SUBLUMINAL AND SUPERLUMINAL SOLUTIONS IN VACUUM OF THE MAXWELL EQUATIONS AND THE MASSLESS DIRAC EQUATION

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We show that Maxwell equations and Dirac equation (with zero mass term) have both subluminal and superluminal solutions in vacuum. We also discuss the possible fundamental physical consequences of our results.

I. INTRODUCTION

According to Bosanac [Bo83] there is no formal proof based only on Maxwell equations that no electromagnetic wave packet can travel faster than the vacuum speed of light c ($c = 1$ in the natural units to be used here). Well, the main purpose of this paper is to show that Maxwell equations (and also the Dirac equation with zero mass) have superluminal solutions ($v > 1$) and also subluminal solutions ($v < 1$) in vacuum.

This paper is organized as follows. In sect.2 we introduce some mathematical tools that will be used. In sect.3 we show how to construct the se called subluminal and superluminal solutions of the free Maxwell equations. In sect.4 the subluminal and superluminal solutions of the massless Dirac equation are discussed. Finally in sect.5 we discuss some of the possible physical implications of these results.

II. MATHEMATICAL PRELIMINARIES

In order to discuss these new solutions of Maxwell and Dirac equations in an unified way we briefly recall how these equations can be written in the Clifford and Spin-Clifford bundles over Minkowski spacetime. Details concerning these theories can be found in [RS93,RS95,RO90].

Let $\mathcal{M} = (M, g, D)$ be Minkowski spacetime. (M, g) is a four dimensional time oriented and space oriented Lorentzian manifold, with $M \simeq I\!\!R^4$ and $g \in \sec(T^*M \times T^*M)$ being a Lorentzian metric of signature (1,3). T^*M $[TM]$ is the cotangent [tangent] bundle. $T^*M = \cup_{x\in M} T^*_x M$ and $TM = \cup_{x\in M} T_x M$, and $T_x M \simeq T^*_x M \simeq I\!\!R^{1,3}$, where $I\!\!R^{1,3}$ is the Minkowski vector space [SW77,RR89]. D is the Levi-Civita connetion of g, i.e., $Dg = 0$, $\boldsymbol{T}(D) = 0$. Also $\boldsymbol{R}(D) = 0$, $\boldsymbol{T}$ and $\boldsymbol{R}$ being respectively the torsion and curvature tensors. Now, the Clifford bundle of differential forms $\mathcal{C}\ell(M)$ is the bundle of algebras $\mathcal{C}\ell(M) = \cup_{x\in M} \mathcal{C}\ell(T^*_x M)$, where $\forall x \in M, \mathcal{C}\ell(T^*_x M) = \mathcal{C}\ell_{1,3}$, the so called spacetime algebra [Lo93,HS87]. Locally as a linear space over the real field $I\!\!R, \mathcal{C}\ell(T^*_x(M)$ is isomorphic to the Cartan algebra $\bigwedge(T^*_x(M)$ of the cotangent space and $\bigwedge(T^*_x M) = \sum_{k=0}^4 \bigwedge^k(T^*_x M)$, where $\bigwedge^k(T^*_x M)$ is the $\binom{4}{k}$ dimensional space of k-forms. The Cartan bundle $\bigwedge(M) = \cup_{x\in M} \bigwedge(T^*_x M)$ can then be thought as "imbedded" in $\mathcal{C}\ell(M)$. In this way sections of $\mathcal{C}\ell(M)$ can be represented as a sum of inhomogeneous differential forms. Let $\{e_\mu = \frac{\partial}{\partial x^\mu}\} \in \sec TM, (\mu = 0, 1, 2, 3)$ be an orthonormal basis $g(e_\mu, e_\nu) = \eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ and let $\{\gamma^\nu = dx^\nu\} \in \sec \bigwedge^1(M) \subset \sec\mathcal{C}\ell(M)$ be the dual basis. Then, the fundamental Clifford product (in what follows to be dennoted by juxtaposition of symbols) is generated by $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}$ and if $\mathcal{C} \in \sec\mathcal{C}\ell(M)$ we have

