Propagation of light in non-inertial reference frames

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Abstract

It is shown that the complete description of the propagation of light in a gravitational field and in non-inertial reference frames in general requires an average coordinate and an average proper velocity of light. The need for an average coordinate velocity of light in non-inertial frames is demonstrated by considering the propagation of two vertical light rays in the Einstein elevator (in addition to the horizontal ray originally discussed by Einstein). As an average proper velocity of light is implicitly used in the Shapiro time delay (as shown in the Appendix) it is explicitly derived and it is shown that for a round trip of a light signal between two points in a gravitational field the Shapiro time delay not only depends on which point it is measured at, but in the case of a parallel gravitational field it is not always a delay effect. The propagation of light in rotating frames (the Sagnac effect) is also discussed and an expression for the coordinate velocity of light is derived. The use of this coordinate velocity naturally explains why an observer on a rotating disk finds that two light signals emitted from a point on the rim of the disk and propagating in opposite directions along the rim do not arrive simultaneously at the same point.

1 Introduction

One of the fundamental facts of modern physics is the constancy of the speed of light. Einstein regarded it as one of the two postulates on which special relativity is based. So far, however, little attention has been paid to the status of this postulate when teaching special relativity. It turns out that the constancy of the speed of light is a direct consequence of the relativity principle, not an independent postulate. To see this let us consider the two postulates of special relativity as formulated by Einstein in his 1905 paper "On the electrodynamics of moving bodies": "the same laws of electrodynamics and optics will be valid for all frames of reference for which the equations of mechanics hold good. We will raise this conjecture (the purport of which will hereafter be called the "Principle of Relativity") to the status of a postulate, and also introduce another postulate, which is only apparently irreconcilable with the former, namely, that light is always propagated in empty space with a definite velocity c which is independent of the state of the motion of the emitting body" [1]. As the principle of relativity states that "the laws of physics are the same in all inertial reference frames" and the constancy of the speed of light means that "the speed of light is the same in all inertial reference frames (regardless of the motion of the source or the observer)" it follow that the second postulate is indeed a consequence of the first - the law describing the propagation of light is the same for all inertial observers.

This becomes even clearer if it is taken into account that the relativity principle is a statement of the impossibility to detect absolute motion. Since all inertial observers (moving with constant velocity) are completely equivalent (none is in absolute motion) according to the principle of relativity, they all observe the same phenomena and describe them by using the same laws of physics. Therefore it does follow that light should propagate with the same speed in all inertial frames; otherwise, if an inertial observer found that the speed of light were not c in his reference frame, that observer would say that he detected his absolute motion.

That all inertial observers are equivalent is also seen from the fact that they are represented by *geodesic* worldlines which in the case of flat spacetime are *straight* worldlines. However, when an observer is accelerating his worldline is not geodesic (not a straight worldline in flat spacetime). Therefore, accelerated motion, unlike motion with constant velocity, is absolute - there is an absolute difference between a geodesic and a non-geodesic worldline. This means that the laws of physics in inertial and non-inertial reference frames are not the same. An immediate consequence is that the speed of light is not constant in non-inertial frames - a non-inertial observer can detect his accelerated motion by using light signals.

It is precisely this corollary of special relativity that received little attention in the courses and books on relativity. In fact, that corollary is regularly used but since it is done implicitly confusions are not always avoided. For instance, an observer in Einstein's thought experiment [3] involving an accelerating elevator can discover his accelerated motion by the deflection of a light ray from its horizontal path. An observer in a rotating reference frame, a rotating disk for example, can detect the disk accelerated motion also by using light: light signals emitted from a point M in opposite directions along the rim of the disk do not arrive at the same time at M (this is the so called Sagnac effect) [2]. It is explicitly stated in general relativity that the *local* speed of light is always c, which implies that the speed of light along a *finite* distance is not necessarily c. However, up to now no average velocity of light propagating between two points has been defined. Before introducing such a velocity let us consider several examples that demonstrate the need for it.

Einstein's thought experiment [3] involving an elevator at rest in a *parallel* [4] gravitational field of strength $\mathbf{g}$ and an elevator accelerating with an acceleration $\mathbf{a} = -\mathbf{g}$ was designed to demonstrate the equivalence of the non-inertial reference frames N^g (elevator at rest in the gravitational field) and N^a (elevator accelerating in space devoid of gravity). Einstein called this equivalence the principle of equivalence: it is not possible by experiment to distinguish between the non-inertial frames N^a and N^g which means that all physical phenomena look the same in N^a and N^g. Therefore if a horizontal light ray propagating in N^a bends, a horizontal light ray propagating in N^g should bend as well.

Although even introductory physics textbooks [5]-[8] have started to discuss Einstein's elevator experiment an obvious question has been overlooked: "Are light rays propagating in an elevator in a vertical direction (parallel and anti-parallel to $\mathbf{a}$ or $\mathbf{g}$) also affected by the accelerated motion of the elevator or its being in a gravitational field?" The answer to this question requires the introduction of an *average coordinate* velocity of light which turns out to be different from c in the case of vertical light rays (see Figure 1 and the detailed discussion in Section 2). It should be stressed that it is the average coordinate velocity of light *between two points* that is different from c; the local speed of light measured at a point is always c.

