects of universal gravitation on the motions and orbits of the primary planets. After a short account of the progress of Physical Astronomy, from Kepler to Laplace, Mrs. Somerville proceeds, in the second chapter, to deduce the Newtonian law of gravity from the three general laws of Kepler.[20] These laws form the very basis of the science; and when the differential equations of motion are formed so as to satisfy them, it is an easy consequence that the force which retains the planets in their orbits is directed to the centre of the sun, and varies in the inverse proportion of the distance of the attracted body from that centre. The most obvious verification of this important result is afforded by the motions of the moon; for the action of terrestrial gravity, which at the surface of the earth causes a body to fall through $16\frac{1}{11}$ feet in the first second of time, being assumed to diminish according to the above law, would cause a body at the distance of the moon to fall through a space which is exactly equal to the moon's deflection from the tangent to her orbit in the same time. All this is explained exactly in the same manner as in the *Mécanique Céleste*.

Having deduced from data furnished by observation the law of the force which regulates the motions of the celestial bodies, it becomes necessary to invert the process, and to form the differential equations of motion on the hypothesis, that all bodies of the solar system attract one another with forces varying directly as their masses, and inversely as the squares of their mutual distances. The equations given in the *Mécanique Céleste* are of the utmost generality; being applicable not only to the law of force which prevails in the solar system, but to any law of attraction which is capable of being expressed in a function of the distance. But it is in the integration of these equations that the real difficulty of Physical Astronomy consists; and this difficulty all the ingenuity of the greatest analysts, and all the resources of the most refined science, have hitherto been unable to overcome. It is only by restricting the hypothesis to particular cases that we can obtain even approximate solutions. The particular constitution of the planetary system fortunately affords considerable facilities in this respect; and by permitting us to decompose it into partial systems, and to estimate successively the influence of the different bodies, enables us to obtain results which it would be impossible to arrive at if it were necessary to compass the general problem, and to consider simultaneously all the causes of perturbation. In the first place, though each planet sustains the action of a multitude of forces, yet its motion is chiefly regulated by the predominant influence of the sun, in comparison of which the attraction of any other body in the system, or even the united force of all of them, is extremely small. In the next place, the planetary orbits differ very little from circles, and are inclined at very small angles to the plane of the ecliptic; and on these accounts the series which express the perturbations converge much more rapidly than would be the case if the orbits were more eccentric, and the inclinations considerable. Lastly, the figures of the planets differ so little from spheres, that at their distances the influence of the figure of the disturbing body entirely disappears, and they attract one another as if their whole mass were united in a point at the centre of gravity. These considerations essentially contribute to diminish the difficulties of the calculus.

[20] See note 3, *Preliminary Dissertation.*

The simplest hypothetical case to which the equations of motion can be applied, is that of a planet obeying the sun's force, and undisturbed by the action of any other body. to this case Mrs. Somerville proceeds in the fourth chapter. The integration, as is well known, gives a line of the second order; the elements of the curve being represented by the arbitrary constant quantities introduced in the double integration. The development of the expressions thus obtained, gives the whole theory of the elliptic motion. In these elementary discussions, the very brief indications of Laplace have been considerably expanded. The subject admitted of no novelty; but the different formulae for finding the radius vector, the eccentric and true anomalies in terms of the mean anomaly, the true and projected longitudes in terms of the mean longitude, the position of the orbit, &c., are demonstrated with much perspicuity and elegance.

After a first approximation to the true path of a planet has been obtained, on the supposition that it obeys the sun's force alone, it is necessary to pass to the infinitely more difficult problem of the perturbations; and to determine how far the previous results are modified by the attraction of the other bodies belonging to the system, This is the famous problem which was begun and prosecuted with so much vigour by the emulous rivalry of the greatest mathematicians of the last century,–Clairaut,[21] d'Alembert,[22] and Euler[23]; and of which the more complete solution has conferred unfading glory on the names of Lagrange[24] and Laplace.[25] A solution in finite terms is indeed impossible; but the approximations have now been carried so far, that the tables computed from theory, give the places of the planets with a precision that rivals observation.

With a view to facilitate the investigation of this intricate subject, geometers have classed the perturbations under two distinct heads; and the distinction does not depend merely on a difference in the form of the analytical expressions, but on certain physical considerations, which may be easily explained. Let us suppose for an instant the planetary orbits to be invariable in form and position. It is evident that the effect produced by the action of one planet on another, must depend on their relative positions in respect of the sun; for the action of the first planet on the second, may either conspire with the sun's attraction, or oppose it; and it can only cause a variation in the longitude, latitude, or distance of the disturbed planet. Now, this disturbing action will always produce the same effects when the two planets occupy the same positions in respect of the sun, which happens after a certain determinate period of time depending on their relative motions. The relative positions of the planets are technically called their configurations, and are consequently periodic; because after a certain determinate time, the same configurations are again restored. Jupiter, for example, performs his sidereal revolution in about twelve years, and Saturn in nearly twenty years; and at the end of sixty years, therefore, these two planets will be again found nearly at the same points of their orbits, and have the same situation relatively to the sun. If they occupied the same positions exactly, and were disturbed by the influence of no other body, the circle of changes depending on their configuration would then be complete. But the orbits themselves are not fixed; on the contrary, they undergo a continual variation, both in respect of form and position. The transverse axes are slowly revolving on the planes of the orbits; the eccentricities are gradually changing; so also are the inclinations and the position of the nodes relatively to an immovable plane. Now these variations in the forms and positions of

[21] See note 14, *Bk. III, Chap. II.*
[22] d'Alembert, Jean Le Rond, (1717-1783), a pioneer in the study of partial differential equations.
[23] See note 6, *Bk. I, Chap. II.*
[24] See note 16, *Preliminary Dissertation.*
[25] See note 4, *Foreword to the Second Edition.*

the orbits, give rise to a second class of inequalities, depending not on the configuration of the planets, but on the relative positions of the major axes, or the configurations of the orbits. Like those of the former class, they are periodic; but their periods are vastly longer, as the revolutions on which they depend are incomparably longer than the revolutions of the planet. The axis major of the earth's orbit accomplishes a revolution in 109,770 years, and that of Jupiter in 197,561 years; hence an idea may be formed of the time required to complete the cycle of inequalities depending on such slow motions. On this account the inequalities depending on the positions of the orbits are called *secular*. Some of them were detected by comparing observations made at distant epochs; but in general they escape observation, by the slowness of their evolutions.

The particular analytical procedure by which the computation of the various inequalities is brought within the power of the calculus, is particularly deserving of attention. It is founded on the supposition that the elements of a planet's orbit are constantly varying; or that the planet only continues to describe the same ellipse during an infinitely small portion of time. The arbitrary quantities , therefore, which enter into the integrated equations of motion, and represent the elements of the elliptic orbits, are considered as variable; their variations being expressed in terms of the partial differentials of the perturbing force. The germ of this method, as of many others of the first importance in analysis, is due to Euler; but the complete theory properly belongs to Lagrange, by whom, after many successive modifications and improvements, it was reduced to its last degree of elegance and generality. As it now stands, the theory of the planetary perturbations is reduced to the integration of a system of linear equations, in which the differential of each elliptic element is expressed by the partial differentials of the perturbing force, multiplied by the element of the time. The great advantage of the method consists in its affording the means of exhibiting, under a single point of view, all the effects arising from the reciprocal actions of the planets, whether secular or periodic, either in their motions of translation or rotation; as well as the derangements that would be produced by a resisting medium, or any other disturbing cause whatever. When the combined action of a great number of forces is to be calculated, there is no more efficient method that this in the whole range of analysis.

At the time the first two volumes of the *Mécanique Céleste* made their appearance, the theory of the variation of arbitrary constants had not reached the degree of perfection it has since attained. In the second volume, Laplace gave expressions for the variations of the eccentricity, the inclination of the orbit, and the longitude of the nodes only; the expressions for the variations of the of the remaining two elements–namely, the longitudes of the perihelion and epoch–are given in the supplement to the third volume. By partially adopting the method of Lagrange, and taking advantage of the more recent discoveries of Poisson,[26] who also has essentially contributed to the perfection of this theory, Pontécoulant has succeeded in rendering the subject greatly more perspicuous; and Mrs. Somerville has judiciously availed herself of the labours of Pontécoulant. The investigation commences with the demonstration of a formula due to Lagrange, for expressing the partial differential of one of the elliptic elements of an orbit, in a linear function of the infinitely small variations of that element, multiplied by certain combinations of the partial differential of the perturbing force, taken with respect to the rectangular co-ordinates of the troubled body. The formula is next applied to the variation of the differential elements in succession, without laying down any restricted hypothesis as to the magnitude of the eccentricities and inclinations; after which, the modifications are pointed out which the expressions receive in consequence of the smallness of the eccentricities and

[26] See note 1, *Bk. I, Chap. VI.*

inclinations of the actual orbits of the planets. All these expressions involve the differential of the function which expresses the perturbing force; the expansion of which into a series, and the determination of the coefficients of its several terms, occupy the remainder of the fifth chapter. This development depends ultimately on that of the irrational factor $(a^2 + 2ab\cos A + b^2)$ –into a series of cosines of the multiples of the angle A;–a subject which seems first to have engaged the attention of Euler in his *Memoir on the Inequalities of Jupiter and Saturn*, and which, on account of its great importance in the theory of the planetary perturbations, has been frequently treated by mathematicians. In certain cases–that is to say, when the ratio of the distances of the disturbed and disturbing planet is very small–the expanded series converges with sufficient rapidity; but when that ratio approached nearer to unity, as happens in the case of Venus and the Earth, or Jupiter and Saturn, the series converges slowly, and it becomes necessary to have recourse to particular artifices in order to obtain the values of its different terms. The labour of computation is, however, greatly facilitated in consequence of a curious relation discovered by Euler to subsist among the terms; which is such, that when any two of them are found, all the others can be determined in a function of these two; hence the difficulty is confined to the determination of the first two terms, and this has been effected in a great many different ways. Mrs. Somerville has taken the development exactly as it is given by Pontécoulant; and though in principle the same, it has the merit of being considerably simpler than that of Laplace.