$$\mathcal{C} = s + v_\mu\gamma^\mu + \frac{1}{2!} b_{\mu\nu}\gamma^\mu\gamma^\nu + \frac{1}{3!} a_{\mu\nu\rho}\gamma^\mu\gamma^\nu\gamma^\rho + p\gamma^5$$

where $\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3 = dx^0 dx^1 dx^2 dx^3$ is the volume element and $s, v_\mu, b_{\mu\nu}, a_{\mu\nu\rho}, p \in \sec \bigwedge^0(M) \subset \sec\mathcal{C}\ell(M)$. For $A_r \in \sec \bigwedge^r(M) \subset \sec\mathcal{C}(M), B_s \in \sec \bigwedge^s(M)$ we define [Lo93,HS87] $A_r \cdot B_s = \langle A_r B_s\rangle_{|r-s|}$ and $A_r \wedge B_s = \langle A_r B_s\rangle_{r+s}$, where $\langle\ \rangle_k$ is the component in $\bigwedge^k(M)$ of the Clifford field.

Besides the vector bundle $\mathcal{C}\ell(M)$ we need also to introduce another vector bundle $\mathcal{C}\ell_{\mathrm{Spin}_+(1,3)}(M)$ [$\mathrm{Spin}_+(1,3) \simeq \mathrm{SL}(2,\mathbb{C})$] called the Spin-Clifford bundle. We can show that $\mathcal{C}\ell_{\mathrm{Spin}_+(1,3)}(M) \simeq \mathcal{C}\ell(M)/\mathcal{R}$, i.e., it is a quotient bundle.

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This means that sections of $\mathcal{C}\ell_{\mathrm{Spin}_+(1,3)}(M)$ are some special equivalence classes of sections of the Clifford bundle, i.e, they are equivalence sections of non-homogeneous differential forms (see eqs.(1,2) below).

Now, as well known an electromagnetic field is represented by $F \in \sec \bigwedge^2(M) \subset \sec\mathcal{C}\ell(M)$. How to represent the Dirac spinor fields in this formalism? We can show that even sections of $\mathcal{C}\ell_{spin_+(1,3)}(M)$, called Dirac-Hestenes spinor fields, do the job. If we fix two orthonormal basis, $\Sigma = \{\gamma^\mu\}$ as before, and $\dot{\Sigma} = \{\dot{\gamma}^\mu = R\gamma^\mu \widetilde{R} = \Lambda^\mu_\nu \gamma^\nu\}$ with $\Lambda^\mu_\nu \in \mathrm{SO}_+(1,3)$ and $R \in \mathrm{Spin}_+(1,3)$, $R\widetilde{R} = \widetilde{R}R = 1$, and where $^\sim$ is the reversion operator in $\mathcal{C}\ell_{1,3}$ [Lo93,HS87], then the representations of an even section $\boldsymbol{\psi} \in \sec\mathcal{C}\ell_{\mathrm{Spin}_+(1,3)}(M)$ are the sections ψ_Σ and $\psi_{\dot{\Sigma}}$ of $\mathcal{C}\ell(M)$ related by

$$\psi_{\dot{\Sigma}} = \psi_\Sigma R \tag{1}$$

and

$$\psi_\Sigma = s + \frac{1}{2!} b_{\mu\nu}\gamma^\mu\gamma^\nu + p\gamma^5 \tag{2}$$

Note that ψ_Σ has the correct number of degrees of freedom in order to represent a Dirac spinor field, which is not the case with the so called Dirac-Kähler spinor field.

Let $\star$ be the Hodge star operator $\star : \bigwedge^k(M) \to \bigwedge^{4-k}(M)$. Then we can show that if $A_p \in \sec \bigwedge^p(M) \subset \sec\mathcal{C}\ell(M)$ we have $\star A = \widetilde{A}\gamma^5$. Let d and δ be respectively the diferential and Hodge codifferential operators acting on sections of $\bigwedge(M)$. If $\omega_p \in \sec \bigwedge^p(M) \subset \sec\mathcal{C}\ell(M)$, then $\delta\omega_p = (-)^p \star^{-1} d \star \omega_p$, with $\star^{-1}\star = $ identity.