A second average velocity of light - an *average proper* velocity of light - is required for the explanation of the Shapiro time delay [9], [10]. It also turns out not to be always c. The fact that it takes more time for a light signal to travel between two points P and Q in a gravitational field than between the same points in flat spacetime as determined by an observer at one of the points indicates that the average velocity of light between the two points is smaller than c. As the proper time of the observer is used in measuring that velocity it seems appropriate to call it average *proper* velocity of light. Unlike the average coordinate velocity the average proper velocity of light between two points depends on which point it is measured at. This fact confirms the dependence of the Shapiro time delay on the point where it is measured and shows, as we shall see in Section 3, that in the case of a parallel gravitational field it is not always a delay effect (in such a field the average proper velocity of light is defined in terms of both the proper distance and proper time of an observer). A light signal will be delayed *only* if it is measured at a point P that is farther from the gravitating mass producing the parallel field; if it is measured at the other point Q, closer to the mass, it will take less time for the signal to travel the same distance which shows that the average proper velocity of the signal determined at Q is greater than that measured at P and greater than c.

Due to the calculation of the average velocities of light in N^a and N^g to verify their agreement with the equivalence principle only a parallel gravitational field will be considered in Sections 2 and 3. That is why the expressions for the average light velocities will be derived for this case. Their generalized expressions for the case of the Schwarzschild metric and other metrics can be easily obtained (see Appendix).

Section 4 deals with the propagation of light in a rotating frame. In the case of light signals propagating in opposite directions along the rim of a rotating disk only the introduction of a coordinate velocity (depending on the centripetal acceleration of the disk) can explain why an observer on the disk discovers that the light signals do not arrive at the same time at the source point.

The introduction of the average velocities of light also sheds some light on a subtle feature of the propagation of light in the vicinity of a massive body - whether or not light falls in its gravitational field. The particle aspect of light seems to entail that a photon, like any other particle, should fall in a gravitational field (due to the mass corresponding to its energy); the deflection of light by a massive body appears to support such a view. And indeed this view is sometimes implicitly or explicitly expressed in papers and books although the correct explanation is given in some books on general relativity (see for instance [11]-[13]). It has been recently claimed that the issue of whether or not a charge falling in a gravitational field radiates can be resolved by assuming that the charge's electromagnetic field is also falling [14]. An electromagnetic field falling in a gravitational field, however, implies that light falls in the gravitational field as well. Even Einstein and Infeld appear to suggest that as a light beam has mass on account of its energy it will fall in a gravitational field: "A beam of light will bend in a gravitational field exactly as a body would if thrown horizontally with a velocity equal to that of light" [3]. This comparison is not quite precise since the vertical component of the velocity of the body will increase as it falls whereas the velocity of the "falling" light beam is decreasing for a non-inertial observer (supported in a gravitational field) , as we shall see below. Statements such as "a beam of light will accelerate in a gravitational field, just like objects that have mass" and therefore "near the surface of the earth, light will fall with an acceleration of 9.81 m/s^2" have found their way in introductory physics textbooks [5]. It will be shown in Section 3 that during its "fall" in a gravitational field light is slowing down - a negative acceleration of 9.81 m/s^2 is decreasing its velocity (at a point P near the Earth's surface as seen from another point above or below P).

2 Average coordinate velocity of light

Why the average velocity of light between two points in a gravitational field is not generally equal to c can be most clearly shown by considering two extra light rays parallel and anti-parallel to the gravitational acceleration $\mathbf{g}$ in the Einstein thought experiment involving an elevator at rest in the Earth's gravitational field (Figure 1).

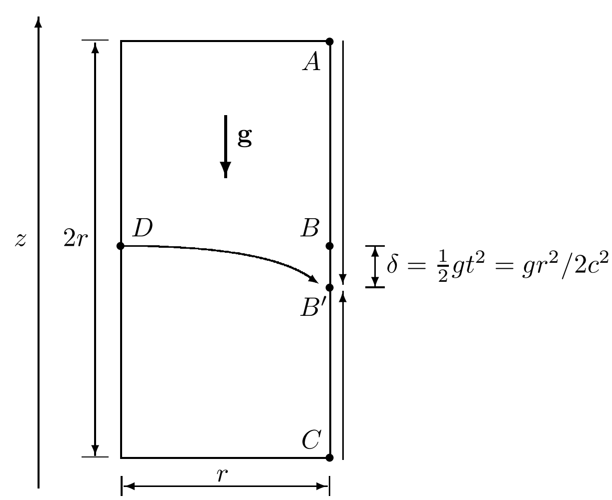

Figure 1 - Propagation of light in the Einstein elevator at rest in a parallel gravitational field.

Three light rays are emitted simultaneously in the elevator which is at rest in the non-inertial reference frame N^g. Two rays are emitted from points A and C towards point B and the third light ray is following the null path from D, spatially directed along constant z towards B, to B'. The emission of the three rays is also simultaneous in the local Lorentz (inertial) frame I which is momentarily at rest with respect to N^g at the moment the light rays are emitted (I and N^g have a common instantaneous three-dimensional space at this moment and therefore common simultaneity). At the next moment, as I starts to fall in the gravitational field, it will appear to an observer in I that the elevator moves upwards with an acceleration $g = |\mathbf{g}|$. Therefore as seen from I the three light rays will arrive simultaneously not at point B, but at B' since for the time $t = r/c$ the elevator moves (from I's viewpoint) at a distance $\delta = gt^2/2 = gr^2/2c^2$. As the simultaneous arrival of the three rays at point B' is an *absolute* (observer-independent) fact due to its being a *single* event, it follows that the rays arrive simultaneously at B' as seen from N^g as well. Since for the *same* coordinate time $t = r/c$ in N^g the three light rays travel different distances $DB' \approx r$, $AB' = r + \delta$, and $CB' = r - \delta$ before arriving simultaneously at B' an observer in N^g concludes (to within terms $\sim c^{-2}$ [15]) that the *average* velocity of the light ray propagating from A to B' is slightly greater than c