It is in the development of the function which expressed the perturbing force, that the two distinct sets of terms arise which respectively represent the periodic and secular inequalities. One part of the expanded function consists of terms having for their argument the sines or cosines of the mean motion and its multiples; while the other terms are entirely independent of the mean motion, being merely functions of the elements of the orbits, and their combinations. The determination of this last set of terms is of the utmost consequence in theoretical astronomy; for if they were susceptible of indefinite increase with the time, the forms of the orbits and the periods of revolution would, in the course of ages, be entirely altered, and the stability of the planetary system destroyed. To this subject Mrs. Somerville addresses herself in the sixth chapter, and examines in detail the terms, independent of the time, which are contained in the variations of each of the elliptic elements.

Of all the elements of a planet's orbit, the axis major is that of which the variations are the most important, on account of the relation subsisting between the mean distance and the mean motion. Accordingly, the efforts of geometers have been particularly directed to this subject, and their successive discoveries distinctly mark the progress of analysis. The first Memoir which Laplace presented to the Academy of Sciences, in 1773, contained the very important discovery that the mean distances and mean motions include no secular inequality, or term increasing with the time, when the approximations are carried to the third powers of the eccentricities and inclinations, and regard is had only to the simple powers of the disturbing force. Stimulated by this remarkable result, Lagrange undertook the investigation of the same subject, and demonstrated, in the Berlin Memoirs of 1776, that on having regard only to the first power of the disturbing force, the differential expression of the major axis can include no term increasing indefinitely with the time, to whatever order of terms the approximations may be carried with regard to the eccentricities and inclination; unless indeed there should exist a commensurable ratio between the mean motions of the disturbed and disturbing planet. Such a condition, however, does not exist in the planetary theory; and therefore the greater axes and mean motions are only susceptible of periodic inequalities depending on the configurations of the planets, and of which the limits may be assigned. But although this approximation is sufficient in regard to the other elements, it is necessary in the case of the major axis to proceed a step farther, and to have regard to the terms depending on the second powers of the disturbing force; because such terms, though multiplied by the squares of the masses, being expressed by second differentials, may acquire in the double integration very small divisors; in consequence of which their values become comparable to those which, in the case of the other elements, depend on the first powers of the masses, and are given by a single integration. Laplace showed, in the sixth book of the *Mécanique Céleste*, that the mean motions of Jupiter and Saturn are not altered by their great inequalities, even when regard is had to the squares of the disturbing forces; but Poisson had the merit of first demonstrating generally, that the terms depending on the squares and products of the perturbing force can introduce no secular inequalities into the expressions of the greater axes or mean motions. This was an important and necessary extension of the great discovery of Lagrange. Mrs. Somerville refers us to a recent paper in the Philosophical Transactions, in which the demonstration of the permanency of the mean motions is said to be carried to *all* the powers of the disturbing masses. This result, if well verified, must be of great interest in regard to analysis, though it is fortunately of no importance to astronomy.

From the consideration of the major axes, Mrs. Somerville passes to that of the other elements of the orbits. In respect of these elements the stability of the system is equally assured as in the case of the mean motions. They are not, indeed, like the mean motions, exempted from the influence of secular perturbations; but their inequalities, though independent of the configurations of the orbits, are nevertheless subject to the law of periodicity, and can never exceed certain small limits. These consequences result from certain relations that subsist among the elements of all the orbits, and limit the increase of their variations. Thus the eccentricities, though subject to slow variations, can never entirely disappear, but must always continue to vibrate about a mean state; subject to the remarkable condition, that 'the sum of the squares of the eccentricities, multiplied by the masses of all the bodies of the system, and by the square roots of the axes of the orbits, remains always the same constant magnitude.' The same condition must be fulfilled with respect to the inclinations of the orbits to a fixed plane. The variations of the longitude of the epoch are extremely important on account of their influence on the mean longitudes of the planets. Theory show that they exist; but they are altogether insensible to observation in the case of the planets. Even in the case of Jupiter and Saturn, the two planets whose mutual perturbations are the most remarkable, the variation of this element amounts to less than the 60^{th} part of a second in a century, and requires no less than 70,414 years to complete its period. The motions of the perihelia are the only elements to the variations of which no limit has yet been assigned; but it is certain that they must always continue to vary with extreme slowness, as they do at the present time.

After having discussed those terms in the variations of the different elements which are independent of the mean motions, and give the secular inequalities, the next step is to return to those which depend on the sines and cosines of the mean motion, and give the periodic inequalities. These being of a simpler kind, had been for the greater part determined by peculiar considerations, before the general method of deducing the inequalities of both kinds from the variations of the elliptic elements had been discovered by Lagrange; but it is of great importance to the progress of the science, that, as all the inequalities are occasioned by the same physical cause, they should also be all comprehended in the same general analysis, and deduced by uniform methods. In reference, however, to the ultimate object of astronomy, that of determining the positions of the planets in space, it is not material to know particularly the alterations which each of the elements of an orbit undergoes; for the periodic variations always remain very small, and have only a transient effect on the orbits. It is sufficient to know the amount of their combined influence on the places of the planets, or the three polar co-ordinates by means of which their positions are fixed, viz. the distances, longitudes and latitudes. Lagrange's method of obtaining these elements in the disturbed orbit, is at once simple and elegant. In the case of elliptic orbits, the radius vector, the longitude and latitude are expressed by series which proceed according to the ascending powers of the eccentricities and inclinations; in these series, therefore, he substitutes for the elliptic elements the same elements corrected for the periodic and secular variations found from the general formulae; and thus obtains correct expressions for the radius vector, the longitude and latitude of the troubled orbit. In this manner the position of the planets, at every instant, may be computed by known rules. Mrs. Somerville has given the development of this method, in the seventh and eighth chapters, from Pontécoulant. The original may be found in Lagrange's Memoirs on the periodic variations of the *Motions of the Planets*, published in the *Berlin Memoir* for 1783.

The method of Lagrange here referred to, though extremely ingenious and important in respect of analysis, is not that which leads most directly to the determination of the periodic variations. When the secular inequalities are left out of view, and particularly when it is not required to extend the approximations beyond the simple powers of the eccentricities and inclinations, the easiest method is to deduce the periodic inequalities directly from the differential equations of the orbit; for in this way we arrive at once at the variations of the longitude, latitude, and radius vector. This method is given in the ninth chapter, and the approximations are carried to the third powers of the eccentricities and inclinations.

Before proceeding farther in this analysis, we cannot avoid expressing our regret that Mrs. Somerville has not given any preliminary explanation of the peculiarities of the analytical methods she exposes, or the principles on which they are founded. In the eighth chapter, the variations of the polar co-ordinates of a planet are given according to Lagrange's method. In the following chapter, a 'second method of finding the perturbations of a planet in longitude, latitude, and distance,' is announced; and the reader, without being informed in what respect the first method is insufficient, or how the second differs from the first, or of any circumstances that can render a second method necessary, is hurried into the midst of an intricate investigation, the uses and object of which he is left to infer, as well as he can, when he arrives at the end of the calculus. This deficiency, or rather entire absence of all explanations or discussion of the peculiarities of the different methods and analytical processes made use of, is the greatest defect of the work, and cannot fail to render its perusal more discouraging and far less instructive than it ought to be, considering the perspicuous arrangement of the subjects. Half the difficulty of a geometrical investigation may be said to be overcome when a distinct perception has been acquired of the object to be attained, and the route to be followed.

Among the terms of the series which expresses the mutual perturbations of two planets, there are some into which the difference between certain multiples of the mean motions enters by integration as a divisor; and if it happens that this difference is very small, or that the mean motions of the two planets are nearly commensurable, such terms, though minute in themselves, may acquire, in consequence of the smallness of their divisors, very considerable values. The mean motions of no two planets in the solar system are exactly commensurable; but those of Jupiter and Saturn approach so nearly to commensurability, that part of the terms belonging to the third and fourth powers of the eccentricities and inclinations, have, in consequence, appreciable values. In the computation of the inequalities of these two planets, therefore, it becomes necessary to push the approximations so far as to include the terms of the fourth order in respect of the eccentricities and inclinations; and likewise to retain those that depend on the square of the perturbing force. On this account the theory of Jupiter and Saturn forms a peculiar, and as it were, a supplementary case of the problem of the planetary perturbations, the solution of which long baffled the efforts of the first mathematicians. The inequalities of their mean motions are so considerable that they had been discovered by Halley[27] from a comparison of observations. Euler had failed in the attempt to connect them with theory; Lagrange only proved that they did not belong to the class of secular inequalities; it was, therefore, for some time supposed that Jupiter and Saturn form an exception to the general principle of the invariability of the mean motions. At length Laplace, with that characteristic sagacity which enabled him on so many other occasions to detect the expression of a physical fact among the mazes of an intricate calculus, discovered the cause of the anomaly, in the near commensurability of the mean motions. The long period of the inequalities in question, namely, 929 years, might easily cause them to be mistaken for secular inequalities. The discovery of their true source and amount, which was necessary to the perfection of theory, has had an important influence on the astronomical tables; the errors of which, in respect of Jupiter and Saturn, hardly now exceed 13″, whereas, not more than twenty years ago, they amounted to a hundred times that quantity.

The theory of Jupiter and Saturn is given in the tenth chapter. We may remark that the computation of the terms depending on the square of the perturbing force is extremely laborious, and that the greatest mathematicians of the present day are not agreed with respect to their exact numerical values.

In the three following chapters Mrs. Somerville discusses the inequalities depending on the ellipticity of the sun, and the action of the satellites, and the data requisite for computing the motions of the planets. The fourteenth chapter is of a very miscellaneous nature,–including the numerical values of the perturbations of Jupiter; remarks on the transits of Venus and Mercury; the perturbations of the Earth, Mars, and the other planets; remarks on the atmospheres of the planets; on the spots and motion of the sun; on the zodiacal light; the influence of the fixed stars in disturbing the system; and the construction and correction of the astronomical tables. This concludes the planetary theory.