The Dirac operator acting on sections of $\mathcal{C}\ell(M)$ is the invariant first order differential operator

$$\partial = \gamma^\mu D_{e_\mu}, \tag{3}$$

and we can show the very important result [Ma90]:

$$\partial = \partial\wedge + \partial\cdot = d - \delta. \tag{4}$$

With these preliminaries we can write Maxwell and Dirac equations as follows [He66]:

$$\partial F = 0, \tag{5}$$

$$\partial\psi_\Sigma\gamma^1\gamma^2 + m\psi_\Sigma\gamma^0 = 0. \tag{6}$$

If $m = 0$ we have the massless Dirac equation

$$\partial\psi_\Sigma = 0, \tag{7}$$

which is Weyl's one when ψ_Σ is reduced to a Weyl spinor field. Note that in this formalism Maxwell equations condensed in a single equation! Also, the specification of ψ_Σ depends on the frame Σ.

If one wants to work in terms of the usual spinor formalism, we can translate our results by choosing, for example, the standard matrix representation of $\{\gamma^\mu\}$, and for ψ_Σ given by eq.(2) we have the following (standard) matrix representation:

$$\psi = \begin{pmatrix} \phi_1 & -\phi_2^* \\ \phi_2 & \phi_1^* \end{pmatrix}, \tag{8}$$

where

$$\phi_1 = \begin{pmatrix} s - ib_{12} & b_{13} - ib_{23} \\ -b_{13} - ib_{23} & s + ib_{12} \end{pmatrix}, \quad \phi_2 = \begin{pmatrix} -b_{03} + ip & -b_{01} + ib_{02} \\ -b_{01} - ib_{02} & b_{03} + ip \end{pmatrix}. \tag{9}$$

Right multiplication by

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

gives the usual Dirac spinor.

Before we present the subluminal and superluminal solutions $F_<$ and $F_>$ we shall define precisely an inertial reference frame (irf) [SW77,RR89]. An irf $I \in \sec TM$ is a timelike vector field pointing into the future such that $g(I,I) = 1$ and $DI = 0$. Each integral line of I is called an observer. A chart $\langle x^\mu \rangle$ of the maximal atlas of M is called naturally adapted to I if $I = \partial/\partial x^0$. Putting $I = e_0$, we can find $e_i = \partial/\partial x^i$ such that $g(e_\mu, e_\nu) = \eta_{\mu\nu}$ and the coordinates $\langle x^\mu \rangle$ are the usual Einstein-Lorentz ones and have a precise operational meaning [RT85]. x^0 is measured by "ideal clocks" at rest synchronized "a la Einstein" and x^i are measured with "ideal rulers".

III. SUBLUMINAL AND SUPERLUMINAL SOLUTIONS OF THE MAXWELL EQUATIONS

Let $A \in \sec \bigwedge^1(M) \subset \sec \mathcal{C}\ell(M)$ be the vector potential. We fix the Lorentz gauge, i.e., $\partial \cdot A = -\delta A = 0$ such that $F = \partial A = (d - \delta)A = dA$. We have the following:

Theorem: Let $\Pi \in \sec \bigwedge^2(M) \subset \sec \mathcal{C}\ell(M)$ be the so called Hertz potential. If Π satisfies the wave equation, i.e, $\partial^2 \Pi = \eta^{\mu\nu}\partial_\mu \partial_\nu \Pi = -(d\delta + \delta d)\Pi = 0$, and if $A = -\delta\Pi$, then $F = \partial A$ satisfies the Maxwell equations $\partial F = 0$.

The proof is trivial. Indeed $A = -\delta\Pi$, then $\delta A = -\delta^2\Pi = 0$ and $F = \partial A = dA$. Now $\partial F = (d - \delta)\ (d - \delta)A = -(d\delta + \delta d)A = \delta d(\delta\Pi) = -\delta^2 d\Pi = 0$ since $\delta d\Pi = -d\delta\Pi$ from $\partial^2\Pi = 0$.