$$c^g_{AB'} = \frac{r+\delta}{t} \approx c\left(1 + \frac{gr}{2c^2}\right).$$

The average velocity $c^g_{CB'}$ of the light ray propagating from C to B' is slightly smaller than c

$$c^g_{CB'} = \frac{r-\delta}{t} \approx c\left(1 - \frac{gr}{2c^2}\right).$$

It is easily seen that to within terms $\sim c^{-2}$ the average velocity of light between A and B is equal to that between A and B', i.e. $c^g_{AB} = c^g_{AB'}$ and also $c^g_{CB} = c^g_{CB'}$:

$$c^g_{AB} = \frac{r}{t - \delta/c} = \frac{r}{t - gt^2/2c} = \frac{c}{1 - gr/2c^2} \approx c\left(1 + \frac{gr}{2c^2}\right) \tag{1}$$

and

$$c^g_{CB} = \frac{r}{t + \delta/c} \approx c\left(1 - \frac{gr}{2c^2}\right). \tag{2}$$

As the average velocities (1) and (2) are not determined with respect to a specific point since the *coordinate* time t is involved in their calculation, it is clear that (1) and (2) represent the average *coordinate* velocities of light between the points A and B and C and B, respectively.

These expressions for the average coordinate velocity of light in N^g can be also obtained from the coordinate velocity of light at a point in a parallel gravitational field. In such a field proper and coordinate times do not coincide whereas proper and coordinate distances are the same [17] as follows from the standard spacetime interval in a *parallel* gravitational field [18]

$$ds^2 = \left(1 + \frac{2gz}{c^2}\right)c^2dt^2 - dx^2 - dy^2 - dz^2$$

which can be also written as [18, p. 173]

$$ds^2 = \left(1 + \frac{gz}{c^2}\right)^2 c^2dt^2 - dx^2 - dy^2 - dz^2. \tag{3}$$

Note that due to the existence of a horizon at $z = -c^2/g$ [18, pp. 169, 172-173] there are constraints on the size of non-inertial reference frames (accelerated or at rest in a parallel gravitational field) which are represented by the metric (3). If the origin of N^g is changed, say to $z_B = 0$ (See Figure 1), the horizon moves to $z = -c^2/g - |z_B|$.

The coordinate velocity of light at a point z can be obtained from (3) (for $ds^2 = 0$)

$$c^g(z) \equiv \frac{dz}{dt} = \pm c\sqrt{\left(1 + \frac{gz}{c^2}\right)^2} = \pm\, c\left(1 + \frac{gz}{c^2}\right). \tag{4}$$

The $+$ and $-$ signs are for light propagating along or against $+z$, respectively. Therefore, the coordinate velocity of light at a point z is locally isotropic in the z direction. It is clear that the coordinate velocity (4) cannot become negative due to the constraints on the size of non-inertial frames which ensure that $|z| < c^2/g$ [18, pp. 169, 172].

As seen from (4) the coordinate velocity of light is a function of z which shows that we can calculate the average coordinate velocity between A and B by taking an average over the distance from A to B. As $c^g(z)$ is not only continuous on the interval $[z_A,\ z_B]$ (for $|z| < c^2/g$), but is also a linear function of z, we can write

$$c^g_{AB} = \frac{1}{z_B - z_A}\int_{z_A}^{z_B} c^g(z)\,dz = c\left(1 + \frac{gz_B}{c^2} + \frac{gr}{2c^2}\right), \tag{5}$$

where we took into account that $z_A = z_B + r$. When the coordinate origin is at point B ($z_B = 0$) the expression (5) coincides with (1).

The coordinate velocity of light $c^g(z)$ is also continuous on the interval $[t_A,\ t_B]$, but in order to calculate c^g_{AB} by taking an average of the velocity of light over the time of its propagation from A to B we should find the dependence of z on t. From (3) we can write (for $ds^2 = 0$):

$$dz = c\left(1 + \frac{gz}{c^2}\right)dt.$$

By integrating and keeping only the terms proportional to c^{-2} we find that $z = ct$ which shows that $c^g(z)$ is also linear in t (to within terms proportional to c^{-2}):

$$c^g(t) = \pm\, c\left(1 + \frac{gt}{c}\right).$$

Therefore for the average coordinate velocity of light between points A and B we have:

$$\begin{aligned} c^g_{AB} &= \frac{1}{t_B - t_A}\int_{t_A}^{t_B} c^g(z)\,dt = \frac{1}{t_B - t_A}\int_{t_A}^{t_B} c\left(1 + \frac{gz}{c^2}\right)dt \\ &= \frac{1}{t_B - t_A}\int_{t_A}^{t_B} c\left(1 + \frac{gt}{c}\right)dt = c\left(1 + \frac{gz_B}{c^2} + \frac{gr}{2c^2}\right), \end{aligned} \tag{6}$$

where the magnitude of $c^g(z)$ was used and it was taken into account that $z_A = z_B + r$ and $z_A = ct_A$ and $z_B = ct_B$. As expected this expression coincides with (5) and for $z_B = 0$ is equal to (1).