The third book belongs to the lunar theory. The problem of finding the lunar perturbations is essentially a problem of the same nature with that of finding the perturbation of a planet; but on account of the great eccentricity of the lunar orbit, and the powerful attraction of the sun, which is in this case the disturbing body, it is necessary to carry the approximations farther than is generally required in the planetary theory. The terms depending on the square of the perturbing force, are not only sensible, but they even double the motion of the lunar perigee; and in computing several of the inequalities, it is necessary to include the fourth, and even the fifth powers of the eccentricity and inclination. It would be extremely difficult to convey any idea of the method employed by Laplace to determine the numerous and complicated inequalities of the moon, without entering into the details of analysis. The lunar theory is certainly the most remarkable portion of the *Mécanique Céleste*, whether we regard it as a mere problem of analysis, or in reference to its important applications in practical astronomy. It unites in itself, says Laplace, all that can give value to discovery–grandeur and utility in the object, fecundity of results, and the merit that attaches to the conquest of great difficulties.

The most remarkable of the lunar inequalities are periodic, and occasioned by the action of the sun; and the difficulty of determining them is chiefly owing to the slow convergence of the

[27] See note 55, *Preliminary Dissertation.*

series. But besides the perturbations which the moon sustains directly from the sun and the planets, her motions are greatly complicated, from the circumstances of her not moving round a fixed centre like the planets, but round a body which is itself in motion, and the elements of whose orbit partake of the general disturbance. All the inequalities that effect the motion of the earth are attended with corresponding effects on the motion of the moon, and are even more sensible in proportion as the moon is further from the common centre of gravity. The variation of the eccentricity of the earth's orbit, for example, introduces secular inequalities into each of the three lunar co-ordinates, namely, the parallax, latitude, and longitude. On the parallax, however, its influence is so small as to be insensible to observation. On the longitude its effects are perceptible, as it occasions that acceleration of the moon's mean motion which had been detected by a comparison of ancient with modern eclipses, and of which the physical cause was only discovered by the powerful analysis of Laplace. On the latitude its effects are manifested in a retrograde motion of the nodes. It effects in a still more sensible degree the motion of the perigee, which becomes slower from century to century. These three inequalities are related to one another in such a manner, that if the variation of the mean motion be called 1, the variation of the nodes is .734, and of the perigee 3, very nearly. The acceleration of the mean motion amounts to $10''.2$ in a century: and it is remarkable, that while the mean motion continues to be accelerated, the motion of the perigee and nodes is retarded.

When these three inequalities shall have developed in the course of ages, and their values determined by a long series of observations, they will lead to a more accurate knowledge than we yet possess of the extent and period of variation of the eccentricity of the terrestrial orbit. This is occasioned principally by the disturbing influence of Mars and Venus; hence if the variations of the earth's eccentricity were correctly known, we should be able to assign an accurate value of the masses of these two planets. It is a striking instance of the intimate dependence of all the phenomena of the planetary system on one another, that by merely observing the moon the astronomer is enabled to determine the quantity of matter in Mars and Venus; and yet science reveals many more wonderful secrets.

Another source of inequality peculiar to the lunar motions, is the non-sphericity of the earth. On account of the moon's proximity, the compression of the earth has a sensible influence on her motions, and occasions two inequalities, to compute which it is necessary to have recourse to the theory of the attraction of spheroids. One of them has for its argument the longitude of the moon's node; the other is an inequality of the motion in latitude, depending on the moon's mean longitude. These two inequalities, determined from a great number of observations, concur in giving an ellipticity of $\frac{1}{305}$ nearly, agreeing in a surprising manner with the results obtained from the measurement of terrestrial degrees, and observations of the pendulum. In all probability they give the most accurate determination of the figure of the earth, being independent of local disturbance. From this result we are enabled to deduce some inferences respecting the interior constitution of the earth. It was demonstrated by Newton that a fluid mass of homogeneous matter, revolving with the same velocity as the earth, would acquire a compression of $\frac{1}{230}$; hence the earth is not homogeneous, but increases in density from the surface towards the centre. Again, if any difference exists in the form or constitution of the two terrestrial hemispheres, it would give rise to a lunar inequality proportional to the cosine of the longitude of the perigee, augmented by twice the longitude of the moon's orbit. Observation has failed in detecting any inequality of this sort; there is consequently no sensible difference of form or constitution in the two hemispheres. It is also to the attraction of the earth that we must refer the rigorous equality that subsists between the mean motions of rotation and revolution of the moon, in virtue of which the same hemisphere is constantly turned towards the earth.

Laplace has likewise investigated the effect that would be produced on the lunar motions by the resistance of a gaseous medium of great rarity occupying the planetary spaces; the existence of which many phenomena, particularly the propagation of light, render extremely probable. The immediate effect of the resistance of a medium on a planet would be to diminish the tangential velocity, and consequently the centrifugal force. This would allow the action of gravity to draw the moon nearer the earth, and cause an acceleration of her angular velocity or mean motion. A similar effect would be produced on the earth, and the other planets; but the effect on the moon would be a hundred times more sensible than on the earth. But the observed acceleration of the moon is perfectly explained from the theory of attraction; and, therefore, if the regions of space are filled with an elastic medium, it must be so rare as to offer no resistance to the planets or satellites. Additional interest has lately been given to this question, from the circumstance that Encke's comet[28] seems to have an accelerated motion, which it is difficult to explain on any other hypothesis; but this body must be observed in many of its future revolutions, before a conclusion of so much importance can be considered as well established. In the mean time, it has been computed by Mazotti, that, if the phenomenon in question is caused by the resistance of an ethereal medium, its rarity must be 360,000 millions of times greater than that of atmospheric air.

Before the true cause of the moon's acceleration was discovered, it had been suggested that the phenomenon might be occasioned by the successive transmission of gravity from the earth to the moon. Laplace also investigated the consequences of this hypothesis, and found that, in order to produce the observed acceleration, the velocity with which gravity is transmitted must be 42 millions of times greater than that of light. But neither the resistance of an ether, not the successive transmission of gravity, can produce the secular variation of the lunar nodes and perigee; these two inequalities consequently afford of themselves the most convincing proof, that all the celestial motions are performed in obedience solely to the Newtonian law of gravity.

'*It is evident*,' says Mrs. Somerville, '*that the lunar motions can be attributed to no other cause than the gravitation of matter: of which the concurring proofs are the motion of the lunar perigee and nodes; the mass of the moon; the magnitude and compression of the earth; the parallax of the sun and moon, and consequently the magnitude of the system; the ratio of the sun's action to that of the moon, and the various secular and periodic inequalities in the moon's motions, every one of which is determined by analysis on the hypothesis of matter attracting inversely as the square of the distance; and the results thus obtained, corroborated by observation, leave not a doubt that the whole obey the law of gravitation. Thus the moon is, of all the heavenly bodies, the best adapted to establish the universal influence of this law of nature; and from the intricacy of her motions, we may form some idea of the powers of analysis,-that marvelous instrument, by the aid of which so complicated a theory has been unraveled*'–

The satellites of Jupiter furnish another case of the problem of attraction, having also its peculiar difficulties. Ever since the discovery of these bodies by Galileo,[29] they have been objects of great interest both to the practical astronomer and to the geometer; to the former, on account of their connexion with the problem of the longitude; and to the latter, on account of the

[28] See *Art. 618, Bk. II.*
[29] See note 1, *Introduction.*

difficulty of submitting their intricate motions to analysis. They exhibit, as it were, a miniature representation of the solar system, in which, by reason of the promptitude of their revolution, all the inequalities arising from their reciprocal action pass through their cycle of changes in comparatively short periods of time. If the figure of the primary planet could be neglected, the problem would be one of five bodies; but the ellipticity of Jupiter has a very powerful influence on the motions of the satellites; and for this reason it becomes necessary to have regard not only to his figure, but also to the inclination of his equator and ecliptic, and the position of his nodes. hence the problem becomes extremely complicated, and is embarrassed with the details of numerous computations. Lagrange was the first who ventured to grapple with it in all its difficulty; but his solution, as is remarked by Delambre,[30] though a wonderful display of his power of analysis, contributed little or nothing towards the amelioration of the tables of the satellites; and it is remained for Laplace to complete the theory, and to substitute formulae rigorously deduced from the differential equations of motion for the empirical equations from which the eclipses has been computed. This theory is contained in the eighth book of the *Mécanique Céleste*, and may be regarded as nearly perfect. All the inequalities have corresponding expressions in the theories of the moon and planets. the mean motion of the first added to twice the mean motion of the third, is rigorously equal to three times the mean motion of the second; and it is extremely remarkable that the secular inequalities of their mean motions, and their motions of rotation, are also subject to the same law.

Before the solution of the problem of the satellites can be rendered of any avail to astronomy, it is necessary to assign values from observation to the quantities which analysis leaves indeterminate. These are, the six elliptic elements of each orbit, the mass of each of the satellites, the ellipticity of Jupiter, the inclination of his equator to his ecliptic, and the position of his nodes–in all thirty-one. This most laborious task was undertaken by Delambre, who computed all the recorded eclipses, amounting to six thousand; and the tables, subsequently improved by Bouvard, now give the positions of the satellites, and the time of the eclipses, more accurately perhaps than direct observation. Great importance was formerly attached to the theory of the satellites, on account of the easy means their eclipses furnish of determining the difference of terrestrial meridians; but since the lunar tables have been brought in a manner to perfection, it has lost much of its interest in practical astronomy. The difficulty of determining the exact instant at which a satellite enters the shadow of Jupiter is such, that it is not rare to find two observations of the same eclipse differing by thirty seconds of time, even in the case of the first satellite, whose motions are by far the most rapid. Various causes concur to produce this uncertainty; among which may be mentioned the ill-defined contour of the shadow of the planet, which renders a satellite longer visible to a good eye, or in a good telescope. Much depends also on the position of Jupiter with respect to the sun, or of the satellite in respect of Jupiter; and perhaps something also to a difference in the physical state of the surface of the satellite, which may render one side of it better fitted to reflect the sun's light than the other. These uncertainties disappear when a great number of observations can be combined; but when one or two observations only can be procured, the certainty of the result is not to be put in comparison with that of one furnished by the lunar occultation of a fixed star.