Now, since our main purpose here is to exhibit the existence of the new solutions we present only particular cases, leaving a complete study for another publication. To show the existence of a stationary solution F_0 ($\partial F_0 = 0$) relative to a given inertial frame I with adapted coordinates $\langle x^\mu \rangle = \langle t, x^i \rangle$ introduce above, we choose the Hertz potential

$$\Pi_0(t, \vec{x}) = \phi(\vec{x}) \exp(\gamma^5 \Omega t)\gamma^1\gamma^2. \tag{10}$$

Since $\partial^2 \Pi_0 = 0$ we get that $\phi(\vec{x})$ satisfies the Helmholtz equation

$$\nabla\phi + \Omega^2\phi = 0 \tag{11}$$

The solutions of this equation are well known. Here we must comment the fact that the scalar wave equation has solutions which travels with speed less than c is known since a long time, being discovered by H. Bateman [Bt15] in 1915 and rediscovered by several people in the last few years, in particular by Barut [BB92] [1]. An elementary solution of eq.(11) with spherical symmetry is

$$\phi(\vec{x}) = C\frac{\sin \Omega r}{r}, \qquad r = \sqrt{x^2 + y^2 + z^2}, \tag{12}$$

where C is an arbitrary constant. From this result we can write F_0 in polar coordinates (r, θ, φ) as

$$\begin{aligned} F_0 = \frac{C}{r^3}[&\sin \Omega t(\alpha\Omega r \sin\theta \sin\varphi - \beta \cos\theta \sin\theta \cos\varphi)\gamma^0\gamma^1 \\ &- \sin\Omega t(\alpha\Omega r \sin\theta\cos\varphi + \beta\sin\theta\cos\theta\sin\varphi)\gamma^0\gamma^2 \\ &+ \sin\Omega t(\beta \sin^2\theta - 2\alpha)\gamma^0\gamma^3 + \cos\Omega t(\beta\sin^2\theta - 2\alpha)\gamma^1\gamma^2 \\ &+ \cos\Omega t(\beta\sin\theta\cos\theta\sin\varphi + \alpha\Omega r\sin\theta\cos\varphi)\gamma^1\gamma^3 \\ &+ \cos\Omega t(-\beta\sin\theta\cos\theta\cos\varphi + \alpha\Omega r \sin\theta\sin\varphi)\gamma^2\gamma^3] \end{aligned} \tag{13}$$

with $\alpha = \Omega r \cos\Omega r - \sin\Omega r$ and $\beta = 3\alpha + \Omega^2 r^2 \sin\Omega r$. Note that F_0 is *regular at the origin and vanishes at infinity.*

Let us rewrite the above solution in terms of the old-fashioned vector algebra. We have that

$$\vec{E}_0 = -\vec{W}\sin\Omega t, \quad \vec{B}_0 = \vec{W}\cos\Omega t, \tag{14}$$

where we defined

$$\vec{W} = C\left(\frac{\alpha\Omega y}{r^3} - \frac{\beta xz}{r^5}, \quad -\frac{\alpha\Omega x}{r^3} - \frac{\beta yz}{r^5}, \quad \frac{\beta(x^2+y^2)}{r^5} - \frac{2\alpha}{r^3}\right). \tag{15}$$

One can explicitly verify that $\text{div}\vec{W} = 0$, so that $\text{div}\vec{E}_0 = \text{div}\vec{B}_0 = 0$, and that $\text{rot}\vec{W} + \Omega\vec{W} = 0$, so that $\text{rot}\vec{E}_0 + \partial\vec{B}_0/\partial t = 0$ and $\text{rot}\vec{B}_0 - \partial\vec{E}_0/\partial t = 0$.

We can show for a given F [Ma90] that $S^0 = \frac{1}{2}\widetilde{F}\gamma^0 F$ is the 1-form representing the energy density and the Poynting vector. We have that $\vec{E}_0 \times \vec{B}_0 = 0$, so that there is *no propagation* and the solution has vanishing angular momentum. The energy density is

$$u = \frac{1}{r^6}[\sin^2\theta(\Omega^2 r^2 \alpha^2 + \cos^2\theta\beta^2) + (\beta\sin^2\theta - 2\alpha)^2] \tag{16}$$

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[1] Barut found also subluminal solutions of Maxwell equations, with a procedure different from the one presented here, and his solutions are also different from the above.

Then $\int u d^3x = \infty$. A finite energy solution can be constructed by considering "wave packets" with a distribution of intrinsic frequencies $f(\Omega)$ [Bt15]. These solutions will be discussed in another publication. It is also very important to see that $F_0^2 = F_0 \cdot F_0 + F_0 \wedge F_0 \neq 0$, i.e, the field invariants are nonvanishing, differently of the usual solutions F_c of Maxwell equations that travel with constant speed $c = 1$ and for which $F_c^2 = 0$.