The fact that $c^g(z)$ is linear in both z and t (to within terms $\sim c^{-2}$) makes it possible to calculate the average coordinate velocity of light propagating between A and B (See Figure 1) by using the values of $c^g(z)$ only at the end points A and B:

$$c^g_{AB} = \frac{1}{2}\left(c^g_A + c^g_B\right) = \frac{1}{2}\left[c\left(1 + \frac{gz_A}{c^2}\right) + c\left(1 + \frac{gz_B}{c^2}\right)\right]$$

and as $z_A = z_B + r$

$$c^g_{AB} = c\left(1 + \frac{gz_B}{c^2} + \frac{gr}{2c^2}\right). \tag{7}$$

This expression coincides with the expressions for c^g_{AB} in (5) and (6).

For the average coordinate velocity of light propagating between B and C we obtain

$$c^g_{BC} = c\left(1 + \frac{gz_B}{c^2} - \frac{gr}{2c^2}\right) \tag{8}$$

since $z_C = z_B - r$. As noted above when the coordinate origin is at point B ($z_B = 0$) the expressions (7) and (8) coincide with (1) and (2).

The average coordinate velocities (7) and (8) correctly describe the propagation of light in N^g yielding the right expression $\delta = gr^2/2c^2$ (See Figure 1). It should be stressed that without these average coordinate velocities the fact that the light rays emitted from A and C arrive not at B, but at B' cannot be explained.

As a coordinate velocity, the average coordinate velocity of light is not determined with respect to a specific point and depends on the choice of the coordinate origin. Also, it is the same for light propagating from A to B and for light travelling in the opposite direction, i.e. $c^g_{AB} = c^g_{BA}$. Therefore, like the coordinate velocity (4) the average coordinate velocity is also isotropic. Notice, however, that the average coordinate velocity of light is isotropic in a sense that the average light velocity between two points is the same in both directions. But as seen from (7) and (8) the average coordinate velocity of light between different pairs of points, whose points are the same distance apart, is different. As a result, as seen in Figure 1, the light ray emitted at A arrives at B *before* the light ray emitted at C.

In an elevator (at rest in the non-inertial reference frame N^a) accelerating with an acceleration $a = |\mathbf{a}|$, where the metric is [18, p. 173]

$$ds^2 = \left(1 + \frac{az}{c^2}\right)^2 c^2 dt^2 - dx^2 - dy^2 - dz^2, \tag{9}$$

the expressions for the average coordinate velocity of light between A and B and B and C, respectively, are

$$c^a_{AB} = c\left(1 + \frac{az_B}{c^2} + \frac{ar}{2c^2}\right) \tag{10}$$

and

$$c^a_{BC} = c\left(1 + \frac{az_B}{c^2} - \frac{ar}{2c^2}\right) \tag{11}$$

in agreement with the equivalence principle.

3 Average proper velocity of light

The average coordinate velocity of light explains the propagation of light in the Einstein elevator and in non-inertial reference frames in general, but cannot be used in a situation where the average light velocity between two points (say a source and an observation point) is determined *with respect to one of the points*. Such situations occur in the Shapiro time delay. As the local velocity of light is c the average velocity of light between a source and an observation point depends on which of the two points is regarded as a reference point with respect to which the average velocity is determined (at the reference point the local velocity of light is always c). The dependence of the average velocity on which point is chosen as a reference point demonstrates that that velocity is anisotropic. This anisotropic velocity can be regarded as an average *proper* velocity of light since it is determined with respect to a given point and therefore its calculation involves the proper time at that point. It is also defined in terms of the proper distance as determined by an observer at the same point (in the case of a parallel gravitational field).

Consider a light source at point B (See Figure 1). To calculate the average proper velocity of light originating from B and observed at A (that is, as seen from A) we have to determine the initial velocity of a light signal at B and its final velocity at A, both with respect to A. As the local velocity of light is c the final velocity of the light signal determined at A is obviously c. Noting that in a parallel gravitational field proper and coordinate distances are the same [17] we can determine the initial velocity of the light signal at B as seen from A

$$c^g_B = \frac{dz_B}{d\tau_A} = \frac{dz_B}{dt}\frac{dt}{d\tau_A}$$

where $dz_B/dt = c^g\left(z_B\right)$ is the coordinate velocity of light (4) at B

$$c^g\left(z_B\right) = c\left(1 + \frac{gz_B}{c^2}\right)$$

and $d\tau_A = ds_A/c$ is the proper time for an observer with constant spatial coordinates at A

$$d\tau_A = \left(1 + \frac{gz_A}{c^2}\right) dt.$$

As $z_A = z_B + r$ and $gz_A/c^2 < 1$ (since for any value of z in N^g there is a restriction $|z| < c^2/g$) for the coordinate time dt we have (to within terms $\sim c^{-2}$)

$$dt \approx \left(1 - \frac{gz_A}{c^2}\right) d\tau_A = \left(1 - \frac{gz_B}{c^2} - \frac{gr}{c^2}\right) d\tau_A.$$

Then for the initial velocity c_B^g at B as seen from A we obtain

$$c_B^g = c\left(1 + \frac{gz_B}{c^2}\right)\left(1 - \frac{gz_B}{c^2} - \frac{gr}{c^2}\right)$$

or keeping only the terms $\sim c^{-2}$

$$c_B^g = c\left(1 - \frac{gr}{c^2}\right). \tag{12}$$

Therefore an observer at A will determine that a light signal is emitted at B with the velocity (12) and during the time of its journey towards A (away from the Earth's surface) will *accelerate* with an acceleration g and will arrive at A with a velocity exactly equal to c.