Mrs. Somerville has treated the subject of the satellites, nearly in the same manner as the lunar theory. The method of Laplace is closely followed, and the equations for the greater part transcribed without alteration. The arrangement is in some respects different; but this is a matter of inferior moment; and as the explanations of the analytical operations are somewhat

[30] See note 54, *Preliminary Dissertation.*

compressed, we apprehend a student would have still more difficulty in mastering this intricate theory from Mrs. Somerville's exposition, from that of the *Mécanique Céleste* itself.

With regard to the satellites of Saturn and Uranus, the difficulty of observing them is so great, that it is only in some instances that their mean distances and periodic times have been ascertained. Their theory can never be of any practical use; but it is interesting to trace the effects of gravitation among those remote and minute bodies, whose existence is but recently known. and which are only discernible in the most powerful telescopes.

The subjects which occupy the remainder of the *Mécanique Céleste* are of great extent and importance; but, in the progress of analysis, other methods have been discovered, by which they may be treated with greater advantage. For example, the methods which Laplace has given for determining the orbits of comets, though stamped with all the characteristics of his powerful mind, is not the most convenient in practice; because it assumes the numerical values of the first and second differential coefficients of the longitude and latitude as functions of the time to be exactly known from observation. The determination of these quantities, however, is often a matter of much difficulty; and when only a few observations can be obtained, as happens in the greater number of cases, cannot be relied on. In other respects, the theory of comets is imperfect; for mathematicians have not been able to represent the perturbations these bodies sustain from the planets by formulae which embrace an indefinite number of revolutions. This may still be considered a *desideratum* in physical astronomy,−the more to be regretted, as the interest attaching to the cometary theory has been vastly increased by recent discoveries. The mathematical theory of the figure of the earth has taken an almost entirely new form in the hands of Ivory and Poisson; so that this portion of the *Mécanique Céleste* is now interesting only as an exercise of analysis. The theory of the tides requires to be entirely recast. Mrs. Somerville has, therefore, wisely selected that department of Physical Astronomy which, in consequence of the degree of perfection it has attained, is most likely to retain its present form. We take leave of her work with the renewed expression of the admiration we have experienced in perusing the proofs which it so strongly affords of high and rare attainments; and of gratitude for what she has done with a view to diffuse the knowledge of those sublime truths which mathematical analysis has so largely revealed.

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THE close of the last century witnessed the successful termination of that great work, commenced by Newton, and prosecuted by a long succession of illustrious mathematicians, by which the movements of the planetary system were reduced under the expression of dynamical laws, and their past and future positions with respect to their common centre and to each other, rendered matter of strict calculation. A wonderful result, which will forever form a principal epoch in the history of mankind, was at length arrived at in the announcement of the fact, that a brief and simple sentence, intelligible to a child of ten years of age, accompanied with a few determinate numbers capable of being written down on half a sheet of paper, comprehends within its meaning the history of all the complicated movements of our globe, and the mighty system to which it belongs—the mazy and mystic dance of the planets and their satellites—'cycle on epicycle, orb on orb'—from the earliest ages of which we have any record, nay, beyond all limits of human tradition, even to the remotest period to which speculation can carry us forward into futurity. By the announcement of this law and the establishment of these data, an indefinite succession of events is thus combined into one great fact, and may be considered as a single feature in creation, independent of the lapse of time, and registered only in the unproductive annals of eternity.

In the course of the investigations which have terminated in this result, another fact, of a no less high and general order, has come to light of which Newton could have formed no anticipation, that, namely, of the stability of our system, and the periodic nature and restricted limits of its fluctuations, which preclude the possibility of such deviations from a mean or average state as may lead to the subversion of any essential feature of that happily balanced order which we observe at present to subsist in it. This noble theorem forms a beautiful and animated comment on the cold and abstract announcement of the general law of gravitation. A thousand systems might have been formed of which the motions would, for a time have been regular and orderly enough, but which would either have ended in a collision of parts subversive of the original conditions, or would pass through a succession of phases or states, endless in variety, among which some would be found no less incompatible with life than such collisions themselves—whether from extreme remoteness or proximity of the source of light and heat, or from violent and sudden alternations of its influence—or in which, at all events, that beautiful and regular succession of seasons—that 'grateful vicissitude' we admire and enjoy, and those orderly and established returns of phenomena which afford at once the opportunity and the inducement to trace their laws, would have been wanting; while in their place might have reigned a succession of changes reducible to no apparent rule; variety without progressive improvement; years of unequal length and seasons of capricious temperature; planets and moons of portentous size and aspect, glaring and disappearing at uncertain intervals, and every part of the system wearing the appearance of anarchy, though, in fact, obeying, to the letter, the same general law of gravitation, which must yet have for ever remained unknown to its inhabitants.

[31] See note 64, *Preliminary Dissertation.*

Among infinite systems equally possible, such, we have no reason to doubt, might exist–but our own is not, nor can it ever, in its own natural progress, pass into such a one. In the choice of its arbitrary constants, (to use the language of geometers,) in the establishment of the relations of magnitude, speed, and distance of its parts, such a case is expressly provided against. In the circulation of its members all in one direction–in the moderate amount of the eccentricities and inclinations of all the planetary orbits, and the extremely small ones of those of its more important bodies, but more especially in the mode in which the general system is broken up into several subordinate ones, and in the individual attachment and allegiance of each member to its immediate superior, we must look to the safeguards of this glorious arrangement.

This last mentioned condition may require some illustration. Had the Earth and Mars, for instance, formed a binary combination separated by an interval no greater than the moon's actual distance from the earth, there is no doubt that such a double planet might have continued to circulate round the sun nearly as the earth and moon do at present. But with such a combination the moon could not have coexisted, without a complete breach of her law of regular periodicity. Its path would alternately by one and the other of its great equipollent centres, whichever, for the moment, occupied the most advantageous position; and should its primitive velocity be so adjusted that it could neither throw itself to a sufficient distance from both to escape from the influential attraction of either, and become a separate planet, nor attach itself so closely to one of them as to be carried about it as a mere appendage, it must continue to wind, for ever, an intricate and sinuous course around and between them, in which occasional collision with one or other would, by no impossible or improbable contingency, afford a tragic epoch in the history of so ill-adjusted a system.

It is, moreover, well worthy of remark, that the mode in which the stability of our system is accomplished is by no nice mathematical adjustment of proportions,–no equilibrated system of counterpoises satisfying an exact equation, and which the slightest deviation in any of the data from its strict geometrical proportion would annul. Such adjustments, it is true, are not incompatible with the law of gravitation, even in a system composed of several bodies. Geometers have demonstrated, for example, that three or even more bodies, exactly adjusted in their weights and distances, and in the velocities and directions in their motions at any one instant, might forever describe conic sections about each other, and about their common centre of gravity. But without supposing any such adjustment of the weights and distances of the members of a system subjected to the law of gravitation, and taking them as they are actually in our own, there is yet another supposition in which the absence of secular perturbation might have been ensured,–that, namely, in which the planetary motions should be performed all in one plane, and all in perfect circles about the sun,–realizing, in fact, the old Aristotelian notion of celestial movements, all which he considered to be of necessity exactly circular. We do not remember to have seen any mention made of the possibility of this case. It follows, however, immediately, from the general proposition demonstrated by Lagrange and Laplace, which establishes an invariable relation among the eccentricities of any number of perturbed orbits; viz. that the sum of the squares of all the eccentricities, each multiplied by an invariable coefficient, is itself invariable, and subject to no change by the mutual action of the parts of the system. For it is evident, that had the orbits been all originally circular, or if at any one instant of time each of the eccentricities were, by some external agency, destroyed, so as to render these orbits at once all circles, after which the system should be abandoned to its own reactions, the sum in question would also vanish at that instant, and therefore at every subsequent instant, which would be impossible, (since none of the coefficients are negative,) unless each several eccentricity were to remain for ever evanescent *per se*, or each several orbit a perfect circle.

If we depart from the law of gravitation, and inquire whether, under other conceivable laws of central force, a system might not exist essentially and mathematically free from the possibility of perturbation, and in which every movement should be performed in undeviating orbits and unalterable periods, we have not far to search. Newton has himself demonstrated, in his *Principia*, or at least it follows almost immediately from the 89th proposition of his first book and its corollary, that this wonderful property belongs to a law of attractive force in the *direct* proportion of the distance; and, however extravagant such a supposition may appear, if we consent to entertain it as a mere mathematical speculation, it is impossible not to be struck with the simplicity and harmony which would obtain in the motions of a system so constituted. Whatever might be the number, magnitudes, figures, or distances of the bodies composing an universe under the dominion of such a law—in whatever planes they might move, and in whatever directions their motions might be performed—each several body would describe about the common centre of gravity of the whole, a perfect ellipse; and all of them, great and small, near and remote, would execute their revolutions in one common period, so that, at the end of every such period, or *annus magnus*, of the system, all its parts would be exactly re-established in their original positions, whence they would set out afresh, to run the same unvarying round forever.