To obtain a solution $F_<$ moving with velocity $0 < v < 1$ relative to I it is necessary only to make a Lorentz boost in the x direction of the solution F_0. Another way to obtain a solution $F'_<$ is to get a new solution $\Phi_<(t, \vec{x})$ of the wave equation ($\partial^2 \Phi_< = 0$), like:

$$\Phi_<(t, \vec{x}) = C \frac{\sin \Omega \xi_<}{\xi_<} \exp[\gamma^5(\omega t - kx)], \tag{17}$$

$$\omega^2 - k^2 = \Omega^2, \tag{18}$$

with

$$\xi_< = (\gamma_<^2 (x - vt)^2 + y^2 + z^2)^{1/2},$$

$$\gamma_< = \frac{1}{\sqrt{1 - v^2}}, \quad v = \frac{d\omega}{dk}$$

Eq.(18) is a dispersion relation for a "particle" moving with velocity (here "group velocity") $v_< = d\omega/dk < 1$. From eq.(17) we can obtain a new solution $F'_<$ moving with $0 < v < 1$ and which satisfies $\partial F'_< = 0$ by taking $F'_< = \partial(\partial \cdot \Pi_<)$ with $\Pi_< = \Phi_< \gamma^1 \gamma^2$. The fact that $F'_<$ satisfies Maxwell equations follows from the above theorem, but it has been explicitly verified using REDUCE 3.5 [2]; unfortunately the explicit form of $F'_<$ is very big to be given here.

Now, to obtain a superluminal solution of Maxwell equations it is enough to observe that the function

$$\Phi_>(t, \vec{x}) = C \frac{\sin \Omega \xi_>}{\xi_>} \exp[\gamma^5(\omega t - kx)], \tag{19}$$

$$\omega^2 - k^2 = -\Omega^2, \tag{20}$$

with

$$\xi_> = (\gamma_>^2 (x - vt)^2 - y^2 - z^2)^{1/2},$$

$$\gamma_> = \frac{1}{\sqrt{v^2 - 1}}, \quad v = \frac{d\omega}{dk},$$

is a solution [BC93] of the wave equation $\partial^2 \Phi_> = 0$, which travels with velocity $v = d\omega/dk > 1$. Writting $\Pi_> = \Phi_> \gamma^1 \gamma^2$ we can obtain $F_> = \partial(\partial \cdot \Pi_>)$ which satisfies $\partial F_> = 0$. This is then a superluminal electromagnetic configuration. The explicit form of $F_>$ is again very big to be reproduced here, but again we verified it explicitly using REDUCE 3.5. Also $F_<^2 \neq 0$ which means that the field invariants are non null. We will discuss more the properties of this and some others extraordinary solutions of Maxwell equations in another paper.

IV. SUBLUMINAL AND SUPERLUMINAL SOLUTIONS OF THE MASSLESS DIRAC EQUATION

In order to find these kinds of solutions of the massless Dirac equation we shall make use of some ideas from supersymmetry. Since a Dirac-Hestenes spinor field ψ_Σ is given by eq(2) we can define a generalized potential for ψ_Σ. Indeed for each ψ_Σ there exists $\mathcal{A} = A + \gamma^5 B$, $A, B \in \sec \bigwedge^1(M) \in \sec \mathcal{C}\ell(M)$, such that

$$\psi_\Sigma = \partial(A + \gamma^5 B) \tag{21}$$

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[2] The use of REDUCE in Clifford Algebras is discussed, for example, in [Va95,Br87].

The Dirac operator ∂ plays here a role analogous to that of supersymmetry operator [Gr94]. In fact, Clifford algebras are Z_2-graded algebras, and $\mathcal{C}\ell^+\mathcal{C}\ell^+ \subset \mathcal{C}\ell^+$, $\mathcal{C}\ell^\pm\mathcal{C}\ell^\mp \subset \mathcal{C}\ell^-$, $\mathcal{C}\ell^-\mathcal{C}\ell^- \subset \mathcal{C}\ell^+$, where $\mathcal{C}\ell^+$ [$\mathcal{C}\ell^-$] denotes the set of elements of $\mathcal{C}\ell$ with even [odd] grade. Since the Dirac operator has vector properties, its action transforms fields of even grade into fields of odd grade, and vice-versa. Representing a spinor field by means of nonhomogeneous forms of even degree is therefore equivalent to find a bosonic representation of a fermionic field.

