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Approximation Methods in Science and Engineering

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There are three types of scientists in the world:
Those who wait for sufficient data,
those who can extrapolate insufficient data

Dedicated to
Mojgan,
Vazan,
Kavosh.

Preface

Since I finished my education in engineering and applied mathematics, I have been observing that there are some subjects that are not being fully covered by textbooks, instructors, or curriculum. The subjects and topics that looked important to me as a curious student were omitted fully or partially from the educational program. Topics such as dimensional analysis, continued fractions, recursive methods, Monte Carlo methods, Ritz and Galerkin methods, applied complex methods, operational mathematics, energy methods, and several other methods are samples of topics that students may nearly miss in their educations. Among those topics, I wish I have been taught "dimensional analysis" to the level that I feel I have mastered model and prototype and nondimensionalization. I have seen that this topic is the lack of knowledge of not only graduates, also most of engineering instructors. Continued fraction is the other topic that has been dropped from the mathematical courses, and there is almost nothing new in textbooks of engineering and applied mathematics. I have learned the continued fractions from the book of Battin, *An introduction to the mathematics and methods of astrodynamics* on astrodynamics and the book of McLachlan, *Theory and Application of Mathieu Functions* in determining stability chart of Mathieu equation. The potential and domain of applicability of continued fraction method has always been a question to be answered. The continued fraction is in connection with several other methods such as series solution, recursive methods, and determinant methods with great advantage in approximation solution of differential equations. This book has been developed to address the lack of knowledge in educational programs of graduates. I covered three topics with a new vision, "dimensional analysis," "continued fractions," and "approximation tools" related to differential equations and continued fractions, plus a new method in determining the stability chart of parametric differential equations called "energy-rate method." I wished to be able to cover more topics like Monte Carlo method, but spending 3 years and around 2500 hours on this book was long enough and the prepared materials were rich enough to be sent for publishing. The purpose of this book is to give the required information to engineers and researchers to become masters on dimensional analysis and some approximate methods for solving differential equations.

Science and engineering students are usually not patient enough to wait for exact mathematical solutions to be discovered. They are also not very well appreciating the complicated mathematical solutions when available. Furthermore, in many cases we are able to show that analytic solution is impossible to develop. They mostly prefer to work with simpler mathematical solutions that are exact enough. The simpler solutions are usually approximate solutions that are exact enough for design and prediction of physical systems. Approximation methods are vast different methods. A few number of approximation methods are general such as Taylor series solution for approximation of functions or solution of differential equations. However, majority of approximation methods are built for particular application and are not considered general enough to be used for a large number of problems.

Dimensional analysis is the topic in the first part of the book. It is divided into static and dynamic dimensional analysis. In static dimensional analysis, we will cover the concepts of physical quantities, units, scales, as well as the principles behind the method of observation, describing, and modeling natural and synthetic phenomena. Dimension, physical quantity, homogeneity, and units are all man-made concepts. There are no such things in nature; however, these concepts help science and engineering students to understand nature better. We see that although dimensional analysis does not provide a solution, it is an exact analytic method that provides the minimum number of variables. In dynamic dimensional analysis, we will cover the practical and engineering use of this area mainly to derive nondimensional constitutive equations, π-groups, and the minimum number of variables toward understanding the model–prototype investigation.

The second part of the book will cover continued fractions, which is divided into two topics: numerical continued fractions and functional continued fractions. In the numerical chapter, we will review the concept and method of continued fractions, its notations, and convergent and fundamental concepts. It also covered the history and understanding of rational, irrational, geometric, and transcendental numbers. In the functional continued fractions, we focus on solution of differential equations in continued fractions as well as the methods of converting a power series into continued fraction, considering continued fractions provide much better approximation than power series.

The third part will cover some approximation tools, in which we use the Mathieu equation and Mathieu functions as the foundation to introduce and compare different methods considering that developing stability chart of parametric differential equation is crucial to engineers and scientists. The book has been written for the technologist, and it is not addressed in any sense to the pure mathematics, for whom I am not qualified to write.

Maybe the most important method of treating differential equation is the "perturbation methods." We have not touched this topic in this book because of two reasons. Firstly, perturbation methods is a topic that is being taught in many universities and is not considered as a missing topic. Secondly, perturbation methods is rich enough to be presented as an individual book.

Level of the Book

This book has been developed from nearly two decades of research and teaching in engineering and mathematics. It is addressed primarily to cover the missing knowledge in the curriculum of graduate students in engineering. Hence, it is an advanced level book that may also be used as a textbook. It provides fundamental and advanced topics needed in computerizing some approximate methods to solve differential equations. The whole book can be covered in one course in 14–16 weeks. Students are required to know the fundamentals of calculus, kinematics, and dynamics, as well as acceptable knowledge in numerical methods and differential equations.

The contents of the book have been kept at a fairly theoretical–practical level. All concepts are deeply explained and their application emphasized, and most of the related theories and formal proofs have been explained. The book places a strong emphasis on the physical meaning and applications of the concepts. Topics that have been selected are of high interest in the field. An attempt has been made to expose students and researchers to the most important topics and applications.

Organization of the Book

The book is organized in such a way that it can be used for teaching or for self-study. Chapter 1 contains basic information on dimensional analysis, physical quantities, and units. It also includes the main theory behind mathematical modeling of physical phenomena. Chapter 2 develops the knowledge of modeling engineering systems and nondimensionalization, develops π-groups, and develops the technics of model–prototype simulation. Chapter 3 introduces a new topic and reviews the concept and methods of numerical continued fractions with a view on number theory. The material in this chapter are required to understand the next chapter. Chapter 4 is about continued fractions that derive approximate solutions of differential equations. It shows that continued fractions are better approximations than power series. Chapter 5 not only covers the theory of Mathieu equations and its stability chart, but it also covers several methods in deriving the transition curves and developing the stability chart of the equation. Chapter 6 introduces a new analytic-numerical method to find stability chart of parametric differential equations with a focus on the Mathieu equation to compare its advantages with respect to the methods in previous chapter.

Method of Presentation

This book uses a "*fact-reason-application*" structure. The "fact" is the main subject we introduce in each section. Then the reason is given as a "proof." The application of the fact is examined in some "examples." The "examples" are a very important part of the book as they show how to implement the "facts." They also cover some other facts that are needed to expand the "fact."

Prerequisites

Since the book is written for researchers and advanced graduate level students of engineering, the assumption is that users are familiar with matrix algebra, numerical analysis, calculus, differential equations, as well as principles of kinematics and dynamics. Therefore, the prerequisites are the fundamentals of kinematics, dynamics, vector analysis, matrix theory, numerical methods, and differential equations.

Unit System

The system of units adopted in this book is, unless otherwise stated, the international system of units (SI). The units of degree (deg) or radian (rad) are utilized for variables representing angular quantities.

Symbols

- Lowercase bold letters indicate vectors. These vectors may be expressed in an n-dimensional Euclidean space. For example:

$$\begin{array}{ccccccccccc} \mathbf{r} & , & \mathbf{s} & , & \mathbf{d} & , & \mathbf{a} & , & \mathbf{b} & , & \mathbf{c} \\ \mathbf{p} & , & \mathbf{q} & , & \mathbf{v} & , & \mathbf{w} & , & \mathbf{y} & , & \mathbf{z} \\ \boldsymbol{\omega} & , & \boldsymbol{\alpha} & , & \boldsymbol{\epsilon} & , & \boldsymbol{\theta} & , & \boldsymbol{\delta} & , & \boldsymbol{\phi} \end{array}$$

- Uppercase bold letters indicate a dynamic vector or a dynamic matrix, such as force and moment. For example:

$$\mathbf{F} \qquad \mathbf{M}$$

- Lowercase letters with a hat indicate unit vectors. These vectors are not bold. For example:

$$\begin{matrix} \hat{\imath} & , & \hat{\jmath} & , & \hat{k} & , & \hat{u} & , & \hat{u} & , & \hat{n} \\ \hat{I} & , & \hat{J} & , & \hat{K} & , & \hat{u}_\theta & , & \hat{u}_\varphi & , & \hat{u}_\psi \end{matrix}$$

- The length of a vector is indicated by a non-bold lowercase letter. For example:

$$r = |\mathbf{r}| \qquad a = |\mathbf{a}| \qquad b = |\mathbf{b}| \qquad s = |\mathbf{s}|$$

- Capital letter B is utilized to denote a body coordinate frame. For example:

$$B(oxyz) \qquad B(Oxyz) \qquad B_1(o_1x_1y_1z_1)$$

- Capital letter G is utilized to denote a global, inertial, or fixed coordinate frame. For example:

$$G \qquad G(XYZ) \qquad G(OXYZ)$$

- An asterisk ★ indicates a more advanced subject or example that is not designed for undergraduate teaching and can be dropped in the first reading.

Melbourne, VIC, Australia — Reza N. Jazar

Contents

Part I
Dimensional Analysis

We compare two physical quantities A and B by a common characteristic, such as length. Then we are able to say, for example, A is longer than B. Then "length" becomes the interested mutual quantity of everything that their length is important to us. We then make an arbitrary and universally constant length to be 1 as the scale to be able to give a number to the length of everything else. Therefore, the dimension of "length" is invented. Similarly, the concept of area, volume, mass, weight, density, temperature, lightness, heat, work, displacement, speed, etc., appeared. However, some of these physical quantities are function of others and, therefore, it becomes necessary to determine the fundamental and basic physical quantities, name their dimension, and define their scales. Then ideally, we will be able to classify, measure, and compare all physical quantities. Equations relationship among physical quantities and employing the concept of physical dimensions gave us the idea to introduce the concept of dimensional homogeneity of equations. This is the story of fundamental and static dimensional analysis.

In the study of physical phenomena we work with inconstant physical quantities and sociably too many variables. This brings the idea of defining the minimum possible and the best variables to describe the behavior of physical phenomena. Hence, the concepts of π numbers and nondimensionalization invented to be able to work with the best variables and also generalize the outcomes of a problem to all similar problems. Then the "similar problem" needed to be clarified to understand how we call two systems to be similar. The study of similarity introduced the science of similitude in which we develop the model and prototype to work identical on physical phenomenon, while their sizes are different. This is the story of applied and dynamic dimensional analysis.

The theory of static and dynamic dimensional analysis are the subjects of this part of the book.

Chapter 1
Static Dimensional Analysis

Science and engineering work with physical equations and mathematical formulas. Equations are based on proportionality, principle of superposition, and dimensional homogeneity. Physical equations must be unit independent as nature does not have preferred units to measure and size the physical quantities.

This chapter reviews the development of the concept of dimensions and shows how to use them in the study of physical phenomena described by mathematical equations. By studying this chapter you will learn why and how the concept of physical dimensions have been invented and expanded, and the method of expressing the relation among physical quantities by mathematical equations. The expansion of the dimensional concept to cover the mathematical equation is the dimensional homogeneity, of which its need and use will be covered in this chapter.

1.1 Base Quantities and Units

Engineering and physical sciences are based on experiments; experiments involve measurements, and measurements need units for comparison. There are seven accepted basic dimensional quantities in engineering and science: Length $[L]$, Mass $[M]$, Time $[T]$, Electric Current $[I]$, Temperature $[\Theta]$, Amount of Substance $[N]$, Luminous $[J]$. They are summarized in Table 1.1.

The base quantities are assumed to be independent. They are the elements of dimensional analysis with their dimension shown in brackets, $[Dimension]$. The dimension of any other physical quantity will be a multiplication of the base dimensions. Any physical quantity whose numerical value depends on the fundamental units is called *dimensional*. Any physical quantity, such as $\pi = 3.14159\cdots$, whose numerical value does not depend on the fundamental units is called *dimensionless*. A physical quantity is dimensional if its numerical value

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Table 1.1 The base physical quantity in International System of Units (SI)

Base physical quantity	Name	Symbol
Length	Meter	m
Time	Second	s
Mass	Kilograms	kg
Electric current	Ampere	A
Thermodynamic temperature	Kelvin	K
Amount of substance	Mole	mol
Luminous intensity	Candela	cd

depends on the utilized scale to measure them. Nondimensional quantities, however, have values that are independent of units.

The definition of the units of the base quantities are supposed to be adapted from unchanging properties of the universe, as explained below.

Meter The meter (m) is the length of the path traveled by light at speed c in vacuum measured in meter/second (m/ s), during a time interval of $1/299792458$ of a second.

It was originally set in 1793 as $1/10000000$ (one ten-millionth) of the distance from north pole to the equator of the Earth. In 1983, physicists measured the speed of light with a great accuracy to fix the length of the meter and make it permanent. Hence, the meter is now defined by an unchanging property of the universe.

Second The second (s) is the duration of time taking the fixed numerical value of the caesium Cs frequency $\Delta\nu_{Cs}$, the unperturbed ground-state hyperfine transition frequency of the caesium 133 atom, to be 9192631770 when expressed in the unit of Hertz $[\text{Hz}] = \left[\text{s}^{-1}\right]$.

The second is the duration of 9192631770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the cesium 133 atom. Second was originally defined as $1/86400$ of a mean solar day. A mean solar day is a measure of time representing the interval between consecutive passages of the Sun across the meridian, averaged over one year, $1\,\text{d} = 24\,\text{h} = 86400\,\text{s}$.

Kilogram The kilogram (kg) is the unit of mass and it is equal to the fixed numerical value of the Planck constant h to be $6.62607015 \times 10^{-34}$ when expressed in the unit of Joule-second J s (equal to $\text{kg}\,\text{m}^2\,\text{s}^{-1}$), where the meter and the second are defined in terms of c and $\Delta\nu_{Cs}$.

This new definition of mass, applied in 2019, is equivalent to fixing the value of Planck constant $h = 6.62607015 \times 10^{-34}\,\text{J s}$ with a standard uncertainty of $0.000000081 \times 10^{-34}$ J s. The Planck constant h is equal to the energy of a quantum of electromagnetic radiation divided by its frequency. Planck assumed that atoms could only vibrate at certain frequencies that are whole number multiples of same base frequency, which he called h. In other words, atoms could vibrate at the frequency h, or $2h$, or $3h$, but not a fraction of h. Therefore, atomic vibrations and energy are quantized, according to $E = h\nu$, where ν (nu) is the frequency in Hz. Albert Einstein (1879–1955) suggested that energy E and mass m are related according to $E = mc^2$, and therefore, matter releases energy in discrete chunks, known as "quanta." Determining exactly how much energy is in a photon let scientists define mass in terms of Planck constant.

A consequence of the new definition of unit mass is that the kilogram is dependent on the definitions of the second and the meter, both based on unchanging properties of the universe.

The kilogram was originally defined as the mass of a liter (cubic decimeter) pure water at the melting point of ice. Similarly, the gram $[\mathrm{g}]$, $1/1000$ of a kilogram, was defined in 1795 as the mass of one cubic centimeter of pure water at the melting point of ice. The kilogram $[\mathrm{kg}]$ manufactured as the international prototype of platinum and iridium alloy cylinder in 1889 equals to the mass of $1\,\mathrm{dm}^3$ of water under atmospheric pressure and at the temperature of its maximum density, 4 °C.

Ampere The ampere [A] is the unit of electric current. It is defined by taking the fixed numerical value of the elementary charge e to be $1.602176634 \times 10^{-19}$ C when expressed in the unit of Coulomb [C], which is equal to Ampere-second [A s], where the second is defined in terms of $\Delta\nu_{Cs}$.

In the past, ampere was defined as the constant current which, if maintained in two straight parallel conductors of infinite length, of negligible circular cross section, and placed 1 meter apart in vacuum, would produce between these conductors a force equal to 2×10^{-7} newton per meter of length.

Kelvin The Kelvin [K] is the unit of thermodynamic temperature. It is defined by taking the fixed numerical value of the Boltzmann constant k to be 1.380649×10^{-23} when expressed in the unit $\mathrm{J\,K^{-1}}$, which is equal to $\mathrm{kg\,m^2\,s^{-2}\,K^{-1}}$, where the kilogram, meter, and second are defined in terms of h, c, and $\Delta\nu_{Cs}$.

Originally, the Kelvin was the fraction $1/273.16$ of the thermodynamic temperature of the triple point of water. In thermodynamics, the triple point of a compressible substance such as water is the unique combination of temperature and pressure at which solid phase, liquid phase, and gaseous phase can all coexist in thermodynamic equilibrium. Figure 1.1 illustrates the PVT surface of water.

The triple point of water has been assigned a value of 273.16 K (0.01 °C, 32.02 °F) and a partial vapor pressure of 611.657 Pa (6.1166 mbar, 0.0060366 atm). At that point, it is possible to change all of the substance to vapor, water, or ice by making arbitrarily small changes in pressure and temperature. The triple point of water, $T_3 = 273.16$ K, was the standard fixed-point temperature for the calibration of thermometers. This agreement sets the size of the Kelvin as $1/273.16$ of the difference between the triple-point temperature of water and absolute zero. The triple state of matter where ice, water, and vapor are in equilibrium is characterized by a temperature called the triple point. The triple state is represented by a line parallel to the PV plane with a characteristic pressure for the substance at variable volume.

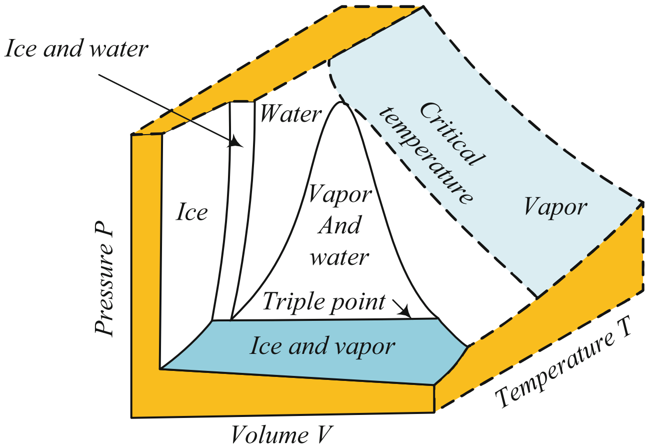

Fig. 1.1 The PVT surface of water

Mole The mole [mol] is the unit of amount of substance. One mole contains exactly $6.02214076 \times 10^{23}$ elementary entities. This number is the fixed numerical value of the Avogadro constant, N_A, when expressed in the unit mol^{-1} and is called the Avogadro number. An elementary entity may be an atom, a molecule, an ion, an electron, or any other particle or specified group of particles.

The mole n [mol] in the past was the amount of substance of a system which contains as many elementary entities as there are atoms in 0.012 kilogram of carbon 12. When the mole is used, the elementary entities must be specified and may be atoms, molecules, ions, electrons, other particles, or specified groups of such particles.

Candela The candela [cd] is the unit of luminous intensity in a given direction. It is defined by taking the fixed numerical value of the luminous efficacy of monochromatic radiation of frequency 540×10^{12} Hz, K_{cd}, to be 683 when expressed in the unit $\mathrm{lm\,W^{-1}}$, which is equal to $\mathrm{cd\,sr\,W^{-1}}$, or $\mathrm{cd\,sr\,kg^{-1}\,m^{-2}\,s^{3}}$, where the kilogram, meter, and second are defined in terms of h, c, and $\Delta\nu_{Cs}$.

The candela was originally defined as the luminous intensity, in a given direction, of a source that emits monochromatic radiation of frequency 540×10^{12} Hz and that has a radiant intensity in that direction of $1/683$ watt [W] per steradian [sr].

Although the number of independent quantities is not necessary to be set to seven, the optimum number of fundamental quantities is the smallest number that is enough to express every other physical quantity. In classical physics and engineering the seven base quantities of length, mass, time, current, temperature, amount of substance, and luminous density are considered independent. The unit of these seven base quantity determines the unit of any other physical quantity. To have a common scientific language, the unit of the base quantities must remain constant and universal. Different unit systems have been introduced and changed during

Table 1.2 SI naturally constant quantities used in the definition of the fundamental units

Physical quantity	Symbol	Value
Unperturbed ground-state hyperfine transition frequency of the caesium 133 atom	$\Delta\nu_{Cs}$	9192631770 Hz
Speed of light in vacuum	c	299792458 m/ s
Planck constant	h	$6.62607015 \times 10^{-34}$ J s
Elementary charge	e	$1.602176634 \times 10^{-19}$ C
Avogadro constant	N_A	$6.02214076 \times 10^{23}$ mol^{-1}
Boltzmann constant	k	1.380649×10^{-23} J/ K
Luminous efficacy of monochromatic radiation of frequency 540×10^{12} Hz	K_{cd}	683 lm/ W

years. There are few commonly used unit systems: MKS Mass System, MKS Force System, CGS System, British Mass System, $USCS$ US Customary.

The most widely accepted unit system is the International System of Units (SI) or MKS Mass System. The letters MKS denotes "meter," "kilogram," and "second." In the MKS Mass System, the kilogram is utilized for the unit of mass. In this system, the unit of force is called "Newton" [N] that is the required force to give an acceleration of 1 m/ s^2 to a mass of 1 kg.

Abstract physical quantity is similar to prime numbers; every number would be either prime or a multiple of prime numbers. The unit system of the seven base quantities determines the units of all other combined physical quantities.

The definition of the fundamental physical quantities are all based on some invariant natural constants, effective from 2019 by the International System of Units, the SI. Table 1.2 summarizes the natural constants used in the definition of the fundamental units.

Example 1 Numeric equation format.

Any numeric equation for a physical quantity needs four elements: the quantity's name, the equality sign (=), the quantity's magnitude, and its dimension, exactly in this order. A person who is 1.8 meter tall, has a height, h, equals to 1.8 meter (m). "Height" is the name, 1.8 is magnitude, "meter" is unit. This numeric equation that assigns a value to a name is shown as:

$$h = 1.8\,\text{m} \tag{1.1}$$

Example 2 Independency of base quantities.

The independency of the seven base quantities is an assumption applicable in classical physics and engineering. For example, we may consider that if we define temperature as the average of kinetic energy of the molecules, then temperature would not be an abstract independent physical quantity. The temperature then would be defined by the units of mass, length, and time.

Example 3 Dimension indicator.

We use brackets [ ] to indicate dimensions in two ways. The following examples show how.

Table 1.3 *SI* and *USCS* units

Base dimension	*SI* system unit	*USCS* system unit
Length L	Meter (m)	Foot (ft)
Time T	Second (s)	Second (s)
Temperature Θ	Kelvin (K)	Rankine (R)
	Celsius (°C)	Fahrenheit (°F)
Electric current Q_e	Ampere (A)	Ampere (A)
Luminous intensity	Candela (cd)	Candle
Amount of substance	Mole (mol)	
Mass M	kilogram (kg)	Pound mass (lb)
Force F	Newton (N)	Pound force (lbf)

Velocity v is measured in meters per second, m/ s. The dimension of meter is Length $[L]$, and dimension of second is Time $[T]$. Therefore, dimension of v is LT^{-1}.

$$[v] = LT^{-1} \tag{1.2}$$

Force F is measured in Newton, N. Force is equal to mass m times acceleration a, $F = ma$. Mass m is measured in kg. Acceleration a is measured in meters per second per second. The dimension of kilogram is $[M]$, and dimension of acceleration is $\left[LT^{-2}\right]$. Therefore, dimension of F is MLT^{-2}.

$$[F] = MLT^{-2} \tag{1.3}$$

If z is a classical mechanic quantity, then dimension of z will be equivalent to $[z]$, and it will be $\left[M^{\alpha}L^{\beta}T^{\gamma}\right]$.

$$[z] = M^{\alpha}L^{\beta}T^{\gamma} \qquad \alpha, \beta, \gamma \in \mathbb{N} \tag{1.4}$$

The dimensions $M, L, T, \cdots$ mean the same with or without brackets.

$$M^{\alpha}L^{\beta}T^{\gamma}\cdots \equiv \left[M^{\alpha}L^{\beta}T^{\gamma}\cdots\right] \tag{1.5}$$

Example 4 *USCS* US Customary System.

Although the *SI* system is the most widely used system of units in the world today, the *USCS* system is one of the several popular engineering systems still in use in the USA. In scientific community, the *SI* is the dominant unit system, and only a few technical journals accept papers using the *USCS* system. However, the *USCS* system is still popular and in use by American people. Table 1.3 indicates *SI* and *USCS* base units.

The greatest difficulty in *USCS* system, for people who are used to *SI* system, arises from the units for mass and force. The unit of mass is the pound-mass (lb) or (lbm), and unit of force the pound-force (lbf). Using Newton's second law, we have:

$$\frac{F}{ma}\left(\frac{\text{lbf}\cdot\text{s}^2}{\text{lb}\cdot\text{ft}}\right) \tag{1.6}$$

which does not equal unity unless we redefine a lbf or a lb, whichever is considered a secondary unit. Consequently, we must modify Newton's second law whenever we use the $USCS$ system as

$$\frac{Fg}{ma} = g\left(\frac{\text{lbf}\cdot\text{s}^2}{\text{lb}\cdot\text{ft}}\right) \tag{1.7}$$

where

$$g\left(\frac{\text{lb}\cdot\text{ft}}{\text{lbf}\cdot\text{s}^2}\right) \tag{1.8}$$

Mass can be either an inertial or a gravitational mass. If mass is gravitational, then a force of one lbf will accelerate a mass of one lb at the rate of $a = 32.174\,\text{ft/s}^2$. Then

$$g = 32.174\left(\frac{\text{lb}\cdot\text{ft}}{\text{lbf}\cdot\text{s}^2}\right) \tag{1.9}$$

and the Newton's law for the $USCS$ system becomes

$$F = \frac{ma}{g} = \frac{ma}{32.174} \tag{1.10}$$

where m must be expressed in lb and the acceleration in ft/s^2 so that the units of force will be lbf.

Example 5 ★Metric Systems.

There are more than one metric system. Some variants of metric systems appeared to address the needs for different applications. The CGS System, MKS Force System, MKS Mass System are the most common variants.

***CGS* System** CGS stands for centimeter, gram, second to indicate the units of length, mass, and time. The CGS system is one of the earliest metric systems. The unit of force in CGS is the dyne, which is defined as the necessary force to give $1\,\text{cm/s}^2$ acceleration to a body of 1 gram mass. The CGS system was developed in the 1860s and found acceptable among the scientific communities. The CGS system is still in use but its base units are too small for everyday industrial application. The electrical and magnetic units were not defined in this system originally.

***MKS* Force System** MKS stands for meter, kilogram, second to indicate the fundamental units of length, mass, and time. Here kilogram is the unit of force. As a consequence, the mass has a derived dimension, which by Newton's second law, the dimension of mass is:

$$mass = \frac{kg}{m/s^2} = \frac{kg\ s^2}{m} \tag{1.11}$$

The force dimension of kilogram in MKS is sometimes written as kgf (kilogram-force) to distinguish it from the mass unit kg in SI. A feature of the MKS system is that mass has no named unit; hence, mass in MKS can only be described by its dimension. The MKS system has been widely used in engineering in non-English-speaking countries, mainly in Europe, and is still in use in many areas.

***MTS* System** MTS stands for meter, tonne, second to indicate the fundamental units of length, mass, and time. MTS system was developed in France for industrial use. It found application in France and the $USSR$ between 1933 and 1955, but is no longer common (Treese 2018).

Example 6 ★Astronomical measurement system.

According to Newton's gravitational law there is an attraction force F between two masses m_1 and m_2 that is inversely proportional to the square of the distance r between the masses. The proportionality ratio is called the gravitational constant G.

$$F = G\frac{m_1 m_2}{r^2} \tag{1.12}$$

In MKS system, the value of the gravitational constant G is:

$$G = 6.67408 \times 10^{-11} \frac{\mathrm{m}^3}{\mathrm{kg\,s}^2} \tag{1.13}$$

Let us make an astronomical units system in which $G = 1$. Assume the unit of length is kilometer and the unit of time is second. The unit of mass in astronomical unit system is called asm. Employing $G = 1$, we have

$$\begin{aligned} G &= 6.67408 \times 10^{-11} \left(\frac{\mathrm{m}^3}{\mathrm{kg\,s}^2}\right) = 6.67408 \times 10^{-11} \left(\frac{0.001^3\,\mathrm{km}^3}{x\ asm\,\mathrm{s}^2}\right) \\ &= \frac{6.67408 \times 10^{-20}}{x} \left(\frac{\mathrm{km}^3}{asm\,\mathrm{s}^2}\right) = 1 \end{aligned} \tag{1.14}$$

$$x = 6.67408 \times 10^{-20} \tag{1.15}$$

and therefore, we find the scale of astronomical mass unit, asm, in terms of kg.

$$1\,\text{kg} = 6.67408 \times 10^{-20}\, asm \tag{1.16}$$

$$1\, asm = 1.498\,3 \times 10^{19}\,\text{kg} \tag{1.17}$$

$$= 4.264 \times 10^{18}\,\text{slug} \tag{1.18}$$

The astronomical unit of force will then be calculated by Newton equation (1.12) for $m_1 = m_2 = 1\, asm$, $r = 1\,\text{km}$

$$F\,(\text{N}) = G\frac{m_1 m_2}{r^2}\left(\frac{\text{kg}\,\text{m}}{\text{s}^2}\right) = 1\frac{\left(6.67408 \times 10^{-20}\right)^2}{\left(0.001^3\right)^2}$$

$$= 4.454\,3 \times 10^{-21}\, asf \tag{1.19}$$

Therefore,

$$1\,\text{N} = 4.454\,3 \times 10^{-21}\, asf \tag{1.20}$$

$$1\, asf = 2.245 \times 10^{20}\,\text{N} \tag{1.21}$$

Example 7 ★Force and mass unit systems.

There is no need to have two different unit systems to accept mass as a base unit in the Mass System and consider force as a base unit in the Force System. To overcome this problem, some scientists suggested to have a unit system in which both mass and force to be fundamental dimension. In such unit system, the Newton equation of motion will be written as:

$$F = kma \tag{1.22}$$

The factor k has the dimension $[k] = FT^2M^{-1}L^{-1}$.

Although such a proposal is correct mathematically, it does not solve many problems and provides no simplicity.

Example 8 ★Length.

The range, accuracy, scale of all measurement units have improved over time along with technology needs and the ability of measurement. Better measurement methods have enabled more accurate measurements and have enabled us to measure smaller and larger things. Measurement of length was the first dimension needed. The US Customary common units of lengths are examples of different scales and their development due to needs and measurement methods. Starting with a base unit, every other unit is usually named as a multiple or fraction of the base unit. However, the multiple or fraction is not always a nice whole number, and also they are not consistent. The US Customary unit systems of lengths are indicated in Table 1.4 (Treese 2018).

Length scales in the past were usually based on parts of the human body. This gave us the units of foot, inch, or cubit (forearm) length. The absolute values of these

Table 1.4 US customary length units

Unit name	Scale
mil	Smallest length unit
line	83.33 mils
inch	12 lines, 1000 mils
hand	4 inches
span	9 inches
foot	12 inches
yard	3 feet
bolt	40 yards
fathom	2 yards
link	$33/50$ foot $\approx 2/3$ foot
rod	25 links
chain	4 rods
furlong	10 chains
mile	80 chains $\equiv$ 8 furlongs $\equiv$ 1760 yards
league	3 miles
cable	120 fathoms

measurements vary for different people. The ability to consistently and accurately measure such units was poor. Therefore, at some points in history, people began using tools to get consistent measurements. A stick with an acceptable standard length or a string with knots at standard lengths might be used. This evolution ended up to the definition of meter and decimal fractions accepted universally (Table 1.4).

It is good to read about the background of a few of mostly British-American old length units.

Inch: originally defined as the width of a man's thumb at the base of the nail in year 1150. Later on in the early fourteenth century, it was redefined as the length of 3 grains of barley, dry and round, placed end to end lengthwise. The old English term "ince', or "ynce," from the Latin "uncia" for the "inch" meant $1/12$. Since 1959 the inch has been defined officially as 2.54 cm.

Foot: originally, this was the length of a man's foot. The Romans brought the division of the foot into 12 inches. At some points, the present absolute value of a foot was developed. This was a Roman unit called a pes or pes pedis. Foot has been used exclusively in English-speaking countries. Foot still is being used in the USA for length and Square Foot in most English-speaking and some other countries for home floor area. Since 1959 the foot has been defined officially as 30.48 cm.

Cubit: it was defined as the length of the forearm to the tip of the middle finger of an average man, this is one of the oldest units. Historically, the actual value of a cubit varied through civilizations from 17 to 21 inches. In ancient Egypt, a cubit was the distance from the elbow to the fingertips. Today a cubit is about 18 inches.

Pace: it was two steps of a man when marching, about 5 feet. The Roman term was passus. The Roman soldiers marched in paces, which were the length of a double step, 1000 paces were a mile.

Mile: it was defined as 1000 paces by the Romans. This has been changed over the years and varied by location. Mile as an English unit of length is equal to 5280 feet, or 1760 yards, and set as exactly 1609.344 meters since 1959.

Example 9 ★The kilogram and weight units.

Among the base units of the International System, the unit of mass is the only one whose name, for historical reasons, contains a prefix. Names and symbols for decimal multiples and submultiples of the unit of mass are formed by attaching prefix names to the unit name "gram" and prefix symbols to the unit symbol "g." 10^{-6} kg = 1 mg (1 milligram) but not 1μ kg (1 microkilogram).

Weight measurement and its unit has a long and rich history of evolution. There are numerous units of weight all around the world. It is good to read about the background of a few of mostly British-American old weight units.

Grain: it was the weight of a wheat seed. This weight definition can be traced back through the Byzantine, Roman, Greek, and Mesopotamia.

Shekel: it is defined as 20 grains today. This measurement also goes back to Mesopotamia. The name comes from the Latin: scrupus, meaning something small.

Dram: it refers to the weight of a Greek drachma coin. It is about $1/16$th ounce.

Ounce: the words ounce and inch have the same origin as $1/12$ of something. However, an ounce is $1/16$th of a pound today.

Pound: this is a basic measurement unit that is equal to 7000 grains or 5250 grains in different old systems.

Ton: ton was the weight of liquid in a large container. The term derives from tunne, which was used in the Scandinavian and Germanic areas. The actual weight represented by a tunne might vary considerably with commodity. Today a metric ton is 1000 kg.

It seems that Persians had the most official standard system of weights and measures in the ancient Persian Empire under the Achaemenid dynasty (550–350BC). The Persian standard of measurement was fixed and enforced throughout the Achaemenid Empire by Darius the Great (522–486BC) around 515BC. Persian measurement system is a key for understanding the ancient systems of measurements as a part of a sexagesimal table (Table 1.5). Their weight scales are shown in Table 1.5 (Bivar 1985; Wikipedia 2019).

Example 10 Metric SI system history.

The name SI is an abbreviated acronym from the French Le Système International d'Unités. In English it is called International System of Units. SI was officially adopted at the 11th General Conference of Weights and Measures in 1960. Although metric dimensions were made legal and widely being used since 1866 in several countries.

In the sixteenth century, the application of a base 10 system for measurement and its usefulness along with non-decimal systems were common in Asia but not

Table 1.5 Sexagesimal units

Unit	Equivalent	Value
halluru	–	0.21 grams
danake	8 halluru	1.39 grams
zwz	3 danake	4.25 grams
shekel	2 zwz, 3 danake	8.40 grams
karsha	10 shekel	84 grams
mina	60 shekel	540 grams
talent	60 mina	30240 grams

in Europe. Flemish mathematician Simon Stevin (1548–1620) in 1586 proposed to create a logical, coherent, and cohesive unit system for decimal weights and measures. In 1668, John Wilkins (1614–1672), the Secretary of the Royal Society of London, proposed the use of the decimal system for length, area, volume, and mass. He proposed the unit of length to be based on the distance traveled by a pendulum with a frequency of one stroke per minute. The good fact was that such a measure could be duplicated anywhere on earth. In 1670, Gabriel Mouton (1618–1694), a French abbot and scientist, and Gottfried Leibniz (1646–1716), a German scientist, proposed decimal measurement systems to have the length scale on a fraction of the length of one of the earth's meridians. In 1791 a Commission of Measures was established in France to adopt a new set of measurements based on unchanging natural phenomena with consistent method. The Commission presented a proposal for a decimal-based system to the French national assembly. The base-10 system and base-12 system have been under discussion and argument for a while until the base-10 supporters prevailed.

Interestingly, one of the metric units suggested by the French Commission of Measures was decimal time, implemented by France in 1793. In the suggested decimal time system, a day was divided into 10 hours, each having 100 min and each minute having 100 s. However, the decimal time was not well-accepted and eventually forgotten.

The metric system in France was initially enacted in 1795 and officially adopted in 1799 and became mandatory throughout the country in 1800. The population at large, however, did not necessarily observe this decree at the beginning.

The original units of measurement in the initial metric system were: Length: meter, 1/10000000 of the length of the earth's meridian from the North Pole to the equator through Dunkirk, France, and Barcelona, Spain. Land Area: are, equal to 100 square meters. Volume of Solids: stére, equal to one cubic meter. Volume of Liquids: liter, equal to one cubic decimeter. Mass: gramme, 1/1000 of a kilogram, derived from a kilogramme, which originally was called a "grave."

Example 11 Other base physical quantities.

There is nothing magical about primary dimensions of length, mass, time, electric current, temperature, amount of substance, and luminous. Their selection is based entirely on practical grounds and experiments. Also, the seven dimensions used in physics are not enough to cover the subjects of other branches of science such as geography, social science, etc. Other dimensions are necessary to be introduced in non-physical sciences. Economy, geography, social sciences are frequently concerned with costs and prices that make it necessary to introduce additional dimension as money or value, indicated by \$. Another measure used in Biology is population P, which will be treated here as a separate dimension (Stahl 1961, 1962). Information Y may be another measure that is being used as independent variable in equations.

Employing the new base quantities of value \$, population P, and information Y, we may introduce other combined variables and find their dimensions such as: population density $\left[PL^{-2}\right]$, land price $\left[\$L^{-2}\right]$, income $\left[\$T^{-1}\right]$, average society information $\left[YP^{-1}\right]$, transport per person $\left[\$L^{-1}Y^{-1}\right]$, etc. (Haynes 1975).

Example 12 Dimensionless quantity.

Dimensionless quantity is a quantity which does not change under a similarity transformation. They are always the ratio of two proportional quantities of the same dimension.

The ratio of

$$e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a} \qquad 0 < e < 1 \tag{1.23}$$

is called the eccentricity of an ellipse and it is a dimensionless quantity as it will not change when all its sizes are increased k times:

$$\frac{kc}{ka} = \frac{c}{a} \tag{1.24}$$

The eccentricity of an ellipse characterizes its shape but not its sizes. Several ellipses that are depicted in Fig. 1.2 all have the fixed length $2a$ of their major axes, whereas their eccentricity e varies, $c = ea$ and $b = a\sqrt{1 - e^2}$. This enables us to see how the eccentricity affects the form of the ellipse: the focuses are drawn together and the minor axis tends to the major one in its length as e decreases.

Passing to the limit as $e \to 0$ we have $e = 0$, $c = 0$, and $b = a$, when we obtain a circle. Consequently, the circle may be regarded as a singular (limiting) case of an ellipse whose focuses merge and coincide with the center of the circle; in this case, the eccentricity is equal to zero. On the contrary, if e approaches 1, the ellipse becomes more and more elongated and it degenerates into a straight line segment in the limiting process (Myŝkis 1972).

Example 13 Maxwell introduced the dimensional symbols.

Clerk Maxwell (1831-1879) used symbols of $[F]$, $[M]$, $[L]$, $[T]$, $[\theta]$ to denote force, mass, length, time, and temperature, respectively (Maxwell 1871, 1894). Maxwell formed any physical quantity as product of these quantities and called the exponent of each of them as their dimensions.

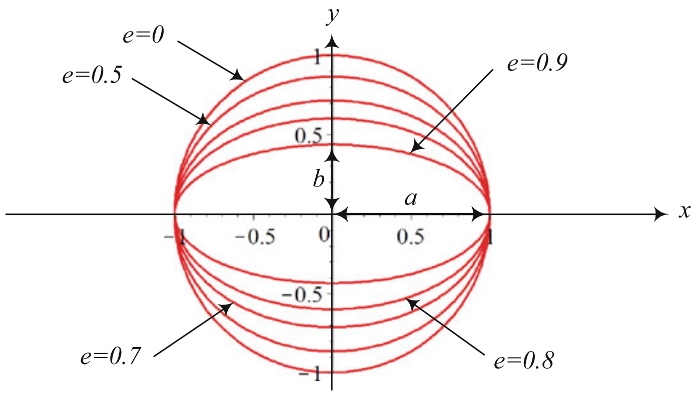

Fig. 1.2 Several ellipses with fixed length $2a$ of their major axes and different eccentricity e

In Maxwell's word: the whole heat conducted during time t in a plate is

$$Q = \frac{abtk}{c}(T_1 - T_2) \tag{1.25}$$

where ab is the area and c the thickness of the plate, t is the time, $T_1 - T_2$ the difference of temperature which causes the flow, and k the specific thermal conductivity of the substance of the plate. It appears, therefore, that the heat conducted is directly proportional to the area of the plate, to the time, to the difference of temperature, to the conductivity, and inversely proportional to the thickness of the plate. From the equation we find

$$k = \frac{cQ}{abt\,(T_1 - T_2)} \tag{1.26}$$

Hence if $[L]$ be the unit of length, $[T]$ the unit of time, $[Q]$ the unit of heat, and $[\theta]$ the unit of temperature, the dimension of k will be $\left[QL^{-1}T^{-1}\theta^{-1}\right]$ (Maxwell 1894).

For a few decades around 1900, scientists always added notes about dimensional analysis in most of their writings. Carlslaw (1921) wrote a book on heat conduction in solids in 1906 for the first edition and discussed the dimensions of the same phenomenon as Maxwell. Consider a solid cylindrical bar of a length d that its two ends are at temperatures T_1 and T_2, where $T_1 > T_2$. When the steady state of temperature has been reached, the quantity Q of heat which flows up through the bar after t seconds over the surface A is equal to:

$$Q = k\frac{T_1 - T_2}{d}At \tag{1.27}$$

where k is at the thermal conductivity of the solid bar material. Strictly speaking, the conductivity k is not constant for the same material, but depends upon the temperature.

$$k = k_0 (1 + aT) \tag{1.28}$$

Since

$$k_0 (1 + aT) = \frac{Qd}{At (T_1 - T_2)} \tag{1.29}$$

the dimension of the heat conductivity k will be

$$[k] = QL^{-1}T^{-1}\theta^{-1} \tag{1.30}$$

and the dimension of coefficient a would be:

$$[a] = \left[k\,\theta^{-1}\right] = \left[QL^{-1}T^{-1}\theta^{-2}\right] \tag{1.31}$$

On the CGS system the unit of heat is the Calorie, the heat quantity which will raise the temperature of 1 gram of water by 1 $°C$.

Example 14 ★Time.

We can measure time physically, in one way only, by counting repeated motions. Apart from physical pulsations we have no natural measure of time. In particular the operation of the astronomical Law of Periodicity supplies us with the principal time units. The primary periodic movements to which we owe our knowledge of time are the two movements of the earth; (1) the rotation of the earth on its axis, which gives us day and night, (2) the revolution of the earth round the sun, which gives us the year and the seasons. A third important periodic motion is (3) the revolution of the moon round the earth, which gives us the month (Philip 1921).

The flow of time seems to be without beginning or end; however, it is cut up periodically by several natural phenomena, namely alternative occurrence of daylight and night, reoccurrence of the moon's phase, reoccurrence of seasons. The day being the smallest unit is usually taken as the fundamental unit of time and the length of months, seasons, and years is expressed in terms of the day. But due to sunrise and sunset in different seasons, the length of the day is not constant. Therefore, gradually we came up to define a day from midnight to midnight, and then came the idea to the mean solar day as the average interval between two successive passages of the sun over the meridian of a fixed place. The astronomers also defined the sidereal day as the period between two successive transits of a fixed star.

The apparent solar time undergoes variations, due to the axial tilt of the Earth as is shown in Fig. 1.3, and the eccentricity of the Earth's orbit. The measurement and scale of time are not supposed to be subject to the variations of the Earth's orbit. As a result, the mean solar time was introduced based on the assumption that

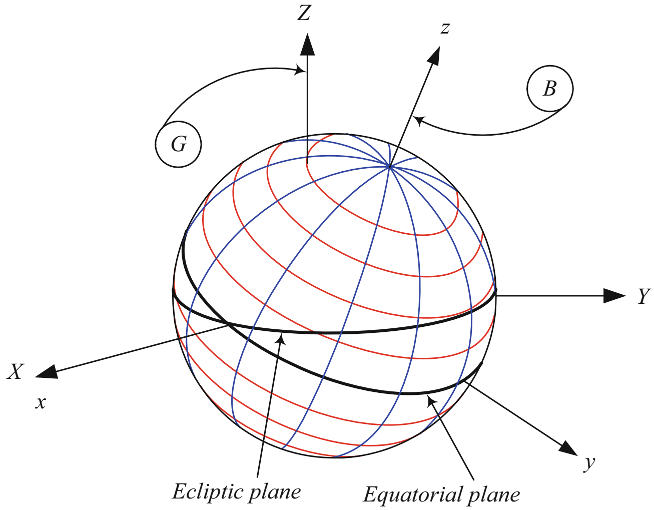

Fig. 1.3 Axial tilt of the Earth

Earth moves uniformly with a constant rate. The rotation of the Earth with respect to the Sun is called Universal Time (UT). It was known that the diurnal rotation with respect to the stars was different from that with respect to the Sun. Hence, a Sidereal time is defined by a measure of the rotation of the Earth with respect to the stars, rather than the Sun.

Scientists discovered that irregularities of the motions of the inner planets and the Moon make the irregularity of the rotation of the Earth (de Sitter 1927; Spencer 1939). Therefore, it was recognized that mean solar time does not satisfy the need for a uniform time scale, and suggested a time scale based on physical laws such as the Newtonian laws of planetary motion (Danjon 1929). Clemence (1948) proposed a uniform fundamental standard time. The proposal was in the tropical year of 1900.0, the period of one complete revolution of the longitude of the Sun with respect to the dynamical equinox, to be the basis for the definition of Ephemeris Time. This proposal was adopted by the International Astronomical Union (IAU) in 1952 and recognized as Ephemeris Time (ET). The second as the unit of time was defined as a fraction of $1/86400$ of the mean solar day until 1960. Then, the second was defined in the International System (SI) of units as a specific fraction of the tropical year after the adoption of ephemeris time. Although this definition was more accurate, it was not constant and measurable with expected accuracy. Following a proposal by Markowitz et al. (1958), a new definition of the "second" was introduced and adopted, in 1967 in terms of a cesium beam frequency which is still valid. The second is the duration of 9192631770 periods of the radiation of the cesium 133 atom. The second is the most accurate measured unit at present, and therefore, along with the speed of light, it is being used to define the unit of length, the meter.

The ISO standard also defines the calendar week number as: ordinal number which identifies a calendar week within its calendar year according to the rule that the first calendar week of a year is that one which includes the first Thursday of that year and that the last calendar week of a calendar year is the week immediately preceding the first calendar week of the next calendar year. An ISO year begins with the Monday between December 29 and January 4 and ends with a Sunday between December 28 and January 3. Accordingly, a year on the ISO calendar consists of 52 or 53 weeks, making the year to be 364 or 371 days long. That is because January 1, 1 (in Gregorian calendar) was a Monday (Dershowitz and Reingold 2008).

The problem of Ephemeris Time was not being consistent with the theory of relativity. In 1976 and then in 1991 the IAU adopted the relationships between space-time coordinates. A Terrestrial Time (TT) has been introduced as the reference time unit of measurement so that it agrees with the SI second on the geoid with a great accuracy. Geocentric Coordinate Time (TCG), Barycentric Dynamical Time (TDB), and Barycentric Coordinate Time (TCB) have also been developed to consider the four-dimensional transformation and the theory of relativity in the definition of time unit (Kovalevsky and Seidelmann 2004; McCarthy and Seidelmann 2009).

The special theory of relativity is based on two postulates: (1) All physical laws and principles are expressed in the same mathematical form in any two systems that move relative to each other with constant velocity. This means we are unable to set any experiment to understand whether the system of reference is at rest or moving with constant velocity. (2) The speed of light in vacuum has the same constant value independent of the velocity of the source or the observer.

Albert Einstein (1879–1955) challenged the thought of absolute space and absolute time concepts, the principles that Newtonian physics are based on. Einstein showed that there is no absolute time and we must view time as a coordinate axis, similar to the space coordinates that can be transformed. The coordinates of space and time can be transformed between two systems of reference moving at constant velocity relative to each other. The length and time, and mass, are then dependent on the relative motion. The space-time transformation equations are known as Lorentz transformations. Let us assume the observer is located in a system of reference G. There is another coordinate frame B that moves at a constant velocity v relative to G, and the speed of light in vacuum is c.

1. A rod resting in system B with length of l_0 will have a shorter length l when remeasured in system G.

$$l = \frac{l_0}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} \tag{1.32}$$

2. If a time period measured in system B is Δt_0, then the observer in system G will measure the longer time Δt,

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} \tag{1.33}$$

3. An object with mass m_0, at rest in system B, has the larger mass m in system G.

$$m = \frac{m_0}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} \tag{1.34}$$

As an example let us assume that a spaceship is launched from a planet P at the beginning of the planet's calendar year of 2020. The spaceship will be traveling at 99.99% of the speed of light for 3 years by the ship's clock. The 3 years in spaceship clock will be 212.1 years in the planet clock, because

$$\frac{3}{\sqrt{1 - \left(\frac{0.9999c}{c}\right)^2}} \simeq 212.1 \tag{1.35}$$

indicating year 2232.

Example 15 ★Einstein's energy–mass relation.

The Einstein equation states that an object's total energy is the product of its mass and the square of the speed of light.

$$E = mc^2 \tag{1.36}$$

As an example the energy radiated every second by the Sun is $\dot{E} = 360 \times 10^{24}$ J/ s and the speed of light is $c \approx 300 \times 10^6$ m/ s. The relation $E = mc^2$ indicates that the Sun is losing 4 million ton of mass per second.

$$\dot{m} = \frac{360 \times 10^{24}}{\left(300 \times 10^6\right)^2} = 4 \times 10^9 \text{ kg/ s} \tag{1.37}$$

If an object with the mass m_0 is accelerated to the velocity v, its mass will be m and then it has been provided with the kinetic energy E_k,

$$E_k = mc^2 - m_0c^2 = m_0 \left[\frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} - 1 \right] c^2 \tag{1.38}$$

This relativistic expression for kinetic energy can be rewritten for very small values of v/c as:

$$E_k \simeq m_0 \left[\frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} - 1 \right] c^2 \simeq m_0c^2 \left[\sqrt{1 - \left(\frac{v}{c}\right)^2} - 1 \right]$$

$$\simeq m_0c^2 \left[1 + \frac{1}{2} \left(\frac{v}{c}\right)^2 - 1 \right] \simeq \frac{1}{2} m_0 v^2 \tag{1.39}$$

which is the classic formula for kinetic energy.

Table 1.6 Dimensions of derived quantities based on the base dimensions of L, T, M, Θ

Category	Physical quantity	Common symbol	Dimensions
Geometry	Area	A	L^2
	Volume	V	L^3
	Second area moment	I, J	L^4
Kinematics	Velocity	V, v, u, c	LT^{-1}
	Acceleration	a	LT^{-2}
	Jerk	j	LT^{-3}
	Angle	α, θ, φ	1
	Angular velocity	ω	T^{-1}
	Frequency	f, ω	T^{-1}
	Quantity of flow	Q	L^3T^{-1}
	Mass flow rate	$\dot{m}$	MT^{-1}
Dynamics	Force	F	MLT^{-2}
	Moment, Torque	M, T, Q	ML^2T^{-2}
	Energy, Work, Heat	E, W, H	ML^2T^{-2}
	Power	P	ML^2T^{-3}
	Pressure, Stress	p, σ, τ	$ML^{-1}T^{-2}$
	Strain	ε	1
Thermofluids	Density	ρ	ML^{-3}
	Viscosity	μ	$ML^{-1}T^{-1}$
	Kinematic viscosity	ν, η	L^2T^{-1}
	Surface tension	σ	MT^{-2}
	Thermal conductivity	k	$MLT^{-3}\Theta^{-1}$
	Specific heat	c_p, c_v	$L^2T^{-2}\Theta^{-1}$
	Bulk modulus	K	$ML^{-1}T^{-2}$

Example 16 Dimensions of derived quantities.

Having identified the base dimensions in physics and engineering, we are able to express all other dimensions of all quantities involved in physical sciences in terms of the base dimensions. Dimensions of most common derived engineering quantities are given in Table 1.6 for future reference.

Example 17 Prefixes used with SI units.

There are a set of prefixes in SI to name large and small fractions of standard units. Table 1.7 shows them for future use.

Example 18 ★Temperature.

Choosing a base unit for temperature is complicated because we can only measure an effect of temperature rather than temperature itself. For example, materials expand when heated, so we may measure temperature by measuring the expansion of a substance such as the height of a column of mercury. We may take a column of mercury, mark its height at the ice–water mixture and label it 0, and then mark its height in boiling water and label it 100, and evenly

Table 1.7 Prefixes using with SI units

Prefix	Symbol	Meaning	Scientific notation
exa	E	1 000 000 000 000 000 000	10^{18}
peta	P	1 000 000 000 000 000	10^{15}
tera	T	1 000 000 000 000	10^{12}
giga	G	1 000 000 000	10^{9}
mega	M	1 000 000	10^{6}
kilo	k	1 000	10^{3}
hecto	h	100	10^{2}
deka	da	10	10^{1}
–	–	1	10^{0}
deci	d	0.1	10^{-1}
centi	c	0.01	10^{-2}
milli	m	0.001	10^{-3}
micro	μ	0.000 001	10^{-6}
nano	n	0.000 000 001	10^{-9}
pico	p	0.000 000 000 001	10^{-12}
femto	f	0.000 000 000 000 001	10^{-15}
atto	a	0.000 000 000 000 000 001	10^{-18}

divide the distance between these two marks into units of measurement called centigrade. Such scale has two problems: First it gets affected by air pressure and indicates different temperatures by pressure fluctuation. Second, the material in use might have different expansion rate at different temperatures. So, one centigrade temperature change at 20 °C may need different amount of energy compared to one centigrade temperature change at 30 °C.

To solve the first problem, the SI convention for temperature has been set based on the observation that water, ice, and water vapor can coexist in equilibrium at only one set of temperature and pressure called triple point of water. By international agreement in 1967, the triple point of water is taken to be at a temperature of 273.15 K at the corresponding pressure 611.2 Pa. In this standard temperature 0 is the absolute zero at which molecules are at the absolute minimum kinetic energy allowed by quantum mechanics. Based on this scale, the freezing temperature of pure water at sea level will happen at 273.15 K and boils at 373.15 K. Figure 1.1 illustrates the equilibrium and the triple point diagram of water.

When writing temperatures in the Kelvin scale, it is the convention to omit the degree symbol and merely use the letter K. The temperature scale is named after the British mathematician and physicist William Thomson Kelvin (1824–1907), who proposed the absolute zero temperature in 1848.

Example 19 ★Modern unit systems.

Since the need of having standard unit systems for national and international trading appeared, many different unit systems have been introduced in different parts of the world each applied for a period of time. We may name The International System of Units (SI), MTS, $MKpS$, $MKSA$, CGS, esu, emu, $Gauss$, International Electrical Units ($IEUS$), Atomic Unit (AU), British Imperial Units, US Customary Units of Length, Foot-pound-second (FPS), etc. There are also many ancient measuring systems applied in the ancient civilizations and empires, namely the old Chinese units system, Indian units system, Egyptian units system, Assyrio-Chaldean-Persian units system, Hebrew units system, Greek units system, Roman units system, Arabic units system, Western and Southern Europe units system, Central and Northern Europe units system, African units system, Middle East and Asia units system, Central and South America units system, North America units system. Most of these units are not applied anymore; however, some of them are still applied partially in different areas. Cardarelli (2003) collected their history, applicability, and conversion.

Example 20 Basic units are all scalars.

All the seven basic dimensional quantities in engineering and science: Length $[L]$, Mass $[M]$, Time $[T]$, Electric Current $[I]$, Temperature $[\Theta]$, Amount of Substance $[N]$, Luminous $[J]$ are scalars. We do not have any basic vector quantity. The scalar quantities are completely given when we know the kind of quantity and how much there is of it. The scalar quantities are specified by giving (1) the unit quantity, (2) the number of units.

The word scalar is used because these quantities can be graphically represented to scale by lengths. Scalae is a Latin word for ladder divided into equal parts by the rungs (Turner 1909). Therefore, if for an example we agree to represent unity by a length of 2 cm, then the number 0 would be represented by a line 10 cm long, and a line of length 21 cm would represent the number 10.5.

Example 21 ★Names of large numbers.

To facilitate reading and remembering large numbers, the digits are usually grouped into groups of three digits from the right, which are separated by comma or blank space. For example, the number 9876543210 is written 9,876,543,210 or 9 876 543 210. Here, the three digits 210 form the first group, 543 the second group, 876 the third group, and 9 the fourth group. Each group has a place value. From right to left, 0 is at units digit position, 1 at tens digit, 2 at hundreds digit, 3 at thousands digit, and so on. The first group makes units, tens, hundreds. The second group makes thousands, the third group millions. After the millions, the American and British standards are different. In American, French, and Russian method, 1000 millions is 1 billion, 1000 billions is 1 trillion, 1000 trillions is 1 quadrillion, 1000 quadrillions is 1 sextillion, and so on for septillions, octillions, noillions, etc. In British and German method, 1000 millions is 1 milliard, 1000 milliards is 1 billion. However, the denomination above billions is 1,000,000 times of the proceeding one, and hence, 1,000,000 billions is 1 trillion, 1,000,000 trillions is 1 quadrillion, etc.

The modern positional system of numeration for illustration of values quantities originated in India and it was widely used by Indians around eighth century, although the positional numeration has been invented by Babylonians in their sexagesimal system, around 5 thousand years ago. However, a decisive role in the spread of Hindu numeration in the world was played by a manual written in Arabic by Persian mathematician, Muhammad ibn Mūsā al-Khwārizmī (780–850AD). Khwārizmī's manual was translated into Latin in Europe in the twelfth century. In less than one century, the Hindu numeration technic became dominant in Italy and by sixteenth century it became dominant in all Europe. Because European took the Hindu numeration technic from Arabic manual of Khwārizmī, they call it Arabic system, a historically incorrect name (Vygodsky 1984).

Introducing "sunia" by Indians, which means "empty position" in Sanskrit, to act as a placeholder was an intelligent idea that made positional numeration applied. "Sunia" translated to "sifr" in Arabic, with the same meaning of "empty position." It was written as "ciphir" in Latin and later on in fifteenth century the term "zero" and "null" appeared to name the empty position in numeration chain.

1.2 Dimensional Homogeneity

Principle of *Dimensional homogeneity* states that: all equations are dimensionally homogenous, and are independent of the measurement unit system.

Dimensional homogeneity means: all terms of an equation must have the same dimensions. Let y be a function of several terms $A_1, A_2, A_3, \cdots$

$$y = A_1 + A_2 + A_3 + \cdots \tag{1.40}$$

in which every term A_i is a function of several variables $x_1, x_2, x_3, \cdots$

$$A_i = f_i\,(x_1, x_2, x_3, \cdots) \tag{1.41}$$

then Eq. (1.40) is dimensionally homogeneous if and only if $y, A_1, A_2, A_3, \cdots,$ have the same dimensions. The symbol f_i indicates a functional operator on the independent variables $x_1, x_2, x_3, \cdots$ to calculate the terms $A_1, A_2, A_3, \cdots$, and eventually the dependent variable y.

Independency of measurement unit system means: All equations must be correct irrelevant of the employed unit system. Let y be a function of several variables $x_1, x_2, x_3, \cdots$

$$y = f\,(x_1, x_2, x_3, \cdots) \tag{1.42}$$

If we change the units such that the variables x_1, x_2, x_3, $\cdots$ take new values of y', $x'_1, x'_2, x'_3, \cdots$, then

$$y' = f\left(x'_1, x'_2, x'_3, \cdots\right) \tag{1.43}$$

where f is the same function as (1.42).

We assume a set of abstract independent base physical quantities that every other physical quantity can be expressed based on them. Then, every other physical quantity will be set as a product of the base quantities, each with an exponent. The exponent indicates the dimension of that quantity. Dimensional homogeneity requires the dimension of each base quantity to be equal in every term and on both sides of a physical equation. A dimension indicates how the numerical value of a quantity changes when the basic units of measurement change. The abstract base physical quantities in the Newtonian mechanics are: Length $[L]$, Time $[T]$, Mass $[M]$, Temperature $[\theta]$. A physical quantity is anything that is expressible in certain units and completely characterized by its numerical value. Dimension of physical quantities are all human made concepts. There is no such thing as dimension in nature. However, these concepts help science and engineering to interpret and model the nature.

Dimensional Analysis will not provide a solution; however, it provides the minimum number of variables in a problem and it is an exact analytic method.

The value of any physical quantity is expressed by its numerical value and its unit of measurement.

$$physical\ quantity = numerical\ value \times unit \tag{1.44}$$

Proof Assume there are 1000 physical equations and 100 of them are dimensionally homogenous naturally. We separate the 100 dimensionally homogenous equations form the rest and make two groups: dimensionally homogeneous equations, H-group, and dimensionally nonhomogeneous equations, N-group.

There is a very useful and productive property among the equations of the H-group. Any mathematical calculation such as derivation, substitution, etc., that only employs equations of the H-group to generate a new equation, will definitely produce a homogeneous equation. To take maximum use of this important property, we take every equation of the N-group and give a proper dimension to its proportionality coefficient to make the equation synthetically dimensionally homogeneous. This equation can now be put in the H-group. Having done the homogenization on all equations of the N-group makes all the 1000 equations to be in H-group and empty the N-group. This homogenization process will also be done on any possible new discovered equation. Therefore, all equations that we have in science and engineering are dimensionally homogamous.

Examples of the H-group are:

Pythagoras' equation: When a triangle has a right angle (90°) and squares are made on each of the three sides, then the biggest square c^2 (the square of the hypotenuse, the side opposite the right angle) has the exact same area as the other two squares put together $a^2 + b^2$.

$$c^2 = a^2 + b^2 \tag{1.45}$$

$$\left[L^2\right] = \left[L^2\right] + \left[L^2\right] \tag{1.46}$$

If we show the dimension of length by $[L]$, then Pythagoras' equation is dimensionally homogenous.

Archimedes volume equation: The Archimedes equation for volume of a sphere is another naturally homogenous equation.

$$V = \frac{4}{3}\pi r^3 \tag{1.47}$$

$$\left[L^3\right] = \left[L^3\right] \tag{1.48}$$

Dimension of volume V is $\left[L^3\right]$, dimension of radius r is $[L]$, and constant numbers are dimensionless.

Examples of the N-group are:

Planck's equation: The energy E contained in a photon is directly proportional to the frequency f of the photon.

$$E = hf \tag{1.49}$$

$$[FL] \neq \left[T^{-1}\right] \tag{1.50}$$

$$F = MLT^{-2} \tag{1.51}$$

If E (force times distance) is given in Joules $[J]$ and f (inverse of time) is given in Hertz $[Hz]$, then $h = 6.626176 \times 10^{-34}$. The proportionality constant h must have the value of energy of a photon in J for one frequency change in Hz. This is the smallest possible "packet" of energy in an electromagnetic wave. If we give the Planck's constant h the dimension of $[E/f] = [FLT]$, then the Planck's electromagnetic quantum radiation energy becomes dimensionally homogeneous.

$$E = hf \tag{1.52}$$

$$[FL] = [FLT]\left[T^{-1}\right] \tag{1.53}$$

Newton's law of cooling: The rate of change of the temperature y of an object is proportional to the difference between its temperature y and the ambient temperature y_o.

$$\frac{dy}{dt} = k\left(y - y_o\right) \tag{1.54}$$

$$\Theta T^{-1} \neq \Theta \tag{1.55}$$

The constant k indicates the temperature rate for one unit of temperature difference. If we give the Newton's cooling constant k the dimension of $\left[T^{-1}\right]$, then the cooling equation becomes dimensionally homogeneous.

$$\frac{dy}{dt} = k\,(y - y_o) \tag{1.56}$$

$$\Theta T^{-1} = T^{-1}\,\Theta \tag{1.57}$$

Let us assume that Eq. (1.42) is an equation in classical mechanics and hence only the dimensions of $[M]$, $[L]$, $[T]$ are involved. Changing the unit system will provide us with

$$\begin{aligned} &1 \text{ original mass unit } M = k_M \text{ new mass unit } M' \\ &1 \text{ original length unit } L = k_L \text{ new length unit } L' \\ &1 \text{ original time unit } T = k_T \text{ new time unit } T' \end{aligned} \tag{1.58}$$

therefore, the value of a physical quantity x_i in the original unit system, and its value in the new unit system x_i' are related as

$$\begin{aligned} x_i \left[M^{a_i} L^{b_i} T^{c_i}\right] &= x_i' \left[k_M^{a_i}\; M^{a_i}\; k_L^{b_i}\; L^{b_i}\; k_T^{c_i}\; T^{c_i}\right] \\ &= x_i' \left[M'^{\,a_i}\; L'^{\,b_i}\; T'^{\,c_i}\right] \end{aligned} \tag{1.59}$$

and therefore the numerical value of the variables in new and original unit systems is related by the conversion factors $k_M^{a_i}\; k_L^{b_i}\; k_T^{c_i}$,

$$x_i' = x_i\; k_M^{a_i}\;\; k_L^{b_i}\;\; k_T^{c_i} \tag{1.60}$$

or in expansion as:

$$\begin{aligned} y' &= y\; k_M^{a}\; k_L^{b}\; k_T^{c} = k\; y \\ x_1' &= x_1\; k_M^{a_1}\; k_L^{b_1}\; k_T^{c_1} = k_1 x_1 \\ \cdots &= \cdots \\ x_n' &= x_n\; k_M^{a_n}\; k_L^{b_n}\; k_T^{c_n} = k_n x_n \end{aligned} \tag{1.61}$$

that indicates

$$k = k_M^{a}\; k_L^{b}\; k_T^{c} \tag{1.62}$$

$$k_i = k_M^{a_i}\, k_L^{b_i}\, k_T^{c_i} \qquad i = 1, 2, \cdots, n \tag{1.63}$$

substitute them back in Eq. (1.42) will show that:

$$kf(x_1, x_2, \cdots, x_n) = f(k_1 x_1, k_2 x_2, \cdots, k_n x_n) \tag{1.64}$$

If Eq. (1.64) is an identity in the variables $x_1, x_2, \cdots, k_M, k_L, k_T$, then Eq. (1.43) fulfills both conditions of the homogeneity and unit independency.

As an example, let us consider the sphere volume equation.

$$V = \frac{4}{3}\pi r^3 \tag{1.65}$$

The volume V, for a radius $r = 2\,\text{m}$ is

$$V = \frac{4}{3}\pi\,(2)^3 = 33.51\,\text{m}^3 \tag{1.66}$$

Changing the unit of length from meter to inches shows that

$$1\,\text{m} = 39.37\,\text{in} \tag{1.67}$$

$$k_L = 39.37 \tag{1.68}$$

$$r\,[L] = r'\,[k_L L] = r'\,[39.37\,L] \tag{1.69}$$

$$r'\,[\text{in}] = k_L r\,[\text{m}] = 39.37\,r\,[\text{m}] \tag{1.70}$$

therefore,

$$V' = k_L^3 V = 39.37^3 V = 39.37^3 \times 33.51 = 2.0449 \times 10^6\,\text{in}^3 \tag{1.71}$$

and hence the equation is unit invariant.

$$V' = \frac{4}{3}\pi r'^3 \tag{1.72}$$

■

Example 22 Simple pendulum period, an illustrative example.

Period T of small oscillations of a simple pendulum shown in Fig. 1.4 with length of string l and tip mass m is:

$$T = 2\pi\sqrt{\frac{l}{g}} \tag{1.73}$$

Let us assume we do not have the period equation and wish to obtain it by considering every possible variables in the dynamics of the system, length l, mass

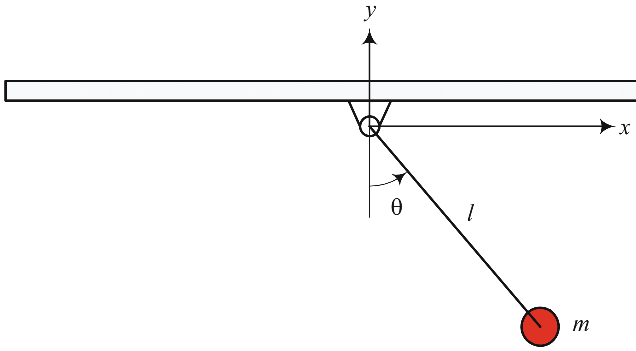

Fig. 1.4 A simple pendulum with length of string l and tip mass of m

m, gravitational acceleration g. Therefore, we will be looking for an equation of the following form:

$$T = f\,(l, m, g) \tag{1.74}$$

or

$$f\,(T, l, m, g) = C \tag{1.75}$$

where C is a dimensionless constant. The dimensional homogeneity implies that:

$$[f\,(T, l, m, g)] = [C] = 1 \tag{1.76}$$

Assuming

$$f\,(T, l, m, g) = f\left(T^a l^b m^c g^d\right) = C \tag{1.77}$$

and applying dimensional homogeneity

$$[f] = T^a L^b M^c \left(LT^{-2}\right)^d = C^0 = 1 \tag{1.78}$$

provides us with 3 equations for 4 unknown exponents.

$$a - 2d = 0 \tag{1.79}$$

$$b + d = 0 \tag{1.80}$$

$$c = 0 \tag{1.81}$$

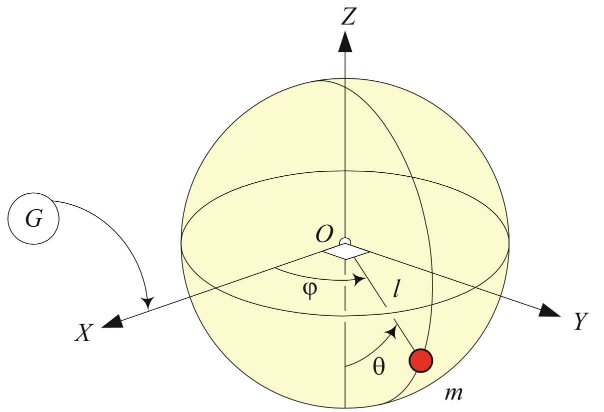

Fig. 1.5 A spherical pendulum

We may pick $a = 1$ and find

$$a = 1 \qquad d = \frac{1}{2} \qquad b = -\frac{1}{2} \qquad c = 0 \tag{1.82}$$

that indicates there is a dimensionless π-group.

$$\pi_1 = T\sqrt{\frac{g}{l}} = C \tag{1.83}$$

$$T = C\sqrt{\frac{l}{g}} \tag{1.84}$$

This example indicates how dimensional analysis works. In model theory when we study a model and prototype to simulate the real system with a larger or smaller model, we must make both systems to have the same π_1.

$$\frac{T_1}{\sqrt{l_1/g}} = \frac{T_2}{\sqrt{l_2/g}} \tag{1.85}$$

Two simple pendulums work similarly as long as $\pi_1 = T/\sqrt{l/g}$ are equal in the two systems. The π_1-number is the only dimensionless variable in linear pendulum dynamics. This π_1-number or π_1-group is nondimensionalized period of the system. This method of analysis based on balance of exponents is called **Rayleigh** method.

Example 23 A spherical pendulum equations of motion.

Figure 1.5 illustrates a spherical pendulum with mass m and length l. The angles φ and θ may be used as generalized describing coordinates $q_1 = \varphi$, $q_2 = \theta$ of the kinematics of the system.

The Cartesian coordinates of m as a function of the generalized coordinates are

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} r\cos\varphi\sin\theta \\ r\sin\theta\sin\varphi \\ -r\cos\theta \end{bmatrix} \tag{1.86}$$

and therefore, the kinetic energy K and potential energy P of the pendulum are:

$$K = \frac{1}{2}m\left(l^2\dot{\theta}^2 + l^2\dot{\varphi}^2\sin^2\theta\right) \tag{1.87}$$

$$P = -mgl\cos\theta \tag{1.88}$$

The Lagrangian function $\mathcal{L} = K - P$ of this system is then equal to:

$$\mathcal{L} = K - P = \frac{1}{2}m\left(l^2\dot{\theta}^2 + l^2\dot{\varphi}^2\sin^2\theta\right) + mgl\cos\theta \tag{1.89}$$

Therefore, every term of the Lagrangian has to have the dimension of energy $[E] = ML^2T^{-2}$.

$$\begin{aligned}
[\mathcal{L}] &= [K - P] = \left[\frac{1}{2}ml^2\dot{\theta}^2\right] + \left[\frac{1}{2}ml^2\dot{\varphi}^2\sin^2\theta\right] + [mgl\cos\theta] \\
&= \left[\frac{1}{2}\right][m]\,[l]^2\left[\dot{\theta}\right]^2 + \left[\frac{1}{2}\right][m]\,[l]^2\,[\dot{\varphi}]^2\,[\sin\theta]^2 + [m]\,[g]\,[l]\,[\cos\theta] \\
&= (1)\,ML^2\left(\frac{1}{T}\right)^2 + (1)\,ML^2\left(\frac{1}{T}\right)^2(1)^2 + M\left(LT^{-2}\right)L\,(1) \\
&= ML^2T^{-2} + ML^2T^{-2} + ML^2T^{-2} = ML^2T^{-2}
\end{aligned} \tag{1.90}$$

Employing the Lagrange equation on this conservative system (Jazar 2011)

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_r}\right) - \frac{\partial\mathcal{L}}{\partial q_r} = 0 \qquad r = 1, 2 \tag{1.91}$$

leads us to the following equations of motion:

$$\ddot{\varphi}\sin^2\theta + 2\dot{\varphi}\dot{\theta}\sin\theta\cos\theta = 0 \tag{1.92}$$

$$\ddot{\theta} - \dot{\varphi}^2\sin\theta\cos\theta + \frac{g}{l}\sin\theta = 0 \tag{1.93}$$

Dimensional homogeneity check shows that

$$\left[\ddot{\varphi}\sin^2\theta\right]+\left[2\dot{\varphi}\dot{\theta}\sin\theta\cos\theta\right]=[\ddot{\varphi}]\,[\sin\theta]^2+[2]\,[\dot{\varphi}]\left[\dot{\theta}\right][\sin\theta]\,[\cos\theta]$$
$$=\left(\frac{1}{T}\right)^2(1)^2+(1)\left(\frac{1}{T}\right)\left(\frac{1}{T}\right)(1)\,(1)$$
$$=T^{-2}+T^{-2}=T^{-2} \tag{1.94}$$

$$\left[\ddot{\theta}\right]-[\dot{\varphi}]^2[\sin\theta]\,[\cos\theta]+\left[\frac{g}{l}\right][\sin\theta]=\left(\frac{1}{T^2}\right)(1)\,(1)+\frac{LT^{-2}}{L}(1)$$
$$=T^{-2}+T^{-2}=T^{-2} \tag{1.95}$$

both equations of motion are dimensionally homogenous as all terms have the same dimension.

A few facts to notice: (1) Every term of both equations of motion has dimension of angular acceleration, T^{-2}. It is not necessary that all equations of motion of a dynamic system have the same dimension as T^{-2}. It depends on the dimension of the generalized coordinates. (2) Transcendental equations, in this case $\sin\theta$ and $\cos\theta$, are dimensionless, $[\sin\theta]=1$, $[\cos\theta]=1$. The argument of all transcendental equations, in this case θ, is also dimensionless $[\theta]=1$.

Example 24 Bernoulli law of fluid dynamics.

In 1738, Daniel Bernoulli (1700–1782) discovered the law of fluid dynamics that states: the total energy of fluid pressure, gravitational potential energy, and kinetic energy of a moving fluid remains constant

$$\frac{v^2}{2}+gz+\frac{p}{\rho}=C \tag{1.96}$$

where v is the fluid velocity, g the acceleration due to gravity, z the elevation of a point in the fluid, p the pressure, ρ the fluid density, and C is a constant. Scientists prior to Bernoulli had understood that a moving body exchanges its kinetic energy for potential energy when the body gains height. Bernoulli realized that, in a similar way, a moving fluid exchanges its kinetic energy for pressure (Pickover 2008).

The Bernoulli equation has 4 terms that their dimensions must be equal.

$$\left[\frac{v^2}{2}\right]+[gz]+\left[\frac{p}{\rho}\right]=[C] \tag{1.97}$$

$$\left(LT^{-1}\right)^2+\left(LT^{-2}\times L\right)+\frac{MLT^{-2}/L^2}{ML^{-3}}=[C] \tag{1.98}$$

$$L^2T^{-2}+L^2T^{-2}+L^2T^{-2}=[C] \tag{1.99}$$

Therefore, $[C]=L^2T^{-2}$.

Example 25 Stokes drag formula.

Consider the Stokes' drag force F_d equation for a sphere moving slowly in a viscous fluid.

$$F_d = 3\pi \mu D v \tag{1.100}$$

The force F_d has dimension of $[F_d] = MLT^{-2}$, the viscosity $[\mu] = L^{-1}MT^{-1}$, the velocity $[v] = LT^{-1}$, and the diameter $[D] = L$. Substituting all dimensions in the Stokes equation

$$MLT^{-2} = \left(L^{-1}MT^{-1}\right)(L)\left(LT^{-1}\right) = MLT^{-2} \tag{1.101}$$

indicates a dimensionally homogeneous equation (Yarin 2012).

Example 26 Dimensional arithmetic.

The product of the dimensions of two quantities Q_1, Q_2 is the dimension of the product of two quantities.

$$[Q_1 Q_2] = [Q_1][Q_2] \tag{1.102}$$

Similarly, the product of the dimensions of a set of n quantities is the dimension of the product of the set of n quantities.

$$[Q_1 Q_2 Q_3 \cdots] = [Q_1][Q_2][Q_3]\cdots \tag{1.103}$$

The quotient of the dimensions of any two quantities Q_1, Q_2 is the dimension of the quotient of the two quantities.

$$\left[\frac{Q_1}{Q_2}\right] = \frac{[Q_1]}{[Q_2]} \tag{1.104}$$

If D_1, D_2, and D_3 are the dimensions of the quantities Q_1, Q_2, and Q_3, respectively, then

$$D_1(D_2 D_3) = (D_1 D_2) D_3 \tag{1.105}$$

The dimension of a power of a quantity is the power of the dimension of that quantity.

$$\left[Q^n\right] = [Q]^n \tag{1.106}$$

To indicate the minor differences between dimensional analysis arithmetic rules and the numerical arithmetic rules, we summarize them here (Bhargava 1991):

1. The dimension of the sum or difference of two functional expressions is the same as the dimension of either of them. The two expressions must have equivalent dimensions.

2. The dimension of the product or quotient of two functional expressions is the product or quotient of the dimensions of the two expressions. A dimensionless expression has dimension of 1, the identity for dimensional multiplication.
3. The dimension of the exponent of an expression is the exponent of its dimension.

4. Any expression of dimensionless functions yields a dimensionless expression.

Example 27 Derivative and integral.

According to dimensional analysis derivative is a division and integration is a multiplication. Dimension of dy/dx is $[dy]/[dx]$ and dimension of $\int y dx$ is $[y][dx]$.

Velocity is derivative of distance with respect to time.

$$v = \frac{dx}{dt} \tag{1.107}$$

The dx is an increment of length and dt is an increment of time. Therefore,

$$[v] = \left[\frac{dx}{dt}\right] = \frac{[dx]}{[dt]} = \frac{L}{T} = LT^{-1} \tag{1.108}$$

Similarly, acceleration is derivative of velocity

$$a = \frac{dv}{dt} \tag{1.109}$$

and therefore,

$$[a] = \left[\frac{dv}{dt}\right] = \left[\frac{d^2x}{dt^2}\right] = \frac{[dv]}{[dt]} = \frac{LT^{-1}}{T} = LT^{-2} \tag{1.110}$$

To match with dimensional homogeneity we indicate the second derivative by d^2x/dt^2 and not d^2x/d^2t or dx^2/dt^2; similarly, the nth derivative is shown by $d^n y/dx^n$.

Example 28 Variable, parameter, constant.

Any term of an equation is made of a product of three items: variable, parameter, constant.

A **variable** can take on different values. Variables usually have dimensions and their value depends on the units system, such as speed, force, etc. If a variable has no dimension, it is always the ratio of a variable and a quantity with the same dimension, such as angle, friction coefficient, etc. A variable which takes on all the numerical values or all the values lying between some limits is called continuous. On the contrary, a variable which assumes certain separated values is called discrete. The set of all numerical values which may be assumed by a variable is called the range of the variable.

A variable x is independent from variables $x_1, x_2, \cdots, x_n$ when the value of x cannot be determined from $x_1, x_2, \cdots, x_n$. For example, the variables $x_1 = mv$ and $x_2 = m^2v^2$ are not independent because if we know either, we know the other. However, the variables $x_1 = mv$ and $x_2 = mv^2$ are independent because we cannot find $x_1 = mv$ by knowing $x_2 = mv^2$ or vice versa. More precisely, we will say x is independent of $x_1, x_2, \cdots, x_n$ if x cannot be written in the form $x = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_3}$ for any choice of the exponents a_i.

A **parameter** is a variable that takes on a constant value in an analysis. Parameters usually have dimensions and their value depends on units system, such as Young's modulus, gravitation acceleration, etc. If a parameter has no dimension, it is always the ratio of a parameter and a quantity with the same dimension, such as Poisson's ratio, friction coefficient, etc.

A **constant** is a number that appears as a coefficient in a term. Constants usually have dimensions, such as speed of light, universal gravitational constant, etc. If the value of the constant does not change by units change, the constant is dimensionless, such as $\pi = 3.1415\cdots$, $e = 2.71828\cdots$, etc. The dimension of a dimensionless quantity, variable, parameter, or constant is indicated by [1]. The value of a dimensionless constant will not change by changing the units system.

For instance, suppose the traction forces of the tires of a 4-wheel vehicle are all equal. The traction force F of the vehicle then would be

$$F = 4C_s \frac{R\omega - v_x}{R\omega} \tag{1.111}$$

In this equation, the force $F\left[LT^{-2}\right]$ is the dependent variable, the vehicle forward velocity $v_x\left[LT^{-1}\right]$ is a variable, the wheel angular velocity $\omega\left[T^{-1}\right]$ is another variable, the radius of the wheel $R\,[L]$ is a parameter, the traction coefficient $C_s\left[LT^{-2}\right]$ is a parameter, the number 4 is a dimensionless constant (Jazar 2017, 2019).

Szirtes (2007) argues that a physical quantity, either a variable, a parameter, or a constant, whose dimension is [1] should not acquire the adjective "dimensionless," or "nondimensional" as they have dimension [1]. The argument is following the fact that in an equation such as

$$y = f(x) = 2 + 3x + x^2 - 6x^3 \tag{1.112}$$

the second term is not called "exponentless," or the third term is not called "coefficientless."

Example 29 Dimensional consistency.

We may summarize the laws for obtaining dimensionally consistent expressions as below (Bhargava 1991).

1. Two functional expressions may be added or subtracted only if they are dimensionally equivalent. Hence, $f + g$ and $f - g$ are dimensionally equivalent if f and g are dimensionally equivalent; $[f] = [g]$.

2. Two functional expressions may be compared for equality or inequality in a conditional expression only if their dimensions are equivalent. Hence, $f = g$, $f > g$, $f < g$, $f \geq g$, $f \leq g$ are valid expressions if f and g are dimensionally equivalent; $[f] = [g]$.
3. Two functional expressions may be multiplied irrespective of their dimensions. Hence, $f \times g$ and f/g are always valid expressions regardless of the dimensions of f and g.
4. Any functional expression may be reciprocated irrespective of its dimension. Hence, $1/f$ is always valid expression regardless of the dimension of f.
5. The exponent of a functional expression must be dimensionless. Hence, f^g is dimensionally consistent if only g is dimensionless; $[f^g] = [f]$.
6. The exponent of a functional expression can be fractional only either if each fundamental unit in the functional expression has a power that is a multiple of the inverse of that fraction or if the functional expression is dimensionless. Hence, $f^{1/n}$, $n \in \mathbb{N}$ is dimensionally consistent if every fundamental unit in f has a power of kn, $k \in \mathbb{N}$. The expression of f^g is dimensionally consistent if the fundamental units of f^g have integer powers; means either g is an integer, or g is dimensionless, or if every fundamental unit in f has a power that is a multiple of $1/g$.

Example 30 Angle dimension $[A] = \left[LL^{-1}\right] = [1]$.

Euclidian definition of angle is: A plane angle is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line (Euclid 2007).

Angle is a variable in any mechanical system in which rotation is a kinematic variable indicated as a generalized coordinate; a pendulum, for example. Angle's dimension is not any one of the usual dimensions $[M]$, or $[L]$, or $[T]$. It can be considered as another independent quantity. Let us show angle's dimension by $[A]$, stands for Angle. We will show that $[A] = [1]$ is the best option for science and engineering application.

Angle is a measure of turn, and so it is closely related to circle. Turn is with respect to a point O as the center of the turn. To measure the turn between two points A and B, we need to measure the turn between lines OA and OB. Therefore, turn is the rotational distance between two rays from a center point. To make a measurement unit of turn, we need a circle, say with unit radius, and divide the circle into n equal radial slices, say $n = 100$. Therefore, a turn α is measurable as a number k of fractional turns of a complete turn, indicated by k/n.

$$angle\ \alpha = \frac{k}{n} \qquad 0 < k < n \tag{1.113}$$

It shows that measuring an angle can be done by percentage, assuming a 100% angle is a complete turn. Percentage inherently indicates a dimensionless quantity unless we synthetically assign a dimension to it. In case a turn of more than a complete round is of interest, then we may use percentage more than 100. Theoretically, dividing a complete turn into fractions can be done by any number such as 360 (degrees), 400 (grads), etc. The necessity of percentage measurement of turns differentiates angles with other dimensional quantities.

The angle is dimensionless not because of the definition of radian as arc length over radius length. As another example of percentage quantities, we can mention efficiency. Strain ε in elasticity is another example that is measured as change of length Δl of an unstressed original length l_0.

$$\varepsilon = \frac{\Delta l}{l_0} = \frac{l - l_0}{l_0} \tag{1.114}$$

As far as we know, Babylonians divided a circle into 360 parts maybe based on the closeness of this number to the length of the year, 365 days. The 360 divisions of a complete turn fitted well with the Babylonian sexagesimal (base 60) numeration system. Today, one section of the 360 sections of a complete turn is called a "degree." The sexagesimal was later adopted by the Greeks and used by Ptolemy in his table of musical chords. The sexagesimal numeration system survived in the division of a circle into 360 degrees in angular measure, as well as in the division of an hour into 60 min and a minute into 60 s (Cajori 1893, 1915, 1928–1929, Maor 1998).

Roger Cotes (1682–1716), an English mathematician, employed the concept of radian and introduced its usefulness (Maor 1998; Cotes 1738) although the term radian first appeared in 1873 by James Thomson (brother of Lord Kelvin). Cotes was the first person who clarified its use as a unit of angular measure. The radian is a unit of angular measure defined such that one radian is the angle opened from the center of a circle with radius R when it produces an arc with length R.

$$angle\,[\,\mathrm{rad}] = \frac{arc\ length}{radius} \tag{1.115}$$

One radian is approximately 57.3 deg.

$$180\,\mathrm{deg} = 3.14\,\mathrm{rad} = \pi \tag{1.116}$$

$$1\,\mathrm{rad} = \frac{180}{\pi}\,\mathrm{deg} = 57.296\,\mathrm{deg} \tag{1.117}$$

The idea of measuring angles by the length of an arc was already in use by other mathematicians before Cotes. For example, the Iranian mathematician, al-Kashi (Jamshid Kashani 1380–1429) used diameter parts as units where one diameter part was $1/60$ radian. He also used sexagesimal subunits of the diameter part (Qurbani 1989; Nejad and Aliabadi 2015). However, the advantage of radian over other angular units remained hidden until calculus was discovered.

The Greeks used the word $\mu o\iota\rho\alpha$ (moira) (means portion), which the Persian physician and translator Hunayn (830–910) translated it to "daraja" (means scale) (Hockey et al. 2007). "Daraja" moved to the Latin as "de gradus," from which came the word "degré" in old French and then "degree." The Greeks called the sixtieth part of a degree the "pars minuta prima" (means first small part), and the sixtieth part of the "pars minuta prima" as "pars minuta secunda" (means second small part), from which came the words "minute" and "second" for $1/60th$ of a degree, and $1/60th$ of a minute, respectively (Maor 1998; Edleston 1850).

Measuring angles using the units of radian confuses us to assume that angles are dimensionless because radian is a ratio of variable length over a constant length. However, this is a true fact that we also measure other physical quantities by treating each of them as a variable over a constant unit of their dimension. Length of a line is measured by ratio of their length over the length of a meter, time is measured as the duration over duration of a second, etc. A unit divisor of the same dimension is a need to make a quantity measurable. However, there is a useful difference between angles and length in modeling and similitude. Two geometrically similar figures will have equal corresponding angles, but they have proportional length ratio. Sedov (1993) correctly argues that if in all unit systems, angles are to be measured in radians, then angle could be treated as a dimensionless quantity. Following the same method, if in all unit systems we keep the unit of length fixed, then length also could be treated as a dimensionless quantity (Sedov 1993). Therefore, all quantities may be assumed dimensionless. However, we invented dimensions to make the science of modeling and similitude.

Giving a dimension, say $[A]$, to angles has been followed, suggested, and developed by several scientists in the past, all trying to come up with a logic to overcome the challenges of dimensional inhomogeneity of equations such as $v\left[LT^{-1}\right] = r\omega\left[LAT^{-1}\right]$, including angles (Romain 1962; Mohr and Phillips 2015; Morikawa and Newbold 2005; Page 196). Regardless that International Standard Organization (ISO 1993) recognizes angles as dimensionless quantity, there were several attempts to introduce a method that not only makes angles dimensional, but also be able to distinguish work and energy dimensionally. All these treatments are based on writing a physical quantity z in the form

$$z = \{z\}\,[z] \tag{1.118}$$

where z is the physical quantity, $[z]$ is the unit of z, and $\{z\}$ is the numerical value of z expressed in the unit $[z]$. As an example, the kinetic energy $K = \frac{1}{2}mv^2$ will be written as

$$\{K\}\,[K] = \frac{1}{2}\,\{m\}\,[m]\,(\{v\}\,[v])^2 \tag{1.119}$$

$$\{K\}\;\mathrm{J} = \frac{1}{2}\,\{m\}\,\{v\}^2\;\mathrm{kg\,m^2\,s^{-2}} = \frac{1}{2}\,\{m\}\,\{v\}^2\;\mathrm{J} \tag{1.120}$$

Radian makes formulae involving circular functions much simpler. In describing rotational motion, a circular arc of angle $\theta\,[\mathrm{rad}]$ provides us with an arc length displacement s:

$$s = R\theta \tag{1.121}$$

where R is the radius of the associated circle to s. If θ is in degrees, then the corresponding formula would be

$$s = \frac{\pi}{180} R\theta \tag{1.122}$$

The area A of a circular sector of angle θ [rad] is

$$A = \frac{1}{2} R^2 \theta \tag{1.123}$$

If θ is in degrees, then the area A of a circular sector would be

$$A = \frac{\pi}{360} R^2 \theta \tag{1.124}$$

Using radians, the derivative formulas for harmonic function, such as sine, are simpler:

$$\frac{d}{d\theta} \sin\theta = \cos\theta \tag{1.125}$$

If θ is in degrees, then

$$\frac{d}{d\theta} \sin\theta = \frac{\pi}{180} \cos\theta \tag{1.126}$$

Similarly, derivative of an arc-angle relation

$$s = r\theta \tag{1.127}$$

provides us with a simple velocity equation.

$$v = \frac{ds}{dt} = r\frac{d\theta}{dt} = r\omega \tag{1.128}$$

If θ is in degrees, then

$$v = \frac{ds}{dt} = \frac{\pi}{180} r \frac{d\theta}{dt} = \frac{\pi}{180} r\omega \tag{1.129}$$

Also if θ was in degrees, then the Euler identity would read

$$e^{i\pi\theta/180} = \cos\theta + i\sin\theta \tag{1.130}$$

and the approximation of $\sin\theta \approx \theta$ for $\theta << 1$ would be

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = \frac{\pi}{180} \tag{1.131}$$

Example 31 Milliradians, Mil, mrad [mrad].

One milliradian (mrad) is the angle at the center of a circle, made by an arc of length 1 with a radius 1000. The milliradian (0.001 rad or 1 mrad) is used in military and targeting. mrad indicates an object of 1 m at a range of 1000 m. At such small angles, the curvature can be considered negligible. Therefore, an object of size a seen at an angle α mrad is at a distance $d = 1000a/\alpha$. If the distance d is known, then the size a of the object is $a = \alpha d/1000$. It means α mrad at 1 km is about 1 m.

A circle is divided into 2000π milliradians. Therefore, a milliradian is $1/6283$ of a circle. Different associations adopted an approximated number close to the exact each $1/6283$ to simplify calculations. $NATO$ uses 1 mrad $\approx 1/6400$ of a circle, Soviet Union and Finland use 1 mrad $\approx 1/6000$ of a circle, Sweden uses 1 mrad $\approx 1/6300$ of a circle.

Example 32 ★ π as a dimensionless constant.

A number is transcendental if it is not algebraic which means it is not the solution of any polynomial equation with integer coefficients. Any real number is either algebraic or transcendental but not both. A real number x is algebraic if it is the solution to some polynomial equations with all the coefficients $a_0, a_1, a, \cdots, a_n$ to be integers.

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0 \qquad n \in \mathbb{N} \tag{1.132}$$

As a few examples, $5/6$ is the solution of $6x - 5 = 0$; $\sqrt{3}$ is the solution of $x^2 - 3 = 0$;

$$x = -\frac{49}{9\sqrt[3]{\frac{370}{27} - \frac{1}{27}\sqrt{27}\sqrt{713}}} - \sqrt[3]{\frac{370}{27} - \frac{1}{27}\sqrt{27}\sqrt{713}} - \frac{7}{3} \tag{1.133}$$

is the solution of $2 + 7x^2 + x^3 = 0$; and $-\frac{1}{2a_2}\left(a_1 + \sqrt{a_1^2 - 4a_0 a_2}\right)$ is the solution of $a_0 + a_1 x + a_2 x^2 = 0$.

Beginning with a unit length representing the number 1, we will be able to make numbers $2, 3, 4, \cdots$, and any rational number such as $1/2$, $5/6$, a/b, $(a, b) \in \mathbb{N}$, as well as irrational lengths involving only square roots such as $\sqrt{2}$, $\sqrt{7}$, etc. Figure 1.6 illustrates a few rational lengths constructed by a unit length.

We will also be able to make their sum, difference, product, or quotient such as

$$\frac{\sqrt{3 + 4\sqrt{2}}}{7 - \sqrt{11 + \sqrt{19 - \sqrt{7}}}} \tag{1.134}$$

however, no transcendental number can be made in this way.

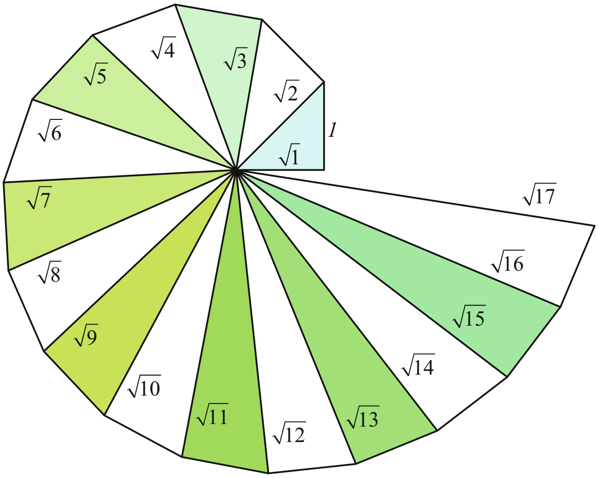

Fig. 1.6 Rational lengths constructed by a unit length

π is the ratio of a circle's circumference to its diameter. π is transcendental, and it means π is not algebraic and thus not constructible. This is equivalent to: the quadrature of the circle is impossible; means circles cannot be squared.

Let us assume that circles can be squared. Using a compass, we may construct a circle having radius $r = 1$. The area A of the circle would be $A = \pi r^2 = \pi$. If circles are quadrable, then we would be able to draw a square with length x and area $A = x^2 = \pi$. Hence the length $x = \sqrt{\pi}$ would be the length of a side of the square. Then $\sqrt{\pi}$ and π would be constructible and hence algebraic. As $\sqrt{\pi}$ and π are not algebraic, the quadrature of the circle is impossible.

Example 33 Exponential decaying force.

Consider a point mass m that is under an exponentially decaying force

$$F(t) = ce^{-t} \tag{1.135}$$

where c is a constant. The Newton equation of motion is:

$$m\frac{dv}{dt} = ce^{-t} \tag{1.136}$$

The velocity of the mass is found by separation of variables and integration

$$m\int_{v_0}^{v} dv = \int_{t_0}^{t} ce^{-t}\, dt \tag{1.137}$$

$$v = v_0 + \frac{c}{m}\left(e^{-t_0} - e^{-t}\right) \tag{1.138}$$

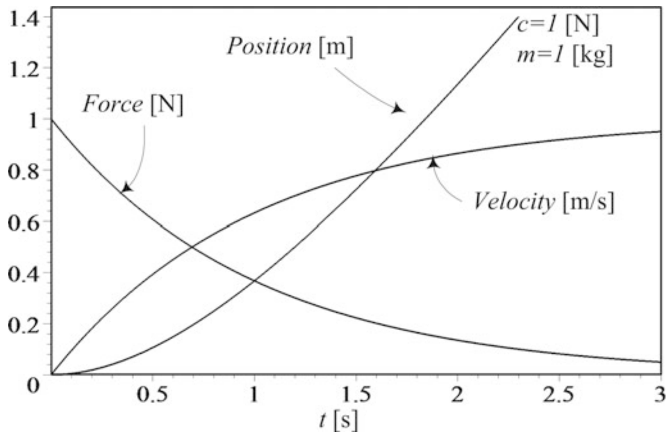

Fig. 1.7 Position x and velocity v for force $F = ce^{-t}$

which by using $v = dx/dt$, we find the position of the mass.

$$\int_{x_0}^{x} dx = \int_{t_0}^{t} \left(v_0 + \frac{c}{m}\left(e^{-t_0} - e^{-t}\right)\right) dt \tag{1.139}$$

$$x = x_0 - \frac{c}{m}\left(1 + t_0 - t\right) e^{-t_0} + v_0\left(t - t_0\right) + \frac{c}{m} e^{-t} \tag{1.140}$$

If the initial time t_0 is assumed to be zero, then the position and velocity of the mass are simplified to

$$x = x_0 - \frac{c}{m}\left(1 - t\right) + v_0 t + \frac{c}{m} e^{-t} \tag{1.141}$$

$$v = v_0 + \frac{c}{m}\left(1 - e^{-t}\right) \tag{1.142}$$

Figure 1.7 illustrates the force, velocity, and position of m for

$$m = 1\,\text{kg} \qquad c = 1\,\text{N} \tag{1.143}$$

$$x_0 = 0 \qquad v_0 = 0 \tag{1.144}$$

Let us review this problem from a dimensional analysis viewpoint. The applied force is

$$F(t) = ce^{-t} \tag{1.145}$$

As the exponential function is transcendental, it must be dimensionless. Therefore,

$$[F] = [c]\left[e^{-t}\right] = [c]\,[1] = [c] \tag{1.146}$$

and hence, dimension of c must be the same as force,

$$[c] = [F] = MLT^{-2} \tag{1.147}$$

The exponential function is dimensionless and it must have dimensionless argument. Therefore the coefficient 1 of t in the exponent of e^{-t} must have a dimension of T^{-1}.

$$F(t) = ce^{-kt} \tag{1.148}$$

$$k = 1 \qquad [k] = T^{-1} \tag{1.149}$$

To determine velocity Eq. (1.137) becomes

$$m\int_{v_0}^{v} dv = \int_{t_0}^{t} ce^{-kt}\, dt \tag{1.150}$$

$$v = v_0 + \frac{c}{mk}\left(e^{-kt_0} - e^{-kt}\right) \tag{1.151}$$

$$[v] = [v_0] + \left[\frac{c}{mk}\right]\left[\left(e^{-kt_0} - e^{-kt}\right)\right] \tag{1.152}$$

$$LT^{-1} = LT^{-1} + \frac{MLT^{-2}}{MT^{-1}}[1] = LT^{-1} + LT^{-1} \tag{1.153}$$

Another integration provides us with the position of the mass.

$$\int_{x_0}^{x} dx = \int_{t_0}^{t}\left(v_0 + \frac{c}{mk}\left(e^{-kt_0} - e^{-kt}\right)\right)dt \tag{1.154}$$

$$x = x_0 - \frac{c}{mk^2}\left(1 + kt_0 - kt\right)e^{-kt_0} + v_0\left(t - t_0\right) + \frac{c}{mk^2}e^{-kt} \tag{1.155}$$

$$[x] = [x_0] - \left[\frac{c}{mk^2}\right][1] + [v_0\left(t - t_0\right)] + \left[\frac{c}{mk^2}\right][1] \tag{1.156}$$

$$L = L - \frac{MLT^{-2}}{MT^{-2}} + LT^{-1}T + \frac{MLT^{-2}}{MT^{-2}} = L - L + L + L \tag{1.157}$$

Substituting the known value of the coefficient $k = 1$ reduces Eq. (1.155) to (1.140). Ignoring the hidden dimension of the coefficient $k = 1$, $[k] = T^{-1}$ in Eq. (1.135) may result in equations that look to be dimensionally nonhomogeneous.

Example 34 Variable, parameter, constant in the period of a pendulum shown in Fig. 1.4.

The Lagrangian of a simple pendulum

$$\mathcal{L} = \frac{1}{2}ml\dot{\theta}^2 - mgl\cos\theta \tag{1.158}$$

provides us with the following equation of motion:

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0 \tag{1.159}$$

If the maximum value of θ is very small

$$\sin\theta = \theta - \frac{1}{6}\theta^3 + \frac{1}{120}\theta^5 + O\left(\theta^7\right) \tag{1.160}$$

we may accept

$$\sin\theta \simeq \theta \tag{1.161}$$

to have a simplified linear model of the pendulum dynamics.

$$\ddot{\theta} + \frac{g}{l}\theta = 0 \tag{1.162}$$

The general solution of this small amplitude model of pendulum is

$$\theta = C_1 \cos\sqrt{\frac{g}{l}}t + C_2 \sin\sqrt{\frac{g}{l}}t \tag{1.163}$$

where $C_1$and C_2 are functions of initial conditions. Assuming

$$\theta\left(0\right) = \theta_0 \qquad \dot{\theta}\left(0\right) = 0 \tag{1.164}$$

we have

$$C_1 = \theta_0 \qquad C_2 = 0 \tag{1.165}$$

and therefore,

$$\theta = \theta_0 \cos\sqrt{\frac{g}{l}}t \tag{1.166}$$

The solution indicates a harmonic oscillation of the pendulum with amplitude θ_0 and frequency $\omega = \sqrt{g/t}\,[\mathrm{rad}/s]$. Converting $[\mathrm{rad}/s]$ to $[cycle/s] = [\mathrm{Hz}]$

$$\omega\,[\mathrm{rad}/s] = 2\pi f\,[\mathrm{Hz}] \tag{1.167}$$

the period of oscillation $T\ [T]$ of the pendulum with length $l\ [L]$ would be

$$T = 2\pi\sqrt{\frac{l}{g}} \tag{1.168}$$

where $g\left[LT^{-2}\right]$ is the gravitational acceleration. The coefficient $2\pi = 6.2832$ is a dimensionless constant as its value will not change by changing the unit system.

Example 35 Newton equation of motion and force.

In Newton's Principia Mathematica we read the following definitions (Chandrasekhar 1995):

Mass m: "The quantity of matter is the measure of the same arising from its density and bulk conjointly." That means Mass m is the amount of material in a body.

The mass of an object is assumed to be constant as the mass of an object never seems to change. As Feynman says: "A spinning top has the same weight as a still one. So a 'law' was invented: mass is constant, independent of speed. That "law" is later found to be incorrect. Mass is found to increase with velocity, but noticeable increases require velocities near that of light" (Feynman et al. 2010).

Momentum p: "The quantity of motion is the measure of the same, arising from the velocity and quantity of matter conjointly." That means momentum of a body is equal to the mass of the body times its velocity.

$$quantity\ of\ motion = mass \times velocity \tag{1.169}$$

$$p = m\,v \tag{1.170}$$

Force F: "A force is an action exerted upon a mass, in order to change its state, either of rest, or of uniform motion in a right line. The quantity of a force is the measure of the same, proportional to the motion which it generates in a given time." That means Force is the reason of change of momentum.

Newton's second law: "The change of motion is proportional to the force impressed and is made in the direction of the right line in which that force is impressed." This law is the equation of motion that states the time rate of momentum is proportional to the applied force.

$$F \propto \frac{d\,(mv)}{dt} = m\frac{dv}{dt} = ma \qquad [F] = MLT^{-2} \tag{1.171}$$

Assuming the velocity is already defined as change of position in a given time.

$$v = \frac{dx}{dt} \qquad [v] = LT^{-1} \tag{1.172}$$

It is the Newton's second law that defines the dimension of force F as $[F] = MLT^{-2}$.

Newton gravitational equation: The attraction force between two masses m_1 and m_2 is proportional to m_1 and m_2 and inversely proportional to the square of the distance r between m_1 and m_2.

$$F \propto \frac{m_1 m_2}{r^2} \tag{1.173}$$

$$MLT^{-2} \neq M^2L^{-2} \quad (1.174)$$

Newton gravitational law is an equation of the N-group, as the dimension of both sides is not the same. A constant of proportionality G makes the proportionality to be an equation. Giving a dimension of

$$[G] = M^{-1}L^3T^{-2} \quad (1.175)$$

makes the gravitational equation dimensionally homogeneous.

$$F = G\frac{m_1 m_2}{r^2} \quad (1.176)$$

$$MLT^{-2} = \left[M^{-1}L^3T^{-2}\right]\left[M^2L^{-2}\right] = MLT^{-2} \quad (1.177)$$

Time t: "Absolute, true and mathematical time, of itself, and from its own nature, flows equably without relation to anything external." Newton followed the assumption on space and time that was generated in Euclidean era and well defined by Descartes. Newtonian world assumes that the universe goes is the three-dimensional space of geometry, and things change in a medium called time. Time exists and flows independently and equably without relation to anything external. Newton's ideas about absolute time were borrowed from Isaac Barrow, his predecessor at Cambridge (Child 1916; Feingold 1990). As Barrow described, quantity of motion cannot be discerned without Time. Time denotes not an actual existence, but a certain capacity or possibility for a continuity of existence; just as space denotes a capacity for intervening length. Time implies motion to be measurable; without motion we could not perceive the passage of Time. Time may be used as a measure of motion; we compare motions with one another by the use of time as an intermediary. Time has analogies with a straight line, and therefore may be represented by a line; as time has length alone.

Example 36 All formulas are approximate.

The laws of nature are all approximate, as the whole nature is only an approximation to the complete truth so far as we know it. For example, it is an assumption that mass of an object is constant and independent of other base physical quantities. The truth is that if an object moves with a speed v will have a mass m such that

$$m = \frac{m_0}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} \quad (1.178)$$

where m_0 is the mass of the object at $v = 0$. When the object is moving less than 1000 km/ s the mass may be considered constant, as it will not change more than five in a million. Therefore, in all engineering applications constant mass approximation is a useful practical law and the correct velocity dependent law adds nothing but complexity. However, having a correct or more correct law must always be appreciated as it expands the limit of our understanding of the world (Feynman et al. 2010).

The knowledge of engineering and science are based on experiment. Experiment is the best judge of scientific truth of a theory, model, formula, equation, to show they are working well or not. Experiments also help us to discover laws. Imagination, experience, and guess are other tools to discover natural laws.

Example 37 Exact, approximate, experimental solutions.

For any problem in science, we have three types of solutions. In order of their importance they are: exact, approximate, experimental. Langhaar (1951) indicates the simplest answers to the question of why exact solution is much better than experimental solution. Assume we wish to determine a function experimentally. A function of one variable may be plotted as a single curve. A function of two variables will be presented as a chart of a family of curves; one curve for different values of the second variable. A function of three variables will be presented by a set of charts, with each chart for a value of the third variable. A function of four variable needs a set of sets of charts. Assuming 10 experiments for each curve, 100 experiments are needed to plot a chart and 1000 for a set of charts and 10000 for a set of sets of charts in order to present a function of four variables. This analysis is too expensive and quickly gets out of hand. Therefore, in case exact and approximate solutions were not available, having the minimum number of variables would be a great advantage. Dimensional analysis will do this (Langhaar 1951).

Example 38 ★Dimensional homogeneity in early years of science.

Dimension was originally associated with geometric length, distance, and line. In Euclidean geometry a point has dimension zero; a shape with length alone has dimension one; an area has dimension two; a volume has dimension three. Although there were some argument among scientists such as Plato and Aristotle on the definition of something (a point) that exists but does not exist (Heath 1956; Euclid 2007).

Geometric dimension was the only physical quantity with clear understanding and definition for centuries.

Euclid himself in "Definitions" of "Book I" defines point and line as:

1. A point is that of which has no part.
2. A line is a length without breadth.
3. The extremities of a line are points.
4. A straight line is a line which lies evenly with the points on itself.
5. A surface is that which has length and breadth only.
6. The extremities of a surface are lines.
7. A plane surface is a surface which lies evenly with the straight lines on itself.

Since thousands of years ago it was known that areas were proportional to the product of two lines, and volumes to the product of three lines. By analogy, the name of "square" and "cube" has been used for a^2 and a^3 of any natural number a. Similarly, every product of two lengths was interpreted as an area or bidimensional, and of three lengths as volume or tridimensional (Martins 1981). The association between dimension and exponent remained in practice until nineteenth century when dimension has no longer been utilized for “power,” “exponent,” “degree.”

Dimensional homogeneity in early stages of civilization appeared as Apples will be added to Apples and Oranges to Oranges. Similarly, it is meaningless to speak of one kilometer equaling one week; neither can these quantities be added or subtracted. All quantities that are to be added, subtracted, or equated must be measured in the same dimensions (Haynes 1975).

Equivalency of dimension that established by Pythagorean made it clear that an area could only be added to an area, length to length, and volume to volume. Therefore, any geometric equation must consist of terms representing the same kind of things.

As we know, it was Descartes who first introduced the principle of homogeneity: the algebraic sum of the exponents of the lengths in each term of an equation must be a constant number. This number is the degree of dimension of the equation (Martins 1981). Fourier took the Descartes' concept of geometric dimension and introduced the physical dimension.

Example 39 Transcendental functions are all dimensionless.

In any physical and engineering formula involving transcendental functions, the argument and the transcendental function are always dimensionless. In equations such as $y = \sin(\omega t)$, $y = \exp(\alpha x)$, etc., the arguments (ωt), (αx) are dimensionless resulting y to be dimensionless too. If $[t] = T$, then $[\omega] = T^{-1}$. Similarly, $[\alpha] = [1/x] = [x]^{-1}$.

Consider the dimensional homogeneity of Euler identity

$$e^{ix} = \cos x + i \sin x \qquad i^2 = -1 \tag{1.179}$$

$$\left[e^{ix}\right] = [\cos x] + [i]\,[\sin x] \tag{1.180}$$

$$[1] = [1] + [i]\,[1] \tag{1.181}$$

indicates that the imaginary unit number, $i^2 = -1$, is dimensionless.

$$[i] = [1] \tag{1.182}$$

Substituting $x = \pi$ provides us with an interesting equation.

$$e^{i\pi} = 1 \tag{1.183}$$

Substituting $x = \pi/2$ provides us with

$$e^{i\frac{\pi}{2}} = i \tag{1.184}$$

$$\left(e^{i\frac{\pi}{2}}\right)^i = i^i \tag{1.185}$$

$$e^{-\frac{\pi}{2}} = i^i \tag{1.186}$$

which is an even more interesting equation, showing that i^i is a real number.

$$i^i = 0.20787957635076190854695561983497877\cdots \tag{1.187}$$

Squaring both sides of Eq. (1.186)

$$e^{-\pi} = i^{2i} = \left(i^2\right)^i = (-1)^i \tag{1.188}$$

provides us with expression of the Gelfond's constant e^{π} (Gelfond 1960).

$$e^{\pi} = (-1)^{-i} = 23.1406926328 \tag{1.189}$$

Example 40 Power series approximation.

Consider a variable y to be a function of variables $x_1, x_2, x_3, \cdots$

$$y = f(x_1, x_2, x_3, \cdots) \tag{1.190}$$

The approximation theory in mathematics states that f can be approximated in a power series form.

$$y = C_1\left(x_1^{a_1} x_2^{b_1} x_3^{c_1} \cdots\right) + C_2\left(x_1^{a_2} x_2^{b_2} x_3^{c_2} \cdots\right) + \cdots \tag{1.191}$$

where C_1, C_2, C_3,$\cdots$ are dimensionless constant coefficients and a_1, a_2,$\cdots$, b_1, b_2,$\cdots$, c_1, c_2,$\cdots$ are dimensionless exponents. Dimensional homogeneity requires that

$$\begin{aligned} a_1 &= a_2 = a_3 = \cdots = a \\ b_1 &= b_2 = b_3 = \cdots = b \\ c_1 &= c_2 = c_3 = \cdots = c \\ &\vdots \quad \vdots \quad \vdots \quad \vdots \end{aligned} \tag{1.192}$$

and therefore y can be rewritten in this form

$$y = (C_1 + C_2 + C_3 + \cdots)\left(x_1^a\, x_2^b\, x_3^c \cdots\right) = C\, x_1^a x_2^b x_3^c \cdots \tag{1.193}$$

$$C = C_1 + C_2 + C_3 + \cdots \tag{1.194}$$

where

$$\begin{aligned} [y] &= \left[C\, x_1^a x_2^b x_3^c \cdots\right] = [C]\left[x_1^a\right]\left[x_2^b\right]\left[x_3^c\right]\cdots = [1]\,[x_1]^a\,[x_2]^b\,[x_3]^c \cdots \\ &= [x_1]^a\,[x_2]^b\,[x_3]^c \cdots \end{aligned} \tag{1.195}$$

and therefore, if the original equation is dimensionally homogeneous, then its power series approximation is also dimensionally homogenous.

As an example, consider a function $y = \exp(kx)/x$, where $[y] = T$, $[x] = T$, therefore, y must be of the form:

$$y = \frac{e^{k_1 x}}{k_2 x} \tag{1.196}$$

$$[k_1 x] = [1] \qquad [k_1] = \frac{1}{[x]} = \frac{1}{T} = T^{-1} \tag{1.197}$$

$$[y] = \left[\frac{e^{k_1 x}}{k_2 x}\right] = \frac{\left[e^{k_1 x}\right]}{[k_2 x]} = \frac{[1]}{[k_2]\,T} \qquad [k_2] = T^{-2} \tag{1.198}$$

where

$$k_2 = 1 \tag{1.199}$$

Taylor series of y is

$$y = \frac{e^{k_1 x}}{k_2 x} = \frac{1}{x k_2} + \frac{k_1}{k_2} + \frac{1}{2} x \frac{k_1^2}{k_2} + \frac{1}{6} x^2 \frac{k_1^3}{k_2} + \frac{1}{24} x^3 \frac{k_1^4}{k_2} + \cdots \tag{1.200}$$

and dimensional examination of its terms shows that

$$\left[\frac{1}{x k_2}\right] = \frac{1}{T T^{-2}} = T \tag{1.201}$$

$$\left[\frac{k_1}{k_2}\right] = \frac{T^{-1}}{T^{-2}} = T \tag{1.202}$$

$$\left[\frac{1}{2} x \frac{k_1^2}{k_2}\right] = T \frac{T^{-2}}{T^{-2}} = T \tag{1.203}$$

$$\left[\frac{1}{6} x^2 \frac{k_1^3}{k_2}\right] = T^2 \frac{T^{-3}}{T^{-2}} = T \tag{1.204}$$

$$\vdots \quad = \quad \vdots \quad \vdots$$

Example 41 Pythagoras' theorem $c^2 = a^2 + b^2$.

Consider a right triangle as of Fig. 1.8. The area A of the right triangle ABC may be determined by its hypotenuse c and the smaller of its acute angle θ.

$$A_c = \frac{1}{2} c^2 \sin\theta \cos\theta = \frac{1}{4} c^2 \sin 2\theta \tag{1.205}$$

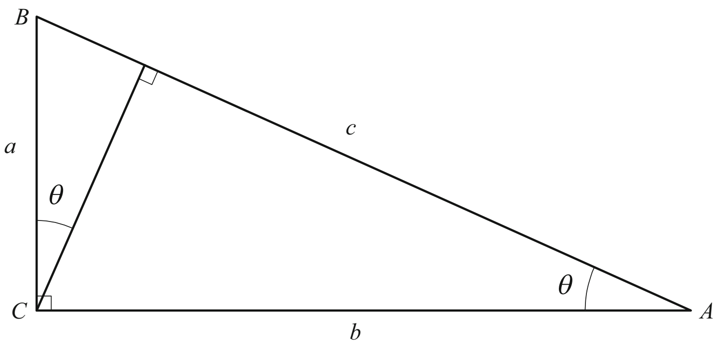

Fig. 1.8 Pythagoras' theorem $c^2 = a^2 + b^2$

Let us assume

$$A_c = f(c, \theta) \tag{1.206}$$

$$[A_c] = L^2 \qquad [c] = L \qquad [\theta] = 1 \tag{1.207}$$

The dimensional analysis says that

$$\pi_1 = \frac{A_c}{c^2} = g(\pi_2) \qquad \pi_2 = \theta \tag{1.208}$$

and therefore,

$$A_c = c^2 g(\theta) \tag{1.209}$$

The altitude perpendicular to the hypotenuse of the triangle divides it into two similar right triangles with hypotenuses a and b. Similar to Eq. (1.209) we must have

$$A_a = a^2 g(\theta) \tag{1.210}$$

$$A_b = b^2 g(\theta) \tag{1.211}$$

However, the sum of the area of the two smaller triangles is equal to the area of the large triangle

$$A_c = A_a + A_b \tag{1.212}$$

and hence

$$c^2 = a^2 + b^2 \tag{1.213}$$

Example 42 Navier–Stokes and continuity equations.

The Navier–Stokes and continuity equations for flows of incompressible fluids are

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla p + \eta \nabla^2 \mathbf{v} \tag{1.214}$$

$$\nabla \cdot \mathbf{v} = 0 \tag{1.215}$$

where $\mathbf{v}$ is the velocity vector, ρ is the density of the continuum media, η is the kinematic viscosity, and p is the pressure. The gradient operator ∇ is derivative with respect to spatial coordinates.

$$\nabla = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k} \tag{1.216}$$

$$\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{1.217}$$

$$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \tag{1.218}$$

$$\nabla \mathbf{v} = \frac{\partial \mathbf{v}}{\partial x}\hat{\imath} + \frac{\partial \mathbf{v}}{\partial y}\hat{\jmath} + \frac{\partial \mathbf{v}}{\partial z}\hat{k} \tag{1.219}$$

$$\mathbf{v} \cdot \nabla \mathbf{v} = v_x \frac{\partial \mathbf{v}}{\partial x} + v_y \frac{\partial \mathbf{v}}{\partial y} + v_z \frac{\partial \mathbf{v}}{\partial z} \tag{1.220}$$

Dimension of the involved variables is:

$$[\mathbf{v}] = LT^{-1} \tag{1.221}$$

$$[\rho] = ML^{-3} \tag{1.222}$$

$$[\eta] = L^2T^{-1} \tag{1.223}$$

$$[p] = ML^{-1}T^{-2} \tag{1.224}$$

$$[t] = T \tag{1.225}$$

Dimensional homogeneity check of the equations indicates that:

$$\left[\frac{\partial \mathbf{v}}{\partial t}\right] + [(\mathbf{v} \cdot \nabla)\, \mathbf{v}] = \left[-\frac{1}{\rho}\right] [\nabla p] + \eta \nabla^2 \mathbf{v} \tag{1.226}$$

$$\frac{LT^{-1}}{T} + LT^{-1}\frac{LT^{-1}}{L} = \frac{1}{ML^{-3}}\frac{ML^{-1}T^{-2}}{L} + \left(L^2T^{-1}\right)\frac{LT^{-1}}{L^2} \tag{1.227}$$

$$LT^{-2} + LT^{-2} = LT^{-2} + LT^{-2} \tag{1.228}$$

Therefore the Navier–Stokes equation is dimensionally homogenous.

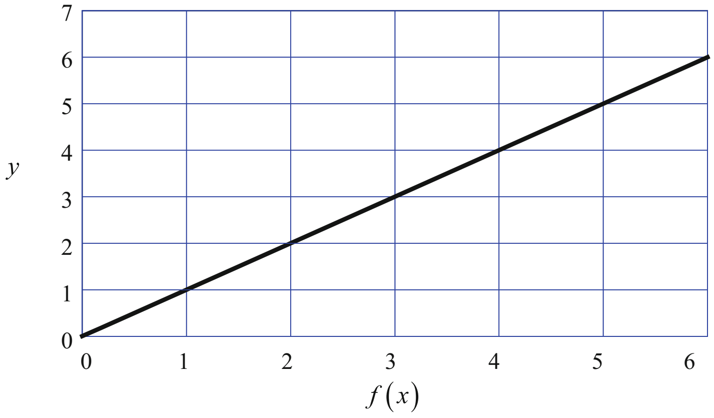

Fig. 1.9 Proportionality of $f(x)$ and y

Example 43 Proportionality and superposition.

Every law of physics is a proportionality. All physical and engineering rules appear in a proportional relationship. If we observe a variable y depends on a variable x, then y is proportional to a certain function of x,

$$y \propto f(x) \tag{1.229}$$

$$k\, y \propto k\, f(x) \qquad k \in \mathbb{R} \tag{1.230}$$

where $f(x)$ is a function to express how variation of x becomes proportional to y. The plot of y versus $f(x)$ is expressed by a straight line as shown in Fig. 1.9.

The proportionality coefficient c is the ratio of y and $f(x)$.

$$\frac{y}{f(x)} = c \tag{1.231}$$

Employing the coefficient ratio c, the proportionality relationship (1.229) becomes an equation.

$$y = c\, f(x) \tag{1.232}$$

This is how we model the nature that every natural law appears to be a proportionality relationship. This is the simplest way to assume the world is governed by linear equation, and the proportionality coefficient remains constant. The proportionality of a natural phenomenon is always applicable for the conditions that:

1. All involved variables are in an acceptable range.
2. All other variables and parameters remain constant.

Similarly, if a variable z depends on variables x and y, then z is proportional to a certain function of x and y,

$$z \propto f(x, y) \tag{1.233}$$

$$k\, z \propto k\, f(x, y) \qquad k \in \mathbb{R} \tag{1.234}$$

where $f(x, y)$ is a function to express how variation of x and y becomes proportional to z, although the variables x and y may not appear in a linear way in the function $f(x, y)$.

One of the early steps of civilization was to discover relations between various magnitudes qualitatively. Some observations such as bigger stones are heavier, or there is a relation between the volume and the weight of a stone. An older tree is taller, a faster runner covers a longer distance at the same time interval. Among all these kinds of relationships, the one that says: the wider a circle is across, the longer it is around, is more important as it is exact and helped in appearance of geometry.

If the volume of a stone is doubled, the weight is doubled; if you double the diameter of a circle, you double its circumference. However, such a rule of proportionality does not always work. A tree twice as old is not twice as tall. The reason is that "the more A, the more B" does not always imply proportionality; or in other words, not every monotonic function is linear. However, the concept of proportionality helped human to recognize pairs of magnitude such that if one was doubled, trebled, quadrupled, halved, or left alone, then the other would also double, treble, quadruple, halve, or show no change. And eventually a great discovery appeared: The greater its range of validity, the greater its significance. They found that if one field feeds half the tribe, two fields will feed the whole tribe. No matter how the two proportional quantities are varied, their ratio remains constant. The proportionality ratio was expressed geometrically at first, as geometry was the first mathematical discipline to make substantial progress.

The following well-known natural laws are illustrative.

Newton equation of motion:

$$F \propto a \qquad F = ma \qquad m = mass \tag{1.235}$$

Applied force on a particle is always proportional to its acceleration. The coefficient of proportionality is called mass and shown by m.

Planck–Einstein relation:

$$E \propto f \qquad E = hf \qquad h = Planck\ constant \tag{1.236}$$

The energy of a photon is proportional to its frequency. The coefficient of proportionality is called Planck number and shown by h.

Newton's Law of Universal Gravitation states that objects attract one another with a force that varies as the product of the masses of the objects and inversely as the square of the distance between the objects.

$$F \propto \frac{m_1 m_2}{r^2} \qquad F = G\frac{m_1 m_2}{r^2} \qquad G = gravitational\ constant \tag{1.237}$$

His Law of Cooling states that the rate of temperature loss of a body is proportional to the difference in temperatures between the body and its surroundings.

$$\frac{d}{dt}T \propto (T - T_0) \qquad \frac{d}{dt}T = k\,(T - T_0) \tag{1.238}$$

Today, we often write this law as

$$T\,(t) = T_0 + (T\,(0) - T_0)\,e^{-kt} \tag{1.239}$$

where T is the temperature, t is time, T_0 is the temperature of the environment, $T\,(0)$ is the initial temperature of the object, and k is a positive constant.

Galileo (1564–1642) once wrote: If two particles are carried with uniform motion, but each with different speed, the distances covered by them during unequal intervals of time to each other the compound ratio of the speeds and time interval.

In today's mathematical language, it means:

$$d = vt \tag{1.240}$$

Example 44 Proportionality sign $\propto$.

William Emerson (1701–1782) was an English mathematician who introduced the sign $\propto$ for proportionality for the first time (Emerson 1768). In his words, "Two variables x and y may be so related that the ratio of one to the other is always the same, x/y =constant," which may be written as:

$$x \propto y \qquad x \text{ is proportional to } y \tag{1.241}$$

Example 45 Lambert's Cosine Law.

Lambert's Law of Emission, also known as Lambert's Cosine Law or the Cosine Law of Emission, states that the intensity (flux per unit solid angle) emitted in any direction from a region of a diffuse radiating surface is proportional to the cosine of the angle between the direction of radiation and the normal to the surface,

$$I_e \propto \cos\theta \tag{1.242}$$

where I_e is the intensity of emitted light, θ is the angle between the observed emitted intensity.

An ideal diffuse surface is usually a rough surface such that small variation in the surface causes an incoming light ray to be reflected in all directions equally. Thus, the region, or element, of the surface that obeys Lambert's Cosine Law will appear equally bright when observed from any direction. Therefore, the total radiant power observed from a radiating surface is proportional to the cosine of the angle θ between the observer's line of sight and a line drawn perpendicular to the surface. The radiating surface appears equally bright regardless of the viewing angle, because purely from geometrical considerations, the apparent size of a portion of the surface is proportional to the cosine of the angle (Pickover 2009).

Example 46 Titius–Bode's law of planet distance.

The German astronomer Johann Daniel Titius (1729–1796) in 1766 discovered an empirical relationship describing the mean distances of the planets from the Sun. Consider the sequence of 0, 3, 6, 12, 24, ⋯, in which each successive number is twice the previous number. Next, add 4 to each number and divide by 10 to form the sequence of 0.4, 0.7, 1.0, 1.6, 2.8, 5.2, 10.0, 19.6, 38.8, 77.2, ⋯. Interestingly Titius law states that this sequence gives the mean distances d of the planets from the Sun, expressed in astronomical units (AU). An AU is the mean distance between Earth and the Sun, which is approximately 149,604,970 km ≈92,960,000 mi. Mercury is approximately at $1/3$ AU from the Sun, and Pluto is at about $39AU$ from the Sun. The Titius–Bode law may be casted as

$$d = \frac{n+4}{10} \tag{1.243}$$

where $n = 0, 3, 6, 12, 24, 48, 96, \cdots$. This equation may also be written as

$$d = 0.4 + 0.3 \times 2^{k-2} \tag{1.244}$$

where $k = 1, 2, 3, 4, \cdots$ working well for $k \geq 2$. Bode (1747–1826) published this law in 1772 (Pickover 2009).

$$\begin{array}{rl}
planet & exact\ AU \simeq 0.4 + 0.3 \times 2^{k-2} = aprox.\ AU \\
Mercury & 0.39\ AU \simeq 0.4 + 0.3 \times 0 = 0.4\ AU \\
Venus & 0.72\ AU \simeq 0.4 + 0.3 \times 2^0 = 0.7\ AU \\
Earth & 1.00\ AU \simeq 0.4 + 0.3 \times 2^1 = 1.0\ AU \\
Mars & 1.52\ AU \simeq 0.4 + 0.3 \times 2^2 = 1.6\ AU \\
Ceres & 2.77\ AU \simeq 0.4 + 0.3 \times 2^3 = 2.8\ AU \\
Jupiter & 5.20\ AU \simeq 0.4 + 0.3 \times 2^4 = 5.2\ AU \\
Saturn & 9.54\ AU \simeq 0.4 + 0.3 \times 2^5 = 10.0\ AU \\
Uranus & 19.19\ AU \simeq 0.4 + 0.3 \times 2^6 = 19.6\ AU \\
Neptune & 30.07\ AU \simeq 0.4 + 0.3 \times 2^7 = 38.8\ AU
\end{array}$$

This law gave a significantly good estimate for the mean distances of the planets that were known, Mercury (0.39), Venus (0.72), Earth (1.00), Mars (1.52), Jupiter (5.20), and Saturn (9.54). Uranus (19.19) was discovered in 1781, and the large asteroid Ceres (2.77) was discovered in 1801. Up to this point the Titus-Bode law surprised everyone, promising new discoveries. Neptune (30.07) was discovered in

1846 and Pluto (39.5) was discovered in 1930. These two planets mean distance from sun are different than Titus law prediction of 38.8 and 77.2, respectively. In 2006, the International Astronomical Union designated Ceres and Pluto as dwarf planets (Pickover 2009). The rule is called Titius–Bode law due to contribution of Johann Elert Bode (1747–1826). Nobody could explain why the Titius–Bode law worked.

Example 47 Proportionality in linear differential equations.

Consider the general second-order linear homogeneous ordinary differential equation as

$$u(x)\,y'' + v(x)\,y' + w(x)\,y = 0 \tag{1.245}$$

$$y' = \frac{dy(x)}{dx} \qquad y'' = \frac{d^2y(x)}{dx^2} \tag{1.246}$$

If $y_1(x)$ and $y_2(x)$ are both solutions of (1.245), then for any two nonzero constants c_1 and c_2,

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \tag{1.247}$$

$$c_1, c_2 \in \mathbb{R} \tag{1.248}$$

is also a solution of (1.245). This is the proportionality in linear systems.

Proof Because $y_1(x)$ and $y_2(x)$ are both solutions of (1.245), we have

$$u(x)\,y_1'' + v(x)\,y_1' + w(x)\,y_1 = 0 \tag{1.249}$$

$$u(x)\,y_2'' + v(x)\,y_2' + w(x)\,y_2 = 0 \tag{1.250}$$

Let us substitute $y(x)$ in Eq. (1.245)

$$\begin{aligned} u(x)\,y'' + v(x)\,y' + w(x)\,y &= \\ u(x)\left(c_1 y_1''(x) + c_2 y_2''(x)\right) + v(x)\left(c_1 y_1'(x) + c_2 y_2'(x)\right) & \\ +w(x)\left(c_1 y_1(x) + c_2 y_2(x)\right) &= \\ \left(u(x)\,y_1'' + v(x)\,y_1' + w(x)\,y_1\right) c_1 & \\ +\left(u(x)\,y_1'' + v(x)\,y_1' + w(x)\,y_1\right) c_2 &= 0 \end{aligned} \tag{1.251}$$

where

$$y'(x) = c_1 y_1'(x) + c_2 y_2'(x) \tag{1.252}$$

$$y''(x) = c_1 y_1''(x) + c_2 y_2''(x) \tag{1.253}$$

Therefore, $y(x)$ satisfies the differential equation (1.245) and is a solution. ■

Example 48 ★ Superposition and proportionality of equation of motion.

Qualitatively, force is whatever changes the motion, and quantitatively, force is whatever that is equal to mass times acceleration. Mathematically, the equation of motion provides a vectorial second-order differential equation:

$$m\ddot{\mathbf{r}} = \mathbf{F}(\dot{\mathbf{r}}, \mathbf{r}, t) \tag{1.254}$$

We assume that the force function may generally be a function of: time t, position $\mathbf{r}$, and velocity $\dot{\mathbf{r}}$. In other words, the Newton equation of motion is correct as long as we show that force is only a function of $\dot{\mathbf{r}}, \mathbf{r}, t$.

If there is a force that depends on acceleration, jerk, or other variables that cannot be reduced to $\dot{\mathbf{r}}, \mathbf{r}, t$, we do not know the equation of motion because

$$\mathbf{F}(\mathbf{r}, \dot{\mathbf{r}}, \ddot{\mathbf{r}}, \dddot{\mathbf{r}}, \cdots, t) \neq m\ddot{\mathbf{r}} \tag{1.255}$$

So, in Newtonian mechanics, we assume that force can only be a function of $\dot{\mathbf{r}}, \mathbf{r}, t$ and nothing else. In the real world, force may be a function of everything; however, we always ignore any other variables than $\dot{\mathbf{r}}, \mathbf{r}, t$, or we make some approximations accordingly.

Because Eq. (1.254) is a linear equation of force $\mathbf{F}$, it accepts the superposition principle. When a mass m is affected by several forces $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3, \cdots$, we may calculate their summation vectorially

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \cdots \tag{1.256}$$

and apply the resultant force on m. So, if a force $\mathbf{F}_1$ provides acceleration $\ddot{\mathbf{r}}_1$ and $\mathbf{F}_2$ provides $\ddot{\mathbf{r}}_2$

$$m\ddot{\mathbf{r}}_1 = \mathbf{F}_1 \tag{1.257}$$

$$m\ddot{\mathbf{r}}_2 = \mathbf{F}_2 \tag{1.258}$$

then the resultant force $\mathbf{F}_3 = \mathbf{F}_1 + \mathbf{F}_2$ provides the acceleration $\ddot{\mathbf{r}}_3$ such that

$$\ddot{\mathbf{r}}_3 = \ddot{\mathbf{r}}_1 + \ddot{\mathbf{r}}_2 \tag{1.259}$$

This is the proportionality and superposition of Newton equation of motion.

To see that the Newton equation of motion is not correct when the force is not only a function of $\dot{\mathbf{r}}, \mathbf{r}, t$, let us assume that a particle with mass m is under two acceleration-dependent forces $F_1(\ddot{x})$ and $F_2(\ddot{x})$ on the x-axis:

$$m\ddot{x}_1 = F_1(\ddot{x}_1) \tag{1.260}$$

$$m\ddot{x}_2 = F_2(\ddot{x}_2) \tag{1.261}$$

The acceleration of m under the action of both forces would be $\ddot{x}_3$

$$m\ddot{x}_3 = F_1(\ddot{x}_3) + F_2(\ddot{x}_3) \tag{1.262}$$

however, we must have

$$\ddot{x}_3 = \ddot{x}_1 + \ddot{x}_2 \tag{1.263}$$

but we have

$$m\left(\ddot{x}_1 + \ddot{x}_2\right) = F_1(\ddot{x}_1 + \ddot{x}_2) + F_2(\ddot{x}_1 + \ddot{x}_2) \neq F_1(\ddot{x}_1) + F_2(\ddot{x}_2) \tag{1.264}$$

Example 49 Inverse dimensional analysis.

Knowing the dimension of a quantity will not allow us to determine the quantity unless the dimension is simple enough to use our experience to name the quantity. Assume we know the dimension of a quantity v to be:

$$[v] = LT^{-1} \tag{1.265}$$

In language of units, v may be measured by length/time such as meter/second, kilometer/hour, etc. As we know, such a quantity is called “speed.” We do not have any other quantity whose dimension is LT^{-1}. Also, we do not have any unit for speed and that is why we use the units of its components: length and time.

Another quantity a has the dimension

$$[a] = LT^{-2} \tag{1.266}$$

and rewriting it as $\left(LT^{-1}\right)T^{-1}$ reminds us that a must be speed/time which is called “acceleration.” However, we may also rewrite it as length/time^2 or meter/second^2. We have no quantity that is meter/second/second, although the dimension of acceleration is (meter/second)/second. In other words, acceleration is speed/time because we measure the change of speed in time to measure acceleration. This example shows that dimension of a quantity is ill to determine the quantity. Acceleration is also another quantity with no particular unit. We use the units of its components, length and time, to read its value, such as 2 meter per second square. We might have defined a unit for speed and acceleration, such as in SI system, displacing one meter in one second to be one “tond” and changing one “tond” in one second to be “haraka.” Then we may say this car has a maximum speed of 100 tond and a maximum acceleration of 10 haraka.

Let look at another quantity shown by f whose dimension is:

$$[f] = MLT^{-2} \tag{1.267}$$

It might be mass$\times$acceleration, or mass$\times$speed/time, or mass$\times$speed^2/length. If it is mass$\times$acceleration, then we have the name of “force” and unit of "Newton” for it. However mass$\times$speed/time is momentum/ time with no name and no unit, similar to mass×speed^2/length that has no name and no unit. We simplify all of them to force and measure them by unit of force.

Now assume the dimension of another quantity, W, is known as:

$$[W] = ML^2T^{-2} \tag{1.268}$$

Although the dimension of W is not very complicated, it is impossible to determine the physical quantity. Using SI system, the unit of W is kilogram meter^2/second^2. Knowing that L/T is speed, W may be interpreted as kilogram speed^2 or $W = mv^2$. Recalling that $(L/T)/T$ is acceleration, W may be shown as mass×length×acceleration or $W = mla$. Employing "force," it will be force×length. Only in this case we have names for it. It may be called "work" or "energy" or "moment" or "torque." In case of "energy," we measure it with a unit of "Joule;" however, in case of "work" or "moment" or "torque," we have not defined any unit and therefore, we measure them by "Newton meter." The quantity may also be more complicated if we assume the given dimension has been simplified after cancellation of some dimensions. It might have been:

$$[W] = ML^2T^{-2} = ML^2T^{-2}TT^{-1} \tag{1.269}$$

which may be interpreted in many ways including momentum×acceleration×time, for example.

Example 50 ★Fractional dimensions.

In classical dimensional analysis, we do not define fractional dimensions because we usually do not work with fractional quantities. However, it is quite possible to have them. As you know, fractional dimension for length has been defined many years ago. Therefore, we may have a quantity x with dimension of $L^{0.2}T^{-1}$ or $M^{0.5}L^{2.1}T^{-1.7}$, for example.