We may please ourselves with such speculations, and enjoy the beauty and harmony of their results, in the very same spirit with which we rejoice in the contemplation of an elegant geometrical truth, or a property of numbers, without presumptuously encroaching on the province of creative wisdom, which alone can judge of what is really in harmonious relation with its own designs. The stability of our actual system, however, rests on a basis far more refined, and far more curiously elaborate. It depends, as we have before observed, on no nice adjustments of quantity, speed and distance. The masses of the planets, and the constants of their motions, might all be changed from what they are, (within certain limits,) yet the same tendency to self-destruction in the *deviations* of the system from a mean state, would still subsist. The actual form of their orbits are not ellipses, but spirals of excessive intricacy, which never return into themselves; yet this intricacy has its laws, which distinguish it from confusion, and its limits, which preserve it from degenerating into anarchy. It is in the conservation of the principle of order in the midst of perplexity—in this ultimate compensation, brought about by the continued action of causes, which appear at first sight pregnant only with subversion and decay—that we trace the master workman, with whom the darkness is even as the light.

This momentous result has been brought to light slowly, and, as it were, piecemeal. The individual propositions of which it consists have presented themselves singly, and at considerable intervals of time, like the buried relics of some of those gigantic animals which geologists speak of, each, as it emerged, becoming a fresh object of wonder and admiration, proportioned to the labour of its extraction, as well as to its intrinsic importance; and those feelings have at length been carried to their climax by finding the disjointed members fit together, and unite into a regular and compact fabric.

It is our continental neighbours, but more especially to the geometers of France, that we owe the disclosure of this magnificent truth: Britain has taken little share in the inquiry. As if content with the glory of originating it, and dazzled and spellbound by the first great achievement of Newton, his countrymen, with few and small exceptions, have stood aloof from the great work of pursuing into its remote details and general principle established by him. We are far from being disposed to attribute this remarkable supineness to the prevalence of any of the meaner or more malignant feelings of national pride, prejudice, or jealousy. Some irritation and distaste for the continental improvements might be, and no doubt were, engendered, and, to a certain extent, continued by the controversies which excited so lively a sensation among the contemporaries of Newton; but, on the other hand, it could not have been, at first, reasonably presumed, (what proved afterwards to have been really the case,) that the applicability of Newton's mode of investigation should terminate almost at the very point where he himself desisted from applying it–still less that algebraic processes, which were regarded by him as mere auxiliaries to geometrical construction and demonstration, should be destined to acquire such strength and consistency as to supercede all others, and leave them on record only as scientific curiosities. It is rather to the barrier thrown by our insular situation in the way of frequent personal communication between our mathematicians and those abroad, to the want of a widely diffused knowledge of the continental languages, and to the consequent indifference in the reading part of the public as to the direction which thought was taking, in the loftier regions of its range, in other lands than our own, that we are inclined to refer what cannot but appear an extraordinary defect of sympathy in so exciting a course of discovery. Much, too, must be attributed to that easy complacency with which human nature is too apt to regard progress already made as all that can be made;–which dwells with admiring and grateful satisfaction on achievements performed and laurels won, while it neglects to body forth the possibilities of a yet richer and more glorious future;–suffers a short breathing time to become prolonged into a state of languor and indifference; and consigns to other and fresher aspirants the toil and the reward of penetrating farther into those thorny and entangled thickets of unexplored research which bound our actual horizon, and by the force of habit and repose come at length to hedge in our thoughts and wishes.

Whatever might be the causes however, it will hardly be denied by any one versed in this kind of reading, that the last twenty years of the eighteenth century were not more remarkable for the triumphs of both the pure and applied mathematics abroad, than for their decline, and, indeed, all but total extinction, at home. From the publication of Waring's[32] profound but cumbrous *Meditationes Algebraicae*, and Landen's[33] researches on the motions of solids, and his remarkable discovery of the rectification of the hyperbola by two ellipses, we may search our libraries in vain for investigations of the slightest moment in the higher analysis, or, indeed, for any evidence of its abstruser parts being so much as known to our mathematical writers. While the academical collections of Turin, Paris, Berlin, and Petersburg, were teeming with the richest treasures of the analytic art, poured forth with unexampled profusion, our own presented the melancholy contrast of entire silence on all the great questions which were then agitating the mathematical world,–a blank, in short, which the respectable names of Vince and Hellins only served to render more conspicuous.

It was with the commencement of the present century that a sense of our deficiencies, and of the astonishing and disreputable distance to which we had fallen behind the general progress of mathematical knowledge in all its branches, began to make itself felt; but to remedy the evil was more difficult than to discover its existence. Great bodies move slowly. It requires time, where national tastes and habits are concerned, to turn the current of thought out of its smooth-worn track into untried and, at first, abrupter channels; and, the means were wanting. A total deficiency of all elementary books in our own language in which the modern improvements could be studied, precluded beginners from obtaining any glimpse beyond the narrow circle in

[32] Waring, Edward, (1734-1798).
[33] Landen, John, (1719-1790).

which their teachers had revolved. The student is guided in his early choice of books by sanction and by usage. He may not, without hazard, venture to chalk out for himself a course of reading unusual and remote; and, rejecting the writers of his own country, choose foreigners for his instructors. To come to such a resolution presupposes a discrimination and a preference which is incompatible with entire unacquaintance with his subjects. It was only, therefore, when, although well instructed and perfect in the usual routine, he found himself arrested at the very first page of any of the elaborate works of the foreign geometers which chance might throw into his hands, that he could acquire the painful but necessary conviction of having all to begin afresh—much even to unlearn;—to forget habits—to change notations—to abandon points of view which had grown familiar—and, in short, put himself once more to school.

The late Professor Woodhouse seems to have been among the first of our countrymen who experienced this inward conviction, with its natural concomitant, the desire to propagate forward to other minds the rising impulse of our own. His papers on the independence of the analytical and geometrical modes of investigation, and the evidence of imaginary symbols, as well as his treatise on the principles of analytical calculation, contributed largely to produce this effect; and in his Trigonometry, in which, for the first time, this important part of geometry was placed before the English reader in a purely analytical form, and with all that peculiar grace and flexibility which belongs to it in that form, he conferred a most essential benefit on the elementary mathematics of his country. We owe also to him a treatise on the Calculus of Variations, not indeed very luminous, nor very extensive, but which had one pre-eminent merit, that of appearing at just the right moment, when the want of any work explanatory of what is merely technical in that calculus was becoming urgent.

An increasing interest in mathematical subjects was now also manifested by the occasional appearance of papers of a higher class in our learned Transactions, (such as that of Dr. Brinkley, now bishop of Cloyne, on the exponential developments of Lagrange, a memoir of curious and elaborate merit, and, though somewhat later in point of time, the curious investigations of Mr. Babbage on the theory of functional equations,) as well as of distinct works on subjects of pure analysis. The most remarkable of these is the *Essay on the various Orders of Logarithmic Transcendants*, by the late W. Spence of Greenock, the first formal essay in our language on any distinct and considerable branch of integral calculus, which had appeared since the publication of Hellins's papers on the *Rectification of the Conic Sections*. A premature death carried off, in Spence, one who might have become the ornament of his country in this department of knowledge. His posthumous essays, which were not, however, collected and published until 1819, prove him to have been both a learned and inventive analyst. He appears to have studied entirely without assistance, and to have formed his taste and strengthened his powers by a diligent perusal of the continental models. In consequence, he was enabled to attack questions which none of his countrymen had entered upon, such as the general integration of equations of finite differences, and others of that difficult and elevated class.

Among our Scottish countrymen, indeed, the torch of abstract science had never burnt so feebly nor decayed so far as in these southern abodes; nor was a high priest of the sublimer muse ever wanting in those ancient shrines, where Gregory and Napier had paid homage to her power. The late Professor Robinson, though his taste for the older geometry led him to undervalue both the evidence and the power of the modern analysis, was yet a mathematician of no inconsiderable note. The remarkable papers of Professor Playfair[34] on Porisms show how deeply

[34] See note 17, *Preliminary Dissertation.*

the mind of that sound mathematician and elegant writer was imbued with the spirit of the analytical methods and a sense of their superior power–a power, however, which he was content to admire and applaud, rather than ready to wield. It may indeed be questioned whether, by any researches of his own, however successful, he could have given a stronger impulse to the public mind in this direction than what his admirable review of the *Mécanique Céleste* communicated.

To this school also we owe the only British geometer who, at this period, seems to have possessed, not only a complete familiarity with the resources of the higher analysis, but also the habit of using them with skill and success in inquiries of moment in the system of the world.–we mean Professor Ivory. The appearance of his *Memoirs on the Attraction of Spheroids*, which are deservedly considered masterpieces of their kind, and which at once placed their author in the high rank among the geometers of Europe which he has ever since maintained, was almost simultaneous with that of Spence's work, a coincidence which might seem to warrant the most sanguine hopes of the speedy re-establishment of our mathematical glories. But the national taste and acquirements had sunk so low, that the stimulus of these examples was yet for a while unfelt. The *Essay on Logarithmic Transcendants* attracted little immediate notice, and the memoirs of Ivory, though received abroad with the respect and admiration they so justly merited, met with slender applause and no imitation at home. Their effect was, to seat their author on a solitary eminence, equally above the sympathy and the comprehension of the world around him. Since that period, however, a change has been slowly but steadily taking place in mathematical education. Students at our universities, fettered by no prejudices, entangled by no habits, and excited by the ardour and emulation of youth, had heard of the existence of masses of knowledge, from which they were debarred by the mere accident of position. There required no more. The *prestige* which magnifies what is unknown, and the attraction inherent in what is forbidden, coincided in their impulse. the books were procured and read, and produced their natural effects. The brows of many a Cambridge moderator were elevated, half in ire, half in admiration, at the unusual answers which began to appear in examination papers. Even moderators are not made of impenetrable stuff; their souls were touched, though fenced with sevenfold Jacquier, and tough bull-hide of Vince and Wood. they were carried away with the stream, in short, or replaced with successors full of their newly acquired powers. The modern analysis was adopted in its largest extent, and at this moment we believe that there exists not throughout Europe a centre from which a richer and purer light of mathematical instruction emanates through a community, than one, at least, of our universities.