For the average proper velocity $\bar{c}_{BA}^g = (1/2)(c_B^g + c)$ of light propagating from B to A as seen from A we have

$$\bar{c}_{BA}^g \,(as\ seen\ from\ A) = c\left(1 - \frac{gr}{2c^2}\right). \tag{13}$$

As the local velocity of light at A (measured at A) is c it follows that if a light signal propagates from A towards B its initial velocity at A is c, its final velocity at B is (12) and therefore, as seen from A, it is subjected to a negative acceleration g and will *slow down* as it "falls" in the Earth's gravitational field. This shows that the average proper speed $\bar{c}_{AB}^g \,(as\ seen\ from\ A)$ of a light signal emitted at A with the initial velocity c and arriving at B with the final velocity (12) will be equal to the average proper speed $\bar{c}_{BA}^g \,(as\ seen\ from\ A)$ of a light signal propagating from B towards A. Thus, as seen from A, the back and forth average proper speeds of light travelling between A and B are the *same*.

Now let us determine the average proper velocity of light between B and A with respect to point B. A light signal emitted at B as seen from B will have an initial (local) velocity c there. The final velocity of the signal at A as seen from B will be

$$c_A^g = \frac{dz_A}{d\tau_B} = \frac{dz_A}{dt}\frac{dt}{d\tau_B}$$

where $dz_A/dt = c^g\,(z_A)$ is the coordinate velocity of light at A

$$c^g\,(z_A) = c\left(1 + \frac{gz_A}{c^2}\right)$$

and $d\tau_B$ is the proper time at B

$$d\tau_B = \left(1 + \frac{gz_B}{c^2}\right) dt.$$

Then as $z_A = z_B + r$ we obtain for the velocity of light at A as determined from B

$$c_A^g = c\left(1 + \frac{gr}{c^2}\right). \tag{14}$$

Using (14) the average proper velocity of light propagating from B to A as determined from B becomes

$$\bar{c}_{BA}^g \,(as\ seen\ from\ B) = c\left(1 + \frac{gr}{2c^2}\right). \tag{15}$$

If a light signal propagates from A to B its average proper speed $\bar{c}^g_{AB}$ ($as\ seen\ from\ B$) will be equal to $\bar{c}^g_{BA}$ ($as\ seen\ from\ B$) - the average proper speed of light propagating from B to A. This demonstrates that for an observer at B a light signal emitted from B with a velocity c will *accelerate* towards A with an acceleration g and will arrive there with the final velocity (14). As determined by the B-observer a light signal emitted from A with the initial velocity (14) will be *slowing down* (with $-\ g$) as it "falls" in the Earth's gravitational field and will arrive at B with a final velocity exactly equal to c. Therefore an observer at B will agree with an observer at A that a light signal will *accelerate* with an acceleration g on its way from B to A and will *decelerate* while "falling" in the Earth's gravitational field during its propagation from A to B but disagree on the velocity of light at the points A and B.

Comparing (13) and (15) demonstrates that the two average proper speeds between the same points A and B are not equal and depend on where they are measured from. As we expected the fact that the local velocity of light at the reference point is c makes the average proper velocity between two points dependant on where the reference point is. An immediate consequence from here is that the Shapiro time delay does not always mean that it takes more time for light to travel a given distance in a parallel gravitational field than the time needed in flat spacetime.

In the case of a parallel gravitational field the Shapiro time effect for a round trip of a light signal propagating between A and B determined from point A will be indeed a delay effect:

$$\Delta\tau_A = \frac{2r}{c\left(1 - gr/2c^2\right)} \approx \Delta t_{flat}\left(1 + \frac{gr}{2c^2}\right),$$

where $\Delta t_{flat} = 2r/c$ is the time for the round trip of light between A and B in flat spacetime. However, an observer at B will determine that it takes less time for a light signal to complete the round trip between A and B:

$$\Delta\tau_B = \frac{2r}{c\left(1 + gr/2c^2\right)} \approx \Delta t_{flat}\left(1 - \frac{gr}{2c^2}\right).$$

However, in the Schwarzschild metric the Shapiro effect is always a delay effect since the average proper speed of light in that metric is always smaller that c as shown in the Appendix.

The average proper velocity of light between A and B can be also obtained by using the average coordinate velocity of light (7) between the same points:

$$c^g_{AB} \equiv \frac{r}{\Delta t} = c\left(1 + \frac{gz_B}{c^2} + \frac{gr}{2c^2}\right)$$

Let us calculate the average proper velocity of light propagating between A and B as determined from point A. This means that we will use A's proper time $\Delta\tau_A = \left(1 + gz_A/c^2\right)\Delta t$:

$$\bar{c}^g_{AB}(as\ seen\ from\ A) = \frac{r}{\Delta\tau_A} = \frac{r}{\Delta t}\frac{\Delta t}{\Delta\tau_A}$$

Noting that $r/\Delta t$ is the average coordinate velocity (7) and $z_A = z_B + r$ we have (to within terms $\sim c^{-2}$)

$$\bar{c}^g_{AB}(as\ seen\ from\ A) \approx c\left(1 + \frac{gz_B}{c^2} + \frac{gr}{2c^2}\right)\left(1 - \frac{gz_A}{c^2}\right) \approx c\left(1 - \frac{gr}{2c^2}\right)$$

which coincides with (13).