1.3 Conversion of Units

If the numerical value of a physical quantity $Q\,[D]$ of dimension D in unit system U is Q, then its numerical value $q\,[d]$ of dimension d in the unit system u will be q,

$$q = k\,Q \tag{1.270}$$

$$q\,[d] = k\left[\frac{d}{D}\right]\,Q\,[D] \tag{1.271}$$

where

$$1 \text{ original unit } D = k \text{ new unit } d \tag{1.272}$$

A force $F = 1\,\text{N}$ in MKS unit system is equal to $F = 0.2248\,\text{lbf}$ in British unit system. In the metric system we have

$$2.248\,\text{lbf} = 10\,\text{N} \tag{1.273}$$

$$2.248\,\text{lbf} = 0.2248\frac{\text{lbf}}{\text{N}} \times 10\,\text{N} \tag{1.274}$$

because

$$1\,\text{N} = 0.2248\,\text{lbf} \tag{1.275}$$

$$k = 0.2248\,\text{lbf/N} \tag{1.276}$$

In general, we have the numerical value of a physical quantity Q in the unit system U and we want to determine its numerical value q in the unit system u. The dimension of Q is $[Q] = D_1^a\ D_2^b\ D_3^c \cdots$ and the dimension of q is $[q] = d_1^a\ d_2^b\ d_3^c \cdots$. If we have conversion coefficients $k_1, k_2, k_3, \cdots$ for

$$\begin{aligned} 1 \text{ original unit } D_1 &= k_1 \text{ new unit } d_1 \\ 1 \text{ original unit } D_2 &= k_2 \text{ new unit } d_2 \\ 1 \text{ original unit } D_3 &= k_3 \text{ new unit } d_3 \end{aligned} \tag{1.277}$$

$$\vdots \tag{1.278}$$

then

$$q = kQ = \left(k_1^a\ k_2^b\ k_3^c \cdots\right) Q \tag{1.279}$$

where

$$q\left[d_1^a\ d_2^b\ d_3^c \cdots\right] = \left(k_1^a \left[\frac{d_1}{D_1}\right] k_2^b \left[\frac{d_2}{D_2}\right] k_3^c \left[\frac{d_3}{D_3}\right] \cdots\right) Q\left[D_1^a\ D_2^b\ D_3^c \cdots\right] \tag{1.280}$$

For example, a quantity $Q = 10\,\text{N}\,\text{m}^2\,\text{s}$ in MKS unit system is equal to $q = 67.213 \times 10^{-4}\,\text{lbf}\,\text{ft}^2\,\text{h}$ in British unit system.

$$67.213 \times 10^{-4}\,\text{lbf}\,\text{ft}^2\,\text{h} = 10\,\text{N}\,\text{m}^2\,\text{s} \tag{1.281}$$

$$67.213 \times 10^{-4}\,\text{lbf}\,\text{ft}^2\,\text{h} = 6.7213 \times 10^{-4}\frac{\text{lbf}\,\text{ft}^2\,\text{h}}{\text{N}\,\text{m}^2\,\text{s}} \times 10\,\text{N}\,\text{m}^2\,\text{s} \tag{1.282}$$

because

$$1\,\text{N} = 0.2248\,\text{lbf} \tag{1.283}$$

$$k_1 = 0.2248\,\text{lbf/N} \tag{1.284}$$

$$1\,\text{m} = 3.2808\,\text{ft} \tag{1.285}$$

$$k_2 = 3.2808\,\text{ft/m} \tag{1.286}$$

$$1\,\text{s} = \frac{1}{3600}\,\text{h} \tag{1.287}$$

$$k_3 = \frac{1}{3600}\,\text{h/s} \tag{1.288}$$

$$k = k_1 k_2^2 k_3 = (0.2248)\,(3.2808)^2\left(\frac{1}{3600}\right) = 6.7213 \times 10^{-4} \tag{1.289}$$

In classical mechanics we usually have quantities from MKS to British or reverse. The conversion will follow the rule of

$$\text{N}^a\,\text{m}^b\,\text{s}^c \approx 4.448^a \times 0.3048^b \times \text{lbf}^a\,\text{ft}^b\,\text{s}^c \tag{1.290}$$

$$\approx 4.448^a \times 0.0254^b \times \text{lbf}^a\,\text{in}^b\,\text{s}^c \tag{1.291}$$

$$\text{lbf}^a\,\text{ft}^b\,\text{s}^c \approx 0.2248^a \times 3.2808^b \times \text{N}^a\,\text{m}^b\,\text{s}^c \tag{1.292}$$

$$\text{lbf}^a\,\text{in}^b\,\text{s}^c \approx 0.2248^a \times 39.37^b \times \text{N}^a\,\text{m}^b\,\text{s}^c \tag{1.293}$$

Proof We know from dimensional analysis that although the dimension of a base unit, such as length, may be expressed in different units, the dimension of the length is always $[L]$. Therefore, we can convert that length to a new set of units by knowing the equivalence of length between the two systems. In most of the engineering problems, we may need to convert a unit from British units lb ft s to the SI unit N m s. In length conversion, for example, we only need to know either how many feet are in a meter or how many meters are in a foot. These factors are usually tabulated in conversion tables. We may use the following relationship:

$$q\,[d] = k\left[\frac{d}{D}\right]\,Q\,[D] \tag{1.294}$$

to express the quantity as Q number of units in measurement system $[D]$ and q the number of units in measurement system $[d]$. So, to convert from system $[D]$ to system $[d]$,

$$k = \frac{[d]}{[D]} \tag{1.295}$$

where

$$1 \text{ original unit } D = k \text{ new unit } d \tag{1.296}$$

This straightforward unit conversion method is called the factor-label method. To apply the factor-label method:

1. Write down the quantity you want to convert along with its measurement units using a fractional format. For example, express $v = 2020\,\mathrm{ft/s}$ (feet per second) as:

$$v = 2020\frac{\mathrm{ft}}{\mathrm{s}} \tag{1.297}$$

2. Write down the conversion or equivalence factors for each unit to your new set of units as a ratio of the units in the original measurement system to the desired system. The dimension of the ratio of the two units in the two systems will be equivalent to [1], so the dimension of the original quantity is not changed (Treese 2018). For example, to convert $v = 2020\,\mathrm{ft/s}$ to $\mathrm{m/h}$ (meters per hour), we know that 1 ft is 0.3048 m and 1 h is 3600 s; therefore,

$$v = 2020\frac{\mathrm{ft}}{\mathrm{s}} \times \frac{0.3048\,\mathrm{m}}{1\,\mathrm{ft}} \times \frac{3600\,\mathrm{s}}{1\,\mathrm{h}} = 2.2165 \times 10^6\,\frac{\mathrm{m}}{\mathrm{h}} \tag{1.298}$$

In general, we may make an equation for conversion of a quantity q that is in dimension of $\mathrm{ft}^a/\mathrm{s}^b$ to $\mathrm{m}^a/\mathrm{h}^b$ as

$$q\,\frac{\mathrm{ft}^a}{\mathrm{s}^b} \times \left(\frac{0.3048\,\mathrm{m}}{1\,\mathrm{ft}}\right)^a \times \left(\frac{3600\,\mathrm{s}}{1\,\mathrm{h}}\right)^b = q \times 0.3048^a \times 3600^b\,\frac{\mathrm{m}^a}{\mathrm{h}^b} \tag{1.299}$$

or from $\mathrm{ft}^a\,\mathrm{s}^b$ to $\mathrm{m}^a\,\mathrm{h}^b$ as

$$q\,\mathrm{ft}^a\,\mathrm{s}^b \times \left(\frac{0.3048\,\mathrm{m}}{1\,\mathrm{ft}}\right)^a \times \left(\frac{1\,\mathrm{h}}{3600\,\mathrm{s}}\right)^b = q \times \frac{0.3048^a}{3600^b}\,\mathrm{m}^a\,\mathrm{h}^b \tag{1.300}$$

■

Example 51 Acceleration of $\mathrm{cm/s^2}$ to $\mathrm{mi/h^2}$.

To convert an acceleration of $1\,\mathrm{cm/s^2}$ to the unit of mile per hour square, we use $1\,\mathrm{cm} = 6.2137 \times 10^{-6}\,\mathrm{mi}$ and $1\,\mathrm{s} = 2.777 \times 10^{-4}\,\mathrm{h}$.

$$1\frac{\mathrm{cm}}{\mathrm{s}^2} = 1\frac{\mathrm{cm}}{\mathrm{s}^2}\frac{6.2137 \times 10^{-6}\,\mathrm{mi}}{1\,\mathrm{cm}}\left(\frac{1\,\mathrm{s}}{2.777 \times 10^{-4}\,\mathrm{h}}\right)^2 = 80.528\frac{\mathrm{mi}}{\mathrm{h}} \tag{1.301}$$

$$\mathrm{cm}^a\,\mathrm{s}^b = \left(6.2137 \times 10^{-6}\right)^a \left(2.7778 \times 10^{-4}\right)^b\,\mathrm{mi\,h} \tag{1.302}$$

$$\begin{aligned} 10\,\mathrm{cm/s^2} &= \left(6.2137 \times 10^{-6}\right)^1 \left(2.7778 \times 10^{-4}\right)^{-2}\,\mathrm{mi/h^2} \\ &= 805.28\,\mathrm{mi/h^2} \end{aligned} \tag{1.303}$$

Example 52 Gravitational constant.

According to Newton gravitational law there is an attraction force F between two masses m_1 and m_2 that is inversely proportional to the square of the distance r between the masses. The proportionality ratio is called the gravitational constant G.

$$F = G\frac{m_1 m_2}{r^2} \tag{1.304}$$

In MKS unit system, the value of the gravitational constant is:

$$G = 6.67408 \times 10^{-11} \left[\frac{\mathrm{m}^3}{\mathrm{kg\,s}^2} \right] \tag{1.305}$$

In CGS unit system, the value of the gravitational constant is:

$$\begin{aligned} G &= 6.67408 \times 10^{-11} \left[\frac{100^3\,\mathrm{cm}^3}{1000\,\mathrm{g\,s}^2} \right] \\ &= 6.67408 \times 10^{-8} \left[\frac{\mathrm{cm}^3}{\mathrm{g\,s}^2} \right] \end{aligned} \tag{1.306}$$

1.4 Chapter Summary

Engineering and physical sciences are based on experiments; experiments involve measurements, and measurements need units for comparison. There are seven accepted basic dimensional quantities in engineering and science: Length $[L]$, Mass $[M]$, Time $[T]$, Electric Current $[I]$, Temperature $[\Theta]$, Amount of Substance $[N]$, Luminous $[J]$.

The classical static dimensional analysis determines the powers of the involved parameters, variables, and constants contributing in a physical equation. This will determine the right equation. For example, the period of oscillation T of a simple pendulum may be proportional to a function of the mass m of the tip mass, length l of the string, gravitational acceleration g.

$$T = f\,(m, l, g, C) \tag{1.307}$$

Assuming all involved parameters are appearing in product, we move all of them in one side of an equation and give them an unknown exponent and set the dimension of the whole equation equal to 1.

$$C^0\, T^a\, m^b\, l^c\, g^d = 0 \tag{1.308}$$

$$[C]^0\, [T]^a\, [m]^b\, [l]^c\, [g]^d = [0] \tag{1.309}$$

$$1 \times T^a M^b L^c \left(LT^{-2}\right)^d = 1 \tag{1.310}$$

To balance the involved dimensions in the equation, in this example, T, L, M, the overall exponent of the basic dimensional quantities must be zero. This fact gives us the maximum 7 algebraic equation for dimensions L, M, T, I, Θ, N, J. The pendulum example gives us 3 equations for the three involved dimensions T, M, L.

$$a - 2d = 0 \tag{1.311}$$

$$b = 0 \tag{1.312}$$

$$c + d = 0 \tag{1.313}$$

We are looking for the solutions when $a = 1$ as required by the assumed Eq. (1.307). Therefore,

$$a = 1 \qquad b = 0 \qquad c = \frac{1}{2} \qquad d = -\frac{1}{2}$$

and the equation of the period of a pendulum will be

$$T = C\sqrt{\frac{l}{g}} \tag{1.314}$$

where the proportionality constant C can be determined by experiment.

The dimensional balance method works well if the guessed involved variables in the equation are right and if the equation is made out of multiplying the involved variables.

Open problem of static dimensional analysis is the "variable dimensions" and "constraint dimensions." It is possible to imagine a physical quantity that changes its dimension by time or by relative size of other variables. At the moment we do not have variable dimension theory.

1.5 Key Symbols

$a \equiv \ddot{x}$	Acceleration
a	Atto
a, b	Semiaxes of ellipses
a, b, c	Length
asf	Astronomical mass force
asm	Astronomical mass unit
atm	Atmospheric pressure
A	Ampere, unit of electric current
A	Area
A	Angle dimension
AD	After death
AU	Astronomical units
B	Body coordinate frame
BC	Before Christ
c	Speed of light in vacuum
c	Hypotenuse of ellipse
c	Centi
c	Proportionality coefficient
c_p	Specific heat
c_v	Specific heat
cd	Candela, unit of luminous intensity
cm	Centimeter
°C	Centigrade, Celsius
C	Coulomb
C	Dimensionless constant
CGS	Centimeter–gram–second system of units
C_s	Traction coefficient
Cs	Caesium
d	Day
d	Displacement, length
d	Deci
da	Deka
deg	Degree
dm	Decimeter
D	Diameter
D	Dimension
e	Elementary charge, $1.602176634 \times 10^{-19}$ C
e	Eccentricity
e	$2.718281828459 \cdots$
E	Energy, work, heat
E_k	Kinetic energy
E	Exa

ET	Ephemeris Time
ft	Foot
f	Function
f	Frequency
f	Force
f	Femto
F, $\mathbf{F}$	Force
°F	Fahrenheit
F_d	Drag force
F_y	Lateral force
g	Gram
g	Function
g $\mathbf{g}$	Gravitational acceleration
G	Gravitational constant
G	Global coordinate gram
G	Observer coordinate gram
G	Giga
h	Planck constant, $h = 6.62607015 \times 10^{-34}$
h	Hecto
h	Hour
H	Energy, work, heat
H-group	Dimensionally homogeneous equations
Hz	Hertz
i	Imaginary unit, $i^2 = -1$
I	Electric current dimension symbol
I	Second area moment
I_e	Intensity of emitted light
IAU	International Astronomical Union
ISO	International Organization for Standardization
j	Jerk
J	Joule
J	Second area moment
J	Luminous
kg	Kilograms, unit of mass
k	Boltzmann constant, $k = 1.380649 \times 10^{-23}$
k	A factor to make f and m fundamental dimensions
k	Specific thermal conductivity
k	Kilo
k	Scale coefficient
K	Kelvin, unit of temperature
K	Kinetic energy
K	Bulk modulus
K_{cd}	Monochromatic radiation of frequency 540×10^{12} Hz
lb	Pound mass
lbf	Pound force

lm	Lumen
l	Length
L	Length dimension symbol
$\mathcal{L}$	Lagrangian
m	Meter, unit of length
m	Mass
m	Mili
$\dot{m}$	Mass flow rate
mg	Milligram
mol	Mole, unit of amount of substance
mbar	Millibar
M	Mass dimension symbol
M	Moment, torque
M	Mega
MKS	Meter–kilogram–second system of units
n	Nano
N	Amount of substance dimension symbol
N	Newton
N_A	Avogadro constant, $6.02214076 \times 10^{23}$ mol^{-1}
N-group	Dimensionally nonhomogeneous equations
Pa	Pascal
p	Tire inflation pressure
p	Pico
p	Momentum
P	Pressure, stress
P	Population dimension symbol
P	Potential energy
P	Power
P	Peta
q	Generalized coordinate
Q	Heat
Q	Quantity of flow
Q	Moment, torque
Q	Physical quantity
Q_e	Electric current
r	Radius
rad	Radian
r	Position vector
R	Rankine
R	Radius
s	Second, unit of time
s	Arc length displacement
sr	Steradian
SI	Systéme International (d'Unités)
t	Time

T	Time dimension symbol, temperature
T	Temperature
T	Moment, torque
T	Tera
T	Period
TCB	Barycentric Coordinate Time
TCG	Geocentric Coordinate Time
TDB	Barycentric Dynamical Time
TT	Terrestrial Time
u	Velocity
U	Unit system
USA	United States of America
$USCS$	United States Customary Units
$USSR$	Union of Soviet Socialist Republics
UT	Universal Time
$v \equiv \dot{x}$, $\mathbf{v}$	Velocity, tread velocity in tireprint
V	Volume
V	Velocity
W	Watt
W	Energy, work, heat
x, y, z	Coordinate axes
X, Y, Z	Global coordinates
x	Displacement
y	Function
y	Temperature
y_o	Ambient temperature
Y	Information dimension symbol
z	Elevation
$	Money dimension symbol
∇	Gradient
$\propto$	Proportional
$\mathbb{N}$	Integer numbers
$\mathbb{R}$	Real numbers
α	Angle
$\alpha, \beta, \gamma, \cdots$	Integer exponents
$\Delta\nu_{Cs}$	Transition frequency of the caesium-133 atom $\Delta\nu_{Cs} = 9192631770\,\text{Hz}$
ε	Strain
η	Kinematic viscosity
θ	Angle
Θ	Temperature dimension symbol
μ	Friction coefficient
μ	Viscosity
μ	Micro
ν	Frequency

ν	Kinematic viscosity
π	$3.14159265359\cdots$
π_i	π_i-number, $i = 1, 2, 3\cdots$
ρ	Density
σ	Pressure, stress
σ	Surface tension
τ	Pressure, stress
φ	Angle
ω	Angular velocity
ω	Frequency

Exercises

1. Lift force of an airplane wing.
   Find a formula for the lift force generated by an airplane wing, in terms of the density of the surrounding air, the area of the wing, and the speed at which it moves through the air.
2. Variable volume Earth.
   Assume the Earth's radius R is fluctuating with frequency ω

$$R = R_0 + 1000000 \sin \omega t \qquad R_0 = 6371000\,\mathrm{m} \tag{1.315}$$

   where R_0 is the current average radius of the Earth. If the mass m of the Earth remains constant, determine the equation of the density rate and its dimension.
3. Centripetal force.
   Beginning with $F = k m^{\alpha} v^{\beta} r^{\gamma}$ determine the correct equation for centripetal force F.
4. ★Physical quantities' dimensions.
   Determine the dimensions of Angular Momentum, in kgm-meter^2/sec. Gravitational Field Strength, in newton/kg. Charge Density, ρ, in coulomb/meter^2, where Charge q is measured in coulomb. Field Intensity, $\mathbf{E}$, in volt/meter. Current Density, j, in amp/meter^2. Permittivity, ε, in farad/meter. Capacitance, C, in farad. Inductance, L, in henry. Resistance, R, ohm. Enthalpy in joule. Entropy, in joule/ K. Gas Constant, in joule/kg- K. Thermal Conductivity, in watt/meter- K. Thermal Diffusivity, in meter^2/sec. Heat Transfer Coefficient, in watt/meter^2- K.
5. Derivative and integrals
   Assume x is a variable that its dimension is

$$[x] = M^a L^b T^c \tag{1.316}$$

   Determine the dimension of $y_1 = dx/dt$, and $y_2 = \int x\, dt$, and $y_3 = \partial^2 x/(\partial z \partial t)$, and $y_2 = \int x\, dV$, where $[t] = T$, $[z] = L$, $[V] = L^3$.
6. Electrical Coulomb's law.
   The basic dimension of electrical charge is determined from Coulomb's law.

$$F = K \frac{Q_1 Q_2}{r^2} \tag{1.317}$$

   Determine the dimension of K.
7. Bulk modulus of elasticity, K.
   The bulk modulus of elasticity is used as a measure of the compressibility of liquids. Liquids can be compressed at very high pressures. If the pressure is

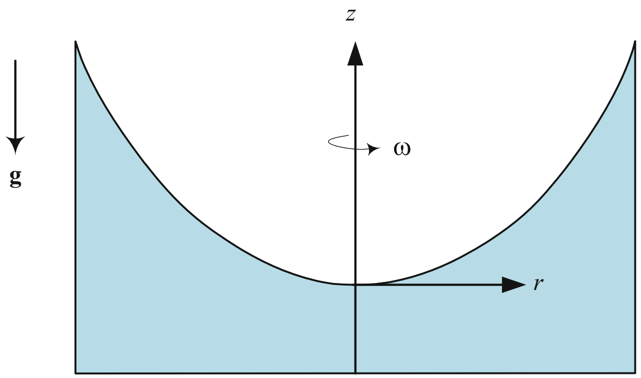

Fig. 1.10 Uniform rotation of a cylinder of liquid

increased isothermally by Δp, it will decrease in volume, $-\Delta V$, such that for any volume V of liquid we have:

$$K = -\left(\frac{V\,dp}{dV}\right) \tag{1.318}$$

Determine the dimension of bulk modulus of elasticity, K.

8. The free-surface shape of rotating liquid.
   A fluid in a cylindrical container rotates at a constant angular speed ω, as shown in Fig. 1.10.
   The free surface of the liquid of density ρ is no longer horizontal and the free surface stays as the equation for a parabola.

$$z = \frac{\omega^2 r^2}{2g} \tag{1.319}$$

   Employing dimensional analysis, show that the parabolic equation is correct.
9. Dimensional homogeneity of elliptic PDE.
   An example of elliptic partial differential equations (PDR)

$$-\nabla \cdot (c\nabla u) + au = f \tag{1.320}$$

   is the Poisson's equation on an area.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = x^2 + y^2 \tag{1.321}$$

   Check the dimensional homogeneity of elliptic partial differential equations.

10. Dimensional homogeneity of parabolic PDE.
An example of parabolic partial differential equations (PDE)

$$\frac{\partial u}{\partial t} - \nabla \cdot (c\nabla u) + au = f \tag{1.322}$$

is the heat equation.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \sin t \tag{1.323}$$

Check the dimensional homogeneity of parabolic partial differential equations.
11. Dimensional homogeneity of hyperbolic PDE.
An example of hyperbolic partial differential equations (PDR)

$$\frac{\partial^2 u}{\partial t^2} - \nabla \cdot (c\nabla u) + au = f \tag{1.324}$$

Check the dimensional homogeneity of hyperbolic partial differential equations.
12. Tension T in string.
Consider a string of length l that connects a ball of mass m to a fixed point. The ball whirls in a circle at speed v. Employ the method of dimensional analysis to show that the tension T in the string is determined by the following equation:

$$\frac{Tl}{mv^2} = cte. \tag{1.325}$$

13. Logarithm function.
The exponent of mathematical functions is supposed to be dimensionless. Explain the mistake and remedy of expressions such as $\ln(r)$, where r is, for example, a radius.
14. Taylor series expansion.
A function $f(x)$ can be expanded into a Taylor series around an ordinary point $x = a$.

$$\begin{aligned} f(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots \\ &= \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = \sum_{k=0}^{\infty} a_k (x-a)^k \end{aligned} \tag{1.326}$$

$$a_k = \frac{f^{(k)}(a)}{k!} \tag{1.327}$$

Determine the dimension of the coefficients a_k if

(a) $[f(x)] = L$, $[x] = L$
(b) $[f(x)] = T$, $[x] = T$
(c) $[f(x)] = L$, $[x] = T$

15. General second-order differential equations.
The general linear second-order differential equation is:

$$f_2(x)\,y'' + f_1(x)\,y' + f_0(x)\,y = f(x) \tag{1.328}$$

which for $f_2(x) \neq 0$ can be transformed into

$$y'' + P(x)\,y' + Q(x)\,y = R(x) \tag{1.329}$$

(a) Determine dimensions of all functions if $[x] = T$, $[y] = L$, $[f(x)] = L$.
(b) The equation may be solved by series solution

$$f(x) = \sum_{k=0}^{\infty} a_k (x-a)^k \qquad a_k = \frac{f^{(k)}(a)}{k!} \tag{1.330}$$

when we substitute

$$P(x) = \sum_{k=0}^{\infty} \frac{P^{(k)}(a)}{k!}(x-a)^k \tag{1.331}$$

$$Q(x) = \sum_{k=0}^{\infty} \frac{Q^{(k)}(a)}{k!}(x-a)^k \tag{1.332}$$

$$R(x) = \sum_{k=0}^{\infty} \frac{R^{(k)}(a)}{k!}(x-a)^k \tag{1.333}$$

into the equation and solving the coefficients of $(x-a)^k$ for a_k. Determine the dimensions of the coefficients $P^{(k)}(a)/k!$, $Q^{(k)}(a)/k!$, $R^{(k)}(a)/k!$, a_k.

16. Hidden dimensional coefficient.

(a) Determine the hidden coefficient and its dimension in the following differential equation:

$$y'' + y = 0 \qquad y' = \frac{dy}{dx} \tag{1.334}$$

if $[x] = T$, $[y] = L$.

(b) The series solution of the equation

$$\ddot{x} + \dot{x} + x^3 = 0 \qquad x(0) = 1 \qquad \dot{x}(0) = 0 \tag{1.335}$$

is:

$$x = 1 - \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{12}t^4 - \frac{1}{24}t^5 - \frac{19}{720}t^6 + \cdots \tag{1.336}$$

Determine the hidden coefficients in the equation and their dimensions. Also determine the dimension of the coefficients of the series solution.

17. Transcendental functions.
As we know, all transcendental functions are dimensionless. Explain how we may have functions such as $y = \exp t$, or $y = \sin t$, where t is time and y is displacement.
18. A parametric second-order differential equation.
Consider the second-order parametric differential equation

$$y'' + y' + \left(1 + x^2\right) y = 0 \tag{1.337}$$

$$y(0) = 2 \qquad y'(0) = 0 \tag{1.338}$$

(a) Determine the hidden coefficients and their dimensions if $[x] = T$, $[y] = L$.
(b) Searching for a series solution of the form

$$y = \sum_{k=0}^{\infty} a_k x^k \tag{1.339}$$

provides

$$2a_2 + a_1 + a_0 = 0 \qquad k = 0 \tag{1.340}$$

$$6a_3 + 2a_2 + a_1 = 0 \qquad k = 1 \tag{1.341}$$

$$(k+2)(k+1)\, a_{k+2} + (k+1)\, a_{k+1} + a_k + a_{k-2} = 0 \qquad k \geq 2 \tag{1.342}$$

by setting the coefficients of x^k, $k = 0, 1, 2, 3, \cdots$ equal to zero. The first equation indicates

$$a_2 = \frac{-a_1 - a_0}{2} \tag{1.343}$$

and the second equation indicates that

$$a_3 = \frac{-2a_2 - a_1}{6} = \frac{a_0}{6} \tag{1.344}$$

and the third equation gives us a recurrence relation to determine all the coefficients of $k \geq 2$.

$$a_{k+2} = \frac{-(k+1)\,a_{k+1} - a_k - a_{k-2}}{(k+2)\,(k+1)} \tag{1.345}$$

Determine the dimensions of a^k, $k = 0, 1, 2, 3, \cdots$.

19. ★Important equations in physics.
The following equations are considered as the equations that changed the world of science. Show that the equations are dimensionally homogeneous.

a. Pythagoras's equation (Pythagoras 530BC).

$$a^2 = b^2 + c^2 \tag{1.346}$$

b. Logarithm

$$\log xy = \log x + \log y \tag{1.347}$$

c. Calculus

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{1.348}$$

d. The square root of minus one (Euler 1750).

$$i^2 = -1 \tag{1.349}$$

e. Wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{1.350}$$

f. Fourier transform

$$f(\omega) = \int_{-\infty}^{\infty} f(x)\; e^{-2\pi i x \omega} dx \tag{1.351}$$

g. Second law of thermodynamics

$$dS \geq 0 \tag{1.352}$$

h. Relativity

$$E = mc^2 \tag{1.353}$$

i. Schrodinger's equation

$$ih \frac{\partial \Psi}{\partial t} = H\Psi \tag{1.354}$$

j. Information theory

$$H = -\sum p(x) \log p(x) \tag{1.355}$$

k. Chaos theory

$$x_{k+1} = k\, x_k \,(1 - x_x) \tag{1.356}$$

l. Black–Scholes equation

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} - rV = 0 \tag{1.357}$$

20. Now look at Newton equation of motion.
Assume the Newton equation of motion is defined by

$$F \propto a^{1.2} \tag{1.358}$$

(a) Determine the dimension of mass.
(b) Analyze the consistency of the Newton gravitational equation with this equation.

21. Work of a planar force on a planar curve.
A planar force **F**,

$$\mathbf{F} = 2xy\hat{\imath} + 3x^2\hat{\jmath} \ \text{N} \tag{1.359}$$

moves a mass m on a planar curve

$$y = x^2 \tag{1.360}$$

from (0, 0) to (10 ft, 50 in). Using

$$dy = 2x\,dx \tag{1.361}$$

calculate the work done by the force in N m:

$${}_1W_2 = \int_{P_1}^{P_2} {}^G\mathbf{F} \cdot d\mathbf{r} = \int_{(0,0)}^{(3,1)} \left(2xy\,dx + 3x^2\,dy\right) \tag{1.362}$$

22. A projectile range.
Consider a projectile with mass $m = 1.12\,\text{lb}$ that is shot with an initial velocity $\mathbf{v}_0$ from the origin of the coordinate frame. The initial conditions of the problem are

$$t_0 = 0 \qquad \mathbf{r}(0) = 0 \qquad \theta = 37\,\text{deg} \tag{1.363}$$

$$\mathbf{v}(0) = \mathbf{v}_0 = (2\,\text{mi/h})\,\hat{\imath} + (17000\,\text{in/s}) \tag{1.364}$$

Knowing that the solution of the equation of motion

$$m\frac{d\mathbf{v}}{dt} = m\mathbf{g} \tag{1.365}$$

$$\mathbf{g} = -g\hat{k} \tag{1.366}$$

$$g = 9.80665\,\text{m/s}^2 \approx 9.81\,\text{m/s}^2 \tag{1.367}$$

is

$$\mathbf{v} = \begin{bmatrix} v_0 \cos\theta \\ 0 \\ v_0 \sin\theta - gt \end{bmatrix} \qquad \mathbf{r} = \begin{bmatrix} v_0 t \cos\theta \\ 0 \\ v_0 t \sin\theta - \frac{1}{2}gt^2 \end{bmatrix} \tag{1.368}$$

determine, all in SI unit system:

(a) the range R of the projectile on a flat ground

$$R = \frac{v_0^2}{g}\sin 2\theta \tag{1.369}$$

(b) the difference between the maximum possible range

$$R_M = \frac{v_0^2}{g} \tag{1.370}$$

and the actual range.

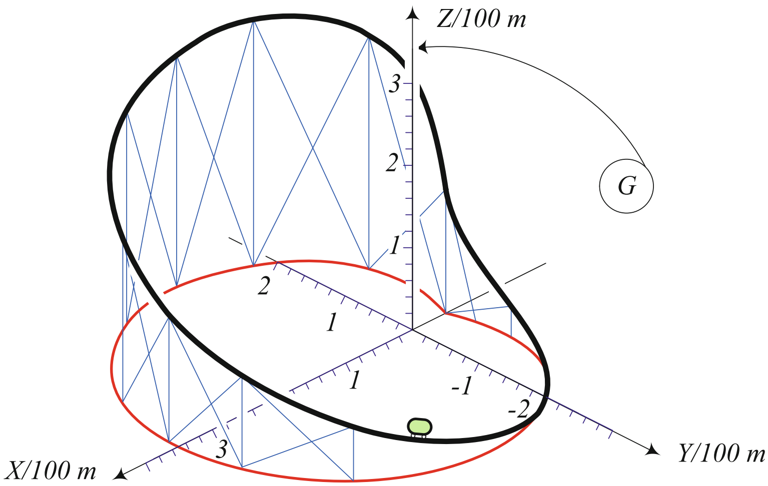

Fig. 1.11 A roller coaster

(c) the range time t_R, at which the projectile reaches R.

$$t_R = 2\frac{v_0}{g}\sin\theta \tag{1.371}$$

(d) the height H and the height time t_H

$$H = \frac{v_0^2}{2g}\sin^2\theta \qquad t_H = \frac{1}{2}t_R = \frac{v_0}{g}\sin\theta \tag{1.372}$$

(e) ★Prove that these results are independent of the mass of the projectile.

23. Length of a roller coaster.
Figure 1.11 illustrates a roller coaster that its mathematical expression in parametric equations is:

$$\begin{aligned} x &= (a + b\sin\theta)\cos\theta \\ y &= (a + b\sin\theta)\sin\theta \\ z &= b + b\cos\theta \end{aligned} \tag{1.373}$$

For

$$a = 600\,\mathrm{ft} \qquad b = 0.150\,\mathrm{mi} \tag{1.374}$$

determine the total length of the roller coaster in (m) by integral of ds for θ from 0 to 2π:

$$s = \int_{\theta_1}^{\theta_2} \sqrt{\frac{d\mathbf{r}}{d\theta} \cdot \frac{d\mathbf{r}}{d\theta}} d\theta$$
$$= \int_{\theta_1}^{\theta_2} \sqrt{\left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2} d\theta \tag{1.375}$$

24. Flight of a bug.
Assume two cars A and B that are initially 50000 ft apart. The cars begin moving toward each other. The speed of car A is 20000 in/h and B is 7 mi/h. At the instant they started moving toward each other, a bug on the bumper of car A starts flying with speed 40 m/s straight toward car B. As soon as the bug reaches the other car it turns and flies back. The bug flies back and forth from one car to the other until the two cars meet. Determine the total length that the bug flies.
25. Relative frequency.
A body B that is moving along the x-axis with a constant velocity u. The body B emits small particles which move with a constant velocity c along the x-axis every T seconds. If f denotes the frequency and λ the distance between two successively emitted particles, then

$$f = \frac{1}{T} = \frac{c-u}{\lambda} \tag{1.376}$$

An observer moves along the x-axis with velocity v. Let us show the number of particles per second that the observer meets by the relative frequency f' and the time between meeting the two successive particles by the relative period T'.

$$f' = \frac{c-v}{\lambda} \tag{1.377}$$

Show that

$$f' \approx f\left(1 - \frac{v-u}{c}\right) \tag{1.378}$$

and provide the solution for dimensional homogeneity.
26. Dimensional homogeneity of the Bernoulli equation.
Bernoulli equation in fluid mechanics is:

$$P + \frac{1}{2}\rho v^2 + \rho g z = C \tag{1.379}$$

Verify the homogeneity of the equation and determine $[C]$. P is pressure, ρ is density, v is velocity, g is gravitational acceleration, z is height.

27. A falling object on a spring on the Moon.
Assume an experiment on the Moon. An object with mass m falls from a height h on a linear spring with stiffness k. We can determine the maximum compression of the spring using the work–energy principle. The gravity force mg and the spring force $-kx$ are the acting forces on m. If x_M is the maximum compression of the spring, then $K_2 = K_1 = 0$ and we have

$$ {}_1W_2 = mg(h + x_M) - \int_0^{x_M} kx\, dx = 0 \tag{1.380}$$

$$ x_M = \frac{mg}{k} + \sqrt{\left(\frac{mg}{k}\right)^2 + \frac{2mgh}{k}} \tag{1.381}$$

If we put m on the spring, it will deflect statically to

$$ x_0 = \frac{mg}{k} \tag{1.382}$$

so we may compare x_M to x_0 and write x_M as

$$ x_M = x_0 \left(1 + \sqrt{1 + \frac{2h}{x_0}}\right) \tag{1.383}$$

(a) Justify the result in Eq. (1.383) must be independent of m and k and g. Therefore this equation and the experiment must provide the same results on any planet, for any value of m and on any spring k.
(b) Knowing that on Earth, $g = 9.81\,\mathrm{m/\,s^2}$, and on Moon, $g = 1.6\,\mathrm{m/\,s^2}$ and

$$ h = 50\,\mathrm{in} \qquad m = 1\,\mathrm{kg} \qquad k = 1000\,\mathrm{lbf/\,ft} \tag{1.384}$$

determine x_0 and x_M on Earth and Moon, and examine the correctness of Eq. (1.383).
(c) Theoretically we can measure g based on such experiment. Show that:

$$ g = \frac{2hk}{m\left(\left(\dfrac{x_M}{x_0} - 1\right)^2 - 1\right)} \tag{1.385}$$

and examine the dimensional homogeneity.

28. Bungee jumper.

(a) Consider a falling bungee jumper from

$$ z\,(0) = z_0 = 0 \qquad \dot{z}\,(0) = \dot{z}_0 = 0 \tag{1.386}$$

who is under the force of gravity and an elastic rope with stiffness k and a free length l. We measure the vertical displacement z downward. The forces applied on the jumper would be:

$$\ddot{z} = \begin{cases} g & z < l \\ g - \dfrac{k}{m}(z - l) & z > l \end{cases} \tag{1.387}$$

(b) Substitute $\dot{z} = p$ and transform the equation to

$$\ddot{z} = p\frac{dp}{dz} \tag{1.388}$$

and find the solution:

$$p = \begin{cases} \sqrt{2gz + C_1} & z < l \\ \dfrac{1}{m}\sqrt{2gm^2 z + C_2 m^2 - kmz^2 + 2klmz} & z > l \end{cases} \tag{1.389}$$

and show that

$$C_1 = 0 \qquad C_2 = -\frac{kl^2}{m} \tag{1.390}$$

(c) Substituting back to $\dot{z} = p$ and find the solution in terms of time t,

$$t = \begin{cases} \dfrac{\sqrt{2z}}{g} + C_3 & z < l \\ \sqrt{\dfrac{m}{k}}\tan^{-1}(AB) + C_4 & z > l \end{cases} \tag{1.391}$$

$$A = \frac{gm - k(z - l)}{-k} \tag{1.392}$$

$$B = \sqrt{\frac{k}{2mgz - k(z - l)^2}} \tag{1.393}$$

and justify

$$C_3 = 0 \tag{1.394}$$

$$C_4 = -\sqrt{\frac{2l}{g}} + \sqrt{\frac{m}{k}}\tan^{-1}\left(\sqrt{\frac{mg}{2kl}}\right) \tag{1.395}$$

(d) Examine the dynamic of the jumper for

$$m = 100\,\text{kg} \qquad k = 80\,\text{N/ft} \qquad l = 500\,\text{in} \tag{1.396}$$

and find t when $z = l$. After this time elapsed, the solution switches to the case $z > l$ that ends when $\dot{z} = 0$. Determine the maximum stretch of the elastic rope z_M.

(e) Determine the maximum stretch of the elastic rope from the conservation of energy equation:

$$mg\, z_M = \frac{1}{2} k\, (z_M - l)^2 \tag{1.397}$$

29. Inverse dimensional analysis.
Try to express the physical quantity or quantities of the following given dimensions, assuming there was no simplification by cancelling out any dimension.

$$a = LT \tag{1.398}$$

$$b = L^2 T \tag{1.399}$$

$$c = L^3 T \tag{1.400}$$

$$d = M^2 LT \tag{1.401}$$

$$e = M^2 L^{-2} T^2 \tag{1.402}$$

$$f = M^3 L^2 T^{-3} \tag{1.403}$$

30. ★Numerical series and transcendental e number.
Show that

$$\sum_{k=0}^{\infty} \frac{1}{k!} = e = 2.71828 \cdots \tag{1.404}$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = \frac{1}{2e} = 0.1839397 \cdots \tag{1.405}$$

$$\sum_{k=1}^{\infty} \frac{k}{(2k+1)!} = \frac{1}{e} = 0.36787 \cdots \tag{1.406}$$

$$\sum_{k=1}^{\infty} \frac{k}{(k+1)!} = 1 \tag{1.407}$$

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Chapter 2
Dynamic Dimensional Analysis

This chapter develops the concept of dimensional analysis to use it in model-prototype similarities, nondimensionalization, and simplification of equations for the most general coverage. The goals of dynamic dimensional analysis are: (1) to minimize the number of variables in analysis of a system, (2) to express the mathematical model of systems by nondimensionalized equations to get the results of the investigation to be unit and size independent, (3) to match the dynamics of model and prototype to do experiments on the model and use the result for the prototype.

In the study of dynamic physical systems we work with several physical equations at the same time, and we have variables that their values are changing. A real system, called prototype, is expensive, too large or too small, time and energy consuming to be used for fine design. If possible, we prefer to work with a manageable size, inexpensive, and a simpler system that only operates on the phenomena of interest. Such a simpler system is called model. Dynamic dimensional analysis is the study of these topics.

2.1 Buckingham pi-Theorem

Buckingham pi-theorem says: If an equation in n arguments is dimensionally homogeneous with respect to m fundamental units, it can be expressed as a relation between $n - m$ independent dimensionless arguments shown by π_i.

Suppose we have a unit free physical law in the form

$$f(x_1, x_2, x_3, \cdots, x_n) = 0 \tag{2.1}$$

where x_i, $i = 1, 2, \cdots, n$ are dimensional variables of m fundamental units. Therefore, dimension of every x_i can be represented as a power product of the m funda-

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mental dimensions. From $x_1, x_2, \cdots, x_n$ we can form k new independent nondimensionalized variables $\pi_1, \pi_2, \cdots, \pi_k$, $k = n - m$. Then $f(x_1, x_2, x_3, \cdots, x_n) = 0$ is equivalent to a physical law of the form

$$g(\pi_1, \pi_2, \cdots, \pi_k) = 0 \tag{2.2}$$

Proof Any mathematical equation in physics and engineering is in the form of

$$f(x_1, x_2, x_3, \cdots, x_n) = 0 \tag{2.3}$$

where f is a function of n variables and parameters x_1, x_2, x_3, $\cdots$, x_n, all measurable dimensional quantities. The n quantities x_1, x_2, x_3, $\cdots$, x_n, involve m fundamental dimensions D_1, D_2, D_3, $\cdots$, D_m, such as Length $[L]$, Time $[T]$, Mass $[M]$, etc. The dimension of any of the n quantities x_1, x_2, x_3, $\cdots$, x_n, will be

$$[x_i] = D_1^{a_{1i}} \, D_2^{a_{2i}} \, \cdots \, D_m^{a_{mi}} \qquad i = 1, 2, \cdots, n \tag{2.4}$$

$$\left\{a_{ij}, i = 1, 2, \cdots, m, j = 1, 2, \cdots, n\right\} \in \mathbb{N} \tag{2.5}$$

where a_{ij} are all known values. We show the dimensions of variables $x_1, x_2, x_3, \cdots$ by the $m \times n$ dimensional matrix $\mathbf{A}$.

$$\begin{array}{c|ccccc} & x_1 & x_2 & x_3 & \cdots & x_n \\ \hline D_1 & a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ D_2 & a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ D_m & a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{array} \tag{2.6}$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \tag{2.7}$$

The top row is the list of variables x_1, x_2, x_3, $\cdots$, x_n in Eq. (2.3) and the left column is the list of dimensions D_1, D_2, D_3, $\cdots$, D_m. Then below of each variable there are the exponents of the associated dimension of the variable.

We are looking for dimensionless variables π

$$\pi = x_1^{b_1} \, x_2^{b_2} \, \cdots \, x_n^{b_n} \tag{2.8}$$

such that

$$[\pi] = D_1^0 \, D_2^0 \, \cdots \, D_m^0 = 1 \tag{2.9}$$

Substituting (2.9) and (2.4) into (2.8) provides us with

$$
\begin{aligned}
D_1^0\, D_2^0\, \cdots\, D_m^0 = & \left(D_1^{a_{11}}\, D_2^{a_{21}}\, \cdots\, D_m^{a_{m1}}\right)^{b_1} \\
& \times \left(D_1^{a_{12}}\, D_2^{a_{22}}\, \cdots\, D_m^{a_{m2}}\right)^{b_2} \\
& \times \left(D_1^{a_{13}}\, D_2^{a_{23}}\, \cdots\, D_m^{a_{m3}}\right)^{b_3} \times \cdots \times \\
& \times \left(D_1^{a_{1n}}\, D_2^{a_{2n}}\, \cdots\, D_m^{a_{mn}}\right)^{b_n} \qquad (2.10)
\end{aligned}
$$

Equating powers of the dimensions D_1, D_2, D_3, $\cdots$, D_m, we will have a set of algebraic equations.

$$D_1 : \qquad 0 = a_{11}b_1 + a_{12}b_2 + \cdots a_{1n}b_n \qquad (2.11)$$

$$D_2 : \qquad 0 = a_{21}b_1 + a_{22}b_2 + \cdots a_{2n}b_n \qquad (2.12)$$

$$\cdots$$

$$D_m : \qquad 0 = a_{m1}b_1 + a_{m2}b_2 + \cdots a_{mn}b_n \qquad (2.13)$$

The coefficients a_{ij}, $i = 1, 2, \cdots, m$, $j = 1, 2, \cdots, n$ are all known and b_i, $i = 1, 2, \cdots, n$ are to be determined, from a set of homogeneous linear system of m equations in n unknown variables b_i, $i = 1, 2, \cdots, n$.

$$
\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = 0 \qquad (2.14)
$$

The determinant of the coefficient matrix **A** must vanish to have nonzero values of the unknowns b_i , $i = 1, 2, \cdots, n$.

$$
|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{21} & a_{31} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{vmatrix} = 0 \qquad (2.15)
$$

A set of homogeneous linear equations in n unknown variables whose matrix of the coefficients exhibits the rank r, has exactly $(n - r)$ linearly independent solutions. So, given n physical quantities x_i, $i = 1, 2, \cdots, n$ with a relation among them, there exist exactly $(n - r)$ independent dimensionless π_j , $j = 1, 2, \cdots, n - r$. We can then mathematically express this theorem as

$$g\left(\pi_1, \pi_2, \cdots, \pi_{n-r}\right) = 0 \qquad (2.16)$$

Because the m number of the dimensions $D_1, D_2, D_3, \cdots, D_m$, are independent, practically the rank of the matrix $\mathbf{A}$ is equal to $r = m$ and therefore, the number of independent π_i variables is $n - m$.

To apply the Buckingham pi-theorem, we may follow the following steps:

1. Identify the physical quantities x_i and the number n.
2. Employ the dimensional system $M, L, T, \cdots$ for every quantity x_i, and determine the number m.
3. Make the dimensional matrix $\mathbf{A}$ and evaluate the rank r which is usually $r = m$.
4. Evaluate $k = n - r$ dimensionless π variables.
5. Apply $g(\pi_1, \pi_2, \cdots, \pi_k) = 0$ to obtain the desired relationship between the π_i variables.

An example will clarify the application. Consider a ship of length l sailing at a constant speed v on the ocean. The energy will be transferred to the water as a result of viscous friction. This energy will induce surface waves as well as to overcome the friction of the turbulent motion of the water. The resistance to the motion of the ship depends on the acceleration of gravity g, the density of water ρ, and the viscosity of water μ.

Step 1 the physical quantities x_i are (l, v, g, ρ, μ) and the number n is $n = 5$.

x_1	x_2	x_3	x_4	x_5
l	v	g	ρ	μ

(2.17)

Step 2 the involved dimensions are M, L, T, and the number m is $m = 3$.

D_1	D_2	D_3
M	L	T

(2.18)

$$[l] = L \tag{2.19}$$

$$[v] = LT^{-1} \tag{2.20}$$

$$[g] = LT^{-2} \tag{2.21}$$

$$[\rho] = ML^{-3} \tag{2.22}$$

$$[\mu] = ML^{-1}T^{-1} \tag{2.23}$$

Step 3 the dimensional matrix $\mathbf{A}$ is formed below and its rank is $r = 3$ which indicates that $r = m = 3$.

	l	v	g	ρ	μ
M	0	0	0	1	1
L	1	1	1	−3	−1
T	0	−1	−2	0	−1

(2.24)

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & -3 & -1 \\ 0 & -1 & -2 & 0 & -1 \end{bmatrix} \tag{2.25}$$

Step 4 there are $k = 2 = n - r$ dimensionless π-variables.

$$\begin{aligned} \pi_i &= M^0 L^0 T^0 = l^{b_1} v^{b_2} g^{b_3} \rho^{b_4} \mu^{b_5} \\ &= L^{b_1} \left(LT^{-1}\right)^{b_2} \left(LT^{-2}\right)^{b_3} \left(ML^{-3}\right)^{b_4} \left(ML^{-1}T^{-1}\right)^{b_5} \\ &= M^{b_4+b_5}\ L^{b_1+b_2+b_3-3b_4-b_5}\ T^{-b_2-2b_3-3b_4-b_5} \end{aligned} \tag{2.26}$$

$$b_4 + b_5 = 0 \tag{2.27}$$

$$b_1 + b_2 + b_3 - 3b_4 - b_5 = 0 \tag{2.28}$$

$$-b_2 - 2b_3 - b_5 = 0 \tag{2.29}$$

We have some freedoms of choice how to solve the three equations for the five unknowns b_1, b_2, b_3, b_4, b_5. Let us firstly assume $b_3 = 0$, $b_4 = 1$, and therefore, we have

$$1 + b_5 = 0 \tag{2.30}$$

$$b_1 + b_2 - 3 - b_5 = 0 \tag{2.31}$$

$$-b_2 - b_5 = 0 \tag{2.32}$$

which provides us with

$$b_5 = -1 \qquad b_2 = 1 \qquad b_1 = 1 \tag{2.33}$$

Therefore,

$$\pi_1 = \frac{\rho v l}{\mu} \tag{2.34}$$

Let us secondly assume $b_3 = 1$, $b_4 = 0$, and therefore we have

$$b_4 + b_5 = 0 \tag{2.35}$$

$$b_1 + b_2 + b_3 - 3b_4 - b_5 = 0 \tag{2.36}$$

$$-b_2 - 2b_3 - b_5 = 0 \tag{2.37}$$

$$b_5 = 0 \tag{2.38}$$

$$b_1 + b_2 + 1 - b_5 = 0 \tag{2.39}$$

$$-b_2 - 2 - b_5 = 0 \tag{2.40}$$

which provide us with

$$b_5 = 0 \qquad b_2 = -2 \qquad b_1 = 1 \tag{2.41}$$

Therefore,

$$\pi_2 = \frac{gl}{v^2} \tag{2.42}$$

Knowing that if π_i is a dimensionless variable then, $1/\pi_i$ and $(\pi_i)^n$ are also dimensionless variables, we usually show the π_1 and π_2 in this problem as

$$\pi_1 = \frac{\rho v l}{\mu} = \mathrm{Re} \tag{2.43}$$

$$\pi_2 = \frac{v}{\sqrt{gl}} = Fr \tag{2.44}$$

$\pi_1 = \rho v l/\mu = \mathrm{Re}$ is called Reynolds number, and $\pi_2 = v/\sqrt{gl} = Fr$ is called Froude number.

Although in this example we are free to choose any two exponents arbitrarily, the results may yield indeterminant solutions or meaningless or trivial solutions such as one with all unknown exponents to be zero.

The Buckingham π-theorem states that any complete physical relationship can be expressed by an equation of a set of independent nondimensional product combinations of the physical quantities concerned. The least possible number of independent nondimensional quantities which appear in the relationship is equal to the number of related physical quantities minus the number of the fundamental units.

In other words, if a problem involves n variables and m independent dimensions, then it can be reduced to a relationship between $n - m$ nondimensional parameters. Hence, every equation is dimensionally homogeneous if and only if it can be set in the form

$$g(\pi_1, \pi_3, \pi_3, \cdots, \pi_k) = 0 \tag{2.45}$$

where g is a function of the dimensionless products of $\pi_1, \pi_3, \pi_3, \cdots, \pi_k$.

Since Buckingham (1914) denoted the dimensionless quantities by π_i, this theorem is referred to as the π-theorem of Buckingham. The π_i variables are the nondimensionalized variables of the problem. Besides the Buckingham π-theorem, there are other techniques such as Rayleigh method that can be used to perform a dimensional analysis. Rayleigh method is the dimensional balance of both sides of an unknown equation with known variables. However, all other methods are similar in principle to the Buckingham technique.

It seems that the π-theorem has been first stated by Vaschy (1892). Buckingham (1914) gave the first proof of the π-theorem in 1914, for some special cases. Martinot-Lagarde (1948) gave a more general proof, and Birkhoff (1956) clarified the proof better (Curtis et al. 1982; Langhaar 1951). ■

Example 53 Escape velocity v from a planet.

Let us assume that the escape velocity v from a planet depends on the mass of the planet m, the planet radius R, and the universal gravitational constant G. Their dimensions are:

$$[v] = LT^{-1} \tag{2.46}$$

$$[m] = M \tag{2.47}$$

$$[R] = L \tag{2.48}$$

$$[G] = L^3T^{-2}M^{-1} \tag{2.49}$$

Let also assume that the escape velocity is a function of the form of

$$v = Cm^a R^b G^c \tag{2.50}$$

where C is a dimensionless constant and a, b, c are the numbers to be determined. Rayleigh's method of dimensional homogeneity of the equation shows that

$$\begin{aligned} [v] = LT^{-1} &= \left[Cm^a R^b G^c\right] = M^a L^b \left(L^3T^{-2}M^{-1}\right)^c \\ &= M^{a-c} L^{b+3c} T^{-2c} \end{aligned} \tag{2.51}$$

Therefore, we must have

$$0 = a - c \tag{2.52}$$

$$1 = b + 3c \tag{2.53}$$

$$-1 = -2c \tag{2.54}$$

with the unique solution of

$$a = 1/2 \qquad b = -1/2 \qquad c = 1/2 \tag{2.55}$$

The escape velocity equation will now become

$$v = C\sqrt{\frac{Gm}{R}} \tag{2.56}$$

To determine the π-functions of the escape velocity problem, we search for the exponents of a, b, c, d such that $[\pi_1] = 1$.

$$\pi_1 = m^a R^b G^c v^d \tag{2.57}$$

$$\begin{aligned}[1] &= \left[m^a R^b G^c v^d\right] = M^a L^b \left(L^3 T^{-2} M^{-1}\right)^c \left(L T^{-1}\right)^d \\ &= M^{a-c} L^{b+3c+d} T^{-2c-d} = 1\end{aligned} \tag{2.58}$$

Therefore,

$$a - c = 0 \tag{2.59}$$

$$b + 3c + d = 0 \tag{2.60}$$

$$-2c - d = 0 \tag{2.61}$$

with the solution of

$$a = -\frac{1}{2}d \qquad b = \frac{1}{2}d \qquad c = -\frac{1}{2}d \tag{2.62}$$

that provides us with a π-function.

$$\pi_1 = v^{-d/2} m^a R^{d/2} G^{-d/2} = \left(v\sqrt{\frac{R}{Gm}}\right)^d \tag{2.63}$$

The value of d is not important in this problem as if $d = 1$ provide us with π_1, then $d = 2$ provide us with π_1^2 and so on. Choosing $d = 1$ makes our dimensionless variable as $\pi_1 = v\sqrt{R/(Gm)}$. According to π-theorem any physical law involving the parameters v, m, R, G can be expressed as $g(\pi_1) = 0$ then $\pi_1 = C$ with a dimensionless constant C is a solution.

Therefore, the scape velocity v from any planet can be calculated from Eq. (2.56) and as a result if the escape velocity from planet 1 is:

$$v_1 = C_1\sqrt{\frac{Gm_1}{R_1}}, \tag{2.64}$$

then the escape velocity from the plant 2 will be:

$$v_2 = C_1\sqrt{\frac{Gm_2}{R_2}} \tag{2.65}$$

and therefore,

$$\frac{v_1^2}{v_2^2} = \frac{m_1}{m_2}\frac{R_2}{R_1} \tag{2.66}$$

Escape velocity is the lowest value of the initial velocity of an object departing from the surface of a celestial body required to escape the body's gravitational attraction. Earth's escape velocity is 11.2 km/ s, and for the Moon it is 2.4 km/ s. To determine the escape velocity we recall that the gravitational potential P between two masses m and M to be

$$P = -\frac{GmM}{r} \tag{2.67}$$

where r is the distance between m and M. If M is a particle, such as a missile, on the surface R of a celestial body such as a planet, then M needs a minimum velocity v to be able to reach the infinity where the potential becomes zero and its speed also be zero. The energy equations determines v.

$$\frac{1}{2}Mv^2 - \frac{GmM}{R} = 0 \tag{2.68}$$

$$v = \sqrt{\frac{2Gm}{R}} \tag{2.69}$$

Interestingly, the scape velocity is not dependent on the mass of the missile. The escape velocity from the Earth with

$$G = 6.67408 \times 10^{-11} \frac{\mathrm{m}^3}{\mathrm{kg\,s}^2} \tag{2.70}$$

$$R \simeq 6.3781 \times 10^6 \,\mathrm{m} \tag{2.71}$$

$$m \simeq 5.972 \times 10^{24} \,\mathrm{kg} \tag{2.72}$$

would be

$$v = \sqrt{\frac{2Gm}{R}} \simeq 11180 \frac{\mathrm{m}}{\mathrm{s}} \tag{2.73}$$

Example 54 Taylor's atomic explosion analysis.

It is a famous story that in the early 1940s a picture of an atomic blast was published on the cover of Life magazine. Sir Geoffrey Ingram Taylor (1886–1975), a fluid mechanic scientist at Cambridge, UK, investigated the amount of energy of the blast using dimensional analysis. In a nuclear explosion there is an instantaneous release of energy E in a small region of space. This explosion produces a strong spherical shock wave, with the pressure inside the shock wave several thousands of times greater than the initial negligible air pressure. Figure 2.1 illustrates the spherical shock wave 15 ms after the explosion (Barenblatt 1996).

To determine the amount of released energy, Taylor (1941, 1950) studied the rate of growth of the radius R of this shock wave with time t. The involved quantities are the energy E, the radius of spherical wave R, the density of air ρ and time t. Thus

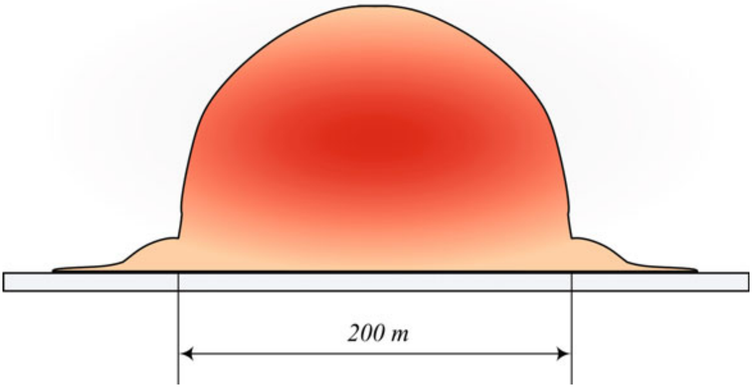

Fig. 2.1 The spherical shock wave 15 ms after the explosion

$$f(R, E, \rho, t) = f\left(R^a E^b \rho^c t^d\right) = 0 \tag{2.74}$$

$$[E] = ML^2T^{-2} \tag{2.75}$$

$$[t] = T \tag{2.76}$$

$$[R] = L \tag{2.77}$$

$$[\rho] = ML^{-3} \tag{2.78}$$

There are $n = 4$ variables and $m = 3$ dimensions and therefore, there exist $m - n = 1$ dimensionless group π_1.

$$\pi_1 = R^a\ E^b\ \rho^c\ t^d \tag{2.79}$$

$$[\pi_1] = [R]^a\ \ [E]^b\ \ [\rho]^c\ \ [t]^d \tag{2.80}$$

$$1 = L^a \left(ML^2T^{-2}\right)^b \left(ML^{-3}\right)^c T^d$$

$$= L^{a+2b+c}\ M^{b+c}\ T^{-2c+d} \tag{2.81}$$

The dimensional balance equation provides us with 3 equations for 4 unknowns.

$$a + 2b + c = 0 \tag{2.82}$$

$$b + c = 0 \tag{2.83}$$

$$-2c + d = 0 \tag{2.84}$$

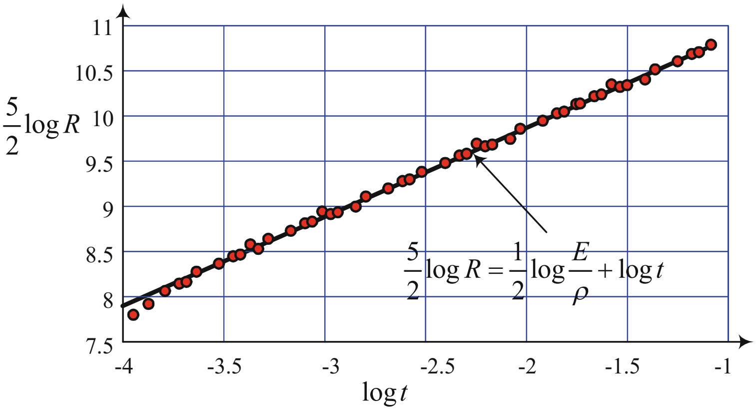

Fig. 2.2 A line with a slope of unity in a plane with logarithmic coordinates of $x = \log t$ and $y = \frac{5}{2}\log R$ illustrates how the radius of the spherical shock wave expands

Assuming $a = 1$, we find

$$b = -\frac{1}{5} \qquad c = \frac{1}{5} \qquad d = -\frac{2}{5} \tag{2.85}$$

which shows:

$$\pi_1 = \frac{R}{E^{1/5} t^{2/5} \rho^{-1/5}} = C \tag{2.86}$$

Therefore, R will be

$$R = C\ E^{1/5} t^{2/5} \rho^{-1/5} \tag{2.87}$$

The equation shows the relationship between the quantities. We may rewrite the equation as

$$\frac{5}{2}\log R = \frac{5}{2}\log C + \frac{1}{2}\log\frac{E}{\rho} + \log t \tag{2.88}$$

If we measure the radius of the spherical shock wave at different instants of time, then the experimental points will lie on a line with a slope of unity in a plane with logarithmic coordinates of $x = \log t$ and $y = \frac{5}{2}\log R$ as illustrated in Fig. 2.2. Using this analysis and mapping several photos of high speed photography, Taylor made very good approximation of the amount of initial E by extrapolation for $R \to 0$ to be $E = 10^{14}$ J = 23900573613.8 kcal. We need 9.81 J energy to lift 1 m a mass of 1 kg up from the Earth surface.

Example 55 Why do we consider $X/(F/k)$ and $r = \omega/\omega_n$ in vibrations?

Consider a forced excited undamped single DOF system:

$$m\ddot{x} + kx = F\sin(\omega t) \tag{2.89}$$

The input, output, and system parameters involved are m, k, F, X, and, $T \equiv 1/\omega$. Based on the Buckingham-π theorem, any complete physical relation of a system can be expressed in terms of a set of independent dimensionless products of its π-terms, which is the products of relevant physical parameters of the system. Because each π-term is dimensionless, we must have a balance for the dimensions involved:

$$X^a k^b m^c F^d \omega^e = L^0 M^0 T^0 \tag{2.90}$$

Let us substitute the parameters X, k, m, F, ω by their dimensions of M, L, T:

$$L^a \left(\frac{M}{T^2}\right)^b M^c \left(\frac{ML}{T^2}\right)^d \left(\frac{1}{T}\right)^e = L^0 M^0 T^0 \tag{2.91}$$

so we have

$$L^{a+d} M^{b+c+d} T^{-2b-2d-e} = L^0 M^0 T^0 \tag{2.92}$$

and

$$a + d = 0 \tag{2.93}$$

$$b + c + d = 0 \tag{2.94}$$

$$-2b - 2d - e = 0 \tag{2.95}$$

These three equations may be solved for any three out of five exponents a, b, c, d, e. The solution by assuming a and e are known as

$$d = -a \qquad b = a - \frac{1}{2}e \qquad c = \frac{1}{2}e \tag{2.96}$$

Substituting them back, we have

$$X^a k^{a-e/2} m^{e/2} F^{-a} \omega^e = L^0 M^0 T^0 \tag{2.97}$$

which indicates

$$\left(\frac{X}{F/k}\right)^a \left(\frac{\omega}{\sqrt{k/m}}\right)^e = L^0 M^0 T^0 \tag{2.98}$$

The dimensionless π-terms are

$$\pi_1 = \frac{X}{F/k} \qquad \pi_2 = \frac{\omega}{\sqrt{k/m}} \tag{2.99}$$

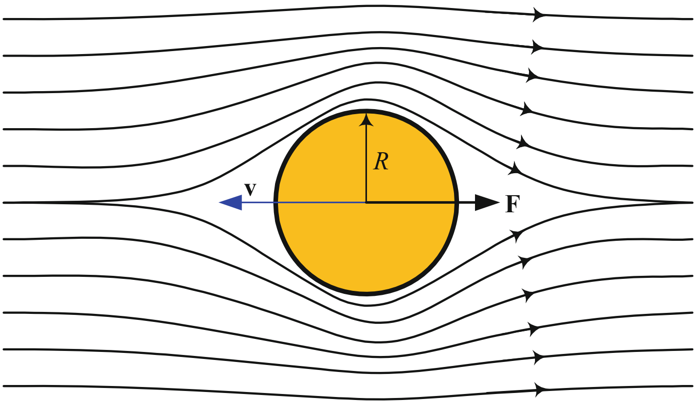

Fig. 2.3 A small spherical ball being pushed by a force F through a viscous fluid and move it with velocity v

The π-theorem states that there is a function f of the π-terms such that

$$f\left(\frac{X}{F/k}, \frac{\omega}{\sqrt{k/m}}\right) = 0 \tag{2.100}$$

So, $X/\left(F/k\right)$ is a function of $\omega/\sqrt{k/m}$.

Example 56 Resistive force F on a ball moving in a viscous fluid.

Consider a small spherical ball being pushed by a force F through a viscous fluid and move it with constant velocity v as is shown in Fig. 2.3. Let us assume that L represent the length dimension of the body, such as the radius of the ball R. Considering the resistance drag force F, length L, viscosity μ, density ρ, and velocity v, we may assume that there is an equation relating all variables to each other such as

$$f\left(F, L, \mu, \rho, v\right) = 0 \tag{2.101}$$

There are 5 variables F, L, μ, ρ, v and there are 3 base dimensions M, L, T. A dimensional balance check

$$[F]^a [L]^b [\mu]^c [\rho]^d [\nu]^e = \left(MLT^{-2}\right)^a L^b \left(ML^{-1}T^{-1}\right)^c \left(ML^{-3}\right)^d \left(LT^{-1}\right)^e = [1] \tag{2.102}$$

indicates that

$$a + c + d = 0 \tag{2.103}$$

$$a + b - c - 3d + e = 0 \tag{2.104}$$

$$-2a - c - e = 0 \tag{2.105}$$

we have a system of 3 equations in 5 unknown exponents a, b, c, d, e.

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & -1 & -3 & 1 \\ -2 & 0 & -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{2.106}$$

Therefore, if we pick 2 of the variables, then we are able to solve for the other 3. Let us take a and e and solve for b, c, d, to obtain

$$b = e \qquad c = -2a - e \qquad d = a + e \tag{2.107}$$

Having the difference between the number of variables and base dimensions, $5 - 3 = 2$ indicates that we will have two π-variables, π_1 and π_2. Assuming $a = 0$ and $e = 1$ we have

$$b = 1 \qquad c = -1 \qquad d = 1 \tag{2.108}$$

and therefore,

$$\pi_1 = \frac{vL\rho}{\mu} \tag{2.109}$$

Assuming $a = 1$ and $e = 0$ we have

$$b = 0 \qquad c = -2 \qquad d = 1 \tag{2.110}$$

and therefore,

$$\pi_2 = \frac{F\rho}{\mu^2} \tag{2.111}$$

We have found two sets of solutions $\mathbf{u}_1$ and $\mathbf{u}_2$

$$\mathbf{u}_1 = \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \qquad \mathbf{u}_2 = \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} \tag{2.112}$$

and any solution of Eq. (2.106) can be written as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$, as $\mathbf{u} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2$, where c_1 and c_2 are scalars. The π-theorem says that the physical law relating these variables can be written in the form

$$h(\pi_1, \pi_2) = 0 \tag{2.113}$$

or

$$\pi_1 = h_1(\pi_2) \tag{2.114}$$

or

$$\pi_2 = h_2(\pi_1) \tag{2.115}$$

for some functions of h_i. Therefore,

$$\frac{vL\rho}{\mu} = h_1\left(\frac{F\rho}{\mu^2}\right) \tag{2.116}$$

or

$$v = \frac{\mu}{L\rho} h_1\left(\frac{F\rho}{\mu^2}\right) \tag{2.117}$$

The equation $\pi_2 = h_2(\pi_1)$ is equivalent to say that the nondimensionalized force $F\rho/\mu^2$ is a function of the nondimensionalized velocity $vL\rho/\mu$.

Let us linearize the function $h_1(x) = h_1\left(F\rho/\mu^2\right)$ for $x \to 0$ and approximate the function for small x as

$$h_1(x) \approx kx \tag{2.118}$$

$$k = \lim_{F\to 0} \frac{\partial h_1}{\partial x} \tag{2.119}$$

and determine an approximate equation for the velocity v at low speed.

$$v \approx \frac{kF}{L\mu} \tag{2.120}$$

The dimensionless variable π_1 is called the Reynolds number. Reynolds number is shown by Re, after Osborne Reynolds (1842–1912). Re is to characterize the flow of a viscous fluid. The Reynolds number is also defined as $\mathrm{Re} = vL/\eta$, where $\eta = \mu/\rho$ is the kinematic viscosity of fluid, L is the representative linear dimension such as the diameter of a sphere, and v is the relative velocity of the object and the fluid.

A result of the above dimensional analysis is that the set S_1 of initial variables

$$S_1 = \{F, L, \mu, \rho, v\} \tag{2.121}$$

is equivalent to the set S_2 when two of the original variables are substituted with π_1 and π_2, such as

$$S_1 = \{\pi_1, L, \mu, \rho, \pi_2\} \tag{2.122}$$

Example 57 Newton's Gravitational equation.

Let us write the Newton Gravitational equation

$$F = G\frac{m_1 m_2}{r^2} \tag{2.123}$$

in the following form

$$\pi_i = G^a m_1^b m_2^c r^d F^e \tag{2.124}$$

$$1 = \left(M^{-1}L^3T^{-2}\right)^a M^b M^c L^d \left(MLT^{-2}\right)^e \tag{2.125}$$

$$1 = M^{-a+b+c+e} L^{3a+d+e} T^{-2a-2e} \tag{2.126}$$

which provides us with the following equations:

$$-a + b + c + e = 0 \tag{2.127}$$

$$3a + d + e = 0 \tag{2.128}$$

$$-2a - 2e = 0 \tag{2.129}$$

There are three equations for five unknowns a, b, c, d, e. We may chose two of the unknowns arbitrarily. Let assume $a = 1$, $b = 1$, then,

$$e = -1 \qquad d = -2 \qquad c = 1 \tag{2.130}$$

and therefore,

$$\pi_1 = \frac{Gm_1 m_2}{Fr^2} \tag{2.131}$$

The exponents e and a cannot be chosen because of $a = -e$. If we pick a value for d, then Eqs. (2.128) and (2.129) are inconsistent unless, $d = -2a$. These problems appear because the dimensional analysis is ill to treat m_1 and m_2 as two different masses.

Example 58 Large elastic deformation.

In the theory of elasticity, deformations are considered large when the relationship between stress and deformation is no longer linear. The variables involved in such problems are the elastic deformation y, force F, characteristic length l, the modulus of elasticity E, the density ρ, and the gravitational acceleration g. Therefore, we may assume an equation of the form:

$$f\left(y, F, l, E, \rho, g\right) = 0 \tag{2.132}$$

Having 6 variables y, F, l, E, ρ, g and 3 dimensions M, L, T provides us with three π-groups,

$$\pi_1 = \frac{y}{l} \qquad \pi_2 = \frac{F}{El^2} \qquad \pi_3 = \frac{\rho g l}{E} \tag{2.133}$$

which can be expressed as

$$\frac{y}{l} = g\left(\frac{F}{El^2}, \frac{\rho g l}{E}\right) \tag{2.134}$$

This is because of

$$[1] = [F]^a\,[l]^b\,[E]^c\,[\rho]^d\,[g]^e\,[y]^f \tag{2.135}$$

$$1 = \left(MLT^{-2}\right)^a L^b \left(ML^{-1}T^{-2}\right)^c \left(ML^{-3}\right)^d \left(LT^{-2}\right)^e L^f \tag{2.136}$$

that yields:

$$a + c + d = 0 \tag{2.137}$$

$$a + b - c - 3d - 2e + f = 0 \tag{2.138}$$

$$-2a - 2c - 2e = 0 \tag{2.139}$$

We can pick three exponents arbitrarily. Let us assume $f = 1$, $d = 0$, $e = 0$ to have

$$c = 0 \qquad a = 0 \qquad b = 1 \tag{2.140}$$

and we get the first π-group.

$$\pi_1 = \frac{y}{l} \tag{2.141}$$

Assuming $f = 0$, $d = 0$, $e = 0$, we have

$$c = -1 \qquad a = 1 \qquad b = -2 \tag{2.142}$$

and we get the second π-group.

$$\pi_2 = \frac{F}{El^2} \tag{2.143}$$

Assuming $e = 1$, $a = 0$, $b = 0$, we have

$$c = -1 \qquad d = 1 \qquad f = 1 \tag{2.144}$$

and we get the third π-group.

$$\pi_3 = \frac{\rho g l}{E} \tag{2.145}$$

Example 59 Pressure drop in a pipe.

Let determine the pressure drop $\Delta p/l$ per unit length of a circular pipe in which an incompressible Newtonian fluid flows. The pressure drop $\Delta p/l$ might be a function of the pipe internal diameter D, the fluid density ρ, the fluid viscosity μ, the mean velocity of the fluid through the pipe v.

$$f\left(D, \rho, \mu, v, \frac{\Delta p}{l}\right) = f\left(D^a \rho^b \mu^c v^d \left(\frac{\Delta p}{l}\right)^e\right) = 0 \tag{2.146}$$

As we know,

$$\begin{gathered} [D] = L \qquad [v] = LT^{-1} \qquad [\rho] = ML^{-3} \\ [\Delta p] = ML^{-1}T^{-2} \qquad [\mu] = ML^{-1}T^{-1} \end{gathered} \tag{2.147}$$

There are 5 variables $(\Delta p/l, D, \rho, \mu, v)$ and 3 dimensions (M, L, T) equal to the rank of the dimensional matrix. The dimensional matrix $\mathbf{A}$ of this system is:

$$\begin{array}{c|ccccc} & D & \rho & \mu & v & \Delta p/l \\ \hline M & 0 & 1 & 1 & 0 & 1 \\ L & 1 & -3 & -1 & 1 & -2 \\ T & 0 & 0 & -1 & -1 & -2 \end{array} \tag{2.148}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & -3 & -1 & 1 & -2 \\ 0 & 0 & -1 & -1 & -2 \end{bmatrix} \tag{2.149}$$

There is at least one nonzero rank 3×3 sub-matrix:

$$\left\{\rho \ \mu \ \Delta p/l\right\} \qquad \begin{vmatrix} 1 & 1 & 1 \\ -3 & -1 & -2 \\ 0 & -1 & -2 \end{vmatrix} = -3.0 \tag{2.150}$$

Therefore, there must be two π-groups. The dimension balance of Eq. (2.146) will be:

$$[1] = [D]^a \, [\rho]^b \, [\mu]^c \, [v]^d \left[\frac{\Delta p}{l}\right]^e \tag{2.151}$$

$$M^0 L^0 T^0 = L^a \left(ML^{-3}\right)^b \left(ML^{-1}T^{-1}\right)^c \left(LT^{-1}\right)^d \left(\frac{ML^{-1}T^{-2}}{L}\right)^e \tag{2.152}$$

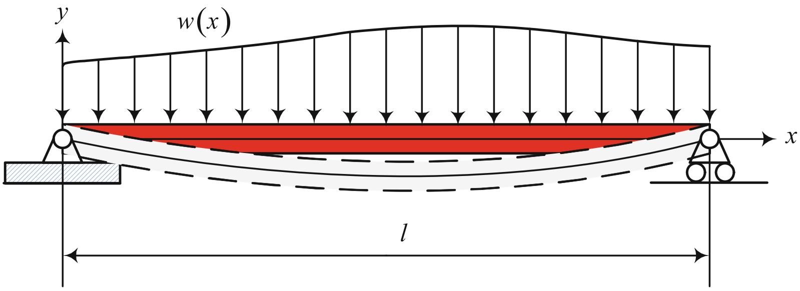

Fig. 2.4 A simply supported beam of length l, Young's modulus E, cross-sectional moment of area I, flexural rigidity EI, a distributed load of $w(x)$

This will give us three equations to determine 5 unknowns a, b, c, d, e to have equal dimensions of M, L, T on both sides.

$$0 = b + c + e \tag{2.153}$$

$$0 = a - 3b - c + d - 2e \tag{2.154}$$

$$0 = -c - d - 2e \tag{2.155}$$

Assuming $e = 0$, $c = 1$, we find

$$b = -1 \qquad d = -1 \qquad a = -1 \tag{2.156}$$

and the first π-group appears

$$\pi_1 = \frac{\mu}{\rho v D} = \frac{1}{\text{Re}} \tag{2.157}$$

Assuming $c = 0$, $e = 1$, we find

$$b = -1 \qquad d = -2 \qquad a = 1 \tag{2.158}$$

and the second π-group will be found.

$$\pi_2 = \frac{(\Delta p/l)\, D}{\rho v^2} = \frac{\Delta p\ D}{\rho v^2 l} \tag{2.159}$$

Example 60 Elastic and small deflection of beams.

Consider a simply supported beam as shown in Fig. 2.4 of length l, Young's modulus E, cross-sectional moment of area I, flexural rigidity EI, a distributed load of $w(x)$. The deflection of the beam $y(x)$ is governed by the following equation and boundary conditions.

$$EI\frac{d^4y(x)}{dx^4} = -w(x) \tag{2.160}$$

$$y(0) = 0 \qquad \frac{d^2y(0)}{dx^2} = 0 \tag{2.161}$$

$$y(l) = 0 \qquad \frac{d^2y(l)}{dx^2} = 0 \tag{2.162}$$

The deflection y of the beam is a function of the variables of l, EI, w.

$$f(y, l, EI, w) = 0 \tag{2.163}$$

For a balance of dimensions, we must have

$$[1] = [y]^a [l]^b [EI]^c [w]^d \tag{2.164}$$

$$1 = L^a L^b \left(ML^{-1}T^{-2}L^4\right)^c \left(MT^{-2}\right)^d \tag{2.165}$$

which provides a set of 3 equations for 4 unknown exponents.

$$c + d = 0 \tag{2.166}$$

$$a + b + 3c = 0 \tag{2.167}$$

$$-2c - 2d = 0 \tag{2.168}$$

Having 4 unknowns y, l, EI, w and 3 dimensions M, L, T, we must have one π-group. The first and third equations are not independent. Assuming $d = 0$, $a = 1$, we get

$$c = 0 \qquad b = -1 \tag{2.169}$$

and the first π-group appears

$$\pi_1 = \frac{y}{l} \tag{2.170}$$

Assuming $d = 1$, $a = 0$ we get

$$c = -1 \qquad b = 3 \tag{2.171}$$

and the second π-group is found.

$$\pi_2 = \frac{wl^3}{EI} \tag{2.172}$$

Therefore, π_1 will be a function of π_2.

$$\pi_1 = g(\pi_2) \tag{2.173}$$

$$\frac{y}{l} = g\left(\frac{wl^3}{EI}\right) \tag{2.174}$$

Example 61 Period of rotation of a body in a circular orbit.

A particle of mass m is moving in a circular orbit of radius R by a central force F. The period of rotation p of such motion will be of the form,

$$p = g(m, R, F) \tag{2.175}$$

or

$$f(p, m, R, F) = 0 \tag{2.176}$$

where

$$\begin{array}{c|cccc} & p & m & R & F \\ \hline M & 0 & 1 & 0 & 1 \\ L & 0 & 0 & 1 & 1 \\ T & 1 & 0 & 0 & -2 \end{array} \tag{2.177}$$

and therefore,

$$[1] = [p]^a [m]^b [R]^c [F]^d \tag{2.178}$$

$$1 = T^a M^b L^c \left(MLT^{-2}\right)^d \tag{2.179}$$

which provides a set of 3 equations for 4 unknown exponents.

$$a - 2d = 0 \tag{2.180}$$

$$b + d = 0 \tag{2.181}$$

$$c + d = 0 \tag{2.182}$$

Assuming $a = 1$, we have

$$d = \frac{1}{2} \qquad b = c = -\frac{1}{2} \tag{2.183}$$

that determines the period equation (Zorich 2011).

$$p = C\sqrt{\frac{mR}{F}} \tag{2.184}$$

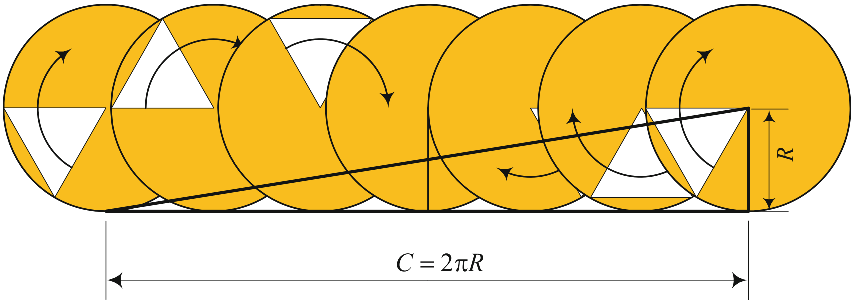

Fig. 2.5 Rolling a circle one complete revolution base

Experiment will show that

$$C = 2\pi \tag{2.185}$$

Example 62 ★ $\pi = 3.141592\cdots$ as the first π-number.

Archimedes (288–212BC) has proven the relationship between the circumference of a circle and the area in the process estimating the value of pi very accurately. $pi = \pi$ is defined as the ratio of the circumference, C, divided by the diameter, D of a circle.

$$\pi = \frac{C}{D} \tag{2.186}$$

$$[\pi] = \left[\frac{C}{D}\right] = \frac{[C]}{[D]} = \frac{L}{L} = 1 \tag{2.187}$$

π is a dimensionless number and as a result, all circles are similar.

$$\frac{C_1}{D_1} = \frac{C_2}{D_2} \tag{2.188}$$

Such a similarity and calculating π for only one circle makes us able to calculate the circumference of any circle by knowing its diameter D or radius R.

$$C = \pi D = 2\pi R \tag{2.189}$$

Archimedes rolled a circle one complete revolution and made a right triangle with base C and height R, as shown in Fig. 2.5. He then deduced such triangle is similar for all circles and suggested the similarity Eq. (2.188).

To prove it, he found out that 6 equilateral triangles fit within the circle, as shown in Fig. 2.6. Increasing the number of triangles embedded within the circle and expanding the circle as shown in Fig. 2.7, Archimedes concluded the area of the circle to be exactly equal to the area of the triangle.

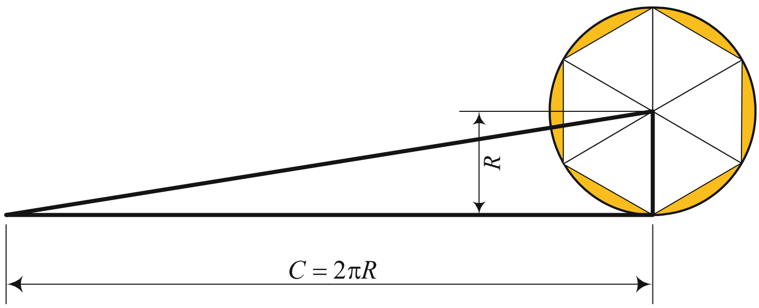

Fig. 2.6 6 equilateral triangles with side R fit within a circle

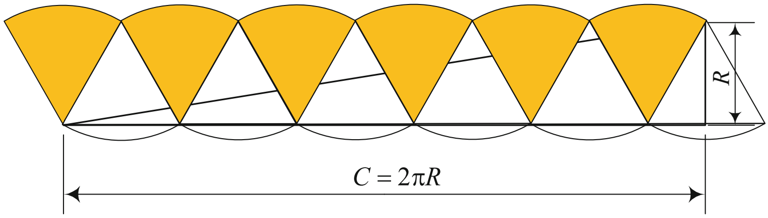

Fig. 2.7 Expanding the circle into infinite number of slices show the area equation

$$A = \frac{CR}{2} \qquad C = \frac{2A}{R} \tag{2.190}$$

and then, he discovered the equation of area.

$$A = \pi R^2 \tag{2.191}$$

Using this approach he increased the number of triangles to 96, and calculated the bounds of $3.1408 < \pi < 3.1429$.

Calculating the value of π is the most fascinating problem in the whole mathematics that kept mathematicians busy for several centuries. Not being able to measure circumference of a circle based on its diameter or vice versa seemed to be a mystery. That was because no one knew the exact area of a circle as a function of its measurable radius. It is interesting to know that it has now been calculated to ten trillion digits (Beckmann 1971; Allen 2014; Dunham 1990; Gullberg 1997).

From the available documents and evidences it is known that the Babylonians, the Egyptians, the Chinese, the Indian, and the Persian were aware of the existence and significance of the constant π as the ratio of circumference C of a circle to its diameter D. As far as we know, before 2000BC, the Babylonians used the approximation of

$$\pi \approx 3\frac{1}{8} \tag{2.192}$$

and Egyptians used the approximation of

$$\pi \approx 4 \left(\frac{8}{9} \right)^2 \tag{2.193}$$

Example 63 Problems of Buckingham theory.

There are several unsolvable problems with application of Buckingham π-theorem. Among them are:

1. The correct involved variables.

   The hardest part of the dimensional analysis is determining what the relevant variables are as the wrong determination may end up in wrong analysis. The below example illustrates the concept.

   We might have assumed that the escape velocity v is dependent on the mass of the planet m_1, the mass of the object m_2, the planet radius R, and the universal gravitational constant G. In this case, we would have

$$\begin{aligned}[v] = LT^{-1} &= \left[C m_1^{a_1} m_2^{a_2} R^b G^c \right] = M^{a_1+a_2} L^b \left(L^3 T^{-2} M^{-1} \right)^c \\ &= M^{a_1+a_2-c} L^{b+3c} T^{-2c}\end{aligned} \tag{2.194}$$

   and the exponents equations would be

$$0 = a_1 + a_2 - c \tag{2.195}$$

$$1 = b + 3c \tag{2.196}$$

$$-1 = -2c \tag{2.197}$$

   with the solution of

$$a_1 + a_2 = 1/2 \qquad b = -1/2 \qquad c = 1/2 \tag{2.198}$$

   that ends up to have a wrong escape velocity equation.

$$v = C \sqrt{\frac{G \sqrt{m_1} \sqrt{m_2}}{R}} \tag{2.199}$$

2. Dimensional analysis informs us that there is a relationship; however, it does not determine what the relationship is, except in the case of a single π_1-group. To determine the relationship we need a theory, such as Newton equation of motion, or experimental data analysis. The Buckingham π-theorem is ill to provide either a unique or a complete solution to a problem.

Example 64 ★Fundamental theorem of dimension theory.

The π Buckingham theorem is sometimes called the fundamental theorem of dimension theory and rephrased as below.

Consider the general case of the dependent variable y and independent variables $x_1, x_2, x_3, \cdots, x_n$. Assume that only the first k independent variables are physically and dimensionally independent.

$$y = f(x_1, x_2, x_3, \cdots, x_n) \tag{2.200}$$

By taking $x_1, x_2, \cdots, x_k$ as the units of measurement of the corresponding quantities. We may use $x_1, x_2, \cdots, x_k$ to change the scales of physical quantities by setting $\alpha_1 = x_1^{-1}, \alpha_2 = x_2^{-1}, \cdots, \alpha_k = x_k^{-1}$ and rewrite Eq. (2.200) relating to the dimensionless quantities.

$$\pi_i = g(1, 1, \cdots, 1, \pi_1, \pi_2, \pi_3, \cdots, \pi_{n-k}) \tag{2.201}$$

$$\pi_i = \frac{y}{x_1^{d_1} x_2^{d_2} \cdots x_k^{d_k}} \tag{2.202}$$

$$\pi_i = \frac{x_{k+i}}{x_1^{d_1} x_2^{d_2} \cdots x_k^{d_k}} \qquad i = 1, 2, \cdots, n-k \tag{2.203}$$

Equation (2.201) may be rewritten in the form

$$y = x_1^{d_1} x_2^{d_2} \cdots x_k^{d_k}\, g(1, 1, \cdots, 1, \pi_1, \pi_2, \pi_3, \cdots, \pi_{n-k}) \tag{2.204}$$

Thus, by using the scale homogeneity of the dependencies between physical quantities we may reduce the number of variables. The possibility of such a transition from the general relation (2.200) to the simpler relation (2.204) forms the content of the π-theorem, which is the fundamental theorem of the theory of dimensions of physical quantities.

2.2 Nondimensionalization

Consider the general equation of the dimensional dependent variable y and dimensional independent variable x.

$$\frac{d^n y}{dx^n} = f\left(x, y, \frac{dy}{dt}, \frac{d^2 y}{dt^2}, \frac{d^3 y}{dt^3}, \cdots, \frac{d^{n-1} y}{dx^{n-1}}\right) \tag{2.205}$$

The equation may be transformed into a nondimensional equation

$$\frac{d^n Y}{dX^n} = \frac{x_0^k}{y_0} f\left(x_0 X, y_0 Y, \frac{y_0}{x_0}\frac{dY}{dX}, \frac{y_0}{x_0^2}\frac{d^2 Y}{dX^2}, \cdots, \frac{y_0}{x_0^{k-1}}\frac{d^{k-1} Y}{dX^{k-1}}\right) \tag{2.206}$$

by introducing dimensionless variables Y and X using dimensional characteristics y_0, x_0.

$$Y = \frac{y}{y_0} \qquad X = \frac{x}{x_0} \tag{2.207}$$

Proof We define the constant dimensional characteristic parameters of y_0 and x_0 to introduce the nondimensional dependent and independent variables Y and X.

$$Y = \frac{y}{y_0} \qquad X = \frac{x}{x_0} \tag{2.208}$$

The derivatives will be redefined as

$$\frac{d}{dx} = \frac{d}{dX}\frac{dX}{dx} = \frac{1}{x_0}\frac{d}{dX} \tag{2.209}$$

$$\frac{d^2}{dx^2} = \frac{d}{dx}\left(\frac{1}{x_0}\frac{d}{dX}\right) = \frac{1}{x_0^2}\frac{d^2}{dX^2} \tag{2.210}$$

$$\frac{d^k}{dx^k} = \frac{1}{x_0^k}\frac{d^k}{dX^k} \tag{2.211}$$

therefore,

$$\frac{dy}{dx} = \frac{y_0}{x_0}\frac{dY}{dX} \tag{2.212}$$

$$\frac{d^k y}{dx^k} = \frac{y_0}{x_0^k}\frac{d^k Y}{dX^k} \tag{2.213}$$

Substituting the dimensional variables will provide us with

$$\frac{y_0}{x_0^k}\frac{d^n Y}{dX^n} = f\left(x_0 X, y_0 Y, \frac{y_0}{x_0}\frac{dY}{dX}, \frac{y_0}{x_0^2}\frac{d^2 Y}{dX^2}, \cdots, \frac{y_0}{x_0^{k-1}}\frac{d^{k-1} Y}{dX^{k-1}}\right) \tag{2.214}$$

Nondimensionalization of equations is the best use of dimensional analysis to reduce the number of variables into the absolute minimum number. The equation may be an algebraic equation or a differential equation. The nondimensionalization of equations is very useful in generalization of the results. Algebraic equations are such as constitutive laws or solution of equations of motion, and differential equations are such as the equations of motion of a dynamic system. Although majority of differential equations are nonlinear without analytic solutions, nondimensionalization of differential equations is the best method for: (1) to reduce the number of variables, (2) to express the numerical or analytical solutions, (3) to make possible transfer of data between model and prototype.

An example of each case will show the idea of nondimensionalization, its usefulness, and reduction of the number of variables.

1-Algebraic Equation Consider the dimensionally homogeneous algebraic equation

$$f(x, y, z, l) = x^2 y - a\frac{xy^4}{z^2} + bz^2 \sin\left(\frac{y}{l}\right) = 0 \tag{2.215}$$

where $[x] = [y] = [z] = [l] = L$ and l is a constant length, a is a dimensionless parameter $[a] = [1]$, and b is a dimensional parameter $[b] = L$. The dimension of f as well as every term of it is L^3. The equation will be made dimensionless by dividing each term by cube of reference length, l^3.

$$\begin{aligned} g\left(\frac{x}{l}, \frac{y}{l}, \frac{z}{l}\right) &= \frac{x^2 y}{l^3} - a\frac{xy^4}{z^2 l^3} + b\frac{z^2}{l^3} \sin\left(\frac{y}{l}\right) \\ &= X^2 Y - a\frac{XY^4}{Z^2} + BZ^2 \sin Y \end{aligned} \tag{2.216}$$

where

$$X = \frac{x}{l} \qquad Y = \frac{y}{l} \qquad Z = \frac{z}{l} \qquad B = \frac{b}{l} \tag{2.217}$$

We have expressed a relationship among six quantities (x, y, z, l, a, b) in terms of five dimensionless quantities $(X = x/l,\ Y = y/l,\ Z = z/l,\ a,\ B = b/l)$. Nondimensionalization is done by dividing every dimensional quantity of the equation with a constant quantity of the system with the same dimension and simplifying the result in terms of new dimensionless quantities.

2-Differential Equation Consider the single degree-of-freedom linear forced vibration system of Fig. 2.8. The equation of motion of the system is:

$$m\ddot{x} + c\dot{x} + kx = F\sin(\omega t) \tag{2.218}$$

The equation of motion has two variables: x and t. Let us define a characteristic length x_0 such as static deflection of the mass m under the constant force F, and a characteristic time t_0,

$$x_0 = \frac{F}{k} \qquad t_0 = \frac{1}{\omega} \tag{2.219}$$

to define nondimensionalized displacement X and time τ

$$X = \frac{x}{x_0} = \frac{x}{F/k} \qquad \tau = \frac{t}{1/\omega} = \omega t \tag{2.220}$$

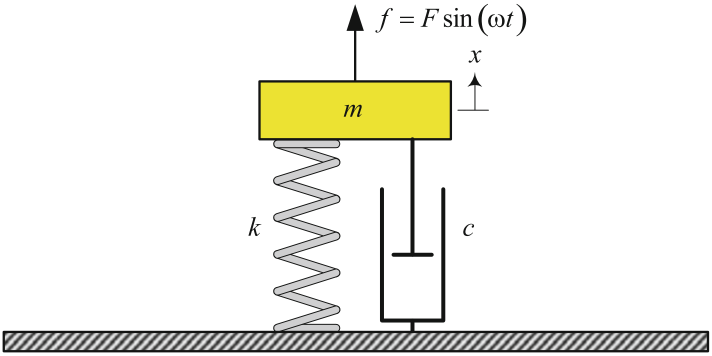

Fig. 2.8 Forced vibration of a linear vibrating system

and replace the actual variables x and t.

$$x = x_0 X = \frac{F}{k} X \qquad t = t_0 \tau = \frac{\tau}{\omega} \tag{2.221}$$

Using nondimensionalized time τ, we may redefine derivatives as

$$\dot{x} = \frac{dx}{dt} = \frac{F}{k}\frac{dX}{dt} = \frac{F}{k}\frac{dX}{d\tau}\frac{d\tau}{dt} = \frac{F}{k}\omega X' \tag{2.222}$$

$$\ddot{x} = \frac{d}{dt}\frac{dx}{dt} = \frac{F}{k}\frac{d}{dt}\left(\omega X'\right) = \frac{F}{k}\frac{d\tau}{dt}\frac{d}{d\tau}\left(\omega X'\right) = \frac{F}{k}\omega^2 X'' \tag{2.223}$$

$$X' = \frac{dX}{d\tau} \qquad X'' = \frac{d^2 X}{d\tau^2} \tag{2.224}$$

The equation of motion may now be rewritten as

$$\frac{F}{k} m\omega^2 X'' + \frac{F}{k} c\omega X' + \frac{F}{k} kX = F \sin(\tau) \tag{2.225}$$

After dividing by F/k

$$m\omega^2 X'' + c\omega X' + kX = k \sin(\tau) \tag{2.226}$$

and dividing by $m\omega^2$

$$X'' + \frac{c}{m\omega} X' + \frac{k}{m\omega^2} X = \frac{k}{m\omega^2} \sin(\tau) \tag{2.227}$$

will simplify the equation to

$$r^2 X'' + 2\xi r X' + X = \sin(\tau) \tag{2.228}$$

$$\pi_2^2 \pi_1'' + 2\pi_3 \pi_2 \pi_1' + \pi_1 = \sin \pi_4 \tag{2.229}$$

where

$$\pi_1 = X = \frac{x}{F/k} \qquad \pi_2 = r = \frac{\omega}{\omega_n} \tag{2.230}$$

$$\pi_3 = \xi = \frac{c}{c_c} \qquad \pi_4 = \tau = \omega t \tag{2.231}$$

$$\omega_n = \sqrt{\frac{k}{m}} \qquad c_c = 2\sqrt{km} \tag{2.232}$$

Therefore, the original Eq. (2.218) with variables x, t, and five parameters m, c, k, F, ω, has been nondimensionalized to Eq. (2.228) with two dimensionless variables X, τ, and two parameters ξ, r.

The number of remaining dimensionless quantities would be the same number determined by Buckingham π-theorem. Nondimensionalization and π-theorem are equivalent in terms of reduction of the number of variables into the absolute minimum number. ■

Example 65 Forced vibration of a linear vibrating system.

The beauty of nondimensionalization appears in the generality of the solution of equations. Consider the linear vibrating system that is shown in Fig. 2.8 with the following equation of motion:

$$m\ddot{x} + c\dot{x} + kx = F \sin(\omega t) \tag{2.233}$$

which will be transformed into the below nondimensionalized differential equation

$$r^2 X'' + 2\xi r X' + X = \sin(\tau) \tag{2.234}$$

$$\pi_1 = X = \frac{x}{F/k} \qquad \pi_2 = r = \frac{\omega}{\omega_n} \qquad \omega_n = \sqrt{\frac{k}{m}} \tag{2.235}$$

$$\pi_3 = \xi = \frac{c}{c_c} \qquad \pi_4 = \tau = \omega t \qquad c_c = 2\sqrt{km} \tag{2.236}$$

using a characteristic length x_0 and a characteristic time t_0,

$$x_0 = \frac{F}{k} \qquad t_0 = \frac{1}{\omega} \tag{2.237}$$

to define nondimensionalized displacement X and time τ.

$$X = \frac{x}{x_0} = \frac{x}{F/k} \qquad \tau = \frac{t}{1/\omega} = \omega t \tag{2.238}$$

Let us search for a harmonic steady state solution for equation of motion (2.234). Because the equation of motion is linear, the steady state solution must be a general form of the excitation.

$$X = A \sin(\tau) + B \cos(\tau) \tag{2.239}$$

$$X' = A \cos(\tau) - B \sin(\tau) \tag{2.240}$$

$$X'' = -A \sin(\tau) - B \cos(\tau) \tag{2.241}$$

Substituting the solution and its derivatives into the equation of motion provides us with

$$r^2 \left(-A \sin(\tau) - B \cos(\tau)\right) + 2\xi r \left(A \cos(\tau) - B \sin(\tau)\right) + A \sin(\tau) + B \cos(\tau) = \sin(\tau) \tag{2.242}$$

Separating sin and cos functions makes two equations

$$2A\xi r - Br^2 + B = 0 \tag{2.243}$$

$$A - 2B\xi r - Ar^2 = 1 \tag{2.244}$$

to find A and B,

$$\begin{bmatrix} 2\xi r & 1 - r^2 \\ 1 - r^2 & -2\xi r \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{2.245}$$

to be equal to:

$$A = \frac{1 - r^2}{\left(1 - r^2\right)^2 + (2\xi r)^2} \tag{2.246}$$

$$B = -\frac{2\xi r}{\left(1 - r^2\right)^2 + (2\xi r)^2} \tag{2.247}$$

Therefore, the steady state solution of Eq. (2.234) is:

$$X = \frac{1 - r^2}{\left(1 - r^2\right)^2 + (2\xi r)^2} \sin(\tau) - \frac{2\xi r}{\left(1 - r^2\right)^2 + (2\xi r)^2} \cos(\tau) \tag{2.248}$$

$$= X_0 \sin(\tau - \varphi) \tag{2.249}$$

or

$$\pi_1 = \pi_5 \sin(\pi_4 - \pi_6) \tag{2.250}$$

where

$$\pi_5 = X_0 = \sqrt{A^2 + B^2} = \frac{1}{\sqrt{\left(1 - r^2\right)^2 + (2\xi r)^2}} \tag{2.251}$$

$$\pi_6 = \tan\varphi = -\frac{B}{A} = \frac{2\xi r}{1 - r^2} \tag{2.252}$$

The π-groups of π_5 and π_6 are not independent as they are functions of π_2 and π_3.

$$\pi_5 = \frac{1}{\sqrt{\left(1 - \pi_2^2\right)^2 + (2\pi_3\pi_2)^2}} \tag{2.253}$$

$$\pi_6 = \frac{2\pi_3\pi_2}{1 - \pi_2^2} \tag{2.254}$$

The solution X of the vibration of the system is a function of single variable τ and two parameters, r, ξ.

$$X = f(r, \xi, \tau) \tag{2.255}$$

All four quantities X, r, ξ, τ are dimensionless. Therefore, Eq. (2.248) is equivalent to

$$\pi_1 = f\left(\pi_2, \pi_3, \pi_4\right) \tag{2.256}$$

$$\pi_1 = X = \frac{x}{F/k} \tag{2.257}$$

$$\pi_2 = r = \frac{\omega}{\omega_n} \tag{2.258}$$

$$\pi_3 = \xi = \frac{c}{c_c} \tag{2.259}$$

$$\pi_4 = \tau = \omega t \tag{2.260}$$

Let us review this problem from the dimensional analysis and π-theorem viewpoint. There are 7 dimensional quantities involved in this vibrating system, the variable displacement x, and time t; and parameters mass m, damping c, stiffness k, force magnitude F, and force frequency ω. There are 3 dimensions in this problem: time T, length L, mass M. As a result, there must be $4 = 7 - 3$ dimensionless π-group, that we may assume to be expressed in the form of $\pi_1 = f\left(\pi_2, \pi_3, \pi_4\right)$. Therefore,

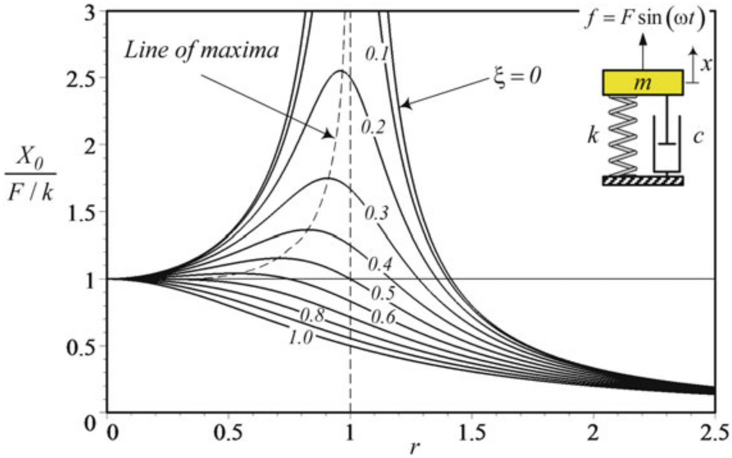

Fig. 2.9 The dimensionless amplitude X_0 as a function of r for different values of ξ

instead of working with 7 variables, we may take advantage of dimensional analysis and only work with 4 variables and fully discover the behavior of the system.

Dimensional analysis is able to determine the π-groups; however, it cannot determine the function f as we did in (2.248). To be able to determine f, we must use another science such as dynamics, differential equations, linear algebra, etc.

The amplitude X_0 of the harmonic steady state response is of interest in engineering. The amplitude X_0 is only a function of two dimensionless variables r, ξ, and therefore it is possible to illustrate their relationship by a surface in a three-dimensional space of (X_0, r, ξ), or plot X_0 as a function of r for different values of ξ as is shown in Fig. 2.9 (Jazar 2014).

Example 66 A two degrees-of-freedom (DOF) system.

Figure 2.10 illustrates a two DOF forced vibrating system such that a secondary system (m_2, k_2, c_2) is attached to a primary forced excited system (m_1, k_1, c_1). The equations of motion of the system are:

$$m_1\ddot{x}_1 + c_1\dot{x}_1 + c_2(\dot{x}_1 - \dot{x}_2) + k_1x_1 + k_2(x_1 - x_2) = F\sin\omega t \tag{2.261}$$

$$m_2\ddot{x}_2 - c_2(\dot{x}_1 - \dot{x}_2) - k_2(x_1 - x_2) = 0 \tag{2.262}$$

which can be found by using the kinetic and potential energies K, P, and the dissipation function D of the system

$$K = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 \tag{2.263}$$

$$P = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2(x_1 - x_2)^2 \tag{2.264}$$

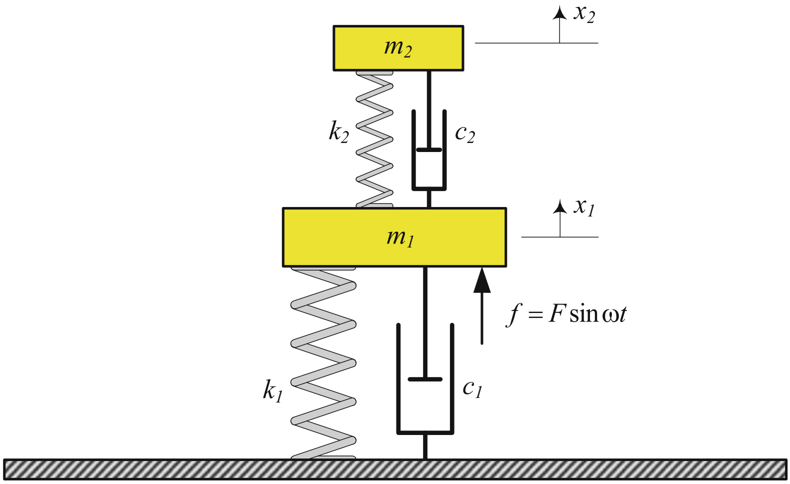

Fig. 2.10 A linear two degrees-of-freedom forced vibrating system

$$D = \frac{1}{2}c_1\dot{x}_1^2 + \frac{1}{2}c_2\left(\dot{x}_1 - \dot{x}_2\right)^2 \tag{2.265}$$

and employing the Lagrange equation,

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{q}_i} + \frac{\partial D}{\partial \dot{q}_i} + \frac{\partial P}{\partial q_i} = F_i \qquad i = 1, 2 \tag{2.266}$$

$$q_1 = x_1 \qquad q_2 = x_2 \tag{2.267}$$

We find the equations of motion as (2.261) and (2.262), or equivalently as:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} = \begin{bmatrix} F \sin \omega t \\ 0 \end{bmatrix} \tag{2.268}$$

Having the differential equations of motion of the system gives us the option to make the equations nondimensionalized and solve them, or solve the equations and then nondimensionalize the results.

We define the characteristic displacement x_0 as the static deflection of the mass m_1 under the constant force F, and a characteristic time t_0

$$x_0 = \frac{F}{k_1} \qquad t_0 = \frac{1}{\omega} \tag{2.269}$$

and define nondimensionalized displacements X_1, X_2, and time τ.

$$X_1 = \frac{x_1}{F/k} \qquad X_2 = \frac{x_2}{F/k_1} \qquad \tau = \omega t \tag{2.270}$$

Now we can replace the actual variables x_1, x_2, t.

$$x_1 = x_0 X_1 = \frac{F}{k_1} X_1 \tag{2.271}$$

$$x_2 = x_0 X_2 = \frac{F}{k_1} X_2 \tag{2.272}$$

$$t = t_0 \tau = \frac{\tau}{\omega} \tag{2.273}$$

Using the nondimensional time τ we may redefine derivatives as

$$\dot{x}_1 = \frac{dx_1}{dt} = \frac{F}{k_1}\frac{dX_1}{dt} = \frac{F}{k_1}\omega\frac{dX_1}{d\tau} = \frac{F}{k_1}\omega X_1' \tag{2.274}$$

$$\dot{x}_2 = \frac{dx_2}{dt} = \frac{F}{k_1}\frac{dX_2}{dt} = \frac{F}{k_1}\omega\frac{dX_2}{d\tau} = \frac{F}{k_1}\omega X_2' \tag{2.275}$$

$$\ddot{x}_1 = \frac{d}{dt}\frac{dx_1}{dt} = \frac{F}{k_1}\frac{d}{dt}\left(\omega X_1'\right) = \frac{F}{k_1}\frac{d\tau}{dt}\frac{d}{d\tau}\left(\omega X_1'\right) = \frac{F}{k_1}\omega^2 X_1'' \tag{2.276}$$

$$\ddot{x}_2 = \frac{d}{dt}\frac{dx_2}{dt} = \frac{F}{k_1}\frac{d}{dt}\left(\omega X_2'\right) = \frac{F}{k_1}\frac{d\tau}{dt}\frac{d}{d\tau}\left(\omega X_2'\right) = \frac{F}{k_1}\omega^2 X_2'' \tag{2.277}$$

$$X' = \frac{dX}{d\tau} \qquad X'' = \frac{d^2X}{d\tau^2} \tag{2.278}$$

The equations of motion (2.261) and (2.262) may now be written as:

$$m_1\omega^2\frac{F}{k_1}X_1'' + c_1\omega\frac{F}{k_1}X_1' + c_2\omega\frac{F}{k_1}\left(X_1' - X_2'\right) + FX_1 + F\frac{k_2}{k_1}\left(X_1 - X_2\right) = F\sin\tau \tag{2.279}$$

$$m_2\omega^2\frac{F}{k_1}X_2'' - c_2\omega\frac{F}{k_1}\left(X_1' - X_2'\right) - F\frac{k_2}{k_1}\left(X_1 - X_2\right) = 0 \tag{2.280}$$

After dividing by $m_1\omega^2 F/k_1$ we will have:

$$X_1'' + \frac{c_1}{m_1\omega}X_1' + \frac{c_2}{m_1\omega}\left(X_1' - X_2'\right) + \frac{k_1}{m_1\omega^2}X_1 + \frac{k_2}{m_1\omega^2}\left(X_1 - X_2\right) = \frac{k_1}{m_1\omega^2}\sin\tau \tag{2.281}$$

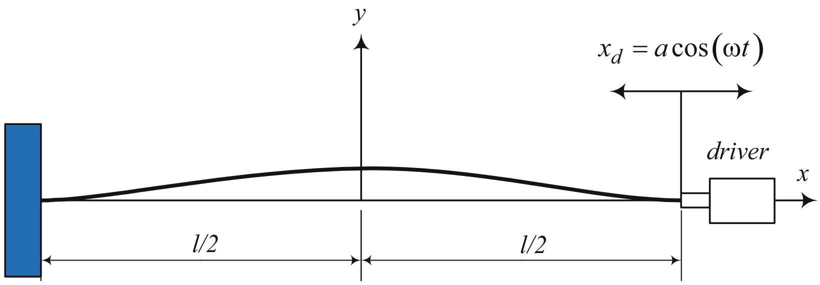

Fig. 2.11 Transverse vibration of a simple beam with length l, under axial tension

$$\frac{m_2}{m_1}X_2'' - \frac{c_2}{m_1\omega}\left(X_1' - X_2'\right) - \frac{k_2}{m_1\omega^2}\left(X_1 - X_2\right) = 0 \tag{2.282}$$

which will be

$$r^2 X_1'' + 2\xi_1 r X_1' + 2\xi_2 r\left(X_1' - X_2'\right) + X_1 + \alpha^2\left(X_1 - X_2\right) = \sin\tau \tag{2.283}$$

$$\varepsilon r^2 X_2'' - 2\xi_2 r\left(X_1' - X_2'\right) - \alpha^2\left(X_1 - X_2\right) = 0 \tag{2.284}$$

using the following nondimensional quantities:

$$\varepsilon = \frac{m_2}{m_1} \qquad \xi_1 = \frac{c_1}{2m_1\omega_1} \qquad \xi_2 = \frac{c_2}{2m_2\omega_2} \qquad r = \frac{\omega}{\omega_1} \tag{2.285}$$

$$\omega_1 = \sqrt{\frac{k_1}{m_1}} \qquad \omega_2 = \sqrt{\frac{k_2}{m_2}} \qquad \alpha = \frac{\omega_2}{\omega_1} = \sqrt{\frac{k_2}{\varepsilon k_1}} \tag{2.286}$$

Nondimensionalization reduced a problem with 3 variables, x_1, x_2, t, and 8 parameters, $m_1, m_2, k_1, k_2, c_1, c_2, F, \omega$, into an equivalent problem with 3 variables, X_1, X_2, τ, and 5 parameters, $\varepsilon, \xi_1, \xi_2, \alpha, r$.

Example 67 Transverse vibration of a longitudinally excited beam.

The equation of transverse vibration of a simple beam with length l, shown in Fig. 2.11, under axial tension is:

$$EI\frac{\partial^4 y}{\partial x^4} + \rho A\frac{\partial^2 y}{\partial t^2} = F\frac{\partial^2 y}{\partial x^2} \qquad -\frac{l}{2} < x < \frac{l}{2} \tag{2.287}$$

where ρ is the mass density of the beam, F is the tension in the beam, EI is the flexural rigidity, and A is the cross-sectional area of the beam.

Let us define the nondimensional variables Y, X, and τ by introducing their dimensional characteristics y_0, x_0, t_0.

$$y_0 = l \qquad x_0 = l \qquad t_0 = \sqrt{\frac{\rho A l^4}{EI}} \tag{2.288}$$

$$Y = \frac{y}{l} \qquad X = \frac{x}{l} \qquad \tau = \frac{t}{\sqrt{\rho A l^4/(EI)}} \tag{2.289}$$

Employing Y, X, τ, we have

$$\frac{\partial^4 y}{\partial x^4} = \frac{1}{l^3}\frac{\partial^4 Y}{\partial X^4} \qquad \frac{\partial^2 y}{\partial x^2} = \frac{1}{l}\frac{\partial^2 Y}{\partial X^2} \tag{2.290}$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{l}{\rho A l^4/(EI)}\frac{\partial^2 Y}{\partial \tau^2} \tag{2.291}$$

that transforms the equation into:

$$\frac{\partial^4 Y}{\partial X^4} + \frac{\partial^2 Y}{\partial \tau^2} = P\frac{\partial^2 Y}{\partial X^2} \qquad -\frac{1}{2} < x < \frac{1}{2} \tag{2.292}$$

where

$$P = \frac{Fl^2}{EI} \tag{2.293}$$

To answer the question how t_0 has been defined, let us assume t_0 is unknown, then we have:

$$\frac{\partial^4 y}{\partial x^4} = \frac{1}{l^3}\frac{\partial^4 Y}{\partial X^4} \qquad \frac{\partial^2 y}{\partial x^2} = \frac{1}{l}\frac{\partial^2 Y}{\partial X^2} \tag{2.294}$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{l}{t_0^2}\frac{\partial^2 Y}{\partial \tau^2} \tag{2.295}$$

and hence Eq. (2.287) becomes:

$$\frac{EI}{l^3}\frac{\partial^4 Y}{\partial X^4} + \frac{\rho A l}{t_0^2}\frac{\partial^2 y}{\partial t^2} = \frac{F}{l}\frac{\partial^2 y}{\partial x^2} \tag{2.296}$$

Dividing by EI/l^3 simplifies the equation.

$$\frac{\partial^4 Y}{\partial X^4} + \frac{\rho A l^4}{EI t_0^2}\frac{\partial^2 y}{\partial t^2} = \frac{Fl^2}{EI}\frac{\partial^2 y}{\partial x^2} \tag{2.297}$$

Because the first term of the equation, $\partial^4 Y/\partial X^4$ is dimensionless, all other terms will also be dimensionless. Therefore, the dimensionless coefficient of the second term indicates that

$$\left[\frac{\rho A l^4}{E I t_0^2}\right] = 1 \tag{2.298}$$

and hence, we must have:

$$\left[\frac{\rho A l^4}{E I}\right] = T^2 \tag{2.299}$$

A dimensional check shows that it is correct.

$$\left[\frac{\rho A l^4}{E I}\right] = \frac{\left(M L^{-3}\right) L^2 L^4}{\left(M L T^{-2} L^{-2}\right) L^4} = T^2 \tag{2.300}$$

As a result we may introduce the characteristic time t_0 as:

$$t_0 = \sqrt{\frac{\rho A l^4}{E I}} \tag{2.301}$$

The last check would be dimension of P.

$$[P] = \left[\frac{F l^2}{E I}\right] = \frac{\left(M L T^{-2}\right) L^2}{\left(M L T^{-2} L^{-2}\right) L^4} = 1 \tag{2.302}$$

Therefore, the dimensional Eq. (2.287) with 3 variables y, x, t and 5 parameters ρ, F, EI, A, and l has been reduced to a nondimensional Eq. (2.292) with 3 variables Y , X, and τ and only one parameter P (Esmailzadeh and Jazar 1997).

Example 68 Simplifying a nondimensionalized differential equation.

The following equation defines the transverse vibration of the beam shown in Fig. 2.11 that is longitudinally excited.

$$\frac{\partial^4 Y}{\partial X^4} + \frac{\partial^2 Y}{\partial \tau^2} = P \frac{\partial^2 Y}{\partial X^2} \qquad -\frac{1}{2} < x < \frac{1}{2} \tag{2.303}$$

Let us apply the separation method and search for a solution of the form:

$$Y = f(\tau)\ g(X) \tag{2.304}$$

where the fixed ends require $g(\pm l/2) = 0$. Considering the fundamental mode of vibration for the spatial function $g(X)$,

$$g(X) = H \cos(\pi X) \tag{2.305}$$

where $H = h/l$ is the nondimensionalized transverse amplitude of the beam at its midpoint. Substituting the separation solution

$$Y = f(\tau)\ H \cos(\pi X) \tag{2.306}$$

in (2.303) provides us with an ordinary differential equation for $f(\tau)$.

$$f(\tau)\pi^4 H \cos(\pi X) + f''(\tau) H \cos(\pi X) = -Pf(\tau)\pi^2 H \cos(\pi X) \tag{2.307}$$

$$f''(\tau) + \pi^2\left(\pi^2 + P\right) f(\tau) = 0 \tag{2.308}$$

The elongation of the beam consists of three components: (1) the initial applied stretch X_0, (2) the driving amplitude X_d, (3) the elongation caused by the transverse deflection δ. The elongation of a deflected beam in the form of (2.305) for $dY/dX << 1$ is:

$$\delta = \int (ds - dX) = \int_{-1/2}^{1/2} \left[\left(1 + \left(\frac{dY}{dX}\right)^2\right)^{1/2} dX - dX\right] \simeq \int_0^{1/2} \left(\frac{dY}{dX}\right)^2 dX = \frac{1}{4}\pi^4 H^2 f^2 \tag{2.309}$$

Therefore, the dimensionless tension P is:

$$P = \frac{Fl^2}{EI} = X_0 + X_d \cos(\omega t) + \delta = X_0 + X_d \cos(\tau) + \frac{1}{4}\pi^4 H^2 f^2 \tag{2.310}$$

Substituting all the results in Eq. (2.308) provides a parametric nonlinear ordinary differential equation to determine the time function $f(\tau)$ (Esmailzadeh and Jazar 1997, 1998; McLachlan 1956).

$$f''(\tau) + \pi^2\left(\pi^2 + X_0 + X_d \cos(\tau)\right) f(\tau) + \frac{1}{4}\pi^6 H^2 f^3(\tau) = 0 \tag{2.311}$$

It is Mathieu-Duffing equation of the form

$$f'' + (k_1 + k_2 \cos(\tau)) f + k_3 f^3 = 0 \tag{2.312}$$

with

$$k_1 = \pi^2\left(\pi^2 + X_0\right) \tag{2.313}$$

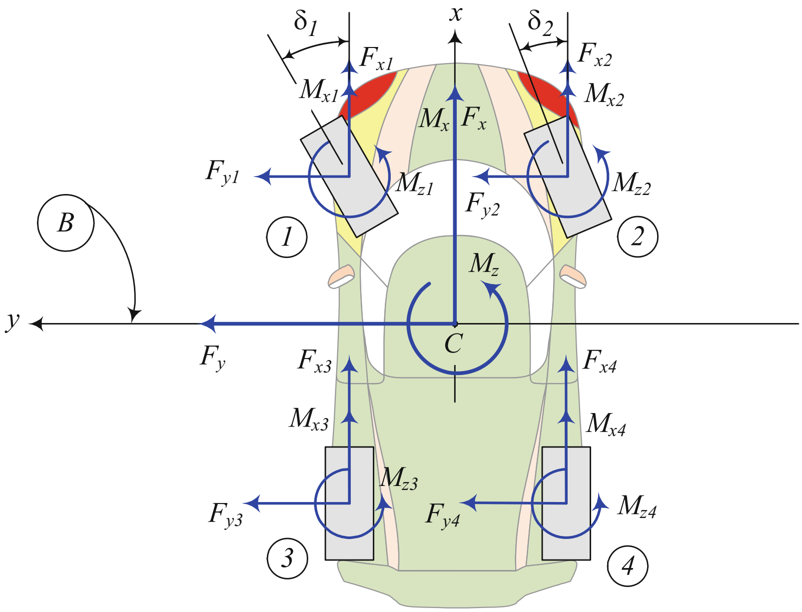

Fig. 2.12 The planar bicycle vehicle dynamics with steerable front wheel

$$k_2 = \pi^2 X_d \cos(\tau) \tag{2.314}$$

$$k_3 = \frac{1}{4}\pi^6 H^2 \tag{2.315}$$

Example 69 Planar model of vehicle dynamics with $v_x = cte$.

Consider the planar bicycle model of vehicle dynamics with steerable front wheel as is shown in Fig. 2.12 (Jazar 2017, 2019).

$$\dot{v}_x = \frac{F_x}{m} + r\, v_y \tag{2.316}$$

$$\dot{v}_y = \frac{1}{m v_x}\left(-a_1 C_{\alpha f} + a_2 C_{\alpha r}\right) r - \frac{1}{m v_x}\left(C_{\alpha f} + C_{\alpha r}\right) v_y + \frac{1}{m} C_{\alpha f} \delta_f - r\, v_x \tag{2.317}$$

$$\dot{r} = \frac{1}{I_z v_x}\left(-a_1^2 C_{\alpha f} - a_2^2 C_{\alpha r}\right) r - \frac{1}{I_z v_x}\left(a_1 C_{\alpha f} - a_2 C_{\alpha r}\right) v_y + \frac{1}{I_z} a_1 C_{\alpha f} \delta_f \tag{2.318}$$

There are 5 variables: forward velocity v_x, lateral velocity v_y, yaw rate r, time t, steer angle of front wheels δ_f. The steer angle variable is dimensionless. There are 7 parameters: vehicle mass m, traction force F_x, distance of front and rear axles from mass center a_1 and a_2, front and rear tires' side slip coefficients $C_{\alpha f}$ and $C_{\alpha r}$, mass moment of the vehicle about the z-axis, I_z, where wheelbase $l = a_1 + a_2$ is not independent (Jazar 2017, 2019).

An applied method of analysis of the planar bicycle model in vehicle dynamics is to assume the forward velocity is constant $v_x = cte$. Based on this assumption we may define the following dimensional characteristics for the remaining dimensional variables v_y, r, t:

$$v_{y_0} = v_x \qquad t_0 = \frac{l}{v_x} \qquad r_0 = \frac{v_x}{l} \tag{2.319}$$

and hence, we define the new nondimensionalized variables.

$$V_y = \frac{v_y}{v_x} \qquad \Gamma = \frac{r}{v_x/l} \qquad \tau = \frac{t}{l/v_x} \tag{2.320}$$

Therefore the derivatives would be:

$$\dot{v}_y = \frac{dv_y}{dt} = v_x \frac{dV_y}{dt} = v_x \frac{dV_y}{d\tau}\frac{d\tau}{dt} = \frac{v_x^2}{l}\frac{dV_y}{d\tau} = \frac{v_x^2}{l}V_y' \tag{2.321}$$

$$\dot{r} = \frac{dr}{dt} = \frac{v_x}{l}\frac{d\Gamma}{dt} = \frac{v_x}{l}\frac{d\Gamma}{d\tau}\frac{d\tau}{dt} = \frac{v_x^2}{l^2}\frac{d\Gamma}{d\tau} = \frac{v_x^2}{l^2}\Gamma' \tag{2.322}$$

Substituting the variables in the second and third equation of motion

$$\begin{aligned}\frac{v_x^2}{l}V_y' &= \frac{1}{mv_x}\left(-a_1 C_{\alpha f} + a_2 C_{\alpha r}\right)\frac{v_x}{l}\Gamma \\ &\quad - \frac{1}{mv_x}\left(C_{\alpha f} + C_{\alpha r}\right) v_x V_y + \frac{1}{m} C_{\alpha f}\delta_f - \frac{v_x}{l}\Gamma\, v_x \end{aligned} \tag{2.323}$$

$$\begin{aligned}\frac{v_x^2}{l^2}\Gamma' &= \frac{1}{I_z v_x}\left(-a_1^2 C_{\alpha f} - a_2^2 C_{\alpha r}\right)\frac{v_x}{l}\Gamma \\ &\quad - \frac{1}{I_z v_x}\left(a_1 C_{\alpha f} - a_2 C_{\alpha r}\right) v_x V_y + \frac{1}{I_z} a_1 C_{\alpha f}\delta_f \end{aligned} \tag{2.324}$$

and simplifying the equations generates the following nondimensionalized equations:

$$\begin{aligned}V_y' &= \frac{1}{mv_x^2}\left(-a_1 C_{\alpha f} + a_2 C_{\alpha r}\right)\Gamma \\ &\quad - \frac{l}{mv_x^2}\left(C_{\alpha f} + C_{\alpha r}\right) V_y + \frac{l}{mv_x^2} C_{\alpha f}\delta_f - \Gamma \end{aligned} \tag{2.325}$$

$$\Gamma' = \frac{l}{I_z v_x^2}\left(-a_1^2 C_{\alpha f} - a_2^2 C_{\alpha r}\right)\Gamma$$
$$-\frac{l^2}{I_z v_x^2}\left(a_1 C_{\alpha f} - a_2 C_{\alpha r}\right)V_y + \frac{l^2}{I_z v_x^2} a_1 C_{\alpha f}\delta_f \tag{2.326}$$

We may rearrange them

$$V_y' = \frac{l}{m v_x^2}\left(-\frac{a_1}{l} C_{\alpha f} + \frac{a_2}{l} C_{\alpha r}\right)\Gamma$$
$$-\frac{l}{m v_x^2}\left(C_{\alpha f} + C_{\alpha r}\right)V_y + \frac{l}{m v_x^2} C_{\alpha f}\delta_f - \Gamma \tag{2.327}$$
$$\Gamma' = \frac{l}{I_z v_x^2}\left(-a_1^2 C_{\alpha f} - a_2^2 C_{\alpha r}\right)\Gamma$$
$$-\frac{l^2}{I_z v_x^2}\left(a_1 C_{\alpha f} - a_2 C_{\alpha r}\right)V_y + \frac{l^2}{I_z v_x^2} a_1 C_{\alpha f}\delta_f \tag{2.328}$$

to become appropriate for final simplification.

$$V_y' = \left(-\frac{a_1}{l}\frac{l C_{\alpha f}}{m v_x^2} + \frac{a_2}{l}\frac{l C_{\alpha r}}{m v_x^2}\right)\Gamma$$
$$-\left(\frac{l C_{\alpha f}}{m v_x^2} + \frac{l C_{\alpha r}}{m v_x^2}\right)V_y + \frac{l C_{\alpha f}}{m v_x^2}\delta_f - \Gamma \tag{2.329}$$
$$\Gamma' = \left(-\frac{a_1^2}{l^2}\frac{l^3 C_{\alpha f}}{I_z v_x^2} - \frac{a_2^2}{l^2}\frac{l^3 C_{\alpha r}}{I_z v_x^2}\right)\Gamma$$
$$-\left(\frac{a_1}{l}\frac{l^3 C_{\alpha f}}{I_z v_x^2} - \frac{a_2}{l}\frac{l^3 C_{\alpha r}}{I_z v_x^2}\right)V_y + \frac{l^3 C_{\alpha f}}{I_z v_x^2}\frac{a_1}{l}\delta_f \tag{2.330}$$

Let us define

$$\varepsilon = \frac{a_1}{l} \qquad \varphi_f = \frac{l C_{\alpha f}}{m v_x^2} \qquad \varphi_r = \frac{l C_{\alpha r}}{m v_x^2}$$
$$\psi_f = \frac{l^3 C_{\alpha f}}{I_z v_x^2} \qquad \psi_r = \frac{l^3 C_{\alpha r}}{I_z v_x^2} \tag{2.331}$$

to rearrange the equations into:

$$V_y' = \left(-\varepsilon\varphi_f + (1-\varepsilon)\varphi_r - 1\right)\Gamma - \left(\varphi_f + \varphi_r\right)V_y + \varphi_f\delta_f \tag{2.332}$$

$$\Gamma' = \left(-\varepsilon^2\psi_f - (1-\varepsilon)^2\psi_r\right)\Gamma - \left(\varepsilon\psi_f - (1-\varepsilon)\psi_r\right)V_y + \varepsilon\psi_f\delta_f \tag{2.333}$$

or in a matrix form:

$$\begin{bmatrix} V_y' \\ \Gamma' \end{bmatrix} = \begin{bmatrix} -\left(\varphi_f+\varphi_r\right) & \varphi_r - \varepsilon\left(\varphi_f+\varphi_r\right) - 1 \\ \psi_r - \varepsilon\left(\psi_f+\psi_r\right) & -\varepsilon^2\psi_f - (1-\varepsilon)^2\psi_r \end{bmatrix} \begin{bmatrix} V_y \\ \Gamma \end{bmatrix} + \begin{bmatrix} \varphi_f \\ \varepsilon\psi_f \end{bmatrix}\delta_f \tag{2.334}$$

Employing the dimensionless parameters and variables, the traction Eq. (2.316) becomes

$$f = \frac{F_x}{mv_x^2/l} = -\Gamma V_y \tag{2.335}$$

Therefore, the dimensional equations of motion of planar vehicle have been nondimensionalized to a set of equations with 5 dimensionless variables: forward velocity V_x, lateral velocity V_y, yaw rate Γ, time τ, steer angle δ_f, and 6 parameters: $\varepsilon, \varphi_f, \varphi_r, \psi_f, \psi_r, f$.

Example 70 Accelerating planar model of vehicle dynamics.

To study transient and accelerated behavior of vehicles we should consider and nondimensionalize all 5 variables: forward velocity v_x, lateral velocity v_y, yaw rate r, time t, steer angle of front wheels δ_f. The equations of motion of planar bicycle model vehicle dynamics with steerable front wheel, shown in Fig. 2.12, are (Jazar 2017, 2019, Milani et al. 2020):

$$\dot{v}_x = \frac{F_x}{m} + r\,v_y \tag{2.336}$$

$$\dot{v}_y = \frac{1}{mv_x}\left(-a_1C_{\alpha f} + a_2C_{\alpha r}\right)r - \frac{1}{mv_x}\left(C_{\alpha f} + C_{\alpha r}\right)v_y + \frac{1}{m}C_{\alpha f}\delta_f - r\,v_x \tag{2.337}$$

$$\dot{r} = \frac{1}{I_zv_x}\left(-a_1^2C_{\alpha f} - a_2^2C_{\alpha r}\right)r - \frac{1}{I_zv_x}\left(a_1C_{\alpha f} - a_2C_{\alpha r}\right)v_y + \frac{1}{I_z}a_1C_{\alpha f}\delta_f \tag{2.338}$$

To nondimensionalize the variables, we may define the following dimensional characteristic parameters:

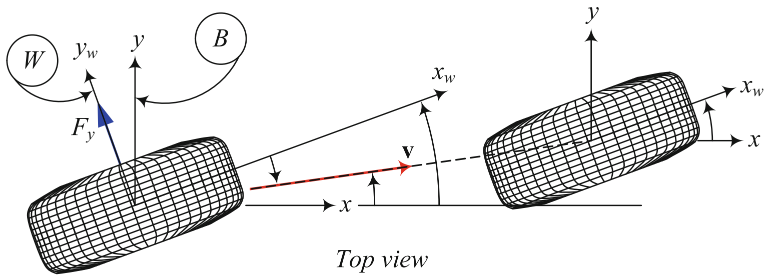

Fig. 2.13 The kinematic indicators of a tire

$$v_0 = \sqrt{\frac{lC_{\alpha f}}{m}} \qquad t_0 = \sqrt{\frac{ml}{C_{\alpha f}}} \qquad r_0 = \sqrt{\frac{C_{\alpha f}}{ml}} \tag{2.339}$$

$$[v_0] = \left[\sqrt{\frac{lC_{\alpha f}}{m}}\right] = \sqrt{\frac{L\left(MLT^{-2}\right)}{M}} = \sqrt{L^2T^{-2}} = LT^{-1} \tag{2.340}$$

$$[t_0] = \left[\sqrt{\frac{ml}{C_{\alpha f}}}\right] = \sqrt{\frac{ML}{MLT^{-2}}} = \sqrt{T^2} = T \tag{2.341}$$

$$[r_0] = \left[\sqrt{\frac{C_{\alpha f}}{ml}}\right] = \sqrt{\frac{MLT^{-2}}{ML}} = \sqrt{T^{-2}} = T^{-1} \tag{2.342}$$

Sideslip coefficient $C_{\alpha f}$ is the parameter that multiplies with tire sideslip angle α to provide lateral force on tire.

$$F_y = -C_{\alpha f}\,\alpha \qquad \alpha = \beta - \delta \tag{2.343}$$

Tire sideslip angle α is the angle between tire plane and the velocity vector of the tire. Vehicle sideslip angle β is the angle between vehicle x-axis and the velocity vector of the tire. The steer angle δ is the angle between vehicle x-axis and tire plane. Figure 2.13 illustrates the kinematic indicators of a tire. The W indicates wheel coordinate frame, set at the center of the wheel, and B indicates the vehicle body coordinate frame set at the mass center of the vehicle, both permanently parallel to the ground.

Employing the proper dimensional characteristic parameters, we define the nondimensionalized variables.

$$V_x = \frac{v_x}{v_0} = \frac{v_x}{\sqrt{\frac{lC_{\alpha f}}{m}}} \qquad V_y = \frac{v_y}{v_0} = \frac{v_y}{\sqrt{\frac{lC_{\alpha f}}{m}}}$$

$$\Gamma = \frac{r}{r_0} = \frac{r}{\sqrt{\frac{C_{\alpha f}}{ml}}} \qquad \tau = \frac{t}{t_0} = \frac{t}{\sqrt{\frac{ml}{C_{\alpha f}}}} \tag{2.344}$$

Therefore the derivatives would be:

$$\dot{v}_x = \frac{dv_x}{dt} = v_0 \frac{dV_x}{dt} = v_0 \frac{dV_x}{d\tau}\frac{d\tau}{dt} = \frac{C_{\alpha f}}{m} V_x' \tag{2.345}$$

$$\dot{v}_y = \frac{dv_y}{dt} = v_0 \frac{dV_y}{dt} = v_0 \frac{dV_y}{d\tau}\frac{d\tau}{dt} = \frac{C_{\alpha f}}{m} V_y' \tag{2.346}$$

$$\dot{r} = \frac{dr}{dt} = \sqrt{\frac{C_{\alpha f}}{ml}}\frac{d\Gamma}{dt} = \sqrt{\frac{C_{\alpha f}}{ml}}\frac{d\Gamma}{d\tau}\frac{d\tau}{dt} = \frac{C_{\alpha f}}{ml}\Gamma' \tag{2.347}$$

Substituting the variables in the equations of motion

$$\frac{C_{\alpha f}}{m} V_x' = \frac{F_x}{m} + \sqrt{\frac{C_{\alpha f}}{ml}}\sqrt{\frac{lC_{\alpha f}}{m}}\Gamma\, V_y \tag{2.348}$$

$$\begin{aligned}\frac{C_{\alpha f}}{m} V_y' = & \frac{1}{m\sqrt{\frac{lC_{\alpha f}}{m}}V_x}\left(-a_1 C_{\alpha f} + a_2 C_{\alpha r}\right)\sqrt{\frac{C_{\alpha f}}{ml}}\Gamma \\ & - \frac{1}{m\sqrt{\frac{lC_{\alpha f}}{m}}V_x}\left(C_{\alpha f} + C_{\alpha r}\right)\sqrt{\frac{lC_{\alpha f}}{m}}V_y \\ & + \frac{1}{m}C_{\alpha f}\delta_f - \sqrt{\frac{C_{\alpha f}}{ml}}\sqrt{\frac{lC_{\alpha f}}{m}}\Gamma\, V_y\end{aligned} \tag{2.349}$$

$$\begin{aligned}\frac{C_{\alpha f}}{ml}\Gamma' = & \frac{1}{I_z\sqrt{\frac{lC_{\alpha f}}{m}}V_x}\left(-a_1^2 C_{\alpha f} - a_2^2 C_{\alpha r}\right)\sqrt{\frac{C_{\alpha f}}{ml}}\Gamma \\ & - \frac{1}{I_z\sqrt{\frac{lC_{\alpha f}}{m}}V_x}\left(a_1 C_{\alpha f} - a_2 C_{\alpha r}\right)\sqrt{\frac{lC_{\alpha f}}{m}}V_y \\ & + \frac{1}{I_z}a_1 C_{\alpha f}\delta_f\end{aligned} \tag{2.350}$$

and simplifying the equation make the following nondimensionalized equations:

$$V_x' = \frac{F_x}{C_{\alpha f}} + \Gamma\, V_y \tag{2.351}$$

$$V_y' = \frac{1}{C_{\alpha f}\, l V_x}\left(-a_1 C_{\alpha f} + a_2 C_{\alpha r}\right)\Gamma - \frac{1}{C_{\alpha f} V_x}\left(C_{\alpha f} + C_{\alpha r}\right) V_y + \delta_f - \Gamma\, V_y \tag{2.352}$$

$$\frac{C_{\alpha f}}{ml}\Gamma' = \frac{1}{I_z\sqrt{\frac{lC_{\alpha f}}{m}}V_x}\left(-a_1^2 C_{\alpha f} - a_2^2 C_{\alpha r}\right)\sqrt{\frac{C_{\alpha f}}{ml}}\Gamma - \frac{1}{I_z\sqrt{\frac{lC_{\alpha f}}{m}}V_x}\left(a_1 C_{\alpha f} - a_2 C_{\alpha r}\right)\sqrt{\frac{lC_{\alpha f}}{m}}V_y + \frac{1}{I_z}a_1 C_{\alpha f}\delta_f \tag{2.353}$$

$$\Gamma' = \frac{m}{I_z C_{\alpha f} V_x}\left(-a_1^2 C_{\alpha f} - a_2^2 C_{\alpha r}\right)\Gamma - \frac{ml}{I_z C_{\alpha f} V_x}\left(a_1 C_{\alpha f} - a_2 C_{\alpha r}\right) V_y + \frac{ml}{I_z C_{\alpha f}} a_1 C_{\alpha f}\delta_f \tag{2.354}$$

We may rearrange them to become appropriate for final simplification.

$$V_x' = \frac{F_x}{C_{\alpha f}} + \Gamma\, V_y \tag{2.355}$$

$$V_y' = \left(-\frac{a_1}{l} + \frac{a_2}{l}\frac{C_{\alpha r}}{C_{\alpha f}}\right)\frac{\Gamma}{V_x} - \left(1 + \frac{C_{\alpha r}}{C_{\alpha f}}\right)\frac{V_y}{V_x} + \delta_f - \Gamma\, V_y \tag{2.356}$$

$$\Gamma' = \left(-\frac{ma_1^2}{I_z} - \frac{ma_2^2}{I_z}\frac{C_{\alpha r}}{C_{\alpha f}}\right)\frac{\Gamma}{V_x} - \left(\frac{mla_1}{I_z} - \frac{mla_2}{I_z}\frac{C_{\alpha r}}{C_{\alpha f}}\right)\frac{V_y}{V_x} + \frac{mla_1}{I_z}\delta_f \tag{2.357}$$

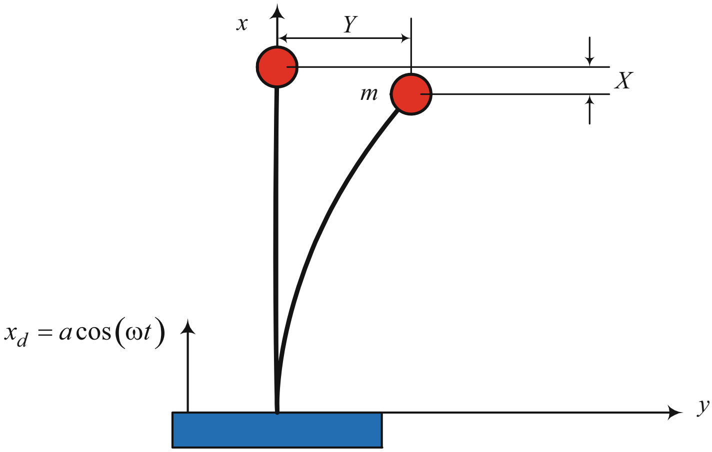

Fig. 2.14 A cantilever beams with base excitation and tip mass

Let us define

$$\varepsilon = \frac{a_1}{l} \qquad c = \frac{C_{\alpha r}}{C_{\alpha f}} \qquad e = \frac{ml^2}{I_z} \qquad p = \frac{F_x}{C_{\alpha f}} \tag{2.358}$$

to rearrange the equations into:

$$V_x' = p + \Gamma\, V_y \tag{2.359}$$

$$V_y' = ((1-\varepsilon)\, c - \varepsilon)\, \frac{\Gamma}{V_x} - (1+c)\, \frac{V_y}{V_x} + \delta_f - \Gamma\, V_y \tag{2.360}$$

$$\Gamma' = -\left((1-\varepsilon)^2\, c + \varepsilon^2\right) e \frac{\Gamma}{V_x} - ((1-\varepsilon)\, c - \varepsilon)\, e \frac{V_y}{V_x} + \varepsilon e \delta_f \tag{2.361}$$

Therefore, the nondimensionalized equations of motion of planar vehicle dynamics has a set of 5 dimensionless variables: forward velocity V_x, lateral velocity V_y, yaw rate Γ, time τ, steer angle δ_f, and 4 parameters: ε, c, e, p.

Example 71 A cantilever with end mass and vertically base excitation.

Cantilever beam with base excitation and tip mass is a good model of many vibrating mechanical and micromechanical systems. Figure 2.14 illustrates the model.

To derive the equations of motion of the system, we may use the Lagrange method. The kinetic energy K of the tip mass m is:

$$K = \frac{1}{2} m \left(\dot{Y}^2 + \left(\dot{x}_d - \dot{X}\right)^2\right) \tag{2.362}$$

where x_d is the vertical displacement of the base and (Y, X) are the coordinates of the mass m with respect to a globally fixed coordinate frame at the base. There is no kinetic energy associated to the massless beam. The total potential energy P of the system consists of the strain energy due to bending of the beam and the gravitational potential energy of the tip mass (Esmailzadeh and Jazar 1998).

$$P = \frac{1}{2} EI \int_0^l \left(\frac{\partial^2 y}{\partial x^2} \right)^2 dx - mgX \tag{2.363}$$

Assuming $dY/dX << 1$ we have

$$X = \int_0^l \left[\left(1 + \left(\frac{dY}{dX} \right)^2 \right)^{1/2} dx - dx \right] \simeq \int_0^l \left(\frac{\partial y}{\partial x} \right)^2 dx \tag{2.364}$$

Employing separation method, we are able to connect the lateral flexural displacement of the beam y to the lateral displacement of the mass, $Y = y(l, t)$,

$$y(x, t) = y(l, t) \; f(x) = Y \; f(x) \tag{2.365}$$

where $f(x)$ is the mode shape of the beam.

$$f(x) = 1 - \cos \left((2n - 1) \frac{\pi x}{2l} \right) \tag{2.366}$$

Substituting (2.365) into (2.363) and (2.364) provides

$$\begin{aligned} P &= \frac{1}{2} EIY^2 \int_0^l \left(\frac{\partial^2 f}{\partial x^2} \right)^2 dx - mgX \\ &= \frac{1}{2} EIAY^2 - \frac{1}{2} mgBY^2 \end{aligned} \tag{2.367}$$

and hence,

$$X = \frac{1}{2} BY^2 \qquad \dot{X} = BY\dot{Y} \tag{2.368}$$

where

$$A = \int_0^l \left(\frac{\partial^2 f}{\partial x^2} \right)^2 dx = \frac{l}{2} \left((2n - 1) \frac{\pi}{2l} \right)^4 \tag{2.369}$$

$$B = \int_0^l \left(\frac{\partial f}{\partial x} \right)^2 dx = \frac{l}{2} \left((2n - 1) \frac{\pi}{2l} \right)^2 \tag{2.370}$$

$$[A] = L^{-3} \qquad [B] = L^{-1} \tag{2.371}$$

The Lagrangian $\mathcal{L}$ will be

$$\mathcal{L} = K - P = \frac{1}{2}\left(\dot{Y}^2 + B^2Y^2\dot{Y}^2 + \dot{x}_d^2 - 2\dot{x}_d BY\dot{Y}\right)$$
$$-\frac{1}{2}EIAY^2 + \frac{1}{2}mgBY^2 \tag{2.372}$$

Applying Lagrange's method, the following ordinary differential equation of motion will be found to express the lateral motion Y of the tip mass:

$$\left(1 + B^2Y^2\right)\ddot{Y} + B^2Y\dot{Y}^2 + \left(\frac{EI}{m}A - gB - B\,\ddot{x}_d\right)Y = 0 \tag{2.373}$$

Considering

$$x_d = a\cos(\omega t) \tag{2.374}$$
$$\ddot{x}_d = -\omega^2 a\cos(\omega t) \tag{2.375}$$

we will have

$$\left(1 + B^2Y^2\right)\ddot{Y} + B^2Y\dot{Y}^2 + \left(\frac{EI}{m}A - gB + B\omega^2 a\cos(\omega t)\right)Y = 0 \tag{2.376}$$

This is a nonlinear differential equation that will be reduced to by linearization.