One of the immediate consequences of the increased demand for a knowledge of the continental analysis, and the manner in which it is made subservient to physical inquiry, was a rapid and abundant supply of elementary works. Lacroix's lesser treatise (we wish it had been greater) has been translated, with note and comment, from the French, and Meier Hirsch's admirable work on the Theory of Algebraic Equations, from the German; and, in addition to these transplanted authorities, (the former of which may be regarded as having greatly contributed, by its numerous examples, to the final *domestication* of the peculiar notation of the differential calculus among us,) a host of indigenous ones on almost every branch of the pure and applied mathematics have emanated chiefly, but by no means entirely, from the press of the Cambridge University, which has thus signalised itself in a manner equally useful to the country and honourable to its directors. many of these works bear, it is true, strong and singular marks of the *transition state* of the science in which they were produced, but, on the whole they contain a copious body of instruction; and although we have still nothing approaching in extent and excellence to the elementary works of Euler,[35] or to the superb digest of analytical knowledge contained in the great work of Lacroix,[36] to which we have alluded, yet, at least, our students can no longer complain of being left wholly without a guide, or without preparation for a profounder course of reading, should they feel disposed to enter upon it.

Another consequence, no less natural and obvious, of this altered state of feeling and instruction has been the gradual formation of what, at length, begins to merit the appellation of a British School of Geometry. We are far indeed from hoping soon to outstrip those who have so much the start of us, but the race is at least less hopeless than heretofore. The interval between the competitors has begun sensibly to diminish, and we need, at least, no longer fear being disgracefully distanced. We no longer perceive the same shyness, on the part of our mathematical champions, in entering on the great and vexed questions of the lunar and planetary perturbations, the theory of the tides, and others relating to the system of the world; nor the same indifference on that of the bystanders whether they are successful or no. The eminent geometer whom we have before named is no longer the only one among us who adventures himself fairly and boldly within the magic circle. On the contrary, we have recently witnessed the publication, by one of our countrymen, of several profound memoirs on the most intricate and profound parts of the terrestrial and planetary theory; on that of another, the novel, and, since Newton's time, the *unique* fact, of a new planetary inequality, not only detected, as so many have been, by British observation, but successfully referred to its origin, and subjected to exact calculation by British analysis, and that by no trifling effort or command of its resources.

We are very sure that in speaking so decidedly as we have felt compelled to do of the long-subsisting superiority of foreign mathematics to our own we run no hazard of wounding any feeling we wish to spare. Had our prospects, indeed, remained in the same deplorable state into which, but a very few years ago, they seem to have settled, we should perhaps preferred silence to the discouraging task of attempting to arouse an apathy so profound—but a better era is evidently advancing. The auguries are favourable. We hail them with delight, and we feel at the same time assured that our Airys, our Lubbocks,[37] our Hamiltons, and our Challises, the hope of our reviving geometry, will bear us out in the view we have taken, and acknowledge with gratitude and pleasure the sources whence they have drawn those principles they are now using so emulously and well.

Meanwhile the anomalous state of our mathematical literature which we have above described explains very naturally, what must have struck most mathematical readers as a remarkable feature in it, —we mean, the scanty supply of English works illustrative of the celestial mechanism, whether in the nature of express commentary and avowed illustration of the immortal work of Laplace, or in the form of independent treatises, calculated to bring the whole subject before the reader in a more compendious and explanatory manner than was compatible with Laplace's object, with the greatness and sweeping generality of his outline, or the close and laboured filling in of his detail. The *Elementary Illustrations of the Celestial Mechanics of Laplace*, by the late celebrated Dr. Young,[38] we hardly, we apprehend, be regarded by any reader as supplying satisfactorily the one of these desiderata; and although the Physical Astronomy of Professor Woodhouse approaches much nearer to what is requisite for the other, yet it by no means satisfies all, or nearly all, the conditions which such a work should accomplish. The

[35] See note 6, *Bk. I, Chap. II.*
[36] See note 26, *Bk. I, Chap. II.*
[37] See note 53, *Bk. II, Chap. VI.*
[38] See note 35, *Preliminary Dissertation.*

details of processes and developments into which it enters, though ample for elucidating the principles of the methods employed, is yet hardly sufficient to give a complete and effective grasp of the subject matter, while the combination of historical detail with theoretical elucidation, which it keeps in sight, tends to embarrass the reader by constantly shifting his point of view, and calling off his attention to inquire how mistakes have heretofore been committed and rectified; a most instructive thing in itself, no doubt–but calculated rather to render such a work a useful companion in a course of original reading, than to enable it to supply the place of many books, and offer, in a moderate compass, a compendium of what is known.

The works whose titles head the present article supply to the English reader, so far as they extend, both these desiderata, and supply them in a manner that leaves little to wish for. They are both, moreover, otherwise extremely remarkable in respect of the quarters from which they emanate. A lady, our own country woman, is the authoress of one; and to an American, by birth and residence, and to the American press, we stand indebted for the other. If anything were wanting to put our geometers effectively on their mettle, it would we think be found in such a coincidence.

Mrs. Somerville is already advantageously known to the philosophical world by her experiments on the magnetising influence of the violet rays of the solar spectrum; a delicate and difficult subject of physical inquiry, which the rarity of opportunities for its prosecution, arising from the nature of our climate, will allow no one to study in this country except at a manifest disadvantage. It is not surprising, therefore, that the feeble, although unequivocal indications of magnetism, which she undoubtedly obtained, should have been regarded by many as insufficient to decide the question at issue. To us their evidence appears entitled to considerable weight; but it is more to our immediate purpose to notice here, the simple and rational manner in which those experiments were conducted–the absence of needless complication and refinement in their plan, and of unnecessary or costly apparatus in their execution–and the perfect freedom from all pretension or affected embarrassment in their statement. The same simplicity of character and conduct, the same entire absence of anything like female vanity or affectation, pervades the whole of the present work. In the pursuit of her object, and in the natural and commendable wish to embody her acquired knowledge in an useful and instructive form for others, she seems entirely to have lost sight of herself; and, although in perfect consciousness of the possession of powers fully adequate to meet every exigency of her arduous undertaking, it yet never appears to have suggested itself to her mind, that the acquisition of such knowledge, or the possession of such powers, by a person of her sex, is in itself anything extraordinary or remarkable. We find, accordingly, beyond the name in the title page, nothing throughout the work introduced to remind us of its coming from a female hand. Even the tempting opportunity of deprecating criticism, which a preface affords, is neglected; nor does anything apologetic, in the tone of her admirably written preliminary discourse, betray a latent consciousness of superiority to the less-gifted of her sex, or a claim either on the admiration or forbearance of ours, beyond what the fair merits of the work itself may justly entitle it to. There is not only good taste, but excellent good sense in this. Whether admiration is due, or allowances needed, we accord both the one and the other, with perfect readiness, when left to the workings of our own good feeling. On the other hand, whenever we see such things as the poems of a minor, or the learning of a lady, introduced by an appeal, direct or indirect, to our good nature, we enter on our task of perusal with no very pleasant impression that this admirable weakness of our disposition is about to be largely taxed–an expectation in which, sooth to say, we are rarely disappointed.

In the present instance, however, we are neither called on for allowances, nor do we find any to make: on the contrary, we know not the geometer in this country who might not reasonably congratulate himself on the execution of such a work. The volume is dedicated to Lord Brougham, and appears to have been originally undertaken, at his instance, for publication by the Society for the Diffusion of Useful Knowledge; but the views of the author extending with its progress, it outgrew its first destination, and assumed an independent form. The nature of these views–the scope and object of the work–will perhaps be best understood from Mrs. Somerville's own words:–

'*A complete acquaintance with Physical Astronomy can only be attained by those who are well versed in the higher branches of mathematical and mechanical science: such alone can appreciate the extreme beauty of the results, and of the means by which these results are obtained. Nevertheless a sufficient skill in analysis to follow the general outline, to see the mutual dependence of the different parts of the system, and to comprehend by what means some of the most extraordinary conclusions have been arrived at, is within the reach of many who shrink from the task, appalled by difficulties, which perhaps are not more formidable than those incident to the study of the elements of every branch of knowledge, and possibly overrating them by not making a sufficient distinction between the degree of mathematical acquirement necessary for making discoveries, and that which is requisite for understanding what others have done. That the study of mathematics and their application to astronomy are full of interest will be allowed by all who have devoted their time and attention to these pursuits, and they only can estimate the delight of arriving at truth, whether it be in the discovery of a world, or of a new property of numbers*.'–p.2 (2^{nd} edition).

Let us now see how far the conduct of Mrs. Somerville's work corresponds with these views. In so doing, it is obvious that we are not to look for original discovery, the ambition of which is disclaimed, and which indeed would be misplaced in a work of the kind–nor even for absolute novelty in the methods of arriving at known results. The subject has been, in fact, so copiously handled, and by such a host of the most profound and accomplished mathematicians, that such novelty is now no longer to be expected, nor indeed desired in any fresh exposition of it. It is sufficient if all the results which it imports to know are clearly and perspicuously derived from their principles–the artifices of calculation on which their deduction rests, distinctly explained, and the processes actually pursued to such an extent as to give the reader a *thorough practical insight* into the development of the subject. This, we think, is fully accomplished in the work before us, for all those parts of the general subject which it professes to embrace, that is to say, the general exposition of the mechanical principles employed–the planetary and lunar theories, and those of Jupiter's satellites with the incidental points arising naturally out of them. The development of the theory of the tides, and the precession of the equinoxes, the attraction of spheroids and the figure of the earth, appear to be reserved for a second volume. A certain degree of inconvenience in incurred by this in the investigation of those irregularities in the motions of the moon and satellites depending on the oblate form of their planets, which compels an anticipation of results not previously demonstrated; but this inconvenience is one more easily perceived than avoided.