The calculation of the average proper velocity of light propagating between A and B, but as seen from B yields the same expression as (15):

$$\bar{c}^g_{AB}(as\ seen\ from\ B) = \frac{r}{\Delta\tau_B} = \frac{r}{\Delta t}\frac{\Delta t}{\Delta\tau_B} \approx c\left(1 + \frac{gz_B}{c^2} + \frac{gr}{2c^2}\right)\left(1 - \frac{gz_B}{c^2}\right) \approx c\left(1 + \frac{gr}{2c^2}\right).$$

As evident from (13) and (15) the average proper velocity of light emitted from a common source and determined at different points around the source is anisotropic in N^g - if the observation point is above the light source the average proper speed of light is slightly smaller than c and smaller than the average proper speed as determined from an observation point below the source. If an observer at point B (See Figure 1) determines the average proper velocities of light coming from A and C he will find that they are also anisotropic - the average proper velocity of light coming from A is greater than that emitted at C and therefore the light from A will arrive at B *before* the light from C (provided that the two light signals from A and C are emitted simultaneously in N^g). However, if the observer at B (See Figure 1) determines the back and forth average proper speeds of light propagating between A and B he finds that they are the same (the back and forth average proper speeds of light between B and C are also the same).

The calculation of the average proper velocities of light in an accelerating frame N^a gives:

$$c_{BA}^{a}\,(as\ seen\ from\ A) = c\left(1 - \frac{ar}{2c^2}\right)$$

and

$$c_{BA}^{a}\,(as\ seen\ from\ B) = c\left(1 + \frac{ar}{2c^2}\right).$$

where $a = |\mathbf{a}|$ is the frame's proper acceleration.

4 The Sagnac effect

The Sagnac effect can de described as follows. Two light signals emitted from a point M on the rim of a rotating disk and propagating along its rim in opposite directions will not arrive simultaneously at M. There still exist people who question special relativity and their main argument has been this effect. They claim that for an observer on the rotating disk the speed of light is not constant - that the Galilean law of velocity addition ($c+v$ and $c-v$, where v is the orbital speed at a point on the disk rim) should be used by the rotating observer in order to explain the time difference in the arrival of the two light signals at M. What makes such claims even more persistent is the lack of a clear position on the issue of the speed of light in non-inertial reference frames. What special relativity states is that the speed of light is constant only in inertial reference frames - this constancy follows from the impossibility to detect absolute motion (more precisely, it follows from the non-existence of absolute motion). Accelerated motion can be detected and for this reason the coordinate velocity of light in non-inertial reference frames is a function of the proper acceleration of the frame. The rotating disk is a non-inertial reference frame and its acceleration can be detected by different means including light signals. That is why it is not surprising that the coordinate velocity of light as determined on the disk depends on the centripetal acceleration of the disk. As we shall see below the coordinate velocity of light calculated on the disk is *not* a manifestation of the Galilean law of velocity addition.

Consider two disks whose centers coincide. One of them is stationary, the other rotates with constant angular velocity ω. As the stationary disk can be regarded as an inertial frame its metric is the Minkowski metric:

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2. \tag{16}$$

To write the interval ds^2 in polar coordinates we use the transformation

$$t = t \qquad x = R\cos\Phi \qquad y = R\sin\Phi \qquad z = z. \tag{17}$$

By substituting (16) in (17) we get

$$ds^2 = c^2dt^2 - dR^2 - R^2d\Phi^2 - dz^2. \tag{18}$$

Let an observer on the rotating disk use the coordinates t, r, φ, and z. The transformation between the coordinates on the stationary and on the rotating disk is obviously:

$$t = t \qquad R = r \qquad \Phi = \varphi + \omega t \qquad z = z. \tag{19}$$

Time does not change in this transformation since the coordinate time on the rotating disk is given by the clock at its center and this clock is at rest with respect to the inertial stationary disk [19]. By substituting (19) in (18) we obtain the metric on the rotating disk:

$$ds^2 = \left(1 - \frac{\omega^2 r^2}{c^2}\right) c^2 dt^2 - dr^2 - r^2 d\varphi^2 - 2\omega r^2 dt d\varphi - dz^2. \tag{20}$$

As light propagates along null geodesics ($ds^2 = 0$) we can calculate the tangential coordinate velocity of light $c^\varphi \equiv r\,(d\varphi/dt)$ from (20) by taking into account that $dr = 0$ and $dz = 0$ for light propagating on the surface of the rotating disk along its rim (of radius r). First we have to determine $d\varphi/dt$. From (20) we can write

$$r^2 \left(\frac{d\varphi}{dt}\right)^2 + 2\omega r^2 \left(\frac{d\varphi}{dt}\right) - \left(1 - \frac{\omega^2 r^2}{c^2}\right) c^2 = 0.$$

The solution of this quadratic equation gives two values for $d\varphi/dt$ - one in the direction in which φ increases ($+\varphi$) (in the direction of the rotation of the disk) and the other in the opposite direction ($-\varphi$):

$$\left(\frac{d\varphi}{dt}\right)^{+\varphi} = -\omega + \frac{c}{r}; \qquad \left(\frac{d\varphi}{dt}\right)^{-\varphi} = -\omega - \frac{c}{r}.$$