Introducing

$$\tau = \omega t \qquad u = BY \tag{2.377}$$
$$u' = \frac{du}{d\tau} = \frac{B}{\omega}\dot{Y} \qquad u'' = \frac{d^2u}{d\tau^2} = \frac{B}{\omega^2}\ddot{Y} \tag{2.378}$$

we have

$$\left(1 + u^2\right)\frac{\omega^2}{B}u'' + Bu\frac{\omega^2}{B^2}u'^2$$
$$+\left(\frac{EI}{m}A - gB + B\omega^2 a\cos(\omega t)\right)\frac{u}{B} = 0 \tag{2.379}$$

that may be rewritten in a nondimensionalized form

$$\left(1 + u^2\right)u'' + uu'^2 + (p + q\cos(\omega t))\,u = 0 \tag{2.380}$$

where

$$p = \frac{1}{\omega^2}\left(\frac{EI}{m}A - gB\right) \qquad q = Ba \tag{2.381}$$

Assuming u and u' are too small then, Eq. (2.379) reduces to a Mathieu equation.

$$u'' + (p + q\cos(\omega t))\,u = 0 \tag{2.382}$$

Example 72 Projectile in air.

Consider a projectile with mass m that is thrown with an initial velocity $\mathbf{v}_0$ from the origin of the coordinate frame in the (x, z)-plane. The air has a resistance force $-c\mathbf{v}$ proportional to the velocity $\mathbf{v}$. Assuming a flat ground with a uniform gravitational acceleration $\mathbf{g}$, the equation of motion of the projectile is:

$$m\frac{d\mathbf{v}}{dt} = -mg\hat{k} - c\mathbf{v} \tag{2.383}$$

$$\mathbf{g} = -g\hat{k} \tag{2.384}$$

$$g = 9.80665\,\mathrm{m/\,s^2} \approx 9.81\,\mathrm{m/\,s^2} \tag{2.385}$$

There are two variables, the velocity $\mathbf{v}$ and time t. To make this problem nondimensionalized, we use the characteristic velocity $\mathbf{v}_0$ and time $t_0 = v_0/g$,

$$\mathbf{v}_0 = v_0\cos\theta\,\hat{\imath} + v_0\sin\theta\,\hat{k} \tag{2.386}$$

where θ is the angle of $\mathbf{v}_0$ with the x-axis. Then, we define the new variables

$$\mathbf{V} = \frac{\mathbf{v}}{\mathbf{v}_0} \qquad \tau = \frac{t}{v_0/g} = \frac{g}{v_0}t \tag{2.387}$$

and,

$$\frac{d\mathbf{v}}{dt} = \frac{v_0}{v_0/g}\frac{d\mathbf{V}}{d\tau} = g\frac{d\mathbf{V}}{d\tau} \tag{2.388}$$

and therefore, we have

$$mg\frac{d\mathbf{V}}{d\tau} = -mg\hat{k} - c\mathbf{v}_0\mathbf{V} \tag{2.389}$$

Dividing the equation with mg will show the nondimensionalized equation

$$\frac{d\mathbf{V}}{d\tau} = -\hat{k} - \frac{c\mathbf{v}_0}{mg}\mathbf{V} = -\hat{k} - \mathbf{p}\mathbf{V} \tag{2.390}$$

$$\mathbf{p} = \frac{c\mathbf{v}_0}{mg} = \frac{cv_0}{mg}\left(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{k}\right) \tag{2.391}$$

$$= p_X\,\hat{\imath} + p_Z\,\hat{k} \tag{2.392}$$

with only one vectorial parameter $\mathbf{p}$ (Jazar 2011).

To solve the equation of motion we may integrate Eq. (2.390)

$$\mathbf{V} = -\tau\hat{k} - \mathbf{p}\left(\mathbf{R} + \mathbf{V}_0\right) \tag{2.393}$$

where

$$\int \mathbf{V} d\tau = \mathbf{R} + \mathbf{V}_0 \tag{2.394}$$

$$\mathbf{R} = \frac{\mathbf{r}}{\mathbf{r}_0} \qquad \mathbf{r}_0 = \frac{\mathbf{v}_0^2}{g} \tag{2.395}$$

Multiplying Eq. (2.393) by $e^{\mathbf{p}\tau}$, we have

$$\frac{d}{d\tau}\left(\mathbf{R}e^{\mathbf{p}\tau}\right) = -\mathbf{p}\mathbf{V}_0 e^{\mathbf{p}\tau} - \tau e^{\mathbf{p}\tau}\hat{k} \tag{2.396}$$

and therefore,

$$\mathbf{R}e^{\mathbf{p}\tau} = -\mathbf{V}_0 e^{\mathbf{p}\tau} - \frac{1}{\mathbf{p}^2} e^{\mathbf{p}\tau}\left(\mathbf{p}\tau - 1\right)\hat{k} + \mathbf{V}_0 - \frac{1}{\mathbf{p}^2}\hat{k} \tag{2.397}$$

We have found the constant of integration such that $\mathbf{r} = 0$ at $t = 0$. The position vector of the projectile can be simplified to

$$\mathbf{R} = \mathbf{V}_0\left(e^{-\mathbf{p}\tau} - 1\right) - \frac{1}{\mathbf{p}^2}\left(e^{-\mathbf{p}\tau} + \mathbf{p}\tau - 1\right)\hat{k} \tag{2.398}$$

and therefore the velocity vector of the projectile is

$$\mathbf{V} = \frac{d\mathbf{R}}{dt} = -\mathbf{p}\mathbf{V}_0 e^{-\mathbf{p}\tau} - \frac{1}{p}\left(1 - e^{-\mathbf{p}\tau}\right)\hat{k} \tag{2.399}$$

Substituting back the variable, we are able to express the results in dimensional variables ($\mathbf{r}$, $\mathbf{v}$, t).

2.3 Model and Prototype Similarity Analysis

The similarity theory of model and prototype is the ultimate goal of dimensional analysis. It is essential to make a laboratory model of a system to be able to do experiment under controlled conditions and fine-tune the parameters to obtain the desired behavior of the real system. The laboratory model of the real system may be in smaller scale, such as model of an airplane, river, ship, or be in larger scale such as model of a MEMS, resonator, watch, etc. Such a model is utilized to determine the best properties under future operating conditions. Many different kinds of measurements are carried out on models: for example, the lift and drag forces of an airfoil model can be measured in a wind tunnel. The similarity method is to scale the results of the experiment carried out on a model up to the full scale of the real prototype.

If a dimensional analysis indicates that a problem is described by a functional relationship between nondimensional parameters $\pi_1, \pi_2, \pi_3, \cdots$, then full similarity requires that these parameters be the same at both prototype scale and model scale. The modeling and similarity of physical phenomena are the central application of the dimensional analysis. Two physical phenomena are similar if the values of their corresponding dimensionless parameters $\pi_1, \cdots, \pi_m$ are identical between them.

The concept of physical similarity is a generalization of the concept of geometric similarity. For example, two triangles are similar if they differ only in the numerical values of their lengths of the sides as their dimensional parameters, while they have equal angles at the vertices as their dimensionless parameters. Two physical phenomena are called similar if they differ only in respect of numerical values of their governing dimensional parameters.

Assume we are going to model a certain phenomenon in a real system. The real system is called *prototype* and the experimental system is called *model*. It is required that the model we wish to use to determine the desired properties of the prototype, be similar to the prototype. Such similarity is guaranteed if all dimensionless parameters π_i have equal values between them. Therefore, we have the following relationship between the parameter a, such as temperature or speed, to be determined and the governing parameters $a_1, \cdots, a_k, b_1, \cdots, b_m$.

$$a = f(a_1, \cdots, a_k, b_1, \cdots, b_m) \tag{2.400}$$

The function f is the same for both model and prototype, although the numerical values of the parameters $a_1, \cdots, a_k, b_1, \cdots, b_m$ and the determined parameter a may differ. Therefore the relationship (2.400) for the prototype takes the form

$$a_P = f\left(a_{P_1}, \cdots, a_{P_k}, b_{P_1}, \cdots, b_{P_m}\right) \tag{2.401}$$

The subscript P will be used to refer to the properties of the prototype. Relation (2.401) for the model is similar in form and different in value.

$$a_M = f\left(a_{M_1}, \cdots, a_{M_k}, b_{M_1}, \cdots, b_{M_m}\right) \tag{2.402}$$

The subscript M will be used to refer to the properties of the model. Employing dimensional analysis, we obtain

$$\pi_P = g\left(\pi_{P_1}, \cdots, \pi_{P_m}\right) \tag{2.403}$$

$$\pi_M = g\left(\pi_{M_1}, \cdots, \pi_{M_m}\right) \tag{2.404}$$

where the function g is the same in both cases. It implies

$$\pi_{P_i} = \pi_{Mi}, \qquad i = 1, 2, \cdots, m \tag{2.405}$$

which is called the *similarity criteria*. Hence,

$$g\left(\pi_{P_1}, \cdots, \pi_{P_m}\right) = g\left(\pi_{M_1}, \cdots, \pi_{M_m}\right) \tag{2.406}$$

Returning to the dimensional parameters $a_{M_1}, \cdots, a_{M_k}$ using (2.401), we find that

$$a_P = a_M \left(\frac{a_{P_1}}{a_{M_1}}\right)^b \cdots \left(\frac{a_{P_m}}{a_{M_m}}\right)^c \tag{2.407}$$

which is the rule for scaling the results of measurements on a similar model up to the prototype.

The model parameters $a_{M_1}, \cdots, a_{M_k}$ may be selected arbitrarily. The conditions for similarity between the model and prototype are:

$$\pi_{P_1} = \pi_{M_1} \qquad b_{M_1} = b_{P_1} \left(\frac{a_{M_1}}{a_{P_1}}\right)^{b_1} \cdots \left(\frac{a_{M_k}}{a_{P_k}}\right)^{c_1} \tag{2.408}$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$\pi_{P_m} = \pi_{M_m} \qquad b_{M_m} = b_{P_1} \left(\frac{a_{M_m}}{a_{P_m}}\right)^{b_m} \cdots \left(\frac{a_{M_k}}{a_{P_k}}\right)^{c_m} \tag{2.409}$$

Proof We may distinguish three fundamental types of similarities between mechanical systems plus another one between thermal systems:

Geometric similarity: the ratio of all corresponding lengths in model and prototype are the same; therefore, model is a shrunk or enlarged copy of the prototype. Two shapes are geometrically similar if with proper magnification the smaller one can be enlarged so that the two shapes can be brought into exact coincidence. Two circles are always geometrically similar, the same is for any two squares or two equilateral triangles. To have a model to produce reasonable result to become transferable to the prototype, geometric similarity is necessary to be kept as close as possible and it is the first step in the model and similarity analysis.

Kinematic similarity: the ratio of all corresponding lengths and times, and hence the ratios of all corresponding velocities, in the model and prototype are the same.

Dynamic similarity: the ratio of all forces in model and prototype are the same. For example, in fluid flow we must have the Reynolds number, Re $=$ (inertial force)/(viscous force) to be the same in model and prototype.

Thermal Similarity: Two systems are thermally similar if at their homologous points, or surfaces, they have equal or homologous temperatures at homologous times. If the heat-flow pattern in two objects is similar, then these objects are considered thermally similar.

The concept of designing similar systems is one of the oldest and most powerful concepts in the natural sciences. This concept was developed mostly for physics and engineering, and then it expanded to other branches of sciences such as biology, ecology, economics, physiology, etc. Today no spaceship, airplane, ship, submarine, and other large or small or expensive engineering product is built that do not have their model first been tested in a wind tunnel, tow tank, sink tank, and other environment similar to what the real system must work in. By building a model, engineers reduce an object or a system to a manageable form. A model is useful when it behaves similar to the actual phenomenon being studied (Granger 1995). ■

Example 73 Model and prototype in engineering.

Generally speaking, model is a dummy, and prototype is the exact product. Below they are more clarification.

Models have several characteristics: (1) Models are not necessarily functional and they do not need to work. (2) Models can be to any scale, although they are usually smaller, but can also be of the original size or bigger. (3) Models are used for display or/and visual demonstration of prototype. (4) Models may consist of only the exterior of the prototype it replicates. (5) Models are relatively cheap to manufacture, unless they are highly sensorized.

Prototypes have several characteristics: (1) Prototypes are fully functional, but not fault-proof. (2) Prototypes are actual version of the intended product. (3) Prototypes are used for performance evaluation and further improvement of product. (4) Prototypes contain complete interior and exterior. (5) Prototypes are relatively expensive to produce.

Example 74 ★Archimedes law of lever.

It seems that Archimedes proposed the first similarity equation in his writing "On the Equilibrium of Planes, Book I," where he introduced the law of the lever: Two weights balance distances reciprocally proportional to their magnitudes (Dijksterhuis 1987; Leath 1897; Assis 2010).

$$m_1 g \times l_1 = m_2 g \times l_2 \tag{2.410}$$

Archimedes law of lever in dimensional analysis language may be written as:

$$\frac{m_1}{m_2} = \frac{l_1}{l_2} \tag{2.411}$$

or

$$\pi_1 = \pi_2 \tag{2.412}$$

$$\pi_1 = \frac{m_1}{m_2} \qquad \pi_2 = \frac{l_1}{l_2} \tag{2.413}$$

as illustrated in Fig. 2.15.

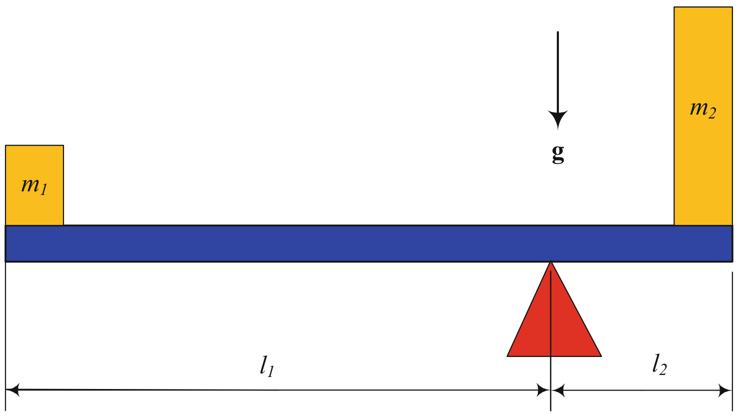

Fig. 2.15 Archimedes law of the lever

The first documented similarity concept belongs to Euclid who defined geometric similarity in Volume VI of his "Elements" as: "Those straight-sided geometric figures are called similar which have equal angles, and whose sides subtending equal angles are proportional." It was in proving the similarity of such geometric figures, that Euclid presented his axiom of the parallel lines (Szucs 1980).

Example 75 Complete and incomplete similarity.

Complete or full similarity happens when all nondimensional π-groups are the same in model and prototype. If complete similarity cannot be achieved, and one or more nondimensional π-groups in model and prototype cannot be kept at the same value, we have incomplete similarity.

As an example of complete similarity, consider the power P of a wind turbine which is a function of blade length l, the wind speed v, the rotational speed ω, the density of the air ρ (Simon et al. 2017).

$$P = f\left(l, v, \omega, \rho\right) \tag{2.414}$$

As we have 5 variables P, l, v, ω, ρ and 3 dimensions M, L, T, dimensional analysis will provide us with two π-groups as

$$\frac{P}{\rho v^3 l^2} = u\left(\frac{v}{l\omega}\right) \tag{2.415}$$

where u is an unknown function. For a complete similarity between the model and full scale prototype to show the same physical behavior, we must have

$$(\pi_i)_P = (\pi_i)_M \tag{2.416}$$

therefore,

$$\pi_1 = \left(\frac{P}{\rho v^3 l^2}\right)_P = \left(\frac{P}{\rho v^3 l^2}\right)_M \tag{2.417}$$

$$\pi_2 = \left(\frac{v}{l\omega}\right)_P = \left(\frac{v}{l\omega}\right)_M \tag{2.418}$$

The relations for the scale factors for pressure P and velocity v between model and prototype are

$$\frac{P_P}{P_M} = \frac{\rho_P}{\rho_M}\frac{v_P^3}{v_M^3}\frac{l_P^2}{l_M^2} \tag{2.419}$$

$$\frac{v_P}{v_M} = \frac{l_P}{l_M}\frac{\omega_P}{\omega_M} \tag{2.420}$$

similarly, we may evaluate different variables in prototype by calculating associated scale factors k_i.

$$P_P = \left(\frac{\rho_P}{\rho_M}\frac{v_P^3}{v_M^3}\frac{l_P^2}{l_M^2}\right) P_M = k_P\ P_M \tag{2.421}$$

$$v_P = \left(\frac{l_P}{l_M}\frac{\omega_P}{\omega_M}\right) v_M = k_v\ v_M \tag{2.422}$$

$$l_p = \left(\frac{v_P}{v_M}\frac{\omega_M}{\omega_P}\right) l_M = k_l\ l_M \tag{2.423}$$

$$\omega_P = \left(\frac{v_P}{v_M}\frac{l_M}{l_P}\right) \omega_M = k_\omega\ \omega_M \tag{2.424}$$

$$\rho_P = \left(\frac{P_P}{P_M}\frac{v_M^3}{v_P^3}\frac{l_M^2}{l_P^2}\right) \rho_M = k_\rho\ \rho_M \tag{2.425}$$

The length l and velocity v may also be calculated from the other π-group.

$$l_p = \sqrt{\frac{P_P}{P_M}\frac{\rho_M}{\rho_P}\frac{v_M^3}{v_P^3}} l_M = k_l\ l_M \tag{2.426}$$

$$v_P = \sqrt[3]{\frac{P_P}{P_M}\frac{\rho_M}{\rho_P}\frac{l_M^2}{l_P^2}} v_M = k_v\ v_M \tag{2.427}$$

Theoretically, we must have the same scale factors k_l and k_v either way, as we must have:

$$\frac{k_P}{k_\rho\ k_v^3\ k_l^2} = 1 \qquad \frac{k_v}{k_l k_\omega} = 1 \tag{2.428}$$

If both, model and prototype, are operating in the same fluid such as air, then

$$k_\rho = 1 \tag{2.429}$$

As an example of incomplete similarity, consider a ship moving on water. The water has viscosity μ and density ρ. The ship has length l and speed v and flow drag resistance force is F. The wave motion is affected by gravity, so the acceleration of gravity g also needs to be included as a variable in dimensional analysis of the system (Simon et al. 2017).

$$F = f\,(l, v, \mu, \rho, g) \tag{2.430}$$

As we have 6 variables F, l, v, μ, ρ, g and 3 dimensions M, L, T, we will have three π-groups as

$$\frac{F}{\rho v^2 l^2} = u\left(\frac{\rho v l}{\mu}, \frac{v^2}{gl}\right) = u\left(\frac{vl}{\eta}, \frac{v^2}{gl}\right) \tag{2.431}$$

where

$$\eta = \frac{\mu}{\rho} \tag{2.432}$$

is the kinematic viscosity of the fluid. The three π-groups are the drag coefficient C_D, the Reynolds number Re, and Froud number Fr, respectively.

$$\pi_1 = C_D = \frac{F}{\rho v^2 l^2} \tag{2.433}$$

$$\pi_2 = \mathrm{Re} = \frac{vl}{\eta} \tag{2.434}$$

$$\pi_3 = Fr = \frac{v^2}{gl} \tag{2.435}$$

To have complete similarity, all dimensionless products for the model and prototype must be equal.

$$\pi_1 = \left(\frac{F}{\rho v^2 l^2}\right)_P = \left(\frac{F}{\rho v^2 l^2}\right)_M \tag{2.436}$$

$$\pi_2 = \left(\frac{vl}{\eta}\right)_P = \left(\frac{vl}{\eta}\right)_M \tag{2.437}$$

$$\pi_3 = \left(\frac{v^2}{gl}\right)_P = \left(\frac{v^2}{gl}\right)_M \tag{2.438}$$

Employing the scale factors k_i, we may rewrite the similarity conditions as

$$\frac{v_P\, l_P}{\eta_P}\frac{k_v\, k_l}{k_\eta} = \frac{v_M\, l_M}{\eta_M} \qquad \frac{k_v\, k_l}{k_\eta} = 1 \tag{2.439}$$

$$\frac{v_P^2}{g_P\, l_P}\frac{k_v^2}{k_g\, k_l} = \frac{v_M^2}{g_M\, l_M} \qquad \frac{k_v^2}{k_g\, k_l} = 1 \tag{2.440}$$

Because the gravitational acceleration is the same for model and prototype, $g_P = g_M$, we have the gravity scale factor to be equal to one, $k_g = 1$. Furthermore, if the model and prototype are both operating in water, then $\eta_P = \eta_M$ and the kinematic viscosity scale factor will also be one, $k_\eta = 1$. Therefore, from (2.439) and (2.440) we must have

$$k_v\, k_l = 1 \qquad \frac{k_v^2}{k_l} = 1 \tag{2.441}$$

that imply $k_l = 1$ and $k_v = 1$. These indicate that the prototype and model must have the same size and be operated at the same speed, which is against the purpose of using scaled down model. To overcome this problem we may accept an incomplete similarity by avoiding the Re number, and only work with C_D and Fr numbers.

Example 76 Nondimensional π-groups in thermofluid mechanics.

Dynamic similarity requires that the ratio of all forces between model and prototype to be the same. The ratio of different forces produces many of the key nondimensional parameters in thermofluid mechanics. Below you will see the most important nondimensional groups frequently appearing in thermofluid sciences (Yarin 2012)

Archimedes number (motion of fluid due to density differences, buoyancy)

$$Ar = \frac{\rho g L^3}{\mu^2}\left(\rho - \rho_f\right) = \frac{gravity\ force}{viscous\ force} \tag{2.442}$$

Biot number (heat transfer)

$$Bi = \frac{hL}{k_s} = \frac{convection\ heat\ transfer}{conduction\ heat\ transfer} \tag{2.443}$$

Bond number (motion of drops and bubbles)

$$Bo = \frac{\rho g L^2}{\sigma} = \frac{gravity\ force}{surface\ tension} \tag{2.444}$$

Brinkman number (viscous flows)

$$Br = \frac{\mu v^2}{k\Delta T} = \frac{heat\ dissipation}{heat\ transferred} \tag{2.445}$$

Capillary number (two-phase flow)

$$Ca = \frac{\mu v}{\sigma} = \frac{viscous\ force}{surface\ tension} \tag{2.446}$$

Damkohler number (momentum, and heat transfer)

$$Da_1 = \frac{WL}{V_m} = \frac{chemical\ reaction\ rate}{bulk\ mass\ flow\ rate} \tag{2.447}$$

$$Da_3 = \frac{qWL}{\rho v c_p \Delta T} = \frac{heat\ released}{convected\ heat} \tag{2.448}$$

Darcy number (flow in porous media)

$$Da_2 = \frac{vL}{D} = \frac{inertia\ force}{permeation\ force} \tag{2.449}$$

Dean number (flow in curved channels and pipes)

$$De = \frac{vR\rho}{\mu}\sqrt{\frac{R}{r}} = \frac{centrifugal\ force}{inertia\ force} \tag{2.450}$$

Deborah number (non-Newtonian hydrodynamics, rheology)

$$De = \frac{\tau_r}{\tau_0} = \frac{Relaxation\ time}{hydrodynamic\ time} \tag{2.451}$$

Eckert number (compressible flows)

$$Ec = \frac{v_\infty^2}{c_p \Delta T} = \frac{kinetic\ energy}{thermal\ energy} \tag{2.452}$$

Ekman number (rotating flows)

$$Ek = \sqrt{\frac{\mu}{2\rho\omega L^2}} = \frac{viscous\ force}{Coriolis\ force} \tag{2.453}$$

Euler number (fluid friction in conduits)

$$Eu = \frac{\rho v^2}{\Delta p} = \frac{pressure\ drop}{dynamic\ pressure} \tag{2.454}$$

Froude number (free-surface flows)

$$Fr = \frac{v}{\sqrt{gL}} = \frac{inertia\ force}{gravitational\ force} \tag{2.455}$$

Grashof number (natural convection)

$$Gr = \frac{\rho^2 g\beta L^3 \Delta T}{\mu^2} = \frac{buoyancy\ force}{viscous\ force} \tag{2.456}$$

Jacob number (boiling)

$$Ja = \frac{c_p\ \rho_f\ \Delta T}{r\rho_V} = \frac{heat\ transfer}{heat\ of\ evaporation} \tag{2.457}$$

Knudsen number (rarefied gas flows and flows in micro- and nanocapillaries)

$$Kn = \frac{\lambda}{L} = \frac{mean\ free\ path}{characteristic\ dimension} \tag{2.458}$$

Kutateladze number (combined heat and mass transfer in evaporation)

$$K = \frac{r_v}{c_p\ \Delta T} = \frac{latent\ heat\ of\ phase\ change}{convective\ heat\ transfer} \tag{2.459}$$

Lewis number (combined heat and mass transfer)

$$Le = \frac{k}{\rho c_p\ D} = \frac{thermal\ diffusivity}{diffusivity} \tag{2.460}$$

Mach number (compressible flows)

$$Ma = \frac{v}{c} = \frac{flow\ speed}{local\ speed\ of\ sound} \tag{2.461}$$

Nusselt number (forced convection)

$$Na = \frac{hL}{k} = \frac{total\ heat\ transfer}{conductive\ heat\ transfer} \tag{2.462}$$

Peclet number (forced convection)

$$Pe = \frac{L\rho v c_p}{k} = \frac{bulk\ heat\ transfer}{conductive\ heat\ transfer} \tag{2.463}$$

Prandtl number (heat transfer in fluid flows)

$$Pr = \frac{\mu c_p}{k} = \frac{momentum\ diffusivity}{thermal\ diffusivity} \tag{2.464}$$

Rayleigh number (natural convection)

$$Ra = \frac{\rho^2 g \beta L^3 c_p}{\mu k} = \frac{thermal\ expansion}{thermal\ diffusivity} \tag{2.465}$$

Reynolds number (viscous flows)

$$\mathrm{Re} = \frac{\rho v L}{\mu} = \frac{inertia\ force}{viscous\ force} \tag{2.466}$$

Richardson number (stratified flow of multilayer systems)

$$Ri = -\left(\frac{g}{\rho}\frac{\partial P}{\partial L_h}\right) \Bigg/ \left(\frac{\partial v}{\partial L_h}\right)_w = \frac{gravity\ force}{inertia\ force} \tag{2.467}$$

Rossby number (geophysical flows)

$$Ro = \frac{v}{L\Omega} = \frac{inertia\ force}{Coriolis\ force} \tag{2.468}$$

Schmidt number (diffusion in flow)

$$Sc = \frac{\mu}{\rho D} = \frac{kinematic\ viscosity}{molecular\ diffusivity} \tag{2.469}$$

Senenov number (convective heat transfer)

$$Se = \frac{h_m}{K} = \frac{intensity\ of\ heat\ transfer}{intensity\ of\ chemical\ reaction} \tag{2.470}$$

Sherwood number (mass transfer)

$$Sh = \frac{h_m L}{D} = \frac{mass\ diffusivity}{molecular\ diffusitivy} \tag{2.471}$$

Stenton number (forced convection)

$$St = \frac{h}{\rho v c_p} = \frac{heat\ transferred}{thermal\ capacity\ of\ fluid} \tag{2.472}$$

Strouhal number (unsteady flow)

$$Sr = \frac{fL}{v} = \frac{time\ scale\ of\ flow}{oscillation\ period} \tag{2.473}$$

Taylor number (rotation on natural convection)

$$Ta = \frac{fL}{v} = \frac{Coriolis\ force}{viscous\ force} \tag{2.474}$$

Weber number (surface tension)

$$We = \frac{\rho v^2 L}{\sigma} = \frac{inertia\ force}{surface\ tension} \tag{2.475}$$

These groups occur regularly when dimensional analysis is applied to thermofluid-dynamical problems. They can be derived by considering forces on a small volume of fluid. Most of them can also be derived by nondimensionalizing the differential equations of thermofluid flow.

Example 77 Water flow as a model for air flow.

Consider a prototype system of flowing air at standard conditions of 15 °C through a circular pipe of diameter $D_P = 1.016\,\text{m}$ at a velocity $v = 6\,\text{m/s}$. What would be the velocity of water flow at standard conditions of 15 °C through a circular pipe of diameter $D_M = 10.16\,\text{cm}$ of a similar model. Then, using experiment on the model and measuring the pressure drop of $\Delta p_M = 1.67\,\text{MPa}$, what would be the pressure drop in the prototype?

To have dynamic similarity of a model and prototype that involve viscous fluid flow having no interface with another fluid, the Reynolds numbers of the prototype and model must be identical.

$$\left(\frac{vD\rho}{\mu}\right)_p = \left(\frac{vD\rho}{\mu}\right)_M \tag{2.476}$$

The density ρ and viscosity μ of air at sea level are $\rho_P = 1.225\,\text{kg/m}^3$ and $\mu_P = 1.802 \times 10^{-5}\,\text{Pa s}$, and the density and viscosity of water at 15 °C are $\rho_M = 999.1\,\text{kg/m}^3$ and $\mu_M = 1.1375 \times 10^{-3}\,\text{Pa s}$. Therefore, to have dynamic similarity between the prototype and model, the velocity of water in the model must be

$$\begin{aligned} v_M &= v_P\left(\frac{\rho_P}{\rho_M}\right)\left(\frac{D_P}{D_M}\right)\left(\frac{\mu_M}{\mu_P}\right) \\ &= 6\left(\frac{1.225}{999.1}\right)\left(\frac{1.016}{10.16\times 10^{-2}}\right)\left(\frac{1.1375\times 10^{-3}}{1.802\times 10^{-5}}\right) \\ &= 4.6438\,\text{m/s} \end{aligned} \tag{2.477}$$

Euler number indicates pressure drop in fluid flow.

$$Eu = \frac{\rho v^2}{\Delta p} \tag{2.478}$$

Equating Euler number between prototype and model

$$\left(\frac{\rho v^2}{\Delta p}\right)_p = \left(\frac{\rho v^2}{\Delta p}\right)_M \tag{2.479}$$

provides an equation to determine the pressure drop Δp_P in prototype by measuring the pressure drop Δp_M in the model.

$$\begin{aligned}\Delta p_P &= \Delta p_M \left(\frac{\rho_P}{\rho_M}\right)\left(\frac{v_P^2}{v_M^2}\right) \\ &= 1.67 \times 10^6 \left(\frac{1.225}{999.1}\right)\left(\frac{6^2}{4.6438^2}\right) \\ &= 3418.2\,\text{Pa} = 3.4182\,\text{kPa}\end{aligned} \tag{2.480}$$

Example 78 Velocity-length inverse relation.

Assuming that similarity between a model and prototype is achieved by equalizing Reynolds number.

$$\text{Re}_P = \text{Re}_M \tag{2.481}$$

$$\left(\frac{vL\rho}{\mu}\right)_p = \left(\frac{vL\rho}{\mu}\right)_M \tag{2.482}$$

Using the same working fluid, $\mu_P = \mu_M$, $\rho_P = \rho_M$, would require a velocity ratio inversely proportional to the length-scale ratio and hence impractically large velocities in the scaled down model is required.

$$(vL)_p = (vL)_M \tag{2.483}$$

$$\frac{v_P}{v_M} = \frac{L_M}{L_P} \tag{2.484}$$

In such cases it is needed to use different fluid to be able to test the model in lower velocities. Let us assume to have a $1/10$ scaled model,

$$\frac{L_M}{L_P} = \frac{1}{10} \tag{2.485}$$

and we wish to conduct an experiment on a model at a velocity of 1/10 of the operational velocity of the prototype.

$$\frac{v_M}{v_P} = \frac{1}{10} \tag{2.486}$$

Therefore, from another form of (2.482)

$$\left(\frac{vL}{\eta}\right)_p = \left(\frac{vL}{\eta}\right)_M \tag{2.487}$$

we have

$$\eta_M = \frac{v_M}{v_P}\frac{L_M}{L_P}\eta_P = \frac{1}{10}\frac{1}{10}\eta_P = \frac{1}{100}\eta_P \tag{2.488}$$

and we need a fluid that its kinematic viscosity $\eta_M = \rho_M/\mu_M$ to be $1/100$ of the kinematic viscosity of prototype operating fluid. Water is the prototype operating fluid that at $t = 20\,°\text{C}$ has $\eta_P = 1.0035 \times 10^{-6}\,\text{m}^2/\,\text{s}$. So, we need a fluid with $\eta_M = 1.0035 \times 10^{-4}\,\text{m}^2/\,\text{s}$. The available fluid for the experiment is air with $\eta_M = 1.0035 \times 10^{-4}\,\text{m}^2/\,\text{s}$ at temperature $t = 650\,°\text{C}$ at one atmosphere. It seems a costly experiment to heat up the air to $t = 650\,°\text{C}$.

Let us use air at room temperature for experiment with $\eta_M = 1.516 \times 10^{-5}\,\text{m}^2/\,\text{s}$ at temperature $t = 20\,°\text{C}$ at one atmosphere. Employing this air as model operating fluid, we must do the experiment at velocity $v_M = 151 v_p$.

$$\frac{v_M}{v_p} = \frac{L_P}{L_M}\frac{\eta_M}{\eta_P} = 10 \times \frac{1.516 \times 10^{-5}}{1.0035 \times 10^{-6}} \tag{2.489}$$

It is still not very practical experiment due to very high velocity of the air. Such an experiment might be more practical when the model is scaled up of a small prototype.

Example 79 Kepler's third law.

In Example 61, it was shown that the period p of rotation of a particle of mass m moving in a circular orbit of radius R by a central force F is:

$$p = 2\pi\sqrt{\frac{mR}{F}} \tag{2.490}$$

Let us accept the Kepler's third law that says square of period of rotation of a planet around the sun is proportional to the cubes of the radius of rotation. Therefore,

$$\left(\frac{p_1}{p_2}\right)^2 = \left(\frac{R_1}{R_2}\right)^3 \tag{2.491}$$

Now assume that the Newton's universal gravitation force is introduced as

$$F = G\frac{m_1 m_2}{R^\alpha} \tag{2.492}$$

From Eq. (2.490), we have

$$\left(\frac{p_1}{p_2}\right)^2 = \frac{m_1 R_1/F_1}{m_2 R_2/F_2} \tag{2.493}$$

Substituting F from (2.492) for planets m_1 and m_2 with respect to a central body of mass M yields:

$$\left(\frac{p_1}{p_2}\right)^2 = \frac{\dfrac{m_1 R_1}{m_1 M/R_1^\alpha}}{\dfrac{m_2 R_2}{m_2 M/R_2^\alpha}} = \left(\frac{R_1}{R_2}\right)^{\alpha+1} \tag{2.494}$$

Accepting Kepler's third law shows that

$$\alpha = 2 \tag{2.495}$$

and hence, the Newton's gravitation equation will be found correctly (Zorich 2011).

$$F = G\frac{m_1 m_2}{R^2} \tag{2.496}$$

Example 80 Constant velocity vehicle dynamics modeling.

Equations of motion of planar bicycle vehicle dynamics with steerable front wheel are:

$$\dot{v}_x = \frac{F_x}{m} + r\, v_y \tag{2.497}$$

$$\dot{v}_y = \frac{1}{m v_x}\left(-a_1 C_{\alpha f} + a_2 C_{\alpha r}\right) r - \frac{1}{m v_x}\left(C_{\alpha f} + C_{\alpha r}\right) v_y + \frac{1}{m} C_{\alpha f}\delta_f - r\, v_x \tag{2.498}$$

$$\dot{r} = \frac{1}{I_z v_x}\left(-a_1^2 C_{\alpha f} - a_2^2 C_{\alpha r}\right) r - \frac{1}{I_z v_x}\left(a_1 C_{\alpha f} - a_2 C_{\alpha r}\right) v_y + \frac{1}{I_z} a_1 C_{\alpha f}\delta_f \tag{2.499}$$

The forward velocity v_x, lateral velocity v_y, and yaw rate r are assumed the main variables of the system. The steer angle of front wheels δ_f and traction force F_x

are true inputs of the system. To simplify the analysis it is traditional to ignore the mechanism of generating the traction force F_x, assume v_x is constant, and consider the dimensionless δ_f is the only input. Therefore, v_y, r, F_x are the outputs of the equations and as a result of this approach, the first Eq. (2.497) becomes independent of the other two equations.

$$F_x = -m\, r\, v_y \tag{2.500}$$

There were 8 parameters: vehicle mass m, traction force F_x, distance of front and axles from mass center a_1 and a_2, wheelbase $l = a_1 + a_2$, front and rear tires' side slip coefficients $C_{\alpha f}$, and $C_{\alpha r}$, mass moment of the vehicle about the z-axis, I_z.

The constant forward velocity is the best option to be used as a characteristic velocity to nondimensionalize the remaining dimensional variables v_y, r, t,

$$v_{y_0} = v_x \qquad t_0 = \frac{l}{v_x} \qquad r_0 = \frac{v_x}{l} \tag{2.501}$$

and define the new nondimensionalized variables.

$$\pi_1 = V_y = \frac{v_y}{v_x} \qquad \pi_2 = \Gamma = \frac{r}{v_x/l} \qquad \pi_3 = \tau = \frac{t}{l/v_x} \tag{2.502}$$

The derivatives are:

$$\dot{v}_y = \frac{dv_y}{dt} = v_x \frac{dV_y}{dt} = v_x \frac{dV_y}{d\tau}\frac{d\tau}{dt} = \frac{v_x^2}{l}\frac{dV_y}{d\tau} = \frac{v_x^2}{l} V_y' = \frac{v_x^2}{l}\pi_1' \tag{2.503}$$

$$\dot{r} = \frac{dr}{dt} = \frac{v_x}{l}\frac{d\Gamma}{dt} = \frac{v_x}{l}\frac{d\Gamma}{d\tau}\frac{d\tau}{dt} = \frac{v_x^2}{l^2}\frac{d\Gamma}{d\tau} = \frac{v_x^2}{l^2}\Gamma' = \frac{v_x^2}{l^2}\pi_2' \tag{2.504}$$

Substituting the variables in the second and third equation of motion, and simplifying the equation provides us with the following nondimensionalized equations:

$$\begin{aligned} V_y' = &\left(-\frac{a_1}{l}\frac{lC_{\alpha f}}{mv_x^2} + \frac{a_2}{l}\frac{lC_{\alpha r}}{mv_x^2}\right)\Gamma \\ &-\left(\frac{lC_{\alpha f}}{mv_x^2} + \frac{lC_{\alpha r}}{mv_x^2}\right) V_y + \frac{lC_{\alpha f}}{mv_x^2}\delta_f - \Gamma \end{aligned} \tag{2.505}$$

$$\begin{aligned} \Gamma' = &\left(-\frac{a_1^2}{l^2}\frac{l^3 C_{\alpha f}}{I_z v_x^2} - \frac{a_2^2}{l^2}\frac{l^3 C_{\alpha r}}{I_z v_x^2}\right)\Gamma \\ &-\left(\frac{a_1}{l}\frac{l^3 C_{\alpha f}}{I_z v_x^2} - \frac{a_2}{l}\frac{l^3 C_{\alpha r}}{I_z v_x^2}\right) V_y + \frac{l^3 C_{\alpha f}}{I_z v_x^2}\frac{a_1}{l}\delta_f \end{aligned} \tag{2.506}$$

Let us define

$$\pi_4 = \varphi_f = \frac{lC_{\alpha f}}{mv_x^2} \qquad \pi_5 = \varphi_r = \frac{lC_{\alpha r}}{mv_x^2} \qquad \pi_6 = \psi_f = \frac{l^3 C_{\alpha f}}{I_z v_x^2}$$
$$\pi_7 = \psi_r = \frac{l^3 C_{\alpha r}}{I_z v_x^2} \qquad \pi_8 = \delta_f \qquad \pi_9 = \varepsilon = \frac{a_1}{l} \tag{2.507}$$

to rearrange the equations into:

$$V_y' = \left(-\varepsilon\varphi_f + (1-\varepsilon)\,\varphi_r - 1\right)\Gamma - \left(\varphi_f + \varphi_r\right)V_y + \varphi_f\delta_f \tag{2.508}$$
$$\Gamma' = \left(-\varepsilon^2\psi_f - (1-\varepsilon)^2\,\psi_r\right)\Gamma - \left(\varepsilon\psi_f - (1-\varepsilon)\,\psi_r\right)V_y + \varepsilon\psi_f\delta_f \tag{2.509}$$

or in a π-group form:

$$\pi_1' = (\pi_5 - \pi_3\,(\pi_4 + \pi_5) - 1)\,\pi_2 - (\pi_4 + \pi_5)\,\pi_1 + \pi_4\pi_8 \tag{2.510}$$
$$\pi_2' = (\pi_7 - \pi_3\,(\pi_6 + \pi_7))\,\pi_2 - \left(\pi_3^2\pi_6 - (1-\pi_3)^2\,\pi_7\right)\pi_1 + \pi_3\pi_6\pi_8 \tag{2.511}$$

They may also be rearranged in a matrix form.

$$\begin{bmatrix}\pi_1'\\ \pi_2'\end{bmatrix} = \begin{bmatrix} -(\pi_4 + \pi_5) & \pi_5 - \pi_9\,(\pi_4 + \pi_5) - 1 \\ \pi_7 - \pi_9\,(\pi_6 + \pi_7) & -\pi_9^2\pi_6 - (1-\pi_9)^2\,\pi_7 \end{bmatrix}\begin{bmatrix}\pi_1\\ \pi_2\end{bmatrix} + \begin{bmatrix}\pi_4\\ \pi_9\pi_6\end{bmatrix}\pi_8 \tag{2.512}$$

Employing the dimensionless parameters and variables, the traction Equation becomes

$$f = \frac{F_x}{mv_x^2/l} = -\Gamma V_y \tag{2.513}$$
$$\pi_{10} = -\pi_1\pi_2 \tag{2.514}$$

To have similarity between model and prototype, we need to match their π-groups.

$$\pi_1 = \frac{v_y}{v_x} \qquad \pi_2 = \frac{r}{v_x/l} \qquad \pi_3 = \frac{t}{l/v_x}$$
$$\pi_4 = \frac{lC_{\alpha f}}{mv_x^2} \qquad \pi_5 = \frac{lC_{\alpha r}}{mv_x^2} \qquad \pi_6 = \frac{l^3 C_{\alpha f}}{I_z v_x^2}$$

$$\pi_7 = \frac{l^3 C_{\alpha r}}{I_z v_x^2} \qquad \pi_8 = \delta_f \qquad \pi_9 = \frac{a_1}{l} \tag{2.515}$$

Full similarity comes from matching all π-groups. The $\pi_{10} = F_x / \left(m v_x^2 / l\right)$ is not independent as it is a function of $\pi_1 = v_y / v_x$ and $\pi_2 = r / (v_x / l)$. The $\pi_3 = t / (l / v_x)$ is only a change in time scale between the model and prototype and will be matched automatically. The $\pi_8 = \delta_f$ is the input measured in degrees or radian and would be equal in both, model and prototype. The $\pi_9 = a_1 / l$ is the position of mass center which must be the same in model and prototype vehicles. The $\pi_1 = v_y / v_x$ and $\pi_2 = r / (v_x / l)\, r$ are the outputs of the equations of motion when they are solved for a set of parameters and input $\pi_8 = \delta_f$. Therefore, to be able to match the output variables $\pi_1 = v_y / v_x$ and $\pi_2 = r / (v_x / l)\, r$ between model and prototype, we must match only 5 π-groups:

$$\begin{gathered} \pi_9 = a_1 / l \qquad \pi_4 = l C_{\alpha f} / \left(m v_x^2\right) \qquad \pi_5 = l C_{\alpha r} / \left(m v_x^2\right) \\ \pi_6 = l^3 C_{\alpha f} / \left(I_z v_x^2\right) \qquad \pi_7 = l^3 C_{\alpha r} / \left(I_z v_x^2\right) \end{gathered} \tag{2.516}$$

Setting $\pi_9 = a_1 / l$ to be the same between model and prototype, we must have scaled down model with the same proportional location of mass center.

$$\left(\frac{a_1}{l}\right)_p = \left(\frac{a_1}{l}\right)_M \tag{2.517}$$

Having set the speed of prototype $(v_x)_P$ and model $(v_x)_M$, we may calculate the required parameters $C_{\alpha f}$, $C_{\alpha r}$, m, I_z of the model using $\pi_4, \pi_5, \pi_6, \pi_7$.

$$\left(C_{\alpha f}\right)_M = \frac{m_M}{m_P} \frac{l_P}{l_M} \frac{\left(v_x^2\right)_M}{\left(v_x^2\right)_P} \left(C_{\alpha f}\right)_P \tag{2.518}$$

$$(C_{\alpha r})_M = \frac{(I_z)_M}{(I_z)_P} \frac{l_P^3}{l_M^3} \frac{\left(v_x^2\right)_M}{\left(v_x^2\right)_P} (C_{\alpha r})_P \tag{2.519}$$

$$m_M = \frac{l_M}{l_P} \frac{(C_{\alpha r})_M}{(C_{\alpha r})_P} \frac{\left(v_x^2\right)_P}{\left(v_x^2\right)_M} m_P \tag{2.520}$$

$$(I_z)_M = \frac{\left(C_{\alpha f}\right)_M}{\left(C_{\alpha f}\right)_P} \frac{l_M^3}{l_P^3} \frac{\left(v_x^2\right)_P}{\left(v_x^2\right)_M} (I_z)_P \tag{2.521}$$

This step might need trial and error to get the consistent set of parameters.

Example 81 Variable velocity vehicle dynamics modeling.

The general theory of planar model of vehicles is governed by the following set of coupled ordinary differential equations. There are 5 variables in the equations:

forward velocity v_x, lateral velocity v_y, yaw rate r, time t, steer angle of front wheels δ_f (Jazar 2017, 2019).

$$\dot{v}_x = \frac{F_x}{m} + r\, v_y \tag{2.522}$$

$$\begin{aligned}\dot{v}_y = &\frac{1}{m v_x}\left(-a_1 C_{\alpha f} + a_2 C_{\alpha r}\right) r \\ &-\frac{1}{m v_x}\left(C_{\alpha f} + C_{\alpha r}\right) v_y + \frac{1}{m} C_{\alpha f}\delta_f - r\, v_x\end{aligned} \tag{2.523}$$

$$\begin{aligned}\dot{r} = &\frac{1}{I_z v_x}\left(-a_1^2 C_{\alpha f} - a_2^2 C_{\alpha r}\right) r \\ &-\frac{1}{I_z v_x}\left(a_1 C_{\alpha f} - a_2 C_{\alpha r}\right) v_y + \frac{1}{I_z} a_1 C_{\alpha f}\delta_f\end{aligned} \tag{2.524}$$

Defining the following dimensional characteristic parameters:

$$v_0 = \sqrt{\frac{l C_{\alpha f}}{m}} \qquad t_0 = \sqrt{\frac{ml}{C_{\alpha f}}} \qquad r_0 = \sqrt{\frac{C_{\alpha f}}{ml}} \tag{2.525}$$

we define the new nondimensionalized variables.

$$\begin{aligned}&\pi_1 = V_y = \frac{v_y}{v_0} = \sqrt{\frac{m}{l C_{\alpha f}}} v_y \qquad \pi_2 = \Gamma = \frac{r}{r_0} = \sqrt{\frac{ml}{C_{\alpha f}}} r \\ &\pi_3 = \tau = \frac{t}{t_0} = \sqrt{\frac{C_{\alpha f}}{ml}} t \qquad \pi_4 = V_4 = \frac{v_x}{v_0} = \sqrt{\frac{m}{l C_{\alpha f}}} v_x\end{aligned} \tag{2.526}$$

and parameters

$$\begin{aligned}&\pi_5 = c = \frac{C_{\alpha r}}{C_{\alpha f}} \qquad \pi_6 = e = \frac{ml^2}{I_z} \qquad \pi_7 = p = \frac{F_x}{C_{\alpha f}} \\ &\pi_8 = \delta_f \qquad \pi_9 = \varepsilon = \frac{a_1}{l}\end{aligned} \tag{2.527}$$

The derivatives are:

$$\dot{v}_x = \frac{d v_x}{dt} = v_0 \frac{d V_x}{d\tau}\frac{d\tau}{dt} = \frac{C_{\alpha f}}{m} V_x' = \frac{C_{\alpha f}}{m}\pi_4' \tag{2.528}$$

$$\dot{v}_y = \frac{d v_y}{dt} == v_0 \frac{d V_y}{d\tau}\frac{d\tau}{dt} = \frac{C_{\alpha f}}{m} V_y' = \frac{C_{\alpha f}}{m}\pi_1' \tag{2.529}$$

$$\dot{r} = \frac{dr}{dt} = r_0 \frac{d\Gamma}{d\tau}\frac{d\tau}{dt} = \frac{C_{\alpha f}}{ml}\Gamma' = \frac{C_{\alpha f}}{ml}\pi_2' \tag{2.530}$$

Substituting the variables in the equations of motion and simplifying the equation provides us with the following nondimensionalized equations:

$$V_x' = \frac{F_x}{C_{\alpha f}} + \Gamma V_y \tag{2.531}$$

$$\begin{aligned} V_y' = & \left(-\frac{a_1}{l} + \frac{a_2}{l}\frac{C_{\alpha r}}{C_{\alpha f}}\right)\frac{\Gamma}{V_x} \\ & - \left(1 + \frac{C_{\alpha r}}{C_{\alpha f}}\right)\frac{V_y}{V_x} + \delta_f - \Gamma V_y \end{aligned} \tag{2.532}$$

$$\begin{aligned} \Gamma' = & \left(-\frac{ma_1^2}{I_z} - \frac{ma_2^2}{I_z}\frac{C_{\alpha r}}{C_{\alpha f}}\right)\frac{\Gamma}{V_x} \\ & - \left(\frac{mla_1}{I_z} - \frac{mla_2}{I_z}\frac{C_{\alpha r}}{C_{\alpha f}}\right)\frac{V_y}{V_x} + \frac{mla_1}{I_z}\delta_f \end{aligned} \tag{2.533}$$

which will be rewritten as:

$$\pi_4' = \pi_7 + \pi_2 \pi_1 \tag{2.534}$$

$$\pi_1' = \left((1-\pi_9)\pi_5 - \pi_9\right)\frac{\pi_2}{\pi_4} - (1+\pi_5)\frac{\pi_1}{\pi_4} + \pi_8 - \pi_2\pi_1 \tag{2.535}$$

$$\begin{aligned} \pi_2' = & -\left((1-\pi_9)^2 \pi_5 + \pi_9^2\right)\pi_6 \frac{\pi_2}{\pi_4} \\ & - \left((1-\pi_9)\pi_5 - \pi_9\right)\pi_6 \frac{\pi_1}{\pi_4} + \pi_9\pi_6\pi_8 \end{aligned} \tag{2.536}$$

Therefore, the equations of motion of planar vehicle dynamics reduce to a set of equations with 5 variables: $\pi_4 = V_4$, $\pi_1 = V_y$, $\pi_2 = \Gamma$, $\pi_3 = \tau$, $\pi_8 = \delta_f$, and 4 parameters: $\pi_5 = \frac{C_{\alpha r}}{C_{\alpha f}}$, $\pi_6 = \frac{ml^2}{I_z}$, $\pi_7 = \frac{F_x}{C_{\alpha f}}$, $\pi_9 = \frac{a_1}{l}$. To make the model and prototype of vehicles to act similarly, we need to match 5 π-groups.

$$\begin{aligned} & \pi_5 = \frac{C_{\alpha r}}{C_{\alpha f}} \qquad \pi_6 = \frac{ml^2}{I_z} \qquad \pi_7 = \frac{F_x}{C_{\alpha f}} \\ & \pi_8 = \delta_f \qquad \pi_9 = \frac{a_1}{l} \end{aligned} \tag{2.537}$$

This set of π-groups allows us to do variable speed experiments on model.

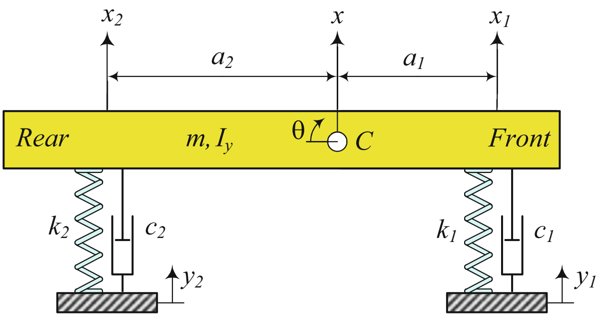

Fig. 2.16 The bicycle model of a car is a beam with mass m and mass moment I, sitting on two springs k_1 and k_2 and dampers c_1 and c_2

Example 82 Ill-nondimensionalization.

Although it is possible and recommended to make all problems nondimensional, it is not an absolute rule. Nondimensionalization will make the result to be applicable to all similar problems regardless of their size and area of science. However, it is also common that investigators prefer to keep the variables, such as displacement and time, as they are, so their values can be sensed as normal physical dimensions. In these cases, the investigator may wish to define new parameters, dimensional or nondimensional, to reform the equations in the way they wish to show something more important to them. When the set of equations of a problem are not fully nondimensionalized, we call it ill-nondimensionalization. Let us review an example to illustrate such ill-nondimensionalization problems.

Figure 2.16 illustrates a beam on two linear suspensions at two ends. Let us ignore the dampers c_1 and c_2 for the moment. The beam with mass m and mass moment I about the mass center C is sitting on two springs k_1 and k_2 to model a car in bounce and pitch motions. The translational coordinate x of C and the rotational coordinate θ are the usual generalized coordinates that we use to measure the kinematics of the beam. Recalling that the equations of motion and the mode shapes of a vibrating system are functions of the chosen coordinates, we find the free vibration equations of motion of the system as:

$$\begin{bmatrix} m & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & a_2 k_2 - a_1 k_1 \\ a_2 k_2 - a_1 k_1 & a_2^2 k_2 + a_1^2 k_1 \end{bmatrix} \begin{bmatrix} x \\ \theta \end{bmatrix} = 0 \tag{2.538}$$

It is also possible to employ the coordinates x_1 and x_2 instead of x and θ and derive the equations of motion of the system in the following form:

$$\begin{bmatrix} \dfrac{m a_2^2 + I}{a_1 + {a_2}^2} & \dfrac{m a_1 a_2 - I}{a_1 + {a_2}^2} \\ \dfrac{m a_1 a_2 - I}{a_1 + {a_2}^2} & \dfrac{m a_1^2 + I}{a_1 + {a_2}^2} \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \tag{2.539}$$

The set of equations in (2.538) has a diagonal mass matrix and a symmetric stiffness matrix, while the set of equations in (2.539) has a symmetric mass matrix and a diagonal stiffness matrix. Having diagonal mass matrix is the preferred way to have the equations of motion to calculate the natural frequencies and mode shapes of a vibrating system. The reason is that the inverse of a diagonal matrix is very easy to calculate and we will prefer to determine the eigenvalues and eigenvectors of

$$A = M^{-1}K = \begin{bmatrix} \dfrac{k_1 + k_2}{m} & -\dfrac{a_1 k_1 - a_2 k_2}{m} \\ -\dfrac{1}{I}\left(a_1 k_1 - a_2 k_2\right) & \dfrac{1}{I}\left(k_1 a_1^2 + k_2 a_2^2\right) \end{bmatrix} \tag{2.540}$$

rather than

$$B = K^{-1}M = \begin{bmatrix} \dfrac{m a_2^2 + I}{k_1\left(a_2^2 + a_1\right)} & -\dfrac{I - m a_1 a_2}{k_1\left(a_2^2 + a_1\right)} \\ -\dfrac{I - m a_1 a_2}{k_2\left(a_2^2 + a_1\right)} & \dfrac{m a_1^2 + I}{k_2\left(a_2^2 + a_1\right)} \end{bmatrix} \tag{2.541}$$

where M is the mass matrix and K is the stiffness matrix of the system (Jazar 2014). However, in this example, let us keep the set (2.539) and introduce the following parameters:

$$\begin{aligned} I &= mr^2 \qquad \Omega_1^2 = \frac{k_1}{m}\beta \qquad \Omega_2^2 = \frac{k_2}{m}\beta \qquad \beta = \frac{l^2}{a_1 a_2} \\ \alpha &= \frac{r^2}{a_1 a_2} \qquad \gamma = \frac{a_2}{a_1} \qquad l = a_1 + a_2 \end{aligned} \tag{2.542}$$

and rewrite the equations as

$$\begin{bmatrix} \alpha + \gamma & 1 - \alpha \\ 1 - \alpha & \alpha + \frac{1}{\gamma} \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} \Omega_1^2 & 0 \\ 0 & \Omega_2^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \tag{2.543}$$

Out of the parameters (2.542), only α, β, γ are dimensionless, while the dimension of radius of gyration r, and characteristic frequencies Ω_1 and Ω_2 are:

$$[r] = L \qquad [\Omega_1] = [\Omega_2] = T^{-1} \tag{2.544}$$

Rewriting the equations as in (2.543) shows an interesting feature. If we set

$$\alpha = \frac{r^2}{a_1 a_2} = 1, \tag{2.545}$$

then the equations will be decoupled.

$$\begin{bmatrix} \alpha+\gamma & 0 \\ 0 & \alpha+\frac{1}{\gamma} \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} \Omega_1^2 & 0 \\ 0 & \Omega_2^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \tag{2.546}$$

Decoupling of the equations makes the two degrees-of-freedom system become equivalent to two single degree-of-freedom systems. This was the first condition for flat ride tuning of vehicle suspensions, suggested by Maurice Olley (1889–1983) and shown by others (Olley 1934, Marzbani et al. 2012, Marzbani and Jazar 2014, Milliken et al. 2002).

The natural frequencies ω_i and mode shapes u_i of the system are

$$\omega_1^2 = \frac{1}{\gamma+1}\Omega_1^2 = \frac{l}{a_2}\frac{k_1}{m} \qquad u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{2.547}$$

$$\omega_2^2 = \frac{\gamma}{\gamma+1}\Omega_2^2 = \frac{l}{a_1}\frac{k_2}{m} \qquad u_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{2.548}$$

2.4 Size Effects

Weight, volume, heat capacity, are proportional to volume, L^3. Strength, heat loss, are proportional to cross-sectional area, or surface area, L^2. Therefore, changing the size of an object will change any property related to the surface-to-volume ratio L^{-1} (or volume-to-surface ratio L). It is the amount of surface area per unit volume of an object, $L^2/L^3 = L^{-1}$. This fact is called *square–cube law*. This law governs many phenomena in all areas of science and engineering. The proportionality of surface-to-volume to L^{-1} not only determines how big or how small an object can be, in many cases there are critical values for the surface-to-volume ratios that changes the phenomena drastically. Understanding this law is important in engineering and science. The square-cube law may be cast in one engineering comment:

Big structure is weak, small structure is strong.

The structure might be a living thing.

Historically, as far as we know, it was Galileo Galilei (1564–1642) who first described the square-cube law in 1638 in his "Two New Sciences" as "the ratio of two volumes is greater than the ratio of their surfaces" (Galileo 1638).

The best way to understand and employ the square-cube law is to study its application and effects in different areas of engineering and science through examples.

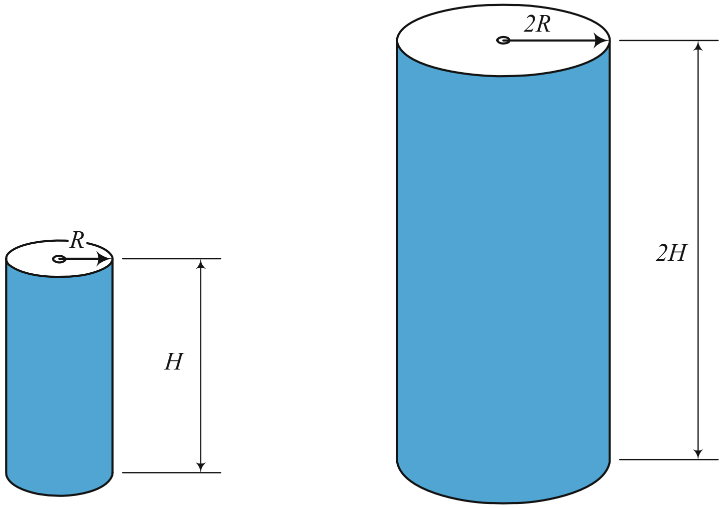

Fig. 2.17 A cylinder of radius R and height H along with a proportionally double sized cylinder or radius $2R$ and height $2H$

Example 83 Stress under a cylindrical concrete column.

Figure 2.17 illustrates a cylinder of radius R and height H along with a double sized cylinder of radius $2R$ and height $2H$.

Consider a cylindrical concrete column of radius R and height $H = 10R$. The volume V of the column would be:

$$V = \pi R^2 H = 10\pi R^3 \tag{2.549}$$

and its cross-sectional area A would be:

$$A = \pi R^2 \tag{2.550}$$

If the density of the concrete is ρ, then its weight W is:

$$W = \rho g V = 10\pi \rho g R^3 \tag{2.551}$$

The compression stress σ at the bottom of the column is the total weight divided by the cross-sectional area.

$$\sigma = \frac{W}{A} = \frac{10\pi \rho g R^3}{\pi R^2} = 10 R g \rho \tag{2.552}$$

The stress is a linear function of the size of the column, represented by R. If the column is magnifying proportionally, the height and radius will grow with the same rate. The cross-sectional area that supports the load will increase proportional to

R^2, while the weight of the column increases proportional to R^3. Therefore, by increasing the size of the column, the stress σ will increase until it reaches the ultimate stress σ_Y and the column will break down.

To design a column that accepts magnification, we must keep the level of normal stress σ constant. Constant normal stress condition may require disproportional expansion of the column. However, having height H to be an independent variable indicates that stress σ is a linear function of H.

$$\sigma = \frac{W}{A} = \frac{\pi \rho g R^2 H}{\pi R^2} = g\rho H \tag{2.553}$$

To keep σ constant, we must keep H constant. As a result, increasing or decreasing the radius of a solid cylinder will not change the stress level if the height of the column remains unchanged.

In practice, we use columns to support external forces. Let us assume there exists an external force F at the center of the cylindrical column along its axis. The stress of such column would be:

$$\sigma = \frac{W + F}{A} = \frac{\pi \rho g R^2 H + F}{\pi R^2} \tag{2.554}$$

Setting the level of σ we are able to determine the relationship between R and H to keep σ constant by solving Eq. (2.554) for R.

$$R = \sqrt{\frac{F}{\pi\left(\sigma - \rho g H\right)}} \tag{2.555}$$

Example 84 The size of animal in polar regions.

Heat loss of any heated object is proportional to its surface area $\left[L^2\right]$. Thermal energy of an object is proportional to its mass m and hence proportional to its volume $\left[L^3\right]$. Therefore, the ratio of the heat loss to the amount of thermal energy of an object is proportional to area/volume $[A/V] = \left[L^2/L^3\right] = L^{-1}$. A mouse is around $l_m = 5\,\text{cm}$ long and polar bear is about $l_b = 2\,\text{m}$ long. Therefore, the ratio of heat loss over mass for a mouse approximated due to similar aspect ratios is $A/V = 1/l_m = 20\,\text{m}^{-1}$ and for a bear is $A/V = 1/l_b = 0.5\,\text{m}^{-1}$. As a result, the heat loss of a mouse with respect to a bear is $20/0.5 = 40/1$, indicating that a mouse loses heat 40 times more or faster than a bear. Hence small animals will not last long in cold environment of polar regions.

Example 85 Bird's wing areas.

The lift forces of a bird to sustain in air is proportional to the wing areas, proportional to $[A] = L^2$ where L is any representative length. The wing loading is the weight of the bird W the bird must carry. Weight is proportional to the bird's volume $[V] = L^3$. Therefore, the wing loading per unit of wing area would be proportional to $[V/A] = L$. A bird is hovering by flapping its wings to move a mass of air downward. The time rate of change of the momentum of that jet of air $\dot{m}v$ is equal to the total lift force on the wings. The lift force must be equal to the bird's weight W. The mass rate $\dot{m}$ of moving air may be estimated in terms of the wing area, A, the jet speed, v, and the air density, ρ (Howison 2005)

$$\dot{m} = \frac{dm}{dt} = \rho A v \tag{2.556}$$

$$W = \dot{m}v = \rho A v^2 \tag{2.557}$$

and therefore,

$$v^2 \propto \frac{W}{A} \propto L \tag{2.558}$$

The power P that the bird uses for hovering would be

$$P = Wv = \rho A v^3 \tag{2.559}$$

$$P \propto L^{7/2} \propto W^{7/6} \tag{2.560}$$

The required power P for any particular state of flight increases with the $7/6 \simeq 1.17$ power of the weight W. Larger birds will have higher wing loadings and are therefore obliged to fly faster.

There are three ways to estimate the available power of a bird to enable it to hover. The heat loss analysis during hovering, analysis of the rate at which bird's heart supplies oxygen, and analysis of the maximum stresses in bird's bones and muscles. Muscles turn chemical energy into mechanical energy at a 25% efficiency rate. The excess heat must be able to exit through the bird's surface area $S \propto L^2$. Hence, to prevent the bird from overheating, the available power must also be proportional to L^2. The oxygen supply is proportional to the time rate of change of the volume of blood delivered by the heart, which in turn is proportional to the cross sectional area L^2. The maximum stress σ in the bird's muscle may be calculated by the work done by a force F in the contracting Δl of the muscle. Employing the principle of conservation of energy, the work must be equal to the change in the kinetic energy K of the limb mass m moved by the muscle,

$$K \propto F\,\Delta l \propto m v^2 \propto L^2 \tag{2.561}$$

where v is the speed of the moving limb. As the available power to the bird is the time rate of the work done, $p = dK/dt$, the power is also proportional to L^2. Therefore, no matter how we calculate the available power of the bird, the power is proportional to the square of the characteristic length L^2. Because the needed power to hover increases faster with the bird size than the available power to the bird, there is a limit to the size of birds (Pennycuick 1968).

Example 86 ★First document on square-cube law.

After a discussion on extruding a cylinder of silver into wires to explain how the surface to volume of the mass of silver will change, Galileo says: "if I determine the ratio between the surfaces of cylinders of the same volume, the problem will be solved. I say then: the areas of cylinders of equal volumes, neglecting the bases, bear to each other a ratio which is the square root of the ratio of their lengths." Galileo then continues: "Since you are to find of these geometrical demonstrations, which carry with them distinct gain, I will give you a companion theorem which answers an extremely interesting query. We have seen above what relations hold between equal cylinders of different heights or lengths; let us now see what holds when the cylinders are equal in area but unequal in height, understanding area to include the curved surface, but not the upper and lower bases. The theorem is:

The volumes of right cylinders having equal curved surfaces are inversely proportional to their altitudes. Let the surfaces of the two cylinders, AE and CF, be equal but let the height of the latter, CD, be greater than that of the former, AB: then I say that the volume of the cylinder AE is to that of the cylinder CF as the height CD is to AB. Now since the surface of CF is equal to the surface of AE, it follows that the volume of CF is less than that of AE; for, if they were equal, the surface of CF would, by the preceding proposition, exceed that of AE, and the excess would be so much the greater if the volume of the cylinder CF were greater than that of AE. Let us now take a cylinder ID having a volume equal to that of AE; then, according to the preceding theorem, the surface of the cylinder ID is to the surface of AE as the altitude IF is to the mean proportional between IF and AB. But since one datum of the problem is that the surface of AE is equal to that of CF, and since the surface ID is to the surface CF as the altitude IF is to the altitude CD, it follows that CD is a mean proportional between IF and AB. Not only so, but since the volume of the cylinder ID is equal to that of AE, each will bear the same ratio to the volume of the cylinder CF; but the volume ID is to the volume CF as the altitude IF is to the altitude CD; hence the volume of AE is to the volume of CF as the length IF is to the length CD, that is, as the length CD."

Following this logic Galileo concludes "the ratio of two volumes is greater than the ratio of their surfaces," and the soon after, the square-cube law appeared.

"Imagine, for example, a die of which the side is two inches long, so that one face will be four square inches, and all six, that is, its whole surface, twenty-four square inches. Next, imagine that the die is sliced with three cuts into eight smaller dice. The side of each of them will be one inch, and each face one square inch, and its whole surface six square inches, whereas the surface of the uncut die contained twenty-four. Now you see that the surface of the little die is one-quarter the surface of the larger one, this being the ratio of six to twenty-four. But the volume of the same die is only one-eighth. Thus the volume, and hence the weight, falls off much more quickly than the surface. If you subdivide the little die into eight others, the whole surface of one of these will be one and one-half square inches, which is one-sixteenth of the surface of the original die, while its volume is only one sixty-fourth. See how in just these two divisions, the volumes have diminished four times as much as have the surfaces; and if we continue the subdivision until the original solid is reduced to fine powder, we shall find the weight of the minute atoms to be diminished hundreds and hundreds of times as much as their surfaces."

Although according to the available documents, Galileo is the first scientist who put the square-cube law on paper, this natural law has been known for centuries before Galileo. Archimedes, for instance, said that "in similar figures the surface increases as the square, and the volume as the cube, of the linear dimensions. If we take the simple case of a sphere, with radius R, the area of its surface is equal to $A = 4\pi R^2$, and its volume to $V = 4\pi R^3/3$ from which it follows that the ratio of its volume to surface, or $V/A = R/3$." That is to say, V/A varies as R; or, in other words, the larger the sphere is the greater will be its volume or its mass, if it be uniformly dense, in comparison with area (Thompson 1942). Taking L to represent any linear dimension, we may write the general equations in the form

$$A \propto L^2 \qquad A = k_A L^2 \tag{2.562}$$

$$V \propto L^3 \qquad V = k_V L^3 \tag{2.563}$$

where k_A and k_V are the proportionality coefficients, and

$$\frac{V}{A} \propto L \qquad \frac{V}{A} = \frac{k_V}{k_A} L = kL \tag{2.564}$$

The square-cube law may also be traced back in eleventh century in writings of Persian scientist, Avicenna (Abu Ali Sina 980–1037), when he was answering questions of Biruni (Abū Rayhān Al-Bīrūnī 937–1050) criticizing various concepts and ideas in Aristotle's "On the Heavens." Avicenna claims that sphere is the natural shape of astrological objects as sphere has the minimum surface area for the same volume or it has the maximum volume for the same surface area (Guthrie 1939; Afnan 1958; Berjak 2005; McGinnis 2010; Daiber 2010).

Stephan (1983) described the square-cube law as: "For a given physical object, say a cube, an increase in the length of a side results in an increase of the surface area and also of the volume. If the new length is ten times the old, the area will be 10^2 or 100 times the old, and the new volume will be 10^3 or 1000 times the old. Thus, the cube-root of the volume will be proportional to the square root of the surface area."

Example 87 Square-cube law and fictions.

Thompson (1942) indicates fictions's mistakes by not considering the square-cube law. The Giant in the fiction "Jack the Giant Killer" was proportionally ten times larger than a normal man. Therefore, the cross section of the leg bones of the giant would be 100 times larger than those of a man, while the giant's mass, proportional to its volume, would be 1000 times larger. As a result, giant's legs would not stand his weight and would break under his weight.

Similarly, in the tale of Gulliver's Travels of Jonathan Swift (1667–1745), Gulliver was proportionally 12 times larger than Lilliputians. A person needs food proportional to their mass which is proportional to their volume. Gulliver's body volume would be $12^3 = 1728$ times more than the volume of one Lilliputian and therefore, he would need 1728 times more food than a Lilliputian.

Example 88 Larger fishes are faster.

Consider a sea animal such as a fish that is immersed in water, then the weight of the fish is counterpoised to the extent of an equivalent volume of water, and is weightless. It is a great advantage to sea animals, so the larger the fish grows the faster would be its speed, because its available energy depends on the mass of its muscles, while the resistance to its motion through the water is water friction. The energy of the fish is proportional to its volume $V \propto L^3$ and the motion resistance is proportional to its area $A \propto L^2$. Assuming other things being equal, the bigger the fish the faster it can go. The energy will be used to do work W to overcome the resistance force F.

$$W \propto F \tag{2.565}$$

The resistance F is proportional to the square of its speed v and area of resistance A.

$$F \propto Av^2 \tag{2.566}$$

Therefore, velocity v is increasing in the ratio of the square root of the increasing length of the fish L.

$$v^2 \propto \frac{W}{A} \propto \frac{L^3}{L^2} \propto L \tag{2.567}$$

$$v \propto \sqrt{L} \tag{2.568}$$

This is the Froude's law, or Froude number, which may be expressed as

$$Fr = \frac{v}{\sqrt{gL}} = \frac{inertia\ force}{gravitational\ force} \tag{2.569}$$

or

$$Fr = \frac{v^2}{gL} = \frac{kinetic\ energy}{potential\ energy} \tag{2.570}$$

Froude's law indicates that the speed attainable by a sea creature or a ship is proportional to the square root of its scale. This result is based on the assumption that the whole body of the fish or the ship is producing energy to move the body in the water. However, the available horse-power of an engine varies as the square of the linear dimensions, because the generated energy depends on the heating-surface of the pistons L^2. Similarly, the rate of supply kinetic energy of a fish varies with the surface of the lung L^2. As a result, small fishes are stronger because they have more power per unit weight than larger fishes. An engineering solution is to increase the heating-surface of the engine, by increasing internal system of tubes or pistons, without increasing its outward dimensions. The Nature has done similarly by increases the respiratory surface of a lung by a complex system of branching tubes in lungs (Thompson 1942).

Example 89 Temperature of warm-blooded animals.

The heat loss of an animal is proportional to the body surface of the animal. The generated heat within the body of the animal must be equal to the heat loss if the temperature of the body keeps constant. This requirement is questionable as the heat loss may vary as the surface area, L^2, and the generated heat by oxidation is proportional to the volume of the body, L^3. Therefore, the ratio of loss to generated heat is like that of surface to volume. So, larger animals must have higher temperature, unless the rate of generation of heat be lower in larger animals. In other words, smaller animals produce more heat per unit of mass than the large animals, in order to keep pace with surface heat loss.

The more heat production needs more energy to spend, which in turn needs more food to consume. The smaller animals need more food and oxygen. Their living rate is faster and they age faster. A human consumes food equal to %2 of its own weight daily. A mouse eats %50 of its own weight per day. A warm-blooded animal much smaller than a mouse cannot exist as it cannot consume and digest enough required food to maintain its constant temperature. The disadvantage of heat loss of small size animals is worse in the Arctic area. Therefore, the size of warm-blooded animals will be larger in colder areas, and small animals may survive better in warmer areas (Thompson 1942).

2.5 Chapter Summary

The dimensional analysis reveals how an equation describing a physical system can be reduced to a function of a set of dimensionless products which are always fewer than the prescribed set of explanatory variables. Buckingham π-theorem and nondimensionalization are keys to develop the required methods to transfer experimental results on a scaled size model to the real size prototype.

Suppose we have a physical law of n arguments $x_i, i = 1, 2, \cdots, n$ of m fundamental units in the form

$$f(x_1, x_2, x_3, \cdots, x_n) = 0 \tag{2.571}$$

This equation can be replaced with another equation with k new independent nondimensionalized variables $\pi_1, \pi_2, \cdots, \pi_k, k = n - m$.

$$g(\pi_1, \pi_2, \cdots, \pi_k) = 0 \tag{2.572}$$

The process to derive the dimensionless π-groups is called nondimensionalization. A problem will be expressed by a set of differential and algebraic equations. They will have a set of independent dimensional variables and parameters. For every single independent variable, we need to define a characteristic constant quantity with the same dimension. Division of independent variables by their associated characteristic quantity, a set of dimensionless variables appear. Substituting the variables with their associated dimensionless variable and simplification provides the required π-groups as well as the new set of dimensionless equations.

An example will review this process. Forced vibration of a linear oscillator is expressed by a second order inhomogeneous differential equation.

$$m\ddot{x} + c\dot{x} + kx = F\sin(\omega t) \tag{2.573}$$

The equation of motion has two variables, x and t. We use the static deflection of the mass m under the constant force F, $x_0 = F/k$ as a characteristic displacement and time $t_0 = 1/\omega$ as a constant characteristic time,

$$x_0 = \frac{F}{k} \qquad t_0 = \frac{1}{\omega} \tag{2.574}$$

Then we define nondimensionalized displacement X and time τ

$$X = \frac{x}{x_0} = \frac{x}{F/k} \qquad \tau = \frac{t}{1/\omega} = \omega t \tag{2.575}$$

to replace the actual variables x and t.

$$x = x_0 X = \frac{F}{k}X \qquad t = t_0\tau = \frac{\tau}{\omega} \tag{2.576}$$

Using nondimensionalized time τ, we may redefine derivatives as

$$\dot{x} = \frac{dx}{dt} = \frac{F}{k}\frac{dX}{dt} = \frac{F}{k}\frac{dX}{d\tau}\frac{d\tau}{dt} = \frac{F}{k}\omega X' \tag{2.577}$$

$$\ddot{x} = \frac{d}{dt}\frac{dx}{dt} = \frac{F}{k}\frac{d}{dt}\left(\omega X'\right) = \frac{F}{k}\frac{d\tau}{dt}\frac{d}{d\tau}\left(\omega X'\right) = \frac{F}{k}\omega^2 X'' \tag{2.578}$$

$$X' = \frac{dX}{d\tau} \qquad X'' = \frac{d^2X}{d\tau^2} \tag{2.579}$$

The equation of motion may now be rewritten as

$$\frac{F}{k}m\omega^2 X'' + \frac{F}{k}c\omega X' + \frac{F}{k}kX = F\sin(\tau) \tag{2.580}$$

that will be simplified to

$$r^2 X'' + 2\xi r X' + X = \sin(\tau) \tag{2.581}$$

$$\pi_2^2 \pi_1'' + 2\pi_3 \pi_2 \pi_1' + \pi_1 = \sin \pi_4 \tag{2.582}$$

where

$$\pi_1 = X = \frac{x}{F/k} \qquad \pi_2 = r = \frac{\omega}{\omega_n} \tag{2.583}$$

$$\pi_3 = \xi = \frac{c}{c_c} \qquad \pi_4 = \tau = \omega t \tag{2.584}$$

$$\omega_n = \sqrt{\frac{k}{m}} \qquad c_c = 2\sqrt{km} \tag{2.585}$$

Therefore, the original equation with variables x, t, and five parameters m, c, k, F, ω, has been simplified to an equation with two dimensionless variables X, τ, and two dimensionless parameters ξ, r.

In case we wish to make an experimental model to simulate a real size prototype, we need to make the two π-groups π_3 and π_4 to be equal in both systems. Then the measured or calculated values of π_1 and π_4 are transferable between model and prototype.

Open problem of static dimensional analysis is "variable dimensions" and "constraint dimensions." It is possible to imagine a physical quantity that changes its dimension by time or by relative size of other variables. At the moment we do not have variable dimension theory.

2.6 Key Symbols

$a \equiv \ddot{x}$	Acceleration
a_1	Distance from the front wheel to the C
a_2	Distance from the rear wheel to the C
a_{ij}	Dimension of D_i in x_i
a_{ij}	Elements of $\mathbf{A}$
a, b, c	Exponent of physical quantities in a π-group
a, b, c	Length
A	Area
A	Angle dimension
$\mathbf{A}$	Dimensional matrix
Ar	Archimedes number
b_i	Exponent of x_i in a π-group
B	Body coordinate frame
Bi	Biot number
Bo	Bond number
Br	Brinkman number
c	Damping coefficient
c	Proportionality coefficient
$c = \frac{C_{\alpha r}}{C_{\alpha f}}$	Sideslip coefficient ratio
c_p	Specific heat
C	Dimensionless constant, mass center
C	Circumference
Ca	Capillary number
C_s	Traction coefficient
C_α	Sideslip coefficient
d	Displacement, length
D	Diameter
Da_i	Damkohler number
De	Dean number
D_i	Fundamental dimensions
$e = \frac{ml^2}{I_z}$	Mass moment ratio
E	Energy, work, heat
E	Modulus of elasticity
Ec	Eckert number
Ek	Ekman number
Eu	Euler number
EI	Flexural rigidity
f	Function
f	Frequency
f	Force
$F, \mathbf{F}$	Force
F_d	Drag force

F_x	Traction force
F_y	Lateral force
Fr	Froude number
g	Function
g $\mathbf{g}$	Gravitational acceleration
G	Gravitational constant
G	Global coordinate frame
Gr	Grashof number
h, h_i	Function
h	Heat
H	Energy, work, heat
H	Height
I	$= mr^2$ Mass moment, second area moment
I_z	Second area moment about the z-axis
Ja	Jacob number
kg	Kilograms, unit of mass
k	Specific thermal conductivity
$k = n - m$	Number of independent nondimensionalized variables
k	Scale coefficient
k_1	Front spring rate
k_2	Rear spring rate
K	Kinetic energy
K	Kutateladze number
Kn	Knudsen number
l	Length, wheelbase
L	Length dimension symbol
$\mathcal{L}$	Lagrangian
Le	Lewis number
m	Meter, unit of length
m	Mass
m	Number of fundamental units
$\dot{m}$	Mass flow rate
M	Mass dimension symbol
M	Moment, torque
M	Mass
M	Model
Ma	Mach number
n	Number of arguments in an equation
N	Newton
Nu	Nusselt number
p	Momentum
p	Force
p	Period
p	Rate of work
p, q	Mathieu equation parameters

$\mathbf{p}$	Vectorial parameter
P	Pressure, stress
P	Potential energy
P	Power
P	Dimensionless force
P	Prototype
Pe	Peclet number
Pr	Prandtl number
q	Generalized coordinate
Q	Heat
Q	Quantity of flow
Q	Moment, torque
Q	Physical quantity
r	Radius, radius of gyration
r	Rank of **A**
r	Yaw rate
$\mathbf{r}$	Position vector
R	Radius
Ra	Rayleigh number
Re	Reynolds number
Ri	Richardson number
Ro	Rossby number
s	Second, unit of time
S	Surface area
S_1	Set of initial variables
Sc	Schmidt number
Se	Senenov number
Sh	Sherwood number
Sr	Strouhal number
St	Stenton number
t	Time
T	Time dimension symbol
T	Temperature
T	Moment, torque
T	Period
Ta	Taylor number
u	Velocity
u	Function
u	Variable
$v \equiv \dot{x}$, $\mathbf{v}$	Velocity, tread velocity in tire print
V	Volume
V	Velocity
$w(x)$	Distributed load of
W	Weight
We	Weber number

x, y, z	Coordinate axes
x_i	Dimensional variables
X, Y, Z	Global coordinates
X, Y, Z	Dimensionless variables
X_i	Dimensionless variables
x	Displacement
x_1	Vertical movement of the front of a vehicle
x_2	Vertical movement of the rear of a vehicle
y	Function
y	Deflection
y_1	Road input to the front wheel
y_2	Road input to the rear wheel
$\propto$	In
$\mathbb{N}$	Integer numbers
$\mathbb{R}$	Real numbers
α, β	Angle
α	$= r^2/a_1 a_2$ Nondimensional parameter
β	$= l^2/a_1 a_2$ Nondimensional parameter
$\alpha, \beta, \gamma, \cdots$	Integer exponents
Γ	Yaw rate
γ	Rate to front length ratio
δ	Steer angle
δ_f	Steer angle of front wheels
ε	Strain
$\varepsilon = \frac{a_1}{l}$	Length ratio
η	Kinematic viscosity
θ	Angle
θ	Pitch angle
μ	Friction coefficient
μ	Viscosity
ν	Frequency
ν	Oscillation period
ν	Kinematic viscosity
π	$3.14159265359\cdots$
π_i	π_i-number, $i = 1, 2, 3\cdots$
ρ	Density
σ	Pressure, stress
σ	Surface tension
τ	Dimensionless time variable
φ	Angle
ω	Angular velocity

ω	Frequency
ω_1	First natural frequency of the system
ω_2	Second natural frequency of the system
Ω	Angular velocity
Ω_1	$= \sqrt{k_1 \beta / m}$ Nondimensional parameter
Ω_2	$= \sqrt{k_2 \beta / m}$ Nondimensional parameter
Δp	Pressure drop

Exercises

1. Escape velocity.
   Knowing the escape velocity in $\left(\frac{\text{km}}{\text{s}}\right)$ and the diameter in (km) of the Moon and some planets

Planet	Diameter	Escape velocity	Planet	Diameter	Escape velocity
Moon	1737.1	2.4	Jupiter	142984	59.6
Earth	12756	11.2	Saturn	120536	35.5
Mercury	4878	4.3	Uranus	51118	21.8
Venus	12104	10.3	Neptune	49532	23.4
Mars	6794	5.0	Pluto	2370	5.4

   determine the mass of them by knowing that the Earth's information are:

$$R \simeq 6.3781 \times 10^6\,\text{m} \qquad m \simeq 5.972 \times 10^{24}\,\text{kg} \tag{2.586}$$

2. Nondimensionalization of pendulum equation of motion.
   Show that the equation of motion of a simple pendulums

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta \tag{2.587}$$

   can be transformed to a simpler form

$$\frac{d^2\theta}{d\tau^2} = -\sin\theta \tag{2.588}$$

   by a proper time scale.
3. Change of variable.
   Show that if the applied force on a particle is

$$F = f(x)\,\dot{x}^n + g(x)\,\dot{x}^2, \tag{2.589}$$

   then the equation of motion becomes linear by $\dot{x}^{2-n} = y$.
4. Forced Duffing equation.
   (a) The forces Duffing equation is:

$$\ddot{x} + k\dot{x} + x^3 = B\cos t \tag{2.590}$$

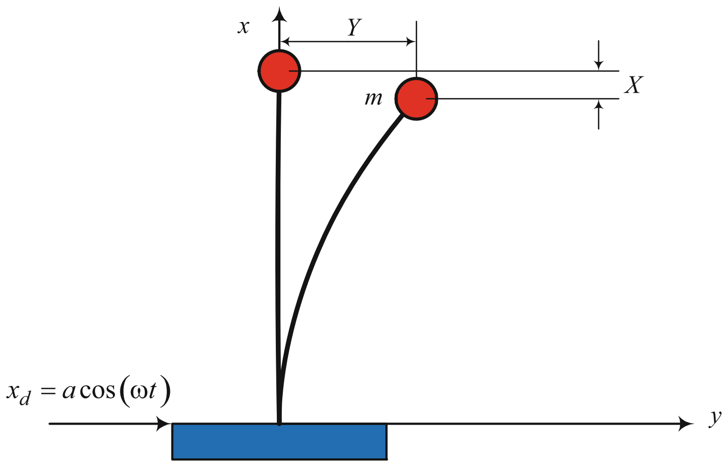

Fig. 2.18 A cantilever with end mass and harmonic base excitation

(b) Check the equation to be dimensionally homogenous and find the missing coefficients.
(c) Make the equation nondimensionalized. To have a characteristic value for x, you may use the steady state x_s when t and all derivatives are zero.

$$x_s = \sqrt[3]{B} \tag{2.591}$$

For the time variable, you may use the frequency constant excitation frequency $\omega = 1$ which you are supposed to found in part a.

5. A cantilever with end mass and harmonic base excitation.

(a) Show that the equation of motion of the cantilever beam of Fig. 2.18 with end mass and horizontally harmonic base excitation is: $\left(1 + B^2Y^2\right)\ddot{Y} + B^2Y\dot{Y}^2 + pY - \ddot{x}_d = 0$
(b) Introducing

$$\tau = \omega t \qquad u = BY \tag{2.592}$$

show that the nondimensionalized form of the equation is:

$$\left(1 + u^2\right)u'' + uu'^2 + pu + q\cos\tau = 0 \tag{2.593}$$

where

$$p = \frac{1}{\omega^2}\left(\frac{EI}{m}A - gB\right) \qquad q = Ba \tag{2.594}$$

6. The projectile problem.
The equation of motion of a projectile of mass m shooting in (x, z)-plane from the origin of a global codominant is

$$m\frac{d\mathbf{v}}{dt} = m\mathbf{g} \qquad \mathbf{g} = -g\hat{k} \tag{2.595}$$

$$g = 9.80665\,\mathrm{m/\,s^2} \approx 9.81\,\mathrm{m/\,s^2} \tag{2.596}$$

An integral of the equation of motion determines the velocity $\mathbf{v}$ of the projectile:

$$\mathbf{v} = -gt\hat{k} + v_0\left(\cos\theta\;\hat{\imath} + \sin\theta\;\hat{k}\right) \tag{2.597}$$

Substituting $\mathbf{v} = d\mathbf{r}/dt$ and integrating provide the position of the projectile as a function of time:

$$\mathbf{r} = -\frac{1}{2}gt^2\hat{k} + v_0\left(\cos\theta\;\hat{\imath} + \sin\theta\;\hat{k}\right)t \tag{2.598}$$

(a) Make the equation of motion nondimensionalized and express the equation in terms of π-groups.
(b) Solve the nondimensionalized to find the position as a function of time.
(c) Determine the path of motion.
(d) Determine the range and height of the projectile as functions of θ.
(e) Derive the equation of envelope parabola that the projectile will touch by changing θ. For the dimensional case, the parabola is:

$$z = \frac{1}{2}\left(\frac{v_0^2}{g} - \frac{g}{v_0^2}x^2\right) \tag{2.599}$$

7. Projectile in air.
Assuming a flat ground with a uniform gravitational attraction $\mathbf{g}$, the equation of motion of a projectile in the air is:

$$m\frac{d\mathbf{v}}{dt} = -mg\hat{k} - c\mathbf{v} \qquad \mathbf{g} = -g\hat{k} \tag{2.600}$$

$$g = 9.80665\,\mathrm{m/\,s^2} \approx 9.81\,\mathrm{m/\,s^2} \tag{2.601}$$

where a mass m is thrown with an initial velocity $\mathbf{v}_0$ from the origin of the coordinate frame. The air provides a resistance force $-c\mathbf{v}$ proportional to the instantaneous velocity $\mathbf{v}$.

(a) Nondimensionalize the equation of motion and determine the π-terms.
(b) Solve the equation to find the dimensionless velocity. The solution of the dimensional equation is:

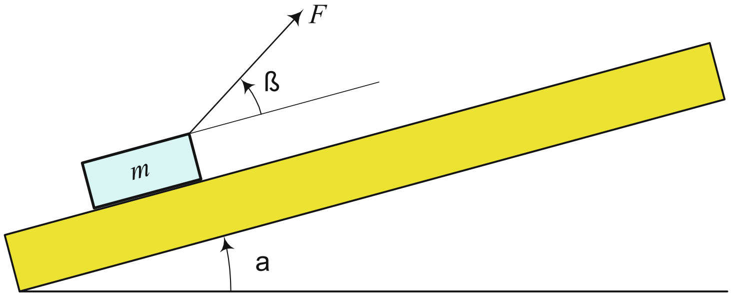

Fig. 2.19 A box of mass m is being pulled by a thread up an inclined plane at an angle β with respect to the plane that is forming an angle α with the horizontal

$$\mathbf{v} = -gt\hat{k} - \frac{c}{m}\mathbf{r} + \mathbf{v}_0 \tag{2.602}$$

(c) Solve the velocity equation to find the dimensionless position. The solution of the dimensional equation is:

$$\mathbf{r} = \frac{m}{c}\mathbf{v}_0\left(1 - e^{-ct/m}\right) + \frac{m^2}{c^2}g\left(1 - e^{-ct/m} - \frac{c}{m}t\right)\hat{k} \tag{2.603}$$

after calculating the constant of integration using the initial conditions of $\mathbf{r} = 0$ at $t = 0$ and simplification.

(d) Determine the maximum height of the projectile. The solution of the dimensional equation is:

$$\begin{aligned} H = z_M = \mathbf{r}\cdot\hat{k} &= \frac{m}{c}\left(\mathbf{v}_0\cdot\hat{k} - gt_H\right) \\ &= \frac{m}{c}\left(\mathbf{v}_0\cdot\hat{k} - \frac{m}{c}g\ln\left(1 + \frac{c}{m}\frac{1}{g}\mathbf{v}_0\cdot\hat{k}\right)\right) \end{aligned} \tag{2.604}$$

a. Determine the range of the projectile. The solution of the dimensional equation is:

$$R = \frac{m}{c}\left(1 - e^{-ct_R/m}\right)\left(\mathbf{v}_0\cdot\hat{\imath}\right) \tag{2.605}$$

8. Minimum tension in pulling thread.
Figure 2.19 illustrates a box of mass m being pulled by a thread up an inclined plane forming an angle α with the horizontal. The coefficient of friction is equal to μ.

(a) Draw the free-body diagram of the system and derive the nondimensionalized equations of equilibrium.

(b) Find the angle β which the thread makes by the inclined plane for the tension of the thread to be minimum when m is moving up with constant speed v.
(c) Find the angle β for the tension of the thread to be minimum when m is moving down with constant speed v.
(d) Find the angle β for the tension of the thread to be minimum when m is moving up with constant acceleration a.

9. Divergence of scale times a vector.
Knowing that the divergence of a vector V is:

$$\operatorname{div} V = \nabla \cdot V = \frac{\partial V}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial V}{\partial y} \tag{2.606}$$

and

$$\operatorname{div}(cV) = c \operatorname{div} V + \nabla c \cdot V \tag{2.607}$$

where c is a scalar function and,

$$V = P(x, y, z)\,\hat{\imath} + P(x, y, z)\,\hat{\jmath} + P(x, y, z)\,\hat{k} \tag{2.608}$$

$$\nabla c = \frac{\partial c}{\partial x}\hat{\imath} + \frac{\partial c}{\partial y}\hat{\jmath} + \frac{\partial c}{\partial y}\hat{k} \tag{2.609}$$

(a) Expand Eq. (2.607) and check the dimensional homogeneity of the equation.
(b) Use a length dimensional characteristic l and make the equation nondimensionalized. How many π-groups are involved?
(c) Expand and make the following equation nondimensionalized:

$$\operatorname{curl}(cV) = c \operatorname{curl} V - \nabla c \cdot V \tag{2.610}$$

10. Nondimensionalization of famous PDE of science.
Most of phenomena in science and engineering are expressed by the following equations.
The Laplace equation

$$\nabla V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial y^2} = 0 \tag{2.611}$$

satisfies the gravitational potential, electrostatic potential in a uniform dielectric, magnetic potential, electric potential, temperature in thermal equilibrium of solids, velocity potential of homogeneous liquid moving irrotationally, and most potential fields.
The wave equation

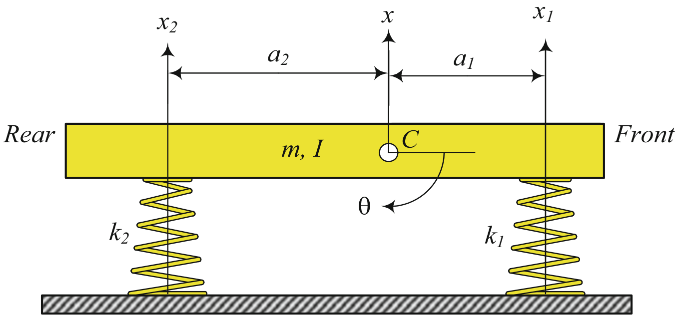

Fig. 2.20 The bounce and pitch model of a car is a beam with mass m and mass moment I, sitting on two springs k_1 and k_2

$$\nabla V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} \tag{2.612}$$

satisfies the mechanical, electric, sound waves, the velocity potential in an ideal gas, and most wave phenomena. The constant c is the speed of the wave in the phenomenon.

The heat conduction equation

$$\nabla V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial y^2} = \frac{1}{k}\frac{\partial V}{\partial t} \tag{2.613}$$

satisfies the temperature distribution of a homogeneous isotropic solid, and most distribution phenomena. The constant c is proportional to the heat conductivity of the body and inversely proportional to its specific heat and density.

The telegraphy equation

$$LC\frac{\partial^2 V}{\partial t^2} + RC\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2} \tag{2.614}$$

satisfies the potential in a telegraph cable where the inductance L, the capacity C, and the resistance R per unit length are given.

(a) Show that these partial differential equations are dimensionally homogenous.
(b) Derived the nondimensionalized form of the equations and list the parameters of the dimensional and nondimensional expression of each equation.

11. Uncoupling the pitch motion of bicycle vehicle model.

(a) Figure 2.20 illustrates a beam model of vehicles as a two degree-of-freedom vibrating system. The beam with mass m and mass moment I about the mass center C is sitting on two springs k_1 and k_2 to model a vehicle in bounce and pitch motions. The translational coordinate x at C and the rotational coordinate θ are the usual generalized coordinates using to measure the kinematics of the beam (Marzbani and Jazar 2014). The equations of free vibration of the system are:

$$\begin{bmatrix} m & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & a_2 k_2 - a_1 k_1 \\ a_2 k_2 - a_1 k_1 & a_2^2 k_2 + a_1^2 k_1 \end{bmatrix} \begin{bmatrix} x \\ \theta \end{bmatrix} = 0 \tag{2.615}$$

(b) Show that employing the coordinates x_1 and x_2 instead of x and θ makes the equations of motion of the system to be:

$$\begin{bmatrix} \dfrac{m a_2^2 + I}{a_1 + a_2{}^2} & \dfrac{m a_1 a_2 - I}{a_1 + a_2{}^2} \\ \dfrac{m a_1 a_2 - I}{a_1 + a_2{}^2} & \dfrac{m a_1^2 + I}{a_1 + a_2{}^2} \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \tag{2.616}$$

(c) Use the following parameters:

$$\begin{aligned} I &= m r^2 \qquad \Omega_1^2 = \frac{k_1}{m}\beta \qquad \Omega_2^2 = \frac{k_2}{m}\beta \qquad \beta = \frac{l^2}{a_1 a_2} \\ \alpha &= \frac{r^2}{a_1 a_2} \qquad \gamma = \frac{a_2}{a_1} \qquad l = a_1 + a_2 \end{aligned} \tag{2.617}$$

and rewrite the equations to be:

$$\begin{bmatrix} \alpha + \gamma & 1 - \alpha \\ 1 - \alpha & \alpha + \frac{1}{\gamma} \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} \Omega_1^2 & 0 \\ 0 & \Omega_2^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \tag{2.618}$$

(d) Show that setting

$$\alpha = 1 \tag{2.619}$$

makes the equations decoupled

$$\begin{bmatrix} \alpha + \gamma & 0 \\ 0 & \alpha + \frac{1}{\gamma} \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} \Omega_1^2 & 0 \\ 0 & \Omega_2^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \tag{2.620}$$

(e) The equations in part b are not fully dimensionless as the variables have not been changes. Define a characteristic time and displacement parameters and make the equations of motion fully nondimensionalized.

(f) Define a new set of parameters to make Eq. (2.615) decoupled.

12. Differential models in ecology.
Ecology studies the interaction of living organisms with the environment. The basic object in ecology is the evolution of populations. Let $x(t)$ be the number of individuals in a population at time t. If A is the number of individuals in the population that are born per unit time and B the number of individuals that die off per unit time, then the rate at which x varies in time is given by

$$\frac{dx}{dt} = A - B \tag{2.621}$$

The problem is to find the dependence of A and B on x. The simplest equation is to assume that the rates A and B are proportional to the population x,

$$A = aX \qquad B = bx \tag{2.622}$$

where a and b are the coefficients of births and deaths of individuals per unit time, respectively. Therefore,

$$\frac{dx}{dt} = (a - b)\,x \tag{2.623}$$

Assuming that at $t = t_0$ the number of individuals in the population is $x = x_0$,

(a) make the differential equation nondimensionalized.
(b) solve the differential equation.
(c) determine the population for $t \to \infty$ based on the ratio of a and b.
(d) All models describing population process are of the form

$$\frac{dx}{dt} = f(x) \tag{2.624}$$

where $f(x)$ is a nonlinear function. Assume

$$\frac{dx}{dt} = ax + bx^2 \tag{2.625}$$

and redo parts (a) to (c).

13. Piecewise linear vibration isolator.
The simplest practical model of a piecewise linear vibration isolator is a system with bilinear stiffness and dumping characteristics as is shown in Fig. 2.21. The first spring and damper that are directly attached to m is called "primary suspension" and the second stage, which is effective beyond the clearance amplitude Δ, is called "secondary suspension." The clearance Δ generates a switch to engage the secondary suspension. This represents a sudden change in the system properties which provides a hard nonlinearity of the piecewise linear system (Pogorilyi et al. 2014, 2015a,b).

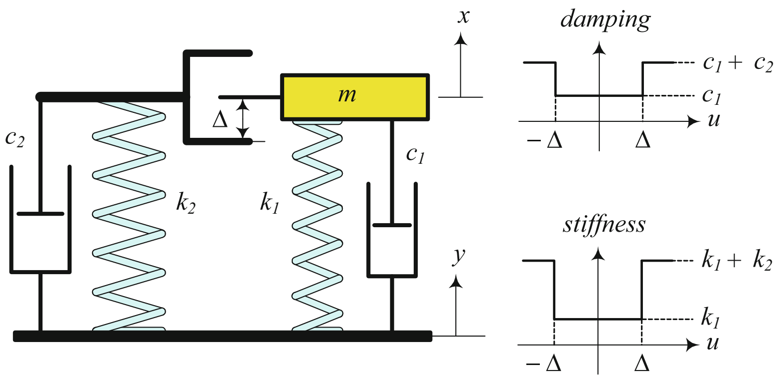

Fig. 2.21 Mechanical model of a piecewise linear system with symmetric constraints

(a) Show that the equations of motion of the system may be written as

$$m\ddot{x} + g_1(x, \dot{x}) = f_1(y, \dot{y}) \tag{2.626}$$

where $g_1(x, \dot{x})$ and $f_1(y, \dot{y})$ are the piecewise linear functions presenting sudden changing characteristics of the system and sudden changing excitation, respectively:

$$g_1(x, \dot{x}) = \begin{cases} (c_1 + c_2)\dot{x} + (k_1 + k_2)x - k_2\Delta & x - y > \Delta \\ c_1\dot{x} + k_1 x & |x - y| < \Delta \\ (c_1 + c_2)\dot{x} + (k_1 + k_2)x + k_2\Delta & x - y < -\Delta \end{cases} \tag{2.627}$$

$$f_1(y, \dot{y}) = \begin{cases} (c_1 + c_2)\dot{y} + (k_1 + k_2)y & x - y > \Delta \\ c_1\dot{y} + k_1 y & |x - y| < \Delta \\ (c_1 + c_2)\dot{y} + (k_1 + k_2)y & x - y < -\Delta \end{cases} \tag{2.628}$$

(b) Show that the equation of motion for the system may also be written in a nondimensional form (Deshpande et al. 2006; Jazar et al. 2007).

$$\begin{cases} \ddot{z} + 2\xi_2\omega_2\dot{z} + \omega_2^2 z = w\omega^2 \sin(\omega t - \varphi) + \omega_3^2 & z > 1 \\ \ddot{z} + 2\xi_1\omega_1\dot{z} + \omega_1^2 z = w\omega^2 \sin(\omega t - \varphi) & |z| < 1 \\ \ddot{z} + 2\xi_2\omega_2\dot{z} + \omega_2^2 z = w\omega^2 \sin(\omega t - \varphi) - \omega_3^2 & z < -1 \end{cases} \tag{2.629}$$

where

$$z = \frac{u}{\Delta} \qquad w = \frac{Y}{\Delta} \qquad u = x - y$$

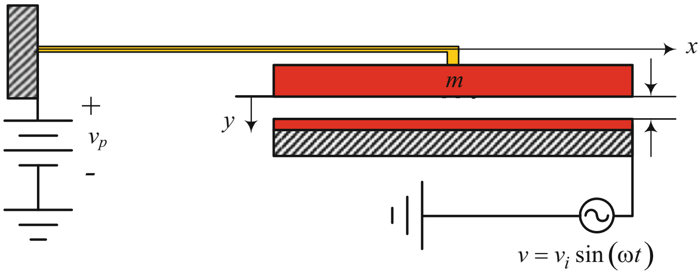

Fig. 2.22 A microcantilever model of microresonators

$$\omega_1^2 = \frac{k_1}{m} \qquad \omega_2^2 = \frac{k_1 + k_2}{m} \tag{2.630}$$
$$\xi_1 = \frac{c_1}{2\sqrt{k_1 m}} \qquad \xi_2 = \frac{c_1 + c_2}{2\sqrt{(k_1 + k_2)m}}$$

(c) Make the time to be nondimensionalized and determine the π-groups of the system.

14. ★Electrostatic force in microresonators.
Figure 2.22 illustrates a microresonator (Younis and Nayfeh 2003; Jazar 2012a,b).

(a) One dimensional electrostatic force, f_e, per unit length of the beam of a microresonator is:

$$f_e = \frac{\varepsilon_0 A \left(v - v_p\right)^2}{2\left(d - w\right)^2} \qquad v = v_i \sin \omega t \tag{2.631}$$

where $\varepsilon_0 = 8.854187817620 \times 10^{-12}\,\mathrm{A}^2\,\mathrm{s}^4/\,\mathrm{kg\,m}^3$ is permittivity in vacuum, A is the area of the microplate, and $w = w(x, t)$ is the lateral displacement of the microbeam. The complete microresonator is composed of a beam resonator, a ground plane underneath, and one (or more) capacitive transducer electrodes. A DC-bias voltage, v_p , is applied to the resonator, while an AC excitation voltage is applied to its underlying ground planes. Check the dimensional homogeneity of the force and make it nondimensionalized.

(b) Including mathematical modeling of squeeze-film effects, the following equation expresses the dynamic behavior of the microresonator behavior:

$$\rho \frac{\partial^2 w}{\partial t^2} + EI \frac{\partial^4 w}{\partial x^4} + c_s w^2 \frac{\partial w}{\partial t} + k_s \left(d - w\right)^2 w \left(\frac{\partial w}{\partial t}\right)^2$$

$$= \frac{Ea_0}{L}\left(\frac{1}{2}\int_0^L \left(\frac{\partial w}{\partial x}\right)^2 dx\right)\frac{\partial^2 w}{\partial x^2} + \frac{\varepsilon_0 a\left(v - v_p\right)^2}{2\left(d - w\right)^2} \tag{2.632}$$

where the parameters involved are: beam length L, width b, thickness h, initial gap d, material Young's modulus E, density $\rho/(hb)$, air viscosity μ, moment of inertia $I = bh^3/12$, Kundsen's number $K = \kappa/w$, and the ambient air pressure P_a, the mean-free path of the gas κ, the size and mass of the microplate a and mass m.
Show that the following variables

$$\tau = \omega_1 t \qquad \omega_1 = \frac{n^2}{L^2}\sqrt{\frac{EI}{\rho}} \qquad z = \frac{x}{L} \qquad y = \frac{w}{d}$$
$$Y = \frac{w_0}{d} \qquad r_x = \frac{\omega}{\omega_x} \qquad r = \frac{\omega}{\omega_1} \qquad a_1 = \frac{\varepsilon_0 a L^4}{2d^3 EI}$$
$$a_3 = \frac{a_0 d^2}{I} \qquad a_4 = \frac{c_s d^2 L^2}{\sqrt{\rho EI}} \qquad a_5 = \frac{k_s d^4}{\rho} \tag{2.633}$$

make the equation of motion dimensionless (Jazar 2006, Jazar et al. 2009).

$$\frac{\partial^2 y}{\partial \tau^2} + \frac{\partial^4 y}{\partial z^4} + a_4 y^2 \frac{\partial y}{\partial \tau} + a_5 \left(1 - y\right)^2 y \left(\frac{\partial y}{\partial \tau}\right)^2$$
$$= a_3 \frac{\partial^2 y}{\partial z^2}\frac{1}{2}\int_0^1 \left(\frac{\partial y}{\partial z}\right)^2 dz + a_1 \frac{\left(v - v_p\right)^2}{\left(1 - y\right)^2} \tag{2.634}$$

(c) Apply a separation solution, $y = Y(\tau)\ \varphi(z)$, and assume a first harmonic function as the mode shape of the deflected microbeam. By accepting a first harmonic shape function, the temporal function $Y(\tau)$ represents the maximum deflection of the microbeam at the tip point for microcantilever. Show that the differential equation describing the evolution of the temporal function $Y(\tau)$ would be

$$\ddot{Y} + Y + \lambda Y^3 + a_4 Y^2 \dot{Y} + a_5 \left(1 - Y\right) Y \dot{Y}^2$$
$$= \frac{(\alpha + \beta) + 2\sqrt{2\alpha\beta}\sin(r\tau) - \beta\cos(2r\tau)}{(1 - Y)^2} \tag{2.635}$$

where

$$\alpha = a_1 v_p^2 \qquad 2\sqrt{2\alpha\beta} = 2a_1 v_p v_i \qquad \beta = \frac{a_1}{2}v_i^2$$
$$\lambda = n^2 a_3 \qquad n^2 = \frac{1}{2}\int_0^1 \left(\frac{\partial \varphi}{\partial z}\right)^2 dz \tag{2.636}$$

The first harmonic mode shape to satisfy the required boundary conditions for a microcantilever is:

$$\varphi(z) = 1 - \cos\left(\frac{\pi z}{2}\right) \qquad n = \frac{\pi}{4} \tag{2.637}$$

15. ★The water clock.
Consider a vessel whose horizontal cross section has an area, A, that is a function of the distance from the bottom of the vessel, z.

$$A = A(z) \qquad A_0 = A(0) \tag{2.638}$$

Suppose that initially at time $t = 0$ the level of the liquid in the vessel is at a height of h meters. Also suppose that the area of the opening in the bottom of the vessel is A_0. There is a hole at the bottom of the vessel. The rate v at which the liquid flows out of the vessel at the moment when the liquid's level is at height x is given by

$$v = k\sqrt{2gz} \tag{2.639}$$

where $g = 9.8\,\mathrm{m/\,s^2}$, and k is the rate constant of the outflow. During an infinitesimal time interval dt the outflow of the liquid may be assumed uniform, when a column of liquid with a height of vdt and a cross-sectional area of A_0 will flow out of the vessel. The outflow causes the level of the liquid to change by $-dz$.

$$kA_0\sqrt{2gz} = -Adz \tag{2.640}$$

(a) Make the differential equation nondimensionalized and determine the π-groups.
(b) Assume the vessel to be cylindrical with a vertical axis six meters high and four meters in diameter, having a constant circular cross section. The radius of the cylinder is $1/12\,\mathrm{m}$. Find how the level of water in the vessel depends on time t and the time it takes all the water to flow out. Use $k = 0.6$ for water.
(c) Assume the vessel is a circular bowl with radius R,

$$R = R(z) \tag{2.641}$$

and you are supposed to determine the shape of the water clock that would ensure the water level lowered at a constant rate. Such clocks were used in ancient Greek and Roman courts to time the lawyers' speeches.

16. A symmetric piecewise linear vibration isolator.
The symmetric piecewise linear vibration isolator is a restricted model of piecewise linear isolators. A symmetric system may easily become asymmetric

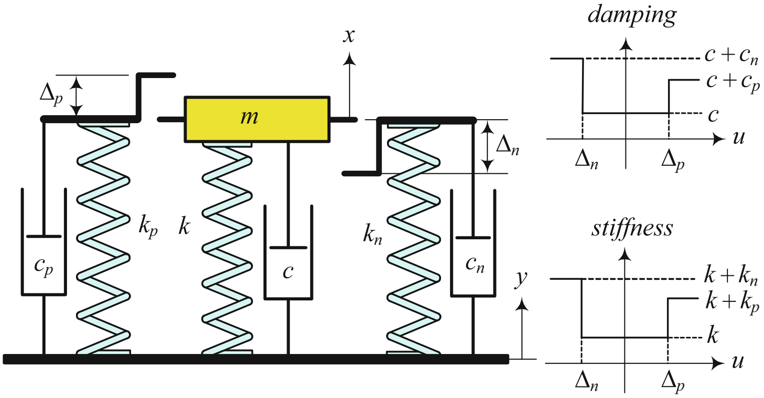

Fig. 2.23 The mathematical model of an asymmetric piecewise linear vibration isolator

in practice. Any change of mass will make the positive and negative gap sizes unequal. The spring may behave different in compression and extension. Similarly, real dampers resist differently in bound and rebound. Figure 2.23 illustrates the mathematical model of an asymmetric piecewise linear vibration isolator (Narimani et al. 2004a,b).

The equations of motion of the system could be written as

$$m\ddot{x} + g_1(x, \dot{x}) = f_1(y, \dot{y}) \tag{2.642}$$

where $g_1(x, \dot{x})$ and $f_1(y, \dot{y})$ are piecewise linear functions presenting nonlinear characteristics of the system and nonlinear excitation respectively:

$$g_1(x, \dot{x}) = \begin{cases} (c + c_p)\dot{x} + (k + k_p)x - k_p\Delta_p & x - y > \Delta_p \\ c\dot{x} + kx & \Delta_n < |x - y| < \Delta_p \\ (c + c_n)\dot{x} + (k + k_n)x + k_n\Delta_n & x - y < -\Delta_n \end{cases} \tag{2.643}$$

$$f_1(y, \dot{y}) = \begin{cases} (c + c_p)\dot{y} + (k + k_p)y & x - y > \Delta_p \\ c\dot{y} + ky & \Delta_n < |x - y| < \Delta_p \\ (c + c_n)\dot{y} + (k + k_n)y & x - y < -\Delta_n \end{cases} \tag{2.644}$$

(a) Define a set of proper characteristic parameters for all variables.
(b) Make the equations nondimensionalized and determine the π-groups of the system.

17. ★Coffee+cream and wait versus coffee and wait plus cream.
Miss A and Miss B ordered coffee and cream in a coffee shop. They were given same coffee in similar cups both simultaneously and proceeded as follows. Miss A poured some of her cream into the coffee, covered the cup with a paper napkin, and waited 10 min. Miss B covered her cup with a napkin and poured the same amount of cream into her coffee after 10 min.
The following equations might be helpful in analytic analysis of the problem. Heat H transferred to the air from a cup is:

$$dH = \eta \frac{T - T_s}{h} A\, dt \tag{2.645}$$

where T is the coffee's temperature at time t, T_s is the room temperature in coffee shop, η is the thermal conductivity of the material of the cup, h is the thickness of the cup, A the area of the cup's lateral surface. The amount of heat given off by the coffee is

$$dH = -cmdT \tag{2.646}$$

where c is the specific heat capacity of the coffee and m the mass of coffee in the cup. The heat balance equation for adding cream to coffee is:

$$cm\,(T_1 - T_2) = c_c m_c\,(T_2 - T_c) \tag{2.647}$$

where T_1 and T_2 is the temperature of coffee before and after adding cream, respectively, T_c is the temperature of cream, c_c is the specific heat capacity of the cream, and m_c is the mass of the added cream.

(a) Whose was hotter when they started drinking after those 10 min?
(b) Make the problem nondimensionalized, so the result is applied to all similar situation, or determine how the result is dependent on π-groups.

18. Heat conduction equation.
The heat conduction equation is governed by

$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T \qquad \kappa = \frac{K}{\rho c_p} \tag{2.648}$$

where T is the temperature and κ is the thermal diffusivity, K is the thermal conductivity, ρ is the density and c_p is the specific heat capacity.
Nondimensionalized the equation by defining a characteristic time and length.

19. ★Time of impact for two masses under gravitational force.
Two masses $m_1$and m_2 are released from rest when they are x meter apart. The masses will start moving by the gravitational force.

$$F = G\frac{m_1 m_2}{r^2} \tag{2.649}$$

Can you derive an equation to determine the time $m_1$and m_2 collide? Nondimensionalize this analysis.

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Part II
Continued Fractions

Continued fractions is a part of mathematics that is usually eliminated from mathematical courses in science and engineering. It seems to be too elementary for college level and advanced for high school and have limited applicability. Theoretically, continued fractions provide better and quicker convergence to the exact value of an irrational number, a series expansion of a function, and the approximate series solution of differential equations. By better we mean more exact, and by quicker we mean less number of terms and less calculations. These advantages become practical when a series expansion works only in a very small domain or does not approach to the desired limit. In this part we introduce the concept of continued fraction and review its applicability in numerical estimation where other methods are not exact enough. Then we review the direct and indirect methods to derive the continued fraction approximation for functions and equations. The most practical method is to derive the series expansion of the desired solution and convert it to continued fractions. We will see that the continued fractions work much more accurate than the series expansion.

Chapter 3
Numerical Continued Fractions

In this chapter we review the method of continued fractions to show its advantage and application. Convergence and its usefulness in working with irrational numbers, as well as converting a numerical series to continued fraction back and forth are the topics in this chapter. This review makes the reader ready to derive and work with solution of differential equations in continued fractions.

3.1 Rational and Irrational Numbers

Any *rational* or *irrational* number x may be expressed by a continued fraction of the form

$$x = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cdots}}} \tag{3.1}$$

that we may show it in a more compact form:

$$x = b_0 + \frac{a_1}{b_1+}\,\frac{a_2}{b_2+}\,\frac{a_3}{b_3+}\cdots \tag{3.2}$$

The sequence of truncations p_k/q_k

$$\frac{p_0}{q_0} = \frac{b_0}{1} \quad \frac{p_1}{q_1} = b_0 + \frac{a_1}{b_1} \quad \frac{p_2}{q_2} = b_0 + \frac{a_1}{b_1+}\,\frac{a_2}{b_2}, \ldots,$$
$$\frac{p_k}{q_k} = b_0 + \frac{a_1}{b_1+}\,\frac{a_2}{b_2+}\,\frac{a_3}{b_3+}\cdots\frac{a_k}{b_k} \tag{3.3}$$

are called the first, second, third, ... *convergents*, respectively.

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If the limit c_k exists,

$$c_k = \lim_{k \to \infty} \frac{p_k}{q_k} \tag{3.4}$$

the associated infinite continued fraction is convergent. Finite continued fractions are always convergent. If the infinite continued fraction converges, all of its remainders converge. Conversely, if at least one of the remainders of a continued fraction diverges, the continued fraction itself diverges (Khinchin 1997).

In general, the numbers a_1, a_2, a_3, ... and b_1, b_2, b_3, ... may be any real or complex numbers, and the number of terms may be finite

$$x = b_0 + \frac{a_1}{b_1+} \frac{a_1}{b_2+} \frac{a_3}{b_3+} \cdots \frac{a_n}{b_n} \tag{3.5}$$

or infinite.

$$x = b_0 + \frac{a_1}{b_1+} \frac{a_2}{b_2+} \frac{a_3}{b_3+} \cdots \frac{a_k}{b_k+} \cdots \tag{3.6}$$

If every a_i is 1 the expression is called a *simple continued fraction*.

$$x = b_0 + \frac{1}{b_1+} \frac{1}{b_2+} \frac{1}{b_3+} \cdots \tag{3.7}$$

The simple continued fraction may also be shown by:

$$x = [b_0, b_1, b_2, b_3, \ldots] \tag{3.8}$$

The numbers b_l, b_2, b_3, ... are called the partial quotients or *quotients*.

In this section we review two tasks: how to find the continued fraction for a given number, and how to find the number for a given continued fraction.

All real numbers are *rational* or *irrational*. Integers and fractions, such as 3, −4.3, 9/4, are *rational* numbers and all other numbers, such as $\sqrt{2}$, e, π, are *irrational*. A rational number is a fraction of the form p/q where p and q are integers with $q \neq 0$. Irrational numbers are either algebraic or transcendental. Algebraic irrational numbers are all non-integral real roots of the algebraic equation with rational coefficients a_i, $(i = 1, 2, \ldots, n)$.

$$x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0 = 0 \tag{3.9}$$

For example, the irrational numbers $x = \sqrt{3}$ and $x = \sqrt[3]{8}$ are roots of $x^5 - 3x^3 - 8x^2 + 24 = 0$. All other irrational numbers are called transcendental that cannot be roots of an algebraic equation with rational coefficients (Yanpolskii and Lyusternik 1965). Both, rational and irrational numbers may be expressed by finite or infinite continued fractions, respectively (Silverman 2011).

Proof Any finite continued fraction represents a rational number, and conversely, any rational number p/q can be represented as a finite continued fraction. To prove this, we may note that for any finite expansion we can always back track and build the expansion into a rational fraction. To prove the converse, we consider p/q, $q > 0$ and divide p by q to obtain

$$\frac{p}{q} = a_1 + \frac{R_1}{q} \tag{3.10}$$

where a_1 is the unique integer to make the remainder $0 < R_1 < q$. The quotient a_1 may be negative, zero, or positive. If the remainder is zero, $R_1 = 0$, the process terminates with the unique result that the continued fraction expansion for p/q is a_1. If the remainder $R_1 \neq 0$, then we rewrite (3.10) as

$$\frac{p}{q} = a_1 + \frac{1}{\frac{q}{R_1}} \qquad 0 < R_1 < q \tag{3.11}$$

and continue the process by dividing q by R_1.

$$\frac{q}{R_1} = a_2 + \frac{R_2}{R_1} \qquad 0 \leq R_2 < R_1 \tag{3.12}$$

The quotient a_2 is the unique integer that makes the remainder $0 < R_2 < R_1$. If the remainder $R_2 = 0$, the process terminates with the unique result that the continued fraction expansion for p/q is $a_1 + \frac{1}{a_2}$. If the remainder $R_2 \neq 0$, then we continue the process

$$\frac{q}{R_1} = a_2 + \frac{1}{\frac{R_1}{R_2}} \qquad 0 \leq R_2 < R_1 \tag{3.13}$$

and find the quotient and remainder for R_1/R_2 and keep repeating the process until a remainder $R_n = 0$. This situation will happen because the sequence of remainders getting smaller.

$$q > R_1 > R_2 > R_3 > \cdots \tag{3.14}$$

■

Example 90 Expansion of rational and irrational numbers to continued fractions.

Let us expand the rational number 96/29 into continued fraction.

$$\frac{96}{29} = 3 + \frac{9}{29} = 3 + \frac{1}{\frac{29}{9}} = 3 + \frac{1}{3 + \frac{2}{9}}$$

$$= 3 + \cfrac{1}{3 + \cfrac{1}{\cfrac{9}{2}}} = 3 + \cfrac{1}{3 + \cfrac{1}{\cfrac{9}{2}}} = 3 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{2}}} \tag{3.15}$$

The continued fraction will end up at a point and it shows the number 96/29 is rational.

Now let us expand an irrational number such as $\sqrt{2}$ to continued fractions.

$$\sqrt{2} = 1 + \sqrt{2} - 1 = 1 + \frac{\left(\sqrt{2}-1\right)\left(\sqrt{2}+1\right)}{\sqrt{2}+1} = 1 + \frac{1}{1+\sqrt{2}}$$
$$= 1 + \cfrac{1}{1 + 1 + \cfrac{1}{1+\sqrt{2}}} = 1 + \cfrac{1}{2 + \cfrac{1}{1+\sqrt{2}}}$$
$$= \cdots = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ddots}}} = 1 + \frac{1}{2+}\frac{1}{2+}\cdots \tag{3.16}$$

The continued fraction will not end and it shows that $\sqrt{2}$ is irrational. If the irrational number is given as a known decimal number such as $\sqrt{2} = 1.414\cdots$, then there are classical methods, such as flip method, to derive their continued fractions. However, if the irrational number is given by a symbol such as $\sqrt{2}$ or π, then there is no standard way to find their continued fractions. The easiest way would be developing an equation in which the symbol also appears in the denominator of a fraction on the right-hand side.

Example 91 Continued fraction individual equations.

For any real number x, the system of equations

$$x = b_0 + a_1 u_1 \qquad 0 \le u_1 \le 1 \tag{3.17}$$
$$\frac{1}{u_1} = b_1 + a_2 u_2 \qquad 0 \le u_2 \le 1 \tag{3.18}$$
$$\frac{1}{u_2} = b_2 + a_3 u_3 \qquad 0 \le u_3 \le 1 \tag{3.19}$$
$$\frac{1}{u_3} = b_3 + a_4 u_4 \qquad 0 \le u_4 \le 1 \tag{3.20}$$
$$\vdots$$

with $a_i, b_i, (i = 1, 2, 3, \ldots)$ to be integers, is known as the continued fraction algorithm to represent x (Battin 1999).

$$x = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + a_4 u_4}}} \tag{3.21}$$

Example 92 Continued fractions notations.

To show the continued fractions in a line in a compact form, several authors have proposed different methods (Lyusternik and Yanupolskii 1965). We use the method that Rogers (1893) proposed,

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots}}} = b_0 + \frac{a_1}{b_1+}\,\frac{a_2}{b_2+}\,\frac{a_3}{b_3+} + \cdots \tag{3.22}$$

However, the method of Pringsheim (1898)

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots}}} = b_0 + \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \frac{a_3|}{|b_3} + \cdots \tag{3.23}$$

and method of Müller (1920)

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots}}} = b_0 \dot{+} \frac{a_1}{b_1} \dot{+} \frac{a_2}{b_2} \dot{+} \frac{a_3}{b_3} + \cdots \tag{3.24}$$

are also alternative ways to represent continued fractions. The fraction a_n/b_n is the nth partial quotient of the continued fraction (3.22). The a_k and b_k are the coefficients of the continued fraction; a_1, a_2, a_3, ... are their partial numerators and b_1, b_2, b_3, ... are its partial denominators. We assume that all coefficients of a continued fraction are finite; all partial denominators of a continued fraction are nonzero.

Another compact notation for continued fractions is also introduced by Alfred Pringsheim (1850–1941).

$$x = \left[b_0; \frac{a_k}{b_k} \right]_1^{\infty} \tag{3.25}$$

This notation is similar to $y = \sum_{k=1}^{\infty} a_k$ for continued summation $y = a_1 + a_2 + a_3 + \cdots$ and suitable to express a function.

The notation p_k/q_k is the kth convergent of the continued fraction (3.22).

$$b_0 + \frac{a_1}{b_1+} \frac{a_2}{b_2+} \frac{a_3}{b_3+} \cdots \frac{a_n}{+b_n} = \frac{p_n}{q_n} \tag{3.26}$$

Therefore,

$$\frac{p_0}{q_0} = \frac{b_0}{1} \qquad \frac{p_1}{q_1} = b_0 + \frac{a_1}{b_1} = \frac{b_0 b_1 + a_1}{b_1} \tag{3.27}$$

$$\frac{p_2}{q_2} = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2}} = \frac{b_0 b_1 b_2 + b_0 a_2 + a_1 b_2}{a_2 + b_1 b_2} = \frac{b_2 p_1 + a_2 p_0}{b_2 q_1 + a_2 q_0} \tag{3.28}$$

and

$$p_n = b_n p_{n-1} + a_n p_{n-2} \tag{3.29}$$

$$q_n = b_n q_{n-1} + a_n q_{n-2} \tag{3.30}$$

or in matrix form as

$$\begin{bmatrix} p_n \\ q_n \end{bmatrix} = b_n \begin{bmatrix} p_{n-1} \\ q_{n-1} \end{bmatrix} + a_n \begin{bmatrix} p_{n-2} \\ q_{n-2} \end{bmatrix} \tag{3.31}$$

then

$$\frac{p_n}{q_n} = \frac{b_n p_{n-1} + a_n p_{n-2}}{b_n q_{n-1} + a_n q_{n-2}} \tag{3.32}$$

Whenever possible, we may show a continued fraction by the symbol $K_{k=k_1}^{k=k_2}$ such as

$$f_k = K_{k=1}^{\infty}\left(\frac{a_k}{b_k}\right) = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots}}} \tag{3.33}$$

$$f_n = K_{k=1}^{n}\left(\frac{a_k}{b_k}\right) = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{\cfrac{\ddots}{b_{n-1} + \cfrac{a_n}{b_n}}}}} \tag{3.34}$$

similar to a continued summation notation $\sum_{k=k_1}^{k=k_2} a_k$,

$$g_k = \sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots \tag{3.35}$$

$$g_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_n \tag{3.36}$$

and similar to continued products $\prod_{k=k_1}^{k=k_2} a_k$.

$$h_k = \prod_{k=1}^{\infty} a_k = a_1\ a_2\ a_3\ \cdots \tag{3.37}$$

$$h_n = \prod_{k=1}^{n} a_k = a_1\ a_2\ a_3\ \cdots\ a_n \tag{3.38}$$

Example 93 Continued fraction of $\sqrt{x}$.

Expanding $\sqrt{x}$ into continued fractions indicate how a function can be expressed by continued fractions.

$$y = \sqrt{x} \tag{3.39}$$

$$\begin{aligned} y = \sqrt{x} = 1 + \sqrt{x} - 1 = 1 + \frac{\left(\sqrt{x}-1\right)\left(\sqrt{x}+1\right)}{1+\sqrt{x}} = 1 + \frac{x-1}{1+\sqrt{x}} \\ = 1 + \cfrac{x-1}{1+1+\cfrac{x-1}{1+\sqrt{x}}} = 1 + \cfrac{x-1}{2+\cfrac{x-1}{1+\sqrt{x}}} \\ = 1 + \frac{x-1}{2+}\ \frac{x-1}{2+}\ \frac{x-1}{2+} \cdots \end{aligned} \tag{3.40}$$

Therefore, we are able to find expansion for radicals.

$$\sqrt{2} = 1 + \frac{1}{2+}\ \frac{1}{2+}\ \frac{1}{2+} \cdots \tag{3.41}$$

$$\sqrt{3} = 1 + \frac{2}{2+}\ \frac{2}{2+}\ \frac{2}{2+} \cdots \tag{3.42}$$

$$\sqrt{5} = 1 + \frac{4}{2+}\ \frac{4}{2+}\ \frac{4}{2+} \cdots \tag{3.43}$$

The continued fraction for the square root of a number that has exact value such as $\sqrt{4}$ can also be expressed by an infinite fractions,

$$\sqrt{4} = 1 + \frac{3}{2+}\,\frac{3}{2+}\,\frac{3}{2+}\cdots \tag{3.44}$$

that converging the exact value.

$$\sqrt{4} = 2.5,\ 1.8571,\ 2.05,\ 1.9836,\ 2.0055,\ 1.9982,$$
$$2.0006,\ 1.9998,\ 2.0001,\ 2.0000,\ 2.0 \tag{3.45}$$

Example 94 Flip method expansion to continued fractions.

Consider a positive number c and let $\lfloor c \rfloor$ denote the largest integer of c. Starting with a positive number c_0 we write

$$c_0 = \lfloor c_0 \rfloor + (c_0 - \lfloor c_0 \rfloor) = \lfloor c_0 \rfloor + \cfrac{1}{\cfrac{1}{c_0 - \lfloor c_0 \rfloor}} \tag{3.46}$$

and then,

$$c_1 = \frac{1}{c_0 - \lfloor c_0 \rfloor} = \lfloor c_1 \rfloor + (c_1 - \lfloor c_1 \rfloor) = \lfloor c_1 \rfloor + \cfrac{1}{\cfrac{1}{c_1 - \lfloor c_1 \rfloor}} \tag{3.47}$$

$$c_2 = \frac{1}{c_1 - \lfloor c_1 \rfloor} = \lfloor c_2 \rfloor + (c_2 - \lfloor c_2 \rfloor) = \lfloor c_2 \rfloor + \cfrac{1}{\cfrac{1}{c_2 - \lfloor c_2 \rfloor}} \tag{3.48}$$

$$\ldots$$

We repeat this process until it stops for rational numbers, or continues infinitely for irrational numbers (Lorentzen and Waadeland 2008).

As an example,

$$c_0 = 1.234 = 1 + \cfrac{1}{\cfrac{1}{0.234}} \tag{3.49}$$

$$c_1 = \frac{1}{0.234} = 4.2735 = 4 + \cfrac{1}{\cfrac{1}{0.2735}} \tag{3.50}$$

$$c_2 = \frac{1}{0.2735} = 3.6563 = 3 + \cfrac{1}{\cfrac{1}{0.6563}} \tag{3.51}$$

$$c_3 = \frac{1}{0.6563} = 1.5237 = 1 + \cfrac{1}{\cfrac{1}{0.5237}} \tag{3.52}$$

$$c_4 = \frac{1}{0.5237} = 1.9095 = 1 + \frac{1}{\frac{1}{0.9095}} \tag{3.53}$$

$$c_5 = \frac{1}{0.9095} = 1.0995 = 1 + \frac{1}{\frac{1}{0.0995}} \tag{3.54}$$

$$c_6 = \frac{1}{0.0995} = 10.05 = 10 + \frac{1}{\frac{1}{0.05}} \tag{3.55}$$

$$c_7 = \frac{1}{0.05} = 20 \tag{3.56}$$

therefore,

$$1.2345 = 1 + \frac{1}{4+}\,\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{10+}\,\frac{1}{20} \tag{3.57}$$

Example 95 Euclid's algorithm for greatest common divisor.

Euclid (third century BC) is the first who reported algorithm for finding the greatest common divisor of two integers a and b, with $a > b$. First, divide b into a to get $a = a_1 b + R$, where a_1 is the quotient and R is the remainder. Hence $a/b = a_1 + R/b$, which can also be written as:

$$\frac{a}{b} = a_1 + \frac{1}{\frac{b}{R}} \tag{3.58}$$

Then, divide R into b and we have $b = a_2 R + R_1$, which we can write it as $b/R = a_2 + R_1/R$. Substituting in (3.58) we get:

$$\frac{a}{b} = a_1 + \frac{1}{a_2 + \frac{1}{\frac{R}{R_1}}} \tag{3.59}$$

Continuing the Euclid's algorithm leads to a continued fraction (Kline 1972).

$$\frac{a}{b} = a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ddots}} \tag{3.60}$$

The Euclid's algorithm process also applies when $a < b$, which in that case a_1 would be zero. In case of a and b are integers, the continued fraction terminates showing a/b is rational.

Example 96 ★History of continued fractions.

Euclid's method for finding the greatest common divisor of two numbers is the basic method of converting a fraction into a continued fractions. Indian mathematician Aryabhata, who died around 550AD has touched the continued fractions in his writing. Aryabhata's work contains one of the earliest attempts at the general solution of a linear indeterminate equation by using continued fractions. The modern theory of continued fractions began with the writings of Rafael Bombelli (1526–1572) and Pietro Antonio Cataldi (1548–1626) from Bologna, Italy. Bombelli's treatise on algebra contains a chapter on calculating square roots by continued fractions (Merzbach and Boyer 2011). In our modern symbolism he showed, something such as

$$\sqrt{13} = 3 + \frac{4}{6+}\,\frac{4}{6+}\cdots \tag{3.61}$$

and it shows that he knew

$$\sqrt{a^2+b} = a + \frac{b}{2a+}\,\frac{b}{2a+}\cdots \tag{3.62}$$

Pietro Antonio Cataldi (1548–1626), in a treatise on the theory of roots (1613), expressed $\sqrt{18}$ in the form

$$\sqrt{18} = 4 + \frac{2}{8+}\,\frac{2}{8+}\cdots \tag{3.63}$$

Daniel Schwenter (1585–1636), who was a professor of mathematics at the University of Altdorf, Germany, in his book, Geometria Practica, found approximation of $177/233$ in the form of continued fractions.

The decades of the 1650s and 1660s were the birth years of some infinite methods including the continued fractions method. John Wallis (1616–1703) was the first person to derive an infinite product fractional definition in 1655, using only rational operations to determine π in his book "Arithmetica infinitorum." It was also Wallis who used the term "continued fractions" for the first time.

$$\pi = 2 \times \frac{2\times 2\times 4\times 4\times 6\times 6\times\cdots}{1\times 1\times 3\times 3\times 5\times 5\times\cdots} \tag{3.64}$$

$$\frac{4}{\pi} = \frac{3\times 3\times 5\times 5\times 7\times 7\times 9\times 9\times\cdots}{2\times 4\times 4\times 6\times 6\times 8\times 8\times 10\times\cdots} \tag{3.65}$$

Lord William Viscount Brouncker (1620–1684), the first president of the Royal Society, transformed Wallis' result into the form of a continued fractions (Beckmann 1971).

$$\frac{4}{\pi} = 1 + \frac{1^2}{2+}\,\frac{3^2}{2+}\,\frac{5^2}{2+}\,\frac{7^2}{2+}\,\frac{9^2}{2+}\cdots \tag{3.66}$$

Christiaan Huygens (1629–1695), as reported in his treatise, Descriptio Automati Planetarii (1698), used continued fractions to approximate the design of the toothed wheels of a planetarium. He was the first to demonstrate a practical application of continued fractions. Roger Cotes (1682–1716) in his Philosophical Transactions of the Royal Society of London showed

$$\frac{e+1}{e-1} = 2 + \frac{1}{6+}\,\frac{1}{10+}\,\frac{1}{14+}\cdots \tag{3.67}$$

and proved e and e^2 are irrational. And here he comes, the Swiss mathematician Leonhard Euler (1707–1783), the king of mathematicians who pushed the boundaries of mathematics, science, and engineering in all directions. Euler took up this subject. In his tint paper "De Fractionibus Continuis," he derived several interesting results, such as

$$e - 1 = 1 + \frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{4+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{6+}\cdots \tag{3.68}$$

Euler's best contribution was showing how to go from a series to a continued fraction, and conversely (Euler 1988). Lagrange used continued fractions and found approximations to the irrational roots of equations, and then, he got approximate solutions of differential equations in the form of continued fractions. He showed that continued fractions are much better approximations than series, however, working with continued fractions is not as easy. Differentiation, integration, and even calculation of continued fractions are challenging. That is the main reason behind putting this subject out of the agenda of mathematical subjects in educations.

Example 97 Flip method and approximation of π by continued fraction.

Let us begin with known value of π.

$$\pi = 3.14159265358979323846264433\cdots \tag{3.69}$$

It may be written as

$$\begin{aligned}\pi &= 3 + 0.14159265358979323846264433\cdots \\ &= 3 + \cfrac{1}{\cfrac{1}{0.14159265358979323846264433\cdots}} \\ &= 3 + \frac{1}{7.0625133059310457697930051\cdots} \\ &= 3 + \frac{1}{7 + 0.0625133059310457697930051\cdots}\end{aligned} \tag{3.70}$$

Ignoring the small fraction $0.0625\cdots$, we may have a reasonable and simple approximation for π with less than 0.2% error.

$$\pi \simeq 3 + \frac{1}{7} = \frac{22}{7} = 3.1429 \tag{3.71}$$

We may continue the flipping process to get a better approximation.

$$\begin{aligned}
\pi &= 3 + \cfrac{1}{7 + 0.0625133059310457697930051\cdots} \\
&= 3 + \cfrac{1}{7 + \cfrac{1}{\cfrac{1}{0.0625133059310457697930051\cdots}}} \\
&= 3 + \cfrac{1}{7 + \cfrac{1}{15.9965944066857198889230 60\cdots}} \\
&= 3 + \cfrac{1}{7 + \cfrac{1}{15 + 0.9965944066857198889230 60\cdots}}
\end{aligned} \tag{3.72}$$

The number 15.99659 is very close to 16 and we may have a better approximate for π with exact value up to 6 decimal digits.

$$\begin{aligned}
\pi &\simeq 3 + \cfrac{1}{7 + \cfrac{1}{16}} = \frac{355}{113} \\
&\simeq 3.14159292035398230088495 57\cdots
\end{aligned} \tag{3.73}$$

The next steps will generate another fraction of approximation.

$$\begin{aligned}
\pi &= 3 + \cfrac{1}{7 + \cfrac{1}{15 + 0.9965944066857198889230 60\cdots}} \\
&= 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{\cfrac{1}{0.9965944066857198889230 60\cdots}}}} \\
&= 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1.0034172310133726034641468\cdots}}}
\end{aligned} \tag{3.74}$$

$$= 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + 0.0034172310133726034641468\cdots}}} \tag{3.75}$$

$$\begin{aligned}
\pi &= 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + 0.0034172310133726034641468\cdots}}} \\
&= 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{\cfrac{1}{0.0034172310133726034641468\cdots}}}}} \\
&= 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292.63459101439547237857177\cdots}}}}
\end{aligned} \tag{3.76}$$

Let us ignore the $0.634\cdots$ and approximate the result with a rational number to compare with π,

$$\begin{aligned}
\pi &\simeq 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292}}}} = \frac{103993}{33102} \\
&\simeq 3.14159265301190260407 22615\cdots
\end{aligned} \tag{3.77}$$

or round it up and get another approximation rational number for π.

$$\begin{aligned}
\pi &\simeq 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{293}}}} = \frac{104348}{33215} \\
&\simeq 3.1415926539214210447087159\cdots
\end{aligned} \tag{3.78}$$

Therefore, either $\frac{103993}{33102}$ or $\frac{104348}{33215}$ may be used to approximate π which are correct up to 9 decimals. To have an estimate of how exact is $\pi \simeq \frac{104348}{33215}$ let us calculate the circumference C_E of the Earth. The radius of the Earth at the equator is $R_e = 6378\,\text{km}$ (= 3963 mi).

$$C_E = 2R_e\pi = 40074155.889\,\text{m} \tag{3.79}$$

$$\simeq 2R_e\frac{104348}{33215} = 40074155.893\,\text{m} \tag{3.80}$$

The error in calculating circumference of the Earth is less than 0.5 cm by using $\pi \simeq \frac{104348}{33215}$.