In Mrs. Somerville's preliminary dissertation, a general view is taken of the consequences of the law of gravitation, so far as they have hitherto been traced, whether as relates to the ecliptic motions and mutual perturbations of the planets and their satellites, and the slow variations in the forms of their orbits thereby produced, or to the figures assumed by each of them individually, in consequence of the combination of their rotations on their axes with the attractions of their particles on each other and that of neighbouring bodies, together with the nutations, precessions, and librations of their axes themselves, arising from external actions, or, lastly, to the equilibrium and oscillations of the waters and atmospheres which cover their surfaces, comprehending the theory of the tides, and the great geological question of the general stability of the ocean. These, and the important points which are essentially dependent on such investigations—their application to those greater operations of geography to which the term geodesy is usually applied—to the determination of standards of weight and measure—to the fixation of chronological epochs—and a multitude of other interesting inquiries, are treated with a condensation, but at the same time a precision and clearness, which render this preliminary dissertation a model of its kind, and a most valuable acquisition to our literature. We have indeed no hesitation in saying, that we consider it by far the best condensed view of the Newtonian philosophy which has yet appeared. We do not, of course, mean to include the *Systême du Monde* of Laplace himself, which embraces a far wider range, both of illustration and detail, and of which Mrs. Somerville's preface may in some sort be regarded as an abstract, but an abstract so vivid and judicious as to have all the merit of originality, and such as could have been produced only by one accustomed to large and general views, as well as perfectly familiar with the particulars of the subject.

As specimens of Mrs. Somerville's style of writing, we shall extract a few sentences almost from the commencement of this discourse:—

'*Science, regarded as the pursuit of truth, which can only be attained by patient and unprejudiced investigation, wherein nothing is too great to be attempted, nothing so minute as to be justly disregarded, must ever afford occupation of consummate interest and of elevated meditation. The contemplation of the worlds of creation elevates the mind to the admiration of whatever is great and noble, accomplishing the object of all study, which in the elegant language of Sir James Mackintosh is to inspire the love of truth, of wisdom, of beauty, especially of goodness, the highest beauty, and of that supreme and eternal mind, which contains all truth and wisdom, all beauty and goodness. By the love or delightful contemplation and pursuit of these transcendent aims for their own sake only, the mind of man is raised from low and perishable objects, and prepared for those high destinies which are appointed for all those who are capable of them*.'

We rejoice at this testimony to the intrinsic worth of scientific pursuits, and the pure and ennobling recompense they carry with them, from such a quarter. The female bosom is true to its impulses, and unwarped in their manifestation by motives which, in the sterner sex, are continually giving a bias to their estimates and conduct. The love of glory, the desire of practical utility, nay, even meaner and more selfish motives, may lead a man to toil in the pursuit of science, and adopt, without deeply feeling, the language of a disinterested worshipper at that sacred shrine—but we can conceive no motive, save immediate enjoyment of the kind so well described in the passage just quoted, which can induce a woman, especially an elegant and accomplished one, to undergo the severe and arduous mental exertion indispensable to the acquisition of a really profound knowledge of the higher analysis and its abstruser applications.

What follows is no less pleasing in another point of view:—

'*The heavens afford the most sublime subject of study which can be derived from science: the magnitude and splendour of the objects, the inconceivable rapidity with which they move, and the enormous distances between them, impress the mind with some notion of the energy that maintains them in their motions with a durability to which we can see no limits. Equally conspicuous is the goodness of the great First Cause in having endowed man with faculties by which he can not only appreciate the magnificence of his works, but trace, with precision, the operation of his laws, use the globe he inhabits as a base wherewith to measure the magnitude and distance of the sun and planets, and make the diameter of the earth's orbit the first step of a scale by which he may ascend to the starry firmament. Such pursuits, while they ennoble the mind, at the same time inculcate humility, by showing that there is a barrier, which no energy, mental or physical, can ever enable us to pass: that however profoundly we may penetrate the depths of space, there still remain innumerable systems, compared with which those which seem so mighty to us must dwindle into insignificance, or even become invisible*.'

We shall extract only one other passage from this discourse, as an example of the manner in which our fair authoress treats the less familiar topics, to which this part of her work is devoted. It is that in which the stability of the equilibrium of the seas and the permanence of the axis of the earth's rotation are considered.

'*It appears from the marine shells found on the tops of the highest mountains, and in almost every part of the globe, that immense continents have been elevated above the ocean, which [ocean]* [39] *must have engulfed others. Such a catastrophe would be occasioned by a variation in the position of the axis of rotation on the surface of the earth; for the seas tending to the new equator would leave some portions of the globe, and overwhelm others.*

But theory proves that neither nutation, precession, nor any of the disturbing forces that affect the system, have the smallest influence on the axis of rotation, which maintains a permanent position on the surface, if the earth be not disturbed in its rotation by some foreign cause, as the collision of a comet which may have happened in the immensity of time. Then indeed, the equilibrium could only have been restored by the rushing of the seas to the new equator, which they would continue to do, till the surface was every where perpendicular to the direction of gravity. But it is probable that such an accumulation of the waters would not be sufficient to restore equilibrium if the derangement had been great; for the mean density of the sea is only about a fifth part of the mean density of the earth, and the mean depth even of the Pacific ocean is not more than four miles, whereas the equatorial radius of the earth exceeds the polar radius by twenty-five or thirty miles; consequently the influence of the sea on the direction of gravity is very small; and as it appears that a great change in the position of the axes is incompatible with the law of equilibrium, the geological phenomena must be ascribed to an internal cause. Thus amidst the mighty revolutions which have swept innumerable races of organized beings from the earth, which have elevated plains, and buried mountains in the ocean, the rotation of the earth, and the position of the axis on its surface, have undergone but slight variations.'

We will only pause to remark here, that an argument, which appears to us much more conclusive against the fact of any disturbance having, in remote antiquity, taken place in the axis of the earth's rotation, is to be found in the amount of the lunar irregularities which depend on

[39] Added by Herschel.

the earth's spheroidal figure. However insufficient the mere transfer of the mass of the ocean from the old to the new equator might be to ensure the permanence of the new axis, the enormous abrasion of the solid matter of such immensely-protuberant continents, as would, on that supposition, be left, by the violent and constant fluctuation of an unequilibrated ocean, would, (according to an ingenious remark of Professor Playfair,[40]) no doubt, in the lapse of some ages, remodel the surface to the spheroidal form; but the lunar theory teaches us that the *internal strata*, as well as the *external outline*, of our globe, are elliptical, their centres being coincident and their axes identical with that of the surface,–a state of things incompatible with a subsequent accommodation of the surface to a new and different state of rotation from that which determined the original distribution of the component matter.

Mrs. Somerville's work is divided into four books, of which the first is devoted to the establishment of those general relations which prevail in the equilibrium or motion of bodies, or systems of bodies, whether solid or fluid, which are necessary to serve as a groundwork for the subsequent investigations;–the second, to the planetary theory, the elliptic motions and mutual perturbations of the bodies of our system, and the secular changes which take place in their orbits. The third book is given to the lunar theory; and the fourth to that of Jupiter's satellites, which is now for the first time introduced in any regular and extensive form to the English reader. From some confusion in the arrangement, or at least the numbering of the chapters in this book, it would seem to have been the original intention of the author to have thrown these two divisions of her subject into one, probably under the general head of the theory of Satellites. The actual arrangement is, on every account, infinitely preferable.

In the treatment of the statical and dynamical principles developed in the first part, the processes of the first book of the *Mécanique Céleste* are pretty closely but by no means servilely adhered to. Laplace's demonstration, for instance, of the fundamental principle of the composition of forces is suppressed, and its place supplied by one more elementary; and again, in the investigation of the equation of continuity of a fluid, the excessive difficulty and complication of the analysis by which he arrives at this result is evaded, and the whole subject in consequence greatly simplified by adopting a different and easier method of estimating the volume of an elementary molecule of the fluid in its displaced position. The whole of this portion of the work is also copiously illustrated by diagrams, which, however readily dispensed with by those with whom long habit has rendered familiar with analytical mechanics, are yet extremely useful in assisting the conception of less experienced readers. We could wish that a little more assistance of this kind had been afforded, and altogether a little more explanatory illustration bestowed on that chapter which treats of the rotatory motion of a solid mass. The subject needs it. There is a difficulty of conception in the notion of an axis of rotation shifting its position within a solid from instant to instant, as well as that of pressures exerted by the revolving matter, on such an imaginary and fugitive line, which is very embarrassing to one not accustomed to such speculations, though easily removed by dilating a little on the subject, and placing it in different and familiar points of view. We have always considered this part of analytical mechanics as among the most beautiful and exquisite of its applications. It is usually, however, regarded by beginners as more abstruse than its real difficulties authorize. This arises partly from the obscurity of conception we have alluded to, but partly, also, from a more technical cause,–the frequent changes of co-ordinates which its analytical treatment involves. This is a difficulty of the same kind as transposition, in a musical performance, from one key to another; and as a

[40] See note 17, *Preliminary Dissertation.*

musician can never expect to become a ready performer till practice has made such difficulty vanish, so the mathematical student can never feel at complete ease in the higher applications, till all such mere technical evolutions cease to be complained of as difficulties, or even felt as inconveniences.

We could have wished, too, that instead of entering, in this part of the work, on the theory of tides, which is by far the most complicated and infinitely the least satisfactory part of the general subject, that of the attractions of spheroids had been traced, at least so far as to demonstrate the theorems which are afterwards taken for granted in the development of those terms of the mercurial and lunar theory, and that of Jupiter's satellites, which depend on the oblate figure of the primary. As it is only a single term in the development of the series expressing the deviation of the law of gravity in the spheroid from that in the sphere which is wanted, this might have been very easily done, and at the same time the reader prepared to enter more fully into this interesting part of the subject, in a more advanced state of his knowledge.

In the second book the planetary theory is given with a fullness commensurate with its importance. Its first chapters are of course devoted to the theory of elliptic motion, which is concisely, but very perspicuously stated. The equations used are the beautiful integrals of the general differential equations first obtained, if we remember rightly, by Lagrange, and used by him with such wonderful effect for ascertaining the variations of the elements. They are the same which Laplace derives in the 18th article of his second book, by a process which we should be inclined to tax with excessive and useless generality, were it not quite necessary to show that this important part of the theory had been probed to the quick, and every resource which analysis could furnish exhausted on it. Mrs. Somerville, however, very properly derives them by the ordinary processes of direct integration. The usual properties of elliptic motion, with the series for the developments of the anomalies and radius vector afterwards required, are there demonstrated, and a few pages added on the determination of the elements.