Then for the tangential coordinate velocities $c^{+\varphi}$ and $c^{-\varphi}$ we obtain

$$c^{+\varphi} \equiv r \left(\frac{d\varphi}{dt}\right)^{+\varphi} = c\left(1 - \frac{\omega r}{c}\right) \tag{21}$$

and

$$c^{-\varphi} \equiv r \left(\frac{d\varphi}{dt}\right)^{-\varphi} = -c\left(1 + \frac{\omega r}{c}\right). \tag{22}$$

As seen from (21) and (22) the tangential coordinate velocities $c^{+\varphi}$ and $c^{-\varphi}$ are *constant* for a given r which means that (21) and (22) also represent the average coordinate velocities of light. The coordinate speed of light propagating in the direction of the rotation of the disk is smaller that the coordinate speed in the opposite direction.

This fact allows an observer on the rotating disk to explain why two light signals emitted from a point M on the disk rim and propagating along the rim in opposite directions will not arrive simultaneously at M - as the coordinate speed of the light signal travelling against the disk rotation is greater that the speed of the other signal it will arrive at M first.

The time it takes a light signal travelling along the rim of the disk in the direction of its rotation to complete one revolution is

$$\Delta t^{+\varphi} = \frac{2\pi r}{c^{+\varphi}} = \frac{2\pi r}{c\,(1 - \omega r/c)} = \frac{2\pi r}{c - \omega r}.$$

The time for the completion of one revolution by the light signal propagating in the opposite direction is:

$$\Delta t^{-\varphi} = \frac{2\pi r}{|c^{-\varphi}|} = \frac{2\pi r}{c\,(1 + \omega r/c)} = \frac{2\pi r}{c + \omega r}.$$

The arrival of the two light signals at M is separated by the time interval:

$$\delta t = \Delta t^{+\varphi} - \Delta t^{-\varphi} = \frac{4\pi\omega r^2}{c^2 - \omega^2 r^2}. \tag{23}$$

The time difference (23) is caused by the different coordinate speeds of light in the $+\varphi$ and $-\varphi$ directions. Here it should be specifically stressed that $c^{+\varphi}$ and $c^{-\varphi}$ are different from c owing to the accelerated motion (rotation) of the disk. In terms of the orbital velocity $v = \omega r$ it appears that the two tangential coordinate velocities can be written as a function of v

$$c^{+\varphi} = c\left(1 - \frac{v}{c}\right) = c - v; \qquad\qquad c^{-\varphi} = c\left(1 + \frac{v}{c}\right) = c + v,$$

which resemble the Galilean law of velocity addition. However it is completely clear that that resemblance is misleading - due to the centripetal (normal) acceleration $a^N = v^2/r$ the *direction* of the orbital velocity constantly changes during the rotation of the disk which means that $c^{+\varphi}$ and $c^{-\varphi}$ depend on the normal acceleration of the disk:

$$c^{+\varphi} = c\left(1 - \frac{\sqrt{a^N r}}{c}\right) \tag{24}$$

and

$$c^{-\varphi} = c\left(1 + \frac{\sqrt{a^N r}}{c}\right). \tag{25}$$

As expected the expressions (24) and (25) are similar to the average coordinate velocities (10) and (11) (for $z_B = 0$) in a sense that all coordinate velocities depend on acceleration, not velocity.

5 Conclusions

The paper revisits the question of the constancy of the speed of light by pointing out that it has two answers - the speed of light is constant in all inertial reference frames but when determined in a non-inertial frame it depends on the frame's proper acceleration. It has been shown that the complete description of the propagation of light in non-inertial frames of reference requires an average coordinate and an average proper velocity of light. The need for an average coordinate velocity was demonstrated in the case of Einstein's thought elevator experiment - to explain the fact that two light signals emitted from points A, and C in Figure 1 meet at B', not at B. It was also shown that an average proper velocity of light is implicitly used in the Shapiro time delay effect; when such a velocity is explicitly defined it follows that in the case of a parallel gravitational field the Shapiro effect is not always a delay effect.

The Sagnac effect was also revisited by defining the coordinate velocity of light in the non-inertial frame of the rotating disk. That velocity naturally explains the fact that two light signals emitted from a point on the rim of the rotating disk and propagating along its rim in opposite directions do not arrive simultaneously at the same point.

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I would like to thank Mark Stuckey for his constructive and helpful comments.

7 Appendix - Shapiro time delay

Although it is recognized that the retardation of light (the Shapiro time delay) is caused by the reduced speed of light in a gravitational field [12, pp. 196, 197], an expression for the average velocity of light has not been derived so far. Now we shall see that the introduction of an average proper velocity of light makes it possible for this effect to be calculated by using this velocity. It is the average proper velocity of light that is needed in the Shapiro time delay since the time measured in this effect is the proper time at a given point.