If we have the numerical value of a number, rational or irrational, we will be able to form the continued fraction by repeatedly flipping and separating off the whole integer part.

Example 98 ★ history of the approximation of π as a transcendental number.

Pi (π) has been the first and the most interesting mathematical object in the history of mathematics and science. It has been studied seriously by every major culture. Although for practical use, no more than six decimal places are needed, π has been calculated to over several quadrillion decimal places (Borwein and Borwein 1987). The Egyptian papyrus from 2000BC gives a value of

$$\pi \simeq \left(\frac{16}{9}\right)^2 = 3.16049 \tag{3.81}$$

A Chinese document of 1000 years old beautifully shows the ratio to be

$$\pi \simeq \frac{355}{113} = 3.1416 \tag{3.82}$$

The first rigorous mathematical calculation of π was due to Archimedes (288–212BC) who used double circumscribed polygons of 6-gons, 12-gons, ..., 96-gons to show that

$$3\frac{10}{71}\,(= 3.14085) < \pi < 3\frac{10}{70}\,(= 3.14286) \tag{3.83}$$

Figure 3.1 illustrates the case of 6-gons, 12-gons.

Archimedes' scheme is the first known true algorithm for approximating π as accurate as we wish. If a_n denotes the length of a circumscribed 6×2^n-gon and b_n denotes the length of the associated inscribed 6×2^n-gon about a circle with unit length diagonal,

$$a_{n+1} = \frac{2a_nb_n}{a_n + b_n} \tag{3.84}$$

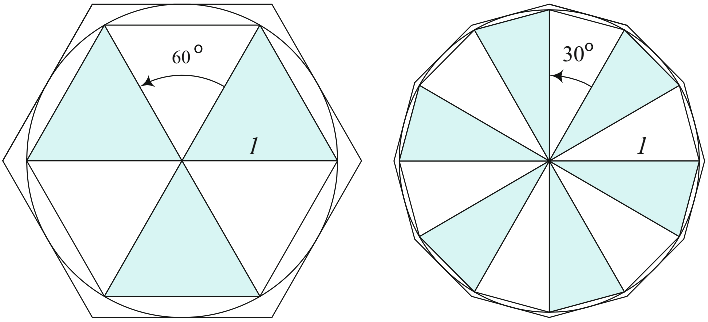

Fig. 3.1 Archimedes' method of computing π with 6-gons and 12-gons

$$b_{n+1} = \sqrt{a_{n+1} b_n} \tag{3.85}$$

$$a_{n+1} - b_{n+1} = \frac{a_{n+1} b_n}{(a_{n+1} - b_{n+1})(a_n + b_n)} (a_n - b_n) \tag{3.86}$$

then these equations may be iterated starting from $a_0 = 2\sqrt{3}$ and $b_0 = 3$, to calculate π. Both a_n and b_n converge to π (Jonathan and Chapman 2015). The fourth iteration provides $a_4 \simeq 3.1429\cdots$ and $b_4 \simeq 3.1408\cdots$ corresponding to the 96-gons (Borwein and Bailey 2008). A thousand years after Archimedes, Jamshid al-Kashi (1380–1429), a Persian mathematician, using 3×2^{28}-gons = 805306368 sides, calculated 2π in sexadecimal that was accurate to seventeen decimal places (Aydin and Hammoudi 2019).

$$2\pi \simeq 6 + \frac{16}{60} + \frac{59}{60^2} + \frac{28}{60^3} + \frac{1}{60^4} + \frac{34}{60^5} + \frac{51}{60^6} + \\ + \frac{46}{60^7} + \frac{14}{60^8} + \frac{50}{60^9} \tag{3.87}$$

Al-Kashi's approximation of π, in 1424, was a remarkable achievement for that time that far surpassed all approximations of π by all previous mathematicians throughout the world. Ludolf van Ceulen obtained the value of π correct to 35 decimal places in 1610.

Francois Viéta (1540–1603) introduced the first infinite expansion formula

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \frac{\sqrt{2 + \sqrt{2}}}{2} \frac{\sqrt{2 + \sqrt{2 + \sqrt{2}}}}{2} \cdots \tag{3.88}$$

and around 100 years later, John Wallis (1616–1703) provided his infinite product to calculate $\pi/2$, similar to (3.64) and (3.65).

$$\frac{\pi}{2} = \frac{2 \times 2 \times 4 \times 4 \times 6 \times 6 \times \cdots}{1 \times 1 \times 3 \times 3 \times 5 \times 5 \times \cdots} \tag{3.89}$$

Lord Brouncker (1620–1684) presented π by continued fractions (Borwein and Borwein 1987).

$$\frac{2}{\pi} = \frac{1}{1+}\,\frac{9}{2+}\,\frac{25}{2+}\,\frac{49}{2+}\,\frac{81}{2+}\cdots \tag{3.90}$$

There are thousands of series expansions and continued fractions that lead to π. Many of them are eye catching due to their symmetry, harmony, or interesting arrangements. As an example of symmetric continued fraction we may include the following one that has been recently founded by Leo Jerome Lange (Bailey and Borwein 2016).

$$\pi = 3 + \frac{1^2}{3+}\,\frac{3^2}{6+}\,\frac{5^2}{6+}\,\frac{7^2}{6+}\cdots \tag{3.91}$$

Example 99 Flip method and approximating $\sqrt{2}$ by continued fraction.

As we know,

$$\sqrt{2} = 1.4142135623730950488 02\cdots \tag{3.92}$$

therefore,

$$\begin{aligned}\sqrt{2} =& 1 + 0.41421356237309504880 2\cdots\\ =& 1 + \cfrac{1}{\cfrac{1}{0.41421356237309504880 2\cdots}}\\ =& 1 + \frac{1}{2.4142135623730950488\cdots}\end{aligned} \tag{3.93}$$

$$\begin{aligned}\sqrt{2} =& 1 + \frac{1}{2 + 0.4142135623730950488\cdots}\\ =& 1 + \cfrac{1}{2 + \cfrac{1}{\cfrac{1}{0.4142135623730950488\cdots}}}\\ =& 1 + \cfrac{1}{2 + \cfrac{1}{2.41421356237309504881 2\cdots}}\end{aligned} \tag{3.94}$$

and finally we have:

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ddots}}} = 1 + \frac{1}{2+}\frac{1}{2+}\cdots = [1, 2, 2, 2, 2, \ldots] \tag{3.95}$$

Example 100 Convergents of $\sqrt{2}$.

Having $\sqrt{2} = [1, 2, 2, 2, 2, \ldots]$ as a simple continued fraction, we may calculate its convergents.

$$\frac{p_0}{q_0} = 1 \qquad \frac{p_1}{q_1} = 1 + \frac{1}{2} = \frac{3}{2} \tag{3.96}$$

$$\frac{p_2}{q_2} = 1 + \cfrac{1}{2 + \cfrac{1}{2}} = \frac{7}{5} \qquad \frac{p_3}{q_3} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2}}} = \frac{17}{12} \tag{3.97}$$

$$\frac{p_4}{q_4} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2}}}} = \frac{41}{29} \tag{3.98}$$

The convergents of a continued fraction approaches the exact value gradually and usually alternatively. The table below indicates convergent of $\sqrt{2}$.

k	0	1	2	3	4	5	6	7	8
$\frac{p_k}{q_k}$	$\frac{1}{1}$	$\frac{3}{2}$	$\frac{7}{5}$	$\frac{17}{12}$	$\frac{41}{29}$	$\frac{99}{70}$	$\frac{239}{169}$	$\frac{577}{408}$	$\frac{1393}{985}$

(3.99)

Example 101 Periodic continued fractions.

A continued fraction A is called periodic if it is in the form:

$$A = [a_0, a_1, a_2, \ldots, a_m, b_0, b_1, b_2, \ldots, b_n, b_0, b_1, b_2, \ldots, b_n, \ldots] \tag{3.100}$$

$$= \left[a_0, a_1, a_2, \ldots, a_m, \overline{b_0, b_1, b_2, \ldots, b_n}\right] \tag{3.101}$$

Any periodic continued fraction

$$A = \left[a_0, a_1, a_2, \ldots, a_m, \overline{b_0, b_1, b_2, \ldots, b_n}\right] \tag{3.102}$$

is equal to a number of the form

$$A = \frac{p + q\sqrt{r}}{s} \tag{3.103}$$

with p, q, r, s integer and $s > 0$.

Conversely, having p, q, r, s integer and $s > 0$, any number of the form

$$A = \frac{p + q\sqrt{r}}{s} \tag{3.104}$$

has a periodic continued fraction.

Consider a continued fraction with repeating quotients such as:

$$A = [b_0, b, b, b, b, \ldots] \tag{3.105}$$

It may also be written as:

$$A = b_0 + \frac{1}{[b, b, b, b, \ldots]} = b_0 + \frac{1}{B} \tag{3.106}$$

$$B = [b, b, b, b, \ldots] \tag{3.107}$$

We may rewrite B in fraction form,

$$B = b_0 + \frac{1}{[b, b, b, b, \ldots]} = b_0 + \frac{1}{B} \tag{3.108}$$

considering that the denominator is also equal to B. Multiplying both sides by B

$$B^2 = bB + 1 \tag{3.109}$$

and using quadratic formula for the positive root, determines B.

$$B = \frac{b + \sqrt{b^2 + 4}}{2} \tag{3.110}$$

Now A would be

$$\begin{aligned} A &= b_0 + \frac{1}{B} = b_0 + \frac{2}{b + \sqrt{b^2 + 4}} \\ &= b_0 + \frac{2}{b + \sqrt{b^2 + 4}} \frac{b - \sqrt{b^2 + 4}}{b - \sqrt{b^2 + 4}} \\ &= b_0 + \frac{b - \sqrt{b^2 + 4}}{2} = \frac{2b_0 - b}{2} + \frac{\sqrt{b^2 + 4}}{2} \end{aligned} \tag{3.111}$$

to have an equation for simple continued fractions with repeating quotients.

$$\frac{2b_0 - b}{2} + \frac{\sqrt{b^2 + 4}}{2} = [b_0, b, b, b, b, \ldots] \tag{3.112}$$

If $b_0 = b$ then,

$$\frac{b + \sqrt{b^2 + 4}}{2} = [b, b, b, b, b, \ldots] \tag{3.113}$$

and if $b = 2b_0$ then,

$$\sqrt{b_0^2 + 1} = [b_0, 2b_0, 2b_0, 2b_0, 2b_0, \ldots] \tag{3.114}$$

As an example,

$$A = [2, 3, 3, 3, 3, \ldots] = [b_0, b, b, b, b, \ldots] \tag{3.115}$$

is equal to:

$$A = \frac{2b_0 - b}{2} + \frac{\sqrt{b^2 + 4}}{2} = \frac{1 + \sqrt{13}}{2} \tag{3.116}$$

and

$$A = [2, 2, 2, 2, 2, \ldots] = [b, b, b, b, b, \ldots] \tag{3.117}$$

is equal to:

$$A = \frac{b + \sqrt{b^2 + 4}}{2} = \frac{2 + \sqrt{8}}{2} = 1 + \sqrt{2} \tag{3.118}$$

and

$$A = [3, 6, 6, 6, 6, \ldots] = [b_0, 2b_0, 2b_0, 2b_0, 2b_0, \ldots] \tag{3.119}$$

is equal to:

$$A = \sqrt{b_0^2 + 1} = \sqrt{9 + 1} = \sqrt{10} \tag{3.120}$$

Example 102 Irrational numbers.

Any number of the form

$$x = \frac{p \pm q\sqrt{r}}{s} \tag{3.121}$$

where p, q, r, s are integer and r is not a perfect square, and $s > 0$ is an irrational number. Such a number is called a geometric quadratic irrational because it is the root of a quadratic equation.

$$s^2x^2 - 2psx + p^2 - rq^2 = 0 \tag{3.122}$$

Any quadratic equation may be solved by continued fractions following these steps:

1. find x^2,

$$x^2 = \frac{2px}{s} - \frac{p^2 - rq^2}{s^2} \tag{3.123}$$

2. divide both sides by x,

$$x = \frac{2p}{s} - \frac{p^2 - rq^2}{s^2 x} \tag{3.124}$$

3. substitute the right-hand side for x into the right-hand side,

$$x = \frac{2p}{s} - \cfrac{p^2 - rq^2}{2ps - \cfrac{p^2 - rq^2}{x}} \tag{3.125}$$

4. keep substituting for x on the right-hand side.

$$x = \frac{2p}{s} - \cfrac{p^2 - rq^2}{2ps - \cfrac{p^2 - rq^2}{\cfrac{2p}{s} - \cfrac{p^2 - rq^2}{2ps - \cfrac{p^2 - rq^2}{\cfrac{2p}{s} - \cdots}}}} \tag{3.126}$$

Example 103 Unusual repeating quotients.

Let us consider a continued fractions with few unrepeating and repeating quotients.

$$A = [6, 2, 2, 3, 11, 21, 21, 21, 21, 21, \ldots] \tag{3.127}$$

To find the associated value to the continued fractions, we pull off the nonrepetitive part,

$$A = 6 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{11 + \cfrac{1}{B}}}}} \qquad B = [21, 21, 21, \ldots] \tag{3.128}$$

then we find the value of the periodic part of the continued fraction.

$$B = 21 + \frac{1}{B} \qquad B = \frac{21 + \sqrt{445}}{2} \tag{3.129}$$

Therefore,

$$A = 6 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{11 + \cfrac{1}{\cfrac{21+\sqrt{445}}{2}}}}}} \simeq 6.4115 \tag{3.130}$$

3.2 Convergents of Continued Fractions

The convergents and partial quotients of continued fractions

$$x = b_0 + \frac{a_1}{b_1+}\,\frac{a_2}{b_2+}\,\frac{a_3}{b_3+}\cdots \tag{3.131}$$

$$= \left[b_0, \frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \ldots, \frac{a_n}{b_n}, \ldots\right] \tag{3.132}$$

have recursive relations to calculate convergents,

$$p_k = b_k\ p_{k-1} + a_k\ p_{k-2} \qquad k \geq 2 \tag{3.133}$$

$$q_k = b_k\ q_{k-1} + a_k\ q_{k-2} \qquad k \geq 2 \tag{3.134}$$

starting with

$$p_0 = b_0 \qquad p_1 = b_0 b_1 + a_1 \tag{3.135}$$

$$q_0 = 1 \qquad q_1 = b_1 \tag{3.136}$$

Proof Recursive formulas are easier to be proven by induction method. We have:

$$\frac{p_0}{q_0} = \frac{b_0}{1} \tag{3.137}$$

$$\frac{p_1}{q_1} = \frac{b_0 b_1 + a_1}{b_1} \tag{3.138}$$

$$\frac{p_2}{q_2} = \frac{a_2 b_0 + a_1 b_2 + b_0 b_1 b_2}{b_1 b_2 + a_2} = \frac{a_2 b_0 + b_2 (b_0 b_1 + a_1)}{b_1 b_2 + a_2}$$
$$= \frac{b_2 p_1 + a_2 p_0}{b_2 q_1 + a_2 q_0} \tag{3.139}$$

therefore,

$$p_2 = b_2 p_1 + a_2 p_0 \tag{3.140}$$
$$q_2 = b_2 q_1 + a_2 q_0 \tag{3.141}$$

Now let us assume that (3.133) and (3.134) are correct for $k = n$,

$$p_n = b_n \; p_{n-1} + a_n \; p_{n-2} \tag{3.142}$$
$$q_n = b_n \; q_{n-1} + a_n \; q_{n-2} \tag{3.143}$$

and use that assumption to prove it is also true for $k = n + 1$.

$$p_{n+1} = b_{n+1} \; p_n + a_{n+1} \; p_{n-1} \tag{3.144}$$
$$q_{n+1} = b_{n+1} \; q_n + a_{n+1} \; q_{n-1} \tag{3.145}$$

We may combine the last two terms of the continued fraction

$$\left[b_0, \frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_{n-1}}{b_{n-1}}, \frac{a_n}{b_n}, \frac{a_{n+1}}{b_{n+1}} \right] \tag{3.146}$$

and rewrite it as a continued fraction with one less term.

$$\left[b_0, \frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_{n-1}}{b_{n-1}}, \frac{a_n b_{n+1}}{b_n b_{n+1} + a_{n+1}} \right] \tag{3.147}$$

Let us rename the continued fraction (3.147)

$$\left[c_0, \frac{c_1}{d_1}, \frac{c_2}{d_2}, \ldots, \frac{c_{n-1}}{d_{m-1}}, \frac{c_n}{d_n} \right] \tag{3.148}$$

where

$$\frac{c_k}{d_k} = \frac{a_k}{b_k} \qquad 0 \le k \le n - 1 \tag{3.149}$$
$$\frac{c_n}{d_n} = \frac{a_n}{b_n +} \frac{a_{n+1}}{b_{n+1}} \tag{3.150}$$

and show the convergents of (3.148) by capital letters.

$$\frac{P_0}{Q_0} = \left[\frac{c_0}{1}\right] = c_0 \tag{3.151}$$

$$\frac{P_1}{Q_1} = \left[c_0, \frac{c_1}{d_1}\right] = \frac{c_0 d_1 + c_1}{d_1} \tag{3.152}$$

$$\frac{P_2}{Q_2} = \left[c_0, \frac{c_1}{d_1}, \frac{c_2}{d_2}\right] = \frac{c_2 d_0 + d_2\,(d_0 d_1 + c_1)}{d_1 d_2 + c_2} \tag{3.153}$$

Based on (3.142) and (3.143), we have

$$P_k = d_k\ P_{k-1} + c_k\ P_{k-2} \qquad 0 \le k \le n \tag{3.154}$$

$$Q_k = d_k\ Q_{k-1} + c_k\ Q_{k-2} \qquad 0 \le k \le n \tag{3.155}$$

and therefore,

$$\left[c_0, \frac{c_1}{d_1}, \frac{c_2}{d_2}, \ldots, \frac{c_n}{d_n}\right] = \frac{P_n}{Q_n} = \frac{d_n\ P_{n-1} + c_n\ P_{n-2}}{d_n\ Q_{n-1} + c_n\ Q_{n-2}} \tag{3.156}$$

Because $c_k = a_k$ and $d_k = b_k$ for $0 \le k \le n-1$, the nth convergents are the same for (3.147) and (3.148) and we may substitute equivalent terms in (3.156).

$$\left[c_0, \frac{c_1}{d_1}, \frac{c_2}{d_2}, \ldots, \frac{c_n}{d_n}\right] = \frac{P_n}{Q_n} = \frac{b_n\ P_{n-1} + a_n\ P_{n-2}}{b_n\ Q_{n-1} + a_n\ Q_{n-2}} \tag{3.157}$$

As we also know

$$\left[c_0, \frac{c_1}{d_1}, \frac{c_2}{d_2}, \ldots, \frac{c_n}{d_n}\right] = \left[b_0, \frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \ldots, \frac{a_{n+1}}{b_{n+1}}\right] \tag{3.158}$$

$$\frac{c_n}{d_n} = \frac{a_n}{b_n +}\ \frac{a_{n+1}}{b_{n+1}} = \frac{a_n b_{n+1}}{b_n b_{n+1} + a_{n+1}} \tag{3.159}$$

we find

$$\begin{aligned}\left[b_0, \frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_{n+1}}{b_{n+1}}\right] &= \frac{d_n\ p_{n-1} + c_n\ p_{n-2}}{d_n\ q_{n-1} + c_n\ q_{n-2}} \\ &= \frac{(b_n b_{n+1} + a_{n+1})\ \ p_{n-1} + a_n b_{n+1}\ p_{n-2}}{(b_n b_{n+1} + a_{n+1})\ \ q_{n-1} + a_n b_{n+1}\ q_{n-2}} \\ &= \frac{b_{n+1}\,(b_n p_{n-1} + a_n p_{n-2}) + a_{n+1}\ p_{n-1}}{b_{n+1}\,(b_n q_{n-1} + a_n q_{n-2}) + a_{n+1}\ q_{n-1}}\end{aligned} \tag{3.160}$$

but

$$p_n = b_n \; p_{n-1} + a_n \; p_{n-2} \tag{3.161}$$

$$q_n = b_n \; q_{n-1} + a_n \; q_{n-2} \tag{3.162}$$

and therefore,

$$\left[b_0, \frac{a_1}{b_1}, \frac{a_2}{b_2}, \dots, \frac{a_{n+1}}{b_{n+1}} \right] = \frac{b_{n+1} \; p_n + a_{n+1} \; p_{n-1}}{b_{n+1} \; q_n + a_{n+1} \; q_{n-1}} \tag{3.163}$$

By definition, we have

$$\left[b_0, \frac{a_1}{b_1}, \frac{a_2}{b_2}, \dots, \frac{a_{n+1}}{b_{n+1}} \right] = \frac{p_{n+1}}{q_{n+1}} \tag{3.164}$$

and hence

$$\frac{p_{n+1}}{q_{n+1}} = \frac{b_{n+1} \; p_n + a_{n+1} \; p_{n-1}}{b_{n+1} \; q_n + a_{n+1} \; q_{n-1}} \tag{3.165}$$

or

$$\frac{p_k}{q_k} = \frac{b_k \; p_{k+1} + a_n \; p_{k-2}}{b_k \; q_{k+1} + a_n \; q_{k-2}} \qquad k \geq 2 \tag{3.166}$$

and the proof is completed.

The convergents and partial quotients of simple continued fractions

$$x = b_0 + \frac{1}{b_1 +} \; \frac{1}{b_2 +} \; \frac{1}{b_3 +} \cdots \tag{3.167}$$

have recursive relationships of

$$p_k = b_k \; p_{k-1} + p_{k-2} \qquad k \geq 2 \tag{3.168}$$

$$q_k = b_k \; q_{k-1} + q_{k-2} \qquad k \geq 2 \tag{3.169}$$

starting with

$$p_0 = b_0 \qquad p_1 = b_0 b_1 + 1 \tag{3.170}$$

$$q_0 = 1 \qquad q_1 = b_1 \tag{3.171}$$

■

Example 104 Transformation of a continued fraction to another.

It is proven by Stolz (1885) that if

$$\lim_{k\to\infty} p_k = \infty \tag{3.172}$$

$$\lim_{k\to\infty} q_k = \infty \qquad q_{k+1} > q_k \tag{3.173}$$

then,

$$\lim_{k\to\infty} \frac{p_k}{q_k} = \lim_{k\to\infty} \frac{p_{k+1} - p_k}{q_{k+1} - q_k} = \lim_{k\to\infty} \frac{p_{k+1} + p_k}{q_{k+1} + q_k} \tag{3.174}$$

provided the limit on the right-hand side exists.

Employing Stolz's theorem and the recurrence relations (3.29) and (3.30),

$$p_n = b_n p_{n-1} + a_n p_{n-2} \tag{3.175}$$

$$q_n = b_n q_{n-1} + a_n q_{n-2} \tag{3.176}$$

we can transform a continued fraction

$$x = b_0 + \frac{a_1}{b_1+}\frac{a_2}{b_2+}\cdots\frac{a_n}{b_n+}\cdots \tag{3.177}$$

into another continued fraction (Lyusternik and Yanupolskii 1965),

$$x = b_0 + \frac{a_1}{b_1 - 1+}\frac{a_2 + b_2 - 1}{b_2 - 1+}\frac{\dfrac{a_3 + b_3 - 1}{a_2 + b_2 - 1}a_2}{b_3 - 1 + \dfrac{a_3 + b_3 - 1}{a_2 + b_2 - 1}+}\cdots$$
$$\frac{\dfrac{a_n + b_n - 1}{a_{n-1} + b_{n-1} - 1}a_{n-1}}{b_n + 1 + \dfrac{a_n + b_n - 1}{a_{n-1} + b_{n-1} - 1}+}\cdots \tag{3.178}$$

or

$$x = b_0 + \frac{a_1}{b_1 + 1-}\frac{b_2 - a_2 + 1}{b_2 + 1+}\frac{\dfrac{b_3 - a_3 + 1}{b_2 - a_2 + 1}a_2}{b_3 + 1 - \dfrac{b_3 - a_3 + 1}{b_2 - a_2 + 1}+}\cdots$$
$$\frac{\dfrac{b_n - a_n + 1}{b_{n-1} - a_{n-1} + 1}a_{n-1}}{b_n + 1 - \dfrac{b_n - a_n + 1}{b_{n-1} - a_{n-1} + 1}+}\cdots \tag{3.179}$$

As an example, we may show that:

$$\sqrt{3} = 1 + \frac{1}{2+}\ \frac{1}{2+}\ \frac{1}{1+}\ \frac{1}{2+} \cdots \tag{3.180}$$

is equal to

$$\sqrt{3} = 1 + \frac{1}{0+}\ \frac{2}{1+}\ \frac{1}{1+}\ \frac{4}{3+}\ \frac{1}{1+}\ \frac{4}{3+} \cdots \tag{3.181}$$

or is equal to

$$\sqrt{3} = 1 + \frac{1}{2-}\ \frac{2}{3+}\ \frac{1}{3+}\ \frac{4}{4+}\ \frac{1}{3+}\ \frac{4}{1+} \cdots \tag{3.182}$$

Example 105 New and equivalent continued fractions.

Consider the nonzero real numbers c_1, c_2, c_3 and a continued fractions x.

$$x = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cdots}}} = b_0 + \frac{a_1}{b_1+}\ \frac{a_2}{b_2+}\ \frac{a_3}{b_3+} \cdots \tag{3.183}$$

Multiplying a_k, b_k, and a_{k+1} by an arbitrary finite number $c_k \neq 0$, $k = 0, 1, 2, \ldots$, the value of the continued fraction is unaltered. Let us keep three fractions and multiply the top and bottom of the first fraction by c_1,

$$x = b_0 + \cfrac{c_1 a_1}{c_1 b_1 + \cfrac{c_1 a_2}{b_2 + \cfrac{a_3}{b_3}}} \tag{3.184}$$

and then multiply the top and bottom of the second fraction by c_2,

$$x = b_0 + \cfrac{c_1 a_1}{c_1 b_1 + \cfrac{c_2 c_1 a_2}{c_2 b_2 + \cfrac{c_2 a_3}{b_3}}} \tag{3.185}$$

and then multiply the top and bottom of the third fraction by c_3,

$$x = b_0 + \cfrac{c_1 a_1}{c_1 b_1 + \cfrac{c_2 c_1 a_2}{c_2 b_2 + \cfrac{c_3 c_2 a_3}{c_3 b_3}}} = b_0 + \frac{c_1 a_1}{c_1 b_1+}\ \frac{c_2 c_1 a_2}{c_2 b_2+}\ \frac{c_3 c_2 a_3}{c_3 b_3} \tag{3.186}$$

we get two equivalent finite continued fractions.

$$b_0 + \frac{a_1}{b_1+} \frac{a_2}{b_2+} \frac{a_3}{b_3} = b_0 + \frac{c_1 a_1}{c_1 b_1+} \frac{c_2 c_1 a_2}{c_2 b_2+} \frac{c_3 c_2 a_3}{c_3 b_3} \tag{3.187}$$

In general for any real numbers $a_1, a_2, a_3, \ldots, b_1, b_2, b_3, \ldots$ and real nonzero constants $c_1, c_2, c_3, \ldots$, we have:

$$\begin{aligned} b_0 &+ \frac{a_1}{b_1+} \frac{a_2}{b_2+} \frac{a_3}{b_3+} \cdots \frac{a_k}{b_k+} \cdots \\ &= b_0 + \frac{c_1 a_1}{c_1 b_1+} \frac{c_2 c_1 a_2}{c_2 b_2+} \frac{c_3 c_2 a_3}{c_3 b_3+} \cdots \frac{c_k c_{k-1} a_k}{c_k b_k+} \cdots \end{aligned} \tag{3.188}$$

let us use $\sqrt{2}$ as an example,

$$\sqrt{2} = 1 + \frac{1}{2+} \frac{1}{2+} \frac{1}{2+} \frac{1}{2+} \cdots \tag{3.189}$$

and constants $c_1, c_2, c_3, \ldots$, equal to $1, 2, 3, 4, 5, \cdots$. We will have:

$$\sqrt{2} = 1 + \frac{1}{2+} \frac{2}{4+} \frac{6}{6+} \frac{12}{8+} \cdots \tag{3.190}$$

or if we choose $c_1, c_2, c_3, \ldots$, equal to 3, then:

$$\sqrt{2} = 1 + \frac{3}{6+} \frac{9}{6+} \frac{9}{6+} \frac{9}{6+} \cdots \tag{3.191}$$

Example 106 ★Length of a year.

Astronomers distinguish 3 years: (1) the sidereal year; the length of the year measured by reference to fixed stars; (2) the anomalistic year; the interval between two successive returns of the earth to perihelion; (3) the tropical year; the interval between two successive returns of the sun to the equinox. It is the tropical year that the seasons depend on Philip (1921). The perihelion is the point in the orbit of a planet that is nearest to the sun. It is the opposite of aphelion, which is the point farthest from the sun. An equinox is the instant of time when the Earth's equator plane passes through the center of the Sun. At that moment the center of the visible Sun is directly above the equator. Equinox occurs twice each year: around 20 March and 23 September. On the day of an equinox, daytime and nighttime are of approximately equal lengths all over the Earth. In the Northern Hemisphere, the March equinox is the spring equinox and the September equinox is autumn equinox.

The length of an astronomical year is approximately equal to:

$$1\,\mathrm{y} \simeq 365\,\mathrm{d}\ 5\,\mathrm{h}\ 48\,\mathrm{min}\ 55\,\mathrm{s} \tag{3.192}$$

and the length of a tropical year is approximately equal to:

$$1\,\mathrm{y} \simeq 365\,\mathrm{d}\;5\,\mathrm{h}\;48\,\mathrm{min}\;45.5\,\mathrm{s} \tag{3.193}$$

The approximation of 1 year to 365 days makes error around 6 h per year. The error accumulates so fast such that the shift of seasons will be noticeable in a few decades (Khrushchev 2008). The exact year in days is:

$$\begin{aligned}
1\,\mathrm{y} &\simeq 365 + \frac{5}{24} + \left(\frac{48}{60}\times\frac{1}{24}\right) + \left(\frac{55}{60}\times\frac{1}{60}\times\frac{1}{24}\right)\\
&= 365 + \frac{20935}{86400}\,\mathrm{d} = 365 + \frac{4187}{17280}\,\mathrm{d} &(3.194)\\
&\simeq 365.2423032407407407407407\cdots\,\mathrm{d} &(3.195)
\end{aligned}$$

The fraction of a day may be expanded to continued fractions say by flip method.

$$\begin{aligned}
\frac{4187}{17280} &= \frac{1}{\frac{17280}{4187}} = \frac{1}{4+}\,\frac{532}{4187} = \frac{1}{4+}\,\frac{1}{7+}\,\frac{463}{532}\\
&= \frac{1}{4+}\,\frac{1}{7+}\,\frac{1}{1+}\,\frac{69}{463} = \frac{1}{4+}\,\frac{1}{7+}\,\frac{1}{1+}\,\frac{1}{6+}\,\frac{49}{69}\\
&= \frac{1}{4+}\,\frac{1}{7+}\,\frac{1}{1+}\,\frac{1}{6+}\,\frac{1}{1+}\,\frac{20}{49}\\
&= \frac{1}{4+}\,\frac{1}{7+}\,\frac{1}{1+}\,\frac{1}{6+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{4+}\,\frac{1}{2} &(3.196)
\end{aligned}$$

The convergents of the continued fraction are:

$$\frac{1}{4}, \frac{7}{29}, \frac{8}{33}, \frac{55}{227}, \frac{63}{260}, \frac{181}{747}, \frac{425}{1754}, \frac{1881}{7763}, \frac{4187}{17280} \tag{3.197}$$

Convergents of a continued fractions approach the exact value alternatively by being greater and less than the exact value alternatively.

The tropical year would be

$$\begin{aligned}
1\,\mathrm{y} &\simeq 365\,\mathrm{d}\;5\,\mathrm{h}\;48\,\mathrm{min}\;45.5\,\mathrm{s}\\
&= 365 + \frac{5}{24} + \left(\frac{48}{60}\times\frac{1}{24}\right) + \left(\frac{91}{120}\times\frac{1}{60}\times\frac{1}{24}\right)\\
&= \frac{41\,851}{172\,800} = 0.242193287037037037037037\,\mathrm{d} &(3.198)
\end{aligned}$$

The length of a year, approximated by $365 + \frac{1}{4} = 365.25$ d is greater than a year, and $365 + \frac{1}{4+}\frac{1}{7} = 365 + \frac{7}{29} = 365.24$ d is less than a year. Therefore, every 4 years we will have a little less than 1 extra day. This adds 1 day (29 February) every leap year in the Julian Calendar, whose length is 365.25 days. The extra day is added to each year that is divisible by 4: $\cdots$, 2008, 2012, 2016, 2020, 2024, 2028, 2032, $\cdots$. However, the error is not fully covered by this formula. We actually need only a little less than 8 days every 33 years or a little less than 97 days every 400 years. To manage the required compensation, the Gregorian Calendar converts $3 = 100 - 97$ leap years within the range of every 400 years into ordinary years. That was why the years 1700, 1800, 1900 were ordinary years and years 1600 and 2000 were leap years. Because

$$\frac{97}{400} - \frac{4187}{17280} = \frac{17}{86400} = 0.000197456\cdots \tag{3.199}$$

the Gregorian calendar makes 17 days error every 86,400 years or about 2 extra days every 10,000 years.

The Gregorian calendar was accepted in 5 October 1582 by Pope Gregory (540–604), in which 1 year is equal to 365.2425 days. At that time the difference between the Julian and Gregorian calendars was already 10 days. To make up the error, the day of 5 October 1582 was announced to be 15 October 1582. Therefore, the days of 4th to 14th of October 1582 do not exist in calendar.

The length of year cannot be counted by exact number of months, weeks, and days (Dershowitz and Reingold 2008). Approximating a year to be 365.24244 days, and a month to be 29.53059 days make their ratio to be:

$$\frac{365.24244}{29.53059} = 12 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{18 + \cfrac{1}{3 + \cdots}}}}}}} \tag{3.200}$$

Example 107 ★Calendar history.

Different interpretations of Moon-based and Sun-based calendars have been used in different places of the world at different ages by different nations. Generally speaking, those nations who were living close to shores and hence their living style were dependent on sea, and therefore on tides and moon, used lunar calendars. Also those nations who were living close to equator, had only one hot season and did not feel the changes of seasons very much, used lunar calendars. On the other hand, people whose living were based on farming and agriculture, needed to plan based on seasons and used solar calendars. However, sun-based calendars were more common and eventually became internationally accepted. Solar years always started at the spring equinox of the northern hemisphere at the moment that assumed to be the birth of sun.

As there is no calendar systems currently in use that perfectly reflects the length of a tropical year, the calendars are compared by their errors. There are calendar systems that are more accurate than the Gregorian calendar. The Persian calendar is the most exact one due to the length and the beginning of its year as it is set based on two natural astronomical factors. The beginning is set by the moment that the center of the Sun and the vernal equinox during the Sun's apparent revolution around the Earth collide. The length is between two successive apparent passages of the Sun's center across that point. Therefore, its beginning is the beginning of natural solar year, its length is the length of solar year, and the length of its months is very close to the time of the Sun's passage across twelve signs of the Zodiac from Farvardin/Aries to Espand/Pisces. The length of a year and the error of different calendars are:

1. Persian calendar, 2nd millennium BC, 1 y $\simeq$ 365.2421986 d, with error of less than 1 s/ y, (1 d in 110000 y).
2. Revised Julian calendar, 1923, 1 y $\simeq$ 365.242222 d, with error of less than 2 s/ y, (1 d in 31250 y).
3. Mayan calendar, 2000BC, 1 y $\simeq$ 365.242036 d, with error of less than 13 s/ y, (1 d in 6500 y).
4. Gregorian calendar, 1582, 1 y $\simeq$ 365.2425 d, with error of less than 27 s/ y, (1 d in 3236 y).
5. Jewish calendar, ninth century, 1 y $\simeq$ 365.246822 d, with error of less than 7 min/ y, (1 d in 216 y).
6. Julian calendar, 45BC, 1 y $\simeq$ 365.25 d, with error of less than 11 min/ y, (1 d in 128 y).
7. Coptic calendar, 25BC, 1 y $\simeq$ 365.25 d, with error of less than 11 min/ y, (1 d in 128 y).

Darius the Great (550–486BC), the third Persian King of the Achaemenid Empire, in 520BC ordered his astronomers to make required corrections in the old Zoroastrian calendar to remove its minor error. They set a year to start at equinox with 12 months, 30 days each, and the left over time till next equinox for celebration, pray, and holiday. Persian at the Darius era did not have weeks, instead every month and every day of months had special name (Idem 1965; Gray 1907; Panaino et al. 1990). Week is a man-made unit with no link to natural phenomena. The seven planets visible to the naked eye inspired the formation of weeks. That is why weekdays are named after celestial bodies. The present universal week is of ancient Greek, Rome, and Hebrew origin, and has been generalized by Jewish, Christian, and Persian persuasion (Saha and Lahiri 1955; Ben-Dov et al. 2012).

Later in 1075, the king Jalaluddin Malekshah, king of the Seljuq Empire from 1072 to 1092, asked Omar Khayyam (1048–1131), the greatest astronomer and mathematician of the time, and seven other mathematicians, to reform the Persian calendar to the most exact one. The reformed calendar, that named Jalali after king

Jalaluddin, is the one that still works as the most accurate world calendar. The Persian calendar was kept safe by Parsees, the Iranian who saved themselves by migrating and taking shelter in India on the conquest of Persia by Islam in seventh to eighth centuries AD (Morony 2012).

Months in Persian Pahlavi are: (1) Frawardīn (Guardian spirits), (2)Ardwahišt (Ultimate Righteousness), (3) Khordād (Perfection), (4) Tīr (Sirius), (5) Murdād (Immortality), (6) Shahrewar (Desirable Dominion), (7) Mihr (Covenant), (8) Ābān (Waters), (9) Ādur (Fire), (10) Day (The Creator), (11) Wahman (Good Spirit), (12) Spandarmad (Holy Devotion).

Months in Roman were: (1) March, (2) April, (3) May, (4) June, (5) Quintilis, (6) Sextilis, (7) September, (8) October, (9) November, (10) December, (11) January, (12) February. In 40BC, the Roman Senate decreed that the month Quintilis should be called July after Julius, because Julius Caesar (July 100–44BC) was born in that month. Also the month Sextilis was named Augustus, because the emperor Augustus was born in that month.

The Romans calendar at the time of Romulus, the first king of Rome, in the 700 s BC had 10 months, 304 days long: Martius, Aprilis, Maius, Junius, Quintilis, Sextilis, September, October, November, and December, plus around 61 days of winter. Arians round 2000BC celebrated the New Year at the Spring equinox, currently around March 21. This natural beginning of the year were adopted by many nations around Persia including Mesopotamia as well as Macedonia and Greece. Romulus celebrated the first Roman triumph, after his victory over the Caeninenses, on 21 days before the beginning the new year in 752BC. To celebrate this event and honor the beginning of the Roman empire, they shifted the 1st of March 21 days back and set the New Year's Day again on the new 1st of March, making the Spring equinox to be on 21st of March.

The Roman ruler Numa Pompilius (753–673BC) added January and February to the calendar at about 700BC. It was in 153BC when Rome started celebrating the New Year on the 1st of January to recognize the newly elected Roman consuls. However, the beginning of the year moved between 1st of March and 1st of January back and forth several times in different parts of Europe. Pope Gregory restored January 1 as the beginning of the New Year according to his reform of the Catholic Liturgical Calendar in 1752, and adopted new improved calendar called Gregorian calendar (Feeney 2007).

It was Julius Caesar who ordered a revision of the calendar in 46BC to the point of dismissing the lunar calendar and adopting a purely solar year. They set a calendar having year-length of 365 days, with a leap year of 366 days every fourth year and called it Julian calendar. The beginning of the year and hence the number of months have moved several times to honor people and celebrate important events. In 9BC, Caesar Augustus' birthday (currently on 29 August) was formally adopted as New Year's Day. Sometimes Macedonian's New Year's Day was in autumn, currently on 28 October, while those nations in Roman area under Persian empire kept the Persian calendar having the beginning of Spring equinox as the New Year's Day (Hannah 2005). The idea to count years from the birth of Jesus Christ was first proposed in the year 525 by Dionysius Exiguus (470–544), a Christian Roman monk. Dionysius tried to calculate the resurrection of Jesus from the dead to set Easter. His calculation showed that the Jesus' birthday was on December 23rd or 25th while some believed it to be on January 6th. Dionysius set Jesus birthday on December 25rd assuming that it fell on a Saturday, and that the moon was full at 13 days old. According to modern calculations, the moon was 18 days old on that day.

3.3 Convergence of Continued Fractions

Convergence of continued fractions requires that the difference of successive convergents becomes smaller and smaller.

$$\frac{p_0}{q_0} - \frac{p_1}{q_1} > \frac{p_1}{q_1} - \frac{p_2}{q_2} \tag{3.201}$$

$$\frac{p_1}{q_1} - \frac{p_2}{q_2} > \frac{p_2}{q_2} - \frac{p_3}{q_3} \tag{3.202}$$

$$\cdots > \cdots$$

$$\frac{p_{k-3}}{q_{k-3}} - \frac{p_{k-2}}{q_{k-2}} > \frac{p_{k-2}}{q_{k-2}} - \frac{p_{k-1}}{q_{k-1}} \tag{3.203}$$

It can be shown that

$$p_{k-1}q_k - q_{k-1}p_k = (-1)^k \qquad k = 1, 2, 3, \cdots \tag{3.204}$$

dividing by $q_{k-1}q_k$ makes

$$\frac{p_{k-1}}{q_{k-1}} - \frac{p_k}{q_k} = \frac{(-1)^k}{q_{k-1}q_k} \qquad k = 1, 2, 3, \cdots \tag{3.205}$$

A continued fraction is to converge if at most a finite number of its denominators q_k vanish, and if the limit c of its sequence of convergents exists and is finite.

$$\lim_{k\to\infty} \frac{p_k}{q_k} = c \tag{3.206}$$

Otherwise, the continued fraction diverges. The value of a continued fraction is defined to be the limit (3.206) of its sequence of convergents. If the $\lim_{n\to\infty} \frac{p_n}{q_n}$ does not exist or

$$\lim_{n\to\infty} \frac{p_n}{q_n} = \infty \tag{3.207}$$

or

$$\lim_{n\to\infty} \frac{p_n}{q_n} = -\infty \tag{3.208}$$

then the continued fraction is divergent (Khovanskii 1963).

Proof It is true for $k = 1$.

$$p_0 q_1 - p_1 q_0 = b_0 b_1 - (b_1 b_0 - 1) = -1 \tag{3.209}$$

Let us assume that (3.204) is true for $k = n$,

$$p_{n-1} q_n - q_{n-1} p_n = (-1)^n \qquad n = 1, 2, 3, \cdots \tag{3.210}$$

and we need to prove it is true for $k = n + 1$.

$$p_n q_{n+1} - q_n p_{n+1} = (-1)^{n+1} \qquad n = 1, 2, 3, \cdots \tag{3.211}$$

By substitution we have:

$$\begin{aligned} p_n q_{n+1} - q_n p_{n+1} &= p_n \left(b_{n+1} q_n + q_{n-1}\right) - q_n \left(b_{n+1} p_n + p_{n-1}\right) \\ &= p_n q_{n-1} - q_n p_{n-1} = -\left(q_n p_{n-1} - p_n q_{n-1}\right) \\ &= -\left(-1\right)^n = (-1)^{n+1} \end{aligned} \tag{3.212}$$

■

Example 108 Series to continued fractions.
If a number x is defined by a series such that

$$x = c_0 + c_1 + c_2 + \cdots + c_n = \sum_{k=0}^{n} c_k \tag{3.213}$$

there exists a continued fractions equivalent to the approximated series.

$$x = c_0 + \frac{c_1}{1+} \frac{-c_2/c_1}{1 + c_2/c_1+} \cdots \frac{-c_m/c_{m-1}}{1 + c_m/c_{m-1}+} \cdots \tag{3.214}$$

Conversely, if a function is defined by a continued fractions

$$x = b_0 + \frac{a_1}{b_1+} \frac{a_2}{b_2+} \cdots \frac{a_n}{b_n+} \cdots \tag{3.215}$$

there exists a series equivalent to the approximated continued fractions (Cuyt et.al. 2008).

$$x = c_0 + \frac{c_1}{1+} \frac{-c_2/c_1}{1 + c_2/c_1+} \cdots \frac{-c_m/c_{m-1}}{1 + c_m/c_{m-1}+} \cdots \tag{3.216}$$

$$c_0 = b_0 \qquad c_k = \frac{(-1)^{k-1}\prod_{j=1}^{k} a_j}{p_k p_{k-1}} \qquad k = 1, 2, 3, \cdots \tag{3.217}$$

A series and a continued fraction were considered to be equal if they were derived from each other by the formal transformation

$$x = b_0 + \frac{a_1}{b_1+}\frac{a_2}{b_2+}\frac{a_3}{b_3+}\cdots = b_0 + \sum_{k=1}^{\infty}\left((-1)^{k+1}\frac{\prod_{i=1}^{k} a_i}{q_k q_{k+1}}\right) \tag{3.218}$$

Example 109 Convergence theorems.

Sleszynski–Pringsheim theorem states that the continued fraction

$$K_{k=1}^{\infty}\left(\frac{a_k}{b_k}\right) = \frac{a_1}{b_1+}\ \frac{a_2}{b_2+}\ \frac{a_3}{b_3+}\cdots \tag{3.219}$$

is convergent if for all n we have

$$|b_n| \geq |a_n| + 1 \tag{3.220}$$

Worpitzky's theorem proves that if for all $n \geq 1$

$$|a_n| \leq \frac{1}{4} \tag{3.221}$$

then

$$K_{k=1}^{\infty}\left(\frac{a_k}{1}\right) = \frac{a_1}{1+}\ \frac{a_2}{1+}\ \frac{a_3}{1+}\cdots \tag{3.222}$$

converges.

Example 110 Ascending continued fractions.

A finite continued fraction represents a rational number. To find the rational number corresponding to the continued fraction such as

$$x = 1 + \frac{1}{2+}\ \frac{1}{2+}\ \frac{1}{2+}\ \frac{1}{2} = \frac{41}{29} \tag{3.223}$$

we may rewrite it, starting from the right-hand side of the expression, in the form of an ascending continued fraction (Kkrushchev 2008).

$$\frac{1}{\frac{1}{\frac{1}{\frac{1}{2}+2}+2}+2}+1=\frac{41}{29} \tag{3.224}$$

Expressing in ascending continued fractions we can show π to be:

$$\pi = 3.1415926\cdots = 3+\frac{1+\frac{4+\frac{1+\frac{5+\cdots}{10}}{10}}{10}}{10} \tag{3.225}$$

Example 111 Continued radicals.

Continued radicals are made by radicals nested inside each other in the following form:

$$f=\sqrt{a+\sqrt{b+\sqrt{c+\sqrt{d+\cdots}}}} \tag{3.226}$$

Continued radicals may be transferred back and forth to continued fractions, although continued radicals are not considered as applied.

The simplest continued radicals is when the numbers under radicals are equal.

$$A=\sqrt{a+\sqrt{a+\sqrt{a+\sqrt{a+\cdots}}}} \tag{3.227}$$

This is how we find the limit A of such a continued radical.

$$A = \sqrt{a+A} \tag{3.228}$$

$$A^2 = A+a \tag{3.229}$$

$$A = \frac{1}{2}\pm\frac{1}{2}\sqrt{4a+1} \tag{3.230}$$

The simplest case for which we have $a = 1$, the continued fraction has the limit of the golden ratio Φ.

$$A=\frac{1+\sqrt{5}}{2}=\Phi \tag{3.231}$$

$$\Phi=\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}}=1+\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\cdots \tag{3.232}$$

Rewriting Eq. (3.229) as

$$A = 1 + \frac{a}{A} \tag{3.233}$$

makes it possible to find the equivalent continued fractions.

$$A = 1 + \frac{a}{1+}\frac{a}{1+}\frac{a}{1+}\frac{a}{1+}\cdots \tag{3.234}$$

By picking a value for the limit A, we may calculate a

$$a = A^2 - A \tag{3.235}$$

and derive the continued radicals. For examples:

$$A = 2 \qquad a = 2 \tag{3.236}$$

$$2 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}} \tag{3.237}$$

$$A = 3 \qquad a = 6 \tag{3.238}$$

$$3 = \sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{6 + \cdots}}}} \tag{3.239}$$

$$A = 4 \qquad a = 12 \tag{3.240}$$

$$4 = \sqrt{12 + \sqrt{12 + \sqrt{12 + \sqrt{12 + \cdots}}}} \tag{3.241}$$

Example 112 ★$\sqrt{2}$ is irrational, π is transcendental.

To prove irrationality of any number such as $\sqrt{2}$, we use the method known to the ancient Greeks as "reductio ad absurdum" that means "reduction to absurdity." We assume the opposite of what we want to prove. Then we show that this assumption leads to a contradiction.

Let us assume that $\sqrt{2}$ is represented by some ratio of whole numbers, p/q where p and q have no common factors, so both cannot be even numbers.

$$\frac{p}{q} = \sqrt{2} \tag{3.242}$$

$$p^2 = 2q^2 \tag{3.243}$$

Therefore, p^2 must be an even number and hence, p is also an even number, say $p = 2s$.

$$p = 2s \tag{3.244}$$

$$p^2 = (2s)^2 = 4s^2 = q^2 \tag{3.245}$$

$$q^2 = 2s^2 \tag{3.246}$$

Therefore, q^2 must also be an even number and hence, q is also an even number, say $q = 2r$. This is the contradiction because we have shown that q must be both even and odd. Hence, no fraction exists that is equal to $\sqrt{2}$. The Greeks called $\sqrt{2}$ an incommensurable length because it could not be represented as the ratio of whole numbers.

Pythagoreans believed in numbers, whole numbers, and nothing else. To them, incommensurable number could not exist. Pythagorean theorem, $a^2 + b^2 = c^2$, showed that lines with length $\sqrt{2}$ exist and they failed to express them by fractions of whole numbers, although they believed there must be such a fraction to be found. The discovery of irrationality of $\sqrt{2}$, probably by Hippasus (530–450BC), was a shock to Pythagoreans that all were sworn to secrecy. It seems that Hippasus was drowned because he revealed this secret (Clawson 1996). It turns out and was proven by the Greeks that any square root of a whole number that is not a perfect square, such as $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, etc., are also incommensurable numbers similar to $\sqrt{2}$, and they belong to the class of radical numbers.

Greek scientists, directed by Pythagoreans, did not consider anything except natural numbers and their ratios, and therefore incommensurable lengths such as the diagonal of a unit square had to be separated from arithmetic. The Greek mathematicians divided mathematics into the geometry and arithmetic. Euclid's theorems were all defined in terms of geometry. The division between arithmetic and geometric lasted for 2000 years until Rene Descartes (1596–1650) combined them together again with analytic geometry and showed that an algebraic equation can have no more real roots than the degree of the equation.

The rational and radical numbers make a bigger set of the algebraic numbers. To understand the algebraic numbers, let us consider the linear equation $ax + b = 0$, $a \neq 0$, where a and b are known integer numbers. The solution for the unknown number x is: $x = b/a$. The value for x will always be a fraction, zero, or integer, and therefore the solution x of linear equations will always be a rational number. Now consider a nonlinear equation of the form $ax^2 - b = 0$, $a \neq 0$, with the solution $x = \sqrt{b/a}$. If a and b or their division is a perfect square, then we will get a rational number for x, otherwise the solution will be a radical number. Therefore, excluding imaginary numbers, if there exist solutions for the equation $ax^2 - b = 0$, then the solutions will be algebraic numbers, either integers, fractions, or radicals. Similarly, any number that fits a polynomial equation with integer coefficients $\{a_0, a_1, \ldots, a_n\} \in \mathbb{N}$ is an algebraic number.

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^x = 0 \tag{3.247}$$

$$n \in \mathbb{N} \tag{3.248}$$

π, however, is another type of numbers. It is irrational similar to $\sqrt{2}$, but dissimilar to $\sqrt{2}$, it is not algebraic. π is a number that transcends the algebraic numbers. It is transcendental. The existence of transcendental numbers has been answered in 1844 in the middle of the nineteenth century by Joseph Liouville (1809–1882) using infinite series.

Consider the series:

$$A = \sum_{n=1}^{\infty} \frac{1}{10^{n!}} = \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^6} + \frac{1}{10^{24}} + \cdots$$
$$= 0.110001000000000000000001000\cdots \tag{3.249}$$

The decimal expression of this series contains zeros except at locations $n!$ from the right of the decimal point, where n are consecutive numbers beginning with 1. Therefore, we have a 1 at positions $1! = 1$, $2! = 2$, $3! = 6$, $4! = 24$, etc. The next digit 1 will appear at 120 places right of the decimal. Liouville showed that A cannot be a solution of any algebraic equation, and hence is not a algebraic number, a transcendental. In 1873 around 30 years after Liouville proved the existence of transcendental numbers, a French mathematician Charles Hermite (1822–1901), proved that the number e is also transcendental. Around 10 years later in 1882, Ferdinand von Lindemann (1852–1939) proved that π is also not the solution to any polynomial equation and therefore is a transcendental number. So, the class of transcendental numbers has been recognized, only less than two century ago.

Interestingly, Euler who were studying the properties of e and knew that e is not an algebraic number could not show that e is transcendental and hence suspected the existence of transcendental numbers (Tabak 2004). Part of the difficulty with understanding and using transcendental numbers is that there is no easy way to express them. Transcendental numbers are not rational and not the root of any algebraic equation. Writing a transcendental number as a decimal, ends up with the sequence of digits that never ends and never forms a repeating pattern. That is the reason we show important transcendental numbers with a symbol such as π, e, Φ, etc. Around the end of the nineteenth century it became evident to mathematicians that definition of numbers and their classifications are still unclear and not precise. They still understood irrational numbers only by knowing that they were not rational. The decimal expansions of irrational numbers do not terminate and do not get a repeating pattern.

The German mathematician Richard Dedekind (1831–1916) took the axiom of geometry, that a line is compact and continuous, and introduced the concept of the real number line. Being able to assign a number to any point on the line as well as assigning a point on the line to any possible real number, Dedekind then suggested that there must be no gaps between numbers. To show this, Dedekind established a logical explanation for correspondence between points on the line and real numbers. Dedekind imagined a line with the position of the rational numbers marked out on it. Then, he cut the line at a point A and divides the line into two parts with the left and right sets of numbers. Every rational number in the left is less than every rational number in the right set, and every rational number in the right set is greater than every rational number in the left set. The number A must be either rational or irrational, as we assumed every number is either rational or irrational. If A is in the left set and if A is rational then A is the largest rational number in the left set. Taking any rational number B in the right set, we are able to find another rational number in the right set that is less than B and greater than A and hence, the right set does not have a smallest rational number. If, however, A is assigned to the right set, and if that set still has no smallest rational number, then A is irrational. It establishes a correspondence between discrete numbers and continuous lines and therefore, the number system varies continuously with no gaps. The complex number system, then, may be imagined as the set of coordinates on a plane, such that both the real and imaginary axes are associated with numbers similarly. Now the answer to the question of what $\sqrt{2}$ is, would be: $\sqrt{2}$ is the position of the cut on number axis such that every rational number, a/b, on the left segment is such that $a^2/b^2 < 2$, and every rational number, a/b, on the right segment is such that $a^2/b^2 > 2$ (Tabak 2004).

The next step in discovery of transcendental numbers paved by Georg Cantor (1845–1918) who clearly defined the concept of infinity ∞. He explained that any addition to an infinite set makes a new infinite set with exactly the same number of elements, $\infty + 1 = \infty$, $2\infty = \infty$. Therefore, there is a one to one association between the elements of positive square integers and the positive integers. He assigned an infinity set of numbers, such as the set of integer numbers, say A_0, and called it aleph-null, a countable set. Cantor showed that the set of all fractions is a countable set with the same number of elements as A_0. Now adding radicals to the set makes the bigger set of algebraic numbers that is again countable, and hence, all algebraic numbers can be put into a one-to-one mapping with the natural numbers. However, the transcendental numbers form a larger infinite and uncountable set. It is impossible to make a one-to-one mapping of the transcendental numbers with the natural numbers. In other words, if we imaginary take out the transcendental numbers from the line and leave only the algebraic numbers in place, then the line would have more holes than number points. This fact is not related to the decimal number system. The π, for example, is transcendental in any number systems that we may choose (Bailey and Borwein 2016). Proving a number is transcendental is not an easy task and there is no classical method for that. For instance, showing that e and π are irrational and transcendental have been done by some great mathematicians only in the past century reported in (Shidlovskii 1989; Hardy and Wright 2008).

Example 113 ★Huygens' planetarium problem.

The Moon's synodic period (the length of a lunar month) is 29.53059 days, or 29 days, 12 h, 44 min, and 2.8 s. Moon's synodic period is the time interval between two consecutive occurrences of full Moon as seen by an observer on Earth. However, 29.53059 days is an average number because due to eccentricity of the lunar orbit around Earth and the Earth's orbit around Sun, the Moon's synodic period may vary up to 7 h during a year. If we wish to design a mechanical clock that shows the Moon phases, there must be a pair of gears between Moon and hour indicator with the ratio of 2×29.53059 to one. This ratio is equivalent to have a gear with $2 \times 29.53059 = 5906118$ teeth mate with a gear with 100,000 teeth. This is a high ratio between two gears. Breaking this ratio into continued fractions makes it possible to have a chain of gears to achieve the final ratio as we need (Lorentzen and Waadeland 2008).

$$\begin{aligned} n &= \frac{5906118}{100000} = 59 + \frac{1}{\frac{100000}{6118}} = 59 + \frac{1}{16 + \frac{2112}{6118}} \\ &= 59 + \frac{1}{16 + \frac{1}{\frac{6118}{2112}}} = 59 + \frac{1}{16 + \frac{1}{2 + \frac{1894}{2112}}} = \\ &= \cdots = 59 + \frac{1}{16+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{8+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{4+}\,\frac{1}{1+}\,\frac{1}{6} \end{aligned} \tag{3.250}$$

Christiaan Huygens (1629–1695), studied and designed a planetarium, which to work accurately, needed a gear ratio of:

$$n = \frac{77708431}{2640858} \tag{3.251}$$

Huygens developed the ratio into the continued fractions (Khinchin 1997).

$$\begin{aligned} n = 29 &+ \frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{5+}\,\frac{1}{1+}\,\frac{1}{4+}\,\frac{1}{1+}\,\frac{1}{1+} \\ &\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{6+}\,\frac{1}{1+}\,\frac{1}{10+}\,\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{3} \end{aligned} \tag{3.252}$$

The successive approximations of convergents of the continued fraction are:

$$\frac{p_0}{q_0} = 29 \quad \frac{p_1}{q_1} = 29 + \frac{1}{2} = \frac{59}{2} = 29.5 \tag{3.253}$$

$$\frac{p_2}{q_2} = 29 + \frac{1}{2+}\,\frac{1}{2} = \frac{147}{5} = 29.4 \tag{3.254}$$

$$\frac{p_3}{q_3} = 29 + \frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{1} = \frac{206}{7} = 29.42857\cdots \tag{3.255}$$

$$\frac{p_4}{q_4} = 29 + \frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{5} = \frac{1177}{40} = 29.425 \tag{3.256}$$

$$\frac{p_5}{q_5} = 29 + \frac{1}{2+}\frac{1}{2+}\frac{1}{1+}\frac{1}{5+}\frac{1}{1} = \frac{1383}{47} = 29.425532\cdots \tag{3.257}$$

$$\frac{p_6}{q_6} = 29 + \frac{1}{2+}\frac{1}{2+}\frac{1}{1}\frac{1}{5+}\frac{1}{1+}\frac{1}{4} = \frac{6709}{228} = 29.4254386\cdots \tag{3.258}$$

$$\frac{p_4}{q_4} = 29 + \frac{1}{2+}\frac{1}{2+}\frac{1}{1+}\frac{1}{5} = \frac{1177}{40} = 29.425 \tag{3.259}$$

$$\frac{p_5}{q_5} = 29 + \frac{1}{2+}\frac{1}{2+}\frac{1}{1+}\frac{1}{5+}\frac{1}{1} = \frac{1383}{47} = 29.425531\cdots \tag{3.260}$$

$$\cdots = \cdots$$

Expansion (3.252) shows that the value of the continued fractions lies between its consecutive convergents p_{k+1}/q_{k+1} and p_k/q_k. To estimate the approximation error Huygens found the following differences of consecutive convergents:

$$\frac{p_1}{q_1} - \frac{p_0}{q_0} = \frac{59}{2} - \frac{29}{1} = \frac{1}{2} = 0.5 \tag{3.261}$$

$$\frac{p_1}{q_1} - \frac{p_2}{q_2} = \frac{59}{2} - \frac{147}{5} = \frac{1}{10} = 0.1 \tag{3.262}$$

$$\frac{p_3}{q_3} - \frac{p_2}{q_2} = \frac{206}{7} - \frac{147}{5} = \frac{1}{35} = 0.028571 \tag{3.263}$$

$$\frac{p_3}{q_3} - \frac{p_4}{q_4} = \frac{206}{7} - \frac{1177}{40} = \frac{1}{280} = 0.0035714 \tag{3.264}$$

$$\frac{p_5}{q_5} - \frac{p_4}{q_4} = \frac{1383}{47} - \frac{1177}{40} = \frac{1}{1880} = 0.00053191 \tag{3.265}$$

$$\cdots = \cdots$$

3.4 Algebraic Equations

Consider a quadratic equation

$$Ax^2 + Bx + C = 0 \tag{3.266}$$

The solution of quadratic equation can be expressed by continued fractions.

$$x = -b - \frac{c}{-b-}\,\frac{c}{-b-}\,\frac{c}{-b-}\cdots \tag{3.267}$$

$$b = \frac{B}{A} \qquad c = \frac{C}{A} \tag{3.268}$$

Proof Rewrite the quadratic equation (3.266) as

$$x^2 = -bx - c \tag{3.269}$$

$$b = \frac{B}{A} \qquad c = \frac{C}{A} \tag{3.270}$$

and divide it by x.

$$x = -b - \frac{c}{x} \tag{3.271}$$

Substituting x into the equation determines the continued fraction solution of the positive root.

$$\begin{aligned} x &= -b - \cfrac{c}{-b - \cfrac{c}{x}} = -b - \cfrac{c}{-b - \cfrac{c}{-b - \cfrac{c}{x}}} \\ &= -b - \cfrac{c}{-b - \cfrac{c}{-b - \cfrac{c}{-b - \cfrac{c}{\ddots}}}} = -b - \frac{c}{-b-}\,\frac{c}{-b-}\,\frac{c}{-b-}\cdots \end{aligned} \tag{3.272}$$

There is no structured method to solve higher degrees of algebraic equations with continued fractions. ■

Example 114 A sample quadratic equation, $x^2 - 5x - 1 = 0$.
The exact solutions of the equation are

$$x = \frac{5}{2} \pm \frac{1}{2}\sqrt{29} = \begin{cases} 5.192\,6 \\ 0.192\,58 \end{cases} \tag{3.273}$$

Let us rewrite the equation as

$$x^2 = 5x + 1 \tag{3.274}$$

and divide it by x.

$$x = 5 + \frac{1}{x} \tag{3.275}$$

Continued substitution gives us the solution.

$$x = 5 + \cfrac{1}{5 + \cfrac{1}{x}} = 5 + \cfrac{1}{5 + \cfrac{1}{5 + \cfrac{1}{\ddots}}} = 5 + \frac{1}{5+}\,\frac{1}{5+}\cdots \tag{3.276}$$

The numerical value of x converges to the exact solution quickly.

$$x = 5,\ 5.2,\ 5.1923,\ 5.1926,\ 5.1926,\ 5.1926, \cdots \tag{3.277}$$

Example 115 A second degree equation, $x^2 - c = 0$.

Solving algebraic equations by continued fraction is a combination of art and mathematics. Different people may solve a problem in different ways. The simplest way for this equation may be to derive continued factions for square root.

$$\begin{aligned} x = \sqrt{c} = 1 + \sqrt{c} - 1 = 1 + \frac{\left(\sqrt{c}-1\right)\left(\sqrt{c}+1\right)}{1+\sqrt{c}} = 1 + \frac{c-1}{1+\sqrt{c}} \\ = 1 + \cfrac{\sqrt{c}-1}{1 + 1 + \cfrac{\sqrt{c}-1}{1+\sqrt{c}}} = 1 + \cfrac{c-1}{2 + \cfrac{c-1}{1+\sqrt{c}}} \\ = 1 + \frac{c-1}{2+}\,\frac{c-1}{2+}\,\frac{c-1}{2+}\,\frac{c-1}{2+}\cdots \end{aligned} \tag{3.278}$$

We may also solve this equation in the following way:

$$x^2 - h^2 = c - h^2 \qquad h^2 < c \tag{3.279}$$

$$(x-h)(x+h) = c - h^2 \tag{3.280}$$

$$x = h + \frac{c-h^2}{h+x} \tag{3.281}$$

and therefore,

$$x = h + \frac{c-h^2}{h+x} = h + \frac{c-h^2}{2h+}\,\frac{c-h^2}{2h+}\,\frac{c-h^2}{2h+}\cdots \tag{3.282}$$

Let us assume $c = 9$. The positive solution of the equation $x^2 - 9 = 0$ will be $x = 3$. Substituting $c = 9$ in Eq. (3.278) makes a continued fraction for the solution $x = 3$.

$$3 = 1 + \frac{8}{2+}\,\frac{8}{2+}\,\frac{8}{2+}\,\frac{8}{2+}\cdots \tag{3.283}$$

Substituting $c = 9$ and $h = 1$ in Eq. (3.282) makes the same continued fractions for the solution $x = 3$.

$$3 = 1 + \frac{8}{2+}\,\frac{8}{2+}\,\frac{8}{2+}\,\frac{8}{2+}\cdots \tag{3.284}$$

However, substituting $c = 9$ and $h = 2$ in Eq. (3.282) makes another continued fractions for $x = 3$.

$$3 = 2 + \frac{5}{4+}\,\frac{5}{4+}\,\frac{5}{4+}\,\frac{5}{4+}\cdots \tag{3.285}$$

This example indicates that different continued fractions may have the same limit. Continued fractions of a number or a function is not unique.

Choosing $c = 16$, for $x = \sqrt{c} = 4$ makes the continued fraction (3.278) to be:

$$4 = 1 + \frac{15}{2+}\,\frac{15}{2+}\,\frac{15}{2+}\,\frac{15}{2+}\cdots \tag{3.286}$$

while choosing $h = 1, 2, 3$ make the following continued fractions for $x = 4$ with different convergence rates.

$$4 = 1 + \frac{15}{2+}\,\frac{15}{2+}\,\frac{15}{2+}\,\frac{15}{2+}\cdots \tag{3.287}$$

$$4 = 2 + \frac{12}{4+}\,\frac{12}{4+}\,\frac{12}{4+}\,\frac{12}{4+}\cdots \tag{3.288}$$

$$4 = 3 + \frac{7}{6+}\,\frac{7}{6+}\,\frac{7}{6+}\,\frac{7}{6+}\cdots \tag{3.289}$$

Example 116 Arithmetic with continued fractions.

Let us consider a general continued fraction.

$$x = b_0 + \frac{a_1}{b_1+}\,\frac{a_2}{b_2+}\,\frac{a_3}{b_3+}\cdots \tag{3.290}$$

$$= \left[b_0, \frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \ldots, \frac{a_n}{b_n}, \ldots\right] \tag{3.291}$$

Dividing both sides into 1 will be:

$$\frac{1}{x} = \frac{1}{b_0+}\,\frac{a_1}{b_1+}\,\frac{a_2}{b_2+}\,\frac{a_3}{b_3+}\cdots \tag{3.292}$$

Then, multiplying both sides by c will show

$$cx = cb_0 + \frac{ca_1}{b_1+}\,\frac{a_2}{b_2+}\,\frac{a_3}{b_3+}\cdots \tag{3.293}$$

$$\frac{c}{x} = \frac{c}{b_0+}\,\frac{a_1}{b_1+}\,\frac{a_2}{b_2+}\,\frac{a_3}{b_3+}\cdots \tag{3.294}$$

Then adding both sides by d yields:

$$x + d = d + b_0 + \frac{a_1}{b_1+}\,\frac{a_2}{b_2+}\,\frac{a_3}{b_3+}\cdots \tag{3.295}$$

$$\frac{1}{x} + d = d + \frac{1}{b_0+}\,\frac{a_1}{b_1+}\,\frac{a_2}{b_2+}\,\frac{a_3}{b_3+}\cdots \tag{3.296}$$

$$cx + d = d + cb_0 + \frac{ca_1}{b_1+}\,\frac{a_2}{b_2+}\,\frac{a_3}{b_3+}\cdots \tag{3.297}$$

$$\frac{c}{x} + d = d + \frac{c}{b_0+}\,\frac{a_1}{b_1+}\,\frac{a_2}{b_2+}\,\frac{a_3}{b_3+}\cdots \tag{3.298}$$

Example 117 Fibonacci numbers.

Consider the equation

$$x^2 - x - 1 = 0 \tag{3.299}$$

with the positive root

$$x = \frac{1+\sqrt{5}}{2} = 1.618033988749894848205\cdots \tag{3.300}$$

and write it in the form

$$x = 1 + \frac{1}{x} \tag{3.301}$$

Substituting this expression for x on the right side of the same expression, we have

$$x = 1 + \frac{1}{1 + \frac{1}{x}} \tag{3.302}$$

Repeating substitution for x, we obtain the simplest continued fractions in mathematics.

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}} = 1 + \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\cdots \tag{3.303}$$

This number $1.61803\cdots$ may now be approximated by a continued fractions as exact as desired, by cutting off the fractions (3.303) at a proper point. The

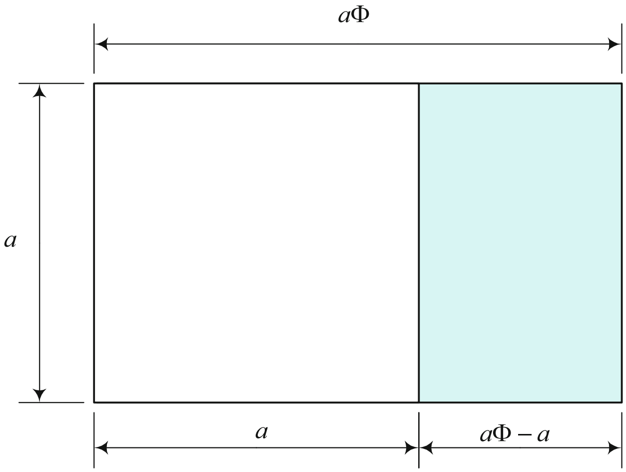

Fig. 3.2 The golden segment $\Phi = \frac{b}{a} = 1.6179775\cdots$

convergents of (3.303) are:

$$1, 2, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \frac{55}{34}, \frac{89}{55}, \frac{144}{89}, \cdots \tag{3.304}$$

The convergents converges quickly.

$$\frac{p_{11}}{q_{11}} = \frac{144}{89} = 1.6179775\cdots \tag{3.305}$$

The numerators and denominators of the convergents form the sequence of

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \cdots \tag{3.306}$$

which after the first two, each number is equal to the sum of the preceding two, making Fibonacci numbers (Olds 1963).

Example 118 ★Golden ratio.

An ancient mathematical problem was to find a rectangle with sides a and $a\Phi$ such that cutting a square of size $a \times a$ results in a rectangle similar to the initial rectangle, as is shown in Fig. 3.2.

$$\frac{a\Phi}{a} = \frac{a}{a\Phi - a} \tag{3.307}$$

$$\Phi^2 - \Phi = 1 \tag{3.308}$$

$$\Phi = \frac{1 + \sqrt{5}}{2} = 1.6179775\cdots \tag{3.309}$$

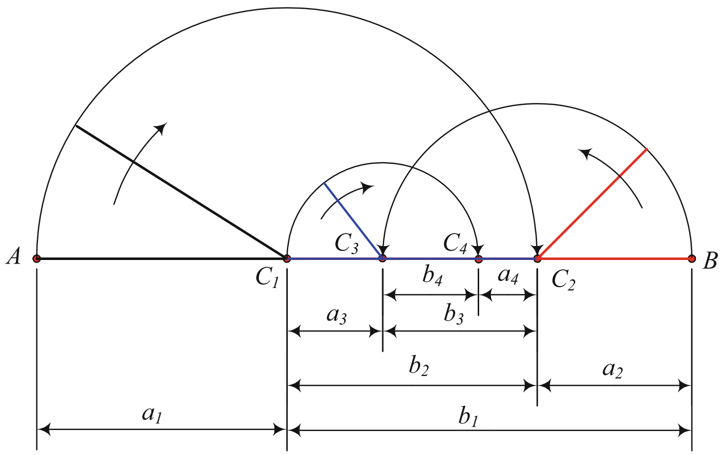

Fig. 3.3 Point C breaks the line AB by golden ratio $\Phi = \frac{b}{l}$. The ratio remains constant in folding process of the short segments, $\frac{a}{b} = \frac{b}{l}$

Because the equation for Φ contains a radical, the resulting number is algebraic, but not rational. It means that Φ is not equal to the ratio of two whole numbers. Its decimal expansion is infinite and nonrepeating, similar to π and e. Dividing Eq. (3.308) by Φ provides us with a recursive equation to develop continued fraction (Herz-Fischler 1987).

$$\Phi = 1 + \frac{1}{\Phi} \tag{3.310}$$

$$= 1 + \frac{1}{1+}\frac{1}{\Phi} = 1 + \frac{1}{1+}\;\frac{1}{1+}\;\frac{1}{1+}\cdots\frac{1}{\Phi} \tag{3.311}$$

The fraction Φ is called the golden ratio and it is expressed by the simplest possible continued fraction. The golden ratio has another interesting story that the Greeks believed the nature and art owed their beauty to this golden fraction. The division of a line segment AB by a point C such that the ratio of the parts a to b is the same as the ratio of b to the whole segment $l = a + b$ as shown in Fig. 3.3,

$$\frac{a}{b} = \frac{b}{a+b} \tag{3.312}$$

and renaming

$$\frac{b}{a} = \Phi \tag{3.313}$$

makes Eq. (3.312) to be equal to (3.310) with the same continued fraction expression as golden ratio. Dividing the line AB into segments a_1 and b_1 such that $l/b_1 = b_1/a_2 = \Phi$

$$\frac{a_1 + b_1}{b_1} = \frac{b_1}{a_1} = \Phi \tag{3.314}$$

$$\frac{1 + b_1/a_1}{b_1/a_1} = 1 + \frac{1}{b_1/a_1} = 1 + \frac{1}{\Phi} = \Phi \tag{3.315}$$

$$\Phi = 1.6179775 \tag{3.316}$$

allows us to fold the longer segment b_1 of the line to make the line AC_1 to be divided into $BC_2 = a_2$ and $C_1C_2 = b_2$ with the same golden ratio.

$$\Phi = \frac{b_1}{a_1} = \frac{a_2 + b_2}{a_1} = \frac{a_2 + b_2}{b_2} = \frac{1}{b_2/a_2} + 1 \tag{3.317}$$

$$\frac{b_2}{a_2} = \frac{1}{\Phi - 1} = \Phi \tag{3.318}$$

The folding process may keep going and providing the same golden ratio Φ every time.

Luca Pacioli (1447–1517) and Leonardo da Vinci (1452–1519) published a book, "Divina Proportione," in 1509 devoted to a study of the number $\varPhi$. Pacioli described thirteen interesting properties of $\varPhi$ in the pentagonal symmetry of certain flowers and marine animals, in the proportions of the human body (Pacioli 1509).

Discovery of the incommensurability of $\sqrt{2}$ as the diagonal of the unit square shocked the Greek mathematicians and Pythagoreans, who only believed in whole numbers in their metaphysics. The next logical extension of the concept was to consider a rectangle with sides equal to 1 and 2 with diagonal $\sqrt{5}$. The diagonal cuts the rectangle into two right triangles. Greeks using one of the triangles, rotated the triangle to form two sides of a new rectangle of $\left(\sqrt{5} + 1\right) \times 2$ (Clawson 1996). The resulting rectangle, shown in Fig. 3.4, has a number of interesting features, and according to the ancient Greeks, a very eye pleasing geometrical shape. The ratio of the two sides of this rectangle is:

$$\frac{1 + \sqrt{5}}{2} = \Phi = 1.6179775 \cdots \tag{3.319}$$

From the Rhind Papyrus, we know that the ancient Egyptians were aware of the golden ratio Φ and worked with it. The ratio of the altitude of a face of the Great Pyramid at Gizeh to half the length of the base is Φ. The golden ratio is well illustrated in different sections of pentagons, vastly used by ancient civilizations.

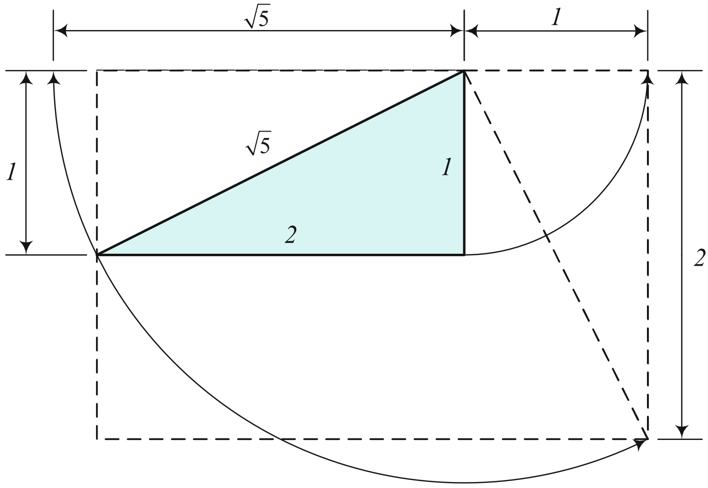

Fig. 3.4 The rectangle with aspect ratio equal to the golden section, $\Phi = \frac{1+\sqrt{5}}{2} = 1.6179775\cdots$

Pythagoreans believed the pentagram was sacred because of appearance of Φ in all sections.

Srinivasa Ramanujan (1887–1920) introduced an interesting equation containing Φ, π and e related by a continued fractions.

$$\left(\sqrt{2+\Phi}-\Phi\right)e^{\frac{2}{5}\pi} = \left(\sqrt{2+\frac{\sqrt{5}+1}{2}}-\frac{\sqrt{5}+1}{2}\right)e^{\frac{2}{5}\pi}$$
$$= \cfrac{1}{1+\cfrac{e^{-2\pi}}{1+\cfrac{e^{-4\pi}}{1+\cdots}}} \tag{3.320}$$

Example 119 SI paper size.

Because $\sqrt{2}$ is twice its inverse,

$$\sqrt{2} = \frac{2}{\sqrt{2}} \tag{3.321}$$

folding a rectangular piece of paper with sides in proportion $\sqrt{2}/1$ makes a new rectangular piece of paper with sides in proportion $\sqrt{2}/1$. The SI standard sizes of an $A0$ paper are defined such that its area to be one square meter and its sides to be proportional to $\sqrt{2}/1$. Assuming a and b as the long and short sides of $A0$, we have

$$ab = 1000000\,\mathrm{mm}^2 \tag{3.322}$$

$$a/b = \sqrt{2}/1 \tag{3.323}$$

which makes an equation to determine a.

$$\frac{a^2}{1000000} = \sqrt{2} \qquad a = 1189.2\,\text{mm} \tag{3.324}$$

$$b = \frac{1000000}{a} = 840.9\,\text{mm} \tag{3.325}$$

As we know

$$\sqrt{2} = 1 + \frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{2+}\cdots \tag{3.326}$$

with convergents

$$1, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \frac{239}{169}, \frac{577}{408}, \frac{1393}{985} \tag{3.327}$$

accepting $a/b = 3/2$, gives

$$b = \sqrt{1000000 \times \left(\frac{3}{2}\right)^{-1}} = 816.5 \tag{3.328}$$

$$a = \frac{1000000}{b} = 1224.7 \tag{3.329}$$

and accepting $a/b = 7/5$, gives

$$b = \sqrt{1000000 \times \left(\frac{7}{5}\right)^{-1}} = 845.15 \tag{3.330}$$

$$a = \frac{1000000}{b} = 1183.2 \tag{3.331}$$

and accepting $a/b = 17/12$, gives

$$b = \sqrt{1000000 \times \left(\frac{17}{12}\right)^{-1}} = 840.17 \tag{3.332}$$

$$a = \frac{1000000}{b} = 1190.2 \tag{3.333}$$

and accepting $a/b = 41/29$, gives

$$b = \sqrt{1000000 \times \left(\frac{41}{29}\right)^{-1}} = 841.02 \tag{3.334}$$

$$a = \frac{1000000}{b} = 1189.0 \tag{3.335}$$

This is a practical approximation. The standard size of $A0$ is 1189 mm × 841 mm which is equivalent to approximating $\sqrt{2}$ as:

$$\sqrt{2} = 1 + \frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{2} \tag{3.336}$$

3.5 Chapter Summary

All real numbers may be divided into rational and irrational. Also they may better be divided into algebraic and transcendental. A rational number can be expressed by a fraction of the form p/q where p and q are integers with $q \neq 0$. Irrational numbers cannot be expressed by a fraction. Numbers also are either algebraic or transcendental. Algebraic numbers are all non-integral real roots of algebraic equations with rational coefficients a_i, $(i = 1, 2, \ldots, n)$.

$$x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0 = 0 \tag{3.337}$$

For example, the irrational numbers $x = \sqrt{3}$ and $x = \sqrt[3]{8}$ are roots of $x^5 - 3x^3 - 8x^2 + 24 = 0$. Transcendental numbers cannot be the root of any algebraic equation, they cannot be expressed by fractions. Although majority of the real number line is occupied by transcendental numbers, only a few of them appear in usual problems of science and engineering. In all those rare cases also we usually approximate them by rational numbers to be able to use them in calculation. π and e are the most famous and most applied transcendental numbers. Because there is no way to show the transcendental numbers numerically, we use symbols to represent them.

Continued fraction is another way to represent approximation for those numbers that are not rational. However, any rational or irrational number x may be expressed by a continued fraction of the form

$$x = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cdots}}} = b_0 + \frac{a_1}{b_1+}\,\frac{a_2}{b_2+}\,\frac{a_3}{b_3+}\cdots \tag{3.338}$$

The sequence of truncations p_k/q_k are called the convergents of the continued fraction.

$$\frac{p_0}{q_0} = \frac{b_0}{1} \qquad \frac{p_k}{q_k} = b_0 + \frac{a_1}{b_1+}\,\frac{a_2}{b_2+}\,\frac{a_3}{b_3+}\cdots\frac{a_k}{b_k} \tag{3.339}$$

Convergence of a continued fraction is equivalent with $\lim_{k\to\infty} p_k/q_k = 0$. There is a recursive equation for p_k and q_k making it possible to calculate them from their previous ones.

$$p_n = b_n p_{n-1} + a_n p_{n-2} \tag{3.340}$$

$$p_n = b_n q_{n-1} + a_n q_{n-2} \tag{3.341}$$

Continued fractions are not unique. A limit may be expressed by different continued fractions. There are also techniques to find equivalent, but different continued fractions from one continued fractions. For example, we can transform a continued fraction

$$x = b_0 + \frac{a_1}{b_1+}\,\frac{a_2}{b_2+}\cdots\frac{a_n}{b_n+}\cdots \tag{3.342}$$

to

$$x = b_0 + \frac{a_1}{b_1 - 1+}\,\frac{a_2 + b_2 - 1}{b_2 - 1+}\,\frac{\frac{a_3 + b_3 - 1}{a_2 + b_2 - 1}a_2}{b_3 - 1 + \frac{a_3 + b_3 - 1}{a_2 + b_2 - 1}+}\cdots$$
$$\frac{\frac{a_n + b_n - 1}{a_{n-1} + b_{n-1} - 1}a_{n-1}}{b_n + 1 + \frac{a_n + b_n - 1}{a_{n-1} + b_{n-1} - 1}+}\cdots \tag{3.343}$$

or we can show for any real numbers $a_1, a_2, a_3, \ldots, b_1, b_2, b_3, \ldots$ and real nonzero constants $c_1, c_2, c_3, \ldots$, we have:

$$b_0 + \frac{a_1}{b_1+}\,\frac{a_2}{b_2+}\,\frac{a_3}{b_3+}\cdots\frac{a_k}{b_k+}\cdots$$
$$= b_0 + \frac{c_1 a_1}{c_1 b_1+}\,\frac{c_2 c_1 a_2}{c_2 b_2+}\,\frac{c_3 c_2 a_3}{c_3 b_3+}\cdots\frac{c_k c_{k-1} a_k}{c_k b_k+}\cdots \tag{3.344}$$

Converting a series to continued fractions is the most useful method in developing continued fractions. If a number x is defined by a series

$$x = c_0 + c_1 + c_2 + \cdots + c_n = \sum_{k=0}^{n} c_k \tag{3.345}$$

then there exists a continued fraction equivalent to the approximated series.

$$x = c_0 + \frac{c_1}{1+} \frac{-c_2/c_1}{1 + c_2/c_1+} \cdots \frac{-c_m/c_{m-1}}{1 + c_m/c_{m-1}+} \cdots \tag{3.346}$$

3.6 Key Symbols

a, b, c, d	Integer
a, b, c	Length
A, B	Real number
A, B	Points on real number axis
A, B, C	Coefficients in algebraic equations
AD	After death
c	Positive number
c_k	$= \lim_{k\to\infty} \frac{p_k}{q_k}$ limit of a continued fraction
$\lfloor c \rfloor$	The largest integer of c
C_E	Circumference of the Earth
d	Day
D	Diameter
e	$2.718281828459\ldots$
f	Function
g	Function
h	Hour
i	Imaginary unit, $i^2 = -1$
k	Integer counter
l	Length
min	Minute
p, q, r, s	Integers
p/q	Fraction
p_k	Numerator of a convergent
p_k/q_k	Convergent of a continued fraction
q_k	Denominator of a convergent
P_k/Q_k	Convergent of a continued fraction
q	Generalized coordinate
R_e	Radius of the Earth at the equator
R	Radius
R	Remainder
s	Second, unit of time
V	Volume
x	Number, unknown number
y	Function
y	Year
$\propto$	Proportional
∞	Infinity
!	Factorial
$\mathbb{N}$	Integer numbers
Φ	Golden ratio
π	$3.14159265359\ldots$

Exercises

1. Your personal rational number.

   Multiply the numbers of your birth year and divide it by sum of the numbers of your birth year to make a rational fraction. Ignore zeros. Express the fraction with continued fractions. For example, if the birth year is 2021, the fraction is $4/5$, and if the birth year is 1991, the fraction is $81/20$.
2. Square root of odd numbers.

   Calculate the continued fractions of the following square roots up to eleven terms.

$$\sqrt{3} = 1.7320508075688772935 27 \tag{3.347}$$

$$\sqrt{5} = 2.2360679774997896964 09 \tag{3.348}$$

$$\sqrt{7} = 2.6457513110645905905 02 \tag{3.349}$$

$$\sqrt{11} = 3.3166247903553998491 15 \tag{3.350}$$

$$\sqrt{13} = 3.6055512754639892931 19 \tag{3.351}$$

   Are they all periodic? Is continued fraction of square roots always periodic?
3. Square root of even numbers.

   Calculate the continued fractions of the following square roots up to eleven terms.

$$\sqrt{2} = 1.4142135623730950488 02 \tag{3.352}$$

$$\sqrt{6} = 2.4494897427831780981 97 \tag{3.353}$$

$$\sqrt{8} = 2.8284271247461900976 04 \tag{3.354}$$

$$\sqrt{10} = 3.1622776601683793319 99 \tag{3.355}$$

$$\sqrt{12} = 3.4641016151377545870 54 \tag{3.356}$$

   Are they all periodic? Is there any difference between periodicity of continued fractions of square root of prime numbers and composite numbers? Is there any difference between periodicity of continued fractions of square root of odd and even numbers?
4. Continued fraction of square root addition.

   (a) Calculate the continued fractions of the following terms.

$$A = \sqrt{2} + \sqrt{3} \tag{3.357}$$

$$B = \sqrt{2} + \sqrt{5} \tag{3.358}$$

$$C = \sqrt{2} + \sqrt{6} \tag{3.359}$$

$$D = \sqrt{2} + \sqrt{7} \tag{3.360}$$

$$E = \sqrt{2} + \sqrt{8} \tag{3.361}$$

$$F = \sqrt{2} + \sqrt{9} \tag{3.362}$$

(b) Calculate the continued fractions of the following terms.

$$A = \sqrt{2} + \sqrt{3} \tag{3.363}$$

$$G = \sqrt{3} + \sqrt{5} \tag{3.364}$$

$$H = \sqrt{5} + \sqrt{6} \tag{3.365}$$

$$I = \sqrt{6} + \sqrt{7} \tag{3.366}$$

$$J = \sqrt{7} + \sqrt{8} \tag{3.367}$$

$$K = \sqrt{8} + \sqrt{9} \tag{3.368}$$

(c) How can you add the continued fractions of the right-hand side numbers to get the combined continued fraction of their sum?

5. $\sqrt{2}$ in a different way.
   Use the identity

$$\sqrt{2} - 1 = \frac{1}{2 + \left(\sqrt{2} - 1\right)} \tag{3.369}$$

and calculate the continued fraction of $\sqrt{2}$.

$$\begin{aligned}\sqrt{2} &= 1 + \frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{2 + \left(\sqrt{2} - 1\right)} \\ &= 1 + \frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{2+} \cdots \end{aligned} \tag{3.370}$$

6. Identity and flip method
   Use the identity

$$\sqrt{1+x} - 1 = \frac{x}{2 + \left(\sqrt{1+x} - 1\right)} \tag{3.371}$$

and apply the flip method to calculate the continued fraction of $\sqrt{1+x}$.

$$\sqrt{1+x} = 1 + \frac{x}{2+}\,\frac{x}{2+}\,\frac{x}{2+}\,\frac{x}{2 + \left(\sqrt{1+x} - 1\right)}$$

$$= 1 + \frac{x}{2+}\,\frac{x}{2+}\,\frac{x}{2+}\,\frac{x}{2+}\,\frac{x}{2+}\cdots \tag{3.372}$$

7. ★Speed of convergence.
   Compare the convergence of the following series expansion and continued fraction, by increasing the number of terms for $x = 0.1$, $x = 0.9$, $x = 1$, $x = 2$.

$$\begin{aligned}\sqrt{1+x} - 1 = &\frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5}{128}x^4 + \frac{7}{256}x^5 \\ &- \frac{21}{1024}x^6 + \frac{33}{2048}x^7 - \frac{429}{32768}x^8 \\ &+ \frac{33}{65536}x^9 - \frac{2431}{262144}x^{10} + \cdots \end{aligned} \tag{3.373}$$

$$\sqrt{1+x} - 1 = \frac{x}{2+}\,\frac{x}{2+}\,\frac{x}{2+}\,\frac{x}{2+}\,\frac{x}{2+}\,\frac{x}{2+}\cdots \tag{3.374}$$

   The series expansion converges only for $|x| < 1$ and diverges when $|x| > 1$.
8. Repeating continued fractions.
   Determine the numerical value of the simple continued fractions.
   (a) $A = [6, 22, 3, 11, 21, 21, 21, 21, 21, \ldots]$
   (b) $A = [2, 5, 7, 15, 15, 4, 7, 15, 15, 4, \ldots]$
9. Continued fractions of π.
   (a) Calculate the continued fractions of the following terms.

$$\pi = 3.141592653589793238463 \tag{3.375}$$

$$\pi^2 = 9.869604401089358618837 \tag{3.376}$$

$$\pi^3 = 31.00627668029982017549 \tag{3.377}$$

$$\pi^4 = 97.40909103400243723648 \tag{3.378}$$

$$\pi^5 = 306.0196847852814532629 \tag{3.379}$$

$$\pi^6 = 961.3891935753044370309 \tag{3.380}$$

   Are they all periodic? Is there any relation between their continued fractions? Is it possible to calculate continued fraction of π^{k+1} from the continued fraction of π^k or previous exponents?
   (b) Calculate the continued fractions of the following terms.

$$\pi = 3.141592653589793238463 \tag{3.381}$$

$$\sqrt{\pi} = 1.772453850905516027298 \tag{3.382}$$

$$\sqrt[3]{\pi} = 1.46459188756152326302 \tag{3.383}$$

$$\sqrt[4]{\pi} = 1.33133536380038971279 \tag{3.384}$$

$$\sqrt[5]{\pi} = 1.25727411566918505938 \tag{3.385}$$

$$\sqrt[6]{\pi} = 1.21020324225376427596 \tag{3.386}$$

Are they all periodic? Is there any relation between their continued fractions? Is it possible to calculate continued fraction of $\pi^{1/(k+1)}$ from the continued fraction of $\pi^{1/k}$ or previous exponents?

10. Continued fractions of e.
    Show that the continued fraction expansion of e is:

$$e = 2 + \frac{1}{1+}\frac{1}{2+}\frac{1}{2+}\frac{1}{3+}\frac{1}{3+}\frac{1}{4+}\frac{1}{4+}\frac{1}{5+}\frac{1}{5+}\cdots \tag{3.387}$$

$$\sqrt{e} = 1 + \frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{5+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{9+}\cdots \tag{3.388}$$

$$\coth\frac{1}{4} = \frac{e+1}{e-1} = 2 + \frac{1}{6+}\frac{1}{10+}\frac{1}{14+}\frac{1}{18+}\frac{1}{22+}\cdots \tag{3.389}$$

11. π as continued radicals.
    Show that

$$\frac{2}{\pi} = \cos\frac{90}{2}\cos\frac{90}{4}\cos\frac{90}{8}\cdots$$

$$= \sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots \tag{3.390}$$

    and covert the continued radical to continued fractions. This equation was found by Francois Viete (1540–1603) in 1593 (Kline 1972).

12. Exponential function.
    Show that

    (a)

$$\frac{e^{\pi}-1}{e^{\pi}+1} = \cfrac{\pi}{2+\cfrac{\pi^2}{6+\cfrac{\pi^2}{10+\cfrac{\pi^2}{14+\cdots}}}} \tag{3.391}$$

(b)

$$e^z = 1 + \frac{z}{1-}\,\frac{z}{2+}\,\frac{z}{3-}\,\frac{z}{2+}\,\frac{z}{5-}\,\frac{z}{2+}\cdots \tag{3.392}$$

(c)

$$\begin{aligned} e^x &= \frac{x^0}{0!}\,\frac{x^1}{1!}\,\frac{x^2}{2!}\,\frac{x^3}{3!}\,\frac{x^4}{4!}\,\frac{x^5}{5!}\cdots \\ &= 1 + \frac{x}{1-}\;\frac{1x}{2+x-}\;\frac{2x}{3+x-}\;\frac{3x}{4+x-}\cdots \end{aligned} \tag{3.393}$$

(d)

$$e^{x/y} == 1 + \frac{2x}{2y-x+}\;\frac{x^2}{6y+}\;\frac{x^2}{10y+}\;\frac{x^2}{14y+}\;\frac{x^2}{18y+}\cdots \tag{3.394}$$

13. Continued radicals of a number.
    Show that

(a)

$$3 = \sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+5\sqrt{1+\cdots}}}}} \tag{3.395}$$

(b)

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2}\,\frac{\sqrt{2+\sqrt{2}}}{2}\,\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\cdots \tag{3.396}$$

(c)

$$2 = \sqrt[3]{6+\sqrt[3]{6+\sqrt[3]{6+\sqrt[3]{6+\sqrt[3]{6+}}}}}\cdots \tag{3.397}$$

14. π number.
    Show that

(a)

$$\pi = \frac{4}{1} - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \cdots = 4\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1} \tag{3.398}$$

(b)

$$\frac{\pi}{2} = \frac{2}{1} \times \frac{2}{3} \times \frac{4}{3} \times \frac{4}{5} \times \frac{6}{5} \times \frac{6}{7} \times \frac{8}{7} \times \frac{8}{9} \cdots$$
$$= \frac{4}{3} \times \frac{16}{15} \times \frac{36}{35} \times \frac{64}{63} \cdots = \prod_{k=1}^{\infty} \frac{(2k)^2}{(2k)^2 - 1} \tag{3.399}$$

(c)

$$\pi = \cfrac{4}{1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \cfrac{9^2}{2 + \cdots}}}}}} \tag{3.400}$$

(d)

$$\pi = 3 + \cfrac{1^2}{6 + \cfrac{3^2}{6 + \cfrac{5^2}{6 + \cfrac{7^2}{6 + \cfrac{9^2}{6 + \cdots}}}}} \tag{3.401}$$

(e)

$$\pi = \cfrac{4}{1 + \cfrac{1^2}{3 + \cfrac{2^2}{5 + \cfrac{3^2}{7 + \cfrac{4^2}{9 + \cdots}}}}} \tag{3.402}$$

(f)

$$\frac{\pi - 3}{4 - \pi} = \frac{1^2}{4+} \frac{2^2}{1+} \frac{3^2}{4+} \frac{4^2}{1+} \frac{5^2}{4+} \frac{6^2}{1+} \frac{7^2}{4+} \cdots \tag{3.403}$$

(g)

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} dx = \pi \tag{3.404}$$

$$\int_{-1}^{1} \sqrt{1-x^2} dx = \frac{\pi}{2} \tag{3.405}$$

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} \tag{3.406}$$

15. Continued fractions and continued radicals.
    Show that

$$\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}}} = \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}}} \tag{3.407}$$

16. Continued radicals of a function.
    Show that

$$1+x = \sqrt{1+x\sqrt{1+(1+x)\sqrt{1+(2+x)\sqrt{1+(3+x)\sqrt{1+\cdots}}}}} \tag{3.408}$$

which as a sample for $x = 2$ will be

$$3 = \sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+5\sqrt{1+\cdots}}}}} \tag{3.409}$$

17. Factorial series.

    (a) Show that

$$\frac{1}{2} = \sum_{k=1}^{\infty} \frac{1}{3^k} = \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4+\cdots} \tag{3.410}$$

(b) Illustrate the series using trisectioning a square or circle.
(c) Convert the series into a continued fractions.

18. A few famous continued fractions.
Show

(a) Schwenter's continued fraction.

$$\frac{177}{233} = \frac{1}{1+}\ \frac{1}{3+}\ \frac{1}{6+}\ \frac{1}{4+}\ \frac{1}{2} \tag{3.411}$$

(b) Euler's continued fractions.

$$\frac{1461}{59} = 24 + \frac{1}{1+}\ \frac{1}{3+}\ \frac{1}{4+}\ \frac{1}{1+}\ \frac{1}{2} \tag{3.412}$$

$$1 = \frac{2}{1+}\ \frac{3}{2+}\ \frac{4}{3+}\ \frac{5}{4+}\ \frac{6}{5+}\ \frac{7}{6+} \cdots \tag{3.413}$$

(c) Smith's continued fraction.

$$\frac{1}{1-}\ \frac{1}{4-}\ \frac{1}{1-}\ \frac{1}{4-} \cdots \frac{1}{4} = \frac{2n}{n+1} \tag{3.414}$$

(d) Ramanujan continued fractions.

$$\frac{4}{3} = \frac{3}{1+}\ \frac{4}{2+}\ \frac{5}{3+}\ \frac{6}{4+}\ \frac{7}{5+} \cdots \tag{3.415}$$

$$\frac{5}{3} = \frac{4}{1+}\ \frac{6}{3+}\ \frac{8}{5+}\ \frac{10}{7+}\ \frac{12}{9+} \cdots \tag{3.416}$$

$$\frac{1}{e-1} = \frac{1}{1+}\ \frac{2}{2+}\ \frac{3}{3+}\ \frac{4}{4+}\ \frac{5}{5+} \cdots \tag{3.417}$$

$$\frac{e^x-1}{e^x+1} = \frac{x}{2+}\ \frac{x^2}{6+}\ \frac{x^2}{8+}\ \frac{x^2}{10+}\ \frac{x^2}{12+} \cdots \tag{3.418}$$

$$\frac{e^\pi-1}{e^\pi+1} = \frac{\pi}{2+}\ \frac{\pi^2}{6+}\ \frac{\pi^2}{8+}\ \frac{\pi^2}{10+}\ \frac{\pi^2}{12+} \cdots \tag{3.419}$$

$$n = \frac{1}{1-n+}\ \frac{2}{2-n+}\ \frac{3}{3-n+} \cdots \frac{n}{0+}\ \frac{n+1}{1+}\ \frac{n+2}{2+} \cdots \tag{3.420}$$

$n \in \mathbb{N}$

$$1 = \frac{x+1}{x+}\;\frac{x+2}{x+1+}\;\frac{x+3}{x+2+}\;\frac{x+4}{x+3+}\cdots \tag{3.421}$$

$$\begin{aligned} 2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{x+2k-1} &= 2\left(\frac{1}{x+1}-\frac{1}{x+3}+\frac{1}{x+5}-\cdots\right) \\ &= \frac{1}{x+}\,\frac{1^2}{x+}\,\frac{2^2}{x+}\,\frac{3^2}{x+}\,\frac{4^2}{x+}\cdots \end{aligned} \tag{3.422}$$

19. Series and continued fractions.
    Show that

$$\begin{aligned} &\frac{1}{b_1}-\frac{1}{b_2}+\frac{1}{b_3}-\frac{1}{b_4}+\cdots = \\ &\quad \frac{1}{b_1+}\;\frac{b_1^2}{b_2-b_1+}\;\frac{b_2^2}{b_3-b_2+}\;\frac{b_3^2}{b_4-b_3+}\cdots \end{aligned} \tag{3.423}$$

20. ★The equivalence transformation.
    Show that

$$b_0+\cfrac{a_1}{b_1+\cfrac{a_2}{b_2+\cfrac{a_3}{b_3+\cfrac{a_4}{b_4+\cdots}}}} = b_0+\cfrac{c_1a_1}{c_1b_1+\cfrac{c_1c_2a_2}{c_2b_2+\cfrac{c_2c_3a_3}{c_3b_3+\cfrac{c_3c_4a_4}{c_4b_4+\cdots}}}} \tag{3.424}$$

21. ★Euler's continued fraction formula.

    (a) Prove that

$$\begin{aligned} &b_0+b_0b_1+b_0b_1b_2+\cdots+b_0b_1b_2\cdots b_n \\ &= \frac{b_0}{1-}\,\frac{b_1}{1+b_1-}\,\frac{b_2}{1+b_2-}\cdots\frac{b_n}{1+b_n} \end{aligned} \tag{3.425}$$

    (b) Prove that if we have

$$x = a-b+c-d+e-f+\cdots \tag{3.426}$$

    then

$$x = \cfrac{a}{1+\cfrac{b}{a-b+\cfrac{ac}{b-c+\cfrac{bd}{c-d+\cfrac{ce}{d-e+\cdots}}}}} \tag{3.427}$$

(c) Prove that if we have

$$x = \frac{1}{a} - \frac{1}{b} + \frac{1}{c} - \frac{1}{d} + \frac{1}{e} - \frac{1}{f} + \cdots \tag{3.428}$$

then

$$x = \cfrac{1}{a + \cfrac{a^2}{b - a + \cfrac{b^2}{c - b + \cfrac{c^2}{d - c + \cfrac{d^2}{e - d + \cdots}}}}} \tag{3.429}$$

(d) Prove that if we have

$$x = \frac{1}{a} - \frac{1}{a-b} + \frac{1}{a-2b} - \frac{1}{a-3b} + \frac{1}{a-4b} - \cdots \tag{3.430}$$

then

$$x = \cfrac{1}{a + \cfrac{a^2}{b + \cfrac{(a+b)^2}{b + \cfrac{(a+2b)^2}{b + \cfrac{(a+3b)^2}{b + \cdots}}}}} \tag{3.431}$$

and Euler (1988).

$$\frac{1}{x} - a = \cfrac{a^2}{b + \cfrac{(a+b)^2}{b + \cfrac{(a+2b)^2}{b + \cfrac{(a+3b)^2}{b + \cdots}}}} \tag{3.432}$$

22. Periodic continued fractions and convergents.
Verify that if

$$x = \frac{1}{a} + \frac{1}{b+}\frac{1}{a+}\frac{1}{b+}\frac{1}{a+}\frac{1}{b+} \cdots \tag{3.433}$$

then

$$p_{n+2} - (ab+2)\, p_n + p_{n-2} = 0 \tag{3.434}$$

23. Numerical series evaluation.
    Show that if $|c| < 1$ then the geometric series

$$1 + c + c^2 + c^3 + \cdots + c^k + \cdots = \frac{1}{1-c} \tag{3.435}$$

converges and has the limit. Get the result of few examples such as:

$$1 + \frac{1}{10} + \frac{1}{100} + \cdots = \frac{1}{1-0.1} = 1\frac{1}{9} \tag{3.436}$$

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{1}{1-0.5} = 2 \tag{3.437}$$

24. Multiplication of summations.
    (a) Prove

$$\left(\sum_{k=0}^{\infty} a_k\right)\left(\sum_{k=0}^{\infty} b_k\right) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k b_{n-k} \tag{3.438}$$

    (b) Prove the Lagrange's identity:

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 = \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) - \sum_{k=1}^{n}\sum_{j=k+1}^{n}\left(a_k b_i - a_j b_k\right)^2 \tag{3.439}$$

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Chapter 4
Functional Continued Fractions

Continued fractions, compared to series solutions, converge much faster, have much larger convergence domain, and are much more accurate. These advantages justify to solve equations with series solution and convert them into continued fractions to get the best result. We will show this procedure and its advantages in this chapter by reviewing the concept of the continued fractions and studying their application in science and engineering.

All mathematical functions may be transformed into continued summation expansion. A function $y = f(x)$ is called *algebraic function* if it satisfies an equation of the form

$$P_0(x) + P_1(x)\,y + P_2(x)\,y^2 + \cdots + P_n(x)\,y^n = 0 \tag{4.1}$$

where each $P_i(x)$ is a polynomial. The algebraic functions including polynomials, trigonometric, inverse trigonometric, exponential, and logarithmic functions, and all other functions that can be constructed from these functions by algebraic operations are called *elementary functions*. Beyond the elementary functions there are the *special functions*. Many of the special functions discovered as solutions of special second-order linear differential equations,

$$y'' + P(x)\,y' + Q(x)\,y = R(x) \tag{4.2}$$

for which there is no general solution, unless the coefficients are constant (Simmons 1991).

In this chapter, we review continued fractions expansion of mathematical functions and differential equations.

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4.1 Power Series Expansion of Functions

A function $f(x)$ may be expressed by a series of function

$$f(x) = f_0(x) + f_1(x) + f_2(x) + \cdots + f_k(x) + \cdots = \sum_{k=0}^{\infty} f_k(x) \tag{4.3}$$

defined at any point x_0 that the series converges. The simplest series of functions is the Maclaurin power series that approximates functions by polynomials around $x = 0$.

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k + \cdots = \sum_{k=0}^{\infty} a_k x^k \tag{4.4}$$

$$a_k = \frac{1}{k!} f^{(k)}(x) \tag{4.5}$$

A power series does not include terms with negative powers. For a power series $f(x) = \sum_{k=0}^{\infty} a_k x^k$ one of the following happens.

1. The series converges at any point in an open interval $(-s, s)$ and diverges at any point out of the interval. At the endpoints, of the interval, the series may be convergent or divergent, $[-s, s]$, $[-s, s)$, $(-s, s]$, $(-s, s)$. The s is called *radius of convergence*.

$$s = \lim_{n \to \infty} \frac{a_n}{a_{n+1}} \tag{4.6}$$

2. The series converges only at $x = 0$.
3. The series converges for $(-\infty, \infty)$.

A more general power series approximation of a function is the Taylor series:

$$P(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n \tag{4.7}$$

Taylor's formula is to approximate a function $f(x)$ around a point $x = c$ such that it matches $f(c)$ and all its derivatives at $x = c$. Suppose that f, f', f'', $\cdots$, $f^{(n)}$ are all continuous for all x within the convergence zone $|x - c| < s$. Then there exists some a between c and x such that

$$f(x) = P(x) + R(x) \tag{4.8}$$

$$P(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots$$

$$+\frac{f^{(n)}(c)}{n!}(x-c)^n \tag{4.9}$$

$$R(x)=\frac{f^{(n+1)}(a)}{(n+1)!}(x-c)^{n+1} \tag{4.10}$$

The function $R(x)$ is called the remainder.

Proof If function $f(x)$ is expressed by a power series,

$$f(x)=a_0+a_1x+a_2x^2+\cdots+a_kx^k+\cdots \tag{4.11}$$

then substituting $x=0$ yields

$$a_0=f(0) \tag{4.12}$$

A differentiation of the function

$$f'(x)=a_1+2xa_2+3x^2a_3+4x^3a_4+\cdots+ka_kx^{k-1}+\cdots \tag{4.13}$$

shows that

$$a_1=f'(0) \tag{4.14}$$

and next derivative shows that

$$f''(x)=2a_2+6xa_3+12x^2a_4+\cdots+k(k-1)a_kx^{k-2}+\cdots \tag{4.15}$$

$$a_2=\frac{1}{2}f''(0) \tag{4.16}$$

Continuing this process indicates that the coefficients of the power series expansion of the function $f(x)$ are:

$$a_k=\frac{1}{k!}f^{(k)}(0) \tag{4.17}$$

and therefore, expansion of $f(x)$ around $x=0$ is a Maclaurin series.

$$f(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\cdots+\frac{f^{(n)}(0)}{n!}x^n \tag{4.18}$$

Any continues function $f(x)$ with continues finite derivatives in the interval $(-s,s)$ may be expanded in the Taylor power series expansion around $x=c$, where $|c|<|s|$,

$$f(x) = P(x) + R(x) \tag{4.19}$$

$$P(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n \tag{4.20}$$

provided for some $0 < a < c$, the Lagrange form of the remainder $R(x)$ approaches zero.

$$R(x) = \frac{f^{(n+1)}(a)}{(n+1)!}(x-c)^{n+1} \tag{4.21}$$

$$\lim_{n\to\infty} R(x) = 0 \tag{4.22}$$

The Taylor series is convergent when

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{f^{(n+1)}(0)}{(n+1)\, f^{(n)}(0)} = 0 \tag{4.23}$$

The convergence radius of a convergent power series is infinity $(-\infty, \infty)$.

$$s = \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} (n+1)\frac{f^{(n)}(0)}{f^{(n+1)}(0)} = \infty \tag{4.24}$$

A function $f(x)$ that has a Taylor series expansion around $x = x_0$

$$f(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k \qquad a_k = \frac{f^{(k)}(x_0)}{k!} \tag{4.25}$$

is analytic at the point $x = x_0$. If $x_0 = 0$, the $f(x)$ is analytic at all points.

There are different, but equivalent, ways to express the power series expansion formula.

$$\begin{aligned} f(x+a) &= f(x) + af'(x) + \frac{a^2}{2!}f''(x) + \frac{a^3}{3!}f'''(x) + \cdots \\ &+ \frac{a^n}{n!}f^{(n)}(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x)}{k!}x^k \end{aligned} \tag{4.26}$$

The power series expansion of a function of two variables

$$z = f(x, y) \tag{4.27}$$

is an expansion of the method for the functions of one variable (Holmes 2009). The expansion of a two variables function around a point $(x, y) = (a, b)$ is:

$$f(x, y) = f(a, b) + \frac{\partial f(a, b)}{\partial x}\frac{(x-a)}{1!} + \frac{\partial f(a, b)}{\partial y}\frac{(y-b)}{1!} + \frac{\partial^2 f(a, b)}{\partial x^2}\frac{(x-a)^2}{2!} + \frac{\partial^2 f(a, b)}{\partial y^2}\frac{(y-b)^2}{2!} + \frac{\partial^2 f(a, b)}{\partial x \partial y}(x-a)(y-b) + \cdots \quad (4.28)$$

■

Example 120 Power series expansion of $x = \exp(-\alpha t)$.

The exponential function with negative exponent appears in analysis of many engineering and physical phenomena. Derivation of power series expansion of a given function is quite easy that involves only differentiation and numerical calculation. The exponential function

$$f(t) = \exp(-\alpha t) \qquad \alpha > 0 \quad (4.29)$$

will be approximated by a power series.

$$f(t) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n \quad (4.30)$$

The first term is a substitution.

$$f(0) = e^{-0} = 1 \quad (4.31)$$

The second term needs a differentiation.

$$f'(t) = -ae^{-at} \qquad \frac{f'(0)}{1!} = -ae^{-0} = -a \quad (4.32)$$

The third term needs a second differentiation.

$$f''(t) = a^2 e^{-at} \qquad \frac{f''(0)}{2!} = \frac{a^2 e^{-0}}{2!} = \frac{a^2}{2!} \quad (4.33)$$

This process may be continued up to the desired number of terms and derive the series expansion of the function.

$$f(t) = e^{-\alpha t} = 1 - \alpha t + \frac{\alpha^2}{2}t^2 - \frac{\alpha^3}{3!}t^3 + \frac{\alpha^4}{4!}t^4 - \frac{\alpha^5}{5!}t^5 + \cdots \quad (4.34)$$

The convergence radius of this power series is infinity $(-\infty, \infty)$ because

$$s = \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{\alpha^n/n!}{\alpha^{n+1}/(n+1)!} = \lim_{n\to\infty} \frac{n+1}{\alpha} = \infty \tag{4.35}$$

Changing the exponent to $\beta = -\alpha$ provides us with the power series of $f(t) = \exp(\beta t)$.

$$f(t) = e^{\beta t} = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + \beta t + \frac{\beta^2}{2!}t^2 + \frac{\beta^3}{3!}t^3 + \frac{\beta^4}{4!}t^4 + \cdots \tag{4.36}$$

To show the simplicity of working with power series let us prove

$$\exp(x+y) = \exp x \ \exp y \tag{4.37}$$

Knowing the binomial identity $(x+y)^n$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \, y^{n-k} \qquad \binom{n}{k} = \frac{n!}{k!\,(n-k)!} \tag{4.38}$$

we have

$$\begin{aligned} \exp x \ \exp y &= \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{m=0}^{\infty} \frac{y^m}{m!} = \sum_{n=0}^{\infty} \sum_{k+m=n}^{\infty} \frac{x^k}{k!} \frac{y^m}{m!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{x^k y^{n-k}}{k!\,(n-k)!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \\ &= \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!} = \exp(x+y) \end{aligned} \tag{4.39}$$

A question would be: how many terms of a series will make a good approximation. Unfortunately the correct answer is infinity. Practically, no power series can provide a good approximation in the large domain of the independent variable. Power series are only good at the very close vicinity to the point that the series expanded about.

Example 121 Derivative, integral and combination of power series.

If $f(x)$ and $g(x)$ are analytic at a point $x = x_0$, then $f(x) + g(x)$, $f(x) \cdot g(x)$ are analytic at $x = x_0$.

If $g(x) \neq 0$, then $f(x)/g(x)$ is also analytic at $x = x_0$.

If the inverse function $f^{-1}(x)$ is continuous, and $f'(x) \neq 0$, then $f^{-1}(x)$ is analytic.

If $g(x)$ is analytic at a point $x = x_0$ and $f(x)$ is analytic at $g(x_0)$, then $f(g(x))$ is analytic at $x = x_0$.

If $f(x) = \sum_{k=0}^{\infty} a_k x^k$ and $f(x) = \sum_{k=0}^{\infty} b_k x^k$, then $a_k = b_k$, $k = 1, 2, 3, \cdots$.
If $f(x) = \sum_{k=0}^{\infty} a_k x^k$ in $|x| < s_1$ and $g(x) = \sum_{k=0}^{\infty} b_k x^k$ in $|x| < s_2$, then

$$f(x) \cdot g(x) = \left(\sum_{k=0}^{\infty} a_k x^k\right)\left(\sum_{k=0}^{\infty} b_k x^k\right) = \sum_{k=0}^{\infty} c_k x^k \tag{4.40}$$

$$c_k = \sum_{k=0}^{n} a_k b_{n-k} \qquad |x| < \min(s_1, s_2) \tag{4.41}$$

A power series can be differentiated and integrated term by term. Assume the power series $f(x) = \sum_{k=0}^{\infty} a_k x^k$ has a radius of convergence $s > 0$, $-s < x < s$.

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k + \cdots = \sum_{k=0}^{\infty} a_k x^k \tag{4.42}$$

Then, $f(x)$ has the derivative

$$f'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1} \tag{4.43}$$

and $f(x)$ has the integral

$$\int_0^x f(x) = \sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1} \tag{4.44}$$

and the series (4.43) and (4.44) both have the same radius of curvature s.

Example 122 Fraction functions $f(x) = 1/(1+x)$ and $f(x) = 1/(1-x)$.
The first term of the power series for

$$f(x) = \frac{1}{1+x} \tag{4.45}$$

is:

$$f(0) = 1 \tag{4.46}$$

The second term would be:

$$f'(x) = \frac{d}{dx}\frac{1}{1+x} = \frac{-1}{(x+1)^2} \qquad f'(0) = -1 \tag{4.47}$$

and the third and fourth terms are:

$$f''(x) = \frac{2}{(x+1)^3} \qquad \frac{f''(0)}{2!} = \frac{2}{2!} = 1 \tag{4.48}$$

$$f'''(x) = \frac{-6}{(x+1)^4} \qquad \frac{f'''(0)}{3!} = \frac{-6}{3!} = -1 \tag{4.49}$$

This process may be continued up to the desired term to derive the power series expansion of the function.

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots = \sum_{k=0}^{\infty} (-1)^k x^k \tag{4.50}$$

Similarly we may check that:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots = \sum_{k=0}^{\infty} x^k \tag{4.51}$$

Example 123 Arctangent function $\arctan x$.

Besides the classical method to show the power series expansion of $\arctan x$ for $-1 \leq x \leq 1$,

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \cdots \tag{4.52}$$

we may use the equation

$$\frac{d}{dx} \arctan x = \frac{1}{1+x^2} \tag{4.53}$$

and

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots = \sum_{k=0}^{\infty} (-1)^k x^k \tag{4.54}$$

to replace x with x^2

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots = \sum_{k=0}^{\infty} (-1)^k x^{2k} \tag{4.55}$$

to get

$$\frac{d}{dx}\arctan x = 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + x^{12} + \cdots \tag{4.56}$$

and therefore,

$$\begin{aligned}\arctan x &= \int \left(1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \cdots\right) dx \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \cdots \end{aligned}\tag{4.57}$$

or similarly

$$\arctan x = \int \frac{1}{1+x^2} = \int \sum_{k=0}^{\infty} (-1)^k x^{2k} dx = \sum_{k=0}^{\infty} (-1)^k \int x^{2k} dx$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} + C \tag{4.58}$$

$$= C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \cdots \tag{4.59}$$

The fact that $\arctan 0 = 0$ indicates that $C = 0$ and Eq. (4.52) recovers. The test for convergence shows that

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{\left|x^{2k+3}\right|/(2k+3)}{\left|x^{2k+1}\right|/(2k+1)} = |x|^2 \lim_{k\to\infty}\frac{2k+1}{2k+3} = |x|^2 \tag{4.60}$$

and the series converges absolutely if $|x| < 1$.

Example 124 Power series of inverse polynomials.

It will be needed to express the inversion of a finite polynomial in power series. It means we may have a function $f(x)$ such that

$$f(x) = \frac{1}{b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots + b_n x^n} \tag{4.61}$$

and we need to express $f(x)$ in a power series.

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \tag{4.62}$$

The process to develop series (4.62) is as usual.

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k + \cdots = \sum_{k=0}^{\infty} a_k x^k \tag{4.63}$$

$$a_k = \frac{1}{k!} f^{(k)}(x) \tag{4.64}$$

A few examples will show the process.

Consider

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots \tag{4.65}$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \frac{x^6}{6!} - \cdots \tag{4.66}$$

and assume we have $f(x)$ which is inverse of the truncated series (4.65) up to term x^6.

$$f(x) = \frac{1}{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!}} \tag{4.67}$$

Expanding $f(x)$ into a power series will address two questions. Firstly, how we determine power series expansion of inverse of a polynomial, and secondly, how close would be the series of inverse of truncated series of e^x to e^{-x}. Determining the coefficients of series (4.63) term by term will show:

$$a_0 = f(0) = 1 \tag{4.68}$$

$$f'(x) = \frac{-4320\left(x^5 + 5x^4 + 20x^3 + 60x^2 + 120x + 120\right)}{\left(x^6 + 6x^5 + 30x^4 + 120x^3 + 360x^2 + 720x + 720\right)^2} \tag{4.69}$$

$$a_1 = f'(0) = -1 \tag{4.70}$$

$$a_2 = \frac{1}{2!} f''(0) = \frac{1}{2} \tag{4.71}$$

$$a_3 = \frac{1}{3!} f'''(0) = -\frac{1}{6} \tag{4.72}$$

$$a_4 = \frac{1}{4!} f''''(0) = \frac{1}{24} \tag{4.73}$$

$$\vdots$$

and we find a series that matches (4.66) very well.

$$f(x) = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \frac{x^6}{6!} - \cdots \tag{4.74}$$

A reasonable question would be what will happen if we expand $f(x)$ further than the term x^6. Would it match to the series of e^{-x} for the terms higher than x^6? The answer is: a function equal to the inverse of a polynomial is an independent function which its power series will have infinity terms. We will theoretically need all terms to converge to the given function. In case that the polynomial is a truncated series of a known function with a known inverse, it is not necessarily true that we get the power series of the inverse function by expanding the inverse polynomial. However, in this example we may expand $f(x)$ further than x^6, to find

$$\begin{aligned} f(x) = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \frac{x^6}{6!} \\ - \frac{x^8}{4 \times 6!} + \frac{x^9}{4 \times 6!} - \frac{19x^{10}}{120 \times 6!} + \frac{x^{11}}{80 \times 5!} - \cdots \end{aligned} \tag{4.75}$$

which does not match the series expansion of e^{-x} after the term x^6.

$$\begin{aligned} e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \frac{x^6}{6!} \\ - \frac{x^7}{7!} + \frac{x^8}{8!} - \frac{x^9}{9!} + \frac{x^{10}}{10!} - \frac{x^{11}}{11!} + \cdots \end{aligned} \tag{4.76}$$

It can be seen that the term proportional to x^7 is missing in the series expansion of $f(x)$.

Example 125 Even series for inverse even polynomials.

It will be needed in future sections to expand functions that are inverse of even or odd polynomials. An even series $f_e(x)$ is a series without terms proportional to odd exponents, x^{2k+1}, $k = 0, 1, 2, 3, \cdots$, such as:

$$f_e(x) = a_0 + a_2 x^2 + a_4 x^4 + a_6 x^6 + \cdots \tag{4.77}$$

Similarly an odd series $f_o(x)$ is a series without terms proportional to even exponents, x^{2k}, $k = 0, 1, 2, 3, \cdots$, such as:

$$f_o(x) = a_1 x + a_3 x^3 + a_5 x^5 + a_7 x^7 + \cdots \tag{4.78}$$

Let us define a function $f(x)$ equal to the inverse of an even polynomial and determine its power series expansion.

$$f(x) = \frac{1}{a_0 + a_2x^2 + a_4x^4 + a_6x^6} \tag{4.79}$$

We have

$$f(0) = \frac{1}{a_0} \tag{4.80}$$

and

$$f'(x) = -\frac{6a_6x^5 + 4a_4x^3 + 2a_2x}{\left(a_6x^6 + a_4x^4 + a_2x^2 + a_0\right)^2} \tag{4.81}$$

$$f'(0) = 0 \tag{4.82}$$

$$\frac{1}{2!}f''(0) = -\frac{2a_2}{a_0^2} \qquad f'''(0) = 0 \tag{4.83}$$

$$\frac{1}{4!}f^{(4)}(0) = \frac{a_2^2 - a_0a_4}{a_0^3} \qquad f^{(5)}(0) = 0 \tag{4.84}$$

$$\frac{1}{6!}f^{(6)}(0) = -\frac{a_6a_0^2 - 2a_4a_0a_2 + a_2^3}{a_0^4} \qquad f^{(5)}(0) = 0 \tag{4.85}$$

to derive the power series of the function.

$$f(x) = \frac{1}{a_0} - \frac{2a_2}{a_0^2}x^2 + \frac{a_2^2 - a_0a_4}{a_0^3}x^4 + \cdots \tag{4.86}$$

It shows that the power series of the inverse of an even polynomial is an even power series.

As an example, we may cut the series expansion of $y = \cos x$, which is an even series,

$$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \cdots \tag{4.87}$$

to define $f(x)$

$$f(x) = \frac{1}{1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}} \tag{4.88}$$

and expand it to a power series.

$$f(x) = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \cdots \tag{4.89}$$

To check if the power series of the inverse of an odd polynomial would be an odd series, let us cut the series expansion of $y = \sin x$, which is an odd series,

$$\sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \cdots \tag{4.90}$$

to define $f(x)$

$$f(x) = \frac{1}{x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040}} \tag{4.91}$$

and expand it to a power series.

$$f(x) = \frac{1}{x} + \frac{1}{6}x + \frac{7}{360}x^3 + \frac{31}{15\,120}x^5 + \cdots \tag{4.92}$$

The power series of the function $f(x)$ is odd, however, it is singular at the point $x - 0$, but well defined everywhere else. This is because the function $f(x)$ is not defined at $x - 0$.

Example 126 ★Singular points of functions.

If a function is singular at a point $x = a$, then the function goes to infinity when $x \to a$. The function $f(x) = 1/x$ is singular at $x = 0$. The function $f(x) = 1/(x-2)$ is singular at $x = 2$. The function $f(x) = 1/\left(x^2 - 4\right)$ is singular at $x = \pm 2$. The function $f(x) = 1/\sin x$ is singular at $x = k\pi$, $k = 0, 1, 2, 3, \cdots$. These functions are analytic at all points other than their singular points.

The function $f(x)$

$$f(x) = \frac{x}{\left(x^2+4\right)\left(x^2-9\right)} \tag{4.93}$$

has four singular points at which it goes to infinity.

$$x = 3, -3, 2i, -2i \tag{4.94}$$

We may locate the singularities on the complex plane, then to determine the radius of convergence is to draw a circle, with center at the origin, passing through the nearest singular point. As an example, the radius of convergence of Eq. (4.93) is:

$$|s| < 2\sqrt{2} \tag{4.95}$$

If we expand the function about another point, say $x = a = 1 + i$, then the center of the circle would be at $x = a = 1 + i$. The closest singular point to $x = 1 + i$ is $x = 2i$ and therefore the radius of convergence would be:

$$|s - (1 + i)| < \sqrt{2} \tag{4.96}$$

Example 127 New power series from known series.

Calculating power series expansion of elementary functions may be utilized to develop series expansion of more complicated functions by substitution, multiplication, division, composition, differentiation, integration, etc. The following power series of elementary functions will be useful for the future analysis.

$$\sin ax = ax - \frac{a^3}{3!}x^3 + \frac{a^5}{5!}x^5 - \frac{a^7}{7!}x^7 + \frac{a^9}{9!}x^9 - \frac{a^{11}}{11!}x^{11} + \cdots \tag{4.97}$$

$$\cos ax = 1 - \frac{a^2}{2!}x^2 + \frac{a^4}{4!}x^4 - \frac{a^6}{6!}x^6 + \frac{a^8}{8!}x^8 - \frac{a^{10}}{10!}x^{10} + \cdots \tag{4.98}$$

$$\tan ax = ax + \frac{a^3}{3}x^3 + \frac{2a^5}{15}x^5 + \frac{17a^7}{315}x^7 + \frac{62a^9}{2835}x^9 + \cdots \tag{4.99}$$

$$\exp ax = 1 + ax + \frac{a^2}{2}x^2 + \frac{a^3}{3!}x^3 + \frac{a^4}{4!}x^4 + \frac{a^5}{5!}x^5 + \cdots \tag{4.100}$$

Let us calculate power series expansion of

$$f(x) = \exp\left(-3x^2\right) \tag{4.101}$$

In Example 120, we found the series expansion for $f(t) = \exp(-\alpha t)$, as:

$$f(t) = e^{-\alpha t} = 1 - \alpha t + \frac{\alpha^2}{2}t^2 - \frac{\alpha^3}{3!}t^3 + \frac{\alpha^4}{4!}t^4 - \frac{\alpha^5}{5!}t^5 + \cdots \tag{4.102}$$

Substituting $a = 1$, and $t = 3x^2$, we have

$$f(x) = e^{-3x^2} = 1 - 3x^2 + \frac{9}{2}x^4 - \frac{9}{2}x^6 + \frac{27}{8}x^8 - \cdots \tag{4.103}$$

Similarly, having the series expansion of $\exp x$, we are able to find the series expansion of functions involving $\exp x$ such as:

$$f(x) = \frac{e^x}{x} = \frac{1}{x}\left(1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots\right)$$
$$= \frac{1}{x} + 1 + \frac{1}{2}x + \frac{1}{3!}x^2 + \frac{1}{4!}x^3 + \frac{1}{5!}x^4 + \frac{1}{6!}x^5 + \cdots \tag{4.104}$$

Example 128 ★Theory of functions.

The theory of functions is a branch of mathematics that tries to represent arbitrary functions by simpler and elementary analytical approximate functions. The class of continued real functions over a specified segment $C[a, b]$ or over $[-\infty, \infty]$ and real periodic functions $C_{2\pi}$

$$f(x + 2\pi) = f(x) \tag{4.105}$$

are the most interested and applied functions in Science and Engineering (Natanson 1964). The first option to approximate the continued real functions C is the algebraic polynomial series expansion with real coefficients

$$P(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_k x^k + \cdots = \sum_{k=0}^{\infty} c_k x^k \tag{4.106}$$

and the first option to approximate the real periodic functions $C_{2\pi}$ is the trigonometric polynomials series expansion with real coefficients.

$$T(x) = a_0 + a_1 \cos x + b_1 \sin x + \cdots + a_n \cos nx + b_n \sin nx \tag{4.107}$$

A polynomial $P(x)$ as an approximate function for a function $f(x) \in C[a, b]$ if for all values of $x \in [a, b]$ the following inequality holds true,

$$|f(x) - P(x)| < \epsilon \tag{4.108}$$

where $\epsilon > 0$ is indicator of the degree of approximation. Similarly, a trigonometric polynomial $T(x)$ is an approximation for a function $f(x) \in C_{2\pi}$, if for all values of $x \in [a, b]$ the following inequality holds true.

$$|f(x) - T(x)| < \epsilon \tag{4.109}$$

The integrals

$$\int_a^b (f(x) - P(x))^2 \, dx \tag{4.110}$$

$$\int_{-\pi}^{\pi} (f(x) - T(x))^2 \, dx \tag{4.111}$$

are the measure of distance of the approximated functions $P(x)$ and $T(x)$ from the function $f(x)$.

Example 129 ★Power series of matrix functions.

Similar to a power series expansion of a function of a scalar variable, $f(x)$,

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \tag{4.112}$$

we may define power series expansion of a function of a matrix, $f(\mathbf{A})$,

$$f(\mathbf{A}) = a_0 \mathbf{I} + a_1 \mathbf{A} + a_2 \mathbf{A}^2 + a_3 \mathbf{A}^3 + \cdots \tag{4.113}$$

where $\mathbf{I}$ is the unit matrix of the same order as $\mathbf{A}$. As an example, exponential function of a matrix is:

$$e^{\mathbf{A}} = \mathbf{I} + \mathbf{A} + \frac{1}{2}\mathbf{A}^2 + \frac{1}{3!}\mathbf{A}^3 + \frac{1}{4!}\mathbf{A}^4 + \frac{1}{5!}\mathbf{A}^5 + \cdots \tag{4.114}$$

Assume $\mathbf{A}$ is a square matrix $n \times n$ whose eigenvalues $\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n$ are associated to its eigenvectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \cdots, \mathbf{u}_n$.

$$\mathbf{A}\mathbf{u}_i = \lambda_i \mathbf{u}_i \qquad i = 1, 2, 3, \cdots, n \tag{4.115}$$

$$\mathbf{u}_i = \begin{bmatrix} u_{1i} \; u_{2i} \; u_{3i} \; \ldots \; u_{ni} \end{bmatrix}^T \tag{4.116}$$

We may combine all equations of (4.115) and write them in an equation,

$$\mathbf{AJ} = \mathbf{\Lambda J} \tag{4.117}$$

where $\mathbf{\Lambda}$ is a diagonal eigenvalue matrix and $\mathbf{J}$ is square eigenvector matrix, assuming all eigenvalues are real and distinct,

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & 0 & \ldots \\ 0 & \lambda_2 & 0 & \ldots \\ 0 & 0 & \lambda_3 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} = diag\,(\lambda_1, \lambda_2, \lambda_3, \cdots) \tag{4.118}$$

$$\mathbf{J} = \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} u_{11} & u_{21} & u_{31} & \ldots \\ u_{12} & u_{22} & u_{32} & \ldots \\ u_{13} & u_{23} & u_{33} & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{4.119}$$

and therefore,

$$\mathbf{J}^{-1}\mathbf{AJ} = \mathbf{\Lambda} \tag{4.120}$$

$$\mathbf{A} = \mathbf{J}\Lambda\mathbf{J}^{-1} \tag{4.121}$$

Now we may have the exponents of **A**,

$$\mathbf{A}^2 = \left[\mathbf{J}\Lambda\mathbf{J}^{-1}\right] \cdot \left[\mathbf{J}\Lambda\mathbf{J}^{-1}\right] = \mathbf{J}\Lambda^2\mathbf{J}^{-1} \tag{4.122}$$

$$\mathbf{A}^2 = \mathbf{A}^2\mathbf{A} = \left[\mathbf{J}\Lambda^2\mathbf{J}^{-1}\right] \cdot \left[\mathbf{J}\Lambda\mathbf{J}^{-1}\right] = \mathbf{J}\mathbf{\Lambda}^3\mathbf{J}^{-1} \tag{4.123}$$

$$\ldots = \ldots$$

$$\mathbf{A}^n = \mathbf{A}^{n-1}\mathbf{A} = \left[\mathbf{J}\Lambda^{n-1}\mathbf{J}^{-1}\right] \cdot \left[\mathbf{J}\Lambda\mathbf{J}^{-1}\right] = \mathbf{J}\mathbf{\Lambda}^n\mathbf{J}^{-1} \tag{4.124}$$

where

$$\mathbf{A}^n = \begin{bmatrix} \lambda_1^n & 0 & 0 & \ldots \\ 0 & \lambda_2^n & 0 & \ldots \\ 0 & 0 & \lambda_3^n & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{4.125}$$

Hence, the power series of a matrix function (4.113) will be:

$$\begin{aligned} f(\mathbf{A}) &= a_0\mathbf{I} + a_1\mathbf{A} + a_2\mathbf{A}^2 + a_3\mathbf{A}^3 + \cdots \\ &= a_0\mathbf{J}\mathbf{I}\mathbf{J}^{-1} + a_1\mathbf{J}\Lambda\mathbf{J}^{-1} + a_2\mathbf{J}\mathbf{\Lambda}^2\mathbf{J}^{-1} + a_3\mathbf{J}\Lambda^3\mathbf{J}^{-1} + \cdots \\ &= \mathbf{J}\left[a_0\mathbf{I} + a_1\mathbf{\Lambda} + a_2\mathbf{\Lambda}^2 + a_3\mathbf{\Lambda}^3 + \cdots\right]\mathbf{J}^{-1} \end{aligned} \tag{4.126}$$

where

$$\begin{aligned} \left[a_0\mathbf{I} + a_1\mathbf{\Lambda} + a_2\mathbf{\Lambda}^2 + \cdots\right] = diag(&a_0 + a_1\lambda_1 + a_2\lambda_1^2 \cdots, \\ &a_0 + a_1\lambda_2 + a_2\lambda_2^2 \cdots, \ldots \\ &a_0 + a_1\lambda_n + a_2\lambda_n^2 \cdots) \end{aligned} \tag{4.127}$$

4.2 Continued Fractions of Functions

Speed and domain of convergence of continued fractions approximation of functions are much better than power series expansions. Hence it is of practical interest to go from a power series to a continued fraction. Assume

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots \tag{4.128}$$

$$= a_{00} + a_{01}x + a_{02}x^2 + a_{03}x^3 + \cdots \tag{4.129}$$

The continued fractions expression of the function $f(x)$ will be:

$$f(x) = a_{00} + \cfrac{a_{01}x}{1 + \cfrac{a_{11}x}{1 + \cfrac{a_{21}x}{1 + \cfrac{a_{31}x}{1 + \cfrac{a_{41}x}{1 + \cdots}}}}} \tag{4.130}$$

where

$$a_{nk} = \frac{1}{k!}\frac{d^k}{dx^k}\left(f_n(0) + 1\right) \tag{4.131}$$

$$f_0(x) = a_{00} + \frac{a_{01}x}{1 + f_1(x)} \tag{4.132}$$

$$f_n(x) = \frac{a_{n1}x}{1 + f_{n+1}(x)} \tag{4.133}$$

Proof To prove (4.130) let us begin with a power series

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \tag{4.134}$$

and rewrite it as

$$\begin{aligned} f_0(x) &= a_0 + a_1 x\left(1 + \frac{a_2}{a_1}x + \frac{a_3}{a_1}x^2 + \cdots\right) \\ &= a_0 + \frac{a_1 x}{1 + f_1(x)} \end{aligned} \tag{4.135}$$

where

$$\begin{aligned} f_1(x) &= \left(1 + \frac{a_2}{a_1}x + \frac{a_3}{a_1}x^2 + \cdots\right)^{-1} - 1 \\ &= b_1 x + b_2 x^2 + b_3 x^3 + \cdots \\ &= b_1 x\left(1 + \frac{b_2}{b_1}x + \frac{b_3}{b_1}x^2 + \cdots\right) = \frac{b_1 x}{1 + f_2(x)} \end{aligned} \tag{4.136}$$

$$b_k = \frac{1}{k!}\frac{d^k}{dx^k}\left(f_1(0) + 1\right) \tag{4.137}$$

The polynomial in $f_1(x)$ begins with 1 and therefore the inverse of the polynomial will be another polynomial beginning with 1 to cancel out the last -1. Then, factoring out the first term of the resultant polynomial makes $f_1(x)$ suitable to be defined by a similar function $f_2(x)$.

$$\begin{aligned} f_2(x) &= \left(1+\frac{b_2}{b_1}x+\frac{b_3}{b_1}x^2+\cdots\right)^{-1}-1 \\ &= c_1x+c_2x^2+c_3x^3+\cdots \\ &= c_1x\left(1+\frac{c_2}{c_1}x+\frac{c_3}{c_1}x^2+\cdots\right)=\frac{c_1x}{1+f_3(x)} \end{aligned} \tag{4.138}$$

$$c_k=\frac{1}{k!}\frac{d^k}{dx^k}\left(f_2(0)+1\right) \tag{4.139}$$

Similar treatment of $f_2(x)$ will make another term of the continued fractions and introduces $f_3(x)$.

$$\begin{aligned} f_3(x) &= \left(1+\frac{c_2}{c_1}x+\frac{c_3}{c_1}x^2+\cdots\right)^{-1}-1 \\ &= d_1x+d_2x^2+d_3x^3+\cdots \\ &= d_1x\left(1+\frac{d_2}{d_1}x+\frac{d_3}{d_1}x^2+\cdots\right)=\frac{d_1x}{1+f_4(x)} \end{aligned} \tag{4.140}$$

$$d_k=\frac{1}{k!}\frac{d^k}{dx^k}\left(f_3(0)+1\right) \tag{4.141}$$

Back substituting $f_k(x)$ will make the continued fractions.

$$f_0(x)=a_0+\cfrac{a_1x}{1+\cfrac{b_1x}{1+\cfrac{c_1x}{1+\cfrac{d_1x}{1+\cfrac{e_1x}{1+\cdots}}}}} \tag{4.142}$$

To make the process suitable to be computerized, we rename the coefficient of the series

$$f(x)=f_0(x)=a_{00}+a_{01}x+a_{02}x^2+a_{03}x^3+\cdots \tag{4.143}$$

to have the continued fractions expression of $f(x)$ as:

$$f(x) = a_{00} + \cfrac{a_{01}x}{1 + \cfrac{a_{11}x}{1 + \cfrac{a_{21}x}{1 + \cfrac{a_{31}x}{1 + \cfrac{a_{41}x}{1 + \cdots}}}}} \tag{4.144}$$

where the coefficients a_{nk} to be:

$$a_{nk} = \frac{1}{k!}\frac{d^k}{dx^k}\left(f_n(0) + 1\right) \tag{4.145}$$

and the functions $f_k(x)$ are:

$$f_0(x) = a_{00} + \frac{a_{01}x}{1 + f_1(x)} \tag{4.146}$$

$$f_n(x) = \frac{a_{n1}x}{1 + f_{n+1}(x)} \qquad n = 1, 2, 3, \cdots \tag{4.147}$$

The last four equations are enough to develop an algorithm to generate continued fractions of given power series.

The convergence behavior of a power or numerical series and that of its corresponding continued fractions may be different. Both may converge, both diverge, or continued fractions may converge while the power series diverges, but not the opposite.

The simplicity of working with power series is the main reason not to employ the continued fractions when power series are available. However, as we will see in the examples, the convergence of continued fractions is much faster and wider than power series for the same number of terms of the expansions. This is the most important advantage of continued fractions compared to series expansions, specially when we consider the accumulated numerical errors. ■

Example 130 Series expansion of $y = \ln(1 + x)$ to continued fractions.