The quantity $\mathcal{A}$ can be interpreted as a kind of potential for the massless Dirac field. Indeed, from eq.(7) with eq.(21) it follows that

$$\partial^2\mathcal{A} = 0, \tag{22}$$

or that

$$\partial^2 A = 0, \qquad \partial^2 B = 0. \tag{23}$$

A simple subluminal solution at rest relative to the inertial frame $I = e_0$ in the coordinates $\langle x^\mu \rangle$ naturally adapted to I is

$$\mathcal{A}_0(t,\vec{x}) = \gamma^0\phi(\vec{x})\exp\left(\gamma^5\Omega t\right) \tag{24}$$

with $\phi(\vec{x})$ given by eq.(12). We have

$$\mathcal{A}_0 = \frac{C}{r}(\sin\Omega r\cos\Omega t\gamma^0 - \sin\Omega r\sin\Omega t\gamma^1\gamma^2\gamma^3) \tag{25}$$

Then

$$\begin{aligned}
&\psi^0_\Sigma = \frac{C}{r^3}[-\Omega r^2\sin\Omega r\sin\Omega t\\
&+\gamma^0\gamma^1\lambda x\cos\Omega t + \gamma^0\gamma^2\lambda y\cos\Omega t\\
&+\gamma^0\gamma^3\lambda z\cos\Omega t - \gamma^1\gamma^2\lambda z\sin\Omega t\\
&+\gamma^1\gamma^3\lambda y\sin\Omega t - \gamma^2\gamma^3\lambda x\sin\Omega t\\
&+\gamma^0\gamma^1\gamma^2\gamma^3\Omega r^2\sin\Omega r\cos\Omega t],
\end{aligned} \tag{26}$$

where $\lambda = \Omega r\cos\Omega r - \sin\Omega r$.

The above solution in the usual formalism reads

$$\psi^0 = \begin{pmatrix} i\sin\Omega t\left(\dfrac{\lambda z}{r^3} + i\dfrac{\Omega}{r}\sin\Omega r\right) \\ i\sin\Omega t\left(\dfrac{x+iy}{r^3}\right)\lambda \\ -\cos\Omega t\left(\dfrac{\lambda z}{r^3} + i\dfrac{\Omega}{r}\sin\Omega r\right) \\ -\cos\Omega t\left(\dfrac{x+iy}{r^3}\right)\lambda \end{pmatrix} \tag{27}$$

and one can explicitly verify that indeed $\partial\psi^0 = 0$.

Other subluminal solutions can be obtainned by appropriated boosts. An explicit superluminal solution $\psi^>_\Sigma$ can be obtained by writting

$$\mathcal{A}_>(t,\vec{x}) = \gamma^0\Phi_>(t,\vec{x}) \tag{28}$$

with $\Phi_>$ given by eq.(19) and $\psi^>_\Sigma = [\partial\Phi_>(t,\vec{x})]\gamma^0$. Again the explicit form of $\Psi^>_\Sigma$ is very big and will be not written here, but it has been verified using REDUCE. Barut [Ba94] has found subluminal solutions of $\partial\Psi = 0$, where Ψ is the usual Dirac spinor field, with a different method from the one used here; our one is much simple since it is representation free and uses elegant tools from supersymmetry.

V. CONCLUSIONS

We want to discuss three possible implications of the results we have shown.