We shall consider the treatment of the Shapiro time delay in [12, Sec. 4.4]. A light (in fact, a radio) signal is emitted from the Earth (at $z_1 < 0$) which propagates in the gravitational field of the Sun, is reflected by a target planet (at $z_2 > 0$), and travels back to Earth. The path of the light signal (parallel to the z axis) is approximated by a straight line [12, pp. 196]. The distance between this line and the Sun (along the x axis) is b. The total proper time from the emission of the light signal to its arrival back on Earth is [12, pp. 197, 198]:

$$\Delta\tau = 2\left(1 - \frac{2GM_\odot}{c^2\sqrt{z_1^2+b^2}}\right)\left(\frac{z_2+|z_1|}{c} + \frac{2GM_\odot}{c^3}\ln\frac{\sqrt{z_2^2+b^2}+z_2}{\sqrt{z_1^2+b^2}-|z_1|}\right). \tag{26}$$

As the approximated distance between the Earth (at $z_1 < 0$) and the target planet (at $z_2 > 0$) is $z_2 + |z_1|$ we can define the average proper velocity of a light signal travelling that distance as determined on Earth:

$$\begin{aligned} \bar{c}^g_{z_1 z_2}(as\ seen\ from\ Earth) &= \frac{z_2+|z_1|}{\Delta\tau_{Earth}} = \frac{z_2+|z_1|}{\Delta t}\frac{\Delta t}{\Delta\tau_{Earth}} \\ &= c^g_{z_1 z_2}\frac{1}{\left(1-\frac{2GM_\odot}{c^2\sqrt{z_1^2+b^2}}\right)}, \end{aligned} \tag{27}$$

where $c^g_{z_1 z_2}=(z_2+|z_1|)/\Delta t$ is the average coordinate velocity of light and it was taken into account that

$$\Delta\tau_{Earth} = \left(1 - \frac{2GM_\odot}{c^2\sqrt{z_1^2+b^2}}\right)\Delta t$$

is the proper time as measured on Earth and obtained from the Schwarzschild metric (the effect of the Earth's gravitational field is neglected).

We have seen in Section 2 that the average coordinate velocity $c^g_{z_1 z_2}$ can be calculated either as an average over time or over distance, so

$$c^g_{z_1 z_2} = \frac{1}{z_2+|z_1|}\int_{z_1}^{z_2} c'(z)\,dz,$$

where

$$c'(z) = c\left(1 - \frac{2GM_\odot}{c^2\sqrt{z^2+b^2}}\right)$$

is the coordinate velocity of light at a point in the case of the Schwarzschild metric. Then

$$\begin{aligned} c^g_{z_1 z_2} &= \frac{c}{z_2+|z_1|}\int_{z_1}^{z_2}\left(1-\frac{2GM_\odot}{c^2\sqrt{z^2+b^2}}\right)dz \\ &= \frac{c}{z_2+|z_1|}\left(z_2+|z_1| - \frac{2GM_\odot}{c^2}\ln\frac{\sqrt{z_2^2+b^2}+z_2}{\sqrt{z_1^2+b^2}-|z_1|}\right) \\ &= c\left(1-\frac{2GM_\odot}{c^2(z_2+|z_1|)}\ln\frac{\sqrt{z_2^2+b^2}+z_2}{\sqrt{z_1^2+b^2}-|z_1|}\right). \end{aligned}$$

By substituting this expression for the average *coordinate* velocity of light in (27) we can obtain the average *proper* velocity of light in the Schwarzschild metric:

$$\bar{c}^g_{z_1 z_2}(as\ seen\ from\ Earth) = \frac{c}{\left(1-\frac{2GM_\odot}{c^2\sqrt{z_1^2+b^2}}\right)}\left(1-\frac{2GM_\odot}{c^2(z_2+|z_1|)}\ln\frac{\sqrt{z_2^2+b^2}+z_2}{\sqrt{z_1^2+b^2}-|z_1|}\right)$$

or

$$\bar{c}^g_{z_1 z_2}(as\ seen\ from\ Earth) \approx c\left(1+\frac{2GM_\odot}{c^2\sqrt{z_1^2+b^2}}-\frac{2GM_\odot}{c^2\left(z_2+|z_1|\right)}\ln\frac{\sqrt{z_2^2+b^2}+z_2}{\sqrt{z_1^2+b^2}-|z_1|}\right).$$

For the total proper time

$$\Delta\tau = \frac{2\left(z_2+|z_1|\right)}{\bar{c}^g_{z_1 z_2}(as\ seen\ from\ Earth)}$$

from the emission of the light signal to its arrival back on Earth we have

$$\begin{aligned}
\Delta\tau &= \frac{2\left(z_2+|z_1|\right)\left(1-\frac{2GM_\odot}{c^2\sqrt{z_1^2+b^2}}\right)}{c\left(1-\frac{2GM_\odot}{c^2(z_2+|z_1|)}\ln\frac{\sqrt{z_2^2+b^2}+z_2}{\sqrt{z_1^2+b^2}-|z_1|}\right)} \\
&\approx 2\left(1-\frac{2GM_\odot}{c^2\sqrt{z_1^2+b^2}}\right)\left(\frac{z_2+|z_1|}{c}+\frac{2GM_\odot}{c^3}\ln\frac{\sqrt{z_2^2+b^2}+z_2}{\sqrt{z_1^2+b^2}-|z_1|}\right)
\end{aligned}$$

and (26) is recovered. The total proper time can be also written (to within terms proportional to c^{-3}) as

$$\Delta\tau \approx 2\left(\frac{z_2+|z_1|}{c}-\frac{2GM_\odot\left(z_2+|z_1|\right)}{c^3\sqrt{z_1^2+b^2}}+\frac{2GM_\odot}{c^3}\ln\frac{\sqrt{z_2^2+b^2}+z_2}{\sqrt{z_1^2+b^2}-|z_1|}\right).$$

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