Compared to power series expansions, the speed of convergence and domain of convergence of continued fractions are advantages that justify using continued fractions instead of series expansions whenever the series expansions cannot converge as quickly as needed. This example shows how we find the continued fractions of $y = \ln(1 + x)$ from its series expansion.

$$y = \ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots \tag{4.148}$$

Let us rewrite this expansion,

$$y_0(x) = \ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$$

$$= x\left(1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \frac{x^4}{5} - \cdots\right)$$

$$= \frac{x}{1 + y_1(x)} \tag{4.149}$$

where

$$y_1(x) = \left(1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \frac{x^4}{5} - \cdots\right)^{-1} - 1$$

$$= \left(1 + \frac{x}{2} - \frac{x^2}{12} + \frac{x^3}{24} - \frac{19x^4}{720} + \frac{71x^5}{480} - \cdots\right) - 1$$

$$= \frac{x}{2}\left(1 - \frac{x}{6} + \frac{x^2}{12} - \frac{19x^3}{360} + \frac{71x^4}{240} - \cdots\right)$$

$$= \frac{x/2}{1 + y_2(x)} \tag{4.150}$$

and

$$y_2(x) = \left(1 - \frac{x}{6} + \frac{x^2}{12} - \frac{19x^3}{360} + \frac{71x^4}{240} - \cdots\right)^{-1} - 1$$

$$= \left(1 + \frac{x}{6} - \frac{x^2}{18} + \frac{4x^3}{135} - \frac{899x^4}{3240} - \cdots\right) - 1$$

$$= \frac{x}{6}\left(1 - \frac{x}{3} + \frac{8x^2}{45} - \frac{899x^3}{540} + \frac{785x^4}{1296} - \cdots\right)$$

$$= \frac{x/6}{1 + y_3(x)} \tag{4.151}$$

and similarly

$$y_3(x) = \left(1 - \frac{x}{3} + \frac{8x^2}{45} - \frac{899x^3}{540} + \frac{785}{1296}x^4 - \cdots\right)^{-1} - 1$$

$$= \left(1 + \frac{x}{3} - \frac{x^2}{15} + \frac{19x^3}{12} + \frac{15\,839x^4}{32\,400} - \cdots\right) - 1$$

$$= \frac{x}{3}\left(1 - \frac{x}{5} + \frac{19x^2}{4} - \frac{15\,839x^3}{10\,800} + \cdots\right)$$

$$= \frac{x/3}{1 + y_4(x)} \tag{4.152}$$

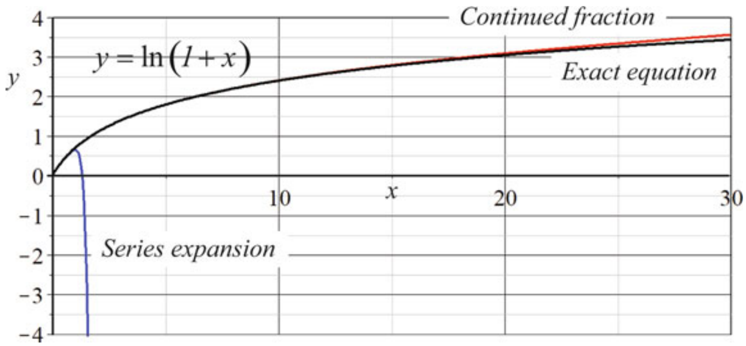

Fig. 4.1 Comparison of $y = \ln(1+x)$ with associated series expansion and continued fraction approximations for the first 10 terms

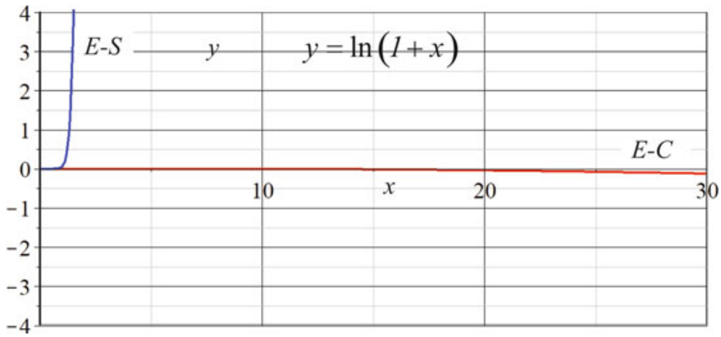

Fig. 4.2 The differences of exact solution minus continued fraction (E−C), and exact solution minus series expansion (E−S) for $y = \ln(1+x)$

Therefore, the associated continued fraction would be:

$$y = \ln(1+x) = \frac{x}{1+} \frac{x/2}{1+} \frac{x/6}{1+} \frac{x/3}{1 + y_4(x)} \tag{4.153}$$

Although this method is not the most efficient method, it indicates a possible conversion of power series to a continued fraction. Figure 4.1 compares $y = \ln(1+x)$ with associated series expansion (4.148) and continued fraction approximations (4.153) for the first 10 terms. It shows well how the continued fraction provides a better approximation compared to series expansion. Figure 4.2 illustrates the differences of exact solution minus continued fraction (E−C), and exact solution minus series expansion (E−S).

Example 131 Continued fraction of $y = \ln(1+x)$ to series expansion.
Let us assume we have the continued fraction of $\ln(1+x)$.

$$y = \ln(1+x) = \frac{x}{1+}\,\frac{x/2}{1+}\,\frac{x/6}{1+}\,\frac{x/3}{1+y_4(x)} \tag{4.154}$$

First we begin with the first two terms.

$$y = \frac{x}{1+\dfrac{x}{2}} = \frac{2x}{x+2} \tag{4.155}$$

Series expansion of this fraction is correct only up to x^2.

$$y = \frac{x}{1+\dfrac{x}{2}} = \frac{2x}{x+2} = x - \frac{1}{2}x^2 + \frac{1}{4}x^3 - \cdots \tag{4.156}$$

The next term of the continued fractions makes the series expansion:

$$y = \frac{x}{1+\dfrac{x}{2+\dfrac{x}{6}}} = \frac{x(x+12)}{7x+12} = x - \frac{x^2}{2} + \frac{7x^3}{24} - \frac{49x^4}{288} + \cdots \tag{4.157}$$

Expansion of the fourth terms of the continued fractions makes the first four terms of the equivalent series expansion.

$$\begin{aligned} y &= \frac{x}{1+\dfrac{x}{2+\dfrac{x}{6+\dfrac{x}{3}}}} = \frac{x(5x+36)}{x^2+23x+36} \\ &= x - \frac{x^2}{2} + \frac{7x^3}{24} - \frac{149x^4}{864} + \frac{3175x^5}{31\,104} - \cdots \end{aligned} \tag{4.158}$$

Continuing the conversion will show that series (4.158) approaches (4.148).

Example 132 Series expansion of $x = \exp(-\alpha t)$ to continued fraction.
Consider the series expansion of $x = \exp(-\alpha t)$.

$$x = e^{-\alpha t} = 1 - \alpha t + \frac{\alpha^2 t^2}{2} - \frac{\alpha^3 t^3}{6} + \frac{\alpha^4 t^4}{24} - \frac{\alpha^5 t^5}{120} + \cdots \tag{4.159}$$

Let us rewrite this expansion,

$$\begin{aligned} x_0(t) = e^{-\alpha t} &= 1 - \alpha t + \frac{\alpha^2 t^2}{2} - \frac{\alpha^3 t^3}{6} + \frac{\alpha^4 t^4}{24} - \frac{\alpha^5 t^5}{120} + \cdots \\ &= 1 - at\left(1 - \frac{\alpha t}{2} + \frac{\alpha^2 t^2}{6} - \frac{\alpha^3 t^3}{24} + \frac{\alpha^4 t^4}{120} - \cdots\right) \\ &= 1 - \frac{\alpha t}{1 + x_1(t)} \end{aligned} \tag{4.160}$$

where

$$\begin{aligned} x_1(t) &= \left(1 - \frac{\alpha t}{2} + \frac{\alpha^2 t^2}{6} - \frac{\alpha^3 t^3}{24} + \frac{\alpha^4 t^4}{120} - \cdots\right)^{-1} - 1 \\ &= \left(1 + \frac{\alpha t}{2} + \frac{\alpha^2 t^2}{12} - \frac{\alpha^4 t^4}{720} - \frac{\alpha^6 t^6}{30240} - \cdots\right) - 1 \\ &= \frac{\alpha t}{2}\left(1 + \frac{\alpha t}{6} - \frac{\alpha^3 t^3}{360} + \frac{\alpha^5 t^5}{15120} - \cdots\right) \\ &= \frac{\alpha t/2}{1 + x_2(x)} \end{aligned} \tag{4.161}$$

and

$$\begin{aligned} x_2(t) &= \left(1 + \frac{\alpha t}{6} - \frac{\alpha^3 t^3}{360} + \frac{\alpha^5 t^5}{15120} - \cdots\right)^{-1} - 1 \\ &= \left(1 - \frac{\alpha t}{6} + \frac{\alpha^2 t^2}{36} - \frac{\alpha^3 t^3}{540} - \frac{\alpha^4 t^4}{6480} + \cdots\right) - 1 \\ &= -\frac{\alpha t}{6}\left(1 - \frac{\alpha t}{6} + \frac{\alpha^2 t^2}{90} + \frac{\alpha^3 t^3}{1080} - \frac{\alpha^4 t^4}{4536} + \cdots\right) \\ &= \frac{-\alpha t/6}{1 + x_3(t)} \end{aligned} \tag{4.162}$$

and similarly

$$\begin{aligned} x_3(t) &= \left(1 - \frac{\alpha t}{6} + \frac{\alpha^2 t^2}{90} + \frac{\alpha^3 t^3}{1080} - \frac{\alpha^4 t^4}{4536} + \cdots\right)^{-1} - 1 \\ &= \left(1 + \frac{\alpha t}{6} + \frac{\alpha^2 t^2}{60} - \frac{\alpha^4 t^4}{8400} + \frac{\alpha^6 t^6}{756000} - \cdots\right) - 1 \\ &= \frac{\alpha t}{6}\left(1 + \frac{\alpha t}{10} - \frac{\alpha^3 t^3}{1400} + \frac{\alpha^5 t^5}{126000} - \cdots\right) \\ &= \frac{\alpha t/6}{1 + x_4(t)} \end{aligned} \tag{4.163}$$

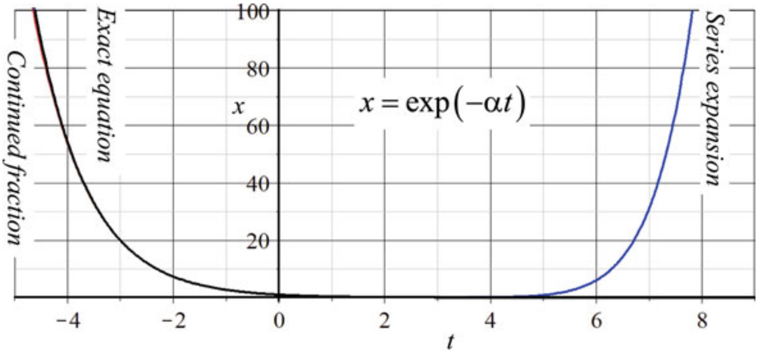

Fig. 4.3 Comparison of $x = \exp(-\alpha t)$ with associated series expansion and continued fraction approximations

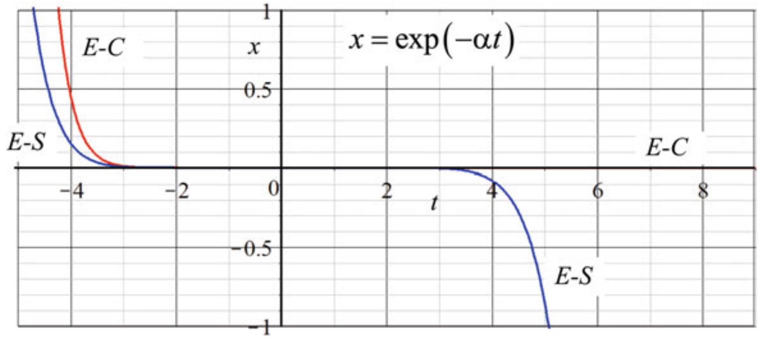

Fig. 4.4 The differences of exact solution minus continued fraction (E−C), and exact solution minus series expansion (E−S) for $x = \exp(-\alpha t)$

Therefore, the associated continued fractions to $x = \exp(-\alpha t)$ would be:

$$x = \exp(-\alpha t) = 1 - \frac{\alpha t}{1+}\,\frac{\alpha t/2}{1+}\,\frac{-\alpha t/6}{1+}\,\frac{\alpha t/6}{1 + x_4(t)} \tag{4.164}$$

Continuing the method will provide:

$$x = e^{-\alpha t} = 1 - \frac{\alpha t}{1+}\,\frac{\alpha t/2}{1-}\,\frac{\alpha t/6}{1+}\,\frac{\alpha t/6}{1-}\,\frac{\alpha t/10}{1+}\,\frac{\alpha t/10}{1-}\cdots \tag{4.165}$$

Figure 4.3 compares $x = \exp(-\alpha t)$ with associated series expansion (4.159) and continued fractions approximation (4.165). It shows very well that the accuracy of series expansion cannot be compared to the well approximation of continued fraction. Figure 4.4 illustrates the differences of exact solution minus continued fraction (E−C), and exact solution minus series expansion (E−S).

Example 133 Even series.

There are even and odd power series that their continued fractions will be needed. The functions $\cos x$ and $\sin x$ are two examples.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \tag{4.166}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \tag{4.167}$$

The presented procedure in (4.130) will fail because we will hit division by zero somewhere in the process of converting even or odd series into continued fractions. Such functions need a reconsideration. To convert such series let us begin with an even power series,

$$f(x) = a_0 + a_2 x^2 + a_4 x^4 + a_6 x^6 + \cdots \tag{4.168}$$

and rewrite it as

$$\begin{aligned} f_0(x) &= a_0 + a_2 x^2 \left(1 + \frac{a_4}{a_2} x^2 + \frac{a_6}{a_2} x^4 + \cdots\right) \\ &= a_0 + \frac{a_2 x^2}{1 + f_2(x)} \end{aligned} \tag{4.169}$$

where

$$\begin{aligned} f_2(x) &= \left(1 + \frac{a_4}{a_2} x^2 + \frac{a_6}{a_2} x^4 + \cdots\right)^{-1} - 1 \\ &= b_2 x^2 + b_4 x^4 + b_6 x^6 + \cdots \\ &= b_2 x^2 \left(1 + \frac{b_4}{b_2} x^2 + \frac{b_6}{b_2} x^4 + \cdots\right) = \frac{b_2 x^2}{1 + f_4(x)} \end{aligned} \tag{4.170}$$

$$b_{2k} = \frac{1}{(2k)!} \frac{d^{2k}}{dx^{2k}} \left(f_2(0) + 1\right) \tag{4.171}$$

then,

$$\begin{aligned} f_4(x) &= \left(1 + \frac{b_4}{b_2} x^2 + \frac{b_6}{b_2} x^4 + \cdots\right)^{-1} - 1 \\ &= c_2 x^2 + c_4 x^4 + c_6 x^6 + \cdots \\ &= c_2 x^2 \left(1 + \frac{c_4}{c_2} x^2 + \frac{c_6}{c_2} x^4 + \cdots\right) = \frac{c_2 x^2}{1 + f_6(x)} \end{aligned} \tag{4.172}$$

$$c_{2k} = \frac{1}{(2k)!} \frac{d^{2k}}{dx^{2k}} \left(f_4(0) + 1\right) \tag{4.173}$$

Similarly we treat $f_6(x)$ to find $f_8(x)$.

$$\begin{aligned} f_6(x) &= \left(1 + \frac{c_4}{c_2} x^2 + \frac{c_6}{c_2} x^4 + \cdots\right)^{-1} - 1 \\ &= d_2 x^2 + d_4 x^4 + d_6 x^6 + \cdots \\ &= d_2 x^2 \left(1 + \frac{d_4}{d_2} x^2 + \frac{d_6}{d_2} x^4 + \cdots\right) = \frac{d_2 x^2}{1 + f_8(x)} \end{aligned} \tag{4.174}$$

$$d_{2k} = \frac{1}{(2k)!} \frac{d^{2k}}{dx^{2k}} \left(f_6(0) + 1\right) \tag{4.175}$$

Back substituting $f_{2k}(x)$ will make the continued fractions.

$$f_0(x) = a_0 + \cfrac{a_2 x^2}{1 + \cfrac{b_2 x^2}{1 + \cfrac{c_2 x^2}{1 + \cfrac{d_2 x^2}{1 + \cfrac{e_2 x^2}{1 + \cdots}}}}} \tag{4.176}$$

Similarly we are able to develop the method to derive the continued fraction of odd power series.

Example 134 Euler method of series to continued fractions.

Euler introduced an alternative method to derive the continued fractions of a power series that is specially easier to use when the coefficients are numbers. If a function $f(x)$ is defined by a power series

$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_k x^k + \cdots = \sum_{k=0}^{\infty} c_k x^k \tag{4.177}$$

there exist a continued fractions equivalent to the approximated series.

$$f(x) = c_0 - \frac{c_1 x}{1-} \frac{\frac{c_2}{c_1} x}{1 + \frac{c_2}{c_1} x -} \cdots \frac{\frac{c_k}{c_{k-1}} x}{1 + \frac{c_k}{c_{k-1}} x -} \cdots \tag{4.178}$$

$$f(x) = c_0 - \frac{c_1 x}{c_0 + c_1 x -} \frac{c_0 c_2 x}{c_1 + c_2 x -} \cdots \frac{c_{k-2} c_k x}{c_{k-1} + c_k x -} \cdots \tag{4.179}$$

To show the Euler method, consider a summation series

$$f = c_0 + c_1 + c_2 + \cdots + c_k + \cdots = \sum_{k=0}^{\infty} c_k \tag{4.180}$$

to be equivalent to a continued fraction

$$f = b_0 + \frac{a_1}{b_1+}\frac{a_2}{b_2+}\frac{a_3}{b_3+}\cdots\frac{a_k}{b_k+}\cdots \tag{4.181}$$

we must have:

$$b_0 = c_0 \qquad a_1 = c_0 + c_1 \qquad b_1 = 1 \tag{4.182}$$

$$b_k = \frac{c_{k-1} + c_k}{c_{k-1}} \qquad a_k = -\frac{c_k}{c_{k-1}} \qquad k = 1, 2, 3, \cdots \tag{4.183}$$

and therefore,

$$\sum_{k=0}^{\infty} c_k = c_0 + \frac{\frac{c_1}{c_0}}{1 + \frac{c_1}{c_0} -} \frac{\frac{c_2}{c_1}}{1 + \frac{c_2}{c_1} -} \frac{\frac{c_3}{c_2}}{1 + \frac{c_3}{c_2} -} \cdots \frac{\frac{c_k}{c_{k-1}}}{1 + \frac{c_k}{c_{k-1}} -} \cdots \tag{4.184}$$

The continued fraction of a power series (4.177) will be expressed in one of the following forms:

$$f(x) = b_0 + \frac{a_1 x}{b_1+}\frac{a_2 x}{b_2+}\frac{a_3 x}{b_3+}\cdots\frac{a_k x}{b_k+}\cdots \tag{4.185}$$

or

$$f(x) = \frac{b_0}{1+}\frac{c_1 x}{d_1+}\frac{c_2 x}{d_2+}\frac{c_3 x}{d_3+}\cdots\frac{c_k x}{d_k+}\cdots \tag{4.186}$$

Example 135 Series to continued fractions, by Euler.

Euler introduced another expression to relate summation series and continued fractions (Euler 1988). We can verify that:

$$\begin{aligned} &a_0 + a_0 a_1 + a_0 a_1 a_2 + a_0 a_1 a_2 a_3 + \cdots + a_0 a_1 a_2 a_3 \cdots a_n \\ &= a_0 + \frac{a_1}{1 + a_1 -}\frac{a_2}{1 + a_2 -}\frac{a_3}{1 + a_3 -}\cdots\frac{a_n}{1 + a_n} \end{aligned} \tag{4.187}$$

Now having the continued summation series expansion of a function as

$$f = c_0 + c_1 + c_2 + c_3 + c_4 + \cdots + c_k + \cdots \tag{4.188}$$

we are able to determine $a_0, a_1, a_2, a_3, \cdots$.

$$a_0 = c_0 \qquad a_1 = \frac{c_1}{c_0} \qquad a_2 = \frac{c_2}{c_1} \qquad a_3 = \frac{c_3}{c_2}$$
$$a_4 = \frac{c_4}{c_3} \qquad \cdots \qquad a_k = \frac{c_k}{c_{k-1}} \tag{4.189}$$

Example 136 Euler method and arctangent function.

The power series expansion of $\arctan x$ for $-1 \le x \le 1$ is:

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \cdots \tag{4.190}$$

Knowing that

$$c_0 = 0 \qquad c_1 = 1 \qquad c_2 = 0 \qquad c_3 = \frac{-1}{3} \qquad c_4 = 0 \tag{4.191}$$
$$c_5 = \frac{1}{5} \qquad c_6 = 0 \qquad c_7 = \frac{-1}{7} \qquad c_8 = 0 \tag{4.192}$$
$$c_9 = \frac{1}{9} \qquad \cdots \qquad c_{2k} = 0 \qquad c_{2k+1} = (-1)^n \frac{1}{2k+1} \tag{4.193}$$

the continued fractions equivalent to the series will be:

$$\arctan x = \frac{x}{1+} \frac{x^2}{3 - x^2+} \frac{9x^2}{5 - 3x^2+} \cdots \frac{(2k+1)^2 x^2}{2k+1-(2k-1)\, x^2+} \cdots \tag{4.194}$$

To employ Eq. (4.194) for $\arctan x$, we may use (4.190) and set

$$a_0 = 0 \qquad a_1 = x \qquad a_2 = \frac{a_1 a_2}{a_1} = \frac{-x^3/3}{x} = -\frac{x^2}{3}$$
$$a_3 = \frac{a_1 a_2 a_3}{a_1 a_2} = \frac{x^5/5}{-x^3/3} = -\frac{3x^2}{5} \cdots \tag{4.195}$$

and recover the continued fractions for $\arctan x$.

$$\arctan x = a_0 + \frac{a_1}{1+} \frac{a_2}{1 + a_2+} \frac{a_3}{1 + a_3+} \cdots \frac{a_n}{1 + a_n}$$
$$= \frac{x}{1+} \frac{x^2}{3 - x^2+} \frac{9x^2}{5 - 3x^2+} \frac{25x^2}{7 - 5x^2+} \cdots \tag{4.196}$$

Example 137 Exponential function $e^{i\theta}$.

If we are able to develop an identity such that a function appears on both sides, usually we are able to express the function in a continued fractions easier. From the

Euler formula

$$e^{i\theta} = \cos\theta + i\,\sin\theta \tag{4.197}$$

$$e^{-i\theta} = \cos\theta - i\,\sin\theta \tag{4.198}$$

we have

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \tag{4.199}$$

$$e^{i\theta} = 2\cos\theta - \frac{1}{e^{i\theta}} \tag{4.200}$$

and therefore,

$$e^{i\theta} = 2\cos\theta - \cfrac{1}{2\cos\theta - \cfrac{1}{2\cos\theta - \cfrac{1}{2\cos\theta - \ddots}}} \tag{4.201}$$

$$\frac{p_0}{q_0} = 2\cos\theta = \frac{\sin 2\theta}{\sin\theta} \tag{4.202}$$

$$\frac{p_1}{q_1} = 2\cos\theta - \frac{1}{2\cos\theta} = \frac{\sin 3\theta}{\sin 2\theta} \tag{4.203}$$

$$\cdots$$

$$\frac{p_k}{q_k} = 2\cos\theta - \frac{1}{2\cos\theta -}\,\frac{1}{2\cos\theta -}\cdots = \frac{\sin(k+2)\,\theta}{\sin(k+1)\,\theta} \tag{4.204}$$

Example 138 Hypergeometric function.

The hypergeometric function $f(a, b, c, x)$ is expressed by the following power series expansion

$$f(a, b, c, x) = 1 + \frac{ab}{c}\frac{x}{1!} + \frac{a(a+1)\,b(b+1)}{c(c+1)}\frac{x^2}{2!} + + \frac{a(a+1)(a+2)\,b(b+1)(b+2)}{c(c+1)(c+2)}\frac{x^3}{3!} + \cdots \tag{4.205}$$

In the special case of $a = 1$, $b = c$, the series (4.205) reduces to a power series or geometric series.

$$f(1, b, b, x) = 1 + x + x^2 + x^3 + \cdots \tag{4.206}$$

That was the reason the function $f(a, b, c, x)$ and series (4.205) have been called hypergeometric function and hypergeometric series. The function is introduced by Gauss and hence, it is also called Gauss function (Gasper and Rahman 1990).

The hypergeometric function $f(a, b, c, x)$ was discovered by Carl Friedrich Gauss (1777–1855). Hypergeometric function includes all elementary functions. For example:

$$\begin{aligned}\sin^{-1} x &= zf\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, x^2\right) \\ &= x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \frac{35}{1152}x^9 + \cdots \end{aligned} \tag{4.207}$$

$$\begin{aligned}\tan^{-1} x &= zf\left(\frac{1}{2}, 1, \frac{3}{2}, -x^2\right) \\ &= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 + \cdots \qquad (4.208) \\ &= \frac{x}{1+}\,\frac{1^2x^2}{3+}\,\frac{2^2x^2}{5+}\,\frac{3^2x^2}{7+}\,\frac{4^2x^2}{9+}\cdots \qquad (4.209)\end{aligned}$$

The parameters a, b, c , $c \neq 0, -1, -2, \cdots$ may be complex constants and x to be a complex variable. The hypergeometric function is the solution of the Gauss differential equation.

$$x(1-x)\frac{d^2y}{dx^2} + (c - (a+b+1)x)\frac{dy}{dx} - aby = 0 \tag{4.210}$$

Assume y is expanded in a power series around the origin.

$$y = \sum_{k=0}^{\infty} A_k x^k \qquad A_0 = 1 \tag{4.211}$$

Comparing the coefficients of x^k for y to be a solution of (4.210), we get a recurrence relation

$$(k+1)(c+k)A_{k+1} = (k+a)(n+\beta)A_k \tag{4.212}$$

and hence,

$$A_{k+1} = \frac{(k+a)(n+\beta)}{(k+1)(c+k)}A_k \tag{4.213}$$

The series converges for $|x| < 1$ and diverges for $|x| > 1$. In case of $x = 1$ it converges when $c > (a + b)$ and in case of $x = -1$ it converges when $(c + 1) > (a + b)$. From the symmetry of the hypergeometric power series we know that

$$f(a, b, c, x) = f(b, a, c, x) \tag{4.214}$$

$$f(a, b, c, x) = (1 - x)^{c-a-b}\ f(c - a, c - b, c, x) \tag{4.215}$$

also we may check that

$$f(a, b, c, x) = f(a, b + 1, c + 1, x) - \frac{a(c - b)}{c(c + 1)} x\ f(a + 1, b + 1, c + 2, x) \tag{4.216}$$

$$f(a, b, c, x) = f(a + 1, b, c + 1, x) - \frac{b(c - a)}{c(c + 1)} x\ f(a + 1, b + 1, c + 2, x) \tag{4.217}$$

$$f(a, b, c, x) = f(a + 1, b + 1, c + 2, x) - \frac{(b + 1)(c - a + 1)}{(c + 1)(c + 2)} x\ f(a + 1, b + 2, c + 3, x) \tag{4.218}$$

employing these three identities we can show

$$\frac{f(a, b, c, x)}{f(a, b + 1, c + 1, x)} = 1 - \frac{a(c - b)}{c(c + 1)} x \frac{f(a + 1, b + 1, c + 2, x)}{f(a, b + 1, c + 1, x)} \tag{4.219}$$

$$\frac{f(a, b + 1, c + 1, x)}{f(a + 1, b + 1, c + 2, x)} = 1 - \frac{(b + 1)(c - a + 1)}{(c + 1)(c + 2)} x \frac{f(a + 1, b + 2, c + 3, x)}{f(a + 1, b + 1, c + 2, x)} \tag{4.220}$$

Let us name the function $g(a, b, c, x)$ as

$$g_i(b, a, c, x) = \frac{f(a, b, c, x)}{f(a, b + 1, c + 1, x)} \tag{4.221}$$

$$g_{i+1}(b, a, c, x) = \frac{f(a, b + 1, c + 1, x)}{f(a + 1, b + 1, c + 2, x)} \tag{4.222}$$

to have

$$g(b,a,c,x) = 1 - \frac{a(c-b)x}{c(c+1)} \frac{1}{g_{i+1}(b,a,c,x)} \tag{4.223}$$

$$g_{i+1}(b,a,c,x) = 1 - \frac{(b+1)(c-a+1)x}{(c+1)(c+2)} \frac{1}{g_{i+2}(b,a,c,x)} \tag{4.224}$$

and therefore,

$$g(b,a,c,x) = 1 + \frac{A_1 x}{1+} \frac{A_2 x}{1+} \cdots \frac{A_k x}{1+} \cdots \tag{4.225}$$

where

$$A_{2k} = -\frac{(b+k)(c-a+k)}{(c+2k-1)(c+2k)} \tag{4.226}$$

$$A_{2k+1} = -\frac{(a+k)(c-b+k)}{(c+2k)(c+2k+1)} \tag{4.227}$$

and these relations make the continued fractions expression of hypergeometric function (Aomoto and Kita 2011).

$$\frac{f(a,b,c,x)}{f(a,b+1,c+1,x)} = 1 + \frac{A_1 x}{1+} \frac{A_2 x}{1+} \cdots \frac{A_k x}{1+} \cdots \tag{4.228}$$

Gauss himself presented the hypergeometric series in continued fraction as follows:

$$\frac{f(a,b+1,c+1,x)}{f(a,b,c,x)} = \frac{1}{1-} \frac{u_1 x}{1-} \frac{v_1 x}{1-} \frac{u_2 x}{1-} \frac{v_2 x}{1-} \cdots \tag{4.229}$$

$$u_k = -\frac{(a+k-1)(c-b+k-1)}{(c+2k-2)(c+2k-1)} \tag{4.230}$$

$$v_k = -\frac{(b+k)(c-b+k)}{(c+2k-1)(c+2k)} \tag{4.231}$$

Based on their Taylor series expansion, many mathematical functions are special cases of the hypergeometric function and hence, (4.228) would express their continued fractions. As a few examples we may check the following functions (Jones and Thron 1980):

$$\begin{aligned}(1+x)^p &= zf(-p,1,1,-x)\\ &= \frac{1}{1+} \frac{(-p)x}{1+} \frac{1(1+p)x}{2+} \frac{1(1-p)x}{3+}\\ &\quad \frac{2(2+p)x}{4+} \frac{2(2-p)x}{5+} \cdots \end{aligned} \tag{4.232}$$

Table 4.1 Approximation of $y = \ln(1+x)$ for $x = 1$ form (4.236) and (4.237)

k	Continued fraction	Series expansion
1	1	1
2	0.666	0.5
3	0.7	0.833
4	0.692	0.583
5	0.6933	0.783
6	0.69312	0.616
7	0.693147	0.7595

$$\exp x = \lim_{a\to\infty} f\left(a, b, b, \frac{x}{a}\right) \tag{4.233}$$

$$\begin{aligned}\ln(1+x)^p &= zf(-p, 1, 1, -x)\\ &= \frac{1}{1+}\,\frac{(-p)\,x}{1+}\,\frac{1\,(1+p)\,x}{2+}\,\frac{1\,(1-p)\,x}{3+}\\ &\quad \frac{2\,(2+p)\,x}{4+}\,\frac{2\,(2-p)\,x}{5+}\cdots\end{aligned} \tag{4.234}$$

Legendre polynomials

$$P_n(x) = f\left(-n, n+1, 1, \frac{1-x}{2}\right) \tag{4.235}$$

Example 139 Convergence speed.

Consider the continued fraction and series expansion of $y = \ln(1+x)$.

$$\begin{aligned}y &= \ln(1+x)\\ &= \frac{x}{1+}\,\frac{x}{2+}\,\frac{x}{3+}\,\frac{2x}{2+}\,\frac{2x}{5+}\,\frac{3x}{2+}\,\frac{3x}{7+}\cdots\frac{kx}{2+}\,\frac{kx}{2+1+}\cdots\end{aligned} \tag{4.236}$$

$$y = \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots \tag{4.237}$$

Let us try the two approximations for $x = 1$. The exact value of $\ln(2)$ is:

$$\ln(2) = 0.6931471805599453094172\cdots \tag{4.238}$$

Table 4.1 compares the continued fraction and series expansion of $y = \ln(2)$ by increasing the number of terms. It can be seen that the continued fraction approaches the acceptable approximation much faster than the series expansion (Brezinski and Wuytack 1992).

Table 4.2 Approximation of $y = \exp(x)$ for $x = 2$

k	Continued fraction	Series expansion
1	1	1
3	0.33	5
5	9	7
7	8	7.3556
9	7.3902	7.3873
10	7.3880597	7.3887125
11	7.38914027	7.38899470
12	7.389055923	7.389046015

As another example, let us consider the continued fraction and series expansion of $y = \exp(x)$.

$$\begin{aligned} y &= \exp(x) \\ &= \frac{1}{1+}\,\frac{x}{-1+}\,\frac{x}{-2+}\,\frac{x}{3+}\,\frac{x}{2+}\,\frac{x}{-5+}\,\frac{x}{-2+}\,\frac{x}{7+}\,\frac{x}{2+}\,\frac{x}{-9+}\cdots \end{aligned} \tag{4.239}$$

$$\begin{aligned} y = \exp(x) &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} \\ &+ \frac{x^7}{7!} + \frac{x^8}{8!} + \frac{x^9}{9!}\cdots \end{aligned} \tag{4.240}$$

The exact value of $\exp(2)$ is:

$$\exp(2) = 7.3890560989306502272 30\cdots \tag{4.241}$$

Examining the two approximations for $x = 2$ provides us with Table 4.2 to compare the continued fractions and series expansion of $y = \exp(2)$ by increasing the number of terms. The continued fractions approach the acceptable approximation much quicker than the series expansion.

Example 140 The perimeter of ellipse.

While the area A of an ellipse has a simple expression,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad a \geq b > 0 \tag{4.242}$$

$$A = \pi ab \tag{4.243}$$

its perimeter P has no simple equation in terms of elementary functions of a and b. The differential of the arc length ds using

$$x = a\cos\theta \qquad y = b\sin\theta$$

is

$$ds^2 = dx^2 + dy^2 = \left(a^2 \sin^2\theta + b^2 \cos^2\theta\right) d\theta^2$$
$$= \left(a^2 - \left(a^2 - b^2\right)\cos^2\theta\right) d\theta^2 \tag{4.244}$$

The arc length formula for calculating the perimeter P

$$P = \int_0^{2\pi} \sqrt{\left(\frac{dx}{x\theta}\right)^2 + \left(\frac{dy}{x\theta}\right)^2} d\theta = 4\int_0^{\pi/2} \sqrt{a^2 \sin^2\theta + b^2 \cos^2\theta} d\theta$$
$$= 4\int_0^{\pi/2} \sqrt{a^2 - \left(a^2 - b^2\right)\cos^2\theta} d\theta \tag{4.245}$$

will be

$$P = 4a \int_0^{\pi/2} \sqrt{1 - e^2 \sin^2\varphi}\, d\varphi = 4aE\,(e) \tag{4.246}$$

if we define $\varphi = \pi/2 - \theta$, and introduce the eccentricity e of the ellipse.

$$a^2 - b^2 = a^2 e^2 \tag{4.247}$$

The perimeter P leads to an equation based on complete elliptic integral of the second kind $E\,(k)$.

$$P = 4aE\,(k) \tag{4.248}$$
$$E\,(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2\theta} d\theta \tag{4.249}$$
$$k^2 = 1 - \frac{b^2}{a^2} \tag{4.250}$$

By expanding the integrand in a series and integrate the series, we get the following approximate formula:

$$P = \pi\,(a+b) \sum_{k=0}^{\infty} \binom{1/2}{k}^2 q^k$$
$$= \pi\,(a+b)\left(1 + \frac{q}{2^2} + \frac{q^2}{2^6} + \frac{q^3}{2^8} + \frac{25q^4}{2^{14}} + \cdots\right) \tag{4.251}$$

$$q = \left(\frac{a-b}{a+b}\right)^2 \tag{4.252}$$

The series may be transformed into a continued fractions.

$$L = \pi \,(a+b) \left(1 + \frac{q/4}{1+} \; \frac{-q/16}{1+} \; \frac{-3q/16}{1+} \; \frac{-3q/16}{1+} \cdots \right) \tag{4.253}$$

Example 141 ★Elliptic functions.

For $0 \le k \le 1$ and $0 < \varphi \le \pi/2$, the elliptic integral of the first kind

$$u = F(x,k) = \int_0^x \frac{dy}{\sqrt{\left(1-y^2\right)\left(1-k^2y^2\right)}} \qquad x \in [-1, 1] \tag{4.254}$$

is also defined as

$$F(\varphi,k) = \int_0^\varphi \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \qquad \varphi \in [-1, 1] \tag{4.255}$$

$$x = \sin\varphi \tag{4.256}$$

For $x = 1$ or $\varphi = \pi/2$, these definite integrals are called complete elliptic integrals of the first kind and denoted by $K\,(k)$:

$$K(k) = \int_0^1 \frac{dy}{\sqrt{\left(1-y^2\right)\left(1-k^2y^2\right)}} = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \tag{4.257}$$

The elliptic integral of the second kind is defined by

$$E(x,k) = \int_0^x \sqrt{\frac{\left(1-k^2y^2\right)}{\left(1-y^2\right)}}\,dy \qquad x \in [-1, 1] \tag{4.258}$$

or

$$E(\varphi,k) = \int_0^\varphi \sqrt{1-k^2\sin^2\theta}\,d\theta \qquad \varphi \in [-1, 1] \tag{4.259}$$

$$x = \sin\varphi \tag{4.260}$$

For $x = 1$ or $\varphi = \pi/2$, these integrals are called complete elliptic integrals of the second kind and denoted by $E\,(k)$:

$$E(k) = \int_0^1 \sqrt{\frac{1-k^2y^2}{1-y^2}}\,dy = \int_0^{\pi/2} \sqrt{1-k^2\sin^2\theta}\,d\theta \tag{4.261}$$

The elliptic integral of the third kind is defined by

$$\Pi(x, k, n) = \int_0^x \frac{dy}{\left(1 + n^2 y^2\right)\sqrt{\left(1 - y^2\right)\left(1 - k^2 y^2\right)}} \qquad x \in [-1, 1] \tag{4.262}$$

or

$$\Pi(\varphi, k, n) = \int_0^\varphi \frac{d\theta}{\left(1 + n^2 \sin^2\theta\right)\sqrt{1 - k^2 \sin^2\theta}} \qquad \varphi \in [-1, 1] \tag{4.263}$$

$$x = \sin\varphi. \tag{4.264}$$

For $x = 1$ or $\varphi = \pi/2$, these integrals are called complete elliptic integrals of the third kind and denoted by $\Pi\,(k, n)$:

$$\begin{aligned}\Pi\,(k, n) &= \int_0^1 \frac{dy}{\left(1 + n^2 y^2\right)\sqrt{\left(1 - y^2\right)\left(1 - k^2 y^2\right)}} \\ &= \int_0^{\pi/2} \frac{d\theta}{\left(1 + n^2 \sin^2\theta\right)\sqrt{1 - k^2 \sin^2\theta}}\end{aligned} \tag{4.265}$$

When a dynamic problem reduces to an elliptic integral, the problem is considered solved. The behavior of elliptic integrals is well-defined.

Example 142 Elliptic functions are the inverse of the elliptic integrals.

There are two standard forms of elliptic functions, known as Jacobi elliptic functions and Weierstrass elliptic functions. Jacobi elliptic functions appear as solutions to differential equations of the form

$$\frac{d^2x}{dt^2} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \tag{4.266}$$

and Weierstrass elliptic functions appear as solutions to differential equations of the form

$$\frac{d^2x}{dt^2} = a_0 + a_1 x + a_2 x^2 \tag{4.267}$$

There are many cases in which $F = F(x)$ in the equation of motion

$$mv\frac{dv}{dx} = F(x) \tag{4.268}$$

is a polynomial or can be expanded as a polynomial,

$$F(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \tag{4.269}$$

Consider the motion of a particle with mass $m = 1$ and with the equation of motion of the form

$$\frac{d^2x}{dt^2} = F(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \tag{4.270}$$

which describes the motion of a particle moving under a force function expanded up to third order in displacement x. The solution of this type of equations can be expressed in terms of Jacobi elliptic functions. Multiplying (4.270) by $\dot{x}$ leads to the first-order differential equation

$$\frac{1}{2}\dot{x}^2 - \left(a_0 x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \frac{a_3}{4}x^4\right) = E_1 \tag{4.271}$$

where E_1 is a constant of motion.

$$E_1 = \frac{1}{2}\dot{x}_0^2 - \left(a_0 x_0 + \frac{a_1}{2}x_0^2 + \frac{a_2}{3}x_0^3 + \frac{a_3}{4}x_0^4\right) \tag{4.272}$$

We may write Eq. (4.271) in the form

$$\dot{x}^2 = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 \tag{4.273}$$

or

$$\dot{x}^2 = b_4 \left(x - \alpha\right)\left(x - \beta\right)\left(x - \gamma\right)\left(x - \delta\right) \tag{4.274}$$

Adrien-Marie Legendre (1752–1833) transformed Eq. (4.274) to

$$\dot{y}^2 = \left(1 - y^2\right)\left(1 - k^2 y^2\right) \tag{4.275}$$

where

$$y^2 = \frac{\left(\beta - \delta\right)\left(x - \alpha\right)}{\left(\alpha - \delta\right)\left(x - \beta\right)} \tag{4.276}$$

$$k^2 = \frac{\left(\beta - \gamma\right)\left(\alpha - \delta\right)}{\left(\alpha - \delta\right)\left(\beta - \delta\right)} \tag{4.277}$$

The solution of Eq. (4.275) is

$$\int dt = \int \frac{dy}{\sqrt{\left(1 - y^2\right)\left(1 - k^2 y^2\right)}} \tag{4.278}$$

Assuming $k^2 < 1$, Legendre transformed the integral

$$F(x,k)=\int_0^x \frac{dy}{\sqrt{\left(1-y^2\right)\left(1-k^2y^2\right)}} \qquad -1\leqslant x\leqslant 1 \tag{4.279}$$

to

$$F(\varphi,k)=\int_0^\varphi \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} \qquad x=\sin\varphi \tag{4.280}$$

which is the elliptic integral of the first kind. The inverse function of the first kind of elliptic integral,

$$x=\mathrm{sn}(x,k)=\sin\varphi=F^{-1}(x,k) \tag{4.281}$$

or

$$\sin\varphi=\mathrm{sn}(u,k)=\sin\varphi=F^{-1}(\varphi,k) \tag{4.282}$$

is called the Jacobi elliptic function $x=x(u,k)$ and is shown by $\mathrm{sn}(u,k)$. So,

$$\mathrm{sn}^{-1}(u,k)=\int_0^\varphi \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}=\int_0^x \frac{dy}{\sqrt{\left(1-y^2\right)\left(1-k^2y^2\right)}} \tag{4.283}$$

The angular variable φ is the amplitude of u and is denoted by $\mathrm{am}(u)$.

$$\varphi=\mathrm{am}(u)=\sin^{-1}\left(\mathrm{sn}(u,k)\right) \tag{4.284}$$

If $x=1$ or $\varphi=\pi/2$, the definite integrals (4.279) and (4.280) are the complete elliptic integrals and are denoted by

$$K(k)=\int_0^1 \frac{dy}{\sqrt{\left(1-y^2\right)\left(1-k^2y^2\right)}}=\int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \tag{4.285}$$

Two other Jacobi elliptic functions are defined by

$$\mathrm{cn}(u,k)=\cos\varphi=\sqrt{1-x^2} \tag{4.286}$$

and

$$\mathrm{dn}(u,k)=\sqrt{1-k^2x^2}=\sqrt{1-k^2\mathrm{sn}^2(u,k)} \tag{4.287}$$

which are the inverse functions of the following integrals, respectively:

$$u = \int_{1}^{\mathrm{cn}(u,k)} \frac{dy}{\sqrt{\left(1-y^2\right)\left(q^2-k^2y^2\right)}} \tag{4.288}$$

$$u = \int_{1}^{\mathrm{dn}(u,k)} \frac{dy}{\sqrt{\left(1-y^2\right)\left(y^2-q^2\right)}} \tag{4.289}$$

The Jacobi elliptic functions are related by

$$\mathrm{sn}^2(u,k) + \mathrm{cn}^2(u,k) = 1 \tag{4.290}$$

$$\mathrm{dn}^2(u,k) + k^2\,\mathrm{sn}^2(u,k) = 1 \tag{4.291}$$

$$\mathrm{dn}^2(u,k) - k^2\,\mathrm{cn}^2(u,k) = 1 - k^2 \tag{4.292}$$

and their derivatives are related by the identities

$$\frac{d}{du}\mathrm{sn}(u,k) = \mathrm{cn}(u,k)\mathrm{dn}(u,k) \tag{4.293}$$

$$\frac{d}{du}\mathrm{cn}(u,k) = -\mathrm{sn}(u,k)\mathrm{dn}(u,k) \tag{4.294}$$

$$\frac{d}{du}\mathrm{dn}(u,k) = -k^2\,\mathrm{sn}(u,k)\mathrm{cn}(u,k) \tag{4.295}$$

The theory of elliptic functions was independently developed by Abel (1802–1829) and Jacobi (1804–1851) in the nineteenth century.

Although elliptic functions only enable us to solve a relatively small class of equations, some important problems, such as pendulum, Duffing equation, and torque-free rigid body motion, belong to this class.

As an application, consider the equation of motion of a pendulum,

$$ml^2\ddot{\theta} + mgl\sin\theta = 0 \tag{4.296}$$

which can be reduced to a first-order equation using the energy integral of motion $K + V = E$:

$$\frac{1}{2}ml^2\dot{\theta}^2 + mg\left(1-\cos\theta\right) = E \tag{4.297}$$

The energy required to raise the pendulum from the lowest position at $\theta = 0$ to the highest position at $\theta = \pi$ is $2mgl$. So, we may write the energy of the system as

$$E = k^2\left(2mgl\right) \qquad k > 0 \tag{4.298}$$

where the value of k can be calculated by the initial conditions.:

$$k = \sqrt{\frac{E_0}{2mgl}} = \frac{1}{2}\sqrt{\frac{1}{gl}\left(2g + l^2\dot{\theta}_0^2 - 2g\cos\theta_0\right)} \tag{4.299}$$

The condition $0 \leq k \leq 1$ is equivalent to the oscillatory motion of the pendulum.

Example 143 Arctangent function, $y = \arctan x$.

The differential equation for $y = \arctan x$ has the form

$$y' = \frac{1}{1+x^2} \qquad y(0) = 0 \tag{4.300}$$

Substituting y with

$$y = \frac{x}{1+z} \tag{4.301}$$

makes a new differential equation.

$$\left(1+x^2\right)xz' + \left(1-x^2\right)z + z^2 = x^2 \qquad z(0) = 0 \tag{4.302}$$

This is another particular case for (4.748), in which

$$\alpha = \alpha' = \beta = \gamma = \delta = 1 \qquad \beta' = -1 \qquad k = 2 \tag{4.303}$$

Therefore the solution (4.751) will be:

$$y = \arctan x = \frac{x}{1+z} = \frac{x}{1+}\frac{x^2}{3+}\frac{4x^2}{5+}\frac{9x^2}{7+}\cdots\frac{n^2x^2}{2n+1}\cdots \tag{4.304}$$

The discovery of this expansion was credited to Lambert (1728–1777). This expansion satisfies the conditions of Stolz's theorem and therefore, Eq. (3.179) makes other form of the continued fractions for the function $y = \arctan x$.

$$y = x - \frac{x^2}{3+}\frac{9x^2}{5+}\frac{4x^2}{7+}\cdots\frac{(2n+1)^2x^2}{4n+1+}\frac{(2n)^2x^2}{4n+3+}\cdots \tag{4.305}$$

Lambert found the continued fractions expansion for $\ln(1+x)$, $\arctan x$, $\tan x$, and clarifies their convergence conditions (Jones and Thron 1980).

Example 144 ★Thiele's formula.

The Taylor series

$$f(x+h) = f(x) + \frac{h}{1!}f'(x) + \frac{h^2}{2!}f''(x) + \cdots + \frac{h^n}{n!}f^{(n)}(x) + \cdots \tag{4.306}$$

is equivalent to the Thiele's continued fraction formula.

$$f(x+h) = f(x) + \frac{h}{r(f(x)) + } \frac{h}{2r(r_1(f(x))) + } \frac{h}{3r(r_2(f(x))) + } \cdots$$
$$\cdots \frac{h}{nr\; r_{n-1}(f(x)) + } \cdots \tag{4.307}$$

Thiele's formula is another method of obtaining the expansions of functions into continued fractions. In Thiele's formula, $r_n(f(x))$ is called the inverse derivative of the nth order of the function $f(x)$ defined as

$$r(f(x)) = \frac{1}{f'(x)} \qquad r_2(f(x)) = f(x) + 2r(r_1(f(x))) \tag{4.308}$$

or in general for $n \geq 2$

$$r_n(f(x)) = r_{n-2}(f(x)) + nr(r_{n-1}(f(x))) \qquad n \geq 2 \tag{4.309}$$

As an example let us consider $f(x) = e^x$

$$r\left(e^x\right) = e^{-x} \qquad r_2\left(e^x\right) = -e^x \qquad r_{2n}\left(e^x\right) = (-1)^n e^x \tag{4.310}$$
$$r_{2n+1}\left(e^x\right) = (-1)^n (n+1) e^x \tag{4.311}$$

and hence

$$(2n+1)\, r\left(r_{2n}\left(e^x\right)\right) = (-1)^n (2n+1) e^{-x} \tag{4.312}$$
$$(2n+1)\, r\left(r_{2n+1}\left(e^x\right)\right) = 2(-1)^{n+1} e^x \tag{4.313}$$

Using these relations and replacing x by 0 and h by x in Thiele's formula (4.307), we find Eq. (4.775).

4.3 Series Solution of Differential Equations

A function $f(x)$ is analytic at a point $x = a$, if it can be expressed as a power series around the point $x = a$.

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$
$$= \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = \sum_{k=0}^{\infty} a_k (x-a)^k \tag{4.314}$$

$$a_k = \frac{f^{(k)}(a)}{k!} \tag{4.315}$$

Consider the general second-order linear differential equation, for which there is no general closed form solution, unless the coefficients are constant (Simmons 1991).

$$y'' + P(x)\, y' + Q(x)\, y = R(x) \tag{4.316}$$

If the functions $P(x)$, $Q(x)$, $R(x)$ are analytic at a point $x = a$, then the solution of the differential equation can be represented by a power series with a finite radius of convergence $s > 0$.

$$y = \sum_{k=0}^{\infty} c_k (x - a)^k \tag{4.317}$$

Such a point $x = a$ is called *ordinary* point. If any of the functions $P(x)$, $Q(x)$, $R(x)$ is not analytic at a point $x = a$ then, the point $x = a$ is a *singular* point.

There are two equivalent methods to derive the series solution of a differential equation: *substituting* and *derivative* methods.

4.3.1 Substituting Method

We assume a power series solution for the unknown function y at the interested point $x = a$,

$$y = \sum_{k=0}^{\infty} c_k (x - a)^k \tag{4.318}$$

and then substitute the power series and its required derivatives in the differential equation, then expand the result and collect the coefficients of $(x - a)^k$ for $k = 1, 2, 3, \cdots$. Equating the coefficients of $(x - a)^k$ to zero determines the coefficients c_k. Depending on the order of the differential equation, all coefficients will be functions of only the same number of coefficients as the order of the equation. The advantage of substituting method is the possibility of having a recursive equation to determine the next coefficients from the previous coefficients.

Proof Let us consider the general form of the second-order linear differential equations

$$f_2(x)\, y'' + f_1(x)\, y' + f_0(x)\, y = f(x) \tag{4.319}$$

where $f_i(x)$ and $f(x)$ are continuous functions of the independent variable x. We are able to determine a solution around an ordinary point c at which

$$f_2(a) \neq 0 \tag{4.320}$$

or around a singular point c at which

$$f_2(a) = 0 \tag{4.321}$$

Equation (4.319) may be simplified to (4.316) at any ordinary point. If $f_2(x) \neq 0$, then there is no singular point and Eq. (4.319) may always be simplified to (4.316).

For analytic functions $P(x)$, $Q(x)$, $R(x)$, we may expand them into power series

$$P(x) = \sum_{k=0}^{\infty} \frac{P^{(n)}(a)}{n!} (x-a)^n \tag{4.322}$$

$$Q(x) = \sum_{k=0}^{\infty} \frac{Q^{(n)}(a)}{n!} (x-a)^n \tag{4.323}$$

$$R(x) = \sum_{k=0}^{\infty} \frac{R^{(n)}(a)}{n!} (x-a)^n \tag{4.324}$$

and find a Taylor series solution

$$y(x) = \sum_{k=0}^{\infty} c_k (x-a)^k \tag{4.325}$$

at an ordinary point by substituting (4.322)–(4.324) into (4.316), and sorting the result for $(x-a)^k$, $k = 0, 1, 2, \cdots$. Setting the coefficients of $(x-a)^k$ equal to zero provides us with a set of algebraic equations to determine c_k, $k = 0, 1, 2, \cdots$, and calculate the series solution. A few examples will show the method. ■

Example 145 Series solution for equation $y'' + y = 0$.

Let us substitute an unknown power series

$$y = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots \tag{4.326}$$

$$y'' = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} = 2a_2 + 6a_3 x + 12a_4 x^2 + \cdots \tag{4.327}$$

in the equation.

$$y'' + y = 0 \tag{4.328}$$

$$2a_2 + 6a_3 x + 12a_4 x^2 + \cdots + a_0 + a_1 x + a_2 x^2 + \cdots = 0 \tag{4.329}$$

Rewriting in a new power series

$$(a_0 + 2a_2) + (a_1 + 6a_3)\,x + (a_2 + 12a_4)\,x^2 + \cdots = 0 \tag{4.330}$$

and equating the coefficients to zero provide us with:

$$a_0 + 2a_2 = 0 \qquad a_2 = -\frac{1}{2}a_0 \tag{4.331}$$

$$a_1 + 6a_3 = 0 \qquad a_3 = -\frac{1}{6}a_1 \tag{4.332}$$

$$a_2 + 12a_4 = 0 \qquad a_4 = -\frac{1}{12}a_2 = \frac{1}{24}a_0 \tag{4.333}$$

$$\cdots$$

Therefore the solution will be:

$$\begin{aligned} y &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots \\ &= a_0 + a_1x - \frac{1}{2}a_0x^2 - \frac{1}{6}a_1x^3 + \frac{1}{24}a_0x^4 + \cdots \\ &= a_0\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \cdots\right) \\ &\quad + a_1\left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \cdots\right) \end{aligned} \tag{4.334}$$

The solution will only depend on two constants a_0 and a_1 which are supposed to be found from the initial conditions. If the initial conditions are

$$y\,(0) = y_0 \qquad y'\,(0) = y'_0 \tag{4.335}$$

then

$$a_0 = y_0 \qquad a_1 = y'_0 \tag{4.336}$$

Figure 4.5 illustrates the series solution of the equation for $y\,(0) = 1$ and $y'\,(0) = 0$ up to different exponents compared to the exact solution.

We may recall that

$$1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \cdots = \cos x \tag{4.337}$$

$$x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \cdots = \sin x \tag{4.338}$$

and therefore, the exact solution of the differential equation is:

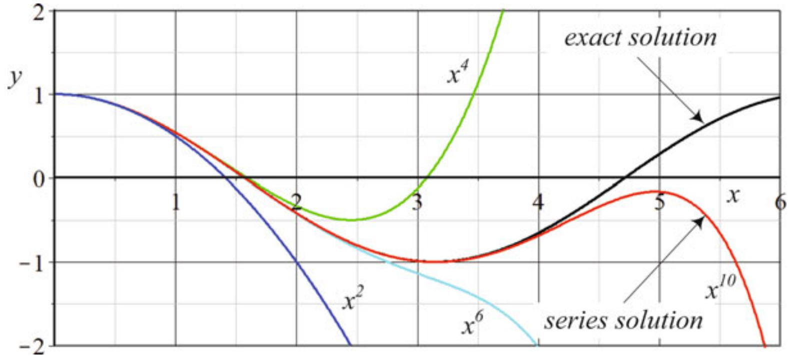

Fig. 4.5 Power series solution of $y'' + y = 0$ for $y(0) = 1$ and $y'(0) = 0$ compared to exact solution

$$y = a_0 \cos x + a_1 \sin x \tag{4.339}$$

A good question is: if we have a convergent power series, is it equal to a known function, and if yes, how can we recognize the function? The answer is: not every power series is equal to a known elementary function, and if it is, then, in general, it is not possible to determine the function. However, we may have a table of power series expansion of elementary function to recognize the associated functions in a power series by comparison. How quick and whether the series solution converges to the actual solution is a more difficult matter to answer. The correct answer is: we need infinity number of terms to get the exact solution. Any truncation eventually makes the series to deviate from the exact solution.

Example 146 Rewriting the summations technic.

Let us find the power series solution around $x = 0$ of the equation,

$$y'' + y' + \left(1 + x^2\right) y = 0 \tag{4.340}$$

$$y(0) = 2 \qquad y'(0) = 0 \tag{4.341}$$

Substituting

$$y = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \tag{4.342}$$

$$y' = \sum_{k=1}^{\infty} k a_k x^{k-1} \qquad y'' = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} \tag{4.343}$$

provides

$$\sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} + \sum_{k=1}^{\infty} k a_k x^{k-1} + \left(1 + x^2\right) \sum_{k=0}^{\infty} a_k x^k = 0 \tag{4.344}$$

We expand and multiply $\left(1 + x^2\right)$ with the summation to rewrite the equation as:

$$\sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} + \sum_{k=1}^{\infty} k a_k x^{k-1} + \sum_{k=0}^{\infty} a_k x^k + \sum_{k=0}^{\infty} a_k x^{k+2} = 0 \tag{4.345}$$

The powers of x in the first, second, and last summations are different from the third summation. To determine the coefficient of each power of x, we need to change the power of x in the first, second, and last terms from $k-2$, $k-1$, and $k+2$ to k. In the first summation we set $m = k - 2$,

$$\sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} = \sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2} x^m \tag{4.346}$$

however, m is a dummy index and may be replaced by any other symbol such as k.

$$\sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2} x^m = \sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2} x^k \tag{4.347}$$

Similarly, setting $m = k - 1$ in the second term and $m = k + 2$ in the last term will make their exponent to start from 0, although not all summations start from $k = 0$.

$$\sum_{k=1}^{\infty} k a_k x^{k-1} = \sum_{m=0}^{\infty} (m+1)a_{m+1} x^m \tag{4.348}$$

$$\sum_{k=0}^{\infty} a_k x^{k+2} = \sum_{m=2}^{\infty} a_{m-2} x^m \tag{4.349}$$

Renaming m back to k in the summations and substitute all summation in (4.345) yields:

$$\sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2} x^k + \sum_{k=0}^{\infty} (k+1)a_{k+1} x^k + \sum_{k=0}^{\infty} a_k x^k + \sum_{k=2}^{\infty} a_{k-2} x^k = 0 \tag{4.350}$$

By assigning $k = 0, 1, 2, 3, \cdots$ and simplifying the equation, we determine the coefficients of $x^0, x, x^2, x^3, \cdots$, all to be equal to zero. The coefficients make a set of algebraic equations to determine the unknown a_k of the solution (4.342).

$$2a_2 + a_1 + a_0 = 0 \qquad k = 0 \tag{4.351}$$

$$6a_3 + 2a_2 + a_1 = 0 \qquad k = 1 \tag{4.352}$$

$$(k+2)(k+1)\,a_{k+2} + (k+1)\,a_{k+1} + a_k + a_{k-2} = 0 \qquad k \geq 2 \tag{4.353}$$

The first equation indicates that

$$a_2 = \frac{-a_1 - a_0}{2} \tag{4.354}$$

and the second equation indicates that

$$a_3 = \frac{-2a_2 - a_1}{6} = \frac{a_0}{6} \tag{4.355}$$

and the third equation gives us a recurrence relation to determine all coefficients of (4.342) for $k \geq 2$.

$$a_{k+2} = \frac{-(k+1)\,a_{k+1} - a_k - a_{k-2}}{(k+2)(k+1)} \tag{4.356}$$

Therefore,

$$a_4 = -\frac{1}{12}a_2 - \frac{1}{12}a_0 - \frac{1}{4}a_3 = -\frac{1}{12}a_0 + \frac{1}{24}a_1 \tag{4.357}$$

$$a_5 = -\frac{1}{20}a_3 - \frac{1}{20}a_1 - \frac{1}{5}a_4 = \frac{1}{120}a_0 - \frac{7}{120}a_1 \tag{4.358}$$

$$a_6 = -\frac{1}{30}a_4 - \frac{1}{30}a_2 - \frac{1}{6}a_5 = \frac{13}{720}a_0 + \frac{1}{40}a_1 \tag{4.359}$$

$$\cdots$$

and the solution will be

$$y = a_0 + a_1 x + \left(\frac{-a_1 - a_0}{2}\right)x^2 + \frac{a_0}{6}x^3 + \left(\frac{1}{24}a_1 - \frac{1}{12}a_0\right)x^4 + \left(\frac{1}{120}a_0 - \frac{7}{120}a_1\right)x^5 + \left(\frac{13}{720}a_0 + \frac{1}{40}a_1\right)x^6 + \cdots \tag{4.360}$$

$$= \left(1 - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \frac{1}{120}x^5 + \frac{13}{720}x^6 + \cdots\right)a_0 + \left(x - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{7}{120}x^5 + \frac{1}{40}x^6 + \cdots\right)a_1 \tag{4.361}$$

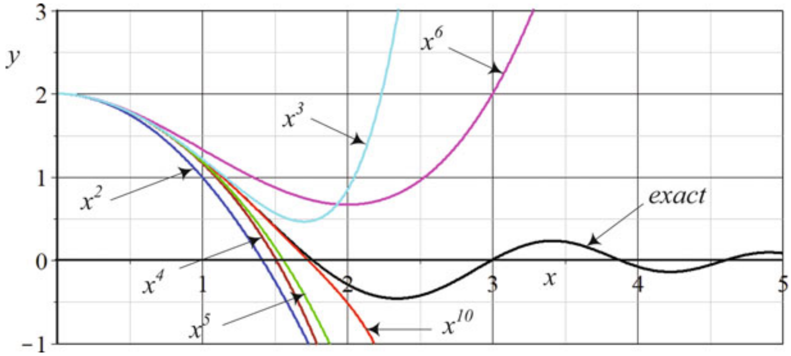

Fig. 4.6 Comparison of series solutions and exact solution of $y'' + y' + \left(1 + x^2\right) y = 0$ for $y(0) = 2$, $y'(0) = 0$

Employing the initial conditions (4.341), we find

$$a_0 = y(0) = 2 \qquad a_1 = y'(0) = 0 \tag{4.362}$$

and the solution simplifies to:

$$y = 2 - x^2 + \frac{1}{3}x^3 - \frac{1}{6}x^4 + \frac{1}{60}x^5 + \frac{13}{360}x^6 - \frac{17}{1260}x^7 + \frac{9}{2240}x^8 - \frac{89}{181440}x^9 - \frac{1}{2520}x^{10} + \cdots \tag{4.363}$$

Figure 4.6 compares the series solutions for different exponents of the series with the exact solution. The series solution is acceptable only for a limited domain. Although theoretically, the series solution will approach the exact solution when the number of terms approaches infinity, it is not practically good enough to approximate the solution away from the point that the series calculated about, in this case $x = 0$.

Example 147 A nonlinear dynamic system.

A spring hardening mass-spring system is governed by equation

$$\ddot{x} + x^3 = 0 \qquad x(0) = 1 \qquad \dot{x}(0) = 0 \tag{4.364}$$

The guess series solution of the equation around $t = 0$ is:

$$x(t) = \sum_{k=0}^{\infty} a_k t^k \tag{4.365}$$

The derivatives of the power series are:

$$\dot{x}(t) = \sum_{k=1}^{\infty} k a_k t^{k-1} \qquad \ddot{x}(t) = \sum_{k=2}^{\infty} k(k-1) a_k t^{k-2} \tag{4.366}$$

Substituting the series in the equation

$$\sum_{k=2}^{\infty} k(k-1) a_k t^{k-2} + \left(\sum_{k=0}^{\infty} a_k t^k\right)^3 = 0 \tag{4.367}$$

and sorting

$$\sum_{k=2}^{\infty} k(k-1) a_k t^{k-2} + \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_i a_j a_k t^i t^j t^k = 0 \tag{4.368}$$

provides the coefficients of $t^0, t, t^2, t^3, \cdots$.

$$2a_2 + a_0^3 = 0 \qquad t^0 \tag{4.369}$$

$$3a_0^2 a1 + 6a_3 = 0 \qquad t \tag{4.370}$$

$$3a_2 a_0^2 + 3a_0 a_1^2 + 12a_4 = 0 \qquad t^2 \tag{4.371}$$

$$3a_3 a_0^2 + 6a_2 a_0 a_1 + a_1^3 + 20a_5 = 0 \qquad t^3 \tag{4.372}$$

$$3a_4 a_0^2 + 6a_3 a_0 a_1 + 3a_0 a_2^2 + 3a_1^2 a_2 + 30a_6 = 0 \qquad t^4 \tag{4.373}$$

$$\begin{aligned} 42a^7 + 3a_5 a_0^2 + 6a_4 a_0 a_1 + 6a_3 a_0 a_2 \\ +3a_3 a_1^2 + 3a_1 a_2^2 = 0 \qquad t^5 \end{aligned} \tag{4.374}$$

Solving the equations for calculating the coefficients $a_2, a_3, a_4, \cdots$ in terms of a_0 and a_1 gives the following:

$$a_2 = -\frac{1}{2} a_0^3 \qquad a_3 = -\frac{1}{2} a_0^2 a_1 \tag{4.375}$$

$$a_4 = \frac{1}{8} a_0^5 - \frac{1}{4} a_1^2 a_0 \qquad a_5 = \frac{9}{40} a_0^4 a_1 - \frac{1}{20} a_1^3 \tag{4.376}$$

$$a_6 = -\frac{3}{80} a_0^7 + \frac{7}{40} a_1^2 a_0^3 \tag{4.377}$$

$$a_7 = -\frac{7}{80} a_0^6 a_1 + \frac{3}{40} a_0^2 a_1^3 \tag{4.378}$$

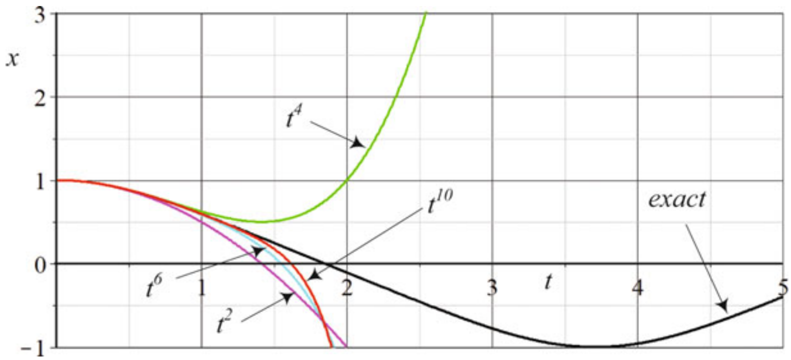

Fig. 4.7 Comparison of series solutions and exact solution of the nonlinear equation $\ddot{x} + x^3 = 0$ for $x(0) = 1, \dot{x}(0) = 0$

$$a_8 = \frac{7}{640}a_0^9 - \frac{3}{32}a_0^5 a_1^2 + \frac{3}{160}a_0 a_1^4 \tag{4.379}$$
$$\cdots$$

The initial condition determines a_0 and a_1,

$$x(0) = a_0 = 1 \qquad \dot{x}(0) = a_1 = 0 \tag{4.380}$$

and that makes the solution

$$x = 1 - \frac{1}{2}t^2 + \frac{1}{8}t^4 - \frac{3}{80}t^6 + \frac{7}{640}t^8 - \frac{61}{19200}t^{10} + \cdots \tag{4.381}$$

Figure 4.7 compares the series solution for different exponents of the series with the exact solution of the differential equation. The series solution practically is not good enough to approximate the solution far from the point $t = 0$ that the series calculated about.

Adding a high damping to the system

$$\ddot{x} + \dot{x} + x^3 = 0 \qquad x(0) = 1 \qquad \dot{x}(0) = 0 \tag{4.382}$$

makes the solution to be:

$$\begin{aligned} x = 1 &- \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{12}t^4 - \frac{1}{24}t^5 - \frac{19}{720}t^6 + \frac{47}{2520}t^7 \\ &+ \frac{173}{40320}t^8 - \frac{61}{10368}t^9 - \frac{197}{453600}t^{10} + \cdots \end{aligned} \tag{4.383}$$

Figure 4.8 compares the series solutions for different terms of the series with the exact solution of the differential equation.

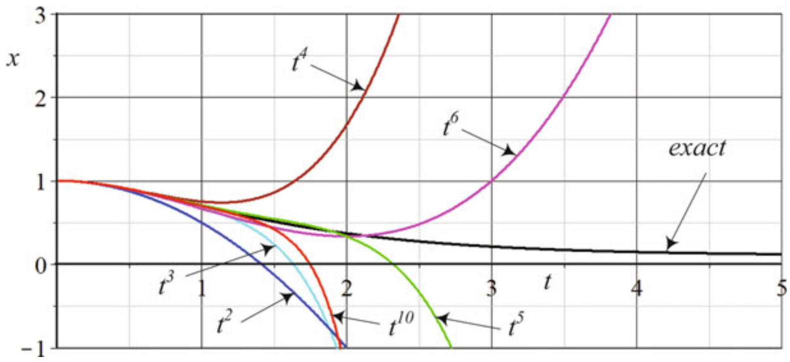

Fig. 4.8 Comparison of series solutions and exact solution of the nonlinear equation $\ddot{x}+\dot{x}+x^3=0$ for $x\,(0)=1$, $\dot{x}\,(0)=0$

Example 148 A variable damping vibrating system.

The equation of motion of free vibration of a linear mass-damper-spring system is

$$m\ddot{x}+c\dot{x}+kx=0 \tag{4.384}$$

Considering the special case of $m=k=1$, and assuming a time hardening damper, such that $c=t$, makes the following system:

$$\ddot{x}+t\dot{x}+x=0 \tag{4.385}$$

$$x(0)=1 \qquad \dot{x}\,(0)=0 \tag{4.386}$$

Assume that we wish to have the series solution of the equation around $t=0$.

$$x\,(t)=\sum_{k=0}^{\infty}a_k t^k \qquad a_0=1 \tag{4.387}$$

The derivatives of the power series solution are:

$$\dot{x}\,(t)=\sum_{k=1}^{\infty}k\;a_k\;t^{k-1} \qquad a_1=0 \tag{4.388}$$

$$\ddot{x}\,(t)=\sum_{k=2}^{\infty}k\,(k-1)\,a_k t^{k-2} \tag{4.389}$$

Substituting the series expansion of $x\,(t)$ into the differential equation,

$$\sum_{k=2}^{\infty} k (k-1) a_k t^{k-2} + t \sum_{k=1}^{\infty} k a_k t^{k-1} + \sum_{k=0}^{\infty} a_k t^k = 0 \tag{4.390}$$

we need to combine them into a single power series.

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} t^k + \sum_{k=1}^{\infty} k a_k t^k + \sum_{k=0}^{\infty} a_k t^k = 0 \tag{4.391}$$

$$\begin{aligned} 2a_2 + \sum_{k=1}^{\infty} (k+2)(k+1) a_{k+2} t^k \\ + \sum_{k=1}^{\infty} k a_k t^k + a_0 + \sum_{k=1}^{\infty} a_k t^k = 0 \end{aligned} \tag{4.392}$$

$$a_0 + 2a_2 + \sum_{k=1}^{\infty} \left((k+2)(k+1) a_{k+2} t^k + (k+1) a_k t^k \right) = 0 \tag{4.393}$$

$$2a_2 + a_0 = 0 \qquad k = 0 \tag{4.394}$$

$$(k+2)(k+1) a_{k+2} + (k+1) a_k = 0 \qquad k \geq 1 \tag{4.395}$$

Applying the initial conditions provided $a_0 = 1$ and $a_1 = 0$. Equation (4.394) gives $a_2 = -1/2$ and the recursive equation (4.395) provides the other coefficients a_i. The resultant power series solution will be:

$$x = 1 - \frac{t^2}{2} + \frac{t^4}{8} - \frac{t^6}{48} + \frac{t^8}{384} - \frac{t^{10}}{3840} + \cdots \tag{4.396}$$

Figure 4.9 illustrates the graphical presentation of the exact and series solution for different exponents.

Example 149 A third order nonlinear nonhomogeneous equation.

Consider a third order nonlinear nonhomogeneous equation

$$y'''(x) + y''(x) + \left(y'(x)\right)^2 + y(x) = \sin x \tag{4.397}$$

$$y(0) = 1 \qquad y'(0) = 1 \qquad y''(0) = 1 \tag{4.398}$$

that has a nonlinear term of y'^2 and excitation $\sin x$.

In differential equations language, a nonhomogeneous equation means an equation with an excitation term of the independent variable. Here, the term $\sin x$ makes

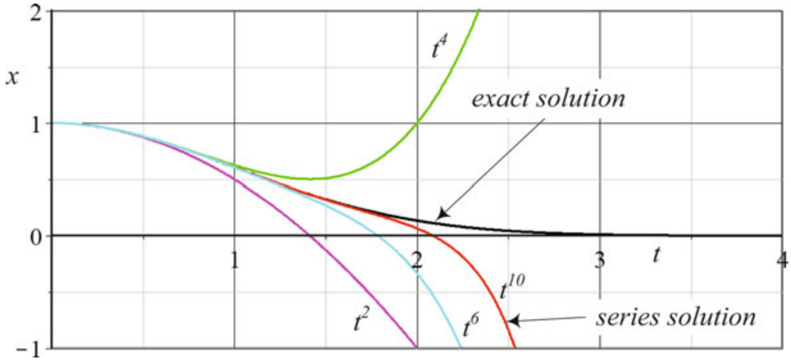

Fig. 4.9 Comparison of series solutions and exact solution of the hardening damping equation $\ddot{x} + t\dot{x} + x = 0$ for $x(0) = 1, \dot{x}(0) = 0$

the equation nonhomogeneous. It is different than dimensional homogeneity in dimensional analysis, indicating inconsistency in scientific terminology.

To derive a series solution for the equation, we substitute the unknown power series for y and its derivatives

$$y = \sum_{k=0}^{\infty} a_k x^k \qquad y' = \sum_{k=1}^{\infty} k a_k x^{k-1} \qquad y'' = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}$$

$$y'' = \sum_{k=3}^{\infty} k(k-1)(k-2) a_k x^{k-3} \tag{4.399}$$

as well as the power series expansion of the excitation function $\sin x$ all in the equation.

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} \tag{4.400}$$

The initial conditions show

$$a_0 = 1 \qquad a_1 = 1 \qquad a_2 = \frac{1}{2} \tag{4.401}$$

and therefore,

$$\begin{aligned} 0 = {} & 3 + 6a_3 + (24a_4 + 6a_3 + 2)x + (60a_5 + 12a_4 + \frac{3}{2} + 6a_3)x^2 \\ & + (7a_3 + 120a_6 + 20a_5 + \frac{1}{6} + 8a_4)x^3 \\ & + (9a_3^2 + 9a_4 + 10a_5 + 30a_6 + 210a_7)x^4 + \cdots \end{aligned} \tag{4.402}$$

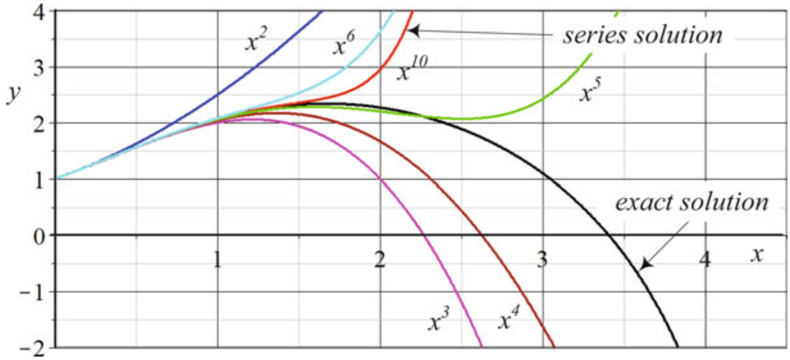

Fig. 4.10 Comparison of series solutions and exact solution of the nonlinear third order equation $y'''(x) + y''(x) + \left(y'(x)\right)^2 + y(x) = \sin x$ for $y(0) = 1$, $y'(0) = 1$, $y''(0) = 1$

To determine the numerical values of a_k, $k \geq 3$, we need to set the coefficients of x^k, $k \geq 0$ to zero, one by one.

$$3 + 6a_3 = 0 \qquad a_3 = -\frac{1}{2} \tag{4.403}$$

$$24a_4 + 6a_3 + 2 = 0 \qquad a_4 = \frac{1}{24} \tag{4.404}$$

$$60a_5 + 12a_4 + \frac{3}{2} + 6a_3 = 0 \qquad a_5 = \frac{1}{60} \tag{4.405}$$

$$\cdots$$

Therefore, the power series of the solution will be found.

$$y = 1 + x + \frac{1}{2}x^2 - \frac{1}{2}x^3 + \frac{1}{24}x^4 + \frac{1}{60}x^5 + \frac{1}{45}x^6$$
$$-\frac{83}{5040}x^7 + \frac{1}{448}x^8 + \frac{1}{12960}x^9 + \frac{13}{16200}x^{10} + \cdots \tag{4.406}$$

Figure 4.10 compares the series solutions up to the term x^{10} with the exact solution of the differential equation.

Example 150 Legendre equation.

Legendre equation with a parameter α appears in physical problems with spherical symmetry.

$$\left(1 - x^2\right) y'' - 2xy' + \alpha(\alpha + 1) y = 0 \qquad \alpha \geq 0 \tag{4.407}$$

Solutions of the equation are called Legendre functions, which are special functions. Comparing the Legendre equation with the general second-order equation

$$y'' + P(x)\,y' + Q(x)\,y = R(x) \tag{4.408}$$

indicates that

$$P(x) = \frac{-2x}{\left(1 - x^2\right)} \qquad Q(x) = \frac{\alpha(\alpha+1)}{\left(1 - x^2\right)} \qquad R(x) = 0 \tag{4.409}$$

Both $P(x)$ and $Q(x)$ are analytic at $x = 0$.

Substituting the powers series solution

$$y = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \tag{4.410}$$

$$y' = \sum_{k=1}^{\infty} k\, a_k\, x^{k-1} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 \cdots \tag{4.411}$$

$$y'' = \sum_{k=2}^{\infty} k(k-1)\,a_k x^{k-2} = 2a_2 + 6a_3 x + 12a_4 x^2 + \cdots \tag{4.412}$$

in the equation,

$$\left(1 - x^2\right)\sum_{k=2}^{\infty} k(k-1)\,a_k x^{k-2} - 2x\sum_{k=1}^{\infty} k a_k x^{k-1} + \alpha(\alpha+1)\sum_{k=0}^{\infty} a_k x^k = 0 \tag{4.413}$$

and rewriting the result will be:

$$\sum_{k=0}^{\infty}(k+2)(k+1)\,a_{k+2}x^k - \sum_{k=2}^{\infty} k(k-1)\,a_k x^k - \sum_{k=1}^{\infty} 2k a_k x^k + \alpha(\alpha+1)\sum_{k=0}^{\infty} a_k x^k = 0 \tag{4.414}$$

Equating each coefficient of x^k to zero gives us the following:

$$2a_2 + \alpha(\alpha+1)\,a_0 = 0 \qquad k = 0 \tag{4.415}$$

$$6a_3 - 2a_1 + \alpha(\alpha+1)\,a_1 = 0 \qquad k = 1 \tag{4.416}$$

$$(k+2)(k+1)\,a_{k+2} - k(k-1)\,a_k - 2ka_k + \alpha(\alpha+1)\,a_k = 0 \qquad k \geq 2 \tag{4.417}$$

From (4.415) and (4.416) we have:

$$a_2 = -\frac{\alpha(\alpha+1)}{2}a_0 \tag{4.418}$$

$$a_3 = \frac{2-\alpha(\alpha+1)}{6}a_1 \tag{4.419}$$

and from (4.417) we get:

$$a_{k+2} = \frac{k(k+1)-\alpha(\alpha+1)}{(k+2)(k+1)}a_k = -\frac{(\alpha-k)(\alpha+k+1)}{(k+2)(k+1)}a_k \tag{4.420}$$

Therefore, we are able to evaluate the coefficients of the solution (4.410).

$$a_2 = -\frac{\alpha(\alpha+1)}{2!}a_0 \qquad a_3 = -\frac{(\alpha-1)(\alpha+2)}{3!}a_1 \tag{4.421}$$

$$a_4 = -\frac{(\alpha-2)(\alpha+3)}{4\times 3}a_2 = \frac{(\alpha-2)\alpha(\alpha+1)(\alpha+3)}{4!}a_0 \tag{4.422}$$

$$a_5 = \frac{(\alpha-3)(\alpha-1)(\alpha+2)(\alpha+4)}{5!}a_1 \tag{4.423}$$

$$a_6 = -\frac{(\alpha-4)(\alpha-2)\alpha(\alpha+1)(\alpha+3)(\alpha+5)}{6!}a_1 \tag{4.424}$$

The coefficients of the even powers of x are all multiples of a_0, while the coefficients of the odd powers of x are all multiples of a_1. The parameters a_0 and a_1 are to be calculated by initial conditions. The solution may be factored as

$$y = \sum_{k=0}^{\infty} a_k x^k = a_0 y_0(x) + a_1 y_1(x) \tag{4.425}$$

where

$$\begin{aligned} y_0(x) = 1 &- \frac{\alpha(\alpha+1)}{2!}x^2 + \frac{(\alpha-2)\alpha(\alpha+1)(\alpha+3)}{4!}x^4 \\ &- \frac{(\alpha-4)(\alpha-2)\alpha(\alpha+1)(\alpha+3)(\alpha+5)}{6!}x^6 + \cdots \end{aligned} \tag{4.426}$$

$$\begin{aligned} y_1(x) = x &- \frac{(\alpha-1)(\alpha+2)}{3!}x^3 \\ &+ \frac{(\alpha-3)(\alpha-1)(\alpha+2)(\alpha+4)}{5!}x^5 - \cdots \end{aligned} \tag{4.427}$$

The series $y_0(x)$ and $y_1(x)$ are linearly independent and are convergent for $-1 < x < 1$ for non-integer α.

Example 151 ★Exact solution of the first-order equations.

The general form of the first-order ordinary differential equation is:

$$Q(x, y)\frac{dy}{dx} + P(x, y) = 0 \tag{4.428}$$

To solve the equation, we rewrite it in expansion form:

$$P(x, y)\, dx + Q(x, y)dy = 0 \tag{4.429}$$

This equation is exact if the left-hand side is an exact differential. The condition to be exact differential is

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \tag{4.430}$$

and that means there is a function $\varphi(x, y)$ such that the equation is exact differential of φ.

$$P(x, y)\, dx + Q(x, y)dy = \frac{\partial \varphi}{\partial x}dx + \frac{\partial \varphi}{\partial y}dy = 0 \tag{4.431}$$

$$\frac{\partial \varphi}{\partial x} = P(x, y) \qquad \frac{\partial \varphi}{\partial y} = Q(x, y) \tag{4.432}$$

The solution of the exact differential would be:

$$\varphi = C \tag{4.433}$$

Equation (4.431) indicates that:

$$\frac{\partial P}{\partial y} = \frac{\partial^2 \varphi}{\partial y \partial x} \qquad \frac{\partial Q}{\partial x} = \frac{\partial^2 \varphi}{\partial x \partial y} \tag{4.434}$$

Assuming that the order of differentiation can be interchanged for a continued function $\varphi(x, y)$, then we have the condition (4.430).

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \tag{4.435}$$

If a first-order differential equation is an exact equation, the function $\varphi(x, y)$ will be found by solving the two first-order partial differential equations (4.432). An integral of $P(x, y)$ gives us

$$\varphi = \int P(x, y)\, dx + f(y) \tag{4.436}$$

which its y-derivative will be equal to the function $Q(x, y)$.

$$\frac{\partial}{\partial y} \int P(x, y)\, dx + f'(y) = Q(x, y) \tag{4.437}$$

$$f'(y) = Q(x, y) - \frac{\partial}{\partial y} \int P(x, y)\, dx \tag{4.438}$$

An y-integral of $f'(y)$ yields:

$$f(y) = \int \left(Q - \frac{\partial}{\partial y} \int P dx \right) dy \tag{4.439}$$

The integrand must be a function only of y. Therefore the derivative of the integrand with respect to x is 0.

$$\begin{aligned} \frac{\partial}{\partial x} \left(Q - \frac{\partial}{\partial y} \int P dx \right) &= \frac{\partial Q}{\partial x} - \frac{\partial^2}{\partial x \partial y} \int P dx \\ &= \frac{\partial Q}{\partial x} - \frac{\partial^2}{\partial y \partial x} \int P dx \\ &= \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 \end{aligned} \tag{4.440}$$

The consistency of these operation indicates that we will be able to determine φ by starting from (4.436).

An example will show the method. Consider the following first-order ordinary differential equation:

$$-\frac{y}{x^2} \cos \frac{y}{x} dx + \frac{1}{x} \cos \frac{y}{x} dy = 0 \tag{4.441}$$

$$P = -\frac{y}{x^2} \cos \frac{y}{x} \qquad Q = \frac{1}{x} \cos \frac{y}{x} + 3y^2 \tag{4.442}$$

The integral (4.436)

$$\begin{aligned} \varphi &= \int P(x, y)\, dx + f(y) = \int -\frac{y}{x^2} \cos \frac{y}{x} dx + f(y) \\ &= \sin \frac{y}{x} + f(y) \end{aligned} \tag{4.443}$$

provides a function that its y-derivative must be equal to Q.

$$\frac{\partial}{\partial y}\left(\sin\frac{y}{x}\right) + f'(y) = \frac{1}{x}\cos\frac{y}{x} + 3y^2 \tag{4.444}$$

The $f'(y)$ will be found

$$f'(y) = \frac{1}{x}\cos\frac{y}{x} + 3y^2 - \frac{1}{x}\cos\frac{y}{x} = 3y^2 \tag{4.445}$$

which provides us with the function $f(y)$ by integration,

$$f(y) = y^3 + C \tag{4.446}$$

and therefore the function φ will be found.

$$\varphi = \sin\frac{y}{x} + y^3 + C \tag{4.447}$$

Example 152 Coupled equations series solution.

Consider two first-order coupled differential equations

$$\dot{x} = a\,x + b\,y \qquad \dot{x} = \frac{dx}{dt} \qquad x(0) = A \tag{4.448}$$

$$\dot{y} = c\,x + d\,y \qquad \dot{y} = \frac{dy}{dt} \qquad y(0) = B \tag{4.449}$$

$$\{a, b, c, d\} \in \mathbb{R} \tag{4.450}$$

To solve the equations in series, we substitute the power series expansions for x and y,

$$x = \sum_{k=0}^{\infty} a_k t^k = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots \tag{4.451}$$

$$y = \sum_{k=0}^{\infty} b_k t^k = b_0 + b_1 t + b_2 t^2 + b_3 t^3 + \cdots \tag{4.452}$$

$$x(0) = a_0 = A \qquad y(0) = b_0 = B \tag{4.453}$$

in the equations.

$$\sum_{k=1}^{\infty} k a_k t^{k-1} = a\sum_{k=0}^{\infty} a_k t^k + b\sum_{k=0}^{\infty} b_k t^k \tag{4.454}$$

$$\sum_{k=1}^{\infty} k b_k t^{k-1} = c \sum_{k=0}^{\infty} a_k t^k + d \sum_{k=0}^{\infty} b_k t^k \tag{4.455}$$

Rewriting the equations will make them ready to determine the coefficients of t^k, $k = 1, 2, 3, \cdots$.

$$\sum_{k=0}^{\infty} (k+1)\, a_{k+1} t^k - a \sum_{k=0}^{\infty} a_k t^k - b \sum_{k=0}^{\infty} b_k t^k = 0 \tag{4.456}$$

$$\sum_{k=0}^{\infty} (k+1)\, b_{k+1} t^k - c \sum_{k=0}^{\infty} a_k t^k - d \sum_{k=0}^{\infty} b_k t^k = 0 \tag{4.457}$$

Assuming

$$a = 10 \qquad b = 2 \qquad c = 3 \qquad d = -4 \tag{4.458}$$

the two series solutions will be

$$\begin{aligned} x = & -2a_0 + a_1 - 10b_0 + (-2a_1 - 10b_1 + 2a_2)t \\ & +(-2a_2 - 10b_2 + 3a_3)t^2 + (-2a_3 - 10b_3 + 4a_4)t^3 \\ & +(-2a_4 - 10b_4 + 5a_5)t^4 + \cdots \end{aligned} \tag{4.459}$$

$$\begin{aligned} y = & \, 4a_0 - 3b_0 + b1 + (4a_1 - 3b_1 + 2b_2)t \\ & +(4a_2 - 3b_2 + 3b_3)t^2 + (4a_3 - 3b_3 + 4b_4)t^3 \\ & +(4a_4 - 3b_4 + 5b_5)t^4 + \cdots \end{aligned} \tag{4.460}$$

Setting the coefficients of t^0 in the equations equal to zero provides two algebraic equations to determine a_1 and b_1.

$$\begin{cases} -2a_0 + a_1 - 10b_0 = 0 \\ \;\; 4a_0 - 3b_0 + b_1 = 0 \end{cases} \tag{4.461}$$

$$a_1 = 2a_0 + 10b_0 \qquad b_1 = -4a_0 + 3b_0 \tag{4.462}$$

Similarly, setting the coefficients of $t, t^2, t^3, \cdots$ in the equations equal to zero gives us series of couple algebraic equations to determine (a_2, b_2), (a_3, b_3), (a_4, b_4), $\cdots$.

$$\begin{cases} -2a_1 - 10b_1 + 2a_2 = 0 \\ \;\; 4a_1 - 3b_1 + 2b_2 = 0 \end{cases} \tag{4.463}$$

$$a_2 = -18a_0 + 25b_0 \qquad b_2 = -10a_0 - \frac{31}{2}b_0 \tag{4.464}$$

$$\begin{cases} -2a_2 - 10b_2 + 3a_3 = 0 \\ 4a_2 - 3b_2 + 3b_3 = 0 \end{cases} \tag{4.465}$$

$$a_3 = -\frac{136}{3}a_0 - 35b_0 \qquad b_3 = 14a_0 + \frac{293}{6}b_0 \tag{4.466}$$

$$\begin{cases} -2a_3 - 10b_3 + 4a_4 = 0 \\ 4a_3 - 3b_3 + 4b_4 = 0 \end{cases} \tag{4.467}$$

$$a_4 = \frac{37}{3}a_0 - \frac{1675}{12}b_0 \qquad b_4 = \frac{335}{6}a_0 - \frac{13}{8}b_0 \tag{4.468}$$

$$\begin{cases} -2a_4 - 10b_4 + 5a_5 = 0 \\ 4a_4 - 3b_4 + 5b_5 = 0 \end{cases} \tag{4.469}$$

$$a_5 = \frac{583}{5}a_0 - \frac{709}{12}b_0 \qquad b_5 = \frac{709}{30}a_0 + \frac{13283}{120}b_0 \tag{4.470}$$

$$\begin{cases} -2a_5 - 10b_5 + 6a_6 = 0 \\ 4a_5 - 3b_5 + 6b_6 = 0 \end{cases} \tag{4.471}$$

$$a_6 = \frac{7043}{90}a_0 + \frac{3955}{24}b_0 \qquad b_6 = -\frac{791}{12}a_0 + \frac{68209}{720}b_0 \tag{4.472}$$

Therefore the power series solutions are:

$$\begin{aligned} x = {} & a_0 + (2a_0 + 10b_0)\,t + (-18a_0 + 25b_0)\,t^2 + \left(-\frac{136}{3}a_0 - 35b_0\right)t^3 \\ & + \left(\frac{37}{3}a_0 - \frac{1675}{12}b_0\right)t^4 + \left(\frac{583}{5}a_0 - \frac{709}{12}b_0\right)t^5 \\ & + \left(\frac{7043}{90}a_0 + \frac{3955}{24}b_0\right)t^6 + \left(-\frac{45239}{630}a_0 + \frac{91939}{504}b_0\right)t^7 + \cdots \end{aligned} \tag{4.473}$$

$$\begin{aligned} y = {} & b_0 + (-4a_0 + 3b_0)\,t + \left(-10a_0 - \frac{31}{2}b_0\right)t^2 \\ & + \left(14a_0 + \frac{293}{6}b_0\right)t^3 + \left(\frac{335}{6}a_0 - \frac{13}{8}b_0\right)t^4 \end{aligned}$$

$$+\left(\frac{709}{30}a_0+\frac{13283}{120}b_0\right)t^5+\left(-\frac{791}{12}a_0+\frac{68209}{720}b_0\right)t^6$$
$$+\left(-\frac{91939}{1260}a_0-\frac{91939}{560}b_0\right)t^7+\cdots \qquad (4.474)$$

We may rearrange the solutions to be series coefficients of a_0 and b_0.

$$x=a_0\left(1+2t-18t^2-\frac{136}{3}t^3+\frac{37}{3}t^4+\frac{583}{15}t^5+\cdots\right)$$
$$+b_0\left(10t+25t^2-35t^3-\frac{1675}{12}t^4-\frac{709}{12}t^5-\cdots\right) \qquad (4.475)$$

$$y=a_0\left(-4t-10t^2+14t^3+\frac{355}{6}t^4+\frac{709}{30}t^5-\cdots\right)$$
$$+b_0\left(1+3t-\frac{31}{2}t^2-\frac{293}{6}t^3-\frac{18}{3}t^4+\frac{13283}{120}t^5+\cdots\right) \qquad (4.476)$$

Assuming

$$x(0)=a_0=1 \qquad y(0)=b_0=1 \qquad (4.477)$$

the solutions are

$$x=1+12t+7t^2-\frac{241}{3}t^3-\frac{509}{4}t^4+\frac{3451}{60}t^5+\frac{87497}{360}t^6$$
$$+\frac{30971}{280}t^7-\frac{375881}{2880}t^8-\frac{25977829}{181440}t^9-\cdots \qquad (4.478)$$

$$y=1-t-\frac{51}{2}t^2-\frac{209}{3}t^3+\frac{1301}{24}t^4+\frac{5373}{40}t^5+\frac{20749}{720}t^6$$
$$-\frac{637729}{5040}t^7-\frac{1381033}{13440}t^8+\frac{8620039}{362880}t^9+\cdots \qquad (4.479)$$

and we may plot the answers in Fig. 4.11 to compare the exact solution of the coupled equations and their series solution up to t^{10}. Figure 4.12 depicts the response of the system in (x, y)-plane, comparing exact and series solution.

The general form of an autonomous system of two simultaneous first-order coupled differential equations is

$$\dot{x}=F(x,y) \qquad \dot{y}=G(x,y) \qquad (4.480)$$

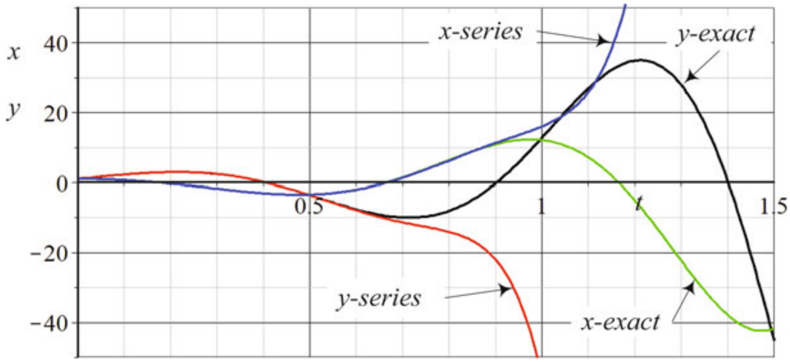

Fig. 4.11 Comparison of series solutions and exact solution of the coupled equations $\dot{x} = ax + by$ and $\dot{y} = cx + dy$ for $x(0) = 1$, $y(0) = 1$

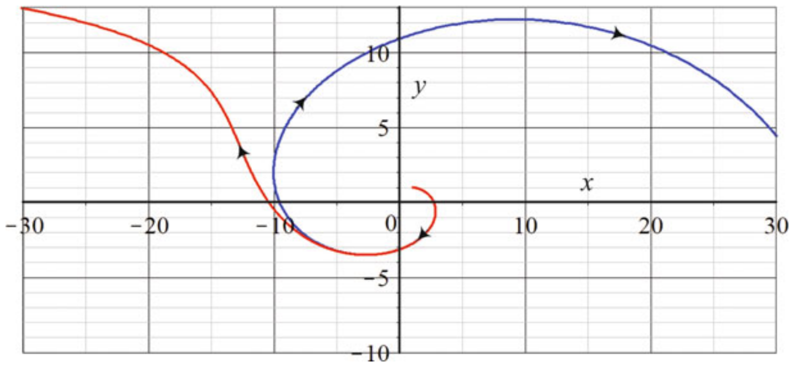

Fig. 4.12 The response of the system in (x, y)-plane, comparing exact and series solution

in which the independent variable t does not appear explicitly. The functions F and G are assumed to be continuous with continuous partial derivatives in a domain in the (x, y)-plane. The system of equations is always along with a given initial condition values.

$$x(t_0) = x_0 \qquad y(t) = y_0 \tag{4.481}$$

The equilibrium or critical points of the system of equations are the points in (x, y)-plane at which $\dot{x} = \dot{y} = 0$.

$$F(x, y) = 0 \qquad G(x, y) = 0 \tag{4.482}$$

Assuming that the origin $(0, 0)$ is a critical point, we may approximate the equations in the neighborhood of the origin $(0, 0)$,

$$\dot{x} = a\,x + b\,y + F_1(x, y) \tag{4.483}$$

$$\dot{y} = c\,x + d\,y + G_1\,(x, y) \tag{4.484}$$

where

$$a = \frac{\partial F\,(0,0)}{\partial x} \qquad b = \frac{\partial F\,(0,0)}{\partial y} \tag{4.485}$$

$$c = \frac{\partial G\,(0,0)}{\partial x} \qquad d = \frac{\partial G\,(0,0)}{\partial y} \tag{4.486}$$

and

$$ad - bc \neq 0 \tag{4.487}$$

If

$$\lim_{r \to 0} \frac{F_1\,(x, y)}{r} \to 0 \qquad \lim_{r \to 0} \frac{G_1\,(x, y)}{r} \to 0 \tag{4.488}$$

$$r = \sqrt{x^2 + y^2} \tag{4.489}$$

the system is called an almost linear system in the neighborhood of the critical point. For an almost linear system, the functions $F_1\,(x, y)$ and $G_1\,(x, y)$ are small compared to the linear terms $ax + by$ and $cx + dy$ in the neighborhood of the critical point and hence, the linear autonomous systems

$$\dot{x} = a\,x + b\,y \qquad \dot{y} = c\,x + d\,y \tag{4.490}$$

are good approximations around the critical point (0, 0) (Makarets and Reshetnyak 1995).

Example 153 Coupled equations combination.

Instead of solving a set of coupled equations in a set of power series, we may combine the equations to make one single higher order equation and find a single power series solution for the combined equation. All other original variables may be calculated from the solution.

Consider the two first-order coupled differential equations that we solved in Example 152.

$$\dot{x} = a\,x + b\,y \qquad \dot{x} = \frac{dx}{dt} \qquad x\,(0) = 1 \tag{4.491}$$

$$\dot{y} = c\,x + d\,y \qquad \dot{y} = \frac{dy}{dt} \qquad y\,(0) = 1 \tag{4.492}$$

$$\{a, b, c, d\} \in \mathbb{R} \tag{4.493}$$

We may calculate y from the first Eq. (4.506)

$$y = \frac{1}{b}\dot{x} - \frac{a}{b}x \tag{4.494}$$

and substitute it in the second Eq. (4.507).

$$\frac{1}{b}\ddot{x} - \frac{a}{b}\dot{x} = c\,x + d\left(\frac{1}{b}\dot{x} - \frac{a}{b}x\right) \tag{4.495}$$

Rearranging the new equation ends up with a second-order homogeneous linear differential equation with constant coefficients.

$$\ddot{x} - (a+d)\,\dot{x} + (ad - bc)\,x = 0 \tag{4.496}$$

Assuming

$$a = 10 \qquad b = 2 \qquad c = 3 \qquad d = -4 \tag{4.497}$$

we have

$$\ddot{x} - 6\dot{x} - 46x = 0 \tag{4.498}$$

$$x\,(0) = 1 \qquad \dot{x}\,(0) = 12 \tag{4.499}$$

because the initial conditions $x\,(0) = 1$ and $y\,(0) = 1$ determine $\dot{x}\,(0)$.

$$\dot{x}\,(0) = ax\,(0) + by\,(0) = 10 + 2 = 12 \tag{4.500}$$

Substituting a power series solution in the equation of motion (4.501),

$$x\,(t) = \sum_{k=0}^{\infty} a_k t^k \qquad x\,(0) = a_0 = 1 \qquad \dot{x}\,(0) = a_1 = 12 \tag{4.501}$$

$$\dot{x}\,(t) = \sum_{k=1}^{\infty} k a_k t^{k-1} \qquad \ddot{x}\,(t) = \sum_{k=2}^{\infty} k\,(k-1)\,a_k t^{k-2} \tag{4.502}$$

we will have:

$$\begin{aligned} 0 = {} & 2a_2 - 118 + (-12a_2 + 6a_3 - 552)t + (-46a_2 - 18a_3 + 12a_4)t^2 \\ & + (-46a_3 - 24a_4 + 20a_5)t^3 + (-46a_4 - 30a_5 + 30a_6)t^4 \\ & + (-46a_5 - 36a_6 + 42a_7)t^5 + \cdots \end{aligned} \tag{4.503}$$

Equating the coefficients of t^k, $k = 1, 2, 3, \cdots$ to zero provides us with the coefficients of the series solution.

$$a_2 = 59 \qquad a_3 = 210 \qquad a_4 = \frac{3247}{6} \qquad a_5 = \frac{5662}{5}$$
$$a_6 = \frac{176597}{90} \qquad a_7 = \frac{306823}{105} \qquad a_8 = \frac{1916909}{504} \qquad \cdots \tag{4.504}$$

$$\begin{aligned} x = {} & 1 + 12t + 59t^2 + 210t^3 + \frac{3247}{6}t^4 + \frac{5662}{5}t^5 \\ & + \frac{176597}{90}t^6 + \frac{306823}{105}t^7 + \frac{1916909}{504}t^8 \\ & + \frac{2773579}{630}t^9 + \frac{519991067}{113400}t^{10} + \cdots \end{aligned} \tag{4.505}$$

Example 154 ★Analytic solution of two coupled equations.

A set of two first-order coupled linear differential equations are:

$$\dot{x} = a\,x + b\,y \qquad \dot{x} = \frac{dx}{dt} \qquad x\,(0) = A \tag{4.506}$$
$$\dot{y} = c\,x + d\,y \qquad \dot{y} = \frac{dy}{dt} \qquad y\,(0) = B \tag{4.507}$$
$$\{a, b, c, d\} \in \mathbb{R} \tag{4.508}$$

The equations have the origin (0, 0) as the only equilibrium point at which $\dot{x} = \dot{y} = 0$. Searching for a solution of exponential form with unknown exponent s,

$$x = Ae^{st} \qquad y = Be^{st} \tag{4.509}$$

we get

$$Ase^{st} = aAe^{st} + bBe^{st} \tag{4.510}$$
$$Bse^{st} = cAe^{st} + dBe^{st} \tag{4.511}$$

and dividing them by e^{st} provides two algebraic equations for A and B.

$$(a - s)\,A + bB = 0 \tag{4.512}$$
$$cA + (d - s)\,B = 0 \tag{4.513}$$

Besides the trivial solution $A = B = 0$ associated to the equilibrium of the system, we will have nontrivial solutions if the determinant of the coefficients is zero.

$$\begin{vmatrix} a - s & b \\ c & d - s \end{vmatrix} = 0 \tag{4.514}$$

Expansion of the determinant gives us the quadratic characteristic equation to determine s.

$$s^2 - (a + d)\, s + (ad - bc) = 0 \tag{4.515}$$

Assuming s_1 and s_2 to be the roots of (4.515),

$$s_{1,2} = \frac{a+d}{2} \pm \frac{1}{2}\sqrt{(a+d)^2 - 4\,(ad - bc)} \tag{4.516}$$

replacing $s = s_1$ in (4.512) and (4.513) gives us a nontrivial solution A_1 and B_1.

$$x = A_1 e^{s_1 t} \qquad y = B_1 e^{s_1 t} \tag{4.517}$$

Similarly, replacing $s = s_2$ in (4.512) and (4.513) gives us another nontrivial solution A_2 and B_2.

$$x = A_2 e^{s_2 t} \qquad y = B_2 e^{s_2 t} \tag{4.518}$$

The two sets of solution must be linearly independent to be able to add them up and make the general solutions of the equations. Depending on s_1 and s_2, three conditions may happen.

1. Distinct real roots.

$$x = C_1 A_1 e^{s_1 t} + C_2 A_2 e^{s_2 t} \tag{4.519}$$

$$y = C_1 B_1 e^{s_1 t} + C_2 B_2 e^{s_2 t} \tag{4.520}$$

2. Distinct complex roots.

$$s_1 = u + iv \qquad s_2 = u - iv \tag{4.521}$$

$$x = e^{ut} \left(C_1 \left(A_1 \cos vt - A_2 \sin vt\right) + C_2 \left(A_1 \sin vt - A_2 \cos vt\right)\right) \tag{4.522}$$

$$y = e^{ut} \left(C_1 \left(B_1 \cos vt - B_2 \sin vt\right) + C_2 \left(B_1 \sin vt - B_2 \cos vt\right)\right) \tag{4.523}$$

3. Equal real roots.

$$s_1 = s_2 = s \tag{4.524}$$

$$x = C_1 A e^{st} + C_2 \left(A_1 + A_2 t\right) e^{st} \tag{4.525}$$

$$y = C_1 B e^{st} + C_2 \left(B_1 + B_2 t\right) e^{st} \tag{4.526}$$

Example 155 Picard's method, successive approximations.

A differential equation in the form

$$y' = f(x, y) \qquad y(x_0) = y_0 \tag{4.527}$$

can be reduced to the equivalent integral equation.

$$y(x) = y_0 + \int_{x_0}^{x} f(x, y(x))\, dx \tag{4.528}$$

This equation may be replaced into itself to generate a recursive approximation solution.

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(x, y_n(x))\, dx \qquad n = 1, 2, 3, \cdots \tag{4.529}$$

The initial y_1 may be an arbitrarily chosen function such as $y_0 = x$ or a number such as $y_0 = 1$. This recursive approximation solution is called the Picard's method.

As an example, let us solve the equation

$$y' = 1 + y^2 \qquad y(0) = 2 \tag{4.530}$$

Starting with

$$y_1(x) = 1 \tag{4.531}$$

we derive y_2

$$y_2(x) = 2 + \int_0^x \left(1 + 1^2\right) dx = 2 + 2x \tag{4.532}$$

Substituting y_2 on the right-hand side and taking the integral will provide y_3.

$$y_3(x) = 2 + \int_0^x \left(1 + (2 + 2x)^2\right) dx = 2 + 5x + 4x^2 + \frac{4}{3}x^3 \tag{4.533}$$

The next step will give us y_4 and y_5.

$$\begin{aligned} y_4(x) = 2 + \int_0^x \left(1 + y_3^2(x)\right) dx &= 2 + 5x + 10x^2 + \frac{41}{3}x^3 \\ &+ \frac{34}{3}x^4 + \frac{88}{15}x^5 + \frac{16}{9}x^6 + \frac{16}{63}x^7 \end{aligned} \tag{4.534}$$

$$\begin{aligned} y_5(x) = 2 + \int_0^x \left(1 + y_4^2(x)\right) dx &= 2 + 5x + 10x^2 + \frac{65}{3}x^3 \\ &+ \frac{116}{3}x^4 + \frac{282}{5}x^5 + \frac{3076}{45}x^6 + \frac{4313}{63}x^7 \end{aligned}$$

$$+\frac{2341}{42}x^8+\frac{34\,324}{945}x^9+\frac{88\,192}{4725}x^{10}+\frac{128\,608}{17\,325}x^{11}$$
$$+\frac{2096}{945}x^{12}+\frac{17\,408}{36\,855}x^{13}+\frac{256}{3969}x^{14}+\frac{256}{59\,535}x^{15} \tag{4.535}$$

The Picard's method gives the power series solution of the differential equation quicker than the Taylor method.

Example 156 Airy equation.

The Airy differential equation

$$\frac{d^2y}{dx^2}-xy=0 \tag{4.536}$$

has been introduced and studied by George Airy (1801–1892) in his investigation on optics. This equation has no elementary solution and its two independent solutions $Ai\,(x)$ and $Bi\,(x)$ are expressed by power series (Goodwine 2011).

$$Ai\,(x)=1+\frac{x^3}{6}+\sum_{k=2}^{\infty}\frac{x^{3k}}{(3k)\,(3k-1)\,(3k-3)\,(3k-4)\cdots(3)\,(2)} \tag{4.537}$$

$$Bi\,(x)=x+\frac{x^4}{12}+\sum_{k=2}^{\infty}\frac{x^{3k+1}}{(3k+1)\,(3k)\,(3k-2)\,(3k-3)\cdots(4)\,(3)} \tag{4.538}$$

Then the solution of the Airy equation would be

$$y=a_0\,Ai\,(x)+a_1\,Bi\,(x) \tag{4.539}$$

To derive the Airy functions, we substitute an assumed power series solution into the equation

$$y\,(x)=\sum_{k=0}^{\infty}a_kx^k \qquad \ddot{y}\,(x)=\sum_{k=2}^{\infty}k\,(k-1)\,a_kx^{k-2} \tag{4.540}$$

and therefore

$$\begin{aligned}0&=\sum_{k=2}^{\infty}k\,(k-1)\,a_kx^{k-2}-x\sum_{k=0}^{\infty}a_kx^k\\&=\sum_{k=2}^{\infty}k\,(k-1)\,a_kx^{k-2}-\sum_{k=0}^{\infty}a_kx^{k+1}\end{aligned} \tag{4.541}$$

which after rearrangement of the indexes will be:

$$\sum_{k=0}^{\infty}(k+2)(k+1)\,a_{k+2}x^k-\sum_{k=1}^{\infty}a_{k-1}x^k=0 \tag{4.542}$$

It will be contracted as

$$2a_2+\sum_{k=1}^{\infty}\left((k+2)(k+1)\,a_{k+2}-a_{k-1}\right)x^k=0 \tag{4.543}$$

that shows

$$a_2=0 \qquad a_{k+2}=\frac{a_{k-1}}{(k+2)(k+1)} \tag{4.544}$$

The recursive equation indicates that the coefficients are related in three steps such that a_0 determines a_3, or a_i determines a_{i+3}. Because of $a_2=0$ we will have

$$a_5=a_8=a_{11}=\cdots=a_{2+3k}=0 \tag{4.545}$$

and

$$a_3=\frac{a_0}{(2)(3)} \qquad a_6=\frac{a_3}{(5)(6)}=\frac{a_0}{(2)(3)(5)(6)}$$

$$a_9=\frac{a_6}{(8)(9)}=\frac{a_0}{(2)(3)(5)(6)(8)(9)} \tag{4.546}$$

$$a_{3k}=\frac{a_0}{(2)(3)(5)(6)\dots(3k-4)(3k-3)(3k-1)(3k)} \tag{4.547}$$

and

$$a_4=\frac{a_0}{(3)(4)} \qquad a_7=\frac{a_4}{(6)(7)}=\frac{a_1}{(3)(4)(6)(7)}$$

$$a_{10}=\frac{a_7}{(9)(10)}=\frac{a_1}{(3)(4)(6)(7)(9)(10)} \tag{4.548}$$

$$a_{3k+1}=\frac{a_0}{(3)(4)(6)(7)\dots(3k-3)(3k-2)(3k)(3k+1)} \tag{4.549}$$

and hence,

$$y=a_0\left(1+\sum_{k=1}^{\infty}\frac{x^{3k}}{(3k)(3k-1)\cdots(3)(2)}\right)$$

$$+a_1\left(x+\sum_{k=1}^{\infty}\frac{x^{3k}}{(3k+1)(3k)\cdots(4)(3)}\right) \tag{4.550}$$

There are several differential equations happening in science and engineering where their power series solutions are introducing new functions that are not expressible by elementary functions. Among them are:

Bessel equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+\left(x^2-a^2\right)y=0 \tag{4.551}$$

Blasius equation

$$\frac{d^3y}{dx^3}+x\frac{dy}{dx}=0 \tag{4.552}$$

Chebyshev equation

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+a^2y=0 \tag{4.553}$$

Hermite equation

$$\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+ay=0 \tag{4.554}$$

Legendre equation

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+a\left(a+1\right)y=0 \tag{4.555}$$

4.3.2 Derivative Method

A general first-order differential equation with given initial condition

$$y'=f\left(x,y\right) \tag{4.556}$$

$$y\left(x_0\right)=y_0 \tag{4.557}$$

may be solved in a Taylor series at a neighborhood of the point $x=x_0$,

$$\begin{aligned}y\left(x\right)&=y\left(x_0\right)+y'\left(x_0\right)\left(x-x_0\right)+\frac{y''\left(x_0\right)}{2!}\left(x-x_0\right)^2\\&+\frac{y'''\left(x_0\right)}{3!}\left(x-x_0\right)^3+\cdots=\sum_{k=0}^{\infty}\frac{y^{(k)}\left(x_0\right)}{k!}\left(x-x_0\right)^k\end{aligned} \tag{4.558}$$

only by using the equation and its derivatives.

Proof The first term of the series solution, $y(x_0)$, is the initial condition. The coefficient of the second term will be calculated directly from the Eq. (4.556).

$$y'(x_0) = f(x_0, y(x_0)) = f(x_0, y_0) \tag{4.559}$$

The coefficient of the third term will be calculated by taking derivative of the Eq. (4.556).

$$y'' = \frac{\partial f(x, y)}{\partial x} + \frac{\partial f(x, y)}{\partial y} y' \tag{4.560}$$

$$y''(x_0) = \frac{\partial f(x_0, y(x_0))}{\partial x} + f(x_0, y_0) \frac{\partial f(x_0, y(x_0))}{\partial y} \tag{4.561}$$

Similarly, subsequent derivatives of the Eq. (4.556) and evaluating them at $x = x_0$ will provide the next coefficients of the series solution. This successive derivation of the differential equation is also called regular expansion method (Polyanin and Zaitsev 2002).

The derivative method may be similarly applied on second and higher order differential equations. The general second-order differential equation with initial conditions may be written as:

$$y'' = f\left(x, y, y'\right) \tag{4.562}$$

$$y(x_0) = y_0 \qquad y'(x_0) = y_1 \tag{4.563}$$

The first term and the coefficients of the second term of the solution (4.558) are calculated from the initial conditions $y(x_0)$ and $y'(x_0)$. The coefficient of the third term will be calculated by differential equation (4.562).

$$y''(x_0) = f\left(x_0, y(x_0), y'(x_0)\right) = f(x_0, y_0, y_1) \tag{4.564}$$

The coefficient of the fourth and higher terms will be calculated by differentiation of differential equation (4.562).

$$y''' = \frac{\partial f(x_0, y_0, y_1)}{\partial x} + \frac{\partial f(x_0, y_0, y_1)}{\partial y} y'(x) + \frac{\partial f(x_0, y_0, y_1)}{\partial y'} y''(x) \tag{4.565}$$

■

Example 157 First-order system free dynamics.

When there is no excitation function, the equation of motion of a first-order linear dynamic system such as a damper-spring system will be

$$\dot{x} + \alpha x = 0 \qquad \dot{x} = \frac{dx}{dt} \tag{4.566}$$

that its motion determined by its initial condition.

$$x\left(0\right)=x_0 \tag{4.567}$$

The series solution of the equation of motion is:

$$x=x\left(0\right)+\dot{x}\left(0\right)t+\frac{\ddot{x}\left(0\right)}{2!}t^2+\frac{\dddot{x}\left(0\right)}{3!}t^3+\cdots \tag{4.568}$$

The first term of the series is the initial condition (4.567). The second term will be determined by Eq. (4.566) itself.

$$\dot{x}=-ax \tag{4.569}$$

$$\dot{x}\left(0\right)=-ax\left(0\right)=-ax_0 \tag{4.570}$$

The third term will be found by taking derivative of the differential equation (4.566).

$$\ddot{x}=-a\dot{x}=a^2x \tag{4.571}$$

$$\ddot{x}\left(0\right)=-a\dot{x}\left(0\right)=a^2x_0 \tag{4.572}$$

Similarly, the next terms will be found by taking another derivative and substituting previous results.

$$\dddot{x}=-a\ddot{x}=-a^3x \tag{4.573}$$

$$\dddot{x}\left(0\right)=-a\ddot{x}\left(0\right)=-a^3x_0 \tag{4.574}$$

$$x^{(4)}=-a\dddot{x}=a^4x \tag{4.575}$$

$$x^{(4)}\left(0\right)=-a\dddot{x}\left(0\right)=a^4x_0 \tag{4.576}$$

$$x^{(n)}=-ax^{(n-1)}=\left(-1\right)^n a^n x \tag{4.577}$$

$$x^{(n)}\left(0\right)=-ax^{(n-1)}\left(0\right)=\left(-1\right)^n a^n x_0 \tag{4.578}$$

Therefore, the series solution would be:

$$x=x_0-ax_0t+\frac{a^2x_0}{2!}t^2-\frac{a^3x_0}{3!}t^3+\frac{a^4x_0}{4!}t^3-\cdots$$
$$+\frac{\left(-1\right)^k a^k x_0}{n!}t^n+\cdots \tag{4.579}$$

$$= x_0 \left(1 - at + \frac{a^2}{2!} t^2 - \frac{a^3}{3!} t^3 + \frac{a^4}{4!} t^4 - \cdots \right) \tag{4.580}$$

Although it is not necessary, we may recognize that the series expansion is the power series of $\exp(-at)$,

$$e^{-\alpha t} = 1 - \alpha t + \frac{\alpha^2}{2} t^2 - \frac{\alpha^3}{6} t^3 + \frac{\alpha^4}{24} t^4 - \frac{\alpha^5}{120} t^5 + \cdots \tag{4.581}$$

and find the exact solution of the equation.

$$x = x_0 \, e^{-\alpha t} \tag{4.582}$$

The equations of motion of a dynamic system will be first order if the system can store energy only in one form and one location. The natural motions of first-order systems are either exponentially decreasing or increasing function of time and they do not show vibrations.

Example 158 Second-order system free dynamics.
Unforced second-order differential equation is

$$y'' + p(x) \, y' + q(x) \, y = 0 \qquad y' = \frac{dy}{dx} \tag{4.583}$$

$$y(0) = y_0 \qquad y'(0) = y'_0 \tag{4.584}$$

The series solution of the equation around $x = 0$ would be:

$$y(x) = \sum_{k=0}^{\infty} \frac{y^{(n)}(0)}{k!} x^k = y_0 + y'_0 x$$
$$+ \frac{1}{2} y''(0) \, x^2 + \frac{1}{6} y'''(0) \, x^3 + \frac{1}{24} y^{(4)}(0) \, x^4 + \cdots \tag{4.585}$$

Using Eq. (4.583), we may calculate $y''(0)$,

$$y'' = -p(x) \, y' - q(x) \, y \tag{4.586}$$

$$y''(0) = -p_0 y'_0 - q_0 y_0 \tag{4.587}$$

$$p_0 = p(0) \qquad q_0 = q(0) \tag{4.588}$$

and therefore, the three terms solution is:

$$y(x) = y_0 + y'_0 x + \frac{1}{2} \left(-p_0 y'_0 - q_0 y_0 \right) x^2 + \cdots$$

The next term of the series solution will be found by differentiating Eq. (4.586).

$$\begin{aligned} y''' &= -p'(x)\,y' - p(x)\,y'' - q'(x)\,y - q(x)\,y' \\ &= -q'y - \left(q + p'\right)y' - py'' = \left(pq - q'\right)y - \left(q + p^2 + p'\right)y' \end{aligned} \tag{4.589}$$

$$\begin{aligned} y'''(0) &= -p_0'y_0' - p_0y_0'' - q_0'y_0 - q_0y_0' \\ &= -p_0'y_0' - p_0\left(-p_0y_0' - q_0y_0\right) - q_0'y_0 - q_0y_0' \\ &= \left(p_0q_0 - q_0'\right)y_0 + \left(p_0^2 - p_0' - q_0\right)y_0' \end{aligned} \tag{4.590}$$

and the solution up to four term is:

$$\begin{aligned} y(x) &= y_0 + y_0'x + \frac{1}{2}\left(-p_0y_0' - q_0y_0\right)x^2 + \\ &+\frac{1}{6}\left(\left(p_0q_0 - q_0'\right)y_0 + \left(p_0^2 - p_0' - q_0\right)y_0'\right)x^3 + \cdots \end{aligned} \tag{4.591}$$

The fifth term of the series solution will be found by another derivative of Eq. (4.589).

$$\begin{aligned} y^{(4)} &= -p''y' - p'y'' - p'y'' - py''' - q''y - q'y' - q'y' - qy'' \\ &= \left(q'p - p^2q + q^2 + 2p'q - q''\right)y \\ &\quad + \left(p^3 - 2q' - p'' + 2pq + 3pp'\right)y' \end{aligned} \tag{4.592}$$

$$\begin{aligned} y^{(4)}(0) &= \left(q_0'p_0 - p_0^2q_0 + q_0^2 + 2p_0'q_0 - q_0''\right)y_0 \\ &\quad + \left(p_0^3 - 2q_0' - p_0'' + 2p_0q_0 + 3p_0p_0'\right)y_0' \end{aligned} \tag{4.593}$$

and the solution up to fifth term is:

$$\begin{aligned} y(x) &= y_0 + y_0'x + \frac{1}{2}\left(-p_0y_0' - q_0y_0\right)x^2 \\ &+\frac{1}{6}\left(\left(p_0q_0 - q_0'\right)y_0 + \left(p_0^2 - p_0' - q_0\right)y_0'\right)x^3 \\ &+\frac{1}{24}\left(\left(q_0'p_0 - p_0^2q_0 + q_0^2 + 2p_0'q_0 - q_0''\right)y_0\right. \\ &\left.+\left(p_0^3 - 2q_0' - p_0'' + 2p_0q_0 + 3p_0p_0'\right)y_0'\right) + \cdots \end{aligned} \tag{4.594}$$

This process may be continued up to any desired term of the solution.

As an example let us solve a second-order dynamic system

$$\ddot{x} - \dot{x} - t^2 x = 0 \qquad x(0) = 1 \qquad \dot{x}(0) = 2 \tag{4.595}$$

around $x = 0$.

$$x(t) = \sum_{k=0}^{\infty} \frac{x^{(n)}(0)}{k!} t^k \tag{4.596}$$

$$= x_0 + \dot{x}_0 t + \frac{1}{2}\ddot{x}(0)\,t^2 + \frac{1}{6}\dddot{x}(0)\,t^3 + \frac{1}{24}x^{(4)}(0)\,t^4 + \cdots \tag{4.597}$$

Employing Eq. (4.595), we determine the $\ddot{x}(0) = \ddot{x}_0$,

$$\ddot{x}(0) = \dot{x}_0 + (0)^2 x_0 = \dot{x}_0 = 2 \tag{4.598}$$

and find the solution up to three terms.

$$x(t) = x_0 + \dot{x}_0 t + \frac{1}{2}\dot{x}_0 t^2 + \cdots = 1 + 2t + t^2 + \cdots \tag{4.599}$$

Taking a derivative of (4.595) provides $\dddot{x}(0) = \dddot{x}_0$, and the solution up to four terms.

$$\dddot{x} = \frac{d}{dt}\left(\dot{x} + t^2 x\right) = \ddot{x} + 2tx + t^2\dot{x} \tag{4.600}$$

$$\dddot{x}(0) = \ddot{x}_0 + 2(0)\,x_0 + (0)^2\,\dot{x}_0 = \dot{x}_0 = 2 \tag{4.601}$$

$$x(t) = x_0 + \dot{x}_0 t + \frac{1}{2}\dot{x}_0 t^2 + \frac{1}{6}\dot{x}_0 t^3 + \cdots$$
$$= 1 + 2t + t^2 + \frac{1}{3}t^3 + \cdots \tag{4.602}$$

The next term of the solution will be found by differentiation from (4.600).

$$x^{(4)} = \frac{d}{dt}\left(\ddot{x} + 2tx + t^2\dot{x}\right) = \dddot{x} + 2x + 4t\dot{x} + t^2\ddot{x} \tag{4.603}$$

$$x^{(4)}(0) = \dddot{x}_0 + 2x_0 + 4(0)\,\dot{x}_0 + (0)^2\,\ddot{x}_0 = \dot{x}_0 + 2x_0 = 4 \tag{4.604}$$

$$x(t) = x_0 + \dot{x}_0 t + \frac{1}{2}\dot{x}_0 t^2 + \frac{1}{6}\dot{x}_0 t^3 + \frac{1}{24}x_0^{(4)} t^4 + \cdots$$
$$= 1 + 2t + t^2 + \frac{1}{3}t^3 + \frac{1}{6}t^4 + \cdots \tag{4.605}$$

and this process may be continued to determine the other terms of the solution.

Example 159 Series solution for two-body problem.

Assume there are only two massive particles in the world that attract each other by Newton gravitational force. The two-body problem is: what would be their motion from a given set of initial conditions. Similarly we may define three and more body problems (Szebehely 1967).

The n-body problem is made of the ordinary differential equations defining the motion of n point masses m_i with absolute positions $\mathbf{X}_i$ and velocities $\dot{\mathbf{X}}_i$, in a globally fixed coordinate frame G, interacting only through the mutual gravitational attraction,

$$m_j\ddot{\mathbf{X}}_j = \sum_{i\neq j} \frac{Gm_i m_j}{\left|\mathbf{X}_i - \mathbf{X}_j\right|^3}\left(\mathbf{X}_i - \mathbf{X}_j\right) \qquad i, j = 1, 2, \cdots, n \tag{4.606}$$

where the dots indicate time derivatives and G is the universal gravitational constant. The n-body problem is an isolated energy conserved system and the mutual gravitational forces have a potential and hence we can define the potential energy P.

$$P = -\sum_{0<i<j<n} \frac{Gm_i m_j}{\left|\mathbf{X}_i - \mathbf{X}_j\right|} \tag{4.607}$$

The kinetic energy K of the system

$$K = \frac{1}{2}\sum_{i=0}^{n} m_i \left|\dot{\mathbf{X}}_i\right|^2 \tag{4.608}$$

makes it also possible to use Lagrangian to derive the equations of motion (Milani and Gronchi 2010).

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{X}}_i} - \frac{\partial \mathcal{L}}{\partial \mathbf{X}_i} = 0 \tag{4.609}$$

$$\mathcal{L} = K - P \tag{4.610}$$

The solution of the problem of two bodies can be represented by a Taylor series. Consider the two point masses m_1 and m_2 shown in Fig. 4.13 that attract each other by a Newtonian gravitational force. Their positions are indicated by vectors $\mathbf{X}_1$ and $\mathbf{X}_2$. The equations of motion of m_1 and m_2 are:

$$\ddot{\mathbf{X}}_1 = -G_2\frac{\mathbf{X}_1 - \mathbf{X}_2}{\left|\mathbf{X}_1 - \mathbf{X}_2\right|^3} \qquad \ddot{\mathbf{X}}_2 = -G_1\frac{\mathbf{X}_2 - \mathbf{X}_1}{\left|\mathbf{X}_2 - \mathbf{X}_1\right|^3} \tag{4.611}$$

$$G_i = Gm_i \qquad i = 1, 2 \tag{4.612}$$

$$G = 6.67259 \times 10^{-11}\ \mathrm{m^3\,kg^{-1}\,s^{-2}} \tag{4.613}$$

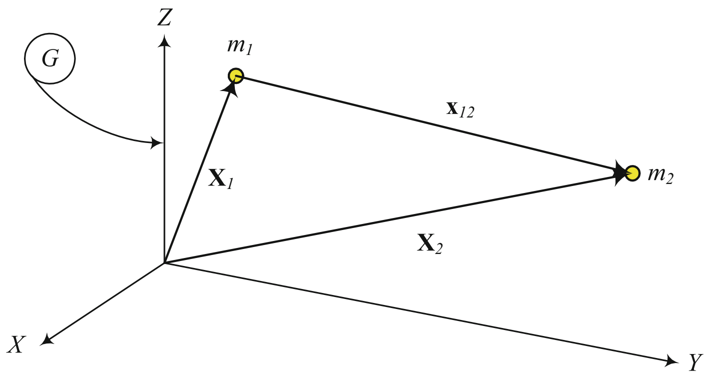

Fig. 4.13 Two bodies in space with position vectors $\mathbf{X}_1(t)$ and $\mathbf{X}_2(t)$ and the initial conditions of $\mathbf{X}_1(t_0)$, $\mathbf{X}_2(t_0)$, $\dot{\mathbf{X}}_1(t_0)$, and $\dot{\mathbf{X}}_2(t_0)$ at a given time t_0

The problem is to determine $\mathbf{X}_1(t)$ and $\mathbf{X}_2(t)$ by having the initial conditions of $\mathbf{X}_1(t_0)$, $\mathbf{X}_2(t_0)$, $\dot{\mathbf{X}}_1(t_0)$, and $\dot{\mathbf{X}}_2(t_0)$ at a given time t_0. Subtracting Eq. (4.611), we reduce the order of the problem and determine the fundamental equation of the two-body problem in terms of the relative position vector $\mathbf{x}$:

$$\ddot{\mathbf{x}} + \mu \frac{\mathbf{x}}{|\mathbf{x}|^3} = 0 \qquad \mathbf{x} = \mathbf{X}_2 - \mathbf{X}_1 \tag{4.614}$$

$$\mu = G_1 + G_2 \tag{4.615}$$

We search for a series solution for (4.614) in the form

$$\mathbf{x}(t) = \mathbf{x}_0 + \dot{\mathbf{x}}_0 (t - t_0) + \ddot{\mathbf{x}}_0 \frac{(t-t_0)^2}{2!} + \dddot{\mathbf{x}}_0 \frac{(t-t_0)^3}{3!} + \cdots \tag{4.616}$$

Equation (4.614) is singular only at $\mathbf{x} = 0$ and therefore, the series (4.616) converges for all nonzero $\mathbf{x}$. The first and second terms of the series are determined by the initial conditions.

$$\mathbf{x}_0 = \mathbf{x}(t_0) \qquad \dot{\mathbf{x}}_0 = \dot{\mathbf{x}}(t_0) \tag{4.617}$$

The coefficient of the third term is calculated by the equation of motion (4.614).

$$\ddot{\mathbf{x}}_0 = \ddot{\mathbf{x}}(t_0) = -\mu \frac{\mathbf{x}_0}{|\mathbf{x}_0|^3} \tag{4.618}$$

The fourth term requires the third derivative of $\mathbf{x}(t)$ from the equation of motion:

$$\dddot{\mathbf{x}} = -\mu \frac{|\mathbf{x}|^3 (d\mathbf{x}/dt) - \mathbf{x} \left(d|\mathbf{x}|^3/dt \right)}{|\mathbf{x}|^6} \tag{4.619}$$

We need to calculate the derivative $d|\mathbf{x}|^3/dt$,

$$\frac{d|\mathbf{x}|^3}{dt} = \frac{d(\mathbf{x}\cdot\mathbf{x})^{3/2}}{dt} = \frac{3}{2}(\mathbf{x}\cdot\mathbf{x})^{1/2}\frac{d(\mathbf{x}\cdot\mathbf{x})}{dt} = \frac{3}{2}|\mathbf{x}|\left(2\mathbf{x}\frac{d\mathbf{x}}{dt}\right) \tag{4.620}$$

to complete the third derivative $d|\mathbf{x}|^3/dt$ to be expressible by initial conditions (Szebehely and Mark 2004).

$$\dddot{\mathbf{x}} = -\mu\left(\frac{1}{|\mathbf{x}|^3}\frac{d\mathbf{x}}{dt} - 3\left(\mathbf{x}\cdot\frac{d\mathbf{x}}{dt}\right)\frac{\mathbf{x}}{|\mathbf{x}|^5}\right) \tag{4.621}$$

At $t = t_0$ we have

$$\dddot{\mathbf{x}}_0 = \dddot{\mathbf{x}}(t_0) = -\mu\left(\frac{\dot{\mathbf{x}}_0}{|\mathbf{x}_0|^3} - 3(\mathbf{x}_0\cdot\dot{\mathbf{x}}_0)\frac{\mathbf{x}_0}{|\mathbf{x}_0|^5}\right) \tag{4.622}$$

The series solution for the first four terms would be:

$$\begin{aligned}\mathbf{x}(t) = \mathbf{x}_0 + \dot{\mathbf{x}}_0(t-t_0) - \mu\frac{\mathbf{x}_0}{|\mathbf{x}_0|^3}\frac{(t-t_0)^2}{2!} \\ -\mu\left(\frac{\dot{\mathbf{x}}_0}{|\mathbf{x}_0|^3} - 3(\mathbf{x}_0\cdot\dot{\mathbf{x}}_0)\frac{\mathbf{x}_0}{|\mathbf{x}_0|^5}\right)\frac{(t-t_0)^3}{3!} + \cdots\end{aligned} \tag{4.623}$$

To determine the series (4.616) to any number of terms, we may take another approach and show that the coefficients $\mathbf{x}_0^{(n)}$ can be determined to any desired number of n (Battin 1999). Defining a new scalar variable

$$\varepsilon = \frac{\mu}{|\mathbf{x}|^3} \tag{4.624}$$

we may rewrite Eq. (4.614) as

$$\ddot{\mathbf{x}}_1 + \varepsilon\mathbf{x} = 0 \tag{4.625}$$

Knowing the time derivative of the absolute value of a vector,

$$\frac{d}{dt}|\mathbf{x}| = \frac{\mathbf{x}\cdot\dot{\mathbf{x}}}{|\mathbf{x}|} \tag{4.626}$$

we find the time derivative of (4.624):

$$\dot{\varepsilon} = -3\mu \frac{\mathbf{x} \cdot \dot{\mathbf{x}}}{|\mathbf{x}|^5} = -3\varepsilon \frac{\mathbf{x} \cdot \dot{\mathbf{x}}}{|\mathbf{x}|^2} = -3\varepsilon\lambda \tag{4.627}$$

$$\lambda = \frac{\mathbf{x} \cdot \dot{\mathbf{x}}}{|\mathbf{x}|^2} \tag{4.628}$$

This equation introduces a new variable λ that we must be able to determine its derivative in the next steps of the solution. The time derivative of λ introduces another variable ψ:

$$\begin{aligned}\dot{\lambda} &= \frac{\dot{\mathbf{x}} \cdot \dot{\mathbf{x}} + \mathbf{x} \cdot \ddot{\mathbf{x}}}{|\mathbf{x}|^2} - 2\frac{(\mathbf{x} \cdot \dot{\mathbf{x}})^2}{|\mathbf{x}|^4} = \frac{\dot{\mathbf{x}} \cdot \dot{\mathbf{x}}}{|\mathbf{x}|^2} - \varepsilon - 2\lambda^2 \\ &= \psi - \varepsilon - 2\lambda^2\end{aligned} \tag{4.629}$$

However the time derivative of ψ is only a function of only ε, λ, and ψ, closing the loop of variable and proving that we are able to calculate derivatives of the series solution to any desired order.

$$\begin{aligned}\dot{\psi} &= 2\frac{\dot{\mathbf{x}} \cdot \ddot{\mathbf{x}}}{|\mathbf{x}|^2} - 2\frac{(\mathbf{x} \cdot \dot{\mathbf{x}})(\dot{\mathbf{x}} \cdot \dot{\mathbf{x}})}{|\mathbf{x}|^4} = -2\varepsilon\lambda - 2\lambda\psi \\ &= -2\lambda(\varepsilon + \psi)\end{aligned} \tag{4.630}$$

The variables ε, λ, and ψ are called *fundamental invariants*. This method of solution and the fundamental invariants have been introduced by Lagrange. They are invariants because they are independent of the coordinate system and form a closed set of time derivatives (Batting 1999, Jazar 2011).

$$\dot{\varepsilon} = -3\varepsilon\lambda \qquad \dot{\lambda} = \psi - \varepsilon - 2\lambda^2 \qquad \dot{\psi} = -2\lambda(\varepsilon + \psi) \tag{4.631}$$

Equation (4.631) guarantees the existence of coefficients of the series solution (4.616). The first eight derivatives of $\mathbf{x}$ are

$$\frac{d\mathbf{x}}{dt} = \dot{\mathbf{x}} \qquad \frac{d^2\mathbf{x}}{dt^2} = -\varepsilon\mathbf{x} \qquad \frac{d^3\mathbf{x}}{dt^3} = 3\varepsilon\lambda\mathbf{x} - \varepsilon\dot{\mathbf{x}} \tag{4.632}$$

$$\frac{d^4\mathbf{x}}{dt^4} = \left(-15\varepsilon\lambda^2 + 3\varepsilon\psi - 2\varepsilon^2\right)\mathbf{x} + 6\varepsilon\lambda\dot{\mathbf{x}} \tag{4.633}$$

$$\begin{aligned}\frac{d^5\mathbf{x}}{dt^5} = &\left(105\varepsilon\lambda^3 - 45\varepsilon\lambda\psi + 30\varepsilon^2\lambda\right)\mathbf{x} \\ &+ \left(-45\varepsilon\lambda^2 + 9\varepsilon\psi - 8\varepsilon^2\right)\dot{\mathbf{x}}\end{aligned} \tag{4.634}$$

$$\frac{d^6\mathbf{x}}{dt^6} = \varepsilon\left(\lambda^2\left(-945\lambda^2 - 420\varepsilon + 630\psi\right) + 66\varepsilon\psi - 22\varepsilon^2 - 45\psi^2\right)\mathbf{x}$$
$$+\varepsilon\lambda\left(420\lambda^2 + 150\varepsilon - 180\psi\right)\dot{\mathbf{x}} \tag{4.635}$$

$$\frac{d^7\mathbf{x}}{dt^7} = \varepsilon\lambda\left(\lambda^2\left(10395\lambda^2 + 6300\varepsilon - 9450\psi\right)\right)\mathbf{x}$$
$$+\varepsilon\lambda\left(-2268\varepsilon\psi + 756\varepsilon^2 + 1575\psi^2\right)\mathbf{x}$$
$$+\varepsilon\left(\lambda^2\left(-4725\lambda^2 - 2520\varepsilon + 3150\psi\right)\right)\dot{\mathbf{x}}$$
$$+\varepsilon\left(+396\varepsilon\psi - 172\varepsilon^2 - 225\psi^2\right)\dot{\mathbf{x}} \tag{4.636}$$

$$\frac{d^8\mathbf{x}}{dt^8} = \varepsilon\left(\lambda^2\left(-135135\lambda^4 + 155925\lambda^2\psi - 103950\varepsilon\lambda^2\right)\right)\mathbf{x}$$
$$+\varepsilon\left(\lambda^2\left(-20160\varepsilon^2 + 60480\varepsilon\psi - 42525\psi^2\right)\right)\mathbf{x}$$
$$+\varepsilon\left(\psi^2\left(1575\psi - 3618\varepsilon\right) + \varepsilon^2\left(2628\psi - 584\varepsilon\right)\right)\mathbf{x}$$
$$+\varepsilon\lambda\left(62370\lambda^4 + 44100\varepsilon\lambda^2 - 56700\lambda^2\psi\right)\dot{\mathbf{x}}$$
$$+\varepsilon\lambda\left(-15876\varepsilon\psi + 9450\psi^2 + 6552\varepsilon^2\right)\dot{\mathbf{x}} \tag{4.637}$$

Substituting the derivatives of $\mathbf{x}$ into the series solution (4.616) and rearranging yields

$$\mathbf{x}(t) = P(t)\,\mathbf{x}(t_0) + Q(t)\,\dot{\mathbf{x}}_0(t_0) = f(\mathbf{x}_0, \dot{\mathbf{x}}_0, t) \tag{4.638}$$

where

$$P(t) = \sum_{i=0}^{\infty} P_i\,(t - t_0)^i \qquad Q(t) = \sum_{i=0}^{\infty} Q_i\,(t - t_0)^i \tag{4.639}$$

The Lagrangian coefficients P and Q are power series of ε, λ, and ψ. The first six terms of P and Q are

$$P_0 = 1 \qquad P_1 = 0 \qquad P_2 = -\frac{1}{2}\varepsilon_0 \qquad P_3 = \frac{1}{2}\varepsilon_0\lambda_0 \tag{4.640}$$

$$P_4 = -\frac{1}{12}\varepsilon_0^2 - \frac{5}{8}\varepsilon_0\lambda_0^2 + \frac{1}{8}\varepsilon_0\psi_0 \tag{4.641}$$

$$P_5 = \frac{1}{4}\varepsilon_0^2\lambda_0 + \frac{7}{8}\varepsilon_0\lambda_0^3 - \frac{3}{8}\varepsilon_0\lambda_0\psi_0 \tag{4.642}$$

$$\begin{aligned} P_6 = &-\frac{11}{360}\varepsilon_0^3 + (-\frac{7}{12}\lambda_0^2 + \frac{11}{120}\psi_0)\varepsilon_0^2 \\ &+(-\frac{1}{16}\psi_0^2 + \frac{7}{8}\lambda_0^2\psi_0 - \frac{21}{16}\lambda_0^4)\varepsilon_0 \end{aligned} \tag{4.643}$$

$$Q_0 = 0 \qquad Q_1 = 1 \qquad Q_2 = 0 \qquad Q_3 = -\frac{1}{6}\varepsilon_0 \tag{4.644}$$

$$Q_4 = \frac{1}{4}\varepsilon_0\lambda_0 \qquad Q_5 = -\frac{1}{15}\varepsilon_0^2 - \frac{3}{8}\varepsilon_0\lambda_0^2 + \frac{3}{40}\varepsilon_0\psi_0 \tag{4.645}$$

$$Q_6 = 5/24\varepsilon_0^2\lambda_0 + (\frac{7}{12}\lambda_0^3 - \frac{1}{4}\lambda_0\psi_0)\varepsilon_0 \tag{4.646}$$

The series solution of the two-body problem was developed by Lagrange (1736–1813). Although this series theoretically converges for all t, it will only provide a suitable approximate solution for a very short time interval. The key point to solve the two-body problem in power series is the ability to reduce the two equations of motion to one equation.