We should have been glad to have found in this part of the work some outline of the powerful and elegant researches of gauss on the determination of the orbits of the celestial bodies, and especially some more practical method of determining those of comets than Laplace's. The subject of the motion of comets is, however, summarily dismissed; and even the beautiful theorem of Lambert,[41] which expresses the time of describing a parabolic arc in terms of the radii vectores of its extremities and its chord, is omitted.

The fine idea of Lagrange, by which the perturbations of a planet are expressed by means of a variable ellipse, and all its inequalities referred to changes in the elliptic elements which are supposed to be in a state of continual fluctuation, has introduced a degree of simplicity and symmetry into the analytical treatment of the planetary theory such as could hardly have been hoped for, and divested it of all that was repulsive and much that was merely laborious in its investigation. It is in this view of the subject alone, that a neat conception can be formed of the distinction between variations truly secular, and those inequalities of long periods which were originally confounded with secular changes. The former class are those which are independent of the mutual configurations of the planets one among the other, and in their theory no other quantities enter than the elements themselves and the time; all those variables on which depend the situations of the planets in their orbits, such as their longitudes, latitudes, and distances from the sun, being excluded. The reactions contemplated in this part of the theory are not so much those of planet on planet, as of orbit on orbit. Nothing can be more exquisite in analysis, nothing more refined in conception, than this investigation, on which depend all those grand propositions

[41] See note 2, *Bk. II, Chap. X.*

respecting the stability of the system to which we have already alluded. In the conduct of this part of her subject, Mrs. Somerville has chiefly adhered to the analysis of Lagrange, as stated by Laplace in the supplement to the third volume of the *Mécanique Céleste*, only in that important and difficult part of it which concerns the invariability of the axes as affected by the squares and products of the disturbing forces, availing herself of the subsequent elaborate investigations of Poisson.

The periodical part of the perturbations of the elements is next investigated, not so much with a view to the ultimate derivation of formulae for the practical computations of the longitudes and latitudes of the disturbed planets, which, though practicable, is not so easy in this view of the subject as in that of Laplace, which depends on the principle of successive approximations from the differential equations of the troubled orbit; and, so to speak, consists in a continual gathering up of the loose and raveled ends of the skein which appear in the form of unperiodic terms out of their proper place. The chief advantage of Lagrange's view of the subject when applied to the periodical terms, consists in the clear insight which it gives us into the nature of those equations of long period, such as, for instance, the secular equations, as they were formerly called, of Jupiter and Saturn, and the secular acceleration of the moon, which appear to alter the mean motion, and therefore to affect the axes of their orbits. They, in fact, do so; but such alterations are all periodical, and no way interfere with the general truth of their ultimate and average invariability. It ought to be remarked, however, that in the case of highly eccentric orbits, such as those of comets, which may approach very near the greater bodies of our system, deviations from the mean motion, and fluctuations of the periodic time may go to such an extent, and the compensation may be put off so long, that, although theoretically true, the proposition of the permanence of the axis may cease to have any useful or practical meaning. This is remarkably exemplified in the comet of Halley,[42] whose periodic return is affected by inequalities of a great many months, nay even whole years.

In the actual development of the perturbation of a planet in longitude there is a term introduced, at the very first step, proportional to the time. This is, in fact, the representative of that part of the planetary action which, like the mean effect of the ablatitious force in Newton's lunar theory, tends to diminish or increase the average intensity of gravitation to the central body, and thereby alter the mean motion and period from what they would be had the disturbing body no existence. The nature of this term, which appears very obscure as it is disposed of in the *Mécanique Céleste*, is placed by Mrs. Somerville in a much clearer light−(p. 381. 2nd edition)

The developments of the perturbations in longitude, latitude, and distance, though tedious, intricate, and laborious, offer no points of real difficulty, except−, in respect of the terms proportional to powers of the time introduced by integration, for the treatment of which we are referred to Laplace's memoir, in which this difficulty was first obviated; secondly, in respect of terms which, from the near commensurability of the mean motions, acquire small divisors by integration. These are, of all which occur in the planetary theory, the most troublesome. In the case of Jupiter and Saturn they give rise to the 'great equation' of those planets, to which Mrs. Somerville has devoted a masterly chapter, where it is treated with much clearness, and in a very compact and well digested form. On the whole, we consider the development of the planetary theory, as we have it thus brought before us, to be extremely well performed, and, in fact, a most useful and valuable summary of the subject.

[42] See *Art. 618, Bk. II.*

The lunar theory differs in many essential points from the planetary. This is owing to the rapid motion of the apsides and nodes of its orbit, in consequence of which it is impossible to treat it, as we do those of the planets, as an ellipse, subject to small and slow variations: this necessitates a totally different analytical treatment of the problem. That which has been universally followed since its first employment by d'Alembert, consists in expressing, not as for the planets, the longitude, &c. in functions of the time, but *vice versâ*, making the moon's longitude itself the independent variable, and expressing the time and the other co-ordinates in terms of this. The reversion of the first series, and substitution of the result in others, will then enable us to express all the co-ordinates in functions of the time.

Nothing, however, can be well imagined more formidable than the actual execution of these operations; at the same time that, when the delicacies of the management of the coefficients depending on the motions of the apsides and nodes are once understood, the whole is little more than a mechanical process, demanding only unwearied patience for its accomplishment. In the treatment, therefore, of this part of the subject, an author, whose object is merely to exhibit a clear view of processes, and a summary of results, is limited to a narrow path, affording little scope for the exercise of any facility but judgement in deciding where to stop. Mrs. Somerville seems to have considered it her duty here to err on the safe side; so that the equations of her lunar theory are, in fact, little else than a transcript, *mutatis mutandis*, of those of Laplace, and co-extensive with his formulae. She has, however, had recourse to the gigantic work of Damoiseau[43] for the expression of the longitude in terms of the time, the deduction of which, by the actual reversion of Laplace's series, would have been a work of infinitely too much labour, and which every one but those who make it their especial project to surpass all who have gone before them in this most intricate inquiry, must be content to receive on his authority.

The last division of the work is devoted to the theory of Jupiter's satellites–a curious and elegant system, in which the near approach to commensurability in the mean motions of the three interior satellites gives rise to peculiarities of a very remarkable nature both in the analysis and its results. In this system also the great ellipticity of the central body causes a material deviation in its attraction from the law of gravity, the effect being to introduce a term in the expression of the *perturbative function*, varying inversely as the cube of the distance. As we have before observed, the investigation of this term is not given, and we must, moreover, take this opportunity to notice that, by an inaccuracy of wording, which is repeated wherever the same point is referred to in other parts of the work, this term is always spoken of as expressing 'the attraction of the excess of matter at the equator of the central body,' whereas, in fact, it expresses no *attractive force* at all, but an artificial quantity, being the significant perturbative term in the development of that useful function in the theory of the attraction of spheroids, which expresses the sum of the molecules of the attracting body, divided each by its respective distance from the point attracted, and which is constantly employed by Laplace in this theory, in preference to the direct expression of the attraction itself, for the convenience and symmetry of analysis. We are the more particular in noticing this point, as the most considerable fault we have to find with the work before us consists in an habitual laxity of language, evidently originating in so complete a familiarity with the *quantities* concerned, as to induce a disregard of the *words* by which they are designated, but which, to any one less intimately conversant with the actual analytical operations than its author, must have infallibly become a source of serious errors, and which at all events, renders it necessary for the reader to be constantly on his guard. It would not be difficult to support this charge (which is rather a grave one) by citations, but we should be extremely

[43] See note 6, *Bk. III, Chap. I.*

unwilling to leave, at the conclusion of our article, any impression less agreeable than that of the unfeigned delight, and we may add, astonishment, with which the perusal of the work has filled us.

We must not, however, stop without saying something of Mr. Bowditch's[44] performance; though what we do say must be short. The idea of undertaking a translation of the whole *Mécanique Céleste*, accompanied throughout with a copious running commentary, is one which savours, at first sight, of the *gigantesque*, and is certainly one which, from what we had hitherto had reason to conceive of the popularity and diffusion of mathematical knowledge on the opposite shores of the Atlantic, we should never have expected to have found originated–or, at least, carried into execution, in that quarter. The first volume only has as yet reached us; and when we consider the great difficulty of printing works of this nature, to say nothing of the heavy and probably unremunerated expense, we are not surprised at the delay of the second. Meanwhile the part actually completed (which contains the first two books of Laplace's work) is, with few and slight exceptions, just what we could have wished to see–an exact and careful translation into very good English–exceedingly well printed, and accompanied with notes appended to each page, which leave no step in the text of moment unsupplied, and hardly any material difficulty either of conception or reasoning unelucidated. To the student of 'Celestial Mechanism' such a work must be invaluable, and we sincerely hope that the success of this volume, which seems thrown out to try the feeling of the public, both American and British, will be such as to induce the speedy appearance of the sequel. Should this unfortunately not be the case, we shall deeply lament that the liberal offer of the American Academy of Arts and Sciences, to print the whole at their expense, was not accepted. Be that as it may, it is impossible to regard the appearance of such a work, even in its present incomplete state, as otherwise than highly creditable to American science, and as the harbinger of future achievements in the loftiest fields of intellectual prowess. Here, at least, is an arena on which we may contend with an emulation unembittered by rivalry.–'Whatever,' says Delambre, 'be the state of political relations, the sciences ought to form, among those who cultivate them, a republic essentially at peace within itself,'–a sentiment applicable, doubtless, to all, but pre-eminently so to that calm, dispassionate pursuit of truth which forms the very essence of the abstract sciences.

[44] See note 3, *Foreword to the Second Edition*.

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