(i) If the superluminal solutions of at least Maxwell equations are realized in Nature we can have a breakdown of Lorentz invariance. Indeed, suppose $I = \partial/\partial t$ is the fundamental reference frame and $I' = (1/\sqrt{1-V^2})\partial/\partial t - V/\sqrt{1-V^2}\partial/\partial x$ is the laboratory frame (an inertial frame). Suppose that $F_>$ is a superluminal solution of Maxwell equations, i.e., $\partial F_> = 0$ $(\omega'^2 - k'^2 = -\Omega^2)$, travelling forward in time according to I. Then, the validity of active Lorentz invariance implies that there exists $R \in \text{Spin}_+(1,3)$ such that $F'_> = RF_>\tilde{R}$ satisfies $\partial F'_> = 0$ with $F'_>$ going backward in time (and carrying negative energy) according to I. This solution can be interpreted as an "anti-field" coming from the past, and is a good solution. However, the physical equivalence of all inertial reference frames implies that according to I' there exists solutions $F''_>$ of Maxwell equations travelling forward in time and carrying positive energy $E' = \omega'$ $(\hbar = 1)$ according to I' but travelling backward in time (and carrying negative energy) according to I. The field $F''_>$ can be absorved, e.g., by a detector in periodic motion in I (it is enough that at the time of absortion the detector has relative to I the velocity V of the I' frame). This generates as is well known a causal paradox [Re86] (Tolman-Regge paradox). The possible solution is to say that I and I' are not physically equivalent. We then have the following: I' cannot send to some observers (integral lines) of the I reference frame a superluminal signal such that $\omega' < (V/\sqrt{1-V^2})\Omega$. When $\omega' = (V/\sqrt{1-V^2})\Omega$ the superluminal generator of I' stops working for k' in some spacetime directions, and an observer in I can calculate his absolute velocity which is $V = \omega'/\sqrt{\omega'^2 + \Omega'^2}$. We must also call the reader's attention that recently Nimtz [HN94,EN93] transmited Mozart's symphony # 40 at 4.7 c through a retangular wave guide, that as is now well known [ML92] acts like a potential barrier for light. Important related results have also been obtained in [St93]. We can show easily that under Nimtz experimental conditions the solution of Maxwell equations in the guide gives a dispersion relation like eq(20), i.e, corresponding to superluminal propagation [Re86]. We shall discuss this issue in details elsewhere.

(ii) The existence of the subluminal solutions $F_<$ are very important for the following reason: Recently [VR93,VR95] we proved that $\partial F = 0$ for $F^2 \neq 0$ is equivalent under certain conditions to a Dirac-Hestenes equation $\partial\psi_\Sigma\gamma^1\gamma^2 + m\psi_\Sigma\gamma^\circ = 0$, where $F = \psi_\Sigma\gamma^1\gamma^2\widetilde{\psi}_\Sigma$. This means that eventually particles are special stationary electromagnetic waves and a de Broglie interpretation of quantum mechanics seems possible [RV93]. We will discuss this issue in details elsewhere.

(iii) Finally the existence of subluminal and superluminal solutions for $\partial\psi_\Sigma = 0$ (which reduces to Weyl equation for ψ_Σ a Weyl spinor) may be important to solve some of the mysteries associated with neutrinos. Indeed if neutrinos can be produced in the subluminal and superluminal modes – see [Ot95,Gn86] for some experimental evidences for superluminal neutrinos – then they can eventually escape detection on earth after leaving the sun. Moreover, for neutrinos in a subluminal mode it would be possible to define a kind of "effective mass". Recently some cosmological evidences that neutrinos may have a nonvanishing mass have been discussed [Pr95]. One such "effective mass" could be responsible for those cosmological evidences, and in such a way that we can still have a left-handed neutrino since it would satisfies the Weyl equation. We are going to study this proposal in a forthcoming paper.

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Note added in proof. After we finished this paper we have been informed by Professor Ziolkowski that he and collaborators found also superluminal solutions of the scalar wave equation and also of Maxwell equations and even Klein-Gordon equation [DZ92,DZ93,Zi93]. Also Dr. Lu and collaborators found a very interesting "superluminal" solution of the scalar wave equation [LG92,LG92a] and Lu even realized an approximation for his solution (the so called X waves) as a nondispersive pressure wave in water which travels with velocity 1.002 c, where c here is the velocity of sound in water!

The solutions found by Lu can be used to construct Hertz potentials for the Maxwell equations and then to generate superluminal electromagnetic field configurations. We believe that it is in principle possible to build such fields with appropriate devices. This is also the opinion of Dr. Lu. Also some of the Ziolkowski solutions, according to his opinion, may be realized in the physical world. We shall discuss these points in another opportunity.

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