Example 160 ★ Series solution for three-body problem.

The basic definition of the problem of three bodies is: three point masses gravitationally attract each other; for a given set of initial conditions, find the resulting motion. It is an old problem, logically follows from the two-body problem. The two-body problem has been introduced and solved by Newton himself (Newton 1687). Newton has been asked about three-body as well as the solar system, a ten-body problem. Newton spent the last 30 years of his life on the three-body problem, hoping to solve the motion of the Moon under the gravitational influence of the Earth and the Sun, without success. The three-body problem has been a challenging problem since. Many giant scientists worked on the problem with several interesting special case solutions, mostly on restricted three-body problem. Euler defined a simplified problem of three bodies in which two massive bodies, called the primeters, are turning on circular orbits about their mutual mass center. A third body will be in the orbital plane of the two primaries and cannot affect their motions. The study of the possible motions of the third body is the restricted three-body problem (Whittaker 1965).

Reduction the two-body problem from two equations into one equation by considering the relative position of the two bodies was a key point helping Lagrange to develop his series solution of the two-body problem (Lagrange 1811). Such a

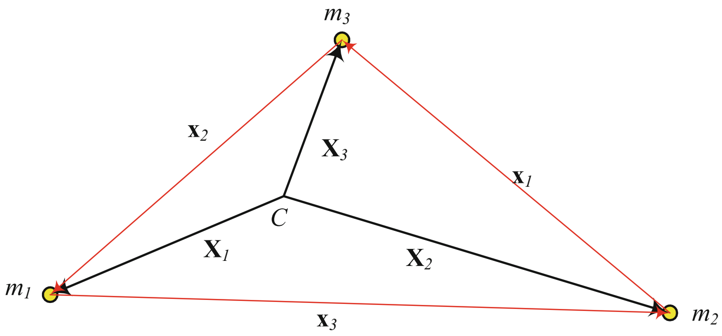

Fig. 4.14 Position vectors for three-body problem

reduction of the order of the problem is not possible for the three-body problem and as a result, the equations of the motions in the general form were not proper for series solution expansion. Roger Broucke (1932–2005) presented the equations of the three bodies in a symmetrical expression using relative position vectors (Broucke 1979; Hestenes 1999). Jazar (2011) employed the Broucke's new defined equations to develop a series solution.

Three point masses m_1, m_2, and m_3 are shown in Fig. 4.14. Their position vectors $\mathbf{X}_1$, $\mathbf{X}_2$, and $\mathbf{X}_3$ are respect to their mass center C. The mass center is assumed to be fixed or moving with a constant speed at a constant direction. The masses attract each other by the Newtonian gravitational force. If the space is assumed Euclidean and there is no other mass in space, the equations of motion of m_1, m_2, and m_3 are:

$$\begin{aligned}\ddot{\mathbf{X}}_1 &= -G_2 \frac{\mathbf{X}_1 - \mathbf{X}_2}{|\mathbf{X}_{21}|^3} - G_3 \frac{\mathbf{X}_1 - \mathbf{X}_3}{|\mathbf{X}_{31}|^3} \\ \ddot{\mathbf{X}}_2 &= -G_3 \frac{\mathbf{X}_2 - \mathbf{X}_3}{|\mathbf{X}_{32}|^3} - G_1 \frac{\mathbf{X}_2 - \mathbf{X}_1}{|\mathbf{X}_{12}|^3} \\ \ddot{\mathbf{X}}_3 &= -G_1 \frac{\mathbf{X}_3 - \mathbf{X}_1}{|\mathbf{X}_{13}|^3} - G_2 \frac{\mathbf{X}_3 - \mathbf{X}_2}{|\mathbf{X}_{23}|^3}\end{aligned} \tag{4.647}$$

where

$$\mathbf{X}_{ij} = \mathbf{X}_j - \mathbf{X}_i \tag{4.648}$$

$$G_i = G m_i \qquad G = 6.67259 \times 10^{-11}\ \mathrm{m^3\,kg^{-1}\,s^{-2}} \tag{4.649}$$

Using the mass center as the origin of coordinate frame implies

$$G_1 \mathbf{X}_1 + G_2 \mathbf{X}_2 + G_3 \mathbf{X}_3 = 0 \tag{4.650}$$

The equations of motion of three bodies have their most symmetric form when expressed in terms of relative position vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$:

$$\begin{aligned} \mathbf{x}_1 &= \mathbf{X}_3 - \mathbf{X}_2 \\ \mathbf{x}_2 &= \mathbf{X}_1 - \mathbf{X}_3 \\ \mathbf{x}_3 &= \mathbf{X}_2 - \mathbf{X}_1 \end{aligned} \tag{4.651}$$

Using the relative position vectors, the kinematic constraint (4.647) reduces to

$$\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 = 0 \tag{4.652}$$

and the absolute position vectors in terms of the relative positions will be

$$\begin{aligned} m\mathbf{X}_1 &= m_3\mathbf{x}_2 - m_2\mathbf{x}_3 \\ m\mathbf{X}_2 &= m_1\mathbf{x}_3 - m_3\mathbf{x}_1 \\ m\mathbf{X}_3 &= m_2\mathbf{x}_1 - m_1\mathbf{x}_2 \end{aligned} \tag{4.653}$$

where

$$m = m_1 + m_2 + m_3 \tag{4.654}$$

Substituting (4.653) in (4.647), we get symmetric form of the equations of motion:

$$\begin{aligned} \ddot{\mathbf{x}}_1 &= -Gm\frac{\mathbf{x}_1}{|\mathbf{x}_1|^3} + G_1\left(\frac{\mathbf{x}_1}{|\mathbf{x}_1|^3} + \frac{\mathbf{x}_2}{|\mathbf{x}_2|^3} + \frac{\mathbf{x}_3}{|\mathbf{x}_3|^3}\right) \\ \ddot{\mathbf{x}}_2 &= -Gm\frac{\mathbf{x}_2}{|\mathbf{x}_2|^3} + G_2\left(\frac{\mathbf{x}_1}{|\mathbf{x}_1|^3} + \frac{\mathbf{x}_2}{|\mathbf{x}_2|^3} + \frac{\mathbf{x}_3}{|\mathbf{x}_3|^3}\right) \\ \ddot{\mathbf{x}}_3 &= -Gm\frac{\mathbf{x}_3}{|\mathbf{x}_3|^3} + G_3\left(\frac{\mathbf{x}_1}{|\mathbf{x}_1|^3} + \frac{\mathbf{x}_2}{|\mathbf{x}_2|^3} + \frac{\mathbf{x}_3}{|\mathbf{x}_3|^3}\right) \end{aligned} \tag{4.655}$$

We are interested in a series solution for Eq. (4.655) in the form

$$\mathbf{x}_i\,(t) = \mathbf{x}_{i_0} + \dot{\mathbf{x}}_{i_0}\,(t - t_0) + \ddot{\mathbf{x}}_{i_0}\frac{(t-t_0)^2}{2!} + \dddot{\mathbf{x}}_{i_0}\frac{(t-t_0)^3}{3!} + \cdots \tag{4.656}$$

$$\mathbf{x}_{i_0} = \mathbf{x}_i\,(t_0) \qquad \dot{\mathbf{x}}_{i_0} = \dot{\mathbf{x}}_i\,(t_0) \qquad i = 1, 2, 3 \tag{4.657}$$

Let us define $\mu = Gm$ and ε_i

$$\mu = Gm \tag{4.658}$$

$$\varepsilon_1 = \frac{1}{|\mathbf{x}_1|^3} \qquad \varepsilon_2 = \frac{1}{|\mathbf{x}_2|^3} \qquad \varepsilon_3 = \frac{1}{|\mathbf{x}_3|^3} \tag{4.659}$$

to rewrite Eq. (4.655).

$$\begin{aligned}
\ddot{\mathbf{x}}_1 &= -\mu\varepsilon_1\mathbf{x}_1 + G_1\left(\varepsilon_1\mathbf{x}_1 + \varepsilon_2\mathbf{x}_2 + \varepsilon_3\mathbf{x}_3\right) \\
\ddot{\mathbf{x}}_2 &= -\mu\varepsilon_2\mathbf{x}_2 + G_2\left(\varepsilon_1\mathbf{x}_1 + \varepsilon_2\mathbf{x}_2 + \varepsilon_3\mathbf{x}_3\right) \\
\ddot{\mathbf{x}}_3 &= -\mu\varepsilon_3\mathbf{x}_3 + G_3\left(\varepsilon_1\mathbf{x}_1 + \varepsilon_2\mathbf{x}_2 + \varepsilon_3\mathbf{x}_3\right)
\end{aligned} \tag{4.660}$$

Also let us define the following three sets of parameters a_{ijk}, b_{ijk}, c_{ijk}.

$$\begin{aligned}
a_{111} &= \frac{\mathbf{x}_1 \cdot \mathbf{x}_1}{|\mathbf{x}_1|^2} = 1 & a_{112} &= \frac{\mathbf{x}_1 \cdot \mathbf{x}_1}{|\mathbf{x}_2|^2} & a_{113} &= \frac{\mathbf{x}_1 \cdot \mathbf{x}_1}{|\mathbf{x}_3|^2} \\
a_{121} &= \frac{\mathbf{x}_1 \cdot \mathbf{x}_2}{|\mathbf{x}_1|^2} & a_{122} &= \frac{\mathbf{x}_1 \cdot \mathbf{x}_2}{|\mathbf{x}_2|^2} & a_{123} &= \frac{\mathbf{x}_1 \cdot \mathbf{x}_2}{|\mathbf{x}_3|^2} \\
a_{131} &= \frac{\mathbf{x}_1 \cdot \mathbf{x}_3}{|\mathbf{x}_1|^2} & a_{132} &= \frac{\mathbf{x}_1 \cdot \mathbf{x}_3}{|\mathbf{x}_2|^2} & a_{133} &= \frac{\mathbf{x}_1 \cdot \mathbf{x}_3}{|\mathbf{x}_3|^2}
\end{aligned} \tag{4.661}$$

$$\begin{aligned}
a_{221} &= \frac{\mathbf{x}_2 \cdot \mathbf{x}_2}{|\mathbf{x}_1|^2} & a_{222} &= \frac{\mathbf{x}_2 \cdot \mathbf{x}_2}{|\mathbf{x}_2|^2} = 1 & a_{223} &= \frac{\mathbf{x}_2 \cdot \mathbf{x}_2}{|\mathbf{x}_3|^2} \\
a_{231} &= \frac{\mathbf{x}_2 \cdot \mathbf{x}_3}{|\mathbf{x}_1|^2} & a_{232} &= \frac{\mathbf{x}_2 \cdot \mathbf{x}_3}{|\mathbf{x}_2|^2} & a_{233} &= \frac{\mathbf{x}_2 \cdot \mathbf{x}_3}{|\mathbf{x}_3|^2} \\
a_{331} &= \frac{\mathbf{x}_3 \cdot \mathbf{x}_3}{|\mathbf{x}_1|^2} & a_{332} &= \frac{\mathbf{x}_3 \cdot \mathbf{x}_3}{|\mathbf{x}_2|^2} & a_{333} &= \frac{\mathbf{x}_3 \cdot \mathbf{x}_3}{|\mathbf{x}_3|^2} = 1
\end{aligned} \tag{4.662}$$

$$\begin{aligned}
b_{111} &= \frac{\dot{\mathbf{x}}_1 \cdot \mathbf{x}_1}{|\mathbf{x}_1|^2} & b_{112} &= \frac{\dot{\mathbf{x}}_1 \cdot \mathbf{x}_1}{|\mathbf{x}_2|^2} & b_{113} &= \frac{\dot{\mathbf{x}}_1 \cdot \mathbf{x}_1}{|\mathbf{x}_3|^2} \\
b_{121} &= \frac{\dot{\mathbf{x}}_1 \cdot \mathbf{x}_2}{|\mathbf{x}_1|^2} & b_{122} &= \frac{\dot{\mathbf{x}}_1 \cdot \mathbf{x}_2}{|\mathbf{x}_2|^2} & b_{123} &= \frac{\dot{\mathbf{x}}_1 \cdot \mathbf{x}_2}{|\mathbf{x}_3|^2} \\
b_{131} &= \frac{\dot{\mathbf{x}}_1 \cdot \mathbf{x}_3}{|\mathbf{x}_1|^2} & b_{132} &= \frac{\dot{\mathbf{x}}_1 \cdot \mathbf{x}_3}{|\mathbf{x}_2|^2} & b_{133} &= \frac{\dot{\mathbf{x}}_1 \cdot \mathbf{x}_3}{|\mathbf{x}_3|^2}
\end{aligned} \tag{4.663}$$

$$\begin{aligned}
b_{211} &= \frac{\dot{\mathbf{x}}_2 \cdot \mathbf{x}_1}{|\mathbf{x}_1|^2} & b_{212} &= \frac{\dot{\mathbf{x}}_2 \cdot \mathbf{x}_1}{|\mathbf{x}_2|^2} & b_{213} &= \frac{\dot{\mathbf{x}}_2 \cdot \mathbf{x}_1}{|\mathbf{x}_3|^2} \\
b_{221} &= \frac{\dot{\mathbf{x}}_2 \cdot \mathbf{x}_2}{|\mathbf{x}_1|^2} & b_{222} &= \frac{\dot{\mathbf{x}}_2 \cdot \mathbf{x}_2}{|\mathbf{x}_2|^2} & b_{223} &= \frac{\dot{\mathbf{x}}_2 \cdot \mathbf{x}_2}{|\mathbf{x}_3|^2} \\
b_{231} &= \frac{\dot{\mathbf{x}}_2 \cdot \mathbf{x}_3}{|\mathbf{x}_1|^2} & b_{232} &= \frac{\dot{\mathbf{x}}_2 \cdot \mathbf{x}_3}{|\mathbf{x}_2|^2} & b_{233} &= \frac{\dot{\mathbf{x}}_2 \cdot \mathbf{x}_3}{|\mathbf{x}_3|^2}
\end{aligned} \tag{4.664}$$

$$b_{311} = \frac{\dot{\mathbf{x}}_3 \cdot \mathbf{x}_1}{|\mathbf{x}_1|^2}\; b_{312} = \frac{\dot{\mathbf{x}}_3 \cdot \mathbf{x}_1}{|\mathbf{x}_2|^2}\; b_{313} = \frac{\dot{\mathbf{x}}_3 \cdot \mathbf{x}_1}{|\mathbf{x}_3|^2}$$
$$b_{321} = \frac{\dot{\mathbf{x}}_3 \cdot \mathbf{x}_2}{|\mathbf{x}_1|^2}\; b_{322} = \frac{\dot{\mathbf{x}}_3 \cdot \mathbf{x}_2}{|\mathbf{x}_2|^2}\; b_{323} = \frac{\dot{\mathbf{x}}_3 \cdot \mathbf{x}_2}{|\mathbf{x}_3|^2} \tag{4.665}$$
$$b_{331} = \frac{\dot{\mathbf{x}}_3 \cdot \mathbf{x}_3}{|\mathbf{x}_1|^2}\; b_{332} = \frac{\dot{\mathbf{x}}_3 \cdot \mathbf{x}_3}{|\mathbf{x}_2|^2}\; b_{333} = \frac{\dot{\mathbf{x}}_3 \cdot \mathbf{x}_3}{|\mathbf{x}_3|^2}$$

$$c_{221} = \frac{\dot{\mathbf{x}}_2 \cdot \dot{\mathbf{x}}_2}{|\mathbf{x}_1|^2}\; c_{222} = \frac{\dot{\mathbf{x}}_2 \cdot \dot{\mathbf{x}}_2}{|\mathbf{x}_2|^2}\; c_{223} = \frac{\dot{\mathbf{x}}_2 \cdot \dot{\mathbf{x}}_2}{|\mathbf{x}_3|^2}$$
$$c_{231} = \frac{\dot{\mathbf{x}}_2 \cdot \dot{\mathbf{x}}_3}{|\mathbf{x}_1|^2}\; c_{232} = \frac{\dot{\mathbf{x}}_2 \cdot \dot{\mathbf{x}}_3}{|\mathbf{x}_2|^2}\; c_{233} = \frac{\dot{\mathbf{x}}_2 \cdot \dot{\mathbf{x}}_3}{|\mathbf{x}_3|^2} \tag{4.666}$$
$$c_{331} = \frac{\dot{\mathbf{x}}_3 \cdot \dot{\mathbf{x}}_3}{|\mathbf{x}_1|^2}\; c_{332} = \frac{\dot{\mathbf{x}}_3 \cdot \dot{\mathbf{x}}_3}{|\mathbf{x}_2|^2}\; c_{333} = \frac{\dot{\mathbf{x}}_3 \cdot \dot{\mathbf{x}}_3}{|\mathbf{x}_3|^2}$$

$$c_{221} = \frac{\dot{\mathbf{x}}_2 \cdot \dot{\mathbf{x}}_2}{|\mathbf{x}_1|^2}\; c_{222} = \frac{\dot{\mathbf{x}}_2 \cdot \dot{\mathbf{x}}_2}{|\mathbf{x}_2|^2}\; c_{223} = \frac{\dot{\mathbf{x}}_2 \cdot \dot{\mathbf{x}}_2}{|\mathbf{x}_3|^2}$$
$$c_{231} = \frac{\dot{\mathbf{x}}_2 \cdot \dot{\mathbf{x}}_3}{|\mathbf{x}_1|^2}\; c_{232} = \frac{\dot{\mathbf{x}}_2 \cdot \dot{\mathbf{x}}_3}{|\mathbf{x}_2|^2}\; c_{233} = \frac{\dot{\mathbf{x}}_2 \cdot \dot{\mathbf{x}}_3}{|\mathbf{x}_3|^2} \tag{4.667}$$
$$c_{331} = \frac{\dot{\mathbf{x}}_3 \cdot \dot{\mathbf{x}}_3}{|\mathbf{x}_1|^2}\; c_{332} = \frac{\dot{\mathbf{x}}_3 \cdot \dot{\mathbf{x}}_3}{|\mathbf{x}_2|^2}\; c_{333} = \frac{\dot{\mathbf{x}}_3 \cdot \dot{\mathbf{x}}_3}{|\mathbf{x}_3|^2}$$

The time derivatives of the ε_i are

$$\dot{\varepsilon}_1 = -3b_{111}\varepsilon_1 \qquad \dot{\varepsilon}_2 = -3b_{222}\varepsilon_2 \qquad \dot{\varepsilon}_3 = -3b_{333}\varepsilon_3 \tag{4.668}$$

and the time derivatives of the a_{ijk}, b_{ijk}, and c_{ijk} are:

$$\dot{a}_{111} = 0 \qquad \dot{a}_{112} = -2b_{222}a_{112} + 2b_{112}$$
$$\dot{a}_{113} = -2b_{333}a_{113} + 2b_{113} \tag{4.669}$$

$$\dot{a}_{121} = -2b_{111}a_{121} + b_{121} + b_{211} \tag{4.670}$$
$$\dot{a}_{122} = -2b_{222}a_{122} + b_{122} + b_{212} \tag{4.671}$$
$$\dot{a}_{123} = -2b_{333}a_{123} + b_{123} + b_{213}$$

$$\dot{a}_{131} = -2b_{111}a_{131} + b_{131} + b_{311}$$
$$\dot{a}_{132} = -2b_{222}a_{132} + b_{132} + b_{312} \tag{4.672}$$
$$\dot{a}_{133} = -2b_{333}a_{133} + b_{133} + b_{313}$$

$$\dot{a}_{221} = -2b_{111}a_{221} + 2b_{221} \qquad \dot{a}_{222} = 0$$
$$\dot{a}_{223} = -2b_{333}a_{233} + 2b_{223} \tag{4.673}$$

$$\dot{a}_{231} = -2b_{111}a_{231} + b_{231} + b_{321}$$
$$\dot{a}_{232} = -2b_{222}a_{232} + b_{232} + b_{322} \tag{4.674}$$
$$\dot{a}_{233} = -2b_{333}a_{233} + b_{233} + b_{323}$$

$$\dot{a}_{331} = -2b_{111}a_{331} + 2b_{331}$$
$$\dot{a}_{332} = -2b_{222}a_{332} + 2b_{332} \qquad \dot{a}_{333} = 0 \tag{4.675}$$

$$\dot{b}_{111} = -2b_{111}^2 + c_{111} - \mu\varepsilon_1 + G_1\left(\varepsilon_1 + \varepsilon_2 a_{211} + \varepsilon_3 a_{311}\right)$$
$$\dot{b}_{112} = -2b_{222}b_{112} + c_{112} - \mu\varepsilon_1 a_{112} + G_1\left(\varepsilon_1 a_{112} + \varepsilon_2 a_{212} + \varepsilon_3 a_{312}\right)$$
$$\dot{b}_{113} = -2b_{333}b_{113} + c_{113} - \mu\varepsilon_1 a_{113}$$
$$+G_1\left(\varepsilon_1 a_{113} + \varepsilon_2 a_{213} + \varepsilon_3 a_{313}\right) \tag{4.676}$$

$$\dot{b}_{121} = -2b_{111}b_{121} + c_{121} - \mu\varepsilon_1 a_{121} + G_1\left(\varepsilon_1 a_{121} + \varepsilon_2 a_{221} + \varepsilon_3 a_{321}\right)$$
$$\dot{b}_{122} = -2b_{222}b_{122} + c_{122} - \mu\varepsilon_1 a_{122} + G_1\left(\varepsilon_1 a_{122} + \varepsilon_2 + \varepsilon_3 a_{322}\right)$$
$$\dot{b}_{123} = -2b_{333}b_{123} + c_{123} - \mu\varepsilon_1 a_{123}$$
$$+G_1\left(\varepsilon_1 a_{123} + \varepsilon_2 a_{223} + \varepsilon_3 a_{323}\right) \tag{4.677}$$

$$\dot{b}_{131} = -2b_{111}b_{131} + c_{131} - \mu\varepsilon_1 a_{131} + G_1\left(\varepsilon_1 a_{131} + \varepsilon_2 a_{231} + \varepsilon_3 a_{331}\right)$$
$$\dot{b}_{132} = -2b_{222}b_{132} + c_{132} - \mu\varepsilon_1 a_{132} + G_1\left(\varepsilon_1 a_{132} + \varepsilon_2 a_{232} + \varepsilon_3 a_{332}\right)$$
$$\dot{b}_{133} = -2b_{333}b_{133} + c_{133} - \mu\varepsilon_1 a_{133}$$
$$+G_1\left(\varepsilon_1 a_{133} + \varepsilon_2 a_{233} + \varepsilon_3\right) \tag{4.678}$$

$$\dot{b}_{211} = -2b_{111}b_{211} + c_{211} - \mu\varepsilon_2 a_{211} + G_2\left(\varepsilon_1 + \varepsilon_2 a_{211} + \varepsilon_3 a_{311}\right)$$
$$\dot{b}_{212} = -2b_{222}b_{212} + c_{212} - \mu\varepsilon_2 a_{212} + G_2\left(\varepsilon_1 a_{112} + \varepsilon_2 a_{212} + \varepsilon_3 a_{312}\right)$$
$$\dot{b}_{133} = -2b_{333}b_{213} + c_{213} - \mu\varepsilon_2 a_{213}$$
$$+G_2\left(\varepsilon_1 a_{113} + \varepsilon_2 a_{213} + \varepsilon_3 a_{313}\right) \tag{4.679}$$

$$\begin{aligned}
\dot{b}_{221} &= -2b_{111}b_{221} + c_{221} - \mu\varepsilon_2 a_{221} + G_2\left(\varepsilon_1 a_{121} + \varepsilon_2 a_{221} + \varepsilon_3 a_{321}\right) \\
\dot{b}_{222} &= -2b_{222}^2 + c_{222} - \mu\varepsilon_2 + G_2\left(\varepsilon_1 a_{122} + \varepsilon_2 + \varepsilon_3 a_{322}\right) \\
\dot{b}_{223} &= -2b_{333}b_{223} + c_{223} - \mu\varepsilon_2 a_{223} \\
&\quad + G_2\left(\varepsilon_1 a_{123} + \varepsilon_2 a_{223} + \varepsilon_3 a_{323}\right)
\end{aligned} \tag{4.680}$$

$$\begin{aligned}
\dot{b}_{231} &= -2b_{111}b_{231} + c_{231} - \mu\varepsilon_2 a_{231} + G_2\left(\varepsilon_1 a_{131} + \varepsilon_2 a_{231} + \varepsilon_3 a_{331}\right) \\
\dot{b}_{232} &= -2b_{222}b_{232} + c_{232} - \mu\varepsilon_2 a_{232} + G_2\left(\varepsilon_1 a_{132} + \varepsilon_2 a_{232} + \varepsilon_3 a_{332}\right) \\
\dot{b}_{233} &= -2b_{333}b_{233} + c_{233} - \mu\varepsilon_2 a_{233} \\
&\quad + G_2\left(\varepsilon_1 a_{133} + \varepsilon_2 a_{233} + \varepsilon_3\right)
\end{aligned} \tag{4.681}$$

$$\begin{aligned}
\dot{b}_{311} &= -2b_{111}b_{311} + c_{311} - \mu\varepsilon_3 a_{311} + G_3\left(\varepsilon_1 + \varepsilon_2 a_{211} + \varepsilon_3 a_{311}\right) \\
\dot{b}_{312} &= -2b_{222}b_{312} + c_{312} - \mu\varepsilon_3 a_{312} + G_3\left(\varepsilon_1 a_{112} + \varepsilon_2 a_{212} + \varepsilon_3 a_{312}\right) \\
\dot{b}_{313} &= -2b_{333}b_{313} + c_{313} - \mu\varepsilon_3 a_{313} \\
&\quad + G_3\left(\varepsilon_1 a_{113} + \varepsilon_2 a_{213} + \varepsilon_3 a_{313}\right)
\end{aligned} \tag{4.682}$$

$$\begin{aligned}
\dot{b}_{321} &= -2b_{111}b_{321} + c_{321} - \mu\varepsilon_3 a_{321} + G_3\left(\varepsilon_1 a_{121} + \varepsilon_2 a_{221} + \varepsilon_3 a_{321}\right) \\
\dot{b}_{322} &= -2b_{222}b_{322} + c_{322} - \mu\varepsilon_3 a_{322} + G_3\left(\varepsilon_1 a_{122} + \varepsilon_2 + \varepsilon_3 a_{322}\right) \\
\dot{b}_{323} &= -2b_{333}b_{323} + c_{323} - \mu\varepsilon_3 a_{323} \\
&\quad + G_3\left(\varepsilon_1 a_{123} + \varepsilon_2 a_{223} + \varepsilon_3 a_{323}\right)
\end{aligned} \tag{4.683}$$

$$\begin{aligned}
\dot{b}_{331} &= -2b_{111}b_{331} + c_{331} - \mu\varepsilon_3 a_{331} + G_3\left(\varepsilon_1 a_{131} + \varepsilon_2 a_{231} + \varepsilon_3 a_{331}\right) \\
\dot{b}_{332} &= -2b_{222}b_{332} + c_{332} - \mu\varepsilon_3 a_{332} + G_3\left(\varepsilon_1 a_{132} + \varepsilon_2 a_{232} + \varepsilon_3 a_{332}\right) \\
\dot{b}_{333} &= -2b_{333}b_{333} + c_{333} - \mu\varepsilon_3 \\
&\quad + G_3\left(\varepsilon_1 a_{133} + \varepsilon_2 a_{233} + \varepsilon_3\right)
\end{aligned} \tag{4.684}$$

$$\begin{aligned}
\dot{c}_{111} &= -2b_{111}c_{111} - 2\mu\varepsilon_1 b_{111} + 2G_1\left(\varepsilon_1 b_{111} + \varepsilon_2 b_{121} + \varepsilon_3 b_{131}\right) \\
\dot{c}_{112} &= -2b_{222}c_{112} - 2\mu\varepsilon_1 b_{112} + 2G_1\left(\varepsilon_1 b_{112} + \varepsilon_2 b_{122} + \varepsilon_3 b_{132}\right) \\
\dot{c}_{113} &= -2b_{333}c_{113} - 2\mu\varepsilon_1 b_{113} \\
&\quad + 2G_1\left(\varepsilon_1 b_{113} + \varepsilon_2 b_{123} + \varepsilon_3 b_{133}\right)
\end{aligned} \tag{4.685}$$

$$\begin{aligned}
\dot{c}_{121} &= -2b_{111}c_{121} - \mu\left(\varepsilon_1 b_{111} + \varepsilon_2 b_{121}\right) + G_1\left(\varepsilon_1 b_{211} + \varepsilon_2 b_{221} + \varepsilon_3 b_{231}\right) \\
&\quad + G_2\left(\varepsilon_1 b_{111} + \varepsilon_2 b_{121} + \varepsilon_3 b_{131}\right) \\
\dot{c}_{122} &= -2b_{222}c_{122} - \mu\left(\varepsilon_1 b_{212} + \varepsilon_2 b_{122}\right) + G_1\left(\varepsilon_1 b_{212} + \varepsilon_2 b_{222} + \varepsilon_3 b_{232}\right) \\
&\quad + G_2\left(\varepsilon_1 b_{112} + \varepsilon_2 b_{122} + \varepsilon_3 b_{132}\right) \\
\dot{c}_{123} &= -2b_{333}c_{123} - \mu\left(\varepsilon_1 b_{213} + \varepsilon_2 b_{123}\right) + G_1\left(\varepsilon_1 b_{213} + \varepsilon_2 b_{223} + \varepsilon_3 b_{233}\right) \\
&\quad + G_2\left(\varepsilon_1 b_{113} + \varepsilon_2 b_{123} + \varepsilon_3 b_{133}\right)
\end{aligned} \tag{4.686}$$

$$\begin{aligned}
\dot{c}_{131} &= -2b_{111}c_{131} - \mu\left(\varepsilon_1 b_{311} + \varepsilon_3 b_{131}\right) + G_1\left(\varepsilon_1 b_{311} + \varepsilon_2 b_{321} + \varepsilon_3 b_{331}\right) \\
&\quad + G_3\left(\varepsilon_1 b_{111} + \varepsilon_2 b_{121} + \varepsilon_3 b_{131}\right) \\
\dot{c}_{132} &= -2b_{222}c_{132} - \mu\left(\varepsilon_1 b_{312} + \varepsilon_3 b_{132}\right) + G_1\left(\varepsilon_1 b_{312} + \varepsilon_2 b_{322} + \varepsilon_3 b_{332}\right) \\
&\quad + G_3\left(\varepsilon_1 b_{112} + \varepsilon_2 b_{122} + \varepsilon_3 b_{132}\right) \\
\dot{c}_{133} &= -2b_{333}c_{133} - \mu\left(\varepsilon_1 b_{313} + \varepsilon_3 b_{133}\right) + G_1\left(\varepsilon_1 b_{313} + \varepsilon_2 b_{323} + \varepsilon_3 b_{333}\right) \\
&\quad + G_3\left(\varepsilon_1 b_{113} + \varepsilon_2 b_{123} + \varepsilon_3 b_{133}\right)
\end{aligned} \tag{4.687}$$

$$\begin{aligned}
\dot{c}_{221} &= -2b_{111}c_{221} - 2\mu\varepsilon_2 b_{221} + 2G_2\left(\varepsilon_1 b_{211} + \varepsilon_2 b_{221} + \varepsilon_3 b_{231}\right) \\
\dot{c}_{222} &= -2b_{222}c_{222} - 2\mu\varepsilon_2 b_{222} + 2G_2\left(\varepsilon_1 b_{212} + \varepsilon_2 b_{222} + \varepsilon_3 b_{232}\right) \\
\dot{c}_{223} &= -2b_{333}c_{223} - 2\mu\varepsilon_2 b_{223} \\
&\quad + 2G_2\left(\varepsilon_1 b_{213} + \varepsilon_2 b_{223} + \varepsilon_3 b_{233}\right)
\end{aligned} \tag{4.688}$$

$$\begin{aligned}
\dot{c}_{231} &= -2b_{111}c_{231} - \mu\left(\varepsilon_1 b_{321} + \varepsilon_2 b_{231}\right) + G_2\left(\varepsilon_1 b_{311} + \varepsilon_2 b_{321} + \varepsilon_3 b_{331}\right) \\
&\quad + G_3\left(\varepsilon_1 b_{211} + \varepsilon_2 b_{221} + \varepsilon_3 b_{231}\right) \\
\dot{c}_{232} &= -2b_{222}c_{232} - \mu\left(\varepsilon_1 b_{322} + \varepsilon_2 b_{232}\right) + G_2\left(\varepsilon_1 b_{312} + \varepsilon_2 b_{322} + \varepsilon_3 b_{332}\right) \\
&\quad + G_3\left(\varepsilon_1 b_{212} + \varepsilon_2 b_{222} + \varepsilon_3 b_{232}\right) \\
\dot{c}_{233} &= -2b_{333}c_{233} - \mu\left(\varepsilon_1 b_{323} + \varepsilon_2 b_{233}\right) + G_2\left(\varepsilon_1 b_{313} + \varepsilon_2 b_{323} + \varepsilon_3 b_{333}\right) \\
&\quad + G_3\left(\varepsilon_1 b_{213} + \varepsilon_2 b_{223} + \varepsilon_3 b_{233}\right)
\end{aligned} \tag{4.689}$$

$$\begin{aligned}
\dot{c}_{331} &= -2b_{111}c_{331} - 2\mu\varepsilon_3 b_{331} + 2G_3\left(\varepsilon_1 b_{311} + \varepsilon_2 b_{321} + \varepsilon_3 b_{331}\right) \\
\dot{c}_{332} &= -2b_{222}c_{332} - 2\mu\varepsilon_3 b_{332} + 2G_3\left(\varepsilon_1 b_{312} + \varepsilon_2 b_{322} + \varepsilon_3 b_{332}\right)
\end{aligned}$$

$$\dot{c}_{333} = -2b_{333}c_{333} - 2\mu\varepsilon_3 b_{333} \\ +2G_3\left(\varepsilon_1 b_{313} + \varepsilon_2 b_{323} + \varepsilon_3 b_{333}\right) \tag{4.690}$$

We have defined 84 fundamental parameters and showed that their time derivatives are independent of the selected coordinate system, making a closed set under the operation of time differentiation. Therefore, employing (4.658)–(4.690) we are able to find the coefficients of series (4.656) to develop the solution for the three-body problem. The first four coefficients of the series are given below:

$$\frac{d\mathbf{x}_i}{dt} = \dot{\mathbf{x}}_i \qquad \frac{d^2\mathbf{x}_i}{dt^2} = -\mu\varepsilon_i\mathbf{x}_i + G_i\sum_{j=1}^{3}\varepsilon_j\mathbf{x}_j \tag{4.691}$$

$$\frac{d^3\mathbf{x}_i}{dt^3} = \mu\varepsilon_i\left(3b_{iii}\mathbf{x}_i - \dot{\mathbf{x}}_i\right) - G_i\sum_{j=1}^{3}\varepsilon_j\left(3b_{jjj}\mathbf{x}_j - \dot{\mathbf{x}}_j\right) \tag{4.692}$$

$$\frac{d^4\mathbf{x}_i}{dt^4} = -3\mu b_{iii}\varepsilon_i\left(3b_{iii}\mathbf{x}_i - \dot{\mathbf{x}}_i\right) + \mu\varepsilon_i Z_i - G_i\sum_{j=1}^{3}S_j \tag{4.693}$$

where

$$Z_i = \left[3b_{iii}\dot{\mathbf{x}}_i + 3\mathbf{x}_i\left(-2b_{iii}^2 + c_{iii} - \mu\varepsilon_i a_{iii} + G_i\sum_{k=1}^{3}\varepsilon_k a_{kii}\right)\right. \\ \left. +\mu\varepsilon_i\dot{\mathbf{x}}_i - G_i\sum_{k=1}^{3}\varepsilon_k\mathbf{x}_k\right] \tag{4.694}$$

$$S_j = -3b_{jjj}\varepsilon_i^2\left(3b_{jjj}\mathbf{x}_j - \dot{\mathbf{x}}_j\right)Z_j \tag{4.695}$$

4.4 Continued Fractions Solution of Differential Equations

The main purpose of our introduction to continued fraction is to be able to express solution of differential equations in continued fractions for better convergence when a closed form solution does not exist. In this section we will investigate possible differential equations by continued fractions. Generally speaking, there is no method to be applied to all differential equations to derive their solutions in continued fractions. We will study this topic employing two methods. Firstly, there are many differential equations whose solution can directly be derived by continued fractions. Secondly, we will study how we can build a continued fractions solution using the series solutions.

4.4.1 Second-Order Linear Differential Equations

The second-order linear equations are the most important class of differential equations in science and engineering that express dynamic behavior of many systems. Consider the general second-order linear differential equation in the form

$$y = P_0(x)\, y' + Q_0(x)\, y'' \tag{4.696}$$

$$y' = \frac{dy}{dx} \tag{4.697}$$

where $P_0(x)$ and $Q_0(x)$ are assumed infinitely differentiable functions of x. The solution of the equation can expressed by continued fractions.

$$y = C \exp F(x) \tag{4.698}$$

$$F'(x) = f(x) \tag{4.699}$$

$$f(x) = P_0 + \frac{Q_0}{P_1+}\frac{Q_1}{P_2+}\frac{Q_2}{P_3+}\cdots \tag{4.700}$$

$$P_n(x) = \frac{P_{n-1}(x) + Q'_{n-1}(x)}{1 - P'_{n-1}(x)} \tag{4.701}$$

$$Q_n(x) = \frac{Q_{n-1}(x)}{1 - P'_{n-1}(x)} \tag{4.702}$$

Proof Let us differentiate the equation and rearrange it to a similar form:

$$y' = P_1(x)\, y'' + Q_1(x)\, y''' \tag{4.703}$$

where

$$P_1(x) = \frac{P_0(x) + Q'_0(x)}{1 - P'_0(x)} \tag{4.704}$$

$$Q_1(x) = \frac{Q_0(x)}{1 - P'_0(x)} \tag{4.705}$$

$$P'_0(x) \neq 1 \tag{4.706}$$

We proceed and find the general equation.

$$y^{(n)} = P_n(x)\, y^{(n+1)} + Q_n(x)\, y^{(n+2)} \tag{4.707}$$

$$P_n(x) = \frac{P_{n-1}(x) + Q'_{n-1}(x)}{1 - P'_{n-1}(x)} \tag{4.708}$$

$$Q_n(x) = \frac{Q_{n-1}(x)}{1 - P'_{n-1}(x)} \tag{4.709}$$

$$P'_{n-1}(x) \neq 1 \tag{4.710}$$

Therefore,

$$\frac{y}{y'} = P_0 + \frac{Q_0}{y'/y''} \tag{4.711}$$

$$\frac{y'}{y''} = P_1 + \frac{Q_1}{y''/y'''} \tag{4.712}$$

$$\vdots$$

$$\frac{y^{(n)}}{y^{(n+1)}} = P_n + \frac{Q_n}{y^{(n+1)}/y^{(n+2)}} \tag{4.713}$$

and we find the continued fraction solution.

$$\frac{y}{y'} = P_0 + \frac{Q_0}{P_1+}\frac{Q_1}{P_2+}\frac{Q_2}{P_3+}\cdots\frac{Q_n}{y^{(n+1)}/y^{(n+2)}} \tag{4.714}$$

If

$$f(x) = P_0 + \frac{Q_0}{P_1+}\frac{Q_1}{P_2+}\frac{Q_2}{P_3+}\cdots \tag{4.715}$$

is convergent, then the solution of the equation is found.

$$y = C \exp F(x) \tag{4.716}$$

$$F'(x) = f(x) \tag{4.717}$$

■

Example 161 A second-order differential equation.

Consider the second-order differential equation

$$y'' + ay' - y = 0 \tag{4.718}$$

which may be rewritten as

$$y = ay' + y'' \tag{4.719}$$

and then by differentiation we have:

$$y' = ay'' + y''' \tag{4.720}$$

$$\vdots$$

$$y^{(n)} = ay^{(n+1)} + y^{(n+2)} \tag{4.721}$$

By dividing each equation by its derivative we get

$$\frac{y}{y'} = a + \frac{1}{y'/y''} \tag{4.722}$$

$$\frac{y'}{y''} = a + \frac{1}{y''/y'''} \tag{4.723}$$

$$\vdots$$

$$\frac{y^{(n)}}{y^{(n+1)}} = a + \frac{1}{y^{(n+1)}/y^{(n+2)}} \tag{4.724}$$

Substituting backward will make a continued fractions (Lorentzen and Waadeland 2008).

$$\frac{y}{y'} = a + \frac{1}{a+}\,\frac{1}{a+}\,\frac{1}{a+}\cdots\frac{1}{a+}\,\frac{1}{y^{(n+1)}/y^{(n+2)}} \tag{4.725}$$

Cutting the last term suggests to approximate the equation to:

$$\frac{y}{y'} \simeq a + \frac{1}{a+}\frac{1}{a+}\frac{1}{a+}\cdots \tag{4.726}$$

The associated number to the continued fraction is A,

$$A = a + \frac{1}{a + B} \qquad B = a + \frac{1}{B} \tag{4.727}$$

$$B = \frac{a + \sqrt{a^2 + 4}}{2} \qquad A = \frac{a - \sqrt{a^2 + 4} + 4a^3}{4a^2 - 2} \tag{4.728}$$

It suggests that

$$\frac{y}{y'} \simeq A = \frac{a - \sqrt{a^2 + 4} + 4a^3}{4a^2 - 2} \tag{4.729}$$

and therefore,

$$\frac{y'}{y} \simeq \frac{1}{A} = \frac{4a^2 - 2}{a - \sqrt{a^2+4} + 4a^3} \tag{4.730}$$

Hence the solution of the differential equation is:

$$y = C \exp\left(\frac{1}{A}\right) = C \exp\left(\frac{4a^2 - 2}{a - \sqrt{a^2+4} + 4a^3}\right) \tag{4.731}$$

Example 162 A parametric linear equation.

Let us derive the solution of the parametric equation

$$y = (a + bx)\, y' + cxy'' \tag{4.732}$$

in continued fractions. A differential equation is called parametric if its coefficients are functions of the independent variable. In this case, the coefficients of y' and y'' are functions of x. Successive differentiation of the equation

$$y' = (a_1 + b_1 x)\, y'' + c_1 x y''' \tag{4.733}$$

$$y'' = (a_2 + b_2 x)\, y''' + c_2 x y^{(4)} \tag{4.734}$$

$$\cdots$$

and rearrangement makes a recursive equation

$$y^{(n)} = (a_n + b_n x)\, y^{(n+1)} + c_n x y^{(n+2)} \qquad n = 1, 2, 3, \cdots \tag{4.735}$$

where

$$a_n = \frac{a + nc}{1 - nb} \qquad b_n = \frac{b}{1 - nb} \qquad c_n = \frac{c}{1 - nc} \qquad n = 1, 2, 3, \cdots \tag{4.736}$$

The recursive equation (4.735) may be written in a new form

$$\frac{y^{(n)}}{y^{(n+1)}} = (a_n + b_n x) + \frac{c_n x\,(a_n + b_n x)}{y^{(n+1)}/y^{(n+2)}} \tag{4.737}$$

to generate a continued fractions for $y^{(n)}/y^{(n+1)}$.

$$\frac{y^{(n)}}{y^{(n+1)}} = (a_n + b_n x) + \frac{c_n x\,(a_n + b_n x)}{(a_{n+1} + b_{n+1} x) +} \frac{c_{n+1} x\,(a_{n+1} + b_{n+1} x)}{(a_{n+2} + b_{n+2} x) +} \cdots \tag{4.738}$$

Assuming

$$f(x) = (a_n + b_n x) + \frac{c_n x\,(a_n + b_n x)}{(a_{n+1} + b_{n+1} x) +} \frac{c_{n+1} x\,(a_{n+1} + b_{n+1} x)}{(a_{n+2} + b_{n+2} x) +} \cdots \tag{4.739}$$

the solution of the differential equation will be found.

$$y = C \exp F(x) \qquad F'(x) = f(x) \tag{4.740}$$

As an example let us consider the parametric equation

$$y = (1 + x)\,y' + x y'' \tag{4.741}$$

$$a = 1 \qquad b = 1 \qquad c = 1 \tag{4.742}$$

Therefore,

$$a_n = \frac{1+n}{1-n} \qquad b_n = \frac{1}{1-n} \qquad c_n = \frac{1}{1-n} \qquad n = 1, 2, 3, \cdots \tag{4.743}$$

$$\frac{y^{(n)}}{y^{(n+1)}} = \frac{1+n+x}{1-n} + \frac{\frac{2+n}{(1-n)^2} x^2}{\frac{1}{2-n}(x+1) +} \frac{\frac{2+n}{(2-n)^2} x\,(x+1)}{\frac{1}{3-n}(x+1) +} \times \frac{\frac{3+n}{(3-n)^2} x\,(x+1)}{\frac{1}{4-n}(x+1) +} \cdots \tag{4.744}$$

and for $n = 0$ we have

$$\frac{y}{y'} = \frac{1+x}{1} + \frac{2x^2}{\frac{1}{2}(x+1) +} \frac{\frac{1}{2} x\,(x+1)}{\frac{1}{3}(x+1) +} \frac{\frac{1}{3} x\,(x+1)}{\frac{1}{4}(x+1) +} \cdots \tag{4.745}$$

$$f(x) = \frac{1+x}{1} + \frac{2x^2}{\frac{1}{2}(x+1) +} \frac{\frac{1}{2} x\,(x+1)}{\frac{1}{3}(x+1) +} \frac{\frac{1}{3} x\,(x+1)}{\frac{1}{4}(x+1) +} \cdots \tag{4.746}$$

$$y = C \exp \int f(x)\,dx \tag{4.747}$$

Example 163 Fundamental differential equation.

Consider the *fundamental differential equation* in y and x,

$$\left(\alpha + \alpha' x^k\right) x y' + \left(\beta + \beta' x^k\right) y + \gamma y^2 = \delta x^k \tag{4.748}$$

$$y(0) = 0 \tag{4.749}$$

$$y' = \frac{dy}{dx} \tag{4.750}$$

that has the solution in the form of continued fractions.

$$\begin{aligned} y = {} & \frac{\delta x^k}{k\alpha + \beta +} \frac{\left((k\alpha + \beta)\left(k\alpha' + \beta'\right) + \gamma\delta\right) x^k}{2k\alpha + \beta +} \\ & \frac{\left(k^2\alpha\alpha' - k\alpha\beta' + k\alpha'\beta + \gamma\delta\right) x^k}{3k\alpha + \beta +} \cdots \\ & \cdots \frac{\left((nk\alpha + \beta)\left(nk\alpha' + \beta'\right) + \gamma\delta\right) x^k}{2nk\alpha + \beta +} \\ & \frac{\left(n^2k^2\alpha\alpha' - nk\alpha\beta' + nk\alpha'\beta + \gamma\delta\right) x^k}{(2n+1)\, k\alpha + \beta +} \cdots \end{aligned} \tag{4.751}$$

In general, we assume

$$\lim_{|x| \to 0} y = u_0 \tag{4.752}$$

and substitute y with y_1 such that

$$y = \frac{u_0}{1 + y_1} \tag{4.753}$$

to obtain a differential equation in y_1 and x. Now assume

$$\lim_{|x| \to 0} y_1 = u_1 \tag{4.754}$$

and substitute y_1 with y_2 such that

$$y_1 = \frac{u_1}{1 + y_2} \tag{4.755}$$

and repeat the process and eventually we arrive at the solution (4.751). The problem of this method is that in general we may not be able to find u_k.

Consider a differential equation in the form

$$\left(\alpha + \alpha' x\right) x y' + \left(\beta + \beta' x\right) y + \gamma y^2 = \delta x \tag{4.756}$$

$$y(0) = 0 \tag{4.757}$$

and let us change the variable y with

$$y = \frac{\delta x}{\alpha + \beta + y_1} \tag{4.758}$$

The equation will be transformed to:

$$\begin{aligned}&\left(\alpha + \alpha' x\right) x y' + \left(\alpha + \beta - \left(\alpha' + \beta'\right) x\right) y_1 + y_1^2 \\ &= \left((\alpha + \beta)\left(\alpha' + \beta'\right) + \gamma\delta\right) x\end{aligned} \tag{4.759}$$

By repeating this process, we find the following continued fraction:

$$\begin{aligned} y = \frac{\delta x}{\alpha + \beta +} & \frac{\left((\alpha + \beta)\left(\alpha' + \beta'\right) + \gamma\delta\right) x}{2\alpha + \beta +} \frac{\left(\alpha\alpha' - \alpha\beta' + \alpha'\beta + \gamma\delta\right) x}{3\alpha + \beta +} \cdots \\ & \cdots \frac{\left((n\alpha + \beta)\left(n\alpha' + \beta'\right) + \gamma\delta\right) x}{2n\alpha + \beta +} \\ & \times \frac{\left(n^2\alpha\alpha' - n\alpha\beta' + n\alpha'\beta + \gamma\delta\right) x}{(2n + 1)\,\alpha + \beta +} \cdots \end{aligned} \tag{4.760}$$

Therefore, the differential equation (4.748) has the solution (4.751).

Example 164 A Lagrange fundamental differential equation.

Consider the equation

$$(1 + x)\, y' = \varphi y \qquad y\,(0) = 0 \qquad y' = \frac{dy}{dx} \tag{4.761}$$

and let us substitute y with

$$y = 1 + \frac{\varphi x}{1 + z} \tag{4.762}$$

to find the new differential equation.

$$(1 + x)\, x z' + (1 - (1 - \varphi)\, x)\, z + z^2 = (1 - \varphi)\, y \tag{4.763}$$

$$z\,(0) = 0 \tag{4.764}$$

This is a particular case of (4.748), in which

$$\begin{gathered} k = \alpha = \alpha' = \beta = \gamma = 1 \\ \beta' = -\,(1 - \varphi) \qquad \delta = 1 - \varphi \end{gathered} \tag{4.765}$$

and therefore, from the solution (4.751) we have:

$$y = (1+x)^{\varphi} = 1 + \frac{\varphi x}{1+} \frac{(1-\varphi)\, x}{2+} \frac{(1+\varphi)\, x}{3+} \frac{(2-\varphi)\, x}{2+} \cdots \frac{(n-\varphi)\, x}{2+} \frac{(n+\varphi)\, x}{2n+1+} \cdots \tag{4.766}$$

or in another form

$$\frac{(1+x)^{\varphi}-1}{\varphi} = \frac{x}{1+} \frac{(1-\varphi)\, x}{2+} \frac{(1+\varphi)\, x}{3+} \frac{(2-\varphi)\, x}{2+} \cdots \frac{(n-\varphi)\, x}{2+} \frac{(n+\varphi)\, x}{2n+1+} \cdots \tag{4.767}$$

This is an expansion of the polynomial function $y = (1+x)^{\varphi}$ into continued fractions (Lyusternik and Yanupolskii 1965).

As an example, we may use $x = 1$, $\varphi = 1/3$ and derive the continued fractions expansion for $\sqrt[3]{2}$,

$$\sqrt[3]{2} = 1 + \frac{1}{3+} \frac{2}{2+} \frac{4}{9+} \frac{5}{2+} \cdots \frac{3n-1}{2+} \frac{3n+1}{3\,(2n+1)} \cdots \tag{4.768}$$

with the following convergents:

$$\frac{p_0}{q_0} = 1 \qquad \frac{p_1}{q_1} = \frac{4}{3} \qquad \frac{p_2}{q_2} = \frac{5}{4} \qquad \frac{p_3}{q_3} = \frac{53}{42} \tag{4.769}$$

Employing Stolz's theorem (3.179), we may also find other form of continued fractions for $y = (1+x)^{\varphi}$.

$$y = (1+x)^{\varphi} = 1 + \frac{2\varphi x}{2+(1-\varphi)\, x-} \frac{\left(1-\varphi^2\right) x^2}{3\,(2+x)-} \frac{\left(4-\varphi^2\right) x^2}{5\,(2+x)-} \cdots \frac{\left(n-\varphi^2\right) x^2}{(2n+1)\,(2+x)-} \cdots \tag{4.770}$$

Example 165 Logarithmic function as a continued fractions.

As we know,

$$\lim_{\varphi \to 0} \frac{(1+x)^{\varphi}-1}{\varphi} = \ln(1+x) \tag{4.771}$$

then continued fractions (4.767)

$$\frac{(1+x)^{\varphi}-1}{\varphi} = \frac{x}{1+} \frac{(1-\varphi)\, x}{2+} \frac{(1+\varphi)\, x}{3+} \frac{(2-\varphi)\, x}{2+}$$

$$\cdots \frac{(n-\varphi)\,x}{2+}\;\frac{(n+\varphi)\,x}{2n+1+}\cdots \tag{4.772}$$

will be

$$\ln(1+x) = \frac{x}{1+}\,\frac{x}{2+}\,\frac{x}{3+}\,\frac{2x}{2+}\,\frac{2x}{5+}\cdots\frac{nx}{2+}\,\frac{nx}{2+1+}\cdots \tag{4.773}$$

These continued fractions are convergent on the complex plane as well as the real axis from $x = -\infty$ to $x = -1$. As an example of its application, let us assume $x = 2$, then we have:

$$\ln 2 = \frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{3+}\,\frac{2}{2+}\,\frac{2}{5+}\cdots\frac{n}{2+}\,\frac{n}{2+1+}\cdots \tag{4.774}$$

Example 166 Exponential function, $y = e^x$.

A continued fraction of exponential function $y = e^x$,

$$e^x = 1 + \frac{x}{1-}\,\frac{x}{2+}\,\frac{x}{3-}\,\frac{x}{2+}\,\frac{x}{5-}\cdots\frac{x}{2+}\,\frac{x}{2n+1-}\cdots \tag{4.775}$$

can be found by replacing x with x/φ in (4.766) to have

$$\left(1+\frac{x}{\varphi}\right)^{\varphi} = 1 + \frac{x}{1+}\,\frac{\frac{1-\varphi}{\varphi}x}{2+}\,\frac{\frac{1+\varphi}{\varphi}x}{3+}\,\frac{\frac{2-\varphi}{\varphi}x}{2+}$$
$$\times\frac{\frac{2+\varphi}{\varphi}x}{5+}\cdots\frac{\frac{n-\varphi}{\varphi}x}{2+}\,\frac{\frac{n+\varphi}{\varphi}x}{2n+1+}\cdots \tag{4.776}$$

and move φ to ∞.

$$\lim_{\varphi\to\infty}\left(1+\frac{x}{\varphi}\right)^{\varphi} = e^x \tag{4.777}$$

Using (3.133) and (3.134), the exponential function in continued fraction is:

$$e^x = 1 + \frac{2x}{2-x+}\,\frac{x^2}{6+}\,\frac{x^2}{10+}\cdots\frac{x^2}{2+(2n+1)}\cdots \tag{4.778}$$

Example 167 Prima's function in continued fractions.

The function

$$y = \int_x^{\infty} t^{a-1}e^{-t}dt \qquad a>0 \qquad x>0 \tag{4.779}$$

is called the Prima's function. The function

$$y = x^{1-a}\, e^{x} \int_{x}^{\infty} t^{a-1}\, e^{-t} dt \qquad a > 0 \qquad x > 0 \tag{4.780}$$

satisfies the differential equation

$$x y' - (1 - a - x)\, y = -x \tag{4.781}$$

By substituting

$$x = \frac{1}{t} \qquad y = \frac{1}{1+u} \tag{4.782}$$

the differential equation takes the form

$$t^2 u' + (1 - (1-a)\, t)\, u = (1-a)\, t \qquad u\,(0) = 0 \tag{4.783}$$

This is a particular case of (4.748) for

$$\begin{aligned} \alpha = 0 \qquad & k = \alpha' = \beta = \gamma = 1 \\ \delta = 1 - a \qquad & \beta' = -\,(1-a) \end{aligned} \tag{4.784}$$

We then find the solution to be:

$$u = \frac{(1-a)\, t}{1+} \frac{t}{1+} \frac{(2-a)\, t}{1+} \cdots \frac{nt}{1} \frac{(n+1-a)\, t}{1+} \cdots \tag{4.785}$$

or in the original notation as:

$$\begin{aligned} y = \int_{x}^{\infty} t^{a-1} e^{-t} dt = \frac{x^a e^{-x}}{x+} \frac{1-a}{1+} \frac{1}{x+} \frac{2-a}{1+} \cdots \\ \cdots \frac{n}{x+} \frac{n+1-a}{1+} \cdots \end{aligned} \tag{4.786}$$

as well as

$$\begin{aligned} y = \int_{x}^{\infty} t^{a-1} e^{-t} dt = \frac{x^a e^{-x}}{x+1-a-} \frac{1-a}{x+3-a-} \frac{2\,(2-a)}{x+5-a-} \cdots \\ \cdots \frac{n\,(n-a)}{x+2n+1-a-} \cdots \end{aligned} \tag{4.787}$$

Example 168 ★Riccati equation.

Riccati equation

$$\frac{du}{dx} + bu^2 = cx^n \tag{4.788}$$

is a special case of the general case

$$x\frac{dy}{dx} - ay + by^2 = cx^n \tag{4.789}$$

by substituting

$$u = \frac{y}{x} \tag{4.790}$$

Riccati equation in general is of the form

$$\frac{dy}{dx} = f_0(x) + f_1(x)\, y + f_2(x)\, y^2 \tag{4.791}$$

The Riccati equation (4.788) is solvable for $n = 2a$. Substituting

$$y = x^a v \tag{4.792}$$

we find

$$x^{a+1}\frac{dv}{dx} + bx^{2a}v^2 = cx^n \tag{4.793}$$

and after dividing by x^{2a} we have:

$$x^{1-a}\frac{dv}{dx} + bv^2 = cx^{n-2a} \tag{4.794}$$

If $n = 2a$, then the equation will be simplified (Boole 1877).

$$x^{1-a}\frac{dv}{dx} + bv^2 = c \tag{4.795}$$

Separation of variables

$$\frac{dv}{c - bv^2} = \frac{dx}{x^{1-a}} \tag{4.796}$$

gives the exact solution

$$\frac{1}{2}\sqrt{\frac{1}{bc}}\ln\left(\frac{v+c\sqrt{\frac{1}{bc}}}{v-c\sqrt{\frac{1}{bc}}}\right)=\frac{x^a}{a}+C \tag{4.797}$$

If we restore v to y/x^a, then Eq. (4.796) becomes an exact differential equation,

$$\frac{x^a dy - ayx^{a-1}dx}{by^2 - cx^{2a}} + x^{a-1}dx = 0 \tag{4.798}$$

with the solution

$$y=\sqrt{-\frac{c}{b}}x^a \tan\left(C-\frac{\sqrt{-bc}x^a}{a}\right) \tag{4.799}$$

The general equation (4.789) is integrable whenever $(n \pm 2a)\,/\,(2n)$ is a positive integer. The equation is reducible to Riccati equation (4.788) if the positive integer condition is fulfilled. Let us define a new variable y_1

$$y = A + \frac{x^n}{y_1} \tag{4.800}$$

with a constant A to be determined. On substitution and arrangement, we have

$$-aA + bA^2 + (n-a+2bA)\frac{x^n}{y_1} + b\frac{x^{2n}}{y_1^2} - \frac{x^{n+1}}{y_1^2}\frac{dy_1}{dx} = cx^n \tag{4.801}$$

If we set

$$-aA + bA^2 = 0 \tag{4.802}$$

then

$$A = \frac{a}{b} \qquad A = 0 \tag{4.803}$$

When $A = \frac{a}{b}$, we have

$$(n+a)\frac{x^n}{y_1} + b\frac{x^{2n}}{y_1^2} - \frac{x^{n+1}}{y_1^2}\frac{dy_1}{dx} = cx^n \tag{4.804}$$

Multiplying by y_1^2/x^n and simplifying make the equation of the general form (4.789) between y and x. The coefficients b and c changed their places and a become $a+n$.

$$x\frac{dy_1}{dx} - (a+n)\,y_1 + cy_1^2 = bx^n \tag{4.805}$$

The transformation has been of the form

$$y = \frac{a}{b} + \frac{x^n}{y_1} \tag{4.806}$$

Now we may have another transformation

$$y_1 = \frac{a+n}{b} + \frac{x^n}{y_2} \tag{4.807}$$

to find a new equation

$$x\frac{dy_2}{dx} - (a+2n)\,y_2 + by_2^2 = cx^n \tag{4.808}$$

The coefficients b and c changed their places again and $a+n$ become $a+2n$. Another k successive transformations of the same series will transform the given equation either to the form

$$x\frac{dy_k}{dx} - (a+kn)\,y_k + by_k^2 = cx^n \tag{4.809}$$

or

$$x\frac{dy_k}{dx} - (a+kn)\,y_k + cy_k^2 = bx^n \tag{4.810}$$

based on k to be odd or even. Looking back at the transformations we have

$$y = \frac{a}{b} + \frac{x^n}{y_1} \tag{4.811}$$

$$y_1 = \frac{a+n}{b} + \frac{x^n}{y_2} \tag{4.812}$$

$$y_2 = \frac{a+2n}{b} + \frac{x^n}{y_3} \tag{4.813}$$

and finally

$$y_k = \frac{a+kn}{e} + \frac{x^n}{y_{k+1}} \tag{4.814}$$

where $e = b$ or c, according to k to be odd or even; and the effect of these transformations is to reduce the given equation to one or the other forms of (4.809) and (4.810). If in the expression for y, we substitute y_1 in terms of y_2 and again, for y_2 with in terms of y_3 and so on, we find a continued fraction solution.

$$y = \frac{a}{b} + \cfrac{x^n}{\cfrac{a+n}{b} + \cfrac{x^n}{\cfrac{a+2n}{b} + \cfrac{x^n}{\cfrac{a+3n}{b} + \cdots}}} \tag{4.815}$$

The last denominator will be $\frac{a+(k-1)n}{e} + \frac{x^n}{y_k}$ where e will be equal to b or c, depending on k to be even or odd.

4.4.2 Series Solution Transformation

There is no classical method to derive the solution of every differential equation in continued fractions. However, we may find the power series solution of the differential equation and transform the power series into continued fractions.

If the power series expression of the solution of a differential equation is in the form

$$f(x) = a_{10} + a_{11}x + a_{12}x^2 + a_{13}x^3 + \cdots \tag{4.816}$$

then it will be transformed to the following continued fractions:

$$f(x) = \frac{a_{10}}{1+}\ \frac{a_{20}x}{a_{10}+}\ \frac{a_{30}x}{a_{20}+}\ \frac{a_{40}x}{a_{30}+\cdots} \tag{4.817}$$

$$a_{nk} = a_{n-2,k+1}\, a_{n-1,0} - a_{n-2,0}\, a_{n-1,k+1} \tag{4.818}$$

$$n = 1, 2, 3, \cdots \qquad k = 0, 1, 2, 3, \cdots \tag{4.819}$$

$$a_{00} = 1 \qquad a_{0k} = 0 \tag{4.820}$$

If the solution of a differential equation is expressed as the ratio of two power series of the form

$$f(x) = \frac{b_0 + b_1x + b_2x^2 + b_3x^3 + \cdots}{a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots} \tag{4.821}$$

then, that will be transformed to the following continued fractions.

$$f(x) = \frac{a_{10}}{a_{00}+}\ \frac{a_{20}x}{a_{10}+}\ \frac{a_{30}x}{a_{20}+}\ \frac{a_{40}x}{a_{30}+\cdots} \tag{4.822}$$

$$a_{nk} = a_{n-2,k+1}\, a_{n-1,0} - a_{n-2,0}\, a_{n-1,k+1} \tag{4.823}$$

$$n = 0, 1, 2, 3, \cdots \qquad k = 0, 1, 2, 3, \cdots \tag{4.824}$$

Proof Let us assume the power series solution of a differential equation is found as:

$$f(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots \tag{4.825}$$

Let us flip the polynomial and rewrite it to find the first term of the continued fractions and a polynomial fraction in the denominator.

$$\begin{aligned} f(x) &= \cfrac{1}{\cfrac{1}{b_0} + \cfrac{1}{b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots} - \cfrac{1}{b_0}} \\ &= \cfrac{b_0}{1 - x\cfrac{c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots}{b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots}} \end{aligned} \tag{4.826}$$

$$c_0 = b_1 \qquad c_1 = b_2 \qquad c_2 = b_3 \qquad c_k = b_{k+1} \tag{4.827}$$

We flip the new fraction in the denominator again and find another term of the continued fractions and a new polynomial.

$$\begin{aligned} f(x) &= \cfrac{b_0}{1 - x\cfrac{1}{\cfrac{b_0}{c_0} + \cfrac{b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots}{c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots} - \cfrac{b_0}{c_0}}} \\ &= \cfrac{b_0}{1 - x\cfrac{c_0}{b_0 - x\cfrac{d_0 + d_1 x + d_2 x^2 + d_3 x^3 + \cdots}{c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots}}} \end{aligned} \tag{4.828}$$

$$\begin{aligned} &d_0 = b_0 c_1 - b_1 c_0 \qquad d_1 = b_0 c_2 - b_2 c_0 \\ &d_2 = b_0 c_3 - b_3 c_0 \qquad d_k = b_0 c_{k+1} - b_{k+1} c_0 \end{aligned} \tag{4.829}$$

The next step of the process will provide another term of the continued fractions and a new polynomial fraction.

$$f(x) = \cfrac{b_0}{1 - \cfrac{c_0 x}{b_0 - x\cfrac{1}{\cfrac{c_0}{d_0} + \cfrac{c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots}{d_0 + d_1 x + d_2 x^2 + d_3 x^3 + \cdots} - \cfrac{c_0}{d_0}}}}$$

$$= \cfrac{a_0}{1 - \cfrac{b_0 x}{b_0 - \cfrac{c_0 x}{c_0 - x\dfrac{d_0 + d_1 x + d_2 x^2 + d_3 x^3 + \cdots}{c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots}}}} \tag{4.830}$$

$$\begin{aligned} d_0 &= b_0 c_1 - b_1 c_0 \qquad & d_1 &= b_0 c_2 - b_2 c_0 \\ d_2 &= b_0 c_3 - b_3 c_0 \qquad & d_k &= b_0 c_{k+1} - b_{k+1} c_0 \end{aligned} \tag{4.831}$$

Continuing the process will generate the associated continued fractions.

$$f(x) = \frac{b_0}{1-}\,\frac{c_0 x}{b_0 -}\,\frac{d_0 x}{c_0 - \cdots} \tag{4.832}$$

Renaming the coefficients as

$$b_0 + b_1 x + b_2 x^2 + \cdots = a_{10} + a_{11} x + a_{12} x^2 + \cdots \tag{4.833}$$

$$c_0 + c_1 x + c_2 x^2 + \cdots = a_{20} + a_{21} x + a_{22} x^2 + \cdots \tag{4.834}$$

$$d_0 + d_1 x + d_2 x^2 + \cdots = a_{30} + a_{31} x + a_{32} x^2 + \cdots \tag{4.835}$$

$$\cdots$$

we will have

$$f(x) = \frac{a_{10}}{1+}\,\frac{a_{20} x}{a_{10}+}\,\frac{a_{30} x}{a_{20}+}\,\frac{a_{40} x}{a_{30} + \cdots} \tag{4.836}$$

$$a_{nk} = a_{n-2,k+1}\, a_{n-1,0} - a_{n-2,0}\, a_{n-1,k+1} \tag{4.837}$$

$$n = 1, 2, 3, \cdots \qquad k = 0, 1, 2, 3, \cdots \tag{4.838}$$

$$a_{00} = 1 \qquad a_{0k} = 0 \tag{4.839}$$

To prove (4.822), let us have a function $f(x)$ expressed by a fraction of polynomials.

$$f(x) = \frac{b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots}{a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots} \tag{4.840}$$

We rewrite the fraction to find

$$f(x) = \cfrac{1}{\dfrac{a_0}{b_0} + \dfrac{a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots}{b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots} - \dfrac{a_0}{b_0}}$$

$$= \frac{b_0}{a_0 + x\dfrac{c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots}{b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots}} \tag{4.841}$$

$$\begin{aligned} c_0 &= a_1 b_0 - a_0 b_1 \qquad & c_1 &= a_2 b_0 - a_0 b_2 \\ c_2 &= a_3 b_0 - a_0 b_3 \qquad & c_k &= a_{k+1} b_0 - a_0 b_{k+1} \end{aligned} \tag{4.842}$$

The new fraction in the denominator will be flipped again.

$$\begin{aligned} f(x) &= \frac{b_0}{a_0 + x\dfrac{1}{\dfrac{b_0}{c_0} + \dfrac{b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots}{c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots} - \dfrac{b_0}{c_0}}} \\ &= \frac{b_0}{a_0 + \dfrac{c_0 x}{b_0 + x\dfrac{d_0 + d_1 x + d_2 x^2 + d_3 x^3 + \cdots}{c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots}}} \end{aligned} \tag{4.843}$$

$$\begin{aligned} d_0 &= b_1 c_0 - b_0 c_1 \qquad & d_1 &= b_2 c_0 - b_0 c_2 \\ d_2 &= b_3 c_0 - b_0 c_3 \qquad & d_k &= b_{k+1} c_0 - b_0 c_{k+1} \end{aligned} \tag{4.844}$$

Therefore the associated continued fractions become:

$$f(x) = \frac{b_0}{a_0 +} \frac{c_0 x}{b_0 +} \frac{d_0 x}{c_0 + \cdots} \tag{4.845}$$

It is better to rename the coefficients

$$a_0 + a_1 x + a_2 x^2 + \cdots = a_{00} + a_{01} x + a_{02} x^2 + \cdots \tag{4.846}$$

$$b_0 + b_1 x + b_2 x^2 + \cdots = a_{10} + a_{11} x + a_{12} x^2 + \cdots \tag{4.847}$$

$$c_0 + c_1 x + c_2 x^2 + \cdots = a_{20} + a_{21} x + a_{22} x^2 + \cdots \tag{4.848}$$

$$d_0 + d_1 x + d_2 x^2 + \cdots = a_{30} + a_{31} x + a_{32} x^2 + \cdots \tag{4.849}$$

$$\cdots$$

to have the continued fractions and recursive formula for the coefficients.

$$f(x) = \frac{a_{10}}{a_{00} +} \frac{a_{20} x}{a_{10} +} \frac{a_{30} x}{a_{20} +} \frac{a_{40} x}{a_{30} + \cdots} \tag{4.850}$$

$$a_{nk} = a_{n-2,k+1}\, a_{n-1,0} - a_{n-2,0}\, a_{n-1,k+1} \tag{4.851}$$

$$n = 0, 1, 2, 3, \cdots \qquad k = 0, 1, 2, 3, \cdots \tag{4.852}$$

To be able to computerize the process, let us begin with a power series again

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \tag{4.853}$$

and rewrite it as

$$\begin{aligned} f_0(x) &= a_0 + a_1 x \left(1 + \frac{a_2}{a_1} x + \frac{a_3}{a_1} x^2 + \cdots\right) \\ &= a_0 + \frac{a_1 x}{1 + f_1(x)} \end{aligned} \tag{4.854}$$

where

$$\begin{aligned} f_1(x) &= \left(1 + \frac{a_2}{a_1} x + \frac{a_3}{a_1} x^2 + \cdots\right)^{-1} - 1 \\ &= b_1 x + b_2 x^2 + b_3 x^3 + \cdots \\ &= b_1 x \left(1 + \frac{b_2}{b_1} x + \frac{b_3}{b_1} x^2 + \cdots\right) = \frac{b_1 x}{1 + f_2(x)} \end{aligned} \tag{4.855}$$

$$b_k = \frac{1}{k!} \frac{d^k}{dx^k} \left(f_1(0) + 1\right) \tag{4.856}$$

The polynomial in $f_1(x)$ begins with 1 and therefore, the inverse of the polynomial will be another polynomial beginning with 1 to cancel out the last -1. The function $f_1(x)$ will introduce another function $f_2(x)$ in denominator that we will treat it similarly to derive a function $f_3(x)$.

$$\begin{aligned} f_2(x) &= \left(1 + \frac{b_2}{b_1} x + \frac{b_3}{b_1} x^2 + \cdots\right)^{-1} - 1 \\ &= c_1 x + c_2 x^2 + c_3 x^3 + \cdots \\ &= c_1 x \left(1 + \frac{c_2}{c_1} x + \frac{c_3}{c_1} x^2 + \cdots\right) = \frac{c_1 x}{1 + f_3(x)} \end{aligned} \tag{4.857}$$

$$c_k = \frac{1}{k!} \frac{d^k}{dx^k} \left(f_2(0) + 1\right) \tag{4.858}$$

$$\begin{aligned} f_3(x) &= \left(1 + \frac{c_2}{c_1} x + \frac{c_3}{c_1} x^2 + \cdots\right)^{-1} - 1 \\ &= d_1 x + d_2 x^2 + d_3 x^3 + \cdots \end{aligned}$$

$$= d_1 x \left(1 + \frac{d_2}{d_1} x + \frac{d_3}{d_1} x^2 + \cdots\right) = \frac{d_1 x}{1 + f_4(x)} \tag{4.859}$$

$$d_k = \frac{1}{k!} \frac{d^k}{dx^k} \left(f_3(0) + 1\right) \tag{4.860}$$

Similar process on $f_3(x)$ will make another term of the continued fractions as well as introducing next function $f_4(x)$.

$$\begin{aligned} f_4(x) &= \left(1 + \frac{d_2}{d_1} x + \frac{d_3}{d_1} x^2 + \cdots\right)^{-1} - 1 \\ &= e_1 x + e_2 x^2 + e_3 x^3 + \cdots \\ &= e_1 x \left(1 + \frac{e_2}{e_1} x + \frac{e_3}{e_1} x^2 + \cdots\right) = \frac{e_1 x}{1 + f_5(x)} \end{aligned} \tag{4.861}$$

$$e_k = \frac{1}{k!} \frac{d^k}{dx^k} \left(f_4(0) + 1\right) \tag{4.862}$$

Repeating this method will provide the continued fractions up to the desired term.

$$f_0(x) = a_0 + \cfrac{a_1 x}{1 + \cfrac{b_1 x}{1 + \cfrac{c_1 x}{1 + \cfrac{d_1 x}{1 + \cfrac{e_1 x}{1 + \cdots}}}}} \tag{4.863}$$

■

Example 169 Free dynamics of a first-order system.

The equation of motion of a first-order linear dynamic system such as a damper-spring system is

$$\dot{x} + \alpha x = 0 \qquad \dot{x} = \frac{dx}{dt} \qquad x(0) = x_0 \tag{4.864}$$

The series solution of the equation of motion is:

$$\begin{aligned} x &= x_0 - a x_0 t + \frac{a^2 x_0}{2!} t^2 - \frac{a^3 x_0}{3!} t^3 + \cdots + \frac{(-1)^k a^k x_0}{n!} t^n + \cdots \\ &= x_0 \left(1 - at + \frac{a^2}{2!} t^2 - \frac{a^3}{3!} t^3 + \frac{a^4}{4!} t^4 - \cdots\right) \end{aligned} \tag{4.865}$$

If we may recognize that:

$$e^{-\alpha t} = 1 - \alpha t + \frac{\alpha^2}{2!} t^2 - \frac{\alpha^3}{3!} t^3 + \frac{\alpha^4}{4!} t^4 - \frac{\alpha^5}{5!} t^5 + \cdots \tag{4.866}$$

then, the exact solution of the equation is:

$$x = x_0 \, e^{-\alpha t} \tag{4.867}$$

Defining the series solution

$$x = 1 - \alpha t + \frac{\alpha^2}{2} t^2 - \frac{\alpha^3}{3!} t^3 + \frac{\alpha^4}{4!} t^4 - \frac{\alpha^5}{5!} t^5 + \cdots \tag{4.868}$$

as

$$f(t) = a_{10} + a_{11} t + a_{12} t^2 + a_{13} t^3 + a_{14} t^4 + \cdots \tag{4.869}$$

with

$$a_{10} = 1 \qquad a_{11} = -\alpha \qquad a_{12} = \frac{\alpha^2}{2} \qquad a_{13} = -\frac{\alpha^3}{3!} \tag{4.870}$$

$$a_{1k} = (-1)^{k+1} \frac{\alpha^{k+1}}{(k+1)!} \tag{4.871}$$

$$a_{00} = 1 \qquad\qquad a_{0k} = 0 \tag{4.872}$$

and employing recursive equation

$$a_{nk} = a_{n-2,k+1} \, a_{n-1,0} - a_{n-2,0} \, a_{n-1,k+1} \tag{4.873}$$

$$n = 1, 2, 3, \cdots \qquad k = 0, 1, 2, 3, \cdots \tag{4.874}$$

we calculate the coefficients of the continued fractions (4.817). The continued fraction only needs the coefficients a_{i0}. We already have $a_{00} = 1$, and $a_{10} = 1$. To calculate a_{i0} we will need some of the $a_{i-1,j}$. The results would be:

$$\begin{aligned} & a_{20} = a_{01} a_{10} - a_{00} a_{11} = \alpha \\ & a_{30} = -\frac{\alpha^2}{2} \qquad a_{40} = -\frac{\alpha^4}{12} \qquad a_{50} = -\frac{\alpha^7}{144} \qquad a_{60} = \frac{\alpha^{12}}{17280} \\ & a_{70} = \frac{\alpha^{20}}{24883200} \qquad a_{80} = \frac{\alpha^{33}}{6019743744000} \end{aligned} \tag{4.875}$$

Figure 4.15 compares the exact solution of the differential equation with series and continued fractions solutions for $x_0 = 1$ and $\alpha = 1$.

Example 170 Overdamped nonlinear spring system.

A vibrating system needs two main elements, inertia m that stores kinetic energy $K = m\dot{x}^2/2$, and stiffness k that stores potential energy $P = kx^2/2$. The system may also have a damping element c that wastes energy $\int c\dot{x} \, dx$, and excitation $f(x, \dot{x}, t)$ that force the system to follow a given command (Jazar 2013).

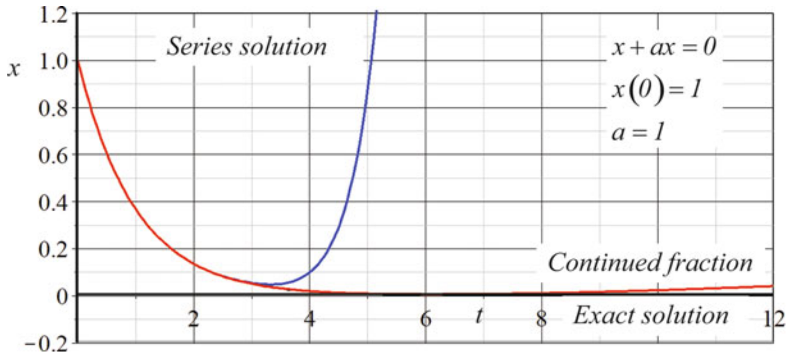

Fig. 4.15 The exact, series, and continued fractions solutions of $\dot{x} + \alpha x = 0$ for $x_0 = 1$ and $\alpha = 1$

$$m\ddot{x} + c\dot{x} + kx = f(x, \dot{x}, t) \tag{4.876}$$

Such a system is a linear vibrating system. In case the excitation is zero or a constant, then there would be an equilibrium $x = f/k$ that the system will approach as long as we have $m > 0$, $c > 0$, $k > 0$.

The damping and stiffness may be nonlinear and dependent on x, or $\dot{x}$, or be time variant. The solution of these cases is more interesting and more complicated. Consider the nonlinear vibrating system

$$\ddot{x} + 2\dot{x} + 0.2x^3 = 0.02 \qquad x(0) = 1 \qquad \dot{x}(0) = 1 \tag{4.877}$$

which has $m = 1$, $c = 2$, $k = 0.2x^2$, $f(x, \dot{x}, t) = 0.02$, and an equilibrium at $x = \sqrt[3]{0.02/0.2} = 0.46416$. The equilibrium positions are the solutions of the resultant algebraic equation after assuming all derivatives to be zero in the differential equation. The power series solution of Eq. (4.877) is:

$$x = 1 + t - \frac{109}{100}t^2 + \frac{47}{75}t^3 - \frac{1853}{6000}t^4 + \frac{1201}{7500}t^5 + \frac{167017}{2250000}t^6 + \frac{672073}{31500000}t^7 + \frac{45793}{14400000}t^8 - \frac{22531823}{2268000000}t^9 + \ldots \tag{4.878}$$

Defining the series solution as

$$f(t) = a_{10} + a_{11}t + a_{12}t^2 + a_{13}t^3 + a_{14}t^4 + \cdots \tag{4.879}$$

and accepting

$$a_{00} = 1 \qquad a_{0k} = 0 \tag{4.880}$$

we use the equation

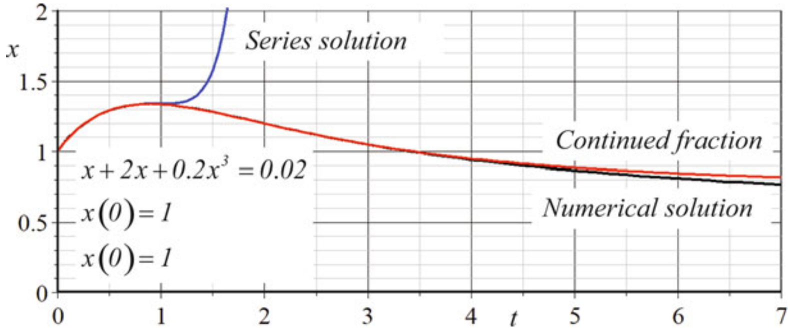

Fig. 4.16 The numerical, series, and continued fractions solutions of $\ddot{x} + 2\dot{x} + 0.2x^3 = 0.02$ for $x(0) = 1$, $\dot{x}(0) = 1$

$$a_{nk} = a_{n-2,k+1}\, a_{n-1,0} - a_{n-2,0}\, a_{n-1,k+1} \tag{4.881}$$

$$n = 1, 2, 3, \cdots \qquad k = 0, 1, 2, 3, \cdots \tag{4.882}$$

to calculate the coefficients of the continued fraction.

$$f(t) = \frac{a_{10}}{1+}\,\frac{a_{20}t}{a_{10}+}\,\frac{a_{30}t}{a_{20}+}\,\frac{a_{40}t}{a_{30}+\cdots} \tag{4.883}$$

We have $a_{00} = 1$, and $a_{10} = 1$, then we calculate a_{ij} to use a_{i0} in the solution. The results are:

$$\begin{aligned}
&a_{00} = 1 \qquad a_{10} = 1 \qquad a_{20} = -1 \qquad a_{30} = -\frac{209}{100} \\
&a_{40} = -\frac{16843}{30000} \qquad a_{50} = -\frac{25544}{140625} \qquad a_{60} = \frac{5345992399}{270000000000} \\
&a_{70} = \frac{209554138270769611}{60750000000000000000} \\
&a_{80} = \frac{7087308691698341368429554431}{89701171875000000000000000000000}
\end{aligned} \tag{4.884}$$

Figure 4.16 compares the numerical solution of the differential equation with series and continued fractions solutions.

Example 171 Derivative check.

In general there is no direct formula to take derivative and integral of a continued fractions as simple as derivative and integral of power series. To compare a continued fraction with its derivative, let us expand $y = \sum_{k=0}^{\infty} a_k x^k$ and $y' = \sum_{k=1}^{\infty} k a_k x^{k-1}$ into continued fractions and compare them employing the rule of derivative.

$$y = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \tag{4.885}$$

$$y' = \sum_{k=1}^{\infty} k a_k x^{k-1} \qquad y'' = \sum_{k=2}^{\infty} k\,(k-1)\,a_k x^{k-2} \tag{4.886}$$

As a practical example, let us work with

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_3 x^4 + a_3 x^5 \tag{4.887}$$

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_3 x^3 + 5a_3 x^4 \tag{4.888}$$

Defining the $y = y\,(x)$ as

$$y = a_{10} + a_{11} x + a_{12} x^2 + a_{13} x^3 + a_{14} x^4 + a_3 x^5 \tag{4.889}$$

and accepting

$$a_{00} = 1 \qquad a_{0k} = 0 \qquad k = 0, 1, 2, \cdots \tag{4.890}$$

we use the recursive equation

$$a_{nk} = a_{n-2,k+1}\ a_{n-1,0} - a_{n-2,0}\ a_{n-1,k+1} \tag{4.891}$$

$$n = 1, 2, 3, \cdots \qquad k = 0, 1, 2, \cdots \tag{4.892}$$

to calculate the coefficients of the continued fraction.

$$y\,(x) = \frac{a_{10}}{1+}\ \frac{a_{20}x}{a_{10}+}\ \frac{a_{30}x}{a_{20}+}\ \frac{a_{40}x}{a_{30}+\cdots} \tag{4.893}$$

Having $a_{00} = 1$, $a_{0i} = 0$, $i = 2, 3, \cdots$, and $a_{10} = 1$, we calculate a_{ij} to use a_{i0} in the solution. The results are:

$$\begin{gathered} a_{00} = 1 \qquad a_{10} = 1 \qquad a_{20} = -a_1 \\ a_{30} = a_0 a_2 - a_1^2 \qquad a_{40} = -a_0 \left(a_2^2 - a_1 a_3\right) \\ a_{50} = -a_0^2 a_1 a_2 a_4 + a_0^2 a_1 a_3^2 + a_0 a_1^3 a_4 - 2a_0 a_1^2 a_2 a_3 + a_0 a_1 a_2^3 \end{gathered} \tag{4.894}$$

$$y\,(x) = \frac{1}{1+}\ \frac{-a_1 x}{1+}\ \frac{\left(a_0 a_2 - a_1^2\right)x}{-a_1+}\ \frac{-a_0\left(a_2^2 - a_1 a_3\right)x}{a_0 a_2 - a_1^2 + \cdots} \tag{4.895}$$

Following the same procedure for

$$\begin{aligned} y' &= a_1 + 2a_2x + 3a_3x^2 + 4a_3x^3 + 5a_3x^4 \\ &= a_{10} + a_{11}x + a_{12}x^2 + a_{13}x^3 + a_{14}x^4 \end{aligned} \tag{4.896}$$

we have

$$\begin{aligned} &a_{00} = 1 \qquad a_{10} = a_1 \qquad a_{20} = -2a_2 \\ &a_{30} = 3a_1a_3 - 4a_2^2 \qquad a_{40} = 8a_1a_2a_4 - 9a_1a_3^2 \\ &a_{50} = -30a_1^2a_2a_3a_5 + 32a_1^2a_2a_4^2 + 40a_1a_2^3a_5 \\ &\qquad -96a_1a_2^2a_3a_4 + 54a_1a_2a_3^3 \end{aligned} \tag{4.897}$$

$$y'(x) = \frac{a_1}{1+}\ \frac{-2a_2x}{a_1+}\ \frac{\left(3a_1a_3 - 4a_2^2\right)x}{-2a_2+}\ \frac{\left(8a_1a_2a_4 - 9a_1a_3^2\right)x}{3a_1a_3 - 4a_2^2 + \cdots} \tag{4.898}$$

Therefore, the continued fraction

$$y'(x) = \cfrac{a_1}{1 + \cfrac{-2a_2x}{a_1 + \cfrac{\left(3a_1a_3 - 4a_2^2\right)x}{-2a_2 + \cfrac{\left(8a_1a_2a_4 - 9a_1a_3^2\right)x}{3a_1a_3 - 4a_2^2 + \cdots}}}} \tag{4.899}$$

is derivative of the following continued fraction:

$$y(x) = \cfrac{1}{1 + \cfrac{-a_1x}{1 + \cfrac{\left(a_0a_2 - a_1^2\right)x}{-a_1 + \cfrac{-a_0\left(a_2^2 - a_1a_3\right)x}{a_0a_2 - a_1^2 + \cdots}}}} \tag{4.900}$$

4.5 Chapter Summary

Speed and domain of convergence of continued fractions approximation of functions are much better than power series expansions. Hence it is of practical interest to go from a power series to a continued fraction. Assuming the power series of a function is given

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots \tag{4.901}$$

$$= a_{00} + a_{01}x + a_{02}x^2 + a_{03}x^3 + \cdots \tag{4.902}$$

we can determine the continued fractions expression of the function $f(x)$ as:

$$f(x) = a_{00} + \cfrac{a_{01}x}{1 + \cfrac{a_{11}x}{1 + \cfrac{a_{21}x}{1 + \cfrac{a_{31}x}{1 + \cfrac{a_{41}x}{1 + \cdots}}}}} \tag{4.903}$$

where

$$a_{nk} = \frac{1}{k!}\frac{d^k}{dx^k}(f_n(0) + 1) \tag{4.904}$$

$$f_0(x) = a_{00} + \frac{a_{01}x}{1 + f_1(x)} \tag{4.905}$$

$$f_n(x) = \frac{a_{n1}x}{1 + f_{n+1}(x)} \tag{4.906}$$

The continued fraction of a power series will be expressed in one of the following forms:

$$f(x) = b_0 + \frac{a_1 x}{b_1+}\frac{a_2 x}{b_2+}\frac{a_3 x}{b_3+}\cdots\frac{a_k x}{b_k+}\cdots \tag{4.907}$$

or

$$f(x) = \frac{b_0}{1+}\frac{c_1 x}{d_1+}\frac{c_2 x}{d_2+}\frac{c_3 x}{d_3+}\cdots\frac{c_k x}{d_k+}\cdots \tag{4.908}$$

Euler introduces another expression to relate summation series and continued fraction interaction. We can verify that:

$$\begin{aligned} & a_0 + a_1 + a_1 a_2 + a_1 a_2 a_3 + \cdots + a_1 a_2 a_3 \cdots a_n \\ &= a_0 + \frac{a_1}{1+}\frac{a_2}{1 + a_2+}\frac{a_3}{1 + a_3+}\cdots\frac{a_n}{1 + a_n} \end{aligned} \tag{4.909}$$

Now having the continued summation series expansion of a function as

$$f = c_0 + c_1 + c_2 + c_3 + c_4 + \cdots + c_k + \cdots \tag{4.910}$$

we are able to determine $a_0, a_1, a_2, a_3, \cdots$.

$$\begin{aligned} & a_0 = c_0 \qquad a_1 = c_1 \qquad a_2 = \frac{c_2}{a_1} \qquad a_3 = \frac{c_3}{a_1 a_2} \\ & a_4 = \frac{c_4}{a_1 a_2 a_3} \qquad \cdots \qquad a_k = \frac{c_k}{a_1 a_2 a_3 \cdots a_{k-1}} \end{aligned} \tag{4.911}$$

Consider the general second-order linear differential equation in the form

$$y = P_0(x)\, y' + Q_0(x)\, y'' \tag{4.912}$$

$$y' = \frac{dy}{dx} \tag{4.913}$$

where $P_0(x)$ and $Q_0(x)$ are assumed infinitely differentiate functions of x. Its solution can be expressed by continued fractions.

$$y = C \exp F(x) \tag{4.914}$$

$$F'(x) = f(x) \tag{4.915}$$

$$f(x) = P_0 + \frac{Q_0}{P_1+}\frac{Q_1}{P_2+}\frac{Q_2}{P_3+}\cdots \tag{4.916}$$

$$P_n(x) = \frac{P_{n-1}(x) + Q'_{n-1}(x)}{1 - P'_{n-1}(x)} \tag{4.917}$$

There is no classical method to derive the solution of every differential equation in continued fraction. The best way is to find the Taylor series solution of the differential equation and transform the power series into continued fraction. The polynomial series expression of the solution of a differential equation

$$f(x) = a_{10} + a_{11}x + a_{12}x^2 + a_{13}x^3 + \cdots \tag{4.918}$$

will be transformed to the following continued fraction:

$$f(x) = \frac{a_{10}}{1+}\frac{a_{20}x}{a_{10}+}\frac{a_{30}x}{a_{20}+}\frac{a_{40}x}{a_{30}+\cdots} \tag{4.919}$$

$$a_{nk} = a_{n-2,k+1}\, a_{n-1,0} - a_{n-2,0}\, a_{n-1,k+1} \tag{4.920}$$

$$n = 1, 2, 3, \cdots \qquad k = 0, 1, 2, 3, \cdots \tag{4.921}$$

$$a_{00} = 1 \qquad a_{0k} = 0 \tag{4.922}$$

In the transformation we will be working with polynomial fractions of the form

$$f(x) = \frac{b_0 + b_1x + b_2x^2 + b_3x^3 + \cdots}{a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots} \tag{4.923}$$

that will be transformed to the following continued fraction:

$$f(x) = \frac{a_{10}}{a_{00}+}\frac{a_{20}x}{a_{10}+}\frac{a_{30}x}{a_{20}+}\frac{a_{40}x}{a_{30}+\cdots} \tag{4.924}$$

$$a_{nk} = a_{n-2,k+1}\, a_{n-1,0} - a_{n-2,0}\, a_{n-1,k+1} \tag{4.925}$$

$$n = 0, 1, 2, 3, \cdots \qquad k = 0, 1, 2, 3, \cdots \tag{4.926}$$

The following method can be computerized. Let us begin with a power series in the form

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \tag{4.927}$$

and rewrite it as

$$\begin{aligned} f_0(x) &= a_0 + a_1 x \left(1 + \frac{a_2}{a_1} x + \frac{a_3}{a_1} x^2 + \cdots \right) \\ &= a_0 + \frac{a_1 x}{1 + f_1(x)} \end{aligned} \tag{4.928}$$

where

$$\begin{aligned} f_1(x) &= \left(1 + \frac{a_2}{a_1} x + \frac{a_3}{a_1} x^2 + \cdots \right)^{-1} - 1 \\ &= b_1 x + b_2 x^2 + b_3 x^3 + \cdots \\ &= b_1 x \left(1 + \frac{b_2}{b_1} x + \frac{b_3}{b_1} x^2 + \cdots \right) = \frac{b_1 x}{1 + f_2(x)} \end{aligned} \tag{4.929}$$

$$b_k = \frac{1}{k!} \frac{d^k}{dx^k} \left(f_1(0) + 1\right) \tag{4.930}$$

The function $f_1(x)$ will introduce another function $f_2(x)$ in denominator that we will treat it similarly to derive a function $f_3(x)$, and then $f_4(x)$, etc.

$$\begin{aligned} f_2(x) &= \left(1 + \frac{b_2}{b_1} x + \frac{b_3}{b_1} x^2 + \cdots \right)^{-1} - 1 \\ &= c_1 x + c_2 x^2 + c_3 x^3 + \cdots \\ &= c_1 x \left(1 + \frac{c_2}{c_1} x + \frac{c_3}{c_1} x^2 + \cdots \right) = \frac{c_1 x}{1 + f_3(x)} \end{aligned} \tag{4.931}$$

$$c_k = \frac{1}{k!} \frac{d^k}{dx^k} \left(f_2(0) + 1\right) \tag{4.932}$$

$$\begin{aligned} f_3(x) &= \left(1 + \frac{c_2}{c_1} x + \frac{c_3}{c_1} x^2 + \cdots \right)^{-1} - 1 \\ &= d_1 x + d_2 x^2 + d_3 x^3 + \cdots \end{aligned}$$

$$= d_1 x \left(1 + \frac{d_2}{d_1}x + \frac{d_3}{d_1}x^2 + \cdots\right) = \frac{d_1 x}{1 + f_4(x)} \tag{4.933}$$

$$d_k = \frac{1}{k!}\frac{d^k}{dx^k}\left(f_3(0) + 1\right) \tag{4.934}$$

$$\begin{aligned} f_4(x) &= \left(1 + \frac{d_2}{d_1}x + \frac{d_3}{d_1}x^2 + \cdots\right)^{-1} - 1 \\ &= e_1 x + e_2 x^2 + e_3 x^3 + \cdots \\ &= e_1 x \left(1 + \frac{e_2}{e_1}x + \frac{e_3}{e_1}x^2 + \cdots\right) = \frac{e_1 x}{1 + f_5(x)} \end{aligned} \tag{4.935}$$

$$e_k = \frac{1}{k!}\frac{d^k}{dx^k}\left(f_4(0) + 1\right) \tag{4.936}$$

Repeating this method will generate the continued fractions.

$$f_0(x) = a_0 + \cfrac{a_1 x}{1 + \cfrac{b_1 x}{1 + \cfrac{c_1 x}{1 + \cfrac{d_1 x}{1 + \cfrac{e_1 x}{1 + \cdots}}}}} \tag{4.937}$$

4.6 Key Symbols

a, b	Semiaxes of tire print area A
a, b	Coefficients
a, b, c, d	Constant derivatives
a_i	Coefficients
A_i	Recurrence coefficients of hypergeometric function
A, B	Coefficients
Ai, Bi	Airy functions
c	Constant point of expansion
c_k	$= \lim_{k\to\infty} \frac{p_k}{q_k}$ limit of a continued fraction
C	Constant coefficients
C, C_i	Constant of integral
$\mathrm{cn}(u, k)$	Jacobi elliptic function
$\mathrm{dn}(u, k)$	Jacobi elliptic function
e	Eccentricity
e	$2.718281828459\cdots$
E	Mechanical energy
E_i	Constant of motion
$E(k)$	Complete elliptic integrals of the second kind
$E(x, k)$	Elliptic integral of the second kind
$E(\varphi, k)$	Elliptic integral of the second kind
f, f_i, F	Function
$F(x, k)$	Elliptic integral of the first kind
$F(\varphi, k)$	Elliptic integral of the first kind
g, g_i, h	Function
$g, \mathbf{g}$	Gravitational acceleration
G	$= 6.67259 \times 10^{-11}\ \mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}$ gravitational constant
G_i	$= Gm_i$
i	Imaginary unit, $i^2 = -1$
k	Integer counter, integer exponent
K	Kinetic energy
$K\ (k)$	Complete elliptic integrals of the first kind
l	Length
L	Function
$\mathcal{L}$	Lagrangian
m	Mass
m, n	Dummy index
n	Exponent for shape and stress distribution of A_P
p	Momentum
p/q	Fraction
p_k	Numerator of a convergent
p_k/q_k	Convergent of a continued fraction
P_k/Q_k	Convergent of a continued fraction

P	Point
P	Arc length
P	Perimeter
P	Algebraic polynomial series
P	Potential energy
P, Q, R	Function
P_i	Function coefficient
q_k	Denominator of a convergent
Q	Geometric tire radius
r, r_i	Inverse derivative in Thiele's continued fraction formula
R	Remainder
s	Reduce of convergence
s_i	Eigenvalue
$\mathrm{sn}(u, k)$	Jacobi elliptic function
t	Time
t	Variable
T	Period
T	Trigonometric polynomials series
u, v, s	Imaginary variables
x, y, z	Coordinate axes
$\mathbf{x}_i$	Relative positions vectors
$\mathbf{X}_i$	Absolute positions vectors
y, z	Functions
α, β, γ	Constant exponent
$\epsilon > 0$	Indicator of the degree of approximation
ε	$= \mu / \lvert\mathbf{x}\rvert^3$
θ, φ	Angular variable
μ	$= G_1 + G_2$
λ	$= (\mathbf{x} \cdot \dot{\mathbf{x}}) / \lvert\mathbf{x}\rvert^2$
φ	Function of two variables
π	$3.14159265359 \cdots$
ψ	$= (\dot{\mathbf{x}} \cdot \dot{\mathbf{x}}) / \lvert\mathbf{x}\rvert^2 =$
$\Pi(k, n)$	Complete elliptic integrals of the third kind
$\Pi(x, k, n)$	Elliptic integral of the third kind
$\Pi(\varphi, k, n)$	Elliptic integral of the third kind

Exercises

1. Logarithm function.
   Find the series expansion of the following functions:

$$y = \log_a x \tag{4.938}$$

   You may use these rules:

$$\frac{d}{dx}\log_b x = \frac{1}{x \ln b} \tag{4.939}$$

$$\int \log_b x \, dx = \frac{x\,(\ln x - 1)}{\ln b} \tag{4.940}$$

2. Power series of $\tan x$ from $\sin x/\cos x$.
   Having

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \tag{4.941}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \tag{4.942}$$

   derive

$$\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \cdots \tag{4.943}$$

   (a) by calculating $\sin x/\cos x$.
   (b) by assuming

$$\tan x = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \tag{4.944}$$

   and calculating $\sin x = \tan x \cos x$.

3. Power series of elementary functions.

   (a) Show that the following power series are correct.

$$(a + x)^n = \sum_{k=1}^{n} \binom{n}{k} x^k a^{n-k} \tag{4.945}$$

$$\frac{1}{a + x} = 1 - x + x^2 - x^3 + \cdots = \sum_{k=1}^{\infty} (-1)^{k-1}\, k x^{k-1} \tag{4.946}$$

$$\frac{1}{(a + x)^2} = 1 - 2x + 3x^2 - 4x^3 + \cdots = \sum_{k=1}^{\infty} (-1)^{k-1}\, x^{k-1} \tag{4.947}$$

$$\sqrt{a+x} = 1 + \frac{x}{2} - \frac{1 \times 1}{2 \times 4}x^2 + \frac{1 \times 1 \times 3}{2 \times 4 \times 6}x^3 - \cdots \tag{4.948}$$

$$\frac{1}{\sqrt{a+x}} = 1 - \frac{x}{2} + \frac{1 \times 3}{2 \times 4}x^2 - \frac{1 \times 3 \times 5}{2 \times 4 \times 6}x^3 + \cdots \tag{4.949}$$

(b) Prove the equation

$$\frac{x}{(a+x)^2} = \sum_{k=1}^{\infty} kx^k \tag{4.950}$$

and determine the radius of convergence of the series.

4. Substitution of series into another series.
Knowing that

$$\ln(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \frac{y^5}{5} - \frac{y^6}{6} + \cdots \tag{4.951}$$

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \tag{4.952}$$

derive the power series expansion of $y = \ln(1+\sin x)$.

$$\ln(1+\sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \frac{x^5}{24} - \frac{x^6}{45} + \frac{61x^7}{5040} - \frac{17x^8}{2520} + \cdots \tag{4.953}$$

5. Hyperbolic trigonometric functions.
Knowing

$$e^{ax} = 1 + ax + \frac{a^2}{2}x^2 + \frac{a^3}{3!}x^3 + \frac{a^4}{4!}x^4 + \frac{a^5}{5!}x^5 + \cdots \tag{4.954}$$

determine the power series expansion of the following functions:

$$\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right) \tag{4.955}$$

$$\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right) \tag{4.956}$$

6. Operations on power series.

(a) Prove the division equation of two power series is:

$$\frac{\sum_{k=0}^{\infty} b_k x^k}{\sum_{k=0}^{\infty} a_k x^k} = \frac{1}{a_0} \sum_{k=0}^{\infty} c_k x^k \tag{4.957}$$

where

$$c_j = b_j - \frac{1}{a_0} \sum_{k=1}^{n} c_{j-k} a_k \tag{4.958}$$

(b) Show the multiplication of power series equation.

$$\sum_{k=0}^{\infty} a_k x^k \sum_{k=0}^{\infty} b_k x^k = \sum_{k=0}^{\infty} c_k x^k \tag{4.959}$$

$$c_j = \sum_{k=1}^{j} a_k b_{j-k} \tag{4.960}$$

(c) Prove the equation of an exponent of power series is:

$$\left(\sum_{k=0}^{\infty} a_k x^k \right)^n = \sum_{k=0}^{\infty} c_k x^k \tag{4.961}$$

where

$$c_0 = a_0^n \tag{4.962}$$

$$c_j = \frac{1}{j a_0} \sum_{k=1}^{j} (kn - j + k)\, a_k c_{j-k} \qquad j \geq 1 \tag{4.963}$$

(d) Show the substitution of one series into another is:

$$y = \sum_{k=1}^{\infty} a_k x^k \tag{4.964}$$

$$\sum_{k=1}^{\infty} b_k y^k = \sum_{k=1}^{\infty} c_k x^k \tag{4.965}$$

where

$$c_1 = a_1 b_1 \qquad c_2 = a_2 b_1 + a_1^2 b_2$$
$$c_3 = a_3 b_1 + 2 a_1 a_2 b_2 + a_1^3 b_3 \tag{4.966}$$

$$c_4 = a_4 b_1 + a_2^2 b_2 + 2a_1 a_3 b_2 + 3a_1^2 a_2 b_3 + a_1^4 b_4$$
$$\dots$$

7. Exponential function to continued fractions.
   Show the following power series and derive the associated continued fractions.

   (a)
$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \tag{4.967}$$

   (b)
$$a^x = \sum_{k=0}^{\infty} \frac{(x \ln a)^k}{k!} \tag{4.968}$$

   (c)
$$e^{-x^2} = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{k!} \tag{4.969}$$

   (d)
$$e^{e^x} = e\left(1 + x + \frac{2x^2}{2!} + \frac{5x^3}{3!} + \frac{15x^4}{4!} + \cdots\right) \tag{4.970}$$

   (e)
$$e^{\cos x} = e\left(1 - \frac{x^2}{2!} + \frac{4x^4}{4!} - \frac{31x^6}{6!} + \cdots\right) \tag{4.971}$$

   (f)
$$e^{\sin x} = 1 + x + \frac{x^2}{2!} - \frac{3x^4}{4!} - \frac{8x^5}{5!} - \frac{3x^6}{6!} + \frac{56x^7}{7!} + \cdots \tag{4.972}$$

8. Series and continued fractions equivalence.
   Is it possible to have
$$a_0 + a_1 x + a_2 x^2 + \cdots = \frac{a_0}{1+}\,\frac{a_1 x}{1+}\,\frac{a_2 x}{1+}\cdots \tag{4.973}$$

9. Expansion of a function into continued fraction.

(a) Show that the function

$$y = \int_0^x \frac{dt}{\left(1 + t^k\right)} \tag{4.974}$$

satisfies the differential equation

$$y' = \frac{1}{1 + t^k} \tag{4.975}$$

then use the change of variable

$$y = \frac{x}{1 + z} \tag{4.976}$$

and show that the new differential equation would be:

$$\left(1 + t^k\right) x z' + \left(1 + t^k\right) z + z^2 = x^k \tag{4.977}$$

(b) Show that the equation is a particular case of (4.748), in which k can be any number and

$$\alpha = \alpha' = \beta = \gamma = \delta = 1 \tag{4.978}$$

$$\beta' = -1 \tag{4.979}$$

then show that the following continued fraction, obtained by Lagrange, is the results of the integral.

$$y = \int_0^x \frac{dt}{\left(1 + t^k\right)} = \frac{x}{1+} \frac{x^k}{k + 1+} \frac{k^2 x^k}{2k + 1+} \frac{(k + 1)^2 x^k}{3k + 1+} \cdots$$

$$\cdots \frac{n^2 k^2 x^k}{2nk + 1+} \frac{(nk + 1)^2 x^k}{(2n + 1) k + 1+} \cdots \tag{4.980}$$

(c) Show when $k = 1$, the expansion becomes the expansion for $\ln(1 + x)$, and when $k = 2$, it becomes the expansion for $\arctan x$.

(d) Show this expansion satisfies the conditions of Stolz's theorem and then substituting x^{m+1} for x in the integral (4.974), and setting $k(m + 1) = s$ to find the new continued fractions as below.

$$\int_0^x \frac{t^m dt}{(1+t^s)} = \frac{x^{m+1}}{m+1+} \frac{(m+1)^2 x^s}{s+1+m+} \frac{s^2 x^s}{2s+1+m+} \cdots$$

$$\cdots \frac{n^2 s^2 x^s}{2ns+1+m+} \frac{(ns+1+m)^2 x^s}{(2n+1)s+1+s+} \cdots \qquad (4.981)$$

10. Expansion of $\tan x$ and $\tanh x$ as continued fractions.
The function $y = \tan x$ satisfies the differential equation of the form

$$y' = 1 + y^2 \qquad y(0) = 0 \qquad (4.982)$$

(a) Use

$$y = \frac{x}{1+z} \qquad (4.983)$$

and change the equation to:

$$xz' + z + z^2 = -x^2 \qquad (4.984)$$

and show that it is a particular case of (4.748) for:

$$\alpha' = \beta' = 0 \qquad \alpha = \beta = \gamma = 1$$
$$\delta = -1 \qquad k = 2 \qquad (4.985)$$

Then you find the Lambert expansion for $\tan x$.

$$\tan x = \frac{x}{1-} \frac{x^2}{3-} \frac{x^2}{5-} \cdots \frac{x^2}{(2n+1)-} \cdots \qquad (4.986)$$

(b) Substitute x with x/i, where $i^2 = -1$, in the continued fractions and multiply it by i to obtain:

$$\tan x = \frac{x}{1+} \frac{x^2}{3+} \frac{x^2}{5+} \cdots \frac{x^2}{(2n+1)+} \cdots \qquad (4.987)$$

11. Quadratic equations.
Solve the following quadratic equations by continued fractions.

$$x^2 - 3x - 1 = 0 \qquad (4.988)$$
$$x^2 - 5x - 1 = 0 \qquad (4.989)$$

12. The perimeter of ellipse.
Derive the continued fraction (4.253) from the series expansion (4.251).

13. Second-order differential equations.

(a) Solve this differential equation in Taylor series around $x = 2$

$$y'' - y = 0 \qquad y(2) = 1 \qquad y'(2) = -1 \tag{4.990}$$

and show that the solution will be

$$\begin{aligned} y(x) &= 1 - (x-2) + \frac{1}{2}(x-2)^2 - \frac{1}{3!}(x-2)^3 + \cdots \\ &= \left(1 + \frac{1}{2}(x-2)^2 + \frac{1}{4!}(x-2)^4 + \cdots\right) \\ &\quad - \left((x-2) + \frac{1}{3!}(x-2)^3 + \frac{1}{5!}(x-2)^5 + \cdots\right) \\ &= \cosh(x-2) - \sinh(x-2) \end{aligned} \tag{4.991}$$

(b) Show that the series solution of the equation

$$y'' - 2xy' + y = 0 \tag{4.992}$$

is

$$\begin{aligned} y &= a_0\left(1 - \frac{x^2}{2!} - \frac{3x^4}{4!} - \frac{3(7)x^6}{6!} + \frac{3(7)(11)x^8}{8!} - \cdots\right) \\ &+ a_1\left(x + \frac{x^3}{3!} + \frac{5x^5}{5!} + \frac{5(9)x^7}{7!} + \frac{5(9)(13)x^9}{9!} - \cdots\right) \end{aligned} \tag{4.993}$$

14. Power series transformation.
If

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots \tag{4.994}$$

show that

$$\sin^2 x = \frac{2x^2}{2!} - \frac{3^3x^4}{4!} + \frac{2^5x^6}{6!} - \frac{2^7x^8}{8!} - \cdots \tag{4.995}$$

15. A first-order parametric equation.
Show that

$$x = \frac{\sin^{-1} t}{\sqrt{1-t^2}} \tag{4.996}$$

satisfies the differential equation

$$\left(1 - t^2\right) \dot{x} - tx = 1 \tag{4.997}$$

and show that

$$x = t + \frac{2}{3}t^3 + \frac{2 \times 4}{3 \times 5}t^5 + \frac{2 \times 4 \times 6}{3 \times 5 \times 7}t^7 + \cdots \tag{4.998}$$

16. Pendulum.

(a) Take a time derivative from the energy equation of a pendulum $K + V = E$,

$$\frac{1}{2}ml^2\dot{\theta}^2 + mg\,(1 - \cos\theta) = E \tag{4.999}$$

and determine its equation of motion.

$$ml^2\ddot{\theta} + mgl\sin\theta = 0 \tag{4.1000}$$

(b) Explain why $0 \leq k \leq 1$ is equivalent to the oscillatory motion of the pendulum, where

$$E = k^2\,(2mgl) \qquad k > 0 \tag{4.1001}$$

and the value of k will be calculated by the initial conditions:

$$k = \sqrt{\frac{E_0}{2mgl}} = \frac{1}{2}\sqrt{\frac{1}{gl}\left(2g + l^2\dot{\theta}_0^2 - 2g\cos\theta_0\right)} \tag{4.1002}$$

17. Two coupled linear first-order equations.
Determine the solutions of

$$\dot{x} = a\,x + b\,y \qquad \dot{x} = \frac{dx}{dt} \qquad x\,(0) = A \tag{4.1003}$$

$$\dot{y} = c\,x + d\,y \qquad \dot{y} = \frac{dy}{dt} \qquad y\,(0) = B \tag{4.1004}$$

$$\{a, b, c, d\} \in \mathbb{R} \tag{4.1005}$$

$$a = 1 \qquad b = -2 \qquad c = 2 \tag{4.1006}$$

for $d = -5, -4, -1, 0, 1, 4, 5$ and plot the solution in (x, y)-plane for:

$$A = 0 \qquad B = -2, -1, 1, 2 \tag{4.1007}$$

and

$$B = 0 \qquad A = -2, -1, 1, 2 \tag{4.1008}$$

18. Solution of the second-order equations in continued fractions.
    Derive the solution of the following differential equations in continued fractions.

    (a)
    $$ay'' + y' - y = 0 \tag{4.1009}$$

    (b)
    $$ay'' + by' - y = 0 \tag{4.1010}$$

    (c)
    $$y'' + xy' - y = 0 \tag{4.1011}$$

    (d)
    $$xy'' + y' - y = 0 \tag{4.1012}$$

    (e)
    $$y'' + y' - xy = 0 \tag{4.1013}$$

19. Riccati equation.
    Derive the solution of the following Riccati equations in continued fractions.

    (a)
    $$\frac{du}{dx} + u^2 = x^2 \tag{4.1014}$$

    (b)
    $$\frac{du}{dx} + u^2 = 2x^4 \tag{4.1015}$$

    (c)
    $$\frac{du}{dx} + 2u^2 = x^4 \tag{4.1016}$$

20. Series to continued fractions.
Knowing

$$\arctan x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} \qquad -1 \le x \le 1 \tag{4.1017}$$

show that

$$\arctan x = \frac{x}{1+}\,\frac{1^2x^2}{3-x^2+}\,\frac{3^2x^2}{5-3x^2+}\,\frac{5^2x^2}{7-5x^2+}\cdots \tag{4.1018}$$

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Part III
Approximation Tools

When we work with differential equation modeling of a physical phenomenon, the equation usually has no closed form solution expressible by elementary functions. We engineers and scientists need to understand the behavior of the phenomenon and wish to extract as much information as possible from the differential equations. In this part, we introduce and review several approximation methods by focusing on one of the most important differential equations for which we do not have elementary solution but still we need to understand its behavior with very good approximation. The Mathieu equation is the simplest parametric equation that directly or indirectly appears in stability analysis of dynamic systems.

A general linear parametric second-order equation is of the form

$$\frac{d^2x}{dt^2} + g\left(t\right)\frac{d^2x}{dt^2} + f\left(t\right)x = 0 \tag{1}$$

$$g\left(t\right) = g\left(t + T\right) \qquad f\left(t\right) = f\left(t + T\right) \tag{2}$$

and a simple case would be

$$f\left(t\right) = a - b\cos t \qquad g\left(t\right) = 0 \tag{3}$$

to have the Mathieu equation.

$$\frac{d^2x}{dt^2} + \left(a - b\cos t\right)x = 0 \tag{4}$$

Direct application of the Mathieu equation is when a dynamic system has a parameter changing with time periodically. A child playing on swing in a park is an example. We model this system as a pendulum with variable length because the length of this pendulum is assumed the distance from fulcrum to the mass center of the child. Indirect application is when we have a periodic solution for a differential equation and we are concerned about the stability of the solution. Studying the stability of the solution ends up to know the stability of Mathieu type equations.

In this part, we show how we investigate the behavior of such an unsolvable equation employing different approximation methods, one of them would be continued fractions. Mathieu equation, although looks simple, is a complicated equation. To provide some simplification in calculations we use the following expression for the Mathieu equation:

$$\frac{d^2x}{dt^2} + (a - 2b\cos 2t)\,x = 0 \tag{5}$$

Chapter 5
Mathieu Equation

Mathieu equation is a linear parametric differential equation. Assuming $b = 0$, the equation reduces to a simple oscillator, such as a mass-spring mechanical system $\ddot{x} + ax = 0$. The simple oscillator system is energy conservative and therefore its response would be oscillation at the natural frequency of $\omega_n = \sqrt{a}$ with a constant amplitude, for $a > 0$ and nonzero initial conditions. Introducing b makes the stiffness of the system to be time variant fluctuating between $a + b$ and $a - b$ around the average a. Depending on the values of a and b, the solution of the equation may be unstable, periodic, or stable. Here, unstable means increasing amplitude infinitely, stable means decreasing amplitude or approaching a constant amplitude, and periodic means oscillation at a constant amplitude. A point in (a, b)-plane provides a periodic response if any small change in a or b puts the system in stable or unstable region. This means the (a, b)-plane will have stable and unstable regions separated by periodic lines. We are interested in the periodic solutions of the Mathieu equation with periods π and 2π as well as the values of (a, b) associated to the periodic solutions.

$$\frac{d^2x}{dt^2} + (a - 2b\cos 2t)\, x = 0 \tag{5.1}$$

5.1 Periodic Solutions of Order $n = 1$

We are interested in the periodic solutions of the Mathieu equation (5.1) which have period π or 2π and happen on the boundary of stability and instability regions in the (a, b)-plane. Such boundaries will be expressed by functions of $a = a(b)$ starting from $a = n^2$, $n = 0, 1, 2, \cdots$. When $b \to 0$ the Mathieu equation approaches $\ddot{x} + n^2 x = 0$ with solutions of $\cos nt$ and $\sin nt$. For $b = 0$ and $n = 1$, either $\cos t$ or $\sin t$ or a combination of them are the solutions of Mathieu equation. By increasing b from zero, the orthogonal solutions $\cos t$ and $\sin t$ begin to change independently

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and grow terms in the power of b each associated to the two different boundaries of $a = a(b)$ starting from $a = 1$.

The solution of the Mathieu equation that reduces to $\cos t$ for $b = 0$ is the *cosine elliptic Mathieu function* ce_1 and the associated $a = a(b)$ is its *characteristic* number a_{ce1}.

$$\begin{aligned} ce_1(t, b) = \cos t - \frac{b}{8}\cos 3t + \frac{b^2}{64}\left(-\cos 3t + \frac{1}{3}\cos 5t\right) \\ -\frac{b^3}{512}\left(\frac{1}{3}\cos 3t - \frac{4}{9}\cos 5t + \frac{1}{18}\cos 7t\right) \\ +\frac{b^4}{4096}\left(\frac{11}{9}\cos 3t + \frac{1}{6}\cos 5t - \frac{1}{12}\cos 7t + \frac{1}{180}\cos 9t\right) \\ +\cdots \end{aligned} \tag{5.2}$$

$$a_{ce_1} = 1 + b - \frac{1}{8}b^2 - \frac{1}{64}b^3 - \frac{1}{1536}b^4 + \frac{11}{36864}b^5 + \cdots \tag{5.3}$$

Similarly, the solution of the Mathieu equation that reduces to $\sin t$ for $b = 0$ is the *sine elliptic Mathieu function* se_1 and the associated $a = a(b)$ is its characteristic number a_{se1}.

$$\begin{aligned} se_1(t, b) = \sin t - \frac{b}{8}\cos 3t + \frac{b^2}{64}\left(\sin 3t + \frac{1}{3}\sin 5t\right) \\ -\frac{b^3}{512}\left(\frac{1}{3}\sin 3t + \frac{4}{9}\sin 5t + \frac{1}{18}\sin 7t\right) \\ +\frac{b^4}{4096}\left(-\frac{11}{9}\cos 3t + \frac{1}{6}\cos 5t + \frac{1}{12}\cos 7t + \frac{1}{180}\cos 9t\right) \\ +\cdots \end{aligned} \tag{5.4}$$

$$a_{se_1} = 1 - b - \frac{1}{8}b^2 + \frac{1}{64}b^3 - \frac{1}{1536}b^4 - \frac{11}{36864}b^5 + \cdots \tag{5.5}$$

Figure 5.1 illustrates the functions a_{ce_1} and a_{se_1} on which the Mathieu equation has the periodic solutions $ce_1(t, b)$ and $se_1(t, b)$.

Proof When $b = 0$, the Mathieu equation becomes $\ddot{x} + ax = 0$ with solutions $\cos nt$ and $\sin nt$ where $n^2 = a$. For integers $n = 1, 2, 3, \ldots$, we will have the solutions of $\cos t$ and $\sin t$ or $\cos 2t$ and $\sin 2t$ or $\cos 3t$ and $\sin 3t$, etc. Periodic solutions for $b \neq 0$ happen only at proper values of (a, b) and hence we may assume there is a relationship between a and b expressed by a polynomial $a = f(b)$ of b.

$$a = n^2 + a_1 b + a_2 b^2 + a_3 b^3 + \cdots \tag{5.6}$$

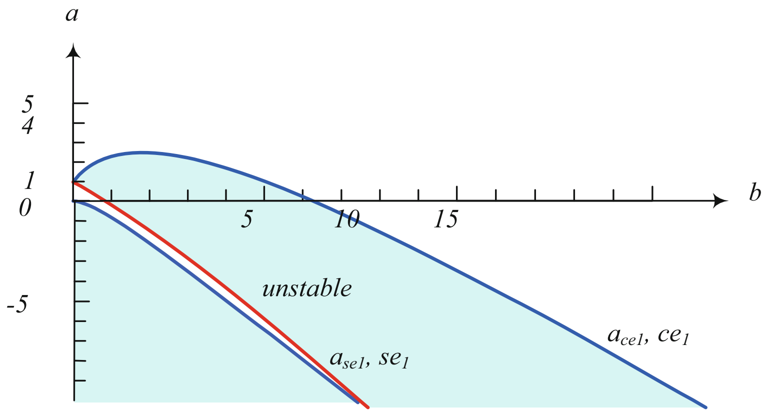

Fig. 5.1 Graphical illustration of the functions a_{ce_1} and a_{se_1} on which the Mathieu equation has the periodic solutions $ce_1(t, b)$ and $se_1(t, b)$

Setting $n = 1$, we assume the power series

$$a = 1 + a_1 b + a_2 b^2 + a_3 b^3 + \cdots \tag{5.7}$$

and because one of the solutions of Mathieu equation is to reduce to $\cos t$ when $b = 0$, we also assume a series solution in the form

$$x = \cos t + b f_1(t) + b^2 f_2(t) + b^3 f_3(t) + \cdots \tag{5.8}$$

Substituting (5.8) and (5.7) in the Mathieu equation (5.1)

$$\frac{d^2x}{dt^2} = -\cos t + b f_1'' + b^2 f_2'' + b^3 f_3'' + \cdots \tag{5.9}$$

$$\begin{aligned} ax &= \cos t + (f_1 + a_1 \cos t)\, b + (f_2 + a_2 \cos t + a_1 f_1)\, b^2 \\ &\quad + (f_3 + a_3 \cos t + a_1 f_2 + a_2 f_1)\, b^3 + \cdots \end{aligned} \tag{5.10}$$

$$\begin{aligned} (-2b \cos 2t)\, x &= -(\cos t + \cos 3t)\, b - 2b^2 f_1 \cos 2t \\ &\quad -2b^3 f_2 \cos 2t + \cdots \end{aligned} \tag{5.11}$$

and rearranging in powers of b,

$$\begin{aligned} 0 &= \left(f_1'' + f_1 - \cos t - \cos 3t + a_1 \cos t\right) b \\ &\quad + \left(f_2'' + f_2 + a_2 \cos t + a_1 f_1 - 2 f_1 \cos 2t\right) b^2 \\ &\quad + \left(f_3'' + f_3 + a_3 \cos t + a_1 f_2 + a_2 f_1 - 2 f_2 \cos 2t\right) b^3 + \cdots \end{aligned} \tag{5.12}$$

we set the coefficients of b^k equal to zero to determine the unknown functions $f_k(t)$ and unknown coefficients a_k. The first nonzero equation is to find $f_1(t)$.

$$\frac{d^2 f_1(t)}{dt^2} + f_1(t) - \cos 3t + (a_1 - 1)\cos t = 0 \tag{5.13}$$

The solution of the equation for zero initial conditions $f_1(0) = 0$, $f_1'(0) = 0$ would be:

$$f_1(t) = \frac{1}{8}\cos 3t + \frac{1}{8}\cos t + (a_1 - 1)\, t \sin t \tag{5.14}$$

The solution $x(t)$ of the Mathieu equation is to be periodic. Therefore, the coefficient of the nonperiodic term $t \sin t$ must be zero. Also, the coefficient of $\cos t$ in the solution must be unity for any b, so we may eliminate any terms including $\cos t$ from the solution. These simplifications determine a_1 and $f_1(t)$.

$$a_1 = 1 \tag{5.15}$$

$$f_1(t) = \frac{1}{8}\cos 3t \tag{5.16}$$

The second equation from the coefficients of b^2 is to find $f_2(t)$.

$$\frac{d^2 f_2(t)}{dt^2} + f_2(t) + f_1(t)(a_1 - 2\cos 2t) + a_2 \cos t = 0 \tag{5.17}$$

The solution of the equation for zero initial conditions $f_2(0) = 0$, $f_2'(0) = 0$, would be:

$$f_2(t) = \frac{1}{192}\cos 5t - \frac{1}{32}\cos 3t + \frac{5}{192}\cos t - \left(\frac{1}{2}a_1 + \frac{1}{16}\right) t \sin t \tag{5.18}$$

Again, to have the solution of the Mathieu equation to be periodic, the coefficient of the nonperiodic terms must be zero and $\cos t$ may be eliminated. Therefore,

$$a_2 = -\frac{1}{8} \tag{5.19}$$

$$f_2(t) = \frac{1}{192}\cos 5t - \frac{1}{64}\cos 3t \tag{5.20}$$

The third equation is to find $f_3(t)$.

$$\frac{d^2 f_3(t)}{dt^2} + f_3(t) + f_2(t)(a_1 - 2\cos 2t) + a_2 f_1(t) + a_3 \cos t = 0 \tag{5.21}$$

The solution for zero initial conditions $f_3(0) = 0$, $f_3'(0) = 0$, is:

$$f_3(t) = -\frac{1}{9216}\cos 7t + \frac{1}{1152}\cos 5t - \frac{1}{1536}\cos 3t - \frac{1}{9216}\cos t - \left(\frac{1}{2}a_1 + \frac{1}{128}\right) t \sin t \tag{5.22}$$

Setting the coefficient of the nonperiodic terms equal to zero and omitting any terms including $\cos t$ provides a_3 and $f_3(t)$.

$$a_3 = -\frac{1}{64} \tag{5.23}$$

$$f_3(t) = -\frac{1}{9216}\cos 7t + \frac{1}{1152}\cos 5t - \frac{1}{1536}\cos 3t \tag{5.24}$$

Substituting $f_k(t)$ in (5.8) gives us a 2π-periodic solution for the Mathieu equation. Let us show the solution by $ce_1(t, b)$ and call it the *cosine elliptic Mathieu function*. Continuing this procedure provides us with other a_k and $f_k(t)$. The resultant a and $ce_1(t, b)$ would be:

$$\begin{aligned} ce_1(t, b) = {} & \cos t - \frac{b}{8}\cos 3t + \frac{b^2}{64}\left(-\cos 3t + \frac{1}{3}\cos 5t\right) \\ & - \frac{b^3}{512}\left(\frac{1}{3}\cos 3t - \frac{4}{9}\cos 5t + \frac{1}{18}\cos 7t\right) \\ & + \frac{b^4}{4096}\left(\frac{11}{9}\cos 3t + \frac{1}{6}\cos 5t - \frac{1}{12}\cos 7t + \frac{1}{180}\cos 9t\right) \\ & + \cdots \end{aligned} \tag{5.25}$$

$$a_{ce_1} = 1 + b - \frac{1}{8}b^2 - \frac{1}{64}b^3 - \frac{1}{1536}b^4 + \frac{11}{36864}b^5 + \cdots \tag{5.26}$$

For a given b we find the characteristic number a_{ce_1} from (5.26) and the cosine elliptic Mathieu function from (5.25). As planned, the Mathieu function $ce_1(t, b)$ reduces to $\cos t$ for $b = 0$ and a to 1.

For $n = 1$, we may also have a second solution in the form

$$x = \sin t + b g_1(t) + b^2 g_2(t) + b^3 g_3(t) + \cdots \tag{5.27}$$

which reduces to $\sin t$ for $b = 0$. We substitute (5.27) and a new series for a

$$a = 1 + c_1 b + c_2 b^2 + c_3 b^3 + \cdots \tag{5.28}$$

in the Mathieu equation (5.1) and rearrange them in powers of b,

$$\begin{aligned}0 &= \left(g_1'' + g_1 + \sin t - \sin 3t + c_1 \sin t\right) b \\ &\quad + \left(g_2'' + g_2 + c_2 \sin t + c_1 g_1 - 2g_1 \cos 2t\right) b^2 \\ &\quad + \left(g_3'' + g_3 + c_3 \sin t + c_1 g_2 + c_2 g_1 - 2g_2 \cos 2t\right) b^3 + \cdots \end{aligned} \tag{5.29}$$

Setting the coefficients of b^k equal to zero, we solve the differential equations to determine the unknown functions $g_k(t)$ and the coefficients c_k. The first nonzero equation is to find $g_1(t)$ and c_1.

$$\frac{d^2 g_1(t)}{dt^2} + g_1(t) - \sin 3t + (c_1 + 1)\sin t = 0 \tag{5.30}$$

The solution of the equation for zero initial conditions $g_1(0) = 0$, $g_1'(0) = 0$, would be:

$$g_1(t) = -\frac{1}{8}\sin 3t - \frac{1}{2}\left(c_1 + \frac{1}{4}\right)\sin t + \frac{1}{2}(c_1 + 1)\, t \cos t \tag{5.31}$$

Having periodic solution $x(t)$ for the Mathieu equation requires the coefficient of the nonperiodic term $t\cos t$ to be zero. Also, the coefficient of $\sin t$ in the solution must be unity for any b. Hence, we may eliminate any terms including $\sin t$ from the solution. These facts determine c_1 and $g_1(t)$.

$$c_1 = -1 \tag{5.32}$$

$$g_1(t) = -\frac{1}{8}\sin 3t \tag{5.33}$$

The second equation from the coefficient of b^2 is to find $g_2(t)$.

$$\frac{d^2 g_2(t)}{dt^2} + g_2(t) + g_1(t)(c_2 - 2\cos 2t) + c_2 \sin t = 0 \tag{5.34}$$

The solution of the equation for zero initial conditions $g_2(0) = 0$, $g_2'(0) = 0$, would be:

$$\begin{aligned} g_2(t) = \frac{1}{192}\sin 5t + \frac{1}{64}\sin 3t - \frac{1}{2}\left(c_2 + \frac{13}{48}\right)\sin t \\ -\frac{1}{2}\left(c_2 + \frac{1}{8}\right) t \sin t \end{aligned} \tag{5.35}$$

To have the solution of the Mathieu equation to be periodic, the coefficient of the nonperiodic terms must be zero and $\sin t$ and $\cos t$ may be eliminated. Therefore,

$$c_2 = -\frac{1}{8} \tag{5.36}$$

$$g_2(t) = \frac{1}{192}\sin 5t + \frac{1}{64}\sin 3t \tag{5.37}$$

The third equation is to find $g_3(t)$.

$$\frac{d^2 g_3(t)}{dt^2} + g_3(t) + g_2(t)(c_1 - 2\cos 2t) + c_2 g_1(t) + c_3\cos t = 0 \tag{5.38}$$

The solution for zero initial conditions $g_3(0) = 0$, $g_3'(0) = 0$, is:

$$\begin{aligned} g_3(t) = &-\frac{1}{9216}\sin(7t) - \frac{1}{1152}\sin 5t - \frac{1}{1536}\sin 3t \\ &-\frac{1}{2}\left(c_3 - \frac{137}{4608}\right)\sin t + \frac{1}{2}\left(c_3 - \frac{1}{64}\right)t\cos t \end{aligned} \tag{5.39}$$

To have a periodic solution for the Mathieu equation, the coefficient of the nonperiodic terms will be set to zero. Also any terms including $\cos t$ and $\cos t$ may be omitted from the solution.

$$c_3 = \frac{1}{64} \tag{5.40}$$

$$g_3(t) = -\frac{1}{9216}\sin 7t - \frac{1}{1152}\sin 5t - \frac{5}{3072}\sin 3t \tag{5.41}$$

Substituting $g_k(t)$ in (5.27) gives us a 2π-periodic solution of the Mathieu equation. Let us show the solution by $se_1(t, b)$ called the *sine elliptic Mathieu function*.

$$\begin{aligned} se_1(t, b) = &\sin t - \frac{b}{8}\cos 3t + \frac{b^2}{64}\left(\sin 3t + \frac{1}{3}\sin 5t\right) \\ &-\frac{b^3}{512}\left(\frac{1}{3}\sin 3t + \frac{4}{9}\sin 5t + \frac{1}{18}\sin 7t\right) \\ &+\frac{b^4}{4096}\left(-\frac{11}{9}\cos 3t + \frac{1}{6}\cos 5t + \frac{1}{12}\cos 7t + \frac{1}{180}\cos 9t\right) \\ &+\cdots \end{aligned} \tag{5.42}$$

Continuing this procedure provides with other c_k and $g_k(t)$. The resultant a_{se_1} and $x(t)$ would be:

$$a_{se_1} = 1 - b - \frac{1}{8}b^2 + \frac{1}{64}b^3 - \frac{1}{1536}b^4 - \frac{11}{36864}b^5 + \cdots \tag{5.43}$$

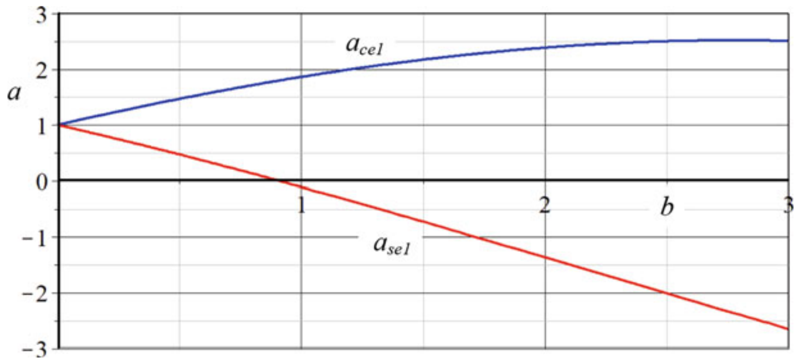

Fig. 5.2 Illustration of the cosine and sine elliptic Mathieu functions $ce_1(t, b)$ and $se_1(t, b)$ for small values of b

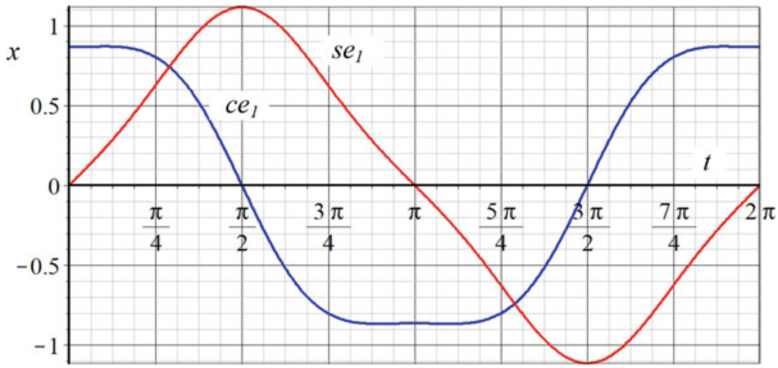

Fig. 5.3 Graphical illustration of the cosine and sine elliptical Mathieu function $ce_1(t, b)$ and $se_1(t, b)$

For a given b we find the characteristic number a_{se_1} from (5.43) and the associate sine elliptic Mathieu function from (5.42).

Inspection of Eqs. (5.43) and (5.42) indicates that the solutions (5.25) and (5.42) do not coexist except when $b = 0$. For $b \neq 0$, the values of a of (5.43) and (5.42) for which the periodic solutions exist are quite different (Richards 1983). The driven power series for a_{se_1} and a_{se_1} are good approximation only for small values of b. The developed solutions (5.26) and (5.43) cannot produce Fig. 5.1 for such a large values of b. Figure 5.2 depicts the characteristic numbers a_{se_1} and a_{se_1} and Fig. 5.3 illustrates the cosine and sine elliptic Mathieu functions $ce_1(t, b)$ and $se_1(t, b)$ based on the above developed solutions. ■

Example 172 A pendulum with harmonically moving support.

Imagine a simple pendulum of mass m hanging at the tip of a massless bar with length l from a pivot A. The point A is moving vertically with a harmonic function and the position of the pendulum is measured by the angle θ between l and gravitational acceleration vector $\mathbf{g}$ pointing downward, as is shown in Fig. 5.4.

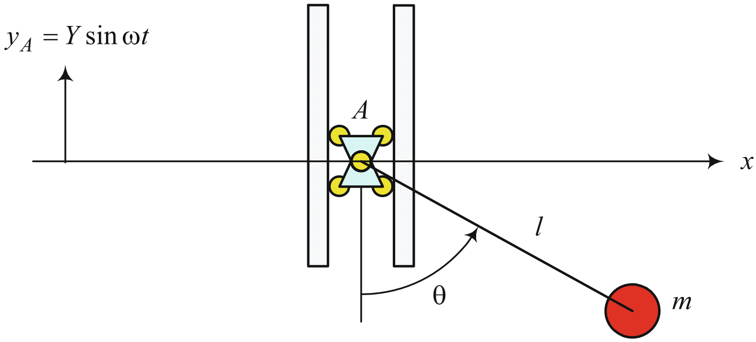

Fig. 5.4 A pendulum of mass m at the tip of a massless bar with length l and moving joint

$$y_A = Y \sin \omega t \tag{5.44}$$

The coordinates of m are:

$$x = l \sin \theta \tag{5.45}$$

$$y = y_A - l \cos \theta = Y \sin \omega t - l \cos \theta \tag{5.46}$$

and the velocity components of m are:

$$\dot{x} = \dot{\theta} l \cos \theta \qquad \dot{y} = Y \omega \cos \omega t + l \dot{\theta} \sin \theta \tag{5.47}$$

The kinetic energy K of the pendulum is:

$$\begin{aligned} K &= \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 \right) \\ &= \frac{1}{2} m \left(Y^2 \omega^2 \cos^2 t\omega + 2 Y l \omega \dot{\theta} \cos t\omega \sin \theta + l^2 \dot{\theta}^2 \right) \end{aligned} \tag{5.48}$$

and the potential energy P of the system is:

$$P = -mgy = -mg \left(Y \sin \omega t - l \cos \theta \right) \tag{5.49}$$

and therefore the Lagrangian $\mathcal{L}$ will be:

$$\begin{aligned} \mathcal{L} = K - P &= \frac{1}{2} m Y^2 \omega^2 \cos^2 \omega t + m Y l \omega \dot{\theta} \cos \omega t \sin \theta \\ &+ \frac{1}{2} m l^2 \dot{\theta}^2 + mg \left(Y \sin \omega t - l \cos \theta \right) \end{aligned} \tag{5.50}$$

Therefore, the equation of motion of the pendulum would be

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0 \tag{5.51}$$

$$\ddot{\theta} + \left(\frac{g}{l} - \frac{Y}{l}\omega^2 \sin \omega t\right) \sin \theta = 0 \tag{5.52}$$

To investigate the stability of the equilibrium solution $\theta = 0$ we would linearize the equation of motion about the equilibrium, and derive an equation of the Mathieu type.

$$\ddot{\theta} + \left(\frac{g}{l} - \frac{Y}{l}\omega^2 \sin \omega t\right) \theta = 0 \tag{5.53}$$

The vertically downward equilibrium position at $\theta = 0$ will be stable or unstable or showing a constant periodic oscillation depending on the values $a = g/l$ and $2b = Y\omega^2/l$. It is good to note that $\theta = 0$ is stable for $\omega = 0$. The vertically oscillation of the base is a way to make the pendulum oscillate and it means we are able to put the pendulum in oscillation with harmonically moving its base up and down.

There is another equilibrium position at $\theta = \pi$ for the pendulum. To study the stability of the vertically upward equilibrium, we may introduce a new variable

$$\varphi = \pi - \theta \tag{5.54}$$

to rewrite the equation of motion (5.52)

$$\ddot{\varphi} - \left(\frac{g}{l} - \frac{Y}{l}\omega^2 \sin \omega t\right) \sin \varphi = 0 \tag{5.55}$$

and linearize it around $\varphi = 0$ and find a similar Mathieu equation.

$$\ddot{\varphi} - \left(\frac{g}{l} - \frac{Y}{l}\omega^2 \sin \omega t\right) \varphi = 0 \tag{5.56}$$

The difference is that the new equation of motion is unstable for $\omega = 0$. Now, the vertically oscillation of the base is a way to make the pendulum stable, and it means we are able to keep the pendulum almost upward with harmonically moving its base up and down.

Example 173 Effect of b on functions $ce_1(t, b)$ and $se_1(t, b)$.

The sine and cosine elliptic Mathieu functions, Eqs. (5.42) and (5.25), are periodic functions of t and dependent on the parameter b. Figure 5.5 illustrates the cosine elliptic Mathieu function $ce_1(t, b)$ for $0 < t < 2\pi$ for different values of b. Figure 5.6 also illustrates the sine elliptic Mathieu functions $se_1(t, b)$ for

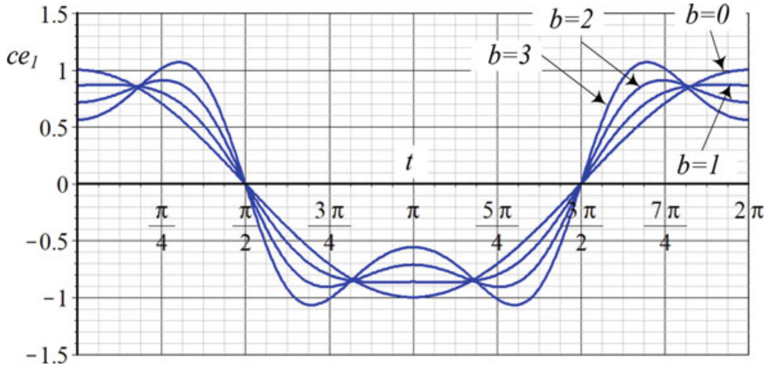

Fig. 5.5 Cosine elliptic Mathieu functions $ce_1(t, b)$ for $0 < t < 2\pi$ for $b = 0, 1, 2, 3$

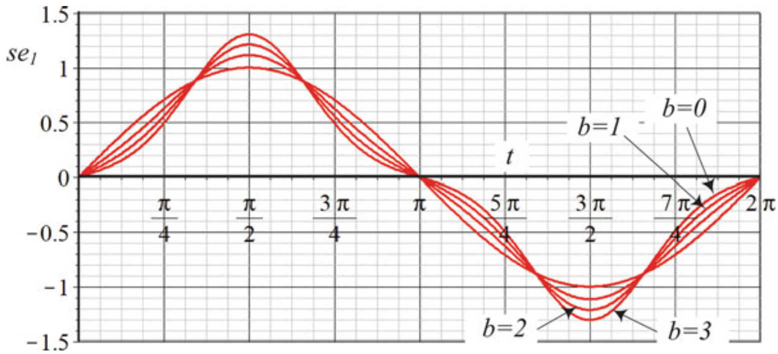

Fig. 5.6 Sine elliptic Mathieu functions $se_1(t, b)$ for $0 < t < 2\pi$ for $b = 0, 1, 2, 3$

$0 < t < 2\pi$ and different values of b. When $b = 0$ the function $ce_1(t, b)$ reduces to $\cos t$ and the function $se_1(t, b)$ reduces to $\sin t$. Figures 5.5 and 5.6 depict how the Mathieu functions deviate from the simple $\cos t$ and $\sin t$ for nonzero b. Figure 5.7 illustrates three-dimensional graphs of $ce_1(t, b)$ and $se_1(t, b)$.

Example 174 Eliminating $\cos t$ and $\sin t$ terms from the solution.

In development of the series solution $ce_1(t, b)$ of the Mathieu equation, we assumed series expansion (5.8) for the function $ce_1(t, b)$ and power series (5.7) for the characteristic number a_{ce_1}. Substituting (5.8) and (5.7) in the Mathieu equation and sorting for exponents of b^k gives us the series (5.12). All coefficients of b^k must be zero to make the series solution satisfy the equation. Each coefficients gives a differential equation to be solved for a_k and $f_k(t)$. The solution of the differential equations for zero initial conditions will have three types of terms. One of them is nonperiodic terms with a time t in the coefficient of a trigonometric function. By setting the coefficients of nonperiodic terms equal to zero, we calculate a_k and we eliminate the noperiodic terms. The second type are periodic functions of $\cos mt$ and $\sin mt$ with $m \neq 1$. These terms are the terms that make $ce_1(t, b)$ and we keep

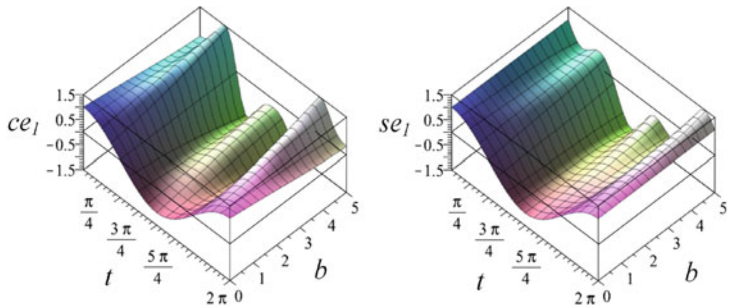

Fig. 5.7 Three-dimensional illustrations of the cosine and sine elliptic Mathieu functions, $ce_1\,(t, b)$ and $se_1\,(t, b)$

them. There is also a third type of terms that are simple functions $\cos t$ and $\sin t$. We dropped these terms with the justification that the cosine elliptic function $ce_1\,(t, b)$ must reduce to $\cos t$ for $b = 0$. In fact keeping the $\cos t$ and $\sin t$ terms or eliminating them is irrelevant. If we keep them, eventually we will have $ce_1\,(t, b)$ including a term of

$$\cos t + b\cos t + b^2\cos t + b^3\cos t + \cdots \tag{5.57}$$

or two terms of

$$\begin{aligned}&\left(\cos t + b\cos t + b^2\cos t + b^3\cos t + \cdots\right)\\&\quad+\left(b\sin t + b^2\sin t + b^3\sin t + \cdots\right)\end{aligned} \tag{5.58}$$

which will become $\cos t$ for $b = 0$ as when as we add all terms of the series for $b \neq 0$. Eliminating the terms $\cos t$ and $\sin t$ from the solutions of the coefficient differential equations makes it possible to cut the series and still have the solution built based on $\cos t$ regardless of the number of terms of the series and the value of b.

Example 175 Damped Mathieu equation.

Adding a damping term to the Mathieu equation makes a mass-damper-spring system where its spring coefficient changes periodically around the average a.

$$\frac{d^2x}{dt^2} + c\frac{dx}{dt} + (a - 2b\cos 2t)\,x = 0 \tag{5.59}$$

Substituting

$$x = e^{-ct}\,y\,(t) \tag{5.60}$$

into the equation, we obtain a new Mathieu equation without damping.

$$\frac{d^2 y}{dt^2} + (h - 2b \cos 2t)\, y = 0 \tag{5.61}$$

$$h = a - c^2 \tag{5.62}$$

Hence the solution and characteristic numbers of the Mathieu equation can be transformed to map the damped Mathieu equation.

Example 176 The importance of Mathieu equation.

Consider the following general nonlinear forced equation:

$$\ddot{x} + f(x, \dot{x}) = e(t) \qquad e(t + T) = e(t) \tag{5.63}$$

where $e(t)$ is a periodic excitation and $f(x, \dot{x})$ is a continued nonlinear differentiable function. We are interested in periodic solutions of the equation. Under the condition that $f(x, \dot{x})$ is bounded and asymptotically stable, the solution must ultimately lead to a periodic solution of which the least period is either the period T of the external force or an integral multiple of T. Suppose that we have a solution $x_0(t)$ for (5.63) such that

$$x_0(t + nT) = x_0(t) \tag{5.64}$$

A small variation from this periodic solution is denoted by y,

$$x(t) = x_0(t) + y(t) \qquad y(t) << 1 \tag{5.65}$$

then substitution of (5.65) in (5.63) leads to a variational equation,

$$\ddot{y} + \frac{\partial f(x_0, \dot{x}_0)}{\partial \dot{x}} \dot{y} + \frac{\partial f(x_0, \dot{x}_0)}{\partial x} y = 0 \tag{5.66}$$

in which we substitute $(x_0, \dot{x}_0)$ after differentiation. The coefficients of y and $\dot{y}$ are periodic functions of T. Equation (5.66) may be rewritten as

$$\ddot{y} + p(t)\, \dot{y} + q(t)\, y = 0 \tag{5.67}$$

$$p(t + T) = p(t) \qquad q(t + T) = q(t) \tag{5.68}$$

Introducing a new variable s

$$y = s \exp\left(-\frac{1}{2} \int p(t)\, dt\right) \tag{5.69}$$

we have

$$\ddot{s} + \left(q(t) - \frac{1}{2} \frac{dp}{dt} - \frac{1}{4} p^2(t)\right) s = 0 \tag{5.70}$$

This is a linear equation in which the coefficient of s is a periodic function of t. In many cases, it is a Mathieu equation or may be approximated by a Mathieu equation

$$\ddot{s} + (A - B\cos\omega t)\,s = 0 \tag{5.71}$$

and therefore, its stability determines the stability of solution $x_0\,(t)$ for Eq. (5.63).

As an example let us consider a Duffing equation

$$\ddot{x} + hx + kx^3 = F\cos\omega t \tag{5.72}$$

and assume we have a harmonic solution for the equation.

$$x_0\,(t) = A\,(\omega)\cos\omega t \tag{5.73}$$

We change the solution $x_0\,(t)$ with a little perturbed function $y\,(t)$,

$$x\,(t) = x_0\,(t) + y\,(t) \qquad y\,(t) << 1 \tag{5.74}$$

and substitute into the Duffing equation (5.72). Therefore,

$$\ddot{x}_0 + \ddot{y} + h\,(x_0 + y) + k\,(x_0 + y)^3 = F\cos\omega t \tag{5.75}$$

that will be expanded to

$$\begin{aligned} &\left(\ddot{x}_0 + hx_0 + kx_0^3 - F\cos\omega t\right) \\ &\quad + \ddot{y} + hy + 3kyx_0^2 + 3ky^2x_0 + ky^3 = 0 \end{aligned} \tag{5.76}$$

Because x_0 is a solution of the equation, the first parenthesis is zero and we will have an equation to determine the perturbation function y.

$$\ddot{y} + hy + 3kyx_0^2 + 3ky^2x_0 + ky^3 = 0 \tag{5.77}$$

Because y is assumed to be a very small perturbation, we may ignore the nonlinear terms to have the following equation:

$$\ddot{y} + hy + 3kyx_0^2 = 0 \tag{5.78}$$

$$\ddot{y} + hy + 3kyA^2\cos^2\omega t = 0 \tag{5.79}$$

This is a parametric equation which after using $\cos^2\omega t = \frac{1}{2}\cos 2t\omega + \frac{1}{2}$ will be a Mathieu equation.

$$\ddot{y} + \left(h + \frac{1}{2}3kA^2 + \frac{3}{2}kA^2\cos 2\omega t\right) y = 0 \tag{5.80}$$

The stability of the Mathieu equation determines the stability of the solution (5.73) for the Duffing equation (5.72).

Example 177 Mathieu equation determines the stability of Mathieu equation.
Assume we have a solution

$$x = x_0 (t) \tag{5.81}$$

for the Mathieu equation.

$$\frac{d^2x}{dt^2} + (a - 2b \cos 2t) x = 0 \tag{5.82}$$

Let us change the solution $x_0 (t)$ with a perturbed function $y (t)$,

$$x (t) = x_0 (t) + y (t) \qquad y (t) << 1 \tag{5.83}$$

and substitute into the Mathieu equation

$$\ddot{x}_0 + \ddot{y} + (a - 2b \cos 2t) (x_0 + y) = 0 \tag{5.84}$$

to determine an equation for $y (t)$.

$$\ddot{y} + (a - 2b \cos 2t) y = 0 \tag{5.85}$$

In order to determine the stability of the solution (5.81), we need to determine the long-term behavior or stability of y from (5.85), which is another Mathieu equation by itself. Therefore, Mathieu equation determines the stability of any solution of the Mathieu equation.

Example 178 First appearance of Mathieu equations.
Mathieu functions appeared in 1868 for the first time in analysis of vibrations of an elliptic drum expressed by elliptic-cylinder coordinates (Mathieu 1868; McLachlan 1947). The Helmholtz wave equation

$$\nabla\varphi + k^2\varphi = 0 \tag{5.86}$$

in two-dimensional plane will be

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + k^2\varphi = 0 \tag{5.87}$$

where k is the constant wave number dependent upon the properties of the medium. Applying a separation of variables

$$\varphi = g (u) f (v)$$

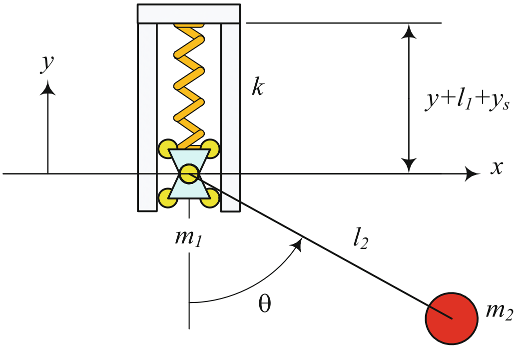

Fig. 5.8 The pendulum m_2 hanging to the mass m_1 attached to the ceiling with a spring of stiffness k

and expressing the wave equation in the elliptical cylinder coordinates (Jazar 2011)

$$x = F \cosh u \cos v \qquad y = F \sinh u \sin v \tag{5.88}$$

provides two equations for u and v.

$$\frac{d^2 f}{dv^2} - (a - 2b \cos 2v)\, f = 0 \tag{5.89}$$

$$\frac{d^2 g}{du^2} - (a - 2b \cosh 2u)\, g = 0 \tag{5.90}$$

Equation (5.89) is called circumferential or angular Mathieu equation, and Equation (5.90) is the radial Mathieu equation. Similarly, the solutions of Eq. (5.89) are called angular Mathieu functions, and the solutions of Eq. (5.90) are called radial Mathieu functions. Mathieu equations and Mathieu functions were introduced by their originator, Emile Leonard Mathieu (1835–1890), in 1868 when he determined the vibrational modes of a membrane (Mathieu 1868). We study only the circumferential Mathieu equation (5.89). Since Mathieu equation is a linear second-order homogenous ordinary differential equation, it has two linearly independent solutions. Therefore, every other solution of the equation is a linear combination of those two solutions.

Example 179 Pendulum subject to elastic guided suspension.

Figure 5.8 illustrates the pendulum m_2 hanging to the mass m_1 that is attached to a spring of stiffness k. The free length of k is l_2, the static deflection of the spring under the mass $m_1 + m_2$ is y_s and the vertical motion of m_1 is measured by y. The motion of mass m_1 is constrained by a vertical guides. The kinetic and potential energies of the system are:

$$K = \frac{1}{2}(m_1 + m_2)\,\dot{y}^2 + \frac{1}{2} m_2 l_2^2 \dot{\theta}^2 - m_2 l_2 \dot{y} \dot{\theta} \sin\theta \tag{5.91}$$

$$P = \frac{1}{2}ky^2 - m_2 g l_2 \cos\theta \tag{5.92}$$

Employing the Lagrange equation, we find two differential equations describing the motion of the system.

$$(m_1 + m_2)\,\ddot{y} - m_2 l_2 \ddot{\theta} \sin\theta - m_2 l_2 \dot{\theta}^2 \cos\theta + ky = 0 \tag{5.93}$$

$$-\ddot{y}\sin\theta + l_2\ddot{\theta} + g\sin\theta = 0 \tag{5.94}$$

Introducing the following nondimensionalized variables and parameters

$$u = \frac{y}{l_2} \qquad \tau = \omega t \qquad \omega = \sqrt{\frac{k}{m_1 + m_2}}$$

$$\gamma = \frac{y_s}{l_2} \qquad \alpha = \frac{m_2}{m_1 + m_2} < 1 \tag{5.95}$$

changes the equations to a new form.

$$u'' + u = \frac{\alpha\theta'^2\cos\theta - \alpha\,(\gamma + u)\sin^2\theta}{1 - \alpha\sin^2\theta} \tag{5.96}$$

$$\theta'' + \gamma\theta = \frac{-u\sin\theta + \alpha\theta'^2\sin\theta\cos\theta - \alpha\gamma\sin^3\theta}{1 - \alpha\sin^2\theta} + \gamma\,(\theta - \sin\theta) \tag{5.97}$$

The unperturbed motion of vertical oscillations of the system is governed by

$$\theta = 0 \qquad u = A\cos\tau \qquad A > 0 \tag{5.98}$$

To study the stability of this motion, we apply a small perturbation φ and v,

$$\theta = 0 + \varphi \qquad u = A\cos\tau + v \tag{5.99}$$

to obtain the perturbation equations:

$$\varphi'' + (\gamma + A\cos\tau)\,\varphi = 0 \tag{5.100}$$

$$v'' + v = 0 \tag{5.101}$$

The stability of unperturbed motion (5.98) is determined by the stability of the trivial solution of Eq. (5.100) which is a Mathieu equation (Starzhinskii 1980).

5.2 Periodic Solutions of Order $n \in \mathbb{N}$

The periodic solutions of the Mathieu equation (5.1) for $a = n^2$ with periods π and 2π on the boundary of stability and instability in the (a, b)-plane may be found similar to the case of $n = 1$ in previous section. For any value of n we will have one cosine elliptic and one sine elliptic functions that each one would have its own characteristic number.

The Mathieu functions and their characteristic numbers in power series of b are:

$$a_{ce_0}(b) = -\frac{1}{2}b^2 + \frac{7b^4}{128} - \frac{29b^6}{2304} + \frac{68687b^8}{18874368} - \frac{123707b^{10}}{104857600} + \frac{8022167579b^{12}}{19568944742400} + \cdots \tag{5.102}$$

$$a_{ce_1}(b) = a_{se_1}(-b) = 1 + b - \frac{b^2}{8} - \frac{b^3}{64} - \frac{b^4}{1536} + \frac{11b^5}{36864} + \frac{49b^6}{589824} + \frac{55b^7}{9437184} - \frac{83b^8}{35389440} + - \frac{12121b^9}{15099494400} - \frac{114299}{1630745395200}b^{10} + \cdots \tag{5.103}$$

$$a_{ce_2}(b) = 4 + \frac{5b^2}{12} - \frac{763b^4}{13824} + \frac{1002401b^6}{79626240} - \frac{1669068401b^8}{458647142400} + \frac{4363384401463b^{10}}{3698530556313600} + \cdots \tag{5.104}$$

$$a_{se_2}(b) = 4 - \frac{b^2}{12} + \frac{5b^4}{13824} - \frac{289b^6}{79626240} + \frac{21391b^8}{458647142400} - \frac{2499767b^{10}}{3698530556313600} + \cdots \tag{5.105}$$

$$a_{ce_3}(b) = a_{se_3}(-b) = 9 + \frac{b^2}{16} + \frac{b^3}{64} + \frac{13b^4}{20480} - \frac{5b^5}{16384} - \frac{1961b^6}{23592960} - \frac{609b^7}{104857600} + \frac{4957199b^8}{2113929216000} + \frac{872713b^9}{1087163596800} + \frac{421511b^{10}}{6012954214400} + \cdots \tag{5.106}$$

$$a_{ce_4}(b) = 16 + \frac{b^2}{30} + \frac{433b^4}{864000} - \frac{5701b^6}{2721600000} - \frac{112236997b^8}{2006581248000000} + \frac{8417126443b^{10}}{3160365465600000000} + \cdots \quad (5.107)$$

$$a_{se_4}(b) = 16 + \frac{b^2}{30} - \frac{317b^4}{864000} + \frac{10049b^6}{2721600000} - \frac{93824197b^8}{2006581248000000} + \frac{21359366443b^{10}}{3160365465600000000} + \cdots \quad (5.108)$$

$$a_{ce_5}(b) = a_{se_5}(-b) = 25 + \frac{b^2}{48} + \frac{11b^4}{774144} + \frac{b^5}{147456} + \frac{37b^6}{891813888} - \frac{7b^7}{339738624} + \frac{63439b^8}{201364441399296} - \frac{1b^9}{2130840649728} - \frac{60609509b^{10}}{5799295912299724800} + \cdots \quad (5.109)$$

$$a_{ce_6}(b) = 36 + \frac{b^2}{70} + \frac{187b^4}{43904000} + \frac{6743617b^6}{92935987200000} - \frac{2337184771b^8}{23315780468736000000} + \frac{107856094183b^{10}}{1595835640971264000000000} + \cdots \quad (5.110)$$

$$a_{se_6}(b) = 36 + \frac{b^2}{70} + \frac{187b^4}{43904000} - \frac{5861633b^6}{92935987200000} + \frac{2825925629b^8}{23315780468736000000} + \frac{45361065433b^{10}}{1595835640971264000000000} + \cdots \quad (5.111)$$

$$ce_0(t,b) = 1 - \frac{b}{2}\cos 2t + \frac{b^2}{32}\cos 4t + \frac{b^3}{128}\left(\cos 2t - \frac{1}{9}\cos 6t\right) + \frac{b^4}{512}\left(\frac{29}{9}\cos 4t + \frac{1}{144}\cos 8t\right) + \cdots \quad (5.112)$$

$$ce_1(t,b) = \cos t - \frac{b}{8}\cos 3t + \frac{b^2}{64}\left(-\cos 3t + \frac{1}{3}\cos 5t\right)$$
$$-\frac{b^3}{512}\left(\frac{1}{3}\cos 3t - \frac{4}{9}\cos 5t + \frac{1}{18}\cos 7t\right)$$
$$+\frac{b^4}{4096}\left(\frac{11}{9}\cos 3t + \frac{1}{6}\cos 5t - \frac{1}{12}\cos 7t + \frac{1}{180}\cos 9t\right)$$
$$+\cdots \tag{5.113}$$

$$se_1(t,b) = \sin t - \frac{b}{8}\cos 3t + \frac{b^2}{64}\left(\sin 3t + \frac{1}{3}\sin 5t\right)$$
$$-\frac{b^3}{512}\left(\frac{1}{3}\sin 3t + \frac{4}{9}\sin 5t + \frac{1}{18}\sin 7t\right)$$
$$+\frac{b^4}{4096}\left(-\frac{11}{9}\cos 3t + \frac{1}{6}\cos 5t + \frac{1}{12}\cos 7t + \frac{1}{180}\cos 9t\right)$$
$$+\cdots \tag{5.114}$$

$$ce_2(t,b) = \cos 2t + \frac{b}{12}\cos 4t + \frac{b^2}{96}\left(-\frac{19}{3}\cos 2t + \frac{1}{4}\cos 6t\right)$$
$$-\frac{b^3}{1152}\left(\frac{11}{4}\cos 4t + \frac{1}{20}\cos 8t\right) + \cdots \tag{5.115}$$

$$se_2(t,b) = \sin 2t - \frac{b}{12}\sin 4t + \frac{b^2}{96}\left(-\frac{1}{3}\sin 2t + \frac{1}{4}\sin 6t\right)$$
$$+\frac{b^3}{1536}\left(\cos 4t - \frac{1}{15}\cos 8t\right) + \cdots \tag{5.116}$$

$$ce_3(t,b) = \cos 3t - \frac{b}{16}\cos 5t + \frac{b^2}{64}\left(-\frac{5}{8}\cos 3t + \frac{1}{10}\cos 7t\right)$$
$$-\frac{b^3}{512}\left(\cos 3t - \frac{11}{80}\cos 5t + \frac{1}{90}\cos 9t\right) + \cdots \tag{5.117}$$

$$se_3(t,b) = \sin 3t - \frac{b}{16}\sin 5t + \frac{b^2}{64}\left(-\frac{5}{8}\sin 3t + \frac{1}{10}\sin 7t\right)$$

$$+\frac{b^3}{512}\left(\sin 3t + \frac{11}{80}\sin 5t - \frac{1}{90}\sin 9t\right) + \cdots \tag{5.118}$$

$$\begin{aligned} ce_4(t,b) = \cos 4t &+ \frac{b}{4}\left(\frac{1}{3}\cos 2t - \frac{1}{5}\cos 6t\right) \\ &+ \frac{b^2}{96}\left(\frac{34}{75}\cos 4t + \frac{1}{10}\cos 8t\right) \\ &+ \frac{b^3}{10}\left(\frac{7}{2880}\cos 2t + \frac{29}{28800}\cos 6t - \frac{1}{8064}\cos 10t\right) + \cdots \end{aligned} \tag{5.119}$$

$$\begin{aligned} se_4(t,b) = \sin 4t &+ \frac{b}{4}\left(\frac{1}{3}\sin 2t - \frac{1}{5}\sin 6t\right) \\ &+ \frac{b^2}{96}\left(-\frac{34}{75}\sin 4t + \frac{1}{10}\sin 8t\right) \\ &+ \frac{b^3}{10}\left(-\frac{1}{160}\sin 2t + \frac{29}{28800}\sin 6t - \frac{1}{8064}\sin 10t\right) + \cdots \end{aligned} \tag{5.120}$$

Proof First we set n to be an integer number. We know that for $b = 0$, the Mathieu equation reduces to $\ddot{x} + n^2 x = 0$ with solutions $\cos nt$ and $\sin nt$. Then, the solution of the Mathieu equation for nonzero b will be found by introducing series solutions based on $\cos nt$ and $\sin nt$, each for its own characteristic number a as a power series of b.

$$x = \cos nt + bf_1(t) + b^2 f_2(t) + b^3 f_3(t) + \cdots \tag{5.121}$$

$$a = n^2 + a_1 b + a_2 b^2 + a_3 b^3 + \cdots \tag{5.122}$$

$$x = \sin nt + bg_1(t) + b^2 g_2(t) + b^3 g_3(t) + \cdots \tag{5.123}$$

$$a = n^2 + c_1 b + c_2 b^2 + c_3 b^3 + \cdots \tag{5.124}$$

We substitute each pair of

$$x = \cos nt + \sum_{k=1}^{\infty} b^k f_k(t) \qquad a = n^2 + \sum_{k=1}^{\infty} a_k b^k \tag{5.125}$$

or

$$x = \sin nt + \sum_{k=1}^{\infty} b^k g_k(t) \qquad a = n^2 + \sum_{k=1}^{\infty} c_k b^k \tag{5.126}$$

in the Mathieu equation (5.1). Then, we sort the resultant equation for powers of b^k and equate the coefficients of b^k to zero. Each coefficient will be a differential equation to determine $f_k(t)$ or $g_k(t)$. Eliminating the nonperiodic terms of the solutions determines a_k and c_k. This method will generate the solutions (5.102)–(5.120). If b is small enough, these formulae may be used with good approximation as the all terms of higher than $O(b^6)$ practically have no contribution in the limit value of the series. But as b becomes larger these terms will affect the value considerably so in general, $a = a(b)$ must be calculated using other ways such as continued fractions, determinant, or energy-rate methods that we will study in the future sections.

The periodic solutions of the Mathieu equation (5.1) can be expressed by Fourier series.

$$ce_{2k}(t, b) = \sum_{j=0}^{\infty} A_{2j}(b) \cos 2jt \tag{5.127}$$

$$ce_{2k+1}(t, b) = \sum_{j=0}^{\infty} A_{2j+1}(b) \cos(2j+1)t \tag{5.128}$$

$$se_{2k+1}(t, b) = \sum_{j=0}^{\infty} B_{2j+1}(b) \sin(2j+1)t \tag{5.129}$$

$$se_{2k+2}(t, b) = \sum_{j=0}^{\infty} B_{2j+2}(b) \sin(2j+2)t \tag{5.130}$$

The Mathieu functions are associated to their characteristic numbers.

$$a = a_{ce_{2k}} \qquad \pi\text{-periodic, even solution} \tag{5.131}$$

$$a = a_{ce_{2k+1}} \qquad 2\pi\text{-periodic, even solution} \tag{5.132}$$

$$a = a_{se_{2k+1}} \qquad 2\pi\text{-periodic, odd solution} \tag{5.133}$$

$$a = a_{se_{2k+2}} \qquad \pi\text{-periodic, odd solution} \tag{5.134}$$

The functions are either π-periodic or 2π-periodic. All Mathieu functions are continuous and convergent for real t. Hence, the series may be differentiated or integrated term by term. According to the nature of the series solutions given by the classical substitution approach, unacceptably huge calculation is generally needed to ensure sufficient convergence to get reasonable approximate answers.

To summarize the series solution method of the Mathieu equation,

$$\frac{d^2x}{dt^2} + (a - 2b\cos 2t)\,x = 0 \tag{5.135}$$

let us repeat that the periodic solutions of the Mathieu equation are well expressed in (a, b)-plane on transition curves $a = a(b)$, branching from $a = n^2$, $n = 1, 2, 3, \cdots$. We are looking for solutions in the form

$$ce_n(b, t) = \cos nt + \sum_{i=1}^{\infty} b^i f_i(t) \qquad a = n^2 + \sum_{i=1}^{\infty} a_{ni} b^i \tag{5.136}$$

$$se_n(b, t) = \sin nt + \sum_{i=1}^{\infty} b^i g_i(t) \qquad a = n^2 + \sum_{i=1}^{\infty} c_{ni} b^i \tag{5.137}$$

The coefficients a_{ni} and c_{ni} are obtained by imposing the functions $ce_n(b, t)$ and $se_n(b, t)$ to be periodic of t. Substituting (5.136) and (5.137) into Eq. (5.135) we obtain the differential equations to determine the functions $f_i(t)$ and $g_i(t)$.

$$\frac{d^2 f_i}{dt^2} + n^2 f_i - 2 f_{i-1} \cos 2t + \sum_{k=1}^{i-1} a_{i-k} f_k = 0 \tag{5.138}$$

$$\frac{d^2 g_i}{dt^2} + n^2 g_i - 2 g_{i-1} \cos 2t + \sum_{k=1}^{i-1} c_{i-k} g_k = 0 \tag{5.139}$$

■

Example 180 Characteristic numbers are boundaries of stability.

Figure 5.9 illustrates the (a, b)-plane showing the line of the characteristic numbers a_{ce_0}, a_{ce_1}, a_{se_s}, a_{ce_2}, a_{se_2}, for $-3 < b < 3$. The series equations of the characteristic numbers provide acceptable approximation for small values of b. Any point on the transition lines will allow the Mathieu equation to have a periodic response. The transition lines separate the stable and unstable regions of the plane. To examine the stability of each region, we may pick a point in the interested region and investigate the response of the Mathieu equation. In the figure, the point 1, $(a, b) = (1.85819999, 1)$, is in an unstable region, point 2, $(a, b) = (1.859108038, 1)$, on a transition line and hence is periodic, and point 3, $(a, b) = (1.85979999, 1)$, is in an stable region. Figures 5.10, 5.11, and 5.12 depict the responses of the Mathieu equation from the initial conditions $x(0) = 1$, $\dot{x}(0) = 0$ for points 1, 2, and 3, respectively.

Example 181 Characteristic curves.

The power series expressions of the characteristic numbers are not suitable to determine the periodic lines for large values of b. We will show how to determine these curves for large values of b by other approximation methods in the future. The stability chart of the Mathieu equation is shown in Fig. 5.13 indicating the elliptic sine and cosine functions as well as the name of the transition curves. The stability chart is symmetric about the a-axis. The curves $a_{ce_{2k}}$ and $a_{se_{2k+2}}$ which are

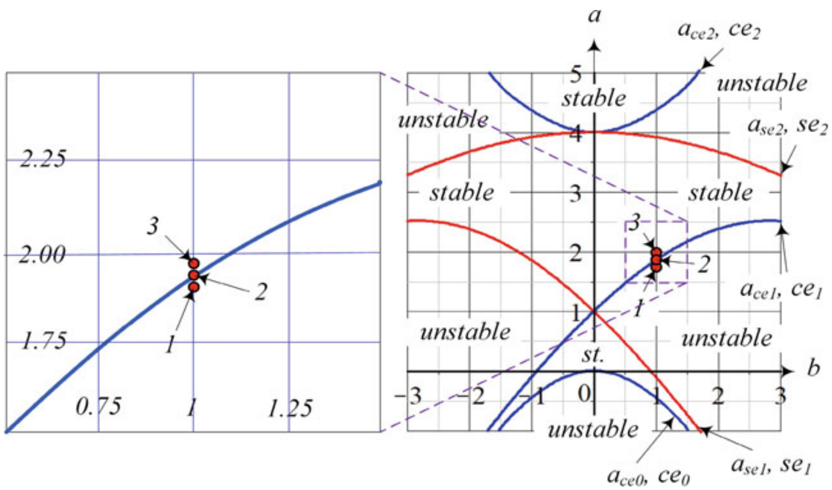

Fig. 5.9 The lines of the characteristic numbers $a_{ce_0}, a_{ce_1}, a_{se_s}, a_{ce_2}, a_{se_2}$, of the Mathieu equation for $-3 < b < 3$ in the (a, b)-plane

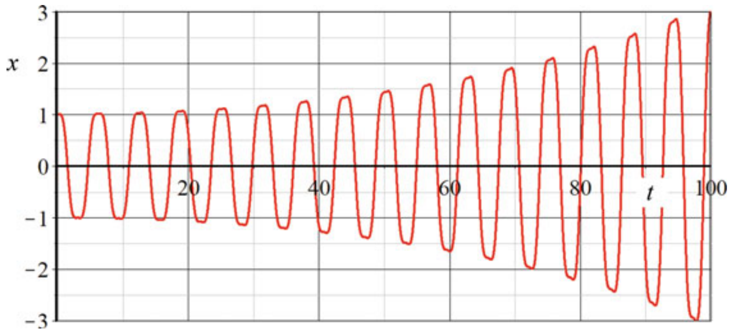

Fig. 5.10 Time response of the Mathieu equation for $(a, b) = (1.85819999, 1)$, a point in unstable region

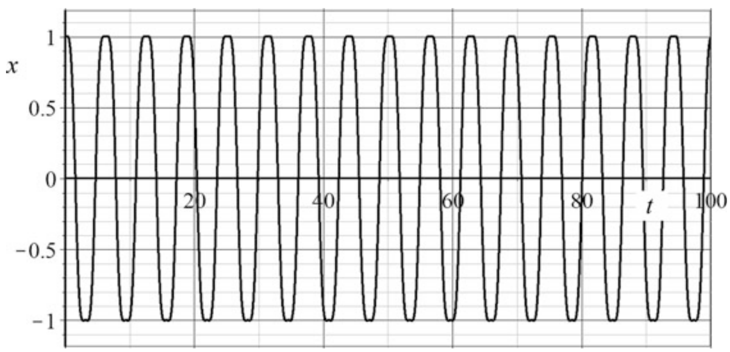

Fig. 5.11 Time response of the Mathieu equation for $(a, b) = (1.859108038, 1)$, a point on characteristic line

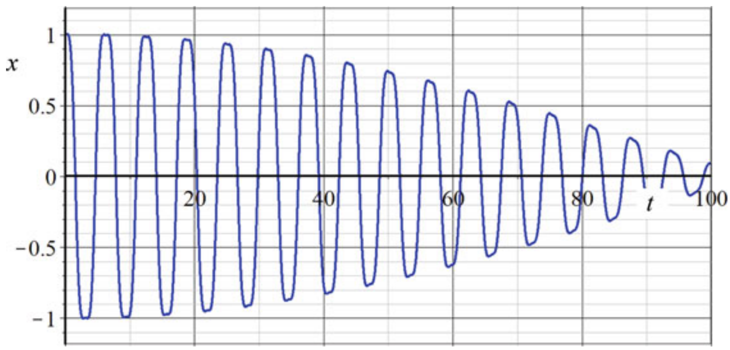

Fig. 5.12 Time response of the Mathieu equation for $(a, b) = (1.859108, 1)$, a point in stable region

π-periodic, are symmetric, but curves $a_{ce_{2k+1}}$ and $a_{se_{2k+1}}$ which are 2π-periodic, are asymmetric, however the overall stability chart remains symmetric as is shown in Fig. 5.14.

Except for a_{ce_0}, all other transition curves will cross the b-axis twice and therefore every characteristic equation has two zeros. For the functions of even order $a_{ce_{2k}}$ and $a_{se_{2k+2}}$ the zeros are equal but opposite. The zeros of $a_{ce_{2k+1}}$ are equal but opposite to those of $a_{se_{2k+1}}$.

Example 182 Problems leading to periodic differential equations.

The ordinary linear differential equations of the second-order are classified according to the number and their singularities. When such an equation has only two separate regular singularities, it takes a more compact form when a trigonometric substitution is made for the independent variable. Consider the equation

$$x\,(x-1)\,y'' + \left(x - \frac{1}{2}\right) y' - (a + bx)\,y = 0 \tag{5.140}$$

which has regular singularities at $x = 0$, $x = 1$, becomes,

$$\frac{d^2y}{dz^2} + (4a + 2b + 2b\cos 2z)\,y = 0 \tag{5.141}$$

if we change the variable$x$ to z.

$$x = \cos^2 z \tag{5.142}$$

The new form of the equation not only is more compact, but it also has no singularities. The new equation is a Mathieu equation.

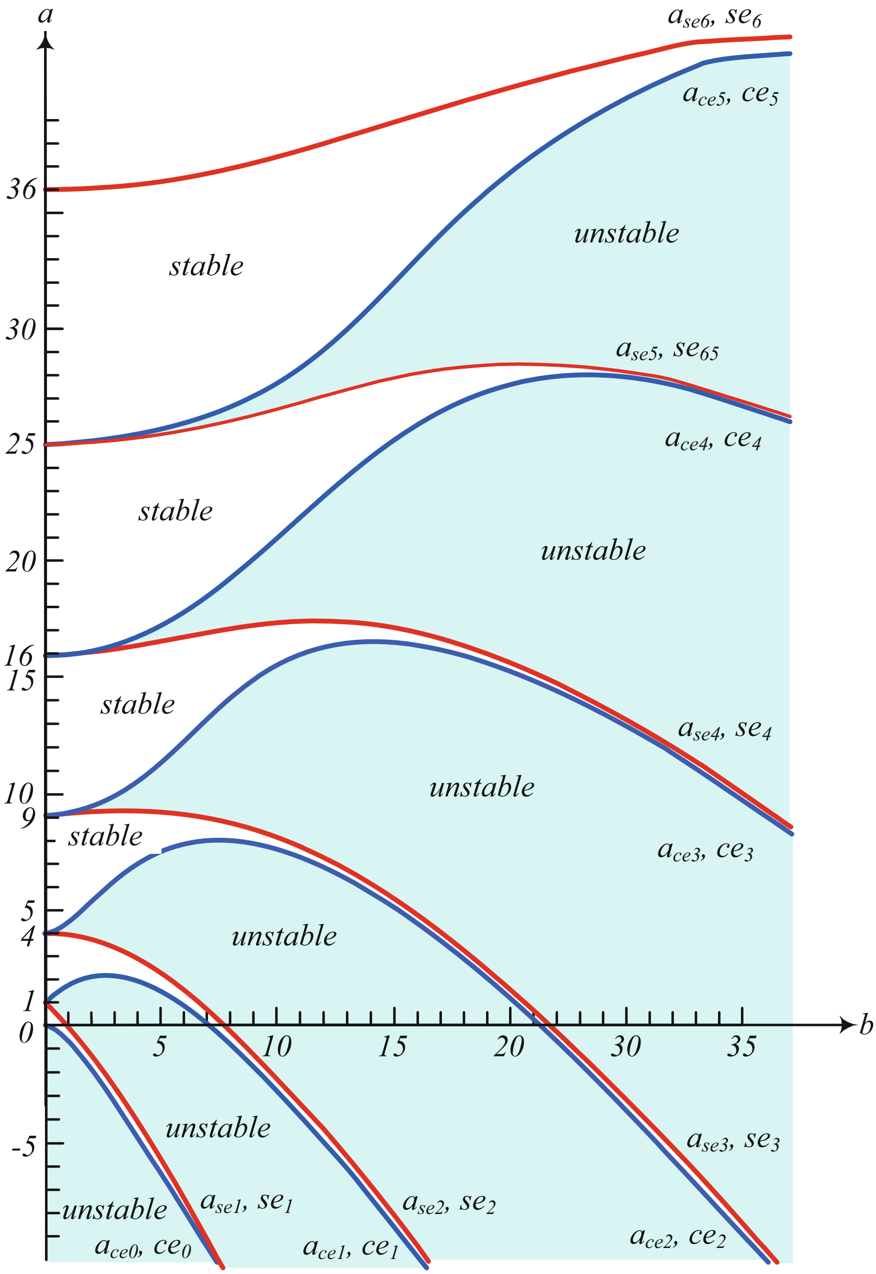

Fig. 5.13 The stability chart of the Mathieu equation

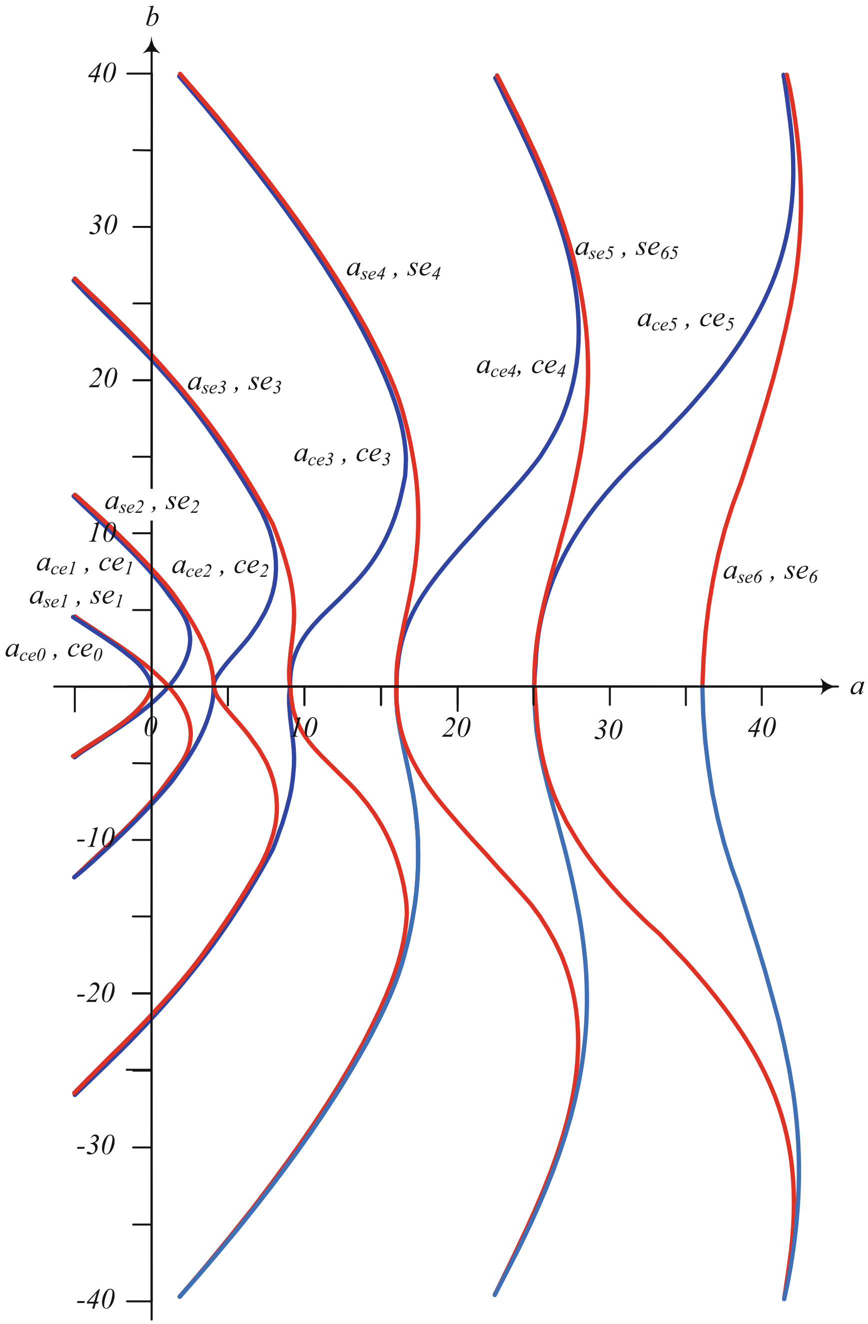

Fig. 5.14 The stability chart of the Mathieu equation, showing the symmetry and behavior of the Mathieu functions

$$\frac{d^2y}{dz^2} + (A + 2B\cos 2z)\,y = 0 \tag{5.143}$$

$$A = 4a + 2b \qquad B = -b \tag{5.144}$$

Example 183 McLachlan's numerical values of characteristic numbers.

McLachlan (1947) reported the data of Table 5.1 to a reasonable accuracy to be used for design as well as plotting the Mathieu stability chart. Figure 5.14 illustrates the table graphically. These data have been calculated by hand and using continued fractions. We will review the continued fractions method in the future sections and show how to computerize the method to increase accuracy of this table. There are other methods besides the continued fractions such as recursive, determinant, and energy-rate methods that are able to provide more accurate results.

Example 184 ★Mathieu functions of fractional order.

If $a = m^2$, $m \notin \mathbb{N}$, $a \neq n^2$, n being an integer, the solution of Mathieu equation will be $x = \cos mt$, and $x = \sin mt$ for $b = 0$. Then the solutions for $b \neq 0$ will be of the form

$$ce_m\,(b,t) = \cos mt + \sum_{i=1}^{\infty} b^i f_i\,(t) \tag{5.145}$$

and

$$se_m\,(b,t) = \sin mt + \sum_{i=1}^{\infty} b^i g_i\,(t) \tag{5.146}$$

with a common characteristic number (Richards 1983).

$$a = m^2 + \sum_{i=1}^{\infty} a_i b^i \tag{5.147}$$

It can be shown that the characteristic number a is the same for both $ce_m\,(b,t)$ and $se_m\,(b,t)$, and therefore both solutions can coexist. Because the fundamental solutions $\cos mt$ and $\sin mt$ are linearly independent, a complete solution will be of the form

$$x\,(t) = A\,ce_m\,(b,t) + B\,se_m\,(b,t) \tag{5.148}$$

where A and B will be determined by initial conditions. The functions $ce_m\,(b,t)$ and $se_m\,(b,t)$ are bounded but not necessarily periodic. Therefore, they are the solutions in the stable regions of the Mathieu stability diagram. Although these solutions are usually nonperiodic, some of them are periodic and it means there are other characteristics but not transition periodic curves in the stable region of the stability charts. Those curves are indicating points where both sides of them are stable. We may put $m = p + q$, with p integer and $0 < q < 1$. If q can be expressed

Table 5.1 McLachlan data for characteristic numbers of Mathieu equation

b	0	1	2	3	4
a_{ce0}	0	−0.4551386	−1.5139569	−2.8343919	−4.2805188
a_{se1}	1	−0.1102488	−1.3906765	−2.7853797	−4.2591829
a_{ce1}	1	1.8591081	2.3791999	2.5190391	2.3180082
a_{se2}	4	3.9170248	3.6722327	3.2769220	2.7468810
a_{ce2}	4	4.3713010,	5.1726651	6.0451969	6.8290748
a_{se3}	9	9.0477393	9.14062277	9.2231328	9.2614461
a_{ce3}	9	9.0783688	9.3703225	9.91155063	10.6710271
a_{se4}	16	16.0329701	16.1276880	16.2727012	16.4520353
a_{ce4}	16	16.0338323	16.1412038	16.3387207	16.6468189
a_{se5}	25	25.0208408	25.0833490	25.1870798	25.3305449
a_{ce5}	25	25.0208543	25.0837778	25.1902855	25.3437576
a_{se6}	36	36.0142899	36.0572070	36.1288712	36.22944114

b	5	6	7	8
a_{ce0}	−5.8000460	−7.3688308	−8.9737425	−10.6067292
a_{se1}	−5.7900806	−7.3639110	−8.9712024	−10.6053681
a_{ce1}	1.8581875	1.2142782	0.4383491	−0.4359436
a_{se2}	2.0994604	1.3513812	0.5175454	−0.3893618
a_{ce2}	7.4491097	7.8700645	8.0866231	8.1152388
a_{se3}	9.2363277	9.1379058	8.9623855	8.7099144
a_{ce3}	11.5488320	12.4656007	13.3584213	14.1818804
a_{se4}	16.6482199	16.8446016	17.0266608	17.1825278
a_{ce4}	17.0965817	17.6887830	18.4166087	19.2527051
a_{se5}	25.5108160	25.7234107	25.9624472	26.2209995
a_{ce5}	25.5499717	25.8172720	26.1561202	26.5777533
a_{se6}	36.3588668	36.5170667	36.7035027	36.9172131

b	9	10	12	14
a_{ce0}	−12.2624142	−13.93698	−17.3320660	−20.7760553
a_{se1}	−12.2616617	−13.9365525	−17.3319184	−20.7760004
a_{ce1}	−1.3867016	−2.3991424	−4.5701329	−6.8934005
a_{se2}	−1.3588101	−2.3821582	−4.5635399	−6.8907007
a_{ce2}	7.9828432	7.7173698	6.8787369	5.7363123
a_{se3}	8.3831192	7.9860691	7.0005668	5.7926295
a_{ce3}	14.9036797	15.5027844	16.3015349	16.5985405
a_{se4}	17.3030110	17.3813807	17.3952497	17.2071153
a_{ce4}	20.1609264	21.1046337	22.9721275	24.6505951
a_{se5}	26.4915472	26.7664264	27.3000124	27.7697667
a_{ce5}	27.0918661	27.7037687	29.2080550	31.0000508
a_{se6}	37.1566950	37.4198588	38.0060087	38.6484719

(continued)

Table 5.1 (continued)

b	16	18	20	24
a_{ce0}	−24.2586795	−27.7728422	−31.3133901	−38.4589732
a_{se1}	−24.2586578	−27.7728332	−31.3133862	−38.4589724
a_{ce1}	−9.33523671	−11.8732425	−14.4913014	−19.9225956
a_{se2}	−9.3341097	−11.8727265	−14.4910633	−19.9225403
a_{ce2}	4.3712326	2.8330567	1.15422829	−2.5397657
a_{se3}	4.3978962	2.8459917	1.1607057	−2.5380779
a_{ce3}	16.4868843	16.0619754	15.3958109	13.5228427
a_{se4}	16.8186837	16.2420804	15.4939776	13.5527965
a_{ce4}	26.0086783	26.9877664	27.5945782	27.8854408
a_{se5}	28.136359	28.3738582	28.4682213	28.2153594
a_{ce5}	32.9308951	34.8530587	36.6449897	39.5125519
a_{se6}	39.3150108	39.9723511	40.5896641	41.6057099

b	28	32	36	40
a_{ce0}	−45.6733696	−52.9422230	−60.2555679	−67.6061522
a_{se1}	−45.6733694	−52.9422229	−60.2555679	−67.6061522
a_{ce1}	−25.5617471	−31.3651544	−37.3026391	−43.3522753
a_{se2}	−25.5617329	−31.3651505	−37.3026380	−43.3522749
a_{ce2}	−6.5880630	−10.9143534	−15.4667703	−20.2079408
a_{se3}	−6.5875850	−10.9142090	−15.4667243	−20.2079254
a_{ce3}	11.1110798	8.2914962	5.1456363	1.7296491
a_{se4}	11.1206227	8.2946721	5.1467375	1.7300456
a_{ce4}	27.2833082	26.0624482	24.3785094	22.3252763
a_{se5}	27.4057488	26.1083526	24.3960665	22.3321485
a_{ce5}	41.2349503	41.9535112	41.9266646	41.3497544
a_{se6}	42.2248415	42.3939428	42.118356	41.4330052

as α/β where α and β are integers with no common factors, then $ce_m(b,t)$ and $se_m(b,t)$ are periodic with period $\pi\beta$ and $2\pi\beta$ depending on α to be even or odd. The characteristic number for fractional m is:

$$a_m = m^2 + \frac{1}{2}\frac{1}{m^2-1}b^2 + \frac{1}{32}\frac{5m^2+7}{\left(m^2-4\right)\left(m^2-1\right)^3}b^4 + \\ + \frac{1}{64}\frac{9m^4+58m^2+29}{\left(m^2-9\right)\left(m^2-4\right)\left(m^2-1\right)^5}b^6 + \cdots \qquad (5.149)$$

For values of a and b sitting in an unstable region of the stability chart the solution of the Mathieu equation are functions of fractional order of the form

$$x(b,\pm t) = \exp(\pm\gamma t)\ \varphi(\pm t) \qquad (5.150)$$

with $\varphi(\pm t)$ periodic with period π and 2π.

Example 185 ★Mathieu-Duffing equation.
The linear Mathieu equation

$$\frac{d^2x}{dt^2} + (a - 2b\cos 2t)\,x = 0 \tag{5.151}$$

has the stiffness coefficient $(a - 2b\cos 2t)$ harmonically alternating around the average a between $a + b$ and $a - b$. Every physical system that we model by linear differential equations is indeed nonlinear. When we model a parametric physical system, depending on the relative importance of the parametric or the nonlinear term, we ignore the nonlinearity or the parametric term and model the system with either Mathieu equation (5.151) or a Duffing equation.

$$\frac{d^2x}{dt^2} + ax + cx^3 = F\cos\omega t \tag{5.152}$$

The parametric equation of Mathieu accepts superposition and its main characteristic is the stability chart in (a, b)-plane in which the stable and unstable regions are separated by the transition periodicity curves. The Duffing equation is a nonlinear equation which its main characteristic is its frequency response with an amplitude as a function of the excitation frequency ω. Duffing equation shows chaotic behavior based on the value of its coefficients, initial conditions, and the forcing term.

There are occasions that neither nonlinearity or periodic terms may be ignored and in those occasions the system must be modeled by Mathieu-Duffing equation.

$$\frac{d^2x}{dt^2} + (a - 2b\cos 2t)\,x + cx^3 = 0 \tag{5.153}$$

The Mathieu-Duffing equation, which yet to be studied in detail, shows all behaviors of Mathieu and Duffing equations combined (Esmailzadeh and Jazar 1997).

As an example we may recall the transverse vibration of a longitudinally excited beam studied in Example 67. Consider a beam with length l, initial stretch x_0, mass density ρ, cross-sectional area A, flexural rigidity EI, and initial axial tension F, that is axially driven by a periodic displacement x_d.

$$x_d = a\cos\omega t \tag{5.154}$$

Such a system is shown in Fig. 5.15. The transverse vibration of the beam is expressed by a partial differential equation (Esmailzadeh and Jazar 1997).

$$EI\frac{\partial^4 y}{\partial x^4} + \rho A\frac{\partial^2 y}{\partial t^2} = F\frac{\partial^2 y}{\partial x^2} \qquad -\frac{l}{2} < x < \frac{l}{2} \tag{5.155}$$

Introducing the following nondimensional parameters and variables

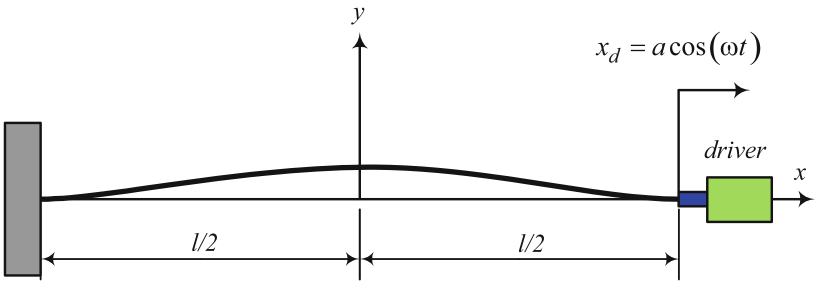

Fig. 5.15 A simple beam with length l, under harmonically axial excitation

$$Y = \frac{y}{l} \qquad X = \frac{x}{l} \qquad X_0 = \frac{x_0}{l} \qquad \tau = \frac{t}{\sqrt{\rho A l^4 / (EI)}}$$

$$P = \frac{Fl^2}{EI} \qquad H = \frac{h}{l} \qquad X_d = \frac{x_d}{l} \tag{5.156}$$

and applying a separation method

$$Y = f(\tau)\ g(X) \qquad g(X) = H\cos(\pi X) \tag{5.157}$$

provides us with an ordinary differential equation for the temporal function $f(\tau)$ (Esmailzadeh and Jazar 1998, McLachlan 1956).

$$f''(\tau) + \pi^2\left(\pi^2 + X_0 + X_d\cos(\omega\tau)\right) f(\tau) + \frac{1}{4}\pi^6 H^2 f^3(\tau) = 0 \tag{5.158}$$

It is a Mathieu-Duffing equation of the form

$$f'' + (k_1 + k_2\cos(\omega\tau))\, f + k_3 f^3 = 0 \tag{5.159}$$

with

$$k_1 = \pi^2\left(\pi^2 + X_0\right) \tag{5.160}$$

$$k_2 = \pi^2 X_d \cos(\omega\tau) \tag{5.161}$$

$$k_3 = \frac{1}{4}\pi^6 H^2 \tag{5.162}$$

Example 186 Transforming linear parametric second-order differential equation to integral equation.

Consider the linear second-order parametric and forced differential equation

$$\ddot{x} + a(t)\,\dot{x} + b(t)\,x = f(t) \tag{5.163}$$

with a general boundary conditions.

$$x(0) = C_1 \qquad x(t_0) = C_2 \tag{5.164}$$

The linearity of the equation allows us to write the solution to be made of two functions

$$x = u + v \tag{5.165}$$

where v is the solution of the inhomogeneous equation with zero initial conditions

$$\ddot{v} + a(t)\,\dot{v} + b(t)\,v = f(t) \tag{5.166}$$

$$v(0) = 0 \qquad \dot{v}(0) = 0 \tag{5.167}$$

and u is the solution of the homogenous equation with nonzero boundary conditions.

$$\ddot{u} + a(t)\,\dot{u} + b(t)\,u = 0 \tag{5.168}$$

$$u(0) = C_1 \qquad u(t_0) = C_2 - v(t_0) \tag{5.169}$$

To develop the complete solutions, we begin with the homogenous boundary conditions:

$$\ddot{u} + a(t)\,\dot{u} + b(t)\,u = 0 \tag{5.170}$$

$$u(0) = C_1 \qquad u(t_0) = C_2 \tag{5.171}$$

A second-order linear equation will have two principal linearly independent solutions. Let u_1 and u_2 be the two principal solutions of the homogeneous equation (5.170) and we use these initial conditions:

$$u_1(0) = 1 \qquad \dot{u}_1(0) = 0 \tag{5.172}$$

$$u_2(0) = 0 \qquad \dot{u}_2(0) = 1 \tag{5.173}$$

Because of the linearity, every solution of (5.170) can be expressed by

$$u(t) = A\,u_1(t) + B\,u_2(t) \tag{5.174}$$

where A and B are constants to be determined by the initial and boundary conditions.

$$A = C_1 \qquad B = \frac{C_2 - C_1 u_1(t_0)}{u_2(t_0)} \qquad u_2(t_0) \neq 0 \tag{5.175}$$

Hence the solution of (5.168) will be (5.174) with proper coefficients.

$$A = C_1 \qquad B = \frac{C_2 - v(t_0) - C_1 u_1(t_0)}{u_2(t_0)} \qquad u_2(t_0) \neq 0 \tag{5.176}$$

The homogenous case of Eq. (5.166) is the same as homogenous equation (5.170) having the same principal solutions u_1 and u_2. To develop the solution of the inhomogeneous equation (5.166), Lagrange invented the method of variation of parameters (Esmailzadeh et al. 1996, Simmons 1991). Let us assume the solution of the inhomogeneous equation to be

$$v(t) = f_1(t)\; u_1(t) + f_2(t)\; u_2(t) \tag{5.177}$$

where $f_1(t)$ and $f_2(t)$ are functions of t to be determined such that (5.177) becomes a proper solution of (5.166). Derivative of (5.177) is:

$$\dot{v}(t) = f_1\,\dot{u}_1 + f_2\,\dot{u}_2 + \dot{f}_1\,u_1 + \dot{f}_2\,u_2 \tag{5.178}$$

Let us set a condition on $f_1(t)$ and $f_2(t)$ such that

$$\dot{f}_1\,u_1 + \dot{f}_2\,u_2 = 0 \tag{5.179}$$

to have

$$\dot{v}(t) = f_1\,\dot{u}_1 + f_2\,\dot{u}_2 \tag{5.180}$$

and

$$\ddot{v}(t) = f_1\,\ddot{u}_1 + f_2\,\ddot{u}_2 + \dot{f}_1\,\dot{u}_1 + \dot{f}_2\,\dot{u}_2 \tag{5.181}$$

Substituting (5.177), (5.180), and (5.181) in (5.168), we have

$$\begin{aligned} f(t) = \ddot{v} + a(t)\,\dot{v} + b(t)\,v = f_1\left(\ddot{u}_1 + a(t)\,\dot{u}_1 + b(t)\,u_1\right) \\ + f_2\left(\ddot{u}_2 + a(t)\,\dot{u}_2 + b(t)\,u_2\right) + \dot{f}_1\,\dot{u}_1 + \dot{f}_2\,\dot{u}_2 \end{aligned} \tag{5.182}$$

and because u_1 and u_2 satisfy the homogenous equation, this relation reduces to:

$$f(t) = \dot{f}_1\,\dot{u}_1 + \dot{f}_2\,\dot{u}_2 \tag{5.183}$$

Now we have two conditions (5.179) and (5.183) on $\dot{f}_1(t)$ and $\dot{f}_2(t)$ to calculate the functions $f_1(t)$ and $f_2(t)$. The solution exists if the determinant of the coefficients is nonzero. The determinant $W(t)$ of this form is called Wronskian (Bellman and Kalaba 1965; Bellman 1970).

$$W(t) = \begin{vmatrix} u_1 & u_2 \\ \dot{u}_1 & \dot{u}_2 \end{vmatrix} = u_1 \dot{u}_2 - \dot{u}_1 u_2 \tag{5.184}$$

Hence,

$$\dot{f}_1(t) = \frac{\begin{vmatrix} 0 & u_2 \\ f(t) & \dot{u}_2 \end{vmatrix}}{W(t)} \qquad \dot{f}_2(t) = \frac{\begin{vmatrix} u_1 & 0 \\ \dot{u}_1 & f(t) \end{vmatrix}}{W(t)} \tag{5.185}$$

and we derive $f_1(t)$ and $f_2(t)$.

$$f_1(t) = -\int_0^t \frac{u_2(s) \;\; f(s)}{W(s)} ds \tag{5.186}$$

$$f_2(t) = \int_0^t \frac{u_1(s) \;\; f(s)}{W(s)} ds \tag{5.187}$$

Now the solution of (5.166) from (5.177) would be

$$v = \int_0^t \frac{u_1(s) u_2(t) - u_2(s) u_1(t)}{W(s)} f(s) \; ds \tag{5.188}$$

It is traditional to show

$$G(t, s) = \frac{u_1(s) u_2(t) - u_2(s) u_1(t)}{W(s)} \tag{5.189}$$

and

$$v = \int_0^t G(t, s) \;\; f(s) \;\; ds \tag{5.190}$$

Therefore, the solution of (5.163) with the boundary value conditions (5.164) is:

$$x = u + v = A\, u_1(t) + B\, u_2(t) + \int_0^t G(t, s) \;\; f(s) \;\; ds \tag{5.191}$$

$$A = C_1 \qquad B = \frac{C_2 - v(t_0) - C_1 u_1(t_0)}{u_2(t_0)} \tag{5.192}$$

To determine Wronskian $W(t)$, we take a derivative of (5.184).

$$\begin{aligned}
\frac{dW}{dt} &= \frac{d}{dt}\begin{vmatrix} u_1 & u_2 \\ \dot{u}_1 & \dot{u}_2 \end{vmatrix} = \begin{vmatrix} \dot{u}_1 & \dot{u}_2 \\ \dot{u}_1 & \dot{u}_2 \end{vmatrix} + \begin{vmatrix} u_1 & u_2 \\ \ddot{u}_1 & \ddot{u}_2 \end{vmatrix} \\
&= \begin{vmatrix} u_1 & u_2 \\ -a(t)\,\dot{u}_1 - b(t)\,u_1 & -a(t)\,\dot{u}_2 - b(t)\,u_2 \end{vmatrix} \\
&= -a(t)\,u_1\dot{u}_2 - b(t)\,u_1u_2 + a(t)\,u_2\dot{u}_1 + b(t)\,u_2u_1 \\
&= -a(t)\begin{vmatrix} u_1 & u_2 \\ \dot{u}_1 & \dot{u}_2 \end{vmatrix} = -a(t)\,W
\end{aligned} \tag{5.193}$$

therefore,

$$W(t) = W(0)\exp\left(-\int_0^t a(s)\,ds\right) \tag{5.194}$$

and hence,

$$f_1(t) = -\int_0^t u_2(s)\ f(s)\exp\left(\int_0^s a(z)\,dz\right)ds \tag{5.195}$$

$$f_2(t) = \int_0^t u_1(s)\ f(s)\exp\left(\int_0^s a(z)\,dz\right)ds \tag{5.196}$$

Now that we have the Wronskian, we can rearrange the solution of

$$\ddot{x} + a(t)\,\dot{x} + b(t)\,x = f(t) \tag{5.197}$$

$$x(0) = C_1 \qquad x(t_0) = C_2 \tag{5.198}$$

as

$$\begin{aligned}
x &= Au_1(t) + Bu_2(t) \\
&\quad + \int_0^t f(s)\exp\left(\int_0^s a(z)\,dz\right)(u_1(s)\,u_2(t) - u_2(s)\,u_1(t))\,ds
\end{aligned} \tag{5.199}$$

Example 187 ★Continued integral solution of Mathieu-Duffing equation.
Let us rewrite the Mathieu-Duffing equation (5.153) as

$$\frac{d^2x}{dt^2} + n^2x = b\left(2x\cos 2t - ex^3\right) = bf(x,t) \tag{5.200}$$

$$e = \frac{c}{b} \tag{5.201}$$

Knowing that $x_1 = \cos nt$ and $x_2 = \sin nt$ are the principal solutions of

$$\frac{d^2x}{dt^2} + n^2 x = 0 \tag{5.202}$$

$$x_1(0) = 1 \qquad \dot{x}_1(0) = 0 \tag{5.203}$$

$$x_2(0) = 0 \qquad \dot{x}_2(0) = 1 \tag{5.204}$$

from (5.199) we have:

$$\begin{aligned} x &= A\cos nt + B\sin nt \\ &\quad + b\int_0^t f(x,s)\left(\cos ns\ \sin nt - \sin ns\ \cos nt\right)\ ds \end{aligned} \tag{5.205}$$

Substituting $f(x,s)$ makes

$$\begin{aligned} x &= A\cos nt + B\sin nt \\ &\quad + b\int_0^t x(s)\left(2\cos 2s - ex^2(s)\right)\left(\cos ns\ \sin nt - \sin ns\ \cos nt\right) ds \\ &= w(t) + b\int_0^t \left(2\cos 2s - ex^2(s)\right) x(s)\sin n(t-s)\ ds \end{aligned} \tag{5.206}$$

where

$$w(t) = A\cos nt + B\sin nt \tag{5.207}$$

Let us call $w(t)$ the first approximate solution $x_0(t)$.

$$x_0 = w(t) = A\cos nt + B\sin nt \tag{5.208}$$

Substituting x_0 into the right-hand side of (5.206) we find the second approximate solution.

$$\begin{aligned} x_1 &= w(t) + b\int_0^t \left(2\cos 2s_1 - ex^2(s_1)\right)\sin n(t-s_1)\, x(s_1)\, ds_1 \\ &= w(t) + b\int_0^t \sin n(t-s_1)\, w(s_1)\left(2\cos 2s_1 - ew^2(s_1)\right) ds_1 \end{aligned} \tag{5.209}$$

Another substitution x_1 for $w(s_1)$ provides us with the third approximate solution.

$$\begin{aligned} x_2 &= w(t) + b\int_0^t \sin n(t-s_1)\left(w(s_1) + bJ_1\right) \times \\ &\quad \left(2\cos 2s_1 - ew^2(s_1) - 2ew(s_1)\, bJ_1 - eb^2 J_1^2\right) ds_1 \end{aligned} \tag{5.210}$$

$$J_1 = \int_0^{s_1} \sin n\,(s_1 - s_2)\, w\,(s_2) \left(2\cos 2s_2 - ew^2\,(s_2)\right) ds_2 \tag{5.211}$$

The next approximate solution will be

$$x_3 = w\,(t) + b \int_0^{t} \sin n\,(t - s_1)\,(w\,(s_1) + bJ_2) \times$$
$$\left(2\cos 2s_1 - ew^2\,(s_1) - 2ebw\,(s_1)\, J_2 - eb^2 J_2^2\right) ds_1 \tag{5.212}$$

$$J_2 = \int_0^{s_1} \sin n\,(s_1 - s_2)\,(w\,(s_2) + bJ_1)$$
$$\left(2\cos 2s_2 - ew^2\,(s_2) - 2ebw\,(s_2)\, J_1 - eb^2 J_1^2\right) ds_2 \tag{5.213}$$

$$J_1 = \int_0^{s_2} \sin n\,(s_2 - s_3)\, w\,(s_3) \left(2\cos 2s_3 - ew^2\,(s_3)\right) ds_3 \tag{5.214}$$

Keep substituting provides us with continued integral approximate solution of the Mathieu-Duffing equation.

$$x_k = w\,(t) + b \int_0^{t} \sin n\,(t - s_1)\,(w\,(s_1) + bJ_2) \times \tag{5.215}$$
$$\left(2\cos 2s_1 - ew^2\,(s_1) - 2ebw\,(s_1)\, J_{k-1} - eb^2 J_{k-1}^2\right) ds_1$$

$$J_i = \int_0^{s_{k-i}} \sin n\,(s_{k-i+1} - s_{k-i})\,(w\,(s_{k-i+1}) + bJ_{i-1}) \tag{5.216}$$
$$\left(2\cos 2s_{k-i+1} - ew\,(s_{k-i+1})\,(w\,(s_{k-i+1}) + 2bJ_{i-1}) - eb^2 J_{n-1}^2\right) ds_{k-i+1}$$
$$1 < i < k \tag{5.217}$$

$$J_1 = \int_0^{s_{i-1}} \sin n\,(s_k - s_{k-1})\, w\,(s_k) \left(2\cos 2s_k - ew^2\,(s_k)\right) ds_k \tag{5.218}$$
$$J_0 = 0 \tag{5.219}$$

5.3 Recursive Method

Recursive equations for the coefficients of Mathieu functions is the right way to computerize generating Fourier series expression of Mathieu functions. Substituting Fourier series

$$ce_{2k}(t,b) = \sum_{j=0}^{\infty} A_{2j}(b)\cos 2jt \tag{5.220}$$

$$ce_{2k+1}(t,b) = \sum_{j=0}^{\infty} A_{2j+1}(b)\cos(2j+1)t \tag{5.221}$$

$$se_{2k+1}(t,b) = \sum_{j=0}^{\infty} B_{2j+1}(b)\sin(2j+1)t \tag{5.222}$$

$$se_{2k+2}(t,b) = \sum_{j=0}^{\infty} B_{2j+2}(b)\sin(2j+2)t \tag{5.223}$$

into the Mathieu equation

$$\frac{d^2x}{dt^2} + (a - 2b\cos 2t)\,x = 0 \tag{5.224}$$

we derive the following recursive relations for the Fourier coefficients A_{2j}, A_{2j+1}, B_{2j+1}, B_{2j+2}.

For $ce_{2k}(t,b)$ with characteristic number $a_{ce_{2k}}$

$$aA_0 - bA_2 = 0 \tag{5.225}$$

$$(a-4)\,A_2 - b\,(2A_0 + A_4) = 0 \tag{5.226}$$

$$\left(a - (2j)^2\right)A_{2j} - b\left(A_{2j-2} + A_{2j+2}\right) = 0 \qquad j \geq 2 \tag{5.227}$$

For $ce_{2k+1}(t,b)$ with characteristic number $a_{ce_{2k+1}}$

$$(a-1)\,A_1 - b\,(A_1 + A_3) = 0 \tag{5.228}$$

$$\left(a - (2j+1)^2\right)A_{2j+1} - b\left(A_{2j-1} + A_{2j+3}\right) = 0 \qquad j \geq 1 \tag{5.229}$$

For $se_{2k+1}(t,b)$ with characteristic number $a_{se_{2k+1}}$

$$(a-1)\,B_1 + b\,(B_1 - B_3) = 0 \tag{5.230}$$

$$\left(a - (2j+1)^2\right)B_{2j+1} - b\left(B_{2j-1} + B_{2j+3}\right) = 0 \qquad j \geq 1 \tag{5.231}$$

For $se_{2k+2}(t,b)$ with characteristic number $a_{se_{2k+2}}$

$$(a-4)\,B_2 - bB_4 = 0 \tag{5.232}$$

$$\left(a - (2j)^2\right) B_{2j} - b\left(B_{2j-2} + B_{2j+2}\right) = 0 \qquad j \geq 2 \tag{5.233}$$

Proof To derive these recursive equations, we may assume a solution for the Mathieu equation in terms of Fourier series

$$x(t, b) = ce_k(t, b) + se_{k+1}(t, b) \tag{5.234}$$

$$= \sum_{j=0}^{\infty} \left(A_j(b) \cos jt + B_{j+1}(b) \sin(j+1)t\right) \tag{5.235}$$

and separate the even and odd solutions to study them independently.

$$x(t, b) = x_1(t, b) + x_2(t, b) \tag{5.236}$$

$$x_1(t, b) = ce_k(t, b) = \sum_{j=0}^{\infty} A_j(b) \cos jt \tag{5.237}$$

$$x_2(t, b) = se_{k+1}(t, b) = \sum_{j=0}^{\infty} B_{j+1}(b) \sin(j+1)t \tag{5.238}$$

To calculate $x_1(t, b) = ce_k(t, b)$ we also need its second derivative

$$\ddot{x}_1(t, b) = -\sum_{j=0}^{\infty} j^2 A_j(b) \cos jt \tag{5.239}$$

to substitute in the Mathieu equation.

$$-\sum_{j=0}^{\infty} j^2 A_j \cos jt + (a - 2b\cos 2t) \sum_{j=0}^{\infty} A_j \cos jt = 0 \tag{5.240}$$

$$\sum_{j=0}^{\infty} \left(a - j^2\right) A_j \cos jt - 2b\cos 2t \sum_{j=0}^{\infty} A_j \cos jt = 0 \tag{5.241}$$

We may insert $\cos 2t$ into the summation by the trigonometric identity

$$\cos A \cos B = \frac{1}{2}\cos(A - B) + \frac{1}{2}\cos(A + B) \tag{5.242}$$

and hence,

$$\sum_{j=0}^{\infty}\left(a-j^{2}\right)A_{j}\cos jt-b\sum_{j=0}^{\infty}A_{j}\left(\cos\left(j-2\right)t+\cos\left(j+2\right)t\right)=0 \tag{5.243}$$

expanding the summation

$$\sum_{j=0}^{\infty}\left(a-j^{2}\right)A_{j}\cos jt$$
$$-b\sum_{j=0}^{\infty}A_{j}\cos\left(j-2\right)t-b\sum_{j=0}^{\infty}A_{j}\cos\left(j+2\right)t=0 \tag{5.244}$$

and renumbering them, we have:

$$\sum_{j=0}^{\infty}\left(a-j^{2}\right)A_{j}\cos jt$$
$$-b\sum_{j=-2}^{\infty}A_{j+2}\cos jt-b\sum_{j=2}^{\infty}A_{j-2}\cos jt=0 \tag{5.245}$$

To have all summations start at $j=2$, we extract a few terms

$$aA_{0}+\left(a-1\right)A_{1}\cos t+\sum_{j=2}^{\infty}\left(a-j^{2}\right)A_{j}\cos jt$$
$$-bA_{0}\cos 2t-bA_{1}\cos t-bA_{2}-bA_{3}\cos t$$
$$-b\sum_{j=2}^{\infty}A_{j+2}\cos jt-b\sum_{j=2}^{\infty}A_{j-2}\cos jt=0 \tag{5.246}$$

but because of the terms aA_0 and $A_0\cos 2t$, we need to take another step to extract the term involving A_0 from the last summation.

$$\left(aA_{0}-bA_{2}\right)+\left(\left(a-1\right)A_{1}-b\left(A_{1}+A_{3}\right)\right)\cos t$$
$$+\left(\left(a-4\right)A_{2}-b\left(2A_{0}+A_{4}\right)\right)\cos 2t \tag{5.247}$$
$$+\sum_{j=3}^{\infty}\left(a-j^{2}\right)A_{j}-b\left(A_{j+2}+A_{j-2}\right)\cos jt=0 \tag{5.248}$$

This equation must be correct for any t so, the constant term as well as the coefficients of cosine functions must all be zero. For the even coefficients A_{2j}, we have

$$aA_0 - bA_2 = 0 \tag{5.249}$$

$$(a-4)\,A_2 - b\,(2A_0 + A_4) = 0 \tag{5.250}$$

$$\left(a - j^2\right) A_j - b\left(A_{j+2} + A_{j-2}\right) = 0 \qquad j = 4, 6, 8, \cdots \tag{5.251}$$

and for the odd coefficients A_{2j+1} we have

$$(a-1)\,A_1 - b\,(A_1 + A_3) = 0 \tag{5.252}$$

$$\left(a - j^2\right) A_j - b\left(A_{j+2} + A_{j-2}\right) = 0 \qquad j = 3, 5, 7, \cdots \tag{5.253}$$

These are consistent with recursive equations of $ce_{2k}\,(t, b)$ and $ce_{2k+1}\,(t, b)$ in Eqs. (5.225)–(5.229). The recursive equations make infinite set of linear equations to determine A_i and B_i.

The recursive equations for the coefficient of $x_2\,(t, b) = se_{k+1}\,(t, b)$ can be proven similarly to derive Equations (5.230)–(5.233). ■

Example 188 Negative b.

If in the Mathieu equation (5.224) we replace t with $\pi/2 - t$, it becomes:

$$\frac{d^2x}{dt^2} + (a + 2b\cos 2t)\,x = 0 \tag{5.254}$$

Then, the solutions of the Mathieu equation with period π, 2π, and b being negative are obtained if we substitute (5.220)–(5.223) in (5.254).

$$\begin{aligned} ce_{2k}\,(t, -b) &= (-1)^k\, ce_{2k}\left(\frac{\pi}{2} - t, b\right) \\ &= (-1)^k \sum_{j=0}^{\infty} (-1)^j\, A_{2j} \cos 2jt \end{aligned} \tag{5.255}$$

$$\begin{aligned} ce_{2k+1}\,(t, -b) &= (-1)^k\, ce_{2k+1}\left(\frac{\pi}{2} - t, b\right) \\ &= \sum_{j=0}^{\infty} (-1)^k\, A_{2j+1} \cos(2j+1)\,t \end{aligned} \tag{5.256}$$

$$\begin{aligned} se_{2k+1}\,(t, -b) &= (-1)^k\, se_{2k+1}\left(\frac{\pi}{2} - t, b\right) \\ &= (-1)^k \sum_{j=0}^{\infty} B_{2j+1} \sin(2j+1)\,t \end{aligned} \tag{5.257}$$

$$se_{2k+2}(t,-b) = (-1)^k\, se_{2k+2}\left(\frac{\pi}{2}-t,b\right)$$
$$= (-1)^k \sum_{j=0}^{\infty} B_{2j+2} \sin(2j+2)\,t \tag{5.258}$$

The multiplier $(-1)^k$ is to insure that when $b=0$ the functions reduce to $\cos nt$ and $\sin nt$.

Example 189 Recursive sequences.

Recursive equations, although very simple to understand and easy to develop, is so practical when we have one. Any sequence is written in the form

$$\{x_k\} = x_1, x_2, x_3, \ldots, x_k, \ldots \tag{5.259}$$

where x_k will be calculated based on the previous members of the sequence, calculated by an equation. The equation is the key to determine the following members. Assume there exist numbers a_1, a_2, a_3, ..., such that starting from a certain number j, we determine the members $j+k$ from a recursion relation.

$$x_{j+k} = a_1 x_{j+k-1} + a_2 x_{j+k-2} + \cdots + a_k x_j \qquad j \geq k \geq 1 \tag{5.260}$$

The name "recursion" is used because we must turn back to the preceding members to obtain the following one.

Recursive relations are imbedded in mathematics, starting with sequence of natural numbers in which every number is the previous number plus one.

$$\mathbb{N} = \{x;\, x_{k+1} = x_k + 1, x_1 = 1\} = \{1, 2, 3, \ldots\} \tag{5.261}$$

We may derive a recursive equation such as (5.260) by doing some calculation:

$$x_{k+1} = x_k + 1 \qquad x_{k+2} = x_{k+1} + 1 \tag{5.262}$$

$$x_{k+1} - x_k = x_{k+2} - x_{k+1} \tag{5.263}$$

$$x_{k+2} = 2x_{k+1} - x_k \tag{5.264}$$

The set of natural numbers is the simplest arithmetic progression in which every member is equal to the previous member plus a constant.

$$A = \{x;\, x_{k+1} = x_k + r\} = \{x_1, r + x_1, 2r + x_1, \ldots\} \tag{5.265}$$

$$x_{k+2} = 2x_{k+1} - x_k \tag{5.266}$$

As can be seen, the recursive equations (5.264) and (5.266) are the same, indicating that in order to start a sequence, we do need to know the first and second members of the sequence, x_1, x_2 to start the sequential chain.

The geometric progression is another familiar recursive set in which every member is equal to the previous member times a number.

$$G = \{x; x_{k+1} = rx_k\} = \left\{x_1, rx_1, r^2x_1, r^3x_1, \ldots\right\} \tag{5.267}$$

$$x_{k+2} = \frac{x_{k+1}^2}{x_k} \tag{5.268}$$

The famous Fibonacci sequence is such that each member is equal to the sum of the previous two members.

$$F = \{x; x_{k+2} = x_k + x_{k+1}\} \tag{5.269}$$

Fibonacci sequence appeared to answer the question of the number of pairs of adult rabbits descending from one pair in a year if it is known that each adult pair of rabbits gives birth to a new pair every month, and the newborn pairs become adult in a month. The sequence whose terms express the total number of adult pairs of rabbits at the initial moment x_1, in a month x_2, in 2 months x_3 and in general after k months is x_{k+1}, starting with $x_1 = 1$. One newborn pair will be added a month later, but the number of adult pairs will still be $x_2 = 1$. In 2 months the little rabbits will become adult and the total number of adult pairs will be $x_3 = 2$, while one new pair was just born to be adult in a month. Hence, in 3 months the number of adult pairs will be 3, while two new pairs are born. Assume that we have calculated the number of adult pairs after $k - 1$ months to be x_k, and after k months, x_{k+1}. By this time the x_k adult pairs that existed before will have an offspring of x_k additional pairs, so that after $k + 1$ months the total number of adult pairs will be

$$x_{k+2} = x_k + x_{k+1} \tag{5.270}$$

showing a sequence of (Markushevich 1983),

$$F = \{1, 1, 2, 3, 5, 8, 13, 21, \ldots\} \tag{5.271}$$

It is good to mention that it is not easy to determine the recursion relation of every sequence. The sequence of prime numbers is an example.

$$P = \{2, 3, 5, 7, 11, 13, 17, 19, 23, \ldots\} \tag{5.272}$$

Example 190 Mathieu functions using recursive equations.

The periodic solution of the Mathieu function $ce_{2k}(t, b)$ is expressed by

$$ce_{2k}(t, b) = \sum_{j=0}^{\infty} A_{2j} \cos 2jt \tag{5.273}$$

where its coefficients are:

$$aA_0 - bA_2 = 0 \tag{5.274}$$

$$(a-4)\,A_2 - b\,(2A_0 + A_4) = 0 \tag{5.275}$$

$$\left(a-(2j)^2\right)A_{2j} - b\left(A_{2j-2} + A_{2j+2}\right) = 0 \qquad j \geq 2 \tag{5.276}$$

The first equation indicates that A_2 can be found as a function of A_0,

$$A_2 = \frac{a}{b}A_0 \tag{5.277}$$

and then, A_4 from the second equation

$$\begin{aligned} A_4 &= \frac{1}{b}\left((a-4)\,A_2 - 2bA_0\right) \\ &= \frac{A_0}{2b^2}\left(a^2 - 2b^2 - 4a\right) \end{aligned} \tag{5.278}$$

and the next coefficients from the recursive equation.

$$\begin{aligned} A_6 &= \frac{1}{b}\left((a-16)\,A_4 - bA_2\right) \\ &= \frac{A_0}{2b^3}\left(a^3 - 4ab^2 - 20a^2 + 64a + 32b^2\right) \end{aligned} \tag{5.279}$$

$$\begin{aligned} A_8 &= \frac{1}{b}\left((a-36)\,A_6 - bA_4\right) \\ &= \frac{A_0}{2b^4}\left(a^4 - 56a^3 - 5a^2b^2 + 784a^2\right) \\ &\quad + \frac{A_0}{2b^4}\left(180ab^2 - 2304a + 2b^4 - 1152b^2\right) \end{aligned} \tag{5.280}$$

The A_{2j}, $j = 1, 2, 3, \ldots$ give implicit functions of $A_{2j} = A_{2j}\,(a, b, A_0)$. Assuming $A_0 = 1$, Fig. 5.16 illustrates the plot of the implicit functions in the (a, b)-plane.

The transition curves from the equations get to the exact position very soon for low values of b. In the figure, the transition curves for A_{12} and A_{20} are not distinguishable for $b < 10$.

In this method, we do not get the transition curves as $a = a\,(b)$ individually. Every coefficients A_{2j} of $\cos 2jt$ is found as an implicit function of $A_{2j} = A_{2j}\,(a, b)$. By setting a value for $a = a_0$, we will be able to calculate the values of b where the varietal line $a = a_0$ intersects with transition lines, assuming the horizontal axis of the stability chart is a and the vertical axis is b. Alternatively, we

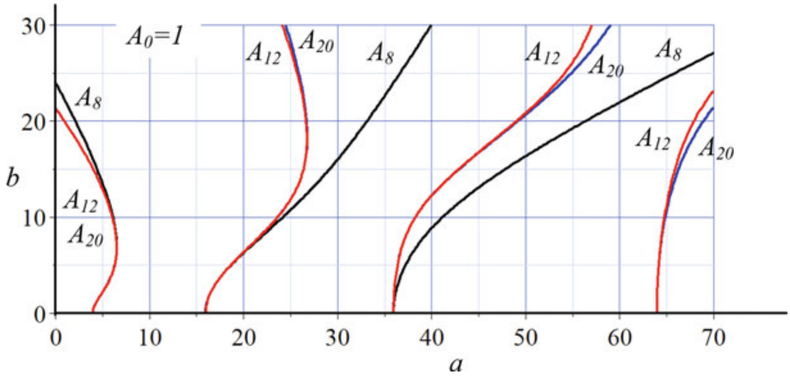

Fig. 5.16 Implicit plot of A_8, A_{12}, A_{20}, for $A_0 = 1$, in (a, b)-plane

Table 5.2 The values of b on transition curves for $a = 0$, from A_{2j}

A_8	A_{10}	A_{12}	A_{14}	A_{20}
24	±21.46625258	±21.30425205	±21.29872544	±21.29863121
		±90.12285419	±72.27778758	±69.42921246
				±147.6986417

Table 5.3 The values of a on transition curves for $b = 5$, from A_{2j}

A_8	A_{12}	A_{14}	A_{20}
6.297273633	6.294538625	6.294538600	6.294538600
18.70845621	18.67614617	18.67614562	18.67614562
37.26885549	36.41057157	36.41052760	36.41052754
	64.20485750	64.19900060	64.19898651
	100.6886967	100.1296740	100.1263693
		144.5649242	144.0874473
			196.0641174
			256.0499575
			324.3667208

may set a value for $b = b_0$, and calculate the values of a where the horizontal line $b = b_0$ intersects with transition lines. As an example, let us set $a = 0$ to calculate where the transition curves hit the vertical axis $b = 0$. Table 5.2 indicates the values of b on transition curves for $a = 0$, from A_{2j}, and Table 5.3 indicates the values of a on transition curves for $b = 5$, from A_{2j}. Increasing j will give longer equations for A_{2j} but better approximation for the points on transition curves.

The periodic solution of the Mathieu function $ce_{2k+1}(t, b)$ is expressed by

$$ce_{2k+1}(t, b) = \sum_{j=0}^{\infty} A_{2j+1} \cos(2j + 1)\, t \tag{5.281}$$

where its coefficients are:

$$(a-1)\,A_1 - b\,(A_1 + A_3) = 0 \tag{5.282}$$

$$\left(a-(2j+1)^2\right)A_{2j+1} - b\left(A_{2j-1}+A_{2j+3}\right) = 0 \qquad j \geq 1 \tag{5.283}$$

Using the first equation we can find A_3 as a function of A_1,

$$A_3 = \frac{1}{b}\,(a-b-1)\,A_1 \tag{5.284}$$

and next coefficients form the recursive equation.

$$\begin{aligned} A_5 &= \frac{1}{b}\,((a-9)\,A_3 - bA_1) \\ &= \frac{A_1}{b^2}\left(a^2 - ab - b^2 - 10a + 9b + 9\right) \end{aligned} \tag{5.285}$$

$$\begin{aligned} A_7 &= \frac{1}{b}\,((a-25)\,A_5 - bA_3) \\ &= \frac{A_1}{b^3}\left(a^3 - a^2b - 35a^2 - 2ab^2 + 34ab\right) \\ &\quad + \frac{A_1}{b^3}\left(259a + b^3 + 26b^2 - 225b - 225\right) \end{aligned} \tag{5.286}$$

$$\begin{aligned} A_9 &= \frac{1}{b}\,((a-36)\,A_7 - bA_5) \\ &= \frac{A_1}{b^4}\left(a^4 - a^3b - 84a^3 - 3a^2b^2 + 83a^2b + 1974a^2\right) \\ &\quad + \frac{A_1}{b^4}\left(2ab^3 + 134ab^2 - 1891ab - 12916a\right) \\ &\quad + \frac{A_1}{b^4}\left(b^4 - 58b^3 - 1283b^2 + 11025b + 11025\right) \end{aligned} \tag{5.287}$$

The A_{2j+1}, $j = 1, 2, \ldots$ also give implicit functions $A_{2j+1} = A_{2j+1}\,(a, b, A_1)$. Assuming $A_1 = 1$, Fig. 5.17 illustrates the plot of the implicit functions in the (a, b)-plane.

The periodic solution of the Mathieu function $se_{2k+1}\,(t, b)$ is expressed by

$$se_{2k+1}\,(t, b) = \sum_{j=0}^{\infty} B_{2j+1} \sin\,(2j+1)\,t \tag{5.288}$$

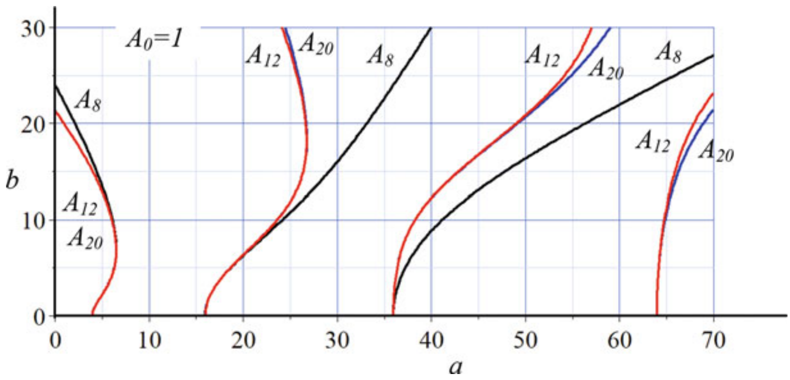

Fig. 5.17 Implicit plot of A_7, A_{11}, A_{19}, for $A_1 = 1$, in (a, b)-plane

and its coefficients are:

$$(a-1)\,B_1 + b\,(B_1 - B_3) = 0 \tag{5.289}$$

$$\left(a-(2j+1)^2\right) B_{2j+1} - b\left(B_{2j-1} + B_{2j+3}\right) = 0 \qquad j \geq 1 \tag{5.290}$$

Using the first equation we find B_3 as a function of B_1,

$$B_3 = \frac{1}{b}\,(a+b-1)\,B_1 \tag{5.291}$$

and other coefficients form the recursive equation.

$$\begin{aligned} B_5 &= \frac{1}{b}\,((a-9)\,B_3 - bB_1) \\ &= \frac{B_1}{b^2}\left(a^2 + ab - 10a - b^2 - 9b + 9\right) \end{aligned} \tag{5.292}$$

$$\begin{aligned} B_7 &= \frac{1}{b}\,((a-25)\,B_5 - bB_3) \\ &= \frac{B_1}{b^3}\left(a^3 + a^2b - 35a^2 - 2ab^2 - 34ab\right) \\ &\quad + \frac{B_1}{b^3}\left(259a - b^3 + 26b^2 + 225b - 225\right) \end{aligned} \tag{5.293}$$

$$B_9 = \frac{1}{b}\,((a-36)\,B_7 - bB_5)$$

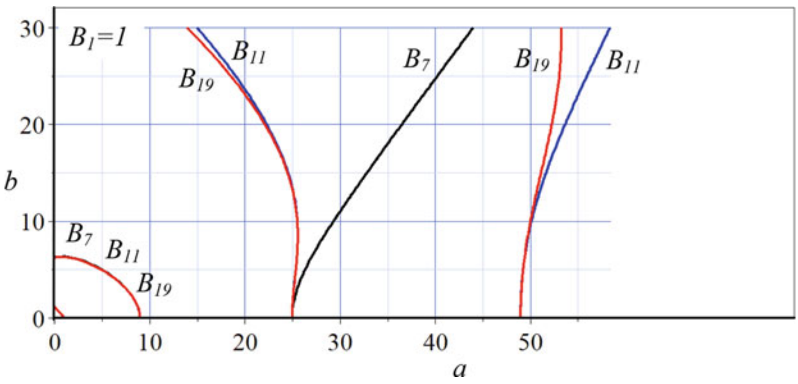

Fig. 5.18 Implicit plot of B_7, B_{11}, B_{19}, for $B_1 = 1$, in (a, b)-plane

$$= \frac{B_1}{b^4}\left(a^4 + a^3 b - 84a^3 - 3a^2 b^2 - 83a^2 b + 1974a^2\right)$$
$$+\frac{B_1}{b^4}\left(-2ab^3 + 134ab^2 + 1891ab - 12916a\right)$$
$$+\frac{B_1}{b^4}\left(b^4 + 58b^3 - 1283b^2 - 11025b + 11025\right) \tag{5.294}$$

The B_{2j+1}, $j = 1, 2, 3, \ldots$ give implicit functions of $B_{2j+1} = B_{2j+1}(a, b, A_1)$. Assuming $B_1 = 1$, Fig. 5.18 illustrates the plot of the implicit functions in the (a, b)-plane.

Similarly, the periodic solution of the Mathieu function $se_{2k+2}(t, b)$ is expressed by

$$se_{2k+2}(t, b) = \sum_{j=0}^{\infty} B_{2j+2} \sin(2j + 2)\, t \tag{5.295}$$

where its coefficients are:

$$(a - 4)\, B_2 - bB_4 = 0 \tag{5.296}$$

$$\left(a - (2j)^2\right) B_{2j} - b\left(B_{2j-2} + B_{2j+2}\right) = 0 \qquad j \geq 2 \tag{5.297}$$

Using the first equation we find B_4 as a function of B_2,

$$B_4 = \frac{1}{b} B_2\,(a - 4)\, B_2 \tag{5.298}$$

and the other coefficients form the recursive equation.

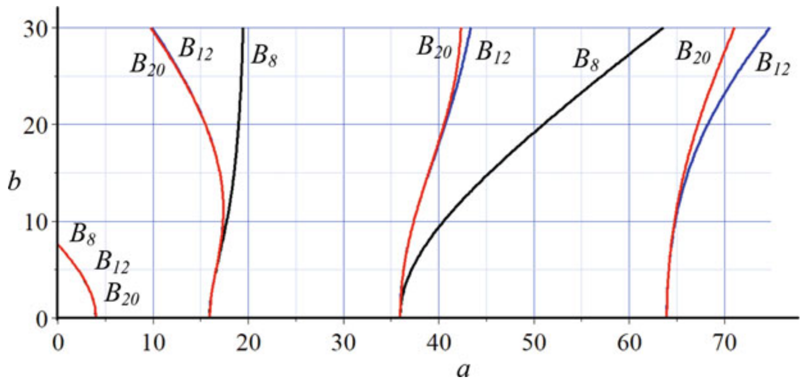

Fig. 5.19 Implicit plot of B_8, B_{12}, B_{20}, for $B_2 = 1$, in (a, b)-plane

$$B_6 = \frac{1}{b}\left((a-16)\,B_4 - bB_2\right) = \frac{B_2}{b^2}\left(a^2 - b^2 - 20a + 64\right) \tag{5.299}$$

$$B_8 = \frac{1}{b}\left((a-36)\,B_6 - bB_4\right) = \frac{B_2}{b^3}\left(a^3 - 2ab^2 - 56a^2 + 40b^2 + 784a - 2304\right) \tag{5.300}$$

$$B_{10} = \frac{1}{b}\left((a-64)\,B_8 - bB_6\right) = \frac{B_2}{b^4}\left(a^4 - 3a^2b^2 + b^4 - 120a^3 + 188ab^2\right) + \frac{B_2}{b^4}\left(4368a^2 - 2624b^2 - 52480a + 147456\right) \tag{5.301}$$

The B_{2j+2}, $j = 2, 3, \ldots$ give implicit functions of $B_{2j+2} = B_{2j+2}(a, b, B_2)$. Assuming $B_2 = 1$, Fig. 5.19 illustrates the plot of the implicit functions in the (a, b)-plane.

To examine how exact the recursive equations can determine the stability chart of the Mathieu equation, we calculated the terms A_{30}, A_{29}, B_{29}, B_{30} and plot them in Fig. 5.20. Although the transient curves are very well for $b < 10$, they are not as well for $b > 20$.

Example 191 Recursive equations proof.

The common method of proof for recursive equations is the induction method in which we check the correctness of the equation for $j = 1, 2, 3$, assume it is correct

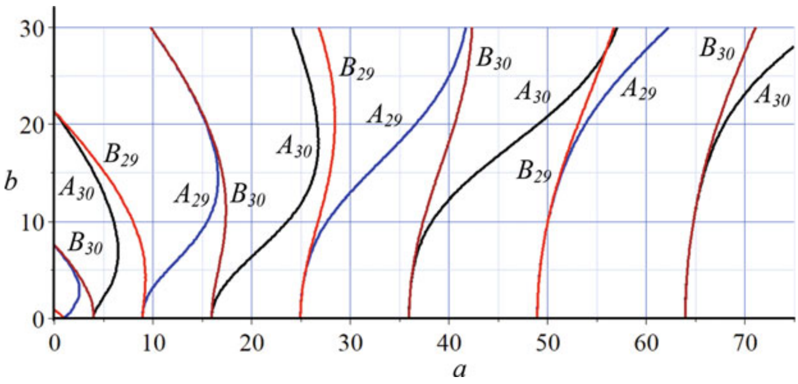

Fig. 5.20 Implicit plot of A_{30}, A_{29}, B_{29}, B_{30}, for $A_1 = 1$, $A_0 = 1$, $B_1 = 1$, $B_2 = 1$, in (a, b)-plane

for $j = k$, then prove that it is correct for $j = k + 1$. As an example, let prove the geometric progression formula.

$$1 + r + r^2 + r^3 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r} \tag{5.302}$$

The equation is correct for $n = 0, 1, 2$. We suppose that it is correct for $n = k$

$$1 + r + r^2 + r^3 + \cdots + r^k = \frac{1 - r^{k+1}}{1 - r} \tag{5.303}$$

and we will prove it is correct for $n = k + 1$.

$$\begin{aligned} \frac{1 - r^{k+1}}{1 - r} + r^{k+1} &= \frac{1 - r^{k+1} + r^{k+1} - r^{k+2}}{1 - r} \\ &= \frac{1 - r^{k+2}}{1 - r} \end{aligned} \tag{5.304}$$

Example 192 Orthogonality of $\cos nt$ and $\sin nt$.

Two real continuous functions $y_1(\tau)$ and $y_2(\tau)$ are said to be orthogonal for the range (τ_1, τ_2) if

$$\int_{\tau_1}^{\tau_2} y_r(\tau)\ y_s(\tau)\, d\tau = \delta_{rs} \tag{5.305}$$

$$\delta_{rs} = \begin{cases} 1 & r = s \\ 0 & r \neq s \end{cases} \tag{5.306}$$

Assume x_1 and x_2 to be two solutions of

$$\frac{d^2x}{dt^2} + (a - 2b\cos 2t)\,x = 0 \tag{5.307}$$

for the same value of b and different a.

$$\frac{d^2x_1}{dt^2} + (a_1 - 2b\cos 2t)\,x_1 = 0 \tag{5.308}$$

$$\frac{d^2x_2}{dt^2} + (a_2 - 2b\cos 2t)\,x_2 = 0 \tag{5.309}$$

Multiplying the first equation by x_2 and the second equation by x_1 and subtracting them, we have:

$$\ddot{x}_1 x_2 - x_1 \ddot{x}_2 = (a_2 - a_1)\,x_1 x_2 \tag{5.310}$$

Integrating both sides of this equation between $t = 0$ and $t = 2\pi$ we get

$$[\dot{x}_1 x_2 - x_1 \dot{x}_2]_0^{2\pi} = (a_2 - a_1)\int_0^{2\pi} x_1 x_2 dt \tag{5.311}$$

The left-hand side will be zero because both $x_1 = ce_r\,(t, b)$ and $x_2 = ce_s\,(t, b)$ are 2π-periodic functions. Therefore,

$$\int_0^{2\pi} x_1 x_2 dt = \int_0^{2\pi} ce_r\,(t, b)\;ce_s\,(t, b)\,dt = \pi\delta_{rs} \tag{5.312}$$

indicating that $ce_r\,(t, b)$ and $ce_s\,(t, b)$ are orthogonal. Similarly, we have:

$$\int_0^{2\pi} se_r\,(t, b)\;se_s\,(t, b)\,dt = \pi\delta_{rs} \tag{5.313}$$

$$\int_0^{2\pi} se_r\,(t, b)\;ce_s\,(t, b)\,dt = 0 \tag{5.314}$$

Example 193 Normalization and limit of Mathieu functions.

It is common to normalize the Mathieu functions such that:

$$\int_0^{2\pi} x^2 dt = \pi \tag{5.315}$$

and as we have seen, the limit of the Mathieu functions for $b \to 0$ are:

$$\lim_{b\to 0} ce_0\,(t) = \frac{1}{\sqrt{2}} \tag{5.316}$$

$$\lim_{b \to 0} ce_r(t) = \cos nt \tag{5.317}$$

$$\lim_{b \to 0} se_r(t) = \sin nt \tag{5.318}$$

5.4 Determinant Method

Employing the recurrence relations (5.225)–(5.233) we may rewrite them in matrix form. The resultant system of algebraic equation may have nontrivial solutions if the determinant of the coefficient matrix is zero. The determinant will be an implicit algebraic equation for a and b which we can solve by assigning b and looking for all a satisfying the equation. Applying this method for all a_{ce} and a_{se} will determine all a on stability curves for that particular b. Repeating the method for another b determines the characteristic numbers.

The recursive equations for $a_{ce_{2k}}$

$$aA_0 - bA_2 = 0 \tag{5.319}$$

$$(a-4)\,A_2 - b\,(2A_0 + A_4) = 0 \tag{5.320}$$

$$\left(a - (2j)^2\right) A_{2j} - b\left(A_{2j-2} + A_{2j+2}\right) = 0 \qquad j \geq 2 \tag{5.321}$$

may be put in matrix form

$$\begin{bmatrix} -a & b & 0 & 0 & \cdots \\ 2b & 2^2 - a & b & 0 & \cdots \\ 0 & b & 4^2 - a & b & \cdots \\ 0 & 0 & b & 6^2 - a & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} A_0 \\ A_2 \\ A_4 \\ A_6 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \tag{5.322}$$

The determinant of the coefficient matrix must be zero in order to have nontrivial solutions for the system of equations. The diagonal terms of $a - (2j)^2$ remind that we will be determining a values on stability curves branched from $a = (2j)^2$. Therefore, we may cut the matrix depending on how many branches of the stability chart we are interested in. The 4×4 determinant of the coefficient matrix of (5.322) is:

$$\begin{vmatrix} -a & b & 0 & 0 \\ 2b & 2^2 - a & b & 0 \\ 0 & b & 4^2 - a & b \\ 0 & 0 & b & 6^2 - a \end{vmatrix} = 0 \tag{5.323}$$

$$a^4 - 56a^3 + \left(784 - 4b^2\right) a^2$$
$$+ \left(144b^2 - 2304\right) a + \left(2b^4 - 1152b^2\right) = 0 \qquad (5.324)$$

The solutions of this equation for $b = 0$ are $a = 0, 4, 16, 36$, and in general $a = (2j)^2$, $j = 1, 2, 3, \cdots$. The solutions for a sample value of b say $b = 1$ are

$$b = 1$$
$$a = -.4609133357, 4.294402271, 16.11642759, 36.01436823 \qquad (5.325)$$

Working with a larger matrix will increase the accuracy of the results and determines more branches of stability curves.

Proof Looking for the periodic solutions of the Mathieu equation, we guessed a Fourier series solution of the form

$$x(t, b) = ce_k(t, b) + se_{k+1}(t, b) \qquad (5.326)$$
$$= \sum_{j=0}^{\infty} \left(A_j(b) \cos jt + B_{j+1}(b) \sin(j+1)t\right) \qquad (5.327)$$

and try to fit this into the Mathieu equation. This process generated a set of algebraic relations among the unknown coefficients A_j and B_j. These equations for $a_{ce_{2k}}$ are:

$$aA_0 - bA_2 = 0 \qquad (5.328)$$
$$(a-4)A_2 - b(2A_0 + A_4) = 0 \qquad (5.329)$$
$$\left(a - (2j)^2\right) A_{2j} - b\left(A_{2j-2} + A_{2j+2}\right) = 0 \qquad j \geq 2 \qquad (5.330)$$

The first and second equations are individual while the third equation indicates a recursive relation among the three consecutive coefficients that enable us to computerize the calculation of the coefficients as many as we wish. We may put these equations in a matrix form

$$\begin{bmatrix} a & -b & 0 & 0 & \cdots \\ -2b & a-2^2 & -b & 0 & \cdots \\ 0 & -b & a-4^2 & -b & \cdots \\ 0 & 0 & -b & a-6^2 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} A_0 \\ A_2 \\ A_4 \\ A_6 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \qquad (5.331)$$

which is equivalent to (5.322). To have nonzero A_{2j} we must have the determinant of the coefficient matrix to be zero. Practically we need to cut the number of equation at some point depending on the desired accuracy and the number of interested branches of the stability curves on the Mathieu stability chart in (a, b)-plane. The determinant of (5.331) provides us with an implicit equation of a and b. The determinant equation for a 4×4 matrix gives us

$$\begin{vmatrix} a & -b & 0 & 0 \\ -2b & a-2^2 & -b & 0 \\ 0 & -b & a-4^2 & -b \\ 0 & 0 & -b & a-6^2 \end{vmatrix} = 0 \tag{5.332}$$

$$\begin{aligned} a^4 - 56a^3 - 4a^2b^2 + 784a^2 + 144ab^2 \\ - 2304a + 2b^4 - 1152b^2 = 0 \end{aligned} \tag{5.333}$$

and a 5×5 matrix gives us

$$\begin{vmatrix} a & -b & 0 & 0 & 0 \\ -2b & a-2^2 & -b & 0 & 0 \\ 0 & -b & a-4^2 & -b & 0 \\ 0 & 0 & -b & a-6^2 & -b \\ 0 & 0 & 0 & -b & a-8^2 \end{vmatrix} = 0 \tag{5.334}$$

$$\begin{aligned} a^5 - 120a^4 - 5a^3b^2 + 4368a^3 + 420a^2b^2 - 52\,480a^2 \\ +5ab^4 - 10\,432ab^2 + 147\,456a - 160b^4 + 73\,728b^2 = 0 \end{aligned} \tag{5.335}$$

The single Eq. (5.333) can provide all stability curves associated to $a_{ce_{2k}}$ branching from $a = 0$, 4, 16, and 36. Figure 5.21 illustrates the implicit plot of (5.333) for $-10 < a < 60$ and $0 < b < 20$. To be able to compare the numerical accuracy of the equation, let us determine the values of a for a fixed value of b, say $b = 5$.

$$\begin{aligned} &b = 5 \\ &a = -6.274585339,\ 6.297273633,\ 18.70845622,\ 37.26885549 \end{aligned} \tag{5.336}$$

Similarly, Eq. (5.335) provides all stability curves associated to $a_{ce_{2k}}$ branching from $a = 0$, 4, 16, 36, and 64. Figure 5.22 illustrates the implicit plot of (5.335) for $-10 < a < 80$ and $0 < b < 20$. The line $b = 5$ intersects the curves at:

$$\begin{aligned} &b = 5 \\ &a = -6.274809847,\ 6.294551463,\ 18.67637134,\ 36.42240081, \\ &\qquad 64.88148624 \end{aligned} \tag{5.337}$$

Comparing the roots (5.336) and (5.337) indicates that working with a larger determinant not only discovers more branches of the stability chart associated to

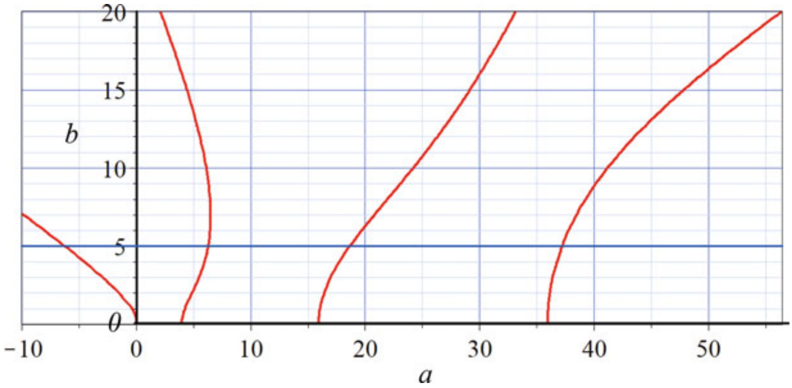

Fig. 5.21 The implicit plot of stability curves associated to $a_{ce_{2k}}$ branching from $a = 0, 4, 16$, and 36 using the determinant of the 4×4 coefficients matrix

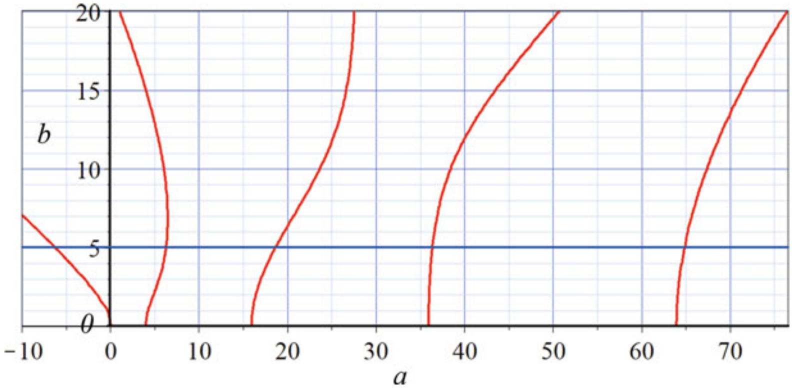

Fig. 5.22 The implicit plot of stability curves associated to $a_{ce_{2k}}$ branching from $a = 0, 4, 16$, and 36 using the determinant of the 4×4 coefficients matrix

higher values of a, the accuracy of the stability curves of the lower branches will improve as well. This fact will help to calculate the characteristic numbers for higher values of b beyond the point that series solution are ill to give a reasonable approximation. Figure 5.23 compares the stability curves for the 4×4 determinant and the 5×5 determinant, as well as a 10×10 determinant. Table 5.4 compares the characteristic points for $b = 0$ and different rank of the coefficient matrices.

The recursive equations for $a_{ce_{2k+1}}$

$$(a - 4)\, A_1 - b\,(A_1 + A_3) = 0 \tag{5.338}$$

$$\left(a - (2j + 1)^2\right) A_{2j+1} - b\left(A_{2j-1} + A_{2j+3}\right) = 0 \qquad j \geq 1 \tag{5.339}$$

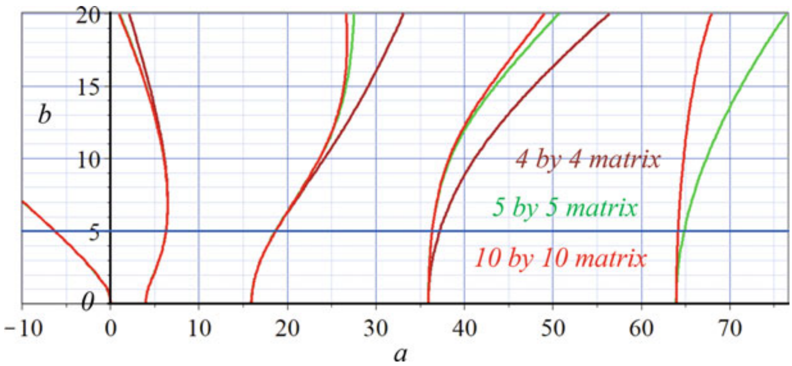

Fig. 5.23 The implicit plot of transition curves of $a_{ce_{2k}}$ branching from $a = 0, 4, 16, 36$ using the determinant of the different order coefficients matrices

Table 5.4 Characteristic values of $a_{ce_{2k}}$ for $b = 5$ from $n \times n$ determinant

	4×4	5×5	10×10
a_{ce0}	−6.274585339	−6.274809847	−6.274810608
a_{ce2}	6.297273633	6.294551463	6.294538600
a_{ce4}	18.70845622	18.67637134	18.67614562
a_{ce6}	37.26885549	36.42240081	36.41052754
a_{ce8}		64.88148624	64.19898651
a_{ce10}			100.1263693
a_{ce12}			144.0874473
a_{ce14}			196.0641161
a_{ce16}			256.0490263
a_{ce18}			324.0393750

make the matrix form equations of

$$\begin{bmatrix} 1+b-a & b & 0 & 0 & \cdots \\ 2b & 3^2-a & b & 0 & \cdots \\ 0 & b & 5^2-a & b & \cdots \\ 0 & 0 & b & 7^2-a & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} A_1 \\ A_3 \\ A_5 \\ A_7 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \tag{5.340}$$

The recursive equations for $a_{se_{2k+1}}$

$$(a-1)\,B_1 + b\,(B_1 - B_3) = 0 \tag{5.341}$$

$$\left(a - (2j+1)^2\right) B_{2j+1} - b\left(B_{2j-1} + B_{2j+3}\right) = 0 \qquad j \geq 1 \tag{5.342}$$

make the matrix form equations of

$$\begin{bmatrix} 1-b-a & b & 0 & 0 & \cdots \\ 2b & 3^2-a & b & 0 & \cdots \\ 0 & b & 5^2-a & b & \cdots \\ 0 & 0 & b & 7^2-a & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} A_1 \\ A_3 \\ A_5 \\ A_7 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \tag{5.343}$$

The recursive equations for $a_{se_{2k+2}}$

$$(a-4)\,B_2 - bB_4 = 0 \tag{5.344}$$

$$\left(a-(2j)^2\right) B_{2j} - b\left(B_{2j-2} + B_{2j+2}\right) = 0 \qquad j \geq 2 \tag{5.345}$$

may also be expressed in matrix form.

$$\begin{bmatrix} 2^2-a & b & 0 & 0 & \cdots \\ b & 4^2-a & b & 0 & \cdots \\ 0 & b & 6^2-a & b & \cdots \\ 0 & 0 & b & 8^2-a & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} A_2 \\ A_4 \\ A_6 \\ A_8 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \tag{5.346}$$

These are straightforward matrix representations of the recursive equations. To have nonzero solutions for the coefficients A_i and B_i, the determinant of the coefficient matrices must be zero. The determinants give implicit equations of a and b of the order equal to the rank of the cut matrix. The four classes of matrix equations will collectively make the whole stability chart of the Mathieu equation. ■

Example 194 ★A comment on numerical characteristic numbers from determinant equations.

Some researchers suggest to substitute $a = n^2 + a_1 b + a_2 b^2 + \cdots$ in the determinant equations and sort them for powers of b. Recalling that the equations must be true for every value of b requires that the coefficients of b^i to be zero. The coefficients are polynomial of a. The roots of the polynomials will determine a_i of the characteristic power series. There is no need and we do not recommend this method. The reason is that we lower the level of accuracy of the implicit determinant equations to the level of the series solutions (5.112)–(5.120), losing the advantages of the determinant method.

Example 195 Modified matrix method.

There are alternative matrix forms to present the recursive equations. As an example we may introduce

$$P_j = \frac{-b}{a - j^2} \tag{5.347}$$

and use the recursive equations for $a_{ce_{2k}}$ (5.319)–(5.319) to become

$$A_0 + P_0 A_2 = 0 \tag{5.348}$$

$$2P_2 A_0 + A_2 + P_2 A_4 = 0 \tag{5.349}$$

$$P_{2j} A_{2j-2} + A_{2j} + P_{2j} A_{2j+2} = 0 \qquad j \geq 2 \tag{5.350}$$

Expanding and rearranging these equation provide a set of algebraic equations.

$$\begin{array}{llllll} & A_0 & +P_0 A_2 & & & = 0 \\ 2P_2 A_0 & & +A_2 & +P_2 A_4 & & = 0 \\ & & P_4 A_2 & +A_4 & +P_4 A_6 & = 0 \\ & & & +P_6 A_4 & +A_6 \; +P_6 A_8 & = 0 \\ & & & \ddots & \ddots \; \ddots & = 0 \end{array} \tag{5.351}$$

The determinant of the coefficients must be zero.

$$\begin{vmatrix} 1 & P_0 & 0 & 0 & 0 & \cdots \\ 2P_2 & 1 & P_2 & 0 & 0 & \cdots \\ 0 & P_4 & 1 & P_4 & 0 & \cdots \\ 0 & 0 & P_6 & 1 & P_6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix} = 0 \tag{5.352}$$

$$\begin{vmatrix} 1 & \frac{-b}{a} & 0 & 0 & 0 & \cdots \\ 2\frac{-b}{a-4} & 1 & \frac{-b}{a-4} & 0 & 0 & \cdots \\ 0 & \frac{-b}{a-16} & 1 & \frac{-b}{a-16} & 0 & \cdots \\ 0 & 0 & \frac{-b}{a-36} & 1 & \frac{-b}{a-36} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix} = 0 \tag{5.353}$$

The parameter b of the Mathieu equation is considered to be known and therefore the determinant is a function of a. The alternative matrix expression of characteristic number $a_{ce_{2k+1}}$, $a_{se_{2k+1}}$ and $a_{se_{2k+2}}$ may be found similarly.

5.5 Continued Fractions of Characteristic Numbers

The periodic solution of the Mathieu equation

$$\frac{d^2x}{dt^2} + (a - 2b\cos 2t)\,x = 0 \tag{5.354}$$

is expressed by Fourier series, classified in four groups

$$ce_{2k}\,(t, b) = \sum_{j=0}^{\infty} A_{2j}\,(a, b)\cos 2jt \tag{5.355}$$

$$ce_{2k+1}\,(t, b) = \sum_{j=0}^{\infty} A_{2j+1}\,(a, b)\cos(2j+1)\,t \tag{5.356}$$

$$se_{2k+1}\,(t, b) = \sum_{j=0}^{\infty} B_{2j+1}\,(a, b)\sin(2j+1)\,t \tag{5.357}$$

$$se_{2k+2}\,(t, b) = \sum_{j=0}^{\infty} B_{2j+2}\,(a, b)\sin(2j+2)\,t \tag{5.358}$$

where coefficients A_{2j}, A_{2j+1}, B_{2j+1}, B_{2j+2} are functions of a and b such that the solution of every group is associated with a transition curve $a = a_{ce_{2k}}(b)$, $a = a_{ce_{2k+1}}(b)$, $a = a_{se_{2k+1}}(b)$, $a = a_{se_{2k+2}}(b)$.

$$a_{ce_{2k}}(b) \qquad \pi\text{-periodic, even solution} \tag{5.359}$$

$$a_{ce_{2k+1}}(b) \qquad 2\pi\text{-periodic, even solution} \tag{5.360}$$

$$a_{se_{2k+1}}(b) \qquad 2\pi\text{-periodic, odd solution} \tag{5.361}$$

$$a_{se_{2k+2}}(b) \qquad \pi\text{-periodic, odd solution} \tag{5.362}$$

The characteristic numbers $a = a\,(b)$ on which the Mathieu equation shows periodic response can be expressed by continued fractions as follows:

Characteristic numbers $a_{ce_{2k}}$

$$\frac{a}{b} = \frac{2}{\dfrac{a-4}{b} -}\,\frac{1}{\dfrac{a-16}{b} -}\,\frac{1}{\dfrac{a-36}{b} -}\,\frac{1}{\dfrac{a-j^2}{b} -}\cdots \tag{5.363}$$

$$j = 2, 4, 6, 8, \cdots \tag{5.364}$$

Characteristic numbers $a_{ce_{2k+1}}$

$$\frac{a-1}{b} - 1 = \frac{1}{\dfrac{a-9}{b} -} \frac{1}{\dfrac{a-25}{b} -} \frac{1}{\dfrac{a-49}{b} -} \frac{1}{\dfrac{a-j^2}{b} -} \cdots \tag{5.365}$$

$$j = 3, 5, 7, 9, \cdots \tag{5.366}$$

Characteristic numbers $a_{se_{2k+1}}$

$$\frac{a-1}{b} + 1 = \frac{1}{\dfrac{a-9}{b} -} \frac{1}{\dfrac{a-25}{b} -} \frac{1}{\dfrac{a-49}{b} -} \frac{1}{\dfrac{a-j^2}{b} -} \cdots \tag{5.367}$$

$$j = 3, 5, 7, 9, \cdots \tag{5.368}$$

Characteristic numbers $a_{se_{2k+2}}$

$$\frac{a-4}{b} = \frac{1}{\dfrac{a-16}{b} -} \frac{1}{\dfrac{a-36}{b} -} \frac{1}{\dfrac{a-64}{b} -} \frac{1}{\dfrac{a-j^2}{b} -} \cdots \tag{5.369}$$

$$j = 4, 6, 8, 10, \cdots \tag{5.370}$$

Proof Substituting (5.355)–(5.358) in the Mathieu equation, one by one, generates the following recursive equation among the coefficients A_{2j}, A_{2j+1}, B_{2j+1}, B_{2j+2}. For the coefficients of $ce_{2k}(t, b)$ we have:

$$aA_0 - bA_2 = 0 \tag{5.371}$$

$$(a-4)\,A_2 - b\,(2A_0 + A_4) = 0 \tag{5.372}$$

$$\left(a - (2j)^2\right) A_{2j} - b\left(A_{2j-2} + A_{2j+2}\right) = 0 \qquad j \geq 2 \tag{5.373}$$

For the coefficients of $ce_{2k+1}(t, b)$ we have:

$$(a-1)\,A_1 - b\,(A_1 + A_3) = 0 \tag{5.374}$$

$$\left(a - (2j+1)^2\right) A_{2j+1} - b\left(A_{2j-1} + A_{2j+3}\right) = 0 \qquad j \geq 1 \tag{5.375}$$

For the coefficients of $se_{2k+1}(t, b)$ we have:

$$(a-1)\,B_1 + b\,(B_1 - B_3) = 0 \tag{5.376}$$

$$\left(a - (2j+1)^2\right) B_{2j+1} - b\left(B_{2j-1} + B_{2j+3}\right) = 0 \qquad j \geq 1 \tag{5.377}$$

For the coefficients of $se_{2k+2}(t,b)$ we have:

$$(a-4)\,B_2 - bB_4 = 0 \tag{5.378}$$

$$\left(a-(2j)^2\right)B_{2j} - b\left(B_{2j-2}+B_{2j+2}\right) = 0 \qquad j \geq 2 \tag{5.379}$$

Let us consider the recursive coefficients of $ce_{2k}(t,b)$ in Eqs. (5.371)–(5.373). We define

$$G_{2j} = \frac{A_{2j}}{A_{2j-2}} \tag{5.380}$$

$$Q_j = \frac{a-j^2}{b} \tag{5.381}$$

to rewrite the recursive equation (5.373).

$$\frac{A_{2j-2}}{A_{2j}} + \frac{A_{2j+2}}{A_{2j}} = \frac{a-(2j)^2}{b} \qquad j \geq 2 \tag{5.382}$$

$$\frac{A_{2j+2}}{A_{2j}} + \frac{1}{\dfrac{A_{2j}}{A_{2j-2}}} = \frac{a-(2j)^2}{b} \tag{5.383}$$

Employing G_{2j} and Q_j yields

$$G_{2j+2} + \frac{1}{G_{2j}} = \frac{a-(2j)^2}{b} = Q_{2j} \tag{5.384}$$

that we can calculate G_{2j}

$$G_{2j} = \frac{1}{Q_{2j} - G_{2j+2}} \tag{5.385}$$

Substituting for G_{2j+2} on the right-hand side

$$G_{2j} = \cfrac{1}{Q_{2j} - \cfrac{1}{Q_{2j+2} - G_{2j+4}}} \tag{5.386}$$

provides us with a continued fraction equation that is applied for $j \geq 2$.

$$G_{2j} = \frac{1}{Q_{2j}-}\ \frac{1}{Q_{2j+2}-}\ \frac{1}{Q_{2j+4}-}\cdots \tag{5.387}$$

Setting $j = 2$ generates the following continued fraction:

$$G_4 = \frac{1}{Q_4-}\,\frac{1}{Q_6-}\,\frac{1}{Q_8-}\,\frac{1}{Q_{10}-}\cdots \tag{5.388}$$

Equations (5.371)–(5.372)

$$G_2 = Q_0 \tag{5.389}$$

$$G_4 = Q_2 - \frac{2}{G_2} \tag{5.390}$$

give us

$$G_2 = \frac{2}{Q_2 - G_4} \tag{5.391}$$

which changes the continued fraction to:

$$Q_0 = \frac{2}{Q_2-}\,\frac{1}{Q_4-}\,\frac{1}{Q_6-}\,\frac{1}{Q_8-}\,\frac{1}{Q_{10}-}\cdots \tag{5.392}$$

Substituting for Q_j leads to a continued fraction between a and b.

$$\frac{a}{b} = \frac{2}{\frac{a-4}{b}-}\,\frac{1}{\frac{a-16}{b}-}\,\frac{1}{\frac{a-36}{b}-}\,\frac{1}{\frac{a-64}{b}-}\cdots \tag{5.393}$$

$$a = \frac{2b^2}{a-4-}\,\frac{b^3}{a-16-}\,\frac{b^4}{a-36-}\,\frac{b^5}{a-64-}\cdots \tag{5.394}$$

Following the same procedure, we will be able to derive the other continued fractions as below (Arscott 1964).
Characteristic numbers $a_{ce_{2k+1}}$

$$Q_1 - 1 = \frac{1}{Q_3-}\,\frac{1}{Q_5-}\,\frac{1}{Q_7-}\,\frac{1}{Q_9-}\,\frac{1}{Q_{11}-}\cdots \tag{5.395}$$

$$\frac{a-1}{b} - 1 = \frac{1}{\frac{a-9}{b}-}\,\frac{1}{\frac{a-25}{b}-}\,\frac{1}{\frac{a-49}{b}-}\cdots$$

$$a - 1 - b = \frac{b^2}{a-9-}\,\frac{b^3}{a-25-}\,\frac{b^4}{a-49-}\cdots \tag{5.396}$$

Characteristic numbers $a_{se_{2k+1}}$

$$Q_1 + 1 = \frac{1}{Q_3-}\frac{1}{Q_5-}\frac{1}{Q_7-}\frac{1}{Q_9-}\frac{1}{Q_{11}-}\cdots \tag{5.397}$$

$$\frac{a-1}{b} + 1 = \frac{1}{\frac{a-9}{b}-}\frac{1}{\frac{a-25}{b}-}\frac{1}{\frac{a-49}{b}-}\cdots \tag{5.398}$$

$$a - 1 + b = \frac{b^2}{a-9-}\frac{b^3}{a-25-}\frac{b^4}{a-49-}\cdots \tag{5.399}$$

Characteristic numbers $a_{se_{2k+2}}$

$$Q_2 = \frac{1}{Q_4-}\frac{1}{Q_6-}\frac{1}{Q_8-}\frac{1}{Q_{10}-}\cdots \tag{5.400}$$

$$\frac{a-4}{b} = \frac{1}{\frac{a-16}{b}-}\frac{1}{\frac{a-36}{b}-}\frac{1}{\frac{a-64}{b}-}\cdots \tag{5.401}$$

$$a - 4 = \frac{b^2}{a-16-}\frac{b^3}{a-36-}\frac{b^4}{a-64-}\cdots \tag{5.402}$$

■

Example 196 Alternative method for continued fractions.

The recursive equations for the coefficients of $ce_{2k}(t, b)$ are:

$$aA_0 - bA_2 = 0 \tag{5.403}$$

$$(a-4)\,A_2 - 2bA_0 - bA_4 = 0 \tag{5.404}$$

$$\left(a - (2j)^2\right) A_{2j} - bA_{2j-2} - bA_{2j+2} = 0 \qquad j \geq 2 \tag{5.405}$$

Let us introduce

$$V_0 = \frac{A_2}{A_0} \qquad V_2 = \frac{A_4}{A_2} \qquad V_0 V_2 = \frac{A_4}{A_0} \tag{5.406}$$

to have the second equation

$$(a-4)\,\frac{A_2}{A_0} - 2b - b\frac{A_4}{A_0} = 0 \tag{5.407}$$

$$(a-4)\,V_0 - 2b - bV_0 V_2 = 0 \tag{5.408}$$

$$-V_0\,(a-4-bV_2)=-2b \tag{5.409}$$

$$-V_0=\frac{\dfrac{b}{2}}{1-\dfrac{a}{4}+\dfrac{b}{4}V_2} \tag{5.410}$$

In the same way,

$$V_{2j-2}=\frac{A_{2j}}{A_{2j-2}}\qquad V_{2j}=\frac{A_{2j+2}}{A_{2j}}\qquad V_{2j-2}V_{2j}=\frac{A_{2j+2}}{A_{2j-2}} \tag{5.411}$$

we may write the third recurrence relations:

$$\left(a-(2j)^2\right)A_{2j}-bA_{2j-2}-bA_{2j+2}=0 \tag{5.412}$$

$$\left(a-(2j)^2\right)V_{2j-2}-b-bV_{2j-2}V_{2j}=0 \tag{5.413}$$

$$-V_{2j-2}=\frac{\dfrac{b}{4j^2}}{1-\dfrac{a}{4j^2}+\dfrac{b}{4j^2}V_{2j}}\qquad j\geq 2 \tag{5.414}$$

Substituting $j=2$ in $-V_{2j-2}$ yields

$$-V_2=\frac{\dfrac{b}{16}}{1-\dfrac{a}{16}+\dfrac{b}{16}V_4} \tag{5.415}$$

and upon replacing this into the equation for $-V_0$ we get

$$-V_0=\frac{\dfrac{b}{2}}{1-\dfrac{1}{4}a-\dfrac{\dfrac{b^2}{64}}{1-\dfrac{a}{16}+\dfrac{b}{16}V_4}} \tag{5.416}$$

Continuing on in this substitution we ultimately get the infinite continued fractions for $-V_0$.

$$-V_0=\frac{\dfrac{b}{2}}{1-\dfrac{a}{4}-}\;\frac{\dfrac{b^2}{64}}{1-\dfrac{a}{16}-}\;\frac{\dfrac{b^2}{576}}{1-\dfrac{a}{36}-}\cdots\frac{\dfrac{b^2}{16j^2\,(j-1)^2}}{1-\dfrac{a}{4j^2}-} \tag{5.417}$$

From (5.406) and (5.403), we have

$$-V_0 = -\frac{A_2}{A_0} = -\frac{a}{b} \tag{5.418}$$

$$V_2 = \frac{A_4}{A_2} \qquad V_{2j} = \frac{A_{2j+2}}{A_{2j}} \tag{5.419}$$

so the continued fractions formula for $-V_0$ becomes

$$a = \frac{\frac{b^2}{2}}{1 - \frac{a}{4} -} \frac{\frac{b^2}{64}}{1 - \frac{a}{16} -} \frac{\frac{b^2}{576}}{1 - \frac{a}{36} -} \cdots \frac{\frac{b^2}{16j^2(j-1)^2}}{1 - \frac{a}{4j^2} -} \tag{5.420}$$

We may modify the equation to be easier for computerization,

$$a = \frac{\frac{b^2}{2}}{1 - \frac{a}{4} -} \frac{1}{4} \frac{\frac{b^2}{16}}{1 - \frac{a}{16} -} \frac{1}{16} \frac{\frac{b^2}{36}}{1 - \frac{a}{36} -} \frac{1}{36} \frac{\frac{b^2}{64}}{1 - \frac{a}{64} -} \cdots \frac{1}{4(j-1)^2} \frac{\frac{b^2}{4j^2}}{1 - \frac{a}{4j^2} - \frac{V_{2j}}{4j^2}} = -2z_1 \tag{5.421}$$

where

$$z_1 = \frac{\frac{b^2}{4}}{1 - \frac{a}{4} - \frac{1}{4} z_2} \qquad z_2 = \frac{\frac{b^2}{16}}{1 - \frac{a}{16} - \frac{1}{16} z_3} \qquad \cdots \tag{5.422}$$

$$z_j = \frac{\frac{b^2}{4j^2}}{1 - \frac{a}{4j^2} - \frac{1}{4j^2} z_{j+1}} \tag{5.423}$$

Because $\lim_{j\to\infty} z_j \to 0$, the continued fraction (5.421) is converging. We assumed and found a and b on c_{e_2} transition curve using the following algorithm.

For the transition curve corresponding to se_{2k+1}, the following recursive equalities will be used to calculate $a_{se_{2k+1}}$:

$$(a-1)\, B_1 + b\,(B_1 - B_3) = 0 \tag{5.424}$$

$$\left(a - (2j+1)^2\right) B_{2j+1} - b\left(B_{2j-1} + B_{2j+3}\right) = 0 \qquad j \geq 1 \tag{5.425}$$

The first equation may be written as:

$$a = 1 - b + bW_1 \tag{5.426}$$

where

$$W_1 = \frac{B_3}{B_1} \qquad W_3 = \frac{B_5}{B_3} \qquad W_1 W_3 = \frac{B_5}{B_1} \tag{5.427}$$

$$W_{2j-1} = \frac{B_{2j+1}}{B_{2j-1}} \tag{5.428}$$

and

$$W_{2j-1} = \frac{b}{a - (2j-1)^2 - bW_{2j+1}} = \frac{-\dfrac{b^2}{(2j+1)^2}}{1 - \dfrac{a}{(2j+1)^2} + \dfrac{W_{2j+1}}{(2j+1)^2}} \tag{5.429}$$

Substituting (5.429) in (5.426) provides the equation we need for calculating the characteristic numbers

$$a = 1 - b - z_1 \tag{5.430}$$

where

$$z_1 = \frac{\dfrac{b^2}{3^2}}{1 - \dfrac{a}{3^2} - \dfrac{1}{3^2} z_2} \qquad z_2 = \frac{\dfrac{b^2}{5^2}}{1 - \dfrac{a}{5^2} - \dfrac{1}{5^2} z_3} \qquad \cdots \tag{5.431}$$

$$z_j = \frac{\dfrac{b^2}{(2j+1)^2}}{1 - \dfrac{a}{(2j+1)^2} - \dfrac{1}{(2j+1)^2} z_{j+1}} \tag{5.432}$$

Using a similar algorithm as above, the transition curves corresponding to se_{2r-1} will be found.

Example 197 Continued fractions algorithm for characteristic numbers.

The characteristic numbers of the Mathieu equation may be calculated by continued fractions method expressed in Eqs. (5.421)–(5.423). To practically employ these equations we have to solve two problems. First, the parameter a is appearing on both sides of the equation and hence, calculating a for a given b is not straightforward.

Second, we need infinity number of terms to get exact value. To solve the second problem, we may truncate the continued fraction at a large number of terms considering $\lim_{j\to\infty} z_j \to 0$. Practically, a few terms of the continued fraction are enough to provide a good approximation. To solve the first problem, we may set b and $a = n^2, n = 0, 1, 2, 3, \cdots$ and calculate enough number of terms to calculate the approximate a for the set value of b. Let us assume $z_{21} = 0$ to cut all terms after the 21th term. The following algorithm may be used to computerize the process of calculating characteristic numbers $a_{ce_{2k}}$.

Continued Fractions Algorithm

1. set b,
2. set $a = n^2, n \in \mathbb{N}$
3. while $E_{av} < \varepsilon$
4. set $z_{20} = \dfrac{b^2/1600}{1 - a/1600}$
5. for i from 1 to 19 set $j = 20 - i$ and calculate $z_j = \dfrac{b^2/\left(4j^2\right)}{1 - \left(a + z_{j+1}\right)/\left(4j^2\right)}$
6. set $a = -2z_1$

5.6 Chapter Summary

Mathieu equation is a linear parametric differential equation,

$$\frac{d^2x}{dt^2} + (a - 2b\cos 2t)\,x = 0 \tag{5.433}$$

that depending on the values of a and b, the solution of the equation may be unstable, periodic, or stable. Therefore, the (a, b)-plane be divided into stable and unstable regions which will be separated by transition curves on which the equation may provide periodic response. The transition curves make the stability chart of the equation.

The Mathieu equation has no solution expressed by elementary functions. Due to the importance of the solution and the stability chart of the equation, many approximation methods have been developed or employed to study the equation that many of them could be used to study similar equations.

Searching periodic solutions in the form

$$x = \cos nt + \sum_{i=1}^{\infty} b^i f_i(t) \qquad a = n^2 + \sum_{i=1}^{\infty} a_i b^i \tag{5.434}$$

or

$$x = \sin nt + \sum_{i=1}^{\infty} b^i g_i\,(t) \qquad a = n^2 + \sum_{i=1}^{\infty} c_i b^i \tag{5.435}$$

will provide solutions called Mathieu function in four groups.

$$ce_{2k}\,(t, b) = \sum_{j=0}^{\infty} A_{2j}^{2k}\,(b) \cos 2jt \tag{5.436}$$

$$ce_{2k+1}\,(t, b) = \sum_{j=0}^{\infty} A_{2j+1}^{2k+1}\,(b) \cos (2j + 1)\, t \tag{5.437}$$

$$se_{2k+1}\,(t, b) = \sum_{j=0}^{\infty} B_{2j+1}^{2k+1}\,(b) \sin (2j + 1)\, t \tag{5.438}$$

$$se_{2k+2}\,(t, b) = \sum_{j=0}^{\infty} B_{2j+2}^{2k+2}\,(b) \sin (2j + 2)\, t \tag{5.439}$$

The Mathieu functions are associated to their characteristic numbers.

$$a = a_{ce_{2k}} \qquad \pi\text{-periodic, even solution} \tag{5.440}$$

$$a = a_{ce_{2k+1}} \qquad 2\pi\text{-periodic, even solution} \tag{5.441}$$

$$a = a_{se_{2k+1}} \qquad 2\pi\text{-periodic, odd solution} \tag{5.442}$$

$$a = a_{se_{2k+2}} \qquad \pi\text{-periodic, odd solution} \tag{5.443}$$

A few of the functions and characteristic numbers are:

$$\begin{aligned} ce_0\,(t, b) = {} & 1 - \frac{b}{2} \cos 2t + \frac{b^2}{32} \cos 4t + \frac{b^3}{128} \left(\cos 2t - \frac{1}{9} \cos 6t \right) \\ & + \frac{b^4}{512} \left(\frac{29}{9} \cos 4t + \frac{1}{144} \cos 8t \right) + \cdots \end{aligned} \tag{5.444}$$

$$\begin{aligned} a_{ce_0}\,(b) = {} & -\frac{1}{2} b^2 + \frac{7b^4}{128} - \frac{29b^6}{2304} + \frac{68687b^8}{18874368} \\ & - \frac{123707b^{10}}{104857600} + \frac{8022167579b^{12}}{19568944742400} + \cdots \end{aligned} \tag{5.445}$$

$$ce_1(t,b) = \cos t - \frac{b}{8}\cos 3t + \frac{b^2}{64}\left(-\cos 3t + \frac{1}{3}\cos 5t\right)$$
$$-\frac{b^3}{512}\left(\frac{1}{3}\cos 3t - \frac{4}{9}\cos 5t + \frac{1}{18}\cos 7t\right)$$
$$+\frac{b^4}{4096}\left(\frac{11}{9}\cos 3t + \frac{1}{6}\cos 5t - \frac{1}{12}\cos 7t + \frac{1}{180}\cos 9t\right)$$
$$+\cdots \tag{5.446}$$

$$se_1(t,b) = \sin t - \frac{b}{8}\cos 3t + \frac{b^2}{64}\left(\sin 3t + \frac{1}{3}\sin 5t\right)$$
$$-\frac{b^3}{512}\left(\frac{1}{3}\sin 3t + \frac{4}{9}\sin 5t + \frac{1}{18}\sin 7t\right)$$
$$+\frac{b^4}{4096}\left(-\frac{11}{9}\cos 3t + \frac{1}{6}\cos 5t + \frac{1}{12}\cos 7t + \frac{1}{180}\cos 9t\right)$$
$$+\cdots \tag{5.447}$$

$$a_{ce_1}(b) = a_{se_1}(-b) = 1 + b - \frac{b^2}{8} - \frac{b^3}{64} - \frac{b^4}{1536} + \frac{11b^5}{36864}$$
$$+\frac{49b^6}{589824} + \frac{55b^7}{9437184} - \frac{83b^8}{35389440} +$$
$$-\frac{12121b^9}{15099494400} - \frac{114299}{1630745395200}b^{10} + \cdots \tag{5.448}$$

$$ce_2(t,b) = \cos 2t + \frac{b}{12}\cos 4t + \frac{b^2}{96}\left(-\frac{19}{3}\cos 2t + \frac{1}{4}\cos 6t\right)$$
$$-\frac{b^3}{1152}\left(\frac{11}{4}\cos 4t + \frac{1}{20}\cos 8t\right) + \cdots \tag{5.449}$$

$$a_{ce_2}(b) = 4 + \frac{5b^2}{12} - \frac{763b^4}{13824} + \frac{1002401b^6}{79626240} - \frac{1669068401b^8}{458647142400}$$
$$+\frac{4363384401463b^{10}}{3698530556313600} + \cdots \tag{5.450}$$

$$se_2(t,b) = \sin 2t - \frac{b}{12}\sin 4t + \frac{b^2}{96}\left(-\frac{1}{3}\sin 2t + \frac{1}{4}\sin 6t\right) + \frac{b^3}{1536}\left(\cos 4t - \frac{1}{15}\cos 8t\right) + \cdots \tag{5.451}$$

$$a_{se_2}(b) = 4 - \frac{b^2}{12} + \frac{5b^4}{13824} - \frac{289b^6}{79626240} + \frac{21391b^8}{458647142400} - \frac{2499767b^{10}}{3698530556313600} + \cdots \tag{5.452}$$

The coefficients of the Mathieu functions may be found by a set of recursive equations. For $ce_{2k}(t,b)$ with characteristic number $a_{ce_{2k}}$

$$aA_0 - bA_2 = 0 \tag{5.453}$$

$$(a-4)A_2 - b(2A_0 + A_4) = 0 \tag{5.454}$$

$$\left(a - (2j)^2\right)A_{2j} - b\left(A_{2j-2} + A_{2j+2}\right) = 0 \qquad j \geq 2 \tag{5.455}$$

For $ce_{2k+1}(t,b)$ with characteristic number $a_{ce_{2k+1}}$

$$(a-1)A_1 - b(A_1 + A_3) = 0 \tag{5.456}$$

$$\left(a - (2j+1)^2\right)A_{2j+1} - b\left(A_{2j-1} + A_{2j+3}\right) = 0 \qquad j \geq 1 \tag{5.457}$$

For $se_{2k+1}(t,b)$ with characteristic number $a_{se_{2k+1}}$

$$(a-1)B_1 + b(B_1 - B_3) = 0 \tag{5.458}$$

$$\left(a - (2j+1)^2\right)B_{2j+1} - b\left(B_{2j-1} + B_{2j+3}\right) = 0 \qquad j \geq 1 \tag{5.459}$$

For $se_{2k+2}(t,b)$ with characteristic number $a_{se_{2k+2}}$

$$(a-4)B_2 - bB_4 = 0 \tag{5.460}$$

$$\left(a - (2j)^2\right)B_{2j} - b\left(B_{2j-2} + B_{2j+2}\right) = 0 \qquad j \geq 2 \tag{5.461}$$

These equations may be found similarly for other differential equations. Having such recursive equations is the base of several other approximate solutions such as determinant and continued fractions. The goal of all approximate methods at this point is to derive the transition curves $a = a(b)$ as accurate as possible.

5.7 Key Symbols

$a \equiv \ddot{x}$	Acceleration
a, b, A, B	Parameters of the Mathieu equation
(a, b)	Points on periodic curves of Mathieu stability chart
$a_{ce_{2k}}$	Characteristic numbers for Mathieu function ce_{2k}
$a_{ce_{2k+1}}$	Characteristic numbers for Mathieu function ce_{2k+1}
A	Cross-sectional area
$\mathbf{A}$	Coefficient matrix
A, B	Coefficient
b	Lateral distance of a wheel from longitudinal x-axis
b_{final}	The final value of b for which E_{av} being calculated
ce	Cosine elliptic Mathieu function
C_1, C_2	Initial conditions
$e(t)$	Periodic function
$e = c/b$	Coefficient of nonlinear term in Mathieu-Duffing
E	Energy
EI	Flexural rigidity
f, g	Function
F	Initial axial tension
g $\mathbf{g}$	Gravitational acceleration
G	Continued fraction parameter of Mathieu functions
H	Heaviside function
H	Dimensionless height
I	Mass moment
$\mathbf{I}$	Identity matrix
J_i	Integral short notation
k	Stiffness
k_1, k_2, k_3	Parameters of Mathieu-Duffing equation
K	Kinetic energy
l	Length
$\mathcal{L}$	Lagrangian
m	Mass
m	Fractional number $a = m^2, m = p + q, p \in \mathbb{N}, 0 < q < 1$
n	Integer number $a = n^2, n \in \mathbb{N}$
o, O	Origin of a coordinate frame
p	Integer number $p \in \mathbb{N}$
P	Potential energy
P	Dimensionless force
$p(t), q(t)$	Periodic functions
q	Non-integer number $0 < q < 1$
Q	Continued fraction parameter of Mathieu functions
se	Sine elliptic Mathieu function
x	Dependent variable of Mathieu equation

$x, y, z, \mathbf{x}$	Displacement
$\mathbf{x}$	Dependent variable vector
x_d	Axially driven by a displacement
X, Y	Dimensionless displacement
t	Independent variable, time
T	Period
u, v	Functions
w	Well of energy-rate surface
W	Continued fraction parameter of Mathieu functions
$W(t)$	Of this form is called Wronskian
y	Function
Y	Amplitude
z	Continued fraction parameter of Mathieu functions
θ	Angle
φ	Function
$\tau = \omega t$	Dimensionless time
ω	Frequency
ρ	Mass density

Exercises

1. Stability of linear differential equations.
   The stability of linear differential equation may be reduced to an eigenvalue problem. Show that

$$\ddot{x} + 2\xi \dot{x} + \eta x = 0 \tag{5.462}$$

   is stable for $\xi > 0$ and $\eta > 0$.
2. Tangent to characteristic curves at $b = 0$.
   (a) Show that the characteristic curve of a_{ce_0} is tangent to the b-axis.
   (b) Determine the slope of the characteristic curves $a_{ce_{2k}}$, $a_{ce_{2k+1}}$, $a_{se_{2k+1}}$, $a_{se_{2k+2}}$, at $b = 0$ and determine their relationship. Is there any curve with zero slope? Is there any trend in the value of slopes by increasing a?
   (c) Determine the slope of the characteristic curves $a_{ce_{2k}}$, $a_{ce_{2k+1}}$, $a_{se_{2k+1}}$, $a_{se_{2k+2}}$, at $b \to \infty$ and determine their relationship. Are there any slope that curves are approaching? Are there any curves to be asymptotically parallel? Is there any trend in the value of slopes for $b \to \infty$ by increasing a?
3. Intersection of the characteristic curves.
   Prove that the characteristic curves of a_{ce_k} and a_{se_k} have no intersections.
4. Maclaurin series solution of Mathieu equation.
   By successive differentiation of the Mathieu equation determine its power series solution around $t = 0$. In general, it is not possible to determine if such series solution is convergent and periodic. It is not also suitable to check the stability of the Mathieu equation for a pair of (a, b). Check the convergence of the power series solution.
5. Elimination of middle term.
   Consider the general linear equation

$$\ddot{x} + p(t)\,\dot{x} + q(t)\,x = 0 \tag{5.463}$$

   Show that a change of variable

$$u = \exp\left(-\frac{1}{2}\int p dt\right) y \tag{5.464}$$

   will change the equation to

$$\ddot{y} + \left(q - \frac{1}{2}\dot{p} - \frac{1}{4}p^2\right) y = 0 \tag{5.465}$$

   without the middle term.

6. Recursive equations for $x_2(t, b)$.
   Prove the recursive relations for the coefficients $se_{2k+1}(t, b)$ and $se_{2k+2}(t, b)$ in Eqs. (5.230)–(5.233).
7. The matrix expression of characteristic numbers.
   Determine the matrix expression of characteristic numbers $a_{ce_{2k+1}}$, $a_{se_{2k+1}}$ and $a_{se_{2k+2}}$ as well as the determinant of the coefficients of the matrix expressions.
8. Sequence of exponents of natural numbers.

   (a) Show that the recursive equation of the sequence of squares of natural numbers,

   $$x_1 = 1^2 \qquad x_2 = 2^3 \qquad x_3 = 3^2 \qquad x_k = k^2 \tag{5.466}$$

   is

   $$x_{k+3} = 3x_{k+2} - 3x_{k+1} + x_k \tag{5.467}$$

   (b) Show that the recursive equation of the sequence of cubes of natural numbers,

   $$x_1 = 1^3 \qquad x_2 = 2^3 \qquad x_3 = 3^3 \qquad x_k = k^3 \tag{5.468}$$

   is

   $$x_{k+4} = 4x_{k+3} - 6x_{k+2} + 4x_{k+1} - x_k \tag{5.469}$$

9. Mathieu s_{ce} recursive coefficients equation.
   Substitute

   $$x_2(t, b) = \sum_{j=0}^{\infty} B_{j+1}(b) \sin(j+1)t \tag{5.470}$$

   into the Mathieu equation

   $$\frac{d^2x}{dt^2} + (a - 2b\cos 2t)x = 0 \tag{5.471}$$

   and derive the recursive equations

   $$(a-1)B_1 + b(B_1 - B_3) = 0 \tag{5.472}$$

   $$\left(a - (2j+1)^2\right) B_{2j+1} - b\left(B_{2j-1} + B_{2j+3}\right) = 0 \qquad j \geq 1 \tag{5.473}$$

   for $se_{2k+1}(t, b)$ and

$$(a-4)\,B_2 - bB_4 = 0 \tag{5.474}$$

$$\left(a-(2j)^2\right)B_{2j} - b\left(B_{2j-2}+B_{2j+2}\right) = 0 \qquad j \geq 2 \tag{5.475}$$

for $se_{2k+2}(t,b)$.

10. Proof by induction method.
    Prove that

$$\sum_{k=1}^{n} k = \frac{n\,(n+1)}{2} \tag{5.476}$$

$$\sum_{k=1}^{n} k^2 = \frac{1}{6}n\,(n+1)\,(2n+1) \tag{5.477}$$

$$\sum_{k=1}^{n} n^3 = \left(\frac{n\,(n+1)}{2}\right)^2 \tag{5.478}$$

$$\sum_{k=1}^{n} k^4 = \frac{1}{30}n\,(n+1)\,(2n+1)\left(3n^2+3n-1\right) \tag{5.479}$$

$$\sum_{k=1}^{n} k^5 = \frac{1}{12}n^2\,(n+1)^2\left(2n^2+2n-1\right) \tag{5.480}$$

$$\sum_{k=1}^{n} k^m = \frac{n^{m+1}}{m+1} + \frac{n^m}{2} + \frac{mn^{m-1}}{12} - \frac{m\,(m-1)\,(m-2)\,n^{m-3}}{720} + \cdots \tag{5.481}$$

$$\sum_{k=1}^{n} k\,(k+1)^2 = \frac{1}{12}n\,(n+1)\,(n+2)\,(3n+5) \tag{5.482}$$

$$\sum_{k=1}^{n} k!\,k = (n+1)! - 1 \tag{5.483}$$

$$\int_0^\infty x^n e^{-x} dx = n! \tag{5.484}$$

11. Transformation of differential equations.

    (a) Show that the substitution

$$x = \sin z \tag{5.485}$$

transforms the equation

$$\left(x^2-1\right)y''+(2c+1)\,y'-\left(a+bx^2\right)y=0 \tag{5.486}$$

to the form

$$\frac{d^2y}{dz^2}-2c\tan z\frac{dy}{dz}+\left(a+b\sin^2 z\right)y=0 \tag{5.487}$$

(b) Show that the substitution

$$x=\sqrt{\frac{c}{b}}\exp\left(2iz\right) \tag{5.488}$$

transforms the equation

$$x^2y''+xy'+\left(a+bx+\frac{c}{x}\right)y=0 \tag{5.489}$$

to the form

$$\frac{d^2y}{dz^2}-4\left(a+2\sqrt{bc}\cos 2z\right)y=0 \tag{5.490}$$

12. Elliptic coordinates and Mathieu's equation.
In the elliptic or ellcylindrical coordinate system (u, v, w), we have:

$$x=c\cosh u\cos v \qquad y=c\sinh u\sin v \qquad z=w \tag{5.491}$$

the curves u =constant are ellipses and the curves v =constant are portions of hyperbolae all having the same foci at $(\pm c, 0)$.

(a) Show that $u=u_0$ makes (x, y) to show an ellipse.
(b) Show that $v=v_0$ makes (x, y) to show a hyperbola.
(c) Transform the two-dimensional wave equation

$$\frac{d^2w}{dx^2}+\frac{d^2w}{dx^2}=-h^2w \tag{5.492}$$

to

$$\frac{d^2w}{du^2}+\frac{d^2w}{dv^2}+\frac{1}{2}c\left(\cosh 2u-\cos 2v\right)=0 \tag{5.493}$$

(d) Assume a separated solution

$$w = f(u)\, g(v) \tag{5.494}$$

and derive the ordinary equations

$$\frac{d^2 f}{du^2} + \left(-a + \frac{1}{2}c^2 h^2 \cosh 2u\right) f = 0 \tag{5.495}$$

$$\frac{d^2 g}{du^2} + \left(a + \frac{1}{2}c^2 h^2 \cos 2v\right) g = 0 \tag{5.496}$$

where a the separation constant.

(e) Show that by substituting

$$u = iz \tag{5.497}$$

Equation (5.495) becomes (5.496).

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Chapter 6
Energy-Rate Method

In this chapter we review a numerical method to determine stable, unstable, and periodic response of differential equations that their stability depends on relation between parameters. We will use the Mathieu equation as the principal example to develop the method.

6.1 Differential Equations

Differential equations is the language of dynamical systems. A differential equation serves as a model for how the rate of change of states depends on the present and maybe the past states of the system. All differential equations expressing real phenomena are nonlinear and majority of them unsolvable. We may simplify most of the models to be expressible by linear or simpler differential equations. The general form of a linear differential equation of the nth order is

$$\frac{d^n x}{dt^n} + p_1(t)\frac{d^{n-1}x}{dt^{n-1}} + p_2(t)\frac{d^{n-2}x}{dt^{n-2}} + \cdots$$
$$+p_{n-1}(t)\frac{dx}{dt} + p_n(t)\,x = f(t) \tag{6.1}$$

where x, is the *dependent* variable, the unknown function to be determined by integration, and t is the *independent* variable. The coefficients p_1, p_2, $\cdots$, p_n are known functions of t and $f(t)$ is a given *forcing function*. Depending on whether $f(t) = 0$ or $f(t) \neq 0$ the linear equation is called *homogeneous* or *nonhomogeneous*. In many science and engineering applications the functions p_1, p_2, $\cdots$, p_n are constant to make the linear differential equation with constant

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coefficients. This is the simplest form of a linear differential equation and the only class of differential equation that its solution is reduced to that of solving an algebraic equation. Linear autonomous ordinary differential equations have the general form

$$\dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) \tag{6.2}$$

where

$$\mathbf{x}(t) = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T \tag{6.3}$$

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} \dot{x}_1 & \dot{x}_2 & \cdots & \dot{x}_n \end{bmatrix}^T \tag{6.4}$$

and $\mathbf{A}$ is an $n \times n$ matrix. For a given initial value $\mathbf{x}(0)$, the solution of (6.2) can be written in the form

$$\mathbf{x}(t) = \mathbf{x}(0)\ \exp(\mathbf{A}t) \tag{6.5}$$

where the matrix exponential $\exp(\mathbf{A}t)$ is defined by the Taylor series of the exponential function

$$\exp(\mathbf{A}t) = \sum_{k=0}^{\infty} \frac{1}{k!}\mathbf{A}^k\, t^k \tag{6.6}$$

$$\mathbf{A}^k = \mathbf{I} \tag{6.7}$$

with $\mathbf{I}$ being the identity matrix.

The solution of linear inhomogeneous ordinary differential equations

$$\dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}(t) \tag{6.8}$$

is

$$\mathbf{x}(t) = \mathbf{x}_0\ \exp(\mathbf{A}(t - t_0)) + \int_{t_0}^{t} \exp(\mathbf{A}(t - s))\,\mathbf{B}(s)\,ds \tag{6.9}$$

$$\mathbf{x}_0 = \mathbf{x}(t_0) \tag{6.10}$$

Proof The solution of linear autonomous ordinary differential equations (6.2) follows the same rules of single differential equation $\dot{x} = Ax$, which is $x = x(0)\,e^{At}$. To show that (6.5) is a solution for (6.2), we only need to show that it satisfies the equation. Knowing that

$$\frac{d}{dt}e^{\mathbf{A}t} = \mathbf{A}e^{\mathbf{A}t} \tag{6.11}$$

$$e^{\mathbf{0}} = \mathbf{I} \tag{6.12}$$

$$\mathbf{IA} = \mathbf{A} \tag{6.13}$$

shows that (6.5) satisfies Eq. (6.2).

$$\frac{d}{dt}\left(\mathbf{x}_0 e^{\mathbf{A}t}\right) = \mathbf{x}_0 \frac{d}{dt}e^{\mathbf{A}t} = \mathbf{x}_0 \mathbf{A}e^{\mathbf{A}t} \tag{6.14}$$

$$\frac{d}{dt}e^{\mathbf{A}t} = \mathbf{A}e^{\mathbf{A}t} \tag{6.15}$$

The derivative (6.11) is a special case of the general rule of derivative of exponential matrix function (Wilcox 1967).

$$\frac{d}{dt}e^{\mathbf{X}(t)} = \int_0^1 e^{s\mathbf{X}(t)} \frac{d\mathbf{X}(t)}{dt} e^{(1-s)\mathbf{X}(t)} ds \tag{6.16}$$

The solution (6.9) for the inhomogeneous linear differential equation (6.8) comes from variation parameter method. We search for a solution in the form

$$\mathbf{x}(t) = \mathbf{g}(t)\, e^{\mathbf{A}t} \tag{6.17}$$

where $\mathbf{g}(t)$ is a differentiable function. Therefore,

$$\frac{d}{dt}\mathbf{x}(t) = \mathbf{A}e^{\mathbf{A}t}\, \mathbf{g}(t) + e^{\mathbf{A}t}\dot{\mathbf{g}}(t) \tag{6.18}$$

Substituting (6.17) and (6.18) into Eq. (6.8) yields:

$$e^{\mathbf{A}t}\dot{\mathbf{g}}(t) = \mathbf{B}(t) \tag{6.19}$$

which implies:

$$\mathbf{g}(t) = \mathbf{C} + \int_{t_0}^{t} e^{-\mathbf{A}s}\, \mathbf{B}(s)\; ds \tag{6.20}$$

where $\mathbf{C}$ is a constant vector. Now the solution is:

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{C} + e^{\mathbf{A}t} \int_{t_0}^{t} e^{-\mathbf{A}s}\, \mathbf{B}(s)\; ds \tag{6.21}$$

The initial condition $\mathbf{x}_0$ determines $\mathbf{C}$.

$$\mathbf{C} = e^{-\mathbf{A}t_0}\mathbf{x}_0 \tag{6.22}$$

Hence, we have

$$\mathbf{x}(t) = \mathbf{x}_0\, e^{-\mathbf{A}(t-t_0)} + \int_{t_0}^{t} e^{\mathbf{A}(t-s)}\mathbf{B}(s)\, ds \tag{6.23}$$

The main feature of linear differential equations is that they obey the principle of superposition. ■

Example 198 Solve $\dot{x} + p(t)\, x = f(t)$.

Assuming $p(t) \neq 0$, $f(t) \neq 0$, we use the identity

$$\frac{d}{dt}\left(xe^{h(t)}\right) = \dot{x}e^{h(t)} + xp(t)\, e^{h(t)} \tag{6.24}$$

$$h(t) = \int p(t)\, dt \neq 0 \tag{6.25}$$

to rewrite the differential equation in a new form

$$\frac{d}{dt}\left(xe^{h(t)}\right) = e^{h(t)} f(t) \tag{6.26}$$

Integrating both sides implies:

$$x = e^{-h(t)}\left(C + \int e^{h(t)} f(t)\, dt\right) \tag{6.27}$$

A simple example of this equation is

$$\dot{x} + ax = b \tag{6.28}$$

for which we have

$$p(x) = a \qquad f(t) = b \tag{6.29}$$

and

$$h(t) = \int a dt = at + C_1 \tag{6.30}$$

and the solution is:

$$x(t) = e^{-at-C_1}\left(C_2 + \int be^{at+C_1}dt\right)$$
$$= e^{-at-C_1}\left(C_2 + \frac{1}{a}be^{C_1+at}\right) = \frac{b}{a} + Ce^{-at} \quad (6.31)$$
$$C = C_2 e^{C_1} \quad (6.32)$$

The steady state condition x_s of the equation is:

$$x_s = \lim_{t\to\infty} x(t) = \begin{cases} \frac{b}{a} & a > 0 \\ \infty & a < 0 \end{cases} \quad (6.33)$$

Linearity of the equation implies that if $a > 0$ and we set the initial condition $x(0)$ of the system to be $x(0) = b/a$, then the system will stay at the initial condition forever.

6.2 Mathieu Stability Chart

Linear differential equations with variable coefficients have a great variety of applications. The well-known Mathieu equation is one of the simplest cases with complicated behavior. The main desired perception of this equation is its stability chart in (a, b)-plane.

The stability of the Mathieu equation

$$\frac{d^2x}{dt^2} + (a - 2b\cos 2t)\,x = 0 \quad (6.34)$$

for a pair of constant parameters (a, b) can be checked by the sign of energy-rate integral

$$E_{av} = \frac{1}{T}\int_0^T \dot{E}dt = \frac{1}{T}\int_0^T (2abx\dot{x}\cos 2t)\,dt \quad (6.35)$$
$$T = 2\pi \quad (6.36)$$

The pair (a, b) is in an unstable region if $E_{av} > 0$, it is on a periodic characteristic curve if $E_{av} = 0$, and it is in a stable region if $E_{av} < 0$.

Proof Let us analyze the following general differential equation:

$$\ddot{x} + f(x) + g(x, \dot{x}, t) = 0 \quad (6.37)$$
$$g(0, 0, t) = 0 \quad (6.38)$$
$$g(x, \dot{x}, t + T) = g(x, \dot{x}, t) \quad (6.39)$$

where $f(x)$ is a single variable odd function, and the function $g(x, \dot{x}, t)$ might be nonlinear and time dependent. The Mathieu equation is an example of (6.37) for which we have:

$$f(x) = ax \tag{6.40}$$

$$g(x, \dot{x}, t) = 2bx \cos 2t \tag{6.41}$$

Let rewrite Eq. (6.37)

$$\ddot{x} + f(x) = -g(x, \dot{x}, t) \tag{6.42}$$

to indicate a model of a unit mass attached to a conservative spring $f(x)$, acted upon by a non-conservative force $-g(x, \dot{x}, t)$. The free motion of the system is governed by the homogeneous equation

$$\ddot{x} + f(x) = 0 \tag{6.43}$$

which would have a periodic motion with constant amplitude depending on the initial conditions. The applied force $-g(x, \dot{x}, t)$ generates or absorbs energy depending on the values of its parameters, time t, and state of the system $(x, \dot{x})$. The stiffness function $f(x)$ is a restoring force, providing potential energy to put the free system into oscillate (Esmailzadeh et al. 1996). Defining a kinetic K, potential P, and mechanical E energies for the system

$$K(\dot{x}) = \frac{1}{2}\dot{x}^2 \tag{6.44}$$

$$P(x) = \int f(x)\, dx \tag{6.45}$$

$$E = K(\dot{x}) + P(x) \tag{6.46}$$

we write an integral of energy rate for Eq. (6.42).

$$\dot{E} = \frac{d}{dt}\left(\frac{1}{2}\dot{x}^2 + \int_0^T f(s)\,\dot{x}\, ds\right) = -\dot{x}g(x, \dot{x}, t) \tag{6.47}$$

This equation represents the energy-rate of absorbed or dissipated by the forcing function $-g(x, \dot{x}, t)$ during the time from 0 to T.

Assume for a set of parameters and a nonzero response $x(t)$, $\dot{E}$ is negative for $T > 0$, then E continuously decreases along the path of $x(t)$ and it will be reflected in decreasing the amplitude $x(t)$. On the contrary if $\dot{E}$ is positive for $T > 0$, then E continuously increases along the path of $x(t)$ and it will be reflected in growing the amplitude $x(t)$. The case of $\dot{E} = 0$ indicates a net zero energy reflecting in steady state constant amplitude oscillation of $x(t)$. A negative energy rate $\dot{E} < 0$ shows that $g(x, \dot{x}, t)$ consumes energy, and $\dot{E} > 0$ indicates that $g(x, \dot{x}, t)$ injects energy into the system in every period T. In either case of nonzero energy rate, $\dot{E} \neq 0$, the amplitude $x(t)$ will change and it may go to infinity, $x \to \infty$, or go to zero, $x \to 0$, or the amplitude changes until a state of $\dot{E} = 0$ appears.

The duration of integral should be equal to the period T we are looking for the system to have; however, the limit of integral $\int_0^T f(s)\,\dot{x}\,ds$ may be nT or T/n, $n = 1, 2, 3, \ldots$, or any other rational or irrational number that we expect to be the period of a constant amplitude oscillation of $x(t)$. To have a periodic oscillation, the time derivative of mechanical energy E must be zero over one period T for systems in a steady state periodic cycle (Jazar et al. 2008). Searching for T-periodic conditions will include any periodic conditions of T/n, $n = 1, 2, 3, \ldots$ as well, since a (T/n)-periodic motion will also be T-periodic.

Integrating the Mathieu equation (6.34), we get

$$\frac{1}{2}\frac{d}{dt}\left(\dot{x}^2 + ax^2\right) = 2bx\dot{x}\cos 2t = \dot{E} \tag{6.48}$$

To find a set of (a, b) to indicate a steady state periodic response, we may choose an arbitrary pair of parameters (a, b) and integrate equation (6.48) numerically to evaluate the averaged energy E_{av} over the period T.

$$E_{av} = \frac{1}{T}\int_0^T \dot{E}dt = \frac{1}{T}\int_0^T (2abx\dot{x}\cos 2t)\,dt \tag{6.49}$$

If $E_{av} > 0$, then the chosen (a, b) belongs to an unstable region in which energy being inserted to the system. However, if $E_{av} < 0$, then the chosen (a, b) belongs to a stable region where energy being extracted from the system. On the common boundary of these two regions, $E_{av} = 0$, indicating that the chosen (a, b) is on a characteristic periodic curve.

To determine a pair (a, b) for periodic response, we fix b and searching for a to be on a boundary that its left-hand side is stable and its right-hand side is unstable. Such characteristic curve would be an a_{ce}. If the chosen parameters show that $E_{av} < 0$, then increasing a increases E_{av}. On the other hand, if $E_{av} > 0$, then decreasing a decreases E_{av}. Employing this method we may find the appropriate value of a such that $E_{av} = 0$ and we find a point on the boundary that has a stable region on its left-hand side and unstable region on its right-hand side. This group of points constitutes a branch of $a_{ce_{2k}}$ with π-periodic or of $a_{ce_{2k+1}}$ with 2π-periodic boundary corresponding to ce_n.

By changing the strategy and looking for a to be on a boundary that its right-hand side is stable and its left-hand side is unstable, we can detect the π-periodic and 2π-periodic branches corresponding to se_n. Now we may change b by some increment and repeat the procedure to determine all points on characteristic curves. We cast the procedure arranged in an algorithm (Jazar 2004).

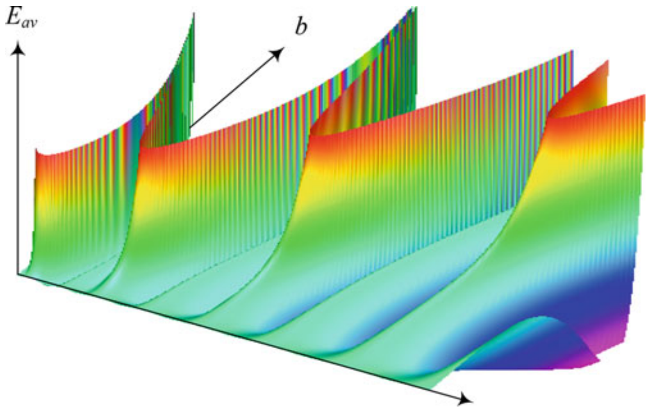

Fig. 6.1 The energy surface $E_{av} = E_{av}(a, b)$ for the Mathieu equation

Stability Chart Algorithm

1. set a equal to one of its special values $n^2 = a$
2. set b equal to some arbitrary small value
3. solve the differential equation numerically
4. evaluate E_{av}
5. decrease (increase) a if $E_{av} > 0$ ($E_{av} < 0$) by some small increment
6. the increment of a must be decreased if $E_{av}(a_i) \cdot E_{av}(a_{i-1}) < 0$
7. save a and b when $E_{av} << 1$
8. while $b < b_{final}$, increase b and go to step 3
9. set a equal to another special value and go to step 2
10. reverse the decision in step 5 and go to step 1

Theoretically the energy-rate method may be applied for any parametric and nonlinear system to determine their stability chart (Esmailzadeh and Jazar 1997; Mahmoudian et al. 2004; Christopherson and Jazar 2005, 2006; Jazar et al. 2006, 2008, 2009; Cveticanin and Kovacic 2007; Sochacki 2008; Sheikhlou et al. 2013; Cveticanin 2014; Platonov 2018). ■

Example 199 Energy surface.

Definition of E_{av} is equivalent to assigning a third dimension to the (a, b)-plane making a surface of energy $E_{av} = E_{av}(a, b)$. Figure 6.1 illustrates such an energy surface for the Mathieu equation. The energy surface will look like a mountain that its intersection with the plane of $E_{av} = 0$ will show the curves on which the response of the equation will be 2π-periodic.

$$E_{av} = \frac{1}{2\pi} \int_0^{2\pi} \dot{E} dt = \frac{1}{2\pi} \int_0^{2\pi} (2abx\dot{x} \cos 2t)\, dt \tag{6.50}$$

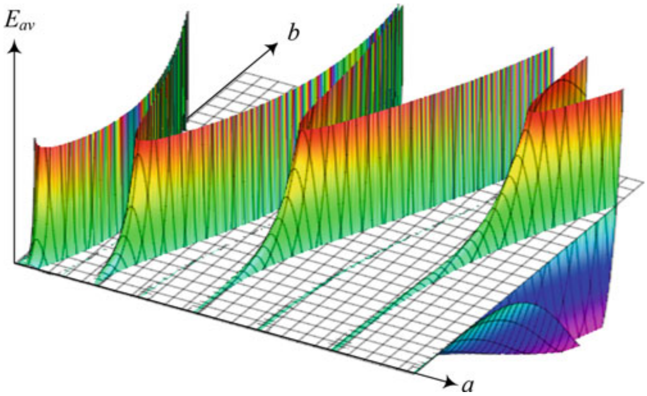

Fig. 6.2 The energy surface $E_{av} = E_{av}(a, b)$ for the Mathieu equation cut by the zero plane $E_{av} = 0$

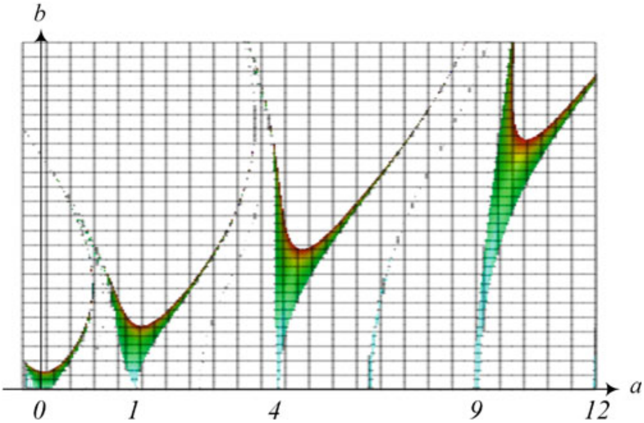

Fig. 6.3 The top view of the energy surface $E_{av} = E_{av}(a, b)$ for the Mathieu equation cut by the zero plane $E_{av} = 0$

Figure 6.2 illustrates the plane $E_{av} = 0$ and the energy surface. Any part of the energy surface that is below the plane $E_{av} = 0$ belongs to the points (a, b) for which the energy rate is negative and hence the Mathieu equation is stable. Similarly, any part of the energy surface above the plane $E_{av} = 0$ belongs to the points (a, b) for which the energy rate is positive and the equation is stable. Figure 6.3 shows a top view of the energy surface to show the intersection with zero energy plane.

Example 200 Stability chart from energy surface.

The energy-rate method provides an energy surface with unstable values associated to $E_{av} > 0$, and stable values associated to $E_{av} < 0$. According to the definition of E_{av} in (6.35), $E_{av} = 0$ indicates the conditions for which the Mathieu equation shows a periodic response with period $T = 2\pi$ as well as

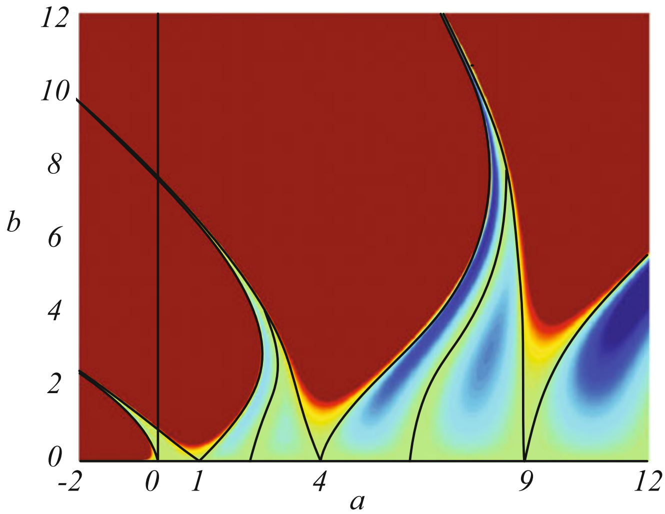

Fig. 6.4 The top view of a higher resolution of the energy E_{av} surface of the Mathieu equation for $-2 < a < 12$ and $0 < b < 12$. The surface is colored proportional to the magnitude of the energy E_{av}, blue the deepest and red the highest value

$T = 2\pi/n$, $n = 1, 2, 3, \ldots$. Using the energy surface, the periodic conditions will be graphically equivalent to the intersection of the energy surface and the zero plane of energy. The energy-rate method and graphical search for the stability chart of parametric differential equation not only show the stability curves, it may reveal other information about the behavior of the equation.

Figure 6.4 illustrates a higher resolution of the energy E_{av} surface of the Mathieu equation for $-2 < a < 12$ and $0 < b < 12$. The surface is colored by the magnitude of the energy E_{av}, blue the deepest and red the highest value. The relative stability of two different points (a_1, b_1) and (a_2, b_2) may be interpreted by their level of E_{av} such that the lower value of E_{av} indicates more stable points. Hence, the blue color indicates where in the map the Mathieu equation is more stable. There are three blue wells in this figure showing where would be more stable areas. The orange and red combination color indicates the unstable regions. The lines of zero energy are drawn in black and to be clearer, we have shown them also in Fig. 6.5.

Besides the periodic characteristic lines that separate the stable and unstable regions, two other single lines can be seen in the stable zones of $1 < a < 4$ and $4 < a < 9$ starting from $b = 0$ and merging to their right periodic curves. Being a single line indicates that the energy surface is under the zero level on both sides of that line and it touches the plane $E_{av} = 0$ on those curves. Therefore, they are 2π-periodic characteristic curves within the stable zones since the energy surface and its zero plane intersection in Figs. 6.4 and 6.5 are the results of integral of energy rate over $0 < t < 2\pi$,

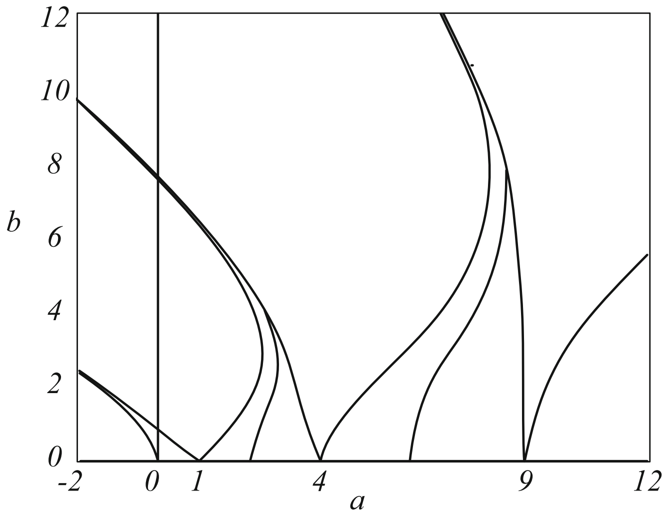

Fig. 6.5 The lines of zero energy of the energy E_{av} surface of the Mathieu equation for $-2 < a < 12$ and $0 < b < 12$

$$E_{av} = \frac{1}{2\pi} \int_0^{2\pi} \dot{E} dt = \frac{1}{2\pi} \int_0^{2\pi} (2abx\dot{x} \cos 2t) \, dt \tag{6.51}$$

the characteristic curves in the figures are all for point (a, b) that provides 2π as well as π-periodic responses. To determine only the π-periodic curves graphically, we need to integrate the energy rate over $0 < t < \pi$. The periodic response for fractional $a = m^2$ has been discussed analytically in Example 185. If $a = m^2$, $a \neq n^2$, n being an integer and $m = p+q$, with p being integer and $0 < q < 1$, then if q is expressed as α/β, where α and β are integers with no common factors, then the coexist $ce_m (b, t)$ and $se_m (b, t)$ are periodic with period $\pi\beta$ and $2\pi\beta$ depending on α to be even or odd. By increasing the limit of the integral to $3\pi, 4\pi, \ldots$ we are able to detect other periodic lines within the stable zones.

Example 201 Enery-rate integral function.

To show how the function of energy rate looks like and how it determines the characteristic and periodic values of (a, b), we plot one curve of $E_{av} = E_{av} (a)$. Figure 6.6 illustrates the Mathieu stability chart based on the McLachlan data reported in Table 5.1. Let us set $b = 5$, which is large enough to be beyond the limit of accuracy of the series solutions. We then walk along the line $b = 5$ on the stability chart from $a = 0$ to $a = 45$ and calculate $E_{av} = \frac{1}{2\pi} \int_0^{2\pi} \dot{E} dt$. To do this, we must set a value for a and then numerically solve the Mathieu equation to have the states $x = x(t)$, $\dot{x} = \dot{x}(t)$, and then calculate $E_{av} = \frac{1}{2\pi} \int_0^{2\pi} (2abx\dot{x} \cos 2t) \, dt$ to determine the energy rate at one point of the line $b = 5$. Repeating the calculation

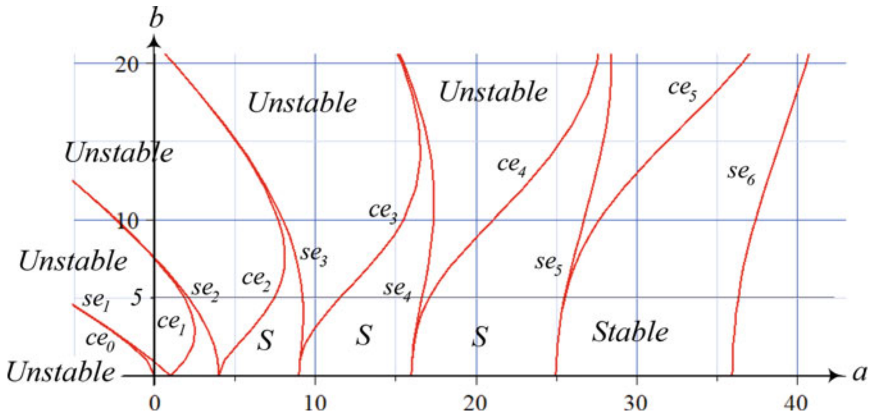

Fig. 6.6 The stability chart of the Mathieu equation based on the characteristic numbers reported by McLachlan in Table 5.1

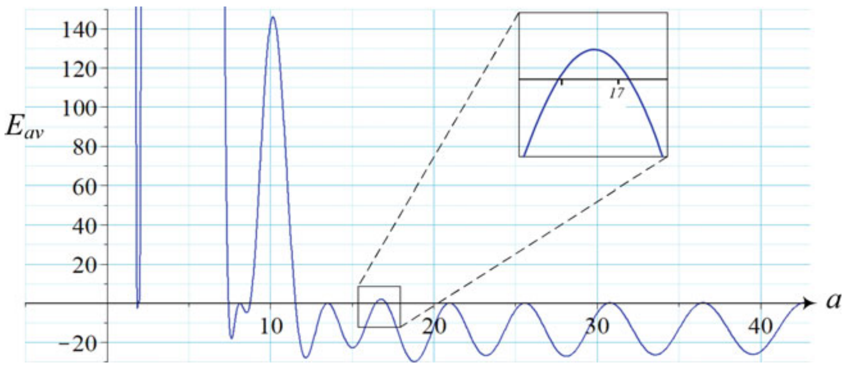

Fig. 6.7 The graph of the $E_{av} = E_{av}(a)$ for $b = 5$

for enough number of points on the line $b = 5$ makes the graph of the $E_{av} = E_{av}(a)$ plotted in Fig. 6.7. Match the graph of E_{av} and the line $b = 5$ indicates that wherever the line $b = 5$ is in unstable region, we have $E_{av} > 0$ and when the line $b = 5$ is in stable region, we get $E_{av} < 0$. The crossing points of $E_{av} = 0$ match with characteristic values of a. A sample of $E_{av} > 0$ has been magnified to show the positive E_{av} and characteristic points.

Example 202 Modified algorithm for exact characteristic numbers.

The energy-rate method provides a fine numerical method, capable of increasing the exactness of the calculation of the characteristic numbers as accurate as desired. In order to find more accurate values than those presented in Table 5.1, we may use the elements of McLachlan's table as an initial value and modify the stability chart algorithm to be able to detect splitting lines accurately. The following algorithm detects splitting lines of the Mathieu equation. It can be extended to cover the other parametric differential equation whose stability chart is two dimensional or could be reduced to two dimensional (Jazar 2004).

Periodic Chart Algorithm

0. set the period of integration T equal to a multiple of 2π
1. set a, equal to a_1, and a_2, two arbitrary values in a stable region
2. set b equal to some arbitrary small value
3. solve the differential equation numerically for both pairs of a and b
4. evaluate $(E_{av})_1$ and $(E_{av})_2$
5. set $a_1 = a_2$
6. increase a_2 by some small increment if

$$|(E_{av})_1| > |(E_{av})_2| \qquad and \qquad a_2 - a_1 > 0$$

   or

$$|(E_{av})_1| < |(E_{av})_2| \qquad and \qquad a_2 - a_1 < 0$$

   else decrease a_2 if

$$|(E_{av})_1| > |(E_{av})_2| \qquad and \qquad a_2 - a_1 < 0$$

   or

$$|(E_{av})_1| < |(E_{av})_2| \qquad and \qquad a_2 - a_1 > 0$$

7. the increment of a must be decreased if a_2 repeats twice
8. save a and b when $a_2 - a_1 << 0$
9. while $b < b_{final}$, increase b and go to step 1
10. set a, equal to a_1, and a_2, two arbitrary values in another stability region and go to step 2

Employing the algorithm we determine the periodicity chart depicted in Fig. 6.8. The numerical values of $a = a(b)$ would be the outcome of the algorithm.

Example 203 Numerical values of characteristic numbers.

We apply the Stability Chart Algorithm to determine the characteristic numbers of the Mathieu equation to compare with the characteristic values reported by McLachlan (1947) in Table 5.1. To increase the efficiency of the calculation and save time of the computer, we may pick numbers of the McLachlan's table and apply the Stability Chart Algorithm to increase the accuracy. The numbers of Table 6.1 are associated with the values in Table 5.1 with better accuracy.

Example 204 More exact characteristic values.

How important is to determine the exact transient curve of a parametric equation depends on many parameters such as the physical system, the application, the time duration the system is supposed to work periodically, etc. If mistakenly the design

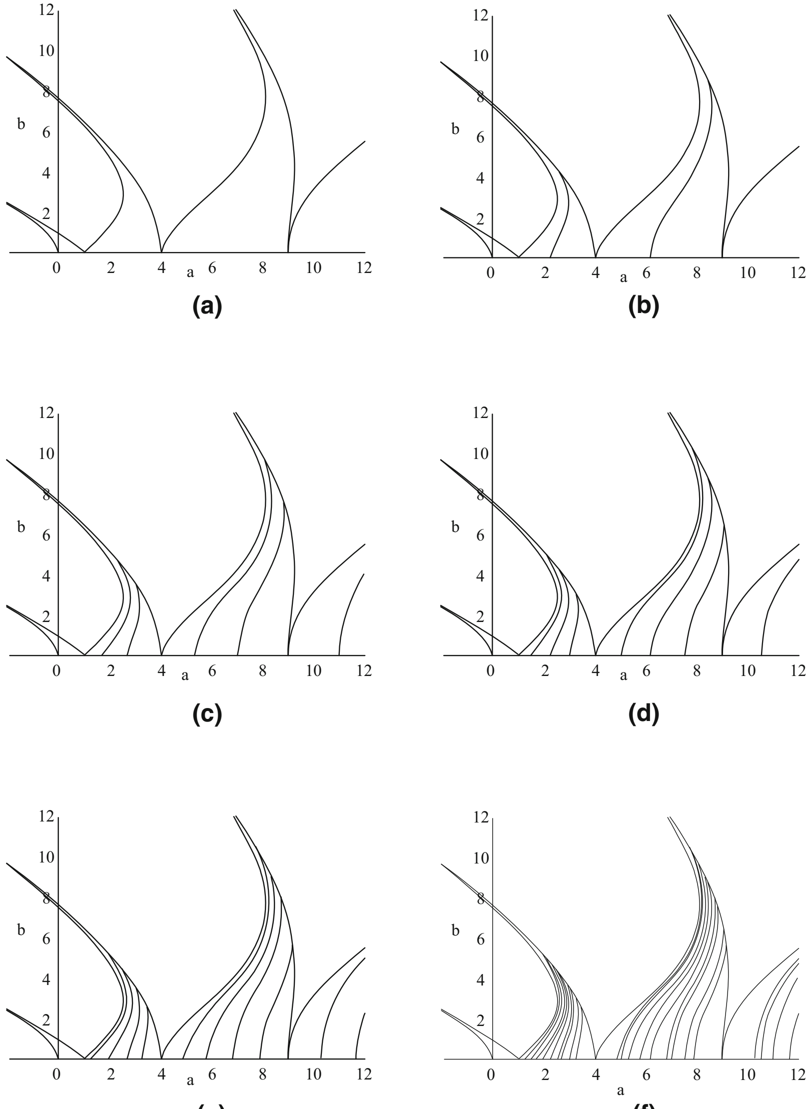

Fig. 6.8 The Mathieu periodicity chart. (**a**) π-periodic lines; (**b**) 2π-periodic lines; (**c**) 3π-periodic lines; (**d**) 4π-periodic lines; (**e**) 5π-periodic lines; (**f**) 6π-periodic lines

Table 6.1 The characteristic numbers of Mathieu equation calculated by the Periodic Chart Algorithm

b	0	1	2	3	4
a_{ce0}	0	−0.45513856	−1.51395682	−2.83439180	−4.28051869
a_{se1}	1	−0.1102491	−1.3906768	−2.7853800	−4.2591832
a_{ce1}	1	1.8591078	2.3791999	2.51903917	2.31800824
a_{se2}	4	3.9170245	3.6722324	3.2769217	2.7468807
a_{ce2}	4	4.3713007	5.1726648	6.0451966	6.8290745
a_{se3}	9	9.0477390	9.14062247	9.2231325	9.2614458
a_{ce3}	9	9.0783685	9.3703222	9.91155033	10.6710268
a_{se4}	16	16.0329698	16.1276877	16.2727009	16.452035
a_{ce4}	16	16.033832	16.1412035	16.3387204	16.6468186
a_{se5}	25	25.0208411	25.0833487	25.1870795	25.3305446
a_{ce5}	25	25.020854	25.0837775	25.1902852	25.3437573
a_{se6}	36	36.0142902	36.0572073	36.1288715	36.22944084

b	5	6	7	8
a_{ce0}	−5.80004589	−7.36883067	−8.97374233	−10.606729
a_{se1}	−5.7900809	−7.3639113	−8.9712027	−10.6053684
a_{ce1}	1.85818761	1.21427824	0.43834918	−0.43594351
a_{se2}	2.0994601	1.3513809	0.5175451	−0.3893621
a_{ce2}	7.4491094	7.87006478	8.08662334	8.11523907
a_{se3}	9.2363274	9.1379055	8.9623852	8.7099141
a_{ce3}	11.5488317	12.4656004	13.358421	14.1818801
a_{se4}	16.6482196	16.8446013	17.0266605	17.1825275
a_{ce4}	17.0965814	17.6887827	18.4166084	19.2527048
a_{se5}	25.5108157	25.7234104	25.9624469	26.2209992
a_{ce5}	25.5499714	25.8172717	26.1561199	26.577753
a_{se6}	36.3588665	36.5170664	36.7035024	36.9172128

b	9	10	12	14
a_{ce0}	−12.2624140	−13.9369798	−17.3320658	−20.7760551
a_{se1}	−12.261662	−13.9365528	−17.3319187	−20.7760007
a_{ce1}	−1.38670147	−2.39914228	−4.57013271	−6.89340038
a_{se2}	−1.3588104	−2.3821585	−4.5635402	−6.890701
a_{ce2}	7.98284339	7.717370027	6.87873702	5.73631258
a_{se3}	8.3831189	7.9860688	7.0005665	5.7926292
a_{ce3}	14.9036794	15.5027841	16.3015352	16.5985408
a_{se4}	17.3030107	17.3813804	17.3952494	17.207115
a_{ce4}	20.1609261	21.1046334	22.9721272	24.6505948
a_{se5}	26.4915469	26.7664261	27.3000121	27.7697664
a_{ce5}	27.0918658	27.7037684	29.2080547	31.0000505
a_{se6}	37.1566947	37.4198585	38.0060084	38.6484716

(continued)

Table 6.1 (continued)

b	16	18	20	24
a_{ce0}	−24.25867922	−27.7728419	−31.3133898	−38.4589729
a_{se1}	−24.2586581	−27.7728335	−31.3133865	−38.45897241
a_{ce1}	−9.33523701	−11.87324232	−14.49130121	−19.92259541
a_{se2}	−9.9.33411	−11.8727268	−14.4910636	−19.9225406
a_{ce2}	4.37123284	2.83305697	1.15422859	−2.5397654
a_{se3}	4.3978959	2.8459914	1.1607054	−2.5380782
a_{ce3}	16.4868846	16.0619757	15.3958112	13.5228430
a_{se4}	16.8186834	16.2420801	15.4939773	13.5527962
a_{ce4}	26.0086780	26.9877667	27.5945785	27.8854411
a_{se5}	28.1363587	28.3738579	28.468221	28.2153591
a_{ce5}	32.9308948	34.8530584	36.6449894	39.5125516
a_{se6}	39.3150105	39.9723508	40.5896638	41.6057096

b	28	32	36	40
a_{ce0}	−45.6733693	−52.9422227	−60.2555676	−67.6061519
a_{se1}	−45.6733697	−52.9422232	−60.2555682	−67.6061525
a_{ce1}	−25.56174683	−31.36515415	−37.3026388	−43.352275
a_{se2}	−25.5617332	−31.3651508	−37.30263823	−43.3522747
a_{ce2}	−6.5880627	−10.9143531	−15.46677	−20.2079405
a_{se3}	−6.5875853	−10.9142093	−15.4667246	−20.2079257
a_{ce3}	11.1110801	8.2914965	5.1456366	1.7296494
a_{se4}	11.1206224	8.2946718	5.1467372	1.7300453
a_{ce4}	27.2833085	26.0624485	24.3785097	22.3252766
a_{se5}	27.4057485	26.1083523	24.3960662	22.3321482
a_{ce5}	41.2349506	41.9535115	41.9266649	41.3497547
a_{se6}	42.2248412	42.3939425	42.1183558	41.4330049

parameters of the system, embedded in a and b, are set in an unstable region instead of being periodic or stable, then system will not be working as expected and may be malfunctioning.

In order to investigate the applicability of the Periodic Chart Algorithm, let us pick two associated values from Tables 5.1 and 6.1 to solve the Mathieu equation for. Due to physical consideration, the first stable zone surrounded a_{ce0} and a_{se1} and the first unstable zone surrounded a_{se1} and a_{ce1} are the most important working zones for dynamical systems (Bolotin 1964). Table 6.2 compares the numerical values of a few characteristic points and their E_{av} on both sides of the first stable zone. Comparing E_{av} (Table 5.1) and E_{av} (Table 6.1) indicates the points of Table 6.1 are closer to the characteristic curves of the Mathieu stability chart. The positive values of E_{av} indicates that all points of (a, b) in Table 6.2 are actually in unstable zone.

Let us compare the points

$$(a_1, b) = (a_{ce0}, 1) = (-0.4551386, 1) \tag{6.52}$$

Table 6.2 The characteristic numbers of Mathieu stability chart in the first stable region surrounded a_{ce0} and a_{se1}

	Table 5.1	Table 6.1	Table 5.1	Table 6.1
b	a_{ce0}	a_{ce0}	E_{av}	E_{av}
1	−0.4551386	−0.455138558	0.000004185	4.0244×10^{-9}
2	−1.5139569	−1.51395682	0.000189536	6.6157×10^{-9}
3	−2.8343919	−2.83439180	0.002687308	0.000002693
4	−4.2805188	−4.28051869	0.022774871	0.000205379
5	−5.8000460	−5.80004580	0.158296344	−0.000023016
6	−7.3688308	−7.36883067	1.018318348	0.000015843
7	−8.9737425	−8.97374233	6.754786779	0.003246531
8	−10.6067292	−10.60672904	25.935317739	0.014145193
9	−12.2624142	−12.26241402	116.91947068	0.157082664
10	−13.93698	−13.93697977	564.37545616	5.003945936

	Table 5.1	Table 6.1	Table 5.1	Table 6.1
b	a_{se1}	a_{se1}	E_{av}	E_{av}
1	−0.1102488	−0.1102491	18.83992146	18.83990077
2	−1.3906765	−1.3906768	231.5834267	−0.00000135
3	−2.7853797	−2.785380	1237.418025	231.582039
4	−4.2591829	−4.2591832	4680.5734	1237.39717
5	−5.7900806	−5.7900809	14,549.4642	4680.38628
6	−7.3639110	−7.3639113	39,711.5358	14,548.2046
7	−8.9712024	−8.9712027	98,632.4669	39,704.5966
8	−10.6053681	−10.6053684	2.2791×10^{5}	98,598.143
9	−12.2616617	−12.261662	4.96789×10^{5}	2.27763×10^{5}
10	−13.9365525	−13.9365528	1.032726×10^{6}	4.96215×10^{5}

Comparing data of Tables 5.1 and 6.1

from Table 5.1 and

$$(a_2, b) = (a_{ce0}, 1) = (-0.455138558, 1) \tag{6.53}$$

from Table 6.1. Figure 6.9 depicts the time response of the Mathieu equation for the two points during $0 < t < 2\pi$. The difference of the two associated values of (a, b) is not clear from the time response of the Mathieu equation in short period of time as both points are close to the periodic line. However, their difference will eventually shows up, as can be seen in Fig. 6.10 that indicates the time response of the equation for the two points during $100\pi < t < 102\pi$.

Figure 6.11 shows the difference between the time responses, $\Delta x = x(a_1) - x(a_2)$ during $0 < t < 10\pi$. Because $\Delta x > 0$ and is growing, point (a_1, b) is more in unstable zone than (a_2, b). Figure 6.12 illustrates how energy rate $\dot{E}$ is varying during $0 < t < 2\pi$ for $(a, b) = (a_{ce0}, 1)$ with zero integral, $E_{av} = 0$.

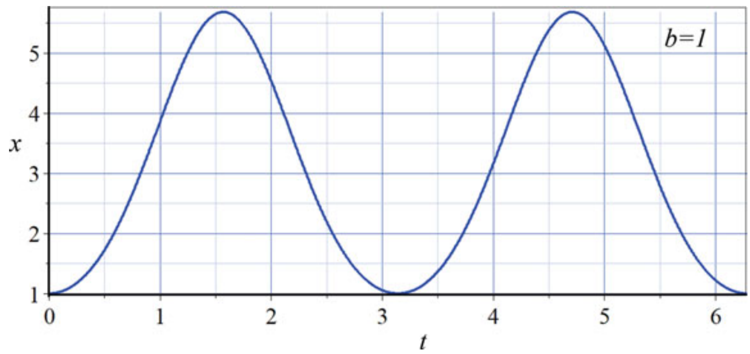

Fig. 6.9 Comparison of the time response of the Mathieu equation for points $(a_1, b) = (-0.4551386, 1)$ and $(a_2, b) = (-0.455138558, 1)$. The response looks identical for the period $0 < t < 2\pi$

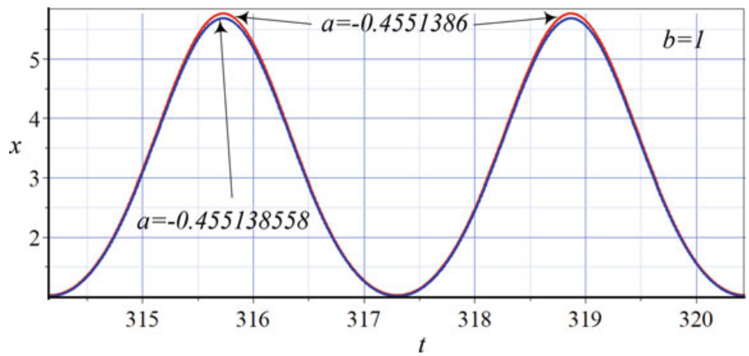

Fig. 6.10 Comparison of the time response of the Mathieu equation for points $(a_1, b) = (-0.4551386, 1)$ and $(a_2, b) = (-0.455138558, 1)$. The response began to deviate for the period $100\pi < t < 102\pi$

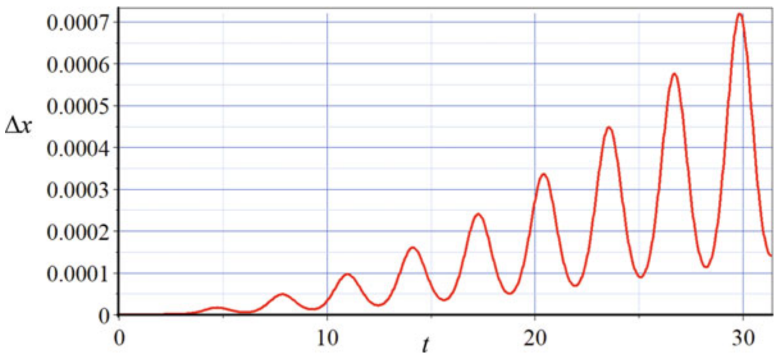

Fig. 6.11 The difference between the time responses, $\Delta x = x(a_1) - x(a_2)$ during $0 < t < 10\pi$

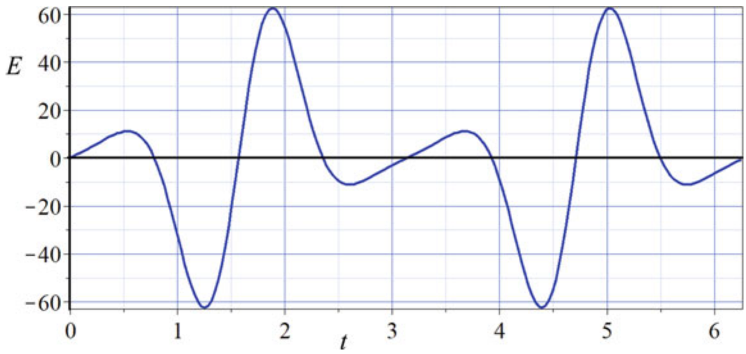

Fig. 6.12 The energy rate $\dot{E}$ variation during $0 < t < 2\pi$ for $(a, b) = (a_{ce0}, 1)$

Example 205 ★Behavior of Mathieu equation in an stable zone.

In this example we apply the energy-rate method and utilize the power spectral density to investigate the behavior of Mathieu equation within a stable zone. Each stable zone is bounded by two transition curves starting from integer roots of $a = n^2, n = 0, 1, 2, 3, \ldots$. Visualizing the behavior of Mathieu equation using energy-rate surface and spectral density will give us a better understanding how different points of a stable zone characterize the system. The first stable zone is bounded between transient curves started from $a = 0$ and $a = 1$; the second zone bounded between transient curves started from $a = 1$, $a = 4$, and so on. We will go across the third stable zone, which is bounded between curves starting from $a = 4$ and $a = 9$, as an example. A three-dimensional view of 2π-periodic surface in the third stable zone is shown in Fig. 6.13. Contours of energy rate of the 2π-periodic energy surface are plotted in Fig. 6.14.

It can be seen in Figs. 6.13 and 6.14, that there are two wells w_1 and w_2 in the third stable zone at points $(a_1 = 8.192, b_1 = 3.084)$ and $(a_2 = 8.198, b_2 = 7.04)$, respectively. The dividing ridge touches the plane $E_{av} = 0$ and indicates a 2π-periodic curve. The depths of the wells shown in Fig. 6.14 are $(E_{av})_1 = -0.047540764$ and $(E_{av})_2 = -0.619065242$. We examine the 2π-periodic energy surface of the third zone to investigate the behavior of the Mathieu equation when there is 2π-periodic ridge. Let will walk on a path from point B on the π-transition curve to point C on 2π-transition curve. The two- and three-dimensional illustration of the walking path is shown in Figs. 6.14 and 6.15, respectively.

Figure 6.16 illustrates E_{av} in the third stable region as a function of a for $b = 3.084$, corresponding to the path B–C passing through w_1. Point 0 is on the π-transition curve, points 1 and 2 are two arbitrary points between the π-transition curve and the local minimum at point 3. Points 4 and 5 are also two sample points between local minimum and the 2π-periodic ridge at point 6. Point 9 is corresponding to the well w_1, and points 7 and 8 are two other sample points between the 2π-periodic ridge and w_1. Point 10 is on the 2π-transition curve of Mathieu stability diagram. Points 0 through 10 are our examining points when

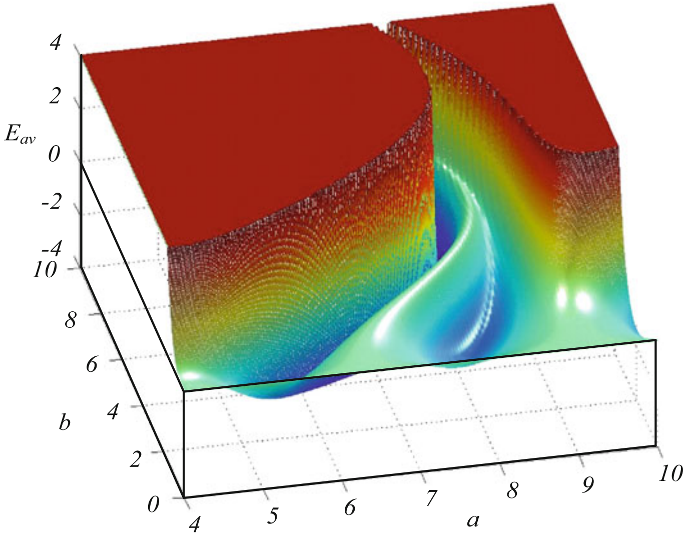

Fig. 6.13 A three-dimensional view of 2π-periodic energy surface in the third stable zone of Mathieu equation

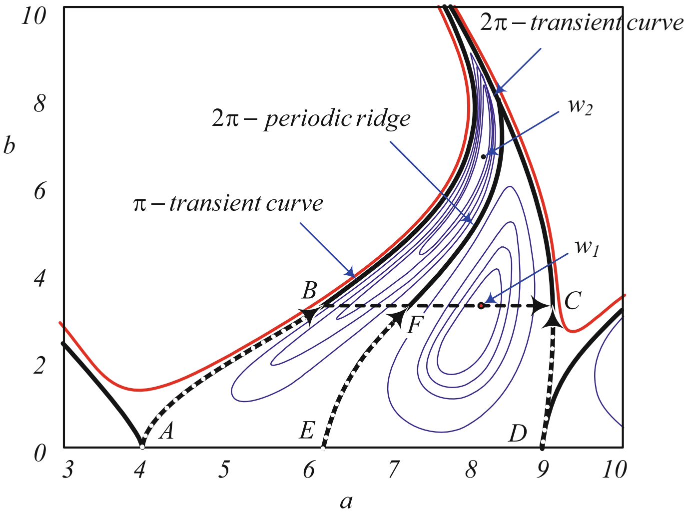

Fig. 6.14 Contours of energy rate of the 2π-stability surface

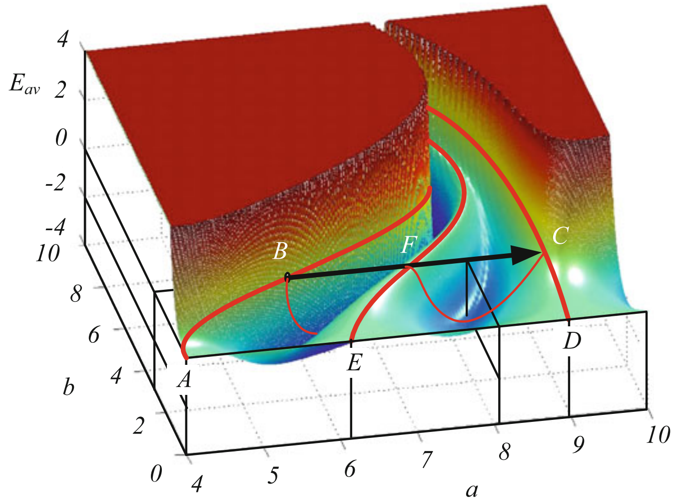

Fig. 6.15 Three-dimensional illustration of the energy-rate surface and the walking path from π-transition to 2π-transition curve

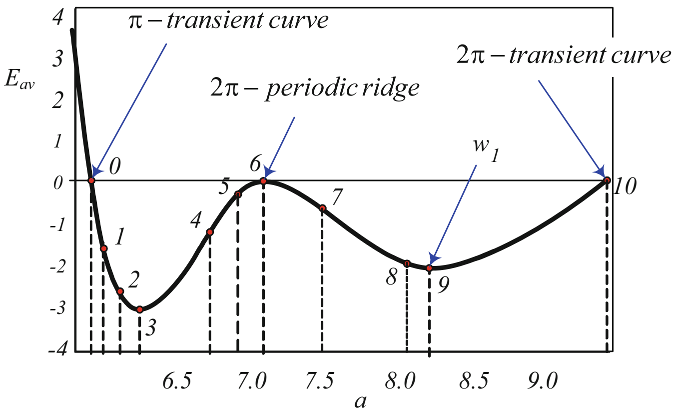

Fig. 6.16 Energy-rate function of Mathieu equation as a function of a for $b = 3.084$

moving on the path BC of Fig. 6.14. The depth of second w_2 is more than the depth of w_1. So, the energy rate at point 3 is lower than the energy rate at the local minimum w_1. The information and characteristics of points 0 to 10 are summarized in Table 6.3. Phase plane, time response, and power spectral density of Mathieu equation for parameters at points 0– 10 are shown in Figs. 6.17, 6.18, 6.19, 6.20, 6.21, 6.22, 6.23, 6.24, 6.25, 6.26, and 6.27, applying initial conditions $x(0) = 1$, $\dot{x}(0) = 0$. The powers are calculated based on fast Fourier transform algorithm over $0 < t < 20\pi$ (Brigham 1974). Time history and phase plane response are plotted for

Table 6.3 The characteristics of points 0–10 of Fig. 6.16

Point	a	b	E_{av}	Table 6.1
0	6.116	3.084	−0.000290415	π-periodic curve
1	6.2	3.084	−0.250140076	
2	6.3	3.084	−0.415556251	
3	6.417	3.084	−0.472598615	local minimum
4	6.7	3.084	−0.293544156	
5	7	3.084	−0.05270861	
6	7.205	3.084	−1.63135E-05	2π-periodic ridge
7	7.5	3.084	−0.084472609	
8	8	3.084	−0.308945302	
9	8.192	3.084	−0.317404257	w_1
10	9.228	3.084	0.000167172	2π-periodic curve

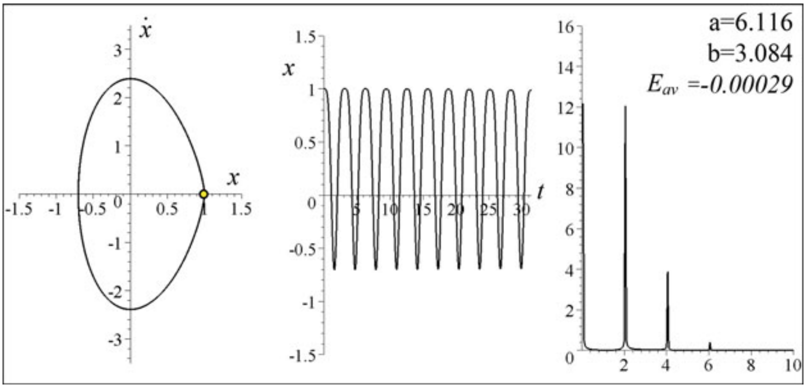

Fig. 6.17 Phase portrait, time response, and power of Mathieu equation of point 0 on path BC

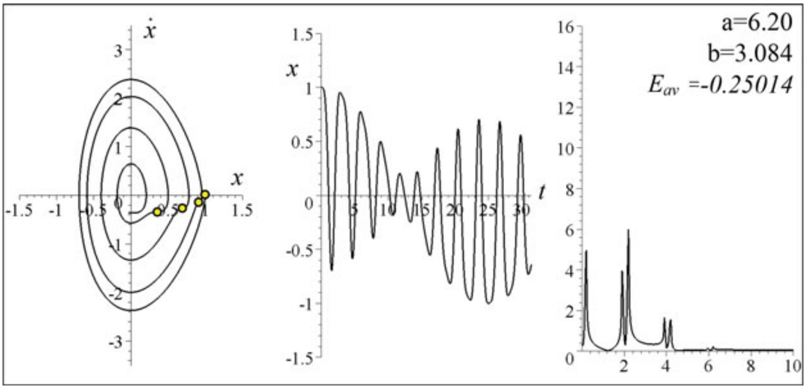

Fig. 6.18 Phase portrait, time response, and power spectral density of Mathieu equation of point 1 on path BC

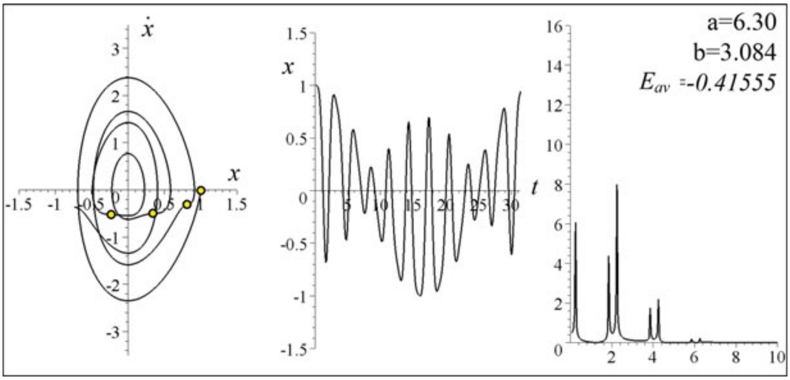

Fig. 6.19 Phase portrait, time response, and power spectral density of Mathieu equation of point 2 on path BC

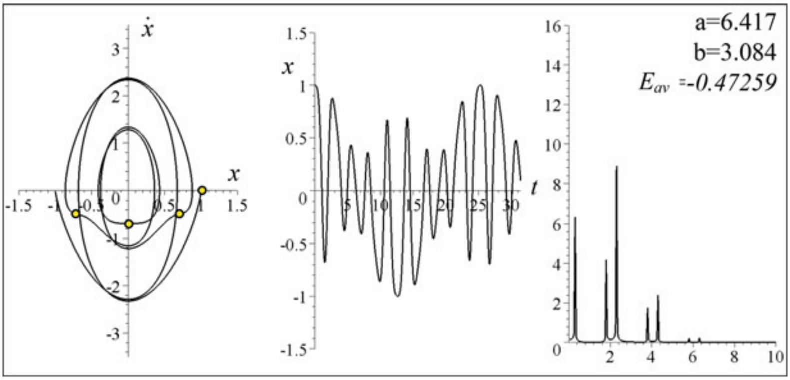

Fig. 6.20 Phase portrait, time response, and power spectral density of Mathieu equation of point 3 on path BC

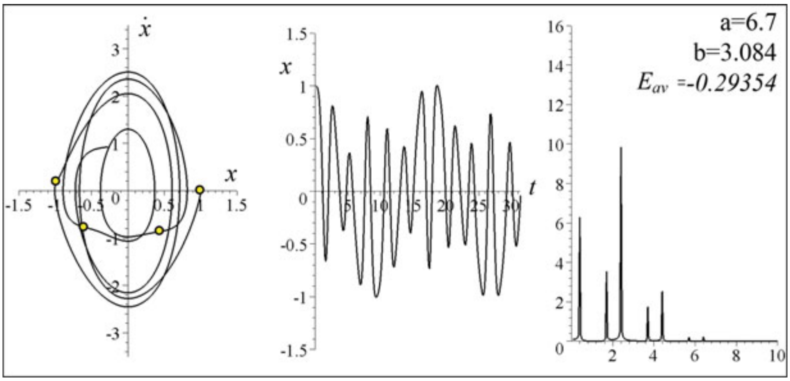

Fig. 6.21 Phase portrait, time response, and power spectral density of Mathieu equation of point 4 on path BC

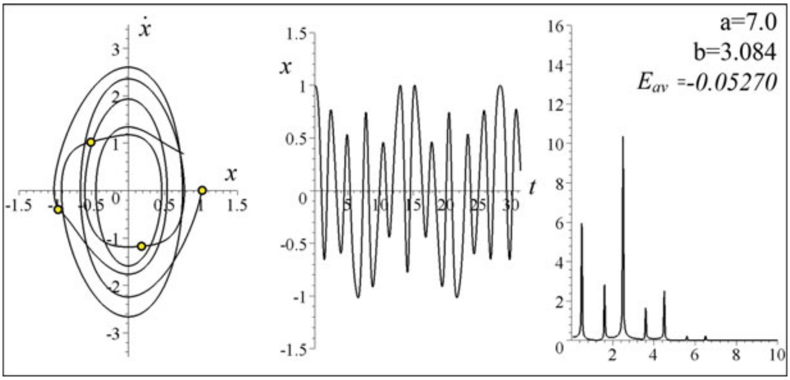

Fig. 6.22 Phase portrait, time response, and power spectral density of Mathieu equation of point 5 on path BC

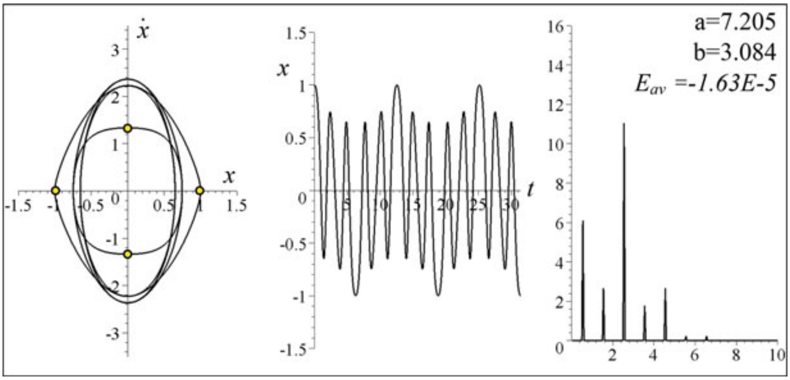

Fig. 6.23 Phase portrait, time response, and power spectral density of Mathieu equation of point 6 on path BC

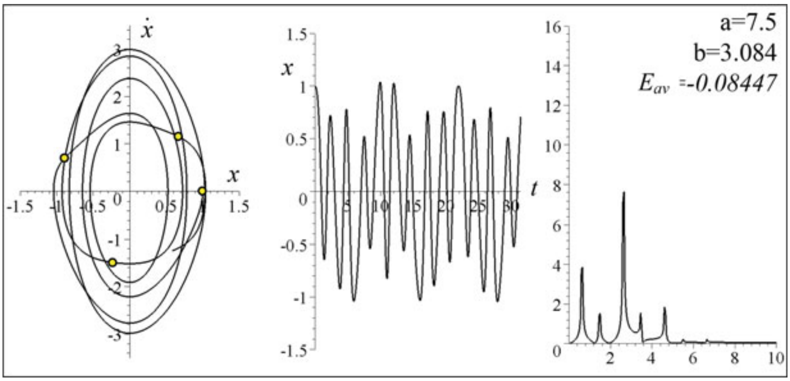

Fig. 6.24 Phase portrait, time response, and power spectral density of Mathieu equation of point 7 on path BC

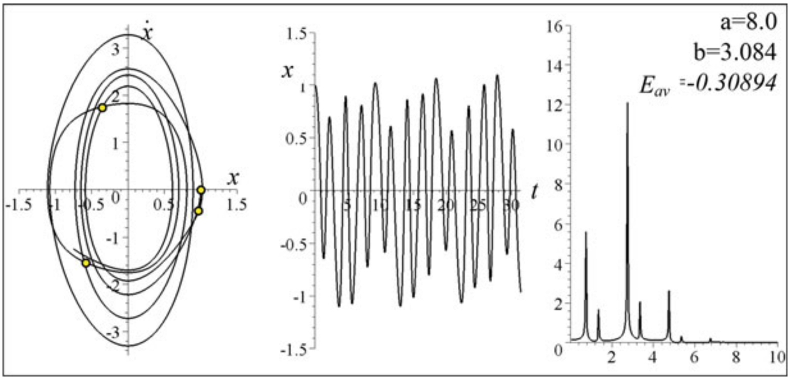

Fig. 6.25 Phase portrait, time response, and power spectral density of Mathieu equation of point 8 on path BC

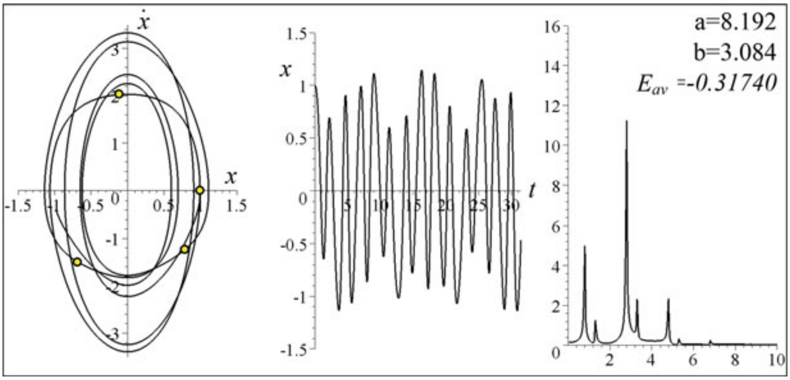

Fig. 6.26 Phase portrait, time response, and power spectral density of Mathieu equation of point 9 on path BC

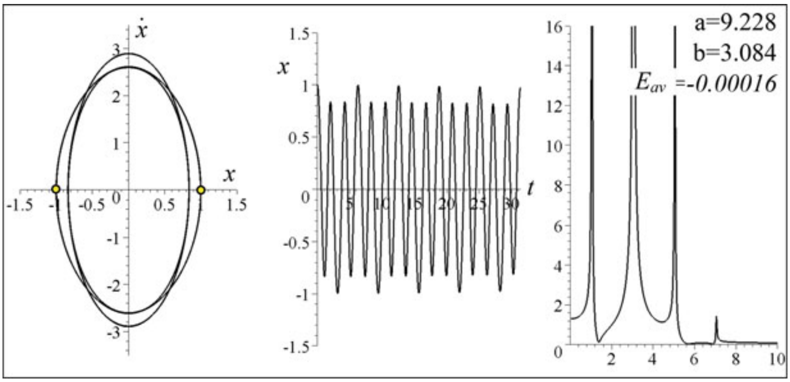

Fig. 6.27 Phase portrait, time response, and power spectral density of Mathieu equation of point 10 on path BC

the first 4π and 10π time units. Also, the first four Poincare points are also indicated on the phase plane response for shooting time $T_p = \pi$ (Argyris et al. 1994).

Time response, phase portrait, and repeating Poincare points shown in Fig. 6.17 indicate that point 0 has a π-periodic response. In addition, power density designates a strong π-periodic spike as well as a $(\pi/2)$-periodic and $(\pi/3)$-periodic sub-harmonics. It also represents existence of a very long period super-harmonic. The power of π-periodic response is much higher than sub-harmonics, indicating that π-periodic response is the dominant period. It is an asymmetric oscillation with positive and negative unequal amplitudes.

By increasing a to 6.20 in Fig. 6.18 a skew symmetric time response with a period close to 16π appears. A long period super-harmonics can also be seen from power spectral graph as well as sub-harmonics close to π, $(\pi/2)$, and $(\pi/3)$. When a is 6.30 a time response with a period close to 10π appears in Fig. 6.19. In Fig. 6.20, a is 6.417 corresponding to the local minimum of energy-rate curve at point 4. The response of the system at this point is a quasi-periodic response with an overall period very close to 10π, indicated in phase plot of the system. Increasing a to 6.7 and 7.0 indicates that response is getting closer to the 2π-periodic ridge as is shown in Figs. 6.21 and 6.22.

Point 6 at $a = 7.205$ in Fig. 6.23 corresponds to the 2π-periodic ridge. The 2π-periodic response is recognizable by investigating the phase plot, Poincare points, and time response of the system in Fig. 6.23. At this point, the spikes of the power spectral appear exactly at points 0.5, 1.5, 2.5, 3.5, 4.5, 5.5, 6.5, which are corresponding to (4π), $(4\pi/3)$, $(4\pi/5)$, $(4\pi/7)$, $(4\pi/9)$, $(4\pi/11)$, and $(4\pi/13)$ periods. The power of (4π) and $(4\pi/5)$ is stronger than the other sub-harmonics, while $(4\pi/11)$ and $(4\pi/13)$ sub-harmonics are the weakest. Figures 6.24, 6.25, 6.26, and 6.27 illustrate how the two harmonics separated from π, $(\pi/2)$ and $(\pi/3)$-harmonics approach each other and get together at (2π), π, $(2\pi/3)$, $(2\pi/5)$, and $(2\pi/7)$ harmonics. Figure 6.27 depicts response of the system at point 10 corresponding to 2π-transition curve.

Investigating power spectral density in Figs. 6.17, 6.18, 6.19, 6.20, 6.21, 6.22, 6.23, 6.24, 6.25, 6.26, and 6.27 presumes that by moving from π-transition curve to 2π-transition curve, the period should gradually change from the period set $\{\pi, \pi/2, \pi/3, \pi/4, \ldots\}$ to the set $\{2\pi, 2\pi/3, 2\pi/5, 2\pi/7, \ldots\}$. Figure 6.28 depicts the power spectral density of Mathieu equation when the investigating point moves on line BC slowly. The resolution of the graph is 10,000 points per unit a. It shows the variation of super and sub-harmonics of Mathieu equation in third stable zone, for $b = 3.084$, and $6.116 < a < 9.228$. It also indicates the power of sub-harmonics is much lower than the power of harmonics around π and $\pi/2$. Projection of the power surface makes a two-dimensional graph indicating the peak values of power on (a, ω)-plane, as shown in Fig. 6.29.

Example 206 ★Energy-rate function for coupled Mathieu equations.

Let $\mathbf{F}$ be a vector field, and the individual components F_1, F_2, F_3,$\cdots$, F_n be differentiable functions. Then, a function $I(\mathbf{x})$ is called the first integral of the differential equation

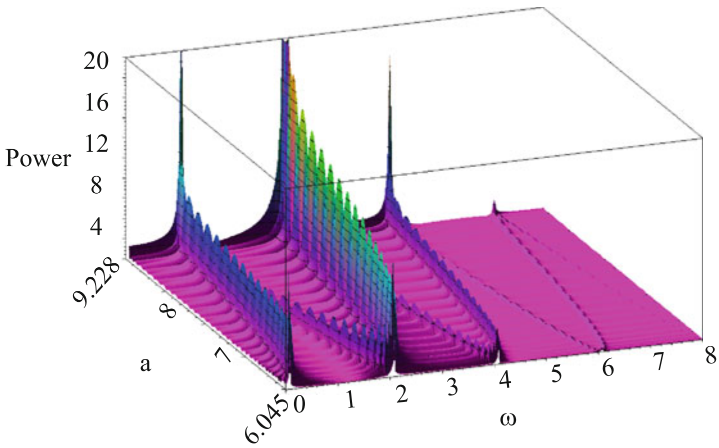

Fig. 6.28 Three-dimensional illustration of power spectral density of Mathieu equation in third stable zone, for $b = 3.084$

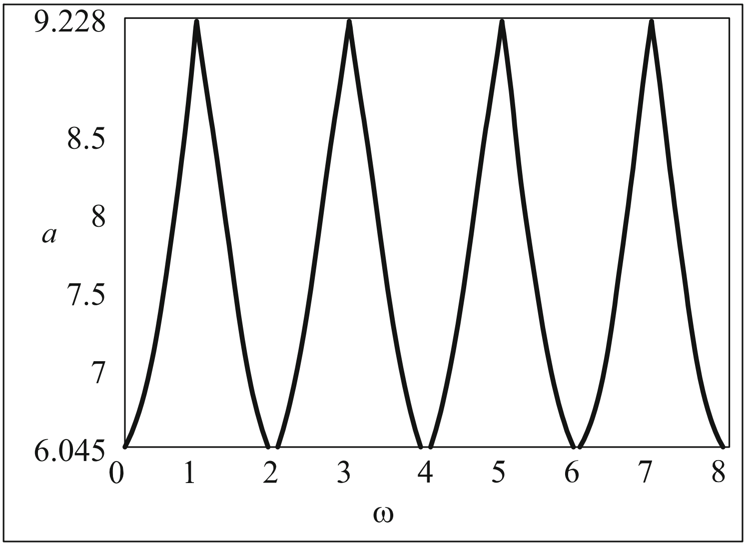

Fig. 6.29 Top view of power spectral density surface of Mathieu equation in third stable region, for b=3.084

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) \tag{6.54}$$

if its Lie derivative along the vector field **F** vanishes.

$$\nabla I \cdot \mathbf{F} = F_1 \frac{\partial I}{\partial x_1} + F_2 \frac{\partial I}{\partial x_2} + \cdots + F_n \frac{\partial I}{\partial x_n} = 0 \tag{6.55}$$

Then, the function $I = I(x_1, x_2, \ldots, x_n)$ remains constant and defines a hypersurface in the phase space where each given trajectory takes place on it. The hypersurface is defined when the initial conditions are set. In mechanics, it is the conservation theorems, which supply first integrals. However, no systematic rule is known which yields an easier derivation of first integrals.

Direct substitution of the function

$$I = \frac{1}{2}u^2 + \frac{1}{2}v^2 + axy \tag{6.56}$$

into the Lie derivative condition (6.55) indicated that I is a first integral of the following coupled equations:

$$\ddot{x} + ay = 0 \qquad \ddot{y} + ax = 0 \tag{6.57}$$

where

$$\mathbf{x} = \begin{bmatrix} x \\ u \\ y \\ v \end{bmatrix} \qquad \mathbf{F} = \begin{bmatrix} u \\ -ay \\ v \\ -ax \end{bmatrix} \tag{6.58}$$

Now the first integral will appear again when the equations are added after multiplying the first equations by $\dot{x}$ and the second by $\dot{y}$,

$$\ddot{x}\dot{x} + ay\dot{x} + \ddot{y}\dot{y} + ax\dot{y} = 2b(y\dot{x} + x\dot{y})\cos 2t \tag{6.59}$$

and the result after taking integral over time will be:

$$\frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 + axy = \int 2b(y\dot{x} + x\dot{y})\cos 2t \tag{6.60}$$

Therefore, the energy-rate function corresponding to Eq. (6.59) is defined by

$$E_{av} = \int_0^{2\pi} 2b(y\dot{x} + x\dot{y})\cos 2t \, dt \tag{6.61}$$

Example 207 ★Fourth-order equations.

The fourth-order homogeneous equation

$$\ddddot{x} + ax = 0 \tag{6.62}$$

has the following general solution:

$$x = C \exp\left(-\sqrt[3]{a}t\right) - \exp\left(-\frac{\sqrt[3]{a}}{2}t\right)(A \sin \omega t - B \cos \omega t) \tag{6.63}$$

where the frequency of the oscillating part is

$$\omega = \sqrt[4]{a} \tag{6.64}$$

Constants of integration are related to initial conditions $x(0) = x_0$, $\dot{x}(0) = \dot{x}_0$, $\ddot{x}(0) = \ddot{x}_0$.

$$A = \frac{\sqrt{3}}{3\sqrt[3]{a^2}}\left(\dot{x}_0\sqrt[3]{a} + \ddot{x}_0\right) \tag{6.65}$$

$$B = \frac{2}{3}x_0 + \frac{1}{3\sqrt[3]{a^2}}\left(-\dot{x}_0\sqrt[3]{a} + \ddot{x}_0\right) \tag{6.66}$$

$$C = \frac{1}{3}x_0 + \frac{1}{3\sqrt[3]{a^2}}\left(\dot{x}_0\sqrt[3]{a} - \ddot{x}_0\right) \tag{6.67}$$

Therefore, Eq. (6.62) may have an oscillatory motion if $C = 0$, and it implies that there be a relationship between the initial conditions.

$$\ddot{x}_0 = x_0\sqrt[3]{a^2} + \dot{x}_0\sqrt[3]{a} \tag{6.68}$$

The oscillatory motion is unstable when $a > 0$, and is asymptotically stable when $a < 0$. Existence of an oscillatory term in general solution of (6.62) indicates that having a forcing term may produce a stable, unstable, as well as a periodic response. Introducing a parametric excitation, $f = 2bx \cos 2t$, will produce resonance at $\omega = n$, $n \in \mathbb{N}$. Hence, the values of $a = 8\sqrt[3]{3}n^3/9$, $n = 1, 2, 3, \cdots$ would be the starting points of instability tongues in the parameter plane (a, b).

Example 208 Damped Mathieu equation.

Although we are able to transform the damped Mathieu equation

$$\frac{d^2x}{dt^2} + (a - 2b\cos 2t)\,x + c\dot{x} = 0 \tag{6.69}$$

into an undamped equation

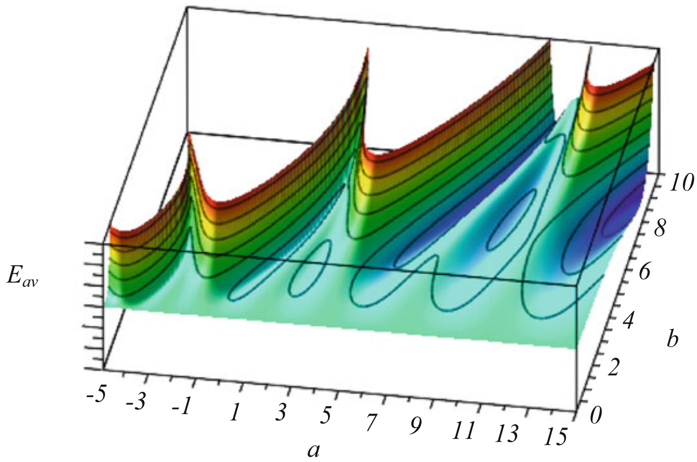

Fig. 6.30 The energy-rate surface of the damped Mathieu equation $\ddot{x} + (a - 2b\cos 2t)\,x + c\dot{x} = 0$ for $c = 0.1$

$$\frac{d^2y}{dt^2} + (h - 2b\cos 2t)\,y = 0 \tag{6.70}$$

$$h = a - c^2 \tag{6.71}$$

using

$$x = e^{-ct}y\,(t) \tag{6.72}$$

it is useful to apply the energy-rate method on a damped Mathieu equation to show two facts: 1—the energy-rate method is capable of determining the stability chart of the damped equation, 2—to see how damping affects the stability chart and expands the stable zone of the Mathieu stability chart.

Introducing a damping term $c\dot{x}$ makes the Mathieu equation to have three parameters (a, b, c). Now the stability chart may be expressed in three different planes if there is such a chart in all of them. It is traditional to assign a constant value to the damping parameter, c, and look at new stability chart in (a, b)-plane.

Figure 6.30 illustrates the energy-rate surface for $c = 0.1$ as an example. Figure 6.31 also shows the cut plane $E_{av} = 0$ and the energy surface. The damping separates the instability tongues from the line $b = 0$ and opens a fully stable region around $b = 0$. The stable area becomes wider by increasing b.

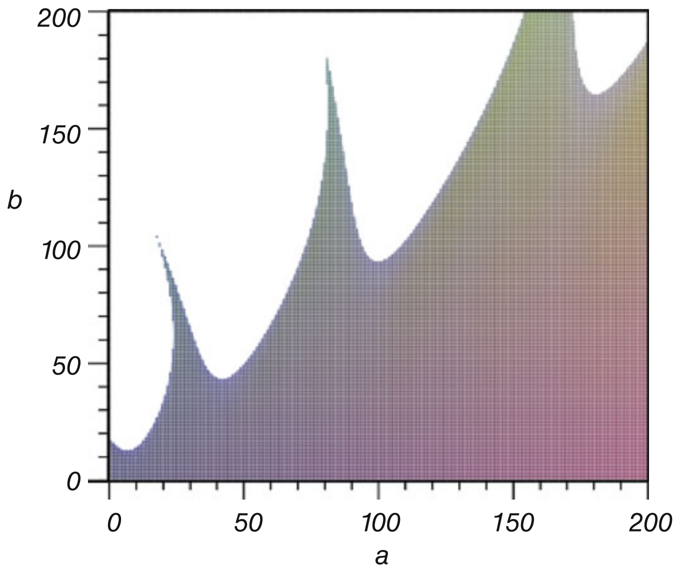

Fig. 6.31 The stability chart of the damped Mathieu equation $\ddot{x} + (a - 2b\cos 2t)\,x + c\dot{x} = 0$ for $c = 0.1$

6.3 Initial Conditions

The energy-rate method works only if the initial conditions of the equation are selected such that it matches with the periodic time response of the equation. The Mathieu equation is linear and hence when it is on its periodic response, it will still show the periodic response for any initial conditions of $x\,(0) = A$, $\dot{x}\,(0) = 0$. Hence the initial condition $x\,(0) = 1$, $\dot{x}\,(0) = 0$ works well to apply the energy-rate method and determine the stability chart of the Mathieu equation.

Proof Let us assume a periodic solution of the form

$$x = B_0 + B_1 \sin 2\,(t + t_0) \tag{6.73}$$

for the Mathieu equation

$$\frac{d^2x}{dt^2} + (a - 2b\cos 2t)\,x = 0 \tag{6.74}$$

where a and b are on a periodic characteristic curve. Having the solution (6.73) we should have

$$\dot{x} = 2B_1 \cos 2\,(t + t_0) \tag{6.75}$$

Looking for initial conditions of

$$x(0) = x_0 \qquad \dot{x}(0) = 0 \tag{6.76}$$

to put the system on (6.73) provides:

$$x_0 = B_0 + B_1 \sin 2t_0 \tag{6.77}$$

$$0 = 2B_1 \cos 2t_0 \tag{6.78}$$

For nonzero B_0 and B_1 we must have

$$x_0 = B_0 + B_1 \qquad t_0 = \frac{\pi}{4} \tag{6.79}$$

and hence the solution would be:

$$x = B_0 + B_1 \sin 2\left(t + \frac{\pi}{4}\right) \tag{6.80}$$

Let us change the initial conditions to

$$x(0) = nx_0 \qquad \dot{x}(0) = 0 \tag{6.81}$$

It gives us

$$nx_0 = B_0 + B_1 \sin 2t_0 \tag{6.82}$$

$$0 = 2B_1 \cos 2t_0 \tag{6.83}$$

that yields

$$x_0 = \frac{1}{n}(B_0 + B_1) \qquad t_0 = \frac{\pi}{4} \tag{6.84}$$

and hence the solution would be:

$$x = \frac{1}{n}\left(B_0 + B_1 \sin 2\left(t + \frac{\pi}{4}\right)\right) = 0 \tag{6.85}$$

Therefore, if the Mathieu equation is at a point on characteristic periodic lines such that the initial conditions $x(0) = x_0$, $\dot{x}(0) = 0$ put it on a periodic solution of the form (6.73), then the initial conditions $nx(0) = x_0$, $\dot{x}(0) = 0$ put it on a proportional periodic solution.

Hence the energy-rate integral remains zero regardless of the value of the initial $x(0)$.

To apply the energy-rate method to establish the stability chart, we need to set the initial condition to be set on a steady state amplitude of the system for the

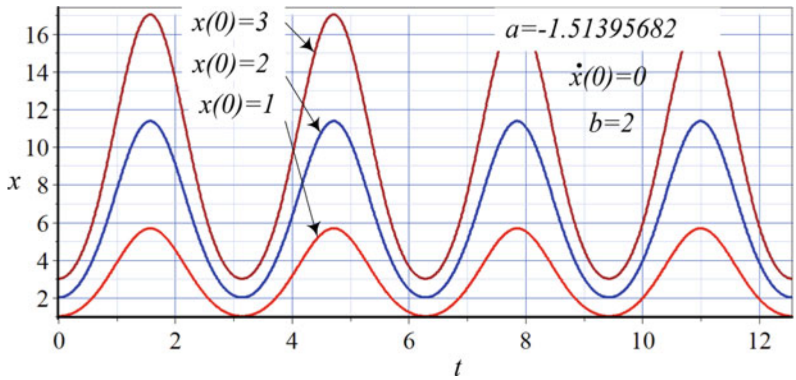

Fig. 6.32 Time history of the Mathieu equation for $a = -1.51395682$, $b = 2$, and initial conditions, $x(0) = 1$, $x(0) = 2$, $x(0) = 3$, $\dot{x}(0) = 0$

frequency of the periodic solution. The steady state amplitude comes from the frequency response of the equation whenever there is such frequency response in case of forced equations. ■

Example 209 Different initial conditions on periodic line.

Let us examine the behavior of the Mathieu equation for a point on a characteristic curve such as

$$a = -1.51395682 \qquad b = 2 \tag{6.86}$$

which is a point on transient line a_{ce0} of the Mathieu equation. Figure 6.32 depicts the time response of the equation for the following initial conditions:

$$x(0) = 1 \qquad \dot{x}(0) = 0 \tag{6.87}$$

$$x(0) = 2 \qquad \dot{x}(0) = 0 \tag{6.88}$$

$$x(0) = 3 \qquad \dot{x}(0) = 0 \tag{6.89}$$

The amplitude of the responses for different initial conditions is proportional.

As another example, let us try another point on a_{ce0} such as

$$a = -0.455138558 \qquad b = 1 \tag{6.90}$$

Figure 6.33 shows that the proportionality of the amplitude and initial conditions holds the same as the other point of a_{ce0}.

Example 210 Wrong initial condition.

Picking a wrong initial conditions may result in wrong evaluation on stability of points on (a, b)-plane. For any point on a periodic characteristic line, the Mathieu equation will have a periodic time response which may be achieved in long term for

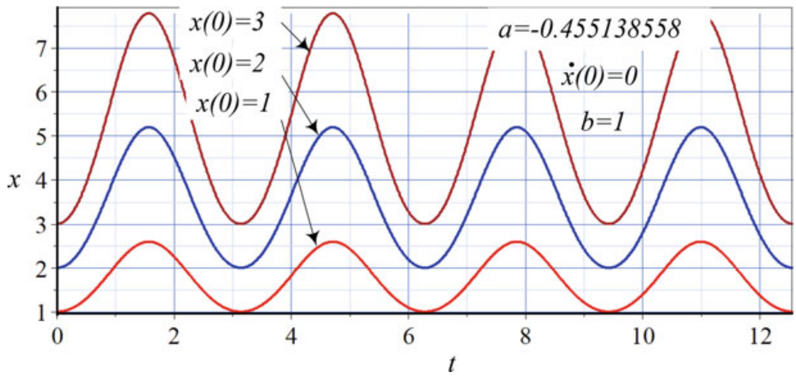

Fig. 6.33 Time history of the Mathieu equation for $a = -0.455138558$, $b = 2$, and initial conditions, $x(0) = 1$, $x(0) = 2$, $x(0) = 3$, $\dot{x}(0) = 0$

proper initial conditions. That means if the initial conditions are not matched with the steady state periodic response, then the first period of numerical solution will not show $E_{av} = 0$. To show this fact let us examine the behavior of the Mathieu equation for a point on a_{ce0} characteristic curve

$$a = -1.51395682 \qquad b = 2 \tag{6.91}$$

and examine the time response of the Mathieu equation for three different initial conditions, two of them not matched with periodic response.

$$x(0) = 1 \qquad \dot{x}(0) = 0 \tag{6.92}$$

$$x(0) = 1 \qquad \dot{x}(0) = 1 \tag{6.93}$$

$$x(0) = 1 \qquad \dot{x}(0) = -1 \tag{6.94}$$

Figure 6.34 illustrates the time response of the Mathieu equation for $0 < t < 10\pi$ and the initial conditions (6.92)–(6.94). The response for the conditions $x(0) = 1$, $\dot{x}(0) = 0$ is periodic for which $E_{av} = 0$. However, $E_{av} \neq 0$ for $x(0) = 1$, $\dot{x}(0) = 1$, and $x(0) = 1$, $\dot{x}(0) = -1$.

Example 211 Frequency response is attractive.

Consider a mass-spring-damper under a harmonic force excitation at frequency ω.

$$m\ddot{x} + c\dot{x} + kx = F \sin \omega t \tag{6.95}$$

The steady state response of the equation would be a harmonic function with a frequency dependent amplitude (Jazar 2013).

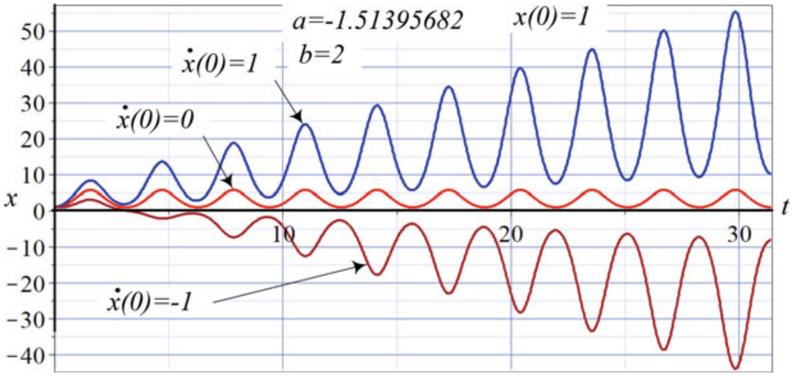

Fig. 6.34 Time history of the Mathieu equation for $a = -0.455138558$, $b = 2$, and initial conditions, $\dot{x}(0) = 0$, $\dot{x}(0) = 1$, $\dot{x}(0) = -1$, $x(0) = 1$

$$x = X \sin\left(\omega t - \varphi_x\right) \tag{6.96}$$

$$X = \frac{F/k}{\sqrt{\left(1 - r^2\right)^2 + (2\xi r)^2}} \qquad \varphi = \tan^{-1}\frac{2\xi r}{1 - r^2} \tag{6.97}$$

$$r = \frac{\omega}{\omega_n} \qquad \omega_n = \sqrt{\frac{k}{m}} \qquad \xi = \frac{c}{2\sqrt{km}} \tag{6.98}$$

Assume a sudden disturbance changes the amplitude to $X + \delta X$ at the same excitation frequency ω and the response of the system changes to $x = x_0 + y$

$$x = x + y = X \sin\left(\omega t - \varphi_x\right) + y \tag{6.99}$$

$$y \ll x \tag{6.100}$$

Substituting x in Eq. (6.95)

$$m\ddot{x} + c\dot{x} + kx + m\,\ddot{y} + c\,\dot{y} + k\,y = F \sin \omega t \tag{6.101}$$

and considering x satisfies the equation, we find the equation to determine y:

$$m\,\ddot{y} + c\,\dot{y} + k\,y = 0 \tag{6.102}$$

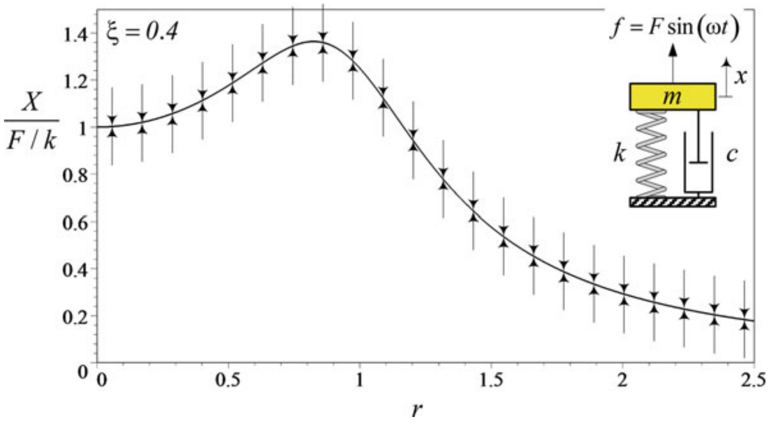

Fig. 6.35 The stability and attraction characteristic of steady state frequency response of linear vibrating system

The solution of this equation with initial conditions

$$y(0) = \delta X \qquad \dot{y}(0) = 0 \tag{6.103}$$

is:

$$y = \frac{1}{2}\frac{-c^2 - c\sqrt{c^2-4km}+4km}{-c^2+4km}\delta X \exp\left(\frac{c-\sqrt{c^2-4km}}{-2m}t\right) + \frac{1}{2}\frac{-c^2 + c\sqrt{c^2-4km}+4km}{-c^2+4km}\delta X \exp\left(\frac{c+\sqrt{c^2-4km}}{-2m}t\right) \tag{6.104}$$

As long as $c^2 > 4km$, we have

$$\lim_{t\to\infty} y = 0 \tag{6.105}$$

and therefore,

$$\lim_{t\to\infty} x = X \sin(\omega t - \varphi) \tag{6.106}$$

Therefore, any disturbance to a steady state vibrating system will disappear after a while and the system will be back to its original steady state vibration. This is the stability characteristic of frequency response and indicates that the frequency response curves are attractive for small amplitude disturbances as is shown in Fig. 6.35 for $\xi = 0.4$.

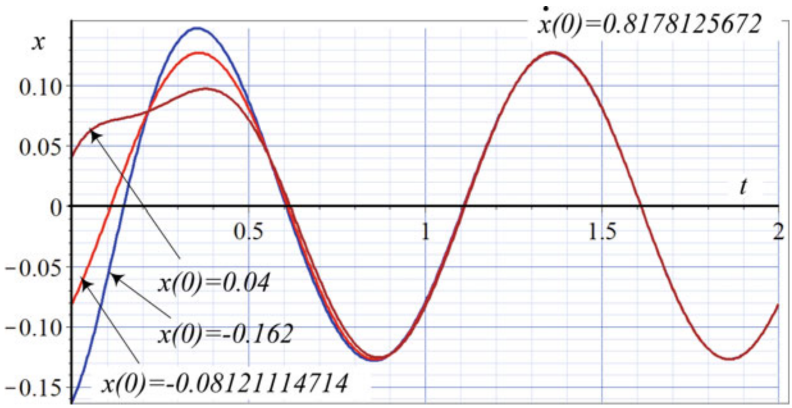

Fig. 6.36 The steady state periodic motion of the system matches with the initial conditions $x_0 = X \sin\left(-\varphi_x\right)$, $\dot{x}_0 = X\omega \cos\left(-\varphi_x\right)$. Any other initial conditions will disappear after a while

As an example let us examine a system with

$$m = 1\,\text{kg} \qquad k = 100\,\text{N/m} \qquad c = 8\,\text{N s/m} \tag{6.107}$$

$$\omega_n = 10\,\text{rad/s} \qquad \xi = 0.4 \tag{6.108}$$

that is under a force harmonic f

$$f = 10 \sin(2\pi t)\ \text{N} \qquad r = \frac{\pi}{5} \tag{6.109}$$

The steady state response of this system (6.96) has the amplitude X and phase φ of:

$$X = 0.1271080028\,\text{m} \qquad \varphi = 0.6930863846\,\text{rad} \tag{6.110}$$

The initial conditions $x_0 = x(0)$ and $\dot{x}_0 = \dot{x}(0)$ will match with the steady state periodic response if we set them as:

$$x_0 = X \sin\left(-\varphi_x\right) = -0.08121114714 \tag{6.111}$$

$$\dot{x}_0 = X\omega \cos\left(-\varphi_x\right) = 0.6143783914 \tag{6.112}$$

Now suppose that, because of a disturbance, x jumps to an amplitude of $2X$, or $-X/2$. Figure 6.36 illustrates the time history of the motions and shows how the system settles down to X after disturbances.

Example 212 Some theorems on Mathieu equation.

Mathieu equation has been under investigations by mathematicians as a general type of parametric equation. There are several theorems on Mathieu equation that some of them worth to repeat here.

1. If $x_1(t)$ is a solution of the Mathieu equation, then $x_1(t \mp k\pi)$, $k = 1, 2, 3, \ldots$ is also a solution.
2. Mathieu equation always has one even and one odd solution.
3. Mathieu equation always has two solutions, $x_1(t)$ and $x_2(t)$ such that: $x_1(t)$ is even and $x_2(t)$ is odd. $x_1(0) = \dot{x}_1(0) = 0$ and $x_2(0) = \dot{x}_2(0) = 0$.

$$x_1(t \mp \pi) = x_1(\pi)\,x_1(t) \pm \dot{x}_1(\pi)\,x_2(t) \tag{6.113}$$

$$x_2(t \mp \pi) = \pm x_2(\pi)\,x_1(t) \pm \dot{x}_2(\pi)\,x_2(t) \tag{6.114}$$

$$x_1(t)\,\dot{x}_2(t) - x_2(t)\,\dot{x}_1(t) = 1 \tag{6.115}$$

$$x_1(\pi) = \dot{x}_2(\pi) \tag{6.116}$$

4. Mathieu equation always has at least one solution $x_1(t)$ such that $x_1(t + \pi) = kx_1(t))$, where k is a constant depending on the parameters of the equation.

6.4 Chapter Summary

The stability of the Mathieu equation

$$\frac{d^2x}{dt^2} + (a - 2b\cos 2t)\,x = 0 \tag{6.117}$$

for a pair of constant parameters (a, b) can be determined by the sign of energy-rate integral

$$E_{av} = \frac{1}{T}\int_0^T \dot{E}dt = \frac{1}{T}\int_0^T (2abx\dot{x}\cos 2t)\,dt \tag{6.118}$$

$$T = 2\pi \tag{6.119}$$

The pair (a, b) is in an unstable region if $E_{av} > 0$, it is on a periodic characteristic curve if $E_{av} = 0$, and it is in a stable region if $E_{av} < 0$. The energy rate provides a surface that its intersection with the plane $E_{av} = 0$ indicates the lines on which the equation provides periodic response. These lines are called the stability chart in (a, b)-plane. The procedure can be arranged in an algorithm.

Stability Chart Algorithm

1. set a equal to one of its special values $n^2 = a$
2. set b equal to some arbitrary small value
3. solve the differential equation numerically

4. evaluate E_{av}
5. decrease (increase) a if $E_{av} > 0$ ($E_{av} < 0$) by some small increment
6. the increment of a must be decreased if $E_{av}(a_i) \cdot E_{av}(a_{i-1}) < 0$
7. save a and b when $E_{av} << 1$
8. while $b < b_{final}$, increase b and go to step 3
9. set a equal to another special value and go to step 2
10. reverse the decision in step 5 and go to step 1

6.5 Key Symbols

$a \equiv \ddot{x}$	Acceleration
a, b	Parameters of the Mathieu equation
(a, b)	Points on periodic curves of Mathieu stability chart
$a_{ce_{2k}}$	Characteristic numbers for Mathieu function ce_{2k}
$a_{ce_{2k+1}}$	Characteristic numbers for Mathieu function ce_{2k+1}
$A \cdots F$	Point indicator on third stable zone of stability chart
$\mathbf{A}$	Coefficient matrix
b	Lateral distance of a wheel from longitudinal x-axis
b_{final}	The final value of b for which E_{av} being calculated
E	Energy
$\dot{E}$	Energy-rate
E_{av}	Integral of energy-rate over one period T
f, g	Function
$\mathbf{I}$	Identity matrix
k	Stiffness
K	Kinetic energy
$\mathcal{L}$	Lagrangian
m	Mass
m	Fractional number $a = m^2$, $m = p + q$, $p \in \mathbb{N}$, $0 < q < 1$
n	Integer number $a = n^2$, $n \in \mathbb{N}$
p	Integer number $p \in \mathbb{N}$
P	Potential energy
q	Noninteger number $0 < q < 1$
x	Displacement, dependent variable of Mathieu equation
$\mathbf{x}$	Dependent variable vector
t	Independent variable
w	Well of energy-rate surface

Exercises

1. Energy-rate function.
   Determine the energy-rate function, $\dot{E}$, for the following equations:

$$\ddot{x} + 2\xi\dot{x} + \eta x = 0 \tag{6.120}$$

$$\ddot{x} + \alpha x^3 = x\cos\omega t \tag{6.121}$$

$$\ddot{x} + \alpha x^3 = x\cos^2\omega t \tag{6.122}$$

$$\ddot{x} + \alpha x^3 = x^2\sin^2\omega t \tag{6.123}$$

$$\ddot{x} + (\alpha + \beta\sinh\omega t)\,x = 0 \tag{6.124}$$

2. ★Coupled undamped Mathieu equations.
   (a) Show that by combining two Mathieu equations

$$\ddot{x} + \alpha x - 2by\cos 2t = 0 \tag{6.125}$$

$$\ddot{y} + \alpha y - 2bx\cos 2t = 0 \tag{6.126}$$

   we find a fourth-order equation.

$$\ddddot{x} + \frac{4\sin 2t}{\cos 2t}\dddot{x} + \left(2a + 4 + \frac{8\sin^2 2t}{\cos^2 2t}\right)\ddot{x} + a\frac{4\sin^2 2t}{\cos^2 2t}\dot{x} + \left(a^2 - 4b^2\cos^2 2t + 4a\left(\frac{2\sin^2 2t}{\cos^2 2t} - 1\right)\right)x = 0 \tag{6.127}$$

   (b) Determine the energy-rate function, $\dot{E}$ for the equation.
   (c) Show that by combining two Mathieu equations

$$\ddot{x} + (\alpha - 2b\cos 2t)\,y = 0 \tag{6.128}$$

$$\ddot{y} + (\alpha - 2b\cos 2t)\,x = 0 \tag{6.129}$$

   we find a fourth-order equation.

$$\ddddot{x} - \frac{8b\sin 2t}{a - 2b\cos 2t}\dddot{x} - (a - 2b\cos 2t)^2 x - \left(\frac{8b\cos 2t}{a - 2b\cos 2t} - \frac{32b^2\sin^2 2t}{(a - 2b\cos 2t)^2}\right)\ddot{x} = 0 \tag{6.130}$$

   (d) Determine the energy-rate function, $\dot{E}$ for this equation.

3. ★Coupled damped Mathieu equations.

(a) Show that by combining two Mathieu equations

$$\ddot{x} + h\dot{x} + (\alpha - 2b\cos 2t)\, y = 0 \tag{6.131}$$

$$\ddot{y} + h\dot{y} + (\alpha - 2b\cos 2t)\, x = 0 \tag{6.132}$$

we find a fourth-order equation.

$$\ddddot{x} + A(a,b,t)\,\dddot{x} + B(a,b,t)\,\ddot{x} + C(a,b,t)\,\dot{x} + D(a,b,t)\,x = 0 \tag{6.133}$$

where

$$A(a,b,t) = \frac{8b\sin 2t}{a - 2b\cos 2t} \tag{6.134}$$

$$B(a,b,t) = \frac{32b^2\sin^2 2t}{(a - 2b\cos 2t)^2} - \frac{8b\cos 2t}{a - 2b\cos 2t} + 4h\frac{1 - b\sin 2t}{a - 2b\cos 2t} + h^2 \tag{6.135}$$

$$C(a,b,t) = h\frac{32b^2\sin^2 2t}{(a - 2b\cos 2t)^2} - h\frac{8b\cos 2t}{a - 2b\cos 2t} - h^2\frac{4b\sin 2t}{a - 2b\cos 2t} \tag{6.136}$$

$$D(a,b,t) = -(a - 2b\cos 2t)^2 \tag{6.137}$$

(b) Is it possible to transform the original equations to eliminate the damping terms?

(c) Is it possible to develop the energy-rate function for this fourth- order parametric equation?

(d) Show that by combining these two Mathieu equations

$$\ddot{x} + h\dot{y} + (\alpha - 2b\cos 2t)\, y = 0 \tag{6.138}$$

$$\ddot{y} + h\dot{x} + (\alpha - 2b\cos 2t)\, x = 0 \tag{6.139}$$

we find a fourth-order equation.

$$\ddddot{x} + A(a,b,t)\,\dddot{x} + B(a,b,t)\,\ddot{x} + C(a,b,t)\,\dot{x} + D(a,b,t)\,x = 0 \tag{6.140}$$

where

$$A(a,b,t) = K_1(a,b,t) \times \left(-16b^2\cos 2t \sin 2t - 8bh\cos 2t + 8ab\sin 2t\right) \tag{6.141}$$

$$\begin{aligned} B(a,b,t) = {} & K_1\left(-4bh^3\sin 2t + 4b^2h^2\cos^2 2t\right) \\ & + K_1\left(8ab\cos 2t - 324b^2\sin^2 2t - 16b^2\cos^2 2t\right) \\ & + K_1\left(a^2h^2 - 4abh^2\cos 2t\right) \end{aligned} \tag{6.142}$$

$$\begin{aligned} C(a,b,t) = {} & K_1\left(-16b^3h\cos^3 2t + 32b^2h^2\sin 2t\cos 2t\right) \\ & + K_1\left(8bh^3\cos 2t - 12a^2bh\cos 2t - 16abh^2\sin 2t\right) \\ & + K_1\left(2a^3h + 24ab^2h\cos^2 2t\right) \end{aligned} \tag{6.143}$$

$$\begin{aligned} D(a,b,t) = {} & K_1\left(-16b^2h^2 - 8a^3b\cos 2t + 24a^2b^2\cos^2 2t\right) \\ & + K_1\left(-32ab^3\cos^3 2t - 32b^3h\sin 2t\cos^2 2t\right) \\ & + K_1\left(-8a^2bh\sin 2t + 8abh^2\cos 2t\right) \\ & + K_1\left(32ab^2h\sin 2t\cos 2t + 16b^4\cos^4 2t + a^4\right) \end{aligned} \tag{6.144}$$

$$K_1(a,b,t) = \frac{1}{4bh\sin 2t - (a - 2b\cos 2t)^2} \tag{6.145}$$

(e) Is it possible to transform the original equations to eliminate the damping terms?
(f) Is it possible to develop the energy-rate function for this fourth- order parametric equation?
(g) Show that by combining these two Mathieu equations

$$\ddot{x} + h\dot{y} + (\alpha - 2b\cos 2t)\,x = 0 \tag{6.146}$$

$$\ddot{y} + h\dot{x} + (\alpha - 2b\cos 2t)\,y = 0 \tag{6.147}$$

we find a fourth-order equation.

$$\dddot{\ddot{x}}+A\left(a,b,t\right)\dddot{x}+B\left(a,b,t\right)\ddot{x}+C\left(a,b,t\right)\dot{x}+D\left(a,b,t\right)x=0 \tag{6.148}$$

where

$$A\left(a,b,t\right)=-\frac{4b\sin 2t}{\alpha-2b\cos 2t} \tag{6.149}$$

$$B\left(a,b,t\right)=2\alpha-4b\cos 2t-h^2 \tag{6.150}$$

$$C\left(a,b,t\right)=4b\sin 2t\left(1+\frac{1}{\alpha-2b\cos 2t}\right) \tag{6.151}$$

$$D\left(a,b,t\right)=\frac{-16b\sin 2t}{\alpha-2b\cos 2t}\left(4b\left(b-a\right)\cos^2 2t+8b\cos 2t+a^2\right) \tag{6.152}$$

(h) Is it possible to transform the original equations to eliminate the damping terms?
(i) Is it possible to develop the energy-rate function for this fourth- order parametric equation?

4. Meissner equation.
The Meissner equation is another linear parametric differential equation similar to the Mathieu equation,

$$\ddot{x}+\left(\alpha-2b\ \operatorname{sgn}\left(\cos 2t\right)\right)x=0 \tag{6.153}$$

where

$$y=\operatorname{sgn}\left(x\right) \tag{6.154}$$

is the *signum* function.

(a) Develop the energy-rate function for the Meissner equation.
(b) ★Develop the energy surface for the Meissner equation.

5. Mathieu square cosine term.
Apply the energy-rate method to determine the stability chart of the Mathieu equation with squares cosine term.

$$\frac{d^2x}{dt^2}+\left(a-2b\cos^2 2t\right)x=0 \tag{6.155}$$

Figure 6.37 would be a view of the energy surface of the equation.

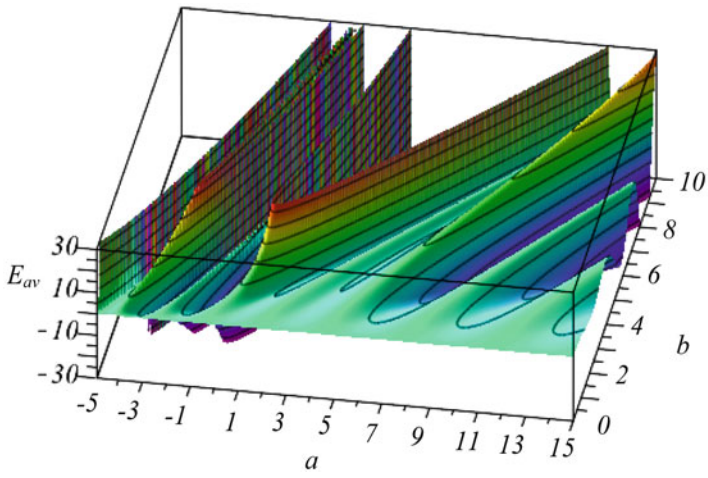

Fig. 6.37 Energy surface of the Mathieu equation with squares cosine term. $\ddot{x} + \left(a - 2b\cos^2 2t\right) x = 0$

6. Mathieu square parametric term.
   Apply the energy-rate method to determine the stability chart of the Mathieu equation with squares parametric term.

$$\frac{d^2x}{dt^2} + (a - 2b\cos 2t)^2 x = 0 \tag{6.156}$$

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Appendix A
Trigonometric Formulas

Definitions in Terms of Exponentials

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} = \cosh(iz) \tag{A.1}$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} = -i \sinh(iz) \tag{A.2}$$

$$\tan z = \frac{e^{iz} - e^{-iz}}{i\left(e^{iz} + e^{-iz}\right)} = \frac{1}{i} \tanh z \tag{A.3}$$

$$e^{iz} = \cos z + i \sin z \tag{A.4}$$

$$e^{-iz} = \cos z - i \sin z \tag{A.5}$$

Angle Sum and Difference

$$\sin(\alpha \pm \beta) = \sin\alpha \cos\beta \pm \cos\alpha \sin\beta \tag{A.6}$$

$$\cos(\alpha \pm \beta) = \cos\alpha \cos\beta \mp \sin\alpha \sin\beta \tag{A.7}$$

$$\sinh(\alpha \pm \beta) = \sinh\alpha \cosh\beta \pm \cosh\alpha \sinh\beta \tag{A.8}$$

$$\cosh(\alpha \pm \beta) = \cosh\alpha \cosh\beta \mp \sinh\alpha \sinh\beta \tag{A.9}$$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha \tan\beta} \tag{A.10}$$

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$$\cot(\alpha \pm \beta) = \frac{\cot\alpha \cot\beta \mp 1}{\cot\beta \pm \cot\alpha} \tag{A.11}$$

$$\tanh(\alpha \pm \beta) = \frac{\tanh\alpha \pm \tanh\beta}{1 \mp \tanh\alpha \tanh\beta} \tag{A.12}$$

Symmetry

$$\sin(-\alpha) = -\sin\alpha \tag{A.13}$$

$$\cos(-\alpha) = \cos\alpha \tag{A.14}$$

$$\tan(-\alpha) = -\tan\alpha \tag{A.15}$$

Multiple Angles

$$\sin(2\alpha) = 2\sin\alpha\cos\alpha = \frac{2\tan\alpha}{1+\tan^2\alpha} \tag{A.16}$$

$$\cos(2\alpha) = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha = \cos^2\alpha - \sin^2\alpha \tag{A.17}$$

$$\tan(2\alpha) = \frac{2\tan\alpha}{1-\tan^2\alpha} \tag{A.18}$$

$$\cot(2\alpha) = \frac{\cot^2\alpha - 1}{2\cot\alpha} \tag{A.19}$$

$$\sin(3\alpha) = -4\sin^3\alpha + 3\sin\alpha \tag{A.20}$$

$$\cos(3\alpha) = 4\cos^3\alpha - 3\cos\alpha \tag{A.21}$$

$$\tan(3\alpha) = \frac{-\tan^3\alpha + 3\tan\alpha}{-3\tan^2\alpha + 1} \tag{A.22}$$

$$\sin(4\alpha) = -8\sin^3\alpha\cos\alpha + 4\sin\alpha\cos\alpha \tag{A.23}$$

$$\cos(4\alpha) = 8\cos^4\alpha - 8\cos^2\alpha + 1 \tag{A.24}$$

$$\tan(4\alpha) = \frac{-4\tan^3\alpha + 4\tan\alpha}{\tan^4\alpha - 6\tan^2\alpha + 1} \tag{A.25}$$

$$\sin(5\alpha) = 16\sin^5\alpha - 20\sin^3\alpha + 5\sin\alpha \tag{A.26}$$

$$\cos(5\alpha) = 16\cos^5\alpha - 20\cos^3\alpha + 5\cos\alpha \tag{A.27}$$

$$\sin(n\alpha) = 2\sin((n-1)\alpha)\cos\alpha - \sin((n-2)\alpha) \tag{A.28}$$

$$\cos(n\alpha) = 2\cos((n-1)\alpha)\cos\alpha - \cos((n-2)\alpha) \tag{A.29}$$

$$\tan(n\alpha) = \frac{\tan((n-1)\alpha) + \tan\alpha}{1 - \tan((n-1)\alpha)\tan\alpha} \tag{A.30}$$

Half Angle

$$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}} \tag{A.31}$$

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}} \tag{A.32}$$

$$\tan\frac{\alpha}{2} = \frac{1-\cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1+\cos\alpha} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} \tag{A.33}$$

$$\sin\alpha = \frac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} \tag{A.34}$$

$$\cos\alpha = \frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} \tag{A.35}$$

Powers of Functions

$$\cos^2\alpha + \sin^2\alpha = 1 \tag{A.36}$$

$$\cosh^2\alpha - \sinh^2\alpha = 1 \tag{A.37}$$

$$\sin^2\alpha = \frac{1}{2}\left(1 - \cos(2\alpha)\right)$$

$$\sin\alpha\cos\alpha = \frac{1}{2}\sin(2\alpha) \tag{A.38}$$

$$\cos^2\alpha = \frac{1}{2}\left(1 + \cos(2\alpha)\right) \tag{A.39}$$

$$\sin^3\alpha = \frac{1}{4}\left(3\sin(\alpha) - \sin(3\alpha)\right) \tag{A.40}$$

$$\sin^2\alpha\cos\alpha = \frac{1}{4}\left(\cos\alpha - 3\cos(3\alpha)\right) \tag{A.41}$$

$$\sin\alpha\cos^2\alpha = \frac{1}{4}\left(\sin\alpha + \sin(3\alpha)\right) \tag{A.42}$$

$$\cos^3\alpha = \frac{1}{4}\left(\cos(3\alpha) + 3\cos\alpha)\right) \tag{A.43}$$

$$\sin^4\alpha = \frac{1}{8}\left(3 - 4\cos(2\alpha) + \cos(4\alpha)\right) \tag{A.44}$$

$$\sin^3\alpha\cos\alpha = \frac{1}{8}\left(2\sin(2\alpha) - \sin(4\alpha)\right) \tag{A.45}$$

$$\sin^2\alpha\cos^2\alpha = \frac{1}{8}\left(1 - \cos(4\alpha)\right) \tag{A.46}$$

$$\sin\alpha\cos^3\alpha = \frac{1}{8}\left(2\sin(2\alpha) + \sin(4\alpha)\right) \tag{A.47}$$

$$\cos^4\alpha = \frac{1}{8}\left(3 + 4\cos(2\alpha) + \cos(4\alpha)\right) \tag{A.48}$$

$$\sin^5\alpha = \frac{1}{16}\left(10\sin\alpha - 5\sin(3\alpha) + \sin(5\alpha)\right) \tag{A.49}$$

$$\sin^4\alpha\cos\alpha = \frac{1}{16}\left(2\cos\alpha - 3\cos(3\alpha) + \cos(5\alpha)\right) \tag{A.50}$$

$$\sin^3\alpha\cos^2\alpha = \frac{1}{16}\left(2\sin\alpha + \sin(3\alpha) - \sin(5\alpha)\right) \tag{A.51}$$

$$\sin^2\alpha\cos^3\alpha = \frac{1}{16}\left(2\cos\alpha - 3\cos(3\alpha) - 5\cos(5\alpha)\right) \tag{A.52}$$

$$\sin\alpha\cos^4\alpha = \frac{1}{16}\left(2\sin\alpha + 3\sin(3\alpha) + \sin(5\alpha)\right) \tag{A.53}$$

$$\cos^5\alpha = \frac{1}{16}\left(10\cos\alpha + 5\cos(3\alpha) + \cos(5\alpha)\right) \tag{A.54}$$

$$\tan^2\alpha = \frac{1 - \cos(2\alpha)}{1 + \cos(2\alpha)} \tag{A.55}$$

Products of sin and cos

$$\cos\alpha\cos\beta = \frac{1}{2}\cos(\alpha-\beta) + \frac{1}{2}\cos(\alpha+\beta) \tag{A.56}$$

$$\sin\alpha\sin\beta = \frac{1}{2}\cos(\alpha-\beta) - \frac{1}{2}\cos(\alpha+\beta) \tag{A.57}$$

$$\sin\alpha\cos\beta = \frac{1}{2}\sin(\alpha-\beta) + \frac{1}{2}\sin(\alpha+\beta) \tag{A.58}$$

$$\cos\alpha\sin\beta = \frac{1}{2}\sin(\alpha+\beta) - \frac{1}{2}\sin(\alpha-\beta) \tag{A.59}$$

$$\sin(\alpha+\beta)\sin(\alpha-\beta) = \cos^2\beta - \cos^2\alpha = \sin^2\alpha - \sin^2\beta \tag{A.60}$$

$$\cos(\alpha+\beta)\cos(\alpha-\beta) = \cos^2\beta + \sin^2\alpha \tag{A.61}$$

Sum of Functions

$$\sin\alpha \pm \sin\beta = 2\sin\frac{\alpha\pm\beta}{2}\cos\frac{\alpha\pm\beta}{2} \tag{A.62}$$

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \tag{A.63}$$

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} \tag{A.64}$$

$$\sinh\alpha \pm \sinh\beta = 2\sinh\frac{\alpha\pm\beta}{2}\cosh\frac{\alpha\pm\beta}{2} \tag{A.65}$$

$$\cosh\alpha + \cosh\beta = 2\cosh\frac{\alpha+\beta}{2}\cosh\frac{\alpha-\beta}{2} \tag{A.66}$$

$$\cosh\alpha - \cosh\beta = -2\sinh\frac{\alpha+\beta}{2}\sinh\frac{\alpha-\beta}{2} \tag{A.67}$$

$$\tan\alpha \pm \tan\beta = \frac{\sin(\alpha\pm\beta)}{\cos\alpha\cos\beta} \tag{A.68}$$

$$\cot\alpha \pm \cot\beta = \frac{\sin(\beta\pm\alpha)}{\sin\alpha\sin\beta} \tag{A.69}$$

$$\frac{\sin\alpha+\sin\beta}{\sin\alpha-\sin\beta} = \frac{\tan\frac{\alpha+\beta}{2}}{\tan\frac{\alpha-+\beta}{2}} \tag{A.70}$$

$$\frac{\sin\alpha + \sin\beta}{\cos\alpha - \cos\beta} = \cot\frac{-\alpha + \beta}{2} \tag{A.71}$$

$$\frac{\sin\alpha + \sin\beta}{\cos\alpha + \cos\beta} = \tan\frac{\alpha + \beta}{2} \tag{A.72}$$

$$\frac{\sin\alpha - \sin\beta}{\cos\alpha + \cos\beta} = \tan\frac{\alpha - \beta}{2} \tag{A.73}$$

Trigonometric Relations

$$\sin^2\alpha - \sin^2\beta = \sin(\alpha + \beta)\sin(\alpha - \beta) \tag{A.74}$$

$$= \cos^2\beta - \cos^2\alpha \tag{A.75}$$

$$\cos^2\alpha - \cos^2\beta = -\sin(\alpha + \beta)\sin(\alpha - \beta) \tag{A.76}$$

$$\cos^2\alpha - \sin^2\beta = \cos(\alpha + \beta)\cos(\alpha - \beta) \tag{A.77}$$

$$= \cos^2\beta - \sin^2\alpha \tag{A.78}$$

Appendix B
Unit Conversions

General Conversion Formulas

$$\begin{aligned}
\text{N}^a\,\text{m}^b\,\text{s}^c &\approx 4.448^a \times 0.3048^b \times \text{lb}^a\,\text{ft}^b\,\text{s}^c \\
&\approx 4.448^a \times 0.0254^b \times \text{lb}^a\,\text{in}^b\,\text{s}^c \\
\text{lb}^a\,\text{ft}^b\,\text{s}^c &\approx 0.2248^a \times 3.2808^b \times \text{N}^a\,\text{m}^b\,\text{s}^c \\
\text{lb}^a\,\text{in}^b\,\text{s}^c &\approx 0.2248^a \times 39.37^b \times \text{N}^a\,\text{m}^b\,\text{s}^c
\end{aligned}$$

Conversion Factors

Acceleration

$$1\,\text{ft/s}^2 \approx 0.3048\,\text{m/s}^2 \quad 1\,\text{m/s}^2 \approx 3.2808\,\text{ft/s}^2$$

Angle

$$1\,\text{deg} \approx 0.01745\,\text{rad} \quad 1\,\text{rad} \approx 57.307\,\text{deg}$$

Area

$$\begin{array}{ll}
1\,\text{in}^2 \approx 6.4516\,\text{cm}^2 & 1\,\text{cm}^2 \approx 0.155\,\text{in}^2 \\
1\,\text{ft}^2 \approx 0.09290304\,\text{m}^2 & 1\,\text{m}^2 \approx 10.764\,\text{ft}^2 \\
1\,\text{acre} \approx 4046.86\,\text{m}^2 & 1\,\text{m}^2 \approx 2.471 \times 10^{-4}\,\text{acre} \\
1\,\text{acre} \approx 0.4047\,\text{hectare} & 1\,\text{hectare} \approx 2.471\,\text{acre}
\end{array}$$

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Damping

$$1\,\mathrm{N\,s/\,m} \approx 6.85218 \times 10^{-2}\,\mathrm{lb\,s/\,ft} \quad 1\,\mathrm{lb\,s/\,ft} \approx 14.594\,\mathrm{N\,s/\,m}$$
$$1\,\mathrm{N\,s/\,m} \approx 5.71015 \times 10^{-3}\,\mathrm{lb\,s/\,in} \quad 1\,\mathrm{lb\,s/\,in} \approx 175.13\,\mathrm{N\,s/\,m}$$

Energy and Heat

$$1\,\mathrm{Btu} \approx 1055.056\,\mathrm{J} \quad 1\,\mathrm{J} \approx 9.4782 \times 10^{-4}\,\mathrm{Btu}$$
$$1\,\mathrm{cal} \approx 4.1868\,\mathrm{J} \quad 1\,\mathrm{J} \approx 0.23885\,\mathrm{cal}$$
$$1\,\mathrm{kW\,h} \approx 3600\,\mathrm{kJ} \quad 1\,\mathrm{MJ} \approx 0.27778\,\mathrm{kW\,h}$$
$$1\,\mathrm{ft\,lbf} \approx 1.355818\,\mathrm{J} \quad 1\,\mathrm{J} \approx 0.737562\,\mathrm{ft\,lbf}$$

Force

$$1\,\mathrm{lb} \approx 4.448222\,\mathrm{N} \quad 1\,\mathrm{N} \approx 0.22481\,\mathrm{lb}$$

Fuel Consumption

$$1\ \ \mathrm{l/100\,km} \approx 235.214583\,\mathrm{mi/\,gal} \quad 1\,\mathrm{mi/\,gal} \approx 235.214583\,\mathrm{l/100\,km}$$
$$1\ \ \mathrm{l/100\,km} = 100\,\mathrm{km/\,l} \quad 1\,\mathrm{km/\,l} = 100\ \ \mathrm{l/100\,km}$$
$$1\,\mathrm{mi/\,gal} \approx 0.425144\,\mathrm{km/\,l} \quad 1\,\mathrm{km/\,l} \approx 2.352146\,\mathrm{mi/\,gal}$$

Length

$$1\,\mathrm{in} \approx 25.4\,\mathrm{mm} \quad 1\,\mathrm{cm} \approx 0.3937\,\mathrm{in}$$
$$1\,\mathrm{ft} \approx 30.48\,\mathrm{cm} \quad 1\,\mathrm{m} \approx 3.28084\,\mathrm{ft}$$
$$1\,\mathrm{mi} \approx 1.609347\,\mathrm{km} \quad 1\,\mathrm{km} \approx 0.62137\,\mathrm{mi}$$

Mass

$$1\,\mathrm{lb} \approx 0.45359\,\mathrm{kg} \quad 1\,\mathrm{kg} \approx 2.204623\,\mathrm{lb}$$
$$1\,\mathrm{slug} \approx 14.5939\,\mathrm{kg} \quad 1\,\mathrm{kg} \approx 0.068522\,\mathrm{slug}$$
$$1\,\mathrm{slug} \approx 32.174\,\mathrm{lb} \quad 1\,\mathrm{lb} \approx 0.03.1081\,\mathrm{slug}$$

Moment and Torque

$$1\,\mathrm{lb\,ft} \approx 1.35582\,\mathrm{N\,m} \quad 1\,\mathrm{N\,m} \approx 0.73746\,\mathrm{lb\,ft}$$
$$1\,\mathrm{lb\,in} \approx 8.85075\,\mathrm{N\,m} \quad 1\,\mathrm{N\,m} \approx 0.11298\,\mathrm{lb\,in}$$

Mass Moment

$$1\,\mathrm{lb\,ft}^2 \approx 0.04214\,\mathrm{kg\,m}^2 \quad 1\,\mathrm{kg\,m}^2 \approx 23.73\,\mathrm{lb\,ft}^2$$

Power

$1\,\text{Btu/h} \approx 0.2930711\,\text{W}$	$1\,\text{W} \approx 3.4121\,\text{Btu/h}$
$1\,\text{hp} \approx 745.6999\,\text{W}$	$1\,\text{kW} \approx 1.341\,\text{hp}$
$1\,\text{hp} \approx 550\,\text{lb ft/s}$	$1\,\text{lb ft/s} \approx 1.8182 \times 10^{-3}\,\text{hp}$
$1\,\text{lb ft/h} \approx 3.76616 \times 10^{-4}\,\text{W}$	$1\,\text{W} \approx 2655.2\,\text{lb ft/h}$
$1\,\text{lb ft/min} \approx 2.2597 \times 10^{-2}\,\text{W}$	$1\,\text{W} \approx 44.254\,\text{lb ft/min}$

Pressure and Stress

$1\,\text{lb/in}^2 \approx 6894.757\,\text{Pa}$	$1\,\text{MPa} \approx 145.04\,\text{lb/in}^2$
$1\,\text{lb/ft}^2 \approx 47.88\,\text{Pa}$	$1\,\text{Pa} \approx 2.0886 \times 10^{-2}\,\text{lb/ft}^2$
$1\,\text{Pa} \approx 0.00001\,\text{atm}$	$1\,\text{atm} \approx 101325\,\text{Pa}$

Stiffness

$1\,\text{N/m} \approx 6.85218 \times 10^{-2}\,\text{lb/ft}$	$1\,\text{lb/ft} \approx 14.594\,\text{N/m}$
$1\,\text{N/m} \approx 5.71015 \times 10^{-3}\,\text{lb/in}$	$1\,\text{lb/in} \approx 175.13\,\text{N/m}$

Temperature

$$^\circ\text{C} = (^\circ\text{F} - 32)/1.8$$
$$^\circ\text{F} = 1.8\,^\circ\text{C} + 32$$

Velocity

$1\,\text{mi/h} \approx 1.60934\,\text{km/h}$	$1\,\text{km/h} \approx 0.62137\,\text{mi/h}$
$1\,\text{mi/h} \approx 0.44704\,\text{m/s}$	$1\,\text{m/s} \approx 2.2369\,\text{mi/h}$
$1\,\text{ft/s} \approx 0.3048\,\text{m/s}$	$1\,\text{m/s} \approx 3.2808\,\text{ft/s}$
$1\,\text{ft/min} \approx 5.08 \times 10^{-3}\,\text{m/s}$	$1\,\text{m/s} \approx 196.85\,\text{ft/min}$

Volume

$1\,\text{in}^3 \approx 16.39\,\text{cm}^3$	$1\,\text{cm}^3 \approx 0.0061013\,\text{in}^3$
$1\,\text{ft}^3 \approx 0.02831685\,\text{m}^3$	$1\,\text{m}^3 \approx 35.315\,\text{ft}^3$
$1\,\text{gal} \approx 3.785\,\text{l}$	$1\,\text{l} \approx 0.2642\,\text{gal}$
$1\,\text{gal} \approx 3785.41\,\text{cm}^3$	$1\,\text{l} \approx 1000\,\text{cm}^3$

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